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On some mathematical problems in fracture dynamics

Candidate:
Maicol Caponi

Advisor:
Prof. Gianni Dal Maso

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Contents

Introduction

This thesis is devoted to the study of several mathematical problems in fracture mechanics for brittle materials. The main ingredient to develop a reasonable model of these phenomena is Griffith's criterion, originally formulated in [31] for the quasi-static setting, namely when the external data vary slowly compared to the elastic wave speed of the material. In this case, Griffith states there is an exact balance between the decrease of elastic energy during the evolution, and the energy used to increase the crack, which is assumed to be proportional to its area.

In sharp-interface models, i.e., when the crack is identified with the discontinuity surface of the displacement in the reference configuration, this principle was turned into a precise definition by Francfort and Marigo in [26]. In the context of small-strain antiplane shear, the following energy functional of Mumford-Shah's type is considered:

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u(x)|^2\,\mathrm{d}x+\mathcal{H}^{d-1}(\Gamma). \tag{1}$$

Here, $\Omega\subset\mathbb{R}^d$ (with $d=2$ being the physically relevant case) is an open bounded set with Lipschitz boundary, which represents the reference configuration of the elastic material, the closed set $\Gamma\subset\overline{\Omega}$ describes the crack, and $u\in H^1(\Omega\setminus\Gamma)$ is the antiplane displacement. The first term represents the stored elastic energy of a homogeneous and isotropic material, while the second one, called surface energy, models the energy used to produce the crack. Here and henceforth all physical constants are normalized to 1.

In this setting, given a time-dependent Dirichlet datum $t\mapsto w(t)$ acting on $\partial\Omega$, a quasi-static evolution is a pair $t\mapsto(u(t),\Gamma_t)$ which at every time t minimizes (1) among all pairs (u^*,Γ^*), where Γ^* is a closed set with $\Gamma^*\supseteq\Gamma_t$, and $u^*\in H^1(\Omega\setminus\Gamma^*)$ with $u^*=w(t)$ on $\partial\Omega\setminus\Gamma^*$. The minimum problem is complemented with the irreversibility condition $\Gamma_s\subseteq\Gamma_t$ for every $s\le t$ (meaning the crack can only increase in time), and with an energy-dissipation balance for every time.

The first rigorous existence result for quasi-static evolutions was due to Dal Maso and Toader in [22] in dimension $d=2$, and with a restriction on the number of connected components of the crack set. Later, Francfort and Larsen in [27] removed these assumptions, by setting the problem in the space SBV of special functions with bounded variation, introduced by De Giorgi and Ambrosio in [25]. More in general, in the context of linear elasticity, the displacement $u\colon\Omega\setminus\Gamma\to\mathbb{R}^d$ is vector-valued, and the term $|\nabla u|^2$ in the elastic energy is replaced by $\mathbb{C}Eu\cdot Eu$, where Eu is the symmetric part of the gradient, namely $Eu:=\frac{1}{2}(\nabla u+\nabla u^T)$, and $\mathbb{C}$ is the elastic tensor. Existence results for quasi-static evolutions in linear elasticity can be found only in dimension $d=2$, see [11] (which works under the same geometric restrictions of [22]) and [29] (for the general case). For related results in dimension $d>2$ we refer to [12, 13]. A detailed analysis of variational models of quasi-static fracture can be found in [5] and in the references therein.

In this thesis, we study several mathematical problems in fracture dynamics. In this setting, the stationarity condition for the displacement has to be replaced by the fact that u solves the elastodynamics system out of the crack, while the crack evolves according to a dynamic version of Griffith's criterion, see [41, 28]. Therefore, any reasonable mathematical model should follow the following principles:

(a) *elastodynamics*: away from the crack set, the displacement u evolves according to the elastodynamics system;

(b) *irreversibility*: the time-dependent crack $t \mapsto \Gamma_t$ is increasing in time with respect to inclusion ($\Gamma_s \subseteq \Gamma_t$ for every $s \leq t$);

(c) *dynamic energy-dissipation balance*: the work done by external forces is balanced by the mechanical energy (sum of kinetic and elastic energy) and the energy dissipated to create a crack;

(d) *maximal dissipation*: if the crack can propagate while balancing energy, then it should propagate.

The last condition, introduced in [35], is needed because a time-independent crack always satisfies the first three principles. So far, it was not possible to prove the existence of a solution for a model satisfying (a)–(d) without stronger a priori regularity conditions on the cracks and their evolutions, which have not mechanical justifications. Some models for a peeling test in dimension 1, based on similar principles, have been recently analyzed in detail in [19, 47], obtaining existence and uniqueness results without a priori regularity assumptions.

The contents of the thesis are organized into four chapters.

Chapter 1: Elastodynamics system in domains with growing cracks

According to the principles stated before, a first step to study the dynamic crack propagation in an elastic material is to solve the elastodynamics system for a prescribed time-dependent crack $\{\Gamma_t\}_{t\in[0,T]}$ satisfying the irreversibility condition. From the mathematical point of view, this means solving a hyperbolic-type system of the form

$$\ddot{u}(t,x) - \operatorname{div}(\mathbb{C}(t,x)Eu(t,x)) = f(t,x) \quad t \in [0,T],\ x \in \Omega \setminus \Gamma_t, \tag{2}$$

supplemented by boundary and initial conditions; the main difficulty in such a problem is that the domain $\Omega \setminus \Gamma_t$ depends on time. In the literature, we can find several different approaches to hyperbolic systems in time-dependent domains. The first one is developed in [16] for the antiplane case, i.e., for the wave equation

$$\ddot{u}(t,x) - \Delta u(t,x) = f(t,x) \quad t \in [0,T],\ x \in \Omega \setminus \Gamma_t, \tag{3}$$

with homogeneous Neumann conditions on the boundary of $\Omega \setminus \Gamma_t$. The existence of a solution with assigned initial data is proved by using a time-discrete approximation and passing to the limit as the time step tends to zero. This construction leads to an existence result under very weak conditions on the cracks $\{\Gamma_t\}_{t\in[0,T]}$. A generalization of this construction to the vector-valued case can be found in the recent work [52]. Unfortunately, the uniqueness of these solutions is still an open problem in this setting.

In [43, 20] the authors use a different technique to study (3), which is based on a suitable change of variables of class C^2 that maps the domain $\{(t,x) \in [0,T] \times \Omega : x \in \Omega \setminus \Gamma_t\}$ into the cylinder $[0,T] \times (\Omega \setminus \Gamma_0)$. In this way, the wave equation (3) is transformed into a new hyperbolic equation in a fixed domain, with coefficients depending also on the change of variables. This method allows proving the existence and uniqueness of a solution, as well as a continuous dependence result. Nevertheless, such a change of variable can be constructed only under very strong regularity assumptions on the cracks (see [20]).

Finally, a third possible approach to the study of (3) can be found in a very recent paper [15]. In this case, existence is proved by means on a suitable approximation of the wave equation via minima of convex functionals (see [48, 49]).

In Chapter 1 (which contains the results of [8]) we study the existence, uniqueness, and continuous dependence on the data for the solutions of the elastodynamics system (2). On the cracks $\{\Gamma_t\}_{t\in[0,T]}$ we assume the irreversibility condition and that they are contained in a given C^2 manifold Γ of dimension $d-1$. The system (2) is supplemented by mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$ and by homogeneous Neumann boundary conditions on the cracks Γ_t (*traction free* case).

These results are obtained by adapting the change of variable method introduced in [20] to the vector-valued case. Since the operator $\mathbb{C}$ is usually defined only on the subspace of symmetric matrices, it is convenient to introduce a new operator $\mathbb{A}$ defined on all $\mathbb{R}^{d\times d}$ as

$$\mathbb{A}(t,x)\xi := \mathbb{C}(t,x)\xi^{sym} \quad \text{for every } \xi \in \mathbb{R}^{d\times d},$$

where ξ^{sym} denotes the symmetric part of ξ. In this way, system (2) can be rephrased as

$$\ddot{u}(t,x) - \operatorname{div}(\mathbb{A}(t,x)\nabla u(t,x)) = f(t,x) \quad t\in[0,T],\, x\in\Omega\setminus\Gamma_t, \tag{4}$$

and the change of variable approach leads to the transformed system

$$\begin{aligned} &\ddot{v}(t,y) - \operatorname{div}(\mathbb{B}(t,y)\nabla v(t,y)) \\ &\quad + \mathbf{p}(t,y)\nabla v(t,y) + \nabla\dot{v}(t,y)b(t,y) = g(t,y) \quad t\in[0,T],\, y\in\Omega\setminus\Gamma_0, \end{aligned} \tag{5}$$

where the new coefficients $\mathbb{B}$, $\mathbf{p}$, b, and g are constructed starting from $\mathbb{A}$ and f. The boundary conditions are also transformed by the change of variables and lead to mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$, and homogeneous Neumann boundary conditions on the fixed crack Γ_0.

The main changes with respect to the paper [20] are in the treatment of the terms involving $\mathbb{B}$. Indeed, in linear elasticity, the natural ellipticity condition on $\mathbb{A}$ is the following:

$$\mathbb{A}(t,x)\xi\cdot\xi \geq \lambda_0|\xi^{sym}|^2 \quad \text{for every } \xi\in\mathbb{R}^{d\times d}, \tag{6}$$

with λ_0 positive constant. Unfortunately, this condition is not inherited by the transformed operator $\mathbb{B}$. To overcome this difficulty, we assume that $\mathbb{B}$ satisfies a weaker ellipticity assumption of integral type (see (1.2.1)), which always holds when $\mathbb{A}$ satisfies (6) and the velocity of the time-dependent diffeomorphisms used in the change of variables is sufficiently small (see (1.2.4)). Another difference with respect to [20] is that we consider also the case of non-homogeneous Neumann boundary conditions on the Neumann part of $\partial\Omega$. This completes the study of [20], including the case of traction forces acting on the boundary.

We first prove the existence and uniqueness of solutions to (5), with assigned initial and boundary conditions. Moreover, we prove an energy equality (see (1.3.2)), which is slightly different from the one in [20], and takes into account the non-homogeneous boundary terms. This energy balance allows us to prove suitable continuity properties with respect to time for the solutions v, which are important in the proof of the main existence result for (4).

Finally, in the last part, we prove the continuous dependence of the solutions on the cracks $\{\Gamma_t\}_{t\in[0,T]}$ and on the manifold Γ. More precisely, given a sequence Γ^n, of manifolds and a sequence $\{\Gamma^n_t\}_{t\in[0,T]}$ of time-dependent cracks contained in Γ^n, we use the energy equality (1.3.2) to prove that, under appropriate convergence conditions, the solutions u^n and v^n to problems (4) and (5) corresponding to $\{\Gamma^n_t\}_{t\in[0,T]}$ converge to the solutions u and v of the limit problems corresponding to $\{\Gamma_t\}_{t\in[0,T]}$.

These results have been used in [18] to prove an existence theorem for a model in fracture dynamics based on (a)–(d) with suitable conditions on the regularity of the cracks.

Chapter 2: Dynamic energy-dissipation balance of a growing crack

Once we are able to solve the elastodynamics system on $\Omega \setminus \Gamma_t$ under suitable assumptions on $\{\Gamma_t\}_{t\in[0,T]}$, the next step towards the solution to the dynamic fracture problem, according to (a)–(d), is to select those cracks $\{\Gamma_t\}_{t\in[0,T]}$ such that the corresponding solutions u satisfy the dynamic energy-dissipation balance.

In Chapter 2 (which contains the results of [9], obtained in collaboration with I. Lucardesi and E. Tasso) we compute the mechanical energy (kinetic + elastic) of the solution corresponding to a sufficiently regular crack evolution $\{\Gamma_t\}_{t\in[0,T]}$ in the antiplane case. We consider as reference configuration a bounded open set Ω of $\mathbb{R}^2$ with Lipschitz boundary and we assume that all the cracks Γ_t are contained in a fixed $C^{3,1}$ curve $\Gamma \subset \overline{\Omega}$ with endpoints on $\partial\Omega$. In this case, Γ_t is determined at time t by the crack-tip position on Γ, described by the arc-length parameter $s(t)$. Here we assume $t \mapsto s(t)$ non-decreasing (irreversibility assumption) and of class $C^{3,1}([0,T])$. Far from the crack set, the displacement u satisfies a wave equation of the form

$$\ddot{u}(t,x) - \operatorname{div}(A(x)\nabla u(t,x)) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t, \tag{7}$$

where A is a suitable matrix field satisfying the usual ellipticity condition. The equation is supplemented by homogeneous mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$, homogeneous Neumann boundary conditions on Γ_t, and initial conditions.

The mechanical energy associated with u at time t is given by

$$\mathcal{E}(t) := \frac{1}{2}\int_{\Omega\setminus\Gamma_t} |\dot{u}(t,x)|^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega\setminus\Gamma_t} A(x)\nabla u(t,x)\cdot\nabla u(t,x)\,\mathrm{d}x. \tag{8}$$

The difficulty of computing (8) is twofold: on one hand, the displacement has a singular behavior near the crack-tip; moreover, the domain of $u(t)$ contains a crack and varies in time. To handle the first issue, a representation result for u is in order: under suitable conditions on the initial data (see Theorems 2.2.4 and 2.2.10) we prove that for every time t the displacement is of class H^1 in a neighborhood of the tip of Γ_t and of class H^2 far from it, namely $u(t)$ is of the form

$$u(t,x) = u^R(t,x) + k(t)\zeta(t,x)S(\Phi(t,x)) \quad x \in \Omega \setminus \Gamma_t, \tag{9}$$

where $u^R(t) \in H^2(\Omega \setminus \Gamma_t)$, $k(t) \in \mathbb{R}$, $\zeta(t)$ is a cut-off function supported in a neighborhood of the moving tip of Γ_t, $S \in H^1(\mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\})$, and $\Phi(t)$ is a diffeomorphism of Ω which maps the tip of Γ_t into the origin. Once fixed ζ, S, and Φ, the function u^R and the constant k are uniquely determined. Actually, the coefficient k only depends on A, Γ, and s. In addition, we provide another decomposition for u which is more explicit and better explains the behavior of the singular part (see Theorem 2.2.10).

The second issue is technical and we overcome it exploiting Geometric Measure Theory techniques. The computation leads to the following formula:

$$\mathcal{E}(t) + \frac{\pi}{4}\int_0^t k^2(\tau)a(\tau)\dot{s}(\tau)\,\mathrm{d}\tau = \mathcal{E}(0) + \int_0^t\int_{\Omega\setminus\Gamma_\tau} f(\tau,x)\dot{u}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \tag{10}$$

for every t, where a is a positive function which depends on A, Γ, and s, and is equal to 1 when A is the identity matrix; see Theorem 2.3.7 for the proof of (10) when $A = Id$, and Remark 2.3.9 for the general case. We compare it with the dynamic energy-dissipation balance, which in this case reads

$$\mathcal{E}(t) + \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \int_0^t\int_{\Omega\setminus\Gamma_\tau} f(\tau,x)\dot{u}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \tag{11}$$

for every t. We deduce that (11) is satisfied if and only if at every time t in which the crack is moving, namely when $\dot{s}(t) > 0$, the function $k(t)$, often called *dynamic stress intensity factor*, is equal to $2/\sqrt{\pi a(t)}$.

We mention that a similar result for a horizontal crack $\Gamma_t := \overline{\Omega} \cap \{(\sigma, 0) \in \mathbb{R}^2 : \sigma \leq ct\}$ moving with constant velocity c (with a suitable boundary datum) can be found in the paper [17, Section 4]. The representation result stated in (9) extends the one of [43] valid for straight cracks and A the identity matrix. Here we adapt their proof to the case of a curved crack and a constant (in time) matrix A, possibly depending on x; moreover, we remove a restrictive assumption on the acceleration $\ddot{s}$.

The main steps in the proof of (10) are the following: by performing four changes of variables, we reduce problem to a second order PDE of the form

$$\ddot{v}(t,x) - \operatorname{div}(\tilde{A}(t,x)\nabla v(t,x)) + l.o.t. = \tilde{f}(t,x) \quad t \in [0,T],\, x \in \tilde{\Omega} \setminus \tilde{\Gamma}_0, \tag{12}$$

with $\tilde{\Omega}$ Lipschitz planar domain and $\tilde{\Gamma}_0$ a $C^{3,1}$ curve which is straight near its tip. The matrix field $\tilde{A}$ has time-dependent coefficients, but at the tip of $\tilde{\Gamma}_0$ it is constantly equal to the identity. Finally, the decomposition result for the solution v to (12), obtained via semi-group theory, leads to (9) for u, the solution to the original problem.

Chapter 3: A dynamic model for viscoelastic materials with growing cracks

When we want to study the dynamic evolution of deformed materials with viscoelastic properties, Kelvin-Voigt's model is the most common one. If no crack is present, this leads in the antiplane case to the damped wave equation

$$\ddot{u}(t,x) - \Delta u(t,x) - \Delta \dot{u}(t,x) = f(t,x) \quad (t,x) \in [0,T] \times \Omega.$$

As it is well known, the solutions to this equation satisfy the dynamic energy-dissipation balance

$$\mathcal{E}(t) + \int_0^t \int_\Omega |\nabla \dot{u}(\tau,x)|^2 \mathrm{d}x\, \mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces}$$

for every t, where in this case $\mathcal{E}(t) := \frac{1}{2}\int_\Omega |\dot{u}(t,x)|^2 \mathrm{d}x + \int_\Omega |\nabla u(t,x)|^2 \mathrm{d}x$. If we consider also the presence of a crack in the viscoelastic material, the damped wave equation becomes

$$\ddot{u}(t,x) - \Delta u(t,x) - \Delta \dot{u}(t,x) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t, \tag{13}$$

and in this case, the dynamic energy-dissipation balance reads

$$\mathcal{E}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) + \int_0^t \int_{\Omega \setminus \Gamma_\tau} |\nabla \dot{u}(\tau,x)|^2 \mathrm{d}x\, \mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces}. \tag{14}$$

For a prescribed crack evolution, this model was already considered by [16] in the antiplane case, and more in general by [52] for the vector-valued case. As proved in the quoted papers, the solutions to (13) satisfy

$$\mathcal{E}(t) + \int_0^t \int_{\Omega \setminus \Gamma_\tau} |\nabla \dot{u}(\tau,x)|^2 \mathrm{d}x\, \mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces}$$

for every time t. This equality implies that (14) cannot be satisfied unless $\Gamma_t = \Gamma_0$ for every t, which means that the crack is not allowed to increase in time. This phenomenon was already well known in mechanics as the viscoelastic paradox, see for instance [51, Chapter 7].

To overcome this problem, in Chapter 3 (which contains the results of [10], obtained in collaboration with F. Sapio) we modify Kelvin-Voigt's model by considering a possibly degenerate viscosity term depending on t and x. More precisely, we study the following equation

$$\ddot{u}(t,x) - \Delta u(t,x) - \operatorname{div}(\Theta^2(t,x)\nabla\dot{u}(t,x)) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t. \tag{15}$$

On the function $\Theta\colon (0,T)\times\Omega \to \mathbb{R}$ we only require some regularity assumptions; a particularly interesting case is when Θ assumes the value zero on some points of Ω, which means that the material has no longer viscoelastic properties in such a zone.

The main result of Chapter 3 is Theorem 3.2.1 (see also Remark 3.3.4), in which we show the existence of a solution to (15) and, more in general, to the analogous problem for the d-dimensional linear elasticity. To this aim, we first perform a time discretization in the same spirit of [16], and then we pass to the limit as the time step goes to zero by relying on energy estimates. As a byproduct, we derive an energy-dissipation inequality (see (3.3.4)), which is used to prove the validity of the initial conditions. By using the change of variables method implemented in [43, 20], we also prove a uniqueness result, but only in dimension $d = 2$ and when $\Theta(t)$ vanishes on a neighborhood of the tip of Γ_t.

We complete the chapter by providing an example in $d = 2$ of a solution to (15) for which the crack can grow while balancing the energy. As we remarked before, this cannot happen for the Kelvin-Voigt's model. More precisely, when the crack Γ_t moves with constant speed along the x_1-axis and $\Theta(t)$ is zero in a neighborhood of the crack-tip, we construct a function u which solves (15) and satisfies

$$\mathcal{E}(t) + \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) + \int_0^t \int_{\Omega\setminus\Gamma_\tau} |\Theta(\tau,x)\nabla\dot{u}(\tau,x)|^2 \mathrm{d}x\,\mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces} \tag{16}$$

for every time t. Notice that (16) is the natural formulation of the dynamic energy-dissipation balance in this setting.

Chapter 4: A phase-field model of dynamic fracture

An alternative approach to the study of crack evolution is based on the so-called phase-field model, which relies on the Ambrosio-Tortorelli's approximation of the energy functional (1). According to Ambrosio and Tortorelli [4], the $(d-1)$-dimensional set Γ is replaced by a phase-field variable $v_\varepsilon\colon \Omega \to [0,1]$ which is close to 0 in an ε-neighborhood of Γ, and close to 1 away from it. Accordingly, the Griffith's functional (1) is replaced by the ε-dependent elliptic functionals

$$\mathscr{E}_\varepsilon(u,v) + \mathscr{H}_\varepsilon(v)$$

for $u, v \in H^1(\Omega)$, where

$$\mathscr{E}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega [(v(x))^2 + \eta_\varepsilon]\,|\nabla u(x)|^2\,\mathrm{d}x,$$
$$\mathscr{H}_\varepsilon(v) := \frac{1}{4\varepsilon}\int_\Omega |1 - v(x)|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla v(x)|^2\,\mathrm{d}x,$$

with $0 < \eta_\varepsilon \ll \varepsilon$. A minimum point $(u_\varepsilon, v_\varepsilon)$ of $\mathscr{E}_\varepsilon + \mathscr{H}_\varepsilon$ provides a good approximation of a minimizer (u,Γ) of (1) as $\varepsilon \to 0^+$, in the sense that u_ε is close to u, v_ε is close to 0 near Γ, and $\mathscr{E}_\varepsilon(u_\varepsilon, v_\varepsilon) + \mathscr{H}_\varepsilon(v_\varepsilon)$ approximates the energy (1). For the corresponding quasi-static evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$, the minimality condition for (1) is replaced by

$$\mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t)) \le \mathscr{E}_\varepsilon(u^*, v^*) + \mathscr{H}_\varepsilon(v^*) \tag{17}$$

among every pair (u^*, v^*) with $v^* \le v_\varepsilon(t)$ and $u^* = w(t)$ on $\partial\Omega$. Notice that the inequality $v^* \le v_\varepsilon(t)$ reflects the inclusion $\Gamma^* \supseteq \Gamma_t$. As before, the minimum problem (17) is complemented with the irreversibility condition $0 \le v_\varepsilon(t) \le v_\varepsilon(s) \le 1$ for every $s \le t$, and with the energy-dissipation balance for every time; we refer to [30] for the convergence of this evolution, as $\varepsilon \to 0^+$, toward the sharp-interface one described at the beginning of the introduction.

In particular, a quasi-static phase-field evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ satisfies:

(Q_1) for every $t \in [0,T]$ the function $u_\varepsilon(t)$ solves $\operatorname{div}([(v_\varepsilon(t))^2 + \eta_\varepsilon]\nabla u_\varepsilon(t)) = 0$ in Ω with suitable boundary conditions;

(Q_2) the map $t \mapsto v_\varepsilon(t)$ is non-increasing ($v_\varepsilon(t) \le v_\varepsilon(s)$ for $0 \le s \le t \le T$) and for every $t \in [0,T]$ the function $v_\varepsilon(t)$ satisfies

$$\mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t)) \le \mathscr{E}_\varepsilon(u_\varepsilon(t), v^*) + \mathscr{H}_\varepsilon(v^*)$$

for every $v^* \le v_\varepsilon(t)$;

(Q_3) for every $t \in [0,T]$ the pair $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies the energy-dissipation balance

$$\mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t)) = \mathscr{E}_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + \mathscr{H}_\varepsilon(v_\varepsilon(0)) + \text{work of external data.}$$

As explained before for the sharp interface model, in the dynamic case the first condition is replaced by the wave equation, while in the energy balance we need to take into account the kinetic energy term. Developing these principles, in [6, 35, 36] the authors propose the following phase-field model of dynamic crack propagation in linear elasticity:

(D_1) u_ε solves $\ddot u_\varepsilon - \operatorname{div}([v_\varepsilon^2 + \eta_\varepsilon]\mathbb{C}Eu_\varepsilon) = 0$ in $(0,T)\times\Omega$ with suitable boundary and initial conditions;

(D_2) the map $t \mapsto v_\varepsilon(t)$ is non-increasing and for every $t \in [0,T]$ the function $v_\varepsilon(t)$ solves

$$\mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t)) \le \mathscr{E}_\varepsilon(u_\varepsilon(t), v^*) + \mathscr{H}_\varepsilon(v^*) \quad \text{for every } v^* \le v_\varepsilon(t);$$

(D_3) for every $t \in [0,T]$ the pair $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies the dynamic energy-dissipation balance

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\dot u_\varepsilon(t)|^2 \mathrm{d}x + \mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t))\\ &= \frac{1}{2}\int_\Omega |\dot u_\varepsilon(0)|^2 \mathrm{d}x + \mathscr{E}_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + \mathscr{H}_\varepsilon(v_\varepsilon(0)) + \text{work of external data,}\end{aligned}$$

where $\mathscr{E}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega [((v(x))^2 + \eta_\varepsilon]\mathbb{C}(x)Eu(x)\cdot Eu(x)\,\mathrm{d}x$ for $u \in H^1(\Omega;\mathbb{R}^d)$ and $v \in H^1(\Omega)$. A solution to this model is approximated by means of a time discretization with an alternate scheme: to pass from the previous time to the next one, one first solves the wave equation for u, keeping v fixed, and then a minimum problem for v, keeping u fixed. This method is used in [36] to prove the existence of a pair (u,v) satisfying (D_1)–(D_3). For technical reasons, a viscoelastic dissipative term is added to (D_1), which means that in [36] the following system is considered

$$\ddot u_\varepsilon - \operatorname{div}([v_\varepsilon^2 + \eta_\varepsilon]\mathbb{C}(Eu_\varepsilon + E\dot u_\varepsilon)) = 0 \quad \text{in } (0,T)\times\Omega.$$

The disadvantage of this term appears when we consider the behavior of the solution as $\varepsilon \to 0^+$, a problem which is out of the scope of this thesis. If we were able to prove the convergence of the solution toward a dynamic sharp-interface evolution, then the dynamic energy-dissipation balance for the damped wave equation in cracked domains of [16, 52] would imply that the limit crack does not depend on time, as explained above in the section about viscoelastic materials.

To avert this problem, in Chapter 4 (which contains the results of [7]) we propose a different model that avoids viscoelastic terms depending on the displacement and consider instead a dissipative term related to the speed of the crack-tips. More precisely, given a natural number $k \in \mathbb{N}\cup\{0\}$, we consider a dynamic phase-field evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ satisfying:

($\tilde{D}_1$) u_ε solves $\ddot{u}_\varepsilon - \operatorname{div}([(v_\varepsilon \vee 0)^2 + \eta_\varepsilon]\mathbb{C}Eu_\varepsilon) = 0$ in $(0,T)\times\Omega$ with suitable boundary and initial conditions;

($\tilde{D}_2$) the map $t \mapsto v_\varepsilon(t)$ is non-increasing and for a.e. $t \in (0,T)$ the function $v_\varepsilon(t)$ solves the variational inequality

$$\mathscr{E}_\varepsilon(u_\varepsilon(t), v^*) - \mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v^*) - \mathscr{H}_\varepsilon(v_\varepsilon(t)) + (\dot{v}_\varepsilon(t), v^* - v_\varepsilon(t))_{H^k(\Omega)} \geq 0$$

for every $v^* \leq v_\varepsilon(t)$;

($\tilde{D}_3$) for every $t \in [0,T]$ the pair $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies the dynamic energy-dissipation balance

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\dot{u}_\varepsilon(t)|^2 \mathrm{d}x + \mathscr{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathscr{H}_\varepsilon(v_\varepsilon(t)) + \int_0^t \|\dot{v}_\varepsilon(\tau)\|^2_{H^k(\Omega)} \mathrm{d}\tau \\ &= \frac{1}{2}\int_\Omega |\dot{u}_\varepsilon(0)|^2 \mathrm{d}x + \mathscr{E}_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + \mathscr{H}_\varepsilon(v_\varepsilon(0)) + \text{work of external data,} \end{aligned} \tag{18}$$

where in this case $\mathscr{E}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega [(v(x) \vee 0)^2 + \eta_\varepsilon]\mathbb{C}(x)Eu(x) \cdot Eu(x)\,\mathrm{d}x$. Notice that for technical reasons the dissipative term $\int_0^t \|\dot{v}_\varepsilon(\tau)\|^2_{H^k(\Omega)} \mathrm{d}\tau$ contains the norm in the Sobolev space $H^k(\Omega)$, rather then the norm in $L^2(\Omega)$, which is more frequently used in the literature. This choice guarantees more regularity in time for the phase-field function, more precisely that $v_\varepsilon \in H^1(0,T;H^k(\Omega))$.

In the quasi-static setting, a condition similar to ($\tilde{D}_2$) can be found in [42, 2], where it defines a unilateral gradient flow evolution for the phase-field function v_ε. In sharp-interface models, a crack-dependent term analogous to $\int_0^t \|\dot{v}_\varepsilon(\tau)\|^2_{H^k(\Omega)} \mathrm{d}\tau$ arises in the study of the so-called vanishing viscosity evolutions, which are linked to the analysis of local minimizers of Griffith's functional (1), see for example [46, 39]. We point out that a similar dissipative term also appears in [37] for a 1-dimensional debonding model.

By adapting the time discretization scheme of [6, 36], we show the existence of a dynamic phase-field evolution $(u_\varepsilon, v_\varepsilon)$ which satisfies ($\tilde{D}_1$)–($\tilde{D}_3$), provided that $k > d/2$, where d is the dimension of the ambient space. This condition is crucial to obtain the validity of the dynamic energy-dissipation balance since in our case the viscoelastic dissipative term used in [36] is not present.

We conclude Chapter 4 by analyzing the dynamic phase-field model (D_1)–(D_3) with no viscous terms. We show the existence of an evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ which satisfies (D_1) and (D_2), but only an energy-dissipation inequality (see (4.4.7)) instead of (D_3).

Notation

Basic notation. The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m = d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$, and the subspace of orthogonal $d \times d$ matrices with determinant equal to 1 by $SO(d)$. We denote by A^T and A^{-1}, respectively, the transpose and the inverse of $A \in \mathbb{R}^{d\times d}$, by A^{-T} the transpose of the inverse, and by A^{sym} the symmetric part, namely $A^{sym} := \frac{1}{2}(A + A^T)$; we use Id to denote the identity matrix in $\mathbb{R}^{d\times d}$. The Euclidian scalar product in $\mathbb{R}^d$ is denoted by $\cdot$ and the corresponding Euclidian norm by $|\cdot|$; the same notation is used also for $\mathbb{R}^{m\times d}$. We denote by $a \otimes b \in \mathbb{R}^{d\times d}$ the tensor product between two vectors $a, b \in \mathbb{R}^d$, and by $a \odot b \in \mathbb{R}^{d\times d}_{sym}$ the symmetrized tensor product, namely the symmetric part of $a \otimes b$.

The d-dimensional Lebesgue measure in $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$, and the $(d-1)$-dimensional Hausdorff measure by $\mathcal{H}^{d-1}$. Given a bounded open set Ω with Lipschitz boundary, we denote by ν the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$-a.e. on the boundary. We use $B_r(x)$ to denote the ball of radius r and center x in $\mathbb{R}^d$, namely $B_r(x) := \{y \in \mathbb{R}^d : |y-x| < r\}$, and id to denote the identity function in $\mathbb{R}^d$, possibly restricted to a subset. Given two numbers $c_1, c_2 \in \mathbb{R}$, we set $c_1 \vee c_2 := \max\{c_1, c_2\}$ and $c_1 \wedge c_2 := \min\{c_1, c_2\}$.

The partial derivatives with respect to the variable x_i are denoted by ∂_i or ∂_{x_i}. Given a function $u\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by ∇u, whose components are $(\nabla u)_{ij} := \partial_j u_i$ for $i = 1, \dots, m$ and $j = 1, \dots, d$. When $u\colon \mathbb{R}^d \to \mathbb{R}^d$, we use Eu to denote its symmetryzed gradient, namely $Eu := \frac{1}{2}(\nabla u + \nabla u^T)$. Given $u\colon \mathbb{R}^d \to \mathbb{R}$, we use Δu to denote its Laplacian, which is defined as $\Delta u := \sum_{i=1}^d \partial_i^2 u$. We set $\nabla^2 u := \nabla(\nabla u)$ and $\Delta^2 u := \Delta(\Delta u)$, and we define inductively $\nabla^k u$ and $\Delta^k u$ for every $k \in \mathbb{N}$, with the convention $\nabla^0 u = \Delta^0 u := u$. For a tensor field $T\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, by $\operatorname{div} T$ we mean its divergence with respect to rows, namely $(\operatorname{div} T)_i := \sum_{j=1}^d \partial_j T_{ij}$ for $i = 1, \dots, m$.

Function spaces. Given two metric spaces X and Y, we use $C^0(X; Y)$ and $\mathrm{Lip}(X; Y)$ to denote, respectively, the space of continuous and Lipschitz functions from X to Y. Given an open set $\Omega \subseteq \mathbb{R}^d$, we denote by $C^k(\Omega; \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued functions with k continuous derivatives; we use $C^k_c(\Omega; \mathbb{R}^m)$ and $C^{k,1}(\Omega; \mathbb{R}^m)$ to denote, respectively, the subspace of functions with compact support in Ω, and of functions whose k-derivatives are Lipschitz. For every $1 \le p \le \infty$ we denote by $L^p(\Omega; \mathbb{R}^m)$ the Lebesgue space of p-th power integrable functions, and by $W^{k,p}(\Omega; \mathbb{R}^m)$ the Sobolev space of functions with k derivatives; for $p = 2$ we set $H^k(\Omega; \mathbb{R}^m) := W^{k,2}(\Omega; \mathbb{R}^m)$, and for $m = 1$ we omit $\mathbb{R}^m$ in the previous spaces. The boundary values of a Sobolev function are always intended in the sense of traces. The scalar product in $L^2(\Omega; \mathbb{R}^m)$ is denoted by $(\cdot, \cdot)_{L^2(\Omega)}$ and the norm in $L^p(\Omega; \mathbb{R}^m)$ by $\|\cdot\|_{L^p(\Omega)}$; a similar notation is valid for the Sobolev spaces. For simplicity, we use $\|\cdot\|_{L^\infty(\Omega)}$ to denote also the supremum norm of continuous and bounded functions.

The norm of a generic Banach space X is denoted by $\|\cdot\|_X$; when X is a Hilbert space, we use $(\cdot, \cdot)_X$ to denote its scalar product. We denote by X' the dual of X, and by $\langle \cdot, \cdot \rangle_{X'}$ the duality product between X' and X. Given two Banach spaces X_1 and X_2, the space of linear and continuous maps from X_1 to X_2 is denoted by $\mathscr{L}(X_1; X_2)$; given $\mathbb{A} \in \mathscr{L}(X_1; X_2)$ and $u \in X_1$, we write $\mathbb{A}u \in X_2$ to denote the image of u under $\mathbb{A}$.

Given an open interval $(a, b) \subseteq \mathbb{R}$ and $1 \le p \le \infty$, we denote by $L^p(a, b; X)$ the space of L^p functions from (a, b) to X; we use $W^{k,p}(a, b; X)$ and $H^k(a, b; X)$ (for $p = 2$) to denote the Sobolev space of functions from (a, b) to X with k derivatives. Given $u \in W^{1,p}(a, b; X)$, we denote by $\dot{u} \in L^p(a, b; X)$ its derivative in the sense distributions. The set of functions from $[a, b]$ to X with k continuous derivatives is denoted by $C^k([a, b]; X)$; we use $C^k_c(a, b; X)$ to denote the subspace of functions with compact support in (a, b). The space of absolutely continuous functions from $[a, b]$ to X is denoted by $AC([a, b]; X)$; we use $C^0_w([a, b]; X)$ to denote the set of weakly continuous functions from $[a, b]$ to X, namely

$$C^0_w([a, b]; X) \coloneqq \{u\colon [a, b] \to X : t \mapsto \langle x', u(t)\rangle_{X'} \text{ is continuous in } [a, b] \text{ for every } x' \in X'\}.$$

When dealing with an element $u \in H^1(a, b; X)$ we always assume u to be the continuous representative of its class. In particular, it makes sense to consider the pointwise value $u(t)$ for every $t \in [a, b]$.

Chapter 1

Elastodynamics system in domains with growing cracks

In this chapter, we prove the existence, uniqueness, and continuous dependence results for the elastodynamics system (4) via the change of variable approach of [43, 20].

The chapter is organized as follows. In Section 1.1 we list the main assumptions on the set Ω, on the geometry of the cracks Γ_t, and on the diffeomorphisms used for the changes of variables. Moreover, in Definitions 1.1.6 and 1.1.9 we specify the notion of weak solution to problems (4) and (5). Section 1.2 deals with the study of the two problems (see Theorems 1.2.2 and 1.2.3). We first show their equivalence (see Theorem 1.1.16), and then we prove an existence and uniqueness result for (5) in a weaker sense (see Theorems 1.2.9). In Section 1.3 we complete the proof of Theorems 1.2.2 and 1.2.3 by showing the energy equality (1.3.2), which ensures that the solution given by Theorem 1.2.9 is indeed a weak solution. Finally, Section 1.4 is devoted to the continuous dependence result, which is proved in Theorem 1.4.1.

The results contained in this chapter have been published in [8].

1.1 Preliminary results

Let T be a positive number, $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$, and $\partial_N\Omega$ be its complement. Throughout this chapter we assume the following hypotheses on the geometry of the crack sets $\{\Gamma_t\}_{t\in[0,T]}$ and on the diffeomorphisms of Ω into itself mapping Γ_0 into Γ_t:

(H1) $\Gamma \subset \mathbb{R}^d$ is a complete $(d-1)$-dimensional C^2 manifold with boundary $\partial\Gamma$ such that $\partial\Gamma \cap \Omega = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(H2) for every $x \in \Gamma \cap \overline{\Omega}$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U\cap\Omega)\setminus\Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(H3) $\{\Gamma_t\}_{t\in[0,T]}$ is a family of (possibly irregular) closed subsets of $\Gamma \cap \overline{\Omega}$ satisfying $\Gamma_s \subseteq \Gamma_t$ for every $0 \le s \le t \le T$;

(H4) $\Phi,\ \Psi\colon [0,T] \times \overline{\Omega} \to \overline{\Omega}$ are two continuous maps and the partial derivatives $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial^2_{ij}\Phi$, $\partial^2_{ij}\Psi$, $\partial_i\partial_t\Phi = \partial_t\partial_i\Phi$, $\partial_i\partial_t\Psi = \partial_t\partial_i\Psi$ exist and are continuous for $i,j = 1,\dots,d$;

(H5) $\Phi(t,\Omega) = \Omega$, $\Phi(t,\Gamma\cap\Omega) = \Gamma\cap\Omega$, $\Phi(t,\Gamma_0) = \Gamma_t$, and $\Phi(t,y) = y$ for every $t \in [0,T]$ and y in a neighborhood of $\partial\Omega$;

(H6) $\Psi(t,\Phi(t,y)) = y$, $\Phi(t,\Psi(t,x)) = x$, and $\Phi(0,y) = y$ for every $t \in [0,T]$ and $x,y \in \overline{\Omega}$;

(H7) $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial^2_{ij}\Phi$, $\partial^2_{ij}\Psi$, $\partial_i\partial_t\Phi$, $\partial_i\partial_t\Psi$ belong to the space $\mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathbb{R}^d))$ for $i,j=1,\dots,d$;

(H8) there exists a constant $L>0$ such that

$$|\partial_i\partial_t\Phi(t,x)-\partial_i\partial_t\Phi(t,y)|\le L|x-y|,\quad |\partial_i\partial_t\Psi(t,x)-\partial_i\partial_t\Psi(t,y)|\le L|x-y|$$

for every $t\in[0,T]$, $x,y\in\overline{\Omega}$, and $i=1,\dots,d$.

By using (H4) and (H6) we derive that $\det\nabla\Phi(t,y)\neq 0$ and $\det\nabla\Psi(t,x)\neq 0$ for every $t\in[0,T]$ and $x,y\in\overline{\Omega}$. In particular, both determinants are positive, since $\nabla\Phi(0,y)=Id$ for every $y\in\overline{\Omega}$.

Assumptions (H1) and (H2) imply the existence of the trace of $\psi\in H^1(\Omega\setminus\Gamma)$ on $\partial\Omega$, and on $\Gamma\cap\Omega$ from both sides. Indeed, we can find a finite number of open sets with Lipschitz boundary $U_j\subseteq\Omega\setminus\Gamma$, $j=1,\dots m$, such that $((\Gamma\cap\Omega)\cup\partial\Omega)\setminus(\Gamma\cap\partial\Omega)\subseteq\cup_{j=1}^m\partial U_j$. Moreover, since $\mathcal{H}^{d-1}(\Gamma\cap\partial\Omega)=0$, there exists a constant $C_{tr}>0$, depending only on Ω and Γ, such that

$$\|\psi\|_{L^2(\partial\Omega)}\le C_{tr}\|\psi\|_{H^1(\Omega\setminus\Gamma)}\quad\text{for every }\psi\in H^1(\Omega\setminus\Gamma). \tag{1.1.1}$$

In a similar way we obtain the embedding $H^1(\Omega\setminus\Gamma)\hookrightarrow L^p(\Omega)$ for every $p\in[1,2^*]$, where $2^*:=\frac{2n}{n-2}$ is the usual critical Sobolev exponent. In particular, there exists a constant $C_p>0$, depending on Ω, Γ, and p, such that

$$\|\psi\|_{L^p(\Omega)}\le C_p\|\psi\|_{H^1(\Omega\setminus\Gamma)}\quad\text{for every }\psi\in H^1(\Omega\setminus\Gamma). \tag{1.1.2}$$

Given a point $y\in\Gamma\cap\Omega$, its trajectory in time is described by the time-dependent map $t\mapsto\Phi(t,y)\in\Gamma$. We infer that its velocity is tangential to the manifold Γ at the point $\Phi(t,y)$, that is $\dot\Phi(t,y)\cdot\nu(\Phi(t,y))=0$, where $\nu(x)$ is the unit normal vector to Γ at x. By combining this equality with the relation

$$\nu(\Phi(t,y))=\frac{\nabla\Phi(t,y)^{-T}\nu(y)}{|\nabla\Phi(t,y)^{-T}\nu(y)|}\quad\text{for }y\in\Gamma\cap\Omega,$$

we deduce

$$(\nabla\Phi(t,y)^{-1}\dot\Phi(t,y))\cdot\nu(y)=\dot\Psi(t,\Phi(t,y))\cdot\nu(y)=0\quad\text{for }y\in\Gamma\cap\Omega, \tag{1.1.3}$$

or equivalently

$$\dot\Phi(t,\Psi(t,x))\cdot\nu(x)=0\quad\text{for }x\in\Gamma\cap\Omega. \tag{1.1.4}$$

In the following lemmas, we investigate some regularity properties of functions defined in $\Omega\setminus\Gamma$, when composed with suitable diffeomorphisms of the domain into itself. Let us specify the class of diffeomorphisms under study.

Definition 1.1.1. We say that $\Lambda\colon[0,T]\times\overline{\Omega}\to\mathbb{R}^d$ is *admissible* if it belongs to the space $C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)$ and for every $t\in[0,T]$ the function $\Lambda(t)$ is a C^2 diffeomorphism of $\overline{\Omega}$ in itself such that $\Lambda(t,\Omega)=\Omega$ and $\Lambda(t,\Gamma\cap\Omega)=\Gamma\cap\Omega$.

Notice that, according to (H4)–(H6), both Φ and Ψ are admissible.

Lemma 1.1.2. *Let f and f^n, $n\in\mathbb{N}$, be elements of $L^2(\Omega)$, and let Λ and Λ^n, $n\in\mathbb{N}$, be admissible diffeomorphisms. Assume there exist $\delta_1,\delta_2>0$ such that $\delta_1<\det\nabla\Lambda^n(t,x)<\delta_2$ for every $t\in[0,T]$, $x\in\overline{\Omega}$, and $n\in\mathbb{N}$. Assume also that for every $t\in[0,T]$*

$$\Lambda^n(t)\to\Lambda(t)\quad\text{in }L^2(\Omega;\mathbb{R}^d),\quad f^n\to f\quad\text{in }L^2(\Omega)\quad\text{as }n\to\infty.$$

Then, for every $t\in[0,T]$ we have

$$f^n(\Lambda^n(t))\to f(\Lambda(t))\quad\text{in }L^2(\Omega)\quad\text{as }n\to\infty.$$

Proof. The proof of this result can be found in [20, Lemma A.7]. □

Lemma 1.1.3. *Let Λ be admissible. There exists a constant $C > 0$ such that for every $\psi \in H^1(\Omega \setminus \Gamma)$ we have*

$$\|\psi(\Lambda(t)) - \psi(\Lambda(s))\|_{L^2(\Omega\setminus\Gamma)} \le C\|\nabla\psi\|_{L^2(\Omega\setminus\Gamma)}|t-s| \quad \textit{for every } 0 \le s \le t \le T.$$

Proof. It is sufficient to repeat the proof of [20, Lemmas A.5], by approximating $\psi \in H^1(\Omega\setminus\Gamma)$ with a sequence of functions $\psi_\varepsilon \in C^\infty(\Omega \setminus \Gamma) \cap H^1(\Omega \setminus \Gamma)$ given by Meyers-Serrin's theorem (see, e.g, [1, Theorem 3.16]), and integrating over $\Omega \setminus \Gamma$. □

Lemma 1.1.4. *Let Λ be admissible and let $t \in [0,T]$ be fixed. Then for every $\psi \in H^1(\Omega \setminus \Gamma)$*

$$\frac{1}{h}[\psi(\Lambda(t+h)) - \psi(\Lambda(t))] \to \nabla\psi(\Lambda(t)) \cdot \dot\Lambda(t) \quad \textit{in } L^2(\Omega \setminus \Gamma) \quad \textit{as } h \to 0.$$

Proof. We argue again as in the proof of [20, Lemmas A.6], by approximating ψ with a sequence of functions $\psi_\varepsilon \in C^\infty(\Omega \setminus \Gamma) \cap H^1(\Omega \setminus \Gamma)$ given by Meyers-Serrin's theorem, and by integrating over $\Omega \setminus \Gamma$. We only have to check that as $h \to 0$

$$T_h(\psi_\varepsilon) := \frac{1}{h}\int_0^h \nabla\psi_\varepsilon(\Lambda(t+\tau)) \cdot \dot\Lambda(t+\tau)\,\mathrm{d}\tau \to L(\psi_\varepsilon) := \nabla\psi_\varepsilon(\Lambda(t)) \cdot \dot\Lambda(t) \quad \text{in } L^2(\Omega \setminus \Gamma).$$

Since $\Lambda\colon [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ is uniformly continuous, for every $\delta > 0$ there exists $\bar\rho > 0$ such that

$$|\Lambda(t+\tau, y) - \Lambda(t,y)| < \delta \quad \text{for every } |\tau| < \bar\rho \text{ and } y \in \overline{\Omega}. \tag{1.1.5}$$

Similarly, fixed $t \in [0,T]$, the map $\Lambda^{-1}(t)\colon \overline{\Omega} \to \mathbb{R}^d$ is uniformly continuous, and so for every $\eta > 0$ there exists $\bar\delta > 0$ such that

$$|\Lambda^{-1}(t,y) - \Lambda^{-1}(t,z)| < \eta \quad \text{for every } y, z \in \overline{\Omega}, \text{ with } |y - z| < \bar\delta. \tag{1.1.6}$$

By combining (1.1.5) and (1.1.6), we get that for every $\eta > 0$ there exists $\bar\rho > 0$ such that

$$\Lambda(t+\tau, A) \subset \Lambda(t, I_\eta(A)) \quad \text{for every set } A \subset \overline{\Omega} \text{ and } |\tau| < \bar\rho,$$

where $I_\eta(A) := \{x \in \overline{\Omega} : \mathrm{dist}(x, A) < \eta\}$ (we recall that $\mathrm{dist}(x,A) := \inf_{y\in A}|x-y|$).

For every $n \in \mathbb{N}$ we define $K_n := \{x \in \Omega \setminus \Gamma : \mathrm{dist}(x, \partial(\Omega \setminus \Gamma)) \ge 1/n\}$. The sets K_n are compact, with $K_n \subset K_{n+1}$, and $\cup_{n=1}^\infty K_n = \Omega \setminus \Gamma$. Fixed $n \in \mathbb{N}$, there exists $\eta > 0$ such that $I_\eta(K_n) \subset\subset \Omega \setminus \Gamma$, which implies that $\Lambda(t, I_\eta(K_n)) \subset\subset \Omega \setminus \Gamma$. Therefore there exists $\bar\rho > 0$ such that for every $|h| < \bar\rho$ and $y \in K_n$

$$|T_h(\psi_\varepsilon)(y)| \le \frac{1}{h}\int_0^h |\nabla\psi_\varepsilon(\Lambda(t+\tau,y)) \cdot \dot\Lambda(t+\tau,y)|\,\mathrm{d}\tau \le C,$$

for a constant $C > 0$ independent of h. Hence, by the dominated convergence theorem we conclude that $\|T_h(\psi_\varepsilon) - L(\psi_\varepsilon)\|_{L^2(K_n)} \to 0$ as $h \to 0$, since $T_h(\psi_\varepsilon)(y) \to L(\psi_\varepsilon)(y)$ for every $y \in \Omega \setminus \Gamma$. Similarly, there exists $\eta > 0$ such that $I_\eta((\Omega \setminus \Gamma) \setminus K_{n+1}) \subset (\Omega \setminus \Gamma) \setminus K_n$, and so we can find $\bar\rho > 0$ such that for every $|h| < \bar\rho$

$$\begin{aligned}\|T_h(\psi_\varepsilon)\|^2_{L^2((\Omega\setminus\Gamma)\setminus K_{n+1})} &\le \frac{1}{h}\int_{(\Omega\setminus\Gamma)\setminus K_{n+1}}\int_0^h |\nabla\psi_\varepsilon(\Lambda(t+\tau,y)) \cdot \dot\Lambda(t+\tau,y)|^2 \mathrm{d}\tau\,\mathrm{d}y \\ &\le C\int_{(\Omega\setminus\Gamma)\setminus K_n} |\nabla\psi_\varepsilon(\Lambda(t,y))|^2 \mathrm{d}y.\end{aligned}$$

Therefore, for every $|h| < \bar\rho$

$$\begin{aligned}
&\|T_h(\psi_\varepsilon) - L(\psi_\varepsilon)\|_{L^2(\Omega\setminus\Gamma)}\\
&\le \|T_h(\psi_\varepsilon) - L(\psi_\varepsilon)\|_{L^2(K_{n+1})} + \|T_h(\psi_\varepsilon)\|_{L^2((\Omega\setminus\Gamma)\setminus K_{n+1})} + \|L(\psi_\varepsilon)\|_{L^2((\Omega\setminus\Gamma)\setminus K_{n+1})}\\
&\le \|T_h(\psi_\varepsilon) - L(\psi_\varepsilon)\|_{L^2(K_{n+1})} + 2C\|\nabla\psi_\varepsilon(\Lambda(t))\|_{L^2((\Omega\setminus\Gamma)\setminus K_n)},
\end{aligned}$$

and consequently $\limsup_{h\to 0}\|T_h(\psi_\varepsilon)-L(\psi_\varepsilon)\|_{L^2(\Omega\setminus\Gamma)} \le 2C\|\nabla\psi_\varepsilon(\Lambda(t))\|_{L^2((\Omega\setminus\Gamma)\setminus K_n)}$ for every $n\in\mathbb{N}$. To conclude it is enough to observe that $\mathcal{L}^d((\Omega\setminus\Gamma)\setminus K_n)\to 0$ as $n\to\infty$, and $\nabla\psi_\varepsilon(\Lambda(t))\in L^2(\Omega\setminus\Gamma;\mathbb{R}^d)$. □

For every $t\in[0,T]$ we introduce the space

$$H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d) := \{\psi\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) : \psi = 0 \text{ on } \partial_D\Omega\}, \tag{1.1.7}$$

where the equality $\psi = 0$ on $\partial_D\Omega$ refers to the trace of ψ on $\partial\Omega$. We have that $H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ is a Hilbert space endowed with the norm of $H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$, and its dual is denoted by $H^{-1}_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$. The canonical isomorphism between $H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ and $[H^1_D(\Omega\setminus\Gamma_t)]^d$ induces an isomorphism of $H^{-1}_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ into $[H^{-1}_D(\Omega\setminus\Gamma_t)]^d$.

The transpose of the natural embedding $H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)\hookrightarrow L^2(\Omega;\mathbb{R}^d)$ induces the embedding of $L^2(\Omega;\mathbb{R}^d)$ into $H^{-1}_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$, which is defined by

$$\langle g,\psi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_t)} := (g,\psi)_{L^2(\Omega)} \quad \text{for } g\in L^2(\Omega;\mathbb{R}^d) \text{ and } \psi\in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d).$$

Given $0\le s\le t\le T$, let $P_{st}\colon H^{-1}_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)\to H^{-1}_D(\Omega\setminus\Gamma_s;\mathbb{R}^d)$ be the transpose of the natural embedding $H^1_D(\Omega\setminus\Gamma_s;\mathbb{R}^d)\hookrightarrow H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$, i.e.,

$$\langle P_{st}(g),\psi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_s)} := \langle g,\psi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_t)} \quad \text{for } g\in H^{-1}_D(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ and } \psi\in H^1_D(\Omega\setminus\Gamma_s;\mathbb{R}^d).$$

The operator P_{st} is continuous, with norm less than or equal to 1, but in general is not injective, since $H^1_D(\Omega\setminus\Gamma_s;\mathbb{R}^d)$ is not dense in $H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$. Notice that $P_{st}(g) = g$ for every $g\in L^2(\Omega;\mathbb{R}^d)$.

Let $\mathbb{C}\colon[0,T]\times\overline{\Omega}\to\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ be a time varying tensor field satisfying

$$\begin{aligned}
&\mathbb{C}\in \mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym}))),\\
&\mathbb{C}(t)\in \mathrm{Lip}(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})), \quad \|\nabla\mathbb{C}(t)\|_{L^\infty(\Omega)}\le C \quad \text{for every } t\in[0,T],\\
&(\mathbb{C}(t,x)\xi_1)\cdot\xi_2 = \xi_1\cdot(\mathbb{C}(t,x)\xi_2) \quad \text{for every } \xi_1,\xi_2\in\mathbb{R}^{d\times d}_{sym},\ t\in[0,T],\ x\in\overline{\Omega},
\end{aligned}$$

where $C>0$ is a constant independent of t. Starting from the operator $\mathbb{C}(t,x)$ it is convienent to define a new operator $\mathbb{A}(t,x)\in\mathscr{L}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})$ as:

$$\mathbb{A}(t,x)\xi := \mathbb{C}(t,x)\xi^{sym} \quad \text{for every } \xi\in\mathbb{R}^{d\times d},\ t\in[0,T],\ x\in\overline{\Omega}.$$

Clearly, $\mathbb{A}$ satisfies

$$\mathbb{A}\in \mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))), \tag{1.1.8}$$

$$\mathbb{A}(t)\in \mathrm{Lip}(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \quad \|\nabla\mathbb{A}(t)\|_{L^\infty(\Omega)}\le C \quad \text{for every } t\in[0,T], \tag{1.1.9}$$

$$(\mathbb{A}(t,x)\xi_1)\cdot\xi_2 = \xi_1\cdot(\mathbb{A}(t,x)\xi_2) \quad \text{for every } \xi_1,\xi_2\in\mathbb{R}^{d\times d},\ t\in[0,T],\ x\in\overline{\Omega}. \tag{1.1.10}$$

Given

$$w\in H^2(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)), \tag{1.1.11}$$

$$f\in L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad F\in H^1(0,T;L^2(\partial_N\Omega;\mathbb{R}^d)), \tag{1.1.12}$$

$$u^0 - w(0) \in H^1_D(\Omega \setminus \Gamma_0; \mathbb{R}^d), \quad u^1 \in L^2(\Omega; \mathbb{R}^d), \tag{1.1.13}$$

we study the linear hyperbolic system

$$\ddot{u}(t) - \operatorname{div}(\mathbb{A}(t)\nabla u(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T], \tag{1.1.14}$$

with boundary conditions formally written as

$$u(t) = w(t) \quad \text{on } \partial_D\Omega,\ t \in [0,T], \tag{1.1.15}$$
$$(\mathbb{A}(t)\nabla u(t))\nu = F(t) \quad \text{on } \partial_N\Omega,\ t \in [0,T], \tag{1.1.16}$$
$$(\mathbb{A}(t)\nabla u(t))\nu = 0 \quad \text{on } \Gamma_t,\ t \in [0,T], \tag{1.1.17}$$

and initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1 \quad \text{in } \Omega \setminus \Gamma_0. \tag{1.1.18}$$

Remark 1.1.5. To give a precise meaning to (1.1.14)–(1.1.18), it is convenient to introduce the following notation. Given $\psi \in H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d)$, its gradient in the sense of distributions is denoted by $\nabla\psi$ and it is an element of $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^{d\times d})$. We extend it to a function in $L^2(\Omega; \mathbb{R}^{d\times d})$ by setting $\nabla\psi = 0$ on Γ_t. Notice that this is not the gradient in the sense of distributions on Ω of the function ψ, considered as defined almost everywhere on Ω; indeed the equality

$$\int_\Omega \nabla\psi(x) \cdot \omega(x)\,\mathrm{d}x = -\int_\Omega \psi(x) \cdot \operatorname{div}\omega(x)\,\mathrm{d}x$$

holds for every $\omega \in C^\infty_c(\Omega \setminus \Gamma_t; \mathbb{R}^{d\times d})$, but in general not for $\omega \in C^\infty_c(\Omega; \mathbb{R}^{d\times d})$. Similarly, we extend $\operatorname{div}\psi \in L^2(\Omega \setminus \Gamma_t)$ to a function in $L^2(\Omega)$ by setting $\operatorname{div}\psi = 0$ on Γ_t.

We recall the notion of solution to (1.1.14)–(1.1.17) given in [20, Definition 2.4]. We consider functions u satisfying the following regularity assumptions:

$$u \in C^1([0,T]; L^2(\Omega; \mathbb{R}^d)), \tag{1.1.19}$$
$$u(t) - w(t) \in H^1_D(\Omega \setminus \Gamma_t; \mathbb{R}^d) \text{ for every } t \in [0,T], \tag{1.1.20}$$
$$\nabla u \in C^0([0,T]; L^2(\Omega; \mathbb{R}^{d\times d})), \tag{1.1.21}$$
$$\dot{u} \in AC([s,T]; H^{-1}_D(\Omega \setminus \Gamma_s; \mathbb{R}^d)) \text{ for every } s \in [0,T), \tag{1.1.22}$$
$$\frac{1}{h}[\dot{u}(t+h) - \dot{u}(t)] \rightharpoonup \ddot{u} \text{ in } H^{-1}_D(\Omega \setminus \Gamma_t; \mathbb{R}^d) \text{ for a.e. } t \in (0,T) \text{ as } h \to 0, \tag{1.1.23}$$
$$\text{the function } t \mapsto \|\ddot{u}(t)\|_{H^{-1}_D(\Omega\setminus\Gamma_t)} \text{ is integrable in } (0,T). \tag{1.1.24}$$

The relationship between $\ddot{u}$ and the distributional time derivative of $\dot{u}$ is explained in [20, Lemma 2.2], which shows that, under assumptions (1.1.19)–(1.1.24), the map $t \mapsto P_{st}(\ddot{u}(t))$ is the distributional derivative of the function $t \mapsto \dot{u}(t)$ from (s,T) to $H^{-1}_D(\Omega\setminus\Gamma_s; \mathbb{R}^d)$. Moreover

$$\dot{u}(t) - \dot{u}(s) = \int_s^t P_{s\tau}(\ddot{u}(\tau))\,\mathrm{d}\tau \quad \text{for every } 0 \le s \le t \le T.$$

Definition 1.1.6. Let $\mathbb{A}$, w, f, and F be as in (1.1.8)–(1.1.12). We say that u is a *weak solution* to the hyperbolic system (1.1.14) with boundary conditions (1.1.15)–(1.1.17) if u satisfies (1.1.19)–(1.1.24), and for a.e. $t \in (0,T)$ we have

$$\langle \ddot{u}(t), \psi \rangle_{H^{-1}_D(\Omega\setminus\Gamma_t)} + (\mathbb{A}(t)\nabla u(t), \nabla\psi)_{L^2(\Omega)} = (f(t), \psi)_{L^2(\Omega)} + (F(t), \psi)_{L^2(\partial_N\Omega)} \tag{1.1.25}$$

for every $\psi \in H^1_D(\Omega \setminus \Gamma_t; \mathbb{R}^d)$, where $\ddot{u}(t)$ is defined in (1.1.23).

Remark 1.1.7. Let us check that (1.1.25) makes sense for a.e. $t \in (0,T)$. Thanks to (1.1.23) we have that $\ddot{u}(t) \in H^{-1}_D(\Omega \setminus \Gamma_t;\mathbb{R}^d)$ for a.e. $t \in (0,T)$, therefore it is in duality with $\psi \in H^1_D(\Omega \setminus \Gamma_t;\mathbb{R}^d)$. Moreover, assumptions (1.1.8) and (1.1.21) implies that $\mathbb{A}(t)\nabla u(t)$ belongs to $L^2(\Omega;\mathbb{R}^{d\times d})$ for every $t \in [0,T]$. Finally, thanks to (1.1.1) the last term of (1.1.25) is well defined for every $t \in [0,T]$.

Remark 1.1.8. Notice that the Neumann boundary conditions (1.1.16) and (1.1.17) are in general only formal. They can be obtained from (1.1.25), by using integration by parts in space, only when $u(t)$ and Γ_t are sufficiently regular.

Following [43], to prove the existence and uniqueness of a weak solution, we perform a change of variable. We denote by

$$v(t,y) := u(t,\Phi(t,y)) \quad t \in [0,T],\, y \in \Omega \setminus \Gamma_0, \tag{1.1.26}$$

where Φ is the diffeomorphism introduced in (H4)–(H8), so that

$$u(t,x) = v(t,\Psi(t,x)) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t. \tag{1.1.27}$$

Notice that $v(t) \in H^1(\Omega \setminus \Gamma_0;\mathbb{R}^d)$ if and only if $u(t) \in H^1(\Omega \setminus \Gamma_t;\mathbb{R}^d)$ and that this change of variables maps the domain $\{(t,x) : t \in [0,T], x \in \Omega \setminus \Gamma_t\}$ into the cylinder $[0,T] \times (\Omega \setminus \Gamma_0)$. The transformed system reads

$$\ddot{v}(t) - \operatorname{div}(\mathbb{B}(t)\nabla v(t)) + \mathbf{p}(t)\nabla v(t) - 2\nabla \dot{v}(t)b(t) = g(t) \quad \text{in } \Omega \setminus \Gamma_0,\, t \in [0,T], \tag{1.1.28}$$

where $\mathbb{B}(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})$, $\mathbf{p}(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$, $b(t,y) \in \mathbb{R}^d$, and $g(t,y) \in \mathbb{R}^d$ are defined for $t \in [0,T]$ and $y \in \overline{\Omega}$ as

$$\mathbb{B}(t,y)\xi := (\mathbb{A}(t,\Phi(t,y))[\xi\nabla\Psi(t,\Phi(t,y))])\,\nabla\Psi(t,\Phi(t,y))^T - \xi b(t,y) \otimes b(t,y), \tag{1.1.29}$$

$$\mathbf{p}(t,y)\xi := -[(\mathbb{B}(t,y)\xi)\nabla(\det\nabla\Phi(t,y)) + \partial_t(\xi b(t,y)\det\nabla\Phi(t,y))]\det\nabla\Psi(t,\Phi(t,y)), \tag{1.1.30}$$

$$b(t,y) := -\dot{\Psi}(t,\Phi(t,y)), \tag{1.1.31}$$

$$g(t,y) := f(t,\Phi(t,y)) \tag{1.1.32}$$

for every $\xi \in \mathbb{R}^{d\times d}$. The system is supplemented by boundary conditions formally written as

$$v(t) = w(t) \quad \text{on } \partial_D\Omega,\, t \in [0,T], \tag{1.1.33}$$

$$(\mathbb{B}(t)\nabla v(t))\nu = F(t) \quad \text{on } \partial_N\Omega,\, t \in [0,T], \tag{1.1.34}$$

$$(\mathbb{B}(t)\nabla v(t))\nu = 0 \quad \text{on } \Gamma_0,\, t \in [0,T], \tag{1.1.35}$$

and initial conditions

$$v(0) = v^0, \quad \dot{v}(0) = v^1 \quad \text{in } \Omega \setminus \Gamma_0, \tag{1.1.36}$$

with initial data

$$v^0 := u^0, \quad v^1 := u^1 + \nabla u^0\dot{\Phi}(0). \tag{1.1.37}$$

To give a precise meaning to the notion of solution to system (1.1.28) with boundary conditions (1.1.33)–(1.1.35), we consider functions v which satisfy the following regularity assumptions:

$$v \in C^1([0,T];L^2(\Omega;\mathbb{R}^d)), \tag{1.1.38}$$

$$v(t) - w(t) \in H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d) \text{ for every } t \in [0,T], \tag{1.1.39}$$

$$\nabla v \in C^0([0,T];L^2(\Omega;\mathbb{R}^{d\times d})), \tag{1.1.40}$$

$$\dot{v} \in AC([0,T];H^{-1}_D(\Omega \setminus \Gamma_0;\mathbb{R}^d)). \tag{1.1.41}$$

Definition 1.1.9. Let $\mathbb{A}$, w, f, and F be as in (1.1.8)–(1.1.12). Let $\mathbb{B}$, $\mathbf{p}$, b, and g be defined according to (1.1.29)–(1.1.32). We say that v is a *weak solution* to the transformed system (1.1.28) with boundary conditions (1.1.33)–(1.1.35), if v satisfies (1.1.38)–(1.1.41) and for a.e. $t \in (0,T)$ we have

$$\begin{aligned}\langle \ddot v(t), \phi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)} &+ (\mathbb{B}(t)\nabla v(t), \nabla\phi)_{L^2(\Omega)} + (\mathbf{p}(t)\nabla v(t), \phi)_{L^2(\Omega)}\\ &+ 2(\dot v(t), \operatorname{div}[\phi\otimes b(t)])_{L^2(\Omega)} = (g(t),\phi)_{L^2(\Omega)} + (F(t),\phi)_{L^2(\partial_N\Omega)}\end{aligned} \tag{1.1.42}$$

for every $\phi \in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$.

Remark 1.1.10. Notice that (H5) and (1.1.3) imply $b(t) = 0$ on the boundary of $\Omega\setminus\Gamma_0$. Hence, in the weak formulation of (1.1.28), it makes sense to pass from $-2(\nabla\dot v(t)b(t), \phi)_{L^2(\Omega)}$ to $2(\dot v(t), \operatorname{div}[\phi\otimes b(t)])_{L^2(\Omega)}$, which can be defined even for $\dot v(t) \in L^2(\Omega;\mathbb{R}^d)$.

Remark 1.1.11. Let v be a function which satisfies (1.1.38)–(1.1.41). Let us check that the scalar products in (1.1.42) are well defined for a.e. $t \in (0,T)$. By (1.1.41) we have $\ddot v(t) \in H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ for a.e. $t\in(0,T)$, therefore it is duality with $\phi \in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$. In view of (1.1.38) and (1.1.40), for every $t\in[0,T]$ the functions $\dot v(t)$ and $\nabla v(t)$ belong to $L^2(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^{d\times d})$, respectively. Hence, to ensure that the scalar products in the left-hand side of (1.1.42) are well defined, we need to show that the coefficients $\mathbb{B}$, $\mathbf{p}$, b, and $\operatorname{div} b$ are essentially bounded in space for almost every time.

Thanks to (H4), (H7), (H8), (1.1.8), and (1.1.9), we derive that the maps $t\mapsto \mathbb{A}(t,\Phi(t))$, $t\mapsto\nabla\Psi(t,\Phi(t))$, $t\mapsto\dot\Psi(t,\Phi(t))$, and $t\mapsto\operatorname{div}(\dot\Psi(t,\Phi(t)))$ are Lipschitz continuous from $[0,T]$ to $L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))$, $L^\infty(\Omega;\mathbb{R}^{d\times d})$, $L^\infty(\Omega;\mathbb{R}^d)$, and $L^\infty(\Omega)$, respectively. Therefore, we get

$$\mathbb{B} \in \operatorname{Lip}([0,T];L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))), \tag{1.1.43}$$

$$b\in\operatorname{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d)),\quad \operatorname{div} b\in\operatorname{Lip}([0,T];L^\infty(\Omega)). \tag{1.1.44}$$

We split the coefficient $\mathbf{p}$ defined in (1.1.30) into the sum $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2$, where the operators $\mathbf{p}_1(t,y), \mathbf{p}_2(t,y)\in\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$ are defined for $t\in[0,T]$ and $y\in\overline\Omega$ as

$$\begin{aligned}\mathbf{p}_1(t,y)\xi &:= -[(\mathbb{B}(t,y)\xi)\nabla(\det\nabla\Phi(t,y)) + \xi b(t,y)\partial_t(\det\nabla\Phi(t,y))]\det\nabla\Psi(t,\Phi(t,y)),\\ \mathbf{p}_2(t,y)\xi &:= -\xi\dot b(t,y)\end{aligned}$$

for every $\xi\in\mathbb{R}^{d\times d}$. In view of the discussion above, $\mathbf{p}_1\in\operatorname{Lip}([0,T];L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)))$, while $\mathbf{p}_2$ is an element of $L^\infty(0,T;L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)))$, being $\dot b$ the distributional derivative of a function in $\operatorname{Lip}([0,T];L^\infty(\Omega;\mathbb{R}^d))$. Moreover, there exists a constant $C>0$ such that $\|\mathbf{p}_2(t)\|_{L^\infty(\Omega)}\le C$ for a.e. $t\in(0,T)$. Finally, the function g defined in (1.1.32) belongs to $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, since $f\in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. Then the right-hand side of (1.1.42) is well defined for a.e. $t\in(0,T)$.

Remark 1.1.12. Thanks to (H1) and (H2), together with a partition of unity, we can integrate by part in $\Omega\setminus\Gamma$ and derive the following formula:

$$\int_\Omega \nabla\psi(x)\cdot h(x)\phi(x)\,\mathrm{d}x = -\int_\Omega \psi(x)\operatorname{div}[h(x)\phi(x)]\,\mathrm{d}x \tag{1.1.45}$$

for every $\psi,\phi\in H^1(\Omega\setminus\Gamma)$, and for every $h\in W^{1,\infty}(\Omega;\mathbb{R}^d)$, with $h\cdot\nu=0$ on $(\Gamma\cap\Omega)\cup\partial\Omega$. Similarly, for every $\psi\in W^{1,1}(\Omega\setminus\Gamma)$ we have

$$\int_\Omega\nabla\psi(x)\cdot h(x)\,\mathrm{d}x = -\int_\Omega\psi(x)\operatorname{div} h(x)\,\mathrm{d}x. \tag{1.1.46}$$

In particular, formulas (1.1.45) and (1.1.46) are satisfied if h is either $\dot\Psi(t,\Phi(t))$ or $\dot\Phi(t,\Psi(t))$, thanks to (1.1.3), (1.1.4), (H4), and (H5).

Let us clarify the relation between problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17), and problem (1.1.28) with boundary conditions (1.1.33)–(1.1.35). We start with the following lemma.

Lemma 1.1.13. *Suppose that u and v are related by (1.1.26) and (1.1.27). Then u satisfies (1.1.19)–(1.1.24) if and only if v satisfies (1.1.38)–(1.1.41).*

Proof. The proof is straightforward by applying Lemmas 2.8 and 2.11 of [20] to the components of u and v, which is possible thanks to our Lemmas 1.1.3 and 1.1.4, and formula (1.1.45). □

By using the identification $H_D^{-1}(\Omega\setminus\Gamma_t;\mathbb{R}^d)=[H_D^{-1}(\Omega\setminus\Gamma_t)]^d$ and Lemmas 2.9 and 2.12 of [20] we derive the following two results.

Lemma 1.1.14. *Assume that u satisfies (1.1.19)–(1.1.24). Then for a.e. $t\in(0,T)$ we have*

$$\begin{aligned}&\langle\ddot v(t),\phi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}\\&=\langle\ddot u(t),\phi(\Psi(t))\det\nabla\Psi(t)\rangle_{H_D^{-1}(\Omega\setminus\Gamma_t)}+(\nabla u(t),\partial_t[\phi(\Psi(t))\otimes\dot\Phi(t,\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}\\&\quad+(\dot u(t),\partial_t[\phi(\Psi(t))\det\nabla\Psi(t)]-\operatorname{div}[\phi(\Psi(t))\otimes\dot\Phi(t,\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}\end{aligned}\tag{1.1.47}$$

for every $\phi\in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$.

Lemma 1.1.15. *Assume that v satisfies (1.1.38)–(1.1.41). Then for a.e. $t\in(0,T)$ we have*

$$\begin{aligned}&\langle\ddot u(t),\psi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_t)}\\&=\langle\ddot v(t),\psi(\Phi(t))\det\nabla\Phi(t)\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}+(\nabla v(t),\partial_t[\psi(\Phi(t))\otimes\dot\Psi(t,\Phi(t))\det\nabla\Phi(t)])_{L^2(\Omega)}\\&\quad+(\dot v(t),\partial_t[\psi(\Phi(t))\det\nabla\Phi(t)]-\operatorname{div}[\psi(\Phi(t))\otimes\dot\Psi(t,\Phi(t))\det\nabla\Phi(t)])_{L^2(\Omega)}\end{aligned}$$

for every $\psi\in H_D^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$.

We can now specify the relation between the two problems.

Theorem 1.1.16. *Under the assumptions of Definition 1.1.6, a function u is a weak solution to problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17), if and only if the corresponding function v introduced in (1.1.26) is a weak solution to (1.1.28) with boundary conditions (1.1.33)–(1.1.35).*

Proof. Let us assume that u is a weak solution to problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17). Thanks to Lemmas 1.1.13 and 1.1.14, the function v satisfies (1.1.38)–(1.1.41) and (1.1.47). Take an arbitrary test function $\phi\in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$. For every $t\in[0,T]$ the function $\phi(\Psi(t))\det\nabla\Psi(t)$ is in $H_D^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$. Thus, by (1.1.25) we have

$$\begin{aligned}&\langle\ddot u(t),\phi(\Psi(t))\det\nabla\Psi(t)\rangle_{H_D^{-1}(\Omega\setminus\Gamma_t)}\\&=-(\mathbb{A}(t)\nabla u(t),\nabla[\phi(\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}+(f(t),\phi(\Psi(t))\det\nabla\Psi(t))_{L^2(\Omega)}\\&\quad+(F(t),\phi(\Psi(t))\det\nabla\Psi(t))_{L^2(\partial_N\Omega)}.\end{aligned}$$

By inserting this expression in (1.1.47) and using assumption (H4), we get

$$\begin{aligned}&\langle\ddot v(t),\phi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}\\&=-(\mathbb{A}(t)\nabla u(t),\nabla[\phi(\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}+(f(t),\phi(\Psi(t))\det\nabla\Psi(t))_{L^2(\Omega)}\\&\quad+(F(t),\phi)_{L^2(\partial_N\Omega)}+(\nabla u(t),\partial_t[\phi(\Psi(t))\otimes\dot\Phi(t,\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}\\&\quad+(\dot u(t),\partial_t[\phi(\Psi(t))\det\nabla\Psi(t)]-\operatorname{div}[\phi(\Psi(t))\otimes\dot\Phi(t,\Psi(t))\det\nabla\Psi(t)])_{L^2(\Omega)}.\end{aligned}\tag{1.1.48}$$

Thanks to (1.1.27), by performing the same computations done in [20] we get

$$\nabla u(t) = \nabla v(t, \Psi(t))\nabla\Psi(t), \quad \dot{u}(t) = \dot{v}(t, \Psi(t)) + \nabla v(t, \Psi(t))\dot{\Psi}(t). \tag{1.1.49}$$

We insert this expression in (1.1.48) and we obtain that v satisfies (1.1.42). Notice that the boundary terms w and F remain the same through the change of variables since the diffeomorphisms Φ and Ψ are the identity in a neighborhood of $\partial\Omega$.

Similarly, by applying Lemmas 1.1.13 and 1.1.15, it is easy to check that if v is a weak solution to problem (1.1.28) with boundary conditions (1.1.33)–(1.1.35), then u is a weak solution to problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17). □

Remark 1.1.17. Given a weak solution u to (1.1.14)–(1.1.17), we can improve the integrability condition (1.1.24). Indeed, by (1.1.25), the Lipschitz regularity of $\mathbb{A}$, the continuity property (1.1.21) of ∇u, and (1.1.1), we infer

$$\|\ddot{u}(t)\|_{H_D^{-1}(\Omega\setminus\Gamma_t)} \le C(1 + \|f(t)\|_{L^2(\Omega)} + \|F(t)\|_{L^2(\partial_N\Omega)}) \quad \text{for a.e. } t \in (0,T),$$

where $C > 0$ is a constant independent of t. Therefore, the function $t \mapsto \|\ddot{u}(t)\|_{H_D^{-1}(\Omega\setminus\Gamma_t)}$ belongs to $L^2(0,T)$, since $f \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and $F \in C^0([0,T];L^2(\partial_N\Omega;\mathbb{R}^d))$. In addition, if $f \in L^p(0,T;L^2(\Omega;\mathbb{R}^d))$, with $p \in (2,\infty]$, then the function $t \mapsto \|\ddot{u}(t)\|_{H_D^{-1}(\Omega\setminus\Gamma_t)}$ belongs to $L^p(0,T)$. The same property is true also for a weak solution v of (1.1.28) with boundary conditions (1.1.33)–(1.1.35), by exploiting the regularity properties of $\dot{v}$ and ∇v, and the regularity of the coefficients (1.1.29)–(1.1.32) discussed in Remark 1.1.11.

1.2 Existence and uniqueness

To prove our existence and uniqueness results, for both problems (1.1.14) and (1.1.28), we require an additional hypothesis on the operator $\mathbb{B}$. We assume that there exist two constants $c_0 > 0$ and $c_1 \in \mathbb{R}$ such that for every $t \in [0,T]$

$$(\mathbb{B}(t)\nabla\phi, \nabla\phi)_{L^2(\Omega)} \ge c_0\|\phi\|^2_{H^1(\Omega\setminus\Gamma_0)} - c_1\|\phi\|^2_{L^2(\Omega)} \quad \text{for every } \phi \in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d). \tag{1.2.1}$$

Notice that assumption (1.2.1) is satisfied whenever the velocity of the diffeomorphism $\dot{\Phi}$ is sufficiently small and $\mathbb{A}$ satisfies the following standard ellipticity condition in linear elasticity:

$$(\mathbb{A}(t,x)\xi)\cdot\xi \ge \lambda_0|\xi^{sym}|^2 \quad \text{for every } \xi \in \mathbb{R}^{d\times d},\ t\in[0,T],\ x \in \overline{\Omega}, \tag{1.2.2}$$

for a suitable constant $\lambda_0 > 0$, independent of t and x. Indeed, thanks to assumptions (H1) and (H2) we can find a finite number of open sets $U_j \subseteq \Omega\setminus\Gamma$, $j = 1,\dots m$, with Lipschitz boundary, such that $\Omega\setminus\Gamma = \cup_{j=1}^m U_j$. By using second Korn's inequality in each U_j (see, e.g., [44, Theorem 2.4]) and taking the sum over j, we can find a constant C_K, depending only on Ω and Γ, such that

$$\|\nabla\psi\|^2_{L^2(\Omega)} \le C_K\left(\|\psi\|^2_{L^2(\Omega)} + \|E\psi\|^2_{L^2(\Omega)}\right) \quad \text{for every } \psi \in H^1(\Omega\setminus\Gamma;\mathbb{R}^d), \tag{1.2.3}$$

where $E\psi$ is the symmetrized gradient of ψ, namely $E\psi := \frac{1}{2}(\nabla\psi + \nabla\psi^T)$. Define

$$M := \max_{(t,y)\in[0,T]\times\overline{\Omega}} \det\nabla\Phi(t,y), \quad m := \min_{(t,y)\in[0,T]\times\overline{\Omega}} \det\nabla\Phi(t,y).$$

For every $t \in [0,T]$ and $\phi \in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ we use the definition of $\mathbb{B}$ and the change of variables formula, together with (1.1.10), (1.2.2), and (1.2.3), to derive

$$M\int_\Omega \mathbb{B}(t,y)\nabla\phi(y)\cdot\nabla\phi(y)\,\mathrm{d}y$$

$$
\begin{aligned}
&\geq \int_\Omega \mathbb{B}(t,y)\nabla\phi(y)\cdot\nabla\phi(y)\det\nabla\Phi(t,y)\,\mathrm{d}y\\
&= \int_\Omega \mathbb{A}(t,x)\nabla(\phi(\Psi(t,x)))\cdot\nabla(\phi(\Psi(t,x)))\,\mathrm{d}x - \int_\Omega |\nabla(\phi(\Psi(t,x)))\dot\Phi(t,\Psi(t,x))|^2\mathrm{d}x\\
&\geq \lambda_0\int_\Omega |E(\phi(\Psi(t,x)))|^2\mathrm{d}x - \int_\Omega |\nabla(\phi(\Psi(t,x)))\dot\Phi(t,\Psi(t,x))|^2\mathrm{d}x\\
&\geq \frac{\lambda_0}{C_K}\int_\Omega |\nabla(\phi(\Psi(t,x)))|^2\mathrm{d}x - \lambda_0\int_\Omega |\phi(\Psi(t,x))|^2\mathrm{d}x - \int_\Omega |\nabla(\phi(\Psi(t,x)))\dot\Phi(t,\Psi(t,x))|^2\mathrm{d}x,
\end{aligned}
$$

since $\phi(\Psi(t))\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)\subset H^1(\Omega\setminus\Gamma;\mathbb{R}^d)$ for every $t\in[0,T]$. Hence, if we assume

$$
|\dot\Phi(t,y)|^2 < \frac{\lambda_0}{C_K}\quad\text{for every } t\in[0,T] \text{ and } y\in\overline{\Omega}, \tag{1.2.4}
$$

then by (H4) we obtain the existence of a constant $\delta>0$ such that

$$
\int_\Omega \mathbb{B}(t,y)\nabla\phi(y)\cdot\nabla\phi(y)\,\mathrm{d}y \geq \frac{m\delta}{M}\int_\Omega |\nabla\phi(y)\nabla\Psi(t,\Phi(t,y))|^2\mathrm{d}y - \frac{m\lambda_0}{M}\int_\Omega |\phi(y)|^2\mathrm{d}y,
$$

which implies (1.2.1).

Remark 1.2.1. Assumption (1.2.4) imposes a condition on the velocity of the growing crack which depends on the geometry of the crack itself.

We have seen that problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17), and problem (1.1.28) with boundary conditions (1.1.33)–(1.1.35) are equivalent. We want to prove the following existence theorem.

Theorem 1.2.2. *Let be given $\mathbb{A}$, w, f, F, u^0, u^1 as in (1.1.8)–(1.1.13). Let $\mathbb{B}$, $\mathbf{p}$, b, g, v^0, v^1 be defined according to (1.1.29)–(1.1.32) and (1.1.37), with $\mathbb{B}$ satisfying (1.2.1). Then problem (1.1.28) with boundary conditions (1.1.33)–(1.1.35) and initial conditions (1.1.36) admits a unique solution v, according to Definition 1.1.9.*

The proof of Theorem 1.2.2 will be postponed at the end of Section 1.3 and it will obtained as a consequence of Theorems 1.2.9 and 1.2.10 below and Proposition 1.3.1. Thanks to Theorem 1.1.16, as corollary we readily obtain the following result.

Theorem 1.2.3. *Let be given $\mathbb{A}$, w, f, F, u^0, u^1 as in (1.1.8)–(1.1.13). Assume that the operator $\mathbb{B}$ defined in (1.1.29) satisfies (1.2.1). Then problem (1.1.14) with boundary conditions (1.1.15)–(1.1.17) and initial conditions (1.1.18) admits a unique solution u, according to Definition 1.1.6.*

Proof. By using Theorems 1.1.16 and 1.2.2 there exists a solution u to (1.1.14)–(1.1.17). Moreover, the initial conditions (1.1.18) follows from the regularity conditions (1.1.19)–(1.1.24) of u and from the initial conditions of v. Finally, the solution is unique since every solution u to (1.1.14) with boundary and initial conditions (1.1.15)–(1.1.18), gives a solution v to (1.1.28) with boundary and initial conditions (1.1.33)–(1.1.36), thanks to Theorem 1.1.16. □

Remark 1.2.4. The existence and uniqueness results of this thesis improve the ones contained in [8]. Indeed, to prove Theorems 1.2.2 and 1.2.3 we only assume that w satisfies (1.1.11), while in [8] is required

$$
w\in H^2(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))\cap L^2(0,T;H^2(\Omega\setminus\Gamma_0;\mathbb{R}^d)).
$$

Moreover, we remove the assumption

$$
(\mathbb{A}(t)\nabla w(t))\nu = 0\quad\text{on } \partial_N\Omega\cup\Gamma_t, t\in[0,T],
$$

which, on the contrary, is needed in [8].

To prove Theorem 1.2.2, we introduce a notion of solution to (1.1.28) which is weaker than the one considered in Definition 1.1.9. Later, in Section 1.3, we will prove an energy equality which ensure that this type of solution is more regular, namely it satisfies the regularity conditions (1.1.38)–(1.1.41).

Definition 1.2.5. Let $\mathbb{A}$, w, f, and F be as in (1.1.8)–(1.1.13). Let $\mathbb{B}$, $\mathbf{p}$, b, and g be defined according to (1.1.29)–(1.1.32). We say that v is a *generalized solution* to (1.1.28) with boundary conditions (1.1.33)–(1.1.35) if

$$v \in L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)),$$
$$v-w \in L^\infty(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)),$$

and for a.e. $t\in(0,T)$ we have

$$\begin{aligned}\langle \ddot v(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)} &+ (\mathbb{B}(t)\nabla v(t),\nabla\phi)_{L^2(\Omega)} + (\mathbf{p}(t)\nabla v(t),\phi)_{L^2(\Omega)}\\ &+2(\dot v(t),\operatorname{div}[\phi\otimes b(t)])_{L^2(\Omega)} = (g(t),\phi)_{L^2(\Omega)} + (F(t),\phi)_{L^2(\partial_N\Omega)}\end{aligned} \tag{1.2.5}$$

for every $\phi\in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$.

Remark 1.2.6. Since $C^\infty_c(0,T)\otimes H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ is dense in $L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, we can recast equality (1.2.5) in the framework of the duality between $L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ and $L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Indeed, it is easy to see that (1.2.5) is equivalent to

$$\begin{aligned}&\int_0^T\Big[\langle \ddot v(t),\varphi(t)\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)} + (\mathbb{B}(t)\nabla v(t),\nabla\varphi(t))_{L^2(\Omega)} + (\mathbf{p}(t)\nabla v(t),\varphi(t))_{L^2(\Omega)}\Big]\,\mathrm{d}t\\ &\quad+2\int_0^T(\dot v(t),\operatorname{div}[\varphi(t)\otimes b(t)])_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T\big[(g(t),\varphi(t))_{L^2(\Omega)} + (F(t),\varphi(t))_{L^2(\partial_N\Omega)}\big]\,\mathrm{d}t\end{aligned}$$

for every $\varphi\in L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$.

Remark 1.2.7. Let us clarify the meaning of the initial conditions (1.1.36) for a generalized solution. We recall that if X,Y are two reflexive Banach spaces, with embedding $X\hookrightarrow Y$ continuous, then

$$C^0_w([0,T];Y)\cap L^\infty(0,T;X) = C^0_w([0,T];X),$$

see for instance [24, Chapitre XVIII, §5, Lemme 6], where $C^0_w([0,T];X)$ and $C^0_w([0,T];Y)$ denote the spaces of weakly continuous functions from $[0,T]$ to X and Y, respectively. In particular, we can apply this result to a generalized solution v. By taking $X=H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ and $Y=L^2(\Omega;\mathbb{R}^d)$ and using

$$v\in C^0([0,T];L^2(\Omega;\mathbb{R}^d))\cap L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),$$

we have $v\in C^0_w([0,T];H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Therefore $v(0)$ is an element of $H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$. Similarly, by taking $X=L^2(\Omega;\mathbb{R}^d)$ and $Y=H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ we get $\dot v\in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$, since

$$\dot v\in C^0([0,T];H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)).$$

Therefore $\dot v(0)$ is an element of $L^2(\Omega;\mathbb{R}^d)$. In particular, the initial conditions (1.1.36) are well defined if $v^0\in H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ and $v^1\in L^2(\Omega;\mathbb{R}^d)$. With a similar argument we also have $v-w\in C^0_w([0,T];H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, which yields $v(t)=w(t)$ on $\partial_D\Omega$ in the sense of traces for every $t\in[0,T]$.

We recall an existence result for evolutionary problems of second order in time, whose proof can be found for example in [24]. Let V, H be two separable Hilbert spaces, with embedding $V\hookrightarrow H$ continuous and dense, and for every $t\in[0,T]$ let $\mathcal{B}(t;\cdot)$, $\mathcal{A}_1(t;\cdot)$, $\mathcal{A}_2(t;\cdot)\colon V\times V\to\mathbb{R}$ be three families of continuous bilinear forms satisfying the following properties:

(i) the bilinear form $\mathcal{B}(t;\cdot)$ is symmetric for every $t \in [0,T]$;

(ii) there exist $c_0 > 0$, $c_1 \in \mathbb{R}$ such that $\mathcal{B}(t;\psi,\psi) \geq c_0\|\psi\|_V^2 - c_1\|\psi\|_H^2$ for every $t \in [0,T]$ and $\psi \in V$;

(iii) for every $\psi,\phi \in V$ the function $t \mapsto \mathcal{B}(t;\psi,\phi)$ is continuously differentiable in $[0,T]$;

(iv) there exists $c_2 > 0$ such that $|\dot{\mathcal{B}}(t;\psi,\phi)| \leq c_2\|\psi\|_V\|\phi\|_V$ for every $t \in [0,T]$ and $\psi,\phi \in V$;

(v) for every $\psi,\phi \in V$ the function $t \mapsto \mathcal{A}_1(t;\psi,\phi)$ is continuous in $[0,T]$;

(vi) there exists $c_3 > 0$ such that $|\mathcal{A}_1(t;\psi,\phi)| \leq c_3\|\psi\|_V\|\phi\|_H$ for every $t \in [0,T]$ and $\psi,\phi \in V$;

(vii) for every $\psi,\phi \in V$ the function $t \mapsto \mathcal{A}_2(t;\psi,\phi)$ is continuous in $[0,T]$;

$(viii)$ there exists $c_4 > 0$ such that $|\mathcal{A}_2(t;\psi,\phi)| \leq c_4\|\psi\|_V\|\phi\|_H$ for every $t \in [0,T]$ and $\psi,\phi \in V$;

where $t \mapsto \dot{\mathcal{B}}(t;\psi,\phi)$ denotes the derivative of $t \mapsto \mathcal{B}(t;\psi,\phi)$ for $\psi,\phi \in V$.

Theorem 1.2.8. *Let $\varepsilon > 0$, $v^0 \in V$, $v^1 \in H$, $g \in L^2(0,T;V')$, and $\mathcal{B}(t;\cdot)$, $\mathcal{A}_1(t;\cdot)$, $\mathcal{A}_2(t;\cdot)$, $t \in [0,T]$, be three families of continuous bilinear forms over $V \times V$ satisfying assumptions (i)–$(viii)$ above. Then there exists a function $v \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H) \cap H^2(0,T;V')$ solution for a.e. $t \in (0,T)$ to*

$$\langle \ddot{v}(t),\phi\rangle_{V'} + \mathcal{B}(t;v(t),\phi) + \mathcal{A}_1(t;v(t),\phi) + \mathcal{A}_2(t;\dot{v}(t),\phi) + \varepsilon(\dot{v}(t),\phi)_V = \langle g(t),\phi\rangle_{V'} \quad (1.2.6)$$

for every $\phi \in V$, with initial conditions $v(0) = v^0$ and $\dot{v}(0) = v^1$.

Proof. See [24, Chapitre XVIII, §5, Théorème 1 and Remarque 4]. □

We are now in a position to state the first existence result.

Theorem 1.2.9. *Let $\mathbb{A}$, w, f, F, u^0, and u^1 be as in (1.1.8)–(1.1.13). Let $\mathbb{B}$, $\mathbf{p}$, b, g, v^0, and v^1 be defined according to (1.1.29)–(1.1.32) and (1.1.37). Assume that $\mathbb{B}$ satisfies (1.2.1). Then there exists a generalized solution to (1.1.28) with boundary conditions (1.1.33)–(1.1.35) satisfying the initial conditions (1.1.36).*

Proof. As in [20, Theorem 3.6], the proof is based on a perturbation argument, following the standard procedure of [24]: we first fix $\varepsilon \in (0,1)$ and we study equation (1.2.5) with the additional term

$$\varepsilon(\dot{v}(t),\phi)_{H^1(\Omega\setminus\Gamma_0)} \quad \text{for } \phi \in H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d),$$

and then we let the viscosity parameter ε tend to zero.

Step 1: the perturbed problem. Let $\varepsilon \in (0,1)$ be fixed. We want to show the existence of a function

$$v_\varepsilon \in H^1(0,T;H^1(\Omega \setminus \Gamma_0;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}_D(\Omega \setminus \Gamma_0;\mathbb{R}^d)),$$

with $v_\varepsilon - w \in H^1(0,T;H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d))$, solution for a.e $t \in (0,T)$ to the equation

$$\begin{aligned}&\langle \ddot{v}_\varepsilon(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)} + (\mathbb{B}(t)\nabla v_\varepsilon(t),\nabla\phi)_{L^2(\Omega)} + (\mathbf{p}(t)\nabla v_\varepsilon(t),\phi)_{L^2(\Omega)}\\ &\quad - 2(\nabla\dot{v}_\varepsilon(t)b(t),\phi)_{L^2(\Omega)} + \varepsilon(\dot{v}_\varepsilon(t),\phi)_{H^1(\Omega\setminus\Gamma_0)} = (g(t),\phi)_{L^2(\Omega)} + (F(t),\phi)_{L^2(\partial_N\Omega)}\end{aligned} \quad (1.2.7)$$

for every $\phi \in H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d)$, and which satisfies the initial conditions $v_\varepsilon(0) = v^0$ and $\dot{v}_\varepsilon(0) = v^1$. To this aim, we regularize our coefficient with respect to time by means of a sequence of mollifiers. Let $\{\rho_n\}_n \subset C_c^\infty(\mathbb{R})$ be a sequence of functions such that $\rho_n \geq 0$, $\mathrm{supp}(\rho_n) \subset [-1/n, 1/n]$, and $\int_{\mathbb{R}} \rho_n \,\mathrm{d}x = 1$ for every $n \in \mathbb{N}$, and let us extend our coefficients $\mathbb{B}$ and $\mathbf{p}$ to all $\mathbb{R}$, as done in [20, Theorem 3.6]. To deal with the boundary terms w and F we introduce the function $\hat{g} \in L^2(0,T; H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ defined for a.e. $t \in (0,T)$ as

$$\begin{aligned}\langle \hat{g}(t), \phi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)} \coloneqq &- (\ddot{w}(t),\phi)_{L^2(\Omega)} - ((\rho_n*\mathbb{B})(t)\nabla w(t), \nabla\phi)_{L^2(\Omega)}\\ &- ((\rho_n*\mathbf{p})(t)\nabla w(t),\phi)_{L^2(\Omega)} + 2(\nabla\dot{w}(t)b(t),\phi)_{L^2(\Omega)}\\ &- \varepsilon(\dot{w}(t),\phi)_{H^1(\Omega\setminus\Gamma_0)} + (g(t),\phi)_{L^2(\Omega)} + (F(t),\phi)_{L^2(\partial_N\Omega)}\end{aligned}$$

for $\phi \in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$; the regularity of $\hat{g}$ is a consequence of (1.1.1), (1.1.11), (1.1.12), and Remark 1.1.11. We apply Theorem 1.2.8 with Hilbert spaces $V = H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$, $H = L^2(\Omega;\mathbb{R}^d)$, bilinear forms $\mathcal{B}(t)(\psi,\phi) \coloneqq ((\rho_n*\mathbb{B})(t)\nabla\psi,\nabla\phi)$, $\mathcal{A}_1(t)(\psi,\phi) \coloneqq (\rho_n*\mathbf{p})(t)\nabla\psi,\phi)$, and $\mathcal{A}_2(t)(\psi,\phi) \coloneqq -2(\nabla\psi b(t),\phi)$ for $\psi,\phi \in V$ and $t\in[0,T]$, forcing term $\hat{g}$, and initial conditions $\hat{v}^0 \coloneqq v^0 - w(0)$ and $\hat{v}^1 \coloneqq v^1 - \dot{w}(0)$. For every $n\in\mathbb{N}$ this leads to a solution $\hat{v}_\varepsilon^n$ to (1.2.6) which satisfies the initial conditions $\hat{v}^0$ and $\hat{v}^1$. In particular, the function $v_\varepsilon^n \coloneqq \hat{v}_\varepsilon^n + w$ solves equation (1.2.7) with $\mathbb{B}(t)$ and $\mathbf{p}(t)$ replaced by $(\rho_n*\mathbb{B})(t)$ and $(\rho_n*\mathbf{p})(t)$, respectively, and initial conditions v^0 and v^1.

We fix $t_0\in(0,T]$. By taking $\dot{v}_\varepsilon^n(t) - \dot{w}(t)$ as test function in (1.2.7) and integrating over $(0,t_0)$, we get

$$\begin{aligned}&\int_0^{t_0}\Big[\langle\ddot{v}_\varepsilon^n, \dot{v}_\varepsilon^n - \dot{w}\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)} + ((\rho_n*\mathbb{B})\nabla v_\varepsilon^n, \nabla\dot{v}_\varepsilon^n - \nabla\dot{w})_{L^2(\Omega)}\Big]\,\mathrm{d}t\\ &\quad+\int_0^{t_0}\big[((\rho_n*\mathbf{p})\nabla v_\varepsilon^n, \dot{v}_\varepsilon^n-\dot{w})_{L^2(\Omega)} - 2(\nabla\dot{v}_\varepsilon^n b, \dot{v}_\varepsilon^n - \dot{w})_{L^2(\Omega)} + \varepsilon(\dot{v}_\varepsilon^n,\dot{v}_\varepsilon^n-\dot{w})_{H^1(\Omega\setminus\Gamma_0)}\big]\,\mathrm{d}t\\ &=\int_0^{t_0}\big[(g,\dot{v}_\varepsilon^n-\dot{w})_{L^2(\Omega)} + (F,\dot{v}_\varepsilon^n - \dot{w})_{L^2(\partial_N\Omega)}\big]\,\mathrm{d}t.\end{aligned}\tag{1.2.8}$$

For the first term we use the integration by parts formula

$$\int_0^{t_0}\langle\ddot{v}_\varepsilon^n - \ddot{w}, \dot{v}_\varepsilon^n - \dot{w}\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}\,\mathrm{d}t = \frac{1}{2}\|\dot{v}_\varepsilon^n(t_0) - \dot{w}(t_0)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|v^1 - \dot{w}(0)\|_{L^2(\Omega)}^2$$

to deduce

$$\begin{aligned}\int_0^{t_0}\langle\ddot{v}_\varepsilon^n, \dot{v}_\varepsilon^n - \dot{w}\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}\,\mathrm{d}t = &\frac{1}{2}\|\dot{v}_\varepsilon^n(t_0)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|v^1\|_{L^2(\Omega)}^2 - (\dot{v}_\varepsilon^n(t_0),\dot{w}(t_0))_{L^2(\Omega)}\\ &+ (v^1,\dot{w}(0))_{L^2(\Omega)} + \int_0^{t_0}(\dot{v}_\varepsilon^n,\ddot{w})_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned}\tag{1.2.9}$$

Similarly, for the second term we have

$$\begin{aligned}&\int_0^{t_0}((\rho_n*\mathbb{B})\nabla v_\varepsilon^n, \nabla\dot{v}_\varepsilon^n - \nabla\dot{w})_{L^2(\Omega)}\,\mathrm{d}t\\ &=\frac{1}{2}((\rho_n*\mathbb{B})(t_0)\nabla v_\varepsilon^n(t_0),\nabla v_\varepsilon^n(t_0))_{L^2(\Omega)} - \frac{1}{2}((\rho_n*\mathbb{B})(0)\nabla v^0,\nabla v^0)_{L^2(\Omega)}\\ &\quad-\int_0^{t_0}\Big[\frac{1}{2}(\partial_t(\rho_n*\mathbb{B})\nabla v_\varepsilon^n,\nabla v_\varepsilon^n)_{L^2(\Omega)} + ((\rho_n*\mathbb{B})\nabla v_\varepsilon^n,\nabla\dot{w})_{L^2(\Omega)}\Big]\,\mathrm{d}t.\end{aligned}\tag{1.2.10}$$

Since $v_\varepsilon^n(t) - w(t) \in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ for every $t\in[0,T]$, by (1.2.1) we derive the following estimate

$$\begin{aligned}((\rho_n*\mathbb{B})(t_0)[\nabla v_\varepsilon^n(t_0) - \nabla w(t_0)], \nabla v_\varepsilon^n(t_0) - \nabla w(t_0))_{L^2(\Omega)} \geq\, &c_0\|v_\varepsilon^n(t_0) - w(t_0)\|_{H^1(\Omega\setminus\Gamma_0)}^2\\ &- c_1\|v_\varepsilon^n(t_0) - w(t_0)\|_{L^2(\Omega)}^2.\end{aligned}$$

In particular, thanks to (1.1.11), (1.1.43), and Young's inequality, there exists a constant $C > 0$, independent of n, ε, and t_0, such that

$$\frac{c_0}{2}\|v_\varepsilon^n(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le ((\rho_n{*}\mathbb{B})(t_0)\nabla v_\varepsilon^n(t_0), \nabla v_\varepsilon^n(t_0))_{L^2(\Omega)} + C(1 + \|v_\varepsilon^n(t_0)\|^2_{L^2(\Omega)}). \quad (1.2.11)$$

In addition, since $v_\varepsilon^n(t_0) = v^0 + \int_0^{t_0} \dot v_\varepsilon^n(t)\,\mathrm{d}t$, we get

$$\|v_\varepsilon^n(t_0)\|^2_{L^2(\Omega)} \le 2\|v^0\|^2_{L^2(\Omega)} + 2T\int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{L^2(\Omega)}\,\mathrm{d}t. \quad (1.2.12)$$

By the regularity properties of $\mathbb{B}$, $\mathbf{p}$, g, and w we obtain another constant $C > 0$, independent of n, ε, and t_0, such that

$$\left|\int_0^{t_0} (\dot v_\varepsilon^n, \ddot w)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{L^2(\Omega)}\,\mathrm{d}t, \quad (1.2.13)$$

$$\left|\int_0^{t_0} \frac{1}{2}(\partial_t(\rho_n{*}\mathbb{B})\nabla v_\varepsilon^n, \nabla v_\varepsilon^n)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \|v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t, \quad (1.2.14)$$

$$\left|\int_0^{t_0} ((\rho_n{*}\mathbb{B})\nabla v_\varepsilon^n, \nabla \dot w)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \|v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t, \quad (1.2.15)$$

$$\left|\int_0^{t_0} ((\rho_n{*}\mathbf{p})\nabla v_\varepsilon^n, \dot v_\varepsilon^n - \dot w)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \left[\|v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)} + \|\dot v_\varepsilon^n\|^2_{L^2(\Omega)}\right]\mathrm{d}t, \quad (1.2.16)$$

$$\left|\int_0^{t_0} (g, \dot v_\varepsilon^n - \dot w)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\left(1 + \int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{L^2(\Omega)}\,\mathrm{d}t\right). \quad (1.2.17)$$

Furthermore, we can use Young's inequality and (1.1.11) to find a further constant $C > 0$, independent of n, ε, and t_0, such that

$$\left|(\dot v_\varepsilon^n(t_0), \dot w(t_0))_{L^2(\Omega)} - (v^1, \dot w(0))_{L^2(\Omega)}\right| \le \frac{1}{4}\|\dot v_\varepsilon^n(t_0)\|^2_{L^2(\Omega)} + C. \quad (1.2.18)$$

By formula (1.1.45) we derive

$$2\int_0^{t_0} (\nabla \dot v_\varepsilon^n b, \dot w)_{L^2(\Omega)}\,\mathrm{d}t = -2\int_0^{t_0} (\dot v_\varepsilon^n, \mathrm{div}[\dot w \otimes b])_{L^2(\Omega)}\,\mathrm{d}t,$$

which implies the existence of $C > 0$, independent of n, ε, and t_0, such that

$$\left|2\int_0^{t_0} (\nabla \dot v_\varepsilon^n b, \dot w)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \|\dot v_\varepsilon^n\|_{L^2(\Omega)}\,\mathrm{d}t. \quad (1.2.19)$$

Thanks to (1.1.46), for every $\phi \in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ we have

$$2(\nabla\phi b(t), \phi)_{L^2(\Omega)} = \int_\Omega b(t,y)\cdot\nabla|\phi(y)|^2\mathrm{d}y = -\int_\Omega \mathrm{div}\, b(t,y)|\phi(y)|^2\mathrm{d}y. \quad (1.2.20)$$

Therefore, by (1.1.44) there exits a constant $C > 0$, independent of n, ε, and t_0, such that

$$\left|2\int_0^{t_0} (\nabla \dot v_\varepsilon^n b, \dot v_\varepsilon^n)_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{L^2(\Omega)}\,\mathrm{d}t. \quad (1.2.21)$$

Since $F \in H^1(0,T;L^2(\partial_N\Omega;\mathbb{R}^d))$, we can integrate the last term in (1.2.8) by parts with respect to time, and we obtain

$$\int_0^{t_0} (F, \dot v_\varepsilon^n - \dot w)_{L^2(\partial_N\Omega)}\,\mathrm{d}t = (F(t_0), v_\varepsilon^n(t_0) - w(t_0))_{L^2(\partial_N\Omega)} - (F(0), v^0 - w(0))_{L^2(\partial_N\Omega)} - \int_0^{t_0} (\dot F, v_\varepsilon^n - w)_{L^2(\partial_N\Omega)}\,\mathrm{d}t.$$

We use the previous identity, together with (1.1.1) and Young's inequality, to deduce the existence of a constant $C > 0$, independent of n, ε, and t_0, such that

$$\left|\int_0^{t_0} (F, \dot v_\varepsilon^n - \dot w)_{L^2(\partial_N\Omega)}\,\mathrm{d}t\right| \le \frac{c_0}{8}\|v_\varepsilon^n(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)} + C\left(1 + \int_0^{t_0} \|v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\right). \quad (1.2.22)$$

Finally, again by Young's inequality

$$\varepsilon\int_0^{t_0} (\dot v_\varepsilon^n, \dot v_\varepsilon^n - \dot w)_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t \ge \frac{\varepsilon}{2}\int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t - \frac{1}{2}\int_0^{t_0} \|\dot w\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t. \quad (1.2.23)$$

By combining (1.2.8)–(1.2.19) with (1.2.21)–(1.2.23), we infer

$$\begin{aligned}&\frac{1}{4}\|\dot v_\varepsilon^n(t_0)\|^2_{L^2(\Omega)} + \frac{c_0}{8}\|v_\varepsilon^n(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)} + \frac{\varepsilon}{2}\int_0^{t_0} \|\dot v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\\ &\le C_1 + C_2\int_0^{t_0}\left[\|\dot v_\varepsilon^n\|^2_{L^2(\Omega)} + \|v_\varepsilon^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t,\end{aligned}$$

for two constants C_1 and C_2 independent of n, ε, and t_0.

Thanks to Gronwall's lemma we conclude that there exists $C > 0$, independent of n, ε, and t_0, such that

$$\|\dot v_\varepsilon^n(t_0)\|^2_{L^2(\Omega)} + \|v_\varepsilon^n(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le C \quad \text{for every } t_0 \in [0,T]. \quad (1.2.24)$$

Hence, we have

$$\begin{aligned}&\{v_\varepsilon^n\}_n \quad \text{is bounded in } L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),\\ &\{\dot v_\varepsilon^n\}_n \quad \text{is bounded in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)),\\ &\{\sqrt{\varepsilon}\dot v_\varepsilon^n\}_n \quad \text{is bounded in } L^2(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),\end{aligned}$$

uniformly with respect to n and ε. From these estimates, by using also the equation solved by v_ε^n and (1.2.20), we derive

$$\{\ddot v_\varepsilon^n\}_n \quad \text{is bounded in } L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d)),$$

uniformly with respect to n and ε. Therefore, up to a not relabeled subsequence, v_ε^n converges weakly to a function v_ε in $H^1(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ and $\ddot v_\varepsilon^n$ converges weakly to $\ddot v_\varepsilon$ in $L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ as $n\to\infty$. Finally, we have $v_\varepsilon - w \in H^1(0,T;H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, since $v_\varepsilon^n - w \in H^1(0,T;H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ for every $n\in\mathbb{N}$.

Let us show that v_ε satisfies (1.2.7). Since $\mathbb{B}$ is symmetric, for every $\phi \in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ we have

$$\begin{aligned}&((\rho_n*\mathbb{B})(t)\nabla v_\varepsilon^n(t), \nabla\phi)_{L^2(\Omega)} = (\nabla v_\varepsilon^n(t), (\rho_n*\mathbb{B})(t)\nabla\phi)_{L^2(\Omega)},\\ &((\rho_n*\mathbf{p})(t)\nabla v_\varepsilon^n(t), \phi)_{L^2(\Omega)} = (\nabla v_\varepsilon^n(t), (\rho_n*\mathbf{p}^\star)(t)\phi)_{L^2(\Omega)},\end{aligned}$$

where $\mathbf{p}^\star(t,y) \in \mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ is the transpose operator of $\mathbf{p}(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$, defined for $t\in[0,T]$ and $y\in\overline{\Omega}$ by

$$(\mathbf{p}^\star(t,y)a)\cdot\xi = (\mathbf{p}(t,y)\xi)\cdot a \quad \text{for every } a\in\mathbb{R}^d \text{ and } \xi\in\mathbb{R}^{d\times d}. \quad (1.2.25)$$

By the regularity properties of $\mathbb{B}$ and $\mathbf{p}$, as $n\to\infty$ we have

$$\begin{aligned}&(\rho_n*\mathbb{B})(t)\nabla\phi \to \mathbb{B}(t)\nabla\phi \quad \text{in } L^2(\Omega;\mathbb{R}^{d\times d}) \quad \text{for every } t\in[0,T],\\ &(\rho_n*\mathbf{p}^\star)(t)\phi \to \mathbf{p}^\star(t)\phi \quad \text{in } L^2(\Omega;\mathbb{R}^{d\times d}) \quad \text{for a.e. } t\in(0,T).\end{aligned}$$

Thanks to the strong convergences above and the weak convergences of v_ε^n, $\dot v_\varepsilon^n$ and $\ddot v_\varepsilon^n$, we can pass to the limit as $n\to\infty$ in the PDE solved by v_ε^n and we obtain that the weak limit v_ε satisfies equation (1.2.7) (see Remark 1.2.6). Furthermore, the bound (1.2.24) and the weak convergences of v_ε^n, $\dot v_\varepsilon^n$ and $\ddot v_\varepsilon^n$, imply for every $t\in[0,T]$

$$v_\varepsilon^n(t)\rightharpoonup v_\varepsilon(t)\quad\text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d),\qquad \dot v_\varepsilon^n(t)\rightharpoonup \dot v_\varepsilon(t)\quad\text{in } L^2(\Omega;\mathbb{R}^d)\quad\text{as } n\to\infty.$$

Hence v_ε satisfies the initial conditions $v_\varepsilon(0)=v^0$ and $\dot v_\varepsilon(0)=v^1$.

Step 2. Vanishing viscosity. As already done in Step 1 for v_ε^n, we take $\dot v_\varepsilon-\dot w$ as test function in (1.2.7), and we integrate in $(0,t_0)$ to derive the energy equality

$$\begin{aligned}
&\frac12\|\dot v_\varepsilon(t_0)\|^2_{L^2(\Omega)}+\frac12(\mathbb{B}(t_0)\nabla v_\varepsilon(t_0),\nabla v_\varepsilon(t_0))_{L^2(\Omega)}+\varepsilon\int_0^{t_0}(\dot v_\varepsilon,\dot v_\varepsilon-\dot w)_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\\
&=\frac12\|v^1\|^2_{L^2(\Omega)}+\frac12(\mathbb{B}(0)\nabla v_\varepsilon(0),\nabla v_\varepsilon(0))_{L^2(\Omega)}\\
&\quad+\int_0^{t_0}\Big[\frac12(\dot{\mathbb{B}}\nabla v_\varepsilon,\nabla v_\varepsilon)_{L^2(\Omega)}+(\mathbb{B}\nabla v_\varepsilon,\nabla\dot w)_{L^2(\Omega)}-(\mathbf{p}\nabla v_\varepsilon,\dot v_\varepsilon-\dot w)_{L^2(\Omega)}\Big]\,\mathrm{d}t\\
&\quad+\int_0^{t_0}\big[-(\dot v_\varepsilon\operatorname{div}b,\dot v_\varepsilon)+2(\dot v_\varepsilon,\operatorname{div}[\dot w\otimes b])_{L^2(\Omega)}+(g,\dot v_\varepsilon^n-\dot w)_{L^2(\Omega)}\big]\,\mathrm{d}t\\
&\quad+\int_0^{t_0}\Big[-(\dot F,v_\varepsilon-w)_{L^2(\partial_N\Omega)}-(\dot v_\varepsilon,\ddot w)_{L^2(\Omega)}\Big]\,\mathrm{d}t+(F(t_0),v_\varepsilon(t_0)-w(t_0))_{L^2(\partial_N\Omega)}\\
&\quad+(\dot v_\varepsilon(t_0),\dot w(t_0))_{L^2(\Omega)}-(F(0),v^0-w(0))_{L^2(\partial_N\Omega)}-(v^1,\dot w(0))_{L^2(\Omega)}.
\end{aligned}\tag{1.2.26}$$

By arguing as before and using the ellipticity condition (1.2.1) of $\mathbb{B}$, we get the following estimate:

$$\begin{aligned}
&\frac14\|\dot v_\varepsilon(t_0)\|^2_{L^2(\Omega)}+\frac{c_0}{8}\|v_\varepsilon(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)}+\frac{\varepsilon}{2}\int_0^{t_0}\|\dot v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\\
&\le C_1+C_2\int_0^{t_0}\Big[\|\dot v_\varepsilon\|^2_{L^2(\Omega)}+\|v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\Big]\,\mathrm{d}t,
\end{aligned}\tag{1.2.27}$$

where C_1 and C_2 are two constants independent of ε and t_0. Therefore, Gronwall's lemma yields

$$\|\dot v_\varepsilon(t_0)\|^2_{L^2(\Omega)}+\|v_\varepsilon(t_0)\|^2_{H^1(\Omega\setminus\Gamma_0)}\le C\quad\text{for every } t_0\in[0,T]\tag{1.2.28}$$

for a constant C independent of ε and t_0. This implies that the sequence $\{v_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ and the sequence $\{\dot v_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$. Moreover, by combining (1.2.27) and (1.2.28), we infer

$$\varepsilon\int_0^T\|\dot v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\le C,\tag{1.2.29}$$

for a constant C independent of ε. By formula (1.1.45) for a.e. $t\in(0,T)$ we have

$$(\nabla\dot v_\varepsilon(t)b(t),\phi)_{L^2(\Omega)}=-(\dot v_\varepsilon(t),\operatorname{div}[\phi\otimes b(t)])_{L^2(\Omega)}\quad\text{for every }\phi\in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d).$$

Thanks to (1.2.7) and the previous estimates we conclude that the sequence $\{\ddot v_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Therefore, there exists a subsequence of ε (not relabeled) and a function

$$v\in L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$$

such that the following convergences hold as $\varepsilon\to0^+$:

$$v_\varepsilon\rightharpoonup v\quad\text{in } L^2(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),\tag{1.2.30}$$

$$\dot v_\varepsilon \rightharpoonup \dot v \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{1.2.31}$$
$$\ddot v_\varepsilon \rightharpoonup \ddot v \quad \text{in } L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)). \tag{1.2.32}$$

Moreover, we have $v-w\in L^\infty(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Notice that, a priori, the weak limit v is not unique, but might depend on the particular subsequence chosen.

Let us show that v_ε solves (1.2.5). We fix a test function $\varphi\in L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ for (1.2.5) (see Remark 1.2.6), and for every $\varepsilon\in(0,1)$ we have

$$\begin{aligned}\int_0^T \Big[\langle \ddot v_\varepsilon,\varphi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)} + (\mathbb{B}\nabla v_\varepsilon,\nabla\varphi)_{L^2(\Omega)} + (\mathbf{p}\nabla v_\varepsilon,\varphi)_{L^2(\Omega)} + 2(\dot v_\varepsilon,\operatorname{div}[\varphi\otimes b])_{L^2(\Omega)}\Big]\,\mathrm{d}t\\ +\varepsilon\int_0^T (\dot v_\varepsilon,\varphi)_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t = \int_0^T [(g,\varphi)_{L^2(\Omega)} + (F,\varphi)_{L^2(\partial_N\Omega)}]\,\mathrm{d}t.\end{aligned} \tag{1.2.33}$$

Thanks to (1.2.29), as $\varepsilon\to 0^+$ we get

$$\begin{aligned}\left|\varepsilon\int_0^T(\dot v_\varepsilon,\varphi)_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\right| &\le \sqrt{\varepsilon}\int_0^T\sqrt{\varepsilon}\|\dot v_\varepsilon\|_{H^1(\Omega\setminus\Gamma_0)}\|\varphi\|_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\\ &\le \sqrt{\varepsilon}\|\varphi\|_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))}\left(\int_0^T\varepsilon\|\dot v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t\right)^{1/2}\le\sqrt{\varepsilon}C\to 0.\end{aligned}$$

The last property, together with (1.2.30)–(1.2.32) and (1.2.33), gives that v solves (1.2.5). Finally, by arguing as in Step 1, for every $t\in[0,T]$ we obtain

$$v_\varepsilon(t)\rightharpoonup v(t) \quad \text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d), \qquad \dot v_\varepsilon(t)\rightharpoonup\dot v(t)\quad\text{in } L^2(\Omega;\mathbb{R}^d)\quad \text{as }\varepsilon\to0^+. \tag{1.2.34}$$

This gives the validity of the initial conditions for v. □

The proof of uniqueness is similar to the one in [20] and relies on a standard technique due to Ladyzenskaya [34], which consists in taking as test function in (1.2.5) the primitive of a solution.

Theorem 1.2.10. *Under the assumptions of Theorem 1.2.9, there exists at most one generalized solution to* (1.1.28) *with boundary conditions* (1.1.33)–(1.1.35) *satisfying the initial conditions* (1.1.36).

Proof. By linearity, it is enough to show that the unique generalized solution v to problem (1.1.28) with

$$g=F=w=v^0=v^1=0$$

is $u=0$. The proof is divided into two steps: first, we show the uniqueness in a small interval $[0,t_0]$; then, by a continuity argument, we deduce the uniqueness in the all $[0,T]$.

Step 1. Let $s\in(0,T]$ be fixed and let $\varphi_s\in L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ be defined as

$$\varphi_s(t):=\begin{cases}-\int_t^s v(\tau)\,\mathrm{d}\tau & \text{if } t\in[0,s],\\ 0 & \text{if } t\in[s,T].\end{cases}$$

Notice that $\varphi_s(s)=\varphi_s(T)=0$. Moreover $\dot\varphi_s\in L^2(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, indeed

$$\dot\varphi_s(t)=\begin{cases}v(t) & \text{if } t\in[0,s),\\ 0 & \text{if } t\in(s,T].\end{cases}$$

By taking φ_s as test function in (1.2.5), we get

$$\begin{aligned}\int_0^s\Big[\langle\ddot v(t),\varphi_s(t)\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)} + (\mathbb{B}(t)\nabla v(t),\nabla\varphi_s(t))_{L^2(\Omega)}\Big]\,\mathrm{d}t\\ +\int_0^s\big[(\mathbf{p}(t)\nabla v(t),\varphi_s(t))_{L^2(\Omega)} + 2(\dot v(t),\operatorname{div}[\varphi_s(t)\otimes b(t)])_{L^2(\Omega)}\big]\,\mathrm{d}t = 0.\end{aligned} \tag{1.2.35}$$

We integrate the first term by parts with respect to time, and we obtain

$$\int_0^s \langle \ddot{v}(t), \varphi_s(t)\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)}\,\mathrm{d}t = -\int_0^s (\dot{v}(t), v(t))_{L^2(\Omega)}\,\mathrm{d}t = -\frac{1}{2}\|v(s)\|^2_{L^2(\Omega)}, \tag{1.2.36}$$

since $v^0 = v^1 = \varphi_s(s) = 0$.

Let us rewrite the second term involving $\mathbb{B}$. By Definition 1.2.5 of generalized solution it is easy to see that $\varphi_s \in \mathrm{Lip}([0,T]; H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Therefore, thanks to (1.1.43), we have that $\mathbb{B}\nabla\varphi_s \in \mathrm{Lip}([0,T]; L^2(\Omega;\mathbb{R}^{d\times d}))$. We perform an integration by parts with respect to time and we use the fact that $\varphi_s(s) = 0$ in $H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ to derive

$$\begin{aligned}
&\int_0^s (\mathbb{B}(t)\nabla v(t), \nabla\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= -\int_0^s (\nabla\dot{\varphi}_s(t), \mathbb{B}(t)\nabla\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= -\frac{1}{2}(\mathbb{B}(0)\nabla\varphi_s(0), \nabla\varphi_s(0))_{L^2(\Omega)} - \frac{1}{2}\int_0^s (\dot{\mathbb{B}}(t)\nabla\varphi_s(t), \nabla\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t,
\end{aligned} \tag{1.2.37}$$

By combining (1.2.35)–(1.2.37) we get

$$\begin{aligned}
&\frac{1}{2}\|v(s)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{B}(0)\nabla\varphi_s(0), \nabla\varphi_s(0))_{L^2(\Omega)}\\
&= \int_0^s \left[-\frac{1}{2}(\dot{\mathbb{B}}\nabla\varphi_s, \nabla\varphi_s)_{L^2(\Omega)} + (\mathbf{p}\nabla v, \varphi_s)_{L^2(\Omega)} + 2(\dot{v}, \operatorname{div}[\varphi_s\otimes b])_{L^2(\Omega)}\right]\mathrm{d}t.
\end{aligned} \tag{1.2.38}$$

Let us bound from above the scalar products in the right-hand side of (1.2.38). By the Lipschitz regularity of $\mathbb{B}$ there is $C > 0$ such that $\|\dot{\mathbb{B}}(t)\|_{L^\infty(\Omega)} \le C$ for a.e. $t \in (0,T)$, and so

$$\left|\int_0^s (\dot{\mathbb{B}}(t)\nabla\varphi_s(t), \nabla\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\right| \le C\int_0^s \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t. \tag{1.2.39}$$

For every $t \in [0,T]$ we split $\operatorname{div}[\varphi_s(t)\otimes b(t)]$ into the sum $\varphi_s(t)\operatorname{div} b(t) + \nabla\varphi_s(t)b(t)$. As already pointed in (1.1.44) we have $\operatorname{div} b \in \mathrm{Lip}([0,T]; L^\infty(\Omega))$, therefore we can repeat the same argument as before. By integrating by parts with respect to time and using the equalities $v^0 = \varphi_s(s) = 0$, we obtain

$$\begin{aligned}
\int_0^s (\dot{v}(t), \varphi_s(t)\operatorname{div} b(t))_{L^2(\Omega)}\,\mathrm{d}t &= -\int_0^s (v(t), v(t)\operatorname{div} b(t) + \varphi_s(t)\operatorname{div}\dot{b}(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&\le C\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t,
\end{aligned} \tag{1.2.40}$$

for some $C > 0$ independent of s. For the other term, we first perform an integration by parts with respect to time exploiting the assumptions $v^0 = \varphi_s(s) = 0$, and then we use formula (1.1.46) and the regularity properties (1.1.44) of b to deduce

$$\begin{aligned}
\int_0^s (\dot{v}(t), \nabla\varphi_s(t)b(t))_{L^2(\Omega)}\,\mathrm{d}t &= \int_0^s \left[\frac{1}{2}(v(t)\operatorname{div} b(t), v(t))_{L^2(\Omega)} - (v(t), \nabla\varphi_s(t)\dot{b}(t))_{L^2(\Omega)}\right]\mathrm{d}t\\
&\le C\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t,
\end{aligned} \tag{1.2.41}$$

for a constant $C > 0$ independent of s.

We now split $\mathbf{p}$ as $\hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2$, where $\hat{\mathbf{p}}_1(t,y), \hat{\mathbf{p}}_2(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$ are defined for $t \in [0,T]$ and $y \in \overline{\Omega}$ as

$$\begin{aligned}
\hat{\mathbf{p}}_1(t,y)\xi &:= \mathbf{p}_1(t,y)\xi - \mathbf{p}_1(0,y)\xi = \mathbf{p}_1(t,y)\xi + \xi b(0,y)\operatorname{div}\dot{\Phi}(0,y),\\
\hat{\mathbf{p}}_2(t,y)\xi &:= \mathbf{p}_2(t,y)\xi + \mathbf{p}_1(0,y)\xi = -\xi[\dot{b}(t,y) + b(0,y)\operatorname{div}\dot{\Phi}(0,y)].
\end{aligned}$$

We have $\hat{\mathbf{p}}_1 \in \mathrm{Lip}([0,T];L^\infty(\Omega;\mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})))$, therefore $\dot{\hat{\mathbf{p}}}_1 \in L^\infty(0,T;L^2(\Omega;\mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})))$ and there exists $C>0$ such that $\|\dot{\hat{\mathbf{p}}}_1(t)\|_{L^\infty(\Omega)} \le C$ for a.e. $t\in(0,T)$. By integrating by parts with respect to time and by exploiting the equalities $\varphi_s(s) = \hat{\mathbf{p}}_1(0) = 0$, we get

$$\begin{aligned}\int_0^s (\hat{\mathbf{p}}_1(t)\nabla v(t), \varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t &= \int_0^s (\nabla\dot\varphi_s(t), \hat{\mathbf{p}}_1^\star(t)\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &= -\int_0^s (\nabla\varphi_s(t), \dot{\hat{\mathbf{p}}}_1^\star(t)\varphi_s(t) + \hat{\mathbf{p}}_1^\star(t)\dot\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &= -\int_0^s (\dot{\hat{\mathbf{p}}}_1(t)\nabla\varphi_s(t), \varphi_s(t)+v(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &\le C\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t,\end{aligned} \tag{1.2.42}$$

for a constant $C>0$ independent of s, where $\hat{\mathbf{p}}_1^\star$ is the transpose operator of $\hat{\mathbf{p}}_1$, defined in a similar way to (1.2.25). On the other hand, by (1.1.44) we have $\operatorname{div}\dot b \in L^\infty(0,T;L^2(\Omega))$ and there exits $C>0$ such that $\|\operatorname{div}\dot b(t)\|_{L^\infty(\Omega)} \le C$ for a.e. $t\in(0,T)$. Furthermore, $b(0)\operatorname{div}\dot\Phi(0) \in \mathrm{Lip}(\overline{\Omega};\mathbb{R}^d)$ thanks to (H8). Hence, by performing an integration by parts with respect to the space variable we obtain

$$\begin{aligned}\int_0^s (\hat{\mathbf{p}}_2(t)\nabla v(t), \varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t &= -\int_0^s (\nabla v(t), \varphi_s(t)\otimes(\dot b(t) + b(0)\operatorname{div}\dot\Phi(0)))_{L^2(\Omega)}\,\mathrm{d}t\\ &= \int_0^s (v(t), \operatorname{div}[\varphi_s(t)\otimes(\dot b(t)+b(0)\operatorname{div}\dot\Phi(0))])_{L^2(\Omega)}\,\mathrm{d}t\\ &\le C\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t,\end{aligned} \tag{1.2.43}$$

for a constant $C>0$ independent of s. To derive (1.2.43) we have used formula (1.1.45) with $h := \dot b(t) + b(0)\operatorname{div}\dot\Phi(0)$. This is possible since the function $\dot b(t)+b(0)\operatorname{div}\dot\Phi(0) \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ for a.e. $t\in(0,T)$ and satisfies $(\dot b(t)+b(0)\operatorname{div}\dot\Phi(0))\cdot\nu = 0$ on $(\Gamma\cap\Omega)\cup\partial\Omega$. Indeed $b(t)\cdot\nu=0$ on $(\Gamma\cap\Omega)\cup\partial\Omega$ for every $t\in[0,T]$ by (1.1.3) and (H5), and $\frac{1}{h}[b(t+h)-b(t)] \to \dot b(t)$ in $C^0(\overline{\Omega};\mathbb{R}^d)$ for a.e. $t\in(0,T)$ by (H7).

Thanks to (1.2.38), the coercivity property (1.2.1) of $\mathbb{B}$, the upper bounds (1.2.39)–(1.2.43), and

$$\|\varphi_s(0)\|^2_{L^2(\Omega)} \le T\int_0^s \|v(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t,$$

we conclude

$$\|v(s)\|^2_{L^2(\Omega)} + c_0\|\varphi_s(0)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le \overline{C}\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t \tag{1.2.44}$$

for a constant $\overline{C}$ independent of the parameter s chosen. Let us consider

$$\zeta(t) := \int_0^t v(\tau)\,\mathrm{d}\tau \quad t\in[0,T].$$

Then we can write $\varphi_s(t) = \zeta(t) - \zeta(s)$ for every $t\in[0,s]$; in particular

$$\|\varphi_s(0)\|_{H^1(\Omega\setminus\Gamma_0)} = \|\zeta(s)\|_{H^1(\Omega\setminus\Gamma_0)}, \tag{1.2.45}$$

$$\int_0^s \|\varphi_s(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t \le 2s\|\zeta(s)\|^2_{H^1(\Omega\setminus\Gamma_0)} + 2\int_0^s \|\zeta(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}t. \tag{1.2.46}$$

By combining (1.2.44)–(1.2.46), we obtain

$$\|v(s)\|^2_{L^2(\Omega)} + (c_0 - 2\overline{C}s)\|\zeta(s)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le 2\overline{C}\int_0^s \left[\|v(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}t.$$

If s is small enough, e.g., $s \le t_0 := \frac{c_0}{4C}$, we can apply Gronwall's lemma and obtain

$$v(s) = 0 \quad \text{for every } s \in [0, t_0].$$

Step 2. Notice that the functions $v\colon [0,T] \to L^2(\Omega;\mathbb{R}^d)$ and $\dot v\colon [0,T] \to H_D^{-1}(\Omega \setminus \Gamma_0;\mathbb{R}^d)$ are continuous, therefore we can define

$$t^* := \sup\{t \in [0,T] : v(s) = 0 \text{ for every } s \in [0,t]\}.$$

Thanks to Step 1 and the continuity of v and $\dot v$, we get that $t^* \ge t_0 > 0$ and $v(t^*) = \dot v(t^*) = 0$. Let us assume by contradiction that $t^* < T$. By repeating the strategy adopted in Step 1, with starting point t^* and initial set $\Omega \setminus \Gamma_{t^*}$, we can find a point $t_1 > t^*$ such that $v(s) = 0$ for every $s \in [t^*, t_1]$, which leads to a contradiction. Therefore $t^* = T$ and so $v(s) = 0$ for every $s \in [0,T]$. □

Remark 1.2.11. Let v be the generalized solution to system (1.1.28) with boundary conditions (1.1.33)–(1.1.35) and initial conditions (1.1.36), and let v_ε be its viscous approximation obtained by solving (1.2.7). By using (1.2.28), (1.2.34), and the weak lower semicontinuity of the norm, there is a constant $C > 0$, independent of t, such that

$$\|\dot v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le C \quad \text{for every } t \in [0,T]. \tag{1.2.47}$$

If we consider u defined by (1.1.27), by using formulas (1.1.49) it is immediate to check that for another constant $C > 0$, independent of t, we have

$$\|\dot u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^1(\Omega\setminus\Gamma_t)} \le C \quad \text{for every } t \in [0,T].$$

1.3 Energy balance

In this section, following [20, Proposition 3.11], we prove an energy equality for the generalized solution v to (1.1.28). In order to state this result, we introduce the following definition for the energy: given a function $z \in C^0_w([0,T]; H^1(\Omega \setminus \Gamma_0;\mathbb{R}^d))$, with distributional derivative $\dot z \in C^0_w([0,T]; L^2(\Omega;\mathbb{R}^d))$, we set

$$\mathcal{E}_{\mathbb{B}}(z;t) := \frac{1}{2}\|\dot z(t)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{B}(t)\nabla z(t), \nabla z(t))_{L^2(\Omega)} \quad \text{for } t \in [0,T], \tag{1.3.1}$$

where $\mathbb{B}$ is the operator defined in (1.1.29).

Proposition 1.3.1. *Under the assumptions of Theorem 1.2.9, let v be the unique generalized solution to (1.1.28) with boundary conditions (1.1.33)–(1.1.35), satisfying the initial conditions (1.1.36). Then the energy $t \mapsto \mathcal{E}_{\mathbb{B}}(v;t)$ is a continuous function from $[0,T]$ to $\mathbb{R}$ and satisfies*

$$\mathcal{E}_{\mathbb{B}}(v;t) = \mathcal{E}_{\mathbb{B}}(v;0) + \mathcal{R}(v;t) \quad \textit{for every } t \in [0,T], \tag{1.3.2}$$

where the remainder $\mathcal{R}$ is defined as

$$\begin{aligned}\mathcal{R}(v;t) := &\int_0^t \left[\frac{1}{2}(\dot{\mathbb{B}}\nabla v, \nabla v)_{L^2(\Omega)} + (\mathbb{B}\nabla v, \nabla \dot w)_{L^2(\Omega)} - (\mathbf{p}\nabla v, \dot v - \dot w)_{L^2(\Omega)}\right] \mathrm{d}t \\ &+ \int_0^t \left[-(\dot v \operatorname{div} b, \dot v)_{L^2(\Omega)} + 2(\dot v, \operatorname{div}[\dot w \otimes b])_{L^2(\Omega)} + (g, \dot v - \dot w)_{L^2(\Omega)}\right] \mathrm{d}t \\ &+ \int_0^t \left[-(\dot F, v - w)_{L^2(\partial_N\Omega)} - (\dot v, \ddot w)_{L^2(\Omega)}\right] \mathrm{d}t + (F(t), v(t) - w(t))_{L^2(\partial_N\Omega)} \\ &+ (\dot v(t), \dot w(t))_{L^2(\Omega)} - (F(0), v(0) - w(0))_{L^2(\partial_N\Omega)} - (\dot v(0), \dot w(0))_{L^2(\Omega)}.\end{aligned}$$

Remark 1.3.2. If the solution v were smooth enough, then (1.3.2) would be straightforward by taking $\dot{v} - \dot{w}$ as test function in (1.2.5). In our case, we follow the proof of [20, Proposition 3.11] by approximating $\dot{v} - \dot{w}$ with $H^1_D(\Omega \setminus \Gamma_0; \mathbb{R}^d)$-valued functions, in the same spirit of [40, Chapter 3, Lemma 8.3].

Proof of Proposition 1.3.1. The function $t \mapsto (F(t), v(t))_{L^2(\partial_N \Omega)}$ is continuous from $[0,T]$ to $\mathbb{R}$, since $F \in C^0([0,T]; L^2(\partial_N\Omega;\mathbb{R}^d))$ and $v \in C^0_w([0,T]; L^2(\partial_N\Omega;\mathbb{R}^d))$, thanks to (1.1.1). Similarly, also the function $t \mapsto (\dot{v}(t), \dot{w}(t))_{L^2\Omega)}$ is continuous from $[0,T]$ to $\mathbb{R}$. Then, to prove that $t \mapsto \mathcal{E}_{\mathbb{B}}(v;t)$ is continuous, it is enough to show that equality (1.3.2) holds.

For $t = 0$ equality (1.3.2) is trivial. Let $t_0 \in (0,T]$ be fixed and let θ_0 denote the characteristic function of the time interval $(0,t_0)$. For every $\delta > 0$, we call $\theta_\delta \colon \mathbb{R} \to \mathbb{R}$ the function equals to 1 in $[\delta, t_0 - \delta]$, 0 outside $[0,t_0]$, and which is linear in $[0,\delta]$ and $[t_0 - \delta, t_0]$; notice that $\theta_\delta \to \delta_0$ in $L^1(\mathbb{R})$ as $\delta \to 0^+$. We also consider a sequence of mollifiers $\{\rho_m\}_m \subset C^\infty_c(\mathbb{R})$.

We want to approximate the function $\theta_0(v - w) \colon [0,T] \to H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d)$ by a suitable sequence of functions in $C^\infty_c(\mathbb{R}; H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d))$. To this aim, we first extend the function $\theta_0(v-w)$ to all $\mathbb{R}$ be setting $\theta_0(v-w) = 0$ outside $[0,T]$. In a similar way we extend every function multiplied by either θ_0 or θ_δ.

For brevity, we set $z \coloneqq v - w$. In view of the above definitions, for every m and δ fixed we have

$$\rho_m * (\theta_\delta z) \in C^\infty_c(\mathbb{R}; H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d)),$$

since $\theta_\delta z \in L^\infty(\mathbb{R}; H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d))$ has compact support, and the regularity follows from

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\rho_m * (\theta_\delta z)) = \left(\frac{\mathrm{d}^k \rho_m}{\mathrm{d}t^k}\right) * (\theta_\delta z).$$

Moreover, we have $\rho_m * (\theta_\delta \dot{z}) \in C^\infty_c(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ and

$$\rho_m * (\theta_\delta \dot{z}) = \dot{\rho}_m * (\theta_\delta z) - \rho_m * (\dot{\theta}_\delta z), \tag{1.3.3}$$

which implies $\rho_m * (\theta_\delta \dot{z}) \in C^\infty_c(\mathbb{R}; H^1_D(\Omega \setminus \Gamma_0;\mathbb{R}^d))$. In a similar way, we can deduce that $\rho_m * (\theta_\delta \ddot{z}) \in C^\infty_c(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$, since $\rho_m * (\theta_\delta \ddot{z}) \in C^\infty_c(\mathbb{R}; H^{-1}_D(\Omega \setminus \Gamma_0;\mathbb{R}^d))$ and

$$\rho_m * (\theta_\delta \ddot{z}) = \dot{\rho}_m * (\theta_\delta \dot{z}) - \rho_m * (\dot{\theta}_\delta \dot{z}). \tag{1.3.4}$$

Since $\rho_m * (\theta_\delta \dot{z}) \in C^\infty_c(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$, we derive

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t} \|\rho_m * (\theta_\delta \dot{z})\|^2_{L^2(\Omega)} \,\mathrm{d}t = 0.$$

By differentiating the integrand and exploiting the properties of the convolution, we get

$$0 = \int_{\mathbb{R}} \left[(\rho_m * (\dot{\theta}_\delta \dot{z}), \rho_m * (\theta_\delta \dot{z}))_{L^2(\Omega)} + (\rho_m * (\theta_\delta \ddot{z}), \rho_m * (\theta_\delta \dot{z}))_{L^2(\Omega)} \right] \mathrm{d}t, \tag{1.3.5}$$

being $\rho_m * (\theta_\delta \ddot{z})$ well defined in $L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$. Let us study separately the behavior of each term in (1.3.5) as $\delta \to 0^+$, keeping m fixed. For the first one we have

$$\begin{aligned} &\lim_{\delta \to 0^+} \int_{\mathbb{R}} (\rho_m * (\dot{\theta}_\delta \dot{z}), \rho_m * (\theta_\delta \dot{z}))_{L^2(\Omega)} \,\mathrm{d}t \\ &= \lim_{\delta \to 0^+} \int_{\mathbb{R}} \dot{\theta}_\delta (\dot{z}, \rho_m * \rho_m * (\theta_0 \dot{z}))_{L^2(\Omega)} \,\mathrm{d}t \\ &= -(\dot{z}(t_0), (\rho_m * \rho_m * (\theta_0 \dot{z}))(t_0))_{L^2(\Omega)} + (\dot{z}(0), (\rho_m * \rho_m * (\theta_0 \dot{z}))(0))_{L^2(\Omega)}. \end{aligned} \tag{1.3.6}$$

To obtain (1.3.6), we have split θ_δ as $(\theta_\delta - \theta_0) + \theta_0$ and used the following facts:

$$\begin{aligned}
&\rho_m{*}(\theta_\delta \dot z) \to \rho_m{*}(\theta_0 \dot z) \quad \text{in } L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d)) \quad \text{as } \delta\to 0^+,\\
&\{\rho_m{*}(\dot\theta_\delta \dot z)\}_\delta \quad \text{is uniformly bounded in } L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d)),\\
&s \mapsto (\dot z(s), (\rho_m{*}\rho_m{*}(\theta_0\dot z))(s))_{L^2(\Omega)} \quad \text{is continuous on } \mathbb{R}.
\end{aligned}$$

The last property is a consequence of the fact that $\rho_m{*}\rho_m{*}(\theta_0\dot z) \in C^0(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ and $\dot z \in C^0_w(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ (see Remark 1.2.7). The second term of (1.3.5) satisfies

$$\lim_{\delta\to 0^+} \int_{\mathbb{R}} (\rho_m{*}(\theta_\delta \ddot z), \rho_m{*}(\theta_\delta \dot z))_{L^2(\Omega)}\,\mathrm{d}t = \int_{\mathbb{R}} (\rho_m{*}(\theta_0 \ddot z), \rho_m{*}(\theta_0 \dot z))_{L^2(\Omega)}\,\mathrm{d}t. \tag{1.3.7}$$

Indeed, the sequence $\{\rho_m{*}(\theta_\delta \ddot z)\}_\delta$ is uniformly bounded in $L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ by (1.3.4) and $\rho_m{*}(\theta_\delta \ddot z)$ converges strongly to $\rho_m{*}(\theta_0 \ddot z)$ in $L^2(\mathbb{R}; H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ as $\delta\to 0^+$. These two facts imply that $\rho_m{*}(\theta_0\ddot z)$ is an element of $L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ and $\rho_m{*}(\theta_\delta \ddot z)$ converges to $\rho_m{*}(\theta_0\ddot z)$ weakly in $L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^d))$ as $\delta\to 0^+$. By combining (1.3.5)–(1.3.7), we get

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho_m{*}(\theta_0 \ddot z), \rho_m{*}(\theta_0 \dot z))_{L^2(\Omega)}\,\mathrm{d}t\\
&= (\dot z(t_0), (\rho_m{*}\rho_m{*}(\theta_0\dot z))(t_0))_{L^2(\Omega)} - (\dot z(0), (\rho_m{*}\rho_m{*}(\theta_0\dot z))(0))_{L^2(\Omega)}.
\end{aligned}$$

In a similar way, we can prove

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho_m{*}(\theta_0 \ddot w), \rho_m{*}(\theta_0 \dot w))_{L^2(\Omega)}\,\mathrm{d}t\\
&= (\dot w(t_0), (\rho_m{*}\rho_m{*}(\theta_0\dot w))(t_0))_{L^2(\Omega)} - (\dot w(0), (\rho_m{*}\rho_m{*}(\theta_0\dot w))(0))_{L^2(\Omega)}.
\end{aligned}$$

From the last two identities we deduce

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho_m{*}(\theta_0 \ddot v), \rho_m{*}(\theta_0 \dot z))_{L^2(\Omega)}\,\mathrm{d}t\\
&= (\dot v(t_0), (\rho_m{*}\rho_m{*}(\theta_0\dot v))(t_0))_{L^2(\Omega)} - (\dot v(0), (\rho_m{*}\rho_m{*}(\theta_0\dot v))(0))_{L^2(\Omega)}\\
&\quad - ((\rho_m{*}\rho_m{*}(\theta_0\dot v))(t_0), \dot w(t_0))_{L^2(\Omega)} - (\dot v(t_0), (\rho_m{*}\rho_m{*}(\theta_0\dot w))(t_0))_{L^2(\Omega)}\\
&\quad + ((\rho_m{*}\rho_m{*}(\theta_0\dot v))(0), \dot w(0))_{L^2(\Omega)} + (\dot v(0), (\rho_m{*}\rho_m{*}(\theta_0\dot w))(0))_{L^2(\Omega)}\\
&\quad + \int_{\mathbb{R}} (\rho_m{*}(\theta_0 \dot v), \rho_m{*}(\theta_0 \ddot w))_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned} \tag{1.3.8}$$

We apply the same argument to the function $(\mathbb{B}\rho_m{*}(\theta_\delta\nabla v), \rho_m{*}(\theta_\delta\nabla v))_{L^2(\Omega)} \in W^{1,\infty}(\mathbb{R})$, which has compact support. Starting from the identity

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\rho_m{*}(\theta_\delta\nabla v), \rho_m{*}(\theta_\delta\nabla v))_{L^2(\Omega)}\,\mathrm{d}t = 0,$$

we infer

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \Big[(\dot{\mathbb{B}}\rho_m{*}(\theta_\delta\nabla v), \rho_m{*}(\theta_\delta\nabla v))_{L^2(\Omega)} + 2(\rho_m{*}(\theta_\delta\mathbb{B}\nabla v), \rho_m{*}(\theta_\delta\nabla \dot v))_{L^2(\Omega)}\Big]\,\mathrm{d}t\\
&\quad + \int_{\mathbb{R}} 2(\rho_m{*}(\theta_\delta\mathbb{B}\nabla v), \rho_m{*}(\dot\theta_\delta\nabla v))_{L^2(\Omega)}\,\mathrm{d}t\\
&\quad + \int_{\mathbb{R}} 2(\mathbb{B}\rho_m{*}(\theta_\delta\nabla v) - \rho_m{*}(\theta_\delta\mathbb{B}\nabla v), \dot\rho_m{*}(\theta_\delta\nabla v))_{L^2(\Omega)}\,\mathrm{d}t,
\end{aligned} \tag{1.3.9}$$

where $\rho_m{*}(\theta_\delta\nabla\dot v)$ is well defined in $L^2(\mathbb{R}; L^2(\Omega;\mathbb{R}^{d\times d}))$ as

$$\rho_m{*}(\theta_\delta\nabla\dot v) := \nabla(\rho_m{*}(\theta_\delta\dot v)) = \dot\rho_m{*}(\theta_\delta\nabla v) - \rho_m{*}(\dot\theta_\delta\nabla v). \tag{1.3.10}$$

Since $\rho_m*(\theta_\delta\dot v)$ is uniformly bounded in $L^2(\mathbb{R};H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ by (1.3.3), and it converges strongly to $\rho_m*(\theta_0\dot v)$ in $L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^d))$ as $\delta\to 0^+$, we get

$$\rho_m*(\theta_\delta\nabla\dot v)\rightharpoonup\rho_m*(\theta_0\nabla\dot v):=\nabla(\rho_m*(\theta_0\dot v))\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+. \tag{1.3.11}$$

Moreover, the operator $\mathbb{B}\in\mathrm{Lip}(\mathbb{R};L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})))$, which yields that $\dot{\mathbb{B}}$ is an element of $L^\infty(\mathbb{R};L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})))$ and there exists $C>0$ such that $\|\dot{\mathbb{B}}(s)\|_{L^\infty(\Omega)}\le C$ for a.e. $s\in\mathbb{R}$. By passing to the limit as $\delta\to 0^+$ in (1.3.9), thanks to (1.3.11) and the properties:

$$\begin{aligned}
&\mathbb{B}\rho_m*(\theta_\delta\nabla v)\to\mathbb{B}\rho_m*(\theta_0\nabla v)\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+,\\
&\dot{\mathbb{B}}\rho_m*(\theta_\delta\nabla v)\to\dot{\mathbb{B}}\rho_m*(\theta_0\nabla v)\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+,\\
&\rho_m*(\theta_\delta\mathbb{B}\nabla v)\to\rho_m*(\theta_0\mathbb{B}\nabla v)\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+,\\
&\rho_m*(\theta_\delta\nabla v)\to\rho_m*(\theta_0\nabla v)\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+,\\
&\dot\rho_m*(\theta_\delta\nabla v)\to\dot\rho_m*(\theta_0\nabla v)\quad\text{in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d}))\quad\text{as }\delta\to 0^+,\\
&\{\rho_m*(\dot\theta_\delta\nabla v)\}_\delta\quad\text{is uniformly bounded in } L^2(\mathbb{R};L^2(\Omega;\mathbb{R}^{d\times d})),\\
&s\mapsto((\rho_m*\rho_m*(\theta_0\mathbb{B}\nabla v))(s),\nabla v(s))_{L^2(\Omega)}\quad\text{is continuous on }\mathbb{R},
\end{aligned}$$

we obtain the following identity

$$\begin{aligned}
&\int_{\mathbb{R}}(\rho_m*(\theta_0\mathbb{B}\nabla v),\rho_m*(\theta_0\nabla\dot z))_{L^2(\Omega)}\,\mathrm{d}t\\
&=((\rho_m*\rho_m*(\theta_0\mathbb{B}\nabla v))(t_0),\nabla v(t_0))_{L^2(\Omega)}-((\rho_m*\rho_m*(\theta_0\mathbb{B}\nabla v))(0),\nabla v(0))_{L^2(\Omega)}\\
&\quad-\int_{\mathbb{R}}\left[\frac12(\dot{\mathbb{B}}\rho_m*(\theta_0\nabla v),\rho_m*(\theta_0\nabla v))_{L^2(\Omega)}+(\rho_m*(\theta_0\mathbb{B}\nabla v),\rho_m*(\theta_0\nabla\dot w))_{L^2(\Omega)}\right]\mathrm{d}t\\
&\quad+\int_{\mathbb{R}}(\rho_m*(\theta_0\mathbb{B}\nabla v)-\mathbb{B}\rho_m*(\theta_0\nabla v),\dot\rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}\tag{1.3.12}$$

Let us consider the function $(\rho_m*(\theta_\delta F),\rho_m*(\theta_\delta z))_{L^2(\partial_N\Omega)}\in C_c^\infty(\mathbb{R})$. We have

$$\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m*(\theta_\delta F),\rho_m*(\theta_\delta z))_{L^2(\partial_N\Omega)}\,\mathrm{d}t=0,$$

which implies

$$\begin{aligned}
0=&\int_{\mathbb{R}}\left[(\rho_m*(\dot\theta_\delta F),\rho_m*(\theta_\delta z))_{L^2(\partial_N\Omega)}+(\rho_m*(\theta_\delta\dot F),\rho_m*(\theta_\delta z))_{L^2(\partial_N\Omega)}\right]\mathrm{d}t\\
&+\int_{\mathbb{R}}\left[(\rho_m*(\theta_\delta F),\rho_m*(\dot\theta_\delta z))_{L^2(\partial_N\Omega)}+(\rho_m*(\theta_\delta F),\rho_m*(\theta_\delta\dot z))_{L^2(\partial_N\Omega)}\right]\mathrm{d}t,
\end{aligned}$$

being $\rho_m*(\theta_\delta\dot z)$ well defined in $L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))$ (see (1.1.1) and (1.3.3)). By the following facts:

$$\begin{aligned}
&\rho_m*(\theta_\delta F)\to\rho_m*(\theta_0 F)\quad\text{in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))\quad\text{as }\delta\to 0^+,\\
&\rho_m*(\theta_\delta\dot F)\to\rho_m*(\theta_0\dot F)\quad\text{in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))\quad\text{as }\delta\to 0^+,\\
&\rho_m*(\theta_\delta z)\to\rho_m*(\theta_0 z)\quad\text{in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))\quad\text{as }\delta\to 0^+,\\
&\rho_m*(\theta_\delta\dot z)\rightharpoonup\rho_m*(\theta_0\dot z)\quad\text{in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))\quad\text{as }\delta\to 0^+,\\
&\{\rho_m*(\dot\theta_\delta F)\}_\delta\quad\text{is uniformly bounded in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d)),\\
&\{\rho_m*(\dot\theta_\delta z)\}_\delta\quad\text{is uniformly bounded in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d)),\\
&s\mapsto(F(s),(\rho_m*\rho_m*(\theta_0 z))(s))_{L^2(\partial_N\Omega)}\quad\text{is continuous on }\mathbb{R},\\
&s\mapsto((\rho_m*\rho_m*(\theta_0 F))(s),z(s))_{L^2(\partial_N\Omega)}\quad\text{is continuous on }\mathbb{R},
\end{aligned}$$

as $\delta \to 0^+$ we get

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho_m*(\theta_0 F), \rho_m*(\theta_0\dot z))_{L^2(\partial_N\Omega)}\,\mathrm{d}t\\
&= ((\rho_m*\rho_m*(\theta_0 F))(t_0), z(t_0))_{L^2(\partial_N\Omega)} + (F(t_0), (\rho_m*\rho_m*(\theta_0 z))(t_0))_{L^2(\partial_N\Omega)}\\
&\quad - ((\rho_m*\rho_m*(\theta_0 F))(0), z(0))_{L^2(\partial_N\Omega)} - (F(0), (\rho_m*\rho_m*(\theta_0 z))(0))_{L^2(\partial_N\Omega)}\\
&\quad - \int_{\mathbb{R}} (\rho_m*(\theta_0\dot F), \rho_m*(\theta_0 z))_{L^2(\partial_N\Omega)}\,\mathrm{d}t.
\end{aligned} \tag{1.3.13}$$

We know that the function v solves

$$\begin{aligned}
&\int_{\mathbb{R}} \Big[\langle\theta_0\ddot v,\varphi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_0)} + (\theta_0\mathbb{B}\nabla v,\nabla\varphi)_{L^2(\Omega)} + (\theta_0\mathbf{p}\nabla v,\varphi)_{L^2(\Omega)} + 2(\theta_0\dot v,\operatorname{div}[\varphi\otimes b])_{L^2(\Omega)}\Big]\,\mathrm{d}t\\
&= \int_{\mathbb{R}} \big[(\theta_0 g,\varphi)_{L^2(\Omega)} + (\theta_0 F,\varphi)_{L^2(\partial_N\Omega)}\big]\,\mathrm{d}t
\end{aligned}$$

for every $\varphi \in L^2(\mathbb{R}; H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ (see Remark 1.2.6). In particular, by considering the function $\varphi := \rho_m*(\rho_m*(\theta_0\dot z))$, which belongs to $L^2(\mathbb{R}; H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ thanks to (1.3.3), and exploiting the properties of the convolution, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \big[(\rho_m*(\theta_0\ddot v), \rho_m*(\theta_0\dot z))_{L^2(\Omega)} + (\rho_m*(\theta_0\mathbb{B}\nabla v), \rho_m*(\theta_0\nabla\dot z))_{L^2(\Omega)}\big]\,\mathrm{d}t\\
&\quad + \int_{\mathbb{R}} \big[(\rho_m*(\theta_0\mathbf{p}\nabla v), \rho_m*(\theta_0\dot z))_{L^2(\Omega)} + 2(\rho_m*(\theta_0\dot v\otimes b), \rho_m*(\theta_0\nabla\dot z))_{L^2(\Omega)}\big]\,\mathrm{d}t\\
&\quad + \int_{\mathbb{R}} 2(\rho_m*(\theta_0\dot v\operatorname{div} b), \rho_m*(\theta_0\dot z))_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_{\mathbb{R}} \big[(\rho_m*(\theta_0 g), \rho_m*(\theta_0\dot z))_{L^2(\Omega)} + (\rho_m*(\theta_0 F), \rho_m*(\theta_0\dot z))_{L^2(\partial_N\Omega)}\big]\,\mathrm{d}t.
\end{aligned} \tag{1.3.14}$$

We combine (1.3.8) and (1.3.12)–(1.3.14) to derive the following identity

$$\begin{aligned}
&(\dot v(t_0), (\rho_m*\rho_m*(\theta_0\dot v))(t_0))_{L^2(\Omega)} + ((\rho_m*\rho_m*(\theta_0\mathbb{B}\nabla v))(t_0), \nabla v(t_0))_{L^2(\Omega)}\\
&- (\dot v(0), (\rho_m*\rho_m*(\theta_0\dot v))(0))_{L^2(\Omega)} - ((\rho_m*\rho_m*(\theta_0\mathbb{B}\nabla v))(0), \nabla v(0))_{L^2(\Omega)} = \mathcal{R}_m(t_0),
\end{aligned} \tag{1.3.15}$$

where

$$\begin{aligned}
\mathcal{R}_m(t_0) :=& \int_{\mathbb{R}} \Big[\frac12(\dot{\mathbb{B}}\rho_m*(\theta_0\nabla v), \rho_m*(\theta_0\nabla v))_{L^2(\Omega)} + (\rho_m*(\theta_0\mathbb{B}\nabla v), \rho_m*(\theta_0\nabla\dot w))_{L^2(\Omega)}\Big]\,\mathrm{d}t\\
&+ \int_{\mathbb{R}} (\rho_m*(\theta_0 g) - \rho_m*(\theta_0\mathbf{p}\nabla v) - 2\rho_m*(\theta_0\dot v\operatorname{div} b), \rho_m*(\theta_0(\dot v-\dot w)))_{L^2(\Omega)}\,\mathrm{d}t\\
&+ \int_{\mathbb{R}} (\mathbb{B}\rho_m*(\theta_0\nabla v) - \rho_m*(\theta_0\mathbb{B}\nabla v), \dot\rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t\\
&- \int_{\mathbb{R}} 2(\rho_m*(\theta_0\dot v\otimes b), \rho_m*(\theta_0(\nabla\dot v-\nabla\dot w)))_{L^2(\Omega)}\,\mathrm{d}t\\
&- \int_{\mathbb{R}} \Big[(\rho_m*(\theta_0\dot F), \rho_m*(\theta_0(v-w)))_{L^2(\partial_N\Omega)} + (\rho_m*(\theta_0\dot v), \rho_m*(\theta_0\ddot w))_{L^2(\Omega)}\Big]\,\mathrm{d}t\\
&+ ((\rho_m*\rho_m*(\theta_0 F))(t_0), v(t_0)-w(t_0))_{L^2(\partial_N\Omega)} + ((\rho_m*\rho_m*(\theta_0\dot v))(t_0), \dot w(t_0))_{L^2(\Omega)}\\
&+ (F(t_0), (\rho_m*\rho_m*(\theta_0(v-w)))(t_0))_{L^2(\partial_N\Omega)} + (\dot v(t_0), (\rho_m*\rho_m*(\theta_0\dot w))(t_0))_{L^2(\Omega)}\\
&- ((\rho_m*\rho_m*(\theta_0 F))(0), v(0)-w(0))_{L^2(\partial_N\Omega)} - ((\rho_m*\rho_m*(\theta_0\dot v))(0), \dot w(0))_{L^2(\Omega)}\\
&- (F(0), (\rho_m*\rho_m*(\theta_0(v-w)))(0))_{L^2(\partial_N\Omega)} - (\dot v(0), (\rho_m*\rho_m*(\theta_0\dot w))(0))_{L^2(\Omega)}.
\end{aligned}$$

Let us now perform the second passage to the limit: we let the index m associated to the convolution ρ_m tend to ∞. Let us study separately the asymptotic of the terms appearing (1.3.15). The left-hand side converges to

$$\frac{1}{2}\|\dot v(t_0)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{B}(t_0)\nabla v(t_0), \nabla v(t_0))_{L^2(\Omega)} - \frac{1}{2}\|\dot v(0)\|^2_{L^2(\Omega)} - \frac{1}{2}(\mathbb{B}(0)\nabla v(0), \nabla v(0))_{L^2(\Omega)}.$$

Here we have used the weak continuity of $\dot v$ and ∇v (see Remark 1.2.7), the presence of θ_0, and the fact that $\rho_m * \rho_m$ is still a smooth even mollifier. Similarly, the last eight terms of $\mathcal{R}_m(t_0)$ converge to

$$(F(t_0), v(t_0))_{L^2(\partial_N\Omega)} + (\dot v(t_0), \dot w(t_0))_{L^2(\partial_N\Omega)} - (F(0), v^0)_{L^2(\partial_N\Omega)} - (v^1, \dot w(0))_{L^2(\partial_N\Omega)}.$$

By the strong approximation property of the convolution and the dominated convergence theorem, it is easy check the following convergences:

$$\begin{aligned}
&\lim_{m\to\infty}\int_{\mathbb{R}}(\dot{\mathbb{B}}\rho_m*(\theta_0\nabla v), \rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\dot{\mathbb{B}}\nabla v, \nabla v)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\mathbb{B}\nabla v), \rho_m*(\theta_0\nabla \dot w))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\mathbb{B}\nabla v, \nabla \dot w)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0 g), \rho_m*(\theta_0(\dot v-\dot w)))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(g, \dot v-\dot w)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\mathbf{p}\nabla v), \rho_m*(\theta_0(\dot v-\dot w)))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\mathbf{p}\nabla v, \dot v-\dot w)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\dot v\operatorname{div} b), \rho_m*(\theta_0\dot v))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\dot v\operatorname{div} b, \dot v-\dot w)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\dot v\otimes b), \rho_m*(\theta_0\nabla \dot w))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\dot v\otimes b, \nabla\dot w)_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\dot F), \rho_m*(\theta_0(v-w)))_{L^2(\partial_N\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\dot F, v-w)_{L^2(\partial_N\Omega)}\,\mathrm{d}t,\\
&\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\dot v), \rho_m*(\theta_0\ddot w))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t_0}(\dot v, \ddot w)_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

For the remaining two terms of $\mathcal{R}_m(t_0)$ we claim

$$\lim_{m\to\infty}\int_{\mathbb{R}}(\rho_m*(\theta_0\dot v\otimes b), \rho_m*(\theta_0\nabla\dot v))_{L^2(\Omega)}\,\mathrm{d}t = -\frac{1}{2}\int_0^{t_0}(\dot v\operatorname{div} b, \dot v)_{L^2(\Omega)}\,\mathrm{d}t, \tag{1.3.16}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}}(\mathbb{B}\rho_m*(\theta_0\nabla v) - \rho_m*(\theta_0\mathbb{B}\nabla v), \dot\rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t = 0. \tag{1.3.17}$$

Once proved the claim we are done: indeed, by combining the previous convergences with identity (1.3.15) we get (1.3.2).

For simplicity we set

$$\zeta_m := \rho_m*(\theta_0\dot v\otimes b) - \rho_m*(\theta_0\dot v)\otimes b, \quad \varphi_m := \rho_m*(\theta_0\dot v).$$

Hence, we may rephrase the integral in (1.3.16) as

$$\begin{aligned}
&\int_{\mathbb{R}}(\rho_m*(\theta_0\dot v\otimes b), \rho_m*(\theta_0\nabla\dot v))_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_{\mathbb{R}}\left[(\varphi_m\otimes b, \nabla\varphi_m)_{L^2(\Omega)} + (\zeta_m, \nabla\varphi_m)_{L^2(\Omega)}\right]\mathrm{d}t.
\end{aligned} \tag{1.3.18}$$

By integrating by parts (recall that b satisfies $b \cdot \nu = 0$ on $\partial\Omega \cup \Gamma_0$) we get

$$\lim_{m\to\infty} \int_{\mathbb{R}} (\varphi_m \otimes b, \nabla\varphi_m)_{L^2(\Omega)} = -\frac{1}{2} \lim_{m\to\infty} \int_{\mathbb{R}} (\varphi_m \operatorname{div} b, \varphi_m)_{L^2(\Omega)} = -\frac{1}{2} \int_0^{t_0} (\dot v \operatorname{div} b, \dot v)_{L^2(\Omega)} \, \mathrm{d}t,$$

since $\varphi_m \to \theta_0 \dot v$ in $L^2(\mathbb{R}; L^2(\Omega; \mathbb{R}^d))$ as $m \to \infty$. Therefore we have to prove that the second term in the right-hand side of (1.3.18) vanishes as $m \to \infty$. Notice that (1.3.10) and (1.3.11) imply

$$\nabla\varphi_m(t) = (\dot\rho_m * (\theta_0 \nabla v))(t) + \rho_m(t - t_0)\nabla v(t_0) - \rho_m(t)\nabla v(0) \quad \text{for } t \in [0, T].$$

Then, we may write

$$\begin{aligned}
&\int_{\mathbb{R}} (\zeta_m(t), \nabla\varphi_m(t))_{L^2(\Omega)} \, \mathrm{d}t \\
&= -\int_{\mathbb{R}} (\dot\zeta_m(t), (\rho_m * (\theta_0 \nabla v))(t))_{L^2(\Omega)} \, \mathrm{d}t + \int_{\mathbb{R}} (\zeta_m(t), \rho_m(t - t_0)\nabla v(t_0) - \rho_m(t)\nabla v(0))_{L^2(\Omega)} \, \mathrm{d}t.
\end{aligned}$$

Since ρ_m and θ_0 have compact support, for m large enough ζ_m and $\dot\zeta_m$ are identically zero outside the interval $I = (-2T, 2T)$. Clearly $\zeta_m \to 0$ in $L^2(I; L^2(\Omega; \mathbb{R}^{d\times d}))$ as $m \to \infty$, and let us check that $\dot\zeta_m$ converges to zero weakly in $L^2(I; L^2(\Omega; \mathbb{R}^{d\times d}))$ as $m \to \infty$. By (H7) we know that $b \in \mathrm{Lip}(\overline{I}; L^\infty(\Omega; \mathbb{R}^d))$, so that $\dot b \in L^\infty(I; L^2(\Omega; \mathbb{R}^d))$ and there exists $C > 0$ such that $\|\dot b(t)\|_{L^\infty(\Omega)} \le C$ for a.e. $t \in I$. Therefore, for a.e. $t \in I$, we may write

$$\begin{aligned}
\dot\zeta_m(t) &= (\dot\rho_m * (\theta_0 \dot v \otimes b))(t) - (\dot\rho_m * (\theta_0 \dot v))(t) \otimes b(t) - (\rho_m * (\theta_0 \dot v))(t) \otimes \dot b(t) \\
&= \int_{\mathbb{R}} \dot\rho_m(t - s)\theta_0(s)\dot v(s) \otimes [b(s) - b(t)] \, \mathrm{d}s - \int_{\mathbb{R}} \rho_m(t - s)\theta_0(s)\dot v(s) \otimes \dot b(t) \mathrm{d}s \\
&= \int_0^{t_0} \dot\rho_m(t - s)\dot v(s) \otimes \frac{[b(s) - b(t)]}{(s - t)}(s - t) \, \mathrm{d}s - \int_0^{t_0} \rho_m(t - s)\dot v(s) \otimes \dot b(t) \, \mathrm{d}s.
\end{aligned}$$

We use the L^∞-boundedness of $t \mapsto \|\dot v(t)\|_{L^2(\Omega)}$, the previous properties of b, and the bounds

$$\int_{\mathbb{R}} |s\dot\rho_m(s)| \, \mathrm{d}s < \infty, \qquad \int_{\mathbb{R}} \rho_m(s) \, \mathrm{d}s < \infty,$$

to deduce that $\{\dot\zeta_m\}_m$ is uniformly bounded in $L^2(I; L^2(\Omega; \mathbb{R}^{d\times d}))$. This gives that $\dot\zeta_m \rightharpoonup 0$ in $L^2(I; L^2(\Omega; \mathbb{R}^{d\times d}))$ as $m \to \infty$, since ζ_m converges to 0 strongly in $L^2(I; L^2(\Omega; \mathbb{R}^{d\times d}))$ as $m \to \infty$. Hence

$$\lim_{m\to\infty} \int_{\mathbb{R}} (\dot\zeta_m, \rho_m * (\theta_0 \nabla v))_{L^2(\Omega)} = 0.$$

Since the embedding $H^1(I; L^2(\Omega; \mathbb{R}^{d\times d})) \hookrightarrow C^0(\overline{I}; L^2(\Omega; \mathbb{R}^{d\times d}))$ is continuous, the sequence $\{\zeta_m\}_m$ is bounded in $C^0(\overline{I}; L^2(\Omega; \mathbb{R}^{d\times d}))$, and

$$\|\zeta_m(t_1) - \zeta_m(t_2)\|_{L^2(\Omega)} \le \|\dot\zeta_m\|_{L^2(\Omega)} |t_1 - t_2|^{1/2} \le C|t_1 - t_2|^{1/2} \quad \text{for every } t_1, t_2 \in \overline{I},$$

with $C > 0$ independent of t_1 and t_2. Then the sequence $\|\zeta_m\|_{L^2(\Omega)} \colon \overline{I} \to \mathbb{R}$, $m \in \mathbb{N}$, is equibounded and equicontinuos. By Ascoli-Arzela's theorem we get that ζ_m converges strongly to zero in $C^0(\overline{I}; L^2(\Omega; \mathbb{R}^{d\times d}))$, since $\|\zeta_m\|_{L^2(\Omega)} \to 0$ in $L^2(I)$. Notice that the function $t \mapsto \rho_m(t - t_0)\nabla v(t_0) - \rho_m(t)\nabla v(0)$ is bounded in $L^1(I; L^2(\Omega; \mathbb{R}^{d\times d}))$, hence

$$\lim_{m\to\infty} \int_{\mathbb{R}} (\zeta_m(t), \rho_m(t - t_0)\nabla v(t_0) - \rho_m(t)\nabla v(0))_{L^2(\Omega)} \, \mathrm{d}t = 0.$$

Similarly, by defining

$$\chi_m := \mathbb{B}\rho_m * (\theta_0 \nabla v) - \rho_m * (\theta_0 \mathbb{B} \nabla v),$$

we may write

$$\int_{\mathbb{R}} (\mathbb{B}\rho_m*(\theta_0\nabla v) - \rho_m*(\theta_0\mathbb{B}\nabla v), \dot\rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t = -\int_{\mathbb{R}} (\dot\chi_m, \rho_m*(\theta_0\nabla v))_{L^2(\Omega)}\,\mathrm{d}t.$$

As before, $\dot\chi_m$ is identically zero outside of I, $\chi_m \to 0$ in $L^2(I;L^2(\Omega;\mathbb{R}^{d\times d}))$, and the sequence $\{\dot\chi_m\}_m$ is bounded in $L^2(I;L^2(\Omega;\mathbb{R}^{d\times d}))$ thanks to the Lipschitz regularity of $\mathbb{B}$. Hence $\dot\chi_m \rightharpoonup 0$ in $L^2(I;L^2(\Omega;\mathbb{R}^{d\times d}))$ and (1.3.17) holds.

This concludes the proof of formula (1.3.2) and implies the desired continuity of the map $t\mapsto \mathcal{E}_{\mathbb{B}}(v;t)$ in $[0,T]$. □

We are now in a position to prove Theorem 1.2.2.

Proof of Theorem 1.2.2. In view of Theorems 1.2.9 and 1.2.10, we know that problem (1.1.28) with boundary and initial conditions (1.1.33)–(1.1.36) admits a unique generalized solution v (cf. Definition 1.2.5). Hence, to conclude the proof, it is enough to show that that every generalized solution v is indeed a weak solution (cf. Definition 1.1.9), more precisely it satisfies (1.1.38)–(1.1.41).

As pointed out in Remark 1.2.7, $v \in C^0([0,T];L^2(\Omega;\mathbb{R}^d)) \cap C^0_w([0,T];H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, while $\dot v \in C^0([0,T];H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)) \cap C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$. In addition, $t\mapsto \mathcal{E}_{\mathbb{B}}(v;t)$ is a continuous function from $[0,T]$ to $\mathbb{R}$, thanks to Proposition 1.3.1. Let us now prove that $t\mapsto \nabla v(t)$ and $t\mapsto \dot v(t)$ are strongly continuous from $[0,T]$ to $L^2(\Omega;\mathbb{R}^{d\times d})$ and $L^2(\Omega;\mathbb{R}^d)$, respectively.

Let $t_0\in[0,T]$ be fixed and let $\{t_k\}_k\subset[0,T]$ be a sequence of points converging to t_0. Since $\dot v$ is weakly continuous, we have

$$\|\dot v(t_0)\|^2_{L^2(\Omega)} \le \liminf_{k\to\infty} \|\dot v(t_k)\|^2_{L^2(\Omega)}.$$

Moreover, condition (1.2.1) implies that $(\mathbb{B}(t_0)\nabla\phi,\nabla\phi)_{L^2(\Omega)} + c_1\|\phi\|^2_{L^2(\Omega)}$, $\phi\in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$, is an equivalent norm on $H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$. Hence, since $z := v - w \in C^0_w([0,T];H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, we have

$$\begin{aligned}
&(\mathbb{B}(t_0)\nabla z(t_0),\nabla z(t_0))_{L^2(\Omega)} + c_1\|z(t_0)\|^2_{L^2(\Omega)}\\
&\le \liminf_{k\to\infty}\Big[(\mathbb{B}(t_0)\nabla z(t_k),\nabla z(t_k))_{L^2(\Omega)} + c_1\|z(t_k)\|^2_{L^2(\Omega)}\Big]\\
&= \liminf_{k\to\infty}(\mathbb{B}(t_0)\nabla z(t_k),\nabla z(t_k))_{L^2(\Omega)} + c_1\|z(t_0)\|^2_{L^2(\Omega)},
\end{aligned}$$

thanks to the strong continuity and the weak continuity of z in $L^2(\Omega;\mathbb{R}^d)$ and $H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$, respectively. In particular, we can use (1.1.11) to derive

$$(\mathbb{B}(t_0)\nabla v(t_0),\nabla v(t_0))_{L^2(\Omega)} \le \liminf_{k\to\infty}(\mathbb{B}(t_0)\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)}.$$

Moreover, by the strong continuity of $t\mapsto\mathbb{B}(t)$ from $[0,T]$ to $L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))$ and the bound (1.2.47) we get

$$\begin{aligned}
&(\mathbb{B}(t_0)\nabla v(t_0),\nabla v(t_0))_{L^2(\Omega)}\\
&\le \liminf_{k\to\infty}\Big[(\mathbb{B}(t_k)\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)} + ((\mathbb{B}(t_0)-\mathbb{B}(t_k))\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)}\Big]\\
&\le \liminf_{k\to\infty}(\mathbb{B}(t_k)\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)} + C\lim_{k\to\infty}\|\mathbb{B}(t_0)-\mathbb{B}(t_k)\|_{L^\infty(\Omega)}\\
&= \liminf_{k\to\infty}(\mathbb{B}(t_k)\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)}.
\end{aligned}$$

Then

$$\begin{aligned}\mathcal{E}_{\mathbb{B}}(v;t_0) &\le \frac{1}{2}\liminf_{k\to\infty}\|\dot v(t_k)\|^2_{L^2(\Omega)} + \frac{1}{2}\liminf_{k\to\infty}(\mathbb{B}(t_k)\nabla v(t_k),\nabla v(t_k))_{L^2(\Omega)}\\ &\le \lim_{k\to\infty}\mathcal{E}_{\mathbb{B}}(v;t_k) = \mathcal{E}_{\mathbb{B}}(v;t_0),\end{aligned}$$

which implies the continuity of $t\mapsto\|\dot v(t)\|^2_{L^2(\Omega)}$ and $t\mapsto(\mathbb{B}(t)\nabla v(t),\nabla v(t))_{L^2(\Omega)}$ in $t_0\in[0,T]$. Thus $\dot v$ and ∇v are strongly continuous from $[0,T]$ to $L^2(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^{d\times d})$, respectively. Therefore, properties (1.1.38)–(1.1.40) are readily verified. Finally, since both $\dot v$ and $\ddot v$ are elements of $L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, we infer that $\dot v\in H^1(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ which is contained in $AC([0,T];H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. This gives (1.1.41) and concludes the proof. □

1.4 Continuous dependence on the data

In this section, following the same procedure adopted in [20, Theorem 4.1], we use the energy equality (1.3.2) to obtain a continuous dependence result on the data, both for problem (1.1.14) with boundary and initial conditions (1.1.15)–(1.1.18), and problem (1.1.28) with boundary and initial conditions (1.1.33)–(1.1.36).

The initial crack Γ_0 is kept fixed. For every $n\in\mathbb{N}$ we consider a family of closed sets $\{\Gamma^n_t\}_{t\in[0,T]}$ and a complete $(d-1)$-dimensional C^2 manifold Γ^n satisfying (H1)–(H8) with diffeomorphisms Ψ^n and Φ^n, and we assume

$$\Gamma^n_0=\Gamma_0 \quad \text{for every } n\in\mathbb{N}. \tag{1.4.1}$$

Moreover, we consider a sequence $\mathbb{A}^n$ of tensor fields, f^n of source terms, w^n of Dirichlet boundary data, F^n of Neumann boundary data, and $(u^{0,n},u^{1,n})$ of initial data. The convergences of the corresponding solutions will be obtained under the assumptions detailed in the following theorem.

Theorem 1.4.1. *Assume that Γ, $\{\Gamma_t\}_{t\in[0,T]}$, Φ, Ψ satisfy (H1)–(H8). Let us consider a tensor field $\mathbb{A}$ which satisfies (1.1.8)–(1.1.10) and such that the transformed operator $\mathbb{B}$ satisfies the ellipticity condition (1.2.1). Let us also consider f, w, F, u^0, and u^1 satisfying (1.1.11)–(1.1.13). Assume that Γ^n, $\{\Gamma^n_t\}_{t\in[0,T]}$, Φ^n, Ψ^n satisfying (H1)–(H8) and condition (1.4.1) for every $n\in\mathbb{N}$. Let us consider a sequence of tensor fields $\mathbb{A}^n$ which satisfy (1.1.8)–(1.1.10) for every $n\in\mathbb{N}$ and such that the operators $\mathbb{B}^n$, constructed starting from $\mathbb{A}^n$, Φ^n, and Ψ^n, satisfy the ellipticity condition (1.2.1) with constants c_0 and c_1 independent of n. Let us consider f^n, w^n, F^n, $u^{0,n}$, and $u^{1,n}$ satisfying (1.1.11)–(1.1.13) for every $n\in\mathbb{N}$.*

We assume there exists of a constant $C>0$ such that every $n\in\mathbb{N}$ and $s,t\in[0,T]$

$$\|\Phi^n(t)-\Phi^n(s)\|_{L^\infty(\Omega)}\le C|t-s|, \quad \|\dot\Phi^n(t)-\dot\Phi^n(s)\|_{L^\infty(\Omega)}\le C|t-s|, \tag{1.4.2}$$

$$\|\partial_i\Phi^n(t)-\partial_i\Phi^n(s)\|_{L^\infty(\Omega)}\le C|t-s|, \quad \|\partial^2_{ij}\Phi^n(t)\|_{L^\infty(\Omega)}\le C, \tag{1.4.3}$$

$$\|\mathbb{A}^n(t)-\mathbb{A}^n(s)\|_{L^\infty(\Omega)}\le C|t-s|, \quad \|\partial_i\mathbb{A}^n(t)\|_{L^\infty(\Omega)}\le C, \tag{1.4.4}$$

for every $i,j=1,\dots,d$. Moreover, we assume the following convergences as $n\to\infty$:

$$\dot\Phi^n(t)\to\dot\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \quad \partial_i\dot\Phi^n(t)\to\partial_i\dot\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \tag{1.4.5}$$

$$\ddot\Phi^n(t)\to\ddot\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \quad \partial^2_{ij}\Phi^n(t)\to\partial^2_{ij}\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \tag{1.4.6}$$

$$\mathbb{A}^n(t)\to\mathbb{A}(t) \quad \text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \tag{1.4.7}$$

$$\partial_i\mathbb{A}^n(t)\to\partial_i\mathbb{A}(t) \quad \text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \tag{1.4.8}$$

$$\dot{\mathbb{A}}^n(t)\to\dot{\mathbb{A}}(t) \quad \text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \tag{1.4.9}$$

$$w^n\to w \quad \text{in } H^2(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)), \tag{1.4.10}$$

$$f^n \to f \quad in\ L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad F^n \to F \quad in\ H^1(0,T;L^2(\partial_N\Omega;\mathbb{R}^d)), \tag{1.4.11}$$
$$u^{0,n} \to u^0 \quad in\ H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d), \quad u^{1,n} \to u^1 \quad in\ L^2(\Omega;\mathbb{R}^d) \tag{1.4.12}$$

for a.e. $t \in (0,T)$ and for every $i,j = 1,\dots,d$. Finally, we assume that (1.4.2), (1.4.3), (1.4.5), and (1.4.6) are true also for the sequence Ψ^n with limit Ψ.

For every $n \in \mathbb{N}$ let u^n be the weak solution to problem (1.1.14) with growing crack Γ_t^n, forcing term f^n, boundary conditions (1.1.15)–(1.1.17) with w, F, and Γ_t replaced by w^n, F^n and Γ_t^n, respectively, and initial data $(u^{0,n}, u^{1,n})$. Similarly, let v^n be the weak solution to (1.1.28) with boundary and conditions (1.1.33)–(1.1.36), where the coefficients (1.1.29)–(1.1.32) and the initial data (1.1.37) are constructed starting from $\Phi^n, \Psi^n, \mathbb{A}^n, f^n, u^{0,n}, u^{1,n}$. Let u and v be the weak solutions to problem (1.1.14) with boundary and initial conditions (1.1.15)–(1.1.18) and problem (1.1.28) with boundary and initial conditions (1.1.33)–(1.1.36), respectively. Under the previous assumptions, for every $t \in [0,T]$ as $n \to \infty$ we have:

$$u^n(t) \to u(t) \quad in\ L^2(\Omega;\mathbb{R}^d), \quad \nabla u^n(t) \to \nabla u(t) \quad in\ L^2(\Omega;\mathbb{R}^{d\times d}), \tag{1.4.13}$$
$$\dot u^n(t) \to \dot u(t) \quad in\ L^2(\Omega;\mathbb{R}^d), \tag{1.4.14}$$
$$v^n(t) \to v(t) \quad in\ H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d), \quad \dot v^n(t) \to \dot v(t) \quad in\ L^2(\Omega;\mathbb{R}^d). \tag{1.4.15}$$

Remark 1.4.2. In the continuous dependence result of [8], both the initial crack and the Dirichlet datum are fixed. In this thesis, we consider also the case of a sequence of Dirichlet data w^n converging to w.

Remark 1.4.3. Since $\Phi^n(0) = id$ for every $n \in \mathbb{N}$, assumption (1.4.5) implies

$$\Phi^n(t) \to \Phi(t), \quad \partial_i\Phi^n(t) \to \partial_i\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } n\to\infty \tag{1.4.16}$$

for every $t \in [0,T]$ and $i = 1,\dots,d$. Moreover, by (1.4.6)–(1.4.9) we also have

$$\dot\Phi^n(t) \to \dot\Phi(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \quad \mathbb{A}^n(t) \to \mathbb{A}(t) \quad \text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})) \quad \text{as } n\to\infty \tag{1.4.17}$$

for every $t \in [0,T]$. Finally, we have $\|\det\nabla\Phi^n(t)\|_{L^\infty(\Omega)} \le C$ and $\|\det\nabla\Psi^n(t)\|_{L^\infty(\Omega)} \le C$ for a constant $C > 0$ independent of n and t. Thus there exists a constant $\delta_0 > 0$, independent of n, such that

$$\det\nabla\Phi^n(t,y) \ge \delta_0, \quad \det\nabla\Psi^n(t,x) \ge \delta_0 \quad \text{for every } t\in[0,T] \text{ and } x,y \in \overline{\Omega}. \tag{1.4.18}$$

Proof of Theorem 1.4.1. We follow the lines of the proof of [20, Theorem 4.1]. As explained in the quoted paper, the statement for the sequence $\{u^n\}_n$ is a consequence of the one for $\{v^n\}_n$. Indeed, let $t \in [0,T]$ be fixed and let us assume that (1.4.15) is satisfied. By (1.1.27), (1.1.49), and the bounds (1.4.2) and (1.4.3) on the diffeomorphisms, we deduce that $\{\nabla u^n(t)\}_n$, $\{u^n(t)\}_n$, and $\{\dot u^n(t)\}_n$ are uniformly bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$, $L^2(\Omega;\mathbb{R}^d)$, and $L^2(\Omega;\mathbb{R}^d)$, respectively. In particular, up to a subsequence, $\nabla u^n(t)$, $u^n(t)$, and $\dot u^n(t)$ converge weakly in these spaces. To determine the weak limits we fix a smooth function $\psi \in C_c^\infty(\Omega;\mathbb{R}^{d\times d})$. By the change of variable formula (1.1.49), we have

$$\begin{aligned}\lim_{n\to\infty}(\nabla u^n(t),\psi)_{L^2(\Omega)} &= \lim_{n\to\infty}(\nabla v^n(t)\nabla\Psi^n(t,\Phi^n(t)),\psi(\Phi^n(t))\det\nabla\Phi^n(t))_{L^2(\Omega)}\\ &= (\nabla v(t)\nabla\Psi(t,\Phi(t)),\psi(\Phi(t))\det\nabla\Phi(t))_{L^2(\Omega)} = (\nabla u(t),\psi)_{L^2(\Omega)}.\end{aligned}$$

Hence $\nabla u^n(t)$ converges weakly to $\nabla u(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. Similarly, by using the convergences (1.4.15) and (1.4.16) we obtain that $\|\nabla u^n(t)\|_{L^2(\Omega)}$ converges to $\|\nabla u(t)\|_{L^2(\Omega)}$ as $n\to\infty$. Then $\nabla u^n(t) \to \nabla u(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$, and the same argument applies to $u^n(t)$ and

$\dot{u}^n(t)$, which converge strongly in $L^2(\Omega;\mathbb{R}^d)$ to $u(t)$ and $\dot{u}(t)$, respectively. This gives (1.4.13) and (1.4.14), since these limits do not depend on the subsequence.

Let us denote by $\mathbb{B}^n$, $\mathbf{p}^n$, b^n, g^n, $v^{0,n}$, and $v^{1,n}$ the coefficients of the system (1.1.28) constructed starting from Φ^n, Ψ^n, $\mathbb{A}^n$, f^n, $u^{0,n}$, and $u^{1,n}$. In view of (1.4.2)–(1.4.4) it is easy to check that for every $n\in\mathbb{N}$ and $i=1,\dots,d$

$$\|\mathbb{B}^n(t)-\mathbb{B}^n(s)\|_{L^\infty(\Omega)}\le C|t-s|,\quad \|b^n(t)-b^n(s)\|_{L^\infty(\Omega)}\le C|t-s| \tag{1.4.19}$$

for every $t,s\in[0,T]$, while

$$\|\partial_i\mathbb{B}^n(t)\|_{L^\infty(\Omega)}\le C,\quad \|\partial_i b^n(t)\|_{L^\infty(\Omega)}\le C,\quad \|\mathbf{p}^n(t)\|_{L^\infty(\Omega)}\le C \tag{1.4.20}$$

for a.e. $t\in(0,T)$, where C is a constant independent of t, s, n, and i. Furthermore, the convergences (1.4.5)–(1.4.9), the lower bounds (1.4.18), and Lemma 1.1.2 imply as $n\to\infty$

$$\dot{\mathbb{B}}^n(t)\to\dot{\mathbb{B}}(t)\quad\text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))\quad\text{for a.e. } t\in(0,T), \tag{1.4.21}$$

$$\mathbf{p}^n(t)\to\mathbf{p}(t)\quad\text{in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d}))\quad\text{for a.e. } t\in(0,T), \tag{1.4.22}$$

$$\partial_i b^n(t)\to\partial_i b(t)\quad\text{in } L^2(\Omega)\quad\text{for a.e. } t\in(0,T) \tag{1.4.23}$$

for every $i=1,\dots,d$. By using (1.4.19), (1.4.20), and Ascoli-Arzela's theorem, we also infer as $n\to\infty$

$$\mathbb{B}^n(t)\to\mathbb{B}(t)\quad\text{in } C^0(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))\quad\text{for a.e. } t\in(0,T), \tag{1.4.24}$$

$$b^n(t)\to b(t)\quad\text{in } C^0(\overline{\Omega};\mathbb{R}^{d})\quad\text{for a.e. } t\in(0,T). \tag{1.4.25}$$

Finally, by (1.4.11) and (1.4.12) we obtain as $n\to\infty$

$$g^n\to g\quad\text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{1.4.26}$$

$$v^{0,n}\to v^0\quad\text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d),\qquad v^{1,n}\to v^1\quad\text{in } L^2(\Omega;\mathbb{R}^d). \tag{1.4.27}$$

In order to prove the validity of (1.4.15), for every $\varepsilon\in(0,1)$ we consider the solution v_ε to the perturbed problem (1.2.7) with coefficients $\mathbb{B}$, $\mathbf{p}$, b, g, w, F, v^0, and v^1, and the solution v^n_ε to the one with coefficients $\mathbb{B}^n$, $\mathbf{p}^n$, b^n, g^n, w^n, F^n, $v^{0,n}$, and $v^{1,n}$. For every $t\in[0,T]$ as $\varepsilon\to0$ we claim

$$v_\varepsilon(t)\to v(t)\quad\text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d),\qquad \dot{v}_\varepsilon(t)\to\dot{v}(t)\quad\text{in } L^2(\Omega;\mathbb{R}^d). \tag{1.4.28}$$

Moreover, we claim that there exists a sequence of parameters $\{\varepsilon_n\}_n\subset(0,1)$, converging to 0 as $n\to\infty$, such that for every $t\in[0,T]$ as $n\to\infty$

$$v^n_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\to0\quad\text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d),\qquad \dot{v}^n_{\varepsilon_n}(t)-\dot{v}_{\varepsilon_n}(t)\to0\quad\text{in } L^2(\Omega;\mathbb{R}^d), \tag{1.4.29}$$

$$v^n_{\varepsilon_n}(t)-v^n(t)\to0\quad\text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d),\qquad \dot{v}^n_{\varepsilon_n}(t)-\dot{v}^n(t)\to0\quad\text{in } L^2(\Omega;\mathbb{R}^d). \tag{1.4.30}$$

Notice that (1.4.28)–(1.4.30) imply (1.4.15). Indeed, by the triangle inequality, as $n\to\infty$ we have

$$\begin{aligned}&\|v^n(t)-v(t)\|_{H^1(\Omega\setminus\Gamma_0)}\\&\le\|v^n(t)-v^n_{\varepsilon_n}(t)\|_{H^1(\Omega\setminus\Gamma_0)}+\|v^n_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|_{H^1(\Omega\setminus\Gamma_0)}+\|v_{\varepsilon_n}(t)-v(t)\|_{H^1(\Omega\setminus\Gamma_0)}\to0\end{aligned}$$

and the same holds true for $\|\dot{v}^n(t)-\dot{v}(t)\|_{L^2(\Omega)}$. To prove (1.4.28)–(1.4.30) we divide the proof into several steps.

Step 1. Strong convergence of v_ε. Let us define $z_\varepsilon := v_\varepsilon - v$. By comparing the two energy equalities (1.2.26) and (1.3.2) we infer that z_ε satisfies

$$\begin{aligned}&\mathcal{E}_{\mathbb{B}}(z_\varepsilon;t)+\varepsilon\int_0^t\|\dot v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\mathrm{d}s\\&=\int_0^t\left[\frac{1}{2}(\dot{\mathbb{B}}\nabla z_\varepsilon,\nabla z_\varepsilon)_{L^2(\Omega)}-(\mathbf{p}\nabla z_\varepsilon,\dot z_\varepsilon)_{L^2(\Omega)}-(\dot z_\varepsilon\operatorname{div}b,\dot z_\varepsilon)_{L^2(\Omega)}\right]\mathrm{d}s+\mathcal{R}_\varepsilon(t),\end{aligned}\tag{1.4.31}$$

where $\mathcal{E}_{\mathbb{B}}$ is defined in according to (1.3.1), and

$$\begin{aligned}\mathcal{R}_\varepsilon(t):=&-(\dot v_\varepsilon(t),\dot v(t))_{L^2(\Omega)}-(\mathbb{B}(t)\nabla v_\varepsilon(t),\nabla v(t))_{L^2(\Omega)}+\|v^1\|^2_{L^2(\Omega)}+(\mathbb{B}(0)\nabla v^0,\nabla v^0)_{L^2(\Omega)}\\&+\int_0^t\left[(\dot{\mathbb{B}}\nabla v_\varepsilon,\nabla v)_{L^2(\Omega)}+(\mathbb{B}(\nabla v_\varepsilon+\nabla v),\nabla\dot w)_{L^2(\Omega)}-(\mathbf{p}\nabla v_\varepsilon,\dot v)_{L^2(\Omega)}\right]\mathrm{d}s\\&+\int_0^t\left[-(\mathbf{p}\nabla v,\dot v_\varepsilon)_{L^2(\Omega)}+(\mathbf{p}(\nabla v_\varepsilon+\nabla v),\dot w)_{L^2(\Omega)}-2(\dot v\operatorname{div}b,\dot v_\varepsilon)_{L^2(\Omega)}\right]\mathrm{d}s\\&+\int_0^t\left[2(\dot v_\varepsilon+\dot v,\operatorname{div}[\dot w\otimes b])_{L^2(\Omega)}+\varepsilon(\dot v_\varepsilon,\dot w)_{H^1(\Omega\setminus\Gamma_0)}+(g,\dot v_\varepsilon+\dot v-2\dot w)_{L^2(\Omega)}\right]\mathrm{d}s\\&+\int_0^t\left[-(\dot F,v_\varepsilon+v-2w)_{L^2(\partial_N\Omega)}-(\dot v_\varepsilon+\dot v,\ddot w)_{L^2(\Omega)}\right]\mathrm{d}s\\&+(F(t),v_\varepsilon(t)+v(t)-2w(t))_{L^2(\partial_N\Omega)}+(\dot v_\varepsilon(t)+\dot v(t),\dot w(t))_{L^2(\Omega)}\\&-2(F(0),v^0-w(0))_{L^2(\partial_N\Omega)}-2(v^1,\dot w(0))_{L^2(\Omega)}\end{aligned}$$

for every $t\in[0,T]$. Thanks to (1.2.28), as $\varepsilon\to0^+$ we have

$$\begin{aligned}\left|\varepsilon\int_0^t(\dot v_\varepsilon,\dot w)_{H^1(\Omega\setminus\Gamma_0)}\mathrm{d}s\right|&\le\sqrt{\varepsilon}\int_0^T\sqrt{\varepsilon}\|\dot v_\varepsilon\|_{H^1(\Omega\setminus\Gamma_0)}\|\dot w\|_{H^1(\Omega\setminus\Gamma_0)}\mathrm{d}s\\&\le\sqrt{\varepsilon}\|\dot w\|_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))}\left(\int_0^T\varepsilon\|\dot v_\varepsilon\|^2_{H^1(\Omega\setminus\Gamma_0)}\mathrm{d}s\right)^{1/2}\le\sqrt{\varepsilon}C\to0.\end{aligned}$$

Therefore, by using also the weak convergences (1.2.34) and the energy equality (1.3.2), we deduce that $\mathcal{R}_\varepsilon(t)\to0$ as $\varepsilon\to0^+$. The uniform bounds on $\dot{\mathbb{B}}$, $\mathbf{p}$, and $\operatorname{div}b$, the ellipticity condition (1.2.1), the estimate

$$\|z_\varepsilon(t)\|^2_{L^2(\Omega)}\le T\int_0^t\|\dot z_\varepsilon(s)\|^2_{L^2(\Omega)}\mathrm{d}s\quad\text{for every }t\in[0,T],\tag{1.4.32}$$

and the identity (1.4.31) imply

$$\|\dot z_\varepsilon(t)\|^2_{L^2(\Omega)}+\|z_\varepsilon(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\le C\left(\mathcal{R}_\varepsilon(t)+\int_0^t\left[\|\dot z_\varepsilon(s)\|^2_{L^2(\Omega)}+\|z_\varepsilon(s)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}s\right)$$

for every $t\in[0,T]$, with $C>0$ independent of t and ε. By applying Fatou's lemma, for every $t\in[0,T]$ we have

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\left[\|\dot z_\varepsilon(t)\|^2_{L^2(\Omega)}+\|z_\varepsilon(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\\&\le C\limsup_{\varepsilon\to0^+}\left(\mathcal{R}_\varepsilon(t)+\int_0^t\left[\|\dot z_\varepsilon(s)\|^2_{L^2(\Omega)}+\|z_\varepsilon(s)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}s\right)\\&\le C\int_0^t\limsup_{\varepsilon\to0^+}\left[\|\dot z_\varepsilon(s)\|^2_{L^2(\Omega)}+\|z_\varepsilon(s)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}s.\end{aligned}$$

Thanks to Gronwall's lemma we conclude

$$\lim_{\varepsilon\to0^+}\left[\|\dot z_\varepsilon(t)\|^2_{L^2(\Omega)}+\|z_\varepsilon(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]=0\quad\text{for every } t\in[0,T],$$

which gives the convergences (1.4.28).

Step 2. Strong convergence of $v^n_{\varepsilon_n}-v_{\varepsilon_n}$. Let $\{\varepsilon_n\}_n\subset(0,1)$ be a sequence of parameters to be fixed. The functions $v^n_{\varepsilon_n}$ and v_{ε_n} satisfy (1.2.7) with different coefficients, but with the same viscosity ε_n. By linearity, the function $z^n:=(v^n_{\varepsilon_n}-v_{\varepsilon_n})-(w^n-w)$ solves for a.e. $t\in(0,T)$

$$\begin{aligned}&\langle\ddot z^n(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}+(\mathbb{B}(t)\nabla z^n(t),\nabla\phi)_{L^2(\Omega)}+(\mathbf{p}(t)\nabla z^n(t),\phi)_{L^2(\Omega)}\\&-2(\nabla\dot z^n(t)b(t),\phi)_{L^2(\Omega)}+\varepsilon_n(\dot z^n(t),\phi)_{H^1(\Omega\setminus\Gamma_0)}=\langle q^n(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}\end{aligned}\tag{1.4.33}$$

for every $\phi\in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$, with initial data $z^{0,n}:=(v^{0,n}-v^0)-(w^n(0)-w(0))$ and $z^{1,n}:=(v^{1,n}-v^1)-(\dot w^n(0)-\dot w(0))$, and right-hand side defined for $t\in[0,T]$ as

$$\begin{aligned}\langle q^n(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}:=&-(\ddot w^n(t)-\ddot w(t),\phi)_{L^2(\Omega)}-(\mathbb{B}(t)(\nabla w^n(t)-\nabla w(t)),\nabla\phi)_{L^2(\Omega)}\\&-(\mathbf{p}(t)(\nabla w^n(t)-\nabla w(t)),\phi)_{L^2(\Omega)}+2((\nabla\dot w^n(t)-\nabla\dot w(t))b(t),\phi)_{L^2(\Omega)}\\&-\varepsilon_n(\dot w^n(t)-\dot w(t),\phi)_{H^1(\Omega\setminus\Gamma_0)}-((\mathbb{B}^n(t)-\mathbb{B}(t))\nabla v^n_{\varepsilon_n}(t),\nabla\phi)_{L^2(\Omega)}\\&-((\mathbf{p}^n(t)-\mathbf{p}(t))\nabla v^n_{\varepsilon_n}(t),\phi)_{L^2(\Omega)}-2(\dot v^n_{\varepsilon_n}(t)\otimes(b^n(t)-b(t)),\nabla\phi)_{L^2(\Omega)}\\&-2((\operatorname{div}b^n(t)-\operatorname{div}b(t))\dot v^n_{\varepsilon_n}(t),\phi)_{L^2(\Omega)}+(g^n(t)-g(t),\phi)_{L^2(\Omega)}\\&+(F^n(t)-F(t),\phi)_{L^2(\Omega)}.\end{aligned}\tag{1.4.34}$$

In particular, the forcing term q^n is an element of $L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Notice that we have used the identity (1.1.46) for both b^n and b to derive formula (1.4.33). By combining the energy equality (1.2.26) with (1.1.1), the uniform ellipticity condition (1.2.1) for $\mathbb{B}^n$, the uniform bounds (1.4.19) and (1.4.20), and the convergences (1.4.10), (1.4.11), (1.4.26), and (1.4.27) we conclude that the sequences $\{v^n_{\varepsilon_n}\}_n$ and $\{\dot v^n_{\varepsilon_n}\}_n$ are uniformly bounded with respect to n in $L^\infty(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ and $L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$, respectively. Moreover, these bounds do not depend on the sequence $\{\varepsilon_n\}_n$. By using (1.1.2), (1.4.19), (1.4.20), and (1.4.22)–(1.4.26) we conclude

$$q^n\to0\quad\text{in } L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))\quad\text{as } n\to\infty,\tag{1.4.35}$$

and the rate of this convergence is independent of the choice of $\{\varepsilon_n\}_n\subset(0,1)$. Notice that, to pass to the limit in the first two terms in the right-hand side of (1.4.34), we have used (1.4.24) and (1.4.25).

Since $z^n\in H^1(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$, we can use $\dot z^n$ as test function in (1.4.33), and by integrating by parts in $(0,t)$ for very $t\in(0,T]$, we get

$$\begin{aligned}&\mathcal{E}_{\mathbb{B}}(z^n;t)+\varepsilon_n\int_0^t\|\dot z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s\\&=\mathcal{E}_{\mathbb{B}}(z^n;0)+\int_0^t\left[\frac12(\dot{\mathbb{B}}\nabla z^n,\nabla z^n)_{L^2(\Omega)}-(\mathbf{p}\nabla z^n,\dot z^n)_{L^2(\Omega)}\,\mathrm{d}s-(\dot z^n\operatorname{div}b,\dot z^n)_{L^2(\Omega)}\right]\mathrm{d}s\\&\quad+\int_0^t\langle q^n,\dot z^n\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}\,\mathrm{d}s.\end{aligned}$$

As in the previous step, the uniform bounds on $\dot{\mathbb{B}}$, $\mathbf{p}$ and $\operatorname{div}b$, the ellipticity condition (1.2.1), and the estimate (1.2.12) applied to z^n imply

$$\begin{aligned}&\frac{1}{2}\|\dot z^n(t)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|z^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}+\varepsilon_n\int_0^t\|\dot z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s\\&\leq\frac{1}{2}\|\dot z^n(0)\|^2_{L^2(\Omega)}+\left(c_1+\frac{1}{2}\right)\|z^n(0)\|^2_{H^1(\Omega\setminus\Gamma_0)}+C\int_0^t\left[\|\dot z^n\|^2_{L^2(\Omega)}+\|z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}s\\&\quad+\int_0^t|\langle q^n,\dot z^n\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}|\,\mathrm{d}s\end{aligned}$$

for a suitable constant $C>0$ independent of n and t. We estimate from above the last term in the previous inequality as

$$\begin{aligned}&\int_0^t|\langle q^n,\dot z^n\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}|\,\mathrm{d}s\\&\leq\frac{1}{2}\|q^n\|_{L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0))}+\frac{1}{2}\|q^n\|_{L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0))}\int_0^t\|\dot z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s.\end{aligned}$$

By choosing $\varepsilon_n\to0^+$ such that

$$\varepsilon_n-\frac{1}{2}\|q^n\|_{L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0))}\geq0\quad\text{for every }n\in\mathbb{N},$$

we obtain then following estimate

$$\|\dot z^n(t)\|^2_{L^2(\Omega)}+c_0\|z^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\leq C_n+2C\int_0^t\left(\|\dot z^n\|^2_{L^2(\Omega)}+\|z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\right)\mathrm{d}s,$$

with

$$C_n:=\|\dot z^n(0)\|^2_{L^2(\Omega)}+(2c_1+1)\|z^n(0)\|^2_{H^1(\Omega\setminus\Gamma_0)}+\|q^n\|_{L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0))}.$$

The convergences (1.4.10), (1.4.27), and (1.4.35) yield that $C_n\to0$ as $n\to\infty$. Therefore, thanks to Fatou and Gronwall's lemmas, we derive

$$\lim_{n\to\infty}\left[\|\dot z^n(t)\|^2_{L^2(\Omega)}+\|z^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]=0\quad\text{for every }t\in[0,T].$$

This fact, together with (1.4.10), proves (1.4.29).

Step 3. Weak convergence of v^n to v. For every $n\in\mathbb{N}$, the function v^n satisfies for a.e. $t\in(0,T)$

$$\begin{aligned}&\langle\ddot v^n(t),\phi\rangle_{H^{-1}_D(\Omega\setminus\Gamma_0)}+(\mathbb{B}^n(t)\nabla v^n(t),\nabla\phi)_{L^2(\Omega)}+(\mathbf{p}^n(t)\nabla v^n(t),\phi)_{L^2(\Omega)}\\&\quad+2(\dot v^n(t),\operatorname{div}[\phi\otimes b^n(t)])_{L^2(\Omega)}=(g^n(t),\phi)_{L^2(\Omega)}+(F^n(t),\phi)_{L^2(\partial_N\Omega)}\end{aligned}\tag{1.4.36}$$

for every $\phi\in H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)$. As shown in (1.2.47),there exists a constant $C>0$ such that

$$\|\dot v^n(t)\|^2_{L^2(\Omega)}+\|v^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\leq C\quad\text{for every }t\in[0,T].\tag{1.4.37}$$

In particular, the constant C can be chosen independent of n, thanks to (1.1.1), the uniform ellipticity condition (1.2.1) for $\mathbb{B}^n$, the bounds (1.4.19), (1.4.20), and the convergences (1.4.10), (1.4.11), (1.4.26), and (1.4.27). By using (1.4.36), we also infer that $\{\ddot v^n\}_n$ is uniformly bounded in $L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. Hence, there exists a function

$$\zeta\in L^\infty(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$$

such that, up to a subsequence (not relabeled), as $n\to\infty$

$$v^n\rightharpoonup\zeta\quad\text{in }L^2(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),\tag{1.4.38}$$

$$\dot{v}^n \rightharpoonup \dot{\zeta} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{1.4.39}$$
$$\ddot{v}^n \rightharpoonup \ddot{\zeta} \quad \text{in } L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)). \tag{1.4.40}$$

Moreover, thanks to (1.4.10), we have $\zeta - w \in L^\infty(0,T;H^1_D(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. By combining the strong convergences (1.4.11) and (1.4.22)–(1.4.27) with the weak convergences (1.4.38)–(1.4.40), we can pass to the limit as $n\to\infty$ in (1.4.36) and we derive that ζ is a generalized solution to the limit problem (1.2.5), with initial conditions v^0 and v^1. In view of Theorem 1.2.10 such solution is unique, therefore $\zeta = v$. Since the result does not depend on the subsequence, we conclude that the whole sequence $\{v^n\}_n$ satisfies as $n\to\infty$

$$\begin{aligned}
&v^n \rightharpoonup v \quad \text{in } L^2(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)),\\
&\dot{v}^n \rightharpoonup \dot{v} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\ddot{v}^n \rightharpoonup \ddot{v} \quad \text{in } L^2(0,T;H^{-1}_D(\Omega\setminus\Gamma_0;\mathbb{R}^d)).
\end{aligned}$$

These convergences, together with the bounds (1.4.37), for every $t\in[0,T]$ imply

$$v^n(t) \rightharpoonup v(t) \quad \text{in } H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d), \qquad \dot{v}^n(t) \rightharpoonup \dot{v}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } n\to\infty. \tag{1.4.41}$$

Step 4. Strong convergence of $v^n_{\varepsilon_n} - v^n$. For every $n\in\mathbb{N}$ we define $z^n := v^n_{\varepsilon_n} - v^n$, where $\varepsilon_n\in(0,1)$ are the parameters chosen in Step 1. Following the same procedure adopted in Step 1, we get

$$\begin{aligned}
&\mathcal{E}_{\mathbb{B}}(z^n;t) + \varepsilon_n \int_0^t \|\dot{v}^n_{\varepsilon_n}\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s\\
&= \int_0^t \left[\frac{1}{2}(\dot{\mathbb{B}}^n\nabla z^n,\nabla z^n)_{L^2(\Omega)} - (\mathbf{p}^n\nabla z^n,\dot{z}^n)_{L^2(\Omega)} - (\dot{z}^n \operatorname{div} b^n,\dot{z}^n)_{L^2(\Omega)}\right]\mathrm{d}s + \mathcal{R}_n(t),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}_n(t) :=& -(\dot{v}^n_{\varepsilon_n}(t),\dot{v}^n(t))_{L^2(\Omega)} - (\mathbb{B}^n(t)\nabla v^n_{\varepsilon_n}(t),\nabla v^n(t))_{L^2(\Omega)} + \|v^{1,n}\|^2_{L^2(\Omega)}\\
&+ (\mathbb{B}^n(0)\nabla v^{0,n},\nabla v^{0,n})_{L^2(\Omega)} + \int_0^t (\dot{\mathbb{B}}^n\nabla v^n_{\varepsilon_n},\nabla v^n)_{L^2(\Omega)}\,\mathrm{d}s\\
&+ \int_0^t \left[(\mathbb{B}^n(\nabla v^n_{\varepsilon_n}+\nabla v^n),\nabla\dot{w}^n)_{L^2(\Omega)} - (\mathbf{p}^n\nabla v^n_{\varepsilon_n},\dot{v}^n)_{L^2(\Omega)} - (\mathbf{p}^n\nabla v^n,\dot{v}^n_{\varepsilon_n})_{L^2(\Omega)}\right]\mathrm{d}s\\
&+ \int_0^t \left[(\mathbf{p}^n(\nabla v^n_{\varepsilon_n}+\nabla v^n),\dot{w}^n)_{L^2(\Omega)} + 2(\dot{v}^n_{\varepsilon_n}+\dot{v}^n,\operatorname{div}[\dot{w}^n\otimes b^n])_{L^2(\Omega)}\right]\mathrm{d}s\\
&+ \int_0^t \left[-2(\dot{v}^n\operatorname{div} b^n,\dot{v}^n_{\varepsilon_n})_{L^2(\Omega)} + \varepsilon_n(\dot{v}^n_{\varepsilon_n},\dot{w}^n)_{H^1(\Omega\setminus\Gamma_0)} + (g^n,\dot{v}^n_{\varepsilon_n}+\dot{v}^n-2\dot{w}^n)_{L^2(\Omega)}\right]\mathrm{d}s\\
&+ \int_0^t \left[-(\dot{F}^n,v^n_{\varepsilon_n}+v^n-2w^n)_{L^2(\partial_N\Omega)} - (\dot{v}^n_{\varepsilon_n}+\dot{v}^n,\ddot{w}^n)_{L^2(\Omega)}\right]\mathrm{d}s\\
&+ (F^n(t),v^n_{\varepsilon_n}(t)+v^n(t)-2w^n(t))_{L^2(\partial_N\Omega)} + (\dot{v}^n_{\varepsilon_n}(t)+\dot{v}^n(t),\dot{w}^n(t))_{L^2(\Omega)}\\
&- 2(F^n(0),v^{0,n}-w^n(0))_{L^2(\partial_N\Omega)} - 2(v^{1,n},\dot{w}^n(0))_{L^2(\Omega)}
\end{aligned} \tag{1.4.42}$$

for $t\in[0,T]$. By using the uniform bounds (1.4.19) and (1.4.20), the ellipticity condition (1.2.1) for $\mathbb{B}^n$ and the estimate (1.4.32) for z^n, we infer

$$\|\dot{z}^n(t)\|^2_{L^2(\Omega)} + \|z^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)} \le C\left(\mathcal{R}_n(t) + \int_0^t \left[\|\dot{z}^n\|^2_{L^2(\Omega)} + \|z^n\|^2_{H^1(\Omega\setminus\Gamma_0)}\right]\mathrm{d}s\right) \tag{1.4.43}$$

for every $t\in[0,T]$, where $C>0$ is a constant independent of n and t.

Let us show that $\mathcal{R}_n(t) \to 0$ as $n \to \infty$. Thanks to Step 1 and 2, we know that $v^n_{\varepsilon_n}(t)$ converges to $v(t)$ strongly in $H^1(\Omega \setminus \Gamma_0; \mathbb{R}^d)$ for every $t \in [0,T]$ as $n \to \infty$, while, $\dot{v}^n_{\varepsilon_n}(t)$ converges to $\dot{v}(t)$ strongly in $L^2(\Omega; \mathbb{R}^d)$. We use now the weak convergences (1.4.41), together with (1.4.16), (1.4.17), (1.4.27), and Lemma 1.1.2, to derive

$$
\begin{aligned}
&\lim_{n\to\infty} \left[(\dot{v}^n_{\varepsilon_n}(t), \dot{v}^n(t))_{L^2(\Omega)} + (\mathbb{B}^n(t)\nabla v^n_{\varepsilon_n}(t), \nabla v^n(t))_{L^2(\Omega)}\right] \\
&= \|\dot{v}(t)\|^2_{L^2(\Omega)} + (\mathbb{B}(t)\nabla v(t), \nabla v(t))_{L^2(\Omega)}, \\
&\lim_{n\to\infty} \left[\|v^{1,n}\|^2_{L^2(\Omega)} + (\mathbb{B}^n(0)\nabla v^{0,n}, \nabla v^{0,n})_{L^2(\Omega)}\right] = \|v^1\|^2_{L^2(\Omega)} + (\mathbb{B}(0)\nabla v^0, \nabla v^0)_{L^2(\Omega)}
\end{aligned}
$$

for every $t \in [0,T]$. Moreover, by arguing as in Step 3, it is easy to check

$$
\varepsilon_n \int_0^T \|\dot{v}^n_{\varepsilon_n}\|^2_{H^1(\Omega\setminus\Gamma_0)} \mathrm{d}s \le C \quad \text{for every } n \in \mathbb{N}
$$

for a constant $C > 0$ independent of n. Therefore, for every $t \in [0,T]$ as $n \to \infty$

$$
\begin{aligned}
\left|\varepsilon_n \int_0^t (\dot{v}^n_{\varepsilon_n}, \dot{w}^n)_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s\right| &\le \sqrt{\varepsilon_n} \int_0^T \sqrt{\varepsilon_n}\|\dot{v}^n_{\varepsilon_n}\|_{H^1(\Omega\setminus\Gamma_0)}\|\dot{w}^n\|_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s \\
&\le \sqrt{\varepsilon}\|\dot{w}^n\|_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))} \left(\int_0^T \varepsilon_n \|\dot{v}^n_{\varepsilon_n}\|^2_{H^1(\Omega\setminus\Gamma_0)}\,\mathrm{d}s\right)^{1/2} \\
&\le \sqrt{\varepsilon_n} C \to 0.
\end{aligned}
$$

since $w^n \to w$ in $L^2(0,T; H^1(\Omega \setminus \Gamma_0; \mathbb{R}^d))$. In view of the previous convergences, (1.4.10), (1.4.11), (1.4.19)–(1.4.27), and the dominated convergence theorem in the time variable, we can pass to the limit as $n \to \infty$ in (1.4.42) and, by using the energy equality (1.3.2), we conclude that $\mathcal{R}_n(t) \to 0$ for every $t \in [0,T]$ as $n \to \infty$. Hence, we can apply Fatou and Gronwall's lemmas to (1.4.43) to derive

$$
\lim_{n\to\infty} \left[\|\dot{z}^n(t)\|^2_{L^2(\Omega)} + \|z^n(t)\|^2_{H^1(\Omega\setminus\Gamma_0)}\right] = 0.
$$

This convergence gives (1.4.30) and concludes the proof. □

Chapter 2

Dynamic energy-dissipation balance of a growing crack

In this chapter, we derive a formula for the mechanical energy (8) associated with the solutions to the wave equation (7), and we derive necessary and sufficient conditions for the validity of the dynamic energy-dissipation balance (11).

The plan of the chapter is the following: in Section 2.1, we fix the standing assumptions on the crack set and the matrix A; moreover, we introduce the changes of variables which transform (7) into (12). Then, in Section 2.2, we prove the decomposition result (9), by adapting the proof of [43, Theorem 4.8] to our more general case, underlying the main differences. Finally, in Section 2.3, we prove the energy balance (10) from which we deduce necessary and sufficient conditions in order to get (11).

The results of this chapter, obtained in collaboration with I. Lucardesi and E. Tasso, are contained in the submitted paper [9].

2.1 Preliminary results

We consider a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, we take a Borel subset $\partial_D\Omega$ of $\partial\Omega$ (possibly empty), and we denote by $\partial_N\Omega$ its complement. We fix a $C^{3,1}$ curve $\gamma\colon [0,\ell] \to \overline{\Omega}$ parametrized by arc-length, with endpoints on $\partial\Omega$; namely, denoting by Γ the support of γ, we assume $\Gamma \cap \partial\Omega = \gamma(0) \cup \gamma(\ell)$. Let T be a positive number, $s\colon [0,T] \to (0,\ell)$ be a non-decreasing function of class $C^{3,1}$, and let us set

$$\Gamma_t := \{\gamma(\sigma) : 0 \le \sigma \le s(t)\} \quad \text{for every } t \in [0,T].$$

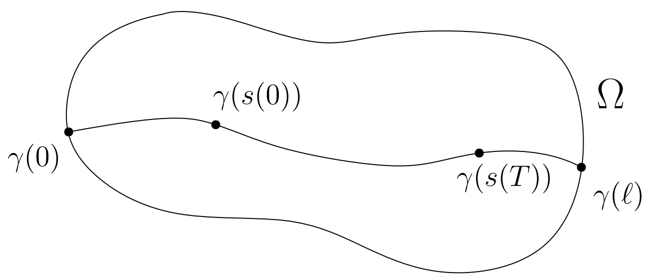

Figure 2.1: *The endpoints of* Γ *are* $\gamma(0)$ *and* $\gamma(\ell)$ *and belong to* $\partial\Omega$*. We study the evolution of the crack along* Γ *from* $\gamma(s(0))$ *to* $\gamma(s(T))$*.*

Let $A \in C^{2,1}(\overline{\Omega}; \mathbb{R}^{2\times 2}_{sym})$ be a matrix field satisfying the ellipticity condition

$$(A(x)\xi)\cdot\xi \ge \lambda_0|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and } x \in \overline{\Omega}, \tag{2.1.1}$$

with $\lambda_0 > 0$ independent of x. Given a function $f \in C^0([0,T];H^1(\Omega)) \cap \mathrm{Lip}([0,T];L^2(\Omega))$ and suitable initial data u^0 and u^1 (for their precise regularity, see Theorems 2.2.4 and 2.2.10), we consider the differential equation

$$\ddot{u}(t) - \operatorname{div}(A\nabla u(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T], \tag{2.1.2}$$

with boundary conditions

$$u(t) = 0 \quad \text{on } \partial_D\Omega,\ t \in [0,T], \tag{2.1.3}$$
$$(A\nabla u(t)) \cdot \nu = 0 \quad \text{on } \partial_N\Omega \cup \Gamma_t,\ t \in [0,T], \tag{2.1.4}$$

where ν denotes the unit normal vector, and initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1 \quad \text{in } \Omega \setminus \Gamma_0. \tag{2.1.5}$$

The equation (2.1.2) has to be intended in the following weak sense: for a.e. $t \in (0,T)$

$$\langle \ddot{u}(t), \psi \rangle_{H_D^{-1}(\Omega\setminus\Gamma_t)} + (A\nabla u(t), \nabla\psi)_{L^2(\Omega)} = (f(t), \psi)_{L^2(\Omega)}$$

for every $\psi \in H_D^1(\Omega \setminus \Gamma_t)$, where $H_D^1(\Omega \setminus \Gamma_t)$ and $H_D^{-1}(\Omega \setminus \Gamma_t)$ are the spaces defined in Chapter 1. We implicitly require $u(t)$ to be in $H_D^1(\Omega \setminus \Gamma_t)$ and $\ddot{u}(t)$ to be in $H_D^{-1}(\Omega \setminus \Gamma_t)$ for a.e. $t \in (0,T)$ (see also [20, Definition 2.4] and Definition 1.1.6).

We assume that the velocity of s is bounded as follows:

$$|\dot{s}(t)|^2 \leq \lambda_0 - \delta \quad \text{for every } t \in [0,T], \tag{2.1.6}$$

for a constant $0 < \delta \leq \lambda_0$. This relation between $\dot{s}$ and the ellipticity constant λ_0 of A is crucial in order to guarantee the resolvability of the problem (see also (2.1.14) in Lemma 2.1.1). (2.1.6) can be interpreted saying that the crack must evolve more slowly than the speed of elastic waves.

2.1.1 The change of variable approach

We fix $t_0, t_1 \in [0,T]$ such that $0 < t_1 - t_0 < \rho$, with ρ sufficiently small. A comment on the value of ρ is postponed to Remark 2.1.3. In the following, we perform 4 changes of variables: first we act on the matrix A, transforming it into the identity on the crack set; then we straighten the crack in a neighborhood of $\gamma(s(t_0))$; then we recall the time-dependent change of variables introduced in [20], that brings Γ_t into Γ_{t_0} for every $t \in [t_0, t_1]$; finally, we perform the last change of variables in a neighborhood of the (fixed) crack-tip, in order to make the principal part of the transformed equation equal to the minus Laplacian. For the sake of clarity, at each step, we use the superscript $i = 1, \dots, 4$, to denote the new objects: the domain $\Omega^{(i)}$, the crack set $\Gamma^{(i)}$, and the time-dependent crack $\Gamma_t^{(i)}$. We will also introduce the matrix fields $A^{(i)}$, which characterize the leading part (with respect to the spatial variables) $-\operatorname{div}(A^{(i)}\nabla v)$ of the PDE (2.1.2) transformed.

Step 1. Thanks to the standing assumptions on A, we may find a matrix field Q of class $C^{2,1}(\overline{\Omega};\mathbb{R}^{2\times 2})$ such that

$$Q(x)A(x)Q(x)^T = Id \quad \text{for every } x \in \Omega, \tag{2.1.7}$$

being Id the identity matrix. In particular we can choose $Q(x)$ to be equal to the square root matrix of $A(x)^{-1}$, namely $Q(x) = Q(x)^T$ and $Q(x)^2 = A(x)^{-1}$. It is easy to prove the existence of a smooth diffeomorphism $\chi \in C^{3,1}(\overline{\Omega};\mathbb{R}^2)$ of Ω into itself which is the identity in a neighborhood of $\partial\Omega$ and satisfies $\nabla\chi(x) = Q(x)$ on $\Gamma \cap V$, being V a suitable neighborhood of $\gamma(s(t_0))$. Notice that the constraint $\nabla\chi = Q$ cannot be satisfied in the whole domain, since the rows of Q in general are not curl free. We set

$$\Omega^{(1)} := \Omega, \quad \Gamma^{(1)} := \chi(\Gamma), \quad \Gamma^{(1)}_t := \chi(\Gamma_t) \quad \text{for } t \in [t_0, t_1],$$
$$A^{(1)} := [\nabla\chi A \nabla\chi^T] \circ \chi^{-1}.$$

Clearly, the matrix $A^{(1)}$ satisfies an ellipticity condition of type (2.1.1) for a suitable positive constant and it equals the identity matrix on $\Gamma^{(1)}$. Moreover, we may easily write an arc-length parametrization $\gamma^{(1)}$ of $\Gamma^{(1)}$ exploiting that of Γ, by setting

$$\gamma^{(1)} := \chi \circ \gamma \circ \beta, \quad \beta^{-1}(\sigma) := \int_0^\sigma \left| \frac{\mathrm{d}}{\mathrm{d}\tau}(\chi \circ \gamma)(\tau) \right| \mathrm{d}\tau.$$

Accordingly, the time-dependent crack $\Gamma^{(1)}_t$ is parametrized by

$$\Gamma^{(1)}_t = \gamma^{(1)}(s^{(1)}(t)) \quad \text{for } t \in [t_0, t_1], \quad s^{(1)} := \beta^{-1} \circ s.$$

The function $s^{(1)}$ is of class $C^{3,1}([t_0, t_1])$ and, thanks to (2.1.7) and (2.1.6), satisfies the following bound:

$$|\dot{s}^{(1)}(t)|^2 = \left| \frac{\mathrm{d}\beta^{-1}}{\mathrm{d}s}(s(t)) \right|^2 |\dot{s}(t)|^2 \le \max_{|\xi|=1,\, x \in \Gamma \cap V} |\nabla\chi(x)\xi|^2 |\dot{s}(t)|^2 \le 1 - c_1^2 \tag{2.1.8}$$

for every $t \in [t_0, t_1]$, where, for brevity, we have set $c_1^2 := \frac{\delta}{\lambda_0}$. Moreover, for the sake of clarity, we also fix a notation for the maximal acceleration: we set c_2 as

$$c_2 := \max_{t \in [t_0, t_1]} |\ddot{s}^{(1)}(t)|. \tag{2.1.9}$$

A direct computation proves that c_2 is bounded and depends on λ_0, δ, $\ddot{s}$, $\ddot{\gamma}$, and $\nabla^2\chi$.

Step 2. We now provide a change of variables Λ of class $C^{2,1}$ which straightens the crack in a neighborhood of $\gamma^{(1)}(s^{(1)}(t_0))$. First, up to further compose Λ with a rigid motion, we may assume that the tip of $\Gamma^{(1)}_{t_0}$ is at the origin, and the tangent vector to $\Gamma^{(1)}$ at the origin is horizontal, namely

$$\gamma^{(1)}(s^{(1)}(t_0)) = 0, \quad \dot{\gamma}^{(1)}(s^{(1)}(t_0)) = e_1 := (1, 0).$$

For brevity, we set $\sigma_0 := s^{(1)}(t_0)$. We begin by transforming a tubular neighborhood U of the crack near 0 into a square: setting

$$U := \{\gamma^{(1)}(\sigma_0 + \sigma) + \tau\nu^{(1)}(\sigma_0 + \sigma) : \sigma \in (-\varepsilon, \varepsilon),\ \tau \in (-\varepsilon, \varepsilon)\},$$

with $\nu^{(1)} := (\dot{\gamma}^{(1)})^\perp$ and $\varepsilon > 0$ such that $U \subset\subset \Omega$, we define $\Lambda \colon U \to (-\varepsilon, \varepsilon)^2$ as the inverse of the function $(\sigma, \tau) \mapsto \gamma^{(1)}(\sigma + \sigma_0) + \tau\nu^{(1)}(\sigma + \sigma_0)$. The global diffeomorphism is obtained by extending Λ to the whole Ω. Accordingly, we set

$$\Omega^{(2)} := \Lambda(\Omega^{(1)}), \quad \Gamma^{(2)} := \Lambda(\Gamma^{(1)}), \quad \Gamma^{(2)}_t := \Lambda(\Gamma^{(1)}_t) \quad \text{for } t \in [t_0, t_1],$$
$$A^{(2)} := [\nabla\Lambda A^{(1)} \nabla\Lambda^T] \circ \Lambda^{-1}.$$

The matrix field $A^{(2)}$ still satisfies an ellipticity condition of type (2.1.1), for a suitable constant.

For $x \in \Gamma^{(2)}$ and in a neighborhood of the origin, setting $y := \Lambda^{-1}(x) \in \Gamma^{(1)}$, we have

$$A^{(2)}(x) = \nabla\Lambda(y) A^{(1)}(y) \nabla\Lambda(y)^T = \nabla\Lambda(y)\nabla\Lambda(y)^T = [(\nabla(\Lambda^{-1})(x))^T \nabla(\Lambda^{-1})(x)]^{-1} = Id.$$

The last equality follows from

$$\frac{\partial(\Lambda^{-1})}{\partial\sigma}(\sigma,\tau)=\dot\gamma^{(1)}(\sigma_0+\sigma)+\tau\dot\nu^{(1)}(\sigma_0+\sigma),\qquad \frac{\partial(\Lambda^{-1})}{\partial\tau}(\sigma,\tau)=\nu^{(1)}(\sigma_0+\sigma),\tag{2.1.10}$$

and the fact that here we consider x of the form $x=(\sigma,0)$. In particular, we may be more precise on the ellipticity constant of $A^{(2)}$ restricted to a neighborhood of the origin: for every $\varepsilon\in(0,1)$, there exists $r>0$ such that

$$(A^{(2)}(x)\xi)\cdot\xi\geq(1-\varepsilon)|\xi|^2\quad\text{for every }\xi\in\mathbb{R}^2\text{ and }|x|\leq r.\tag{2.1.11}$$

Finally, we underline that if $\rho:=t_1-t_0$ is small enough (see also Remark 2.1.3), the whole set $\Gamma^{(1)}_{t_1}\setminus\Gamma^{(1)}_{t_0}$ is contained in U, so that the time-dependent crack $\Gamma^{(2)}_t$ satisfies

$$\Gamma^{(2)}_t=\Gamma^{(2)}_{t_0}\cup\{(\sigma,0)\in\mathbb{R}^2:0\leq\sigma\leq s^{(1)}(t)-s^{(1)}(t_0)\}\quad\text{for every }t\in[t_0,t_1].$$

Step 3. Here we introduce a family of 1-parameter C^2 diffeomorphisms $\Psi(t)$, $t\in[t_0,t_1]$, which transform the time-dependent domain $\Omega^{(2)}\setminus\Gamma^{(2)}_t$ into $\Omega^{(2)}\setminus\Gamma^{(2)}_{t_0}$. All in all, we map the domain $\{(t,x):t\in[t_1,t_2],\,x\in\Omega^{(2)}\setminus\Gamma^{(2)}_t\}$ into the cylinder $[t_0,t_1]\times(\Omega^{(2)}\setminus\Gamma^{(2)}_{t_0})$. This construction can be found in [43] and [20, Example 2.14], thus we limit ourselves to recall the main properties: the diffeomorphism $\Psi\colon[t_0,t_1]\times\overline{\Omega}^{(2)}\to\overline{\Omega}^{(2)}$ satisfies

$$\Psi(t_0)=id,\quad \Psi(t)|_{\partial\Omega^{(2)}}=id,\quad \Psi(t,\Gamma^{(2)}_t)=\Gamma^{(2)}_{t_0}\quad\text{for }t\in[t_0,t_1].$$

The corresponding matrix field is

$$A^{(3)}(t):=[\nabla\Psi(t)A^{(2)}\nabla\Psi(t)^T-\dot\Psi(t)\otimes\dot\Psi(t)]\circ\Psi^{-1}(t)\quad\text{for }t\in[t_0,t_1].$$

Notice that $A^{(2)}$ does not depend on time, while $A^{(3)}$ does. For $t\in[t_0,t_1]$ and x in a neighborhood of the origin we have

$$\Psi(t,x)=x-(s^{(1)}(t)-s^{(1)}(t_0))e_1,\quad \Psi^{-1}(t,x)=x+(s^{(1)}(t)-s^{(1)}(t_0))e_1,\tag{2.1.12}$$

so that $\nabla\Psi(t)=Id$, $\dot\Psi(t)=-\dot s^{(1)}(t)e_1$, and for $x=(\sigma,0)$, with σ small enough in modulus

$$A^{(3)}(t,x)=\begin{pmatrix}1-|\dot s^{(1)}(t)|^2 & 0\\ 0 & 1\end{pmatrix}.$$

Step 4. In this last step we apply a change of variables P near the origin (namely the tip of $\Gamma^{(2)}_{t_0}$), in order to make the matrix field $A^{(4)}$, constructed as in the previous steps, satisfy $A^{(4)}(t,0)=Id$ for every $t\in[t_0,t_1]$. To this aim, we recall the construction introduced in [43, Section 4].

We define $\alpha\colon[t_0,t_1]\to[0,\infty)$ and $d\colon[t_0,t_1]\times\overline{\Omega}^{(2)}\to[0,c_1]$ as

$$\begin{aligned}\alpha(t)&:=\sqrt{1-|\dot s^{(1)}(t)|^2},\\ d(t,x)&:=\alpha(t)k_\eta(|x|)+(1-k_\eta(|x|))c_1,\end{aligned}$$

where k_η is the following cut-off function:

$$k_\eta(\tau):=\begin{cases}1 & \text{if }0\leq\tau<\eta/2,\\ \left(2\frac{\tau}{\eta}-2\right)^2\left(4\frac{\tau}{\eta}-1\right) & \text{if }\eta/2\leq\tau<\eta,\\ 0 & \text{if }\tau\geq\eta.\end{cases}\tag{2.1.13}$$

Here η is a positive parameter, whose precise value will be specified later, small enough such that the ball $B_\eta(0) \subset \Omega^{(2)}$. Eventually, we set

$$P(t,x) := \left(\frac{x_1}{d(t,x)}, x_2\right) \quad t \in [t_0,t_1],\, x \in \Omega^{(2)}.$$

For every $t \in [t_0,t_1]$ the map $P(t)$ defines a diffeomorphism of $\Omega^{(2)}$ into its dilation in the horizontal direction

$$\Omega^{(4)} := \left\{\left(\frac{x_1}{c_1}, x_2\right) : x \in \Omega^{(2)}\right\},$$

which maps 0 in 0 and $\Gamma^{(2)}_{t_0}$ into a fixed set $\Gamma^{(4)}_{t_0}$, horizontal near the origin. This chain of transformations maps the Dirichlet part $\partial_D\Omega$ into $\partial_D\Omega^{(4)} := \{(\Lambda_1(x)/c_2, \Lambda_2(x)) : x \in \partial_D\Omega\}$ and the Neumann one $\partial_N\Omega$ into $\partial_N\Omega^{(4)} := \{(\Lambda_1(x)/c_2, \Lambda_2(x)) : x \in \partial_N\Omega\}$.

The matrix field $A^{(4)}$ associated to P reads

$$A^{(4)}(t) := [\nabla P(t) A^{(3)}(t) \nabla P(t)^T - \dot P(t) \otimes \dot P(t) - 2\nabla P(t) \dot\Psi(t, \Psi^{-1}(t)) \odot \dot P(t)] \circ P^{-1}(t)$$

for $t \in [t_0,t_1]$. The properties of $A^{(4)}$ are gathered in the following lemma.

Lemma 2.1.1. *There exists a constant $\lambda_4 > 0$ such that for every $t \in [t_0,t_1]$ and $x \in \Omega^{(4)}$*

$$(A^{(4)}(t,x)\xi)\cdot\xi \geq \lambda_4|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2. \tag{2.1.14}$$

Moreover, for every $t \in [t_0,t_1]$, there holds

$$A^{(4)}(t,0) = Id. \tag{2.1.15}$$

Finally, there exists a vector field $W\colon \partial_N\Omega^{(4)} \cup \Gamma^{(4)}_{t_0} \to \mathbb{R}^2$ such that for every $t \in [t_0,t_1]$ and $x \in \partial_N\Omega^{(4)} \cup \Gamma^{(4)}_{t_0}$

$$A^{(4)}(t,x)^T\nu(x) = W(x), \tag{2.1.16}$$

and $W(x) = \nu(x) = e_2 := (0,1)$ in a neighborhood of the tip of $\Gamma^{(4)}_{t_0}$.

Proof. Let $t \in [t_0,t_1]$ and $x \in \Omega^{(4)}$ be fixed. By setting $y := P^{-1}(t,x) \in \Omega^{(2)}$, we distinguish the three cases: $|y| < \eta/2$, $\eta/2 \leq |y| \leq \eta$, and $|y| > \eta$, where η is the constant introduced in (2.1.13). Without loss of generality, up to take η smaller, by recalling (2.1.12) we may assume that if $y \in B_\eta(0)$

$$\nabla\Psi(t,\Psi^{-1}(t,y)) = Id, \quad \dot\Psi(t,\Psi^{-1}(t,y)) = -\dot s^{(1)}(t)e_1,$$

so that

$$A^{(3)}(t,P^{-1}(t,x)) = A^{(3)}(t,y) = A^{(2)}(y) - |\dot s^{(1)}(t)|^2 e_1 \otimes e_1.$$

Moreover, we take $\eta < r$, where r is the radius associated to $\varepsilon = c_1^2/2$ as in (2.1.11), so that the ellipticity constant of $A^{(2)}$ in $B_\eta(0)$ is $1 - c_1^2/2$.

If $|y| < \eta/2$ we have

$$\nabla P(t,y) = \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \dot P(t,y) = \begin{pmatrix} -y_1\frac{\dot\alpha(t)}{\alpha^2(t)} \\ 0 \end{pmatrix},$$

thus

$$A^{(4)}(t,x) = \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix} A^{(2)}(y) \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{|\dot s^{(1)}(t)|^2}{\alpha(t)^2} + y_1\frac{2\dot s^{(1)}(t)\dot\alpha(t)}{\alpha^3(t)} + y_1^2\frac{\dot\alpha^2(t)}{\alpha^4(t)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $P^{-1}(t,0)=0$ and $A^{(2)}(0)=Id$, we immediately get (2.1.15). For ξ arbitrary vector of $\mathbb{R}^2$, we have

$$(A^{(4)}(t,x)\xi)\cdot\xi \geq \left(\frac{1-c_1^2/2-|\dot{s}^{(1)}(t)|^2}{\alpha^2(t)} - 2y_1\frac{\dot{s}^{(1)}(t)\dot{\alpha}(t)}{\alpha^3(t)} - y_1^2\frac{\dot{\alpha}^2(t)}{\alpha^4(t)}\right)\xi_1^2 + (1-c_1^2/2)\xi_2^2.$$

In view of the bounds (2.1.6), (2.1.9), and (2.1.8), we get

$$|\dot{\alpha}(t)| \leq \frac{c_2}{c_1}, \quad c_1 \leq |\alpha(t)| \leq 1,$$

in particular

$$(A^{(4)}(t,x)\xi)\cdot\xi \geq \left(\frac{c_1^2}{2} - 2\eta\frac{c_2}{c_1^4} - \eta^2\frac{c_2^2}{c_1^6}\right)\xi_1^2 + \frac{\xi_2^2}{2}.$$

The coefficient of ξ_1 is bounded from below, provided that η is small enough. This gives the statement (2.1.14) for $y \in B_{\eta/2}(0)$.

Let now $\eta/2 < |y| < \eta$. In this case we have

$$\nabla P(t,y) = \frac{1}{d^2(t,y)}\begin{pmatrix} d(t,y) - y_1\partial_1 d(t,y) & -y_1\partial_2 d(t,y) \\ 0 & d^2(t,y)\end{pmatrix},$$
$$\dot{P}(t,y) = \frac{1}{d^2(t,y)}\begin{pmatrix} -y_1\dot{d}(t,y) \\ 0 \end{pmatrix}.$$

Again, by exploiting the ellipticity of $A^{(2)}$ with constant $(1-c_1^2/2) \geq \frac{1}{2}$ and setting

$$m := y_1^2\dot{d}(t,y)^2 + 2y_1\dot{s}^{(1)}(t)\dot{d}(t,y)(d(t,y)-y_1\partial_1 d(t,y)),$$
$$p := d(t,y) - y_1\partial_1 d(t,y), \quad q := -y_1\partial_2 d(t,y), \quad d := d(t,y),$$

we get

$$\begin{aligned}(A^{(4)}(t,x)\xi)\cdot\xi &\geq \frac{1}{2}|\nabla P(t,y)^T\xi|^2 - \frac{m}{d^4}\xi_1^2 \\ &= \frac{1}{2d^4}\left[(p^2+q^2-2m)\xi_1^2 + 2qd^2\xi_1\xi_2 + d^4\xi_2^2\right] \\ &\geq \frac{1}{2}\left[p^2 - \left(\frac{1}{\varepsilon}-1\right)q^2 - 2|m|\right]\xi_1^2 + \frac{1}{2}(1-\varepsilon)\xi_2^2,\end{aligned} \tag{2.1.17}$$

where in the last inequality we have used $d \leq 1$ and Young's inequality with $0<\varepsilon<1$, whose precise value will be fixed later. Let us prove that, if η and ε are well chosen, the coercivity of $A^{(4)}$ is guaranteed. The identities

$$\nabla d(t,y) = (\alpha(t)-c_1)\frac{y}{|y|}\dot{k}_\eta(|y|), \quad \dot{d}(t,y) = -\frac{\dot{s}^{(1)}(t)\ddot{s}^{(1)}(t)k_\eta(|y|)}{\alpha(t)},$$

together with the bounds

$$0 \leq k_\eta(|y|) \leq 1, \quad c_1 \leq d(t,y) \leq \alpha(t) \leq 1, \quad -\frac{3}{\eta} \leq \dot{k}_\eta(|y|) \leq 0,$$

give

$$\frac{1}{d^4} \geq 1, \quad p = d(t,y) + \frac{y_1^2}{|y|}(\alpha(t)-c_1)|\dot{k}_\eta(|y|)| \geq d(t,y) \geq c_1,$$
$$q^2 = (\alpha(t)-c_1)^2\frac{y_1^2y_2^2}{|y|^2}\dot{k}_\eta(|y|)^2 \leq 9(1-c_1)^2, \quad |m| \leq \frac{42c_2(1-c_1^2)}{c_1}\eta + \frac{c_2^2(1-c_1^2)}{c_1^2}\eta^2.$$

By inserting these estimates into (2.1.17), we infer

$$(A^{(4)}(t,x)\xi)\cdot\xi \geq \left[\frac{c_1^2}{2} - \frac{9}{2}\left(\frac{1}{\varepsilon}-1\right)(1-c_1)^2 - \frac{42c_2(1-c_1^2)}{c_1}\eta - \frac{c_2^2(1-c_1^2)}{c_1^2}\eta^2\right]\xi_1^2 + \frac{1-\varepsilon}{2}\xi_2^2.$$

By taking

$$\varepsilon := \frac{9(1-c_1)^2}{c_1^2/2 + 9(1-c_1)^2} \in (0,1)$$

we have

$$\frac{c_1^2}{2} - \frac{9}{2}\left(\frac{1}{\varepsilon}-1\right)(1-c_1)^2 = \frac{c_1^2}{4}.$$

Thus, by choosing η small enough, we obtain the desired coercivity of $A^{(4)}$.

Finally, if $|y| > \eta$ we have

$$\nabla P(t,y) = \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \dot{P}(t,y) = 0,$$

and condition (2.1.14) is readily satisfied in view of the ellipticity of $A^{(3)}$.

The assertion (2.1.16) is clearly verified for $A^{(2)}$: the matrix field does not depend on time and equals to the identity on the crack, in a neighborhood of the origin. The last diffeomorphisms Ψ and P both act in a neighborhood of the origin modifying the set only in the horizontal component; in particular they do not modify the normal to the crack in a neighborhood of the origin. As for the external boundary, Ψ is the identity and P acts as a constant dilation, so that

$$W(x) = \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix} A^{(2)}(c_1x_1, x_2) \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix} \nu(x) \quad \text{on } \partial_N\Omega^{(4)}.$$

This concludes the proof of the lemma. □

Remark 2.1.2. The idea of the proof of Lemma 2.1.1 is taken from [43, Lemma 4.1]. Let us underline the main differences: in [43] the authors deal with the identity matrix as starting matrix field (here instead we have $A^{(3)}$) and consider only the dynamics for which the acceleration of the crack-tip is bounded by a precise constant depending on c_1 (in place of our bound c_2, not fixed a priori). We also point out that in [43] the study of the ellipticity of the transformed matrix field, in the annulus $\eta/2 < |y| < \eta$, is carried out forgetting the coefficients out of the diagonal.

Remark 2.1.3. In our construction, a control on the maximal amplitude ρ of the time interval $[t_0, t_1]$ is needed only in Step 2: roughly speaking, in order to straighten the set $\Gamma_{t_1}^{(1)} \setminus \Gamma_{t_0}^{(1)}$ and to remain inside Ω, we need to have enough room. A sufficient condition is that the length of the set, which is at most $\rho \max_{t\in[0,T]} \dot{s}^{(1)}(t)$, has to be less than or equal to the distance of the crack-tip $\gamma^{(1)}(s^{(1)}(t))$ from the boundary $\partial\Omega$, which is, thanks to the assumption $\Gamma_T^{(1)} \setminus \Gamma_0^{(1)} \subset\subset \Omega$, bounded from below by a positive constant. Notice that if we considered also a further diffeomorphism which is the identity in a neighborhood of $\Gamma_T^{(1)} \setminus \Gamma_0^{(1)}$ and stretches Ω near the boundary, then our results could be stated for every time $t \in [0,T]$.

2.2 Representation result

In this section we derive the decomposition (9) locally in time, namely in a time interval $[t_0, t_1]$ small enough (see Section 2.1 and Remark 2.1.3). Finally, in Theorem 2.2.10 we give a global representation of u, valid in the whole time interval $[0, T]$.

Here we recall some classical facts of semigroup theory. Standard references on the subject are the books [45] and [33]. Let X be a Banach space and $\mathcal{A}(t)\colon \mathrm{D}(\mathcal{A}(t)) \subseteq X \to X$ a differential operator. Consider the evolution problem

$$\dot{V}(t) + \mathcal{A}(t)V(t) = G(t), \tag{2.2.1}$$

with initial condition $V(0) = V_0$ and G forcing term (the boundary conditions are encoded in the function space X).

Definition 2.2.1. A triplet $\{\mathcal{A}; X, Y\}$ consisting of a family $\mathcal{A} = \{\mathcal{A}(t),\, t \in [0, T]\}$ and a pair of real separable Banach spaces X and Y is called a *constant domain system* if the following conditions hold:

(i) the space Y is embedded continuously and densely in X;

(ii) for every t the operator $\mathcal{A}(t)$ is linear and has constant domain $\mathrm{D}(\mathcal{A}(t)) \equiv Y$;

(iii) the family $\mathcal{A}$ is a stable family of (negative) generators of strongly continuous semigroups on X;

(iv) the operator $\dot{\mathcal{A}}$ is essentially bounded from $[0, T]$ to the space of linear functionals from Y to X.

Theorem 2.2.2. *Let $\{\mathcal{A}; X, Y\}$ form a constant domain system. Let us assume that $V^0 \in Y$ and $G \in \mathrm{Lip}([0, T]; X)$. Then there exists a unique solution $V \in C^0([0, T]; Y) \cap C^1([0, T]; X)$ to (2.2.1) with $V(0) = V^0$.*

2.2.1 Local representation result

We fix $t_0, t_1 \in [0, T]$ such that $0 < t_1 - t_0 < \rho$. The chain of transformations introduced in Section 2.1 defines the family of time-dependent diffeomorphisms

$$\Phi(t) := P(t) \circ \Psi(t) \circ \Lambda \circ \chi, \quad \Phi(t) : \overline{\Omega} \to \overline{\Omega}^{(4)}, \tag{2.2.2}$$

which map Γ into $\Gamma^{(4)}$, Γ_t into $\Gamma^{(4)}_{t_0}$ for every $t \in [t_0, t_1]$, $\partial\Omega$ into $\partial\Omega^{(4)}$, the Dirchlet part $\partial_D\Omega$ into $\partial_D\Omega^{(4)}$, and the Neumann one $\partial_N\Omega$ into $\partial_N\Omega^{(4)}$. For the sake of clarity, we denote by x the variables in Ω and by y the new variables in $\Omega^{(4)}$.

Looking for a solution u to (2.1.2) in $[t_0, t_1]$ is equivalent to look for $v := u \circ \Phi^{-1}$ solution to the equation

$$\ddot{v}(t) - \mathrm{div}(A^{(4)}(t)\nabla v(t)) + p(t) \cdot \nabla v(t) - 2b(t) \cdot \nabla \dot{v}(t) = g(t) \quad \text{in } \Omega^{(4)} \setminus \Gamma^{(4)}_{t_0},\ t \in [t_0, t_1], \tag{2.2.3}$$

supplemented by the boundary conditions

$$v = 0 \quad \text{on } \partial_D\Omega^{(4)},\ t \in [t_0, t_1], \tag{2.2.4}$$

$$\partial_W v = 0 \quad \text{on } \partial_N\Omega^{(4)} \cup \Gamma^{(4)}_{t_0},\ t \in [t_0, t_1], \tag{2.2.5}$$

and by suitable initial conditions v^0 and v^1 (see [20]). Here W is the vector field introduced in (2.1.16) of Lemma 2.1.1, and for $t \in [t_0, t_1]$ and $y \in \overline{\Omega}^{(4)}$

$$p(t, y) := -[A^{(4)}(t, y)\nabla(\det \nabla\Phi^{-1}(t, y)) + \partial_t(b(t, y) \det \nabla\Phi^{-1}(t, y))] \det \nabla\Phi(t, \Phi^{-1}(t, y)),$$

$$b(t,y) := -\dot{\Phi}(t,\Phi^{-1}(t,y)),$$
$$g(t,y) := f(t,\Phi^{-1}(t,y)).$$

The equation (2.2.3) has to be intended in the weak sense, namely valid for a.e. $t \in (0,T)$ in duality with an arbitrary test function in the space

$$H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}) := \{v \in H^1(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}) : v = 0 \text{ on } \partial_D\Omega^{(4)}\}.$$

We implicitly require $v(t)$ and $\dot{v}(t)$ to be in $H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}(t_0))$, and $\ddot{v}(t)$ to be in the dual $H^{-1}_D(\Omega^{(4)} \setminus \Gamma^{(4)}(t_0))$ for a.e. $t \in (0,T)$.

The characterization of u will follow from that of v, slightly easier to be derived. The advantages in dealing with problem (2.2.3) are essentially 3: first of all, the domain is cylindrical and constant in time; then, the fracture set is straight near the tip; finally, even if the coefficients depend on space and time, the principal part of the spatial differential operator is constant at the crack-tip. Before stating the result, we define

$$\mathcal{H} := \{v \in H^2(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}) : (2.2.5) \text{ hold true}\} \oplus \{k\zeta S : k \in \mathbb{R}\},$$

where ζ is a cut-off function supported in neighborhood of the origin and

$$S(y) := Im(\sqrt{y_1 + iy_2}) = \frac{y_2}{\sqrt{2}\sqrt{|y| + y_1}} \quad y \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\}, \tag{2.2.6}$$

with Im denoting the imaginary part of a complex number.

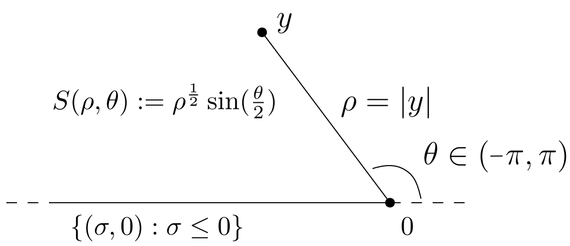

Figure 2.2: *In polar coordinates, the function S reads $S(\rho,\theta) = \rho^{\frac{1}{2}} \sin(\frac{\theta}{2})$, where ρ is the distance from the origin and $\theta \in (-\pi,\pi)$ is the angle which has a discontinuity on the horizontal half line $\{(\sigma,0) : \sigma \leq 0\}$.*

Proposition 2.2.3. *Take $v^0 \in \mathcal{H}$, $v^1 \in H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0})$, and $g \in \mathrm{Lip}([t_0,t_1]; L^2(\Omega^{(4)}))$. Then there exists a unique solution v to (2.2.3)–(2.2.5) with $v(t_0) = v^0$, $\dot{v}(t_0) = v^1$ in the class*

$$v \in C([t_0,t_1];\mathcal{H}) \cap C^1([t_0,t_1]; H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0})) \cap C^2([t_0,t_1]; L^2(\Omega^{(4)})).$$

Proof. Once we show that the triplet $\{\mathcal{A}; X; Y\}$ defined by

$$\mathcal{A}(t) := \begin{pmatrix} 0 & -1 \\ -\operatorname{div}(A^{(4)}(t)\nabla(\cdot)) + p(t)\cdot\nabla(\cdot) & -2b(t)\cdot\nabla(\cdot) \end{pmatrix},$$
$$X := H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}) \times L^2(\Omega^{(4)}),$$
$$Y := \mathcal{H} \times H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}),$$

is a constant domain system in $[t_0,t_1]$ (cf. Definition 2.2.1), we are done. Indeed, we are in a position to apply Theorem 2.2.2 with

$$G(t) := \begin{pmatrix} 0 \\ g(t) \end{pmatrix},$$

and the searched v is the second component of the solution V to (2.2.1).

The detailed proof of properties (i)–(iv) in Definition 2.2.1 can be found in [43, Theorem 4.7], with the appropriate modifications (see Remark 2.1.2). Here we limit ourselves to list the main ingredients. First of all, the domain of $\operatorname{div}(A^{(4)}(t)\nabla(\cdot))$ is constant in time: in view (2.1.15), its principal part, evaluated at the crack-tip, is the Laplace operator for every t, thus the domain of $\operatorname{div}(A^{(4)}(t)\nabla(\cdot))$ can be decomposed as the sum $\{v \in H^2(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}) : (2.2.5) \text{ holds true}\} \oplus \{k\zeta S : k \in \mathbb{R}\} = \mathcal{H}$ (cf. [32, Theorem 5.2.7]). Moreover, in view of (2.1.16), the boundary conditions (2.2.5) do not depend on time. Other key points are the equicoercivity in time of the bilinear form

$$(\phi_0, \phi_1) \mapsto (A^{(4)}(t)\nabla\phi_0) \cdot \nabla\phi_1 \quad \phi_0, \phi_1 \in H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0}),$$

which is guaranteed by (2.1.14) and the property

$$\int_{\Omega^{(4)}\setminus\Gamma^{(4)}_{t_0}} \phi(y)\nabla\phi(y) \cdot b(t, y)\,\mathrm{d}y = -\frac{1}{2}\int_{\Omega^{(4)}\setminus\Gamma^{(4)}_{t_0}} \phi^2(y) \operatorname{div} b(t, y)\,\mathrm{d}y,$$

valid for every $\phi \in H^1_D(\Omega^{(4)} \setminus \Gamma^{(4)}_{t_0})$. Finally, the needed continuity of the differential operator is ensured by the following regularity properties of the coefficients: for every $i, j = 1, 2$ we have

$$\begin{aligned}
&A^{(4)}_{ij}(t) \in C^0(\overline{\Omega}^{(4)}) \quad \text{for every } t \in [t_0, t_1],\\
&A^{(4)}_{ij},\ p_i,\ b_i \in \operatorname{Lip}([t_0, t_1]; L^\infty(\Omega^{(4)})),\\
&\|\nabla A^{(4)}_{ij}(t)\|_{L^\infty(\Omega^{(4)})} \le C, \quad \|\operatorname{div} b(t)\|_{L^\infty(\Omega^{(4)})} \le C \quad \text{for every } t \in [t_0, t_1],
\end{aligned}$$

for a suitable constant $C > 0$ independent of t. □

We are now in a position to state the following representation result for u.

Theorem 2.2.4. *Let $f \in C^0([t_0, t_1]; H^1(\Omega)) \cap \operatorname{Lip}([t_0, t_1]; L^2(\Omega))$. Consider u^0 and u^1 of the form*

$$u^0 - k^0\zeta S(\Phi(t_0)) \in H^2(\Omega \setminus \Gamma_{t_0}), \tag{2.2.7}$$

$$u^1 - \nabla u^0 \cdot \Big(\nabla\Phi^{-1}(t_0, \Phi(t_0))\dot{\Phi}(t_0)\Big) \in H^1(\Omega \setminus \Gamma_{t_0}), \tag{2.2.8}$$

with u^0 satisfying the boundary conditions (2.1.3) and (2.1.4), $u^1 = 0$ on $\partial_D\Omega$, ζ cut-off function supported in neighborhood of $\gamma(s(t_0))$, and $k^0 \in \mathbb{R}$. Then there exists a unique solution to (2.1.2)–(2.1.4) with initial conditions $u(t_0) = u^0$, $\dot{u}(t_0) = u^1$ of the form

$$u(t, x) = u^R(t, x) + k(t)\zeta(t, x)S(\Phi(t, x)) \quad t \in [t_0, t_1],\ x \in \Omega \setminus \Gamma_t, \tag{2.2.9}$$

where $\zeta(t)$, $t \in [t_0, t_1]$, is a C^2 (in time) family of cut-off functions supported in neighborhood of $\gamma(s(t))$, and k is a C^2 function in $[t_0, t_1]$ such that $k(t_0) = k^0$. Moreover, $u^R(t) \in H^2(\Omega\setminus\Gamma_t)$ for every $t \in [t_0, t_1]$, and

$$u^R \in C^2([t_0, t_1]; L^2(\Omega)), \quad \nabla u^R \in C^1([t_0, t_1]; L^2(\Omega; \mathbb{R}^2)), \quad \nabla^2 u^R \in C^0([t_0, t_1]; L^2(\Omega; \mathbb{R}^{2\times 2})).$$

Remark 2.2.5. Notice that the equality $u(t, x) = v(t, \Phi(t, x))$ implies

$$u^0 = v^0(\Phi(t_0)), \quad u^1 = v^1(\Phi(t_0)) + \nabla v^0(\Phi(t_0)) \cdot \dot{\Phi}(t_0),$$

where the last term reads $\dot{\Phi}(t_0) = [\dot{P}(t_0, \Psi(t_0)) + \nabla P(t_0, \Psi(t_0))\dot{\Psi}(t_0)] \circ \Lambda \circ \chi$. A priori, the function ∇v^0 is just in L^2 in a neighborhood of the origin and its gradient behaves like $|y|^{-3/2}$; nevertheless, since $\dot{P}(t,y) \sim (y_1, 0)$, we recover the L^2 integrability of the gradient of $\nabla v^0(\Phi(t_0)) \cdot [\dot{P}(t_0, \Psi(t_0)) \circ \Lambda \circ \chi]$. The same reasoning does not apply for the term $\nabla v^0(\Phi(t_0)) \cdot [(\nabla P(t_0, \Psi(t_0))\dot{\Psi}(t_0)) \circ \Lambda \circ \chi]$, since the singularity of ∇v^0 in a neighborhood of the orgin is not compensated by $\nabla P \cdot \dot{\Psi}$. Therefore we are not free to take $u^1 \in H^1_D(\Omega \setminus \Gamma_{t_0})$ (as, on the contrary, is done in [43]).

Remark 2.2.6. Notice that the solution u to problem (2.1.2)–(2.1.5) displays a singularity only at the crack-tip. Clearly, the fracture is responsible for this lack of regularity. On the other hand, the Dirichlet-Neumann boundary conditions do not produce any further singularity, due to the compatible initial data chosen.

2.2.2 Global representation result

We conclude the section by showing an alternative representation formula which can be expressed for every time. This is done providing another expression for the singular function, as in [38], whose computation does not require to straighten the crack. To simplify the notation we reduce ourselves to the case $A = Id$, so that the diffeomorphism χ coincides with the identity.

The chosen singular part of the solution to (2.1.2)–(2.1.5) is a suitable reparametrization of the function S introduced in (2.2.6). More precisely, fixed $t_0, t_1 \in [0,T]$, with $0 < t_1 - t_0 < \rho$, for every $t \in [t_0, t_1]$ and x in a neighborhood of $r(t) := \gamma(s(t))$, the singular part reads

$$S\left(\frac{\Lambda_1(x) - (s(t) - s(t_0))}{\sqrt{1 - |\dot{s}(t)|^2}}, \Lambda_2(x)\right). \tag{2.2.10}$$

To compute (2.2.10) it is necessary to know the expression of Λ, which is explicit only for small time and locally in space. We hence provide a more explicit formula for the singular part, which has also the advantage of being defined for every time: for every $t \in [0,T]$ we set

$$\hat{S}(t,x) := Im\left(\sqrt{\frac{(x - r(t)) \cdot \dot{\gamma}(s(t))}{\sqrt{1 - |\dot{s}(t)|^2}} + i\,(x - r(t)) \cdot \nu(s(t))}\right), \tag{2.2.11}$$

where $\nu(\sigma) \perp \dot{\gamma}(\sigma)$ and $\hat{S}(t)$ is given by the unique continuous determination of the complex square function such that in $x = r(t) + \sqrt{1 - |\dot{s}(t)|^2}\dot{\gamma}(s(t))$ takes value 1 and its discontinuity set lies on Γ_t. Roughly speaking, if we forget the term $\sqrt{1 - |\dot{s}(t)|^2}$, the function (2.2.11) is the determination of $Im(\sqrt{y_1 + iy_2})$ in the orthonormal system with center $\gamma(s(t))$ and axes $\dot{\gamma}(s(t))$ and $\nu(s(t))$.

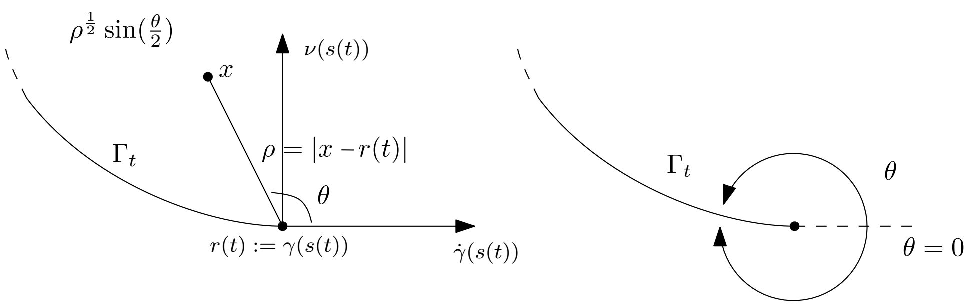

Figure 2.3: *A possible choice of determination of* $Im(\sqrt{y_1 + iy_2})$*, centered in* $r(t) = \gamma(s(t))$ *with axes* $\dot{\gamma}(s(t))$ *and* $\nu(s(t))$*, and with* Γ_t *as discontinuity set.*

For every $t \in [0,T]$ we consider the matrix $R(t) \in SO(2)$ that rotates the orthonormal system with axes $\dot{\gamma}(s(t))$ and $\nu(s(t))$ in the one with axes e_1 and e_2. Thanks to our construction of Λ, and in particular to (2.1.10), the matrix $R(t)$ coincides with $\nabla\Lambda(r(t))$ in $[t_0, t_1]$.

By setting for $t \in [0,T]$

$$L(t) := \begin{pmatrix} \frac{1}{\sqrt{1-|s(t)|^2}} & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\Phi}(t,x) := L(t)R(t)(x-r(t)) \quad \text{for } x \in \Omega,$$
$$\tilde{\Omega}_t := \tilde{\Phi}(t,\Omega), \quad \tilde{\Gamma}_t := \tilde{\Phi}(t,\Gamma_t),$$

we may also write $\hat{S}(t,x) = \tilde{S}(t,\tilde{\Phi}(t,x))$ for $t \in [0,T]$ and $x \in \Omega \setminus \Gamma_t$, where $\tilde{S}(t)$ is given by the continuous determination of $Im(\sqrt{y_1 + iy_2})$ in $\tilde{\Omega}_t \setminus \tilde{\Gamma}_t$ such that in $y = (1,0)$ takes the value 1.

Lemma 2.2.7. *Let $\zeta(t)$, $t \in [t_0,t_1]$, be a C^2 (in time) family of cut-off functions with support in a neighborhood of $r(t)$. Let us define the function*

$$w(t,x) := \zeta(t,x)S(\Phi(t,x)) - \hat{S}(t,x) \quad \text{for } t \in [t_0,t_1],\ x \in \Omega \setminus \Gamma_t\,. \tag{2.2.12}$$

Then $w(t) \in H^2(\Omega \setminus \Gamma_t)$ for every $t \in [t_0,t_1]$.

Proof. Let us fix $t \in [t_0,t_1]$. The function $w(t)$ is of class C^2 in $\Omega \setminus \Gamma_t$ and belongs to the space $H^1(\Omega \setminus \Gamma_t) \cap H^2((\Omega \setminus \Gamma_t) \setminus B_\varepsilon(r(t)))$ for every $\varepsilon > 0$. Hence it remains to prove the L^2 integrability of its second spatial derivatives in $B_\varepsilon(r(t))$. Let us choose $\varepsilon > 0$ so small that $\zeta(t) = 1$ on $B_\varepsilon(r(t))$. In $B_\varepsilon(r(t)) \setminus \Gamma_t$ we have

$$\begin{aligned}
\partial^2_{ji} w(t) &= \sum_{h=1}^{d} [\partial_h S(\Phi(t))\partial^2_{ji}\Phi_h(t) - \partial_h \tilde{S}(t,\tilde{\Phi}(t))\partial^2_{ji}\tilde{\Phi}_h(t)] \\
&\quad + \sum_{h,k=1}^{d} [\partial^2_{hk} S(\Phi(t))\partial_j\Phi_k(t)\partial_i\Phi_h(t) - \partial^2_{hk}\tilde{S}(t,\tilde{\Phi}(t))\partial_j\tilde{\Phi}_k(t)\partial_i\tilde{\Phi}_h(t)] \\
&=: I_1(t) + I_2(t)
\end{aligned}$$

for every $i,j = 1,2$.

Notice that $\nabla S(\Phi(t)), \nabla\tilde{S}(t,\tilde{\Phi}(t)) \in L^2(B_\varepsilon(r(t));\mathbb{R}^2)$, while $\nabla^2\Phi(t)$ and $\nabla^2\tilde{\Phi}(t)$ are uniformly bounded in Ω. Therefore $I_1(t) \in L^2(B_\varepsilon(r(t)))$ and there exists a positive constant C, independent of t, such that

$$|I_1(t,x)| \le C|x-r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t,$$

provided that $\varepsilon > 0$ is small enough.

As for $I_2(t)$, we estimate it from above as

$$\begin{aligned}
|I_2(t)| \le &\sum_{h,k=1}^{d} |\partial^2_{hk}S(\Phi(t)) - \partial^2_{hk}\tilde{S}(t,\tilde{\Phi}(t))||\partial_j\tilde{\Phi}_k(t)||\partial_i\tilde{\Phi}_h(t)| \\
&+ \sum_{h,k=1}^{d} |\partial^2_{hk}S(\Phi(t))||\partial_j\Phi_k(t)\partial_i\Phi_k(t) - \partial_j\tilde{\Phi}_k(t)\partial_i\tilde{\Phi}_h(t)|.
\end{aligned} \tag{2.2.13}$$

Let us study the right-hand side of (2.2.13). By choosing ε small enough and using the definitions of $\Phi(t)$ and $\tilde{\Phi}(t)$, for every $x \in B_\varepsilon(r(t))$ we deduce

$$\begin{aligned}
&|\partial_j\Phi_k(t,x)\partial_i\Phi_h(t,x) - \partial_j\tilde{\Phi}_k(t,x)\partial_i\tilde{\Phi}_h(t,x)| \\
&\le \frac{2}{c_1^2}\|\nabla\Lambda\|_{L^\infty(\Omega^{(1)})}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|x-r(t)|,
\end{aligned} \tag{2.2.14}$$

since $\|\nabla\Phi(t)\|_{L^\infty(\Omega)} \leq \frac{1}{c_1}\|\nabla\Lambda\|_{L^\infty(\Omega^{(1)})}$, $\|\nabla\tilde{\Phi}(t)\|_{L^\infty(\Omega)} \leq \frac{1}{c_1}\|\nabla\Lambda\|_{L^\infty(\Omega^{(1)})}$, and

$$|\nabla\Phi(t,x) - \nabla\tilde{\Phi}(t,x)| \leq \frac{1}{c_1}|\nabla\Lambda(x) - R(t)| \leq \frac{1}{c_1}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|x - r(t)|.$$

Moreover, the function S satisfies $|\nabla^2 S(y)| \leq M|y|^{-\frac{3}{2}}$ for $y \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\}$, with M positive constant, while Λ is invertible and $|P(t,x)| \geq |x|$. This allows us to conclude

$$|\partial^2_{hk}S(\Phi(t,x))| \leq M\|\nabla\Lambda^{-1}\|^{\frac{3}{2}}_{L^\infty(\Omega^{(2)})}|x - r(t)|^{-\frac{3}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t. \tag{2.2.15}$$

For the second term in the right-hand side of (2.2.13), we fix $x \in B_\varepsilon(r(t))$ and we consider the segment $[\Phi(t,x), \tilde{\Phi}(t,x)] := \{\lambda\Phi(t,x) + (1-\lambda)\tilde{\Phi}(t,x) : \lambda \in [0,1]\}$ and the function $d(t,x) := \mathrm{dist}([\Phi(t,x), \tilde{\Phi}(t,x)], 0)$. We claim that we can choose $\varepsilon > 0$ so small that

$$d(t,x) \geq \frac{1}{2}|x - r(t)| \quad \text{for every } x \in B_\varepsilon(r(t)). \tag{2.2.16}$$

Indeed let $y \in [\Phi(t,x), \tilde{\Phi}(t,x)]$ be such that $|y| = d(t,x)$, then

$$|\tilde{\Phi}(t,x)| \leq |y| + |\tilde{\Phi}(t,x) - y| \leq |y| + |\tilde{\Phi}(t,x) - \Phi(t,x)|.$$

Since $|P(t,x)| \geq |x|$ and $R(t)$ is a rotation, for ε small we deduce $|\tilde{\Phi}(t,x)| \geq |x - r(t)|$. On the other hand, by Lagrange's theorem there exists $z = z(t,x) \in B_\varepsilon(r(t))$ such that

$$\begin{aligned}\Phi(t,x) &= \Phi(t,r(t)) + \nabla\Phi(t,r(t))(x - r(t)) + \nabla^2\Phi(t,z)(x - r(t)) \cdot (x - r(t)) \\ &= \tilde{\Phi}(t,x) + \nabla^2\Phi(t,z)(x - r(t)) \cdot (x - r(t)).\end{aligned}$$

Hence we derive the estimate

$$|\Phi(t,x) - \tilde{\Phi}(t,x)| \leq \frac{1}{c_1}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|x - r(t)|^2 \quad \text{for every } x \in B_\varepsilon(r(t)), \tag{2.2.17}$$

which implies

$$d(t,x) \geq |x - r(t)| - \frac{1}{c_1}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|x - r(t)|^2 \quad \text{for every } x \in B_\varepsilon(r(t)).$$

In particular we obtain (2.2.16) by choosing $\varepsilon < c_1/(2\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})})$. Notice that ε does not depend on $t \in [t_0, t_1]$.

Let us now fix $x \in B_\varepsilon(r(t)) \setminus \Gamma_t$. Thanks to our construction of Φ and $\tilde{\Phi}$, it is possible to find two other determinations $S^\pm(t)$ of $Im(\sqrt{y_1 + iy_2})$ in $\mathbb{R}^2$ such that their discontinuity sets $\Gamma^\pm(t)$ do not intersect the segment $[\Phi(t,x), \tilde{\Phi}(t,x)]$, which is far way from 0. Moreover, we choose them in such a way that $S^+(t)$ is positive along $\{(\sigma,0) : \sigma \leq 0\}$, while $S^-(t)$ is negative, and $S(\Phi(t,x)) = S^\pm(t,\Phi(t,x))$ if and only if $\tilde{S}(t,\tilde{\Phi}(t,x)) = S^\pm(t,\tilde{\Phi}(t,x))$; notice that $|\nabla^3 S^\pm(t,y)| \leq M|y|^{-\frac{5}{2}}$ for a positive constant M and for every $y \in \mathbb{R}^2 \setminus \Gamma^\pm(t)$. By using Lagrange's theorem, (2.2.16), and (2.2.17), we deduce

$$\begin{aligned}|\partial^2_{hk}S(\Phi(t,x)) - \partial^2_{hk}\tilde{S}(t,\tilde{\Phi}(t,x))| &= |\partial^2_{hk}S^\pm(t,\Phi(t,x)) - \partial^2_{hk}S^\pm(t,\tilde{\Phi}(t,x))| \\ &\leq |\nabla^3 S^\pm(t,z)||\Phi(t,x) - \tilde{\Phi}(t,x)| \\ &\leq \frac{M}{c_1}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|d(t,x)|^{-\frac{5}{2}}|x - r(t)|^2 \\ &\leq \frac{4\sqrt{2}M}{c_1}\|\nabla^2\Lambda\|_{L^\infty(\Omega^{(1)})}|x - r(t)|^{-\frac{1}{2}},\end{aligned} \tag{2.2.18}$$

where $z = z(t,x) \in [\Phi(t,x), \tilde\Phi(t,x)]$. Hence, by combining (2.2.13) with (2.2.14), (2.2.15), and (2.2.18), we obtain the existence of a positive constant C such that

$$|I_2(t,x)| \le C|x - r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t.$$

In particular we get the following bound for $\nabla^2 w$:

$$|\nabla^2 w(t,x)| \le C|x - r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t, \tag{2.2.19}$$

and consequently $w(t) \in H^2(\Omega \setminus \Gamma_t)$ for every $t \in [t_0, t_1]$. □

In the following two lemmas, we investigate the regularity in time for w.

Lemma 2.2.8. *Under the same assumptions of Lemma 2.2.7, the function w introduced in (2.2.12) is an element of $C^0([t_0,t_1]; L^2(\Omega))$. Moreover, $\nabla w \in C^0([t_0,t_1]; L^2(\Omega;\mathbb{R}^2))$ and $\nabla^2 w \in C^0([t_0,t_1]; L^2(\Omega;\mathbb{R}^{2\times 2}))$.*

Proof. The function $\zeta(S\circ\Phi)$ is continuous from $[0,T]$ to $L^2(\Omega)$, since S belongs to the space $C^2(\mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \le 0\}) \cap L^2_{\rm loc}(\mathbb{R}^2)$ and Φ is continuous in $[t_0,t_1] \times \overline{\Omega}$. We also claim that $\hat S = \tilde S \circ \tilde\Phi \in C^0([t_0,t_1] \times (\Omega \setminus \Gamma)) \cap L^\infty((t_0,t_1) \times \Omega)$. Indeed, let $(t^*,x^*) \in [t_0,t_1] \times (\Omega\setminus\Gamma)$ and let $\{(t_j,x_j)\}_j \subset [t_0,t_1] \times (\Omega \setminus \Gamma)$ be a sequence of points converging to (t^*,x^*) as $j \to \infty$. Thanks to the convergence $\tilde\Phi(t_j,x_j) \to \tilde\Phi(t^*,x^*) \in \tilde\Omega_{t^*} \setminus \tilde\Gamma_{t^*}$ as $j \to \infty$, there exists $\bar j \in \mathbb{N}$ such that

$$\tilde S(t_j, \tilde\Phi(t_j,x_j)) = \tilde S(t^*, \tilde\Phi(t_j,x_j)) \quad \text{for every } j \ge \bar j.$$

This allows us to conclude that $\hat S(t_j,x_j) \to \hat S(t^*,x^*)$ as $j \to \infty$, since the function $\tilde S(t^*)$ is continuous in $\tilde\Omega_{t^*} \setminus \tilde\Gamma_{t^*}$. Furthermore, there exists $M > 0$ such that $|\hat S(t,x)| \le M|\tilde\Phi(t,x)|^{\frac{1}{2}}$ for $x \in \Omega\setminus\Gamma$ and $t \in [t_0,t_1]$, which yields that $\hat S$ is uniformly bounded in $\Omega\setminus\Gamma$. We hence derive the claim, which implies $\hat S \in C^0([t_0,t_1]; L^2(\Omega))$, by the dominated convergence theorem.

Arguing as before, we can easily derive that $\nabla(\zeta(S\circ\Phi))$ belongs to $C^0([t_0,t_1]; L^2(\Omega;\mathbb{R}^2))$, while $\nabla \hat S = \nabla\tilde\Phi^T(\nabla\tilde S \circ \tilde\Phi) \in C^0([t_0,t_1] \times (\Omega\setminus\Gamma); \mathbb{R}^2)$. Therefore, thanks to the estimate $|\nabla \tilde S(t, \tilde\Phi(t,x))| \le M|\tilde\Phi(t,x)|^{-\frac{1}{2}}$ for $x \in \Omega \setminus \Gamma$ and $t \in [t_0,t_1]$, and the dominated converge theorem, we conclude that $\nabla\hat S \in C^0([t_0,t_1]; L^2(\Omega;\mathbb{R}^2))$.

Finally, notice that the function $\nabla^2 w$ is continuous in $[t_0,t_1] \times (\Omega\setminus\Gamma)$. Let us now fix $t^* \in [t_0,t_1]$ and let $\{t_j\}_j$ be a sequence of points in $[t_0,t_1]$ such that $t_j \to t^*$ as $j \to \infty$. Thanks to the estimate (2.2.19), we can find $\bar j \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$|\nabla^2 w(t_j,x)| \le C|x - r(t_j)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t_j)) \setminus \Gamma \text{ and } j \ge \bar j,$$

with C independent of j. Here we have used the fact that the constant in (2.2.19) can be chosen uniform in time. Furthermore, the functions $\nabla^2 w(t_j)$ are uniformly bounded with respect to j outside the ball $B_\varepsilon(r(t_j))$. We can hence apply the generalized dominated convergence theorem to deduce that $\nabla^2 w(t_j)$ converges strongly to $\nabla^2 w(t^*)$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$, which implies $\nabla w^2 \in C^0([t_0,t_1]; L^2(\Omega;\mathbb{R}^{2\times 2}))$. □

Lemma 2.2.9. *Under the same assumptions of Lemma 2.2.7, the function w introduced in (2.2.12) is an element of $C^2([0,T]; L^2(\Omega))$; moreover $\nabla w \in C^1([0,T]; L^2(\Omega;\mathbb{R}^2))$.*

Proof. For every $x \in \Omega \setminus \Gamma$ the function $t \mapsto w(t,x)$ is differentiable in $[t_0,t_1]$ and

$$\dot w(t,x) = \dot\zeta(t,x) S(\Phi(t,x)) + \zeta(t,x)\nabla S(\Phi(t,x)) \cdot \dot\Phi(t,x) - \nabla\tilde S(t, \tilde\Phi(t,x)) \cdot \dot{\tilde\Phi}(t,x).$$

Indeed, fixed $(t^*,x^*) \in [t_0,t_1] \times (\Omega\setminus\Gamma)$, we can find $\bar h > 0$ such that for every $|h| \le \bar h$

$$\frac{\tilde S(t^*+h, \tilde\Phi(t^*+h,x^*)) - \tilde S(t^*, \tilde\Phi(t^*,x^*))}{h} = \frac{\tilde S(t^*, \tilde\Phi(t^*+h,x^*)) - \tilde S(t^*, \tilde\Phi(t^*,x^*))}{h},$$

thanks to the fact that $\tilde{\Phi}(t^*+h,x^*) \to \tilde{\Phi}(t^*,x^*) \in \tilde{\Omega}_{t^*} \setminus \tilde{\Gamma}_{t^*}$ for every $x^* \in \Omega \setminus \Gamma$ as $h \to 0$. In particular $\frac{1}{h}[\tilde{S}(t^*+h,\tilde{\Phi}(t^*+h,x^*)) - \tilde{S}(t^*,\tilde{\Phi}(t^*,x^*))] \to \nabla\tilde{S}(t^*,\tilde{\Phi}(t^*,x^*)) \cdot \dot{\tilde{\Phi}}(t^*,x^*)$, since $\tilde{S}(t^*) \in C^2(\tilde{\Omega}_{t^*} \setminus \tilde{\Gamma}_{t^*})$. Hence, for every $(t,x) \in [t_0,t_1] \times (\Omega \setminus \Gamma)$ and $h \in \mathbb{R}$ such that $t+h \in [t_0,t_1]$ we may write

$$\frac{w(t+h,x)-w(t,x)}{h} = \frac{1}{h}\int_t^{t+h} \dot{w}(\tau,x)\,\mathrm{d}\tau.$$

By arguing as in the proof of the previous lemma we deduce that $\dot{w} \in C^0([t_0,t_1];L^2(\Omega))$. Therefore we obtain that as $h \to 0$

$$\frac{1}{h}\int_t^{t+h} \dot{w}(\tau)\,\mathrm{d}\tau \to \dot{w}(t) \quad \text{in } L^2(\Omega) \quad \text{for every } t \in [t_0,t_1],$$

and consequently $\frac{1}{h}[w(t+h)-w(t)] \to \dot{w}(t)$ in $L^2(\Omega)$.

Similarly, for every $x \in \Omega \setminus \Gamma$ the map $t \mapsto \dot{w}(t,x)$ is differentiable in $[t_0,t_1]$ and

$$\begin{aligned}
\ddot{w}(t,x) &= \ddot{\zeta}(t,x)S(\Phi(t,x)) + 2\dot{\zeta}(t,x)\nabla S(\Phi(t,x)) \cdot \dot{\Phi}(t,x) \\
&\quad + \zeta(t,x)\nabla S(\Phi(t,x)) \cdot \ddot{\Phi}(t,x) - \nabla\tilde{S}(t,\tilde{\Phi}(t,x)) \cdot \ddot{\tilde{\Phi}}(t,x) \\
&\quad + \zeta(t,x)\nabla^2 S(\Phi(t,x)) \cdot [\dot{\Phi}(t,x) \otimes \dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x) \otimes \dot{\tilde{\Phi}}(t,x)] \\
&\quad + [\zeta(t,x)\nabla^2 S(\Phi(t,x)) - \nabla^2\tilde{S}(t,\tilde{\Phi}(t,x))]\dot{\tilde{\Phi}}(t,x) \otimes \dot{\tilde{\Phi}}(t,x).
\end{aligned}$$

We may find $\varepsilon > 0$ so small that $|\dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x)| \le C|x-r(t)|$ in $B_\varepsilon(r(t))$ for every $t \in [t_0,t_1]$ and for a positive constant C. Therefore, we can proceed as in the proof of Lemma 2.2.7 to obtain that $\ddot{w}(t) \in L^2(\Omega)$ for every $t \in [t_0,t_1]$, with

$$|\ddot{w}(t,x)| \le C|x-r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t.$$

In particular, by arguing as in Lemma 2.2.8, this uniform estimate implies that $\ddot{w}$ belongs to $C^0([t_0,t_1];L^2(\Omega))$. We can hence repeat the same procedure adopted before for $\dot{w}$ to conclude that as $h \to 0$

$$\frac{\dot{w}(t+h)-\dot{w}(t)}{h} \to \ddot{w}(t) \quad \text{in } L^2(\Omega) \quad \text{for every } t \in [t_0,t_1],$$

which gives that $w \in C^2([t_0,t_1];L^2(\Omega))$.

Finally, also the function $t \mapsto \nabla w(t,x)$ is differentiable in $[t_0,t_1]$ for every $x \in \Omega \setminus \Gamma$, with derivative

$$\begin{aligned}
\nabla\dot{w}(t,x) &= \nabla\dot{\zeta}(t,x)S(\Phi(t,x)) + \nabla\zeta(t,x)\nabla S(\Phi(t,x)) \cdot \dot{\Phi}(t,x) + \dot{\zeta}(t,x)\nabla\Phi(t,x)^T\nabla S(\Phi(t,x)) \\
&\quad + \zeta(t,x)\nabla\dot{\Phi}(t,x)^T\nabla S(\Phi(t,x)) - \nabla\dot{\tilde{\Phi}}(t,x)^T\nabla\tilde{S}(t,\tilde{\Phi}(t,x)) \\
&\quad + [\zeta(t,x)\nabla\Phi(t,x)^T - \nabla\tilde{\Phi}(t,x)^T]\nabla^2 S(\Phi(t,x))\dot{\Phi}(t,x) \\
&\quad + \zeta(t,x)\nabla\tilde{\Phi}(t,x)^T\nabla^2 S(\Phi(t,x))[\dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x)] \\
&\quad + \nabla\tilde{\Phi}(t,x)^T[\zeta(t,x)\nabla^2 S(\Phi(t,x)) - \nabla^2\tilde{S}(t,\tilde{\Phi}(t,x))]\dot{\tilde{\Phi}}(t,x).
\end{aligned}$$

Moreover there exists $\varepsilon > 0$ so small that for every $t \in [t_0,t_1]$

$$|\nabla\dot{w}(t,x)| \le C|x-r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\varepsilon(r(t)) \setminus \Gamma_t,$$

which implies the continuity of the map $t \mapsto \nabla\dot{w}(t)$ from $[t_0,t_1]$ to $L^2(\Omega;\mathbb{R}^2)$. Therefore, as $h \to 0$ we get

$$\frac{\nabla w(t+h)-\nabla w(t)}{h} \to \nabla\dot{w}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^2) \quad \text{for every } t \in [t_0,t_1],$$

and in particular $\nabla w \in C^1([t_0,t_1];L^2(\Omega;\mathbb{R}^2))$. □

Thanks to previous lemmas we derive the following decomposition result.

Theorem 2.2.10. *Let $f \in C^0([0,T];H^1(\Omega)) \cap \mathrm{Lip}([0,T];L^2(\Omega))$. Consider u^0 and u^1 of the form*

$$u^0 - k^0\hat{S}(0) \in H^2(\Omega \setminus \Gamma_0),$$
$$u^1 - k^0\dot{\hat{S}}(0) \in H^1(\Omega \setminus \Gamma_0),$$

with u^0 satisfying (2.1.3) and (2.1.4), $u^1 = 0$ on $\partial_D\Omega$, and $k^0 \in \mathbb{R}$. Then there exists a unique solution u to (2.1.2)–(2.1.4) with initial condition $u(0) = u^0$ and $\dot{u}(0) = u^1$ of the form

$$u(t,x) = \hat{u}^R(t,x) + k(t)\hat{S}(t,x) \quad t \in [0,T],\ x \in \Omega \setminus \Gamma_t, \tag{2.2.20}$$

where $k \in C^2([0,T])$ and $\hat{u}^R(t) \in H^2(\Omega \setminus \Gamma_t)$ for every $t \in [0,T]$. Moreover

$$\hat{u}^R \in C^2([0,T];L^2(\Omega)), \quad \nabla\hat{u}^R \in C^1([0,T];L^2(\Omega;\mathbb{R}^2)), \quad \nabla^2\hat{u}^R \in C^0([0,T];L^2(\Omega;\mathbb{R}^{2\times 2})). \tag{2.2.21}$$

In particular the function k does not depend on the choice of Φ, but only on Γ and s.

Proof. Thanks to our assumptions on f, u^0 and u^1 we can apply Theorem 2.2.4 with $t_0 = 0$. Indeed, in view of the computations done before, we have

$$\hat{S}(0) - \zeta(0)S(\Phi(0)) \in H^2(\Omega \setminus \Gamma_0),$$

which gives (2.2.7). In particular

$$\nabla u^0 - k^0\zeta(0)\nabla\Phi(0)^T\nabla S(\Phi(0)) \in H^1(\Omega \setminus \Gamma_0),$$

from which we derive

$$k^0\dot{\hat{S}}(0) - \nabla u^0 \cdot \Big(\nabla\Phi^{-1}(0,\Phi(0))\dot{\Phi}(0)\Big) \in H^1(\Omega \setminus \Gamma_0),$$

since $\dot{\hat{S}}(0) - \zeta(0)\nabla\Phi(0)^T\nabla S(\Phi(0)) \in H^1(\Omega \setminus \Gamma_0)$, by arguing again as in the previous lemmas. Therefore, also condition (2.2.8) is satisfied. This implies the representation formula (2.2.9) in $[0,t_1]$, with $t_1 < \rho$. By combining (2.2.9) with Lemma 2.2.7, we deduce (2.2.20) in $[0,t_1]$. Indeed, we can write

$$u(t) = \hat{u}^R(t) + k(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma_t,\ t \in [0,t_1],$$

where $\hat{u}^R(t) := u^R(t) + k(t)[\zeta(t)S(\Phi(t)) - \hat{S}(t)] \in H^2(\Omega \setminus \Gamma_t)$.

We can repeat this construction starting from t_1 and we find a finite number of times $\{t_i\}_{i=0}^n$, with $0 =: t_0 < t_1 < \cdots < t_{n-1} < t_n := T$ such that the solution u to (2.1.2)–(2.1.4) with initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$ can be written for $i = 1,\ldots,n$ as

$$u(t) = \hat{u}_i^R(t) + k_i(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma_t,\ t \in [t_{i-1},t_i].$$

Define $k\colon [0,T] \to \mathbb{R}$ and $\hat{u}^R\colon [0,T] \to H^2(\Omega \setminus \Gamma)$ as $k(t) := k_i(t)$ and $\hat{u}^R := \hat{u}_i^R$ in $[t_{i-1},t_i]$ for every $i = 1,\ldots,n$, respectively. The functions k and $\hat{u}^R$ are well defined and do not depend on the particular choice of $\{t_i\}_{i=0}^n$. Indeed, if we have

$$u(t) = \hat{u}_1^R(t) + k_1(t)\hat{S}(t) = \hat{u}_2^R(t) + k_2(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma_t$$

for a time $t \in [0,T]$, then we derive

$$\hat{u}_1^R(t) - \hat{u}_2^R(t) = [k_2(t) - k_1(t)]\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma_t.$$

Since the left-hand side belongs to $H^2(\Omega\setminus\Gamma_t)$ while $\hat{S}(t)$ is an element of $H^1(\Omega\setminus\Gamma_t)\setminus H^2(\Omega\setminus\Gamma_t)$, such identity can be true if and only if $k_1(t) = k_2(t)$ and $\hat{u}_1^R(t) = \hat{u}_2^R(t)$. Hence, we deduce that $k \in C^2([0,T])$ and that u satisfies the decomposition result (2.2.20) in $[0,T]$.

Finally, by combining the regularity in time of w, proved in Lemmas 2.2.8 and 2.2.9, with the definition of $\hat{u}^R$, we conclude that $\hat{u}^R$ satisfies (2.2.21). □

Remark 2.2.11. When $A \neq Id$ all the previous results are still true if we define

$$\hat{S}(t,x) := Im\left(\sqrt{\frac{[A(r(t))^{-1}(x-r(t))]\cdot\dot{\gamma}(s(t))}{c_{A,\dot{\gamma}}(t)\sqrt{1-|c_{A,\dot{\gamma}}(t)|^2|\dot{s}(t)|^2}}+i\frac{(x-r(t))\cdot\nu(s(t))}{c_{A,\nu}(t)}}\right), \tag{2.2.22}$$

where $c_{A,\dot{\gamma}}(t) := |A(r(t))^{-1/2}\dot{\gamma}(s(t))|$, $c_{A,\nu}(t) := |A(r(t))^{1/2}\nu(s(t))|$, with $A^{1/2}$ and $A^{-1/2}$ the square root matrices of A and A^{-1}, respectively, and where the function $\hat{S}(t)$ is given by the unique continuous determination of the complex square function such that in the point $x = r(t) + \sqrt{1/|c_{A,\dot{\gamma}}(t)|^2 - |\dot{s}(t)|^2}\dot{\gamma}(s(t))$ takes the value 1 and its discontinuity set lies on Γ_t. Indeed, by exploiting the following identities in $[t_0,t_1]$

$$\dot{\gamma}^{(1)}(s^{(1)}(t)) = \frac{A(r(t))^{-1/2}\dot{\gamma}(s(t))}{|A(r(t))^{-1/2}\dot{\gamma}(s(t))|}, \quad \nu^{(1)}(s^{(1)}(t)) = \frac{A(r(t))^{1/2}\nu(s(t))}{|A(r(t))^{1/2}\nu(s(t))|},$$
$$\dot{s}^{(1)}(t) = |A(r(t))^{-1/2}\dot{\gamma}(s(t))|\dot{s}(t), \quad \nabla\chi(r(t)) = A(r(t))^{-1/2},$$

where $\dot{\gamma}^{(1)}$ and $\nu^{(1)}$ are, respectively, the tangent and the unit normal vectors to the curve $\Gamma^{(1)}$ in the point $\gamma^{(1)}(s^{(1)}(t))$, the function (2.2.22) can be rewritten as

$$Im\left(\sqrt{\frac{[\nabla\chi(r(t))\,(x-r(t))]\cdot\dot{\gamma}^{(1)}(s^{(1)}(t))}{\sqrt{1-|\dot{s}^{(1)}(t)|^2}}+i[\nabla\chi(r(t))\,(x-r(t))]\cdot\nu^{(1)}(s^{(1)}(t))}\right).$$

In this case, it is enough to set $\tilde{\Phi}(t,x) := L(t)R(t)\nabla\chi(r(t))(x-r(t))$ for $t \in [0,T]$ and $x \in \Omega$, where L and R are constructed starting from $\gamma^{(1)}$ and $s^{(1)}$, and we can proceed again as in Lemmas 2.2.7–2.2.9, thanks to the fact that for every $t \in [t_0,t_1]$ and $x \in B_\varepsilon(r(t))$

$$|\Phi(t,x)-\tilde{\Phi}(t,x)| \le C|x-r(t)|^2, \; |\nabla\Phi(t,x)-\nabla\tilde{\Phi}(t,x)| \le C|x-r(t)|,$$
$$|\dot{\Phi}(t,x)-\dot{\tilde{\Phi}}(t,x)| \le C|x-r(t)|.$$

We hence obtain the decomposition result (2.2.20) with singular part (2.2.22). As a byproduct, arguing as in Theorem 2.2.10, we derive that the values of k do not depend on the particular construction of Φ, but only on A, Γ, and s.

We point out that the condition $|\dot{s}(t)|^2 < 1/|c_{A,\dot{\gamma}}(t)|^2$, which is necessary in order to define $\hat{S}$, is implied by (2.1.6). Indeed

$$1 = \nabla\chi(r(t))A(r(t))\nabla\chi(r(t))^T\dot{\gamma}(s(t))\cdot\dot{\gamma}(s(t)) \ge \lambda_0|A(r(t))^{-1/2}\dot{\gamma}(s(t))|^2 = \lambda_0|c_{A,\dot{\gamma}}(t)|^2.$$

2.3 Dynamic energy-dissipation balance

In this section we derive formula (10) for the energy

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega |\dot{u}(t,x)|^2\mathrm{d}x + \frac{1}{2}\int_\Omega A(x)\nabla u(t,x)\cdot\nabla u(t,x)\,\mathrm{d}x \quad t \in [0,T]$$

associated to u, solution to (2.1.2)–(2.1.4) with initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$.

The computation is divided into three steps: first, in Proposition 2.3.5 we consider straight cracks when A is the identity matrix; then, in Theorem 2.3.7 we adapt the techniques to curved fractures; finally, in Remark 2.3.9 we generalize the former results to $A \neq Id$. To this aim, some preliminaries are in order: first, in Remark 2.3.1 we compute the partial derivatives of u in a more convenient way, then in Lemmas 2.3.2 and 2.3.3 we provide two key results, based on Geometric Measure Theory. Once this is done, we deduce formula (10) in the time interval $[t_0,t_1]$ where the decomposition (2.2.9) holds.

For brevity of notation, in this section we consider $[t_0,t_1] = [0,1]$. All the results can be easily extended to the general case. The global result in $[0,T]$ easily follows by iterating the procedure a finite number of steps, and using both the additivity of the integrals and the fact that k depends only on A, Γ, and s (see Theorem 2.2.10 and Remark 2.2.11).

Remark 2.3.1. Let us focus our attention on a fracture which is straight in a neighborhood of the tip. Without loss of generality, we may fix the origin so that for every $t \in [0,1]$

$$\Gamma_t \setminus \Gamma_0 = \{(\sigma,0) \in \mathbb{R}^2 : 0 < \sigma \le s(t) - s(0)\}.$$

The diffeomorphisms χ and Λ introduced in Section 2.1 can be both taken equal to the identity, so that, in a neighborhood of the origin, the diffeomorphisms $\Phi(t)$ defined in (2.2.2) simply read

$$\Phi(t,x) = \left(\frac{x_1 - (s(t) - s(0))}{\sqrt{1 - |\dot{s}(t)|}}, x_2 \right) \quad \text{for } t \in [0,1] \text{ and } x \in \Omega.$$

Accordingly, the decomposition result in Theorem 2.2.4 states that the solution u to (2.1.2)–(2.1.5) with suitable initial conditions can be decomposed as

$$u(t,x) = u^R(t,x) + k(t)\zeta(t,x)\bar{S}(t,x) \quad \text{for } t \in [0,1] \text{ and } x \in \Omega \setminus \Gamma_t,$$

where, for brevity, we have set $\bar{S}(t,x) := S(\Phi(t,x))$. We recall that $u^R(t) \in H^2(\Omega \setminus \Gamma_t)$ for every $t \in [0,1]$ and $S(y) = \frac{y_2}{\sqrt{2}\sqrt{|y|+y_1}}$ for $y \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \le 0\}$.

Let us now compute the partial derivatives of u. For $t \in [0,1]$ and $x \in \Omega \setminus \Gamma_t$ we get

$$\nabla u(t,x) = \nabla u^R(t,x) + k(t)\nabla\zeta(t,x)\bar{S}(t,x) + k(t)\zeta(t,x)\nabla\bar{S}(t,x), \tag{2.3.1}$$

$$\dot{u}(t,x) = \dot{u}^R(t,x) + \dot{k}(t)\zeta(t,x)\bar{S}(t,x) + k(t)\dot{\zeta}(t,x)\bar{S}(t,x) + k(t)\zeta(t,x)\dot{\bar{S}}(t,x). \tag{2.3.2}$$

Since in $\mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \le 0\}$ we have

$$\begin{aligned}
\partial_1 S(y) &= -\frac{y_2}{2\sqrt{2}|y|\sqrt{|y|+y_1}}, \quad \partial_2 S(y) = \frac{\sqrt{|y|+y_1}}{2\sqrt{2}|y|},\\
\partial^2_{11} S(y) &= \frac{2y_1y_2 + y_2|y|}{4\sqrt{2}|y|^3\sqrt{|y|+y_1}}, \quad \partial^2_{22} S(y) = -\frac{2y_1y_2 + y_2|y|}{4\sqrt{2}|y|^3\sqrt{|y|+y_1}},\\
\partial^2_{12} S(y) &= \partial^2_{21} S(y) = \frac{\sqrt{|y|+y_1}(|y| - 2y_1)}{4\sqrt{2}|y|^3},
\end{aligned}$$

we claim

$$\dot{u}(t)\nabla u(t) - k^2(t)\zeta^2(t)\dot{\bar{S}}(t)\nabla\bar{S}(t) \in W^{1,1}(\Omega \setminus \Gamma_t;\mathbb{R}^2) \quad \text{for every } t \in [0,1].$$

Indeed, $\nabla u^R(t)$, $\zeta(t)\bar{S}(t)$, $\dot{u}^R(t)$, $\zeta(t)\bar{S}(t)$, and $k(t)\dot{\zeta}(t)\bar{S}(t)$ are functions in $H^1(\Omega \setminus \Gamma_t)$ for every $t \in [0,1]$; by the Sobolev embeddings theorem we deduce that they belong to $L^p(\Omega)$ for every $p \ge 1$. By using also the explicit form of $\bar{S}(t)$ and $\dot{\bar{S}}(t)$, one can check that these functions are elements of $W^{1,4/3}(\Omega \setminus \Gamma_t)$. Therefore, we can easily conclude that the product of each term in (2.3.1) with each term in (2.3.2), except for $k^2(t)\zeta^2(t)\dot{\bar{S}}(t)\nabla\bar{S}(t)$, is a function in $W^{1,1}(\Omega \setminus \Gamma_t;\mathbb{R}^2)$ for every $t \in [0,1]$.

Lemma 2.3.2. *Let $a, b \in \mathbb{R}$, with $a < 0$ and $b > 0$, and define $H^+ := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \ge 0\}$ to be the upper half plane in $\mathbb{R}^2$. Let $g\colon H^+ \to \mathbb{R}$ be bounded, continuous at the origin, and call ω a modulus of continuity for g at $x = 0$. Then*

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \int_0^\varepsilon \left(\int_a^b g(x_1,x_2) \frac{x_2}{x_1^2 + x_2^2} \,\mathrm{d}x_1 \right) \mathrm{d}x_2 - \pi g(0,0) \right| \\
&\le \|g\|_{L^\infty(H^+)} \left(2\varepsilon^{1/2}|b-a| + \theta(\varepsilon) \right) + \pi\omega(\varepsilon^{1/4}),
\end{aligned} \tag{2.3.3}$$

where

$$\theta(\varepsilon) := \left|\pi - \int_0^1 \left[\arctan\left(\frac{b}{\varepsilon x_2}\right) - \arctan\left(\frac{a}{\varepsilon x_2}\right)\right] \mathrm{d}x_2\right|.$$

In particular, for every $g\colon H^+ \to \mathbb{R}$ *bounded and continuous at the origin, we have*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^\varepsilon \left(\int_a^b g(x_1,x_2)\frac{x_2}{x_1^2+x_2^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 = \pi g(0,0).$$

Proof. After a change of variable on the integral in (2.3.3), we can rewrite it as

$$\int_0^1 \left(\int_a^b g(x_1,\varepsilon x_2)\frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2.$$

Notice that

$$\int_a^b \frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1 = \int_a^b \partial_1 \arctan\left(\frac{x_1}{\varepsilon x_2}\right)\mathrm{d}x_1 = \arctan\left(\frac{b}{\varepsilon x_2}\right) - \arctan\left(\frac{a}{\varepsilon x_2}\right),$$

therefore

$$\begin{aligned}
&\left|\int_0^1 \left(\int_a^b g(x_1,\varepsilon x_2)\frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 - \pi g(0,0)\right|\\
&\le \left|\int_0^1 \left(\int_a^b [g(x_1,\varepsilon x_2)-g(0,0)]\frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2\right| + g(0,0)\theta(\varepsilon)\\
&\le \left|\int_0^1 \left(\int_{(a,b)\setminus(-\varepsilon^{1/4},\varepsilon^{1/4})} [g(x_1,\varepsilon x_2)-g(0,0)]\frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2\right| + \pi\omega(\varepsilon^{1/4}) + g(0,0)\theta(\varepsilon).
\end{aligned}$$

By using the estimate

$$\sup_{x\in[(a,b)\setminus(-\varepsilon^{1/4},\varepsilon^{1/4})]\times(0,1)} \frac{\varepsilon x_2}{(x_1^2+(\varepsilon x_2)^2)} \le \frac{\varepsilon^{1/2}}{1+\varepsilon^{3/2}} \le \varepsilon^{1/2},$$

valid for every $\varepsilon\in(0,1)$, we can continue the above chain of inequalities and we get

$$\begin{aligned}
&\left|\int_0^1 \left(\int_a^b g(x_1,\varepsilon x_2)\frac{\varepsilon x_2}{x_1^2+(\varepsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 - \pi g(0,0)\right|\\
&\le \varepsilon^{1/2}\int_0^1 \left(\int_{(a,b)\setminus(-\varepsilon^{1/4},\varepsilon^{1/4})} |g(x_1,\varepsilon x_2)-g(0,0)|\,\mathrm{d}x_1\right)\mathrm{d}x_2 + \pi\omega(\varepsilon^{1/4}) + g(0,0)\theta(\varepsilon)\\
&\le 2\varepsilon^{1/2}\|g\|_{L^\infty(H^+)}|b-a| + \pi\omega(\varepsilon^{1/4}) + g(0,0)\theta(\varepsilon),
\end{aligned}$$

which is (2.3.3). □

Lemma 2.3.3. *Let* $\Omega\subset\mathbb{R}^2$ *be open and let* $\gamma\colon[0,\ell]\to\Omega$ *be a Lipschitz curve. Let us set* $\Gamma := \{\gamma(\sigma) : \sigma\in[0,\ell]\}$, *and for every* $\varepsilon>0$ *let us define* $\varphi_\varepsilon(x) := \frac{\mathrm{dist}(x,\Gamma)}{\varepsilon}\wedge 1$ *for* $x\in\Omega$. *Then for every* $u\in W^{1,1}(\Omega\setminus\Gamma)$ *and for every* $v\colon\Omega\to\mathbb{R}$ *bounded and satisfying*

$$\lim_{x\to\bar x} v(x) = v(\bar x) \quad \textit{for every } \bar x\in\Gamma,$$

we have

$$\lim_{\varepsilon\to 0^+}\int_{\{\mathrm{dist}^+(x,\Gamma)<\varepsilon\}} u(x)v(x)|\nabla\varphi_\varepsilon(x)|\,\mathrm{d}x = \int_\Gamma u^+(x)v(x)\,\mathrm{d}\mathcal{H}^1(x),$$

where

$$\{\mathrm{dist}^+(x,\Gamma)<\varepsilon\} := \bigcup_{\sigma\in[0,\ell]}\left(B_\varepsilon(\gamma(\sigma))\cap\{x\in\Omega : x\cdot(\dot\gamma(\sigma))^\perp>0\}\right),$$

and u^+ *is the trace on* Γ *from* $\{\mathrm{dist}^+(x,\Gamma)<\varepsilon\}$. *Equivalently,*

$$\lim_{\varepsilon\to 0^+}\int_{\{\mathrm{dist}^-(x,\Gamma)<\varepsilon\}} u(x)v(x)|\nabla\varphi_\varepsilon(x)|\,\mathrm{d}x = \int_\Gamma u^-(y)v(y)\,\mathrm{d}\mathcal{H}^1(y),$$

where

$$\{\mathrm{dist}^-(x,\Gamma)<\varepsilon\} := \bigcup_{\sigma\in[0,\ell]}\Big(B_\varepsilon(\gamma(\sigma))\cap\{x\in\Omega : x\cdot(\dot\gamma(\sigma))^\perp<0\}\Big),$$

and u^- *is the trace on* Γ *from* $\{\mathrm{dist}^-(x,\Gamma)<\varepsilon\}$.

Proof. It is enough to apply the coarea formula to the Lipschitz maps φ_ε. □

Remark 2.3.4. In what follows we compute the energy balance in the case of homogeneous Neumann conditions on the whole $\partial\Omega$. However, the same proof applies with no changes to the case of Dirichlet boundary conditions. For example, to treat the homogeneous Dirichlet condition on $\partial_D\Omega\subseteq\partial\Omega$, it is enough to check that the time derivative of the solution $\dot u(t)$ has still zero trace on $\partial\Omega$, in such a way that it still remains an admissible test function. But this is simply because the incremental quotient in time $\frac{1}{h}[u(t+h)-u(t)]$ converges to $\dot u(t)$ as $h\to 0$, strongly in H^1 in a sufficiently small neighborhood of $\partial_D\Omega$, so that $\dot u$ has still zero trace on the Dirichlet part of the boundary.

Analogously, if we prescribe a regular enough non-homogeneous Dirichlet boundary condition, we can rewrite the wave equation changing the forcing term f appearing in its right-hand side, and turn the non-homogeneous Dirichlet condition into a homogeneous one. Also in this case, the computations follow unchanged.

Proposition 2.3.5. *Let* $\Omega\subset\mathbb{R}^2$ *be an open bounded set with Lipschitz boundary and let* $\{\Gamma_t\}_{t\in[0,1]}$ *be a family of rectilinear cracks inside* Ω *of the form*

$$\Gamma_t := \overline{\Omega}\cap\{(\sigma,0)\in\mathbb{R}^2 : \sigma\le s(t)\},$$

where $s\in C^2([0,1])$ *and* $\dot s(t)\ge 0$ *for every* $t\in[0,1]$. *Suppose that a function* $u\colon[0,1]\times\Omega\to\mathbb{R}$ *can be decomposed as in* (2.2.9) *for* $t\in[0,1]$ *and satisfies the wave equation*

$$\ddot u(t)-\Delta u(t)=f(t)\quad \text{in } \Omega\setminus\Gamma_t,\ t\in[0,1], \tag{2.3.4}$$

with homogeneous Neumann boundary conditions on the boundary and on the cracks. Then u *satisfies the dynamic energy-dissipation balance* (11) *for every* $t\in[0,1]$ *if and only if the stress intensity factor* k *is constantly equal to* $\frac{2}{\sqrt{\pi}}$ *in the set* $\{t\in[0,1]:\dot s(t)>0\}$.

Proof. By hypothesis we can decompose the function u as $u(t,x)=u^R(t,x)+k(t)\zeta(t,x)\bar S(t,x)$ for $t\in[0,1]$ and $x\in\Omega\setminus\Gamma_t$, where $u^R(t)\in H^2(\Omega\setminus\Gamma_t)$, $\zeta(t)$ is a cut-off function supported in a neighborhood of the moving tip of Γ_t, and

$$\bar S(t,x):=S\left(\frac{x_1-(s(t)-s(0))}{\sqrt{1-|\dot s(t)|^2}},x_2\right),$$

with $S(y)=\frac{y_2}{\sqrt2\sqrt{|y|+y_1}}$ for $y\in\mathbb{R}^2\setminus\{(\sigma,0):\sigma\le 0\}$.

Let us fix $t^*\in[0,1]$ and for every $\varepsilon>0$ let us define $\varphi_\varepsilon(x):=\frac{\mathrm{dist}(x,\Gamma_{t^*}\setminus\Gamma_0)}{\varepsilon}\wedge 1$ for $x\in\Omega$. Since $\varphi_\varepsilon\dot u(t)$ belongs to $H^1(\Omega\setminus\Gamma_t)$ for every $t\in[0,t^*]$, we can use it as test function in (2.3.4), and we get

$$\begin{aligned}\int_0^{t^*}\langle\ddot u(t),\varphi_\varepsilon\dot u(t)\rangle_{(H^1(\Omega\setminus\Gamma_t))'}\,\mathrm{d}t+\int_0^{t^*}(\nabla u(t),\varphi_\varepsilon\nabla\dot u(t))_{L^2(\Omega)}\,\mathrm{d}t\\ +\int_0^{t^*}(\nabla u(t),\nabla\varphi_\varepsilon\dot u(t))_{L^2(\Omega)}\,\mathrm{d}t=\int_0^{t^*}(f(t),\varphi_\varepsilon\dot u(t))_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned}\tag{2.3.5}$$

By using integration by parts with the fact that $t \mapsto (\dot{u}(t), \varphi_\varepsilon \dot{u}(t))_{L^2(\Omega)}$ is absolutely continuous, we obtain

$$\begin{aligned}\int_0^{t^*} \langle \ddot{u}(t), \varphi_\varepsilon \dot{u}(t)\rangle_{(H^1(\Omega\setminus\Gamma_t))'} \,\mathrm{d}t &= \frac{1}{2}\int_0^{t^*} \frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}(t), \varphi_\varepsilon \dot{u}(t))_{L^2(\Omega)} \,\mathrm{d}t \\ &= \frac{1}{2}(\dot{u}(t^*), \varphi_\varepsilon \dot{u}(t^*))_{L^2(\Omega)} - \frac{1}{2}(\dot{u}(0), \varphi_\varepsilon \dot{u}(0))_{L^2(\Omega)},\end{aligned}$$

and by passing to the limit as $\varepsilon \to 0^+$ by the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} \langle \ddot{u}(t), \varphi_\varepsilon \dot{u}(t)\rangle_{(H^1(\Omega\setminus\Gamma_t))'} \,\mathrm{d}t = \frac{1}{2}\|\dot{u}(t^*)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\dot{u}(0)\|^2_{L^2(\Omega)}.$$

Analogously, we take the limit as $\varepsilon \to 0^+$ in the right-hand side of (2.3.5) and in the second term in the left-hand side, and we get

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \int_0^{t^*} (\nabla u(t), \nabla \dot{u}(t)\varphi_\varepsilon)_{L^2(\Omega)} \,\mathrm{d}t = \frac{1}{2}\|\nabla u(t^*)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega)}, \\ &\lim_{\varepsilon\to 0^+} \int_0^{t^*} (f(t), \dot{u}(t)\varphi_\varepsilon)_{L^2(\Omega)} \,\mathrm{d}t = \int_0^{t^*} (f(t), \dot{u}(t))_{L^2(\Omega)} \,\mathrm{d}t.\end{aligned}$$

The most delicate term is the third one in the left-hand side of (2.3.5). First of all, we write the partial derivatives explicitly:

$$\begin{aligned}&\nabla[k(t)\zeta(t,x)\bar{S}(t,x)] = k(t)\nabla\zeta(t,x)\bar{S}(t,x) + k(t)\zeta(t,x)\nabla\bar{S}(t,x), \\ &\partial_t[k(t)\zeta(t,x)\bar{S}(t,x)] = \dot{k}(t)\zeta(t,x)\bar{S}(t,x) + k(t)\dot{\zeta}(t,x)\bar{S}(t,x) + k(t)\zeta(t,x)\dot{\bar{S}}(t,x).\end{aligned}$$

Moreover, if we consider $\Phi_1(t,x) = \frac{x_1 - s(t)}{\sqrt{1-|\dot{s}(t)|^2}}$, we have

$$\nabla\bar{S}(t,x) = \left(\frac{\partial_1 S(\Phi_1(t,x), x_2)}{\sqrt{1-|\dot{s}(t)|^2}}, \partial_2 S(\Phi_1(t,x), x_2)\right)$$

and

$$\begin{aligned}\dot{\bar{S}}(t,x) &= \frac{-\dot{s}(t)(1-|\dot{s}(t)|^2) + \dot{s}(t)\ddot{s}(t)(x_1 - (s(t)-s(0)))}{(1-|\dot{s}(t)|^2)^{3/2}}\partial_1 S(\Phi_1(t,x), x_2) \\ &= \dot{\Phi}_1(t,x)\sqrt{1-|\dot{s}(t)|^2}\,\partial_1 \bar{S}(t,x).\end{aligned}$$

Thanks to Remark 2.3.1, the only contribution to the limit as $\varepsilon \to 0^+$ is given by

$$\int_0^{t^*} k^2(t)(\zeta^2(t)\nabla\bar{S}(t), \nabla\varphi_\varepsilon \dot{\bar{S}}(t))_{L^2(\Omega)} \,\mathrm{d}t.$$

Therefore, we need to compute

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} \left(\int_{\{\mathrm{dist}(x,\Gamma_t\setminus\Gamma_0)<\varepsilon\}} k^2(t)\zeta^2(t,x)\nabla\bar{S}(t,x)\cdot\nabla\varphi_\varepsilon(x)\dot{\bar{S}}(t,x)\,\mathrm{d}x\right)\mathrm{d}t. \tag{2.3.6}$$

To this aim, we set

$$I_\varepsilon(t) := \int_{\{\mathrm{dist}(x,\Gamma_t\setminus\Gamma_0)<\varepsilon\}} k^2(t)\zeta^2(t,x)\nabla\bar{S}(t,x)\cdot\nabla\varphi_\varepsilon(x)\dot{\bar{S}}(t,x)\,\mathrm{d}x,$$

and we decompose I_ε as $I_\varepsilon^+ + I_\varepsilon^-$, where I_ε^+ is the integral restricted to the upper half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and I_ε^- is the integral restricted to the lower half plane $\{x \in \mathbb{R}^2 : x_2 < 0\}$.

Let us focus on $I^+_\varepsilon(t)$. For brevity, we write $r(t) \coloneqq (s(t)-s(0),0)$ for every $t\in[0,1]$. The gradient of φ_ε reads

$$\nabla\varphi_\varepsilon(x) = \begin{cases} \frac{e_2}{\varepsilon} & \text{in } \{x\in\mathbb{R}^2 : 0\le x_1\le s(t^*)-s(0),\ 0\le x_2<\varepsilon\},\\ \frac{x}{\varepsilon|x|} & \text{in } \{x\in\mathbb{R}^2 : x\in B_\varepsilon(0),\ x_1<0,\ x_2\ge 0\},\\ \frac{x-r(t^*)}{\varepsilon|x-r(t^*)|} & \text{in } \{x\in\mathbb{R}^2 : x\in B_\varepsilon(r(t^*)),\ x_1>s(t^*)-s(0),\ x_2\ge 0\},\\ 0 & \text{otherwise.}\end{cases}$$

Thus we get

$$\begin{aligned} I^+_\varepsilon(t) &= \frac{1}{\varepsilon}\int_{[0,s(t^*)-s(0)]\times(0,\varepsilon]} k^2(t)\sqrt{1-|\dot s(t)|^2}\zeta^2(t,x)\partial_2\bar S(t,x)\dot\Phi_1(t,x)\partial_1\bar S(t,x)\,\mathrm{d}x\\ &+\frac{1}{\varepsilon}\int_{B_\varepsilon(0)\cap\{x_1<0,\,x_2\ge0\}} k^2(t)\sqrt{1-|\dot s(t)|^2}\zeta^2(t,x)\nabla\bar S(t,x)\cdot\frac{x}{|x|}\dot\Phi_1(t,x)\partial_1\bar S(t,x)\,\mathrm{d}x\\ &+\frac{1}{\varepsilon}\int_{B_\varepsilon(r(t^*))\cap\{x_1>s(t^*)-s(0),\,x_2\ge0\}} k^2(t)\sqrt{1-|\dot s(t)|^2}\zeta^2(t,x)\nabla\bar S(t,x)\cdot\frac{x-r(t^*)}{|x-r(t^*)|}\dot\Phi_1(t,x)\partial_1\bar S(t,x)\,\mathrm{d}x. \end{aligned} \tag{2.3.7}$$

We notice that the last two terms in (2.3.7) have integrands which are bounded on the domains of integration, and so passing to the limit as ε goes to 0 they do not give any contribution. Thus we only have to analyze the first term of (2.3.7). By recalling that $\bar S(t,x) = S(\Phi_1(t,x),x_2)$, $\Phi_1(t,x) = \frac{x_1-(s(t)-s(0))}{\sqrt{1-|\dot s(t)|^2}}$, and that $\zeta(t,x)=\zeta(\Phi_1(t,x),x_2)$, with ζ cut-off function supported in a neighborhood of the origin, and making the change of variable $x_1'\sqrt{1-|\dot s(t)|^2} = x_1-(s(t)-s(0))$, we rewrite the first term of (2.3.7) as

$$\begin{aligned} &-\frac{k^2(t)\dot s(t)}{\varepsilon}\int_0^\varepsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\partial_1 S(x_1,x_2)\partial_2 S(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &+\frac{k^2(t)\dot s(t)\ddot s(t)}{\varepsilon\sqrt{1-|\dot s(t)|^2}}\int_0^\varepsilon\int_{a_t}^{b_t} x_1\zeta^2(x_1,x_2)\partial_1 S(x_1,x_2)\partial_2 S(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \end{aligned} \tag{2.3.8}$$

where the interval (a_t,b_t) denotes the segment

$$(a_t,b_t) \coloneqq \left(\frac{s(0)-s(t)}{\sqrt{1-|\dot s(t)|^2}}, \frac{s(t^*)-s(t)}{\sqrt{1-|\dot s(t)|^2}}\right).$$

Notice that

$$\begin{aligned} &-\frac{k^2(t)\dot s(t)}{\varepsilon}\int_0^\varepsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\partial_1 S(x_1,x_2)\partial_2 S(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &=\frac{k^2(t)\dot s(t)}{\varepsilon}\int_0^\varepsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\frac{x_2}{8|x|^2}\,\mathrm{d}x_1\,\mathrm{d}x_2, \end{aligned}$$

and that the map $(x_1,x_2)\mapsto\zeta^2(x_1,x_2)$ is bounded and continuous in $(0,0)$, with $\zeta(0,0)=1$. Therefore we are in a position to apply Lemma 2.3.2, which gives

$$\lim_{\varepsilon\to0^+}\frac{k^2(t)\dot s(t)}{\varepsilon}\int_0^\varepsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\frac{x_2}{8|x|^2}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{\pi}{8}k^2(t)\dot s(t)\zeta^2(0,0) = \frac{\pi}{8}k^2(t)\dot s(t).$$

By arguing in the very same way, we can show that the limit as $\varepsilon\to0^+$ of the second term of (2.3.8), thanks to the presence of x_1, is zero. This means that the limit of $I^+_\varepsilon(t)$ is

$$\lim_{\varepsilon\to0^+} I^+_\varepsilon(t) = \frac{\pi}{8}k^2(t)\dot s(t),$$

and, similarly,

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^-(t) = \frac{\pi}{8}k^2(t)\dot{s}(t).$$

All in all,

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(t) = \lim_{\varepsilon\to 0^+} (I_\varepsilon^+(t) + I_\varepsilon^-(t)) = \frac{\pi}{4}k^2(t)\dot{s}(t).$$

Thanks to the estimate in (2.3.3), we infer that the family of functions $\{I_\varepsilon^+(t)\}_{\varepsilon>0}$ are dominated on $[0,1]$ by a bounded function, and the same holds for $\{I_\varepsilon^-(t)\}_{\varepsilon>0}$; by the dominated convergence theorem, we can pass the limit in (2.3.6) inside the integral in time, and we can write

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} I_\varepsilon(t)\,\mathrm{d}t = \int_0^{t^*} \frac{\pi}{4}k^2(t)\dot{s}(t)\,\mathrm{d}t.$$

So we deduce that the dynamic energy-dissipation balance (11) holds for every $t \in [0,1]$ if and only if the stress intensity factor $k(t)$ is equal to $\frac{2}{\sqrt{\pi}}$ whenever $\dot{s}(t) > 0$. □

Remark 2.3.6. We underline that our approach is different to that of Dal Maso, Larsen, and Toader in [17, Section 4]: in order to derive the energy balance associated to a horizontal crack opening with constant velocity c, they prove that the mechanical energy of $u(t)$ is constant in the moving ellipse $E_r(t) := \{(x_1,x_2) \in \mathbb{R}^2 : (x_1-ct)^2/(1-c^2)+x_2^2 \le r^2\}$ centered at the crack-tip $(ct, 0)$, for some small $r > 0$, and they make the explicit computation of the energy in $\mathbb{R}^2 \setminus E_r(t)$.

We now generalize the previous result to non straight cracks.

Theorem 2.3.7. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary and let $\{\Gamma_t\}_{t\in[0,1]}$ be a family of growing cracks inside Ω. Assume that there exists a bi-Lipschitz map $\Lambda\colon \Omega \to \Omega$ with the following properties:*

(i) $\Lambda(\Gamma_t \setminus \Gamma_0) = \{(\sigma,0) \in \mathbb{R}^2 : 0 < \sigma \le s(t) - s(0)\}$, *where* $s \in C^2([0,1])$ *and* $\dot{s}(t) \ge 0$ *for every* $t \in [0,1]$,

(ii) $\mathcal{H}^1(\Lambda(\Gamma_t \setminus \Gamma_0)) = \mathcal{H}^1(\Gamma_t \setminus \Gamma_0)$ *for every* $t \in [0,1]$;

(iii) $\lim_{x\to\bar{x}} \nabla\Lambda(x) = \nabla\Lambda(\bar{x}) \in SO(2)^+$, *for every* $\bar{x} \in \overline{\Gamma_1 \setminus \Gamma_0}$.

Suppose that a function $u\colon [0,1]\times\Omega \to \mathbb{R}$ can be decomposed as in (2.2.9) for $t \in [0,1]$ and satisfies the wave equation

$$\ddot{u}(t) - \Delta u(t) = f(t) \quad \text{in } \Omega\setminus\Gamma_t,\ t \in [0,1], \tag{2.3.9}$$

with homogeneous Neumann boundary conditions on the boundary and on the cracks. Then u satisfies the dynamic energy-dissipation balance (11) for every $t \in [0,1]$ if and only if the stress intensity factor k is constantly equal to $\frac{2}{\sqrt{\pi}}$ in the set $\{t \in [0,1] : \dot{s}(t) > 0\}$.

Proof. In view of (2.2.9), we have $u(t,x) = u^R(t,x) + k(t)\zeta(t,\Lambda(x))\bar{S}(t,\Lambda(x))$, for $t \in [0,1]$ and $x \in \Omega\setminus\Gamma_t$, with $u^R(t) \in H^2(\Omega\setminus\Gamma_t)$, $\zeta(t)\circ\Lambda$ cut-off function supported in a neighborhood of the moving tip of Γ_t, and

$$\bar{S}(t,\Lambda(x)) := S\left(\frac{\Lambda_1(x) - (s(t)-s(0))}{\sqrt{1-|\dot{s}(t)|^2}}, \Lambda_2(x)\right),$$

being $S(y) = \frac{y_2}{\sqrt{2}\sqrt{|y|+y_1}}$ for $y \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \le 0\}$.

As in the proof of Proposition 2.3.5, we fix $t^* \in [0,1]$ and for every $\varepsilon > 0$ we define the function $\varphi_\varepsilon(x) := \frac{\mathrm{dist}(x,\Gamma_{t^*}\setminus\Gamma_0)}{\varepsilon} \wedge 1$ for $x \in \Omega$. Since $\varphi_\varepsilon \dot u(t) \in H^1(\Omega\setminus\Gamma_t)$, we can use it as test function in (2.3.9), and we get

$$\begin{aligned}\int_0^{t^*} \langle \ddot u(t), \varphi_\varepsilon \dot u(t)\rangle_{(H^1(\Omega\setminus\Gamma_t))'}\,\mathrm{d}t + \int_0^{t^*} (\nabla u(t), \varphi_\varepsilon \nabla \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t \\ + \int_0^{t^*} (\nabla u(t), \nabla\varphi_\varepsilon \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t^*} (f(t), \varphi_\varepsilon \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{2.3.10}$$

By integrating by parts, we easily obtain

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} \langle \ddot u(t), \varphi_\varepsilon \dot u(t)\rangle_{(H^1(\Omega\setminus\Gamma_t))'}\,\mathrm{d}t = \frac12 \|\dot u(t^*)\|^2_{L^2(\Omega)} - \frac12 \|\dot u(0)\|^2_{L^2(\Omega)}, \tag{2.3.11}$$

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} (\nabla u(t), \varphi_\varepsilon \nabla \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t = \frac12 \|\nabla u(t^*)\|^2_{L^2(\Omega)} - \frac12 \|\nabla u(0)\|^2_{L^2(\Omega)}, \tag{2.3.12}$$

$$\lim_{\varepsilon\to 0^+} \int_0^{t^*} (f(t), \varphi_\varepsilon \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^{t^*} (f(t), \dot u(t))_{L^2(\Omega)}\,\mathrm{d}t. \tag{2.3.13}$$

The asymptotics as $\varepsilon \to 0^+$ of the third term in the left-hand side of (2.3.10) is more delicate to handle. To simplify the notation, we set

$$\bar\zeta(t,x) := \zeta(t,\Lambda(x)), \quad \overline{\varphi}_\varepsilon(x) := \varphi_\varepsilon(\Lambda^{-1}(x)) \quad \text{for } t \in [0,1] \text{ and } x \in \Omega.$$

By using Lemma 2.3.3 and Remark 2.3.1, as in the proof of the previous proposition for the rectilinear case, we have that the only contribution to the limit as $\varepsilon \to 0^+$ is given by the term

$$\begin{aligned}&\int_\Omega k^2(t)\bar\zeta^2(t,x)\nabla\bar S(t,\Lambda(x))\cdot\nabla\varphi_\varepsilon(x)\dot{\bar S}(t,\Lambda(x))\,\mathrm{d}x\\ &= \int_\Omega k^2(t)\alpha(t)[\nabla\Lambda(x)^T\nabla\bar S(t,\Lambda(x))]\cdot\nabla\varphi_\varepsilon(x)\bar\zeta^2(t,x)\dot\Phi_1(t,\Lambda(x))\partial_1\bar S(t,\Lambda(x))\,\mathrm{d}x\\ &= \int_\Omega k^2(t)\alpha(t)[\nabla\Lambda(\Lambda^{-1}(x))^T\nabla\bar S(t,x)]\cdot\nabla\varphi_\varepsilon(\Lambda^{-1}(x))\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\bar S(t,x)|J\Lambda^{-1}(x)|\,\mathrm{d}x\\ &= \int_\Omega k^2(t)\alpha(t)\nabla\bar S(t,x)\cdot[B(\Lambda^{-1}(x))\nabla\overline{\varphi}_\varepsilon(x)]\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\bar S(t,x)|J\Lambda^{-1}(x)|\,\mathrm{d}x,\end{aligned} \tag{2.3.14}$$

where $\Phi_1(t,x) := \frac{x_1-(s(t)-s(0))}{\sqrt{1-|\dot s(t)|^2}}$, $B(x) := \nabla\Lambda(x)\nabla\Lambda(x)^T$, $J\Lambda^{-1}(x) := \det\nabla\Lambda^{-1}(x)$, and $\alpha(t) := \sqrt{1-|\dot s(t)|^2}$ for $t \in [0,1]$ and $x \in \Omega$. In the last equality we have used the coarea formula applied with the Lipschitz change of variables Λ^{-1}.

Thanks to our construction of Λ, for any x in a suitable small neighborhood of the tip of $\Lambda(\Gamma_1)$ we have

$$B(\Lambda^{-1}(x)) = \begin{pmatrix} b_{11}(x) & 0 \\ 0 & 1\end{pmatrix},$$

where $b_{11}\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with $b_{11}(x_1,0) = 1$. The last term in (2.3.14) can be split as

$$\begin{aligned}&\int_\Omega k^2(t)\alpha(t)b_{11}(x)\partial_1\overline{\varphi}_\varepsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)(\partial_1\bar S(t,x))^2|J\Lambda^{-1}(x)|\,\mathrm{d}x\\ &\quad+\int_\Omega k^2(t)\alpha(t)\partial_2\bar S(t,x)\partial_2\overline{\varphi}_\varepsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\bar S(t,x)|J\Lambda^{-1}(x)|\,\mathrm{d}x.\end{aligned}$$

By construction of Λ, each line parallel to $\{x \in \mathbb{R}^2 : x_2 = 0\}$ is mapped by Λ^{-1} into a level set of φ_ε; more precisely $\varphi_\varepsilon(\Lambda^{-1}(\{x \in \mathbb{R}^2 : x_2 = s\})) = \frac{s}{\varepsilon} \wedge 1$, and this means that on the set of points $\{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Lambda(\Gamma_1)) \le \varepsilon\}$, we have

$$\nabla\overline{\varphi}_\varepsilon(x) = \begin{cases} \frac{e_2}{\varepsilon} & \text{in } \{x \in \mathbb{R}^2 : 0 \le x_1 \le s(t^*) - s(0),\ 0 \le x_2 < \varepsilon\}, \\ \frac{x}{\varepsilon|x|} & \text{in } \{x \in \mathbb{R}^2 : x \in B_\varepsilon(0),\ x_1 < 0,\ x_2 \ge 0\}, \\ \frac{x - r(t^*)}{\varepsilon|x - r(t^*)|} & \text{in } \{x \in \mathbb{R}^2 : x \in B_\varepsilon(r(t^*)),\ x_1 > s(t^*) - s(0),\ x_2 \ge 0\}, \\ 0 & \text{otherwise,} \end{cases}$$

where, for brevity, we have set $r(t) \coloneqq (s(t) - s(0), 0)$ for every $t \in [0, 1]$.

Since Λ is a bi-Lipschitz map, $J\Lambda^{-1}$ is bounded, thus by hypothesis (iii) we have

$$\lim_{x \to (s(t)-s(0),0)} |J\Lambda^{-1}(x)| = 1,$$

for every $t \in [0, 1]$. Moreover, in view of assumption (iii), we have that $|J\Lambda^{-1}|$ is continuous on the compact set $\overline{\Gamma_1 \setminus \Gamma_0}$, hence uniformly continuous; therefore, proceeding exactly as in the proof of Proposition 2.3.5, we can write

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_\Omega k^2(t)\alpha(t)\partial_2\bar{S}(t,x)\partial_2\overline{\varphi}_\varepsilon(x)\zeta^2(t,x)\dot{\Phi}_1(t,x)\partial_1\bar{S}(t,x)|J\Lambda^{-1}(x)|\,\mathrm{d}x \\ &= \frac{\pi}{4}k^2(t)\dot{s}(t). \end{aligned} \tag{2.3.15}$$

Again by hypothesis (iii), we can apply estimate (2.3.3) and deduce that the sequence of integrands in (2.3.15) is dominated in t, so that we can apply the dominated convergence theorem to deduce

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \int_0^{t^*} \left(\int_\Omega k^2(t)\alpha(t)\partial_2\bar{S}(t,x)\partial_2\overline{\varphi}_\varepsilon(x)\zeta^2(t,x)\dot{\Phi}_1(t,x)\partial_1\bar{S}(t,x)|J\Lambda^{-1}(x)|\,\mathrm{d}x \right) \mathrm{d}t \\ &= \int_0^{t^*} \frac{\pi}{4}k^2(t)\dot{s}(t)\,\mathrm{d}t. \end{aligned} \tag{2.3.16}$$

By combining (2.3.10) with (2.3.11)–(2.3.13) and (2.3.16), we infer that

$$\mathcal{E}(t) - \mathcal{E}(0) + \frac{\pi}{4}\int_0^t k^2(\tau)\dot{s}(\tau)\,\mathrm{d}\tau = \int_0^t (f(\tau), \dot{u}(\tau))_{L^2(\Omega)}\,\mathrm{d}\tau \quad \text{for every } t \in [0, 1]. \tag{2.3.17}$$

Hence, the dynamic energy-dissipation balance (11) is satisfied if and only if

$$\int_0^t \frac{\pi}{4}k^2(\tau)\dot{s}(\tau)\,\mathrm{d}\tau = \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) = \mathcal{H}^1(\Lambda(\Gamma_t \setminus \Gamma_0)) = s(t) \quad \text{for every } t \in [0, 1],$$

which is true if and only if $k(t)$ is equal to $\frac{2}{\sqrt{\pi}}$ whenever $\dot{s}(t) > 0$. This concludes the proof. □

Remark 2.3.8. Our approach is constructive and allows us to show the existence of time-dependent pairs $t \mapsto (\Gamma_t, u(t))$ satisfying the dynamic energy-dissipation balance (11). Under the standing assumptions on Γ_t, it is enough to take the forcing term f associated to $\frac{2}{\sqrt{\pi}}\zeta(\Phi(t))S(\Phi(t))$ (which of course is a solution $u(t)$), where ζ is a suitable cut-off function supported in a neighborhood of the origin. In order to ensure the homogeneous Neumann condition on the fracture, we choose ζ satisfying $\partial_2\zeta(y_1, 0) = 0$ for every $y_1 \in \mathbb{R}$. This can be achieved, e.g., by taking $\zeta(y_1, y_2) = \varphi(y_1)\varphi(y_2)$, where $\varphi \in C_c^\infty(\mathbb{R})$ has compact support contained in $(-\varepsilon, \varepsilon)$ and satisfies $\varphi = 1$ in $(-\varepsilon/2, \varepsilon/2)$, for some $\varepsilon > 0$.

Remark 2.3.9. When in equation (2.1.2) the matrix A is (possibly) not the identity, an energy balance similar to (2.3.17) is still valid: for every $t \in [0,1]$ there holds

$$\mathcal{E}(t) - \mathcal{E}(0) + \frac{\pi}{4}\int_0^t k^2(\tau)a(\tau)\dot{s}(\tau)\mathrm{d}\tau = \int_0^t (f(\tau),\dot{u}(\tau))_{L^2(\Omega)}\mathrm{d}\tau, \tag{2.3.18}$$

where a is a function depending only on A, Γ, and s, and it is given by

$$a(t) := |A(r(t))^{-1/2}\dot{\gamma}(s(t))| \cdot |A(r(t))^{1/2}\nu(s(t))| \cdot \sqrt{\det A(r(t))}.$$

Here $A^{1/2}$ and $A^{-1/2}$ denote the square root of the symmetric and positive definite matrices A and A^{-1}, respectively, and $\dot{\gamma}(s(t))$ and $\nu(s(t))$ are the tangent and unit normal vectors to Γ at the point $r(t) := \gamma(s(t))$, respectively. In this case, the dynamic energy-dissipation balance (11) holds true if and only if the stress intensity factor $k(t)$ satisfies

$$k(t) = \frac{2}{\sqrt{\pi a(t)}}$$

during the crack opening, namely when $\dot{s}(t) > 0$.

In order to derive formula (2.3.18), we use the decomposition result (2.2.9) rewritten as

$$u(t,x) = u^R(t,x) + k(t)\zeta(t,x)\bar{S}(t,\chi(x)),$$

where $\bar{S}(t,x)$ is the singular part of the solution relative to the transformed curve $\Gamma^{(1)} = \chi(\Gamma)$. Then we proceed as in the previous theorem and proposition: we test the PDE with $\varphi_\varepsilon \dot{u}(t)$, where $\varphi_\varepsilon(x) := \frac{\mathrm{dist}(x,\Gamma_{t^*}\setminus\Gamma_0)}{\varepsilon} \wedge 1$ for $x \in \Omega$, and, as before, we notice that the only delicate term is the one that converges to the integral in the left hand-side of (2.3.18):

$$\lim_{\varepsilon\to 0^+}\int_0^{t^*} k^2(t)\left(\int_\Omega \zeta^2(t,x)[A(x)\nabla\bar{S}(t,\chi(x))]\cdot\nabla\varphi_\varepsilon(x)\dot{\bar{S}}(t,\chi(x))\,\mathrm{d}x\right)\mathrm{d}t.$$

By applying the change of variables χ^{-1}, we can rewrite the space integral in the previous expression as follows:

$$\int_\Omega \zeta^2(t,x)([\nabla\chi A\nabla\chi^T](\chi^{-1}(x))\nabla\bar{S}(t,x))\cdot\nabla\chi(x)^{-T}\nabla\varphi_\varepsilon(x)(\chi^{-1}(x))\dot{\bar{S}}(t,x)|\det\nabla\chi^{-1}(x)|\,\mathrm{d}x.$$

Finally, we work on the transformed curve $\Gamma^{(1)}$, exactly as in the previous theorem, by using the property of the singular part $\bar{S}(t,x)$ together with the following facts: by construction, $[\nabla\chi A\nabla\chi^T]\circ\chi^{-1}$ is a continuous function which agrees with the identity on the points of $\Gamma^{(1)}$; moreover $\nabla\chi(x)^{-T}\nabla\varphi_\varepsilon(\chi^{-1}(x))$ is a continuous function which is equal to $\frac{1}{\varepsilon}|A(r(t))^{1/2}\nu(s(t))|\nu^{(1)}(s^{(1)}(t))$ on the points $\gamma^{(1)}(s^{(1)}(t))$ of $\Gamma^{(1)}$; the velocity $\dot{s}^{(1)}$ of the curve $\Gamma^{(1)}$ satisfies $\dot{s}^{(1)}(t) = |A(r(t))^{-1/2}\dot{\gamma}(s(t))|\dot{s}(t)$; finally, $|\det\nabla\chi^{-1}(x)|$ is a continuous function equal to $\sqrt{\det A(r(t))}$ on the points $\gamma^{(1)}(s^{(1)}(t))$ of $\Gamma^{(1)}$.

Remark 2.3.10. By combining Theorems 2.2.4 and 2.2.10 with Theorem 2.3.7 and Remarks 2.3.4 and 2.3.9 we deduce that if f, u^0, and u^1 satisfy the assumptions of Theorem 2.2.10, then the unique solution u to (2.1.2)–(2.1.5) satisfies (10) for every $t \in [0,T]$. This formula gives an important quantitative information on the functions k and s which satisfy the dynamic energy-dissipation balance (11): for every $t \in [0,T]$

$$\left(\frac{2}{\sqrt{\pi a(t)}} - k(t)\right)\dot{s}(t) = 0.$$

In particular, in the set $\{t \in [0,T] : \dot{s}(t) > 0\}$ the stress intensity factor k coincides with the function $2/\sqrt{\pi a}$.

Chapter 3

A dynamic model for viscoelastic materials with growing cracks

In this chapter we prove an existence and uniqueness result for equation (15) and the analogous problem in linear elasticity, that is system (3.1.12).

This chapter is organized as follows. In Section 3.1 we fix the notation adopted throughout the chapter, we list the main assumptions on the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$ and on the function Θ, and we specify the notion of solution to (3.1.12). In Section 3.2 we state our main existence result (Theorem 3.2.1), which is obtained by means of a time discretization scheme. We conclude the proof of Theorem 3.2.1 in Section 3.3, where we show the validity of the initial conditions (3.1.21) and the energy-dissipation inequality (3.3.4). Section 3.4 deals with the uniqueness problem. Under stronger regularity assumptions on the cracks sets, in Theorem 3.4.4 we prove the uniqueness, but only when the space dimension is $d = 2$. To this aim, we assume also that the function Θ is zero in a neighborhood of the crack-tip. We conclude with Section 3.5, where, in dimension $d = 2$ and for an antiplane evolution, we show an example of a moving crack which satisfies the dynamic energy-dissipation balance (16).

The results presented here are obtained in collaboration with F. Sapio and are contained in the submitted paper [10].

3.1 Preliminary results

Let T be a positive real number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$ and let $\partial_N\Omega$ be its complement. We assume the following hypotheses on the geometry of the crack sets $\{\Gamma_t\}_{t\in[0,T]}$:

(E1) $\Gamma \subset \overline{\Omega}$ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(E2) for every $x \in \Gamma$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(E3) $\{\Gamma_t\}_{t\in[0,T]}$ is a family of closed subsets of Γ satisfying $\Gamma_s \subseteq \Gamma_t$ for every $0 \le s \le t \le T$.

Remark 3.1.1. Assumptions (E1)–(E3) are a weaker version of assumptions (H1)–(H3) of Chapter 1. In particular, we do need Γ to be a C^2 manifold of dimension $(d-1)$, since we are not interested in define the trace of $\psi \in H^1(\Omega \setminus \Gamma)$ on Γ.

Thanks (E1)–(E3) the space $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ coincides with $L^2(\Omega; \mathbb{R}^m)$ for every $t \in [0, T]$ and $m \in \mathbb{N}$. In particular, we can extend a function $\psi \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ to a function in $L^2(\Omega; \mathbb{R}^m)$ by setting $\psi = 0$ on Γ_t. Moreover, by arguing as for (1.1.1), the trace of

$\psi \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ on $\partial\Omega$ is well defined and there exists a constant $C_{tr} > 0$, depending only on Ω and Γ, such that

$$\|\psi\|_{L^2(\partial\Omega)} \leq C_{tr} \|\psi\|_{H^1(\Omega\setminus\Gamma)} \quad \text{for every } \psi \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{3.1.1}$$

Hence, for every $t \in [0,T]$ we can define the space $H^1_D(\Omega \setminus \Gamma_t; \mathbb{R}^d)$ as done in (1.1.7), and we denote its dual by $H^{-1}_D(\Omega \setminus \Gamma_t; \mathbb{R}^d)$. Similarly, by proceeding as for (1.2.3), we can find a constant C_K, depending only on Ω and Γ, such that

$$\|\nabla\psi\|^2_{L^2(\Omega)} \leq C_K \left(\|\psi\|^2_{L^2(\Omega)} + \|E\psi\|^2_{L^2(\Omega)} \right) \quad \text{for every } \psi \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{3.1.2}$$

Let $\mathbb{C}, \mathbb{D} \colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ be two fourth-order tensors satisfying:

$$\mathbb{C}, \mathbb{D} \in L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})), \tag{3.1.3}$$
$$(\mathbb{C}(x)\xi_1) \cdot \xi_2 = \xi_1 \cdot (\mathbb{C}(x)\xi_2) \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym} \text{ and for a.e. } x \in \Omega, \tag{3.1.4}$$
$$(\mathbb{D}(x)\xi_1) \cdot \xi_2 = \xi_1 \cdot (\mathbb{D}(x)\xi_2) \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym} \text{ and for a.e. } x \in \Omega. \tag{3.1.5}$$

We require that $\mathbb{C}$ and $\mathbb{D}$ satisfy the following ellipticity condition, which is standard in linear elasticity:

$$\mathbb{C}(x)\xi \cdot \xi \geq \lambda_1 |\xi|^2, \quad \mathbb{D}(x)\xi \cdot \xi \geq \lambda_2 |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^{d\times d}_{sym} \text{ and for a.e. } x \in \Omega, \tag{3.1.6}$$

for two positive constants λ_1, λ_2 independent of x. By combining the ellipticity condition for $\mathbb{C}$ with the Korn's inequality (3.1.2), we can find two constants $c_0 > 0$ and $c_1 \in \mathbb{R}$ such that

$$(\mathbb{C}E\psi, E\psi)_{L^2(\Omega)} \geq c_0 \|\psi\|^2_{H^1(\Omega\setminus\Gamma_T)} - c_1 \|\psi\|^2_{L^2(\Omega)} \quad \text{for every } \psi \in H^1(\Omega \setminus \Gamma_T; \mathbb{R}^d). \tag{3.1.7}$$

Let us consider a function $\Theta \colon (0,T) \times \Omega \to \mathbb{R}$ satisfying

$$\Theta \in L^\infty((0,T) \times \Omega), \quad \nabla\Theta \in L^\infty((0,T) \times \Omega; \mathbb{R}^d). \tag{3.1.8}$$

Given

$$w \in H^2(0,T; L^2(\Omega; \mathbb{R}^d)) \cap H^1(0,T; H^1(\Omega \setminus \Gamma_0; \mathbb{R}^d)), \tag{3.1.9}$$
$$f \in L^2(0,T; L^2(\Omega; \mathbb{R}^d)), \quad F \in H^1(0,T; L^2(\partial_N\Omega; \mathbb{R}^d)), \tag{3.1.10}$$
$$u^0 - w(0) \in H^1_D(\Omega \setminus \Gamma_0; \mathbb{R}^d), \quad u^1 \in L^2(\Omega; \mathbb{R}^d), \tag{3.1.11}$$

we want to find a solution to the viscoelastic dynamic system

$$\ddot{u}(t) - \operatorname{div}(\mathbb{C}Eu(t)) - \operatorname{div}(\Theta^2(t)\mathbb{D}E\dot{u}(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T], \tag{3.1.12}$$

satisfying the following boundary conditions

$$u(t) = w(t) \quad \text{on } \partial_D\Omega,\ t \in [0,T], \tag{3.1.13}$$
$$(\mathbb{C}Eu(t) + \Theta^2(t)\mathbb{D}E\dot{u}(t))\nu = F(t) \quad \text{on } \partial_N\Omega,\ t \in [0,T], \tag{3.1.14}$$
$$(\mathbb{C}Eu(t) + \Theta^2(t)\mathbb{D}E\dot{u}(t))\nu = 0 \quad \text{on } \Gamma_t,\ t \in [0,T], \tag{3.1.15}$$

and initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1 \quad \text{in } \Omega \setminus \Gamma_0. \tag{3.1.16}$$

As pointed out in Chapter 1, the Neumann boundary conditions (3.1.14) and (3.1.15) are only formal, and their meaning will be specified later in Definition 3.1.5.

Throughout the chapter we always assume that the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$ satisfies (E1)–(E3), as well as $\mathbb{C}$, $\mathbb{D}$, Θ, f, w, F, u^0, and u^1 satisfy (3.1.3)–(3.1.11). In the following, we want to specify the notion of solution to problem (3.1.12)–(3.1.15). As pointed out in Chapter 1, the main difficulty is to give a meaning to $\ddot{u}(t)$. For this reason, we follow the definition given in [23, Definition 2.7], which does not require the second derivative of u in time. To simplify the notation, let us define the following three functional spaces:

$$\begin{aligned}
\mathcal{V} &\coloneqq \{\varphi \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)) : \dot\varphi \in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\qquad\qquad \varphi(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ for a.e. } t\in(0,T)\},\\
\mathcal{V}_D &\coloneqq \{\varphi\in\mathcal{V} : \varphi(t)\in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ for a.e. } t\in(0,T)\},\\
\mathcal{W} &\coloneqq \{u\in\mathcal{V} : \Theta\dot u \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)),\\
&\qquad\qquad \Theta(t)\dot u(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ for a.e. } t\in(0,T)\}.
\end{aligned}$$

Remark 3.1.2. In the classical viscoelastic case, namely when Θ is identically equal to 1, the solution u to system (3.1.12) has derivative $\dot u(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$ with $E\dot u\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$. For a generic Θ we expect to have $\Theta E\dot u\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$. Therefore $\mathcal{W}$ is the natural setting where looking for a solution to (3.1.12). Indeed, from a distributional point of view we have

$$\Theta(t)E\dot u(t) = E(\Theta(t)\dot u(t)) - \nabla\Theta(t)\odot\dot u(t) \quad \text{in } \Omega\setminus\Gamma_t \quad \text{for a.e. } t\in(0,T),$$

and $E(\Theta\dot u), \nabla\Theta\odot\dot u\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ if $u\in\mathcal{W}$, thanks to (3.1.8).

Remark 3.1.3. The set $\mathcal{W}$ coincides with the collection of functions $u\in H^1(0,T;L^2(\Omega;\mathbb{R}^d))$ such that $u(t)$ and $\Theta(t)\dot u(t)$ belong to $H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$ and

$$\int_0^T \|u(t)\|^2_{H^1(\Omega\setminus\Gamma_t)} + \|\Theta(t)\dot u(t)\|^2_{H^1(\Omega\setminus\Gamma_t)}\,\mathrm{d}t < \infty. \tag{3.1.17}$$

Indeed the functions $t\mapsto u(t)$ and $t\mapsto\Theta(t)\dot u(t)$ are strongly measurable from $(0,T)$ to $H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$, which gives that (3.1.17) is well defined and u and $\Theta\dot u$ are elements of $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$. To prove the strong measurability of the two maps, it is enough to observe that $t\mapsto u(t)$ and $t\mapsto\Theta(t)\dot u(t)$ are weakly measurable from $(0,T)$ to $L^2(\Omega;\mathbb{R}^d)$, which is a separable Hilbert space. Moreover, $t\mapsto Eu(t)$ and $t\mapsto E(\Theta(t)\dot u(t))$ are weakly measurable from $(0,T)$ to $L^2(\Omega;\mathbb{R}^{d\times d})$, since for every $\varphi\in C_c^\infty(\Omega\setminus\Gamma_T)$ the maps

$$\begin{aligned}
t &\mapsto \int_{\Omega\setminus\Gamma_T} Eu(t,x)\varphi(x)\,\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} u(t,x)\odot\nabla\varphi(x)\,\mathrm{d}x,\\
t &\mapsto \int_{\Omega\setminus\Gamma_T} E(\Psi(t,x)\dot u(t,x))\varphi(x)\,\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} \Psi(t,x)\dot u(t,x)\odot\nabla\varphi(x)\,\mathrm{d}x
\end{aligned}$$

are measurable from $(0,T)$ to $\mathbb{R}$, and $C_c^\infty(\Omega\setminus\Gamma_T)$ is dense in $L^2(\Omega)$.

Lemma 3.1.4. *The spaces $\mathcal{V}$ and $\mathcal{W}$ are Hilbert spaces with respect to the following norms:*

$$\begin{aligned}
\|\varphi\|_{\mathcal{V}} &\coloneqq (\|\varphi\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma_T))} + \|\dot\varphi\|^2_{L^2(0,T;L^2(\Omega))})^{\frac12} \quad \textit{for } \varphi\in\mathcal{V},\\
\|u\|_{\mathcal{W}} &\coloneqq (\|u\|^2_{\mathcal{V}} + \|\Theta\dot u\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma_T))})^{\frac12} \quad \textit{for } u\in\mathcal{W}.
\end{aligned}$$

Moreover, $\mathcal{V}_D$ is a closed subspace of $\mathcal{V}$.

Proof. It is clear that $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$ are norms on $\mathcal{V}$ and $\mathcal{W}$, respectively, induced by scalar products. We just have to check the completeness of such spaces with respect to these norms.

Let $\{\varphi_k\}_k \subset \mathcal{V}$ be a Cauchy sequence. Then, $\{\varphi_k\}_k$ and $\{\dot\varphi_k\}_k$ are Cauchy sequences in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, respectively, which are complete Hilbert spaces. Thus there exists $\varphi \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ with $\dot\varphi \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ such that $\varphi_k \to \varphi$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $\dot\varphi_k \to \dot\varphi$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. In particular there exists a subsequence $\{\varphi_{k_j}\}_j$ such that $\varphi_{k_j}(t) \to \varphi(t)$ in $H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Since $\varphi_{k_j}(t) \in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$ we deduce that $\varphi(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Hence $\varphi\in\mathcal{V}$ and $\varphi_k\to\varphi$ in $\mathcal{V}$. With a similar argument, it is easy to check that $\mathcal{V}_D \subseteq \mathcal{V}$ is a closed subspace.

Let us now consider a Cauchy sequence $\{u_k\}_k \subset \mathcal{W}$. We have that $\{u_k\}_k$ and $\{\Theta\dot u_k\}_k$ are Cauchy sequences in $\mathcal{V}$ and $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$, respectively, which are complete Hilbert spaces. Thus there exist $u\in\mathcal{V}$ and $z\in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ such that $u_k\to u$ in $\mathcal{V}$ and $\Theta\dot u_k \to z$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$. Since $\dot u_k\to\dot u$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and Θ belongs to $L^\infty((0,T)\times\Omega)$, we derive that $\Theta\dot u_k\to\Theta\dot u$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, which gives $z=\Theta\dot u$. Finally let us prove that $\Theta(t)\dot u(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Thanks to the fact that $\Theta\dot u_k\to\Theta\dot u$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$, we can find a subsequence $\{\Theta\dot u_{k_j}\}_j$ such that $\Theta(t)\dot u_{k_j}(t)\to\Theta(t)\dot u(t)$ in $H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Since $\Theta(t)\dot u_{k_j}(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$, we deduce that $\Theta(t)\dot u(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Hence $u\in\mathcal{W}$ and $u_k\to u$ in $\mathcal{W}$. □

We are in a position to define the notion of solution to (3.1.12)–(3.1.15).

Definition 3.1.5. We say that a function $u\in\mathcal{W}$ is a *solution* to system (3.1.12) with boundary conditions (3.1.13)–(3.1.15) if $u-w\in\mathcal{V}_D$ and

$$\begin{aligned}
&-\int_0^T(\dot u(t),\dot\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_0^T(\mathbb{C}Eu(t),E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&\quad+\int_0^T(\mathbb{D}[E(\Theta(t)\dot u(t))-\mathbb{D}\nabla\Theta(t)\odot\dot u(t)],\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&=\int_0^T(f(t),\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_0^T(F(t),\varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t
\end{aligned}\tag{3.1.18}$$

for every $\varphi\in\mathcal{V}_D$ such that $\varphi(0)=\varphi(T)=0$.

Remark 3.1.6. When $\dot u$ is enough regular, for example $\dot u\in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ with $\dot u(t)\in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$, we can write $\Theta E\dot u = E(\Theta\dot u)-\nabla\Theta\odot\dot u$. Therefore (3.1.18) is coherent with the strong formulation (3.1.12). In particular, for a function $u\in\mathcal{W}$ we can define

$$\Theta E\dot u := E(\Theta\dot u)-\nabla\Theta\odot\dot u\in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})),\tag{3.1.19}$$

so that equation (3.1.18) can be written as

$$\begin{aligned}
&-\int_0^T(\dot u(t),\dot\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_0^T(\mathbb{C}Eu(t),E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&\quad+\int_0^T(\mathbb{D}[\Theta(t)E\dot u(t)],\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&=\int_0^T(f(t),\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_0^T(F(t),\varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t
\end{aligned}\tag{3.1.20}$$

for every $\varphi\in\mathcal{V}_D$ such that $\varphi(0)=\varphi(T)=0$.

Remark 3.1.7. Notice that equation (3.1.20), which is formally obtained by integrating the PDE (3.1.12) in time and space and using the integration by part formula, can be rephrased pointwise in time, as done in the previous chapters (see, e.g., Definition 1.1.6). Indeed, by arguing as in [23, Theorem 2.17], it is possible to show that every solution u to (3.1.12)–(3.1.15) satisfies

$$\begin{aligned}&\langle \ddot u(t),\psi\rangle_{H_D^{-1}(\Omega\setminus\Gamma_t)}+(\mathbb{C}Eu(t),E\psi)_{L^2(\Omega)}+(\mathbb{D}[\Theta(t)E\dot u(t)],\Theta(t)E\psi)_{L^2(\Omega)}\\&=(f(t),\psi)_{L^2(\Omega)}+(F(t),\psi)_{L^2(\partial_N\Omega)}\end{aligned}$$

for every $\psi\in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$, where the second derivative $\ddot u$ is defined similarly to (1.1.23) (see [23, Proposition 2.13]).

The notion of solution given in Definition 3.1.5 requires less regularity in time with respect to the ones given in the previous chapters. In particular, the functions u and $\dot u$ may not be defined pointwise. Therefore, we define the initial conditions (3.1.16) as in [16].

Definition 3.1.8. We say that $u\in\mathcal{W}$ satisfies the initial conditions (3.1.16) if

$$\lim_{h\to0^+}\frac1h\int_0^h\Big(\|u(t)-u^0\|^2_{H^1(\Omega\setminus\Gamma_t)}+\|\dot u(t)-u^1\|^2_{L^2(\Omega)}\Big)\,\mathrm{d}t=0. \tag{3.1.21}$$

3.2 Existence

We now state our main existence result, whose proof will be given at the end of Section 3.3.

Theorem 3.2.1. *There exists a solution $u\in\mathcal{W}$ to system (3.1.12)–(3.1.15), according to Definition 3.1.5, which satisfies the initial conditions $u(0)=u^0$ and $\dot u(0)=u^1$ in the sense of (3.1.21). Moreover, we have*

$$\begin{aligned}&u\in C^0_w([0,T];H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)),\\&\dot u\in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^{-1}_D(\Omega\setminus\Gamma_0)),\end{aligned}$$

and as $t\to0^+$

$$u(t)\to u^0\quad\text{in } H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d),\qquad \dot u(t)\to u^1\quad\text{in } L^2(\Omega;\mathbb{R}^d).$$

To prove the existence of a solution to (3.1.12)–(3.1.15), we use a time discretization scheme in the same spirit of [16]. Let us fix $n\in\mathbb{N}$ and set

$$\tau_n:=\frac{T}{n},\quad u_n^0:=u^0,\quad u_n^{-1}:=u^0-\tau_n u^1.$$

We define

$$\begin{aligned}&F_n^k:=F(k\tau_n),\quad w_n^k:=w(k\tau_n)\quad\text{for } k=0,\dots,n,\\&f_n^k:=\frac1{\tau_n}\int_{(k-1)\tau_n}^{k\tau_n}f(s)\,\mathrm{d}s,\quad \Theta_n^k:=\frac1{\tau_n}\int_{(k-1)\tau_n}^{k\tau_n}\Theta(s)\,\mathrm{d}s\quad\text{for } k=1,\dots,n,\\&\delta F_n^k:=\frac{F_n^k-F_n^{k-1}}{\tau_n},\quad \delta w_n^k:=\frac{w_n^k-w_n^{k-1}}{\tau_n}\quad\text{for } k=1,\dots,n,\\&\delta w_n^0:=\dot w(0),\quad \delta^2w_n^k:=\frac{\delta w_n^k-\delta w_n^{k-1}}{\tau_n}\quad\text{for } k=1,\dots,n.\end{aligned}$$

For every $k=1,\dots,n$ let $u_n^k-w_n^k\in H^1_D(\Omega\setminus\Gamma_{k\tau_n};\mathbb{R}^d)$, be the solution to

$$\begin{aligned}&(\delta^2u_n^k,\psi)_{L^2(\Omega)}+(\mathbb{C}Eu_n^k,E\psi)_{L^2(\Omega)}+(\mathbb{D}[\Theta_n^kE\delta u_n^k],\Theta_n^kE\psi)_{L^2(\Omega)}\\&=(f_n^k,\psi)_{L^2(\Omega)}+(F_n^k,\psi)_{L^2(\partial_N\Omega)}\end{aligned} \tag{3.2.1}$$

for every $\psi \in H^1_D(\Omega \setminus \Gamma_{k\tau_n};\mathbb{R}^d)$, where

$$\delta u_n^k := \frac{u_n^k - u_n^{k-1}}{\tau_n} \quad \text{for } k = 0,\dots,n, \quad \delta^2 u_n^k := \frac{\delta u_n^k - \delta u_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n.$$

The existence of a unique solution u_n^k to (3.2.1) is a consequence of Lax-Milgram's theorem.

Remark 3.2.2. Since $\delta u_n^k \in H^1(\Omega \setminus \Gamma_{k\tau_n};\mathbb{R}^d)$, we have $\Theta_n^k E\delta u_n^k = E(\Theta_n^k u_n^k) - \nabla\Theta_n^k \odot u_n^k$. Hence, the discrete equation (3.2.1) is coherent with the weak formulation given in (3.1.18).

In the next lemma we show a uniform estimate for the family $\{u_n^k\}_{k=1}^n$ with respect to n that will be used later to pass to the limit in the discrete equation (3.2.1).

Lemma 3.2.3. *There exists a constant $C > 0$, independent of n, such that*

$$\max_{i=1,\dots,n}\{\|\delta u_n^i\|_{L^2(\Omega)} + \|u_n^i\|_{H^1(\Omega\setminus\Gamma_T)}\} + \sum_{i=1}^n \tau_n \|\Theta_n^i E\delta u_n^i\|^2_{L^2(\Omega)} \le C. \tag{3.2.2}$$

Proof. We fix $n \in \mathbb{N}$. To simplify the notation, for every $\psi,\phi \in H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$ we set

$$a(\psi,\phi) := (\mathbb{C}E\psi, E\phi)_{L^2(\Omega)}, \quad b_n^k(\psi,\phi) := (\mathbb{D}[\Theta_n^k E\psi], \Theta_n^k E\phi)_{L^2(\Omega)} \quad \text{for } k = 1,\dots,n.$$

By taking as test function $\psi := \tau_n(\delta u_n^k - \delta w_n^k) \in H^1_D(\Omega\setminus\Gamma_{k\tau_n};\mathbb{R}^d)$ in (3.2.1), we obtain

$$\|\delta u_n^k\|^2_{L^2(\Omega)} - (\delta u_n^{k-1}, \delta u_n^k)_{L^2(\Omega)} + a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) = \tau_n L_n^k$$

for $k = 1,\dots,n$, where

$$\begin{aligned} L_n^k :=& (f_n^k, \delta u_n^k - \delta w_n^k)_{L^2(\Omega)} + (F_n^k, \delta u_n^k - \delta w_n^k)_{L^2(\partial_N\Omega)} \\ &+ (\delta^2 u_n^k, \delta w_n^k)_{L^2(\Omega)} + a(u_n^k, \delta w_n^k) + b_n^k(\delta u_n^k, \delta w_n^k). \end{aligned}$$

Thanks to the following identities

$$\begin{aligned} &\|\delta u_n^k\|^2_{L^2(\Omega)} - (\delta u_n^{k-1}, \delta u_n^k)_{L^2(\Omega)} = \frac12\|\delta u_n^k\|^2_{L^2(\Omega)} - \frac12\|\delta u_n^{k-1}\|^2_{L^2(\Omega)} + \frac{\tau_n^2}{2}\|\delta^2 u_n^k\|^2_{L^2(\Omega)}, \\ &a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) = \frac12 a(u_n^k, u_n^k) - \frac12 a(u_n^{k-1}, u_n^{k-1}) + \frac{\tau_n^2}{2} a(\delta u_n^k, \delta u_n^k), \end{aligned}$$

and by omitting the terms with τ_n^2, which are non negative, we derive

$$\frac12\|\delta u_n^k\|^2_{L^2(\Omega)} - \frac12\|\delta u_n^{k-1}\|^2_{L^2(\Omega)} + \frac12 a(u_n^k, u_n^k) - \frac12 a(u_n^{k-1}, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \tau_n L_n^k.$$

We fix $i \in \{1,\dots,n\}$ and sum over $k = 1,\dots,i$ to obtain the discrete energy inequality

$$\frac12\|\delta u_n^i\|^2_{L^2(\Omega)} + \frac12 a(u_n^i, u_n^i) + \sum_{k=1}^i \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \mathcal{E}_0 + \sum_{k=1}^i \tau_n L_n^k, \tag{3.2.3}$$

where $\mathcal{E}_0 := \frac12\|u^1\|^2_{L^2(\Omega)} + \frac12(\mathbb{C}Eu^0, Eu^0)_{L^2(\Omega)}$. Let us now estimate the right-hand side in (3.2.3) from above. By (3.1.1) and (3.1.3) we have

$$\left|\sum_{k=1}^i \tau_n (f_n^k, \delta u_n^k)_{L^2(\Omega)}\right| \le \frac12\|f\|^2_{L^2(0,T;L^2(\Omega))} + \frac12\sum_{k=1}^i \tau_n\|\delta u_n^k\|^2_{L^2(\Omega)}, \tag{3.2.4}$$

$$\left|\sum_{k=1}^i \tau_n (f_n^k, \delta w_n^k)_{L^2(\Omega)}\right| \le \frac12\|f\|^2_{L^2(0,T;L^2(\Omega))} + \frac12\|\dot w\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.2.5}$$

$$\left|\sum_{k=1}^{i} \tau_n (F_n^k, \delta w_n^k)_{L^2(\partial_N \Omega)}\right| \le \frac{1}{2}\|F\|^2_{L^2(0,T;L^2(\partial_N\Omega))} + \frac{C_{tr}^2}{2}\|\dot w\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))}, \tag{3.2.6}$$

$$\left|\sum_{k=1}^{i} \tau_n a(u_n^k, \delta w_n^k)\right| \le \frac{1}{2}\|\mathbb{C}\|_{L^\infty(\Omega)}\left(\|\dot w\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))} + \sum_{k=1}^{i}\tau_n\|u_n^k\|^2_{H^1(\Omega\setminus\Gamma_T)}\right). \tag{3.2.7}$$

For the other term involving F_n^k, we perform the following discrete integration by parts

$$\begin{aligned}\sum_{k=1}^{i} \tau_n (F_n^k, \delta u_n^k)_{L^2(\partial_N\Omega)} &= (F_n^i, u_n^i)_{L^2(\partial_N\Omega)} - (F(0), u^0)_{L^2(\partial_N\Omega)} \\ &\quad - \sum_{k=1}^{i}\tau_n(\delta F_n^k, u_n^{k-1})_{L^2(\partial_N\Omega)}.\end{aligned} \tag{3.2.8}$$

Hence for every $\varepsilon \in (0,1)$, by using (3.1.1) and Young's inequality, we get

$$\begin{aligned}\left|\sum_{k=1}^{i}\tau_n(F_n^k, \delta u_n^k)_{L^2(\partial_N\Omega)}\right| &\le \frac{1}{2\varepsilon}\|F_n^i\|^2_{L^2(\partial_N\Omega)} + \frac{\varepsilon}{2}\|u_n^i\|^2_{L^2(\partial_N\Omega)} + \|F(0)\|_{L^2(\partial_N\Omega)}\|u^0\|_{L^2(\partial_N\Omega)} \\ &\quad + \sum_{k=1}^{i}\tau_n\|\delta F_n^k\|_{L^2(\partial_N\Omega)}\|u_n^{k-1}\|^2_{L^2(\partial_N\Omega)} \\ &\le C_\varepsilon + \frac{\varepsilon C_{tr}^2}{2}\|u_n^i\|^2_{H^1(\Omega\setminus\Gamma_T)} + \frac{C_{tr}^2}{2}\sum_{k=1}^{i}\tau_n\|u_n^k\|^2_{H^1(\Omega\setminus\Gamma_T)},\end{aligned} \tag{3.2.9}$$

where C_ε is a positive constant depending on ε. Similarly to (3.2.8), we can say

$$\begin{aligned}\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta w_n^k)_{L^2(\Omega)} &= (\delta u_n^i, \delta w_n^i)_{L^2(\Omega)} - (\delta u_n^0, \delta w_n^0)_{L^2(\Omega)} \\ &\quad - \sum_{k=1}^{i}\tau_n(\delta u_n^{k-1}, \delta^2 w_n^k)_{L^2(\Omega)},\end{aligned} \tag{3.2.10}$$

from which we deduce that for every $\varepsilon > 0$

$$\begin{aligned}\left|\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta w_n^k)_{L^2(\Omega)}\right| &\le \frac{1}{2\varepsilon}\|\delta w_n^i\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|\delta u_n^i\|^2_{L^2(\Omega)} + \|u^1\|_{L^2(\Omega)}\|\dot w(0)\|_{L^2(\Omega)} \\ &\quad + \sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|_{L^2(\Omega)}\|\delta^2 w_n^k\|_{L^2(\Omega)} \\ &\le \bar C_\varepsilon + \frac{\varepsilon}{2}\|\delta u_n^i\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|^2_{L^2(\Omega)},\end{aligned} \tag{3.2.11}$$

where $\bar C_\varepsilon$ is a positive constant depending on ε. We estimate from above the last term in right-hand side of (3.2.3) in the following way

$$\begin{aligned}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k, \delta w_n^k) &\le \sum_{k=1}^{i}\tau_n[b_n^k(\delta u_n^k, \delta u_n^k)]^{\frac{1}{2}}[b_n^k(\delta w_n^k, \delta w_n^k)]^{\frac{1}{2}} \\ &\le \frac{1}{2}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k, \delta u_n^k) + \frac{1}{2}\|\mathbb{D}\|_{L^\infty(\Omega)}\|\Theta\|^2_{L^\infty((0,T)\times\Omega)}\|\dot w\|^2_{L^2(0,T;H^1(\Omega\setminus\Gamma_0))}.\end{aligned} \tag{3.2.12}$$

By considering (3.2.3)–(3.2.12) and using (3.1.7) we obtain

$$\left(\frac{1-\varepsilon}{2}\right)\|\delta u_n^i\|^2_{L^2(\Omega)}+\frac{c_0-\varepsilon C_{tr}^2}{2}\|u_n^i\|^2_{H^1(\Omega\setminus\Gamma_t)}-\frac{c_1}{2}\|u_n^i\|^2_{L^2(\Omega)}+\frac{1}{2}\sum_{k=1}^i\tau_n b_n^k(\delta u_n^k,\delta u_n^k)$$
$$\le\hat{C}_\varepsilon+\hat{C}\sum_{k=1}^i\tau_n\left(\|\delta u_n^k\|^2_{L^2(\Omega)}+\|u_n^k\|^2_{H^1(\Omega\setminus\Gamma_T)}\right),$$

where $\hat{C}_\varepsilon$ and $\hat{C}$ are two positive constants, with $\hat{C}_\varepsilon$ depending on ε. We can now choose $\varepsilon<\min\left\{\frac{1}{2},\frac{c_0}{2C_{tr}^2}\right\}$ and use the inequality

$$\|u_n^i\|^2_{L^2(\Omega)}\le\left(\|u_0\|_{L^2(\Omega)}+\sum_{k=1}^i\tau_n\|\delta u_n^k\|_{L^2(\Omega)}\right)^2\le 2\|u^0\|^2_{L^2(\Omega)}+2T\sum_{k=1}^i\tau_n\|\delta u_n^k\|^2_{L^2(\Omega)},$$

to derive the following estimate

$$\begin{aligned}&\frac{1}{4}\|\delta u_n^i\|^2_{L^2(\Omega)}+\frac{1}{4}\|u_n^i\|^2_{H^1(\Omega\setminus\Gamma_T)}+\frac{1}{2}\sum_{k=1}^i\tau_n b_n^k(\delta u_n^k,\delta u_n^k)\\&\le C_1+C_2\sum_{k=1}^i\tau_n\left(\|\delta u_n^k\|^2_{L^2(\Omega)}+\|u_n^k\|^2_{H^1(\Omega\setminus\Gamma_T)}\right),\end{aligned}\tag{3.2.13}$$

where C_1 and C_2 are two positive constants depending only on u^0, u^1, f, F, and w. Thanks to a discrete version of Gronwall's lemma (see, e.g., [3, Lemma 3.2.4]) we deduce the existence of a constant $C_3>0$, independent of i and n, such that

$$\|\delta u_n^i\|_{L^2(\Omega)}+\|u_n^i\|_{H^1(\Omega\setminus\Gamma_T)}\le C_3\quad\text{for every } i=1,\dots,n \text{ and for every } n\in\mathbb{N}.$$

By combining this last estimate with (3.2.13) and (3.1.6) we finally get (3.2.2) and we conclude. □

We now want to pass to the limit in (3.2.1) to obtain a solution to problem (3.1.12)–(3.1.15). To this aim, we define the piecewise affine interpolants $u_n\colon[0,T]\to H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$ of $\{u_n^j\}_{j=1}^n$ and $u_n'\colon[0,T]\to L^2(\Omega;\mathbb{R}^d)$ of $\{\delta u_n^j\}_{j=1}^n$ as

$$u_n(t):=u_n^j+(t-j\tau_n)\delta u_n^j\quad\text{for } t\in[(j-1)\tau_n,j\tau_n],\ j=1,\dots,n,$$
$$u_n'(t):=\delta u_n^j+(t-j\tau_n)\delta^2 u_n^j\quad\text{for } t\in[(j-1)\tau_n,j\tau_n],\ j=1,\dots,n.$$

We also define the backward interpolants $\overline{u}_n\colon[0,T]\to H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$, $\overline{u}_n'\colon[0,T]\to L^2(\Omega;\mathbb{R}^d)$ and the forward interpolants $\underline{u}_n\colon[0,T]\to H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$, $\underline{u}_n'\colon[0,T]\to L^2(\Omega;\mathbb{R}^d)$ in the following way:

$$\overline{u}_n(t):=u_n^j\quad\text{for } t\in((j-1)\tau_n,j\tau_n],\ j=1,\dots,n,\quad \overline{u}_n(0)=u_n^0,$$
$$\overline{u}_n'(t):=\delta u_n^j\quad\text{for } t\in((j-1)\tau_n,j\tau_n],\ j=1,\dots,n,\quad \overline{u}_n'(0)=\delta u_n^0,$$
$$\underline{u}_n(t):=u_n^{j-1}\quad\text{for } t\in[(j-1)\tau_n,j\tau_n),\ j=1,\dots,n,\quad \underline{u}_n(T)=u_n^n,$$
$$\underline{u}_n'(t):=\delta u_n^{j-1}\quad\text{for } t\in[(j-1)\tau_n,j\tau_n),\ j=1,\dots,n,\quad \underline{u}_n'(T)=\delta u_n^n.$$

Notice that $u_n\in H^1(0,T;L^2(\Omega;\mathbb{R}^d))$ with $\dot{u}_n(t)=\delta u_n^k=\overline{u}_n'(t)$ for $t\in((k-1)\tau_n,k\tau_n)$ and $k=1,\dots,n$. Let us also approximate Θ and w by

$$\overline{\Theta}_n(t):=\Theta_n^k\quad\text{for } t\in((k-1)\tau_n,k\tau_n],\ k=1,\dots,n,\quad \overline{\Theta}_n(0):=\Theta_n^0,$$
$$\overline{w}_n(t):=w_n^k\quad\text{for } t\in((k-1)\tau_n,k\tau_n],\ k=1,\dots,n,\quad \overline{w}_n(0):=w_n^0,$$
$$\underline{\Theta}_n(t):=\Theta_n^{k-1}\quad\text{for } t\in[(k-1)\tau_n,k\tau_n),\ k=1,\dots,n,\quad \overline{\Theta}_n(T):=\Theta_n^n,$$
$$\underline{w}_n(t):=w_n^{k-1}\quad\text{for } t\in[(k-1)\tau_n,k\tau_n),\ k=1,\dots,n,\quad \overline{w}_n(T):=w_n^n.$$

Lemma 3.2.4. *There exists a function $u \in \mathcal{W}$, with $u - w \in \mathcal{V}_D$, and a subsequence of n, not relabeled, such that the following convergences hold as $n \to \infty$:*

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), \quad u_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.2.14}$$

$$\overline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), \quad \overline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.2.15}$$

$$\underline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), \quad \underline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.2.16}$$

$$E(\overline{\Theta}_n \overline{u}_n') \rightharpoonup E(\Theta\dot{u}) \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \tag{3.2.17}$$

$$\nabla\overline{\Theta}_n \odot \overline{u}_n' \rightharpoonup \nabla\Theta \odot \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})). \tag{3.2.18}$$

Proof. By Lemma 3.2.3 the sequences $\{u_n\}_n \subset L^\infty(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$ and $\{\overline{u}_n\}_n \subset L^\infty(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d))$ are uniformly bounded. Therefore, there exist two functions $u \in L^\infty(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$ and $z \in L^\infty(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ such that, up to a not relabeled subsequence, as $n \to \infty$

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), \quad \dot{u}_n \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$
$$\overline{u}_n \rightharpoonup z \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)).$$

Moreover, we have $u = z$, since we can find a constant $C > 0$, independent of n, such that

$$\|u_n - \overline{u}_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le C\tau_n \to 0 \quad \text{as } n \to \infty.$$

Since $\underline{u}_n(t) = \overline{u}_n(t-\tau_n)$ for $t \in (\tau_n, T)$, $\overline{u}_n'(t) = \dot{u}_n(t)$ for a.e. $t \in (0,T)$, and $\underline{u}_n'(t) = \overline{u}_n'(t-\tau_n)$ for $t \in (\tau_n, T)$, we deduce

$$\underline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), \quad \overline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$
$$\underline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)).$$

By using (3.2.2) we derive that the sequences $\{E(\overline{\Theta}_n\overline{u}_n')\}_n \subset L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ and $\{\nabla\overline{\Theta}_n \odot \overline{u}_n'\}_n \subset L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ are uniformly bounded. Indeed, there exists a constant $C > 0$ independent of n such that

$$\begin{aligned}
\|\nabla\overline{\Theta}_n \odot \overline{u}_n'\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|\nabla\Theta_n^k \odot \delta u_n^k\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\le \|\nabla\Theta\|^2_{L^\infty((0,T)\times\Omega)} \sum_{k=1}^n \tau_n \|\delta u_n^k\|^2_{L^2(\Omega)} \le C, \\
\|E(\overline{\Theta}_n\overline{u}_n')\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|E(\Theta_n^k \delta u_n^k)\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&= \sum_{k=1}^n \tau_n \|\Theta_n^k E\delta u_n^k + \nabla\Theta_n^k \odot \delta u_n^k\|^2_{L^2(\Omega)} \\
&\le 2\sum_{k=1}^n \tau_n \|\Theta_n^k E\delta u_n^k\|^2_{L^2(\Omega)} + 2\sum_{k=1}^n \tau_n \|\nabla\Theta_n^k \odot \delta u_n^k\|^2_{L^2(\Omega)} \le C.
\end{aligned}$$

Therefore, there exists $z_1, z_2 \in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ such that, up to a further not relabeled subsequence, as $n \to \infty$ we have

$$\nabla\overline{\Theta}_n \odot \overline{u}_n' \rightharpoonup z_1 \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), \quad E(\overline{\Theta}_n\overline{u}_n') \rightharpoonup z_2 \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})).$$

We want to identify the limits z_1 and z_2. We fix a function $\varphi \in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$, and we have

$$\int_0^T (\nabla\overline{\Theta}_n \odot \overline{u}_n', \varphi)_{L^2(\Omega)}\,\mathrm{d}t = \frac{1}{2}\int_0^T (\overline{u}_n', \varphi\nabla\overline{\Theta}_n)_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}\int_0^T (\overline{u}_n', \varphi^T\nabla\overline{\Theta}_n)_{L^2(\Omega)}\,\mathrm{d}t$$
$$= \int_0^T (\overline{u}_n', \varphi^{sym}\nabla\overline{\Theta}_n)_{L^2(\Omega)}\,\mathrm{d}t,$$

being $\varphi^{sym} := \frac{\varphi+\varphi^T}{2}$. Since $\overline{u}_n' \rightharpoonup \dot{u}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and $\varphi^{sym}\nabla\overline{\Theta}_n \to \varphi^{sym}\nabla\Theta$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ as $n\to\infty$ by dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^T (\nabla\overline{\Theta}_n \odot \overline{u}_n', \varphi)_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (\dot{u}, \varphi^{sym}\nabla\Theta)_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (\nabla\Theta \odot \dot{u}, \varphi)_{L^2(\Omega)}\,\mathrm{d}t,$$

and so $z_1 = \nabla\Theta\odot\dot{u}$. Moreover, fixed $\phi \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ we have

$$\lim_{n\to\infty}\int_0^T (\overline{\Theta}_n\overline{u}_n', \phi)_{L^2(\Omega)}\,\mathrm{d}t = \lim_{n\to\infty}\int_0^T (\overline{u}_n', \overline{\Theta}_n\phi)_{L^2(\Omega)}\,\mathrm{d}t$$
$$= \int_0^T (\dot{u}, \Theta\phi)_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (\Theta\dot{u}, \phi)_{L^2(\Omega)}\,\mathrm{d}t,$$

thanks to the fact that $\overline{u}_n' \rightharpoonup \dot{u}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and $\overline{\Theta}_n\phi \to \Theta\phi$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ as $n\to\infty$, again by the dominated convergence theorem. Therefore $\overline{\Theta}_n\overline{u}_n' \rightharpoonup \Theta\dot{u}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ as $n\to\infty$, from which we get that $E(\overline{\Theta}_n\overline{u}_n') \rightharpoonup E(\Theta\dot{u})$ in the sense of distributions as $n\to\infty$, that gives $z_2 = E(\Theta\dot{u})$. In particular, $\Theta\dot{u} \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$.

Let us check that the limit point u is an element of $\mathcal{W}$. To this aim we define the set

$$E := \{v \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)) : v(t) \in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ for a.e. } t\in(0,T)\}.$$

Notice that E is a weakly closed subset of $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$, since it is closed and convex. Moreover, we have $\{\underline{u}_n\}_n, \{\underline{\Theta}_n\underline{u}_n'\}_n \subset E$. Indeed

$$\underline{u}_n(t) = u_n^{k-1} \in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) \quad \text{for } t\in[(k-1)\tau_n, k\tau_n),\ k=1,\dots,n,$$
$$\underline{\Theta}_n(t)\underline{u}_n'(t) = \Theta_n^{k-1}\delta u_n^{k-1} \in H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d) \quad \text{for } t\in[(k-1)\tau_n, k\tau_n),\ k=1,\dots,n.$$

Since $\underline{u}_n \rightharpoonup u$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $\underline{\Theta}_n\underline{u}_n' \rightharpoonup \Theta\dot{u}$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$, we conclude that $u, \Theta\dot{u} \in E$. Finally, to show that $u - w \in \mathcal{V}_D$ we observe

$$\underline{u}_n(t) - \underline{w}_n(t) = u_n^{k-1} - w_n^{k-1} \in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d) \quad \text{for } t\in[(k-1)\tau_n, k\tau_n),\ k=1,\dots,n.$$

Therefore

$$\{\underline{u}_n - \underline{w}_n\}_n \subset \{v \in L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)) : v(t) \in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d) \text{ for a.e. } t\in(0,T)\},$$

which is a closed convex subset of $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$, and so it is weakly closed. Since $\underline{u}_n \rightharpoonup u$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $\underline{w}_n \to w$ in $L^2(0,T;H^1(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ as $n\to\infty$, we get that $u(t) - w(t) \in H^1_D(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ for a.e. $t\in(0,T)$, which implies $u - w \in \mathcal{V}_D$. □

We now use Lemma 3.2.4 to pass to the limit in the discrete equation (3.2.1).

Lemma 3.2.5. *The function $u \in \mathcal{W}$ given by Lemma 3.2.4 is a solution to* (3.1.12)*–*(3.1.15)*, according to Definition 3.1.5.*

Proof. We only need to prove that $u \in \mathcal{W}$ satisfies (3.1.18). Fixed $n \in \mathbb{N}$, we consider a function $\varphi \in C_c^1(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d))$ such that $\varphi(t) \in H_D^1(\Omega \setminus \Gamma_t;\mathbb{R}^d)$ for every $t \in (0,T)$, and we set

$$\varphi_n^k := \varphi(k\tau_n) \quad \text{for } k = 0, \dots, n, \quad \delta\varphi_n^k := \frac{\varphi_n^k - \varphi_n^{k-1}}{\tau_n} \quad \text{for } k = 1, \dots, n.$$

By using $\tau_n \varphi_n^k \in H_D^1(\Omega \setminus \Gamma_{k\tau_n};\mathbb{R}^d)$ as test function in (3.2.1) and summing over $k = 1, \dots, n$ we get

$$\begin{aligned}
&\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k)_{L^2(\Omega)} + \sum_{k=1}^n \tau_n(\mathbb{C}Eu_n^k, E\varphi_n^k)_{L^2(\Omega)} + \sum_{k=1}^n \tau_n(\mathbb{D}[\Theta_n^k E\delta u_n^k], \Theta_n^k E\varphi_n^k)_{L^2(\Omega)} \\
&= \sum_{k=1}^n \tau_n(f_n^k, \varphi_n^k)_{L^2(\Omega)} + \sum_{k=1}^n \tau_n(F_n^k, \varphi_n^k)_{L^2(\partial_N\Omega)}.
\end{aligned} \tag{3.2.19}$$

Let us define the approximating sequences

$$\overline{\varphi}_n(t) := \varphi_n^k, \quad \overline{\varphi}_n'(t) := \delta\varphi_n^k \quad \text{for } t \in ((k-1)\tau_n, k\tau_n],\ k = 1, \dots, n.$$

Thanks to the identity

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k)_{L^2(\Omega)} = -\sum_{k=1}^n \tau_n(\delta u_n^{k-1}, \delta\varphi_n^k)_{L^2(\Omega)} = -\int_0^T (\underline{u}_n'(t), \overline{\varphi}_n'(t))_{L^2(\Omega)}\, \mathrm{d}t,$$

from (3.2.19) we deduce the following equality

$$\begin{aligned}
&-\int_0^T (\underline{u}_n', \overline{\varphi}_n')_{L^2(\Omega)}\, \mathrm{d}t + \int_0^T (\mathbb{C}E\overline{u}_n, E\overline{\varphi}_n)_{L^2(\Omega)}\, \mathrm{d}t \\
&+ \int_0^T (\mathbb{D}[E(\overline{\Theta}_n \overline{u}_n') - \nabla\overline{\Theta}_n \odot \overline{u}_n'], E\overline{\varphi}_n)_{L^2(\Omega)}\, \mathrm{d}t \\
&= \int_0^T (\overline{f}_n, \overline{\varphi}_n)_{L^2(\Omega)}\, \mathrm{d}t + \int_0^T (\overline{F}_n, \overline{\varphi}_n)_{L^2(\partial_N\Omega)}\, \mathrm{d}t,
\end{aligned} \tag{3.2.20}$$

where $\overline{f}_n$ and $\overline{F}_n$ are the backward interpolants of $\{f_n^k\}_{k=1}^n$ and $\{F_n^k\}_{k=1}^n$, respectively. Notice that as $n \to \infty$ we have

$$\begin{aligned}
&\overline{\varphi}_n \to \varphi \quad \text{in } L^2(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d)), &&\overline{\varphi}_n' \to \dot{\varphi} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\overline{f}_n \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), &&\overline{F}_n \to F \quad \text{in } L^2(0,T;L^2(\partial_N\Omega;\mathbb{R}^d)).
\end{aligned}$$

By (3.2.14)–(3.2.18) and the above convergences we can pass to the limit as $n \to \infty$ in (3.2.20), and we get that $u \in \mathcal{W}$ satisfies (3.1.18) for every $\varphi \in C_c^1(0,T;H^1(\Omega \setminus \Gamma_T;\mathbb{R}^d))$ such that $\varphi(t) \in H_D^1(\Omega \setminus \Gamma_t;\mathbb{R}^d)$ for every $t \in (0,T)$. Finally, we can use a density argument (see [23, Remark 2.9]) to conclude that $u \in \mathcal{W}$ is a solution to (3.1.12)–(3.1.15). □

3.3 Initial conditions

To complete our existence result, it remains to prove that the solution $u \in \mathcal{W}$ to (3.1.12)–(3.1.15) given by Lemma 3.2.4 satisfies the initial conditions (3.1.16) in the sense of (3.1.21).

We start by showing that the second distributional derivative $\ddot{u}$ belongs to the space $L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$. By using the discrete equation (3.2.1), for every $v \in H_D^1(\Omega\setminus\Gamma_0;\mathbb{R}^d)$ with $\|v\|_{H^1(\Omega\setminus\Gamma_0)} \le 1$ we have

$$\begin{aligned}
|(\delta^2 u_n^k, v)_{L^2(\Omega)}| \le{}& \|\mathbb{C}\|_{L^\infty(\Omega)}\|Eu_n^k\|_{L^2(\Omega)} + \|\mathbb{D}\|_{L^\infty(\Omega)}\|\Theta\|_{L^\infty((0,T)\times\Omega)}\|\Theta_n^k E\delta u_n^k\|_{L^2(\Omega)}\\
&+ \|f_n^k\|_{L^2(\Omega)} + C_{tr}\|F_n^k\|_{L^2(\partial_N\Omega)},
\end{aligned}$$

and taking the supremum over v, we conclude

$$\|\delta^2 u_n^k\|^2_{H_D^{-1}(\Omega\setminus\Gamma_0)} \le C(\|Eu_n^k\|^2_{L^2(\Omega)} + \|\Theta_n^k E\delta u_n^k\|^2_{L^2(\Omega)} + \|f_n^k\|^2_{L^2(\Omega)} + \|F_n^k\|^2_{L^2(\partial_N\Omega)})$$

for a positive constant C independent of n. We multiply this inequality by τ_n, we sum over $k = 1,\dots,n$, and we use (3.2.2) to obtain

$$\sum_{k=1}^n \tau_n \|\delta^2 u_n^k\|^2_{H_D^{-1}(\Omega\setminus\Gamma_0)} \le \tilde{C} \quad \text{for every } n \in \mathbb{N}, \tag{3.3.1}$$

where $\tilde{C}$ is a positive constant independent of n. The above estimate implies that the sequence $\{u_n'\}_n \subset H^1(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ is uniformly bounded ($\dot{u}_n'(t) = \delta^2 u_n^k$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1,\dots,n$). Up to extract a further subsequence (not relabeled) from the one of Lemma 3.2.4, we deduce that there is $z_3 \in H^1(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$ such that

$$u_n' \rightharpoonup z_3 \quad \text{in } H^1(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d)) \quad \text{as } n\to\infty. \tag{3.3.2}$$

By using the following estimate

$$\|u_n' - \overline{u}_n'\|_{L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0))} \le \tau_n \|\dot{u}_n'\|_{L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0))} \le \tilde{C}\tau_n \to 0 \quad \text{as } n\to\infty,$$

we conclude that $z_3 = \dot{u}$, which gives $\ddot{u} \in L^2(0,T;H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d))$.

The solution $u \in \mathcal{W}$ given by Lemma 3.2.4 satisfies

$$u \in L^\infty(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)), \quad \dot{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)),$$

and recalling Remark 1.2.7 we derive

$$u \in C_w^0([0,T];H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)), \quad \dot{u} \in C_w^0([0,T];L^2(\Omega;\mathbb{R}^d)).$$

Therefore, by (3.2.14) and (3.3.2) for every $t\in[0,T]$ we obtain

$$u_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d), \quad u_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } H_D^{-1}(\Omega\setminus\Gamma_0;\mathbb{R}^d) \quad \text{as } n\to\infty, \tag{3.3.3}$$

so that $u(0) = u^0$ and $\dot{u}(0) = u^1$, being $u_n(0) = u^0$ and $u_n'(0) = u^1$ for every $n\in\mathbb{N}$.

To prove

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left(\|u(t) - u^0\|^2_{H^1(\Omega\setminus\Gamma_t)} + \|\dot{u}(t) - u^1\|^2_{L^2(\Omega)}\right) \mathrm{d}t = 0$$

we will actually show as $t \to 0^+$

$$u(t) \to u^0 \quad \text{in } H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d), \quad \dot{u}(t) \to u^1 \quad \text{in } L^2(\Omega;\mathbb{R}^d).$$

This is a consequence of an energy-dissipation inequality which holds for the solution $u \in \mathcal{W}$ to (3.1.12)–(3.1.15) given by Lemma 3.2.4. Let us define the mechanical energy of u as

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)} \quad \text{for } t \in [0,T].$$

Notice that $\mathcal{E}(t)$ is well defined for every $t\in[0,T]$ since $u \in C_w^0([0,T];H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $\dot{u} \in C_w^0([0,T];L^2(\Omega;\mathbb{R}^d))$, and that $\mathcal{E}(0) = \frac{1}{2}\|u^1\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{C}Eu^0, Eu^0)_{L^2(\Omega)}$.

Theorem 3.3.1. *The solution $u \in \mathcal{W}$ to (3.1.12)–(3.1.15) given by Lemma 3.2.4 satisfies for every $t \in [0,T]$ the following energy-dissipation inequality*

$$\mathcal{E}(t) + \int_0^t (\mathbb{D}[\Theta E\dot{u}], \Theta E\dot{u})_{L^2(\Omega)}\, \mathrm{d}s \le \mathcal{E}(0) + \mathcal{W}_{tot}(t), \tag{3.3.4}$$

where $\Theta E\dot{u}$ is the function defined in (3.1.19) and $\mathcal{W}_{tot}(t)$ is the total work at time $t \in [0,T]$, which is given by

$$\begin{aligned}\mathcal{W}_{tot}(t) := &\int_0^t \left[(f, \dot{u} - \dot{w})_{L^2(\Omega)} + (\mathbb{C}Eu, E\dot{w})_{L^2(\Omega)} + (\mathbb{D}[\Theta E\dot{u}], \Theta E\dot{w})_{L^2(\Omega)}\right] \mathrm{d}s \\ &- \int_0^t \left[(\dot{u}, \ddot{w})_{L^2(\Omega)} + (\dot{F}, u - w)_{L^2(\partial_N\Omega)}\right] \mathrm{d}s + (\dot{u}(t), \dot{w}(t))_{L^2(\Omega)} \\ &+ (F(t), u(t) - w(t))_{L^2(\partial_N\Omega)} - (u^1, \dot{w}(0))_{L^2(\Omega)} - (F(0), u^0 - w(0))_{L^2(\partial_N\Omega)}.\end{aligned}$$

Remark 3.3.2. The total work $\mathcal{W}_{tot}(t)$ is well defined for every $t \in [0,T]$, since we have $F \in C^0([0,T]; L^2(\partial_N\Omega; \mathbb{R}^d))$, $\dot{w} \in C^0([0,T]; L^2(\Omega; \mathbb{R}^d))$, $u \in C^0_w([0,T]; H^1(\Omega \setminus \Gamma_T; \mathbb{R}^d))$, and $\dot{u} \in C^0_w([0,T]; L^2(\Omega; \mathbb{R}^d))$. In particular, the function $t \mapsto \mathcal{W}_{tot}(t)$ is continuous from $[0,T]$ to $\mathbb{R}$.

Proof of Theorem 3.3.1. Fixed $t \in (0,T]$, for every $n \in \mathbb{N}$ there exists a unique $j \in \{1, \dots, n\}$ such that $t \in ((j-1)\tau_n, j\tau_n]$. After setting $t_n := j\tau_n$, we can write (3.2.3) as

$$\begin{aligned}&\frac{1}{2}\|\overline{u}_n'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{C}E\overline{u}_n(t), E\overline{u}_n(t))_{L^2(\Omega)} + \int_0^{t_n} (\mathbb{D}[\overline{\Theta}_n E\overline{u}_n'], \overline{\Theta}_n E\overline{u}_n')_{L^2(\Omega)}\, \mathrm{d}s \\ &\le \mathcal{E}(0) + \mathcal{W}_n(t),\end{aligned} \tag{3.3.5}$$

where

$$\begin{aligned}\mathcal{W}_n(t) := &\int_0^{t_n} \left[(\overline{f}_n, \overline{u}_n' - \overline{w}_n')_{L^2(\Omega)} + (\mathbb{C}E\overline{u}_n, E\overline{w}_n')_{L^2(\Omega)} + (\mathbb{D}[\overline{\Theta}_n E\overline{u}_n'], \overline{\Theta}_n E\overline{w}_n')_{L^2(\Omega)}\right] \mathrm{d}s \\ &+ \int_0^{t_n} \left[(\dot{u}_n', \overline{w}_n')_{L^2(\Omega)} + (\overline{F}_n, \overline{u}_n' - \overline{w}_n')_{L^2(\partial_N\Omega)}\right] \mathrm{d}s.\end{aligned}$$

We want to pass to the limit as $n \to \infty$ in (3.3.5) and we start studying the left-hand side. Thanks to (3.2.2) and (3.3.1), as $n \to \infty$ we have

$$\begin{aligned}&\|u_n(t) - \overline{u}_n(t)\|_{L^2(\Omega)} \le \tau_n \|\delta u_n^j\|_{L^2(\Omega)} \le C\tau_n \to 0, \\ &\|u_n'(t) - \overline{u}_n'(t)\|^2_{H_D^{-1}(\Omega\setminus\Gamma_0)} \le \tau_n^2 \|\delta^2 u_n^j\|^2_{H_D^{-1}(\Omega\setminus\Gamma_0)} \le \tau_n \sum_{k=1}^n \tau_n \|\delta^2 u_n^k\|^2_{H_D^{-1}(\Omega\setminus\Gamma_0)} \le \tilde{C}\tau_n \to 0.\end{aligned}$$

The last two convergences, together (3.3.3), imply

$$\overline{u}_n(t) \rightharpoonup u(t) \quad \text{in } L^2(\Omega; \mathbb{R}^d), \qquad \overline{u}_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } H_D^{-1}(\Omega \setminus \Gamma_0; \mathbb{R}^d) \quad \text{as } n \to \infty,$$

and since $\|\overline{u}_n(t)\|_{H^1(\Omega\setminus\Gamma_T)} + \|\overline{u}_n'(t)\|_{L^2(\Omega)} \le C$ for every $n \in \mathbb{N}$, we conclude

$$\overline{u}_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega \setminus \Gamma_T; \mathbb{R}^d), \qquad \overline{u}_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega; \mathbb{R}^d) \quad \text{as } n \to \infty. \tag{3.3.6}$$

Hence, we get

$$\|\dot{u}(t)\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|\overline{u}_n'(t)\|^2_{L^2(\Omega)}, \tag{3.3.7}$$

$$(\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)} \le \liminf_{n\to\infty} (\mathbb{C}E\overline{u}_n(t), E\overline{u}_n(t))_{L^2(\Omega)}. \tag{3.3.8}$$

Thanks to Lemma 3.2.4 and (3.1.19), as $n \to \infty$ we obtain

$$\overline{\Theta}_n E\overline{u}_n' = E(\overline{\Theta}_n \overline{u}_n') - \nabla\overline{\Theta}_n \odot \overline{u}_n' \rightharpoonup E(\Theta\dot{u}) - \nabla\Theta \odot \dot{u} = \Theta E\dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})),$$

so that

$$\begin{aligned}\int_0^t (\mathbb{D}[\Theta E\dot{u}], \Theta E\dot{u})_{L^2(\Omega)}\,\mathrm{d}s &\le \liminf_{n\to\infty} \int_0^t (\mathbb{D}[\overline{\Theta}_n E\overline{u}_n'], \overline{\Theta}_n E\overline{u}_n')_{L^2(\Omega)}\,\mathrm{d}s\\ &\le \liminf_{n\to\infty} \int_0^{t_n} (\mathbb{D}[\overline{\Theta}_n E\overline{u}_n'], \overline{\Theta}_n E\overline{u}_n')_{L^2(\Omega)}\,\mathrm{d}s,\end{aligned} \tag{3.3.9}$$

since $\varphi \mapsto \int_0^t (\mathbb{D}E\varphi(s), E\varphi(s))_{L^2(\Omega)}\,\mathrm{d}s$ is lower semicontinuous on $L^2(0,T;H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $t \le t_n$.

Let us study the right-hand side of (3.3.5). By the following convergences as $n \to \infty$:

$$\overline{f}_n \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad \overline{u}_n' - \overline{w}_n' \rightharpoonup \dot{u} - \dot{w} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$

we deduce

$$\lim_{n\to\infty} \int_0^{t_n} (\overline{f}_n, \overline{u}_n' - \overline{w}_n')_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (f, \dot{u} - \dot{w})_{L^2(\Omega)}\,\mathrm{d}s. \tag{3.3.10}$$

In a similar way, we can prove

$$\lim_{n\to\infty} \int_0^{t_n} (\mathbb{C}E\overline{u}_n, E\overline{w}_n')_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (\mathbb{C}Eu, E\dot{w})_{L^2(\Omega)}\,\mathrm{d}s, \tag{3.3.11}$$

$$\lim_{n\to\infty} \int_0^{t_n} (\mathbb{D}[\overline{\Theta}_n E\overline{u}_n'], \overline{\Theta}_n E\overline{w}_n')_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (\mathbb{D}[\Theta E\dot{u}], \Theta E\dot{w})_{L^2(\Omega)}\,\mathrm{d}s, \tag{3.3.12}$$

since the following convergences hold as $n \to \infty$:

$$\begin{aligned}&E\overline{w}_n' \to E\dot{w} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), &&\mathbb{C}E\overline{u}_n \rightharpoonup \mathbb{C}Eu \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})),\\ &\overline{\Theta}_n E\overline{w}_n' \to \Theta E\dot{w} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})), &&\overline{\Theta}_n E\overline{u}_n' \rightharpoonup \Theta E\dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})).\end{aligned}$$

Now, we use formula (3.2.10) to derive

$$\int_0^{t_n} (\dot{u}_n', \overline{w}_n')_{L^2(\Omega)}\,\mathrm{d}s = (\overline{u}_n'(t), \overline{w}_n'(t))_{L^2(\Omega)} - (u^1, \dot{w}(0))_{L^2(\Omega)} - \int_0^{t_n} (\underline{u}_n', \dot{w}_n')_{L^2(\Omega)}\,\mathrm{d}s.$$

By arguing as before, we can deduce

$$\lim_{n\to\infty} \int_0^{t_n} (\dot{u}_n', \overline{w}_n')_{L^2(\Omega)}\,\mathrm{d}s = (\dot{u}(t), \dot{w}(t))_{L^2(\Omega)} - (u^1, \dot{w}(0))_{L^2(\Omega)} - \int_0^t (\dot{u}, \ddot{w})_{L^2(\Omega)}\,\mathrm{d}s, \tag{3.3.13}$$

thanks to (3.3.6) and the following convergences as $n \to \infty$

$$\begin{aligned}&\dot{w}_n' \to \ddot{w} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad \underline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\ &\|\overline{w}_n'(t) - \dot{w}(t)\|_{L^2(\Omega)} \le \frac{1}{\tau_n} \int_{(j-1)\tau_n}^{j\tau_n} \|\dot{w}(s) - \dot{w}(t)\|_{L^2(\Omega)}\,\mathrm{d}s \to 0.\end{aligned}$$

Notice that in the last convergence we have used $\dot{w} \in C^0([0,T];L^2(\Omega;\mathbb{R}^d))$. Similarly we have

$$\begin{aligned}\int_0^{t_n} (\overline{F}_n, \overline{u}_n' - \overline{w}_n')_{L^2(\partial_N\Omega)}\,\mathrm{d}s &= (\overline{F}_n(t), \overline{u}_n(t) - \overline{w}_n(t))_{L^2(\partial_N\Omega)} - (F(0), u^0 - w(0))_{L^2(\partial_N\Omega)}\\ &\quad - \int_0^{t_n} (\dot{F}_n, \underline{u}_n - \underline{w}_n)_{L^2(\partial_N\Omega)}\,\mathrm{d}s,\end{aligned}$$

and by using (3.1.1), (3.3.6), the continuity of F from $[0,T]$ in $L^2(\partial_N\Omega;\mathbb{R}^d)$, and

$$\dot{F}_n \to \dot{F} \quad \text{in } L^2(0,T;L^2(\partial_N\Omega;\mathbb{R}^d)), \quad \underline{u}_n - w_n \rightharpoonup u - w \quad \text{in } L^2(0,T;L^2(\partial_N\Omega;\mathbb{R}^d))$$

as $n\to\infty$, we get

$$\begin{aligned}\lim_{n\to\infty}\int_0^{t_n}(\overline{F}_n,\overline{u}_n'-\overline{w}_n')_{L^2(\partial_N\Omega)}\,\mathrm{d}s &= (F(t),u(t)-w(t))_{L^2(\partial_N\Omega)}-(F(0),u^0-w(0))_{L^2(\partial_N\Omega)}\\ &\quad -\int_0^t(\dot{F},u-w)_{L^2(\partial_N\Omega)}\,\mathrm{d}s. \end{aligned} \tag{3.3.14}$$

By combining (3.3.7)–(3.3.14), we deduce the energy-dissipation inequality (3.3.4) for every $t\in(0,T]$. Finally, for $t=0$ the inequality trivially holds since $u(0)=u^0$ and $\dot{u}(0)=u^1$. □

Lemma 3.3.3. *The solution $u\in\mathcal{W}$ to (3.1.12)–(3.1.15) given by Lemma 3.2.4 satisfies*

$$u(t)\to u^0 \quad \text{in } H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d), \quad \dot{u}(t)\to u^1 \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } t\to 0^+. \tag{3.3.15}$$

In particular, u satisfies the initial conditions (3.1.16) in the sense of (3.1.21).

Proof. By sending $t\to0^+$ in the energy-dissipation inequality (3.3.4) and using the fact that $u\in C_w^0([0,T];H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d))$ and $\dot{u}\in C_w^0([0,T];L^2(\Omega;\mathbb{R}^d))$ we deduce

$$\mathcal{E}(0)\le\liminf_{t\to0^+}\mathcal{E}(t)\le\limsup_{t\to0^+}\mathcal{E}(t)\le\mathcal{E}(0),$$

since the right-hand side of (3.3.4) is continuous in t, $u(0)=u^0$, and $\dot{u}(0)=u^1$. Therefore there exists $\lim_{t\to0^+}\mathcal{E}(t)=\mathcal{E}(0)$. We combine this fact with the lower semicontinuity properties of $t\mapsto\|\dot{u}(t)\|^2_{L^2(\Omega)}$ and $t\mapsto(\mathbb{C}Eu(t),Eu(t))_{L^2(\Omega)}$ to derive

$$\lim_{t\to0^+}\|\dot{u}(t)\|^2_{L^2(\Omega)}=\|u^1\|^2_{L^2(\Omega)},\quad \lim_{t\to0^+}(\mathbb{C}Eu(t),Eu(t))_{L^2(\Omega)}=(\mathbb{C}Eu^0,Eu^0)_{L^2(\Omega)}.$$

Finally, since we have

$$\dot{u}(t)\rightharpoonup u^1 \quad \text{in } L^2(\Omega;\mathbb{R}^d), \quad Eu(t)\rightharpoonup Eu^0 \quad \text{in } L^2(\Omega;\mathbb{R}^{d\times d}) \quad \text{as } t\to0^+,$$

we deduce (3.3.15). In particular, we derive that the functions $u\colon[0,T]\to H^1(\Omega\setminus\Gamma_T;\mathbb{R}^d)$ and $\dot{u}\colon[0,T]\to L^2(\Omega;\mathbb{R}^d)$ are continuous at $t=0$, which implies (3.1.21). □

We are now in a position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. The proof is a consequence of Lemmas 3.2.5 and 3.3.3. □

Remark 3.3.4. We have proved Theorem 3.2.1 for the d-dimensional linear elastic case, namely when the displacement u is a vector-valued function. The same result is true with identical proofs in the antiplane case, that is when the displacement u is a scalar function and satisfies (15).

3.4 Uniqueness

In this section we investigate the uniqueness properties of system (3.1.12) with boundary and initial conditions (3.1.13)–(3.1.16). To this aim, we need to assume stronger regularity assumptions on the cracks $\{\Gamma_t\}_{t\in[0,T]}$ and the function Θ. Moreover, we have to restrict our problem to the dimensional case $d=2$, since in our proof we need to build a suitable family of diffeomorphisms which maps the time-dependent crack Γ_t into a fixed set, and this construction is explicit only for $d = 2$ (see [20, Example 2.14]).

We proceed in two steps; first, in Lemma 3.4.2 we prove a uniqueness result in every dimension d, but when the cracks are not increasing, that is $\Gamma_T = \Gamma_0$. Next, in Theorem 3.4.4 we combine Lemma 3.4.2 with the uniqueness result of [23] and the finite speed propagation result of [18] to prove the uniqueness in the case of a moving crack in $d = 2$.

Let us start with the following lemma, whose proof is analogous to the one of Proposition 2.10 of [23].

Lemma 3.4.1. *Let $u \in \mathcal{W}$ be a solution to (3.1.12)–(3.1.15), according Definition 3.1.5, satisfying the initial condition $\dot{u}(0) = 0$ in the following sense:*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot{u}(t)\|^2_{L^2(\Omega)} = 0.$$

Then u satisfies

$$\begin{aligned}
&-\int_0^T (\dot{u}(t),\dot{\varphi}(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&\quad+\int_0^T (\mathbb{D}[\Theta(t)E\dot{u}(t)],\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&=\int_0^T (f(t),\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T (F(t),\varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t
\end{aligned}$$

for every $\varphi \in \mathcal{V}_D$ such that $\varphi(T) = 0$, where $\Theta E\dot{u}$ is the function defined in (3.1.19).

Proof. We fix $\varphi \in \mathcal{V}_D$ with $\varphi(T) = 0$ and for every $\varepsilon > 0$ we define the function

$$\varphi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon}\varphi(t) & t\in[0,\varepsilon],\\ \varphi(t) & t\in[\varepsilon,T].\end{cases}$$

We have that $\varphi_\varepsilon \in \mathcal{V}_D$ and $\varphi_\varepsilon(0) = \varphi_\varepsilon(T) = 0$, so we can use φ_ε as test function in (3.1.18). By proceeding as in [23, Proposition 2.10] we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_0^T (\dot{u}(t),\dot{\varphi}_\varepsilon(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (\dot{u}(t),\dot{\varphi}(t))_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{\varepsilon\to 0^+}\int_0^T (\mathbb{C}Eu(t),E\varphi_\varepsilon(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (\mathbb{C}Eu(t),E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t,\\
&\lim_{\varepsilon\to 0^+}\int_0^T (f(t),\varphi_\varepsilon(t))_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T (f(t),\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

It remains to consider the terms involving $\mathbb{D}$ and F. We have

$$\begin{aligned}
&\int_0^T (\mathbb{D}[\Theta(t)E\dot{u}(t)],\Theta(t)E\varphi_\varepsilon(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_0^\varepsilon (\mathbb{D}[\Theta(t)E\dot{u}(t)],\frac{t}{\varepsilon}\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_\varepsilon^T (\mathbb{D}[\Theta(t)E\dot{u}(t)],\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t,
\end{aligned}$$

and by the dominated convergence theorem we get as $\varepsilon \to 0^+$

$$\begin{aligned}
&\left|\int_0^\varepsilon (\mathbb{D}[\Theta(t)E\dot{u}(t)],\frac{t}{\varepsilon}\Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\right|\\
&\le \|\mathbb{D}\|_{L^\infty(\Omega)}\|\Theta\|_{L^\infty((0,T)\times\Omega)}\int_0^\varepsilon \|\Theta(t)E\dot{u}(t)\|_{L^2(\Omega)}\|E\varphi(t)\|_{L^2(\Omega)}\,\mathrm{d}t \to 0,
\end{aligned}$$

$$\int_\varepsilon^T (\mathbb{D}[\Theta(t)E\dot u(t)], \Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t \to \int_0^T (\mathbb{D}[\Theta(t)E\dot u(t)], \Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t.$$

Similarly, we have

$$\int_0^T (F(t), \varphi_\varepsilon(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t = \int_0^\varepsilon (F(t), \frac{t}{\varepsilon}\varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t + \int_\varepsilon^T (F(t), \varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t,$$

and as $\varepsilon \to 0^+$

$$\left|\int_0^\varepsilon (F(t), \frac{t}{\varepsilon}\varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t\right| \le \int_0^\varepsilon \|F(t)\|_{L^2(\partial_N\Omega)}\|\varphi(t)\|_{L^2(\partial_N\Omega)}\,\mathrm{d}t \to 0,$$
$$\int_\varepsilon^T (F(t), \varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t \to \int_0^T (F(t), \varphi(t))_{L^2(\partial_N\Omega)}\,\mathrm{d}t.$$

By combining together the previous convergences we get the thesis. □

We prove now the uniqueness result in the case of a fixed domain, that is $\Gamma_T = \Gamma_0$, by following the same procedure adopted in Theorem 1.2.10 of Chapter 1. On the function Θ we assume

$$\Theta \in \mathrm{Lip}([0,T]\times\overline{\Omega}), \quad \nabla\dot\Theta \in L^\infty((0,T)\times\Omega;\mathbb{R}^d), \tag{3.4.1}$$

while on $\Gamma_T = \Gamma_0$ we require only (E1)–(E3).

Lemma 3.4.2. *Assume that Θ satisfies (3.4.1) and $\Gamma_T = \Gamma_0$. Then the viscoelastic dynamic system (3.1.12) with boundary conditions (3.1.13)–(3.1.15) has a unique solution, according to Definition 3.1.5, satifying $u(0) = u^0$ and $\dot u(0) = u^1$ in the sense of (3.1.21).*

Proof. Let $u_1, u_2 \in \mathcal{W}$ be two solutions to (3.1.12)–(3.1.15) with initial conditions (3.1.16). The function $u := u_1 - u_2$ satisfies

$$\lim_{h\to0^+}\frac{1}{h}\int_0^h (\|u(t)\|^2_{H^1(\Omega\setminus\Gamma_t)} + \|\dot u(t)\|^2_{L^2(\Omega)})\,\mathrm{d}t = 0, \tag{3.4.2}$$

hence by Lemma 3.4.1 it solves

$$\begin{aligned}-\int_0^T (\dot u(t), \dot\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T (\mathbb{C}Eu(t), E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\ + \int_0^T (\mathbb{D}[\Theta(t)E\dot u(t)], \Theta(t)E\varphi(t))_{L^2(\Omega)}\,\mathrm{d}t = 0\end{aligned} \tag{3.4.3}$$

for every $\varphi \in \mathcal{V}_D$ such that $\varphi(T) = 0$. We fix $s \in (0,T]$ and we consider the function

$$\varphi_s(t) := \begin{cases} -\int_t^s u(\tau)\,\mathrm{d}\tau & t\in[0,s],\\ 0 & t\in[s,T].\end{cases}$$

Since $\varphi_s \in \mathcal{V}_D$ and $\varphi_s(T) = 0$, we can use it as test function in (3.4.3) to obtain

$$\begin{aligned}-\int_0^s (\dot u(t), u(t))_{L^2(\Omega)}\,\mathrm{d}t + \int_0^s (\mathbb{C}E\dot\varphi_s(t), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\ + \int_0^s (\mathbb{D}[\Theta(t)E\dot u(t)], \Theta(t)E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t = 0.\end{aligned}$$

In particular we deduce

$$\begin{aligned}-\frac12\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac12\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{C}E\varphi_s(t), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t\\ + \int_0^s (\mathbb{D}[\Theta(t)E\dot u(t)], \Theta(t)E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t = 0,\end{aligned}$$

which implies

$$\frac{1}{2}\|u(s)\|^2_{L^2(\Omega)} + \frac{1}{2}(\mathbb{C}E\varphi_s(0), E\varphi_s(0))_{L^2(\Omega)} = \int_0^s (\mathbb{D}[\Theta(t)E\dot{u}(t)], \Theta(t)E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t, \quad (3.4.4)$$

since $u(0) = \varphi_s(s) = 0$. From the distributional point of view, the following identity holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Theta E u) = \dot{\Theta}Eu + \Theta E\dot{u} \in L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})). \quad (3.4.5)$$

Indeed, for every $\varphi \in C_c^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ we have

$$\begin{aligned}
&\int_0^T (\frac{\mathrm{d}}{\mathrm{d}t}(\Theta(t)Eu(t)), \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= -\int_0^T (\Theta(t)Eu(t), \dot{\varphi}(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= -\int_0^T (E(\Theta(t)u(t)) - \nabla\Theta(t)\odot u(t), \dot{\varphi}(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_0^T (E(\dot{\Theta}(t)u(t)) + E(\Theta(t)\dot{u}(t)) - \nabla\dot{\Theta}(t)\odot u(t) - \nabla\Theta(t)\odot \dot{u}(t), \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_0^T (\dot{\Theta}(t)Eu(t) + \Theta(t)E\dot{u}(t), \varphi(t))_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

In particular $\Theta E u \in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$, so that by (3.4.2)

$$\begin{aligned}
\|\Theta(0)Eu(0)\|^2_{L^2(\Omega)} &= \lim_{h\to 0^+}\frac{1}{h}\int_0^h \|\Theta(t)Eu(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\
&\le \|\Theta\|^2_{L^\infty((0,T)\times\Omega)}\lim_{h\to 0^+}\frac{1}{h}\int_0^h \|u(t)\|^2_{H^1(\Omega\setminus\Gamma_t)}\,\mathrm{d}t = 0,
\end{aligned}$$

which yields $\Theta(0)Eu(0) = 0$. Thanks to (3.4.5) and to $\Theta u \in H^1(0,T;L^2(\Omega;\mathbb{R}^d))$, we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{D}[\Theta Eu], \Theta E\varphi_s)_{L^2(\Omega)}\\
&= 2(\mathbb{D}[\Theta Eu], \dot{\Theta}E\varphi_s)_{L^2(\Omega)} + (\mathbb{D}[\Theta E\dot{u}], \Theta E\varphi_s)_{L^2(\Omega)} + (\mathbb{D}[\Theta Eu], \Theta E\dot{\varphi}_s)_{L^2(\Omega)},
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^s (\mathbb{D}[\Theta E\dot{u}], \Theta E\varphi_s)_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_0^s \left[\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{D}[\Theta Eu], \Theta E\varphi_s)_{L^2(\Omega)} - 2(\mathbb{D}[\Theta Eu], \dot{\Theta}E\varphi_s)_{L^2(\Omega)} - (\mathbb{D}[\Theta Eu], \Theta E\dot{\varphi}_s)_{L^2(\Omega)}\right]\mathrm{d}t\\
&\le (\mathbb{D}[\Theta(s)Eu(s)], \Theta(s)E\varphi_s(s))_{L^2(\Omega)} - (\mathbb{D}[\Theta(0)Eu(0)], \Theta(0)E\varphi_s(0))_{L^2(\Omega)}\\
&\quad + \int_0^s \left[2\big((\mathbb{D}[\Theta Eu], \Theta Eu)_{L^2(\Omega)}\big)^{\frac{1}{2}}\big((\mathbb{D}[\dot{\Theta}E\varphi_s], \dot{\Theta}E\varphi_s)_{L^2(\Omega)}\big)^{\frac{1}{2}} - (\mathbb{D}[\Theta Eu], \Theta E\dot{\varphi}_s)_{L^2(\Omega)}\right]\mathrm{d}t\\
&\le \int_0^s \left[(\mathbb{D}[\Theta Eu], \Theta Eu)_{L^2(\Omega)} + (\mathbb{D}[\dot{\Theta}E\varphi_s], \dot{\Theta}E\varphi_s)_{L^2(\Omega)} - (\mathbb{D}[\Theta Eu], \Theta E\dot{\varphi}_s)_{L^2(\Omega)}\right]\mathrm{d}t\\
&\le \|\mathbb{D}\|_{L^\infty(\Omega)}\|\dot{\Theta}\|^2_{L^\infty((0,T)\times\Omega)}\int_0^s \|E\varphi_s\|^2_{L^2(\Omega)}\mathrm{d}t,
\end{aligned}$$

since $E\varphi_s(s) = \Theta(0)Eu(0) = 0$ and $E\dot{\varphi}_s = Eu$ in $(0,s)$. By combining the previous inequality with (3.4.4) and using the coercivity of the tensor $\mathbb{C}$, we derive

$$\begin{aligned}
\frac{\lambda_1}{2}\|E\varphi_s(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(s)\|^2_{L^2(\Omega)} &\le \frac{1}{2}(\mathbb{C}E\varphi_s(0), E\varphi_s(0))_{L^2(\Omega)} + \frac{1}{2}\|u(s)\|^2_{L^2(\Omega)}\\
&\le \|\mathbb{D}\|_{L^\infty(\Omega)}\|\dot{\Theta}\|^2_{L^\infty((0,T)\times\Omega)}\int_0^s \|E\varphi_s(t)\|^2_{L^2(\Omega)}\mathrm{d}t.
\end{aligned}$$

Let us now set $\zeta(t) := \int_0^t u(\tau)\,\mathrm{d}\tau$, then

$$\|E\varphi_s(0)\|^2_{L^2(\Omega)} = \|E\zeta(s)\|^2_{L^2(\Omega)},$$
$$\|E\varphi_s(t)\|^2_{L^2(\Omega)} = \|E\zeta(t) - E\zeta(s)\|^2_{L^2(\Omega)} \leq 2\|E\zeta(t)\|^2_{L^2(\Omega)} + 2\|E\zeta(s)\|^2_{L^2(\Omega)},$$

from which we deduce

$$\frac{\lambda_1}{2}\|E\zeta(s)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(s)\|^2_{L^2(\Omega)} \leq C\int_0^s \|E\zeta(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + Cs\|E\zeta(s)\|^2_{L^2(\Omega)}, \tag{3.4.6}$$

where $C := 2\|\mathbb{D}\|_{L^\infty(\Omega)}\|\dot{\Theta}\|^2_{L^\infty((0,T)\times\Omega)}$. Therefore, if we set $s_0 := \frac{\lambda_1}{4C}$, for all $s \leq s_0$ we obtain

$$\frac{\lambda_1}{4}\|E\zeta(s)\|^2_{L^2(\Omega)} \leq \left(\frac{\lambda_1}{2} - Cs\right)\|E\zeta(s)\|^2_{L^2(\Omega)} \leq C\int_0^s \|E\zeta(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

By Gronwall's lemma the last inequality implies $E\zeta(s) = 0$ for all $s \leq s_0$. Hence, thanks to (3.4.6) we get $\|u(s)\|^2_{L^2(\Omega)} \leq 0$ for all $s \leq s_0$, which yields $u(s) = 0$ for all $s \leq s_0$. Since s_0 depends only on $\mathbb{C}$, $\mathbb{D}$, and Θ, we can repeat this argument starting from s_0, and with a finite number of steps we obtain $u = 0$ on $[0,T]$. □

We now are in a position to prove the uniqueness in the case of a moving crack. We consider the dimensional case $d = 2$, and we require the following assumptions:

(F1) there exists a $C^{2,1}$ simple curve $\Gamma \subset \overline{\Omega} \subset \mathbb{R}^2$, parametrized by arc-length $\gamma\colon [0,\ell] \to \overline{\Omega}$, such that $\Gamma \cap \partial\Omega = \gamma(0) \cup \gamma(\ell)$ and $\Omega \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary;

(F2) $\Gamma_t = \{\gamma(\sigma) : 0 \leq \sigma \leq s(t)\}$, where $s\colon [0,T] \to (0,\ell)$ is a non-decreasing function of class $C^{1,1}$;

(F3) $|\dot{s}(t)|^2 < \frac{\lambda_1}{C_K}$ for every $t \in [0,T]$, where λ_1 is the ellipticity constant of $\mathbb{C}$ and C_K is the constant that appears in Korn's inequality (3.1.2).

Remark 3.4.3. Notice that hypotheses (F1) and (F2) imply (E1)–(E3). Moreover, by (F2) we have $\Gamma_T \setminus \Gamma_0 \subset\subset \Omega$.

We also assume that Θ satisfies (3.4.1) and that there exists a constant $\varepsilon > 0$, independent of t, such that

$$\Theta(t,x) = 0 \quad \text{for every } t \in [0,T] \text{ and } x \in \{y \in \overline{\Omega} : |y - \gamma(s(t))| < \varepsilon\}. \tag{3.4.7}$$

Theorem 3.4.4. *Assume $d = 2$, (F1)–(F3), and that Θ satisfies (3.4.1) and (3.4.7). Then the system (3.1.12) with boundary conditions (3.1.13)–(3.1.15) has a unique solution $u \in \mathcal{W}$, according to Definition (3.1.5), satisfying $u(0) = u^0$ and $\dot{u}(0) = u^1$ in the sense of (3.1.21).*

Proof. As before let $u_1, u_2 \in \mathcal{W}$ be two solutions to (3.1.12)–(3.1.15) with initial conditions (3.1.16). Then $u := u_1 - u_2$ satisfies (3.4.2) and (3.4.3) for every $\varphi \in \mathcal{V}_D$ such that $\varphi(T) = 0$. Let us define

$$t_0 := \sup\{t \in [0,T] : u(s) = 0 \text{ for every } s \in [0,t]\},$$

and assume by contradiction that $t_0 < T$. Consider first the case in which $t_0 > 0$. By (F1), (F2), (3.4.1), and (3.4.7) we can find two open sets A_1 and A_2, with $A_1 \subset\subset A_2 \subset\subset \Omega$, and a number $\delta > 0$ such that for every $t \in [t_0 - \delta, t_0 + \delta]$ we have $\gamma(s(t)) \in A_1$, $\Theta(t,x) = 0$ for every $x \in \overline{A}_2$, and $(A_2 \setminus A_1) \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary. Let us define

$$\hat{V}^1 := \{u \in H^1((A_2 \setminus A_1) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial A_1 \cup \partial A_2\}, \quad \hat{H}^1 := L^2(A_2 \setminus A_1; \mathbb{R}^2).$$

Since every function in $\hat{V}^1$ can be extended to a function in $H^1_D(\Omega \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2)$, by standard results for linear hyperbolic equations (se, e.g., [24]) we deduce that $\ddot{u} \in L^2(t_0-\delta, t_0+\delta; (\hat{V}^1)')$ and u satisfies for a.e. $t \in (t_0 - \delta, t_0 + \delta)$

$$\langle \ddot{u}(t), \psi \rangle_{(\hat{V}^1)'} + (\mathbb{C}Eu(t), E\psi)_{\hat{H}^1} = 0 \quad \text{for every } \psi \in \hat{V}^1.$$

Moreover, we have $u(t_0) = 0$ as element of $\hat{H}^1$ and $\dot{u}(t_0) = 0$ as element of $(\hat{V}^1)'$, since $u(t) = 0$ in $[t_0 - \delta, t_0)$, $u \in C^0([t_0 - \delta, t_0]; \hat{H}^1)$, and $\dot{u} \in C^0([t_0 - \delta, t_0]; (\hat{V}^1)')$. We are now in a position to apply the finite speed propagation result of [18, Theorem 6.1]. This theorem ensures the existence of a third open set A_3, with $A_1 \subset\subset A_3 \subset\subset A_2$, such that, up to choose a smaller δ, we have $u(t) = 0$ on ∂A_3 for every $t \in [t_0, t_0 + \delta]$, and both $(\Omega \setminus A_3) \setminus \Gamma$ and $A_3 \setminus \Gamma$ are union of two disjoint open sets with Lipschitz boundary.

In $\Omega \setminus A_3$ the function u solves

$$\begin{aligned} &-\int_{t_0-\delta}^{t_0+\delta} \int_{\Omega\setminus A_3} \dot{u}(t,x) \cdot \dot{\varphi}(t,x) \,\mathrm{d}x \,\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta} \int_{\Omega\setminus A_3} \mathbb{C}(x)Eu(t,x) \cdot E\varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_{t_0-\delta}^{t_0+\delta} \int_{\Omega\setminus A_3} \mathbb{D}(x)\Theta(t,x)E\dot{u}(t,x) \cdot \Theta(t,x)E\varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t = 0 \end{aligned}$$

for every $\varphi \in L^2(t_0 - \delta, t_0 + \delta; \hat{V}^2) \cap H^1(t_0 - \delta, t_0 + \delta; \hat{H}^2)$ such that $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$, where

$$\hat{V}^2 := \{u \in H^1((\Omega \setminus A_3) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega \cup \partial A_3\}, \quad \hat{H}^2 := L^2(\Omega \setminus A_3; \mathbb{R}^2).$$

Since $u(t) = 0$ on $\partial_D\Omega \cup \partial A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$ and $u(t_0 - \delta) = \dot{u}(t_0 - \delta) = 0$ in the sense of (3.1.21) (we recall that $u = 0$ in $[t_0 - \delta, t_0)$), we can apply Lemma 3.4.2 to deduce that $u(t) = 0$ in $\Omega \setminus A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$.

On the other hand in A_3, by setting

$$\hat{V}^3_t := \{u \in H^1(A_3 \setminus \Gamma_t; \mathbb{R}^2) : u = 0 \text{ on } \partial A_3\}, \quad \hat{H}^3 := L^2(A_3; \mathbb{R}^2),$$

we get that the function u solves

$$-\int_{t_0-\delta}^{t_0+\delta} \int_{A_3} \dot{u}(t,x) \cdot \dot{\varphi}(t,x) \,\mathrm{d}x \,\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta} \int_{A_3} \mathbb{C}(x)Eu(t,x) \cdot E\varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t = 0$$

for every function $\varphi \in L^2(t_0-\delta, t_0+\delta; \hat{V}^3_{t_0+\delta}) \cap H^1(t_0-\delta, t_0+\delta; \hat{H}^3)$ such that $\varphi(t) \in \hat{V}^3_t$ for a.e. $t \in (t_0 - \delta, t_0 + \delta)$ and $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$. Here we would like to apply the uniqueness result contained in [23, Theorem 4.3] (which is a slightly generalization of Theorem 1.2.10 of Chapter 1) for the spaces $\{\hat{V}^3_t\}_{t\in[t_0-\delta,t_0+\delta]}$ and $\hat{H}^3$, endowed with the usual norms, and for the bilinear form

$$a(u, v) := \int_{A_3} \mathbb{C}(x)Eu(x) \cdot Ev(x) \,\mathrm{d}x \quad \text{for } u, v \in \hat{V}^3_{t_0+\delta}. \tag{3.4.8}$$

As show in [20, Example 2.14] we can construct two maps $\Phi, \Psi \in C^{1,1}([t_0 - \delta, t_0 + \delta] \times \overline{A}_3; \mathbb{R}^2)$ such that for every $t \in [0, T]$ the function $\Phi(t)\colon \overline{A}_3 \to \overline{A}_3$ is a diffeomorfisms of A_3 in itself with inverse $\Psi(t)\colon \overline{A}_3 \to \overline{A}_3$. Moreover, $\Phi(0, y) = y$ for every $y \in \overline{A}_3$, $\Phi(t, \Gamma \cap \overline{A}_3) = \Gamma \cap \overline{A}_3$ and $\Phi(t, \Gamma_{t_0-\delta} \cap \overline{A}_3) = \Gamma_t \cap \overline{A}_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$. For every $t \in [t_0 - \delta, t_0 + \delta]$, the maps

$(Q_t u)(y) := u(\Phi(t,y))$, $u \in \hat{V}_t^3$ and $y \in A_3$, and $(R_t v)(x) := v(\Psi(t,x))$, $v \in \hat{V}^3_{t_0-\delta}$ and $x \in A_3$, provide a family of linear and continuous operators which satisfies assumptions (U1)–(U8) of [23, Theorem 4.3]. The only condition to check is (U5), which is a ensured by (F3). Indeed, the bilinear form a satisfies the following ellipticity condition:

$$a(u,u) \geq \lambda_1 \|Eu\|^2_{L^2(A_3)} \geq \frac{\lambda_1}{\hat{C}_k}\|u\|^2_{\hat{V}^3_{t_0+\delta}} - \lambda_1\|u\|^2_{\hat{H}^3} \quad \text{for every } u \in \hat{V}^3_{t_0+\delta}, \tag{3.4.9}$$

where $\hat{C}_K$ is the constant in Korn's inequality in $\hat{V}^3_{t_0+\delta}$, namely

$$\|\nabla u\|^2_{L^2(A_3)} \leq \hat{C}_K\left(\|u\|^2_{L^2(A_3)} + \|Eu\|^2_{L^2(A_3)}\right) \quad \text{for every } u \in \hat{V}^3_{t_0+\delta}.$$

Therefore have to show that Φ satisfies

$$|\dot{\Phi}(t,y)|^2 < \frac{\lambda_1}{\hat{C}_K} \quad \text{for every } t \in [t_0-\delta, t_0+\delta] \text{ and } y \in \overline{A}_3,$$

which is analogous to the condition (1.2.4) appearing in Chapter 1. Thanks to (F3), we can construct the maps Φ and Ψ in such a way that

$$|\dot{\Phi}(t,y)|^2 < \frac{\lambda_1}{C_K} \quad \text{for every } t \in [t_0-\delta, t_0+\delta] \text{ and } y \in \overline{A}_3,$$

as explained in [20, Example 3.1]. Moreover, every function in $\hat{V}^3_{t_0+\delta}$ can be extended to a function in $H^1(\Omega\setminus\Gamma;\mathbb{R}^d)$. Hence, we can use for Korn's inequality in $\hat{V}^3_{t_0+\delta}$ the same constant C_K of (3.1.2). This allows us to apply the uniqueness result [23, Theorem 4.3], which implies $u(t) = 0$ in A_3 for every $t \in [t_0, t_0+\delta]$. In the case $t_0 = 0$, it is enough to argue as before in $[0,\delta]$, by exploiting (3.4.2). Therefore $u(t) = 0$ in Ω for every $t \in [t_0, t_0+\delta]$, which contradicts the maximality of t_0. Hence $t_0 = T$, that yields $u(t) = 0$ in Ω for every $t \in [0,T]$. □

Remark 3.4.5. Also Theorem 3.4.4 is true in the antiplane case, i.e, for (15), with identical proof. Notice that, when the displacement is scalar, we do not need to use Korn's inequality in (3.4.9) to get the coercivity in $\hat{V}^3_{t_0+\delta}$ of the bilinear form a defined in (3.4.8). Therefore, in this case in (F3) it is enough to assume $|\dot{s}(t)|^2 < \lambda_1$.

3.5 An example of a growing crack

We conclude this chapter with an example of a moving crack $\{\Gamma_t\}_{t\in[0,T]}$ and a solution to system (3.1.12)–(3.1.16) which satisfy the dynamic energy-dissipation balance (16), similarly to the purely elastic case of [17].

In dimension $d = 2$ we consider an antiplane evolution, which means that the displacement u is scalar, and we take $\Omega := B_R(0) \subset \mathbb{R}^2$, with $R > 0$. We fix a constant $0 < c < 1$ such that $cT < R$, and we set

$$\Gamma_t := \overline{\Omega} \cap \{(\sigma, 0) \in \mathbb{R}^2 : \sigma \leq ct\} \quad \text{for every } t \in [0,T].$$

Let us define the following function

$$S(x_1,x_2) := Im(\sqrt{x_1 + ix_2}) = \frac{x_2}{\sqrt{2}\sqrt{|x| + x_1}} \quad x \in \mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\},$$

where Im denotes the imaginary part of a complex number. Notice that the function S belongs to $H^1(\Omega\setminus\Gamma_0) \setminus H^2(\Omega\setminus\Gamma_0)$, and it is a solution to

$$\begin{cases} \Delta S = 0 & \text{in } \Omega\setminus\Gamma_0, \\ \nabla S \cdot \nu = \partial_2 S = 0 & \text{on } \Gamma_0. \end{cases}$$

We consider the function

$$u(t,x) := \frac{2}{\sqrt{\pi}} S\left(\frac{x_1 - ct}{\sqrt{1-c^2}}, x_2\right) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t,$$

and we define by $w(t)$ its restriction to $\partial\Omega$. Since $u(t)$ has a singularity only at the crack-tip $(ct, 0)$, the function $w(t)$ can be seen as the trace on $\partial\Omega$ of a function belonging to $H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega\setminus\Gamma_0))$, still denoted by $w(t)$. It is easy to see that u solves the wave equation

$$\ddot{u}(t) - \Delta u(t) = 0 \quad \text{in } \Omega\setminus\Gamma_t,\, t \in [0,T],$$

with boundary conditions

$$\begin{aligned} &u(t) = w(t) \quad \text{on } \partial\Omega,\, t\in[0,T],\\ &\frac{\partial u}{\partial \nu}(t) = \nabla u(t)\cdot\nu = 0 \quad \text{on } \Gamma_t,\, t \in [0,T], \end{aligned}$$

and initial data

$$\begin{aligned} u^0(x_1,x_2) &:= \frac{2}{\sqrt{\pi}} S\left(\frac{x_1}{\sqrt{1-c^2}}, x_2\right) \in H^1(\Omega\setminus\Gamma_0),\\ u^1(x_1,x_2) &:= -\frac{2}{\sqrt{\pi}}\frac{c}{\sqrt{1-c^2}}\partial_1 S\left(\frac{x_1}{\sqrt{1-c^2}}, x_2\right) \in L^2(\Omega). \end{aligned}$$

Let us consider a function Θ satisfying the regularity assumptions (3.4.1) and condition (3.4.7), namely

$$\Theta(t) = 0 \quad \text{on } B_\varepsilon(t) := \{x\in\mathbb{R}^2 : |x-(ct,0)| < \varepsilon\} \text{ for every } t\in[0,T],$$

with $0 < \varepsilon < R - cT$. In this case u is a solution, according to Definition 3.1.5, to the damped wave equation

$$\ddot{u}(t) - \Delta u(t) - \operatorname{div}(\Theta^2(t)\nabla\dot{u}(t)) = f(t) \quad \text{in } \Omega\setminus\Gamma_t,\, t\in[0,T],$$

with forcing term f given by

$$f := -\operatorname{div}(\Theta^2\nabla\dot{u}) = -\nabla\Theta\cdot 2\Theta\nabla\dot{u} - \Theta^2\Delta\dot{u} \in L^2(0,T;L^2(\Omega)),$$

and boundary and initial conditions

$$\begin{aligned} &u(t) = w(t) \quad \text{on } \partial\Omega,\, t\in[0,T],\\ &\frac{\partial u}{\partial\nu}(t) + \Theta^2(t)\frac{\partial\dot{u}}{\partial\nu}(t) = 0 \quad \text{on } \Gamma_t,\, t\in[0,T],\\ &u(0) = u^0, \quad \dot{u}(0) = u^1 \quad \text{in } \Omega\setminus\Gamma_0. \end{aligned}$$

Notice that for the homogeneous Neumann boundary conditions on Γ_t we have used the fact that $\frac{\partial\dot{u}}{\partial\nu}(t) = \nabla\dot{u}(t)\cdot\nu = \partial_2\dot{u}(t) = 0$ on Γ_t. By the uniqueness result proved in the previous section, the function u coincides with the solution given by Theorem 3.2.1. Thanks to the computations done in [17, Section 4], we know that u satisfies for every $t\in[0,T]$ the following dynamic energy-dissipation balance for the undamped equation, where ct coincides with the length of $\Gamma_t\setminus\Gamma_0$:

$$\begin{aligned} &\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + ct\\ &= \frac{1}{2}\|\dot{u}(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega)} + \int_0^t (\frac{\partial u}{\partial\nu}(s), \dot{w}(s))_{L^2(\partial\Omega)}\,\mathrm{d}s. \end{aligned} \tag{3.5.1}$$

Moreover, we have

$$\int_0^t (\frac{\partial u}{\partial \nu}(s), \dot{w}(s))_{L^2(\partial\Omega)} \,\mathrm{d}s = \int_0^t (\nabla u(s), \nabla \dot{w}(s))_{L^2(\Omega)} \,\mathrm{d}s - \int_0^t (\dot{u}(s), \ddot{w}(s))_{L^2(\Omega)} \,\mathrm{d}s \\ + (\dot{u}(t), \dot{w}(t))_{L^2(\Omega)} - (\dot{u}(0), \dot{w}(0))_{L^2(\Omega)}. \tag{3.5.2}$$

For every $t \in [0, T]$ we compute

$$\begin{aligned}(f(t), \dot{u}(t) - \dot{w}(t))_{L^2(\Omega)} &= -\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Theta^2(t,x)\nabla\dot{u}(t,x)](\dot{u}(t,x) - \dot{w}(t,x)) \,\mathrm{d}x \\ &= -\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Theta^2(t,x)\nabla\dot{u}(t,x)(\dot{u}(t,x) - \dot{w}(t,x))] \,\mathrm{d}x \\ &\quad + \int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \Theta^2(t,x)\nabla\dot{u}(t,x) \cdot (\nabla\dot{u}(t,x) - \nabla\dot{w}(t,x)) \,\mathrm{d}x.\end{aligned}$$

If we denote by $\dot{u}^+(t)$ and $\dot{w}^+(t)$ the traces of $\dot{u}(t)$ and $\dot{w}(t)$ on Γ_t from above and by $\dot{u}^-(t)$ and $\dot{w}^-(t)$ the trace from below, thanks to the divergence theorem we have

$$\begin{aligned}&\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Theta^2(t,x)\nabla\dot{u}(t,x)(\dot{u}(t,x) - \dot{w}(t,x))] \,\mathrm{d}x \\ &= \int_{\partial\Omega} \Theta^2(t,x)\frac{\partial \dot{u}}{\partial \nu}(t,x)(\dot{u}(t,x) - \dot{w}(t,x)) \,\mathrm{d}x + \int_{\partial B_\varepsilon(t)} \Theta^2(t,x)\frac{\partial \dot{u}}{\partial \nu}(t,x)(\dot{u}(t,x) - \dot{w}(t,x)) \,\mathrm{d}x \\ &\quad - \int_{(\Omega\setminus B_\varepsilon(t))\cap\Gamma_t} \Theta^2(t,x)\partial_2\dot{u}^+(t,x)(\dot{u}^+(t,x) - \dot{w}^+(t,x)) \,\mathrm{d}\mathcal{H}^1(x) \\ &\quad + \int_{(\Omega\setminus B_\varepsilon(t))\cap\Gamma_t} \Theta^2(t,x)\partial_2\dot{u}^-(t,x)(\dot{u}^-(t,x) - \dot{w}^-(t,x)) \,\mathrm{d}\mathcal{H}^1(x) = 0,\end{aligned}$$

since $u(t) = w(t)$ on $\partial\Omega$, $\Theta(t) = 0$ on $\partial B_\varepsilon(t)$, and $\partial_2\dot{u}(t) = 0$ on Γ_t. Therefore for every $t \in [0, T]$ we get

$$(f(t), \dot{u}(t) - \dot{w}(t))_{L^2(\Omega)} = \|\Theta(t)\nabla\dot{u}(t)\|^2_{L^2(\Omega)} - (\Theta(t)\nabla\dot{u}(t), \Theta(t)\nabla\dot{w}(t))_{L^2(\Omega)}. \tag{3.5.3}$$

By combining (3.5.1)–(3.5.3) we deduce that u satisfies for every $t \in [0, T]$ the following dynamic energy-dissipation balance

$$\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + ct + \int_0^t \|\Theta(s)\nabla\dot{u}(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \\ = \frac{1}{2}\|\dot{u}(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega)} + \mathcal{W}_{tot}(t), \tag{3.5.4}$$

where in this case the total work takes the form

$$\begin{aligned}\mathcal{W}_{tot}(t) := &\int_0^t \left[(f(s), \dot{u}(s) - \dot{w}(s))_{L^2(\Omega)} + (\nabla u(s), \nabla\dot{w}(s))_{L^2(\Omega)}\right] \mathrm{d}s \\ &+ \int_0^t \left[(\Theta(s)\nabla\dot{u}(s), \Theta(s)\nabla\dot{w}(s))_{L^2(\Omega)} - (\dot{u}(s), \ddot{w}(s))_{L^2(\Omega)}\right] \mathrm{d}s \\ &+ (\dot{u}(t), \dot{w}(t))_{L^2(\Omega)} - (\dot{u}(0), \dot{w}(0))_{L^2(\Omega)}.\end{aligned}$$

Notice that equality (3.5.4) gives (16). This show that in this model the dynamic energy-dissipation balance can be satisfied by a moving crack, in contrast with the case $\Theta = 1$, which always leads to (14).

Chapter 4

A phase-field model of dynamic fracture

In this chapter we prove an existence result for the dynamic phase-field model of fracture with a crack-dependent dissipation $(\tilde{D}_1)$–$(\tilde{D}_3)$.

The chapter is organized as follows: in Section 4.1 we list the main assumptions on our model, and in Theorem 4.1.5 we state our existence result. Section 4.2 is devoted to the study of the time discretization scheme. We construct an approximation of our evolution by solving, with an alternate minimization procedure, problems $(\tilde{D}_1)$ and $(\tilde{D}_2)$. Next, we show that this discrete evolution satisfies the estimate (4.2.17), which allows us to pass to the limit as the time step tends to zero. For every $k \in \mathbb{N} \cup \{0\}$ we obtain the existence of a dynamic evolution $t \mapsto (u(t), v(t))$ which satisfies $(\tilde{D}_1)$ and $(\tilde{D}_2)$, and the energy-dissipation inequality (4.2.32). We complete the proof of Theorem 4.1.5 in Section 4.3, where we prove that for $k > d/2$ our evolution is more regular in time, and it satisfies the dynamic energy-dissipation balance (18). Finally, in Section 4.4 we study the dynamic phase-field model without dissipative terms (D_1)–(D_3).

The results contained in this chapter are the basis of the submitted paper [7].

4.1 Preliminary results

Let T be a positive number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. We fix two (possibly empty) Borel subsets $\partial_{D_1}\Omega$, $\partial_{D_2}\Omega$ of $\partial\Omega$, and we denote by $\partial_{N_1}\Omega$, $\partial_{N_2}\Omega$ their complements. We introduce the spaces

$$H^1_{D_1}(\Omega;\mathbb{R}^d) := \{\psi \in H^1(\Omega;\mathbb{R}^d) : \psi = 0 \text{ on } \partial_{D_1}\Omega\},$$
$$H^1_{D_2}(\Omega) := \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \partial_{D_2}\Omega\},$$

and we denote by $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$ the dual space of $H^1_{D_1}(\Omega;\mathbb{R}^d)$. The transpose of the natural embedding $H^1_{D_1}(\Omega;\mathbb{R}^d) \hookrightarrow L^2(\Omega;\mathbb{R}^d)$ induces the embedding of $L^2(\Omega;\mathbb{R}^d)$ into $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$, which is defined by

$$\langle g, \psi\rangle_{H^{-1}_{D_1}(\Omega)} := (g, \psi)_{L^2(\Omega)} \quad \text{for } g \in L^2(\Omega;\mathbb{R}^d) \text{ and } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d).$$

Let $\mathbb{C}\colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ be a fourth-order tensor field satisfying the following natural assumptions in linear elasticity:

$$\mathbb{C} \in L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})), \tag{4.1.1}$$
$$(\mathbb{C}(x)\xi_1)\cdot\xi_2 = \xi_1\cdot(\mathbb{C}(x)\xi_2) \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym}, \tag{4.1.2}$$

$$\mathbb{C}(x)\xi\cdot\xi\geq\lambda_0|\xi|^2\quad\text{for a.e. } x\in\Omega \text{ and for every } \xi\in\mathbb{R}^{d\times d}_{sym}, \tag{4.1.3}$$

for a constant $\lambda_0>0$. Thanks to second Korn's inequality there exists a constant $C_K>0$, depending on Ω, such that

$$\|\nabla\psi\|_{L^2(\Omega)}\leq C_K(\|\psi\|_{L^2(\Omega)}+\|E\psi\|_{L^2(\Omega)})\quad\text{for every } \psi\in H^1(\Omega;\mathbb{R}^d).$$

By combining Korn's inequality with (4.1.3), we obtain that $\mathbb{C}$ satisfies the following ellipticity condition of integral type:

$$(\mathbb{C}E\psi,E\psi)_{L^2(\Omega)}\geq c_0\|\psi\|^2_{H^1(\Omega)}-c_1\|\psi\|^2_{L^2(\Omega)}\quad\text{for every } \psi\in H^1(\Omega;\mathbb{R}^d), \tag{4.1.4}$$

for two constants $c_0>0$ and $c_1\in\mathbb{R}$.

We fix $\varepsilon>0$ and we consider a map $b\colon\mathbb{R}\to[0,+\infty)$ satisfying

$$b\in C^1(\mathbb{R}) \text{ is convex and non-decreasing}, \tag{4.1.5}$$

$$b(s)\geq\eta \text{ for every } s\in\mathbb{R} \text{ and some } \eta>0. \tag{4.1.6}$$

We define the functionals elastic energy $\mathscr{E}\colon H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega)\to[0,\infty]$ and surface energy $\mathscr{H}\colon H^1(\Omega)\to[0,\infty)$ in the following way:

$$\mathscr{E}(u,v):=\frac{1}{2}\int_\Omega b(v(x))\mathbb{C}(x)Eu(x)\cdot Eu(x)\,\mathrm{d}x,$$

$$\mathscr{H}(v):=\frac{1}{4\varepsilon}\int_\Omega|1-v(x)|^2\,\mathrm{d}x+\varepsilon\int_\Omega|\nabla v(x)|^2\,\mathrm{d}x$$

for $u\in H^1(\Omega;\mathbb{R}^d)$ and $v\in H^1(\Omega)$. We also define the kinetic energy $\mathscr{K}: L^2(\Omega;\mathbb{R}^d)\to[0,\infty)$ and the dissipative energy $\mathscr{G}\colon H^k(\Omega)\to[0,\infty)$ for every $k\in\mathbb{N}\cup\{0\}$ as

$$\mathscr{K}(w):=\frac{1}{2}\int_\Omega|w(x)|^2\,\mathrm{d}x,\quad \mathscr{G}(\sigma):=\sum_{i=0}^k\alpha_i\int_\Omega|\nabla^i\sigma(x)|^2\mathrm{d}x$$

for $w\in L^2(\Omega;\mathbb{R}^d)$ and $\sigma\in H^k(\Omega)$, where α_i, $i=0,\dots,k$, are non negative numbers with $\alpha_0,\alpha_k>0$ (we recall that $H^0(\Omega):=L^2(\Omega)$). Notice that, by [1, Corollary 4.16], the functional $\mathscr{G}$ induces a norm on $H^k(\Omega)$ which is equivalent to the standard one. In particular, there exist two constants $\beta_0,\beta_1>0$ such that

$$\beta_0\|\sigma\|^2_{H^k(\Omega)}\leq\mathscr{G}(\sigma)\leq\beta_1\|\sigma\|^2_{H^k(\Omega)}\quad\text{for every } \sigma\in H^k(\Omega).$$

Finally, we define the total energy $\mathscr{F}\colon H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^d)\times H^1(\Omega)\to[0,\infty]$ as

$$\mathscr{F}(u,w,v):=\mathscr{K}(w)+\mathscr{E}(u,v)+\mathscr{H}(v)$$

for $u\in H^1(\Omega;\mathbb{R}^d)$, $w\in L^2(\Omega;\mathbb{R}^d)$, and $v\in H^1(\Omega)$.

Throughout the chapter we always assume that $\mathbb{C}$ and b satisfy (4.1.1)–(4.1.3), (4.1.5), and (4.1.6), and that ε is a fixed positive number. Given

$$w_1\in H^2(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{4.1.7}$$

$$w_2\in H^1(\Omega)\cap H^k(\Omega) \text{ with } w_2\leq 1 \text{ on } \partial_{D_2}\Omega, \tag{4.1.8}$$

$$f\in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\quad g\in H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \tag{4.1.9}$$

$$u^0-w_1(0)\in H^1_{D_1}(\Omega;\mathbb{R}^d),\quad u^1\in L^2(\Omega;\mathbb{R}^d), \tag{4.1.10}$$

$$v^0-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega) \text{ with } v^0\leq 1 \text{ in } \Omega, \tag{4.1.11}$$

we search a pair (u, v) which solves the *elastodynamics system*

$$\ddot{u}(t) - \operatorname{div}[b(v(t))\mathbb{C}Eu(t)] = f(t) + g(t) \quad \text{in } \Omega, \quad t \in [0, T], \tag{4.1.12}$$

with boundary conditions formally written as

$$u(t) = w_1(t) \quad \text{on } \partial_{D_1}\Omega, \quad t \in [0, T], \tag{4.1.13}$$

$$v(t) = w_2 \quad \text{on } \partial_{D_2}\Omega, \quad t \in [0, T], \tag{4.1.14}$$

$$(b(v(t))\mathbb{C}Eu(t))\nu = 0 \quad \text{on } \partial_{N_1}\Omega, \quad t \in [0, T], \tag{4.1.15}$$

and initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1, \quad v(0) = v^0 \quad \text{in } \Omega. \tag{4.1.16}$$

In addition, we require the *irreversibility condition*:

$$v(t) \le v(s) \quad \text{in } \Omega \quad \text{for } 0 \le s \le t \le T, \tag{4.1.17}$$

and for a.e. $t \in (0, T)$ the following *crack stability condition*:

$$\begin{aligned} &\mathscr{E}(u(t), v^*) - \mathscr{E}(u(t), v(t)) + \mathscr{H}(v^*) - \mathscr{H}(v(t)) \\ &\quad + \sum_{i=0}^{k} \alpha_i (\nabla^i \dot{v}(t), \nabla^i v^* - \nabla^i v(t))_{L^2(\Omega)} \ge 0 \end{aligned} \tag{4.1.18}$$

among all $v^* - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $v^* \le v(t)$. Notice that the space $H^1(\Omega) \cap H^k(\Omega)$ coincides with either $H^1(\Omega)$ (when $k = 0$) or $H^k(\Omega)$ (for $k \ge 1$). Finally, for every $t \in [0, T]$ we ask the dynamic energy-dissipation balance:

$$\mathscr{F}(u(t), \dot{u}(t), v(t)) + \int_0^t \mathscr{G}(\dot{v}(s)) \, \mathrm{d}s = \mathscr{F}(u^0, u^1, v^0) + \mathscr{W}_{tot}(u, v; 0, t), \tag{4.1.19}$$

where $\mathscr{W}_{tot}(u, v; t_1, t_2)$ is the *total work* over the time interval $[t_1, t_2] \subseteq [0, T]$, defined as

$$\begin{aligned} \mathscr{W}_{tot}(u, v; t_1, t_2) &:= \int_{t_1}^{t_2} \left[(f(s), \dot{u}(s) - \dot{w}_1(s))_{L^2(\Omega)} + (b(v(s))\mathbb{C}Eu(s), E\dot{w}_1(s))_{L^2(\Omega)} \right] \mathrm{d}s \\ &\quad - \int_{t_1}^{t_2} \left[(\dot{u}(s), \ddot{w}_1(s))_{L^2(\Omega)} + \langle \dot{g}(s), u(s) - w_1(s) \rangle_{H^{-1}_{D_1}(\Omega)} \right] \mathrm{d}s \\ &\quad + (\dot{u}(t_2), \dot{w}_1(t_2))_{L^2(\Omega)} + \langle g(t_2), u(t_2) - w_1(t_2) \rangle_{H^{-1}_{D_1}(\Omega)} \\ &\quad - (\dot{u}(t_1), \dot{w}_1(t_1))_{L^2(\Omega)} - \langle g(t_1), u(t_1) - w_1(t_1) \rangle_{H^{-1}_{D_1}(\Omega)}. \end{aligned}$$

Remark 4.1.1. A simple prototype for the function b is given by

$$b(s) := (s \vee 0)^2 + \eta \quad \text{for } s \in \mathbb{R}.$$

In this case, the elastic energy becomes

$$\mathscr{E}(u, v) = \frac{1}{2} \int_\Omega [(v(x) \vee 0)^2 + \eta] \mathbb{C}(x) Eu(x) \cdot Eu(x) \, \mathrm{d}x \tag{4.1.20}$$

for $u \in H^1(\Omega; \mathbb{R}^d)$ and $v \in H^1(\Omega)$, which corresponds to the phase-field model of dynamic fracture $(\tilde{D}_1)$–$(\tilde{D}_3)$ considered in the introduction. Usually, in the phase-field setting, the elastic energy functional is defined as

$$\frac{1}{2} \int_\Omega [(v(x))^2 + \eta] \mathbb{C}(x) Eu(x) \cdot Eu(x) \, \mathrm{d}x$$

for $u \in H^1(\Omega; \mathbb{R}^d)$ and $v \in H^1(\Omega)$, with v satisfying $0 \le v \le 1$. In our case, due to the presence of the dissipative term introduced in $(\tilde{D}_2)$ and $(\tilde{D}_3)$, we need to consider phase-field functions v which may assume negative values. Therefore, we have to slightly modify the elastic energy functional by considering (4.1.20).

Remark 4.1.2. We give an idea of the meaning of the term $\mathscr{G}(\dot{v})$ in the phase-field setting, by comparing it with a dissipation, in the sharp-interface case, which depends on the velocity of the crack-tips. We consider just an example in the particular case $d = 2$ and $k = 0$ of a rectilinear crack $\Gamma_t := \{(\sigma, 0) : \sigma \le s(t)\}$, $t \in [0, T]$, moving along the x_1-axis, with $s \in C^1([0, T])$, $s(0) = 0$, and $\dot{s}(t) \ge 0$ for every $t \in [0, T]$. In view of the analysis done in [4], the sequence $v_\varepsilon(t)$ which best approximate Γ_t takes the following form:

$$v_\varepsilon(t, x) := \Psi\left(\frac{\operatorname{dist}(x, \Gamma_t)}{\varepsilon}\right) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^2.$$

Here, $\Psi\colon \mathbb{R} \to [0, 1]$ is a C^1 function satisfying $\Psi(s) = 0$ for $|s| \le \delta$, with $0 < \delta < 1$, and $\Psi(s) = 1$ for $|s| \ge 1$. The function $v_\varepsilon \in C^1([0, T] \times \mathbb{R}^2)$ is constantly 0 in a $\varepsilon\delta$-neighborhood of Γ_t, and takes the value 1 outside a ε-neighborhood of Γ_t. Moreover, its time derivative satisfies

$$\dot{v}_\varepsilon(t, x) = -\frac{\dot{s}(t)}{\varepsilon}\partial_1\Phi\left(\frac{x - (s(t), 0)}{\varepsilon}\right) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^2,$$

where $\Phi(y) := \Psi(\operatorname{dist}(y, \Gamma_0))$ for $y \in \mathbb{R}^2$. In particular for every $t \in [0, T]$ we deduce

$$\|\dot{v}_\varepsilon(t)\|^2_{L^2(\Omega)} = \frac{\dot{s}(t)^2}{\varepsilon^2}\int_{\mathbb{R}^2}\left|\partial_1\Phi\left(\frac{x - (s(t), 0)}{\varepsilon}\right)\right|^2 \mathrm{d}x = \dot{s}(t)^2\int_{\mathbb{R}^2}|\partial_1\Phi(y)|^2\,\mathrm{d}y = C_\Phi\dot{s}(t)^2.$$

Therefore, this term can be used to detect the dissipative effects due to the velocity of the moving crack. With similar computations, if there are m crack-tips with different velocities $\dot{s}_i(t)$, $i = 1, \dots, m$, then the term $\|\dot{v}_\varepsilon(t)\|^2_{L^2(\Omega)}$ corresponds to a dissipation of the form $\sum_{i=1}^m C_i\dot{s}_i^2(t)$, with C_i positive constants.

To precise the notion of solution to problem (4.1.12)–(4.1.19), we consider a pair of functions (u, v) satisfying the following regularity assumptions:

$$u \in C^0([0, T]; H^1(\Omega; \mathbb{R}^d)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^d)) \cap H^2(0, T; H^{-1}_{D_1}(\Omega; \mathbb{R}^d)), \tag{4.1.21}$$

$$u(t) - w_1(t) \in H^1_{D_1}(\Omega; \mathbb{R}^d) \text{ for every } t \in [0, T], \tag{4.1.22}$$

$$v \in C^0([0, T]; H^1(\Omega)) \cap H^1(0, T; H^k(\Omega)), \tag{4.1.23}$$

$$v(t) - w_2 \in H^1_{D_2}(\Omega) \text{ and } v(t) \le 1 \text{ in } \Omega \text{ for every } t \in [0, T]. \tag{4.1.24}$$

Definition 4.1.3. Let w_1, w_2, f, and g be as in (4.1.7)–(4.1.9). We say that (u, v) is a *weak solution* to the elastodynamics system (4.1.12) with boundary conditions (4.1.13)–(4.1.15), if (u, v) satisfies (4.1.21)–(4.1.24), and for a.e. $t \in (0, T)$ we have

$$\langle\ddot{u}(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)} + (b(v(t))\mathbb{C}Eu(t), E\psi)_{L^2(\Omega)} = (f(t), \psi)_{L^2(\Omega)} + \langle g(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)} \tag{4.1.25}$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$.

Remark 4.1.4. Since b satisfies (4.1.5) and (4.1.6), and $v(t) \le 1$ for every $t \in [0, T]$, the function $b(v(t))$ belongs to $L^\infty(\Omega)$ for every $t \in [0, T]$. Hence, equation (4.1.25) makes sense for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$. Moreover, if (u, v) satisfies (4.1.21)–(4.1.24), then the function $(t_1, t_2) \mapsto \mathscr{W}_{tot}(u, v; t_1, t_2)$ is well defined and continuous, thanks to the previous assumptions on $\mathbb{C}$, b, w_1, f, and g.

We state now our main result, whose proof will be given at the end of Section 4.3.

Theorem 4.1.5. *Let $k > d/2$ and let w_1, w_2, f, g, u^0, u^1, and v^0 be as in (4.1.7)–(4.1.11). Then there exists a weak solution (u, v) to problem (4.1.12) with boundary conditions (4.1.13)–(4.1.15) and initial conditions (4.1.16). Moreover, the pair (u, v) satisfies the irreversibility condition (4.1.17), the crack stability condition (4.1.18), and the dynamic energy-dissipation balance (4.1.19).*

Remark 4.1.6. According to Griffith's dynamic criterion (see [41]), we expect the sum of kinetic and elastic energy to be dissipated during the evolution, while it is balanced when we take into account the surface energy associated to the phase-field function v. This happens in our case if we also consider $\int_0^t \mathscr{G}(\dot{v})\,\mathrm{d}s$. The presence of this term takes into account the rate at which the function v is decreasing and it is a consequence of the crack stability condition (4.1.18).

We need $k > d/2$ in order to obtain the energy equality (4.1.19). Indeed, in this case the embedding $H^k(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is continuous and compact (see, e.g., [1, Theorem 6.2]), which implies that $\dot{v}(t) \in C^0(\overline{\Omega})$ for a.e. $t \in (0,T)$. This regularity is crucial, since we obtain (4.1.19) throughout another energy balance (see (4.3.20)), which is well defined only when $\dot{v}(t) \in L^\infty(\Omega)$.

Remark 4.1.7. In Theorem 4.1.5 we consider only the case of zero Neumann boundary data. Anyway, the previous result can be easily adapted to Neumann boundary conditions of the form

$$(b(v(t))\mathbb{C}Eu(t))\nu = F(t) \quad \text{on } \partial_{N_1}\Omega, \quad t \in [0,T], \tag{4.1.26}$$

provided that $F \in H^1(0,T;L^2(\partial_{N_1}\Omega;\mathbb{R}^d))$. In this case a *weak solution* to problem (4.1.12) with Dirichlet boundary conditions (4.1.13) and (4.1.14), and Neumann boundary condition (4.1.26) is a pair (u,v) satisfying (4.1.21)–(4.1.24) and for a.e. $t \in (0,T)$ the equation

$$\langle \ddot{u}(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)} + (b(v(t))\mathbb{C}Eu(t), E\psi)_{L^2(\Omega)} = (f(t),\psi)_{L^2(\Omega)} + \langle \tilde{g}(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$, where the term $\tilde{g}(t) \in H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$ is defined for $t \in [0,T]$ as

$$\langle \tilde{g}(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)} := \langle g(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)} + \int_{\partial_{N_1}\Omega} F(t,x)\cdot\psi(x)\mathrm{d}\mathscr{H}^{d-1}(x) \quad \text{for } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d).$$

Since $\tilde{g} \in H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$, we can apply Theorem 4.1.5 with $\tilde{g}$ instead of g, and we derive the existence of a weak solution (u,v) to (4.1.12)–(4.1.14) with Neumann boundary condition (4.1.26).

In the next lemma we show that for $k > d/2$ the dynamic energy-dissipation balance can be rephrased in the following identity:

$$\partial_v\mathscr{E}(u(t),v(t))[\dot{v}(t)] + \partial\mathscr{H}(v(t))[\dot{v}(t)] + \mathscr{G}(\dot{v}(t)) = 0 \quad \text{for a.e. } t \in (0,T), \tag{4.1.27}$$

where the derivatives $\partial_v\mathscr{E}$ and $\partial\mathscr{H}$ take the form

$$\partial_v\mathscr{E}(u,v)[\chi] = \frac{1}{2}\int_\Omega \dot{b}(v)\chi\mathbb{C}Eu\cdot Eu\,\mathrm{d}x \quad \text{for } u \in H^1(\Omega;\mathbb{R}^d) \text{ and } v,\chi \in H^1(\Omega)\cap L^\infty(\Omega),$$
$$\partial\mathscr{H}(v)[\chi] = \frac{1}{2\varepsilon}\int_\Omega (v-1)\chi\,\mathrm{d}x + 2\varepsilon\int_\Omega \nabla v\cdot\nabla\chi\,\mathrm{d}x \quad \text{for } v,\chi \in H^1(\Omega).$$

Lemma 4.1.8. *Let $k > d/2$ and let w_1, w_2, f, g, u^0, u^1, and v^0 be as in (4.1.7)–(4.1.11). Assume that (u,v) is a weak solution to problem (4.1.12)–(4.1.15) with initial conditions (4.1.16). Then the dynamic energy-dissipation balance (4.1.19) is equivalent to identity (4.1.27).*

Proof. We follow the same techniques of [21, Lemma 2.6]. Let us fix $0 < h < T$ and let us define the function

$$\psi_h(t) := \frac{u(t+h)-u(t)}{h} - \frac{w_1(t+h)-w_1(t)}{h} \quad \text{for } t \in [0,T-h].$$

We use $\psi_h(t)$ as test function in (4.1.25) first at time t, and then at time $t+h$. By summing the two expressions and integrating in a fixed time interval $[t_1,t_2]\subseteq[0,T-h]$, we obtain the identity

$$\begin{aligned}&\int_{t_1}^{t_2}\langle\ddot u(t+h)+\ddot u(t),\psi_h(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}(b(v(t+h))\mathbb{C}Eu(t+h)+b(v(t))\mathbb{C}Eu(t),E\psi_h(t))_{L^2(\Omega)}\,\mathrm{d}t\\&=\int_{t_1}^{t_2}(f(t+h)+f(t),\psi_h(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_{t_1}^{t_2}\langle g(t+h)+g(t),\psi_h(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t.\end{aligned}\tag{4.1.28}$$

We study these four terms separately. By performing an integration by parts, the first one becomes

$$\begin{aligned}&\int_{t_1}^{t_2}\langle\ddot u(t+h)+\ddot u(t),\psi_h(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t\\&=-\int_{t_1}^{t_2}(\dot u(t+h)+\dot u(t),\dot\psi_h(t))_{L^2(\Omega)}\,\mathrm{d}t+(\dot u(t_2+h)+\dot u(t_2),\psi_h(t_2))_{L^2(\Omega)}\\&\quad-(\dot u(t_1+h)+\dot u(t_1),\psi_h(t_1))_{L^2(\Omega)}\\&=-\frac1h\int_{t_2}^{t_2+h}\|\dot u(t)\|^2_{L^2(\Omega)}\mathrm{d}t+\frac1h\int_{t_1}^{t_1+h}\|\dot u(t)\|^2_{L^2(\Omega)}\mathrm{d}t\\&\quad+\frac1h\int_{t_1}^{t_2}(\dot u(t+h)+\dot u(t),\dot w_1(t+h)-\dot w_1(t))_{L^2(\Omega)}\,\mathrm{d}t\\&\quad+(\dot u(t_2+h)+\dot u(t_2),\psi_h(t_2))_{L^2(\Omega)}-(\dot u(t_1+h)+\dot u(t_1),\psi_h(t_1))_{L^2(\Omega)}.\end{aligned}$$

Since $u,w_1\in C^1([0,T];L^2(\Omega;\mathbb{R}^d))$, by sending $h\to0^+$ we deduce

$$\begin{aligned}&\lim_{h\to0^+}\left[-\frac1h\int_{t_2}^{t_2+h}\|\dot u(t)\|^2_{L^2(\Omega)}\mathrm{d}t+\frac1h\int_{t_1}^{t_1+h}\|\dot u(t)\|^2_{L^2(\Omega)}\mathrm{d}t\right]\\&=-\|\dot u(t_2)\|^2_{L^2(\Omega)}+\|\dot u(t_1)\|^2_{L^2(\Omega)},\end{aligned}\tag{4.1.29}$$

$$\begin{aligned}&\lim_{h\to0^+}\left[(\dot u(t_2+h)+\dot u(t_2),\psi_h(t_2))_{L^2(\Omega)}-(\dot u(t_1+h)+\dot u(t_1),\psi_h(t_1))_{L^2(\Omega)}\right]\\&=2\|\dot u(t_2)\|^2_{L^2(\Omega)}-2(\dot u(t_2),\dot w_1(t_2))_{L^2(\Omega)}-2\|\dot u(t_1)\|^2_{L^2(\Omega)}+2(\dot u(t_1),\dot w_1(t_1))_{L^2(\Omega)}.\end{aligned}\tag{4.1.30}$$

Notice that $\frac1h[\dot w_1(\,\cdot+h)-\dot w_1]$ converges strongly to $\ddot w_1$ in $L^2(t_1,t_2;L^2(\Omega;\mathbb{R}^d))$ as $h\to0^+$, since $\dot w_1$ belongs to $H^1(0,T;L^2(\Omega;\mathbb{R}^d))$. Therefore, there exist a sequence $h_m\to0^+$ as $m\to\infty$, and a function $\kappa\in L^2(t_1,t_2)$ such that for a.e. $t\in(t_1,t_2)$

$$\begin{aligned}&\frac1{h_m}(\dot u(t+h_m)+\dot u(t),\dot w_1(t+h_m)-\dot w_1(t))_{L^2(\Omega)}\to2(\dot u(t),\ddot w_1(t))_{L^2(\Omega)}\quad\text{as }m\to\infty,\\&\left|\frac1{h_m}(\dot u(t+h_m)+\dot u(t),\dot w_1(t+h_m)-\dot w_1(t))_{L^2(\Omega)}\right|\le2\|\dot u\|_{L^\infty(0,T;L^2(\Omega))}\kappa(t)\quad\text{for every }m\in\mathbb{N}.\end{aligned}$$

By the dominated convergence theorem we derive

$$\lim_{h\to0^+}\frac1h\int_{t_1}^{t_2}(\dot u(t+h)+\dot u(t),\dot w_1(t+h)-\dot w_1(t))_{L^2(\Omega)}\,\mathrm{d}t=2\int_{t_1}^{t_2}(\dot u(t),\ddot w_1(t))_{L^2(\Omega)}\,\mathrm{d}t,\tag{4.1.31}$$

since the limit does not depend on the subsequence $\{h_m\}_m$. For the term involving f, we observe that $f(\,\cdot+h)\to f$ and $\psi_h\to\dot u-\dot w_1$ in $L^2(t_1,t_2;L^2(\Omega;\mathbb{R}^d))$ as $h\to0^+$. Hence, we have

$$\lim_{h\to0^+}\int_{t_1}^{t_2}(f(t+h)+f(t),\psi_h(t))_{L^2(\Omega)}\,\mathrm{d}t=2\int_{t_1}^{t_2}(f(t),\dot u(t)-\dot w_1(t))_{L^2(\Omega)}\,\mathrm{d}t.\tag{4.1.32}$$

By using the identity

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle g(t+h)+g(t), \psi_h(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t \\
&= \frac{2}{h}\int_{t_2}^{t_2+h} \langle g(t), u(t)-w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t - \frac{2}{h}\int_{t_1}^{t_1+h} \langle g(t), u(t)-w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t \\
&\quad - \frac{1}{h}\int_{t_1}^{t_2} \langle g(t+h)-g(t), u(t+h)+u(t)-w_1(t+h)-w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t,
\end{aligned}
$$

and proceeding as before, we also deduce

$$
\begin{aligned}
&\lim_{h\to 0^+}\int_{t_1}^{t_2} \langle g(t+h)+g(t), \psi_h(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t \\
&= 2\langle g(t_2), u(t_2)-w_1(t_2)\rangle_{H^{-1}_{D_1}(\Omega)} - 2\langle g(t_1), u(t_1)-w_1(t_1)\rangle_{H^{-1}_{D_1}(\Omega)} \\
&\quad - 2\int_{t_1}^{t_2} \langle \dot g(t), u(t)-w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t.
\end{aligned} \tag{4.1.33}
$$

It remains to study the last term, that can be rephrased in the following way

$$
\begin{aligned}
&\int_{t_1}^{t_2} (b(v(t+h))\mathbb{C}Eu(t+h)+b(v(t))\mathbb{C}Eu(t), E\psi_h(t))_{L^2(\Omega)}\,\mathrm{d}t \\
&= \frac{1}{h}\int_{t_2}^{t_2+h} (b(v(t))\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t - \frac{1}{h}\int_{t_1}^{t_1+h} (b(v(t))\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad - \frac{1}{h}\int_{t_1}^{t_2} ([b(v(t+h))-b(v(t))]\mathbb{C}Eu(t), Eu(t+h))_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad - \frac{1}{h}\int_{t_1}^{t_2} (b(v(t+h))\mathbb{C}Eu(t+h)+b(v(t))\mathbb{C}Eu(t), Ew_1(t+h)-Ew_1(t))_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
$$

Since $H^k(\Omega)\hookrightarrow C^0(\overline{\Omega})$, we deduce that v belongs to the space $C^0([0,T];C^0(\overline{\Omega}))$. This property, together with $b\in C^1(\mathbb{R})$ and $u\in C^0([0,T];H^1(\Omega;\mathbb{R}^d))$, implies

$$
\begin{aligned}
&\lim_{h\to 0^+}\left[\frac{1}{h}\int_{t_2}^{t_2+h} (b(v(t))\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t - \frac{1}{h}\int_{t_1}^{t_1+h} (b(v(t))\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t\right] \\
&= (b(v(t_2))\mathbb{C}Eu(t_2), Eu(t_2))_{L^2(\Omega)} - (b(v(t_1))\mathbb{C}Eu(t_1), Eu(t_1))_{L^2(\Omega)}.
\end{aligned} \tag{4.1.34}
$$

Moreover, the sequence $\frac{1}{h}[v(\,\cdot+h)-v]$ converges strongly to $\dot v$ in $L^2(t_1,t_2;C^0(\overline{\Omega}))$ as $h\to 0^+$. Therefore, there exist a subsequence $h_m\to 0^+$ as $m\to\infty$ and a function $\kappa\in L^2(t_1,t_2)$ such that for a.e. $t\in(t_1,t_2)$

$$
\begin{aligned}
&\frac{v(t+h_m)-v(t)}{h_m}\to \dot v(t) \quad \text{in } C^0(\overline{\Omega}) \quad \text{as } m\to\infty, \\
&\left\|\frac{v(t+h_m)-v(t)}{h_m}\right\|_{L^\infty(\Omega)} \le \kappa(t) \quad \text{for every } m\in\mathbb{N}.
\end{aligned}
$$

Thanks to (4.1.5), we can apply Lagrange's theorem to derive for a.e. $t\in(t_1,t_2)$

$$
\begin{aligned}
&\frac{1}{h_m}(b(v(t+h_m))-b(v(t))\mathbb{C}Eu(t), Eu(t+h_m))_{L^2(\Omega)} \\
&\to (\dot b(v(t))\dot v(t)\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)},
\end{aligned}
$$

as $m \to \infty$, while for every $m \in \mathbb{N}$

$$\begin{aligned}&\left|\frac{1}{h_m}([b(v(t+h_m)) - b(v(t))]\mathbb{C}Eu(t), Eu(t+h_m))_{L^2(\Omega)}\right|\\ &\le \dot{b}(1)\|\mathbb{C}\|_{L^\infty(\Omega)}\|Eu\|^2_{L^\infty(0,T;L^2(\Omega))}\kappa(t),\end{aligned}$$

since $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^d))$ and $v(t) \le 1$ for every $t \in [0,T]$. The dominated convergence theorem yields

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\int_{t_1}^{t_2}([b(v(t+h)) - b(v(t))]\mathbb{C}Eu(t), Eu(t+h))_{L^2(\Omega)}\,\mathrm{d}t\\ &= \int_{t_1}^{t_2}(\dot{b}(v(t))\dot{v}(t)\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t,\end{aligned} \tag{4.1.35}$$

being the limit independent of the subsequence $\{h_m\}_m$. Finally, $\frac{1}{h}[Ew_1(\,\cdot\,+h) - Ew_1]$ converges strongly to $E\dot{w}_1$ in $L^2(t_1,t_2;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $h \to 0^+$. By arguing as in (4.1.31), this fact gives

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\int_{t_1}^{t_2}(b(v(t+h))\mathbb{C}Eu(t+h) + b(v(t))\mathbb{C}Eu(t), Ew_1(t+h) - Ew_1(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &= 2\int_{t_1}^{t_2}(b(v(t))\mathbb{C}Eu(t), E\dot{w}_1(t))_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{4.1.36}$$

We combine together (4.1.28)–(4.1.36) to derive

$$\begin{aligned}&\mathscr{K}(\dot{u}(t_2)) + \mathscr{E}(u(t_2),v(t_2)) - \frac{1}{2}\int_{t_1}^{t_2}(\dot{b}(v(t))\dot{v}(t)\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &= \mathscr{K}(\dot{u}(t_1)) + \mathscr{E}(u(t_1),v(t_1)) + \mathscr{W}_{tot}(u,v;t_1,t_2)\end{aligned}$$

for every $t_1,t_2 \in [0,T)$ with $t_1 < t_2$. Since all terms in the previous equality are continuous with respect to t_2, we deduce that a weak solution to (4.1.12)–(4.1.15) with initial conditions (4.1.16) satisfies the energy balance

$$\begin{aligned}&\mathscr{K}(\dot{u}(t_2)) + \mathscr{E}(u(t_2),v(t_2)) - \frac{1}{2}\int_{t_1}^{t_2}(\dot{b}(v(t))\dot{v}(t)\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t\\ &= \mathscr{K}(\dot{u}(t_1)) + \mathscr{E}(u(t_1),v(t_1)) + \mathscr{W}_{tot}(u,v;t_1,t_2)\end{aligned} \tag{4.1.37}$$

for every $t_1,t_2 \in [0,T]$ with $t_1 < t_2$.

Let us assume now (4.1.27). Since $v \in H^1(0,T;H^k(\Omega))$, the function $t \mapsto \zeta(t) := \mathscr{H}(v(t))$ is absolutely continuous on $[0,T]$, with $\dot{\zeta}(t) = \partial\mathscr{H}(v(t))[\dot{v}(t)]$ for a.e. $t \in (0,T)$. By integrating (4.1.27) over $[t_1,t_2] \subseteq [0,T]$, we obtain

$$-\frac{1}{2}\int_{t_1}^{t_2}(\dot{b}(v(t))\dot{v}(t)\mathbb{C}Eu(t), Eu(t))_{L^2(\Omega)}\,\mathrm{d}t = \mathscr{H}(v(t_2)) - \mathscr{H}(v(t_1)) + \int_{t_1}^{t_2}\mathscr{G}(v(t))\,\mathrm{d}t. \tag{4.1.38}$$

This identity, together with (4.1.37), implies the dynamic energy-dissipation balance (4.1.19). On the other hand, if (4.1.19) is satisfied, by comparing it with (4.1.37) we deduce (4.1.38) for every interval $[t_1,t_2] \subseteq [0,T]$, from which (4.1.27) follows. □

Remark 4.1.9. When $k > \frac{d}{2}$, the crack stability condition (4.1.18) is equivalent for a.e. $t \in (0,T)$ to the following variational inequality

$$\partial_v\mathscr{E}(u(t),v(t))[\chi] + \partial\mathscr{H}(v(t))[\chi] + \sum_{i=0}^{k}\alpha_i(\nabla^i\dot{v}(t), \nabla^i\chi)_{L^2(\Omega)} \ge 0 \tag{4.1.39}$$

among all $\chi \in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $\chi \le 0$. Indeed, for every $s \in (0,1]$ we can take $v(t)+s\chi$ as test function in (4.1.18). After some computations and by dividing by s, we deduce

$$\begin{aligned}&\frac{\mathscr{E}(u(t),v(t)+s\chi)-\mathscr{E}(u(t),v(t))}{s}+\partial\mathscr{H}(v(t))[\chi]+\sum_{i=0}^k\alpha_i(\nabla^i\dot v(t),\nabla^i\chi)_{L^2(\Omega)}\\&+s\left[\frac{1}{4\varepsilon}\|\chi\|^2_{L^2(\Omega)}+\varepsilon\|\nabla\chi\|^2_{L^2(\Omega)}\right]\ge 0.\end{aligned}\tag{4.1.40}$$

Let us fix $x\in\Omega$. By Lagrange's theorem there exists $z_s(t,x)\in[v(t,x)+s\chi(x),v(t,x)]$ such that

$$\frac{b(v(t,x)+s\chi(x))-b(v(x))}{s}=\dot b(z_s(t,x))\chi(x),$$

since $b\in C^1(\mathbb{R})$. In particular, we have

$$\begin{aligned}&\lim_{s\to0^+}\frac{b(v(t,x)+s\chi(x))-b(v(x))}{s}=\dot b(v(t,x))\chi(x),\\&\left|\frac{b(v(t,x)+s\chi(x))-b(v(x))}{s}\right|\le\dot b(1)|\chi(x)|,\end{aligned}$$

because $\dot b\in C^0(\mathbb{R})$ is non negative, non-decreasing, and $z_s(t,x)\le v(t,x)\le 1$. Then, the dominated convergence theorem yields

$$\lim_{s\to0^+}\frac{\mathscr{E}(u(t),v(t)+s\chi)-\mathscr{E}(u(t),v(t))}{s}=\frac12\int_\Omega\dot b(v(t))\chi\mathbb{C}Eu(t)\cdot Eu(t)\,\mathrm{d}x=\partial_v\mathscr{E}(u(t),v(t))[\chi].$$

By sending $s\to0^+$ in (4.1.40) we hence deduce (4.1.39). On the other hand, it is easy to check that (4.1.39) implies (4.1.18), by exploiting the convexity of $v^*\to\mathscr{E}(u(t),v^*)+\mathscr{H}(v^*)$ and taking $\chi:=v^*-v(t)$ for every $v^*-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $v^*\le v(t)$.

The inequality (4.1.39) gives that for a.e. $t\in(0,T)$ the distribution

$$-\frac12\dot b(v(t))\mathbb{C}Eu(t)\cdot Eu(t)-\frac{1}{2\varepsilon}(v(t)-1)+2\varepsilon\Delta v(t)-\sum_{i=0}^k\alpha_i(-1)^i\Delta^i\dot v(t)$$

is positive on Ω. Therefore it coincides with a positive Radon measure $\mu(t)$ on Ω, by Riesz's representation theorem. In particular, since $H^k(\Omega)\hookrightarrow C^0(\overline\Omega)$, for a.e. $t\in(0,T)$ we deduce

$$\langle\zeta(t),\chi\rangle_{(H^k(\Omega))'}:=\partial_v\mathscr{E}(u(t),v(t))[\chi]+\partial\mathscr{H}(v(t))[\chi]+\sum_{i=0}^k\alpha_i(\nabla^i\dot v(t),\nabla^i\chi)_{L^2(\Omega)}=-\int_\Omega\chi\,\mathrm{d}\mu(t)$$

for every function $\chi\in H^k(\Omega)$ with compact support in Ω. We combine this fact with identity (4.1.27) to derive for our model an analogous of the classical activation rule in Griffith's criterion: for a.e. $t\in(0,T)$ the positive measure $\mu(t)$ must vanish on the set of points $x\in\Omega$ where $\dot v(t,x)>0$. Indeed, let us consider a sequence $\{\varphi_m\}_m\subset C_c^\infty(\Omega)$ such that $0\le\varphi_m\le\varphi_{m+1}\le1$ in Ω for every $m\in\mathbb{N}$, and $\varphi_m(x)\to1$ for every $x\in\Omega$ as $m\to\infty$. The function $\dot v(t)$ is admissible in (4.1.39) for a.e. $t\in(0,T)$, since $\frac1h[v(t+h)-v(t)]\in H^1_{D_2}(\Omega)$ converges strongly to $\dot v(t)$ in $H^k(\Omega)$ as $h\to0^+$, and $t\mapsto v(t)$ is non-decreasing in $[0,T]$. Therefore, thanks to (4.1.27) and (4.1.39), for a.e. $t\in(0,T)$ we get

$$\begin{aligned}0=\langle\zeta(t),\dot v(t)\rangle_{(H^k(\Omega))'}&=\langle\zeta(t),\dot v(t)\varphi_m\rangle_{(H^k(\Omega))'}+\langle\zeta(t),\dot v(t)(1-\varphi_m)\rangle_{(H^k(\Omega))'}\\&\ge\langle\zeta(t),\dot v(t)\varphi_m\rangle_{(H^k(\Omega))'}=-\int_\Omega\dot v(t)\varphi_m\mathrm{d}\mu(t)\ge0,\end{aligned}$$

because $\dot v(t)\varphi_m\in H^k(\Omega)$ has compact support. Hence, for a.e. $t\in(0,T)$ we have

$$0=\lim_{m\to\infty}\int_\Omega\dot v(t)\varphi_m\mathrm{d}\mu(t)=\int_\Omega\dot v(t)\,\mathrm{d}\mu(t),$$

by the monotone convergence theorem, which implies our activation condition.

4.2 The time discretization scheme

In this section we show some general results that are true for every $k \in \mathbb{N} \cup \{0\}$. In particular, we prove that problem (4.1.12)–(4.1.16) admits a solution (u, v) (in a weaker sense) which satisfies the irreversibility condition (4.1.17) and the crack stability condition (4.1.18). Throughout this section, we always assume that the functions w_1, w_2, f, g, u^0, u^1, and v^0 satisfy (4.1.7)–(4.1.11).

We start by introducing the following notion of solution, which requires less regularity on the time variable.

Definition 4.2.1. The pair (u, v) is a *generalized solution* to (4.1.12)–(4.1.15) if

$$u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \tag{4.2.1}$$

$$u(t) - w_1(t) \in H^1_{D_1}(\Omega;\mathbb{R}^d) \text{ for every } t \in [0,T], \tag{4.2.2}$$

$$v \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^k(\Omega)), \tag{4.2.3}$$

$$v(t) - w_2 \in H^1_{D_2}(\Omega) \text{ and } v(t) \le 1 \text{ in } \Omega \text{ for every } t \in [0,T], \tag{4.2.4}$$

and for a.e. $t \in (0,T)$ equation (4.1.25) holds.

Remark 4.2.2. By arguing as in Remark 1.2.7, we derive that a generalized solution (u, v) to problem (4.1.12)–(4.1.15) satisfies $u \in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$, $\dot u \in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$, and $v \in C^0_w([0,T];H^1(\Omega))$. Therefore, the initial conditions (4.1.16) makes sense, since the functions $u(t)$, $\dot u(t)$, and $v(t)$ are uniquely defined for every $t \in [0,T]$ as elements of $H^1(\Omega;\mathbb{R}^d)$, $L^2(\Omega;\mathbb{R}^d)$, and $H^1(\Omega)$, respectively.

To show the existence of a generalized solution to (4.1.12)–(4.1.16), we approximate our problem by means of a time discretization with an alternate scheme, as done in [6, 36]. We divide the time interval $[0,T]$ by introducing n equispaced nodes, and in each of them we first solve the elastodynamics system (4.1.4), keeping v fixed, and then the crack stability condition (4.1.18), keeping u fixed. Finally, we consider some interpolants of the discrete solutions and, thanks to an a priori estimate, we pass to the limit as $n \to \infty$.

We fix $n \in \mathbb{N}$, and we set

$$\begin{aligned}
&\tau_n = \frac{T}{n}, \quad u_n^0 := u^0, \quad u_n^{-1} := u^0 - \tau_n u^1, \quad v_n^0 := v^0,\\
&g_n^j := g(j\tau_n), \quad w_n^j := w_1(j\tau_n) \quad \text{for } j = 0,\dots,n,\\
&f_n^j := \frac{1}{\tau_n}\int_{(j-1)\tau_n}^{j\tau_n} f(s)\,\mathrm{d}s \quad \text{for } j = 1,\dots,n.
\end{aligned}$$

For $j = 1,\dots,n$ we consider the following two minimum problems:

(*i*) $u_n^j - w_n^j \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ is the minimizer of

$$u^* \mapsto \frac{1}{2\tau_n^2}\left\|u^* - 2u_n^{j-1} - u_n^{j-2}\right\|^2_{L^2(\Omega)} + \mathscr{E}(u^*, v_n^{j-1}) - (f_n^j, u^*)_{L^2(\Omega)} - \langle g_n^j, u^* - w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}$$

among every $u^* - w_n^j \in H^1_{D_1}(\Omega;\mathbb{R}^d)$;

(*ii*) $v_n^j - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $v_n^j \le v_n^{j-1}$ is the minimizer of

$$v^* \mapsto \mathscr{E}(u_n^j, v^*) + \mathscr{H}(v^*) + \frac{1}{2\tau_n}\mathscr{G}(v^* - v_n^{j-1})$$

among every $v^* - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $v^* \le v_n^{j-1}$.

Since $\mathbb{C}$ and b satisfy assumptions (4.1.1)–(4.1.3), (4.1.5), and (4.1.6), the discrete problems (i) and (ii) are well defined. In particular, for every $j = 1, \dots, n$ there exists a unique pair $(u_n^j, v_n^j) \in H^1(\Omega; \mathbb{R}^d) \times (H^1(\Omega) \cap H^k(\Omega))$ solution to (i) and (ii).

Let us define

$$\delta u_n^j := \frac{u_n^j - u_n^{j-1}}{\tau_n} \quad \text{for } j = 0, \dots, n,$$
$$\delta^2 u_n^j := \frac{\delta u_n^j - \delta u_n^{j-1}}{\tau_n}, \quad \delta v_n^j := \frac{v_n^j - v_n^{j-1}}{\tau_n} \quad \text{for } j = 1, \dots, n.$$

For $j = 1, \dots, n$ the minimality of u_n^j implies

$$(\delta^2 u_n^j, \psi)_{L^2(\Omega)} + (b(v_n^{j-1})\mathbb{C}Eu_n^j, E\psi)_{L^2(\Omega)} = (f_n^j, \psi)_{L^2(\Omega)} + \langle g_n^j, \psi\rangle_{H^{-1}_{D_1}(\Omega)} \tag{4.2.5}$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$, which is the discrete counterpart of (4.1.25). Moreover, we can characterize the function v_n^j in the following way.

Lemma 4.2.3. *For $j = 1, \dots, n$ the function $v_n^j - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $v_n^j \le v_n^{j-1}$ is the unique solution to the variational inequality*

$$\mathscr{E}(u_n^j, v^*) - \mathscr{E}(u_n^j, v_n^j) + \partial\mathscr{H}(v_n^j)[v^* - v_n^j] + \sum_{i=0}^k \alpha_i(\nabla^i \delta v_n^j, \nabla^i v^* - \nabla^i v_n^j)_{L^2(\Omega)} \ge 0 \tag{4.2.6}$$

among all $v^* - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $v^* \le v_n^{j-1}$. In particular, we have $v_n^j \le 1$ in Ω and

$$\frac{\mathscr{E}(u_n^j, v_n^j) - \mathscr{E}(u_n^j, v_n^{j-1})}{\tau_n} + \partial\mathscr{H}(v_n^j)[\delta v_n^j] + \mathscr{G}(\delta v_n^j) \le 0. \tag{4.2.7}$$

Finally, if $k = 0$, $w_2 \ge 0$ on $\partial_{D_2}\Omega$, $v^0 \ge 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then $v_n^j \ge 0$ in Ω for every $j = 1, \dots, n$.

Proof. Let v_n^j be the solution to (ii) and let $v^* - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ be such that $v^* \le v_n^{j-1}$. For every $s \in (0,1]$ the function $v_n^j + s(v^* - v_n^j)$ is a competitor for (ii). Hence, by exploiting the minimality of v_n^j and dividing by s, we deduce the following inequality

$$\begin{aligned}&\frac{\mathscr{E}(u_n^j, v_n^j + s(v^* - v_n^j)) - \mathscr{E}(u_n^j, v_n^j)}{s} + \partial\mathscr{H}(v_n^j)[v^* - v_n^j] + \sum_{i=0}^k \alpha_i(\nabla^i \delta v_n^j, \nabla^i v^* - \nabla^i v_n^j)_{L^2(\Omega)}\\ &\quad + s\left[\frac{1}{4\varepsilon}\|v^* - v_n^j\|^2_{L^2(\Omega)} + \varepsilon\|\nabla v^* - \nabla v_n^j\|^2_{L^2(\Omega)} + \frac{1}{2\tau_n}\mathscr{G}(v^* - v_n^j)\right] \ge 0.\end{aligned} \tag{4.2.8}$$

Notice that

$$\frac{\mathscr{E}(u_n^j, v_n^j + s(v^* - v_n^j)) - \mathscr{E}(u_n^j, v_n^j)}{s} \le \mathscr{E}(u_n^j, v^*) - \mathscr{E}(u_n^j, v_n^j) \quad \text{for every } s \in (0,1], \tag{4.2.9}$$

since the difference quotients are non-decreasing in $s \in (0,1]$, being b is convex. By combining (4.2.8) with (4.2.9) and passing to the limit as $s \to 0^+$, we derive (4.2.6). On the other hand, it is easy to see that every solution to (4.2.6) satisfies (ii), thanks to the convexity of $\mathscr{H}$ and $\mathscr{G}$. Finally, for every $j = 1, \dots, n$ we have $v_n^j \le v^0 \le 1$ in Ω, and the inequality (4.2.7) is obtained by taking $v^* = v_n^{j-1}$ in (4.2.6) and dividing by τ_n.

Let us assume that $k = 0$, $w_2 \ge 0$ on $\partial_{D_2}\Omega$, $v_0 \ge 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$. The function $v_n^1 \vee 0$ is a competitor for (ii) and satisfies

$$\mathscr{E}(u_n^1, v_n^1 \vee 0) + \mathscr{H}(v_n^1 \vee 0) + \frac{1}{2\tau_n}\mathscr{G}((v_n^1 \vee 0) - v^0) \le \mathscr{E}(u_n^1, v_n^1) + \mathscr{H}(v_n^1) + \frac{1}{2\tau_n}\mathscr{G}(v_n^1 - v^0),$$

being $\mathscr{E}(u_n^1, v_n^1 \vee 0) = \mathscr{E}(u_n^1, v_n^1)$, $\mathscr{H}(v_n^1 \vee 0) \le \mathscr{H}(v_n^1)$, and $|(v_n^1 \vee 0) - v_0| \le |v_n^1 - v_0|$ in Ω, which is a consequence of $v_0 \ge 0$. Hence, the function $v_n^1 \vee 0$ solves (ii). This fact implies $v_n^1 = (v_n^1 \vee 0) \ge 0$ in Ω, since the minimum point is unique (the L^2 norm is strictly convex). We now proceed by induction: if $v_n^{j-1} \ge 0$ in Ω, we can argue as before to get

$$\mathscr{E}(u_n^j, v_n^j \vee 0) + \mathscr{H}(v_n^j \vee 0) + \frac{1}{2\tau_n}\mathscr{G}((v_n^j \vee 0) - v_n^{j-1}) \le \mathscr{E}(u_n^j, v_n^j) + \mathscr{H}(v_n^j) + \frac{1}{2\tau_n}\mathscr{G}(v_n^j - v_n^{j-1}),$$

which gives $v_n^j = (v_n^j \vee 0) \ge 0$ in Ω for every $j = 1 \dots, n$. □

As done in [36], we combine equation (4.2.5) with inequality (4.2.7) to derive a discrete energy inequality for the family $\{(u_n^j, v_n^j)\}_{j=1}^n$.

Lemma 4.2.4. *The family $\{(u_n^j, v_n^j)\}_{j=1}^n$, solution to problems (i) and (ii), satisfies for every $j = 1, \dots, n$ the discrete energy inequality*

$$\begin{aligned}
&\mathscr{F}(u_n^j, \delta u_n^j, v_n^j) + \sum_{l=1}^j \tau_n \mathscr{G}(\delta v_n^l) + \sum_{l=1}^j \tau_n^2 D_n^l \\
&\le \mathscr{F}(u^0, u^1, v^0) + \sum_{l=1}^j \tau_n \Big[(f_n^l, \delta u_n^l - \delta w_n^l)_{L^2(\Omega)} + (b(v_n^{l-1})\mathbb{C}Eu_n^l, E\delta w_n^l)_{L^2(\Omega)}\Big] \\
&\quad - \sum_{l=1}^j \tau_n \Big[(\delta u_n^{l-1}, \delta^2 w_n^l)_{L^2(\Omega)} - \langle \delta g_n^l, u_n^{l-1} - w_n^{i-1}\rangle_{H_{D_1}^{-1}(\Omega)}\Big] + (\delta u_n^j, \delta w_n^j)_{L^2(\Omega)} \\
&\quad + \langle g_n^j, u_n^j - w_n^j\rangle_{H_{D_1}^{-1}(\Omega)} - (u^1, \dot w_1(0))_{L^2(\Omega)} - \langle g(0), u^0 - w_1(0)\rangle_{H_{D_1}^{-1}(\Omega)},
\end{aligned} \tag{4.2.10}$$

where $\delta w_n^0 := \dot w_1(0)$, $\delta w_n^j := \frac{1}{\tau_n}(w_n^j - w_n^{j-1})$, $\delta^2 w_n^j := \frac{1}{\tau_n}(\delta w_n^j - \delta w_n^{j-1})$, $\delta g_n^j := \frac{1}{\tau_n}(g_n^j - g_n^{n-1})$ for $j = 1, \dots, n$, and the dissipation D_n^j are defined for $j = 1, \dots, n$ as

$$D_n^j := \frac{1}{2}\|\delta^2 u_n^j\|_{L^2(\Omega)}^2 + \frac{1}{2}(b(v_n^{j-1})\mathbb{C}E\delta u_n^j, E\delta u_n^j)_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|\delta v_n^j\|_{L^2(\Omega)}^2 + \varepsilon\|\nabla \delta v_n^j\|_{L^2(\Omega)}^2.$$

Proof. By using $\psi := \tau_n(\delta u_n^j - \delta w_n^j) \in H_{D_1}^1(\Omega; \mathbb{R}^d)$ as test function in (4.2.5), for every $j = 1, \dots, n$ we deduce the following identity

$$\begin{aligned}
&\tau_n(\delta^2 u_n^j, \delta u_n^j)_{L^2(\Omega)} + \tau_n(b(v_n^{j-1})\mathbb{C}Eu_n^j, E\delta u_n^j)_{L^2(\Omega)} \\
&= \tau_n(f_n^j, \delta u_n^j - \delta w_n^j)_{L^2(\Omega)} + \tau_n\langle g_n^j, \delta u_n^j - \delta w_n^j\rangle_{H_{D_1}^{-1}(\Omega)} \\
&\quad + \tau_n(\delta^2 u_n^j, \delta w_n^j)_{L^2(\Omega)} + \tau_n(b(v_n^{j-1})\mathbb{C}Eu_n^j, E\delta w_n^j)_{L^2(\Omega)}.
\end{aligned} \tag{4.2.11}$$

Thanks to the identity $|a|^2 - a \cdot b = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a-b|^2$ for $a, b \in \mathbb{R}^d$, we can write the first term as

$$\begin{aligned}
\tau_n(\delta^2 u_n^j, \delta u_n^j)_{L^2(\Omega)} &= \|\delta u_n^j\|_{L^2(\Omega)}^2 - (\delta u_n^{j-1}, \delta u_n^j)_{L^2(\Omega)} \\
&= \mathscr{K}(\delta u_n^j) - \mathscr{K}(\delta u_n^{j-1}) + \frac{\tau_n^2}{2}\|\delta^2 u_n^j\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.2.12}$$

Similarly, we have

$$\begin{aligned}
\tau_n(b(v_n^{j-1})\mathbb{C}Eu_n^j, E\delta u_n^j)_{L^2(\Omega)} &= \mathscr{E}(u_n^j, v_n^j) - \mathscr{E}(u_n^{j-1}, v_n^{j-1}) + \frac{\tau_n^2}{2}(b(v_n^{j-1})\mathbb{C}E\delta u_n^j, E\delta u_n^j)_{L^2(\Omega)} \\
&\quad + \frac{1}{2}([b(v_n^{j-1}) - b(v_n^j)]\mathbb{C}Eu_n^j, Eu_n^j)_{L^2(\Omega)}.
\end{aligned} \tag{4.2.13}$$

We use (4.2.7) to estimate from below the last term in the previous inequality as

$$\begin{aligned}
&\frac{1}{2}([b(v_n^{j-1})-b(v_n^j)]\mathbb{C}Eu_n^j,Eu_n^j)_{L^2(\Omega)}\\
&\geq\frac{\tau_n}{2\varepsilon}(v_n^j-1,\delta v_n^j)_{L^2(\Omega)}+2\varepsilon\tau_n(\nabla v_n^j,\nabla\delta v_n^j)_{L^2(\Omega)}+\tau_n\mathscr{G}(\delta v_n^j)\\
&=\mathscr{H}(v_n^j)-\mathscr{H}(v_n^{j-1})+\tau_n\mathscr{G}(\delta v_n^j)+\frac{\tau_n^2}{4\varepsilon}\|\delta v_n^j\|_{L^2(\Omega)}^2+\varepsilon\tau_n^2\|\nabla\delta v_n^j\|_{L^2(\Omega)}^2.
\end{aligned}\tag{4.2.14}$$

By combining (4.2.11)–(4.2.14), for every $j=1,\dots,n$ we obtain

$$\begin{aligned}
&\mathscr{F}(u_n^j,\delta u_n^j,v_n^j)-\mathscr{F}(u_n^{j-1},\delta u_n^{j-1},v_n^{j-1})+\tau_n\mathscr{G}(\delta v_n^j)+\tau_n^2D_n^j\\
&\leq\tau_n(f_n^j,\delta u_n^j-\delta w_n^j)_{L^2(\Omega)}+\tau_n\langle g_n^j,\delta u_n^j-\delta w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}\\
&\quad+\tau_n(\delta^2u_n^j,\delta w_n^j)_{L^2(\Omega)}+\tau_n(b(v_n^{j-1})\mathbb{C}Eu_n^j,E\delta w_n^j)_{L^2(\Omega)}.
\end{aligned}$$

Finally, we sum over $l=1,\dots,j$ for every $j\in\{1,\dots,n\}$, and we use the identities

$$\begin{aligned}
\sum_{l=1}^j\tau_n\langle g_n^l,\delta u_n^l-\delta w_n^l\rangle_{H^{-1}_{D_1}(\Omega)}&=\langle g_n^j,u_n^j-w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}-\langle g(0),u^0-w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)}\\
&\quad-\sum_{l=1}^j\tau_n\langle\delta g_n^l,u_n^{l-1}-w_n^{l-1}\rangle_{H^{-1}_{D_1}(\Omega)},
\end{aligned}\tag{4.2.15}$$

$$\begin{aligned}
\sum_{l=1}^j\tau_n(\delta^2u_n^l,\delta w_n^l)_{L^2(\Omega)}&=(\delta u_n^j,\delta w_n^j)_{L^2(\Omega)}-(u^1,w_1(0))_{L^2(\Omega)}\\
&\quad-\sum_{l=1}^j\tau_n(\delta u_n^{l-1},\delta^2w_n^l)_{L^2(\Omega)},
\end{aligned}\tag{4.2.16}$$

to deduce the discrete energy inequality (4.2.10). □

The first consequence of (4.2.10) is the following a priori estimate.

Lemma 4.2.5. *There exists a constant $C>0$, independent of n, such that*

$$\max_{j=1,\dots,n}\{\|\delta u_n^j\|_{L^2(\Omega)}+\|u_n^j\|_{H^1(\Omega)}+\|v_n^j\|_{H^1(\Omega)}\}+\sum_{j=1}^n\tau_n\|\delta v_n^j\|_{H^k(\Omega)}^2+\sum_{j=1}^n\tau_n^2D_n^j\leq C.\tag{4.2.17}$$

Proof. Thanks to (4.1.4) and (4.1.6) we can estimate from below the left-hand side of (4.2.10) in the following way

$$\begin{aligned}
&\mathscr{F}(u_n^j,\delta u_n^j,v_n^j)+\sum_{l=1}^j\tau_n\mathscr{G}(\delta v_n^l)+\sum_{l=1}^j\tau_n^2D_n^l\\
&\geq\frac{1}{2}\|\delta u_n^j\|_{L^2(\Omega)}^2+\frac{\eta c_0}{2}\|u_n^j\|_{H^1(\Omega)}^2-\frac{\eta c_1}{2}\|u_n^j\|_{L^2(\Omega)}^2
\end{aligned}\tag{4.2.18}$$

for every $j=1,\dots,n$. Let us now bound from above the right-hand side of (4.2.18). We define

$$L_n:=\max_{j=1,\dots,n}\|\delta u_n^j\|_{L^2(\Omega)},\qquad M_n:=\max_{j=1,\dots,n}\|u_n^j\|_{H^1(\Omega)},$$

and we use (4.1.7)–(4.1.11) to derive for every $j=1,\dots,n$ the following estimates:

$$\sum_{l=1}^j\tau_n(f_n^l,\delta u_n^l-\delta w_n^l)_{L^2(\Omega)}\leq C_1L_n+C_2,\tag{4.2.19}$$

$$(\delta u_n^j, \delta w_n^j)_{L^2(\Omega)} - (u^1, w_1(0))_{L^2(\Omega)} - \sum_{l=1}^{j} \tau_n (\delta u_n^{l-1}, \delta^2 w_n^l)_{L^2(\Omega)} \leq C_1 L_n + C_2, \tag{4.2.20}$$

$$\langle g_n^j, u_n^j - w_n^j \rangle_{H^{-1}_{D_1}(\Omega)} - \langle g(0), u^0 - w_1(0) \rangle_{H^{-1}_{D_1}(\Omega)} - \sum_{l=1}^{j} \tau_n \langle \delta g_n^l, u_n^{l-1} - w_n^{l-1} \rangle_{H^{-1}_{D_1}(\Omega)} \leq C_1 M_n + C_2, \tag{4.2.21}$$

for two positive constants C_1 and C_2 independent of n. Moreover, since $\mathbb{C}$ belongs to $L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{d\times d}; \mathbb{R}^{d\times d}))$, b is non-decreasing, and $v_n^{j-1} \leq 1$, we get

$$\sum_{l=1}^{j} \tau_n (b(v_n^{l-1}) \mathbb{C} E u_n^l, E \delta w_n^l)_{L^2(\Omega)} \leq b(1) \|\mathbb{C}\|_{L^\infty(\Omega)} \sqrt{T} \|E \dot{w}_1\|_{L^2(0,T;L^2(\Omega))} M_n \tag{4.2.22}$$

for every $j = 1, \dots, n$. By combining (4.2.10) with (4.2.18)–(4.2.22) and the following estimate

$$\|u_n^j\|_{L^2(\Omega)} \leq \sum_{l=1}^{n} \tau_n \|\delta u_n^l\|_{L^2(\Omega)} + \|u^0\|_{L^2(\Omega)} \leq T L_n + \|u^0\|_{L^2(\Omega)} \quad \text{for every } j = 1, \dots, n,$$

we obtain the existence of two positive constants $\tilde{C}_1$ and $\tilde{C}_2$, independent of n, such that

$$(L_n + M_n)^2 \leq \tilde{C}_1 (L_n + M_n) + \tilde{C}_2 \quad \text{for every } n \in \mathbb{N}.$$

This implies that L_n and M_n are uniformly bounded in n. In particular, there exists a constant $C > 0$, independent of n, such that

$$\mathscr{K}(\delta u_n^j) + \mathscr{E}(u_n^j, v_n^j) + \mathscr{H}(v_n^j) + \sum_{l=1}^{j} \tau_n \mathscr{G}(\delta v_n^l) + \sum_{l=1}^{j} \tau_n^2 D_n^l \leq C \quad \text{for every } j = 1, \dots, n.$$

Finally, for $j = 1, \dots, n$ we have

$$\min\left\{\varepsilon, \frac{1}{4\varepsilon}\right\} \|v_n^j - 1\|^2_{H^1(\Omega)} \leq \mathscr{H}(v_n^j) \leq C, \quad \beta_0 \sum_{j=1}^{n} \tau_n \|\delta v_n^j\|^2_{H^k(\Omega)} \leq \sum_{j=1}^{n} \tau_n \mathscr{G}(\delta v_n^j) \leq C,$$

which gives the remaining estimates. □

Remark 4.2.6. By combining together (4.2.5) and (4.2.17) we also obtain

$$\sum_{j=1}^{n} \tau_n \|\delta^2 u_n^j\|^2_{H^{-1}_{D_1}(\Omega)} + \max_{j=1,\dots,n} \|v_n^j\|_{H^k(\Omega)} \leq C$$

for a positive constant C independent of n. Indeed, by (4.2.5), for every $j = 1, \dots, n$ we have

$$\begin{aligned} \|\delta^2 u_n^j\|_{H^{-1}_{D_1}(\Omega)} &= \sup_{\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d),\, \|\psi\|_{H^1(\Omega)} \leq 1} \left| (\delta^2 u_n^j, \psi)_{L^2(\Omega)} \right| \\ &\leq b(1) \|\mathbb{C}\|_{L^\infty(\Omega)} \|E u_n^j\|_2 + \|f_n^j\|_{L^2(\Omega)} + \|g_n^j\|_{H^{-1}_{D_1}(\Omega)}. \end{aligned}$$

Hence, thanks to (4.1.9) and (4.2.17), there exists a constant $C > 0$, independent of n, such that

$$\sum_{j=1}^{n} \tau_n \|\delta^2 u_n^j\|^2_{H^{-1}_{D_1}(\Omega)} \leq C(1 + \|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;H^{-1}_{D_1}(\Omega))}).$$

Finally, also $\|v_n^j\|_{H^k(\Omega)}$ is uniformly bounded with respect to j and n, since

$$\|v_n^j\|_{H^k(\Omega)} \leq \sqrt{T} \left(\sum_{l=1}^{n} \tau_n \|\delta v_n^l\|^2_{H^k(\Omega)} \right)^{1/2} + \|v^0\|_{H^k(\Omega)} \quad \text{for every } j = 1, \dots, n.$$

We now use the family $\{(u_n^j, v_n^j)\}_{j=1}^n$ to construct a generalized solution to (4.1.12)–(4.1.18). As in Chapter 3, we denote by $u_n\colon [0,T]\to H^1(\Omega;\mathbb{R}^d)$ and $u_n'\colon [0,T]\to L^2(\Omega;\mathbb{R}^d)$ the piecewise affine interpolants of $\{u_n^j\}_{j=1}^n$ and $\{\delta u_n^j\}_{j=1}^n$, respectively, which are defined in the following way:

$$u_n(t) := u_n^j + (t-j\tau_n)\delta u_n^j \quad \text{for } t\in[(j-1)\tau_n, j\tau_n],\ j=1,\dots,n,$$
$$u_n'(t) := \delta u_n^j + (t-j\tau_n)\delta^2 u_n^j \quad \text{for } t\in[(j-1)\tau_n, j\tau_n],\ j=1,\dots,n.$$

We also define the backward interpolants $\overline{u}_n\colon [0,T]\to H^1(\Omega;\mathbb{R}^d)$ and $\overline{u}_n'\colon [0,T]\to L^2(\Omega;\mathbb{R}^d)$, and the forward interpolants $\underline{u}_n\colon [0,T]\to H^1(\Omega;\mathbb{R}^d)$ and $\underline{u}_n'\colon [0,T]\to L^2(\Omega;\mathbb{R}^d)$ as:

$$\overline{u}_n(t) := u_n^j \quad \text{for } t\in((j-1)\tau_n, j\tau_n],\ j=1,\dots,n, \quad \overline{u}_n(0)=u_n^0,$$
$$\overline{u}_n'(t) := \delta u_n^j \quad \text{for } t\in((j-1)\tau_n, j\tau_n],\ j=1,\dots,n, \quad \overline{u}_n'(0)=\delta u_n^0,$$
$$\underline{u}_n(t) := u_n^{j-1} \quad \text{for } t\in[(j-1)\tau_n, j\tau_n),\ j=1,\dots,n, \quad \underline{u}_n(T)=u_n^n,$$
$$\underline{u}_n'(t) := \delta u_n^{j-1} \quad \text{for } t\in[(j-1)\tau_n, j\tau_n),\ j=1,\dots,n, \quad \underline{u}_n'(T)=\delta u_n^n.$$

In a similar way, we can define the piecewise affine interpolant $v_n\colon [0,T]\to H^1(\Omega)$ of $\{v_n^j\}_{j=1}^n$, as well as the backward interpolant $\overline{v}_n\colon [0,T]\to H^1(\Omega)$, and the forward interpolant $\underline{v}_n\colon [0,T]\to H^1(\Omega)$. Notice that $u_n\in H^1(0,T;L^2(\Omega;\mathbb{R}^d))$, $u_n'\in H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$, and $v_n\in H^1(0,T;H^k(\Omega))$, with $\dot u_n(t)=\overline{u}_n'(t)=\delta u_n^j$, $\dot u_n'(t)=\delta^2 u_n^j$, and $\dot v_n(t)=\delta v_n^j$ for $t\in((j-1)\tau_n, j\tau_n)$ and $j=1,\dots,n$.

Lemma 4.2.7. *There exist a subsequence of n, not relabeled, and two functions*

$$u\in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),$$
$$v\in L^\infty(0,T;H^1(\Omega))\cap H^1(0,T;H^k(\Omega)),$$

such that the following convergences hold as $n\to\infty$:

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)), && u_n' \rightharpoonup \dot u \quad \text{in } H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&u_n \to u \quad \text{in } C^0([0,T];L^2(\Omega;\mathbb{R}^d)), && u_n' \to \dot u \quad \text{in } C^0([0,T];H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&\overline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \overline{u}_n' \rightharpoonup \dot u \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\underline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \underline{u}_n' \rightharpoonup \dot u \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&v_n \rightharpoonup v \quad \text{in } H^1(0,T;H^k(\Omega)), && v_n \to v \quad \text{in } C^0([0,T];L^2(\Omega)),\\
&\overline{v}_n \rightharpoonup v \quad \text{in } L^2(0,T;H^1(\Omega)), && \underline{v}_n \rightharpoonup v \quad \text{in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

Proof. Thanks to (4.2.17) the sequence $\{u_n\}_n\subset L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$ is uniformly bounded. Hence, by Aubin-Lions's lemma (see [50, Corollary 4]) there exist a subsequence of n, not relabeled, and a function

$$u\in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)),$$

such that

$$u_n\rightharpoonup u \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)), \quad u_n\to u \quad \text{in } C^0([0,T];L^2(\Omega;\mathbb{R}^d)) \quad \text{as } n\to\infty.$$

Moreover, the sequence $\{\overline{u}_n\}_n\subset L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))$ is uniformly bounded, and satisfies

$$\|u_n(t)-\overline{u}_n(t)\|_{L^2(\Omega)}\le \tau_n\|\dot u_n\|_{L^\infty(0,T;L^2(\Omega))}\le C\tau_n \quad \text{for every } t\in[0,T] \text{ and } n\in\mathbb{N}, \tag{4.2.23}$$

where C is a positive constant independent of n and t. Therefore, there exists a further subsequence, not relabeled, such that

$$\overline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), \quad \overline{u}_n \to u \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

Similarly, we have

$$\underline{u}_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), \quad \underline{u}_n \to u \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

Let us now consider the sequence $\{u_n'\}_n \subset L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$. Since it is uniformly bounded with respect to n, we can apply again the Aubin-Lions's lemma and we deduce the existence of

$$z \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$$

such that, up to a further (not relabeled) subsequence

$$u_n' \rightharpoonup z \quad \text{in } H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \quad u_n' \to z \quad \text{in } C^0([0,T];H^{-1}_{D_1}(\Omega;\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

Furthermore, we have

$$\|u_n'(t) - \dot{u}_n(t)\|_{H^{-1}_{D_1}(\Omega)} = \|u_n'(t) - \overline{u}_n'(t)\|_{H^{-1}_{D_1}(\Omega)} \leq \sqrt{\tau_n}\|\dot{u}_n'\|_{L^2(0,T;H^{-1}_{D_1}(\Omega))} \leq C\sqrt{\tau_n} \qquad (4.2.24)$$

for every $t \in [0,T]$ and $n \in \mathbb{N}$, with $C > 0$ independent of n and t. This fact implies that $z = \dot{u}$, and

$$\overline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad \overline{u}_n' \to \dot{u} \quad \text{in } L^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

In a similar way, we get

$$\underline{u}_n' \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \quad \underline{u}_n' \to \dot{u} \quad \text{in } L^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

Finally, the thesis for the sequences $\{v_n\}_n$, $\{\overline{v}_n\}_n$, and $\{\underline{v}_n\}_n$ is obtained as before, by using (4.2.17) and the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. □

Remark 4.2.8. As already observed in Remark 4.2.2, we have $u \in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$, $u \in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$, and $v \in C^0_w([0,T];H^1(\Omega))$. By using the estimate (4.2.17), we get

$$\|u_n(t)\|_{H^1(\Omega)} + \|u_n'(t)\|_{L^2(\Omega)} \leq C \quad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}$$

for a constant $C > 0$ independent of n and t. Hence, for every $t \in [0,T]$ we derive

$$u_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d), \quad u_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } n \to \infty,$$

thanks to the previous convergences. In particular, for every $t \in [0,T]$ we can use (4.2.23) and (4.2.24) to obtain

$$\begin{aligned} \overline{u}_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d), \quad \overline{u}_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } n \to \infty, \\ \underline{u}_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d), \quad \underline{u}_n'(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{as } n \to \infty. \end{aligned}$$

With a similar argument, for every $t \in [0,T]$ we have

$$v_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\Omega), \quad \overline{v}_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\Omega), \quad \underline{v}_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\Omega) \quad \text{as } n \to \infty.$$

We are now in a position to pass to the limit in the discrete problem (4.2.5).

Lemma 4.2.9. *The pair (u, v) given by Lemma 4.2.7 is a generalized solution to problem (4.1.12)–(4.1.15). Moreover, (u, v) satisfies the initial conditions (4.1.16) and the irreversibility condition (4.1.17). Finally, if $k = 0$, $w_2 \geq 0$ on $\partial_{D_2}\Omega$, $v^0 \geq 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then $v(t) \geq 0$ in Ω for every $t \in [0, T]$.*

Proof. The pair (u, v) given by Lemma 4.2.7 satisfies (4.2.1), (4.2.3), and the initial conditions (4.1.16), since $u^0 = u_n(0) \rightharpoonup u(0)$ in $H^1(\Omega; \mathbb{R}^d)$, $u^1 = u_n'(0) \rightharpoonup \dot{u}(0)$ in $L^2(\Omega; \mathbb{R}^d)$, and $v^0 = v_n(0) \rightharpoonup v(0)$ in $H^1(\Omega)$ as $n \to \infty$. If we consider the piecewise affine interpolant w_n of $\{w_n^j\}_{j=1}^n$, for every $t \in [0, T]$ we have $u_n(t) - w_n(t) \in H^1_{D_1}(\Omega; \mathbb{R}^d)$ for every $n \in \mathbb{N}$ and $w_n(t) \to w_1(t)$ in $H^1(\Omega; \mathbb{R}^d)$ as $n \to \infty$. Therefore, the function u satisfies (4.2.2). Similarly, $v_n(t) - w_2 \in H^1_{D_2}(\Omega)$ and $v_n(t) \leq v_n(s) \leq 1$ in Ω for every $0 \leq s \leq t \leq T$ and $n \in \mathbb{N}$, which give (4.2.4) and (4.1.17). Finally, if $k = 0$, $w_2 \geq 0$ on $\partial_{D_2}\Omega$, $v^0 \geq 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then for every $t \in [0, T]$ we deduce $v_n(t) \geq 0$ in Ω, by Lemma 4.2.3, which implies $v(t) \geq 0$ in Ω.

It remains to prove the validity of (4.1.25) for a.e. $t \in (0, T)$. For every $j = 1, \dots, n$ we know that (u_n^j, v_n^j) satisfies (4.2.5). In particular, by integrating it in $[t_1, t_2] \subseteq [0, T]$ and using the previous notation, we derive

$$\begin{aligned} &\int_{t_1}^{t_2} \langle \dot{u}_n'(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt + \int_{t_1}^{t_2} (b(\underline{v}_n(t)) \mathbb{C} E \overline{u}_n(t), E\psi)_{L^2(\Omega)} \, dt \\ &= \int_{t_1}^{t_2} (\overline{f}_n(t), \psi)_{L^2(\Omega)} \, dt + \int_{t_1}^{t_2} \langle \overline{g}_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt \end{aligned} \tag{4.2.25}$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$, where $\overline{f}_n$ and $\overline{g}_n$ are the backward interpolants of $\{f_n^j\}_{j=1}^n$ and $\{g_n^j\}_{j=1}^n$, respectively. We now pass to the limit as $n \to \infty$ in (4.2.25). For the first term we have

$$\lim_{n \to \infty} \int_{t_1}^{t_2} \langle \dot{u}_n'(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt = \int_{t_1}^{t_2} \langle \ddot{u}(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt,$$

since $\dot{u}_n' \rightharpoonup \ddot{u}$ in $L^2(0, T; H^{-1}_{D_1}(\Omega; \mathbb{R}^d))$ as $n \to \infty$. Moreover, as $n \to \infty$ it is easy to check that $\overline{f}_n$ converges strongly to f in $L^2(0, T; L^2(\Omega; \mathbb{R}^d))$, while $\overline{g}_n$ converges strongly to g in $L^2(0, T; H^{-1}_{D_1}(\Omega; \mathbb{R}^d))$, which implies

$$\begin{aligned} &\lim_{n \to \infty} \left[\int_{t_1}^{t_2} (\overline{f}_n(t), \psi)_{L^2(\Omega)} \, dt + \int_{t_1}^{t_2} \langle \overline{g}_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt \right] \\ &= \int_{t_1}^{t_2} (f(t), \psi)_{L^2(\Omega)} \, dt + \int_{t_1}^{t_2} \langle g(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt. \end{aligned}$$

It remains to analyze the second term of (4.2.25). By the previous remark and using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we get that $\underline{v}_n(t) \to v(t)$ in $L^2(\Omega)$ as $n \to \infty$ for every $t \in [0, T]$. Thanks to the estimate

$$|b(\underline{v}_n(t, x)) \mathbb{C}(x) E\psi(x)| \leq b(1) \|\mathbb{C}\|_{L^\infty(\Omega)} |E\psi(x)| \quad \text{for every } t \in [0, T] \text{ and a.e. } x \in \Omega$$

and the dominated convergence theorem, we conclude that $b(\underline{v}_n) \mathbb{C} E\psi$ converges strongly to $b(v) \mathbb{C} E\psi$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))$. Hence, we obtain

$$\lim_{n \to \infty} \int_{t_1}^{t_2} (b(\underline{v}_n(t)) \mathbb{C} E \overline{u}_n(t), E\psi)_{L^2(\Omega)} \, dt = \int_{t_1}^{t_2} (b(v(t)) \mathbb{C} E u(t), E\psi)_{L^2(\Omega)} \, dt,$$

since $E\overline{u}_n \rightharpoonup Eu$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))$. Therefore, the pair (u, v) solves

$$\begin{aligned} &\int_{t_1}^{t_2} \langle \ddot{u}(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt + \int_{t_1}^{t_2} (b(v(t)) \mathbb{C} E u(t), E\psi)_{L^2(\Omega)} \, dt \\ &= \int_{t_1}^{t_2} (f(t), \psi)_{L^2(\Omega)} \, dt + \int_{t_1}^{t_2} \langle g(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \, dt \end{aligned}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $[t_1,t_2]\subseteq[0,T]$. We fix a countable dense set $\mathscr{D}\subset H^1_{D_1}(\Omega;\mathbb{R}^d)$. By Lebesgue's differentiation theorem, we derive that the pair (u,v) solves (4.1.25) for a.e. $t\in(0,T)$ and for every $\psi\in\mathscr{D}$. Finally, we use the density of $\mathscr{D}$ in $H^1_{D_1}(\Omega;\mathbb{R}^d)$ to conclude that the equation (4.1.25) is satisfied for every $\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d)$. □

In the next lemma we exploit the inequality (4.2.6) to prove (4.1.18).

Lemma 4.2.10. *The pair (u,v) given by Lemma 4.2.7 satisfies for a.e. $t\in(0,T)$ the crack stability condition* (4.1.18)*.*

Proof. For every $j=1,\dots,n$ the pair (u_n^j,v_n^j) satisfies the inequality (4.2.6), that can be rephrased in

$$\begin{aligned}&\mathscr{E}(\overline{u}_n(t),v^*)-\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))+\partial\mathscr{H}(\overline{v}_n(t))[v^*-\overline{v}_n(t)]\\&\quad+\sum_{i=0}^k\alpha_i(\nabla^i\dot{v}_n(t),\nabla^i v^*-\nabla^i\overline{v}_n(t))_{L^2(\Omega)}\geq 0\end{aligned}\tag{4.2.26}$$

for a.e. $t\in(0,T)$ and for every $v^*-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $v^*\leq\underline{v}_n(t)$. Given $\chi\in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $\chi\leq 0$, the function $\chi+\overline{v}_n(t)$ is admissible for (4.2.26). After an integration in $[t_1,t_2]\subseteq[0,T]$, we deduce the following inequality

$$\begin{aligned}&\int_{t_1}^{t_2}[\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t))-\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))]\,\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}\partial\mathscr{H}(\overline{v}_n(t))[\chi]\,\mathrm{d}t+\sum_{i=0}^k\alpha_i\int_{t_1}^{t_2}(\nabla^i\dot{v}_n(t),\nabla^i\chi)_{L^2(\Omega)}\,\mathrm{d}t\geq 0.\end{aligned}\tag{4.2.27}$$

Let us send $n\to\infty$. We have

$$\lim_{n\to\infty}\sum_{i=0}^k\alpha_i\int_{t_1}^{t_2}(\nabla^i\dot{v}_n(t),\nabla^i\chi)_{L^2(\Omega)}\,\mathrm{d}t=\sum_{i=0}^k\alpha_i\int_{t_1}^{t_2}(\nabla^i\dot{v}(t),\nabla^i\chi)_{L^2(\Omega)}\,\mathrm{d}t,\tag{4.2.28}$$

since $\dot{v}_n\rightharpoonup\dot{v}$ in $L^2(0,T;H^k(\Omega))$. Moreover $\overline{v}_n\rightharpoonup v$ in $L^2(0,T;H^1(\Omega))$, which implies

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\partial\mathscr{H}(\overline{v}_n(t))[\chi]\,\mathrm{d}t=\int_{t_1}^{t_2}\partial\mathscr{H}(v(t))[\chi]\,\mathrm{d}t.\tag{4.2.29}$$

The function $\phi(x,y,\xi):=\frac{1}{2}[b(y)-b(\chi(x)+y)]\mathbb{C}(x)\xi^{sym}\cdot\xi^{sym}$, $(x,y,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}$, satisfies the assumptions of Ioffe-Olech's theorem (see, e.g., [14, Theorem 3.4]). Thus, for every $t\in[0,T]$ we derive

$$\begin{aligned}\mathscr{E}(u(t),v(t))-\mathscr{E}(u(t),\chi+v(t))&=\int_\Omega\phi(x,v(t,x),Eu(t,x))\,\mathrm{d}x\\&\leq\liminf_{n\to\infty}\int_\Omega\phi(x,\overline{v}_n(t,x),E\overline{u}_n(t,x))\,\mathrm{d}x\\&=\liminf_{n\to\infty}[\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t))-\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))],\end{aligned}$$

since $\overline{v}_n(t)\to v(t)$ in $L^2(\Omega)$ and $E\overline{u}_n(t)\rightharpoonup Eu(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ for every $t\in[0,T]$. By Fatou's lemma, we conclude

$$\begin{aligned}&\int_{t_1}^{t_2}[\mathscr{E}(u(t),v(t))-\mathscr{E}(u(t),\chi+v(t))]\,\mathrm{d}t\\&\leq\int_{t_1}^{t_2}\liminf_{n\to\infty}[\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))-\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t))]\,\mathrm{d}t\\&\leq\liminf_{n\to\infty}\int_{t_1}^{t_2}[\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))-\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t))]\,\mathrm{d}t,\end{aligned}$$

which gives

$$\begin{aligned}&\int_{t_1}^{t_2}[\mathscr{E}(u(t),\chi+v(t))-\mathscr{E}(u(t),v(t))]\,\mathrm{d}t\\&\geq\limsup_{n\to\infty}\int_{t_1}^{t_2}[\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t))-\mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))]\,\mathrm{d}t.\end{aligned}\tag{4.2.30}$$

By combining (4.2.27)–(4.2.30) we obtain the following inequality

$$\int_{t_1}^{t_2}\left[\mathscr{E}(u(t),\chi+v(t))-\mathscr{E}(u(t),v(t))+\partial\mathscr{H}(v(t))[\chi]+\sum_{i=0}^{k}\alpha_i(\nabla^i\dot{v}(t),\nabla^i\chi)_{L^2(\Omega)}\right]\mathrm{d}t\geq 0.$$

We choose now a countable dense set $\mathscr{D}\subset\{\chi\in H^1_{D_2}(\Omega)\cap H^k(\Omega):\chi\leq 0\}$. Thanks to Lebesgue's differentiation theorem for a.e. $t\in(0,T)$ we derive

$$\mathscr{E}(u(t),\chi+v(t))-\mathscr{E}(u(t),v(t))+\partial\mathscr{H}(v(t))[\chi]+\sum_{i=0}^{k}\alpha_i(\nabla^i\dot{v}(t),\nabla^i\chi)_{L^2(\Omega)}\geq 0\tag{4.2.31}$$

for every $\chi\in\mathscr{D}$. Finally, we use a density argument and the dominated convergence theorem to deduce that (4.2.31) is satisfied for every $\chi\in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $\chi\leq 0$. In particular, for a.e. $t\in(0,T)$ we get

$$\mathscr{E}(u(t),v^*)-\mathscr{E}(u(t),v(t))+\partial\mathscr{H}(v(t))[v^*-v(t)]+\sum_{i=0}^{k}\alpha_i(\nabla^i\dot{v}(t),\nabla^i v^*-\nabla^i v(t))_{L^2(\Omega)}\geq 0,$$

for every $v^*-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega)$ with $v^*\leq v(t)$, by taking $\chi:=v^*-v(t)$. This implies the crack stability condition (4.1.18), since the map $v^*\mapsto\mathscr{H}(v^*)$ is convex. □

We conclude this section by showing that the pair (u,v) given by Lemma 4.2.7 satisfies an energy-dissipation inequality. Notice that also for a generalized solution (u,v) the total work $\mathscr{W}_{tot}(u,v;t_1,t_2)$ is well defined for every $t_1,t_2\in[0,T]$. Indeed, we have $u\in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$ and $\dot{u}\in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$, which gives that $u(t)-w_1(t)$ and $\dot{u}(t)$ are uniquely defined for every $t\in[0,T]$ as elements of $H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. Moreover, by combining the weak continuity of u and $\dot{u}$, with the strong continuity of g, w_1, and $\dot{w}_1$, it is easy to see that the function $(t_1,t_2)\to\mathscr{W}_{tot}(t_1,t_2,u,v)$ is continuous.

Lemma 4.2.11. *The pair (u,v) given by Lemma 4.2.7 satisfies for every $t\in[0,T]$ the energy-dissipation inequality*

$$\mathscr{F}(u(t),\dot{u}(t),v(t))+\int_0^t\mathscr{G}(\dot{v}(s))\,\mathrm{d}s\leq\mathscr{F}(u^0,u^1,v^0)+\mathscr{W}_{tot}(u,v;0,t).\tag{4.2.32}$$

Proof. Let g_n, w_n, and w'_n be the piecewise affine interpolants of $\{g_n^j\}_{j=1}^n$, $\{w_n^j\}_{j=1}^n$, and $\{\delta w_n^j\}_{j=1}^n$, respectively, and let $\overline{w}_n,\overline{w}'_n$ and $\underline{w}_n,\underline{w}'_n$ be the backward and the forward interpolants of $\{w_n^j\}_{j=1}^n$ and $\{\delta w_n^j\}_{j=1}^n$, respectively.

For $t=0$ the inequality (4.2.32) trivially holds thanks to our initial conditions (4.1.16). We fix $t\in(0,T]$ and for every $n\in\mathbb{N}$ we consider the unique $j\in\{1,\dots,n\}$ such that $t\in((j-1)\tau_n,j\tau_n]$. As done before, we use the previous interpolants and (4.2.10) to write

$$\begin{aligned}&\mathscr{F}(\overline{u}_n(t),\overline{u}'_n(t),\overline{v}_n(t))+\int_0^{t_n}\mathscr{G}(\dot{v}_n)\,\mathrm{d}s\\&\leq\mathscr{F}(u^0,u^1,v^0)+\int_0^{t_n}[(\overline{f}_n,\overline{u}'_n-\overline{w}'_n)_{L^2(\Omega)}+(b(\underline{v}_n)\mathbb{C}E\overline{u}_n,E\overline{w}'_n)_{L^2(\Omega)}]\,\mathrm{d}s\\&\quad-\int_0^{t_n}[\langle\dot{g}_n,\underline{u}_n-\underline{w}_n\rangle_{H^{-1}_{D_1}(\Omega)}+(\underline{u}'_n,\dot{w}'_n)_{L^2(\Omega)}]\,\mathrm{d}s+\langle\overline{g}_n(t),\overline{u}_n(t)-\overline{w}_n(t)\rangle_{H^{-1}_{D_1}(\Omega)}\\&\quad+(\overline{u}'_n(t),\overline{w}'_n(t))_{L^2(\Omega)}-\langle g(0),u^0-w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)}-(u^1,w_1(0))_{L^2(\Omega)},\end{aligned}\tag{4.2.33}$$

where we have set $t_n := j\tau_n$, and we have neglected the terms D_n^j, which are non negative. It easy to see that the following convergences hold as $n \to \infty$:

$$\overline{f}_n \to f \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \qquad \dot{g}_n \to \dot{g} \quad \text{in } L^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),$$
$$\underline{w}_n \to w_1 \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), \qquad \overline{w}_n' \to \dot{w}_1 \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)),$$
$$\dot{w}_n' \to \ddot{w}_1 \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)).$$

By using also the ones of Lemma 4.2.7 and observing that $t_n \to t$ as $n \to \infty$, we deduce

$$\lim_{n\to\infty} \int_0^{t_n} (\overline{f}_n(s), \overline{u}_n'(s) - \overline{w}_n'(s))_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (f(s), \dot{u}(s) - \dot{w}_1(s))_{L^2(\Omega)}\,\mathrm{d}s, \tag{4.2.34}$$

$$\lim_{n\to\infty} \int_0^{t_n} \langle \dot{g}_n(s), \underline{u}_n(s) - \underline{w}_n(s)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}s = \int_0^t \langle \dot{g}(s), u(s) - w_1(s)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}s, \tag{4.2.35}$$

$$\lim_{n\to\infty} \int_0^{t_n} (\underline{u}_n'(s), \dot{w}_n'(s))_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (\dot{u}(s), \ddot{w}_1(s))_{L^2(\Omega)}\,\mathrm{d}s. \tag{4.2.36}$$

Moreover, the strong continuity of g, w_1, and $\dot{w}_1$ in $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$, $H^1(\Omega;\mathbb{R}^d)$, and $L^2(\Omega;\mathbb{R}^d)$, respectively, and the convergences of Remark 4.2.8, imply

$$\lim_{n\to\infty} \langle \overline{g}_n(t), \overline{u}_n(t) - \overline{w}_n(t)\rangle_{H^{-1}_{D_1}(\Omega)} = \langle g(t), u(t) - w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}, \tag{4.2.37}$$

$$\lim_{n\to\infty} (\overline{u}_n'(t), \overline{w}_n'(t))_{L^2(\Omega)} = (\dot{u}(t), \dot{w}_1(t))_{L^2(\Omega)}. \tag{4.2.38}$$

It is easy to check that $b(\underline{v}_n)\mathbb{C}E\overline{w}_n' \to b(v)\mathbb{C}E\dot{w}_1$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$, thanks to the dominated convergence theorem. By combining it with $E\overline{u}_n \rightharpoonup Eu$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$, we conclude

$$\lim_{n\to\infty} \int_0^{t_n} (b(\underline{v}_n(s))\mathbb{C}E\overline{u}_n(s), E\overline{w}_n'(s))_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (b(v(s))\mathbb{C}Eu(s), E\dot{w}_1(s))_{L^2(\Omega)}\,\mathrm{d}s. \tag{4.2.39}$$

If we now consider the left-hand side of (4.2.33), we get

$$\mathscr{K}(\dot{u}(t)) \le \liminf_{n\to\infty} \mathscr{K}(\overline{u}_n'(t)), \qquad \mathscr{H}(v(t)) \le \liminf_{n\to\infty} \mathscr{H}(\overline{v}_n(t)), \tag{4.2.40}$$

since $\overline{u}_n'(t) \rightharpoonup \dot{u}(t)$ in $L^2(\Omega,\mathbb{R}^d)$ and $\overline{v}_n(t) \rightharpoonup v(t)$ in $H^1(\Omega)$. Furthermore, we have $\dot{v}_n \rightharpoonup \dot{v}$ in $L^2(0,T;H^k(\Omega))$ and $t \le t_n$, which gives

$$\int_0^t \mathscr{G}(\dot{v}(s))\,\mathrm{d}s \le \liminf_{n\to\infty} \int_0^t \mathscr{G}(\dot{v}_n(s))\,\mathrm{d}s \le \liminf_{n\to\infty} \int_0^{t_n} \mathscr{G}(\dot{v}_n(s))\,\mathrm{d}s. \tag{4.2.41}$$

Finally, let us consider the function $\phi(x,y,\xi) := \frac{1}{2}b(y)\mathbb{C}(x)\xi^{sym}\cdot\xi^{sym}$, $(x,y,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}$. As in the previous lemma, the function ϕ satisfies the assumption of Ioffe-Olech's theorem, while $\overline{v}_n(t) \to v(t)$ in $L^2(\Omega)$ and $E\overline{u}_n(t) \rightharpoonup Eu(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. Thus, we obtain

$$\begin{aligned}\mathscr{E}(u(t), v(t)) &= \int_\Omega \phi(x, v(t,x), Eu(t,x))\,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_\Omega \phi(x, \overline{v}_n(t,x), E\overline{u}_n(t,x))\,\mathrm{d}x = \liminf_{n\to\infty} \mathscr{E}(\overline{u}_n(t), \overline{v}_n(t)).\end{aligned} \tag{4.2.42}$$

By combining (4.2.33) with (4.2.34)–(4.2.42) we deduce the inequality (4.2.32) for every $t \in (0,T]$. □

4.3 Proof of the main result

In this section we show that for $k > d/2$ the generalized solution (u, v) given by Lemma 4.2.7 is a weak solution and satisfies the identity (4.1.27). To this aim we need several lemmas: we start by proving that, given a function $v \in H^1(0,T;C^0(\overline{\Omega}))$ satisfying (4.1.17), there exists a unique solution u to equation (4.1.25). As a consequence, we deduce that the mechanical energy associated to u satisfies formula (4.3.20) for *every* $t \in [0,T]$, which guarantees that the function u is more regular in time, namely $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^d)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^d))$. Finally, we use the crack stability condition (4.1.18) and the energy-dissipation inequality (4.2.32) to obtain (4.1.19) from (4.3.20).

Lemma 4.3.1. *Let w_1, f, g, u^0, and u^1 be as in (4.1.7), (4.1.9), and (4.1.10). Let us assume that $\sigma \in H^1(0,T;C^0(\overline{\Omega}))$ satisfies (4.1.17). Then there exists a unique function z which satisfies (4.2.1), (4.2.2), the initial conditions $z(0) = u^0$ and $\dot z(0) = u^1$, and which solves for a.e. $t \in (0,T)$ the following equation:*

$$\langle \ddot z(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)} + (b(\sigma(t))\mathbb{C}Ez(t), E\psi)_{L^2(\Omega)} = (f(t),\psi)_{L^2(\Omega)} + \langle g(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)} \tag{4.3.1}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$.

Proof. To prove the existence of a solution z to (4.3.1), we proceed as before. We fix $n \in \mathbb{N}$ and we define

$$\tau_n := \frac{T}{n}, \quad z_n^0 := u^0, \quad z_n^{-1} := u^0 - \tau_n u^1, \quad \sigma_n^j := \sigma(j\tau_n) \quad \text{for } j = 0,\dots,n.$$

For $j = 1,\dots,n$ we consider the unique solution $z_n^j - w_n^j \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ to

$$(\delta^2 z_n^j, \psi)_{L^2(\Omega)} + (b(\sigma_n^{j-1})\mathbb{C}Ez_n^j, E\psi)_{L^2(\Omega)} = (f_n^j,\psi)_{L^2(\Omega)} + \langle g_n^j,\psi\rangle_{H^{-1}_{D_1}(\Omega)} \tag{4.3.2}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$, where we have set $\delta z_n^j := \frac{1}{\tau_n}(z_n^j - z_n^{j-1})$ for $j = 0,\dots,n$, and $\delta^2 z_n^j := \frac{1}{\tau_n}(\delta z_n^j - \delta z_n^{j-1})$ for $j = 1,\dots,n$. By using $\psi := \tau_n(\delta z_n^j - \delta w_n^j)$ as test function in (4.3.2) and proceeding as in Lemma 4.2.4, we get that the function z_n^j satisfies for $j = 1,\dots,n$

$$\begin{aligned}
&[\mathscr{K}(\delta z_n^j) + \mathscr{E}(z_n^j,\sigma_n^j)] - [\mathscr{K}(\delta z_n^{j-1}) + \mathscr{E}(z_n^{j-1},\sigma_n^{j-1})] - \frac{1}{2}([b(\sigma_n^j) - b(\sigma_n^{j-1})]\mathbb{C}Ez_n^j, Ez_n^j)_{L^2(\Omega)}\\
&\le \tau_n(f_n^j, \delta z_n^j - \delta w_n^j)_{L^2(\Omega)} + \tau_n\langle g_n^j, \delta z_n^j - \delta w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}\\
&\quad + \tau_n(\delta^2 z_n^j, \delta w_n^j)_{L^2(\Omega)} + \tau_n(b(v_n^{j-1})\mathbb{C}Ez_n^j, E\delta w_n^j)_{L^2(\Omega)}.
\end{aligned}$$

In particular, we can sum over $l = 1,\dots,j$ for every $j \in \{1,\dots,n\}$ and use the identities (4.2.15) and (4.2.16) to derive the discrete energy inequality

$$\begin{aligned}
&\mathscr{K}(\delta z_n^j) + \mathscr{E}(z_n^j,\sigma_n^j) - \frac{1}{2}\sum_{l=1}^{j}([b(\sigma_n^l) - b(\sigma_n^{l-1})]\mathbb{C}Ez_n^l, Ez_n^l)_{L^2(\Omega)}\\
&\le \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \sum_{l=1}^{j}\tau_n[(f_n^l, \delta z_n^l - \delta w_n^l)_{L^2(\Omega)} + (b(\sigma_n^{l-1})\mathbb{C}Ez_n^l, E\delta w_n^l)_{L^2(\Omega)}]\\
&\quad - \sum_{l=1}^{j}\tau_n[\langle \delta g_n^l, z_n^{l-1} - w_n^{l-1}\rangle_{H^{-1}_{D_1}(\Omega)} + (\delta z_n^{l-1}, \delta^2 w_n^l)_{L^2(\Omega)}] + \langle g_n^j, z_n^j - w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}\\
&\quad + (\delta z_n^j, \delta w_n^j)_{L^2(\Omega)} - \langle g(0), u^0 - w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)} - (u^1, w_1(0))_{L^2(\Omega)}.
\end{aligned} \tag{4.3.3}$$

Since $\sigma_n^j \leq \sigma_n^{j-1}$ and b is non-decreasing, the last term in the left-hand side is non negative. Hence, by arguing as in Lemma 4.2.5 and in Remark 4.2.6, we can find a constant $C > 0$, independent of n, such that

$$\max_{j=1,\dots,n} \left[\|\delta z_n^j\|_{L^2(\Omega)} + \|z_n^j\|_{H^1(\Omega)}\right] + \sum_{j=1}^n \tau_n \|\delta^2 z_n^j\|^2_{H^{-1}_{D_1}(\Omega)} \leq C.$$

Let z_n, z_n', $\overline{z}_n$, $\overline{z}_n'$, and $\underline{z}_n$, $\underline{z}_n'$ be the piecewise affine, the backward, and the forward interpolants of $\{z_n^j\}_{j=1}^n$ and $\{\delta z_n^j\}_{j=1}^n$, respectively. As in Lemma 4.2.7, the above estimate implies the existence of a subsequence of n, not relabeled, and function z satisfying (4.2.1), (4.2.2) and the initial conditions $z(0) = u^0$ and $\dot z(0) = u^1$, such that the following convergences hold as $n \to \infty$:

$$\begin{aligned}
&z_n \rightharpoonup z \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)), && z_n' \rightharpoonup \dot z \quad \text{in } H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&z_n \to z \quad \text{in } C^0([0,T];L^2(\Omega;\mathbb{R}^d)), && z_n' \to \dot z \quad \text{in } C^0([0,T];H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&\overline{z}_n \rightharpoonup z \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \overline{z}_n', \rightharpoonup \dot z \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\underline{z}_n \rightharpoonup z \quad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \underline{z}_n' \rightharpoonup \dot z \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)).
\end{aligned}$$

Let us now define the backward interpolant $\overline{\sigma}_n$ and the forward interpolant $\underline{\sigma}_n$ of $\{\sigma_n^j\}_{j=1}^n$. By integrating the equation (4.3.2) in the time interval $[t_1,t_2] \subseteq [0,T]$, we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2} \langle \dot z_n'(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2} (b(\underline{\sigma}_n(t))\mathbb{C}E\overline{z}_n(t), E\psi)_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_{t_1}^{t_2} (\overline{f}_n(t),\psi)_{L^2(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2} \langle \overline{g}_n(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t
\end{aligned}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$. Thanks to the fact that $\sigma \in H^1(0,T;C^0(\overline{\Omega}))$ and the previous convergences, we can pass to the limit as $n \to \infty$ as done in Lemma 4.2.9, and we deduce

$$\begin{aligned}
&\int_{t_1}^{t_2} \langle \ddot z(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2} (b(\sigma(t))\mathbb{C}Ez(t), E\psi)_{L^2(\Omega)}\,\mathrm{d}t\\
&= \int_{t_1}^{t_2} (f(t),\psi)_{L^2(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2} \langle g(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t
\end{aligned}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$. By Lebesgue's differentiation theorem and a density argument we can conclude that the function z solves (4.3.1) for a.e. $t \in (0,T)$ and for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$.

To prove the uniqueness, we use the same technique adopted in Theorem 1.2.10 and Lemma 3.4.2. Let z_1 and z_2 be two solutions to (4.3.1) satisfying (4.2.1), (4.2.2), and the initial conditions u^0 and u^1. The function $z =: z_1 - z_2$ belongs to the space

$$L^\infty(0,T;H^1_{D_1}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),$$

and for a.e. $t \in (0,T)$ solves

$$\langle \ddot z(t), \psi\rangle_{H^{-1}_{D_1}(\Omega)} + (b(\sigma(t))\mathbb{C}Ez(t), E\psi)_{L^2(\Omega)} = 0 \quad \text{for every } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d),$$

with initial conditions $z(0) = \dot z(0) = 0$. We fix $s \in (0,T]$, and we consider the function

$$\varphi_s(t) = \begin{cases} -\int_t^s z(r)\,\mathrm{d}r & \text{if } t \in [0,s],\\ 0 & \text{if } t \in [s,T]. \end{cases}$$

Clearly, we have $\varphi_s \in C^0([0,T]; H^1_{D_1}(\Omega;\mathbb{R}^d))$ and $\varphi_s(s) = 0$. Moreover

$$\dot\varphi_s(t) = \begin{cases} z(t) & \text{if } t \in [0,s), \\ 0 & \text{if } t \in (s,T], \end{cases}$$

which implies $\dot\varphi_s \in L^\infty(0,T; H^1_{D_1}(\Omega;\mathbb{R}^d))$. We use $\varphi_s(t)$ as test function in (4.3.1) and we integrate in $[0,s]$ to deduce

$$\int_0^s \langle \ddot z(t), \varphi_s(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t + \int_0^s (b(\sigma(t))\mathbb{C}Ez(t), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t = 0. \tag{4.3.4}$$

By integration by parts, the first term becomes

$$\int_0^s \langle \ddot z(t), \varphi_s(t)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t = -\int_0^s (\dot z(t), z(t))_{L^2(\Omega)}\,\mathrm{d}t = -\frac{1}{2}\|z(s)\|^2_{L^2(\Omega)},$$

since $\varphi_s(s) = \dot z(0) = z(0) = 0$. Moreover, the function $t \mapsto (b(\sigma(t))\mathbb{C}E\varphi_s(t), E\varphi_s(t))_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$, because $\varphi_s \in H^1(0,T; H^1_{D_1}(\Omega;\mathbb{R}^d))$ and $\sigma \in H^1(0,T; C^0(\overline\Omega))$. Hence, we can integrate by parts the second terms of (4.3.4) to obtain

$$\begin{aligned}&\int_0^s (b(\sigma(t))\mathbb{C}E(z(t)), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t \\ &= -\frac{1}{2}\int_0^s (\dot b(\sigma(t))\dot\sigma(t)\mathbb{C}E\varphi_s(t), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t - \frac{1}{2}(b(\sigma(0))\mathbb{C}E\varphi_s(0), E\varphi_s(0))_{L^2(\Omega)},\end{aligned}$$

since $\varphi_s(s) = 0$. These two identities imply that z and φ_s satisfy

$$\|z(s)\|^2_{L^2(\Omega)} + (b(\sigma(0))\mathbb{C}E\varphi_s(0), E\varphi_s(0))_{L^2(\Omega)} = -\int_0^s (\dot b(\sigma(t))\dot\sigma(t)\mathbb{C}E\varphi_s(t), E\varphi_s(t))_{L^2(\Omega)}\,\mathrm{d}t.$$

In particular, we get

$$\begin{aligned}&\|z(s)\|^2_{L^2(\Omega)} + \eta\lambda_0\|E\varphi_s(0)\|^2_{L^2(\Omega)} \\ &\le \dot b(\|\sigma\|_{L^\infty(0,T;C^0(\overline\Omega))})\|\mathbb{C}\|_{L^\infty(\Omega)}\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\|E\varphi_s(t)\|^2_{L^2(\Omega)}\mathrm{d}t,\end{aligned}$$

since $\dot b$ is non-decreasing. Let us define $\zeta(t) := \int_0^t z(r)\,\mathrm{d}r$ for $t \in [0,s]$. Since $\varphi_s(t) = \zeta(t) - \zeta(s)$ for $t \in [0,s]$, we deduce that $\|E\varphi_s(0)\|_{L^2(\Omega)} = \|E\zeta(s)\|_{L^2(\Omega)}$ and

$$\begin{aligned}&\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\|E\varphi_s(t)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\le 2\|E\zeta(s)\|^2_{L^2(\Omega)}\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\,\mathrm{d}t + 2\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\|E\zeta(t)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\le 2\sqrt{s}\|\dot\sigma\|_{L^2(0,T;C^0(\overline\Omega))}\|E\zeta(s)\|^2_{L^2(\Omega)} + 2\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\|E\zeta(t)\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\|z(s)\|^2_{L^2(\Omega)} + \left[\eta\lambda_0 - 2\dot b(\|\sigma\|_{L^\infty(0,T;C^0(\overline\Omega))})\|\mathbb{C}\|_{L^\infty(\Omega)}\|\dot\sigma\|_{L^2(0,T;C^0(\overline\Omega))}\sqrt{s}\right]\|E\zeta(s)\|^2_{L^2(\Omega)} \\ &\le 2\dot b(\|\sigma\|_{L^\infty(0,T;C^0(\overline\Omega))})\|\mathbb{C}\|_{L^\infty(\Omega)}\int_0^s \|\dot\sigma(t)\|_{L^\infty(\Omega)}\|E\zeta(t)\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Let us set

$$t_0 := \left(\frac{\eta\lambda_0}{4\dot b(\|\sigma\|_{L^\infty(0,T;C^0(\overline\Omega))})\|\mathbb{C}\|_{L^\infty(\Omega)}\|\dot\sigma\|_{L^2(0,T;C^0(\overline\Omega))}}\right)^2.$$

By the previous estimate, for every $s \in [0, t_0]$ we derive

$$\begin{aligned}&\|z(s)\|^2_{L^2(\Omega)} + \frac{\eta\lambda_0}{2}\|E\zeta(s)\|^2_{L^2(\Omega)}\\&\leq 2\dot{b}(\|\sigma\|_{L^\infty(0,T;C^0(\overline{\Omega}))})\|\mathbb{C}\|_{L^\infty(\Omega)}\int_0^s \|\dot{\sigma}(t)\|_{L^\infty(\Omega)}\|E\zeta(t)\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Thanks to Gronwall's lemma (see, e.g., [24, Chapitre XVIII, §5, Lemme 1]), this inequality implies that $z(s) = E\zeta(s) = 0$ for every $s \in [0, t_0]$. Since t_0 depends only on $\mathbb{C}$, b, and σ, we can repeat this procedure starting from t_0 and, with a finite number of steps, we obtain that $z = 0$ on the whole interval $[0, T]$. □

Corollary 4.3.2. *Let w_1, f, g, u^0, u^1, and σ be as in Lemma 4.3.1. Then the unique solution z to (4.3.1) with initial conditions $z(0) = u^0$ and $\dot{z}(0) = u^1$ satisfies for every $t \in [0, T]$ the following energy-dissipation inequality*

$$\begin{aligned}&\mathscr{K}(\dot{z}(t)) + \mathscr{E}(z(t), \sigma(t)) - \frac{1}{2}\int_0^t (\dot{b}(\sigma(s))\dot{\sigma}(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s\\&\leq \mathscr{K}(u^1) + \mathscr{E}(u^0, \sigma(0)) + \mathscr{W}_{tot}(z, \sigma; 0, t).\end{aligned} \tag{4.3.5}$$

Proof. For $t = 0$ the inequality (4.3.5) is trivially true, thanks to the initial conditions of z. We fix $t \in (0, T]$ and we write the inequality (4.3.3) as

$$\begin{aligned}&\mathscr{K}(\overline{z}'_n(t)) + \mathscr{E}(\overline{z}_n(t), \overline{\sigma}_n(t)) - \frac{1}{2\tau_n}\int_0^{t_n} ([b(\overline{\sigma}_n) - b(\underline{\sigma}_n)]\mathbb{C}E\overline{z}_n, E\overline{z}_n)_{L^2(\Omega)}\,\mathrm{d}s\\&\leq \mathscr{K}(u^1) + \mathscr{E}(u^0, \sigma(0)) + \int_0^{t_n} [(\overline{f}_n, \overline{z}'_n - \overline{w}'_n)_{L^2(\Omega)} + (b(\underline{\sigma}_n)\mathbb{C}E\overline{z}_n, E\overline{w}'_n)_{L^2(\Omega)}]\,\mathrm{d}s\\&\quad - \int_0^{t_n} [\langle \dot{g}_n, \underline{z}_n - \underline{w}_n\rangle_{H^{-1}_{D_1}(\Omega)} + (\underline{z}'_n, \dot{w}'_n)_{L^2(\Omega)}]\,\mathrm{d}s + \langle \overline{g}_n(t), \overline{z}_n(t) - \overline{w}_n(t)\rangle_{H^{-1}_{D_1}(\Omega)}\\&\quad + (\overline{z}'_n(t), \overline{w}'_n(t))_{L^2(\Omega)} - \langle g(0), u^0 - w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)} - (u^1, w_1(0))_{L^2(\Omega)},\end{aligned} \tag{4.3.6}$$

where $t_n := j\tau_n$, and j is the unique element in $\{1, \dots, n\}$ for which $t \in ((j-1)\tau_n, j\tau_n]$. To pass to the limit as $n \to \infty$ in (4.3.6), we follow the same procedure adopted in Lemma 4.2.11. Notice that $\overline{z}_n(t) \rightharpoonup z(t)$ in $H^1(\Omega; \mathbb{R}^d)$ and $\overline{z}'_n(t) \rightharpoonup \dot{z}(t)$ in $L^2(\Omega; \mathbb{R}^d)$, by arguing as in Remark 4.2.8, while $\overline{\sigma}_n(t) \to \sigma(t)$ in $C^0(\overline{\Omega})$. Hence, we derive

$$\mathscr{K}(\dot{z}(t)) \leq \liminf_{n\to\infty} \mathscr{K}(\overline{z}'_n(t)), \quad \mathscr{E}(z(t), \sigma(t)) \leq \liminf_{n\to\infty} \mathscr{E}(\overline{z}_n(t), \overline{\sigma}_n(t)). \tag{4.3.7}$$

Similarly, we combine the convergences given by the previous lemma, with $\underline{\sigma}_n(s) \to \sigma(s)$ in $C^0(\overline{\Omega})$ for every $s \in [0, T]$ and $t_n \to t$ as $n \to \infty$, to deduce

$$\lim_{n\to\infty} \int_0^{t_n} (\overline{f}_n(s), \overline{z}'_n(s) - \overline{w}'_n(s))_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (f(s), \dot{z}(s) - \dot{w}_1(s))_{L^2(\Omega)}\,\mathrm{d}s, \tag{4.3.8}$$

$$\lim_{n\to\infty} \int_0^{t_n} (b(\underline{\sigma}_n(s))\mathbb{C}E\overline{z}_n(s), E\overline{w}'_n(s))_{L^2(\Omega)}]\,\mathrm{d}s = \int_0^t (b(\sigma(s))\mathbb{C}Ez(s), E\dot{w}(s))_{L^2(\Omega)}\,\mathrm{d}s, \tag{4.3.9}$$

$$\lim_{n\to\infty} \int_0^{t_n} (\underline{z}'_n(s), \dot{w}'_n(s))_{L^2(\Omega)}\,\mathrm{d}s = \int_0^t (\dot{z}(s), \ddot{w}_1(s))_{L^2(\Omega)}\,\mathrm{d}s, \tag{4.3.10}$$

$$\lim_{n\to\infty} \int_0^{t_n} \langle \dot{g}_n(s), \underline{z}_n(s) - \underline{w}_n(s)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}s = \int_0^t \langle \dot{g}(s), z(s) - w_1(s)\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}s, \tag{4.3.11}$$

$$\lim_{n\to\infty} (\overline{z}'_n(t), \overline{w}'_n(t))_{L^2(\Omega)} = (\dot{z}(t), \dot{w}_1(t))_{L^2(\Omega)}, \tag{4.3.12}$$

$$\lim_{n\to\infty} \langle \overline{g}_n(t), \overline{z}_n(t) - \overline{w}_n(t)\rangle_{H^{-1}_{D_1}(\Omega)} = \langle g(t), z(t) - w_1(t)\rangle_{H^{-1}_{D_1}(\Omega)}. \tag{4.3.13}$$

Finally, for a.e. $s \in (0,T)$ we have

$$\left\| \frac{\overline{\sigma}_n(s) - \underline{\sigma}_n(s)}{\tau_n} - \dot{\sigma}(s) \right\|_{L^\infty(\Omega)} \le \frac{1}{\tau_n} \int_{s-\tau_n}^{s+\tau_n} \|\dot{\sigma}(r) - \dot{\sigma}(s)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } n \to \infty, \tag{4.3.14}$$

since $\dot{\sigma} \in L^2(0,T;C^0(\overline{\Omega}))$. Let us fix $s \in (0,T)$ for which (4.3.14) holds. By Lagrange's theorem for every $x \in \Omega$ there exists a point $r_n(s,x) \in [\overline{\sigma}_n(s,x), \underline{\sigma}_n(s,x)]$ such that

$$\frac{b(\overline{\sigma}_n(s,x)) - b(\underline{\sigma}_n(s,x))}{\tau_n} = \dot{b}(r_n(s,x)) \frac{\overline{\sigma}_n(s,x) - \underline{\sigma}_n(s,x)}{\tau_n}.$$

Notice that $r_n(s,x) \to \sigma(s,x)$ as $n \to \infty$ for every $x \in \Omega$. Hence, for a.e. $s \in (0,T)$ we get

$$\lim_{n\to\infty} \frac{b(\overline{\sigma}_n(s,x)) - b(\underline{\sigma}_n(s,x))}{\tau_n} = \dot{b}(\sigma(s,x))\dot{\sigma}(s,x) \quad \text{for every } x \in \Omega.$$

Furthermore, thanks to (4.3.14) there is a constant $C_s > 0$, which may depend on s, but it is independent of n, such that for every $x \in \Omega$

$$\begin{aligned}\left| \frac{b(\overline{\sigma}_n(s,x)) - b(\underline{\sigma}_n(s,x))}{\tau_n} \right| &\le \dot{b}(\|\sigma\|_{L^\infty(0,T;C^0(\overline{\Omega}))}) \left\| \frac{\overline{\sigma}_n(s) - \underline{\sigma}_n(s)}{\tau_n} \right\|_{L^\infty(\Omega)} \\ &\le \dot{b}(\|\sigma\|_{L^\infty(0,T;C^0(\overline{\Omega}))}) C_s.\end{aligned}$$

Therefore, for a.e. $s \in (0,T)$ we can apply the dominated convergence theorem to deduce

$$\frac{b(\overline{\sigma}_n(s)) - b(\underline{\sigma}_n(s))}{\tau_n} \to \dot{b}(\sigma(s))\dot{\sigma}(s) \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty.$$

The function $\phi(x,y,\xi) := \frac{1}{2}|y|\,\mathbb{C}(x)\xi^{sym} \cdot \xi^{sym}$, $(x,y,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{d\times d}$, satisfies the assumptions of Ioffe-Olech's theorem, while $E\overline{z}_n(s) \rightharpoonup Ez(s)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ for every $s \in [0,T]$. Then, we have

$$\begin{aligned}&-\frac{1}{2}(\dot{b}(\sigma(s))\dot{\sigma}(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)} \\ &= \int_\Omega \phi(x, \dot{b}(\sigma(s))\dot{\sigma}(s,x), Ez(s,x))\,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_\Omega \phi\left(x, \frac{b(\overline{\sigma}_n(s,x)) - b(\underline{\sigma}_n(s,x))}{\tau_n}, E\overline{z}_n(s,x)\right)\mathrm{d}x \\ &= \liminf_{n\to\infty} \left[-\frac{1}{2\tau_n}([b(\overline{\sigma}_n(s)) - b(\underline{\sigma}_n(s))]\mathbb{C}E\overline{z}_n(s), E\overline{z}_n(s))_{L^2(\Omega)} \right]\end{aligned}$$

for a.e. $s \in (0,T)$, being $b(\overline{\sigma}_n(s)) \le b(\underline{\sigma}_n(s))$ in Ω. In particular, thanks to Fatou's lemma we get

$$\begin{aligned}&-\frac{1}{2}\int_0^t (\dot{b}(\sigma(s))\dot{\sigma}(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &\le \int_0^t \liminf_{n\to\infty} \left[-\frac{1}{2\tau_n}([b(\overline{\sigma}_n(s)) - b(\underline{\sigma}_n(s))]\mathbb{C}E\overline{z}_n(s), E\overline{z}_n(s))_{L^2(\Omega)} \right] \mathrm{d}s \\ &\le \liminf_{n\to\infty} \left[-\frac{1}{2\tau_n} \int_0^{t_n} ([b(\overline{\sigma}_n(s)) - b(\underline{\sigma}_n(s))]\mathbb{C}E\overline{z}_n(s), E\overline{z}_n(s))_{L^2(\Omega)}\,\mathrm{d}s \right],\end{aligned} \tag{4.3.15}$$

since $t \le t_n$. By combining (4.3.6)–(4.3.13) with (4.3.15) we deduce the inequality (4.3.5) for every $t \in (0,T]$. □

The other inequality, at least for a.e. $t \in (0,T)$, is a consequence of equation (4.3.1).

Lemma 4.3.3. *Let w_1, f, g, u^0, u^1, and σ be as in Lemma 4.3.1. Then the unique solution z to (4.3.1) with initial conditions $z(0) = u^0$ and $\dot z(0) = u^1$ satisfies for a.e. $t \in (0,T)$*

$$\begin{aligned} &\mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t)) - \frac{1}{2}\int_0^t (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &\geq \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \mathscr{W}_{tot}(z,\sigma;0,t). \end{aligned} \tag{4.3.16}$$

Proof. It is enough to proceed as done in Lemma 4.1.8, by using Lebesgue's differentiation theorem and the fact that $z \in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$ and $\dot z \in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$. This ensures that z satisfies

$$\begin{aligned} &\mathscr{K}(\dot z(t_2)) + \mathscr{E}(z(t_2),\sigma(t_2)) - \frac{1}{2}\int_{t_1}^{t_2} (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &= \mathscr{K}(\dot z(t_1)) + \mathscr{E}(z(t_1),\sigma(t_1)) + \mathscr{W}_{tot}(z,\sigma;t_1,t_2) \end{aligned}$$

for a.e. $t_1, t_2 \in (0,T)$ with $t_1 < t_2$. Since the right-hand side is lower semicontinuous with respect to t_1, while the left-hand side is continuous, sending $t_1 \to 0^+$ we deduce (4.3.16). □

By combining the two previous results we obtain that the solution z to (4.3.1) satisfies

$$\begin{aligned} &\mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t)) - \frac{1}{2}\int_0^t (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &= \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \mathscr{W}_{tot}(z,\sigma;0,t). \end{aligned} \tag{4.3.17}$$

for a.e. $t \in (0,T)$. Actually, this is true for every time, as shown in the following lemma.

Lemma 4.3.4. *Let w_1, f, g, u^0, u^1, and σ be as in Lemma 4.3.1. Then the unique solution z to (4.3.1) with initial conditions $z(0) = u^0$ and $\dot z(0) = u^1$ satisfies equality (4.3.17) for every $t \in [0,T]$. In particular, the function $t \mapsto \mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t))$ is continuous from $[0,T]$ to $\mathbb{R}$ and*

$$z \in C^0([0,T];H^1(\Omega;\mathbb{R}^d)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^d)). \tag{4.3.18}$$

Proof. We may assume that σ, w_1, f, and g are defined on $[0,2T]$ and satisfy the hypotheses of Lemma 4.3.1 with T replaced by $2T$. As for w_1 and σ, we can set $w_1(t) := 2w_1(T) - w_1(2T-t)$ and $\sigma(t) := \sigma(T)$ for $t \in (T,2T]$, respectively. By Lemma 4.3.1, the solution z on $[0,T]$ can be extended to a solution on $[0,2T]$ still denoted by z. Thanks to Corollary 4.3.2 and Lemma 4.3.3, the function z satisfies equality (4.3.17) for a.e. $t \in (0,2T)$, and inequality (4.3.5) for every $t \in [0,2T]$. By contradiction assume the existence of a point $t_0 \in [0,T]$ such that

$$\begin{aligned} &\mathscr{K}(\dot z(t_0)) + \mathscr{E}(z(t_0),\sigma(t_0)) - \frac{1}{2}\int_0^{t_0} (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &< \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \mathscr{W}_{tot}(z,\sigma;0,t_0). \end{aligned}$$

We have $z(t_0) - w(t_0) \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $\dot z(t_0) \in L^2(\Omega;\mathbb{R}^d)$, since $z \in C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$ and $\dot z \in C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$. Then we can consider the solution z_0 to (4.3.1) in $[t_0,2T]$ with these initial conditions. The function defined by z in $[0,t_0]$ and z_0 in $[t_0,2T]$ is still a solution to (4.3.1) in $[0,2T]$ and so, by uniqueness, we have $z = z_0$ in $[t_0,2T]$. Furthermore, in view of (4.3.5) we deduce

$$\begin{aligned} &\mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t)) - \frac{1}{2}\int_{t_0}^t (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s \\ &\leq \mathscr{K}(z(t_0)) + \mathscr{E}(z(t_0),\sigma(t_0)) + \mathscr{W}_{tot}(z,\sigma;t_0,t) \end{aligned}$$

for every $t \in [t_0, 2T]$. By combining the last two inequalities, we get

$$\begin{aligned}
&\mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t)) - \frac{1}{2}\int_0^t (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s\\
&\le \mathscr{K}(z(t_0)) + \mathscr{E}(z(t_0),\sigma(t_0)) + \mathscr{W}_{tot}(z,\sigma;t_0,t) - \frac{1}{2}\int_0^{t_0} (\dot b(\sigma(s))\dot\sigma(s)\mathbb{C}Ez(s), Ez(s))_{L^2(\Omega)}\,\mathrm{d}s\\
&< \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \mathscr{W}_{tot}(z,\sigma;0,t_0) + \mathscr{W}_{tot}(z,\sigma;t_0,t)\\
&= \mathscr{K}(u^1) + \mathscr{E}(u^0,\sigma(0)) + \mathscr{W}_{tot}(z,\sigma;0,t)
\end{aligned}$$

for every $t \in [t_0, 2T]$, which contradicts (4.3.17). Therefore, equality (4.3.17) holds for every $t \in [0, T]$, which implies the continuity of the function $t \mapsto \mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t))$ from $[0,T]$ to $\mathbb{R}$.

Let us now prove (4.3.18). We fix $t_0 \in [0,T]$ and we consider a sequence of points $\{t_m\}_m$ converging to t_0 as $m \to \infty$. Since $z \in C^0_w([0,T]; H^1(\Omega;\mathbb{R}^d))$ and $\dot z \in C^0_w([0,T]; L^2(\Omega;\mathbb{R}^d))$, we have

$$\mathscr{K}(\dot z(t_0)) \le \liminf_{m\to\infty} \mathscr{K}(\dot z(t_m)), \quad \mathscr{E}(z(t_0),\sigma(t_0)) \le \liminf_{m\to\infty} \mathscr{E}(z(t_m),\sigma(t_0)).$$

Moreover, $\sigma \in C^0([0,T]; C^0(\overline\Omega))$ and $b \in C^1(\mathbb{R})$, which implies as $m \to \infty$

$$\begin{aligned}
&|\mathscr{E}(z(t_m),\sigma(t_0)) - \mathscr{E}(z(t_m),\sigma(t_m))|\\
&\le \frac{1}{2}\dot b(\|\sigma\|_{L^\infty(0,T;C^0(\overline\Omega))})\|\mathbb{C}\|_{L^\infty(\Omega)}\|Ez\|^2_{L^\infty(0,T;L^2(\Omega))}\|\sigma(t_0)-\sigma(t_m)\|_{L^\infty(\Omega)} \to 0.
\end{aligned}$$

In particular, we deduce

$$\mathscr{E}(z(t_0),\sigma(t_0)) \le \liminf_{m\to\infty} \mathscr{E}(z(t_m),\sigma(t_m)).$$

The above inequalities and the continuity of $t \mapsto \mathscr{K}(\dot z(t)) + \mathscr{E}(z(t),\sigma(t))$ gives

$$\begin{aligned}
\mathscr{K}(\dot z(t_0)) + \mathscr{E}(z(t_0),\sigma(t_0)) &\le \liminf_{m\to\infty} \mathscr{K}(\dot z(t_m)) + \liminf_{m\to\infty} \mathscr{E}(z(t_m),\sigma(t_m))\\
&\le \lim_{m\to\infty} [\mathscr{K}(\dot z(t_m)) + \mathscr{E}(z(t_m),\sigma(t_m))]\\
&= \mathscr{K}(\dot z(t_0)) + \mathscr{E}(z(t_0),\sigma(t_0)),
\end{aligned}$$

which implies the continuity of $t \mapsto \mathscr{K}(\dot z(t))$ and $t \mapsto \mathscr{E}(z(t),\sigma(t))$ in $t_0 \in [0,T]$. In particular, we derive that the functions $t \mapsto \|\dot z(t)\|_{L^2(\Omega)}$ and $t \mapsto \|z(t)\|_{H^1(\Omega)}$ are continuous from $[0,T]$ to $\mathbb{R}$. By combining this fact with the weak continuity of $\dot z$ and z, we get (4.3.18). □

We are now in a position to prove Theorem 4.1.5.

Proof of Theorem 4.1.5. By Lemmas 4.2.9 and 4.2.10, there exists a generalized solution (u,v) to (4.1.12)–(4.1.15) satisfying the initial conditions (4.1.16), the irreversibility condition (4.1.17), and the crack stability condition (4.1.18). Clearly, the function v satisfies (4.1.23), since $k \ge 1$. Moreover, the function $v = \sigma$ is admissible in Lemmas 4.3.1 and 4.3.4, since $H^k(\Omega) \hookrightarrow C^0(\overline\Omega)$. Therefore, $u = z$ satisfies (4.1.21), which gives that (u,v) is a weak solution to (4.1.12)–(4.1.15).

It remains to prove that (u,v) satisfies the dynamic energy-dissipation balance (4.1.19). As observed in Remark 4.1.9, for $k > d/2$ the crack stability condition (4.1.18) is equivalent to the variational inequality (4.1.39) for a.e. $t \in (0,T)$ and the function $\dot v(t) \in H^k(\Omega)$ is admissible in (4.1.39). Therefore, we have

$$\partial_v \mathscr{E}(u(t),v(t))[\dot v(t)] + \partial\mathscr{H}(v(t))[\dot v(t)] + \mathscr{G}(\dot v(t)) \ge 0 \quad \text{for a.e. } t \in (0,T).$$

By integrating the above inequality in $[0,t_0]$ for every $t_0\in[0,T]$, we get

$$\int_0^{t_0}\partial_v\mathscr{E}(u(t),v(t))[\dot v(t)]\,\mathrm{d}t+\mathscr{H}(v(t_0))-\mathscr{H}(v^0)+\int_0^{t_0}\mathscr{G}(\dot v(t))\,\mathrm{d}t\ge 0. \tag{4.3.19}$$

Thanks to Lemma 4.3.4, for every $t_0\in[0,T]$ the pair (u,v) satisfies

$$\begin{aligned}&\mathscr{K}(\dot u(t_0))+\mathscr{E}(u(t_0),v(t_0))-\frac12\int_0^{t_0}(\dot b(v(t))\dot v(t)\mathbb{C}Eu(t),Eu(t))_{L^2(\Omega)}\,\mathrm{d}t\\&=\mathscr{K}(u^1)+\mathscr{E}(u^0,v^0)+\mathscr{W}_{tot}(u,v;0,t_0).\end{aligned} \tag{4.3.20}$$

Hence, by combining (4.3.19) and (4.3.20), we deduce

$$\mathscr{F}(u(t_0),\dot u(t_0),v(t_0))+\int_0^{t_0}\mathscr{G}(\dot v(t))\,\mathrm{d}t\ge\mathscr{F}(u^0,u^1,v^0)+\mathscr{W}_{tot}(u,v;0,t_0)$$

for every $t_0\in[0,T]$. This inequality, together with (4.2.32), implies (4.1.19) and concludes the proof. □

4.4 The case without dissipative terms

We conclude the chapter by analyzing the dynamic phase-field model of crack propagation without dissipative terms. Given w_1, w_2, f, g, u^0, u^1, and v^0 satisfying (4.1.7)–(4.1.11) and

$$v^0\in\arg\min\{\mathscr{E}(u^0,v^*)+\mathscr{H}(v^*):v^*-w_2\in H^1_{D_2}(\Omega),\ v^*\le v^0\text{ in }\Omega\}, \tag{4.4.1}$$

we search a pair (u,v) which solves the elastodynamics system (4.1.12) with boundary and initial conditions (4.1.13)–(4.1.16), the *irreversibility condition* (4.1.17), and the following *crack stability condition* for every $t\in[0,T]$

$$\mathscr{E}(u(t),v(t))+\mathscr{H}(v(t))\le\mathscr{E}(u(t),v^*)+\mathscr{H}(v^*) \tag{4.4.2}$$

among all $v^*-w_2\in H^1_{D_2}(\Omega)$ with $v^*\le v(t)$.

Remark 4.4.1. We need the compatibility conditions (4.4.1) for the initial data (u^0,v^0), since we want that (4.4.2) is satisfied for every time. Notice that, given $u^0\in H^1(\Omega;\mathbb{R}^d)$, an admissible v^0 can be constructed by minimizing $v^*\mapsto\mathscr{E}(u^0,v^*)+\mathscr{H}(v^*)$ among all $v^*-w_2\in H^1_{D_2}(\Omega)$ with $v^*\le 1$ in Ω.

In this section we consider the following notion of solution, which is a slightly modification of Definition 4.2.1.

Definition 4.4.2. Let w_1, w_2, f, and g be as in (4.1.7)–(4.1.9). The pair (u,v) is a *generalized solution* to (4.1.12)–(4.1.15) if

$$u\in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \tag{4.4.3}$$

$$u(t)-w_1(t)\in H^1_{D_1}(\Omega;\mathbb{R}^d)\text{ for every }t\in[0,T], \tag{4.4.4}$$

$$v\colon[0,T]\to H^1(\Omega)\text{ with }v\in L^\infty(0,T;H^1(\Omega)), \tag{4.4.5}$$

$$v(t)-w_2\in H^1_{D_2}(\Omega)\text{ and }v(t)\le 1\text{ in }\Omega\text{ for every }t\in[0,T], \tag{4.4.6}$$

and for a.e. $t\in(0,T)$ equation (4.1.25) holds.

Remark 4.4.3. By exploiting the regularity properties (4.4.3), we deduce that u belongs to $C^0_w([0,T];H^1(\Omega;\mathbb{R}^d))$, while $\dot u$ is an element of $C^0_w([0,T];L^2(\Omega;\mathbb{R}^d))$. Therefore, it makes sense to evaluate u and $\dot u$ at time 0. On the other hand, we require v to be defined pointwise for every $t\in[0,T]$, so that we can consider its precise value at 0. Hence, the initial condition (4.1.16) are well defined.

The main result of this section is the following theorem.

Theorem 4.4.4. *Assume that w_1, w_2, f, g, u^0, u^1, and v^0 satisfy (4.1.7)–(4.1.11) and (4.4.1). Then there exists a generalized solution (u,v) to problem (4.1.12)–(4.1.15) satisfying the initial condition (4.1.16), the irreversibility condition (4.1.17), and the crack stability condition (4.4.2). Moreover, the pair (u,v) satisfies for every $t \in [0,T]$ the following energy-dissipation inequality*

$$\mathscr{F}(u(t),\dot{u}(t),v(t)) \leq \mathscr{F}(u^0,u^1,v^0) + \mathscr{W}_{tot}(u,v;0,t). \tag{4.4.7}$$

Finally, if $w_2 \geq 0$ on $\partial_{D_2}\Omega$, $v^0 \geq 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then we can take $v(t) \geq 0$ in Ω for every $t \in [0,T]$.

Remark 4.4.5. By choosing $b(s) = (s \vee 0) + \eta$ for $s \in \mathbb{R}$ we deduce the existence of a dynamic phase-field evolution (u,v) satisfying (D_1) and (D_2), since we can take $v(t) \geq 0$ in Ω and

$$\int_\Omega [(v(x) \vee 0)^2 + \eta]\mathbb{C}(x)Eu(x) \cdot Eu(x)\,\mathrm{d}x \leq \int_\Omega [(v(x))^2 + \eta]\mathbb{C}(x)Eu(x) \cdot Eu(x)\,\mathrm{d}x$$

for every $u \in H^1(\Omega;\mathbb{R}^d)$ and $v \in H^1(\Omega)$. Without adding a dissipative term to the model, we are not able to show the dynamic energy-dissipation balance (D_3). However, we can always select a solution (u,v) which satisfies (4.4.7) for every $t \in [0,T]$.

To prove Theorem 4.4.4 we perform a time discretization, as done in the previous sections. From now on we assume that w_1, w_2, f, g, u^0, u^1, and v^0 satisfy (4.1.7)–(4.1.11) and (4.4.1). We fix $n \in \mathbb{N}$ and for every $j = 1,\dots,n$ we define inductively:

(i) $u_n^j - w_n^j \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ is the minimizer of

$$u^* \mapsto \frac{1}{2\tau_n^2}\left\|u^* - 2u_n^{j-1} - u_n^{j-2}\right\|^2_{L^2(\Omega)} + \mathscr{E}(u^*,v_n^{j-1}) - (f_n^j,u^*)_{L^2(\Omega)} - \langle g_n^j, u^* - w_n^j\rangle_{H^{-1}_{D_1}(\Omega)}$$

among every $u^* - w_n^j \in H^1_{D_1}(\Omega;\mathbb{R}^d)$;

(ii) $v_n^j - w_2 \in H^1_{D_2}(\Omega)$ with $v_n^j \leq v_n^{j-1}$ is the minimizer of

$$v^* \mapsto \mathscr{E}(u_n^j,v^*) + \mathscr{H}(v^*)$$

among every $v^* - w_2 \in H^1_{D_2}(\Omega)$ with $v^* \leq v_n^{j-1}$.

As before, for every $j = 1,\dots,n$ there exists a unique pair $(u_n^j,v_n^j) \in H^1(\Omega;\mathbb{R}^d) \times H^1(\Omega)$ solution to problems (i) and (ii). Moreover, the function u_n^j solves (4.2.5), while the function v_n^j satisfies

$$\mathscr{E}(u_n^j,v^*) - \mathscr{E}(u_n^j,v_n^j) + \partial\mathscr{H}(v_n^j)[v^* - v_n^j] \geq 0 \tag{4.4.8}$$

among all $v^* - w_2 \in H^1_{D_2}(\Omega)$ with $v^* \leq v_n^{j-1}$, arguing as in Lemma 4.2.3. In particular, if $w_2 \geq 0$ on $\partial_{D_2}\Omega$, $v^0 \geq 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then for every $j = 1,\dots,n$ we can use $v_n^j \vee 0 \in H^1(\Omega)$ as a competitor in (ii) to derive that $v_n^j = v_n^j \vee 0 \geq 0$ in Ω.

Lemma 4.4.6. *The family $\{(u_n^j,v_n^j)\}_{j=1}^n$, solution to (i) and (ii), satisfies for $j = 1,\dots,n$ the discrete energy inequality*

$$\begin{aligned}
&\mathscr{F}(u_n^j,\delta u_n^j,v_n^j) + \sum_{l=1}^j \tau_n^2 D_n^l\\
&\leq \mathscr{F}(u^0,u^1,v^0) + \sum_{l=1}^j \tau_n[(f_n^l,\delta u_n^l - \delta w_n^l)_{L^2(\Omega)} + (b(v_n^{l-1})\mathbb{C}Eu_n^l, E\delta w_n^l)_{L^2(\Omega)}]\\
&\quad - \sum_{l=1}^j \tau_n[(\delta u_n^{l-1},\delta^2 w_n^l)_{L^2(\Omega)} - \langle \delta g_n^l, u_n^{l-1} - w_n^{i-1}\rangle_{H^{-1}_{D_1}(\Omega)}] + (\delta u_n^j,\delta w_n^j)_{L^2(\Omega)}\\
&\quad + \langle g_n^j, u_n^j - w_n^j\rangle_{H^{-1}_{D_1}(\Omega)} - (u^1,\dot{w}_1(0))_{L^2(\Omega)} - \langle g(0), u^0 - w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)}.
\end{aligned}$$

In particular, there exists a constant $C>0$, independent of n, such that

$$\max_{j=1,\dots,n}\left[\|\delta u_n^j\|_{L^2(\Omega)}+\|u_n^j\|_{H^1(\Omega)}+\|v_n^j\|_{H^1(\Omega)}\right]+\sum_{j=1}^n\tau_n\|\delta^2u_n^j\|^2_{H^{-1}_{D_1}(\Omega)}+\sum_{j=1}^n\tau_n^2D_n^j\le C. \quad (4.4.9)$$

Proof. It is enough to proceed as in Lemmas 4.2.4 and 4.2.5, and Remark 4.2.6. □

As done in Section 4.2, we use the family $\{(u_n^j,v_n^j)\}_{j=1}^n$ and the estimate (4.4.9) to construct a generalized solution (u,v) to (4.1.12)–(4.1.16). Let u_n, u_n', $\overline{u}_n$, $\overline{u}_n'$, and $\underline{u}_n$, $\underline{u}_n'$ be the piecewise affine, the backward, and the forward interpolants of $\{u_n^j\}_{j=1}^n$ and $\{\delta u_n^j\}_{j=1}^n$, respectively. Moreover, we consider the backward interpolant $\overline{v}_n$ and the forward interpolant $\underline{v}_n$ of $\{v_n^j\}_{j=1}^n$.

Before passing to the limit as $n\to\infty$, we recall the following Helly's type result for vector-valued functions.

Lemma 4.4.7. *Let $[a,b]\subset\mathbb{R}$ and let $\varphi_m\colon[a,b]\to L^2(\Omega)$, $m\in\mathbb{N}$, be a sequence of functions satisfying*

$$\varphi_m(s)\le\varphi_m(t)\quad\text{in }\Omega\quad\text{for every }a\le s\le t\le b\text{ and }m\in\mathbb{N}.$$

Assume there exists a constant C, independent of m, such that

$$\|\varphi_m(t)\|_{L^2(\Omega)}\le C\quad\text{for every }t\in[a,b]\text{ and }m\in\mathbb{N}.$$

Then there is a subsequence of m, not relabeled, and a function $\varphi\colon[a,b]\to L^2(\Omega)$ such that for every $t\in[a,b]$

$$\varphi_m(t)\rightharpoonup\varphi(t)\quad\text{in }L^2(\Omega)\quad\text{as }m\to\infty.$$

Moreover, we have $\|\varphi(t)\|_{L^2(\Omega)}\le C$ for every $t\in[a,b]$ and

$$\varphi(s)\le\varphi(t)\quad\text{in }\Omega\quad\text{for every }a\le s\le t\le b. \quad (4.4.10)$$

Proof. Let us consider a countable dense set $\mathscr{D}\subset\{\chi\in L^2(\Omega):\chi\ge0\}$ and let us fix $\chi\in\mathscr{D}$. For every $m\in\mathbb{N}$ the map $t\mapsto\int_\Omega\varphi_m(t,x)\chi(x)\,\mathrm{d}x$ is non-decreasing and uniformly bounded in $[a,b]$, since

$$\left|\int_\Omega\varphi_m(t,x)\chi(x)\,\mathrm{d}x\right|\le C\|\chi\|_{L^2(\Omega)}\quad\text{for every }t\in[a,b]. \quad (4.4.11)$$

By applying the Helly's theorem, we can find a subsequence of m, not relabeled, and a function $a_\chi\colon[a,b]\to\mathbb{R}$ such that for every $t\in[a,b]$

$$\int_\Omega\varphi_m(t,x)\chi(x)\,\mathrm{d}x\to a_\chi(t)\quad\text{as }m\to\infty.$$

Moreover, thanks to a diagonal argument, the subsequence of m can be chosen independent of $\chi\in\mathscr{D}$.

We now fix $t\in[a,b]$ and $\chi\in L^2(\Omega)$ with $\chi\ge0$. Given $h>0$, there is $\chi_h\in\mathscr{D}$ such that $\|\chi-\chi_h\|_{L^2(\Omega)}<h$. Moreover, thanks to the previous convergence we can find $\bar m\in\mathbb{N}$ such that for every $m,l>\bar m$

$$\left|\int_\Omega\varphi_m(t,x)\chi_h(x)\,\mathrm{d}x-\int_\Omega\varphi_j(t,x)\chi_h(x)\,\mathrm{d}x\right|<h.$$

Therefore, the sequence $\int_\Omega\varphi_m(t,x)\chi(x)\,\mathrm{d}x$, $m\in\mathbb{N}$, is Cauchy in $\mathbb{R}$. Indeed, for every $h>0$ there exists $\bar m\in\mathbb{N}$ such that for every $m,l>\bar m$

$$\begin{aligned}&\left|\int_\Omega\varphi_m(t,x)\chi(x)\,\mathrm{d}x-\int_\Omega\varphi_l(t,x)\chi(x)\,\mathrm{d}x\right|\\&\le 2C\|\chi-\chi_h\|_{L^2(\Omega)}+\left|\int_\Omega\varphi_m(t,x)\chi_h(x)\,\mathrm{d}x-\int_\Omega\varphi_j(t,x)\chi_h(x)\,\mathrm{d}x\right|\\&<(2C+1)h.\end{aligned}$$

Hence, we can find an element $a_\chi(t) \in \mathbb{R}$ such that

$$\int_\Omega \varphi_m(t,x)\chi(x)\,\mathrm{d}x \to a_\chi(t) \quad \text{as } m \to \infty.$$

In particular, for every $t \in [a,b]$ and $\chi \in L^2(\Omega)$ we have as $m \to \infty$

$$\begin{aligned}\int_\Omega \varphi_m(t,x)\chi(x)\,\mathrm{d}x &= \int_\Omega \varphi_m(t,x)\chi_+(x)\,\mathrm{d}x - \int_\Omega \varphi_m(t,x)\chi_-(x)\,\mathrm{d}x\\ &\to a_{\chi_+}(t) - a_{\chi_-}(t) =: a_\chi(t),\end{aligned}$$

where we have set $\chi_+ := \chi \vee 0$ and $\chi_- := (-\chi) \vee 0$. For every $t \in [a,b]$ fixed, let us consider the functional $\zeta(t)\colon L^2(\Omega) \to \mathbb{R}$ defined by

$$\zeta(t)(\chi) := a_\chi(t) \quad \text{for } \chi \in L^2(\Omega).$$

We have that $\zeta(t)$ linear and continuous on $L^2(\Omega)$. Indeed, by (4.4.11) we deduce

$$|\zeta(t)(\chi)| \le C\|\chi\|_{L^2(\Omega)} \quad \text{for every } \chi \in L^2(\Omega).$$

Hence, Riesz's representation theorem implies the existence of a function $\varphi(t) \in L^2(\Omega)$ such that

$$a_\chi(t) = \int_\Omega \varphi(t,x)\chi(x)\,\mathrm{d}x \quad \text{for every } \chi \in L^2(\Omega).$$

In particular, for every $t \in [a,b]$ we deduce that $\varphi_m(t) \rightharpoonup \varphi(t)$ in $L^2(\Omega)$ as $m \to \infty$ and $\|\varphi(t)\|_{L^2(\Omega)} \le C$. Finally observe that $\{\chi \in L^2(\Omega) : \chi \ge 0\}$ is a weakly closed subset of $L^2(\Omega)$. Therefore, we derive (4.4.10), since $\varphi_m(t) - \varphi_m(s) \rightharpoonup \varphi(t) - \varphi(s)$ in $L^2(\Omega)$ as $m \to \infty$ and $\varphi_m(t) - \varphi_m(s) \in \{\chi \in L^2(\Omega) : \chi \ge 0\}$ for every $m \in \mathbb{N}$ and $a \le s \le t \le b$. □

Lemma 4.4.8. *There exist a subsequence of n, not relabeled, and two functions*

$$\begin{aligned}&u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\ &v\colon [0,T] \to H^1(\Omega) \textit{ with } v \in L^\infty(0,T;H^1(\Omega)),\end{aligned}$$

such that as $n \to \infty$

$$\begin{aligned}
&u_n \rightharpoonup u \quad \textit{in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)), && u_n' \rightharpoonup \dot u \quad \textit{in } H^1(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&u_n \to u \quad \textit{in } C^0([0,T];L^2(\Omega;\mathbb{R}^d)), && u_n' \to \dot u \quad \textit{in } C^0([0,T];H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\
&\overline{u}_n \rightharpoonup u \quad \textit{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \overline{u}_n' \rightharpoonup \dot u \quad \textit{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\underline{u}_n \rightharpoonup u \quad \textit{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)), && \underline{u}_n' \rightharpoonup \dot u \quad \textit{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
&\overline{v}_n \to v \quad \textit{in } L^2(0,T;L^2(\Omega)), && \overline{v}_n \rightharpoonup v \quad \textit{in } L^2(0,T;H^1(\Omega)),\\
&\underline{v}_n \to v \quad \textit{in } L^2(0,T;L^2(\Omega)), && \underline{v}_n \rightharpoonup v \quad \textit{in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

Moreover, for every $t \in [0,T]$ as $n \to \infty$ we have

$$\overline{v}_n(t) \to v(t) \quad \textit{in } L^2(\Omega), \qquad \overline{v}_n(t) \rightharpoonup v(t) \quad \textit{in } H^1(\Omega).$$

Proof. The existence of a limit point u and the related convergences can be obtained by arguing as in Lemma 4.2.7. Let us now consider the sequence $\{\overline{v}_n\}_n$. For every $n \in \mathbb{N}$ the functions $\overline{v}_n\colon [0,T] \to L^2(\Omega)$ are non-increasing in $[0,T]$, that is

$$\overline{v}_n(t) \le \overline{v}_n(s) \quad \text{in } \Omega \quad \text{for every } 0 \le s \le t \le T,$$

and, in view of Lemma 4.4.6, there exists $C > 0$, independent of n, such that

$$\|\overline{v}_n(t)\|_{H^1(\Omega)} \le C \quad \text{for every } t \in [0,T] \text{ and } n \in \mathbb{N}. \tag{4.4.12}$$

Therefore, we can apply Lemma 4.4.7. Up to extract a subsequence (not relabeled), we obtain the existence of a non-increasing function $v\colon [0,T] \to L^2(\Omega)$ such that as $n \to \infty$

$$\overline{v}_n(t) \rightharpoonup v(t) \quad \text{in } L^2(\Omega) \quad \text{for every } t \in [0,T].$$

Moreover, by (4.4.12) for every $t \in [0,T]$ we derive that $v(t) \in H^1(\Omega)$ and as $n \to \infty$

$$\overline{v}_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\Omega), \quad \overline{v}_n(t) \to v(t) \quad \text{in } L^2(\Omega),$$

thanks to Rellich's theorem. Notice that the function $v\colon [0,T] \to H^1(\Omega)$ is strongly measurable. Indeed, it is weak measurable, since it is non-increasing, and with values in a separable Hilbert space. In particular, we have $v \in L^\infty(0,T;H^1(\Omega))$, since $\|v(t)\|_{H^1(\Omega)} \le C$ for every $t \in [0,T]$. By the dominated convergence theorem, as $n \to \infty$ we conclude

$$\overline{v}_n \to v \quad \text{in } L^2(0,T;L^2(\Omega)), \quad \overline{v}_n \rightharpoonup v \quad \text{in } L^2(0,T;H^1(\Omega)).$$

Finally, as $n \to \infty$ we have

$$\underline{v}_n \to v \quad \text{in } L^2(0,T;L^2(\Omega)), \quad \underline{v}_n \rightharpoonup v \quad \text{in } L^2(0,T;H^1(\Omega)),$$

since $\underline{v}_n(t) = \overline{v}_n(t-\tau_n)$ for a.e. $t \in (\tau_n, T)$. □

Remark 4.4.9. As pointed out in Remark 4.2.8, for every $t \in [0,T]$ we have as $n \to \infty$

$$\begin{aligned}
&\overline{u}_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d), \quad \overline{u}'_n(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d),\\
&\underline{u}_n(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d), \quad \underline{u}'_n(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d).
\end{aligned}$$

We are now in a position to prove Theorem 4.4.4.

Proof of Theorem 4.4.4. Thanks to the previous lemma there exists a pair (u,v) satisfying (4.4.3)–(4.4.6), since $u_n(t) - w_n(t) \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $\overline{v}_n(t) - w_2 \in H^1_{D_2}(\Omega)$ for every $t \in [0,T]$ and $n \in \mathbb{N}$. Moreover, (u,v) satisfies the irreversibility condition (4.1.17) and the initial conditions (4.1.16), thanks to (4.4.10) and the fact that $u^0 = u_n(0) \rightharpoonup u(0)$ in $H^1(\Omega;\mathbb{R}^d)$, $u^1 = u'_n(0) \rightharpoonup \dot{u}(0)$ in $L^2(\Omega;\mathbb{R}^d)$, and $v^0 = \overline{v}_n(0) \rightharpoonup v(0)$ in $H^1(\Omega)$ as $n \to \infty$.

For every $n \in \mathbb{N}$ and $j = 1,\dots,n$ the pair (u_n^j, v_n^j) solves equation (4.2.5). In particular, by integrating it over the time interval $[t_1,t_2] \subseteq [0,T]$, we deduce

$$\begin{aligned}
&\int_{t_1}^{t_2} \langle \dot{u}'_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t + \int_{t_1}^{t_2} (b(\underline{v}_n(t))\mathbb{C}E\overline{u}_n(t), E\psi)_{L^2(\Omega)} \,\mathrm{d}t\\
&= \int_{t_1}^{t_2} (\overline{f}_n(t), \psi)_{L^2(\Omega)} \,\mathrm{d}t + \int_{t_1}^{t_2} \langle \overline{g}_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t
\end{aligned}$$

for every $\psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$. Let us pass to the limit as $n \to \infty$. We have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{t_1}^{t_2} \langle \dot{u}'_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t = \int_{t_1}^{t_2} \langle \ddot{u}(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t,\\
&\lim_{n\to\infty} \int_{t_1}^{t_2} (\overline{f}_n(t), \psi)_{L^2(\Omega)} \,\mathrm{d}t = \int_{t_1}^{t_2} (f(t), \psi)_{L^2(\Omega)} \,\mathrm{d}t,\\
&\lim_{n\to\infty} \int_{t_1}^{t_2} \langle \overline{g}_n(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t = \int_{t_1}^{t_2} \langle g(t), \psi \rangle_{H^{-1}_{D_1}(\Omega)} \,\mathrm{d}t,
\end{aligned}$$

since $\dot{u}_n' \rightharpoonup \ddot{u}$ in $L^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$, $\overline{g}_n \to g$ in $L^2(0,T;H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$, and $f_n \to f$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ as $n\to\infty$. Moreover, the dominated convergence theorem yields that $b(\underline{v}_n)\mathbb{C}E\psi \to b(v)\mathbb{C}E\psi$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $n\to\infty$, being

$$|b(\underline{v}_n(t,x))\mathbb{C}(x)E\psi(x)| \le b(1)\|\mathbb{C}\|_{L^\infty(\Omega)}|E\psi(x)| \quad \text{for every } t\in[0,T] \text{ and a.e. } x\in\Omega,$$

and $\underline{v}_n \to v$ in $L^2(0,T;L^2(\Omega))$. Therefore, we derive

$$\lim_{n\to\infty}\int_{t_1}^{t_2}(b(\underline{v}_n(t))\mathbb{C}E\overline{u}_n(t),E\psi)_{L^2(\Omega)}\,\mathrm{d}t = \int_{t_1}^{t_2}(b(v(t))\mathbb{C}Eu(t),E\psi)_{L^2(\Omega)}\,\mathrm{d}t,$$

because $E\overline{u}_n \rightharpoonup Eu$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $n\to\infty$. These facts imply that the pair (u,v) solves

$$\begin{aligned}&\int_{t_1}^{t_2}\langle\ddot{u}(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2}(b(v(t))\mathbb{C}Eu(t),E\psi)_{L^2(\Omega)}\,\mathrm{d}t\\ &= \int_{t_1}^{t_2}(f(t),\psi)_{L^2(\Omega)}\,\mathrm{d}t + \int_{t_1}^{t_2}\langle g(t),\psi\rangle_{H^{-1}_{D_1}(\Omega)}\,\mathrm{d}t\end{aligned}$$

for every $\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $[t_1,t_2]\subseteq[0,T]$. By Lebesgue's differentiation theorem and a density argument we hence obtain (4.1.25) for a.e. $t\in(0,T)$.

For $t=0$ the crack stability condition (4.4.2) trivially holds, since (u,v) satisfies the initial conditions (4.1.16) and the compatibility condition (4.4.1). We fix $t\in(0,T]$ and we use the variational inequality (4.4.8) to derive

$$\mathscr{E}(\overline{u}_n(t),v^*) - \mathscr{E}(\overline{u}_n(t),\overline{v}_n(t)) + \partial\mathscr{H}(\overline{v}_n(t))[v^*-\overline{v}_n(t)] \ge 0 \tag{4.4.13}$$

among all $v^*-w_2\in H^1_{D_2}(\Omega)$ with $v^*\le\overline{v}_n(t-\tau_n)$. Given $\chi\in H^1_{D_2}(\Omega)$, with $\chi\le 0$ in Ω, the function $\chi+\overline{v}_n(t)$ is admissible for (4.4.13). Hence, we have

$$\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t)) - \mathscr{E}(\overline{u}_n(t),\overline{v}_n(t)) + \partial\mathscr{H}(\overline{v}_n(t))[\chi] \ge 0.$$

Let us send $n\to\infty$. Since $\overline{v}_n(t)\rightharpoonup v(t)$ in $H^1(\Omega)$, we deduce

$$\lim_{n\to\infty}\partial\mathscr{H}(\overline{v}_n(t))[\chi] = \partial\mathscr{H}(v(t))[\chi].$$

Moreover, $E\overline{u}_n(t)\rightharpoonup Eu(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ and $\overline{v}_n(t)\to v(t)$ in $L^2(\Omega)$ as $n\to\infty$, which implies

$$\mathscr{E}(u(t),\chi+v(t)) - \mathscr{E}(u(t),v(t)) \ge \limsup_{n\to\infty}\,[\mathscr{E}(\overline{u}_n(t),\chi+\overline{v}_n(t)) - \mathscr{E}(\overline{u}_n(t),\overline{v}_n(t))]$$

by Ioffe-Olech's theorem, as in Lemma 4.2.10. If we combine these two results, for every $t\in(0,T]$ we get

$$\mathscr{E}(u(t),\chi+v(t)) - \mathscr{E}(u(t),v(t)) + \partial\mathscr{H}(v(t))[\chi] \ge 0$$

for every $\chi\in H^1_{D_2}(\Omega)$ with $\chi\le 0$ in Ω. This implies (4.4.2), since the map $v^*\mapsto\mathscr{H}(v^*)$ is convex.

It remains to prove the energy-dissipation inequality (4.4.7) for every $t\in[0,T]$. For $t=0$ we have actually the equality, thanks to the initial conditions (4.1.16). We now fix $t\in(0,T]$ and we use (4.3.3) to write

$$\begin{aligned}&\mathscr{F}(\overline{u}_n(t),\overline{u}_n'(t),\overline{v}_n(t))\\ &\le \mathscr{F}(u^0,u^1,v^0) + \int_0^{t_n}[(\overline{f}_n,\overline{u}_n'-\overline{w}_n')_{L^2(\Omega)} + (b(\underline{v}_n)\mathbb{C}E\overline{u}_n,E\overline{w}_n')_{L^2(\Omega)}]\,\mathrm{d}s\\ &\quad - \int_0^{t_n}[\langle\dot{g}_n,\underline{u}_n-\underline{w}_n\rangle_{H^{-1}_{D_1}(\Omega)} + (\underline{u}_n',\dot{w}_n')_{L^2(\Omega)}]\,\mathrm{d}s + \langle\overline{g}_n(t),\overline{u}_n(t)-\overline{w}_n(t)\rangle_{H^{-1}_{D_1}(\Omega)}\\ &\quad + (\overline{u}_n'(t),\overline{w}_n'(t))_{L^2(\Omega)} - \langle g(0),u^0-w_1(0)\rangle_{H^{-1}_{D_1}(\Omega)} - (u^1,w_1(0))_{L^2(\Omega)}\end{aligned}$$

for every $n \in \mathbb{N}$, where t_n is the same number defined in Lemma 4.2.11. By using the fact that $\overline{v}_n(t) \rightharpoonup v(t)$ in $H^1(\Omega)$ as $n \to \infty$, we deduce

$$\mathscr{H}(v(t)) \leq \liminf_{n\to\infty} \mathscr{H}(\overline{v}_n(t)).$$

Similarly, thanks to Ioffe-Olech's theorem we derive

$$\mathscr{E}(u(t), v(t)) \leq \liminf_{n\to\infty} \mathscr{E}(\overline{u}_n(t), \overline{v}_n(t)),$$

since $\overline{v}_n(t) \to v(t)$ in $L^2(\Omega)$ and $E\overline{u}_n(t) \rightharpoonup Eu(t)$ in $L^2(\Omega; \mathbb{R}^{d\times d})$. Finally, we can argue as in Lemma 4.2.11 to derive that the remaining terms converge to $\mathscr{W}_{tot}(u, v; 0, t)$ as $n \to \infty$. By combining the previous results, we deduce (4.4.7) for every $t \in (0, T]$.

Finally, if $w_2 \geq 0$ on $\partial_{D_2}\Omega$, $v^0 \geq 0$ in Ω, and $b(s) = (s \vee 0)^2 + \eta$ for $s \in \mathbb{R}$, then we have $\overline{v}_n(t) \geq 0$ in Ω for every $t \in [0, T]$, which implies $v(t) \geq 0$ in Ω. □

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