An orbit space of a nonlinear involution of S2 × S2 with nonnegative sectional curvature

AN ORBIT SPACE OF A NONLINEAR INVOLUTION OF $S^2 \times S^2$ WITH NONNEGATIVE SECTIONAL CURVATURE

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Abstract: We describe a construction of Riemannian metrics of nonnegative sectional curvature on a closed smooth nonorientable 4-manifold with fundamental group of order two that realizes a homotopy class that was not previously known to contain nonnegatively curved manifolds. The procedure yields new metrics of nonnegative sectional curvature on any 2-sphere bundle with base space the 2-sphere or the real projective plane.

1. Introduction and main result

Let (M, g) be a Riemannian manifold, where we assume the metric g to be smooth unless stated otherwise. We say that the manifold and/or the metric is nonnegatively curved if the sectional curvature of (M, g) satisfies $\sec_g \geq 0$, and consider cut-and-paste constructions of nonnegatively curved manifolds as in the following principle.

Principle A. *Let (U, g_U) and (V, g_V) be compact Riemannian manifolds with nonempty boundaries $(\partial U, g_{\partial U})$ and $(\partial V, g_{\partial V})$ equipped with the induced metrics $g_{\partial U} := g_U|_{\partial U}$ and $g_{\partial V} := g_V|_{\partial V}$, and for which there is an isometry*

$$\varphi : (\partial U, g_{\partial U}) \to (\partial V, g_{\partial V}). \tag{1.1}$$

If $\sec_{g_U} \geq 0$ and $\sec_{g_V} \geq 0$, then the closed manifold

$$M(U, V, \varphi) := (U, g_U) \cup_\varphi (V, g_V) \tag{1.2}$$

has a C^1- Riemannian metric of nonnegative sectional curvature.

Additional care is required to guarantee that the metric obtained in this way on $M(U, V, \varphi)$ is smooth. Principle A is a well-known procedure to equip manifolds with nonnegatively curved metrics; see the surveys of Ziller [20] and Wilking [19] for details on all the known constructions of nonnegatively curved manifolds. Cheeger showed that the connected sums of two compact rank one symmetric spaces have a metric of nonnegative sectional curvature [1]. Grove-Ziller showed that Principle A can be applied to cohomogeneity one G-manifolds with codimension two singular orbits [7, Theorem E], where U and V are tubular neighborhoods $G \times_{K_\pm} D^2$ (for $K_\pm$ isotropy groups) that are determined by the slice theorem. Among the myriad of examples that are within the range of Grove-Ziller's method, one finds orbit spaces of all linear and nonlinear $\mathbb{Z}/2$-involutions on the 5-sphere, i.e., the four closed smooth 5-manifolds that are homotopy equivalent to the real projective 5-space $\mathbb{R}P^5$. These manifolds realize two homeomorphism classes and four diffeomorphism classes. A $\mathbb{Z}/2$-involution $T : M \to M$ is a fixed-point free $\mathbb{Z}/2$-action on M and we say that it is nonlinear if T is not topologically conjugate to a linear one.

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In this short note, we apply Principle A to produce a new example of a nonnegatively curved 4-manifold.

Theorem A. *There exists a closed nonnegatively curved 4-manifold that is the orbit space of a nonlinear orientation-reversing $\mathbb{Z}/2$-involution on $S^2 \times S^2$, which is not homotopy equivalent to any of the known examples of nonnegatively curved 4-manifolds with fundamental group of order two.*

The isometry group of the product of two round 2-spheres

$$\mathrm{Isom}((S^2, g_{S^2}) \times (S^2, g_{S^2})) = (\mathrm{O}(3) \times \mathrm{O}(3)) \rtimes \mathbb{Z}/2 \tag{1.3}$$

contains four conjugacy classes of fixed-point free $\mathbb{Z}/2$ actions up to conjugation [10, Section 12.2]. The orbit spaces of these linear involutions give rise to the four 2-sphere bundles over the real projective plane [10, Section 12.3], and they realize four out of the five homotopy equivalence classes of closed smooth 4-manifolds with fundamental group of order two and whose universal cover is $S^2 \times S^2$ as shown using work of Hambleton-Kreck [8], Kim-Kojima-Raymond [13], and Hambleton-Kreck-Teichner [9]. The following result is an immediate consequence.

Corollary B. *There is a closed smooth nonnegatively curved 4-manifold within each homotopy equivalence class of orbit space of a $\mathbb{Z}/2$-involution on $S^2 \times S^2$.*

The organization of this note is as follows. Several deconstructions and assemblages of 4-manifolds are described in Section 2.1. They cover all 2-sphere bundles over either the 2-sphere or the real projective plane and include the example of Theorem A. Section 4.1 contains a description of the choices of Riemannian metrics. Pairing these metrics with the deconstructions of 4-manifolds yield new C^1-Riemannian metrics of nonnegative sectional curvature on every such 2-sphere bundle as we describe in Section 4.3, and any of these metrics is smooth as we discuss in Section 4.2. The manifold of Theorem A is distinguished from all previously known nonnegatively curved four dimensional examples with fundamental group of order two in Section 2.2 and Section 3.

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2. Surgical constructions and homotopy types of orbit spaces

2.1. **Cut-and-paste constructions of 4-manifolds along 2-spheres and real projective planes: Gluck twists.** It is well-known that the non-trivial S^2-bundle over S^2 is obtained from $S^2 \times S^2$ through an application of a Gluck twist [3] along an embedded 2-sphere

$$S^2 \times \{pt\} \hookrightarrow S^2 \times S^2, \tag{2.1}$$

and we now describe the procedure. A smooth closed simply connected 4-manifold is constructed as

$$M(\varphi) := (D^2 \times S^2) \cup_\varphi (D^2 \times S^2) \tag{2.2}$$

where the diffeomorphism

$$\varphi : S^1 \times S^2 \longrightarrow S^1 \times S^2 \tag{2.3}$$

that is used to identify the boundaries together is either the identity id or the map

$$\varphi_\alpha([t/a], x) = ([t/a], \alpha([t/a]) \cdot x) \tag{2.4}$$

where the map

$$\alpha([t/a]) : S^2 \to S^2 \tag{2.5}$$

for $x \in S^2$ is rotation of the 2-sphere through an angle $[t/a] \in \mathbb{R}/\mathbb{Z} = S^1$ and about the axis that goes through the north and south poles [3, Section 6]. The latter yields an essential map $\alpha : S^1 \to \mathrm{SO}(3)$ [18, Section 16]. Other choices of axis yield diffeomorphic manifolds. Identification of the pieces using the identity map yields the double of the trivial 2-disk bundle $M(\mathrm{id}) = S^2 \times S^2$. A handlebody argument quickly reveals that $M(\varphi_\alpha)$ is diffeomorphic to the nontrivial S^2-bundle over S^2. The latter is known to be diffeomorphic to the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ of a copy of the complex projective plane and a copy of its underlying smooth manifold taken with the opposite orientation.

A similar situation arises in the nonorientable realm and we use it to construct the example in Theorem A. Consider the nonorientable compact 4-manifold

$$D^2 \tilde{\times} \mathbb{R}P^2 = (D^2 \times S^2)/(r, \mathbb{A}) = (D^2 \times S^2)/\mathbb{Z}/2, \tag{2.6}$$

where

$$r : D^2 \to D^2 \tag{2.7}$$

is a rotation by π radians and

$$\mathbb{A} : S^2 \to S^2 \tag{2.8}$$

is the antipodal map.

Lemma 1. *Let $\gamma \subset \mathbb{R}P^4$ be the loop that represents the homotopy class of the generator of $\pi_1(\mathbb{R}P^4) = \mathbb{Z}/2$. The compact nonorientable manifold $D^2 \tilde{\times} \mathbb{R}P^2$ is diffeomorphic to the complement $\mathbb{R}P^4 \backslash \nu(\gamma)$ of a tubular neighborhood $\nu(\gamma)$ of γ.*

Moreover, the boundary is diffeomorphic to the nonorientable 2-sphere bundle over the circle $\partial(D^2 \tilde{\times} \mathbb{R}P^2) = S^2 \tilde{\times} S^1$.

Proof. Deconstruct the 4-sphere as

$$S^4 = \partial D^5 = \partial(D^2 \times D^3) = (D^2 \times S^2) \cup_{\mathrm{id}} (S^1 \times D^3) \tag{2.9}$$

and consider the antipodal involution $\mathbb{A} : S^4 \to S^4$ with orbit space $\mathbb{R}P^4$. The involution $\mathbb{A}$ and the decomposition (2.9) induce a decomposition

$$\mathbb{R}P^4 = (D^2 \tilde{\times} \mathbb{R}P^2) \cup_{\mathrm{id}} (D^3 \tilde{\times} S^1), \tag{2.10}$$

where $D^3 \tilde{\times} S^1$ is the nonorientable 3-disk bundle over the circle. The tubular neighborhood $\nu(\gamma)$ is diffeomorphic to the normal bundle of the loop, which is $D^3 \tilde{\times} S^1$. It follows from (2.10) that

$$\mathbb{R}P^4 \backslash \nu(\gamma) = D^2 \tilde{\times} \mathbb{R}P^2 \tag{2.11}$$

and

$$\partial(D^2 \tilde{\times} \mathbb{R}P^2) = \partial(D^3 \tilde{\times} S^1) = S^2 \tilde{\times} S^1 \tag{2.12}$$

as claimed.

□

Build the closed smooth nonorientable 4-manifold

$$P(\varphi') := (D^2 \widetilde{\times} \mathbb{R}P^2) \cup_{\varphi'} (D^2 \widetilde{\times} \mathbb{R}P^2) \tag{2.13}$$

where the diffeomorphism

$$\varphi' : S^2 \widetilde{\times} S^1 \longrightarrow S^2 \widetilde{\times} S^1 \tag{2.14}$$

is either the identity id or the map analogous to (2.4) given by

$$\varphi'_\alpha(x, [t/a]) = (\alpha([t/a]) \cdot x, [t/a]) \tag{2.15}$$

where α is a diffeomorphism analogous (2.5), i.e., the diffeomorphism φ'_α rotates every $x \in S^2$ in the 2-sphere fiber about the north-south pole axis through an angle $[t/a] \in \mathbb{R}/\mathbb{Z} = S^1$ in the circle base of the nontrivial bundle [3, Section 6]. In particular, the double $P(\mathrm{id})$ of the 2-disk bundle over the real projective plane is the nonorientable nontrivial S^2-bundle over $\mathbb{R}P^2$.

The diffeomorphism classes of the manifolds $M(\varphi)$ and $P(\varphi')$ depend on the isotopy classes of the diffeomorphisms (2.3) and (2.14), respectively.

Proposition 1. *Let*

$$\phi : S^1 \times S^2 \to S^1 \times S^2 \tag{2.16}$$

be a diffeomorphism isotopic to φ_α in (2.3). The manifold $M(\varphi_\alpha)$ is diffeomorphic to

$$M(\phi) := (S^2 \times S^2 - \nu(S^2)) \cup_\phi (D^2 \times S^2). \tag{2.17}$$

Let

$$\phi' : S^2 \widetilde{\times} S^1 \to S^2 \widetilde{\times} S^1 \tag{2.18}$$

be a diffeomorphism isotopic to φ'_α in (2.14). The manifold $P^4(\varphi'_\alpha)$ is diffeomorphic to

$$P(\phi') := (P(\mathrm{id}) - \nu(\mathbb{R}P^2)) \cup_{\phi'} (D^2 \widetilde{\times} \mathbb{R}P^2). \tag{2.19}$$

Proposition 1 says that $M(\varphi_\alpha) = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is obtained by performing surgery on $S^2 \times S^2$ along an embedded homologically essential 2-sphere with tubular neighborhood $\nu(S^2)$ diffeomorphic to $D^2 \times S^2$, while the manifold $P(\varphi'_\alpha)$ is obtained from the nonorientable 2-sphere bundle over $\mathbb{R}P^2$, $P(\mathrm{id})$, by performing surgery along an embedded real projective plane with tubular neighborhood $\nu(\mathbb{R}P^2)$ diffeomorphic to $D^2 \widetilde{\times} \mathbb{R}P^2$.

2.2. **Homotopy classes of orbit spaces of smooth free $\mathbb{Z}/2$-involutions on $S^2 \times S^2$.** Hambleton-Kreck's classification up to homeomorphism of closed smooth orientable 4-manifolds with finite fundamental group [8] and Kim-Kojima-Raymond computation of homotopy invariants for closed smooth nonorientable 4-manifolds with fundamental group of order two [13] (cf [9]) imply the following proposition.

Proposition 2. *There are five homotopy types of closed smooth 4-manifolds that are orbit spaces of a $\mathbb{Z}/2$-involution on $S^2 \times S^2$. These homotopy types are realized by*

1. *two orientable total spaces of S^2-bundles over $\mathbb{R}P^2$,*
2. *two nonorientable total spaces of S^2-bundles over $\mathbb{R}P^2$, and*
3. *the nonorientable manifold $P(\varphi'_\alpha)$ of (2.13).*

Proof. The homotopy type of a closed smooth orientable 4-manifold with fundamental group of order two is determined by its Euler characteristic, its signature, and its w_2-type [8, Theorem C]. Both orientable S^2-bundles over $\mathbb{R}P^2$ have Euler characteristic equal to two and signature equal to zero. One of them is w_2-type (II), and the other one is type (III). The homotopy type of closed nonorientable smooth is determined by its Euler characteristic, its w_2-type, w_1^4 and in the case of w_2-type (III), by the Arf invariant [9, Corollary 1] [13]. Straightforward computations yield that the trivial bundle $S^2 \times \mathbb{R}P^2$, the nontrivial nonorientable 2-sphere bundle over $\mathbb{R}P^2$, $P(\mathrm{id})$, and $P(\varphi'_\alpha)$ realize all the possible combinations of these invariants. Notice that the manifold $P(\mathrm{id})$ is denoted by $S(2\gamma \oplus \mathbb{R})$ and $P(\varphi'_\alpha)$ is denoted by $\mathbb{R}P^4 \#_{S^1} \mathbb{R}P^4$ in [9]. A quadratic function on $\pi_2 \otimes \mathbb{Z}/2 \to \mathbb{Z}/4$ was used in [13] to show that the nonorientable manifolds $P(\mathrm{id})$ and $P(\varphi'_\alpha)$ are not homotopy equivalent. □

3. Comparison to the known constructions of metrics of nonnegative sectional curvature on orbit spaces of $\mathbb{Z}/2$-involutions of $S^2 \times S^2$

The bundles of Items (1) and (2) of Proposition 2 can be equipped with a metric of nonnegative sectional curvature by taking products, as homogeneous spaces, and through cohomogeineity one G-actions. We now discern the 4-manifold $P(\varphi'_\alpha)$ of (2.13) from the known nonnegatively curved examples.

Lemma 2. *The manifold $P(\varphi'_\alpha)$ is not homotopy equivalent to a connected sum of compact rank one symmetric spaces nor to a biquotient.*

Proof. Since $P(\varphi_{\alpha'})$ is the orbit space of a $\mathbb{Z}/2$-involution on $S^2 \times S^2$, its fundamental group is $\mathbb{Z}/2$ and the higher homotopy groups are

$$\pi_k(P(\varphi'_\alpha)) = \pi_k(S^2 \times S^2) \tag{3.1}$$

for $k \geq 2$. Compact rank one symmetric spaces in dimension four are

$$\{S^4, \mathbb{R}P^4, \mathbb{CP}^2, \overline{\mathbb{CP}^2}\}. \tag{3.2}$$

Besides $\mathbb{R}P^4 \# \mathbb{CP}^2$, the claim follows by comparing homotopy groups. The proof of the lemma concludes by noticing that the universal cover of the $\mathbb{R}P^2$-bundle over S^2, $\mathbb{R}P^4 \# \mathbb{CP}^2$, is not homotopy equivalent to $S^2 \times S^2$. The arguments cover the only four-dimensional biquotients as listed in [17], [12].

□

Parker's work imply that $P(\varphi'_\alpha)$ can not be equipped with a metric of nonnegative sectional curvature by using the results of Grove-Ziller [7].

Proposition 3. *Parker* [14]. *Let M be a closed smooth 4-manifold that arises as the orbit space of a $\mathbb{Z}/2$-involution on $S^2 \times S^2$ and that admits a cohomogeneity one G-action. Then M is the total space of an S^2-bundle over $\mathbb{R}P^2$.*

Notice that the omission in [14] that was corrected in [11] does not concern $S^2 \times S^2$. Lemma 2 and Proposition 3 cover all but one of the known methods to construct nonnegatively curved metrics; see Remark 3.

4. Riemannian metrics of nonnegative sectional curvature

4.1. **Nonnegatively curved disk bundles.** Our first step to apply Principle A is to choose nonnegatively curved metrics on the building blocks (2.2) and (2.13) for which the diffeomorphisms (2.4) and (2.15) are isotopic to an isometry of the metric induced on their boundaries. We build on the analysis of Gromoll-Tapp in [5], where they classified nonnegatively curved metrics on $S^2 \times \mathbb{R}^2$.

Proposition 4. *There is a Riemannian metric $(D^2 \times S^2, g)$ with $\sec_g \geq 0$ such that the boundary $(S^1 \times S^2, h)$ equipped with the induced intrinsic metric has positive-definite second fundamental form and for which the diffeomorphism (2.4) is an isometry.*

There is a Riemannian metric $(D^2 \widetilde{\times} \mathbb{R}P^2, g)$ with $\sec_g \geq 0$ such that the boundary $(S^2 \widetilde{\times} S^1, h)$ equipped with the induced intrinsic metric has positive-definite second fundamental form and for which the diffeomorphism (2.15) is an isometry.

The abuse in our notation is justified by the fact that the metrics are locally isometric.

Proof. Take a rotationally symmetric metric $(\mathbb{R}^2, g_f)$ and denote by $\hat{\Theta}$ the vector field that corresponds to a rotation at one radian per unit speed. Equip the 2-sphere with its round metric (S^2, g_{S^2}) and let $X \in \mathfrak{X}(S^2)$ be the Killing vector field such that the flow along X is given by the isometry $p \mapsto \alpha_s(p)$, where α_s is rotation of the 2-sphere along an axis at a time s for every $p \in S^2$ (cf. (2.5)). Consider the product Riemannian manifold $(\mathbb{R}^2, g_f) \times (S^2, g_{S^2})$ and its Killing vector field $\hat{\Theta} + X \in \mathfrak{X}(\mathbb{R}^2 \times S^2)$. Notice that the flow along this vector field restricted to the circle bundle coincides with the diffeomorphism (2.3). Take the quotient metric

$$((\mathbb{R}^2, g_f) \times (S^2, g_{S^2}) \times (\mathbb{R}, dt^2))/\mathbb{R} \to (\mathbb{R}^2 \times S^2, g) \tag{4.1}$$

under the isometric $\mathbb{R}$ action induced by the flow along $\hat{\Theta} + X$. Proposition 4 is phrased by considering the disk bundle $(D^2 \times S^2, g)$. Let $S^1(r) \subset (\mathbb{R}^2, g_f)$ denote the circle of circumference equal to $2\pi r$. The induced isometric $\mathbb{R}$ action on $(S^1(r), g_f|_{S^1(r)}) \times (S^2, g_{S^2}) \times (\mathbb{R}, dt^2)$ yields the following diagram (cf. [5])

$$(S^1 \times S^2, h) \xleftarrow{\varphi_\alpha} ((S^1(r), g_f|_{S^1(r)}) \times (S^2, g_{S^2}) \times (\mathbb{R}, dt^2))/\mathbb{R} \xrightarrow{\Pi}$$
$$\xrightarrow{\Pi} ((S^2, g_{S^2})/\mathbb{R} \xrightarrow{\delta} (S^2, g_\Sigma). \tag{4.2}$$

The metric $(S^1 \times S^2, h)$ is defined as the metric for which the diffeomorphism φ_α (as in (2.4)) is an isometry. The diffeomorphism δ is defined by $[p, [t/a]] \mapsto \varphi_\alpha(p)$ for $p \in S^2$ and (S^2, g_Σ) is the metric for which δ is an isometry. The map Π is defined by $[p, s, t] \mapsto [p, t]$ and it is a Riemannian submersion. In particular, the circle bundle map

$$\phi : \delta \circ \Pi \circ {\varphi_\theta}^{-1} : (S^1 \times S^2, h) \to (S^2, g_\Sigma) \tag{4.3}$$

is a Riemannian submersion given by $(p, s) \mapsto p$, and (S^2, g_Σ) is the soul of the metric (4.1) [2], [6, Chapter 3] (cf. (4.1) and (4.2)). The second fundamental form of $(S^1 \times S^2, h)$ is positive-definite since it can be described as the boundary of a convex set [6, Section 3.1]. Indeed, the boundary of a metric ball about $(S^2, g_\Sigma) \subset (\mathbb{R}^2 \times S^2, g)$ equipped with the intrinsic metric is $(S^1 \times S^2, h)$, hence its second fundamental form is positive-definite. Moreover, the Gauss equation implies $\sec_h \geq 0$. The metric h is obtained from the product metric $(S^1(r), g_f|_\partial) \times (S^2, g_{S^2})$

by rescaling along the Killing field $\hat{\Theta} + X$ for the vector field $\hat{\Theta}$ that is tangent to the circle factor and which arises as rotation at one radian per unit speed [5, Proof Claim 2.2]. The metric on the soul (S^2, g_Σ) is obtained from (S^2, g_{S^2}) by rescaling along the Killing field X. The universal cover of $(S^1 \times S^2, h)$ is isometric to $(\mathbb{R}, dt^2) \times (S^2, g_{S^2})$ with the pullback metric according to the splitting theorem, and we have

$$(\mathbb{R}, dt^2) \times (S^2, g_{S^2}) \xrightarrow{f} (S^1 \times S^2, h) \xrightarrow{\pi} (S^2, g_\Sigma), \tag{4.4}$$

where f is the Riemannian covering and π is projection onto the 2-sphere factor. The canonical isometric $\mathbb{Z}$ action associated to the Riemannian covering map f yields an isometry

$$(\mathbb{R}, dt^2) \times (S^2, g_{S^2})/\mathbb{Z} \to (S^1 \times S^2, h) \tag{4.5}$$

given by

$$[t, p] \mapsto ([t/a], \rho_{-[t/a]}(p)) \tag{4.6}$$

for $[t/a] \in \mathbb{R}/\mathbb{Z} = S^1$, $p \in S^2$, and where $\rho_{-[t/a]}$ is the flow associated to the Killing field X on (S^2, g_{S^2}) for $[t/a] \in S^1$. The isometric $\mathbb{R}$ action of (4.1) gives rise to the isometry $(\mathbb{R}, dt^2) \times (S^2, g_{S^2})/\mathbb{R} \to (S^2, g_\Sigma)$ given by $[p, t] \mapsto \rho_{-[t/a]}(p)$. The flow $\rho_{-[t/a]}$ associated to the Killing vector field X yields a rotation by an angle $-[t/a] \in S^1$ about an axis as in diffeomorphism (2.4) cf. [6, Example 3.6.1]. The involution $(r, \mathbb{A})$ is an isometry of the metric and using (2.6) we obtain a Riemannian submersion

$$\pi : (D^2 \times S^2, g) \longrightarrow (D^2 \tilde{\times} \mathbb{R}P^2, g). \tag{4.7}$$

Since we have an isometric action by a finite group, which explains the abuse in our notation, we conclude that the sectional curvature of g is nonnegative. Moreover, the second fundamental form of $(S^2 \tilde{\times} S^1, h)$ is positive-definite.

□

Remark 1. *The metrics of Proposition 4 are invariant under other involutions besides (2.6). In particular, they are invariant under orientation-preserving involutions and yield metrics on* $(D^2 \times S^2)/\mathbb{Z}/2$.

4.2. **Obtaining smooth Riemannian metrics from Principle A.** The description given in Section 4.1 makes it clear that the of the metric $(D^2 \times S^2, g)$ is not a product metric near the boundary, i.e., $(S^1 \times S^2, h)$ is not a product metric. To argue that the metrics we construct are smooth, we use a generalization of the well-known situation of warped product metrics (cf. [6, Theorem 2.7.1 and Remark 2.7.1]). In such a scenario and in terms of Principle A, the choice of metrics is $g_U = dt^2 + f^2(t)g = g_V$ and they yield a C^1-metric. A modification of the warping function f around a small neighborhood of the seam of the surgery allows one to smooth out the C^1-metric and conclude the existence of a smooth metric.

Our first step is to describe the metrics of Proposition 4 as warped connection metrics, which are determined by the following data $\{g_\Sigma, \theta, \psi\}$ as in [15, Definition 2.1].

• The metric (S^2, g_Σ) is obtained by rescaling the round sphere (S^2, g_{S^2}) along the Killing field X. As it was mentioned in the previous section, the projection onto the 2-sphere factor of (4.4) is a Riemannian submersion. In particular we have the decomposition

$$T_x(S^1 \times S^2) = \mathcal{V}_x \oplus \mathcal{H}_x = \hat{\Theta} \oplus \mathcal{H}_x \tag{4.8}$$

for every point $x \in S^1 \times S^2$.

- A principal connection θ whose kernel is spanned by

$$\left\{ Y, -X + \frac{2\pi}{a^2} |X|^2_{g_\Sigma} \hat{\Theta} \right\} \tag{4.9}$$

for Y a vector field on the 2-sphere that is perpendicular to the Killing field X. The kernel of θ is the horizontal space $\mathcal{H}$ (4.8) of the Riemannian submersion (4.4).

- A warping function $\psi : S^2 \to \mathbb{R}^+$ given by

$$\psi(p) = \frac{1}{2\pi} \cdot \left(\frac{1}{a^2} - \frac{1}{a^4} |X(p)|^2_{g_\Sigma} \right)^{-1/2} \tag{4.10}$$

for

$$|X|^2_{g_\Sigma} = \frac{a^2 |X|^2_{g_{S^2}}}{a^2 + |X|^2_{g_{S^2}}} \tag{4.11}$$

and

$$a^2 = \frac{4\pi^2 r^2}{1 + r^2}. \tag{4.12}$$

The warping function ψ is consistent with the uniform rescaling of the vertical space $\hat{\Theta}$ so that the length of the fibers equals (4.10). A modification of the warping function (4.10) with respect to the radius r of the fiber in (4.12) as it is done in the case of warped metrics that was described in the beginning of this section allows us to conclude that the metrics of Proposition 4 yield smooth Riemannian metrics under Principle A.

4.3. **New nonnegatively curved examples.** Proposition 4 is now coupled with the decompositions of manifolds that were considered in Section 2.1 to construct new examples of metrics of nonnegative sectional curvature.

Proposition 5. *Principle A and the metrics discussed in Section 4.1 yield Riemannian metrics of nonnegative sectional curvature on*

1) every total sphere of a 2-sphere bundle over either the 2-sphere or the real projective plane, and

2) the nonorientable closed smooth 4-manifold $P(\varphi'_\alpha)$.

Remark 2. *It is well-known that both total spaces of an S^2-bundle over S^2 admit Riemannian metrics of nonnegative sectional curvature* [1, 20]*. Wilking discusses* [19, p. 55] *an argument due to Bruce Kleiner that shows that the moduli space of metrics of nonnegative sectional curvature on $S^2 \times S^2$ is larger than expected. Tapp has produced large families of nonnegatively curved metrics on both bundles in* [16, Example 1.4]*.*

We now prove Proposition 5.

Proof. We first prove Item 1). Let g and h be the metrics of Proposition 4 and set

$$(U, g_U) = (D^2 \times S^2, g) = (V, g_V) \tag{4.13}$$

and

$$(\partial U, g_{\partial U}) = (S^1 \times S^2, h) = (\partial V, g_{\partial V}) \tag{4.14}$$

in terms of Principle A. The discussion at the beginning of Section 2.1 explains how $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is obtained by identifying two copies of $D^2 \times S^2$ using a diffeomorphism isotopic to (2.4), which is an isometry for our choice of metrics. We obtain a C^1 metric on the nontrivial 2-sphere bundle over the 2-sphere. As it was discussed in Section 4.2, a modification to the warping function (4.10) yields a smooth metric of nonnegative sectional curvature on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. The procedure that we have just described yields a metric of nonnegative sectional curvature on the double of $D^2 \times S^2$ and we denote such a metric by $(S^2 \times S^2, \bar{g})$. Moreover, every 2-sphere bundle over $\mathbb{R}P^2$ arises as a double of a 2-disk bundle over the real projective plane $(D^2 \times S^2)/\mathbb{Z}/2$.

We now prove Item 2) by setting

$$(U, g_U) = (D^2 \widetilde{\times} \mathbb{R}P^2, g) = (V, g_V) \tag{4.15}$$

and

$$(\partial U, g_{\partial U}) = (S^2 \widetilde{\times} S^1, h) = (\partial V, g_{\partial V}) \tag{4.16}$$

in terms of Principle A. As it was discussed in Section 2.1 and Proposition 1, the manifold $P(\varphi'_\alpha)$ is diffeomorphic to a manifold constructed by gluing together two copies of the bundle $D^2 \widetilde{\times} \mathbb{R}P^2$ with a diffeomorphism of the boundary that is isotopic to (2.15). The latter is an isometry of the metric h by Proposition 4. Notice that there is a Riemannian submersion

$$(D^2 \times S^2, g) \to (D^2 \widetilde{\times} \mathbb{R}P^2, g) \tag{4.17}$$

with respect to the metrics of Proposition 4. In particular, there is a Riemannian submersion

$$(S^2 \times S^2, \bar{g}) \to (P(\varphi'_\alpha), \bar{g}). \tag{4.18}$$

As it was mentioned before, we justify our abuse of notation for the metrics since they are locally isometric. ☐

4.4. **Proof of Theorem A.** A Riemannian metric of nonnegative sectional curvature on the closed smooth nonorientable 4-manifold $P(\varphi')$ was constructed in Proposition 5. Proposition 2 states that $P(\varphi'_\alpha)$ is not homotopy equivalent to a homogeneous space. Lemma 2 says it is not homotopy equivalent to a compact rank one symmetric space nor to a connected sum of two of them nor to a biquotient. As stated in Proposition 3, Parker has shown that $P(\varphi'_\alpha)$ does not admit a cohomogeneity one G-action.

Remark 3. *Wilking has generalized Grove-Ziller's cohomogeneity one construction of nonnegatively curved metrics; see* [19, Theorem 2.8]. *Goette-Kerin-Shankar have shown that Wilking's construction yields further examples of nonnegatively curved manifolds, including all homotopy 7-spheres* [4, Theorem A]. *It is unclear to the author of this note if Wilking's construction can be used to equip $P(\varphi'_\alpha)$ with a metric of nonnegative sectional curvature.* *We thank an anonymous referee for kindly pointing this out.*

References

[1] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Diff. Geom. 9 (1972), 623 - 628.

[2] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. 96 (1972), 413 - 443.

[3] H. Gluck, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. 104 (1962), 308 - 333.

[4] S. Goette, M. Kerin, and K. Shankar, *Highly connected 7-manifolds and non-negative sectional curvature*, (2017), arXiv:1705.05895v1.

[5] D. Gromoll and K. Tapp, *Nonnegatively curved metrics on* $S^2 \times \mathbb{R}^2$, Geom. Dedicata 99 (2003), 127 - 136.

[6] D. Gromoll and G. Walschap, *Metric foliations and curvature*, Progress in Mathematics, 268. Birkhaüser Verlag, Basel, 2009. vii + 174 pp.

[7] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. 152 (2000), 331 - 367.

[8] I. Hambleton and M. Kreck, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math., 444 (1993), 79 - 100.

[9] I. Hambleton, M. Kreck, and P. Teichner, *Nonorientable 4-manifolds with fundamental group of order 2*, Trans. Amer. Math. Soc. 344 (1994), 649 - 665.

[10] J. Hillman, *Four-manifolds, geometries and knots*, Geom. & Topol. Monographs vol. 5, Geom. & Top. Publications, Coventry, 2002. xiv + 379 pp.

[11] C. A. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, PhD Thesis - University of Pennsylvania, 2007.

[12] V. Kapovitch and W. Ziller, *Biquotients with singly generated rational cohomology*, Geom. Dedicata 104 (2004), 149 - 160.

[13] M. H. Kim, S. Kojima, and F. Raymond, *Homotopy invariants of nonorientable 4-manifolds*, Trans. Amer. Math. Soc. 333 (1992), 71 - 81.

[14] J. Parker, *4-dimensional G-manifolds with 3-dimensional orbits.* Pacific J. Math. 125 (1986), 187 - 204.

[15] K. Shankar, K. Tapp, and W. Tuschmann, *Nonnegatively and positively curved invariant metrics on circle bundles*, Proc. Amer. Math. Soc. 133 (2005), 2449 - 2459.

[16] K. Tapp, *Rigidity for nonnegatively curved metrics on* $S^2 \times \mathbb{R}^3$, Ann. Global. Anal. Geom. 25 (2004), 273 - 287.

[17] B. Totaro, *Cheeger manifolds and the classifications of biquotients*, J. Differ. Geom. 61 (2002), 397 - 451.

[18] C. T. C. Wall, *Surgery on compact manifolds* Second Ed. Edited and with a foreword by A. A. Ranicki. Mathematical Surveys and Monographs, 69. Amer. Math. Soc., Providence, RI, 1999. xvi + 203 pp.

[19] B. Wilking, *Nonnegatively and positively curved manifolds*, Metric and comparison geometry, Surv. Differ. Geom. 11 ed. K. Grove and J. Cheeger, International Press, Sommervile, MA, 2007, 25 - 62.

[20] W. Ziller, *Examples of Riemannian manifolds with non-negative sectional curvature*, Metric and comparison Geometry. Surv. Differ. Geom. 11 ed. K. Grove and J. Cheeger, International Press, Sommervile, MA, 2007, 63 - 102.

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