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Geometric and analytic properties of metric measure spaces through Sobolev calculus

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Introduction

The Sobolev space theory had a huge impact during the XXth century in the analysis of partial differential equations (PDEs). A consolidated principle in the modern analysis is by-now the fact that enlarging the space of functions from smooth to Sobolev ones gives, in turn, a better understanding of problems with groundbreaking results especially concerning the *regularity* of solutions to PDEs.

When moving from classical (e.g. Euclidean space, Riemannian manifolds...) to the singular framework of *metric measure spaces* the same principle certainly applies. Here, by metric measure space we mean a structure $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ composed of a metric space (X, d) equipped with a *reference measure* $\mathfrak{m}$ that plays the role of the volume measure. In this situation, it is by no means trivial to derive a notion of the Sobolev space $W^{1,p}(\mathrm{X})$, as the underlying lack of smooth coordinates makes unclear what smooth functions should be. Several definitions have been proposed during the past years, here we mention [64], [178] and [21, 20] (recall also the previous work [134, 118] and the manuscripts [114, 117]) and we recall that they turned to be equivalent [20] (truth to be said, in [178] it has been previously proved the equivalence with [64]). In all the aforementioned references, the notion of $W^{1,p}(\mathrm{X})$ comes with an approach to define the 'modulus of the distributional differential' and, given that these approaches are interchangeable, there is an advanced understanding at disposal of the space $W^{1,p}(\mathrm{X})$ and the object $|Df|_p$.

It is worth to describe here informally the approach of [21, 20] that will be the predominant one of this work. Roughly speaking, a function $f \in L^p(\mathfrak{m})$ is Sobolev and $|Df|_p \in L^p(\mathfrak{m})$ is its *minimal p-weak upper gradient*, provided the inequality

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 |Df|_p(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t, \tag{0.0.1}$$

holds true among almost every curve γ and $|Df|_p$ is the $\mathfrak{m}$-a.e. minimal $L^p(\mathfrak{m})$ function obeying the above inequality. This requirement, which is true and defining in the smooth category, should be interpreted as the duality between the distributional differential and the velocity of a curve. More in details, the correct way to require the Sobolev condition, is to superpose the duality expressed in (0.0.1) with the help of the notion of *q-test plan*. Roughly, a q-test plan, with q conjugate exponent of p, is a probability π concentrated on q-absolutely continuous curve with values in X that do not accumulate mass with respect to the reference measure

$$\pi(\{\gamma\colon \gamma_t \in B\}) \leq \mathrm{Comp}(\pi)\mathfrak{m}(B), \qquad \text{for every } t \in [0,1],\ B \subset \mathrm{X} \text{ Borel},$$

where $\mathrm{Comp}(\pi)$ is the least constant $C > 0$ for the above to hold. In particular, with this powerful notion at hand, a solid *nonsmooth calculus* has been achieved [21, 95] that revealed to be capable of backing up many important calculations in the field.

Without any further assumptions on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, requiring a function to be Sobolev in any of the aforementioned senses may be also a void condition (for instance a Sobolev function in a metric space equipped with a Dirac mass must be constant). Therefore, many properties of the classical Sobolev space on Euclidean domains should not be expected at this high level of generality. To mention only a few that are particularly relevant for this Thesis, it is in general false that $W^{1,p}(\mathrm{X})$ is reflexive, separable and that the object $|Df|_p$ is independent on the integrable exponent p.

Moreover, this dependence [82] maybe incredibly pathological going against the intuition given by the classical setting (where, evidently, such dependence is not even an issue). Nevertheless, when a *regularity* assumption is enforced to X, the space $W^{1,p}(\mathrm{X})$ and its calculus inherits many extremely useful properties that are crucial to pursuing geometric investigations of the space X.

In this fashion, we are going to see that the framework of metric measure spaces plays in geometric applications the same role that the Sobolev space played in the analysis of PDEs. The analogy indeed carries on as, allowing the presence of singular spaces in the theory, makes it possible to get access to a novelty of 'implicit' techniques capable of dealing also with regularity questions from a new viewpoint.

This Thesis is devoted to the geometric analysis of metric measure spaces through the non-smooth calculus. The main goal of this note is to study the Sobolev space and its calculus in combination with the *regularity* of the underlying metric measure space. By this, we mean that:

A) we look for specific regularity assumptions of $(\mathrm{X},\mathsf{d},\mathfrak{m})$ ensuring a well-behaved Sobolev calculus;

but also, and somewhat conversely, we mean that:

B) we study the Sobolev space $W^{1,p}(\mathrm{X})$ to shed light on the regularity of the underlying geometric structure of X.

A remarkable instance of the A-principle is the thorough investigation conducted by many authors concerning metric measure spaces satisfying a Doubling condition and supporting a weak Poincaré inequality (Doubling & Poincaré). When a metric measure space meets these two conditions, Sobolev functions enjoy a number of fine properties that are widely employed in applications. We refer to the monographs [114, 117, 42] for a presentation of those and to the references therein. Here we only mention that Cheeger in [64] showed - among many deep results around Lipschitz and Sobolev functions - that a metric measure space satisfying Doubling & Poincaré has a precise first order differentiable structure. Indeed, with the goal of proving a generalized Rademacher Theorem for Lipschitz functions, he obtained a *concrete* description of the cotangent bundle where naturally differentials of Lipschitz functions live (see also the recent work [88]).

On the other hand, an *abstract* notion of (measurable section of the) cotangent bundle in the language of *normed modules* can be always built on arbitrary metric measure spaces [97] *through* the Sobolev calculus. Even though at this high level of generality, the notion of differentials and cotangent bundle is *analytical* rather than *concrete and geometric*, these module structures are capable of detecting many important geometric properties. For instance, given a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we can wonder whether the underlying geometry at small scales looks *Hilbertian*, meaning that there is a *hidden* scalar product so to compute angles between attached cotangent vectors at $\mathfrak{m}$-a.e. points. Addressing positively this question requires studying the Sobolev space as understood in [95], where the notion of 'infinitesimal Hilbertianity' was derived. In particular, what one should (equivalently) do is try to prove that $W^{1,2}(\mathrm{X})$ is Hilbert (a fact that is not true in general). This is a remarkable instance of the B-principle, since the study of the space $W^{1,2}(\mathrm{X})$ can give in turn a better understanding of the geometry of X. In this direction, we register growing efforts in studying the infinitesimal Hilbertianity property in different settings of interests (weighted Euclidean spaces, weighted Riemannian manifolds, sub-Riemannian manifolds...)[103, 144, 84, 81], but also in more sophisticated frameworks [79].

The typical regularity that will be encountered in the sequel is the one of curvature bounds of a metric or metric measure space. We begin now a brief *detour* of such settings as they will be predominant in this work. Nevertheless, we mention that the frameworks that will be faced are not limited to this one, meaning that part of the main results of this work holds without curvature assumptions.

The synthetic approach to curvature bounds

The study of synthetic treatments of curvature bounds initiated in the late '40s by the work of A. D. Alexandrov [8] where the first brick was posed towards a theory that is now called

Alexandrov geometry. The concept of Alexandrov space has been later considerably developed [52] in the context of metric spaces to speak about *sectional curvature bounds*. By synthetic, we mean that an 'implicit' notion is worked out on an abstract setting without relying on the 'explicit' smoothness of any kind of the underlying space. Moreover, this notion is *coherent*, i.e. when we restrict the attention to only Riemannian manifolds, it is equivalent to the smooth one. This is the case of Alexandrov spaces, whose definition builds upon the classical Toponogov's Theorem that encodes a sectional curvature bound via a suitable triangle comparison condition.

In this manuscript, we shall only deal with *upper bounds* $\kappa \in \mathbb{R}$ on the sectional curvature and the resulting metric structure is commonly called a $\mathsf{CAT}(\kappa)$ space[1]. We do not enter now into further details of such definition and postpone it to Chapter 2. Here, we recommend [47, 51, 113, 39, 6, 7] for a broad picture on Alexandrov geometry.

Moving to Ricci lower bounds, the setting of metric measure spaces very naturally comes into play. The reason being that the notion of a distance is no more sufficient to encode Ricci bounds, hence the need to decouple the metric and measure structure in the triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$.

The theory of *Ricci limit spaces*, developed in [65] and further investigated in [66, 67, 68] is certainly a first step towards this direction. It is remarkable that on Ricci limits, that are measure Gromov-Hausdorff (mGH) limits of Riemannian manifolds with uniform Ricci lower bounds, many important structural results persist. We mention in this direction, the Cheeger-Gromoll splitting theorem [69], brought to the context of Ricci limit spaces in [65] and the constancy of the dimension of a Ricci limit space [70]. Nevertheless, even though this class certainly posses singularities, their study comes from the knowledge of their smooth approximations and progress have been later put to capture the essence of Ricci lower bounds via a fully synthetic theory.

In the independent works [143] and [180, 181], it has been clarified for the first time what a Ricci lower bounds for a metric measure space should be. The authors, by means of Optimal Transport techniques, gave birth to the celebrated *curvature dimension condition* thus producing the notion of $\mathsf{CD}(K,N)$-spaces, namely metric measure spaces having 'Ricci bounded from below by $K \in \mathbb{R}$' and 'dimension bounded from above by $N \geq 1$'. Again, this synthetic definition is coherent with the classical one. Indeed, it is formulated by prescribing suitable convexity property of entropies, a condition that on a Riemannian manifold encodes Ricci lower bounds as understood in [71, 188]. Moreover, the goodness of this notion is reflected by many instances as, to mention only the most relevant ones for our discussion, the following ones:

- ▷ the curvature dimension condition is *stable* for mGH-convergence [180, 181, 143, 100]. In particular, it is a fully synthetic theory containing Ricci limit spaces;

- ▷ important geometric inequalities are available in this class, such as the Bishop-Gromov monotonicity formula, the Brunn-Minkowski inequality, the Bonnet-Meyers diameter estimates [181] and the Isoperimetric Inequalities à la Lévy-Gromov [58];

- ▷ important functional inequalities are available in this class, such as the sharp Lichnerowitz spectral gap inequality [143], a weak local $(1,1)$-local Poincaré inequality [174] and Laplacian comparison estimates [95].

However, a CD-space may not look Riemannian, as the basic example of the Euclidean space $\mathbb{R}^n$ equipped with a norm not arising from a scalar product reveals. With the underlying goal of proposing a synthetic theory of Ricci limits, a notion of 'Riemannian curvature dimension condition' was proposed to single out Finsler geometries and goes under the name of $\mathsf{RCD}(K,N)$-condition (the letter R stands for Riemannian). This definition, derived first in the infinite dimensional case $(N=\infty)$ in [22] and later in the finite dimensional case $(N<\infty)$ in [95], is essentially obtained by coupling the curvature dimension condition with the infinitesimal Hilbertianity [95] (recall, the fact that the Sobolev space $W^{1,2}(\mathrm{X})$ is Hilbert, false in general on CD-spaces). It is precisely this

[1] Gromov [111] in 1987 coined the acronym CAT: the letters are in honor of E. Cartan, A. Topogonov and A. D. Alexandrov

latter condition that detects the typical geometry of a Riemannian manifold at small scales. It has to be said that the RCD-condition, as it is commonly presented nowadays, builds upon the key contributions [33, 23, 18, 26, 87, 56] that filled the gaps with the CD-theory and identified important equivalent formulations. For brevity reasons (and since we shall never need other formulations of the RCD-class here), we will not discuss them and refer to the aforementioned literature and the surveys [10] and [186, 184] for more references and insight on synthetic treatments of Ricci lower bounds.

As of today, it is well established that the RCD-class is the suitable one among the currently available synthetic notions to re-produce and push further typical results valid on Riemannian manifolds. We briefly list some important instances reflecting this fact:

- ▷ the validity of many 'splitting' type of results, such as the Cheeger-Gromoll Splitting theorem in RCD-class in [93, 94] and the fact that equality in the 1-Bakry-Émery inequality forces a splitting [11];
- ▷ the validity of many *rigidity* theorems, such as the Cheng's rigidity of the maximal diameter [132], the Obata rigidity Theorem [131] and the 'volume cone to metric cone' theorem [75];
- ▷ the RCD-class enjoys compactness property in the mGH-topology [180, 181, 143, 91, 22, 100] (and thanks to [113]). Therefore, many *rigidty* theorems can be turned in *almost rigidity* theorems providing new results even in the smooth situation. We mention, among many, the almost rigidity of the Lévy-Gromov isoperimetric inequality [58] and of the Obata Theorem [159].

Additionally, there has been an incredible research effort in recent years to develop a satisfactory *structure theory* of RCD-spaces and they turned out to possess a strong $\mathfrak{m}$-rectifiable structure [103, 129, 76] (improving on [158]) with constant dimension [50]. Nowadays, this tendency took a significant turn towards a structure theory in *codimension one* after the proof of the De Giorgi structure theorem for sets of finite perimeter on finite dimensional RCD-space [49]and it culminated recently with the proof of the constancy of dimension for boundaries of finite perimeter sets in [48]. We mention finally the recent work [53] where an existence and uniqueness theory of parallel transport has been developed (in a suitable subclass of RCD space, extending the previous work [105]). All in all, we believe that the RCD-class has proved over the years, not only to encode from a synthetic point of view a Riemannian manifold but also to provide new powerful tools for its study.

Finally, before passing to introduce the original contributions of this Thesis, we explain the plan we are going to pursue throughout this note.

We will address the problem of defining new differential objects with associated calculus rule that behaves like in the smooth category; we aim at studying many aspects of the Sobolev space and of functions with bounded variations that are 'fundamental', meaning that they are obtained - not assuming regularity of any kind - of the underlying metric measure space; we aim at singling out a *first order condition* of a metric measure space that ensures a *strong* independence of $|Df|_p$ on the integrable exponent. Also, it is our task also to show that this condition is shared by a large class of spaces that play a central role in the current literature; we also aim at studying spaces with second-order curvature bounds and achieve original results. In this direction, we prove new rigidity and almost rigidity results on RCD-spaces by the study of (non) compact Sobolev embeddings.

Main contributions

Gradient flows on $\mathsf{CAT}(\kappa)$ spaces and applications

Here we introduce the main results of the joint work [102] with N. Gigli.

The theory of gradient flows in metric spaces has been initiated by De Giorgi and collaborators [74], [73] (see also the more recent [19]): a basic feature of the approach is to provide a very general existence theory - at this level uniqueness is typically lost - without neither curvature assumptions on the space nor semiconvexity of the functional.

In this setting gradient flow trajectories (x_t) of E (or curves of maximal slopes) are defined by imposing the maximal rate of dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(x_t) = -|\dot{x}_t|^2 = -|\partial^-\mathsf{E}|^2(x_t), \qquad a.e.\ t,$$

where here $|\dot{x}_t|$ is the metric speed of the curve and $|\partial^-\mathsf{E}|$ is the descending slope of E. It has been later understood [19, 22],[95], [189],[166],[161] that if E is λ-convex and the metric space has some form of some Hilbert-like structure at small scales, then an equivalent formulation can be given via the so-called Evolution Variational Inequality (EVI)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathsf{d}^2(x_t,y)}{2} + \mathsf{E}(x_t) + \frac{\lambda}{2}\mathsf{d}^2(x_t,y) \leq \mathsf{E}(y) \qquad \text{a.e. } t \tag{0.0.2}$$

for any choice of point y on the space. We refer to [161] for the precise definition and a thorough structural study of the EVI condition. Moving to the setting of our main interests, it is reasonable to expect that the EVI-condition is achievable on $\mathsf{CAT}(\kappa)$ spaces. The reason being that a CAT-space looks Hilbertian as many properties persist in this setting (see e.g. the monograph [39] and the survey [38]).

Our contribution aims at completing the previously available results of [153, 124], [170, 146, 166] around gradient flows on $\mathsf{CAT}(\kappa)$ spaces. More precisely, we show that a *differential* theory of gradient flows as

$$\begin{cases} x_t'^{+} \in -\partial^-\mathsf{E}(x_t) & \forall t > 0, \\ \lim_{t\downarrow 0} x_t = x, \end{cases} \tag{0.0.3}$$

is possible on (locally) $\mathsf{CAT}(\kappa)$ spaces X for semi-convex and lower semicontinuous functionals $\mathsf{E}\colon \mathrm{X} \to [0,\infty]$. In particular, we study the notion of the 'minus subdifferential set' appearing and prove the equivalence between (0.0.2) and (0.0.3).

Finally, using the notion of 'Korevaar-Shcoen' energy of an L^2-map derived in [109] (revisiting the original paper [133]), we produce a notion of *Laplacian* for a sufficiently regular Sobolev map $u\colon \mathrm{X} \to \mathrm{Y}$ with RCD-source and $\mathsf{CAT}(0)$ target building on top of (0.0.3). Basic calculus rules associated with this Laplacian are also derived. The motivation behind this application comes from the study of the *Lipschitz regularity* of harmonic maps in a fully synthetic setting, a program initiated by Gigli and collaborators (see [79, 106, 108, 109, 80] for the full treatment) aiming also at stating a full *Bochner-Ells-Sampson inequality* [86] in a nonsmooth context. Knowing what a Laplacian of a $\mathsf{CAT}(0)$ valued map is, is a crucial step for this program.

A first-order condition for the independence on p of weak gradients

Here we introduce the main results of the joint work [101] with N. Gigli.

The classical Sobolev space defined on the Euclidean space $\mathbb{R}^d$ requires a functions $f \in L^p$ to posses integrable *weak derivatives* $\partial_i f \in L^p$ satisfying

$$\int f\partial_i\varphi\,\mathrm{d}\mathscr{L}^d = -\int f\partial_i\varphi\,\mathrm{d}\mathscr{L}^d, \qquad \forall \varphi \in C^\infty,$$

for all $i = 1, ..., d$. As these are defined in distribution, it is evident that they do not depend on p.

However, moving to the setting of metric measure spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, this is definitely not the case. Assuming for simplicity $\mathfrak{m}(\mathrm{X}) = 1$, one should always expect that for $p_1 \leq p_2$, if $f \in W^{1,p_2}(\mathrm{X})$ then $f \in W^{1,p_1}(\mathrm{X})$ and also

$$|Df|_{p_1} \leq |Df|_{p_2} \qquad \mathfrak{m}\text{-a.e.}. \tag{0.0.4}$$

Nevertheless, without any further assumption on X, when $p_2 \leq p_1$ the picture is generally more complicated. Here we recall the work [82], where a detailed study on weighted Euclidean spaces has been performed to build a family of metric measure spaces for which only strict inequality may occur in (0.0.4).

On the contrary, the independence of weak gradients on p is strictly linked to the *regularity* of the underlying metric measure space and efforts have been put to rule out both of these situations. In [64], it has been also shown that equality in (0.0.4) is ensured on doubling spaces supporting a weak and local Poincaré inequality. More recently, in [99] the RCD-condition has revealed to be a positive framework for the independence of $|Df|_p$ on p especially suggesting that a *stronger* kind of independence is possible.

Our motivation is twofold: on one hand, we aim at completing the picture on this topic, on the other, we propose advances around it discussing a new point of view, the guideline being:

> to define the object $|Df|_p$, only the *first-order* differential structure of $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is involved. Hence, its independence on p should require the regularity of the underlying metric measure space at *first-order*.

More in details, we propose a condition that we call Bounded Interpolation Property (BIP) ensuring the following kind of stronger independence: for every $p_1, p_2 \in (1, \infty)$ it holds

$$f \in W^{1,p_1}(\mathrm{X}) \text{ with } f, |Df|_{p_1} \in L^{p_2}(\mathfrak{m}) \quad \Rightarrow \quad \begin{array}{l} f \in W^{1,p_2}(\mathrm{X}), \\ |Df|_{p_1} = |Df|_{p_2}, \quad \mathfrak{m}\text{-a.e.}. \end{array}$$

This condition, inspired by [174], is defined by requiring suitable compression estimates of transportation geodesics and is stable for (pointed) Gromov Hausdorff convergence of metric measure structures. Moreover, we show that it is shared by a broad class of spaces satisfying (different type of) curvature dimension conditions and, in these settings, it extends the previously available weaker independence based on the Doubling & Poincaré condition.

Finally, building on top of the recent analysis developed in [169], we prove that a single test plan, called master test plan, is enough to test the Sobolev property on every metric measure space. By this, we mean that it is capable to establish whether $f \in W^{1,p}(\mathrm{X})$ and to detect the object $|Df|_p$ by quantifying the exceptional curves for which (0.0.1) fails. On spaces satisfying the (BIP), this can be also taken concentrated on geodesics curves.

On master test plans for the space of BV functions

Here we introduce the main results of the joint work [162] with E. Pasqualetto and T. Schultz.

The first notion of a *function of bounded variation* (or just a *BV function* for short) in the setting of metric measure spaces dates back to a paper by M. Miranda Jr., published almost 20 years ago [156]. Since then, several equivalent definitions of BV function have been introduced and studied in the literature. The most relevant ones for the purposes of the present paper are the notions proposed in [13], which we are going to describe informally. While the original approach in [156] is of 'Eulerian' nature as it is based upon a relaxation procedure using Lipschitz functions, the definition in [13] is 'Lagrangian' as it ultimately looks at the behaviour of functions along well-chosen curves. The motivation of the latter approach comes from the BV-theory in the classical Euclidean space where it is well established that BV functions behave well under one dimensional restrictions (see [16] for a thorough discussion). In the absence of smooth coordinates, the key concept to mimic this characterization is that of a ∞-test plan (the extreme case $q = \infty$ for the Sobolev case). Following [78], we define the space $BV^*(\mathrm{X})$ as the collection of $f \in L^1(\mathfrak{m})$ for which there exists a constant $\mathsf{C} > 0$ such that

$$\int f(\gamma_1) - f(\gamma_0)\, \mathrm{d}\pi \le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mathsf{C}, \qquad \text{for all } \infty\text{-test plan } \pi, \tag{0.0.5}$$

where $\mathrm{Lip}(\pi)$ is the least constant $L > 0$ for which π is concentrated on L-Lipschitz curves (and it is finite, as a defining requirement of an ∞-test plan). Then, by appealing to the equivalent definitions of a BV function [13], it is actually possible to achieve a 'true' total variation measure $|\boldsymbol{D}f|$ so that $|\boldsymbol{D}f|(\mathrm{X})$ is the least constant that can be chosen in (0.0.5).

The main goal of this part is to push the analysis performed in [101] in the extreme case of functions of bounded variations. Our main result is that:

> on an arbitrary metric measure space, a countable collection of ∞-test plans is sufficient to detect a BV function and its total variation measure $|\boldsymbol{D}f|$. Moreover, on CD spaces (non branching with finite reference measure), we are also able to require these test plans to be concentrated on geodesics.

However, due to the 'lack of the linearity w.r.t. π' on the right-hand side of (0.0.5), it is not possible to directly employ the techniques of [169] to reduce the countable collection to a single plan (observe that this is not an issue in the Sobolev case when integrating (0.0.1) w.r.t. π). To this aim, we derive yet another definition of a 'curvewise BV space' denoted $BV^{\mathsf{cw}}(\mathrm{X})$ by adapting the 'AM-modulus approach' of [151, 152] with the concept of a test plan. We show that this approach is compatible with the ones of [156, 13] and roughly requires the existence of $(g_n) \subset L^1(\mathfrak{m})$ with $\sup_n \|g_n\|_{L^1} < \infty$, called *curvewise bound* for f, that controls from above the total variations of the one dimensional restrictions of f. With this notion at hand, we are able to reduce to a singleton the countable family of ∞-test plan detecting the BV space. We thus derive the notion of *master test plan* in the curvewise sense that, by construction, can be also required to be concentrated on geodesics in the (non branching with finite measure) CD class.

To conclude, we mention that the present result is relevant for the analysis of [48], where the authors pushed the constancy of dimension result for RCD-spaces up to codimension one.

Rigidity and almost rigidity of Sobolev inequalities

Here we introduce the main results of the joint work [163] with I. Y. Violo.

The standard Sobolev inequality in sharp form reads as

$$\|u\|_{L^{p^*}} \leq \mathsf{Eucl}(n,p)\|\nabla u\|_{L^p}, \qquad \forall u \in W^{1,p}(\mathbb{R}^n),$$

where $p \in (1,n)$, $p^* := \frac{pn}{n-p}$ is the Sobolev conjugate exponent and $\mathsf{Eucl}(n,p)$ is the smallest positive constants for which the above inequality is valid. Its precise value was computed independently by Aubin [31] and Talenti [182] (see also [72]). When moving to the setting of Riemannian manifolds with positive Ricci lower bounds, it is not straightforward what an optimal constant should be and, the notion and value of optimal Sobolev constant is one of the main object of study of the so-called AB-program (see [115] and [85] for a thorough discussion).

Postponing a proper discussion to Chapter 6, here we just mention that on a compact n-dimensional manifold M, a typical Sobolev inequality reads as

$$\|u\|_{L^{2^*}(M)} \leq A\|\nabla u\|_{L^2(M)} + B\|u\|_{L^2(M)}, \qquad \forall u \in W^{1,2}(M).$$

Then, the value $A^{\rm opt}(M)$ of the optimal Sobolev constant in the above can be suitably defined optimizing on constants $A, B > 0$ (that is the reason of the name AB-program). It turns out that this value is completely characterized in the case of the n-sphere $M = \mathbb{S}^n$ with $n \geq 3$, after [30, 40] while, under $\mathsf{Ric} \geq n-1$ we have [122]

$$A^{\rm opt}(M) \leq A^{\rm opt}(\mathbb{S}^n). \tag{0.0.6}$$

The very same notion of optimal Sobolev constant have been brought to different nonsmooth context and the above comparison has been generalized in the works [35, 173, 59]. In particular, it holds true on compact RCD spaces.

Our main results are then the *rigidity* and *almost rigidity* of Sobolev inequalities on compact RCD-spaces, both results are new even in the smooth setting. Considering, for simplicity, a closed n dimensional manifold M with $n \geq 3$ and $\mathsf{Ric} \geq n-1$ we prove:

equality (resp. almost equality) occurs in (0.0.6) if and only if M is isometric to $\mathbb{S}^n$ (resp. mGH-close to a spherical suspension)

Remarkably, the smoothness of the objects involved plays no role as the sought rigidity will follow from the study of the noncompact embedding

$$W^{1,2} \hookrightarrow L^{2^*}.$$

The idea of the proof consists of a fine geometric investigation of extremizing sequences, namely bounded sequences in $W^{1,2}$ for which the optimal Sobolev inequality saturates (note, the fact that this sequence is smooth is irrelevant). Therefore, by taking care of the technicalities arising from a synthetic approach, the proof reads *verbatim* on compact RCD-spaces where we can take advantage of specific *compactness* properties of the RCD-condition to produce the sought *almost rigidity* statement. To conclude, we just mention two key tools for our proof and an application of independent interest:

- ▷ a new concentration compactness dichotomy principle in the spirit of [140, 141] under mGH-converging RCD-spaces to study the shape of almost extremal functions on varying spaces;
- ▷ a new Polya-Szego principle in the spirit of [159] for *Euclidean monotone rearrangements* of Sobolev functions on finite dimensional CD-spaces without nonbranching assumptions;
- ▷ an existence theory of solutions of the Yamabe equation set on compact RCD-spaces and an application concerning the mGH-continuity of the generalize Yamabe constant pushing further [121].

Structure of the Thesis

This Thesis is divided into two parts and ends with an Appendix.

- ▷ In Part I, we collect all the needed preliminary material in two different chapters.

  In Chapter 1, we start presenting basics facts about metric measure spaces concerning their analysis and geometry at a first-order level with special emphasis on the theory of gradient flow and optimal transport of (geodesic) metric spaces and on the theory of normed modules. Then, we move to the presentation of the nonsmooth calculus via the test plan approach and derive the notion of Sobolev functions, Sobolev maps and BV functions. Next, we consider metric measure spaces asking for regularity conditions, namely Doubling & Poincaré and infinitesimal Hilbertianity. Finally, we study how the nonsmooth calculus benefits from these kinds of assumptions.

  In Chapter 2, we then face the preliminary materials concerning the geometry of nonsmooth spaces at a second-order level and we consider several synthetic notions of curvature bounds. We start presenting the CAT-condition of metric spaces and their *concrete* first-order calculus. Then we move to the CD-condition and list important geometric and functional inequalities, as well as the existence of test plans concentrated on transportation geodesics. We shall then discuss the RCD-condition and recall some important characterizations of Sobolev maps available in this setting, the fact that weak gradients do not depend on p and present some rigidity theorems. We end Part I by discussing convergence and compactness results of spaces satisfying curvature dimension conditions and by listing other possible curvature conditions via Optimal Transport.
- ▷ In Part II, we present the main contributions in Chapters 3,4,5,6 following the order of appearance in this introduction. Each of them is organized as follows: a specific introduction is given to present the main results and possibly the main statements of the chapter. We end all of them with a brief structure of the chapter that serves as a guideline for the reader.
- ▷ In Appendix A, we extend the interpolation estimates of [174] and [57] for Wasserstein geodesics on spaces satisfying synthetic Ricci curvature bounds. These estimates, achieved in the 2-Wasserstein space are to be used in this Thesis with generalized exponent $q \in (1, \infty)$.

List of included papers

The main results of this Thesis are reported, eventually with simplification and minor changes, from the following research papers:

i) N. Gigli and F. Nobili, *A Differential Perspective on Gradient Flows on* $\mathsf{CAT}(\kappa)$*-Spaces and Applications*, J. Geom. Anal., (2021), pp. 1–39.

ii) F. Nobili and I. Y. Violo, *Rigidity and almost rigidity of Sobolev inequalities on compact spaces with lower Ricci curvature bounds.* arXiv:2108.02135, (2021).

iii) N. Gigli and F. Nobili, *A first order condition for the independence on p of weak gradients.* ongoing.

iv) F. Nobili, E. Pasqualetto, and T. Schultz, *On master test plans for the space of BV functions.* arXiv:2109.04980, (2021).

Part I

Preliminaries

1 First-order analysis of metric measure spaces

Contents

1.1 Metric measure spaces

1.1.1 Notation and basics

A metric measure space is a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ where

$$\begin{array}{ll} (\mathrm{X}, \mathsf{d}) & \text{is a complete and separable metric space,} \\ \mathfrak{m} \neq 0 & \text{is non negative and boundedly finite Borel measure.} \end{array}$$

In this Thesis, metric measure spaces will play a fundamental role. For this reasons, we will refer to $\mathfrak{m}$ as the *reference measure* on X. Moreover, we will be working under the simplification condition (see (1.1.1) below for the definition of support) that

$$\mathrm{supp}(\mathfrak{m}) = \mathrm{X}.$$

This choice will let us avoid some technicalities during the note. Often, we will need the (weighted) Euclidean space as a model metric measure space. We denote then by $\mathscr{L}^1$ the one dimensional Lebesgue measure on $\mathbb{R}$. We start now recalling basic facts and the main notation around metric measure spaces.

We denote the ball of radius r and centred at x by

$$B_r(x) := \{y \in \mathrm{X} \colon \mathsf{d}(x,y) < r\}, \qquad \forall x \in \mathrm{X}, r > 0,$$

and, to avoid confusion when it may occur, we will write $B_r^{\mathrm{X}}(x)$ to emphasize the space where the ball belongs. We define, for $E \subset \mathrm{X}$, the diameter of E, and write $\mathrm{diam}(E) \in [0,\infty]$, the possibly infinite quantity $\sup_{x,y\in E} \mathsf{d}(x,y)$ (with $\mathrm{diam}(\emptyset) = 0$ by convention). We say that $E \subseteq \mathrm{X}$ is bounded, provided $\mathrm{diam}(E) < \infty$. We recall that compact sets must necessarily be closed and bounded while we say that (X,d) is *proper*, provided the converse also holds. The metric space (X,d) is called *metrically doubling* provided there exists $\mathsf{C} \in \mathbb{N}$ so that any open ball B or radius $r > 0$ can be covered by C-many open balls centred in B of radius $r/2$.

If not differently stated, X will be always thought as a measurable space with the Borel sigma-algebra $\mathcal{B}(\mathrm{X})$ and we denote by $\mathscr{M}_b^+(\mathrm{X})$ the space of finite Borel positive measures over X. Given $\mu \in \mathscr{M}_b^+(\mathrm{X})$, we define

$$\mathrm{supp}(\mu) := \bigcap\{C \subset \mathrm{X} \colon C \text{ closed and } \mu(\mathrm{X} \setminus C) = 0\}, \tag{1.1.1}$$

and say that a Borel set E is μ-negligible, if $\mu(E) = 0$. We say then that μ is concentrated on C if $\mathrm{X} \setminus C$ is μ-negligible and we notice that, thanks to the separability of X, μ is concentrated on its support. Also, we write $\mathscr{P}(\mathrm{X})$ for the space of probability measures over X and equip it the following weak topology: given any sequence $(\mu_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq \mathscr{P}(\mathrm{X})$, we say that μ_n weakly converges to μ_∞ as $n \to \infty$ (or, briefly, $\mu_n \rightharpoonup \mu_\infty$ as $n \to \infty$) provided $\int \varphi \,\mathrm{d}\mu_\infty = \lim_{n\to\infty} \int \varphi \,\mathrm{d}\mu_n$ for all $\varphi \in C_b(\mathrm{X})$, where $C_b(\mathrm{X})$ stands for the space of bounded, continuous real-valued functions. Notice that there exists a complete and separable distance $\mathsf{d}_{\mathscr{P}}$ which metrizes this topology. This choice is motivated by the fact that, when X is compact, this topology reduces to the standard weak-* topology. A family $\mathcal{K} \subset \mathscr{P}(\mathrm{X})$ is called *tight*, provided

$$\forall \epsilon > 0, \exists K_\epsilon \subset \mathrm{X} \text{ compact so that } \quad \mu(\mathrm{X} \setminus K_\epsilon) \le \epsilon, \qquad \forall \mu \in \mathcal{K}.$$

For later use, we report without proof a well known characterization of compactness in the weak topology.

Theorem 1.1.1 (Prokhorov)**.** *Let (X,d) be a complete and separable metric space and $\mathcal{K} \subset \mathscr{P}(\mathrm{X})$. The following are equivalent:*

i) $\mathcal{K}$ *is precompact in the weak topology;*

ii) $\mathcal{K}$ *is tight;*

iii) *There exists a functional $\psi \colon \mathrm{X} \to [0,\infty]$ with compact sublevels so that*

$$\sup_{\mu\in\mathcal{K}} \int \psi \,\mathrm{d}\mu < \infty.$$

Finally, if $\varphi \colon \mathrm{X} \to \mathrm{Y}$ is a Borel map between two metric spaces and $\mu \in \mathscr{P}(\mathrm{X})$, the set-value map $\mathcal{B}(\mathrm{Y}) \ni B \mapsto \varphi_\sharp\mu(B) := \mu(\varphi^{-1}(B))$ belongs to $\mathscr{P}(\mathrm{Y})$ and it is called the *pushforward* measure of μ via ϕ. When φ is continuous, the operation $\varphi_\sharp$ is *weakly* continuous.

Two metric measure spaces $(\mathrm{X}_i, \mathsf{d}_i, \mathfrak{m}_i)_{i=1,2}$ are said to be *isomorphic*, and we write $\mathrm{X}_1 \simeq \mathrm{X}_2$ for short, if there exists an isometry $\iota : \mathrm{X}_1 \to \mathrm{X}_2$ such that $(\iota)_\sharp\mathfrak{m}_1 = \mathfrak{m}_2$.

Let $L^0(\mathfrak{m})$ be the space of equivalence classes up to $\mathfrak{m}$-a.e. equality of Borel functions on X equipped with the topology of local convergence in measure. Then, for a given exponent $p \in [1,\infty]$, we denote by $L^p(\mathfrak{m})$ the subset of $L^0(\mathfrak{m})$ of usual Lebegue p-integrable functions equipped with the norm

$$\|f\|_{L^p(\mathfrak{m})} := \begin{cases} \left(\int |f|^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} & \text{if } p < \infty, \\ \text{ess sup } |f| & \text{if } p = \infty. \end{cases}$$

Then, the space $L^p_{loc}(\mathfrak{m})$ is the subset of $L^0(\mathfrak{m})$ consisting of (equivalence class of) functions which are p-integrable when restricted to bounded sets. For a given non negligible Borel set $E \subset \mathrm{X}$, an analogous definition leads to the space $L^p(\mathfrak{m}_{|_E})$, sometimes simply written $L^p(E)$ and, for $\Omega \subset \mathrm{X}$ open, $L^p_{loc}(E)$ is then the subset of (the equivalence class of) Borel functions $f \in L^2(C)$ for every $C \subset \Omega$ closed. Here, the restriction measure is defined as $\mathfrak{m}_{|_E}(B) := \mathfrak{m}(E \cap B)$ for all $B \in \mathcal{B}(\mathrm{X})$. Alternatively, denoting χ_E the (equivalence class of the) function $\mathfrak{m}$-a.e. equal to 1 on E and zero otherwise, we have obviously $\mathfrak{m}_{|_E} = \chi_E \mathfrak{m}$, here having denoted $f\mathfrak{m}(B) := \int_B f \,\mathrm{d}\mathfrak{m}$ for every non negative $f \in L^0(\mathfrak{m})$. In the case when $\mathfrak{m}(E) < \infty$ and $u : E \to [0,+\infty)$ a non-negative Borel function we recall that the *distribution* function $\mu : [0,+\infty) \to [0, \mathfrak{m}(E)]$, is defined as

$$\mu(t) := \mathfrak{m}(\{u > t\}), \qquad \forall t \geq 0. \tag{1.1.2}$$

In the sequel, we will consider distribution functions of $u \in L^0(\mathfrak{m}_{|_E})$. By this, we mean that it is possible to select a Borel representative to define (1.1.2); but it is evident that the outcome μ is independent on this choice.

Next, we recall without proof the following well known fact of Lebesgue spaces. For any $\mu \in \mathscr{P}(\mathrm{X})$, we have that $\|f\|_{L^p(\mu)} \leq \|f\|_{L^{p'}(\mu)}$ for every $p, p' \in [1,\infty]$ with $p \leq p'$ and $f \colon \mathrm{X} \to [0,+\infty)$ Borel, as a consequence of Hölder's inequality. Moreover, we have that

$$\|f\|_{L^\infty(\mu)} = \lim_{q\to\infty} \|f\|_{L^p(\mu)}, \qquad \text{for every } \mu \in \mathscr{P}(\mathrm{X}) \text{ and } f \colon \mathrm{X} \to [0,+\infty) \text{ Borel.} \tag{1.1.3}$$

1.1.2 Lipschitz class

We shall denote as usual by $\mathrm{Lip}(\mathrm{X})$, $\mathrm{Lip}_{bs}(\mathrm{X})$ and $\mathrm{Lip}_c(\mathrm{X})$ the spaces of Lipschitz functions, boundedly supported Lipschitz functions and compactly supported Lipschitz functions. For a given open subset $\Omega \subset \mathrm{X}$, we also write $\mathrm{Lip}_c(\Omega)$ and $\mathrm{Lip}_{loc}(\Omega)$ the space of Lipschitz with compact support and locally Lipschitz functions on Ω, respectively. Here, by *locally Lipschitz functions* on Ω, we mean $f \colon \Omega \to \mathbb{R}$ so that, for any $x \in \Omega$, there exists a ball B centred at x so that $f \in \mathrm{Lip}(B)$. We will also consider metric valued Lipschitz maps $u \colon \mathrm{X}_1 \to \mathrm{X}_2$ from two metric spaces $\mathrm{X}_1, \mathrm{X}_2$ and we denote by $\mathrm{Lip}(u)$ the analogous Lipschitz constant in this case. A special situation is when we consider curves γ defined on $[0,1]$ and valued on a metric measure space; we shall then say that γ is Lipschitz writing $\gamma \in \mathrm{Lip}([0,1],\mathrm{X})$ and that γ is L-Lipschitz, $L > 0$, if $\mathrm{Lip}(\gamma) \leq L$.

Moreover, if $f \in \mathrm{Lip}(\mathrm{X})$, we call $\mathrm{Lip}(f)$ its Lipschitz constant and we can define

$$\mathrm{lip}\ f(x) := \varlimsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(x,y)},$$

(set to be 0 is x is isolated) the *local Lipschitz constant*.

In what follows, we are going also to deal with Lusin-Lipschitz functions. We recall that a function $f \colon \mathrm{X} \to \mathbb{R}$ is Lusin-Lipschitz, provided there are N, K_n Borel, $n \in \mathbb{N}$, with N negligible and K_n compacts, so that $\mathrm{X} := N \cup (\cup_n K_n)$ and $f_{|_{K_n}}$ is Lipschitz for every $n \in \mathbb{N}$.

Conversely, suppose we are given a functions $f \colon E \to \mathbb{R}$ with $E \subset \mathrm{X}$ that is Lipschitz, it is well known that an extension $\bar{f} \colon \mathrm{X} \to \mathbb{R}$ with $\bar{f} = f$ on E and $\mathrm{Lip}(\bar{f}) = \mathrm{Lip}(f)$ exists. A way to produce such extension, called *McShane extension*, is

$$\bar{f}(x) := \inf\{f(y) + \mathrm{Lip}(f)\mathsf{d}(x,y) \colon y \in E\}, \qquad \forall x \in \mathrm{X}.$$

1.1.3 Absolutely continuous curves

We recall some basic fact about continuous curves valued in a complete metric space.

We denote by $C([0,1],\mathrm{X})$ the space of continuous and X-valued curves, and equip it with the uniform distance

$$\mathsf{d}_{\sup}(\gamma,\eta) := \max_{t\in[0,1]} \mathsf{d}(\gamma_t,\eta_t), \qquad \gamma,\eta \in C([0,1],\mathrm{X}).$$

When (X,d) is complete and separable, then $(C([0,1],\mathrm{X}),\mathsf{d}_{\sup})$ is also complete and separable. The evaluation map is the assignment $\mathsf{e}_t\colon C([0,1],\mathrm{X}) \to \mathrm{X}$ defined via $\mathsf{e}_t(\gamma) := \gamma_t$.

Then, given $q \in [1,\infty]$, the set of q-absolutely continuous curves, denoted by $AC^q([0,1],\mathrm{X})$, is the subset of $\gamma \in C([0,1],\mathrm{X})$ so that there exists $g \in L^q(0,1)$ satisfying

$$\mathsf{d}(\gamma_t,\gamma_s) \le \int_s^t g(r)\,\mathrm{d}r, \qquad \forall s \le t \text{ in } [0,1]. \tag{1.1.4}$$

It is evident that the notion of AC^∞ space is coincident with the one of Lipschitz curves, i.e.

$$\mathrm{Lip}([0,1],\mathrm{X}) = AC^\infty([0,1],\mathrm{X}).$$

We will use both notations interchangeably. Also, by Hölder inequality, we have $\mathrm{Lip}([0,1],\mathrm{X}) \subseteq AC^q([0,1],\mathrm{X}) \subseteq AC([0,1],\mathrm{X})$ holds for every $q \in (1,\infty)$. We recall that, for any $\gamma \in AC^q([0,1],\mathrm{X})$, there exists a minimal a.e. object playing the role of absolute value of the velocity.

Theorem 1.1.2 (Metric speed)**.** *Let (X,d) be a metric space and $q \in [1,\infty]$. Then, for every $\gamma \in AC^q([0,1],\mathrm{X})$ there exists the limit*

$$\lim_{h\downarrow 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{h} \qquad a.e.\ t \in [0,1],$$

which we denote by $|\dot\gamma_t|$ and call metric speed*. Moreover, it is the least, in the a.e. sense, function $g \in L^q(0,1)$ that can be taken in (1.1.4).*

See, for the proof, [19, Theorem 1.1.2]. More generally, we can define the *metric speed* functional $\mathsf{ms}(\gamma,t)\colon C([0,1],\mathrm{X}) \times [0,1] \to [0,\infty]$ as follows:

$$\mathsf{ms}(\gamma,t) := \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{h}, \qquad \text{if } \gamma \in AC([0,1],\mathrm{X}) \text{ and } \exists \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{h},$$

setting $\mathsf{ms}(\gamma,t) := \infty$ otherwise. Then, we have that ms is Borel regular (see, e.g., [104]). Then, for finite q's, we define the Kinetic energy functional $C([0,1],\mathrm{X}) \ni \gamma \mapsto \mathsf{Ke}_q(\gamma) := \int_0^1 \mathsf{ms}(\gamma,t)^q\,\mathrm{d}t = \int_0^1 |\dot\gamma_t|^q\,\mathrm{d}t$, if $\gamma \in AC^q([0,1],\mathrm{X})$ and ∞ otherwise. We recall the following well known lemma, whose proof we give for completeness.

Lemma 1.1.3. *Let (X,d) be a metric space, $q \in (1,\infty)$ and $(\gamma^n) \subseteq \mathrm{AC}^q([0,1],\mathrm{X})$ uniformly converging to $\gamma \in C([0,1],\mathrm{X})$ with $\sup_n \mathsf{Ke}_q(\gamma^n) < \infty$.*

Then, $\gamma \in \mathrm{AC}^q([0,1],\mathrm{X})$ and $\mathsf{Ke}_q(\gamma) \le \underline{\lim}_n \mathsf{Ke}_q(\gamma^n)$. Moreover, if $\mathsf{Ke}_q(\gamma^n) \to \mathsf{Ke}_q(\gamma)$ (i.e. there is conservation of the Kinetic Energy), then one also recovers $|\dot\gamma^n_\cdot| \to |\dot\gamma_\cdot|$ in $L^q(0,1)$.

Proof. For the first part, it suffices to notice that any weak-L^q limit G (possibly along a not relabeled suitable subsequence) of $|\dot\gamma^n_\cdot|$ in $L^q(0,1)$ satisfies

$$\mathsf{d}(\gamma_t,\gamma_s) = \lim_n \mathsf{d}(\gamma^n_t,\gamma^n_s) \le \int_s^t G(r)\,\mathrm{d}t, \qquad \forall s,t \in [0,1].$$

Thus, $\gamma \in AC^q([0,1],\mathrm{X})$ and, by minimality of $|\dot\gamma_t|$ and weak lower semicontinuity of L^q-norms, one has

$$\mathsf{Ke}_q(\gamma) \le \int_0^1 G^q(t)\,\mathrm{d}t \le \varliminf_{n\to\infty} \mathsf{Ke}_q(\gamma^n)$$

Moreover, under the hypotheses of the second claim, the above becomes a chain of equalities ensuring that $G = |\dot\gamma_t|$ is a strong limit in $L^q(0,1)$. □

We face now two important compactness criterions for collection of absolutely continuous curves. The first deals with $q < \infty$ while the second deals with the limit case.

Lemma 1.1.4 ([100])**.** *Let* (X, d) *be a complete and separable metric space,* $q \in (1, \infty)$ *and* $\psi \colon \mathrm{X} \to [0, \infty]$ *be a functional with compact sublevels. Then, the lifted functional* $\Psi \colon C([0,1], \mathrm{X}) \to [0, \infty]$ *defined via*

$$\gamma \mapsto \Psi(\gamma) := \int_0^1 \psi(\gamma_t) + |\dot\gamma_t|^q \, \mathrm{d}t, \qquad \text{if } \gamma \in AC^q([0,1], \mathrm{X}), \quad \infty \text{ otherwise},$$

has compact sublevels in $C([0,1], \mathrm{X})$.

In [100, Lemma 5.8], the above Lemma is proven only for $q = 2$, but the proof works with straightforward modification for $q \in (1, \infty)$. Lemma 1.1.4 will be often frequently used in combination with Prokhorov's theorem to lift functionals to the space of continuous curves.

When dealing with Lipschitz curve, we instead recall the following variant of the Arzelà–Ascoli theorem (see [167, Proposition 2.1]).

Theorem 1.1.5 (Arzelà–Ascoli theorem revisited)**.** *Let* (X, d) *be a metric space. Let* $\mathcal{K}$ *be a closed subset of* $C([0,1], \mathrm{X})$ *which satisfies the following properties:*

i) $\mathcal{K} \subseteq \mathrm{Lip}([0,1], \mathrm{X})$ *and* $\sup \big\{ \mathrm{Lip}(\gamma) \,:\, \gamma \in \mathcal{K} \big\} < +\infty$.

ii) *Given any* $\varepsilon > 0$*, there exists a compact set* $K_\varepsilon \subseteq \mathrm{X}$ *such that*

$$\mathscr{L}^1\big(\big\{ t \in [0,1] \,:\, \gamma_t \notin K_\varepsilon \big\} \big) \le \varepsilon, \qquad \text{for every } \gamma \in \mathcal{K}.$$

Then the set $\mathcal{K}$ *is compact in* $C([0,1], \mathrm{X})$.

Geodesics and tangent cone of directions

A special role in this Thesis will be played by geodesics and geodesic metric spaces. We fix then a complete metric space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$: a curve $[0,1] \ni t \mapsto \gamma_t \in \mathrm{Y}$ is a minimizing constant speed geodesic (or simply a geodesic) if $\mathsf{d}_\mathrm{Y}(\gamma_t, \gamma_s) = |t - s| \mathsf{d}_\mathrm{Y}(\gamma_0, \gamma_1)$, for every $t, s \in [0,1]$. Throughout this manuscript, geodesics are always considered minimizing with constant speed defined on $[0,1]$. We say that Y is a geodesic metric space provided for any couple of points, there exists a constant speed geodesic joining them. Sometimes we shall deal also with non branching geodesic spaces, i.e. geodesic spaces for which we have

$$\gamma, \theta \text{ geodesic with } \gamma|_{[0,t]} = \theta|_{[0,t]} \text{ for some } t \in (0,1) \quad \Rightarrow \quad \gamma \equiv \theta \text{ on } [0,1].$$

Basic examples of non branching spaces are Euclidean spaces, Riemannian manifolds but also Banach spaces with strictly convex norms. More sophisticated examples will be then faced during this note.

Whenever the geodesic connecting y to z is unique, we shall denote it by G_y^z. Fix $y \in \mathrm{Y}$, and denote by $\mathsf{Geo}_y\mathrm{Y}$ the space of constant speed minimizing curve emanating from y, and defined on some right neighbourhood of 0. We endow this space with the pseudo-distance d_y defined as following:

$$\mathsf{d}_y(\gamma, \eta) := \varlimsup_{t \downarrow 0} \frac{\mathsf{d}_\mathrm{Y}(\gamma_t, \eta_t)}{t} \qquad \forall \gamma, \eta \in \mathsf{Geo}_y\mathrm{Y}. \tag{1.1.5}$$

It is easy to see that d_y naturally induces an equivalence relation on $\mathsf{Geo}_y\mathrm{Y}$, by simply imposing $\gamma \sim \eta$ if $\mathsf{d}_y(\gamma, \eta) = 0$. By construction, d_y passes to the quotient $\mathsf{Geo}_y\mathrm{Y}/\sim$ and with (a common) abuse of notation, we still denote d_y the distance on the quotient space. The equivalence class of the geodesic γ under this relation will be denoted γ_0'. In particular this applies to the geodesics G_y^z defined on $[0,1]$, whose corresponding element in $\mathsf{Geo}_y\mathrm{Y}/\sim$ will be denoted by $(\mathsf{G}_y^z)_0'$. Even at this high generality, it is possible to speak about tangent cones from a metric viewpoint as follows.

Definition 1.1.6 (Tangent cone)**.** *Let* Y *be a geodesic space and* $y \in \mathrm{Y}$*. The* tangent cone $(\mathrm{T}_y\mathrm{Y}, \mathsf{d}_y)$*, is the completion of* $(\mathsf{Geo}_y\mathrm{Y}/\sim, \mathsf{d}_y)$*. Moreover, we call* $0_y \in \mathrm{T}_y\mathrm{Y}$*, the equivalence class of the steady geodesic at* y*.*

Without any other regularity assumption, it is clear that the above definition is purely abstract. Nevertheless, in Section 2.1 it will provide a good framework to build a first order differential calculus under curvature constraints in the Alexandrov's sense.

1.1.4 Optimization and gradient flows on geodesic metric spaces

Let us first clarify the notion of *semiconvexity* on a geodesic metric space.

Definition 1.1.7 (Semiconvex function)**.** *Let* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ *be a geodesic metric space and* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R}\cup\{+\infty\}$*. We say that* E *is* λ-convex*,* $\lambda \in \mathbb{R}$*, if for any geodesic* γ *it holds*

$$\mathsf{E}(\gamma_t) \le (1-t)\mathsf{E}(\gamma_0) + t\mathsf{E}(\gamma_1) - \frac{\lambda}{2}t(1-t)\mathsf{d}_\mathrm{Y}^2(\gamma_0,\gamma_1) \qquad \forall t \in [0,1].$$

If $\lambda = 0$*, then we simply speak of convex functions. We shall denote by* $D(\mathsf{E}) \subset \mathrm{Y}$ *the set of* y*'s such that* $\mathsf{E}(y) < \infty$*.*

Next, we define the (descending) slope $|\partial^-\mathsf{E}|$ of a functional E setting, for every $y \in D(\mathsf{E})$

$$|\partial^-\mathsf{E}|(y) := \varlimsup_{z\to y} \frac{(\mathsf{E}(y)-\mathsf{E}(z))^+}{\mathsf{d}_\mathrm{Y}(y,z)}. \tag{1.1.6}$$

The domain of the slope is the collection of points where the slope is finite and will be denoted by $D(|\partial^-\mathsf{E}|) \subset D(\mathsf{E})$. It is easy to prove that for λ-convex functionals, the slope admits the following 'global' formulation. The proof is taken from [19, Theorem 2.4.9]).

Lemma 1.1.8. *Let* Y *be a geodesic space and* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R}\cup\{+\infty\}$ *be* λ*-convex,* $\lambda \in \mathbb{R}$*, and lower semicontinuous. Then, for every* $y \in D(\mathsf{E})$*,*

$$|\partial^-\mathsf{E}|(y) = \sup_{z\neq y}\left(\frac{\mathsf{E}(y)-\mathsf{E}(z)}{\mathsf{d}_\mathrm{Y}(y,z)} + \frac{\lambda}{2}\mathsf{d}_\mathrm{Y}(y,z)\right)^+.$$

Moreover, $y \mapsto |\partial^-\mathsf{E}|(y)$ *is a lower semicontinuous function.*

Proof. The inequality $|\partial^- E|(y) \le \sup_{z\neq y}\left(\frac{E(y)-E(z)}{\mathsf{d}_\mathrm{Y}(y,z)} + \frac{\lambda}{2}\mathsf{d}_\mathrm{Y}(y,z)\right)^+$ is of immediate verification. On the other hand, by definition of geodesic, we have, $\frac{E(y)-E((\mathsf{G}_y^z)_t)}{\mathsf{d}_\mathrm{Y}(y,(\mathsf{G}_y^z)_t)} = \frac{E(y)-E((\mathsf{G}_y^z)_t)}{t\mathsf{d}_\mathrm{Y}(y,z)}$, for every $t \in [0,1]$. Notice that, the semiconvexity implies

$$|\partial^- E|(y) \ge \left(\frac{E(y)-E((\mathsf{G}_y^z)_t)}{t\mathsf{d}_\mathrm{Y}(y,z)}\right)^+ \ge \left(\frac{(E(y)-E(z))}{\mathsf{d}_\mathrm{Y}(y,z)} + \frac{\lambda}{2}\mathsf{d}_\mathrm{Y}(y,z)\right)^+,$$

and, by taking the suprimum over $z \in \mathrm{Y}$, the first part of the lemma is shown. Finally, from this news characterization, lower semicontinuity follows immediately. □

We now consider a notion of gradient flow on a metric space defined by means of the so-called Evolution Variational Inequality (EVI).

Definition 1.1.9 (EVI gradient flow)**.** *Let* Y *be a geodesic metric space,* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R}\cup\{+\infty\}$*,* $y \in \mathrm{Y}$ *and* $\lambda \in \mathbb{R}$*.*

A curve $[0,\infty) \ni t \mapsto y_t \in \mathrm{Y}$ *is* gradient flow *in the EVI sense starting at* $y \in \mathrm{Y}$*, provided it is locally absolutely continuous on* $(0,\infty)$ *with* $y_t \to y$ *as* $t \downarrow 0$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathsf{d}_\mathrm{Y}^2(y_t,z)}{2} + \mathsf{E}(y_t) + \frac{\lambda}{2}\mathsf{d}_\mathrm{Y}^2(y_t,z) \le \mathsf{E}(z) \qquad a.e.\ t>0, \tag{EVI}$$

for every $z \in \mathrm{Y}$*.*

We are not interested in discussing, at the highest level of generality, existence and different formulations of gradient flows in this Thesis. We are going to deal with gradient flows only on Hilbert spaces (Section 1.4.3) and Geodesic metric space satisfying a suitable curvature constraint in the sense of Alexandrov (Chapter 3). We thus refer to [19] for a much more complete picture and [161] for a structural analysis of the EVI-condition.

1.1.5 Optimal transport

In this subsection we recall some basic notions in Optimal Transport theory. The problem of optimal transportation dates back to G. Monge [160] and seeks for optimal transport maps minimising the transportation cost of two probability measures in the Euclidean space. Here instead we consider the more general formulation of L. Kantorovich [126] in terms of optimal plans and follow modern approaches to the theory on complete and separable metric spaces (X, d), referring for instance to [185] (see also [17]) for a thorough presentation of this topic.

Let $q \in (1, \infty]$ and denote by $\mathscr{P}_q(\mathrm{X})$ the set of probabilities $\mu \in \mathscr{P}(\mathrm{X})$ with finite q-moment, i.e. so that $\mathsf{d}(\cdot, x_0) \in L^q(\mu)$ for some (and thus, any) $x_0 \in \mathrm{X}$. By Hölder inequality, we also have $\mathscr{P}_{q'}(\mathrm{X}) \subseteq \mathscr{P}_q(\mathrm{X})$ for every $q, q' \in [1, \infty]$ with $q \le q'$. Also, for a (possibly countable) cartesian product of the space X, denote by P^i and $P^{1,\dots,i}$ the canonical projections onto the i-th factor and the first i factors, respectively. For $q < \infty$, we equip $\mathscr{P}_q(\mathrm{X})$ with the *Wasserstein distance*

$$W_q(\mu_0, \mu_1) := \Big(\inf_{\alpha \in \mathrm{Adm}(\mu_0,\mu_1)} \int_{\mathrm{X}\times\mathrm{X}} \mathsf{d}^q(x, y)\, \mathrm{d}\alpha(x, y) \Big)^{1/q}, \qquad \text{for every } \mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X}), \tag{1.1.7}$$

while in the limit case $q = \infty$, the ∞-Wasserstein distance W_∞ on $\mathscr{P}_\infty(\mathrm{X})$ is defined as

$$W_\infty(\mu_0, \mu_1) := \inf_{\alpha \in \mathrm{Adm}(\mu_0,\mu_1)} \operatorname*{ess\,sup}_{\alpha\text{-a.e. } (x,y)} \mathsf{d}(x, y), \qquad \text{for every } \mu_0, \mu_1 \in \mathscr{P}_\infty(\mathrm{X}). \tag{1.1.8}$$

where $\mathrm{Adm}(\mu_0, \mu_1) := \{\alpha \in \mathscr{P}(\mathrm{X} \times \mathrm{X})\colon P^1_\sharp\alpha = \mu_0, P^2_\sharp\alpha = \mu_1\}$ is the set of *admissible plans* between μ_0 and μ_1. We denote by $\mathrm{Opt}_q(\mu_0, \mu_1)$ the set of all optimal plans between μ_0 and μ_1, i.e. of all minimisers of (1.1.7) or (1.1.8). By using the direct method in the calculus of variations, one can readily show that $\mathrm{Opt}_q(\mu_0, \mu_1) \neq \emptyset$ for every $q \in [1, \infty]$ and $\mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X})$. Also, it is well-known that W_∞ is the monotone limit of W_q as q tends to infinity:

$$W_\infty(\mu_0, \mu_1) = \lim_{q\to\infty} W_q(\mu_0, \mu_1), \qquad \text{for every } \mu_0, \mu_1 \in \mathscr{P}_\infty(\mathrm{X}). \tag{1.1.9}$$

For a proper treatment of the ∞-Wasserstein distance we refer to [110, 63]. In the sequel, we refer to $(\mathscr{P}_q(\mathrm{X}), W_q)$ as the *Wasserstein space* and we report now some basic facts about convergence and compactness under Wasserstein convergence. Let $(\mu_n) \subseteq \mathscr{P}_q(\mathrm{X})$ with $q < \infty$ and recall that

$$W_q(\mu_n, \mu) \to 0 \quad \Leftrightarrow \quad \begin{array}{l} \mu_n \rightharpoonup \mu \\ \int \mathsf{d}^q(x, x_0)\, \mathrm{d}\mu_n \to \int \mathsf{d}^q(x, x_0)\, \mathrm{d}\mu \end{array} \tag{1.1.10}$$

as n goes to infinity for $\mu \in \mathscr{P}_q(\mathrm{X})$ and for some (hence, any) $x_0 \in \mathrm{X}$. Moreover, $(\mathscr{P}_q(\mathrm{X}), W_q)$ is complete and separable if X is and, for $q < \infty$ we recall that a family $\mathcal{K}$ is compact with respect to the topology induced by W_q if and only if is tight and q-uniformly integrable.

In the sequel, we shall also need the dynamical formulation of the Optimal Transport problem. The dynamical viewpoint was first discovered by R. McCann [154], where he derived the notion of *displacement interpolations* in the Euclidean space. Here instead, we shall work on the more general setting of complete and separable geodesic metric spaces and consider *optimal dynamical plans* [17, Theorem 2.10] (whose proof is inspired by [142] and extends the previous works [143, 185] for compact and locally compact spaces). It is well known that the q-Wasserstein space is geodesic if X is and thus, given $\mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X})$ and $q \in (1, \infty]$, we can define the set of q-*dynamical optimal plans* between μ_0 and μ_1 as

$$\mathrm{OptGeo}_q(\mu_0, \mu_1) := \{\pi \in \mathscr{P}(\mathsf{Geo}(\mathrm{X}))\colon (\mathsf{e}_i)_\sharp\pi = \mu_i,\ i = 0, 1,\ (\mathsf{e}_0, \mathsf{e}_1)_\sharp\pi \in \mathrm{Opt}_q(\mu_0, \mu_1)\}.$$

Remark 1.1.10. If $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ it is easy to see that $t \mapsto (\mathsf{e}_t)_\sharp \pi$ is a W_q-geodesic and that

$$W_q^q(\mu_0, \mu_1) = \iint_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\pi, \quad \text{if } q < \infty, \tag{1.1.11a}$$

$$W_\infty(\mu_0, \mu_1) = \|\mathsf{ms}\|_{L^\infty(\pi \otimes \mathscr{L}^1)}, \quad \text{if } q = \infty. \tag{1.1.11b}$$

In other words, the transportation cost can be equivalently evaluated as the superposition of kinetic energies of transportation geodesic in the support of optimal dynamical plans. These facts are well known for $q < \infty$, while for the limit case $q = \infty$ consider taking $s, t \in [0,1]$ with $s \leq t$ and observe that it holds

$$W_\infty\big((\mathsf{e}_s)_\sharp \pi, (\mathsf{e}_t)_\sharp \pi\big) \leq \operatorname*{ess\,sup}_{\pi\text{-a.e. } \gamma} \mathsf{d}(\gamma_s, \gamma_t) = (t-s) \operatorname*{ess\,sup}_{\pi\text{-a.e. } \gamma} \mathsf{d}(\gamma_0, \gamma_1) = (t-s) W_\infty(\mu_0, \mu_1),$$

where in the first inequality we used the fact that $(\mathsf{e}_s, \mathsf{e}_t)_\sharp \pi \in \mathrm{Adm}\big((\mathsf{e}_s)_\sharp \pi, (\mathsf{e}_t)_\sharp \pi\big)$. This yields that $[0,1] \ni t \mapsto (\mathsf{e}_t)_\sharp \pi \in \mathscr{P}_\infty(\mathrm{X})$ is a W_∞-geodesic. By plugging $s = 0$ and $t = 1$ in the previous estimates, we obtain the identity in (1.1.11b). ■

Finally, assuming further that the underlying metric measure space is non branching, the behavior of geodesic in $\mathscr{P}_q(\mathrm{X})$ is particularly nice. Indeed, (see e.g. [17, Proposition 2.16] for the proof with $q = 2$, but it works with standard modification for finite q) in this case we have

$$\text{If } (\mathrm{X}, \mathsf{d}) \text{ is geodesic and non branching} \quad \Rightarrow \quad \mathrm{OptGeo}_q(\mu_0, \mu_1) \text{ is a singleton}, \tag{1.1.12}$$

for every $\mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X})$ and $q \in (1, \infty)$. Sometimes we will consider the Wasserstein space also over the complete and separable space $(C([0,1], \mathrm{X}), \mathsf{d}_{\sup})$. To avoid confusion, we will denote the corresponding Wasserstein space as $\big(\mathscr{P}_q\big(C([0,1], \mathrm{X})\big), \mathcal{W}_q\big)$.

1.1.6 Gromov-Hausdorff convergence

A very weak notion of distance and convergence of sets in a metric space (X, d) is the one of Hausdorff convergence. For simplicity, given $\varepsilon > 0$ and $B \subset \mathrm{X}$, we denote by $B^\epsilon := \{x \in \mathrm{X} \colon \mathsf{d}(x, B) \leq \epsilon\}$ the ε-enlargement of the set B. Then, given $A, B \subset \mathrm{X}$, the *Hausdorff distance* between A and B, is the quantity

$$\mathsf{d}_H(A, B) := \inf\{\varepsilon > 0 \colon A \subset B^\varepsilon, B \subset A^\varepsilon\} \in [0, \infty],$$

where, as costumary, the above is set ∞ when no competitors exist. This may happen when A, B are unbounded, however it is finite when they are bounded. We shall not recall basics fact about this distance and refer e.g. to [51] to a complete presentation.

Next, we move to basic facts about convergence and limits of metric measure spaces as introduced in [113] (see also [180], in this Thesis instead we follow the extrinsic approach described in [100]). Instead, for a thorough presentation of Gromov-Hausdorff convergence and topology, we refer again to [51].

A pointed metric measure space, is a quadruple $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x)$, where $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a metric measure space and $x \in \mathrm{X}$. Also, we put $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Definition 1.1.11 (pmGH-convergence)**.** *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$*,* $n \in \bar{\mathbb{N}}$*, be a sequence of pointed metric measure spaces. We say that that* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ pointed-measure Gromov Hausdorff-converges *(pmGH-converges for short) to* $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ *provided there exists a complete and separable metric measure space* (Z, d) *and isometric embeddings*

$$\iota_n \colon (\mathrm{X}_n, \mathsf{d}_n) \to (\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}),$$
$$\iota_\infty \colon (\mathrm{X}_\infty, \mathsf{d}_\infty) \to (\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}),$$

such that $(\iota_n)(x_n) \to \iota_\infty(x_\infty)$ *and*

$$(\iota_n)_\sharp \mathfrak{m}_n \rightharpoonup (\iota_\infty)_\sharp \mathfrak{m}_\infty, \qquad \textit{in duality with } C_{bs}(\mathrm{Z}).$$

In this case, we write for brevity $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$*.*

For the above, recall that we are assuming reference measures to be of full support. In the sequel, we shall identify the spaces X_n, $n \in \bar{\mathbb{N}}$, with their isomorphic images in Z via *extrisinc approach* [100]. Details of this formulation will be given later in the appropriate context.

Also, we will be frequently consider just the case of compact (with uniformly bounded diameter) metric measure spaces, for which we can reduce the above convergence to the so-called *measure Gromov Hausdorff convergence*, mGH-convergence for short. In this case, we will simply consider non pointed sequences $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ and write $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. It can be checked that Z realizing the convergence can be also taken to be compact.

1.1.7 The language of normed modules

We recall the algebraic notion of a normed module over $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and discuss a nonsmooth differential calculus on metric measure spaces as defined in [97]. We will sometimes also refer to the more recent works [104, 168].

In the following definition, we denote by $\hat{1}$ the equivalence class up to $\mathfrak{m}$-negligible set of the function constantly equal to one.

Definition 1.1.12 ($L^0(\mathfrak{m})$-normed module)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. We call a* $L^0(\mathfrak{m})$-normed module *the quadruple* $(\mathscr{M}, \tau, \cdot, |\cdot|)$*, where*

i) $(\mathscr{M}, \tau)$ *is a topological vector space;*

ii) $\cdot\colon L^0(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ *is a bilinear map satisfying the product's axioms*
$$g \cdot (f \cdot v) = (fg) \cdot v, \quad \hat{\mathbf{1}} \cdot v = v, \quad \forall f, g \in L^0(\mathfrak{m}), v \in \mathscr{M};$$

iii) *The map* $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$*, called* pointwise norm*, satisfying* $|v| \geq 0$ *and* $|f \cdot v| = |f||v|$ $\mathfrak{m}$*-a.e. for every* $f \in L^0(\mathfrak{m}), v \in \mathscr{M}$*, is s.t. the function* $\mathsf{d}_{\mathscr{M}}\colon \mathscr{M} \times \mathscr{M} \to [0, \infty]$ *defined via*
$$\mathsf{d}_{\mathscr{M}}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}', \quad \textit{for some chosen, fixed } \mathfrak{m}' \textit{ so that } \mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m},$$
is a complete distance on $\mathscr{M}$ *inducing the topology* τ*.*

It is possible to check that point iii) does not make such definition ill posed, since the metric $\mathsf{d}_{\mathscr{M}}$ may depend on the choice of $\mathfrak{m}'$, but the induced topology τ on $\mathscr{M}$ does not. We recall that a module isomorphism between two $L^0(\mathfrak{m})$-normed modules $\mathscr{M}, \mathscr{N}$ is a bijection $\Phi : \mathscr{M} \to \mathscr{N}$ preserving the module's operations, namely the pointwise isometry $|\Phi(v)| = |v|$ $\mathfrak{m}$-a.e. and the multiplication $\Phi(f \cdot v) = f\Phi(v)$, for every $v \in \mathscr{M}$ and $f \in L^0(\mathfrak{m})$.

We face now basic construction of normed modules that are going to be used during this note.

Dual of a module

Motivated by the need to discuss in this note also tangent structures over a metric measure space it is natural to give a definition of *dual* of a module.

Definition 1.1.13 (Dual of $L^0(\mathfrak{m})$-normed module)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $\mathscr{M}$ *be a* $L^0(\mathfrak{m})$*-normed module. Then, we define its* dual module $\mathscr{M}^*$ *as*
$$\mathscr{M}^* := \{L\colon \mathscr{M} \to L^0(\mathfrak{m})\colon L \textit{ is } L^0(\mathfrak{m})\textit{-linear and continuous}\},$$
equipped with the following operations
$$\begin{aligned} (L + L')(v) &:= L(v) + L'(v), \\ (f \cdot L)(v) &:= L(f \cdot v), \\ |L|_* &:= \operatorname{ess\,sup} \{L(v)\colon v \in \mathscr{M}, |v| \leq 1 \ \mathfrak{m}\textit{-a.e.}\}, \end{aligned}$$
for any $f \in L^0(\mathfrak{m})$*,* $L, L' \in \mathscr{M}^*, v \in \mathscr{M}$*.*

It is an easy task to check that $\mathscr{M}^*$ has a natural $L^0(\mathfrak{m})$-normed module structure.

Pullbacks

There are situations in which, given two metric measure spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mu)$ and a Borel map $u\colon \mathrm{X}\to\mathrm{Y}$, one wants to carry a differential structure on Y directly over X. This can be done, provided a suitable compressibility condition is satisfied. The reason is that μ-a.e. defined objects (e.g. functions in $L^0(\mu)$) are pulled back to $\mathfrak{m}$-a.e. defined ones (e.g. functions in $L^0(\mathfrak{m})$), provided μ-negligible sets are also $u_\sharp\mathfrak{m}$-negligible.

Proposition 1.1.14. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mu)$ be two metric measure spaces, $\mathscr{M}$ a $L^0(\mu)$-normed module, $u\colon \mathrm{X}\to\mathrm{Y}$ Borel so that $u_\sharp\mathfrak{m}\ll\mu$.*

Then, there exists a unique couple (up to a unique isomorphism) $(u^*\mathscr{M},u^*)$ so that $u^*\mathscr{M}$ is a $L^0(\mathfrak{m})$-normed module and $u^*\colon\mathscr{M}\to u^*\mathscr{M}$ is linear, continuous and so that $|u^*v|=|v|\circ u$ $\mathfrak{m}$-a.e. for every $v\in\mathscr{M}$. Moreover, the set $\{u^*v\colon v\in\mathscr{M}\}$ generates $u^*\mathscr{M}$.

Then, we call $u^*\mathscr{M}$ the *pulled back module* while the map u^* is called pullback map.

Restrictions and extensions

There are two canonical ways to localize or extend a given normed modules, namely restrictions and extensions.

Given a $L^0(\mathfrak{m})$-normed modules $\mathscr{M}$ and a Borel non negligible set $E\subset\mathrm{X}$, we define the *restriction* $\mathscr{M}_{|_E}:=\{\chi_E v\colon v\in\mathscr{M}\}$ and, when equipped it with the module operations inherited from $\mathscr{M}$, it is straightforward to see that the resulting structure is a $L^0(\mathfrak{m}_{|_E})$-normed module.

Conversely, defining $\nu:=\mathfrak{m}_{|_E}$ and given a $L^0(\nu)$-normed modules $\mathscr{M}$, we can canonically associate a $L^0(\mathfrak{m})$-normed module denoted $\mathrm{Ext}(\mathscr{M})$. We stress what the intuition is: the module $\mathrm{Ext}(\mathscr{M})$ 'should coincide' with $\mathscr{M}$ on E, therefore its elements are going for convenience to be set equal to 'zero' outside E. This discussion can be made rigorous recalling the extension operator: given $E\subset\mathrm{X}$ Borel, we consider the map $\mathrm{ext}\colon L^0(\mathfrak{m}_{|_E})\to L^0(\mathfrak{m})$, sending $f\in L^0(\mathfrak{m}_{|_E})$ to the (equivalence class up to $\mathfrak{m}$-a.e. equality of the) function $\mathrm{ext}(f)$ agreeing with f on E and set zero on $\mathrm{X}\setminus E$. Then, given a $L^0(\mathfrak{m}_{|_E})$-normed module $\mathscr{M}$, we define the set $\mathrm{Ext}(\mathscr{M}):=\mathscr{M}$ as a set and we equipp it with the operation $f\cdot v:=\mathrm{proj}(f)\cdot v\in\mathscr{M}$, for every $f\in L^0(\mathfrak{m})$ and $v\in\mathrm{Ext}(\mathscr{M})=\mathscr{M}$, where $\mathrm{proj}(f)\in L^0(\mathfrak{m}_{|_E})$ is the projection operator given by passage to the $\mathfrak{m}$-a.e. equality on E. We also consider on the pointwise norm $|\mathrm{ext}(v)|:=\mathrm{ext}(|v|)\in L^0(\mathfrak{m})$. It can be directly checked that the resulting structure is the one of an $L^0(\mathfrak{m})$-normed module. Moreover, the extension of a module commutes with the dual operation in a trivial way noticing that

$$\mathrm{Ext}(\mathscr{M}^*)\sim\mathrm{Ext}(\mathscr{M})^*,\quad\text{via}\quad \mathrm{ext}(L)(\mathrm{ext}(v))=\mathrm{ext}(L(v)),$$

for every $v\in\mathscr{M},L\in\mathscr{M}^*$.

Hilbert modules

An $L^0(\mathfrak{m})$-normed module $\mathscr{H}$ is called a *Hilbert module* provided

$$2|v|^2+2|w|^2=|v-w|^2+|v+w|^2,\qquad \mathfrak{m}\text{-a.e.},$$

holds for any $v,w\in\mathscr{H}$. Then, by polarization, it can be readily checked that the formula

$$\langle v,w\rangle:=\frac{1}{2}(|v-w|^2-|v|^2-|w|^2)\in L^0(\mathfrak{m}),\tag{1.1.13}$$

defines a $L^0(\mathfrak{m})$-bilinear map on its entries, called *pointwise scalar product*, satisfying

$$\begin{aligned}&\langle v,w\rangle=\langle w,v\rangle,\\ &\big|\langle v,w\rangle\big|\le|v||w|\qquad\mathfrak{m}\text{-a.e.},\\ &\langle v,v\rangle=|v|^2,\end{aligned}$$

for every $v, w \in \mathscr{H}$. It can be readily checked that a Hilbert module is preserved by the action of restriction, extension and pullback. Also, a direct check shows that a module is Hilbert if and only if its dual is Hilbert. Indeed, the two can be identified via a suitable Riesz isomorphism of $L^0(\mathfrak{m})$-normed modules $\mathcal{R} : \mathscr{H} \to \mathscr{H}^*$, sending $v \mapsto \mathcal{R}(v) =: L$, where $L \in \mathscr{H}^*$ is the unique element satisfying

$$|L(v)| = |v|^2 = |L|_*^2, \qquad \mathfrak{m}\text{-a.e.}.$$

Finally, a Hilbert base of a Hilbert module $\mathscr{H}$ is a collection $(e_i)_{i\in\mathbb{N}} \subset \mathscr{H}$ satisfying

$$\{e_i : i \in \mathbb{N}\} \text{ generates } \mathscr{H}, \qquad \langle e_i, e_j \rangle = \delta_{i,j} \ \mathfrak{m}\text{-a.e.}, \ \forall i, j \in \mathbb{N}.$$

We say that a Hilbert module $\mathscr{H}$ has dimension $d \in \mathbb{N}$, provided a base of d-elements can be taken as in the above.

$L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules

Given a $L^0(\mathfrak{m})$-normed modules, we can restrict our attention to elements which are not just Borel, but integrable. To this aim, fix $p \in [1, \infty]$ and, given a $L^0(\mathfrak{m})$-normed module $\mathscr{M}$, we can consider the set

$$\mathscr{M}_p := \{v \in \mathscr{M} : |v| \in L^p(\mathfrak{m})\}.$$

Differently from the original structure, that is *topological*, the latter can be turned into a Banach space by declaring

$$\|v\|_{\mathscr{M}_p} := \||v|\|_{L^p(\mathfrak{m})}, \qquad \forall v \in \mathscr{M}_p.$$

It is also easy to see that $\mathscr{M}_p$ is a module over the commutative ring of L^∞ functions as the product operation (inherited from the $\mathscr{M}$) between a bounded function with an integrable object is easily seen to be closed.

We shall not discuss further these spaces (and refer to [97] for a complete treatment), but we mention that the resulting structure goes under the name of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules. Truth to be told, in the theory developed in [97], $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules are the main object of study employed to build a nonsmooth differential calculus over metric measure spaces. Then, the author introduces $L^0(\mathfrak{m})$-normed modules *a posteriori* via a L^0-completion construction.

1.2 Nonsmooth calculus at first order

1.2.1 Test plans

We follow the definition of Sobolev spaces proposed in [20] (for earlier approaches see the original work [64] of Cheeger and the one [178] of Shanmugalingam).

Definition 1.2.1 (*q*-test plan)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $q \in (1, \infty)$. *A measure* $\pi \in \mathscr{P}(C(\mathrm{X}, [0,1]))$ *is said to be a* q-test plan, *provided*

i) *There exists* $C > 0$ *so that* $(\mathsf{e}_t)_\sharp \pi \le C\mathfrak{m}$ *for every* $t \in [0, 1]$*;*

ii) *We have* $\int \mathsf{Ke}_q(\gamma)\, \mathrm{d}\pi < \infty$.

Moreover, we say that π *is an* ∞-test plan *if, instead of* ii)*, we require*

ii') π *is concentrated on Lipschitz curves with* $\mathrm{Lip}(\gamma) \in L^\infty(\pi)$.

We usually refer to i) as the 'compression condition' and denote by $\mathrm{Comp}(\pi)$ the smallest constant C satisfying i) (and call it compression constant of π). Moreover, we shall sometimes deal with plans π having bounded support: this is equivalent to say that the *trace*

$$[\pi] := \bigcup_{t\in[0,1]} \mathrm{supp}\big((\mathsf{e}_t)_\sharp \pi\big) = \{\gamma_t : \gamma \in \mathrm{supp}(\pi), \ t \in [0,1]\} \subset \mathrm{X}, \tag{1.2.1}$$

is bounded. We define the kinetic energy of a q-test plan π as follows:

$$\begin{aligned} \mathrm{Ke}_q(\pi) &:= \int \mathsf{Ke}_q(\gamma)\,\mathrm{d}\pi, \qquad &\text{if } q \in (1,\infty),\\ \mathrm{Ke}_\infty(\pi) &:= \|\mathsf{ms}\|_{L^\infty(\pi\otimes\mathscr{L}^1)}, \qquad &\text{if } q=\infty, \end{aligned}$$

For $q<\infty$, we recall that

$$\pi \mapsto \mathrm{Ke}_q(\pi) \qquad \text{is weakly lower semicontinuous.} \tag{1.2.2}$$

This can be easily seen as the integrand $\mathsf{Ke}_q(\gamma)$ appearing in the definition of $\mathrm{Ke}_q(\cdot)$ is a lower semicontinuous on $C([0,1],\mathrm{X})$ by Lemma 1.1.3 and bounded from below, therefore it admits the representation as supremum of continuous and bounded functions. For $q=\infty$, we introduce an alternative terminology that reflects the extreme case, namely

$$\mathrm{Lip}(\pi) := \mathrm{Ke}_\infty(\pi), \qquad \forall \pi\ \infty\text{-test plan}.$$

Remark 1.2.2. Given any ∞-test plan π on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, the quantity $\mathrm{Lip}(\pi)$ can be equivalently characterised as the minimal $L\geq 0$ such that π is concentrated on L-Lipschitz curves.

Indeed, by Fubini's theorem we know that π-a.e. γ is Lipschitz and satisfies $|\dot\gamma_t|\leq \mathrm{Lip}(\pi)$ for $\mathscr{L}^1$-a.e. $t\in[0,1]$, thus accordingly $\mathsf{d}(\gamma_t,\gamma_s)\leq \int_s^t|\dot\gamma_r|\,\mathrm{d}r \leq \mathrm{Lip}(\pi)|t-s|$ for every $s,t\in[0,1]$ with $s\leq t$, which shows that γ is $\mathrm{Lip}(\pi)$-Lipschitz. On the other hand, if π is concentrated on L-Lipschitz curves for some $L\geq 0$, then for π-a.e. curve γ we have that

$$|\dot\gamma_t| = \lim_{h\to 0}\frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} \leq \lim_{h\to\infty}\frac{L|(t+h)-t|}{|h|} = L, \qquad \text{a.e. } t\in[0,1],$$

whence it follows that $\mathrm{Lip}(\pi)=\|\mathsf{ms}\|_{L^\infty(\pi\otimes\mathcal{L}_1)}\leq L$. Therefore, the claim is achieved. ■

Let us clarify now a weak continuity of plans under the pushforward of the evaluation map.

Remark 1.2.3. We point out that, given any $t\in[0,1]$, it holds that

$$\mathscr{P}\big(C([0,1],\mathrm{X})\big) \ni \pi \mapsto (\mathsf{e}_t)_\sharp\pi \in \mathscr{P}(\mathrm{X}), \qquad \text{is continuous,}$$

where both the domain and the target are endowed with the weak topology. Indeed, the continuity of e_t ensures that $\varphi\circ\mathsf{e}_t\in C_b\big(C([0,1],\mathrm{X})\big)$ whenever $\varphi\in C_b(\mathrm{X})$, so that if we assume $\pi_n\rightharpoonup\pi$, then we have that $\int\varphi\,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_n\to\int\varphi\,\mathrm{d}(\mathsf{e}_t)_\sharp\pi$ for every $\varphi\in C_b(\mathrm{X})$.

In particular, if we further assume that π and $(\pi_n)_{n\in\mathbb{N}}$ are q-test plans for some $q\in[1,\infty]$ and $\mathrm{C}:=\varliminf_{n\to\infty}\mathrm{Comp}(\pi_n)<+\infty$, then it holds that $\mathrm{Comp}(\pi)\leq\mathrm{C}$. This can be proved by observing that, chosen a subsequence $(n_i)_{i\in\mathbb{N}}$ which satisfies $\mathrm{C}=\lim_{i\to\infty}\mathrm{Comp}(\pi_{n_i})$, we have $\int\varphi\,\mathrm{d}(\mathsf{e}_t)_\sharp\pi=\lim_i\int\varphi\,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_{n_i}\leq\mathrm{C}\int\varphi\,\mathrm{d}\mathfrak{m}$ for every $t\in[0,1]$ and $\varphi\in C_b(\mathrm{X})^+$. ■

The kinetic q-energies can be extended to functionals $\mathrm{Ke}_q\colon\mathscr{P}\big(C([0,1],\mathrm{X})\big)\to[0,+\infty]$ and $\mathrm{Lip}\colon\mathscr{P}\big(C([0,1],\mathrm{X})\big)\to[0,+\infty]$ by declaring that $\mathrm{Ke}_q(\pi):=+\infty$ whenever π is not a q-test plan and $\mathrm{Lip}(\pi)=\infty$ when π is not an ∞-test plan. Finally, we conclude this part by recalling two well known closed operations within the class of q-test plans, namely restrictions and rescalings. Given their importance in this manuscript, we give their proof for completeness.

Lemma 1.2.4 (Restriction and rescaling)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, $q\in(1,\infty]$ and π a q-test plan π. Let also $\Gamma\subset C([0,1],\mathrm{X})$ be Borel with $\pi(\Gamma)>0$ and $s,t\in[0,1]$ with $s<t$.*

Then, the probabilities

$$\frac{\pi|_\Gamma}{\pi(\Gamma)} \qquad \textit{and} \qquad (\mathsf{Restr}_s^t)_\sharp\pi, \tag{1.2.3}$$

are q-test plans, where $\mathsf{Restr}_s^t\colon C([0,1],\mathrm{X})\to C([0,1],\mathrm{X})$ is defined via $\mathsf{Restr}_s^t(\gamma):=\gamma_{(1-\cdot)s+\cdot t}$.

Proof. Let us start observing that, for every $r \in [0,1]$, we have

$$\pi(\Gamma)^{-1}(\mathsf{e}_r)_\sharp \pi_{|_\Gamma} \leq \pi(\Gamma)^{-1}\mathrm{Comp}(\pi)\mathfrak{m},$$
$$(\mathsf{e}_r)_\sharp(\mathsf{Restr}_s^t)_\sharp \pi = (\mathsf{e}_r \circ \mathsf{Restr}_s^t)_\sharp \pi = (\mathsf{e}_{(1-r)s+rt})_\sharp \pi \leq \mathrm{Comp}(\pi)\mathfrak{m}.$$

Next, for $q < \infty$, we estimate

$$\mathrm{Ke}_q(\pi(\Gamma)^{-1}\pi_{|_\Gamma}) \leq \pi(\Gamma)^{-1}\mathrm{Ke}_q(\pi) < \infty,$$
$$\mathrm{Ke}_q((\mathsf{Restr}_s^t)_\sharp \pi) \leq |t-s|^{q-1}\mathrm{Ke}_q(\pi) < \infty,$$

having used in the second inequality the change of variable formula $\mathsf{Ke}_q(\mathsf{Restr}_s^t(\gamma)) = |t-s|^{q-1}\mathsf{Ke}_q(\gamma)$ valid for every $\gamma \in C([0,1],\mathrm{X})$. While, if $q = \infty$, it is immediate to see that both the plans are concentrated on equiLipschitz curve and they satisfy

$$\mathrm{Lip}(\pi(\Gamma)^{-1}\pi_{|_\Gamma}) \leq \mathrm{Lip}(\pi), \tag{1.2.4}$$
$$\mathrm{Lip}((\mathsf{Restr}_s^t)_\sharp \pi) \leq |t-s|\mathrm{Lip}(\pi). \tag{1.2.5}$$

The proof is then concluded as we checked i)-ii)/ii') in Definition 1.2.1. □

1.2.2 Sobolev functions

We are finally ready to present the definition of Sobolev class via duality with q-test plans.

Definition 1.2.5 (Sobolev class)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and* $p \in (1,\infty)$*. A Borel function* f *belongs to* $S^p(\mathrm{X})$*, provided there exits* $G \in L^p(\mathfrak{m})$*,* $G \geq 0$*, called* p-weak upper gradient *so that*

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi, \qquad \forall \pi \text{ } q\text{-test plan.} \tag{1.2.6}$$

Let us comment on the well-posedness of the definition.

Remark 1.2.6. First, we notice that, given a q-test plan π, the composition $G \circ \gamma$ is π-a.e. independent on the chosen representative. Indeed, given $G_1, G_2 \colon \mathrm{X} \to \mathbb{R}$ be two Borel representatives and notice $A := \{G_1 \neq G_2\}$ is $\mathfrak{m}$-negligible. Then, by the compression condition, we have $\pi(\{\gamma_t \in A\}) = 0$ for every $t \in [0,1]$ and ultimately that

$$0 = \int_0^1 \pi(\{\gamma_t \in A\})\,\mathrm{d}t = \iint_0^1 \chi_A \circ \mathsf{e}_t\,\mathrm{d}t\mathrm{d}\pi,$$

having used Fubini's Theorem. Therefore, we deduce that π-a.e. γ it holds $\int_0^1 \chi_A(\gamma_t)\,\mathrm{d}t = 0$. For any such γ, this means that $\gamma_t \notin A$ a.e. $t \in [0,1]$ and, accordingly, that $G_1(\gamma_t) = G_2(\gamma_t)$ a.e.. Being the collection of these γ of full measure, the claim is proved.

Secondly, fixing with abuse of notation any Borel representative G, we notice that the assignment $(t,\gamma) \mapsto G(\gamma_t)|\dot\gamma_t|$ is Borel (see, e.g. [104]) and the right hand side in (1.2.6) is finite. Indeed, the properties of any q-test plan π ensure

$$\begin{aligned}\iint_0^1 G(\gamma_t))|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi &\leq \left(\iint_0^1 G^p(\gamma_t)\,\mathrm{d}t\mathrm{d}\pi\right)^{1/p}\left(\iint_0^1 |\dot\gamma_t|^q\,\mathrm{d}t\mathrm{d}\pi\right)^{1/q} \\ &\leq \mathrm{Comp}(\pi)\|G\|_{L^p(\mathfrak{m})}\mathrm{Ke}_q^{1/q}(\pi) < +\infty.\end{aligned} \tag{1.2.7}$$

■

The calculation (1.2.7) shows at the same time finiteness of the integral and that

$$L^p(\mathfrak{m}) \ni G \mapsto \iint_0^1 G(\gamma_t))|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi \qquad \text{is linear and continous.} \tag{1.2.8}$$

Let us give a simple example of functions belonging to the Sobolev class. The inequality $|f(\gamma_1) - f(\gamma_0)| \le \int_0^1 \mathrm{lip}\ f(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t$ for every $f \in \mathrm{Lip}_{bs}(\mathrm{X})$ and $\gamma \in C([0,1],\mathrm{X})$ reveals that

$$\mathrm{Lip}_{bs}(\mathrm{X}) \subset S^p(\mathrm{X}), \qquad \forall p \in (1,\infty).$$

and lip $f \in L^p(\mathfrak{m})$ is a p-weak upper gradient.

Next, we recall two important properties of weak upper gradients and, in the following proposition, a lower semicontinuity property.

Convexity. For every $f, g \in S^p(\mathrm{X})$ and $\lambda \in (0,1)$, it holds

$$|D((1-\lambda)f + \lambda g)|_p \le (1-\lambda)|Df|_p + \lambda|Dg|_p, \qquad \mathfrak{m}\text{-a.e..}$$

Minimum. For every $f \in S^p(\mathrm{X})$, if G_1, G_1 are p-weak upper gradient of f, we have that

$$\min\{G_1, G_2\}, \qquad \text{is a } p\text{-weak upper gradient.}$$

Proposition 1.2.7 (Closure of p-weak upper gradients)**.** *Let $(f_n) \subset S^p(\mathrm{X})$ and $f\colon \mathrm{X} \to \mathbb{R}$ so that $f_n \to f$ $\mathfrak{m}$-a.e. and let (G_n) be a sequence of p-weak upper gradients. Assume that $G_n \rightharpoonup G$ for some non negative $G \in L^p(\mathfrak{m})$ as n goes to infinity.*

Then, $f \in S^p(\mathrm{X})$ and G is a p-weak upper gradient of f.

Proof. Observe that, by assumptions, f is Borel and $\{\lim_n f_n(x) \ne f(x)\}$ is Borel negligible hence, we have

$$(\mathsf{e}_t)_\sharp\pi(\{\lim_n f_n(x) \ne f(x)\}) \le C\mathfrak{m}(\{\lim_n f_n(x) \ne f(x)\}) = 0, .$$

for all $t \in [0,1]$. This in turn implies that $f_n(\gamma_t) \to f(\gamma_t)$ for π-a.e. γ and, by Fatou's Lemma, ultimately that

$$\begin{aligned}\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi &\le \varliminf_{n\to+\infty} \int |f_n(\gamma_1) - f_n(\gamma_0)|\,\mathrm{d}\pi \\ &\le \lim_{n\to+\infty} \iint_0^1 G_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi \overset{(1.2.8)}{=} \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi.\end{aligned}$$

□

Then, Proposition 1.2.7 and the two properties before ensures that the set of weak upper gradients of a given function is convex and closed in $L^p(\mathfrak{m})$ and has a lattice structure. This leads then to the well-posedness of the following key definition.

Definition 1.2.8 (The object $|Df|_p$)**.** *The* p-minimal weak upper gradient *of $f \in S^p(\mathrm{X})$, denoted by $|Df|_p$, is the minimal normed element in $L^p(\mathfrak{m})$ of the set of weak upper gradient of f. Moreover, $|Df|_p$ is minimal also in the $\mathfrak{m}$-a.e. sense.*

Remark 1.2.9. Differently from the smooth category, if $p_1 \ne p_2$ and $f \in S^{p_1}(\mathrm{X}) \cap S^{p_2}(\mathrm{X})$ then we should not expect directly from Definition 1.2.8 that $|Df|_{p_1}$ agrees with $|Df|_{p_2}$. Let us clarify on their relation: for $p_1 \le p_2$, Hölder inequality easily implies $\{q_1\text{-test plans}\} \subseteq \{q_2\text{-test plans}\}$. Therefore the implication

$$p_1 \le p_2 \text{ and } \begin{array}{l} f \in S^{p_2}(\mathrm{X}) \\ |Df|_{p_1} \in L^{p_1}(\mathfrak{m}) \end{array} \quad \Rightarrow \quad \begin{array}{l} f \in S^{p_1}(\mathrm{X}) \\ |Df|_{p_2} \text{ is a } p_1\text{-weak upper gradient} \\ |Df|_{p_1} \le |Df|_{p_2} \quad \mathfrak{m}\text{-a.e.} \end{array} \tag{1.2.9}$$

can be easily seen to be true on arbitrary an metric measure space. In general, there are examples [82] showing that strict inequality may occur. ■

We recall next three important properties of minimal weak upper gradients and refer to [21, 95] for the proof.

LOCALITY. For every $f, g \in S^p(\mathrm{X})$, it holds

$$|Df|_p = |Dg|_p \qquad \mathfrak{m}\text{-a.e. on } \{f = g\}.$$

CHAIN RULE (INEQ.). Let $\varphi \in \mathrm{Lip}(\mathrm{X})$ and $f \in S^p(\mathrm{X})$, then

$$\varphi \circ f \in S^p(\mathrm{X}) \quad \text{and} \quad |D(\varphi \circ f)|_p \leq \mathrm{Lip}(\varphi)|Df|_p, \qquad \mathfrak{m}\text{-a.e.}.$$

LEIBNIZ RULE. For $f, g \in S^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ we have $fg \in S^p(\mathrm{X})$ with

$$|D(fg)|_p \leq |f||Dg|_p + |g||Df|_p, \qquad \mathfrak{m}\text{-a.e.}. \tag{1.2.10}$$

Before passing to the definition of the full Sobolev space, we recall that arguing by restriction and rescalings [21] (see, also [104]), it is possible to see (indeed, it is equivalent) that if $f \in S^p(\mathrm{X})$ then for every q-test plan π it holds that

$$\pi\text{-a.e. } \gamma \quad f \circ \gamma \in W^{1,1}(0,1) \quad \text{and} \quad (f \circ \gamma)'_t \leq |Df|_p(\gamma_t)|\dot{\gamma}_t| \quad \text{a.e. } t. \tag{1.2.11}$$

We now are going to the define the full Sobolev space and, to this aim, we define the p-Cheeger energy as the functional $\mathsf{Ch}_p \colon L^p(\mathfrak{m}) \to [0, \infty]$ defined by

$$\mathsf{Ch}_p(f) := \int |Df|_p^p \, \mathrm{d}\mathfrak{m},$$

if f is (the equivalence class of a Borel function) belonging to $S^p(\mathrm{X})$, and $+\infty$ otherwise. It is immediate to see, using the convexity of minimal weak upper gradients and applying Proposition 1.2.7 with $G := |Df_n|_p$ for a L^p-converging sequence (f_n), that

$$L^p(\mathfrak{m}) \ni f \mapsto \mathsf{Ch}_p(f) \qquad \text{is convex and weakly lower semicontinuous.} \tag{1.2.12}$$

Hence, requiring as customary a function in the Sobolev class to be also p-integrable leads to the definition of the full Sobolev space.

Definition 1.2.10 (Sobolev space $W^{1,p}(\mathrm{X})$)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $p \in (1, \infty)$. The Sobolev space, denoted by $W^{1,p}(\mathrm{X})$, is $L^p(\mathfrak{m}) \cap S^p(\mathrm{X})$ as a set, equipped with the norm*

$$\|f\|_{W^{1,p}(\mathrm{X})} := \left(\|f\|^p_{L^p(\mathfrak{m})} + \mathsf{Ch}_p(f)^p\right)^{\frac{1}{p}}, \qquad \forall f \in W^{1,p}(\mathrm{X}).$$

To be more precise in the above, an element $f \in L^p(\mathfrak{m}) \cap S^p(\mathrm{X})$ is (the equivalence class of a) function in $L^p(\mathfrak{m})$ that admits a Borel representatives that is in the Sobolev class $S^p(\mathrm{X})$. Proposition 1.2.7 and standard arguments grant that $W^{1,p}(\mathrm{X})$ is a Banach space but it is in general false that it is reflexive. This will occur under a suitable doubling assumption of the space (see Theorem 1.3.5 below), especially implying that Lipschitz functions are dense. However, on arbitrary metric measure spaces, a weaker type of density of the Lipschitz class can be deduced [20].

Theorem 1.2.11 (Density in energy of Lipschitz functions)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $p \in (1, \infty)$. Then, the class* $\mathrm{Lip}_{bs}(\mathrm{X})$ *is* dense in energy *in $W^{1,p}(\mathrm{X})$, i.e. for every $f \in W^{1,p}(\mathrm{X})$, there exists a sequence $(f_n) \subset \mathrm{Lip}_{bs}(\mathrm{X})$ so that*

$$f_n \to f, \quad \mathrm{lip}\, f_n \to |Df|_p, \qquad \textit{in } L^p(\mathfrak{m}).$$

We finish this part by recalling the definition of *local* Sobolev class and Sobolev space defined on open subsets with possibly homogeneous boundary conditions.

Definition 1.2.12. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space,* $p \in (1, \infty)$*. A Borel function* f *belongs to* $S^p_{loc}(\mathrm{X})$*, provided there is* $G \in L^p_{loc}(\mathfrak{m})$*,* $G \geq 0$*, such that for any* $k > 0$ *and* $\eta \in \mathrm{Lip}_{bs}(\mathrm{X})$*, we have* $\eta f^k \in S^p(\mathrm{X})$*, where* $f^k := k \wedge (f \vee -k)$*, with*

$$|D(\eta f^k)| \leq |\eta| G, \qquad \mathfrak{m}\text{-a.e. on } \{\eta = 1\}.$$

In this case, we define $|Df|_p \in L^p_{loc}(\mathrm{X})$ *via*

$$|Df|_p = |D(\eta f^k)|_p \qquad \mathfrak{m}\text{-a.e. on } \{\eta = 1\} \cap \{|f| < k\},$$

for every η *and* k *as before.*

We point out that, the locality of the minimal p-weak upper gradient grants that the above definition is well-posed and the object $|Df|_p$ is $\mathfrak{m}$-a.e. well defined. It can also be proven (see [95]) that $f \in S^p_{loc}(\mathrm{X})$ if and only if for some $G \in L^p_{loc}(\mathrm{X})$ non-negative (1.2.6) holds. In the sequel, we shall also deal with Sobolev functions defined on open subset $\Omega \subset \mathrm{X}$ of the space. Following [95], we can define:

Definition 1.2.13 (The spaces $W^{1,p}(\Omega), W^{1,p}_0(\Omega)$)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $\Omega \subset \mathrm{X}$ *be open. We define* $W^{1,p}_{loc}(\Omega)$ *as the subset of* $f \in L^p_{loc}(\Omega)$ *so that* $\eta f \in W^{1,p}(\mathrm{X})$ *for every* $\eta \in \mathrm{Lip}_{bs}(\mathrm{X})$ *with* $\mathrm{supp}(\eta) \subset \Omega$ *and, in this case, we define*

$$|Df|_p := |D(\eta f)|_p \qquad \mathfrak{m}\text{-a.e. on } \{\eta = 1\}.$$

Then, we say that $f \in W^{1,p}(\Omega)$ *if* $f \in W^{1,p}_{loc}(\Omega)$ *with* $f, |Df|_p \in L^p(\Omega)$ *and define* $\|f\|^p_{W^{1,p}(\Omega)} := \|f\|^p_{L^p(\Omega)} + \||Df|_p\|^p_{L^p(\Omega)}$*. Finally, we set*

$$W^{1,p}_0(\Omega) := \overline{\{f \in W^{1,p}(\Omega) \colon \mathrm{supp}(f) \subset \Omega\}}^{W^{1,p}(\Omega)}.$$

The above definition, again, is well-posed in light of the *locality* property of the minimal p-weak upper gradient. Moreover, it can be readily checked that $W^{1,p}(\Omega)$, and consequently $W^{1,p}_0(\Omega)$, are Banach when considered with the defined norm. Finally, standard arguments show that calculus tools such as chain and Leibniz rule are in place.

1.2.3 Sobolev maps

We recall that for an open subset $\Omega \subseteq \mathrm{X}$ (possibly $\Omega = \mathrm{X}$) of a metric measure space, and for $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$ a complete (not necessarily separable) metric space, the space $L^0(\Omega, \mathrm{Y})$ is the collection of Borel maps $u \colon \mathrm{X} \to \mathrm{Y}$ (identified up to $\mathfrak{m}$-a.e. equality) which are *essentially separately valued*, i.e. there exists a $\mathfrak{m}$-negligible set $N \subset \mathrm{X}$ so that $u(\mathrm{X} \setminus N) \subset \mathrm{Y}$ is separable. Then, for $p \in (1, \infty)$ and a pointed complete metric space $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \bar{y})$, we define $L^p(\Omega, \mathrm{Y}_{\bar{y}}) \subset L^0(\mathrm{X}, \mathrm{Y})$ as the subset of (equivalence class of) maps so that $\int_\Omega \mathsf{d}_{\mathrm{Y}}(u(x), \bar{y}) \, \mathrm{d}\mathfrak{m}(x) < \infty$. Obviously, if $\mathfrak{m}(\Omega) < \infty$, this space is independent on the choice of $\bar{y}$, and in this situations we drop the subscript simply writing $L^p(\Omega, \mathrm{Y})$. We consider

$$\mathsf{d}^p_{L^p}(u, v) := \int_\Omega \mathsf{d}^p_{\mathrm{Y}}(u, v) \, \mathrm{d}\mathfrak{m}, \qquad \forall u, v \in L^p(\Omega, \mathrm{Y}_{\bar{y}}),$$

and observe that it is a metric turning $L^p(\Omega, \mathrm{Y}_{\bar{y}})$ into a complete metric space with density of finite range maps.

Following [119], we recall the definition of a metric valued Sobolev map. Notice that in such paper, the case $p = 2$ has been extensively discussed, but is is also remarked that for general $p \in (1, \infty)$ all the statements directly extends without difficulties.

Definition 1.2.14 (Sobolev class $S^p(\mathrm{X}, \mathrm{Y})$)**.** *The set* $S^p(\mathrm{X}, \mathrm{Y})$*, is the collection of all Borel and essentially separately valued maps* $u \colon \mathrm{X} \to \mathrm{Y}$ *so that there is* $G \in L^p(\mathfrak{m}), G \geq 0$ *called p-weak upper gradient of* u*, such that for any* $\varphi \in \mathrm{Lip}(\mathrm{Y})$*, we have* $\varphi \circ u \in S^p(\mathrm{X})$ *and*

$$|D(\varphi \circ u)|_p \leq \mathrm{Lip}(\varphi) G \qquad \mathfrak{m}\text{-a.e.}.$$

The least, in the $\mathfrak{m}$-a.e. sense, function G for which the above holds will be denoted $|Du|_p$ and called minimal p-weak upper gradient.

The object $|Du|_p$ in the above definition is well-posed, as the set of p-weak upper gradient of a given map u is a closed lattice. Moreover, it can be computed by

$$|Du|_p = \text{ess sup}|D(\varphi \circ u)|_p, \tag{1.2.13}$$

as φ varies in $\mathrm{Lip}(\mathrm{Y})$ with $\mathrm{Lip}(\varphi) \leq 1$. See [106] for this claim. Next, we report for later use the following.

Proposition 1.2.15 ([106]). *Let $u \in S^p(\mathrm{X}, \mathrm{Y})$ and let $f \in S^p(\mathrm{Y})$. Then, there is $g \in S^p(\mathrm{X})$ so that $g = f \circ u$ $\mathfrak{m}$-a.e. on $\{|Du|_p > 0\}$ with*

$$|\mathrm{d}_p g| \leq |\mathrm{d}_p f| \circ u |Du|_p, \qquad \mathfrak{m}\text{-a.e.}.$$

More precisely, there is a sequence $(f_n) \subset \mathrm{Lip}_{bs}(\mathrm{Y})$ so that

$$\begin{aligned} f_n &\to f \quad \mu_p\text{-a.e.} & \mathrm{lip}_a(f_n) &\to |\mathrm{d}_p f| & &\text{in } L^p(\mu_p), \\ f_n \circ u &\to g \quad \mathfrak{m}\text{-a.e.} & \mathrm{lip}_a(f_n) \circ u |Du|_p &\to |\mathrm{d}_p f| \circ u |Du|_p & &\text{in } L^p(\mathfrak{m}). \end{aligned}$$

Finally, requiring a $L^p(\mathrm{X}, \mathrm{Y}_{\bar{y}})$-integrability of a map in $S^p(\mathrm{X}, \mathrm{Y})$ yields certainly a notion metric Sobolev space $W^{1,p}(\mathrm{X}, \mathrm{Y}_{\bar{y}})$ which, for the sake of generality, can be suitably adapted for maps defined on open domain as follows.

Definition 1.2.16 (The space $W^{1,p}(\Omega, \mathrm{Y}_{\bar{y}})$). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $\Omega \subset \mathrm{X}$ be open, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \bar{y})$ be a pointed complete metric space and $p \in (1, \infty)$.*

We say that $u \in L^p(\Omega, \mathrm{Y}_{\bar{y}})$ is a Sobolev map, and we write $u \in W^{1,p}(\Omega, \mathrm{Y}_{\bar{y}})$, if there is $G \in L^p(\mathfrak{m}_{|\Omega})$ positive so that for every 1-Lipschitz $\varphi \colon \mathrm{Y} \to \mathbb{R}$ with $\varphi(\bar{y}) = 0$, we have $\varphi \circ u \in W^{1,p}(\Omega)$ with $|D(\varphi \circ u)|_p \leq G$ $\mathfrak{m}$-a.e. on Ω.

However, for the applications considered in the sequel, we shall need instead a definition of Sobolev map in the sense of 'Korevaar-Schoen' [133] (a notion considered first with smooth domains and metric targets). We postpone this definition to the setting where it will be actually considered and linked with the above definition.

1.2.4 BV functions and sets of finite perimeter

We introduce the space of functions of bounded variation and sets finite perimeter following [156, 14].

Definition 1.2.17 (BV-functions). *A function $f \in L^1(\mathfrak{m})$ is of bounded variation, and we write $f \in BV(\mathrm{X})$, provided there exists a sequence of locally Lipschitz functions $f_n \to f$ in $L^1(\mathfrak{m})$ such that*

$$\varlimsup_{n\to\infty} \int \mathrm{lip}\ f_n \,\mathrm{d}\mathfrak{m} < \infty.$$

By localizing this definition, we can define accordingly

$$|\boldsymbol{D}f|(A) := \inf\Big\{ \varliminf_{n\to\infty} \int_A \mathrm{lip}\ f_n \,\mathrm{d}\mathfrak{m} \colon f_n \subset \mathrm{Lip}_{loc}(A), f_n \to f \text{ in } L^1(A) \Big\}, \tag{1.2.14}$$

for every open $A \subset \mathrm{X}$. Often BV functions are defined with the *approximate* Lipschitz constant rather than the local Lipschitz constant used here. However, the two approaches are perfectly compatible, see e.g. [29, Lemma 2.9].

It turns out (see [14] and also the previous work [156] for locally compact spaces) that the map $A \mapsto |\boldsymbol{D}f|(A)$ is the restriction to open sets of a non-negative finite Borel measure called the *total* *variation* of f, obtained via a *Carathédoroty extension* and still denoted $|\boldsymbol{D}f|$ by common abuse of notation. Directly from the definition, we see that

$$|\boldsymbol{D}f|(A) \leq \int_A \mathrm{lip}\ f\, \mathrm{d}\mathfrak{m} \qquad \forall f \in \mathrm{Lip}_{bs}(\mathrm{X}), A \subset \mathrm{X} \text{ open.} \tag{1.2.15}$$

hence in particular $|\boldsymbol{D}f| \leq \mathrm{lip}\ f\mathfrak{m}$ for every f as in the above. Let us comment about the type of convergence given by (1.2.14).

Remark 1.2.18. Let $f \in BV(\mathrm{X})$ and consider, by definition, a sequence $(f_n) \subset \mathrm{Lip}(\mathrm{X})$ so that $f_n \to f$ in $L^1(\mathfrak{m})$ and $\int \mathrm{lip}\ f_n\, \mathrm{d}\mathfrak{m} \to |\boldsymbol{D}f|(\mathrm{X})$ as n goes to infinity. We claim that

$$\mathrm{lip}\ f_n \mathfrak{m} \rightharpoonup |\mathbf{D}f| \qquad \text{in duality with } C_b(\mathrm{X}). \tag{1.2.16}$$

Indeed, for every $\varphi \in C_b(\mathrm{X}), \varphi \geq 0$, setting $\mu_n := \mathrm{lip}\ f_n\mathfrak{m}$, by Cavalieri's formula (applied twice), we have

$$\varliminf_{n\to\infty} \int \varphi \mathrm{lip}\ f_n\, \mathrm{d}\mathfrak{m} \geq \int_0^\infty \varliminf_{n\to\infty} \mu_n(\{\varphi > t\})\, \mathrm{d}t \geq \int_0^\infty |\mathbf{D}f|(\{\varphi > t\})\, \mathrm{d}t = \int \varphi\, \mathrm{d}|\mathbf{D}f|,$$

having used, in the central inequality, that the set $\{\varphi > t\}$ is open and that (f_n) is a competitor sequence in the definition of the quantity $|\mathbf{D}f|(\{\varphi > t\})$. Now, taking instead $\phi := \sup \varphi - \varphi$, we get by optimality of the sequence (f_n) for the quantity $|\mathbf{D}f|(\mathrm{X})$ that

$$\varliminf_{n\to\infty} \int -\varphi \mathrm{lip}\ f_n\, \mathrm{d}\mathfrak{m} \geq -\int \varphi \mathrm{d}|\mathbf{D}f|,$$

which proves (1.2.16).

We point out that the principle hidden behind the above verification is that, given $\mu \in \mathscr{M}_b^+(\mathrm{X})$ and a sequence of measure $(\mu_n) \in \mathscr{M}_b^+(\mathrm{X})$ satisfying $\mu(A) \leq \varliminf_n \mu_n(A)$ for every A open and $\mu_n(\mathrm{X}) \to \mu(\mathrm{X})$, then necessarily $\mu_n \rightharpoonup \mu$ weakly in duality with $C_b(\mathrm{X})$. ■

Let us then clarify from this point a notation to be kept in the sequel:

> **Notation**. In the sequel, whenever $|\boldsymbol{D}f| \ll \mathfrak{m}$, we denote by $|D^{ac}f|$ the density of $|\boldsymbol{D}f|$ with respect to $\mathfrak{m}$.
> Moreover, for $BV(0,1)$ we denote the *classical* space of functions with bounded variation on the real line and, for $\phi \in BV(0,1)$, we denote by $|D\phi|$ the total variation of the measure $D\phi$ defined in distribution.

If we suitably modify Definition 1.2.17 for functions in $L^1_{loc}(\mathfrak{m})$ we can choose $f = \chi_E$ for any $E \subset \mathrm{X}$ Borel and define:

Definition 1.2.19 (Perimeter and finite perimeter sets)**.** *Let E be Borel and A open subset of* X. *The perimeter of E in A, written* $\mathrm{Per}(E, A)$ *is defined as*

$$\mathrm{Per}(E, A) := \inf \Big\{ \varliminf_{n\to\infty} \int_A \mathrm{lip}\ u_n\, \mathrm{d}\mathfrak{m} \colon u_n \subset \mathrm{Lip}_{loc}(A), u_n \to \chi_E \text{ in } L^1_{loc}(A) \Big\}.$$

Moreover, we say that E is a set of finite perimeter if $\mathrm{Per}(E, \mathrm{X}) < \infty$.

Again, (see, e.g. [9, 14, 156]), when E has finite perimeter, it holds that $A \mapsto \mathrm{Per}(E, A)$ is the restriction of a non-negative finite Borel measure to open sets, which we denote by $\mathrm{Per}(E, \cdot)$. Moreover, as a common convention, when $A = \mathrm{X}$ we simply write $\mathrm{Per}(E)$ instead of $\mathrm{Per}(E, \mathrm{X})$.

For a Borel set $E \subset \mathrm{X}$ of finite measure we also define its Minkowski content as:

$$\mathfrak{m}^+(E) = \varliminf_{\delta\to 0^+} \frac{\mathfrak{m}(E^\delta) - \mathfrak{m}(E)}{\delta},$$

where $E^\delta := \{x \in \mathrm{X} \ : \ \mathsf{d}(x, E) < \delta\}$. In general we only have $\mathrm{Per}(E) \leq \mathfrak{m}^+(E)$, however the following approximation result is valid:

Proposition 1.2.20 ([15])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $E \subset B_r(x)$ *be Borel with finite perimeter and* $\mathfrak{m}(E) < +\infty$*. Then for every* $r' > r$ *there exists a sequence* $E_n \subset B_{r'}(x)$ *of closed sets such that* $\chi_{E_n} \to \chi_E$ *in* $L^1(\mathfrak{m})$ *and*

$$\mathrm{Per}(E) = \lim_{n\to\infty} \mathfrak{m}^+(E_n).$$

Proof. The result is contained in [15], however since it does not appear in this exact form we provide some details. The result follows observing that there exists a sequence $f_n \in \mathrm{Lip}(\mathrm{X})$ with $\mathrm{supp}(f_n) \subset B_{r'}(x)$ so that $f_n \to \chi_E$ in $L^1(\mathfrak{m})$ and $\mathrm{Per}(E) = \lim_n \int \mathrm{lip}\ f_n \,\mathrm{d}\mathfrak{m}$. Indeed from this fact, the conclusion follows arguing as in the end of the proof of [15, Theorem 3.6].

To construct the sequence (f_n) we known that from the definition of perimeter there exist $g_n \in \mathrm{Lip}_{loc}(\mathrm{X})$ so that $g_n \to \chi_E$ in $L^1(\mathfrak{m})$ and $\mathrm{Per}(E) = \lim_n \int \mathrm{lip}\ g_n \,\mathrm{d}\mathfrak{m}$. Moreover we can build a cut-off function $\eta \in \mathrm{Lip}(\mathrm{X})$ such that $\eta = 1$ in $B_r(x)$, $0 \le \eta \le 1$, $\mathrm{supp}(\eta) \subset B_{r'}(\mathrm{X})$ and $\mathrm{Lip}(\eta) \le 2(r'-r)^{-1}$. Then we simply take $f_n := g_n\eta$. Clearly $f_n \to \chi_E$. Moreover

$$\mathrm{Per}(E) \le \varliminf_{n\to\infty} \int \mathrm{lip}\ f_n \,\mathrm{d}\mathfrak{m} \le \varlimsup_{n\to\infty} \int \mathrm{lip}\ g_n \,\mathrm{d}\mathfrak{m} + \frac{2}{r'-r}\int_{E^c} g_n \,\mathrm{d}\mathfrak{m} = \mathrm{Per}(E),$$

that is what we wanted. $\square$

We recall that the following *coarea formula* is valid for locally compact space [156, Proposition 4.2], but the proof works more generally on arbitrary metric measure spaces.

Theorem 1.2.21 (Coarea formula)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $f \in BV(\mathrm{X})$*. Then the set* $\{f > t\}$ *is of finite perimeter for a.e.* $t \in \mathbb{R}$ *and given any Borel function* $g : \mathrm{X} \to [0, \infty)$*, it holds that*

$$\int_{\{s\le u<t\}} g \,\mathrm{d}|\boldsymbol{D}f| = \int_s^t \int g \,\mathrm{dPer}(\{f > t\}, \cdot)\,\mathrm{d}t, \qquad \forall s, t \in [0, \infty),\ s < t. \tag{1.2.17}$$

For later use, we recall a key property of distribution functions in the Lipschitz class.

Lemma 1.2.22 (Derivative of the distribution function, ([159]))**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $\Omega \subseteq X$ *be an open subset with* $\mathfrak{m}(\Omega) < +\infty$*. Assume that* $u \in \mathrm{Lip}_c(\Omega)$ *is non-negative and* $|D^{ac}u|(x) \ne 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \{u > 0\}$*. Then the distribution function* $\mu\colon [0, \infty) \to [0, \mathfrak{m}(\Omega)]$ *(as defined in* (1.1.2)*) is absolutely continuous. Moreover it holds*

$$\mu'(t) = -\int \frac{1}{|D^{ac}u|} \,\mathrm{dPer}(\{u > t\}, \cdot) \qquad a.e., \tag{1.2.18}$$

where the quantity $1/|D^{ac}u|$ *is defined to be* 0 *whenever* $|D^{ac}u| = 0$*.*

Equivalent definitions of BV-functions

In Definition 1.2.17, a BV function and its total variation measure are given through a relaxation procedure in the L^1-topology. Even tough this approach is very effective in applications (see Chapter 6), in this Thesis we are also going to conduct an investigation related to the BV space and fill the gap with previously derived approaches in the literature.

We recall next two equivalent characterization (close in spirit to the approach of Sobolev functions defined in duality with test plans).

Theorem 1.2.23 ([13])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then, the space* $BV(\mathrm{X})$ *admits the following characterizations: let* $f \in L^1(\mathfrak{m})$*, then it holds that*

a) $f \in BV(\mathrm{X})$ *if and only if for every* ∞*-test plan* π *it holds*

   i) $f \circ \gamma \in BV(0, 1)$ π*-a.e.* γ*;*

ii) *there exists a finite Borel meausure* $\mu \in \mathscr{M}_b^+(\mathrm{X})$ *so that*

$$\int \gamma_\sharp |D(f\circ\gamma)|(B)\,\mathrm{d}\pi \le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mu(B), \qquad \forall B \in \mathcal{B}(\mathrm{X});$$

b) $f \in BV(\mathrm{X})$ *if and only if there exists* $\mathsf{C} > 0$ *so that*

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi \le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mathsf{C}, \qquad \forall \pi \ \infty\text{-test plan.}$$

Moreover, if $f \in BV(\mathrm{X})$*, then the minimal measure* μ *satisfying* ii) *agrees on open sets with (the set value map defined in* (1.2.14)*)* $|\boldsymbol{D}f|$ *and the least constant* C *in b) is equal to* $|\boldsymbol{D}f|(\mathrm{X})$*.*

Remark 1.2.24. We point out three facts concerning the above theorem.

- ▷ We notice that a) is a well posed condition. Indeed, if $f \in L^1(\mathfrak{m})$, then the bounded compressibility of any plan π ensures that $f\circ\gamma \in L^1(0,1)$ is π-a.e. independent of the chosen representative of f (see Remark 1.2.6 for details). Moreover, the family of B for which $\gamma \mapsto \gamma_\sharp|D(f\circ\gamma)|(B)$ is Borel measurable can be directly checked to be coincident with the Borel σ-field $\mathcal{B}(\mathrm{X})$ by appealing to the definition of total variation measure of a function of bounded variation on the real line and the π-λ theorem. Finally, a minimal $\mu \in \mathscr{M}_b^+(\mathrm{X})$ satisfying ii) always exists [13] (provided the set of μ's is not empty), and can be achieved by taking the least upper bound, in the complete and separable lattice $\mathscr{M}_b^+(\mathrm{X})$, of the family of measures
$$B \mapsto \frac{\int \gamma_\sharp|D(f\circ\gamma)|(B)\,\mathrm{d}\pi}{\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)}$$
as π runs over all the class of ∞-test plans. Well posedness of b) is instead obvious.

- ▷ It has to be said that in [13] it is actually proven:

  a') $f \in BV(\mathrm{X})$ if and only if for every ∞-test plan π it holds
    i) $f\circ\gamma \in BV(0,1)$ and $|f(\gamma_1) - f(\gamma_0)| \le |D(f\circ\gamma)|(0,1)$ π-a.e. γ;
    ii) there exists a finite Borel meausure $\mu \ge 0$ so that

  $$\int \gamma_\sharp |D(f\circ\gamma)|(B)\,\mathrm{d}\pi \le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mu(B), \qquad \forall B \in \mathcal{B}(\mathrm{X}).$$

  It is clear that the characterization a') implies the one of a). So the validity of Theorem 1.2.23 follows by showing the converse. We are then left to show that the condition $|f(\gamma_1)-f(\gamma_0)| \le |D(f\circ\gamma)|(0,1)$ for π-a.e. γ is redundant; this is a direct consequence of Lemma 1.2.26.

- ▷ Let us clarify how to improve the last part of the statement, as notice in [13]. A set value map $\Omega \mapsto \nu(\Omega)$ defined only for $\Omega \subset \mathrm{X}$ open can be extended to the Borel σ-field according to the *Carathéodory extension*

  $$\tilde{\nu}(B) := \inf\{\nu(\Omega) \colon B \subset \Omega \text{ open}\}, \qquad \forall B \in \mathcal{B}(\mathrm{X}).$$

  However, in general it need not be a measure. If $f \in BV(\mathrm{X})$, extending as above the set value map $|\boldsymbol{D}f|$ as defined in (1.2.14) provides instead a Borel measure as proven in [156, 13] and thus satisfies
  $$|\boldsymbol{D}f|(\Omega) = \mu(\Omega), \qquad \forall \Omega \subset \mathrm{X} \text{ open},$$
  where $\mu \in \mathscr{M}_b^+(\mathrm{X})$ is the minimal measure satisfying a)-ii). Given that μ is *outer regular*, we see that (the Charathéodory extension of) $|\boldsymbol{D}f|$ agrees with μ on the whole σ-field $\mathcal{B}(\mathrm{X})$ of Borel sets. Thus, in what follows we will say, with slight abuse of notation, that $|\boldsymbol{D}f|$ is the minimal measure satisfying a)-ii) for every $f \in BV(\mathrm{X})$. In light of this clarification, we have no ambiguity in the sequel.

■

With the characterization a) of the BV space, it is easy to see that Sobolev functions are BV functions.

Proposition 1.2.25. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p\in(1,+\infty)$. If $f\in S^p(\mathrm{X})$ is so that $f,|Df|_p\in L^1(\mathfrak{m})$, then $f\in BV(\mathrm{X})$ and*

$$|\boldsymbol{D}f|\leq |Df|_p\mathfrak{m}. \tag{1.2.19}$$

Proof. Observe that any ∞-test plan is a q-test plan for every $q\in(1,+\infty]$. Hence, for every π a ∞-test plan, we have by (1.2.11) that for π-a.e. γ it holds that $f\circ\gamma\in W^{1,1}(0,1)$ and $|(f\circ\gamma)'_t|\leq |Df|_p(\gamma_t)|\dot\gamma_t|$ for a.e. t. Thus, recalling $|D(f\circ\gamma)|(I)=\int_I|(f\circ\gamma)'_t\,\mathrm{d}t$ for every $I\subset[0,1]$ Borel, we have a)-i) in Theorem 1.2.23. Finally, for every $B\in\mathcal{B}(\mathrm{X})$, observe that

$$\begin{aligned}\int\gamma_\sharp|D(f\circ\gamma)|(B)\,\mathrm{d}\pi&=\iint_0^1\chi_{\gamma^{-1}(B)}(t)(f\circ\gamma)'_t\,\mathrm{d}t\mathrm{d}\pi\\&\leq\iint_0^1\chi_B(\gamma_t)|Df|_p(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi\\&\leq\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\int_B|Df|_p\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

The above computation concludes the proof, as $\mu(B):=\int_B|Df|_p\,\mathrm{d}\mathfrak{m}$ is a competitor measure in a)-ii), hence (1.2.19) follows. □

Lemma 1.2.26. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let π be a ∞-test plan on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Suppose $f\in L^1(\mathfrak{m})$ satisfies $f\circ\gamma\in\mathrm{BV}(0,1)$ for π-a.e. γ. Fix any $\bar t\in[0,1]$. Then it holds*

$$|D(f\circ\gamma)|(\{\bar t\})=0,\qquad \textit{for } \pi\textit{-a.e. } \gamma.$$

In particular, it holds that $\big|f(\gamma_1)-f(\gamma_0)\big|\leq|D(f\circ\gamma)|(0,1)$ for π-a.e. γ.

Proof. Let us just prove the statement in the case where $\bar t=0$, since the other $\bar t$'s can be treated in a similar way. To prove it amounts to showing that

$$\lim_{t\searrow0}\fint_0^t\big|f(\gamma_s)-f(\gamma_0)\big|\,\mathrm{d}s=0,\qquad\text{for }\pi\text{-a.e. }\gamma. \tag{1.2.20}$$

We know [16, Theorems 3.27 and 3.28] that there exists $\big\{\lambda_\gamma\,:\,\gamma\in C([0,1],\mathrm{X})\big\}\subseteq\mathbb{R}$ such that

$$\lim_{t\searrow0}\fint_0^t\big|f(\gamma_s)-\lambda_\gamma\big|\,\mathrm{d}s=0,\qquad\text{for }\pi\text{-a.e. }\gamma. \tag{1.2.21}$$

Now fix any sequence $t_n\searrow0$. Given any $\varepsilon>0$, we can find a function $f_\varepsilon\in\mathrm{LIP}_{bs}(\mathrm{X})$ such that $\|f-f_\varepsilon\|_{L^1(\mathfrak{m})}\leq\varepsilon$. Then for any $n\in\mathbb{N}$ we may estimate

$$\begin{aligned}&\int\fint_0^{t_n}\big|f(\gamma_s)-f(\gamma_0)\big|\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)\\&\leq\int\fint_0^{t_n}|f-f_\varepsilon|\circ\mathsf{e}_s\,\mathrm{d}s\,\mathrm{d}\pi+\int\fint_0^{t_n}\big|f_\varepsilon(\gamma_s)-f_\varepsilon(\gamma_0)\big|\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)+\int|f_\varepsilon-f|\circ\mathsf{e}_0\,\mathrm{d}\pi\\&\leq2\,\mathrm{Comp}(\pi)\|f-f_\varepsilon\|_{L^1(\mathfrak{m})}+\mathrm{Lip}(f_\varepsilon)\int\fint_0^{t_n}\mathsf{d}(\gamma_s,\gamma_0)\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)\\&\leq2\,\mathrm{Comp}(\pi)\varepsilon+\mathrm{Lip}(f_\varepsilon)\mathrm{Lip}(\pi)t_n.\end{aligned}$$

By first letting $n\to\infty$ and then $\varepsilon\searrow0$, we see that $\lim_{n\to\infty}\int\fint_0^{t_n}\big|f(\gamma_s)-f(\gamma_0)\big|\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)=0$. Hence, up to taking a not relabelled subsequence, we have $\lim_{n\to\infty}\fint_0^{t_n}\big|f(\gamma_s)-f(\gamma_0)\big|\,\mathrm{d}s=0$ for π-a.e. γ. Recalling (1.2.21), we conclude that $\lambda_\gamma=f(\gamma_0)$ and $\lim_{t\searrow0}\fint_0^t\big|f(\gamma_s)-f(\gamma_0)\big|\,\mathrm{d}s=0$ hold for π-a.e. γ. This proves the validity of the first π-a.e. identity in (1.2.20), as desired. □

1.2.5 Nonsmooth differential structures

In this part, we exploit the machinery of $L^0(\mathfrak{m})$-normed modules developed in Section 1.1.7 to speak about differential calculus over arbitrary metric measure spaces. This has been developed in [97], but we will sometimes refer also to [104, 168].

We recall the existence and uniqueness theorem of a suitable cotangent structure over a metric measure space in the language of $L^0(\mathfrak{m})$-normed modules.

Theorem 1.2.27. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p\in(1,\infty)$. Then, there is a unique couple $(L^0_p(T^*\mathrm{X}),\mathrm{d}_p)$ where $L^0_p(T^*\mathrm{X})$ is a $L^0(\mathfrak{m})$-normed module and $\mathrm{d}_p\colon S^p_{loc}(\mathrm{X})\to L^0_p(T^*\mathrm{X})$ is linear and satisfying*

- i) *For any $f\in S^p_{loc}(\mathrm{X})$, it holds $|\mathrm{d}_p f|=|Df|_p$ $\mathfrak{m}$-a.e.;*
- ii) *The space $\{\mathrm{d}_p f\colon f\in W^{1,p}(\mathrm{X})\}$ generates $L^0_p(T^*\mathrm{X})$.*

Here, uniqueness is intended up to unique module isomorphism, i.e. if $(\mathscr{M},L)$ is another couple with the same properties, then there is a unique isomorphism $\Phi\colon\mathscr{M}\to L^0_p(T^*\mathrm{X})$ so that $\Phi\circ L=\mathrm{d}_p$.

In the above statement, by generating, we mean that simple L^0-linear combinations are $\mathsf{d}_{L^0_p(T^*\mathrm{X})}$-dense. Moreover, writing $\mathrm{d}_p f$ for $f\in W^{1,p}(\mathrm{X})$ causes no trouble (recall that we defined a differential for a Borel map) as the object $\mathrm{d}_p f$ is built from the equivalence class of f up to $\mathfrak{m}$-a.e. equality.

Definition 1.2.28. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $p\in(1,\infty)$. We define the p-cotangent module and the p-differential, as $L^0_p(T^*\mathrm{X})$ and d_p given by Theorem 1.2.27. Moreover, we define the q-tangent module by duality*

$$L^0_q(T\mathrm{X}):=\big(L^0_p(T^*\mathrm{X})\big)^*,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

We sometimes informally call elements of the above modules ‘Borel covector fields’ and ‘Borel vector fields’, respectively. In the sequel, we shall sometimes require to work with p-integrable covector and vector fields among the Borel ones. In these situations, we restrict the attention to the spaces

$$L^p(T^*\mathrm{X}):=\{\omega\in L^0_p(T^*\mathrm{X})\colon|v|\in L^p(\mathfrak{m})\},\quad \|\omega\|^p_{L^p(T^*\mathrm{X})}:=\int_\mathrm{X}|\omega|^p\,\mathrm{d}\mathfrak{m},$$
$$L^q(T\mathrm{X}):=\{X\in L^0_q(T\mathrm{X})\colon|X|_*\in L^q(\mathfrak{m})\},\quad \|X\|^q_{L^q(T\mathrm{X})}:=\int_\mathrm{X}|X|^q_*\,\mathrm{d}\mathfrak{m},$$

that have the natural structure of $L^p(\mathfrak{m})$-normed (resp. $L^q(\mathfrak{m})$-normed) $L^\infty(\mathfrak{m})$-module. Again, we will not discuss such structure and refer to [97] for a thorough presentation.

Following [106], we recall next the notion of differential of a metric valued Sobolev map. Fix then $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ a complete metric space and a map $u\in S^p(\mathrm{X},\mathrm{Y})$. We consider, as previously done, equipping $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ with the measure $\mu_p:=u_\sharp(|Du|_p^p\mathfrak{m})$. Ideally, the p-differential $\mathrm{d}_p u$ should be a $L^0(\mathfrak{m})$-linear map from $L^0_p(T\mathrm{X})$ to the pullback $u^*L^0_p(T\mathrm{Y})$. However, the map u satisfies suitable compressibility condition only between the spaces $(\mathrm{X},\mathsf{d},|Du|_p^p\mathfrak{m})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mu_p)$, therefore the pullback $u^*L^0_p(T\mathrm{Y})$ may be defined according to Proposition 1.1.14 as a $L^0(|Du|_p^p\mathfrak{m})$-normed module. A way to overcome this issue, is by noticing that the set $\{|Du|_p=0\}$ is irrelevant, as any definition of $\mathrm{d}_p u$ should be zero on such set. Thus, appealing to the extensor functor of normed modules we can define:

Definition 1.2.29 (p-differential of a map)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ a complete metric space. The differential $\mathrm{d}_p u$ of $u\in S^p(\mathrm{X},\mathrm{Y})$ is defined as the operator*

$$\mathrm{d}_p u\colon L^0_p(T\mathrm{X})\to\mathrm{Ext}\big((u^*L^0_p(T^*\mathrm{Y}))^*\big)$$

sending any $v \in L^0(T\mathrm{X})$ to the object $\mathrm{d}_p u(v)$ characterized by: for every $f \in S^p(\mathrm{Y})$, considering $g \in S^p(\mathrm{X})$ given by Proposition 1.2.15, we have

$$\mathrm{ext}(u^*\mathrm{d}_p f)(\mathrm{d}_p u(v)) = \mathrm{d}_p g(v), \qquad \mathfrak{m}\text{-}a.e.. \tag{1.2.22}$$

Again, in [106], it is remarked that the above definition is well-posed (given for $p = 2$, but working for arbitrary p), meaning that it does not depend on the particular function g given by Proposition 1.2.15, but only on its properties. Moreover, for $u \in S^p(\mathrm{X}, \mathrm{Y})$, we have [106]

$$\mathrm{d}_p u \text{ is } L^0(\mathfrak{m})\text{-linear and continuous} \qquad \text{and} \qquad |\mathrm{d}_p u| = |Du|_p \quad \mathfrak{m}\text{-a.e..}$$

1.3 Doubling and Poincaré

Definition 1.3.1 (Doubling & Poincaré)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We say that*

i) *it is* uniformly locally doubling *provided, for every $R > 0$, there exists a constant $\mathsf{C} := \mathsf{C}(R)$ so that*

$$\mathfrak{m}(B_{2r}(x)) \le \mathsf{C}\mathfrak{m}(B_r(x)), \qquad \forall x \in \mathrm{X}, r \in (0, R).$$

For brevity, we shall also say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a doubling metric measure space or that $\mathfrak{m}$ is a doubling reference measure.

ii) *it supports a* weak local $(1,1)$-Poincaré inequality*, provided for every $R > 0$ there exists $\tau, \Lambda > 0$ so that for any $f \colon \mathrm{X} \to \mathbb{R}$ Lipschitz it holds*

$$⨍_{B_r(x)} |f - f_{B_r(x)}| \,\mathrm{d}\mathfrak{m} \le \tau r ⨍_{B_{\Lambda r}(x)} |\mathrm{lip}\ f| \,\mathrm{d}\mathfrak{m}, \qquad \forall r \in (0, R), x \in \mathrm{X},$$

with the convention $f_B := ⨍_B f \,\mathrm{d}\mathfrak{m}$ for every $B \in \mathcal{B}(\mathrm{X})$.

The notion of Poincaré inequality is often given on a metric measure space with the concept of *upper gradient*, rather than local lipschitz constant. Nevertheless, we shall never use this concept in the present note and we remark that, by appealing to Theorem 1.2.11 (the main result of [20]) and [64] (see Theorem 1.3.4 below), the two approaches to define the Poincaré type inequality are in any case fully equivalent.

It is clear that, in the above definition, we are asking first order constraints to hold on a metric measure structure. In partiuclar, it is the Poincaré assumption that involves the first order differential structures, as it contains the notion of local lipschitz constant (or in general, a notion of weak gradient). Next we are going to face the deep consequences related to the the Doubling & Poincaré assumptions.

1.3.1 Lebesgue differentiation

In general, on an arbitrary metric measure space, there may be a lot of points in a Borel set E for which the following limit

$$D_E(x) := \lim_{r \downarrow 0} \frac{\mathfrak{m}(B_r(x) \cap E)}{\mathfrak{m}(B_r(x))} \tag{1.3.1}$$

computed at $x \in \mathrm{X}$ ($= \mathrm{supp}(\mathfrak{m})$ under our assumptions) may not exists or it is different to 1 (i.e. porosity may occur). It is trivial to see that $D_E(x) \in [0, 1]$ when it exists and it will be called *density of E at $x \in \mathrm{X}$*. This, contrary to the Euclidean setting, is an obstacle to the study fine property of functions where *measure-theoretic* notions of limits naturally enters into play. A way to avoid this is by working with doubling reference measures:

Theorem 1.3.2. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure spaces. Then, for every Borel set $E \subset \mathrm{X}$, there exists $D_E(x) = 1$ $\mathfrak{m}$-a.e..*

Proof. It is well known (see, e.g., [119]) that, a doubling metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a Vitali-space and the following Lebesgue differentiation theorem holds: namely for every $f \in L^1_{loc}(\mathfrak{m})$, there exists the limit

$$\lim_{r\downarrow 0} \fint_{B_r(x)} f \,\mathrm{d}\mathfrak{m} = f(x), \qquad \mathfrak{m}\text{-a.e.}.$$

To conclude, just consider $f = \chi_E$ as a $L^1_{loc}(\mathfrak{m})$ function and apply the above. □

We start now recalling the notion of approximate limits. Denote $\bar{\mathbb{R}} := \mathbb{R} \cup \pm\infty$ and, given $f\colon \mathrm{X} \to \bar{\mathbb{R}}$ Borel, we define the *approximate liminf and limsup* as the quantities

$$\begin{aligned}
\text{ap-}\varlimsup_{y\to x} f(x) &:= \inf\{t \in \mathbb{R}\colon D_{\{f\le t\}}(x) = 1\},\\
\text{ap-}\varliminf_{y\to x} f(x) &:= \sup\{t \in \mathbb{R}\colon D_{\{f\ge t\}}(x) = 1\},
\end{aligned}$$

where, as customary, we set the first $+\infty$ and the latter $-\infty$, when no such t's exist (here, a competitor t is also intended to be so that the limit in (1.3.1) exists).

Differently from the concept of upper and lower limits, the above notions are measure-theoretic: consider changing f in $\mathfrak{m}$-negligible set, then it can be readily checked that the above values do not change. Moreover, it is also evident that in general we have $\text{ap-}\varliminf_{y\to x} f(x) \le \text{ap-}\varlimsup_{y\to x} f(x)$ and strict inequality may also occur. When equality occurs at $x \in \mathrm{X}$, we shall say that the *approximate limit* of f at x exists and we shall write

$$\text{ap-}\lim_{y\to x} f(x) := \text{ap-}\varliminf_{y\to x} f(x) = \text{ap-}\varlimsup_{y\to x} f(x).$$

In this case, we say that f is *approximate continuous* at x. Also, we can relax the notion of *approximate local Lipschitz constant*, defined as

$$\text{ap-lip}\, f(x) := \text{ap-}\varlimsup_{y\to x} \frac{|f(y) - f(x)|}{\mathsf{d}(x,y)},$$

and 0 is x is isolated.

Remark 1.3.3. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a Doubling metric measure space and $f, g\colon \mathrm{X} \to \mathbb{R}$ be two Borel functions. From the previous consideration, we see that the approximate Lipschitz constant is *local*, namely

$$\text{ap-lip}\, f = \text{ap-lip}\, g, \qquad \mathfrak{m}\text{-a.e. on } \{f = g\} \tag{1.3.2}$$

and that

$$\text{ap-lip}\, f = \text{lip}\, f, \qquad \mathfrak{m}\text{-a.e.} \quad \forall f : \mathrm{X} \to \mathbb{R} \text{ Lipschitz}, \tag{1.3.3}$$

for every $f, g\colon \mathrm{X} \to \bar{\mathbb{R}}$ Borel. The proof of the latter fact can be found e.g. in [109, Proposition 2.5]. ■

1.3.2 Fine properties of Sobolev functions

We start recalling a deep implication of the seminal work by Cheeger [64] concerning the Sobolev calculus on metric measure spaces under doubling and Poincaré assumptions.

Theorem 1.3.4 ([64])**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space supporting a weak local $(1,1)$-Poincaré inequality. Then, for all $p \in (1,\infty)$ it holds that*

$$|Df|_p = \text{lip}\, f, \qquad \mathfrak{m}\text{-}a.e., \quad \forall f \in \mathrm{Lip}_{bs}(\mathrm{X}). \tag{1.3.4}$$

Another remarkable result holding in this class is the reflexivity of the Sobolev space. Nevertheless, the Poincaré assumption has been later removed in [12], thus we give for the sake of generality the following statement.

Theorem 1.3.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metrically doubling metric measure space. Then, for every* $p \in (1,\infty)$*, the space* $W^{1,p}(\mathrm{X})$ *is reflexive.*

In particular, the above holds for doubling metric measure space. A simple but crucial Corollary is the following, taken from [12].

Corollary 1.3.6 (Strong density of Lipschitz functions)**.** *Let* $p \in (1,\infty)$ *and* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space so that* $W^{1,p}(\mathrm{X})$ *is reflexive (in particular, under the assumption of Theorem 1.3.5).*
Then, $\mathrm{Lip}_{bs}(\mathrm{X})$ *is a strongly dense subset.*

We also recall a well known fine property of Sobolev functions in this context.

Theorem 1.3.7. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be doubling and* $p \in (1,\infty)$*. Then, any* $f \in W^{1,p}(\mathrm{X})$ *is Lusin-Lipschitz, i.e. there are* N, K_n *Borel,* $n \in \mathbb{N}$*, with* N *negligible and* K_n *compacts, so that* $\mathrm{X} := N \cup (\cup_n K_n)$ *and* $f_{|_{K_n}}$ *is Lipschitz for every* $n \in \mathbb{N}$*.*

The proof of this fact essentially follows from the characterization of the space $W^{1,p}(\mathrm{X})$ with the so-called Hajłasz Sobolev space under doubling condition (see, e.g., [114, Theorem 3.2], then argue e.g. as in [53, Proposition 3.8]). We omit the details.

1.3.3 Independence of weak upper gradients

In this part, we face a first remarkable result of independence on the integrable exponent of the minimal p-weak upper gradient under Doubling & Poincaré. This is well known [64] but, given the importance of this result in this work, we provide the reader with a proof for completeness.

Theorem 1.3.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a doubling metric measure space supporting a weak local* $(1,1)$*-Poincaré inequality. Then for every* $p_1, p_2 \in (1,\infty)$ *and* $f \in W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$*, we have*

$$|Df|_{p_1} = |Df|_{p_2} \qquad \mathfrak{m}\text{-}a.e..$$

Proof. We claim that

$$|Df|_p = \text{ap-lip } f, \qquad \mathfrak{m}\text{-a.e.}, \quad \forall f \in W^{1,p}(\mathrm{X}),\ p \in (1,\infty).$$

As the right hand side is independent on p, this concludes the proof. To see the claim start recalling that any $f \in W^{1,p}(\mathrm{X})$ has the Lusin-Lipschitz property by Theorem 1.3.7. Let $K_n \subset \mathrm{X}$ a sequence of increasing compact sets so that $f_{|_{K_n}} \in \mathrm{Lip}(K_n)$, $f_{|_{K_n}} = f$ $\mathfrak{m}$-a.e.. and $\mathfrak{m}(\mathrm{X} \setminus \cup_n K_n) \to 0$. Then, call f_n a Lipschitz extension of $f_{|_{K_n}}$ to all X with bounded support, for instance using the McShane extension theorem and a cut-off argument. We then have $f_n \in \mathrm{Lip}_{bs}(\mathrm{X})$ for every n and by locality of minimal p-weak upper gradients

$$|Df|_p = |Df_n|_p \overset{(1.3.4)}{=} \text{lip } f_n, \quad \mathfrak{m}\text{-a.e. on } K_n,$$

Recalling (1.3.2) and (1.3.3) we deduce that

$$|Df|_p = \text{ap-lip } f, \quad \mathfrak{m}\text{-a.e. on } K_n$$

and since the K_n's cover all X up to $\mathfrak{m}$-negligible sets, we conclude. □

In light of the above theorem, we make the following clarification.

> **Notation**. When the conclusion of Theorem 1.3.8 holds on a metric measure space, we omit without notice the p-subscript in minimal p-weak upper gradients and simply write $|Df|$ for every $f \in S^p(\mathrm{X})$ and $p \in (1,\infty)$ as no confusion may arise.

1.4 Infinitesimal Hilbertianity

As we have already observed, the space $W^{1,p}(\mathrm{X})$ need not to be reflexive. If on one hand, a doubling condition as shown in [12] (see Theorem 1.3.5) ensures such result, it is in general false that the particular choice $p = 2$ leads to a Hilbert space. Following [95], we have the following definition.

Definition 1.4.1. *We say that a metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is* infinitesimal Hilbertian *if* $W^{1,2}(\mathrm{X})$ *is Hilbert.*

Moreover, we say that a complete metric space (X, d) *is* universally *infinitesimal Hilbertian, provided it is infinitesimal Hilbertian when equipped with any nonnegative and boundedly finite Borel Radon measure* μ.

1.4.1 Equivalent definitions

We summarize in the next Proposition (some of the) equivalent definitions of infinitesimal Hilbertianity [95] (see also [97, 104]).

Proposition 1.4.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. The following are equivalent:*

i) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimal Hilbertian;*

ii) *For every* $f, g \in W^{1,2}(\mathrm{X})$ *, it holds*

$$2|Df|_2^2 + 2|Dg|_2^2 = |D(f+g)|_2^2 + |D(f-g)|_2^2, \qquad \mathfrak{m}\text{-}a.e.; \tag{1.4.1}$$

iii) $L^0_2(T^*\mathrm{X})$ *and* $L^0_2(T\mathrm{X})$ *are Hilbert modules;*

iv) *For every* $f, g \in W^{1,2}(\mathrm{X})$*, there exists the limit in* $L^1(\mathfrak{m})$ *of* $\lim_{\varepsilon\to 0} \frac{|D(f+\varepsilon g)|_2^2 - |Df|_2^2}{\varepsilon}$ *and it is a* $L^0(\mathfrak{m})$*-bilinear map on* f, g.

Moreover, if any of the above holds, the pointwise scalar product $\langle \nabla f, \nabla g \rangle$ *(defined via polarization (1.1.13)) satisfies*

$$\langle \nabla f, \nabla g \rangle = \lim_{\varepsilon\to 0} \frac{|D(f+\varepsilon g)|_2^2 - |Df|_2^2}{\varepsilon}, \qquad \mathfrak{m}\text{-}a.e., \tag{1.4.2}$$

for every $f, g \in W^{1,2}(\mathrm{X})$.

Given $\Omega \subset \mathrm{X}$ an open subset of an infinitesimal Hilbertian metric measure space, we would like to compute the pointwise scalar product $\langle \nabla g, \nabla g \rangle \in L^1(\Omega)$ between $f, g \in W^{1,2}(\Omega)$. To do so, we can formally define for every $\eta \in \mathrm{Lip}_{bs}(\mathrm{X})$ with $\mathrm{supp}(\eta) \subset \Omega$, te coupling

$$\langle \nabla f, \nabla g \rangle := \langle \nabla f, \nabla(\eta g) \rangle, \qquad \mathfrak{m}\text{-a.e. on } \{\eta = 1\}, \tag{1.4.3}$$

for every $f \in \mathrm{Lip}_{bs}(\Omega), g \in W^{1,2}(\Omega)$. Thanks to Definition 1.2.13 (and the choice of f), observe that the right-hand side is perfectly defined as both $f, \eta g \in W^{1,2}(\mathrm{X})$. This especially grants that the object is *local*.

Remark 1.4.3. Suppose we are given an infinitesimal Hilbertian metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and a universally infinitesimal Hilbertian complete metric space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$. Then, for every $u \in W^{1,2}(\mathrm{X}, \mathrm{Y})$, it is possible to speak about the *Hilbert-Schmidt* pointwise norm of the differential $\mathrm{d}_2 u$ as follows: by assumptions, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mu_2 := u_\sharp(|Du|_2^2\mathfrak{m}))$ is infinitesimal Hilbertian, then recall that $\mathrm{d}_2 u$ is a $L^0(\mathfrak{m})$-continuous and linear operator between $L^0_2(T\mathrm{X})$ and $\mathrm{Ext}\big((u^*L^0_2(T^*\mathrm{Y}))^*\big)$ that are Hilbert modules. Since $L^0_2(T^*\mathrm{Y})$ is a $L^0(\mu_2)$-normed Hilbert modules, then $\mathrm{Ext}\big((u^*L^0_2(T^*\mathrm{Y}))^*\big)$ is a $L^0(\mathfrak{m})$-normed Hilbert module. Therefore, by appealing to the construction of tensor products between $L^0(\mathfrak{m})$-normed modules [97], $\mathrm{d}_2 u$ can be equivalently thought as an element of $L^0_2(T^*\mathrm{X}) \otimes \mathrm{Ext}\big((u^*L^0_2(T^*\mathrm{Y}))^*\big)$ that, being a tensor product of Hilbert module, is again a Hilbert module. We call $|\mathrm{d}_2 u|_{\mathsf{HS}} \in L^2(\mathfrak{m})$ the pointwise norm of this object seen in such product.

For the case of a map $u \in W^{1,2}(\Omega,\mathrm{Y})$ defined on an open subset Ω of X, we sketch the construction of the 'formal' object $|\mathrm{d}_2 u|_{\mathsf{HS}} \in L^2(\mathfrak{m}_{|\Omega})$: for every $C \subset \Omega$ closed with $\mathsf{d}(C,\Omega) > 0$, we can suitably define a Sobolev map u_C with a cut-off argument which is global on X, takes values in the copy of Y inside a Banach space (e.g. via the Kuratowski embedding) and is 'zero' outside Ω. With this procedure (see, e.g., [108, Proposition 5.6] for the details) and for what has been said above, imposing

$$|\mathrm{d}_2 u|_{\mathsf{HS}} := |\mathrm{d}_2 u_C|_{\mathsf{HS}}, \qquad \mathfrak{m}\text{-a.e. on } C,$$

turns out to be a well defined object by locality and arbitrariness of C. Moreover, by construction (that independent on the embedding chosen) it vanishes outside Ω and thus, up to passing to a $\mathfrak{m}_{|\Omega}$-a.e. equality, we have $|\mathrm{d}_2 u|_{\mathsf{HS}} \in L^2(\mathfrak{m}_{|\Omega})$. ■

1.4.2 Linear Laplacian

Here we recall the notion of measure-valued Laplacian as introduced in [95].

Definition 1.4.4. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an infinitesimal Hilbertian metric measure space.*

We say that $f \in W^{1,2}(\mathrm{X})$ has a measure valued Laplacian and write $f \in D(\mathbf{\Delta})$, provided there is a Radon measure μ such that

$$\int g\,\mathrm{d}\mu = -\int \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m}, \qquad \forall g \in \mathrm{Lip}_{bs}(\mathrm{X}).$$

Assume that X is also proper and $\Omega \subset \mathrm{X}$ is open and bounded. We say that $f \in W^{1,2}(\Omega)$ has a measure valued Laplacian in Ω, and write $f \in D(\mathbf{\Delta},\Omega)$, provided there is a Borel measure μ on Ω such that

$$\int_\Omega g\,\mathrm{d}\mu = -\int_\Omega \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m} \qquad \forall g \in \mathrm{Lip}_c(\Omega).$$

These unique measures will be denoted by $\mathbf{\Delta} f$ and $\mathbf{\Delta} f_{|\Omega}$, respectively.

Differently from [95], here the presentation of Laplacian in Ω greatly simplifies by the assumption of proper and/or infinitesimal Hilbertian metric measure spaces. We shall never need to work in full generality. It is also clear that this definition is well posed and, thanks to the infinitesimal Hilbertianity assumption, we see also that the assignments $f \mapsto \mathbf{\Delta} f, \mathbf{\Delta} f_{|\Omega}$ are linear.

Finally, we shall need the following criterium from [95] to check whether $f \in D(\mathbf{\Delta},\Omega)$: for $f \in W^{1,2}(\Omega)$ and $h \in L^1(\mathfrak{m}_{|\Omega})$ we have

$$-\int_\Omega \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m} \geq \int_\Omega gh\,\mathrm{d}\mathfrak{m} \quad \forall g \in \mathrm{Lip}_c(\Omega)^+ \quad \Rightarrow \quad f \in D(\mathbf{\Delta},\Omega) \text{ and } \mathbf{\Delta} f_{|\Omega} \geq h\mathfrak{m}, \tag{1.4.4}$$

having used the formal definition (1.4.3) of the pointwise scalar product in the left hand most integral.

1.4.3 Linear Heat flow

In this section we recall basic facts on the Heat flow of the Cheeger energy on infinitesimal Hilbertian metric measure spaces. This *detour* will be a guidline for the presentation in Chaper 3.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and we start recalling that the 2-Cheeger energy is a convex and lower semicontinuous functional on the Hilbert space $L^2(\mathfrak{m})$ with *domain of finiteness* $D(\mathsf{Ch}_2) := W^{1,2}(\mathrm{X})$, by definition. Hence, standard gradient flow theory (see, e.g., [19]) applies ensuring for every $f \in L^2(\mathfrak{m})$ the existence of a *gradient flow trajectory* $[0,\infty) \ni t \mapsto f_t \in L^2(\mathfrak{m})$, i.e. a locally absolutely continuous satisfying

$$\begin{aligned} \dot f_t &\in -\partial^- \mathsf{Ch}_2(f_t) && \forall t > 0, \\ f &= \lim_{t \to 0} f_t && \text{in } L^2(\mathfrak{m}), \end{aligned} \tag{1.4.5}$$

where

$$\partial^{-}\mathsf{Ch}_2(f) := \left\{ v \in L^2(\mathfrak{m})\colon \mathsf{Ch}_2(f) + \int v(g-f)\,\mathrm{d}\mathfrak{m} \le \mathsf{Ch}_p(g) \text{ for all } g \in L^2(\mathfrak{m}) \right\}$$

is the *subdifferential set of* Ch_2 at f, and $\dot{f}_t := \lim_{h\to 0} \frac{f_{t+h}-f_t}{h}$ is the velocity of a Hilbert-valued absolutely continuous map (see, e.g., [104]). Here, $\dot{f}_t$ exists *a priori* only for a.e. t but, in light of the regularization property of the gradient flows, it turns out that it exists *for every* $t > 0$.

Remark 1.4.5. We point out that, the existence of a gradient flow trajectory $t \mapsto f_t$ satisfying (1.4.6) immediately implies that (f_t) is an EVI gradient flow, as defined in Definition 1.1.9. This follows directly from the definition of the subdifferential and the fact that a Hilbert norm is *differentiable*, namely that, for every $g \in L^2(\mathfrak{m})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\,\|g-f_t\|^2_{L^2(\mathfrak{m})} = -\int \dot{f}_t(g-f_t)\,\mathrm{d}\mathfrak{m}, \qquad \text{a.e..}$$

■

Until now, the above discussion did never assume the metric measure space to be infinitesimal Hilbertian. This assumption will be now employed to link (1.4.5) with a suitably stated *heat equation* (thus, justifying the name heat flow) thanks to the following results [95](see also [104]).

Proposition 1.4.6. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a infinitesimal Hilbertian metric measure space and let* $f \in L^2(\mathfrak{m})$*. The following are equivalent:*

i) $f \in D(\mathbf{\Delta})$ *with* $\mathbf{\Delta} f \ll \mathfrak{m}$ *and* $\Delta f := \frac{\mathrm{d}\mathbf{\Delta} f}{\mathrm{d}\mathfrak{m}} \in L^2(\mathfrak{m})$

ii) $\frac{f_h - f}{h}$ *admits a strong limit* $g \in L^2(\mathfrak{m})$ *as* h *goes to zero.*

If any of the two holds true, we have $\Delta f = g$*.*

As a corollary, we see that, since the heat flow in (1.4.5) is regularizing, meaning that at each time $t > 0$, there exists the strong limit $\dot{f}_t := \lim_{h\to 0}\frac{f_{t+h}-f_t}{h}$ in $L^2(\mathfrak{m})$, the above proposition ensures that $f_t \in D(\mathbf{\Delta})$ and consequently (1.4.5) becomes

$$\begin{aligned} \dot{f}_t &= \Delta f_t & \qquad & \forall t > 0,\\ f &= \lim_{t\to 0} f_t & & \text{in } L^2(\mathfrak{m}), \end{aligned} \tag{1.4.6}$$

recalling that $\mathbf{\Delta} f_t = \Delta f_t \mathfrak{m}$ by definition for every $t > 0$. From this discussion, it is also evident that (1.4.5) grants that $\Delta f_t \in -\partial^{-}\mathsf{Ch}_2(f_t)$. This means that, the analytical Laplacian defined via integration by parts in Definition 1.4.4 turns out to lies in the 'minus subdifferential set' $-\partial^{-}\mathsf{Ch}_2(f_t)$ along the flow (actually, it is the minimal normed element of the subdifferential and it is selected from the gradient flow trajectory).

To conclude, we mention that in Chapter 3, we will follow this approach, but reversed. Namely, in a genuinely setting of metric valued Sobolev maps, where the absence of linearity in the target makes impossible to formulate properly an integration by parts formula, we will derive a differential notion of 'harmonic map heat flow' by means of a subdifferential inclusion in the spirit of (1.4.5). This will make possible to speak of a Laplacian of a sufficiently regular Sobolev map satisfying in turn a basic integration by parts formula/inequality.

2 Notions of curvature bounds

Contents

In this part, we present the main definitions and features of *synthetic curvature bounds* in a metric or metric measure setting. The words synthetic refers to the principle that the main object of this chapter, i.e. curvature, is going to be treated via an 'implicit' approach, rather than an 'explicit' one that builds upon the smoothness of Riemannian manifolds.

Let us start with a simple, yet well known, example to explain the principle behind the synthetic approach. The prototype of synthetic definition is *convexity*: given $f\colon \mathbb{R}^n \to \mathbb{R}$ smooth and $x, y \in \mathbb{R}^n$ arbitrary, it holds

$$\mathrm{Hess} f \succeq 0 \qquad \text{if and only if} \qquad f\big((1-t)x + ty\big) \leq (1-t)f(x) + tf(y), \quad \forall t \in [0,1].$$

This is a remarkable instance of the fact that convexity is a condition that can be fully captured by a zeroth-order implicit condition, i.e. no regularity requirements. Nevertheless, for smooth f and Euclidean underlying space, this property can be equivalently (and explicitely, involving second-order derivatives) expressed in terms of a sign constraints on the spectrum of the Hessian matrix.

Passing to curvature bounds but keeping in mind the same philosophy, any reasonable synthetic definition must asks suitable conditions to the metric and measure structure of the underlying space. Moreover, when looking back to the smooth category, these requirements must meet with the classical definition. In the sequel, we will be focusing with upper bounds on the sectional curvature and lower bounds on the Ricci curvature to give rise to effective singular classes.

2.1 The CAT-condition

The study of synthetic bounds on the sectional curvature goes back to the fundamental works of A. D. Alexandrov started in 1941 with the work [8]. Here, the celebrated Alexandrov's embedding theorem and the Gluing theorem were established. Then, a systematic investigation started later (see, e.g. [5, 52]) leading to what are nowadays called Alexandrov spaces, i.e. metric spaces satisfying sectional curvature bounds either from above or below.

For the purposes of this work, it will be sufficient to recall only spaces with upper bounds on the sectional curvature, but we refer to [47, 51, 113, 39, 6, 7] for a full picture on Alexandrov geometry. Also, we recommend the references therein for a complete overview of the deep contributions made by many authors to the theory.

Let us begin from the very basics in Riemannian geometry, so to fix some notation. Recall that Riemannian manifolds with *constant* curvature are completely characterized. Given $\kappa \in \mathbb{R}$, we call $\mathbb{M}_\kappa$, the *model space* of curvature κ, i.e. the simply connected, complete 2-dimensional manifold with constant curvature κ, and d_κ the distance induced by the metric tensor. This restricts $(\mathbb{M}_\kappa, \mathsf{d}_\kappa)$ to only three possibilities: the hyperbolic space $\mathbb{H}^2_\kappa$ of constant sectional curvature κ, if $\kappa < 0$, the plane $\mathbb{R}^2$ with usual Euclidean metric, if $\kappa = 0$, and the sphere $\mathbb{S}^2_\kappa$ of constant sectional curvature κ, if $\kappa > 0$. Also, set $D_\kappa := \mathrm{diam}(\mathbb{M}_\kappa)$, i.e.

$$D_\kappa = \begin{cases} \infty & \text{is } \kappa \leq 0, \\ \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0. \end{cases}$$

We refer to [47, Chapter I.2] for a detailed study of the model spaces $\mathbb{M}_\kappa$.

In order to speak of a $\mathsf{CAT}(\kappa)$ condition (recall that the letters in the acronym CAT refer to E. Cartan, A. Topogonov and A.D. Alexandrov), i.e. κ-upper bound of the sectional curvature in a geodesic metric space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$, we shall enforce a metric comparison property to geodesic triangles of Y, the intuition being that they are 'thinner' than in $\mathbb{M}_\kappa$. To define them we start by recalling that if $a, b, c \in \mathrm{Y}$ is a triple of points satisfying $\mathsf{d}_\mathrm{Y}(a,b) + \mathsf{d}_\mathrm{Y}(b,c) + \mathsf{d}_\mathrm{Y}(c,a) < 2D_\kappa$, then there are points, unique up to isometries of the ambient space and called *comparison points*, $\bar{a}, \bar{b}, \bar{c} \in \mathbb{M}_\kappa$ such that

$$\mathsf{d}_\kappa(\bar{a}, \bar{b}) = \mathsf{d}_\mathrm{Y}(a,b), \qquad \mathsf{d}_\kappa(\bar{b}, \bar{c}) = \mathsf{d}_\mathrm{Y}(b,c), \qquad \mathsf{d}_\kappa(\bar{c}, \bar{a}) = \mathsf{d}_\mathrm{Y}(c,a).$$

In the case where Y is geodesics (and this will be always assumed), we refer to $\triangle(a,b,c)$ as the geodesic triangle in Y consisting in three points a, b, c, the *vertices*, and a choice of three corresponding geodesics, the *edges*, linking pairwise the points. By $\triangle^\kappa(\bar{a}, \bar{b}, \bar{c})$ we denote the so built geodesic triangle in $\mathbb{M}_\kappa$, which from now on we call comparison triangle. A point $d \in \mathrm{Y}$ is said to be intermediate between $b, c \in \mathrm{Y}$ provided $\mathsf{d}_\mathrm{Y}(b,d) + \mathsf{d}_\mathrm{Y}(d,c) = \mathsf{d}_\mathrm{Y}(b,c)$ (this means that d lies on a geodesic joining b and c). The *comparison point of* d is the (unique, once we fix the comparison triangle) point $\bar{d} \in \mathbb{M}_\kappa$, such that

$$\mathsf{d}_\kappa(\bar{d}, \bar{b}) = \mathsf{d}_\mathrm{Y}(d,b), \qquad \mathsf{d}_\kappa(\bar{d}, \bar{c}) = \mathsf{d}_\mathrm{Y}(d,c).$$

Definition 2.1.1 ($\mathsf{CAT}(\kappa)$-spaces)**.** *A metric space* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ *is called a* $\mathsf{CAT}(\kappa)$*-space if it is complete, geodesic and satisfies the following triangle comparison principle: for any* $a, b, c \in \mathrm{Y}$ *satisfying* $\mathsf{d}_\mathrm{Y}(a,b) + \mathsf{d}_\mathrm{Y}(b,c) + \mathsf{d}_\mathrm{Y}(c,a) < 2D_\kappa$ *and any intermediate point* d *between* b, c*, denoting by* $\triangle^\kappa(\bar{a}, \bar{b}, \bar{c})$ *the comparison triangle and by* $\bar{d} \in \mathbb{M}_\kappa$ *the corresponding comparison point (as said,* $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ *are unique up isometries of* $\mathbb{M}_\kappa$*), it holds*

$$\mathsf{d}_\mathrm{Y}(a,d) \leq \mathsf{d}_\kappa(\bar{a}, \bar{d}). \tag{2.1.1}$$

A metric space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ *is said to be locally* $\mathsf{CAT}(\kappa)$ *if it is complete, geodesic and every point in* Y *has a neighbourhood which is a* $\mathsf{CAT}(\kappa)$*-space with the inherited metric.*

Let us first discuss the situation when Y is actually a Riemannian manifold. The next theorem (due to [5] but the proof can be found e.g. in [47, Theorem 1A.6]) shows that the above definition is consistent with the classical setting.

Theorem 2.1.2. *Let M be a smooth Riemannian manifold. Then M is locally $\mathsf{CAT}(\kappa)$ if and only if it satisfies $\mathsf{Sect} \leq \kappa$.*

Notice that balls of radius $< D_\kappa/2$ in the model space $\mathbb{M}_\kappa$ are convex, i.e. meaning that geodesics with endpoint lies entirely inside them. Hence the comparison property (2.1.1) grants that the same is true on $\mathsf{CAT}(\kappa)$-spaces (see [47, Proposition II.1.4.(3)] for the rigorous proof of this fact). It is then easy to see that, for the same reasons, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ is locally $\mathsf{CAT}(\kappa)$ provided every point has a neighbourhood U where the comparison inequality (2.1.1) holds for every triple of points $a, b, c \in U$, where the geodesics connecting the points (and thus the intermediate points) are allowed to exit the neighbourhood U.

Let us fix the following notation: if $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ is a local $\mathsf{CAT}(\kappa)$-space, for every $y \in \mathrm{Y}$ we set

$$\mathsf{r}_y := \sup \big\{ r \leq D_\kappa/2 \ : \ \bar{B}_r(y) \text{ is a } \mathsf{CAT}(\kappa)\text{-space} \big\}.$$

Notice that in particular $B_{\mathsf{r}_y}(y)$ is a $\mathsf{CAT}(\kappa)$-space. The definition trivially grants that $\mathsf{r}_y \geq \mathsf{r}_z - \mathsf{d}(y, z)$ and thus in particular $y \mapsto \mathsf{r}_y$ is continuous.

Next, we remark the important fact which will be exploited in the sequel

> On $\mathsf{CAT}(\kappa)$-spaces, geodesics with endpoint at distance $< D_\kappa$ are unique (up to reparametrization) and vary continuously with respect to the endpoints.

For a quantitative version of this fact, see [79, Lemma 2.2]. Finally, it will be important to examine the case of global $\mathsf{CAT}(0)$-spaces, as they naturally arise as tangent structures of $\mathsf{CAT}(\kappa)$-spaces (see Theorem 2.1.4 below) and also because we are going to examine $\mathsf{CAT}(0)$-valued maps in Section 3.3. Since $\mathbb{M}_0$ is the Euclidean plane $\mathbb{R}^2$ equipped with the Euclidean norm, for Y $\mathsf{CAT}(0)$ and $a, b, c \in \mathrm{Y}$ as in Definition 2.1.1, the defining inequality (2.1.1) reads

$$\mathsf{d}_\mathrm{Y}(\gamma_t, a) \leq \|(1-t)\bar{b} + t\bar{c} - \bar{a}\|,$$

for every $t \in [0, 1]$, where γ_t is the constant speed geodesic connecting b to c and $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}^2$ are comparison points. By squaring and expanding the right hand side, we easily obtain the condition

$$\mathsf{d}^2_\mathrm{Y}(\gamma_t, a) \leq (1-t)\mathsf{d}^2_\mathrm{Y}(\gamma_0, a) + t\mathsf{d}^2_\mathrm{Y}(\gamma_1, a) - t(1-t)\mathsf{d}^2_\mathrm{Y}(\gamma_0, \gamma_1), \tag{2.1.2}$$

for every $t \in [0, 1]$. Inequality (2.1.2) (which can be equivalently used to define $\mathsf{CAT}(0)$-spaces) is to be understood as a synthetic deficit of the curvature of Y, with respect to the Euclidean plane $\mathbb{R}^2$ (where it holds with equality). In other words, it quantifies how much the triangle $\triangle(a, b, c)$ is 'thin' compared to $\triangle^0(\bar{a}, \bar{b}, \bar{c})$ in the Euclidean plane. The advantage of (2.1.2) is to be more practical to work with in convex analysis and optimization. We conclude recalling that locally $\mathsf{CAT}(\kappa)$-spaces are universally infinitesimal Hilbertian (recall Definition 1.4.1).

Theorem 2.1.3 ([79])**.** *Let Y be a locally $\mathsf{CAT}(\kappa)$-space, then it is universally infinitesimal Hilbertian.*

2.1.1 Metric calculus on tangent cones

We recall here the notion of tangent structures on a $\mathsf{CAT}(\kappa)$-space, referring to the above-mentioned bibliography for a much more complete discussion. We consider here tangent cones as defined in Definition 1.1.6 and see how this construction will benefit from the local $\mathsf{CAT}(\kappa)$-condition making a suitable calculus possible. Recall that tangent cones $(\mathrm{T}_y\mathrm{Y}, \mathsf{d}_y)$ are obtained by completions starting from the (suitably quotiented) space of directions $\mathsf{Geo}_y\mathrm{Y}$.

Let Y be a local $\mathsf{CAT}(\kappa)$ space and notice that, for every $y \in \mathrm{Y}, \gamma, \eta \in \mathsf{Geo}_y\mathrm{Y}$, the limsup in (1.1.5) is actually a limit and it will be also useful to notice that

$$\text{if Y is } \mathsf{CAT}(0),\ t \mapsto \frac{\mathsf{d}_\mathrm{Y}(\gamma_t, \eta_t)}{t} \text{ is non-decreasing} \qquad \forall \gamma, \eta \in \mathsf{Geo}_y\mathrm{Y}, \tag{2.1.3}$$

a property which is directly implied by (2.1.2). A well known (see e.g. [47, Theorem II-3.19]) and useful fact is that tangent cones at local $\mathsf{CAT}(\kappa)$ spaces are $\mathsf{CAT}(0)$ spaces:

Theorem 2.1.4. *Let* Y *be locally* $\mathsf{CAT}(\kappa)$*. Then, for every* $y \in \mathrm{Y}$*, the tangent cone* $(\mathrm{T}_y\mathrm{Y}, \mathsf{d}_y)$ *is a* CAT*(0)-space.*

We now build a calculus on the tangent cone that resembles the one of Hilbert spaces.

- *Multiplication by a positive scalar.* Let $\lambda \geq 0$. Then the map sending $t \mapsto \gamma_t$ to $t \mapsto \gamma_{\lambda t}$ is easily seen to pass to the quotient in $\mathsf{Geo}_y\mathrm{Y}/\sim$ and to be λ-Lipschitz. Hence it can be extended by continuity to a map defined on $\mathrm{T}_y\mathrm{Y}$, called multiplication by λ.

- *Norm.* $|v|_y := \mathsf{d}_y(v, 0)$.

- *Scalar product.* $\langle v, w\rangle_y := \frac{1}{2}\big[|v|_y^2 + |w|_y^2 - \mathsf{d}_y^2(v, w)\big]$.

- *Sum.* $v \oplus w := 2m$, where m is the midpoint of v, w (well-defined because $\mathrm{T}_y\mathrm{Y}$ is a $\mathsf{CAT}(0)$-space).

We next report the following fact (see, e.g., [79, Theorem 2.9] for the proof):

$$\text{for } \mathcal{D} \text{ dense in } B_{\mathsf{r}_y}(y) \text{ we have that } \{\alpha(\mathsf{G}_y^w)'_0 \colon \alpha \in \mathbb{Q}^+,\ w \in \mathcal{D}\} \text{ is dense in } \mathrm{T}_y\mathrm{Y}. \tag{2.1.4}$$

Moreover, we recall the following proposition:

Proposition 2.1.5 (Basic calculus on the tangent cone)**.** *Let* Y *be locally* $\mathsf{CAT}(\kappa)$ *and* $y \in \mathrm{Y}$*. Then, the four operations defined above are continuous in their variables. The 'sum' and the 'scalar product' are also symmetric. Moreover:*

$$\mathsf{d}_y(\lambda v, \lambda w) = \lambda \mathsf{d}_y(v, w), \tag{2.1.5a}$$
$$\langle \lambda v, w\rangle_y = \langle v, \lambda w\rangle_y = \lambda\langle v, w\rangle_y, \tag{2.1.5b}$$
$$|\langle v, w\rangle_y| \leq |v|_y|w|_y, \tag{2.1.5c}$$
$$\langle v, w\rangle_y = |v|_y|w|_y \quad \textit{if and only if} \quad |w|_y v = |v|_y w, \tag{2.1.5d}$$
$$\mathsf{d}_y^2(v, w) + |v \oplus w|_y^2 \leq 2(|v|_y^2 + |w|_y^2), \tag{2.1.5e}$$
$$\langle v_1 \oplus v_2, w\rangle_y \geq \langle v_1, w\rangle_y + \langle v_2, w\rangle_y \tag{2.1.5f}$$

for any $v, v_1, v_2, w \in \mathrm{T}_y\mathrm{Y}$ *and* $\lambda \geq 0$*.*

Proof. The continuity of 'norm', 'scalar product' and 'multiplication by a scalar' are obvious by definition, the one of 'sum' then follows from the continuity of the midpoint of a geodesic as a function of the extremal points.

Points (2.1.5a), (2.1.5b), (2.1.5c), (2.1.5d), (2.1.5e) are well known and recalled, e.g., in [79, Proposition 2.11]. The concavity property (2.1.5f) is also well known. A way to prove it is to notice that from (2.1.5b) and letting m be the midpoint of v_1, v_2 we get that

$$\langle v_1 \oplus v_2, w\rangle_y = 2\varepsilon^{-1}\langle \varepsilon m, w\rangle_y = \varepsilon^{-1}\big(\varepsilon^2|m|_y^2 + |w|_y^2 - \mathsf{d}_y^2(\varepsilon m, w)\big) \qquad \forall \varepsilon > 0.$$

From the fact that $\mathrm{T}_y\mathrm{Y}$ is $\mathsf{CAT}(0)$ and the fact that εm is the midpoint of $\varepsilon v_1, \varepsilon v_2$ (consequence of (2.1.5a)) we get that $\mathsf{d}_y^2(\varepsilon m, w) \leq \frac{1}{2}\mathsf{d}_y^2(\varepsilon v_1, w) + \frac{1}{2}\mathsf{d}_y^2(\varepsilon v_2, w)$ and plugging this in the above we get

$$\begin{aligned}\langle v_1 \oplus v_2, w\rangle_y &\geq \varepsilon^{-1}\big(\tfrac{1}{2}\big(|w|_y^2 - \mathsf{d}_y^2(\varepsilon v_1, w)\big) + \tfrac{1}{2}\big(|w|_y^2 - \mathsf{d}_y^2(\varepsilon v_2, w)\big)\big) \\ &= \langle v_1, w\rangle_y + \langle v_2, w\rangle_y - \tfrac{\varepsilon}{2}(|v_1|_y^2 + |v_2|_y^2) \qquad \forall \varepsilon > 0\end{aligned}$$

and the conclusion follows letting $\varepsilon \downarrow 0$. $\square$

It will also be useful to know that

$$\alpha(\mathsf{G}_y^z)'_0 \oplus \beta(\mathsf{G}_y^w)'_0 = \lim_{t\downarrow 0} \frac{2}{\varepsilon}(\mathsf{G}_y^{m_t})'_0, \tag{2.1.6}$$

for $z, w \in B_{\mathsf{r}_y}(y) \setminus \{y\}$, where m_t is the midpoint of $(\mathsf{G}_y^z)_{\alpha t}$ and $(\mathsf{G}_y^z)_{\beta t}$, see for instance [47, II-Theorem 3.19] for the simple proof.

We conclude recalling that on $\mathsf{CAT}(\kappa)$-spaces not only a notion of metric derivative is in place for absolutely continuous curves, but it is possible to speak about right (or left) derivatives in the following sense, as proved in [145]:

Proposition 2.1.6 (Right derivatives)**.** *Let* Y *be locally* $\mathsf{CAT}(\kappa)$ *and* (y_t) *an absolutely continuous curve. Then, for a.e. t, the tangent vectors* $\frac{1}{h}(\mathsf{G}_{y_t}^{y_{t+h}})'_0 \in \mathrm{T}_{\gamma_t}\mathrm{Y}$ *have a limit* $y_t'^{+}$ *in* $\mathrm{T}_{\gamma_t}\mathrm{Y}$ *as* $h \downarrow 0$*.*

For us such concept will be useful in particular in connection with the well known first-order variation of the squared distance:

Proposition 2.1.7. *Let* Y *be a* $\mathsf{CAT}(\kappa)$*-space,* (y_t) *an absolutely continuous curve and* $z \in \mathrm{Y}$*. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\mathsf{d}_{\mathrm{Y}}^2(y_t, z) = -\big\langle y_t'^{+}, (\mathsf{G}_{y_t}^z)'_0\big\rangle_{y_t} \qquad \textit{a.e. } t.$$

To prove the above proposition, see e.g. [79, Propositions 2.17 and 2.20], one needs to introduce the notion of angle between geodesics and study its monotonicity properties, its behaviour along absolutely continuous curve and finally its connection with the inner product we introduced. Nevertheless, even if we omit the proof, in the sequel we shall use the following fact (see [79, Lemma 2.19]): let Y be $\mathsf{CAT}(\kappa)$, (y_t) be an absolutely continuous curve and $z \in \mathrm{Y}$. Then, for the time t s.t. $|\dot{\gamma}_t|$ exists and it is positive, we have

$$\begin{gathered}
-\big\langle \tfrac{1}{h}(\mathsf{G}_{y_t}^{y_{t+h}})'_0, (\mathsf{G}_{y_t}^z)'_0\big\rangle_{y_t} \le -\mathsf{d}_{\mathrm{Y}}(y_t, z)\frac{\mathsf{d}_{\mathrm{Y}}(y_{t+h}, y_t)}{h}\cos(\angle_{y_t}^{\kappa}(y_{t+h}, z)), \quad \forall h > 0 \text{ s.t. } y_{t+h} \in B_{\mathsf{r}_{y_t}}(y_t) \\
\varliminf_{h\downarrow 0} -\cos(\angle_{y_t}^{\kappa}(y_{t+h}, z)) = \varliminf_{h\downarrow 0} \frac{\mathsf{d}_{\mathrm{Y}}(y_{t+h}, z) - \mathsf{d}_{\mathrm{Y}}(y_t, z)}{h|\dot{\gamma}_t|},
\end{gathered} \tag{2.1.7}$$

where $\angle_{y_t}^{\kappa}(y_{t+h}, z)$ is the angle at $\bar{y}$ in $\mathbb{M}_k$ of the comparison triangle $\triangle^{\kappa}(\bar{y}, \bar{y}_h, \bar{z})$. The first of these is an obvious consequence of the definition of $\angle_y^{\kappa}(z_1, z_2)$ together with the fact that $\kappa \mapsto \angle_y^{\kappa}(z_1, z_2)$, and thus $\kappa \mapsto -\cos(\angle_y^{\kappa}(z_1, z_2))$, is increasing, while the second one follows from the Taylor expansion of $\cos(\angle_y^{\kappa}(z_1, z_2))$ for $\mathsf{d}_{\mathrm{Y}}(y, z_1)$ small (notice that the explicit formula for $\cos(\angle_y^{\kappa}(z_1, z_2))$ in terms of $\mathsf{d}_{\mathrm{Y}}(y, z_1), \mathsf{d}_{\mathrm{Y}}(y, z_2), \mathsf{d}_{\mathrm{Y}}(z_1, z_2)$ can be obtained by the cosine rule).

2.1.2 Weak convergence

In this section, we recall following [38] the concept of weak convergence in a $\mathsf{CAT}(0)$-space, highlighting the similarities with weak convergence on a Hilbert setting.

Still, it is important to underline that although a well-behaved notion of 'weakly converging sequence' exists, in [38] it is stressed that the existence of a well-behaved weak topology inducing such convergence is an open challenge. Recently, a definition has been proposed in [147] but for our goals we shall only recall the operative definition of weak convergence of sequences within its properties.

Notice that, if Y is $\mathsf{CAT}(0)$ and $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R}^+ \cup \{+\infty\}$ is 2-convex and lower semicontinuous, then it admits a unique minimizer. To see this, we argue as for Proposition 3.2.4 and prove that any minimizing sequence $(y_n) \subset \mathrm{Y}$ is Cauchy: let $I := \inf \mathsf{E} \ge 0$, $y_{n,m}$ the midpoint of y_n, y_m and notice that

$$I \le \mathsf{E}(y_{n,m}) \le \frac{1}{2}\big(\mathsf{E}(y_n) + \mathsf{E}(y_m)\big) - \frac{1}{4}\mathsf{d}_{\mathrm{Y}}^2(y_n, y_m) \qquad \forall n, m \in \mathbb{N},$$

so that rearranging and passing to the limit we get

$$\frac{1}{4}\varlimsup_{n,m\to\infty}\mathsf{d}^2_{\mathrm{Y}}(y_n,y_m)\le\varlimsup_{n,m\to\infty}\frac{1}{2}\big(\mathsf{E}(y_n)+\mathsf{E}(y_m)\big)-I=0,$$

giving the claim. The first example of 2-convex functional we have encountered is the squared distance from a point in a $\mathsf{CAT}(0)$-space, as a consequence of inequality (2.1.2). Hence, for $(y_n)\subset\mathrm{Y}$ be a bounded sequence, we can consider the mapping

$$\mathrm{Y}\ni y\mapsto\omega(y;(y_n)):=\varlimsup_{n}\mathsf{d}^2_{\mathrm{Y}}(y,y_n).$$

and notice that, as a limsup of a sequence of 2-convex and locally equiLipschitz functions, it is still 2-convex and locally Lipschitz. By the above remark, it has a unique minimizer.

Definition 2.1.8 (Asymptotic center and weak convergence)**.** *Let* Y *be* CAT*(0)-space and* (y_n) *be a bounded sequence. We call the minimizer of* $\omega(\cdot,(y_n))$ *the* asymptotic center *of* (y_n).

We say that a sequence $(y_n)\subset\mathrm{Y}$ weakly converges *to* y*, and write* $y_n\rightharpoonup y$*, if* y *is the asymptotic center of every subsequence* (y_{n_k}) *of* (y_n).

In analogy with the Hilbert setting, we shall sometimes say that (y_n) converges strongly to y if $\mathsf{d}_{\mathrm{Y}}(y_n,y)\to0$. The main properties of weak convergence are collected in the following statement:

Proposition 2.1.9. *Let* Y *be a* CAT*(0)-space. Then, the following holds:*

i) *If* (y_n) *converges to* y *strongly, then it converges weakly.*

ii) $y_n\to y$ *if and only if* $y_n\rightharpoonup y$ *and for some* $z\in\mathrm{Y}$ *we have* $\mathsf{d}_{\mathrm{Y}}(y_n,z)\to\mathsf{d}_{\mathrm{Y}}(y,z)$.

iii) *Any bounded sequence admits a weakly converging subsequence.*

iv) *If* $C\subset\mathrm{Y}$ *is convex and closed, then it is sequentially weakly closed.*

v) *If* $\mathsf{E}\colon\mathrm{Y}\to\mathbb{R}\cup\{+\infty\}$ *is a convex and lower semicontinuous function, then it is sequentially weakly lower semicontinuous.*

Moreover, at the tangent cone $\mathrm{T}_y\mathrm{Y}$ *at* $y\in\mathrm{Y}$ *(which is also a* CAT*(0)-space by Theorem 2.1.4) we also have*

vi) *Let* $(v_n),(w_n)\subset\mathrm{T}_y\mathrm{Y}$ *be such that* $v_n\to v$ *and* $w_n\rightharpoonup w$ *for some* $v,w\in\mathrm{T}_y\mathrm{Y}$*. Then* $\varlimsup_{n\to\infty}\langle v_n,w_n\rangle_y\le\langle v,w\rangle_y$.

Proof. i) is obvious, as a strong limit is trivially the asymptotic center of the full sequence. For ii),iii),iv) see [39, Proposition 3.1.6], [39, Proposition 3.1.2] and [39, Proposition 3.2.1] respectively. v) follows trivially from iv) by considering the strongly closed and convex sublevels of E. Finally, for vi) we let $C:=\sup_n|w_n|_y<\infty$ and notice that for every $\varepsilon>0$ it holds

$$\begin{aligned}2\varepsilon\langle v_n,w_n\rangle_y=\langle v_n,2\varepsilon w_n\rangle_y&\le|v_n|_y^2+|2\varepsilon w|_y^2-\mathsf{d}_y^2(v,2\varepsilon w_n)+\big(\mathsf{d}_y^2(v,2\varepsilon w_n)-\mathsf{d}_y^2(v_n,2\varepsilon w_n)\big)+4\varepsilon^2C^2\\&\le|v_n|_y^2+|2\varepsilon w|_y^2-\mathsf{d}_y^2(v,2\varepsilon w_n)+4\varepsilon C\mathsf{d}_y(v,v_n)(|v|_y+|v_n|_y)+4\varepsilon^2C^2\end{aligned}$$

and that $\varepsilon w_n\rightharpoonup\varepsilon w$ (by (2.1.5a)). Sending $n\to\infty$ and using the sequential weak lower semicontinuity of $\mathsf{d}_y^2(v,\cdot)$ (consequence of v)) we obtain that

$$2\varepsilon\varlimsup_{n\to\infty}\langle v_n,w_n\rangle_y\le|v|_y^2+|2\varepsilon w|_y^2-\mathsf{d}_y^2(v,2\varepsilon w)+4\varepsilon^2C^2=2\varepsilon\langle v,w\rangle_y+4\varepsilon^2C^2$$

and the claim follows dividing by $\varepsilon>0$ and letting $\varepsilon\downarrow0$. □

2.1.3 Geometric tangent bundle

In this section we briefly recall some concepts from [79] about the construction of the Geometric Tangent Bundle $\mathrm{T}_G\mathrm{Y}$ of a given *separable* local $\mathsf{CAT}(\kappa)$-space Y. From now on, $\mathcal{B}(\mathrm{Y})$ is the Borel σ-algebra on Y. As a set, the space $\mathrm{T}_G\mathrm{Y}$ is defined as

$$\mathrm{T}_G\mathrm{Y} := \big\{(y,v)\colon y \in \mathrm{Y},\ v \in \mathrm{T}_y\mathrm{Y}\big\}.$$

Such set is equipped with a σ-algebra $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$, called Borel σ-algebra (with a slight abuse of terminology, because there is no topology inducing it), defined as the smallest σ-algebra such that the following maps are measurable:

i) the canonical projection $\pi_{\mathrm{Y}}\colon \mathrm{T}_G\mathrm{Y} \to \mathrm{Y}$

ii) the maps $\pi_{\mathrm{Y}}^{-1}(B_{r_{\bar y}}\bar y) \ni (y,v) \mapsto \big\langle v, (\mathsf{G}_y^z)_0'\big\rangle_y \in \mathbb{R}$ for every $\bar y \in \mathrm{Y}, z \in B_{r_{\bar y}}(\bar y)$.

It turns out that $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$ is countably generated and that, rather than asking ii) for every $z \in \mathrm{Y}$, one can require it only for a dense set of points (notice that in the axiomatization chosen in [79] one speaks about the differential of the distance function rather than of scalar product with vectors of the form $(\mathsf{G}_y^z)_0'$, but the two approaches are actually trivially equivalent thanks to the explicit expression of the differential of the distance in terms of such scalar product which is hidden in Proposition 2.1.7). We also recall that

$$\text{the map } \mathrm{T}_G\mathrm{Y} \ni (y,v) \mapsto |v|_y \in \mathbb{R} \text{ is Borel.} \tag{2.1.8}$$

A *section* of $\mathrm{T}_G\mathrm{Y}$ is a map $\mathsf{s}\colon \mathrm{Y} \to \mathrm{T}_G\mathrm{Y}$ such that $\mathsf{s}_y \in \mathrm{T}_y\mathrm{Y}$ for every Y. A section is said Borel if it is measurable with respect to $\mathcal{B}(\mathrm{Y})$ and $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$. Among the various sections, *simple* ones play a special role, similar to the one played by finite-ranged functions in the theory of Bochner integration: s is a simple section provided there are $(y_n) \subset \mathrm{Y}$, $(\alpha_n) \subset \mathbb{R}^+$ and (E_n) Borel partition of Y such that $y_n \in B_{\mathsf{r}_y}(y)$ for every $y \in E_n$ and $\mathsf{s}_{|_{E_n}} = \alpha_n(\mathsf{G}_\cdot^{y_n})_0'$. If this is the case we write $\mathsf{s} = \sum_n \chi_{E_n}\alpha_n(\mathsf{G}_\cdot^{y_n})_0'$, although the 'sum' here is purely formal. The following basic result - obtained in [79] - will be useful, we report the proof for completeness:

Proposition 2.1.10. *Let Y be separable and locally $\mathsf{CAT}(\kappa)$. Then, simple sections of $\mathrm{T}_G\mathrm{Y}$ as defined above are Borel.*

Proof. It is sufficient to prove that for any given $\bar y \in \mathrm{Y}, z \in B_{r_{\bar y}}(\bar y)$ and $\alpha \in \mathbb{R}^+$ the assignment $B_{r_{\bar y}}(\bar y) \ni y \mapsto \mathsf{s}_y := \alpha(\mathsf{G}_y^z)_0'$ is Borel and to this aim, by the very definition of $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$, it is sufficient to check that $\pi_{\mathrm{Y}} \circ \mathsf{s}\colon \mathrm{Y} \to \mathrm{Y}$ is Borel - which it is, being this map the identity on Y - and, for any $w \in B_{r_{\bar y}}(\bar y)$, the map $B_{r_{\bar y}}(\bar y) \ni y \mapsto \big\langle \mathsf{s}_y, (\mathsf{G}_y^w)_0'\big\rangle_y$ is Borel. Thus fix w and notice that thanks to (2.1.8) and to the definition of scalar product on $\mathrm{T}_y\mathrm{Y}$ to conclude it is sufficient to check that $y \mapsto \mathsf{d}_y(\mathsf{s}_y, (\mathsf{G}_y^w)_0')$ is Borel. We have

$$\mathsf{d}_y(\mathsf{s}_y, (\mathsf{G}_y^w)_0') = \mathsf{d}_y(\alpha(\mathsf{G}_y^z)_0', (\mathsf{G}_y^w)_0') = \lim_{t\downarrow 0} \frac{\mathsf{d}_{\mathrm{Y}}\big((\mathsf{G}_y^z)_{\alpha t}, (\mathsf{G}_y^w)_t\big)}{t}.$$

From the continuous dependence of geodesics on their endpoints we deduce that $y \mapsto \mathsf{d}_{\mathrm{Y}}\big((\mathsf{G}_y^z)_{\alpha t}, (\mathsf{G}_y^w)_t\big)$ is a continuous function for every $t \in (0, 1 \wedge \alpha^{-1})$. The conclusion then follows from the fact that a pointwise limit of continuous functions is Borel. $\square$

It has been proved in [79] that simple sections are dense among Borel ones (see also Lemma 3.3.6 below in the case $\mathrm{X} = \mathrm{Y}$ and $u = $ Identity). Moreover, the operations on a single tangent space $\mathrm{T}_y\mathrm{Y}$ induce in a natural way operations on the space of Borel sections of $\mathrm{T}_G\mathrm{Y}$: these are Borel regular, as recalled in the next statement (see [79, Proposition 3.6] for the proof).

Proposition 2.1.11. *Let Y be separable and locally $\mathsf{CAT}(\kappa)$, s, t Borel sections of $\mathrm{T}_G\mathrm{Y}$ and $f\colon \mathrm{Y} \to \mathbb{R}^+$ Borel. Then, the maps from Y to $\mathbb{R}$ sending y to $|\mathsf{s}_y|_y, \mathsf{d}_y(\mathsf{s}_y, \mathsf{t}_y), \big\langle \mathsf{s}_y, \mathsf{t}_y\big\rangle_y$ are Borel and the sections $y \mapsto f(y)\mathsf{s}_y, \mathsf{s}_y \oplus \mathsf{t}_y$ are Borel as well.*

2.2 The CD-condition

Synthetic treatments of Ricci lower bounds have been initiated with the independent seminal works by K.-T. Sturm [180, 181] and J. Lott and C. Villani [143] where the authors give birth to the celebrated *curvature dimension condition.* Before giving more details, we remark that, differently from the Alexandrov setting, a bound on the Ricci tensor must be formulated in the abstract framework of metric measure spaces. This fact was firstly noticed in 1991 by Gromov, who wrote in [112] that a synthetic theory of Ricci lowerbound needed to be still achieved at the time and it must be formulated in the context of metric measure spaces, rather than simply metric spaces. The intuition is that, as reflected by the main properties of this framework (see (2.2.5) and (2.2.8) below), *volumes* naturally enters into play in the theory.

As a motivation for what comes next, we report a necessary and sufficient condition for a Riemannian manifold to satisfy Ricci lowerbounds. Before giving the actual statement, we denote by Vol the volume measure and by Ric the Ricci tensor of a manifold and we consider the Shannon relative entropy on a Riemannian manifold M as the functional

$$\mathrm{Ent}(\mu) := \int \rho \log \rho \, \mathrm{dVol}, \tag{2.2.1}$$

if $\mu = \rho\mathrm{Vol} + \mu^s$ with $\mu^s \perp \mathrm{Vol}$. Then, as shown in [71, Theorem 6.2] and [188, Theorem 0.1], we have the equivalence:

Theorem 2.2.1. *For any smooth connected Riemannian manifold M and $K \in \mathbb{R}$, the following are equivalent*

i) *it holds* $\mathsf{Ric} \geq K$*;*

ii) *for every $(\mu_t) \subset \mathscr{P}_2(M)$ that is a W_2-geodesic between $\mu_0, \mu_1 \ll \mathrm{Vol}$, it holds*

$$\mathrm{Ent}(\mu_t) \leq (1-t)\mathrm{Ent}(\mu_0) + t\mathrm{Ent}(\mu_1) - K\frac{t(1-t)}{2}W_2(\mu_0, \mu_1), \qquad \forall t \in [0,1].$$

In the above theorem, the dimension of M is not encoded in condition ii). Therefore, we will see how turning ii) into a definition will yield a definition of curvature dimension condition that is dimension-free. Since in this Thesis we shall work both in finite and infinite dimensional setting, we start now recalling the main objects.

For $\mu \in \mathscr{P}(\mathrm{X})$ and $N \in [1, \infty)$, we define the *N-Rényi relative entropy* with respect to $\mathfrak{m}$ by

$$\mathfrak{U}_N(\mu|\mathfrak{m}) := -\int \rho^{1-\frac{1}{N}} \, \mathrm{d}\mathfrak{m}, \qquad \text{if } \mu = \rho\mathfrak{m} + \mu^s, \quad \mu^s \perp \mathfrak{m},$$

and the *Shannon entropy* (the analogous of (2.2.1) on metric measure spaces) by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \int \rho \log \rho \, \mathrm{d}\mathfrak{m}, \qquad \text{if} \mu = \rho\mathfrak{m}, \quad \infty \text{ otherwise}.$$

The distorsion coefficients are, for every $K \in \mathbb{R}, N \in [0, \infty), t \in [0,1]$, defined as

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})}, & \text{if } 0 < K\theta^2 < N\pi^2, \\ t, & \text{if } K\theta^2 < 0 \text{ and } N = 0 \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}, & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases}$$

Set also, for $N \geq 1$, $\tau^{(t)}_{K,N}(\theta) := t^{\frac{1}{N}}\sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}$ while $\tau^{(t)}_{K,1}(\theta) = t$ if $K \leq 0$ and $\tau^{(t)}_{K,1}(\theta) = \infty$ if $K > 0$.

Next, we provide a synthetic definition of spaces with Ricci curvature bounded from below and dimension bounded from above given independently by [143] and [180, 181]. It was given for the exponent $q = 2$, but requiring the analogous convexity properties of the so-defined entropy functionals along W_q-geodesics we can define:

Definition 2.2.2 (CD_q-spaces)**.** *Let $q \in (1,\infty)$, $K \in \mathbb{R}$ and $N \in [1,\infty]$. We say that a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the* curvature dimension condition $\mathsf{CD}_q(K,N)$ *if, for every $\mu_0,\mu_1 \in \mathscr{P}_q(\mathrm{X})$ absolutely continuous , there exists $\pi \in \mathrm{OptGeo}_q(\mu_0,\mu_1)$ so that $\mu_t := (\mathsf{e}_t)_\sharp\pi \ll \mathfrak{m}$ and*

▷ *if $N < \infty$, for every $t \in [0,1]$ and $N' \geq N$*

$$\mathcal{U}_{N'}(\mu_t|\mathfrak{m}) \leq -\int \left(\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_1(\gamma_1))^{-\frac{1}{N}}\right) \mathrm{d}\pi(\gamma), \quad (2.2.2)$$

having denoted $\rho_i = \frac{\mathrm{d}\mu_i}{\mathrm{d}\mathfrak{m}}$ for $i = 0,1$;

▷ *if $N = \infty$, for every $t \in [0,1]$*

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leq (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}t(1-t)W_q^2(\mu_0,\mu_1). \quad (2.2.3)$$

In the sequel of this Thesis, to avoid a heavy notation in most of its part, we adopt the following convention.

> **Notation**. In concordance with standard notation in the literature, we shall stick to the convention
> $$\mathsf{CD}(K,N) := \mathsf{CD}_2(K,N).$$

For arbitrary q, the class of CD_q-spaces was investigated e.g. in [127, 128]:. We also recall the following important result, which was proved in [1, Theorem 1.1]. It states that on non-branching spaces (whose reference measure is finite), the $\mathsf{CD}_q(K,N)$ condition is in fact independent of q.

Theorem 2.2.3 (Equivalence of CD_q on $q > 1$)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a non-branching $\mathsf{CD}(K,N)$ space, for some $K \in \mathbb{R}$ and $N \in (1,\infty)$. Suppose that the measure $\mathfrak{m}$ is finite. Then $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}_q(K,N)$ space for every $q \in (1,\infty)$.*

The above results in fact hold under a weaker assumption, called q-essential non-branching; see [175] for the definition of such condition. We shall never need the extremal case $q = 1$, but an analogous theorem has been shown in the setting of CD_1-spaces in [54].

We recall the notion of one-dimensional model space for the $\mathsf{CD}(N-1,N)$ condition:

Definition 2.2.4 (One dimensional model space)**.** *For every $N > 1$ we define $I_N := ([0,\pi],|.|,\mathfrak{m}_N)$, where $|.|$ is the Euclidean distance restricted on $[0,\pi]$ and*

$$\mathfrak{m}_N := \tfrac{1}{c_N}\sin^{N-1}\mathscr{L}^1|_{[0,\pi]},$$

with $c_N := \int_{[0,\pi]}\sin(t)^{N-1}\,\mathrm{d}t$.

In the sequel, we shall consider Sobolev spaces on weighted Euclidean intervals (or half lines). It is worth to remark on the consistency of the metric and classical definition of Sobolev spaces in this case. Consider $(\mathrm{X},\mathsf{d},\mathfrak{m}) = ([a,b],|.|,h\mathscr{L}^1)$, for $a,b \in \mathbb{R}$ with $a < b$ where $h \in L^1([a,b])$ is so that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ so that $h \geq c_\varepsilon$ $\mathscr{L}^1$-a.e. in $[a+\varepsilon,b-\varepsilon]$. Let us write $W^{1,p}([a,b],|.|,h\mathscr{L}^1)$ for the p-Sobolev space over the weighted interval according to the metric definition given above, while simply write $W^{1,p}_{loc}(a,b)$ for the classical definition via integration by parts of locally Sobolev functions in the standard sense. Then, using the definition via test plan, it can be shown that

$$f \in W^{1,p}([a,b],|.|,h\mathscr{L}^1) \quad \Longleftrightarrow \quad f \in W^{1,1}_{loc}(a,b) \text{ with } f,f' \in L^p(h\mathscr{L}^1) \quad (2.2.4)$$

and, in this case, it holds that $|Df|_p = |f'|$ at $\mathscr{L}^1$-a.e. point. We omit the details.

2.2.1 Geometric inequalities

On $\mathsf{CD}(K,N)$ spaces the Bishop-Gromov inequality holds (see [181]):

$$\frac{\mathfrak{m}(B_R(x))}{v_{K,N}(R)} \le \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}, \quad \text{for any } 0<r<R\le \pi\sqrt{\frac{N-1}{K^+}} \text{ and any } x\in \mathrm{X}, \tag{2.2.5}$$

where the quantities $v_{K,N}(r)$, $N\in[1,\infty)$ $K\in\mathbb{R}$ are defined as

$$v_{K,N}(r) := \sigma_{N-1}\int_0^r |s_{K,N}(t)|^{N-1}\,\mathrm{d}t,$$

and $s_{K,N}(t)$ is defined as $\sin\left(t\sqrt{\frac{K}{N-1}}\right)$, if $K>0$, $\sinh\left(t\sqrt{\frac{|K|}{N-1}}\right)$, if $K<0$ and t if $K=0$.

In particular we have (see for the proof, e.g., [180, 181]):

$$\mathsf{CD}(K,N)\text{-spaces are doubling} \qquad \forall K\in\mathbb{R}, N\in[1,\infty). \tag{2.2.6}$$

As a direct consequence of (2.2.6), it is easy to see that when $N<\infty$ they are also *proper*, i.e. closed and bounded sets are compact.

We also note that in the case $K=0$ this implies that the limit

$$\mathsf{AVR}(\mathrm{X}) := \lim_{r\to+\infty}\frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}$$

exists finite and does not depend on the point $x\in\mathrm{X}$. We call the quantity $\mathsf{AVR}(\mathrm{X})$ *asymptotic volume ratio* of X and if $\mathsf{AVR}(\mathrm{X})>0$ we say that X has *Euclidean-volume growth.* A key-role in the note will be played by the following quantities:

$$\theta_{N,r}(x) := \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}, \quad \theta_N(x) := \lim_{r\to 0^+}\theta_{N,r}(x), \qquad \forall\, r>0, x\in\mathrm{X}.$$

Observe that the above limit exists thanks to the Bishop-Gromov inequality and the fact that $\lim_{r\to 0^+}\frac{\omega_N r^N}{v_{K,N}(r)}=1$ for every $K\in\mathbb{R}$, $N\in[1,\infty)$, which in particular grants that

$$\theta_N(x) = \lim_{r\to 0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} = \sup_{r>0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}. \tag{2.2.7}$$

This and the fact that $\mathfrak{m}(\partial B_r(x))=0$ for every $r>0$ and $x\in\mathrm{X}$ (which follows from the Bishop-Gromov inequality), implies that $\theta_N(x)$ is a *lower-semicontinuous* function of x. Therefore, when X is compact, there exists $\min_{x\in\mathrm{X}}\theta_N(x)$.

Next we recall the Brunn-Minkowski inequality.

Theorem 2.2.5 ([181])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}(K,N)$ space with $N\in[1,\infty)$, $K\in\mathbb{R}$. For any couple of Borel sets $A_0,A_1\subset\mathrm{X}$ it holds that*

$$\mathfrak{m}(A_t)^{\frac{1}{N}} \ge \sigma_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{\frac{1}{N}} + \sigma_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{\frac{1}{N}}, \qquad \forall t\in[0,1], \tag{2.2.8}$$

where $A_t := \{\gamma_t \ :\ \gamma \text{ geodesic such that } \gamma_0\in A_0,\, \gamma_1\in A_1\}$ and

$$\theta := \begin{cases}\inf_{(x_0,x_1)\in A_0\times A_1}\mathsf{d}(x_0,x_1), & \text{if } K\ge 0,\\ \sup_{(x_0,x_1)\in A_0\times A_1}\mathsf{d}(x_0,x_1), & \text{if } K<0,\end{cases}$$

We remark that (2.2.8) is actually weaker than the statement appearing in [181] and it holds for the (a-priori) larger class of $\mathrm{CD}^*(K,N)$ spaces (see [33]).

We report the Bonnet-Myers diameter-comparison theorem for CD-spaces from [181]:

$$\begin{matrix}(\mathrm{X},\mathsf{d},\mathfrak{m}) \text{ is a } \mathsf{CD}(K,N) \text{ space}\\ \text{for some } K>0, N\in(1,\infty)\end{matrix} \quad\Rightarrow\quad \mathrm{diam}(\mathrm{X}) \le \pi\sqrt{\tfrac{N-1}{K}}. \tag{2.2.9}$$

2.2.2 Functional inequalities

Spectral gap

The Lichnerowitz 2-spectral gap inequality is valid also in the CD-setting. To state it we recall the notion of first non-trivial Neumann eigenvalue of the Laplacian (or 2-spectral gap) in metric measure spaces.

Definition 2.2.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with finite measure. We define the first non trivial* 2*-eigenvalue* $\lambda^{1,2}(\mathrm{X})$ *as the non-negative number given by*

$$\lambda^{1,2}(\mathrm{X}) := \inf \left\{ \frac{\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}}{\int |f|^2 \,\mathrm{d}\mathfrak{m}} \ : \ f \in \mathrm{Lip}(\mathrm{X}) \cap L^2(\mathfrak{m}),\ f \neq 0,\ \int f \,\mathrm{d}\mathfrak{m} = 0 \right\}. \tag{2.2.10}$$

Clearly, in light of [20], in the above definition one can equivalently take the infimum among all $f \in W^{1,2}(\mathrm{X})$. In the sequel will use this fact without further notice.

Then the spectral-gap inequality as proven in [143] (see also [123]) says that:

$$\lambda^{1,2}(\mathrm{X}) \geq N, \qquad \text{for every } \mathsf{CD}(N-1, N)\text{-space } \mathrm{X},$$

with N ranging in $(1, \infty)$.

Sobolev-Poincaré inequality

From [174], we report the following well known fact

$$\begin{array}{l}\mathsf{CD}(K,N) \text{ spaces support a} \\ \text{weak local } (1,1)\text{-Poincaré inequality}\end{array} \qquad \forall K \in \mathbb{R}, N \in [1, \infty]. \tag{2.2.11}$$

Then, a well-established fact which goes back to the seminal work [114], is that a $(1,p)$-Poincaré inequality on a doubling metric measure space, improves to a (q,p)-Poincaré inequality with $q > 1$. On $\mathsf{CD}(K,N)$ spaces this translates in the following result.

Theorem 2.2.7 ((p^*, p)-Poincaré inequality)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{CD}(K,N)$ *space for some* $N \in (1, \infty)$*,* $K \in \mathbb{R}$*. Fix also* $p \in (1, N)$ *and* $r_0 > 0$*. Then, for every* $B_r(x) \subset \mathrm{X}$ *with* $r \leq r_0$ *it holds*

$$\Big(\fint_{B_r(x)} |u - u_{B_r(x)}|^{p^*} \,\mathrm{d}\mathfrak{m} \Big)^{\frac{1}{p^*}} \leq C(K,N,p,r_0) r \Big(\fint_{B_{2r}(x)} |Du|^p \,\mathrm{d}\mathfrak{m} \Big)^{\frac{1}{p}}, \qquad \forall\, u \in \mathrm{Lip}(\mathrm{X}), \tag{2.2.12}$$

where $p^* := pN/(N-p)$ *and* $u_{B_r(x)} := \fint_{B_r(x)} u \,\mathrm{d}\mathfrak{m}$.

Proof. From (2.2.11), we know that X also supports a strong $(1,p)$-Poincaré inequality for every $p \in [1, \infty)$, by Hölder inequality. Moreover, for every $x_0 \in \mathrm{X}$, $r \leq r_0$ and $x \in B_{r_0}(x_0)$, from the Bishop-Gromov inequality (2.2.5) it holds that

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_{r_0}(x_0))} \geq C(K,N,r_0) \Big(\frac{r}{r_0} \Big)^N.$$

Then (2.2.12) follows from [114, Theorem 5.1] (see also [43, Theorem 4.21]). □

We end this part recalling the sharp Sobolev-inequality on the N model space I_N (see Def. 2.2.4) for $N \in (2, \infty)$ (see e.g. [138]):

$$\|u\|^2_{L^q(\mathfrak{m}_N)} \leq \frac{q-2}{N} \, \||Du|\|^2_{L^2(\mathfrak{m}_N)} + \|u\|^2_{L^2(\mathfrak{m}_N)}, \quad \forall u \in W^{1,2}([0,\pi], |.|, \mathfrak{m}_N), \tag{2.2.13}$$

for every $q \in (2, 2^*]$, with $2^* = 2N/(N-2)$.

Polya-Szego inequality

The Polya-Szego inequality, namely the fact that the Dirichlet energy decreases under *decreasing rearrangements*, dates back to Faber and Krahn and was successively formalized in [172]. Later, in [41], this collection of ideas was brought to the context of manifolds with Ricci lower bounds to achieve applications concerning the rigidity of the 2-spectral gap.

In this part we recall the Polya-Szego inequality for essentially non branching $\mathrm{CD}(K,N)$ spaces proven in [159]. We will also collect some additional technical results and definitions from [159] that will be used in Section 6.3.1 to prove an Euclidean-variant of this inequality.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, $\Omega\subset\mathrm{X}$ open with $\mathfrak{m}(\Omega)<\infty$. Given $u\colon\Omega\to[0,\infty)$ Borel and non-negative and denoting μ its distribution function (see 1.1.2), we let $u^{\#}$ be the generalized inverse of μ, defined by

$$u^{\#}(s):=\begin{cases}\operatorname{ess\,sup}u & \text{if } s=0,\\ \inf\{t:\mu(t)<s\} & \text{if } s>0.\end{cases}$$

It can be checked that $u^{\#}$ is non-increasing and left-continuous.

Then, we define the *monotone rearrangement* into $I_N=([0,\pi],|.|,\mathfrak{m}_N)$ (see Definition 2.2.4 for the N-model interval) as follows: first, we consider $r>0$ so that $\mathfrak{m}(\Omega)=\mathfrak{m}_N([0,r])$ and define $\Omega^*:=[0,r]$, then we define the monotone rearrangement function $u^*_N:\Omega^*\to\mathbb{R}^+$ as

$$u^*_N(x):=u^{\#}(\mathfrak{m}_N([0,x])),\qquad\forall x\in[0,r].$$

In the sequel, whenever u and Ω are fixed, Ω^* and u^*_N will be implicitly defined as above.

Theorem 2.2.8 (Polya-Szego inequality, [159])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an essentially non braching $\mathsf{CD}(N-1,N)$ space for some $N\in(1,\infty)$ and $\Omega\subseteq\mathrm{X}$ be open. Then, for every $p\in(1,\infty)$, the monotone rearrangement in I_N maps $L^p(\Omega)$ (resp. $W^{1,p}_0(\Omega)$) into $L^p(\Omega^*)$ (resp. $W^{1,p}(\Omega^*,|.|,\mathfrak{m}_N)$) and satisfies:*

$$\|u\|_{L^p(\Omega)}=\|u^*_N\|_{L^p(\Omega^*)},\qquad\forall u\in L^p(\Omega)\tag{2.2.14}$$

$$\int_\Omega|Du|^p\,\mathrm{d}\mathfrak{m}\geq\int_{\Omega^*}|Du^*_N|^p\,\mathrm{d}\mathfrak{m}_N,\qquad\forall u\in W^{1,p}_0(\Omega).\tag{2.2.15}$$

2.2.3 Wasserstein interpolation with bounded compression

On $\mathsf{CD}(K,\infty)$-spaces, a deep analysis conducted in [174] showed that it is possible under the curvature dimension condition to interpolate between bounded and boundedly supported probability densities via an optimal geodesic plan that is also a test plan. We give here the precise statement.

Theorem 2.2.9. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}(K,\infty)$-space for some $K\in\mathbb{R}$. For any $D>0$ and $\rho_0,\rho_1\in L^\infty(\mathfrak{m})$ probability densities with diam(supp(ρ_0) $\cup$ supp(ρ_1)) $<D$, there exists $\pi\in\mathrm{OptGeo}_2(\rho_0\mathfrak{m},\rho_1\mathfrak{m})$ satisfying $\mu_t:=(\mathsf{e}_t)_\sharp\pi\ll\mathfrak{m}$. Moreover, writing $\mu_t:=\rho_t\mathfrak{m}$, we have*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})}\leq e^{K^-D^2/12}\|\rho_0\|_{L^\infty(\mathfrak{m})}\vee\|\rho_1\|_{L^\infty(\mathfrak{m})},\qquad\forall t\in[0,1].$$

We will see the proof in Appendix A, where we will also extend this Theorem to all exponent $q\neq2$ for CD_q-spaces. All the arguments from [174] are easily seen to be transported, the main step being the concept of *spreading of mass* under the curvature dimension condition.

2.3 The RCD-condition

If on one hand, the CD-theory encodes many remarkable properties valid in the smooth category of Riemannian manifolds, on the other it allows the presence of other geometries. A basic example of CD-space (that is evidently not a Riemannian manifold) is infact the standard Euclidean space equipped with a norm not arising from a scalar product and the standard Lebesgue measure. In particular, it is not even a Ricci limit space as guaranteed by the 'almost splitting' result of [65].

Consequently, progress have been put to formulate a *Riemannian* synthetic theory of Ricci lowerbounds to rule out *Finsler geometries*. In this work, a motivation to discuss this different class of spaces is that they are more suitable to derive the *rigidty results* typical of Riemannian manifolds.

The *Riemannian curvature dimension condition*, called shortly RCD-condition, appeared for the first time in [22] in the infinite dimensional case, giving thus birth to the notion of $\mathsf{RCD}(K,\infty)$-spaces. This has been essentially achieved by adding the linearity of the heat flow to the CD-theory. In the finite dimensional case, the $\mathsf{RCD}(K,N)$ condition was instead proposed in [95] in combination with the investigation of Infinitesimal Hilbertianity.

We mention additional important contributions [33, 23, 18, 26, 87, 56] that lead to the developments to the theory as it is nowadays presented. Here, we shall not discuss other equivalent formulations and, for brevity reason, we stick to the following definition.

Definition 2.3.1 (RCD-spaces)**.** *A metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *satisfies the* $\mathsf{RCD}(K,N)$ *condition for some* $K\in\mathbb{R}$ *and* $N\in[1,\infty]$*, provided it is an infinitesimal Hilbertian* $\mathsf{CD}(K,N)$ *metric measure space.*

Remark 2.3.2. For a reason that we now make clear, we avoided introducing the exponent's subscript in the definition of RCD-space (i.e. a RCD_q-condition), even though the two requirements characterizing the above definition are strictly related to the exponent $q=2$. Indeed, the first motivation is that, recently in [77], it has been proven that

$$\mathsf{RCD}(K,N) \text{ spaces with } N<\infty \text{ are non branching.} \tag{2.3.1}$$

Therefore, by recalling Theorem 2.2.3, it is clear that *a posteriori* in Definition 2.3.1 the choice $q=2$ is irrelevant at least for finite reference measure). While, the second motivation is that in the work [99] it has been already shown that $\mathsf{RCD}(K,\infty)$-spaces posse p-independent weak upper gradients (see Theorem 2.3.6). Again, this implies *a posteriori* that one can equivalently require the defining parallelogram rule (1.4.1) with arbitrary minimal p-weak upper gradients. ■

In the sequel, structural properties of RCD-spaces will play a minor role as we will focus mostly on characterization of Sobolev functions/maps and rigidity results attached to functional inequalities in this setting. However, here we just mention that RCD-spaces have been proved in [50] to have *constant dimension*. This is a deep result in the literature that, for convenience of this work, we shall just mention in the following form due to [50, Corollary 3.10].

Theorem 2.3.3 (Essential dimension of $\mathsf{RCD}(K,N)$-spaces)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(K,N)$*-space for some* $N\in[1,\infty)$ *and* $K\in\mathbb{R}$*. Then, there is* $d\le N$ *called* essential dimension *of* X *so that*

$$L^0_2(T^*\mathrm{X}), L^0_2(T\mathrm{X}) \text{ are Hilbert modules of dimension } d,$$

namely they admit d*-dimensional orthonormal generating basis.*

2.3.1 Sobolev maps: Korevaar-Schoen revisited

When considering maps defined defined on a RCD-space X and valued in $\mathsf{CAT}(0)$-space $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$, we have the following characterization (obtained in [109], see [133] for the original definition of 'Korevaar-Schoen' space).

Theorem 2.3.4 (The Korevaar-Schoen space and energy)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(K,N)$ *space for some* $K\in\mathbb{R}$*,* $N\in[1,\infty)$*),* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\bar y)$ *a pointed* CAT*(0)-space,* $\Omega\subset\mathrm{X}$ *open and* $u\in L^2(\Omega,\mathrm{Y}_{\bar y})$*. Then the following are equivalent:*

i) $u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar{y}})$, *namely defining* $\mathsf{ks}_{2,r}[u,\Omega] \colon \Omega \to \mathbb{R}^+$ *by*

$$\mathsf{ks}_{2,r}[u,\Omega](x) := \begin{cases} \left| \fint_{B_r(x)} \frac{\mathsf{d}^2_{\mathrm{Y}}(u(x),u(\tilde{x}))}{r^2} \, \mathrm{d}\mathfrak{m}(\tilde{x}) \right|^{1/2} & \text{if } B_r(x) \subset \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

and defining the energy $\mathsf{E}^{\mathsf{KS}}(u)$ *by*

$$\mathsf{E}^{\mathsf{KS}}(u) := \varlimsup_{r \downarrow 0} \frac{1}{2} \int_\Omega \mathsf{ks}^2_{2,r}[u,\Omega] \, \mathrm{d}\mathfrak{m}, \tag{2.3.2}$$

we have $\mathsf{E}^{\mathsf{KS}}(u) < \infty$;

ii) $u \in W^{1,2}(\Omega, \mathrm{Y}_{\bar{y}})$, *as defined in Definition 1.2.16.*

If any of these hold, the 'energies at scale r' $\mathsf{ks}_{2,r}[u,\Omega]$ *converge to* $(d+2)^{-\frac{1}{2}} |\mathrm{d}_2 u|_{\mathsf{HS}}$ *in* $L^2(\mathfrak{m}_{|_\Omega})$ *as* $r \downarrow 0$, *where* d *is the essential dimension of* X. *In particular, the* $\varlimsup$ *in* (2.3.2) *is actually a limit and the energy admits the representation*

$$\mathsf{E}^{\mathsf{KS}}(u) = \frac{1}{2(d+2)} \int_\Omega |\mathrm{d}_2 u|^2_{\mathsf{HS}} \, \mathrm{d}\mathfrak{m}.$$

Finally, the functional $\mathsf{E}^{\mathsf{KS}} \colon L^2(\Omega, \mathrm{Y}_{\bar{y}}) \to [0, +\infty]$ *is convex and lower semicontinuous.*

Remark 2.3.5. Let us give few remarks related to the above Theorem.

- ▷ The notion of the 'Korevaar-Schoen' space $\mathsf{KS}^{1,p}(\Omega, \mathrm{Y}_{\bar{y}}) \subset L^p(\Omega, \mathrm{Y}_{\bar{y}})$ (with straightforward definition when $p \neq 2$) was derived (with a $\underline{\lim}$ in (2.3.2)) in [133] for maps defined on (open subset of) smooth manifolds and valued to arbitrary complete metric space Y. Nevertheless, this notion can be given for arbitrary metric measure spaces but, in general, $\mathsf{KS}^{1,p}(\Omega, \mathrm{Y}_{\bar{y}}) \neq W^{1,p}(\Omega, \mathrm{Y}_{\bar{y}})$. In [109], the authors revisited [133] (see also [108]) proving in the above theorem that these two notions are actually equivalent for RCD-domains (indeed, under the weaker hypothesis of $\mathfrak{m}$-strong rectifiability and Doubling & Poincaré).
- ▷ Since Y is a $\mathsf{CAT}(0)$-space, it is universally infinitesimal Hilbertian by Theorem 2.1.3. Then, recalling Remark 1.4.3, the differential $\mathrm{d}_2 u$ belongs to a suitable Hilbert module and the object $|\mathrm{d}_2 u|_{\mathsf{HS}} \in L^2(\mathfrak{m}_{|_\Omega})$ is well defined.
- ▷ It should be noticed that the minimal 2-weak upper gradient $|Du|_2$ is not the Hilbert-Schmidt norm $|\mathrm{d}_2 u|_{\mathsf{HS}}$ of the differential $\mathrm{d}_2 u$, but rather the (pointwise) operator norm of $\mathrm{d}_2 u$. The two quantities are nevertheless comparable [109], i.e. one controls the other up to multiplication with a dimensional constant. This especially grants that $L^2(\Omega, \mathrm{Y}_{\bar{y}}) \ni u \mapsto \mathsf{E}^{\mathsf{KS}}(u)$ is lower semicontinuous in light of (1.2.13).
- ▷ The $\mathsf{CAT}(0)$ condition (2.1.2) ensures the convexity of the assignment $u \mapsto \mathsf{E}^{\mathsf{KS}}(u)$ along L^2-geodesic interpolation of maps (see (2.3.4) below).

■

In the sequel, we shall also consider the notion of Korevaar-Schoen map with prescribed boundary values. To this aim, we recall from [109] that for $u, v \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar{y}})$ we always have $\mathsf{d}_{\mathrm{Y}}(u,v) \in W^{1,2}(\Omega)$. Therefore it makes sense to ask whether u, v attain the same boundary value by checking whether or not we have $\mathsf{d}_{\mathrm{Y}}(u,v) \in W^{1,2}_0(\Omega)$.

Then given $\bar{u} \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar{y}})$ the 'energy $\mathsf{E}^{\mathsf{KS}}_{\bar{u}} \colon L^2(\Omega, \mathrm{Y}) \to [0,\infty]$ with $\bar{u}$ as prescribed boundary value' can be defined as

$$\mathsf{E}^{\mathsf{KS}}_{\bar{u}}(u) := \begin{cases} \mathsf{E}^{\mathsf{KS}}(u) & \text{if } u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar{y}}) \text{ and } \mathsf{d}_{\mathrm{Y}}(u,\bar{u}) \in W^{1,2}_0(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We shall denote the domain of $\mathsf{E}^{\mathsf{KS}}_{\bar u}$ by $\mathsf{KS}^{1,2}_{\bar u}(\Omega,\mathrm{Y}_{\bar y})\subset L^2(\Omega,\mathrm{Y}_{\bar y})$ and recall [109] that

$$\mathsf{E}^{\mathsf{KS}}_{\bar u}\colon L^2(\Omega,\mathrm{Y}_{\bar y})\to[0,+\infty]\qquad\text{is convex and lower semicontinuous,}\tag{2.3.3}$$

moreover it admits a unique minimizer, called harmonic map with $\bar u$ as boundary value.

For later use we recall that the convexity of both E^{KS} and $\mathsf{E}^{\mathsf{KS}}_{\bar u}$ can be improved to the following inequality:

$$\mathsf{E}^{\mathsf{KS}}((\mathsf{G}^v_u)_t)+t(1-t)\mathsf{E}^{\mathsf{KS}}(d)\le(1-t)\mathsf{E}^{\mathsf{KS}}(u)+t\mathsf{E}^{\mathsf{KS}}(v)\qquad\forall t\in[0,1],\tag{2.3.4}$$

where $d(x):=\mathsf{d}(u(x),v(x))$. Such inequality has been proved for the case $t=\frac12$ in [109] (imitating the arguments in [133]), the general case follows along the same arguments. It is worth to underline that in the above the maps $u,v,(\mathsf{G}^v_u)_t$ are Y-valued, while d is real valued. In this sense the energy of $\mathsf{E}^{\mathsf{KS}}(d)$ of d has a different meaning w.r.t. the energy of the other maps. Still, we recall (see [109] and [133]) that for a constant $c(d)$ depending only on the essential dimension $d\le N$ of X (recall Theorem 2.3.3) we have $\mathsf{E}^{\mathsf{KS}}(f)=c(d)\mathsf{Ch}_2(f)$ for any $f\in L^2(\mathfrak{m}_{|_\Omega})$, where Ch_2 is the standard 2-Cheeger energy on X.

2.3.2 Independence of weak upper gradients

We recall that in general, minimal p-weak upper gradients may depend on p on a arbitrary metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Moreover, in Remark 1.2.9 we saw the kind of link should always be expected between p-minimal weak upper gradients with respect to different p's. Moreover, at least when $N<\infty$, thanks to (2.2.6) and (2.2.11) we know that $\mathsf{RCD}(K,N)$ are doubling and supports a weak local $(1,1)$-Poincaré inequality hence, Proposition 1.3.8 certainly applies.

Nevertheless, the analysis conducted in [99] showed that this kind of independence should be expected to hold also in the infinite dimensional setting and to be *stronger* (as will be axiomatizated later Chapter 4.

Theorem 2.3.6 ([99])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(K,\infty)$-space for some $K\in\mathbb{R}$. For any $p_1,p_2\in(1,\infty)$, if $f\in S^{p_1}(\mathrm{X})$ is so that $|Df|_{p_1}\in L^{p_2}(\mathfrak{m})$, then*

$$f\in S^{p_2}(\mathrm{X})\qquad\textit{and}\qquad|Df|_{p_2}=|Df|_{p_1}\quad\mathfrak{m}\text{-a.e.}.$$

As previously stated, from now on we will omit the p-subscript, $p\in(1,\infty)$, and simply write $|Df|$ without further notice on proper $\mathsf{RCD}(K,\infty)$-spaces. Let us comment the above theorem in relation to this Thesis work.

Remark 2.3.7. The proof of the above theorem relies on heat-flow regularization techniques (which are well understood and at hand in this class, see e.g. [21]). One of the key ingredients is a Bakry-Émery contraction rate for Lipschitz functions and local Lipschitz constants, as proved in [176]. This makes it possible to show that weak upper gradients do not depends on p in the class of Lipschitz functions. Then, given that the Lipschitz class is *large* enough in the RCD-setting in every Sobolev space $W^{1,p}(\mathrm{X})$, the conclusion comes appealing to the lower semicontinuity of minimal weak upper gradients (Proposition 1.2.7).

The arguments do not carry to the setting of CD-spaces, as the lack of infinitesimal Hilbertianity and Bakry-Émery contration rates makes impossible to follow the very same line of the proof. In Chapter 4, we will see how a completely different proof, via Optimal Transportation, yields the very same statement of Theorem 2.3.6 on CD-spaces. ■

A second deep result following from the analysis of [99], is the resolution to the problem of independence of the minimal weak upper gradient also in the *extreme case* $p=1$. Recall that in Proposition 1.2.25, we showed that Sobolev functions are of bounded variation. Nevertheless, given $f\in S^p(\mathfrak{m})$, it is by no means true that $|\boldsymbol{D}f|=|Df|_p\mathfrak{m}$ and we shall only expect in general the inequality (1.2.19). In the RCD-setting, this is actually true as shown in [99, Remark 3.5] under an additional properness assumption.

Theorem 2.3.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper* $\mathsf{RCD}(K,\infty)$*-space for some* $K \in \mathbb{R}$*. Let* $f \in BV(\mathrm{X})$ *and assume that, for some* $p \in (1,\infty)$*, it holds* $|\boldsymbol{D}f| \ll \mathfrak{m}$ *and* $\frac{\mathrm{d}|\boldsymbol{D}f|}{\mathrm{d}\mathfrak{m}} \in L^p(\mathfrak{m})$*. Then,*

$$f \in S^p(\mathrm{X}) \qquad \text{and} \qquad \frac{\mathrm{d}|\boldsymbol{D}f|}{\mathrm{d}\mathfrak{m}} = |Df|_p, \quad \mathfrak{m}\text{-a.e.}.$$

Next, we finish by stating a well-known key implication of the above result. In the setting of proper $\mathsf{RCD}(K,\infty)$ spaces, the Coarea formula of Theorem 1.2.21 yields

$$\int \operatorname{lip} f \,\mathrm{d}\mathfrak{m} = |\boldsymbol{D}f|(\mathrm{X}) = \int_0^\infty \operatorname{Per}(\{f > t\})\,\mathrm{d}t,$$

for every positive $f \in \operatorname{Lip}_{bs}(\mathrm{X})$.

2.3.3 Some rigidity theorems

We conclude this part recalling some rigidity and stability statements for $\mathsf{RCD}(K,N)$ spaces and to this aim we need to define the notion of spherical suspension over a metric measure space. For any $N \in [1,\infty)$ the N-spherical suspension over a metric measure space $(\mathrm{Z}, \mathfrak{m}_{\mathrm{Z}}, \mathsf{d}_{\mathrm{Z}})$ is defined to be the space $([0,\pi] \times^N_{\sin} \mathrm{Z}) := \mathrm{Z} \times [0,\pi]/(\mathrm{Z} \times \{0,\pi\})$ endowed with the following distance and measure

$$\begin{aligned}
&\mathsf{d}((t,z),(s,z')) := \cos^{-1}\big(\cos(s)\cos(t) + \sin(s)\sin(t)\cos\left(\mathsf{d}_{\mathrm{Z}}(z,z') \wedge \pi\right)\big),\\
&\mathfrak{m} := \sin^{N-1}(t)\mathrm{d}t \otimes \mathfrak{m}_{\mathrm{Z}}.
\end{aligned}$$

It turns out that the RCD condition is stable under the action of taking spherical suspensions, more precisely it has been proven in [132] that

$$\begin{array}{lcl} [0,\pi] \times^N_{\sin} \mathrm{Z} \text{ is } \mathsf{RCD}(N-1,N) & \iff & Z \text{ is } \mathrm{RCD}(N-2,N-1) \text{ with} \\ \text{for some } N \geq 2 & & \operatorname{diam}(\mathrm{Z}) \leq \pi. \end{array} \tag{2.3.5}$$

We can now recall the two main rigidity statements that we will use in the note: the maximal diameter theorem and the Obata theorem for $\mathsf{RCD}(K,N)$ spaces:

Theorem 2.3.9 ([131])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(N-1,N)$ *space with and* $N \in [2,\infty)$ *and suppose that* $\operatorname{diam}(\mathrm{X}) = \pi$*. Then* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathsf{RCD}(N-2,N-1)$ *space* $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with* $\operatorname{diam}(\mathrm{Z}) \leq \pi$ *satisfying* $\mathrm{X} \simeq [0,\pi] \times^N_{\sin} \mathrm{Z}$*.*

Theorem 2.3.10 ([132])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(N-1,N)$ *space with and* $N \in [2,\infty)$ *and suppose that* $\lambda^{1,2}(\mathrm{X}) = N$*. Then* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathsf{RCD}(N-2,N-1)$ *space* $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with* $\operatorname{diam}(\mathrm{Z}) \leq \pi$ *satisfying* $\mathrm{X} \simeq [0,\pi] \times^N_{\sin} \mathrm{Z}$*.*

We will also need the following rigidity of the Polya-Szego inequality proven in [159, Theorem 5.4].

Theorem 2.3.11. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(N-1,N)$ *space for some* $N \in [2,\infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and* $p \in (1,\infty)$*. Let* $\Omega \subset \mathrm{X}$ *be an open set and assume that there exists a non-negative and non-constant function* $u \in W^{1,p}_0(\Omega)$ *achieving equality in the Polya-Szego inequality* (2.2.15)*.*

Then $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathsf{RCD}(N-2,N-1)$ *space* $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with* $\mathfrak{m}_{\mathrm{Z}}(\mathrm{Z}) = 1$ *so that* $\mathrm{X} \simeq [0,\pi] \times^N_{\sin} \mathrm{Z}$*.*

Remark 2.3.12. Observe that in Theorem 2.3.11 we did not assume that $\mathfrak{m}(\Omega) < 1$, assumption that is actually present in Theorem 5.4 of [159]. This is intentional, since we will need to apply Theorem 2.3.11 precisely in the case $\Omega = \mathrm{X}$. This is possible since the arguments in [159] work also in the case $\Omega = \mathrm{X}$ without modification. The only part where the argument does not cover explicitly the case $\Omega = \mathrm{X}$ is the proof of the approximation Lemma 3.6 in [159], which however can be easily adapted (see Lemma 2.3.13 below). ■

Recall for the next lemma, the notation $|D^{ac}f| := \frac{\mathrm{d}|\boldsymbol{D}f|}{\mathrm{d}\mathfrak{m}}$ for every $f \in BV(\mathrm{X})$ with $|\boldsymbol{D}f| \ll \mathfrak{m}$.

Lemma 2.3.13 (Approximation with non-vanishing gradients)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K, N)$ metric measure space with $N < +\infty$, and let $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. Then for any non-negative $u \in \mathrm{Lip}_c(\Omega)$ there exists a sequence of non-negative $u_n \in \mathrm{Lip}_c(\Omega)$ satisfying $|D^{ac}u_n| \neq 0$ $\mathfrak{m}$-a.e. in $\{u_n > 0\}$ and such that $u_n \to u$ in $W^{1,p}(\mathrm{X})$.*

Proof. The case $\Omega \neq \mathrm{X}$ has been proven in [159, Lemma 3.6 and Corollary 3.7]. The proof presented there, as it is written, does not cover the case $\Omega = \mathrm{X}$ with X compact and $\mathrm{supp}(u) = \mathrm{X}$. However, the argument can be easily adapted by considering a sequence $\varepsilon_n \to 0$ such that $\mathfrak{m}(\{\mathrm{lip}(u_n) = \varepsilon_n\}) = 0$ and taking

$$u_n := u + \varepsilon_n v,$$

with $v(x) := \mathsf{d}(x_0, x)$, for an arbitrary fixed point $x_0 \in \mathrm{X}$. Since $v \in \mathrm{Lip}(\mathrm{X})$ and $\mathrm{lip}(v) = 1$ $\mathfrak{m}$-a.e. in X, arguing exactly as in [159, Lemma 3.6] we get that $u_n \to u$ in $W^{1,p}(\mathrm{X})$ and $\mathrm{lip}(u_n) \neq 0$ $\mathfrak{m}$-a.e. in $\{u_n > 0\}$. To get the claimed non-vanishing of $|D^{ac}u_n|$, as in [159, Corollary 3.7] we use the existence of a constant $c > 0$ such that

$$|D^{ac}u| \geq c\,\mathrm{lip}(u), \quad \mathfrak{m}\text{-a.e.},$$

for every $u \in \mathrm{Lip}_{loc}(\mathrm{X})$, which holds from the results in [27] and the fact that $\mathsf{CD}(K, N)$ spaces are locally doubling and supports a local-Poincaré inequality. □

The above lemma will be also used in Section 6.3.1.

2.4 Convergence and compactness under mGH-convergence

In this part we report some properties concerning sequences of pointed metric measure spaces satisfying synthetic lower Ricci bounds. We will follow the characterization of Definition 1.1.11 that it is not the classical one (see e.g. [51, 113]), but it is equivalent in the case of a sequence of uniformly locally doubling metric measure spaces, thanks to the results in [100].

We start recalling the notation of extended natural numbers $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and that of a pointed metric measure space, namely a quadruple $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x)$ consisting of a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and a point $x \in \mathrm{X}$. In the case of a sequence of uniformly locally doubling spaces (as in the case of $\mathsf{CD}(K, N)$-spaces for fixed $K \in R, N < \infty$) we can also take $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$ in Definition 1.1.11 to be proper. Moreover, again for a class of uniformly locally doubling spaces, in [100] it is proven that the pmGH-convergence is metrizable with a distance which we call d_{pmGH}.

It will be also convenient to adopt, thanks to Definition 1.1.11, the so-called *extrinsic approach*, where the spaces X_n are identified as subsets of a common *proper* metric space $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$, $\mathrm{X}_n \subset \mathrm{Z}$, $\mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$, $\mathsf{d}_{\mathrm{Z}}|_{\mathrm{X}_n \times \mathrm{X}_n} = \mathsf{d}_n$ for all $n \in \bar{\mathbb{N}}$, and $\mathsf{d}_{\mathrm{Z}}(x_n, x_\infty) \to 0$, $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_{bs}(\mathrm{Z})$. Any such space $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$ (together with an the identification of $\mathrm{X}_n \subset \mathrm{Z}$) is called *realization of the convergence* and (in the case of geodesic uniformly locally doubling spaces) can be taken so that $\mathsf{d}_H^{\mathrm{Z}}(B_R^{\mathrm{X}_n}(x_n), B_R^{\mathrm{X}_\infty}(x_\infty)) \to 0$ for every $R > 0$, where $\mathsf{d}_H^{\mathrm{Z}}$ is the Hausdorff distance in Z. To avoid confusion when dealing with this identification, we shall sometimes write $B_r^{\mathrm{X}_n}(x)$ with $x \in \mathrm{X}_n$, $r > 0$, to denote the set $B_r^{\mathrm{Z}}(x) \cap \mathrm{X}_n$.

After the above clarification, we shall now state a crucial precompactness theorem: after the works in [180, 181, 143, 91, 22, 100] and thanks to the Gromov's precompactness theorem [113] it holds:

Theorem 2.4.1. *Let $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ be a sequence of pointed $\mathsf{CD}(K_n, N_n)$ (resp. $\mathsf{RCD}(K_n, N_n)$) spaces, $n \in \bar{\mathbb{N}}$, with $\mathfrak{m}(B_1(x_n)) \in [v^{-1}, v]$, for $v > 1$ and $K_n \to K \in \mathbb{R}, N_n \to N \in [1, \infty)$. Then, there exists a subsequence (n_k) and a pointed $\mathsf{CD}(K, N)$ (resp. $\mathsf{RCD}(K, N)$) space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ satisfying*

$$\lim_{k \to \infty} \mathsf{d}_{pmGH}\big((\mathrm{X}_{n_k}, \mathsf{d}_{n_k}, \mathfrak{m}_{n_k}, x_{n_k}), (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)\big) = 0.$$

We will be frequently consider the case of compact (with uniformly bounded diameter) metric measure spaces which is the natural setting for the Sobolev embedding of this note, for which we can reduce the above convergence to the so-called *measure Gromov Hausdorff convergence*, mGH-convergence for short, where we simply ignore the convergence of the base points. Also in this case, on every class of uniformly doubling metric measure spaces with uniformly bounded diameter, the mGH-convergence can be metrized by a distance that we denote by d_{mGH}. The extrinsic approach applies verbatim as well, with the exception that the common ambient space Z can be also taken to be compact.

We now recall some stability and convergence results of functions along pmGH-convergence. For additional details and analogous results we refer to [120, 100, 24]. For brevity reasons in what follows we consider fixed a sequence of pointed CD(K,N) spaces $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, for $n \in \bar{\mathbb{N}}$, so that $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$.

Definition 2.4.2. *Let $p \in (1,\infty)$, we say that*

i) $f_n \in L^p(\mathfrak{m}_n)$ converges L^p-weak *to* $f_\infty \in L^p(\mathfrak{m}_\infty)$, *provided* $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$ *and* $f_n\mathfrak{m}_n \rightharpoonup f_\infty \mathfrak{m}_\infty$ *in duality with* $C_{bs}(\mathrm{Z})$,

ii) $f_n \in L^p(\mathfrak{m}_n)$ converges L^p-strong *to* $f_\infty \in L^p(\mathfrak{m}_\infty)$, *provided it converges L^p-weak and* $\overline{\lim}_n \|f_n\|_{L^p(\mathfrak{m}_n)} \le \|f_\infty\|_{L^p(\mathfrak{m}_\infty)}$,

iii) $f_n \in W^{1,2}(\mathrm{X}_n)$ converges $W^{1,2}$-weak *to* $f_\infty \in W^{1,2}(\mathrm{X})$ *provided it converges L^2-weak and* $\sup_{n\in\mathbb{N}} \||Df_n|\|_{L^2(\mathfrak{m}_n)} < \infty$,

iv) $f_n \in W^{1,2}(\mathrm{X}_n)$ converges $W^{1,2}$-strong *to* $f_\infty \in W^{1,2}(\mathrm{X})$ *provided it converges L^2-strong and* $\||Df_n|\|_{L^2(\mathfrak{m}_n)} \to \||Df_\infty|\|_{L^2(\mathfrak{m}_\infty)}$.

Moreover, we say that f_n is uniformly bounded in L^p if $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$. In the following statement we collect a list of useful properties of L^p-convergence.

Proposition 2.4.3 (Properties of L^p-convergence)**.** *For all $p \in (1,\infty)$, it holds*

i) *If f_n converges L^p-strong to f_∞, then $\phi(f_n)$ converges L^p-strong to $\phi(f_\infty)$ for every $\phi \in$ $\mathrm{Lip}(\mathbb{R})$ with $\phi(0) = 0$,*

ii) *If f_n (resp. g_n) converges L^p-strong to f_∞ (resp. g_∞), then $f_n + g_n$ converges L^p-strong to $f_\infty + g_\infty$,*

iii) *if f_n converges L^p-weak to f, then $\|f_\infty\|_{L^p(\mathfrak{m}_\infty)} \le \underline{\lim}_n \|f_n\|_{L^p(\mathfrak{m}_n)}$,*

iv) *suppose that $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < +\infty$, then up to a subsequence f_n converges L^p-weak to some $f_\infty \in L^p(\mathfrak{m}_\infty)$,*

v) *If f_n converges L^p-strong (resp. L^p-weak) to f_∞, then ϕf_n converges L^p-strong (resp. L^p-weak) to ϕf_∞, for all $\phi \in C_b(\mathrm{Z})$,*

vi) *for every $f \in L^p(\mathfrak{m}_\infty)$ there exists a sequence $f_n \in L^p(\mathfrak{m}_n)$ converging L^p-strong to f,*

vii) *if f_n are non-negative and converge in L^p-strong to f, then for every $q \in (1,\infty)$, $f_n^{p/q}$ converge L^q-strong to $f^{p/q}$,*

viii) *Fix $p, q \in (1,\infty]$ so that $p < q$. If the sequence (f_n) is uniformly bounded in L^q and converges L^p-strong to f_∞, then it converges also L^r-strong to f_∞ for every $r \in [p,q)$,*

Proof. For the proof of the items i) up to v) we refer to [24, Prop. 3.3]. vi) can instead be found in [100] (see also [120]). vii) follows immediately from the characterization of L^p-strong convergence via convergence of graph (see e.g. [24, Remark 3.2]). For viii) , the case $q = \infty$ follows immediately from item i) (see also [24, e) of Prop. 3.3]), hence we can assume $q < +\infty$. Fix $r \in [p,q)$. Clearly from the Hölder inequality f_n is uniformly bounded in L^r, hence by definition

f_n converges L^r-weakly to f_∞. Moreover from item iii) we known that $f_\infty \in L^r(\mathfrak{m}_\infty)$, therefore by truncation and diagonalization we can suppose that $f \in L^\infty(\mathfrak{m}_\infty)$. From vi) then there exists a sequence $g_n \in L^r(\mathfrak{m}_n)$ converging to f_∞ in L^r-strong and by item i) we can also assume that g_n are uniformly bounded in L^∞. Then, from viii) in the case $q = \infty$ we have that g_n converge also in L^p-strong to f_∞. Then by ii) we have that $g_n - f_n$ converges to 0 in L^p-strong and in particular $\|f_n - g_n\|_{L^p(\mathfrak{m}_n)} \to 0$. Finally by the Hölder inequality (since f_n, g_n are both uniformly bounded in L^q) we have that $\|f_n - g_n\|_{L^r(\mathfrak{m}_n)} \to 0$. In particular $\lim_n \|f_n\|_{L^r(\mathfrak{m}_n)} = \lim_n \|g_n\|_{L^r(\mathfrak{m}_n)} = \|f_\infty\|_{L^r(\mathfrak{m}_\infty)}$, which concludes the proof. □

We now pass to some convergence and stability results related to Sobolev spaces. We start with the following generalized version of the compact embedding of $W^{1,2} \hookrightarrow L^2$ (reported here specifically for compact metric measure spaces, but works e.g. on sequences of congerging uniformly bounded $\mathsf{CD}(K,\infty)$ spaces):

Proposition 2.4.4 ([100]). *Suppose that* X_n, $n \in \bar{\mathbb{N}}$ *are compact and assume that* $(f_n) \in W^{1,2}(\mathrm{X}_n)$ *are uniformly bounded in* $W^{1,2}$, *i.e.* $\sup_n \|f_n\|_{W^{1,2}(\mathrm{X}_n)} < +\infty$. *Then* (f_n) *has a* L^2*-strongly convergent subsequence.*

Next, we recall the Γ-convergences result of the 2-Cheeger energies proven in [100]:

- Γ-$\underline{\lim}$: for every $f_n \in L^2(\mathfrak{m}_n)$ L^2-strong converging to $f_\infty \in L^2(\mathfrak{m}_\infty)$, it holds

$$\int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty \le \varliminf_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n; \tag{2.4.1}$$

- Γ-$\overline{\lim}$: for every $f_\infty \in L^2(\mathfrak{m}_\infty)$, there exists a sequence $f_n \in L^2(\mathfrak{m}_n)$ converging L^2-strong to f_∞ so that

$$\varlimsup_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty. \tag{2.4.2}$$

We will also need the Γ-$\overline{\lim}$ inequality also for the p-Cheeger energies as proved in [24, Theorem 8.1]: for every $p \in (1,\infty)$ and every $f_\infty \in L^p(\mathfrak{m}_\infty)$, there exists $f_n \in L^p(\mathfrak{m}_n)$ converging L^p-strong to f_∞ so that

$$\varlimsup_{n\to\infty} \int |Df_n|^p \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^p \,\mathrm{d}\mathfrak{m}_\infty.$$

The above is stated in [24] only for a sequence of $\mathsf{RCD}(K,\infty)$ spaces, but it easily seen that the proof works without modification also in the case of $\mathsf{CD}(K,\infty)$ spaces.

We end this part recalling a well known continuity result of the spectral gap (see [100] and [25]): if X_n, $n \in \bar{\mathbb{N}}$, are all compact it holds

$$\lambda^{1,2}(\mathrm{X}_\infty) = \lim_{n\to\infty} \lambda^{1,2}(\mathrm{X}_n). \tag{2.4.3}$$

We mention that the continuity of the spectral gap was previously obtained in the setting of Ricci-limit spaces by Cheeger and Colding [66].

2.5 The MCP-condition

Here, we report the definition of the *measure contraction property* introduced independently in [164] and [181]. The two definitions are slighty different on arbitrary metric measure space, however they coincides under the non branching assumption (actually, under the weaker 2-essentially non branching assumption, a technical property investigated in [175] that we shall never employ in this note).

Definition 2.5.1 (MCP spaces)**.** *We say that a metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *satisfies the* measure contraction property $\mathsf{MCP}(K, N)$ *for some* $K \in \mathbb{R}, N \in [1, \infty)$ *if for any* $o \in \mathrm{supp}(\mathfrak{m})$ *and* $\mu_0 \in \mathcal{P}_2(\mathrm{X})$ *with* $\mu := \mathfrak{m}(A)^{-1}\mathfrak{m}|_A$ *for some Borel set* $A \subset \mathrm{X}$ *(or* $A \subset B_{\pi\sqrt{(N-1)/K}}(o)$ *if* $K > 0$*) with* $\mathfrak{m}(A) \in (0, \infty)$*, there exists* $\pi \in \mathrm{OptGeo}_2(\mu, \delta_o)$ *so that*

$$(\mathsf{e}_t)_\sharp(\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0, o))^N \pi) \leq \frac{\mathfrak{m}}{\mathfrak{m}(A)}, \qquad \forall t \in [0, 1].$$

The reason why, in this note, we will couple the MCP-class with the *non branching* assumption is that they enjoys good properties of Wasserstein geodesics that in turns implies deep analytical consequences. A non branching $\mathsf{MCP}(K, N)$ metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is locally uniformly doubling and supports a weak local $(1, 1)$-Poincaré inequality [164, 187]. In particular, we know that it is also proper.

Remark 2.5.2. Notice that Definition 2.5.1 is *independent* on the particular choice $q = 2$ (and this explains also the reason why we did not define a MCP_q-condition).

The first observation is that, if $o \in \mathrm{X}$, the set $\mathrm{OptGeo}_q(\mu_0, \delta_o)$ is independent on q when $\mu_0 = \rho_0\mathfrak{m}$ with $\rho_0 \in L^\infty(\mathfrak{m})$ and of bounded support. Indeed, in this case

$$\mu_0 \in \mathscr{P}_q(\mathrm{X}) \qquad \text{and} \qquad W_q^q(\mu_0, \delta_o) = \int \mathsf{d}^q(x, o)\, \mathrm{d}\mu_0(x), \quad \forall q \in (1, \infty).$$

The verification being that the plan $\mu_0 \otimes \delta_0$ is the only admissible coupling between the two marginal, and therefore it must be optimal for any q. Then, it is straightforward to see that if $\pi \in \mathrm{OptGeo}_2(\mu_0, \delta_0)$, then $(\mathsf{e}_1)_\sharp\pi = \delta_o$ and therefore $\pi \in \mathrm{OptGeo}_q(\mu_0, \delta_0)$ for all $q \in (1, \infty)$. This discussion automatically shows that the choice of $q = 2$ in Definition 2.5.1 is irrelevant.

■

When $N < \infty, K \in \mathbb{R}$ it is also known that the

$$\text{A non branching } \mathsf{CD}_q(K, N) \text{ space satisfies the } \mathsf{MCP}(K, N) \text{ condition} \tag{2.5.1}$$

The proof of this fact is due to [56, Lemma 6.11] for $q = 2$, but it is remarked in [1] that the proof works for any $q \in (1, \infty)$.

2.5.1 Wasserstein interpolation with degenerating compression

We take from [57] (see also [128]) a key property that is similar in spirit (and in the proof) to Theorem 2.2.9.

Theorem 2.5.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a non branching* $\mathsf{MCP}(K, N)$*-space for some* $K \in \mathbb{R}, N \in [1, \infty)$*. Then, for every* $D > 0$ *and* $\mu_0, \mu_1 \in \mathscr{P}_2(\mathrm{X})$ *with* $\mu_0 = \rho_0\mathfrak{m}, \rho_0 \in L^\infty(\mathfrak{m})$*, and diam*$(\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1)) < D$*, there exists* $\pi \in \mathrm{OptGeo}_2(\mu_0, \mu_1)$ *with* $\mu_t := (\mathsf{e}_t)_\sharp\pi \ll \mathfrak{m}$ *and*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})} \leq \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \|\rho_0\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0, 1), \tag{2.5.2}$$

having set $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$ *for* $t < 1$*.*

We will provide a proof in Appendix A, where we will also extend this Theorem to all exponent $q \neq 2$.

Part II

Main contributions

3 Gradient flows on CAT-spaces: differential viewpoint and applications

Contents

3.1 Introduction

The geometry of the metric space and the convexity properties of the functional under consideration greatly affect the kind of results one can obtain for gradient flows. For the purpose of this Chapter, the works [153], [124] are particularly relevant: it is showed that the classical Crandall-Liggett generation theorem can be generalized to the metric setting of $\mathsf{CAT}(0)$ spaces to produce a satisfactory theory of gradient flows for semi-convex and lower semicontinuous functionals.

If the metric space one is working on admits some nicely-behaved tangent spaces/cones, one might hope to give a meaning to the classical defining formula

$$x_t' \in -\partial^-\mathsf{E}(x_t) \qquad a.e.\ t$$

or to its more precise variant

$$x_t'^+ = \text{ the element of minimal norm in } -\partial^-\mathsf{E}(x_t) \qquad \forall t > 0. \tag{3.1.1}$$

This has been done in [146], where previous approaches in [170] have been generalized. Here, notably, the basic assumptions on the metric space are of first order in nature (and refer precisely to the structure of tangent cones) and the energy functional is assumed to be semiconvex and locally Lipschitz. While the convexity assumption is very natural when studying gradient flows (all in all, even in the Hilbert setting many fundamental results rely on such hypothesis), asking for Lipschitz continuity is a bit less so: it certainly covers many concrete examples, for instance of functionals built upon distance functions on spaces satisfying some one-sided curvature bound, but from the analytic perspective it may be not satisfying: already the Dirichlet energy as a functional on L^2 is not Lipschitz, and the same holds for the Korevaar-Schoen energy we aim to study here.

Our motivation to study this topic comes from the desire of providing a notion of Laplacian for $\mathsf{CAT}(0)$-valued Sobolev maps defined on a RCD-space, where here 'Sobolev' is intended in the sense of Korevaar-Schoen [133] (recall the caracterization of Theorem 2.3.4 obtained in [109] and the associated definition of Korevaar-Schoen energy E^{KS}). Imitating one of the various equivalent definitions for the Laplacian in the classical smooth and linear setting, one is lead to define the Laplacian of u as the element of minimal norm in $-\partial^{-}\mathsf{E}^{\mathsf{KS}}(u)$. This approach of course carries at least two tasks: to define what $-\partial^{-}\mathsf{E}$ is and to show that it is not empty for a generic convex and lower semicontinuous functional E. Providing a reasonable definition for $-\partial^{-}\mathsf{E}$ is not that hard (see Definition 3.2.5), but is less obvious to show that this object is not-empty (in particular, minimizing $\mathsf{E}(\cdot)+\frac{\mathsf{d}^2(\cdot,x)}{2\tau}$ is of no help here, see the discussion in Remark 3.2.6). It is here that the theory of gradient flows comes to help:

> our main result is that, for semiconvex and lower semicontinuous functions on a $\mathsf{CAT}(\kappa)$ space, the analogue of (3.1.1) holds, see Theorem 3.2.9.

As a byproduct, we deduce that the domain of $-\partial^{-}\mathsf{E}$ is dense in the one of E. A result similar to ours has been obtained in [62] under some additional geometric assumptions on the base space, which in some sense tell that there is the opposite of any tangent vector.

As said, we then apply this result to study the Laplacian of $\mathsf{CAT}(0)$-valued Sobolev maps. Let us remark that in this case the relevant metric space $L^2(\Omega,\mathrm{Y}_{\bar{y}})$ is that of L^2 maps from some open subset Ω of a metric measure space X to a pointed $\mathsf{CAT}(0)$ space $(\mathrm{Y},\bar{y})$ and the energy functional is the Korevaar-Schoen energy E^{KS}: it is well known that $L^2(\Omega,\mathrm{Y}_{\bar{y}})$ is a $\mathsf{CAT}(0)$ space and that E^{KS} is convex and lower semicontinuous, but certainly not Lipschitz, whence the need to generalize Lytchak results to cover also this case.

Once we have a notion for $-\partial^{-}\mathsf{E}^{\mathsf{KS}}$ we enrich the analysis with:

i) the actual definition of Laplacian Δu of a $\mathsf{CAT}(0)$-valued map u (Definition 3.3.9), which pays particular attention to the link between the tangent cones in $L^2(\Omega,\mathrm{Y}_{\bar{y}})$, where $-\partial^{-}\mathsf{E}^{\mathsf{KS}}$ lives, and the tangent cones in Y, where we think 'variations' of u should live, see in particular Propositions 3.3.5 and 3.3.8,

ii) a basic, weak, integration by parts formula, see Proposition 3.3.10, which is sufficient to show that our approach is compatible with the classical one valid in the smooth category,

iii) a presentation of a simple and concrete example (Example 3.3.17) showing why Δu seems to be very much linked to the geometry of Y, but less so to Sobolev calculus on it.

Structure of the Chapter. This Chapter is organized as follows:

In Section 3.2, we recall the equivalent definitions of gradient flows on locally $\mathsf{CAT}(\kappa)$ spaces and list the main available structural properties. Then, we move to the study of subdifferential sets by exloiting a well established concrete first order calculus of CAT-spaces. Then, we move to the well-posedness of a subdifferential definition of gradient flows in this setting and prove the equivalence with previously available notions.

Finally, in Section 3.3, we consider applying the previous theory to define a notion of Laplacian of a Sobolev map with RCD domain and $\mathsf{CAT}(0)$ target. We start by building the right framework for the Laplacian defining the space of integrable section of the pullback geometric tangent bundle of a $\mathsf{CAT}(0)$ space via a map u. Then, we set naturally the sought Laplacian in the subdifferential of the energy functional and establish basic calculus rule and integration by-parts type of formula.

3.2 Gradient flows on $\mathsf{CAT}(\kappa)$-spaces

3.2.1 Metric approach and structural properties

We start facing various equivalent definitions of gradient flows on locally $\mathsf{CAT}(\kappa)$-spaces. The equivalence between the first two notions below is due to the convexity assumption, while the equivalence of these with the EVI is due to the geometric properties of $\mathsf{CAT}(\kappa)$-spaces, and in particular their Hilbert-like structure at small scales.

Theorem 3.2.1 (Gradient flows on locally $\mathsf{CAT}(\kappa)$-spaces: equivalent definitions)**.** *Let* Y *be a locally* $\mathsf{CAT}(\kappa)$*-space,* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *a* λ*-convex and lower semicontinuous functional,* $\lambda \in \mathbb{R}$*,* $y \in \mathrm{Y}$ *and* $(0, \infty) \ni t \mapsto y_t \in \mathrm{Y}$ *a locally absolutely continuous curve such that* $y_t \to y$ *as* $t \downarrow 0$*. Then, the following are equivalent:*

i) Energy Dissipation Inequality *We have*

$$-\partial_t \mathsf{E}(y_t) \geq \frac{1}{2}|\dot{y}_t|^2 + \frac{1}{2}|\partial^- \mathsf{E}|^2(y_t)$$

where the derivative in the left hand side is intended in the sense of distributions.

ii) Sharp dissipation rate $t \mapsto \mathsf{E}(y_t)$ *is locally absolutely continuous and*

$$\lim_{h \downarrow 0} \frac{\mathsf{E}(y_t) - \mathsf{E}(y_{t+h})}{h} = |\dot{y}_t^+|^2 = |\partial^- \mathsf{E}|^2(y_t) \qquad \textit{for every } t > 0, \tag{3.2.1}$$

where $|\dot{y}_t^+| := \lim_{h \downarrow 0} \frac{\mathsf{d}_{\mathrm{Y}}(y_{t+h}, y_t)}{h}$ *is the right metric speed, which in this case exists for every* $t > 0$*.*

iii) Evolution Variational Inequality (y_t) *is a EVI gradient flow (in the sense of Definition 1.1.9, i.e. for every* $z \in \mathrm{Y}$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathsf{d}_{\mathrm{Y}}^2(y_t, z)}{2} + \mathsf{E}(y_t) + \frac{\lambda}{2} \mathsf{d}_{\mathrm{Y}}^2(y_t, z) \leq \mathsf{E}(z) \qquad \textit{a.e. } t > 0. \tag{3.2.2}$$

Proof. The fact that ii) implies i) is obvious. The converse implication has been proved in [19] as a consequence of the so called *strong upper gradient property* of the slope. The implication iii) $\to$ ii) is proved in [161] (the argument in [161] has been also reported in [92]). The fact that on locally $\mathsf{CAT}(\kappa)$-spaces ii) implies iii) has also been proved in [161] (see in particular Theorems 4.2 and 3.14 there). More precisely, in [161] only the 'global' case of $\mathsf{CAT}(\kappa)$-spaces has been considered, but the arguments there can be quickly adapted to cover our case by noticing that:

- arguing as for the proof of (3.2.6) below, we see that (3.2.2) holds at some t if and only if it holds at t for z varying only in a neighbourhood of y_t,
- property (3.2.1) is local by nature,
- if $B \subset \mathrm{Y}$ is closed, convex and $\mathsf{CAT}(\kappa)$, then a curve $I \ni t \mapsto y_t \in B$ satisfies ii) (resp. iii)) in B if and only if it satisfies ii) (resp. iii)) in Y.

□

A curve satisfying any of the equivalent conditions in this last theorem will be called *gradient flow trajectory*. Moreover, we define the *gradient flow map* $\mathsf{GF}^{\mathsf{E}}\colon (0, \infty) \times \mathrm{Y} \to \mathrm{Y}$ via $\mathsf{GF}_t^{\mathsf{E}}(y) := y_t$ for every $t \in (0, \infty), y \in \mathrm{Y}$, where, evidently, y_t is the gradient flow trajectory starting at y and associated to the functional E evaluated at time t. Some of their main properties are collected in the following statement:

Theorem 3.2.2 (Gradient flows on locally $\mathsf{CAT}(\kappa)$-spaces: some basic properties)**.** *Let* Y *be a locally* $\mathsf{CAT}(\kappa)$*-space,* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *a* λ*-convex and lower semicontinuous functional. Then, the following holds:*

▷ Existence

For every $y \in \overline{D(\mathsf{E})}$ *there exists a gradient flow trajectory for* E *starting from* y*.*

▷ Uniqueness and λ-contraction

For any two gradient flow trajectories $(y_t), (z_t)$ starting from y, z respectively we have

$$\mathsf{d}_{\mathrm{Y}}(y_t, z_t) \le e^{-\lambda(t-s)} \mathsf{d}_{\mathrm{Y}}(y_s, z_s) \qquad \forall t \ge s > 0 \tag{3.2.3}$$

▷ MONOTONICITY PROPERTIES *For (y_t) gradient flow trajectory for E starting from y we have that*

$$t \mapsto y_t \text{ is locally Lipschitz in } (0, +\infty) \text{ with values in } D(|\partial^- \mathsf{E}|) \subset D(\mathsf{E}),$$

$$\begin{aligned} &t \mapsto \mathsf{E}(y_t) \text{ is nonincreasing in } [0, +\infty), \\ &t \mapsto e^{\lambda t} |\partial^- \mathsf{E}|(y_t) \text{ is nonincreasing in } [0, +\infty). \end{aligned} \tag{3.2.4}$$

Proof. In the $\mathsf{CAT}(0)$ case, existence of a limit of the so-called minimizing movements scheme in this setting has been proved in [153] and [124]. The fact that the limit curve obtained in this way satisfies the EVI condition has been proved in [19]. The contractivity property, also at the level of the discrete scheme, has been proved in [153] and [124] (at least in the case $\lambda = 0$, the general case can be found e.g. in [19] as a consequence of the EVI condition). Then, uniqueness is directly implied by (3.2.3) and the last claims are a consequence of (3.2.1) and the contraction property.

The $\mathsf{CAT}(\kappa)$ case has been treated in [166], at least under some compactness assumptions on the sublevels of the functional. Such compactness assumption has been removed in [161]. Finally, the case of locally $\mathsf{CAT}(\kappa)$ spaces can be dealt with as in the proof of Theorem 3.2.1 above. □

Finally, we conclude the section with an *a priori* estimate, a variant of the ones investigated in [161], concerning contraction properties along the gradient flow trajectories at different times. The proof is inspired by the one of [171, Lemma 2.1.4] in the context of CBB-spaces.

Lemma 3.2.3 (A priori estimates)**.** *Let Y be locally $\mathsf{CAT}(\kappa)$ and $\mathsf{E}\colon \mathrm{Y} \to [0, \infty]$ be a λ-convex and lower semicontinuous functional, $\lambda \in \mathbb{R}$. Let $y, z \in \mathrm{Y}$ and consider the gradient flow trajectories $(y_t), (z_t)$ associated with E.*

Then, for any $t \ge s > 0$, it holds

$$\begin{aligned} \mathsf{d}^2_{\mathrm{Y}}(y_t, z_s) \le e^{-2\lambda s}\Big(&\mathsf{d}^2_{\mathrm{Y}}(y, z) + 2(t-s)(\mathsf{E}(z) - \mathsf{E}(y)) \\ &+ 2|\partial^- \mathsf{E}|^2(y) \int_0^{t-s} \theta_\lambda(r)\,\mathrm{d}r - \lambda \int_0^{t-s} \mathsf{d}^2_{\mathrm{Y}}(y_r, z)\,\mathrm{d}r\Big), \end{aligned} \tag{3.2.5}$$

where $\theta_\lambda(t) := \int_0^t e^{-2\lambda r}\,\mathrm{d}r$.

Proof. We start fixing $t > 0$. First, we notice that, in light of ii) of Theorem 3.2.1 and the basic properties in Theorem 3.2.2, we have for any $r > 0$ (and not a.e. r),

$$-\mathsf{E}(y_r) + \mathsf{E}(y) = \int_0^r e^{-2\lambda q} e^{+2\lambda q} |\partial^- \mathsf{E}|^2(y_q)\,\mathrm{d}q \le |\partial^- \mathsf{E}|^2(y)\theta_\lambda(r).$$

Thus, we can integrate from 0 to t the EVI condition (3.2.2) to get

$$\begin{aligned} \frac{1}{2}(\mathsf{d}^2_{\mathrm{Y}}(y_t, z) - \mathsf{d}^2_{\mathrm{Y}}(y, z)) &\le \int_0^t \mathsf{E}(z) - \mathsf{E}(y_r) - \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y_r, z)\,\mathrm{d}r \\ &\le t(\mathsf{E}(z) - \mathsf{E}(y)) + |\partial^- \mathsf{E}|^2(y) \int_0^t \theta_\lambda(r)\,\mathrm{d}r - \frac{\lambda}{2}\int_0^t \mathsf{d}^2_{\mathrm{Y}}(y_r, z)\,\mathrm{d}r. \end{aligned}$$

Finally, for general $t \ge s > 0$, we can reduce to above case by appealing to property (3.2.3). □

3.2.2 The object $-\partial^-\mathsf{E}(y)$

In this section we introduce the key object $-\partial^-\mathsf{E}(y)$ of this Chapter associated to a semiconvex and lower semicontinuous functional E over a local $\mathsf{CAT}(\kappa)$ space. As the notation suggests, and as will be clear from Definition 3.2.5, for functionals on Hilbert spaces this corresponds to $\{-v : v \in \partial^-\mathsf{E}(y)\}$.

We start recalling the following well known fact:

Proposition 3.2.4 (Metric projection)**.** *Let* Y *be a* $\mathsf{CAT}(0)$*-space and* $C \subset \mathrm{Y}$ *be a closed convex subset. Then, for every* $y \in \mathrm{Y}$*, there is a unique* $\mathrm{Pr}_C(y) \in C$*, called* metric projection *of* y *onto* C*, such that* $\mathsf{d}_{\mathrm{Y}}(y, \mathrm{Pr}_C(y)) = \inf_C \mathsf{d}_{\mathrm{Y}}(y, \cdot)$*.*

Proof. Since the function to be minimized is continuous and C closed, it is sufficient to prove that any minimizing sequence (c_n) for $I := \inf_{c\in C} \mathsf{d}^2_{\mathrm{Y}}(c, y)$ (which is equivalent to be minimizing for $\inf_C \mathsf{d}_{\mathrm{Y}}(y, \cdot)$) is Cauchy. Fix such sequence and, for every $n, m \in \mathbb{N}$, let $c_{m,n}$ be the mid-point between c_n and c_m. Observe that since C is convex, $c_{n,m}$ belongs to C and thus is a competitor for the minimization problem. Condition (2.1.2) therefore implies

$$I \leq \mathsf{d}^2_{\mathrm{Y}}(c_{n,m}, y) \leq \frac{1}{2}\mathsf{d}^2_{\mathrm{Y}}(c_n, y) + \frac{1}{2}\mathsf{d}^2_{\mathrm{Y}}(c_m, y) - \frac{1}{4}\mathsf{d}^2_{\mathrm{Y}}(c_n, c_m),$$

for every $n, m \in \mathbb{N}$. Rearranging terms, and taking the limsup as n, m go to infinity we observe

$$\varlimsup_{n,m\to+\infty} \frac{1}{4}\mathsf{d}^2_{\mathrm{Y}}(c_n, c_m) \leq \varlimsup_{n,m\to+\infty} \frac{1}{2}\mathsf{d}^2_{\mathrm{Y}}(c_n, y) + \frac{1}{2}\mathsf{d}^2_{\mathrm{Y}}(c_m, y) - I = 0,$$

i.e. (c_n) is Cauchy, as desired. □

We remark that the metric projection can be also shown to be 1-Lipschitz and to satisfy a 'Pythagoras' inequality' (see [39, Theorem 2.1.12]), but we will not make use of this fact. Finally, we are ready to give an effective definition of (opposite of the) subdifferential of E as a subset of the tangent cone.

Definition 3.2.5 (Minus-subdifferential)**.** *Let* Y *be locally* $\mathsf{CAT}(\kappa)$*,* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *be a* λ*-convex and lower semicontinuous functional,* $\lambda \in \mathbb{R}$*, and* $y \in D(\mathsf{E})$*. We define the* minus-subdifferential *of* E *at* y*, denoted by* $-\partial^-\mathsf{E}(y)$*, as the collection of* $v \in \mathrm{T}_y\mathrm{Y}$ *satisfying the subdifferential inequality*

$$\mathsf{E}(y) - \left\langle v, \gamma'_0 \right\rangle_y + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y, z) \leq \mathsf{E}(z),$$

for every $z \in \mathrm{Y}$*, and some geodesic* γ *from* y *to* z*. Moreover, by* $D(-\partial^-\mathsf{E})$*, we denote the collection of* $y \in \mathrm{Y}$ *for which* $-\partial^-\mathsf{E}(y) \neq \emptyset$*.*

Notice that $v \in -\partial^-\mathsf{E}(y)$ if and only if

$$-\left\langle v, \gamma'_0 \right\rangle_y \leq \lim_{t\downarrow 0} \frac{\mathsf{E}(\gamma_t) - \mathsf{E}(y)}{t} \qquad \forall z \in \mathrm{Y},\ \text{for some geodesic } \gamma \text{ from } y \text{ to } z. \tag{3.2.6}$$

so that in particular the definition of $-\partial^-\mathsf{E}(y)$ does not depend on λ. Indeed the 'if' is obvious by λ-convexity while for the 'only if' we apply the defining inequality with $z_t := \gamma_t$ in place of z and, for t small enough, rearrange to get

$$-\big\langle v, (\mathsf{G}^{z_t}_y)'_0 \big\rangle_y + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y, z_t) \leq \mathsf{E}(z_t) - \mathsf{E}(y)$$

so that the conclusion follows noticing that $\mathsf{d}^2_{\mathrm{Y}}(y, z_t) = t^2\mathsf{d}^2_{\mathrm{Y}}(y, z)$, $(\mathsf{G}^{z_t}_y)'_0 = t\gamma'_0$ (because for $t \ll 1$ the geodesic from y to z_t in unique), then dividing by t and letting $t \downarrow 0$. The same arguments also show that both in Definition 3.2.5 and in (3.2.6) we can take γ to be *any* geodesic from y to z.

It is also worth to point out that

$$\begin{gathered}\text{For } \mathsf{E} \text{ convex and lower semicontinuous we have that:}\\ x \text{ is a minimum point for } \mathsf{E} \text{ if and only if } 0 \in -\partial^- \mathsf{E}(x).\end{gathered} \tag{3.2.7}$$

The proof of this fact being obvious.

Remark 3.2.6. It would certainly be possible to define the analogous notion of subdifferential $\partial^- \mathsf{E}$ by replacing $-\langle v, \gamma'_0 \rangle_y$ with $\langle v, \gamma'_0 \rangle_y$ in the defining formula, however, since the tangent cone is only a cone and not a space, there is no obvious relation between the two definitions.

For our purposes, $-\partial^- \mathsf{E}$ is the correct object to work with because, as discussed in the introduction, we aim at showing the existence of the Laplacian of a $\mathsf{CAT}(0)$-valued Sobolev map by looking at the gradient flow of the Korevaar-Schoen energy E^{KS}. Thus, we notice on one hand that, by definition and imitating what happens in the smooth category, the Laplacian of u has to be introduced as (the element of minimal norm in) $-\partial^- \mathsf{E}^{\mathsf{KS}}(u)$, and on the other hand that in the gradient flow equation (3.1.1) it is $-\partial^- \mathsf{E}$ who appears.

In this direction, it is interesting to point out that the classical procedure of minimizing

$$y \quad \mapsto \quad \mathsf{E}(y) + \frac{\mathsf{d}^2_{\mathrm{Y}}(y, \bar{y})}{2\tau},$$

which is the cornerstone of most existence results about gradient flows in the metric setting (see e.g. [19]), produces a (unique, if $\tau > 0$ is small enough) point y_τ for which we have $\frac{1}{\tau}(\mathsf{G}^{\bar{y}}_{y_\tau})'_0 \in \partial^- \mathsf{E}(y_\tau)$. In particular it gives no informations about whether $-\partial^- \mathsf{E}(y_\tau)$ is not empty. In our approach this latter fact, and the related one that the slope at y coincides with the norm of the element of least norm in $-\partial^- \mathsf{E}(y)$, will be a consequence of the fact that gradient flow trajectories satisfy an analogue of (3.1.1), see Theorem 3.2.9. ■

It will be important to know that in $-\partial^- \mathsf{E}(y)$ there is always an element of minimal norm:

Proposition 3.2.7. *Let* Y *be a locally* $\mathsf{CAT}(\kappa)$*-space,* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *be a* λ*-convex and lower semicontinuous functional,* $\lambda \in \mathbb{R}$*, and* $y \in \mathrm{Y}$*. Then,* $-\partial^- \mathsf{E}(y)$ *is a closed and convex subset of* $\mathrm{T}_y\mathrm{Y}$*. In particular, if this set is not empty, the optimization problem*

$$\inf_{v \in -\partial^- \mathsf{E}(y)} |v|_y$$

admits a unique minimiser.

Proof. Recalling that $\mathrm{T}_y\mathrm{Y}$ is $\mathsf{CAT}(0)$, by Proposition 3.2.4 the existence of a unique minimizer in $-\partial^- \mathsf{E}(y)$ for the norm, i.e. of a unique metric projection of 0_y onto $-\partial^- \mathsf{E}(y)$, will follow once we show that $-\partial^- \mathsf{E}(y)$ is closed and convex.

The fact that it is closed follows from the definition and the consideration already stated in Proposition 2.1.5 that the scalar product $\langle \cdot, \cdot \rangle_y$ is continuous on $\mathrm{T}_y\mathrm{Y}$. The convexity follows from the inequality

$$-\langle (\mathsf{G}^{v_2}_{v_1})_t, w \rangle_y \leq -(1-t)\langle v_1, w \rangle_y - t\langle v_2, w \rangle_y \qquad \forall v_1, v_2, w \in \mathrm{T}_y\mathrm{Y},\ t \in [0,1],$$

which is a direct consequence of (2.1.5b) and (2.1.5f). □

3.2.3 Subdifferential approach and equivalence

Here we prove the main results of this note, namely Theorem 3.2.9 and Corollary 3.2.10 below. We shall use the following preliminary result (notice that the fact that equality holds in (3.2.8) will be obtained in (3.2.9)):

Proposition 3.2.8. *Let* Y *be locally* $\mathsf{CAT}(\kappa)$ *and* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *be a* λ*-convex and lower semicontinuous functional,* $\lambda \in \mathbb{R}$*. Then, for every* $y \in D(-\partial^- \mathsf{E})$*, we have*

$$|\partial^- \mathsf{E}|(y) \le \inf_{v \in -\partial E(y)} |v|_y. \tag{3.2.8}$$

In particular, $D(-\partial^- \mathsf{E}) \subset D(|\partial^- \mathsf{E}|)$*.*

Proof. Let $v \in -\partial^- \mathsf{E}(y)$ and notice that

$$\mathsf{E}(y) - \mathsf{E}(z) + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y, z) \le |\langle v, (\mathsf{G}^z_y)'_0 \rangle_y| \overset{(2.1.5c)}{\le} |v|_y \mathsf{d}_{\mathrm{Y}}(y, z), \qquad \forall z \in \mathrm{Y}$$

which in turns implies

$$\left(\frac{\mathsf{E}(y) - \mathsf{E}(z)}{\mathsf{d}_{\mathrm{Y}}(y, z)} + \frac{\lambda}{2}\mathsf{d}_{\mathrm{Y}}(y, z) \right)^+ \le |v|_y \qquad \forall z \in \mathrm{Y},\ z \ne y.$$

Taking the supremum over $z \ne y$ and recalling Lemma 1.1.8 we conclude. $\square$

We now come to the main result of this Chapter, namely the existence of right incremental ratios of the flow for all time.

Theorem 3.2.9 (Right derivatives of the flow)**.** *Let* Y *be locally* $\mathsf{CAT}(\kappa)$ *and* $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ *be a* λ*-convex and lower semicontinuous functional,* $\lambda \in \mathbb{R}$*. Let* $y \in \overline{D(\mathsf{E})}$*, and* (y_t) *be the gradient flow trajectory starting from* y *(recall Theorem 3.2.2).*

Then, for every $t > 0$*, the right 'difference quotient'* $\frac{1}{h}(\mathsf{G}^{y_{t+h}}_{y_t})'_0$ *strongly converges to the element of minimal norm in* $-\partial^- \mathsf{E}(y_t) \subset \mathrm{T}_{y_t}\mathrm{Y}$ *(i.e. to* $\mathrm{Pr}_{-\partial^- \mathsf{E}(y_t)}(0_{y_t})$*) as* h *goes to* 0^+*. The same holds for* $t = 0$ *if (and only if) we have* $y \in D(|\partial^- \mathsf{E}|)$*.*

Moreover, $D(-\partial^- \mathsf{E}) = D(|\partial^- \mathsf{E}|)$ *and the identity*

$$|\partial^- \mathsf{E}|(y) = \min_{v \in -\partial^- \mathsf{E}(y)} |v|_y \qquad \forall y \in \mathrm{Y} \tag{3.2.9}$$

holds, where as customary the minimum of the empty set is declared to be $+\infty$*. In particular,* $D(-\partial^- \mathsf{E})$ *is dense in* $D(\mathsf{E})$*.*

Proof. By the semigroup property ensured by the uniqueness of gradient flow trajectories and taking into account that $y_t \in D(|\partial^- E|)$ for every $t > 0$ (recall (3.2.2)), it suffices to show the claim for $t = 0$ under the assumption $y \in D(|\partial^- \mathsf{E}|)$. Suppose y is not a minimum point for E, otherwise there is nothing to prove. In particular, ii) of Theorem 3.2.1 ensures that $|\dot{y}_0|$ exists and it is positive. Also, notice that the continuity at time $t = 0$ of the gradient flow trajectory ensures that for $\epsilon > 0$ sufficiently small we have $y_h \in B_{\mathsf{r}_y}(y)$ for every $h \in (0, \epsilon)$. In particular for such h the tangent vector $v_h := \frac{1}{h}(\mathsf{G}^{y_h}_y)'_0 \in \mathrm{T}_y\mathrm{Y}$ is well defined and the statement makes sense. Fix such $\epsilon > 0$.

Step 1. For every $h \in (0, \epsilon)$ we have

$$|v_h|_y = \frac{\mathsf{d}_{\mathrm{Y}}(y_h, y)}{h} \le \fint_0^h |\dot{y}_t| \,\mathrm{d}t \overset{(3.2.1)}{=} \fint_0^h |\partial^- \mathsf{E}|(y_t) \,\mathrm{d}t \overset{(3.2.4)}{\le} |\partial^- \mathsf{E}|(y) \fint_0^h e^{-\lambda t} \,\mathrm{d}t. \tag{3.2.10}$$

Hence $\sup_{h \in (0, \epsilon)} |v_h|_y < \infty$, therefore point (iii) of Proposition 2.1.9 gives that for every sequence $h_n \downarrow 0$ there is a subsequence, not relabelled, such that $v_{h_n} \rightharpoonup v$ for some $v \in \mathrm{T}_y\mathrm{Y}$.

Fix such sequence and such weak limit v. To conclude it is sufficient to prove that the convergence is strong and that v is the element of minimal norm in $-\partial^- \mathsf{E}(y)$, as this in particular grants that the limit is independent on the particular subsequence chosen.

Step 2. We claim that $v \in -\partial^- \mathsf{E}(y)$. To see this, integrate (3.2.2) from 0 to h and divide by h to obtain

$$\frac{\mathsf{d}^2_{\mathrm{Y}}(y_h, z) - \mathsf{d}^2_{\mathrm{Y}}(y, z)}{2h} + \fint_0^h \mathsf{E}(y_t) + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y_t, z) \,\mathrm{d}t \le \mathsf{E}(z) \qquad \forall z \in \mathrm{Y},\ h \in (0, \epsilon).$$

Letting $h = h_n \downarrow 0$ and recalling that E is lower semicontinuous we deduce that

$$\varliminf_{n\to\infty} \frac{\mathsf{d}^2_{\mathrm{Y}}(y_{h_n}, z) - \mathsf{d}^2_{\mathrm{Y}}(y, z)}{2h_n} + \mathsf{E}(y) + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(y, z) \leq \mathsf{E}(z) \qquad \forall z \in \mathrm{Y}. \tag{3.2.11}$$

Next, fix $z \in \mathrm{Y}$, let $\gamma \in \mathsf{Geo}_y\mathrm{Y}$ with $\gamma_1 = z$, denote $z_s := \gamma_s$ and notice that, for s sufficiently small, $z_s, y_{h_n} \in B_{\mathsf{r}_y}(y)$. Now (2.1.7) yields

$$\begin{aligned}\varliminf_{n\to\infty} -\langle v_{h_n}, (\mathsf{G}_y^{z_s})'_0\rangle_y &\leq \mathsf{d}_{\mathrm{Y}}(y, z) \varliminf_{n\to\infty} \frac{\mathsf{d}(y_{h_n}, y)}{h_n} \frac{\mathsf{d}_{\mathrm{Y}}(y_{h_n}, z_s) - \mathsf{d}_{\mathrm{Y}}(y, z_s)}{|\dot{y}_0| h_n} \\ &= \varliminf_{n\to\infty} \frac{\mathsf{d}^2_{\mathrm{Y}}(y_{h_n}, z_s) - \mathsf{d}^2_{\mathrm{Y}}(y, z_s)}{2h_n},\end{aligned}$$

having used the fact that $\varliminf_n a_n b_n = a \varliminf_n b_n$ if $\lim_n a_n = a > 0$ and $(a_n), (b_n) \subset \mathbb{R}$ are bounded, and a chain rule argument in the last equality. Thus, recalling the weak upper semicontinuity of the scalar product proved in point (vi) of Proposition 2.1.9 we get

$$\varliminf_{n\to\infty} \frac{\mathsf{d}^2_{\mathrm{Y}}(y_{h_n}, z_s) - \mathsf{d}^2_{\mathrm{Y}}(y, z_s)}{2h_n} \geq -\langle v, (\mathsf{G}_y^{z_s})'_0\rangle_y.$$

Now, combine with (3.2.11) to get

$$\mathsf{E}(y) - \langle v, (\mathsf{G}_y^{z_s})'_0\rangle_y + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(z_s, y) \leq \mathsf{E}(z_s) \leq (1-s)\mathsf{E}(y) + s\mathsf{E}(z) - \frac{\lambda}{2}s(1-s)\mathsf{d}^2_{\mathrm{Y}}(z, y).$$

Finally, using that $(\mathsf{G}_y^{z_s})'_0 = s\gamma'_0$, $\mathsf{d}^2_{\mathrm{Y}}(z_s, y) = s^2\mathsf{d}^2_{\mathrm{Y}}(y, z)$ and (2.1.5b), we can rearrange terms and take the limit as $s \downarrow 0$ to get

$$\mathsf{E}(y) - \langle v, \gamma'_0\rangle_y + \frac{\lambda}{2}\mathsf{d}^2_{\mathrm{Y}}(z, y) \leq \mathsf{E}(z) \qquad \text{for every } \gamma \text{ geodesic from } y \text{ to } z.$$

Given that z was arbitrary, we conclude.

STEP 3. Since $|\cdot|_y^2 : \mathrm{T}_y\mathrm{Y} \to \mathbb{R}$ is convex and continuous, by point (v) of Proposition 2.1.9 we get

$$|v|_y^2 \leq \varliminf_{n\to\infty} |v_{h_n}|_y^2 \leq \varlimsup_{n\to\infty} |v_{h_n}|_y^2 \overset{(3.2.10)}{\leq} |\partial^- E|^2(y) \overset{(3.2.8)}{\leq} \inf_{w\in -\partial^-\mathsf{E}(y)} |w|_y^2 \leq |v|_y^2,$$

and thus all the inequalities must be equalities. This proves at once the strong convergence of (v_{h_n}) to v (by the convergence of norms and point ii) of Proposition 2.1.9) and that v is the element of minimal norm in $-\partial^-\mathsf{E}(y)$.

The argument also proves that if $y \in D(|\partial^-\mathsf{E}|)$, then $y \in D(-\partial^-\mathsf{E})$ and in this case the equality in (3.2.9) holds. Taking into account Proposition 3.2.8 we conclude that $D(|\partial^-\mathsf{E}|) = D(-\partial^-\mathsf{E})$ that (3.2.9) holds for every $y \in \mathrm{Y}$, as desired.

The last claim then follows from the existence of gradient flow trajectories starting from points in $D(\mathsf{E})$ (Theorem 3.2.2) and (3.2.1). □

As a direct consequence of the above result, we see that we can characterize gradient flow trajectories by means of the classical differential inclusion $x'_t \in -\partial^-\mathsf{E}(x_t)$ which can be used to define such evolution in the Hilbert setting:

Corollary 3.2.10. *Let Y be locally $\mathsf{CAT}(\kappa)$ and $\mathsf{E}\colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ be a λ-convex and lower semicontinuous functional, $\lambda \in \mathbb{R}$. Let $y \in \overline{D(\mathsf{E})}$, and $(0, \infty) \ni t \mapsto y_t \in D(\mathsf{E})$ be a locally absolutely continuous curve. Then, the following are equivalent:*

i) *(y_t) is a gradient flow trajectory for E starting from y, i.e. satisfies any of the three equivalent conditions in Theorem 3.2.1.*

ii) *The right derivative* $y_t'^+$ *exists for every* $t > 0$ *and*

$$\begin{cases} y_t'^+ \in -\partial^-\mathsf{E}(y_t) & \forall t > 0 \quad \textit{and is the element of minimal norm,} \\ \lim_{t\downarrow 0} y_t = y. \end{cases}$$

If $y \in D(|\partial^-\mathsf{E}|) = D(-\partial^-\mathsf{E})$ *then the above holds also at* $t = 0$.

iii) *It holds*

$$\begin{cases} y_t'^+ \in -\partial^-\mathsf{E}(y_t) & \textit{a.e. } t > 0, \\ \lim_{t\downarrow 0} y_t = y. \end{cases}$$

Proof. The implication i) $\Rightarrow$ ii) is proved in Theorem 3.2.9 above and the one ii)) $\Rightarrow$ iii) is obvious. The fact that iii) implies i) (in the form of the Evolution Variation Inequality) is a direct consequence of Proposition 2.1.7 (applied in a $\mathsf{CAT}(\kappa)$ neighbourhood of y_t in combination with arguments similar to those outlined in the proof of Theorem 3.2.1 to cover the case of a local $\mathsf{CAT}(\kappa)$ space) and the definition of $-\partial^-\mathsf{E}$. □

Remark 3.2.11. In the setting of Alexandrov geometry it is more customary to study the gradient flow of semi*concave* functions F, thus studying (a properly interpreted version of) $y_t' \in \partial^+\mathsf{F}$. Let E be semiconvex on a $\mathsf{CAT}(\kappa)$-space Y and put $\mathsf{F} := -\mathsf{E}$. Then it is clear that the slope $|\partial^-\mathsf{E}|$ as we defined it coincides with the *absolute gradient* $|\nabla\mathsf{F}|$ as defined in [146, Definition 4.1], therefore, taking into account the characterization (3.2.1), we see that up to a different choice of parametrization, our notion of gradient flow trajectory coincides with the one of gradient-like curve studied in [146, Definition 6.1].

The property $\frac{\mathrm{d}}{\mathrm{d}t^+}\mathsf{F}(y_t) = -\frac{\mathrm{d}}{\mathrm{d}t^+}\mathsf{E}(y_t) = |\partial^-\mathsf{E}|^2(y_t) = |v_t|^2_{y_t}$, where $v_t \in -\partial^-\mathsf{E}(y_t)$ is the element of minimal norm, together with the existence of the right derivative of y_t and the characterization (3.2.6) show that the element of minimal norm in $-\partial^-\mathsf{E}(y)$ coincides with $\nabla\mathsf{F}(y)$ as defined in [6, Definition 11.4.1] on spaces with curvature bounded from *below*. This shows that our 'differential' perspective on gradient flows is compatible with the one studied in [6] on CBB spaces. ■

3.3 Application: Laplacian of $\mathsf{CAT}(0)$-valued maps

3.3.1 Pullback geometric tangent bundle

In this part, we perform a pullback metric construction related to concept of Borel section of the Geometric tangent bundle $\mathrm{T}_G\mathrm{Y}$ over a locally $\mathsf{CAT}(\kappa)$ space Y as defined in Section 2.1.3. This can be seen as the pullback related to space $L^2(\mathrm{T}_G\mathrm{Y};\mu)$ of $L^2(\mu)$ Borel section of $\mathrm{T}_G\mathrm{Y}$ as defined in [79], when μ is an arbitrary positive Borel and boundedly finite measure on Y. Nevertheless, the author dealt only with separable $\mathsf{CAT}(\kappa)$ spaces, hence our work refinds and extends these results simply when considering the trivial operation of pullback via the identity map.

We distinguish two situations: Y separable or not.

The general non-separable case

Let us fix a pointed $\mathsf{CAT}(0)$ space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \bar y)$, a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and an open subset $\Omega \subset \mathrm{X}$. For $u, v \in L^2(\Omega, \mathrm{Y}_{\bar y})$ a direct computation shows that $t \mapsto (\mathsf{G}_u^v)_t \in L^2(\Omega, \mathrm{Y}_{\bar y})$, where $(\mathsf{G}_u^v)_t(x) := (\mathsf{G}_{u(x)}^{v(x)})_t$, is a geodesic from u to v (the fact that $(\mathsf{G}_u^v)_t \colon \Omega \to \mathrm{Y}$ is Borel follows from the continuous dependence of the - unique - geodesics on Y w.r.t. the endpoints). Also, by appealing to the equivalent characterization (2.1.2) of $\mathsf{CAT}(0)$-spaces, the computation

$$\begin{aligned} \mathsf{d}^2_{L^2}((\mathsf{G}_u^v)_t, w) &= \int \mathsf{d}^2_\mathrm{Y}((\mathsf{G}_{u(x)}^{v(x)})_t, w(x))\,\mathrm{d}\mathfrak{m}(x) \\ &\overset{(2.1.2)}{\leq} \int (1-t)\mathsf{d}^2_\mathrm{Y}(u(x), w(x)) + t\mathsf{d}^2_\mathrm{Y}(v(x), w(x)) - t(1-t)\mathsf{d}^2_\mathrm{Y}(u(x), v(x))\,\mathrm{d}\mathfrak{m}(x) \\ &= (1-t)\mathsf{d}^2_{L^2}(u, w) + t\mathsf{d}^2_{L^2}(v, w) - t(1-t)\mathsf{d}^2_{L^2}(u, v) \end{aligned}$$

valid for any $w \in L^2(\Omega, \mathrm{Y}_{\bar{y}})$ and every $t \in [0,1]$, reveals that $L^2(\Omega, \mathrm{Y}_{\bar{y}})$ is a $\mathsf{CAT}(0)$-space as well and thus G_u^v is the only geodesic from u to v.

In particular, given $u \in L^2(\Omega, \mathrm{Y}_{\bar{y}})$ we have a well defined tangent cone $\mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar{y}})$ containing what we may think of as the set of 'infinitesimal variations' of u. Intuitively, these variations should correspond to a collection, for $\mathfrak{m}$-a.e. $x \in \Omega$, of a variation of $u(x) \in \mathrm{Y}$, i.e. to a collection of elements of $\mathrm{T}_{u(x)}\mathrm{Y}$.

We now want to make this intuition rigorous and, due to the fact that $\mathsf{CAT}(0)$-spaces are typically studied in non separable environments, we first discuss this case, postponing to the next sections the separable case and its relations with the Borel structure on $\mathrm{T}_G\mathrm{Y}$ seen in Section 2.1.3. Fix $u \in L^2(\Omega, \mathrm{Y}_{\bar{y}})$ and a Borel representative of it, which by abuse of notation we shall continue to denote by u. By $u^*\mathrm{T}_G\mathrm{Y}$ we intend the set

$$u^*\mathrm{T}_G\mathrm{Y} := \big\{(x, y, v) \colon x \in \Omega,\ y = u(x),\ v \in \mathrm{T}_y\mathrm{Y}\big\} \subset \mathrm{X} \times \mathrm{T}_G\mathrm{Y}$$

A section of $u^*\mathrm{T}_G\mathrm{Y}$ is a map $\mathsf{S} \colon \Omega \to u^*\mathrm{T}_G\mathrm{Y}$ such that $\pi_{\mathrm{X}}(\mathsf{S}(x)) = x$, where $\pi_{\mathrm{X}} \colon u^*\mathrm{T}_G\mathrm{Y} \to \mathrm{X}$ is

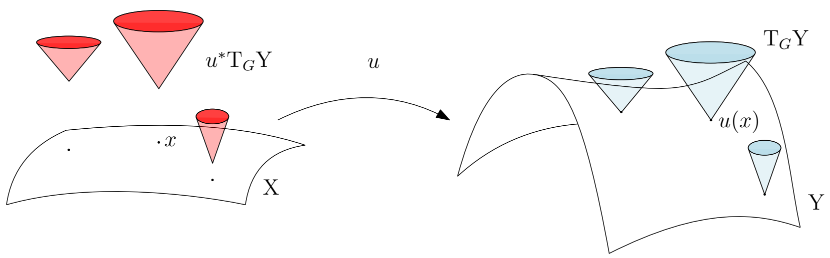

Figure 3.1: Pullback geometric tangent bundle $u^*\mathrm{T}_G\mathrm{Y}$ via $u \colon \mathrm{X} \to \mathrm{Y}$.

the canonical projection. Given such a section S we write $\mathsf{S}(x) = (x, u(x), \mathsf{S}_x)$ for any $x \in \Omega$. We shall denote by $\mathsf{0}$ the zero section defined by $\mathsf{0}_x := 0_{u(x)} \in \mathrm{T}_{u(x)}\mathrm{Y}$.

Then given another $v \in L^2(\Omega, \mathrm{Y}_{\bar{y}})$ and a Borel representative of it, still denoted by v, and $\alpha \geq 0$, we can consider the section S of $u^*\mathrm{T}_G\mathrm{Y}$ given by $x \mapsto (x, u(x), \alpha(\mathsf{G}_{u(x)}^{v(x)})_0')$. We then have the following simple and useful lemma.

Lemma 3.3.1. *Let $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \bar{y})$ be a pointed $\mathsf{CAT}(0)$-space, $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ a metric measure space, $\Omega \subset \mathrm{X}$ an open subset, $u, v^1, v^2 \colon \Omega \to \mathrm{Y}$ Borel representatives of maps in $L^2(\Omega, \mathrm{Y}_{\bar{y}})$. Also, let $\alpha^1, \alpha^2 \in \mathbb{R}^+$ and consider the sections S^i of $u^*\mathrm{T}_G\mathrm{Y}$ given by $\mathsf{S}_x^i := \alpha_i(\mathsf{G}_{u(x)}^{v_i(x)})_0'$, $i = 1, 2$.*

Then the maps $\Omega \ni x \mapsto |\mathsf{S}_x^1|_{u(x)}, \mathsf{d}_{u(x)}(\mathsf{S}_x^1, \mathsf{S}_x^2), \langle \mathsf{S}_x^1, \mathsf{S}_x^2 \rangle_{u(x)}$ are Borel.

Proof. It is sufficient to prove that $\Omega \ni x \mapsto \mathsf{d}_{u(x)}(\mathsf{S}_x^1, \mathsf{S}_x^2) \in \mathbb{R}$ is Borel, as then the other Borel regularities will follow. We have already noticed that the maps $x \mapsto (\mathsf{G}_u^{v^i})_{\alpha^i t}(x) \in \mathrm{Y}$, $i = 1, 2$, are Borel, hence so is the map

$$x \mapsto \frac{\mathsf{d}_{\mathrm{Y}}\big((\mathsf{G}_u^{v^1})_{\alpha^1 t}(x), (\mathsf{G}_u^{v^2})_{\alpha^2 t}(x)\big)}{t}$$

for any $0 < t \ll 1$. Since these maps pointwise converge to $x \mapsto \mathsf{d}_{u(x)}(\mathsf{S}_x^1, \mathsf{S}_x^2)$ as $t \downarrow 0$, the claim follows. $\square$

In particular, for $\mathsf{S}^1, \mathsf{S}^2$ as in the above statement, the quantity

$$\mathsf{d}_{L^2}\big(\mathsf{S}^1, \mathsf{S}^2\big) := \sqrt{\int_\Omega \mathsf{d}_{u(x)}^2(\mathsf{S}_x^1, \mathsf{S}_x^2)\,\mathrm{d}\mathfrak{m}(x)}, \tag{3.3.1}$$

is well defined. Standard arguments then show that d_{L^2} is symmetric, satisfies the triangle inequality and $\mathsf{d}_{L^2}(\mathsf{S}, \mathsf{S}) = 0$ (but it might happen that $\mathsf{d}_{L^2}(\mathsf{S}^1, \mathsf{S}^2) = 0$ for $\mathsf{S}^1 \neq \mathsf{S}^2$ and that $\mathsf{d}_{L^2}(\mathsf{S}^1, \mathsf{S}^2) = +\infty$).

We then give the following definitions:

Definition 3.3.2 ($\mathcal{L}^2$ sections of $u^*\mathrm{T}_G\mathrm{Y}$)**.** *Let* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\bar y)$ *be a pointed* CAT*(0)-space,* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *a metric measure space,* $\Omega\subset\mathrm{X}$ *an open subset and* u *a Borel representative of a map in* $L^2(\Omega,\mathrm{Y}_{\bar y})$*. Then,* $\mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *is the collection of sections* S *of* $u^*\mathrm{T}_G\mathrm{Y}$ *such that:*

i) For any $\alpha\in\mathbb{R}^+$ *and* $v\colon\Omega\to\mathrm{Y}$ *Borel and essentially separably valued we have that* $x\mapsto \mathsf{d}_{u(x)}(\mathsf{S}_x,\alpha(\mathsf{G}_{u(x)}^{v(x)})'_0)$ *is a Borel function.*

ii) There is a sequence $(\alpha_n)\subset\mathbb{R}^+$ *and maps* $v_n\colon\Omega\to\mathrm{Y}$ *Borel and essentially separably valued such that for the sections* S^n *given by* $\mathsf{S}^n_x:=\alpha_n(\mathsf{G}_{u(x)}^{v_n(x)})'_0$ *we have*

$$\begin{aligned}\sup_{n\in\mathbb{N}}\mathsf{d}_{L^2}(\mathsf{S}^n,0)&<\infty,\\ \lim_{n\to\infty}\mathsf{d}_{u(x)}(\mathsf{S}^n_x,\mathsf{S}_x)&=0\qquad\forall x\in\Omega.\end{aligned}\tag{3.3.2}$$

It is clear from the definitions that for $\mathsf{S}^1,\mathsf{S}^2\in\mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ the map $x\mapsto\mathsf{d}_{u(x)}(\mathsf{S}^1_x,\mathsf{S}^2_x)$ is Borel and $L^2(\mathfrak{m}_{|_\Omega})$-integrable, therefore $\mathsf{d}_{L^2}(\mathsf{S}^1,\mathsf{S}^2)$ is well defined by (3.3.1) and finite.

Definition 3.3.3 (L^2 sections of $u^*\mathrm{T}_G\mathrm{Y}$)**.** *Let* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\bar y)$ *be a pointed* CAT*(0)-space,* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *a metric measure space,* $\Omega\subset\mathrm{X}$ *an open subset and* u *a Borel representative of a map in* $L^2(\Omega,\mathrm{Y}_{\bar y})$*. We define* $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *as the quotient of* $\mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *with respect to the relation* $\mathsf{S}^1\sim\mathsf{S}^2$ *if* $\mathsf{d}_{L^2}(\mathsf{S}^1,\mathsf{S}^2)=0$*.*

It is obvious that the relation indicated in the previous definition is an equivalence relation, so that $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ is well defined. Also, the quantity d_{L^2} passes to the quotient and defines a distance, still denoted by d_{L^2}, on $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ and standard considerations show that the resulting object is a complete metric space.

Now let $\tilde u\colon\Omega\to\mathrm{Y}$ be Borel and $\mathfrak{m}$-a.e. equal to u and consider the identification $I\colon\mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})\to\mathcal{L}^2(\tilde u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ sending S to the section $I(\mathsf{S})$ defined by

$$I(\mathsf{S})_x:=\begin{cases}\mathsf{S}_x, & \text{if } u(x)=\tilde u(x),\\ 0_{\tilde u(x)}, & \text{if } u(x)\neq\tilde u(x).\end{cases}$$

It is clear that this map passes to the quotients and thus induces a map, still denoted by I, from $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ to $L^2(\tilde u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$. Also, the fact that $u=\tilde u$ $\mathfrak{m}$-a.e. trivially implies that such I is an isometry.

Thanks to these considerations, it makes sense to consider the space $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ for $u\in L^2(\Omega,\mathrm{Y}_{\bar y})$, i.e. even when u is only given up to $\mathfrak{m}$-a.e. equality: it is just sufficient to pick any Borel representative of u, consider the corresponding space of L^2-sections up to $\mathfrak{m}$-a.e. equality and notice that such space does not depend on the representative of u chosen.

The basic properties of the space $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ are collected in the following statement.

Proposition 3.3.4 (Properties of $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$)**.** *Let* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\bar y)$ *be a pointed* CAT*(0)-space,* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *a metric measure space,* $\Omega\subset\mathrm{X}$ *an open subset and* $u\in L^2(\Omega,\mathrm{Y}_{\bar y})$*.*

Then:

i) *For every* $\mathsf{S}^1,\mathsf{S}^2\in L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *the functions* $\Omega\ni x\mapsto\mathsf{d}_{u(x)}(\mathsf{S}^1_x,\mathsf{S}^2_x),|\mathsf{S}^1_x|_{u(x)},\langle\mathsf{S}^1_x,\mathsf{S}^2_x\rangle_{u(x)}$ *are (equivalence classes up to* $\mathfrak{m}$*-a.e. equality of) Borel functions.*

ii) *For every* $\mathsf{S}^1,\mathsf{S}^2\in L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *the section* $\mathsf{S}^1\oplus\mathsf{S}^2$ *given by the (equivalence class of the) map* $x\mapsto(x,u(x),\mathsf{S}^1_x\oplus\mathsf{S}^2_x)$ *belongs to* $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$*.*

(iii) For every $\mathsf{S}\in L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$ *and* $f\in L^\infty(\mathfrak{m}_{|_\Omega})$ *the section* $f\mathsf{S}$ *given by the (equivalence class of the) map* $x\mapsto(x,u(x),f(x)\mathsf{S}^1_x)$ *belongs to* $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|_\Omega})$*.*

Proof. The Borel regularity of $x \mapsto \mathsf{d}_{u(x)}(\mathsf{S}^1_x, \mathsf{S}^2_x)$ has already been noticed. Then the one of $|\mathsf{S}^1_x|_{u(x)}$ follows from the fact that the zero section $\mathsf{0}$ belongs to $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ and thus the one of $\langle \mathsf{S}^1_x, \mathsf{S}^2_x \rangle_{u(x)}$ follows by the definition of scalar product.

We pass to ii) and first consider the case of $\mathsf{S}^1, \mathsf{S}^2 \in \mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ of the form $\mathsf{S}^1_x = \alpha(\mathsf{G}^{v(x)}_{u(x)})'_0$ and $\mathsf{S}^2_x = \beta(\mathsf{G}^{w(x)}_{u(x)})'_0$ for $v, w\colon \Omega \to \mathrm{Y}$ Borel and essentially separably valued and $\alpha, \beta \geq 0$. Put $T := \min\{\alpha^{-1}, \beta^{-1}\} \in (0, \infty]$ and for $t \in (0, T)$ put $v_t := (\mathsf{G}^v_u)_{\alpha t}$, $w_t := (\mathsf{G}^w_u)_{\beta t}$ and let $m_t(x)$ be the midpoint of $v_t(x), w_t(x)$ for every $x \in \Omega$. From the continuity of the 'midpoint' operation and the triangle inequality it easily follows that $x \mapsto m_t(x)$ is Borel, essentially separably valued and in $L^2(\Omega, \mathrm{Y}_{\bar y})$. Then, define the section M_t as $\mathsf{M}_{t,x} := \frac{1}{t}(\mathsf{G}^{m_t(x)}_{u(x)})'_0$ and recall (2.1.6) to see that $\mathsf{M}_{t,x} \to \frac{1}{2}(\mathsf{S}^1 \oplus \mathsf{S}^2)_x$ in $\mathrm{T}_{u(x)}\mathrm{Y}$ as $t \downarrow 0$ for every $x \in \Omega$: this proves that $\frac{1}{2}(\mathsf{S}^1 \oplus \mathsf{S}^2)$ satisfies the requirement i) in Definition 3.3.2. The same convergence together with the bound

$$|\mathsf{M}_{t,\cdot}|_{u(\cdot)} \leq \tfrac{1}{t}\mathsf{d}_\mathrm{Y}(u, m_t) \leq \tfrac{2}{t}\big(\mathsf{d}_\mathrm{Y}(u, v_t) + \mathsf{d}_\mathrm{Y}(u, w_t)\big) \leq 2\big(\alpha \mathsf{d}_\mathrm{Y}(u, v) + \beta \mathsf{d}_\mathrm{Y}(u, w)\big) \quad \text{on } \Omega$$

valid for every $t \in (0, T)$ shows that $\frac{1}{2}(\mathsf{S}^1 \oplus \mathsf{S}^2)$ satisfies also the requirement ii) in Definition 3.3.2.

Now the fact that $\frac{1}{2}(\mathsf{S}^1 \oplus \mathsf{S}^2) \in \mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ for generic $\mathsf{S}^1, \mathsf{S}^2 \in \mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ follows by approximation (recall point ii) in Definition 3.3.2) and the continuity of the 'sum' operation noticed in Proposition 2.1.5, then the analogous properties for elements of $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ trivially follow.

Finally, the fact that $\frac{1}{2}(\mathsf{S}^1 \oplus \mathsf{S}^2) \in L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ implies $\mathsf{S}^1 \oplus \mathsf{S}^2 \in L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ is trivial from the definitions (see also the arguments below).

For (iii) we notice that it is sufficient to prove that $f\mathsf{S}$ is in $\mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ whenever $f\colon \Omega \to \mathbb{R}$ is Borel and bounded and $\mathsf{S} \in \mathcal{L}^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$. In this case the fact that $f\mathsf{S}$ satisfies the requirement i) in Definition 3.3.2 is obvious. For ii) we consider sections $\mathsf{S}^n_x = \alpha_n(\mathsf{G}^{v_n(x)}_{u(x)})'_0$ for which (3.3.2) hold and put $\tilde{\mathsf{S}}^n_x := \alpha_n \|f\|_{L^\infty}(\mathsf{G}^{w_n(x)}_{u(x)})'_0$, where $w_n(x) := (\mathsf{G}^{v_n(x)}_{u(x)})_{f(x)/\|f\|_{L^\infty}}$. The fact that the w_n's are Borel representatives of maps in $L^2(\Omega, \mathrm{Y}_{\bar y})$ can be easily checked from the definition while the fact that (3.3.2) holds for $f\mathsf{S}$ and $(\tilde{\mathsf{S}}^n)$ is obvious. $\square$

Let us now come back to the initial discussion and, for given $u \in L^2(\Omega, \mathrm{Y}_{\bar y})$, let us define the map $\iota\colon \mathsf{Geo}_u L^2(\Omega, \mathrm{Y}_{\bar y}) \to L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ as follows. For $v \in L^2(\Omega, \mathrm{Y}_{\bar y})$ and $\alpha \geq 0$ we send the geodesic $t \mapsto (\mathsf{G}^v_u)_{\alpha t}$ to the (equivalence class of the) section given by $x \mapsto \alpha(\mathsf{G}^{v(x)}_{u(x)})'_0$. The relation between $\mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar y})$ and $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ is then described by the following result:

Proposition 3.3.5 ($L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ and $\mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar y})$)**.** *Let $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \bar y)$ be a pointed* CAT*(0)-space, $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ a metric measure space, $\Omega \subset \mathrm{X}$ an open subset and $u \in L^2(\Omega, \mathrm{Y}_{\bar y})$.*

Then, the map $\iota\colon \mathsf{Geo}_u L^2(\Omega, \mathrm{Y}_{\bar y}) \to L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ passes to the quotient and induces a map, still denoted ι, from $\mathsf{Geo}_u L^2(\Omega, \mathrm{Y}_{\bar y})/\sim$ to $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ that can be uniquely extended by continuity to a bijective isometry, again denoted ι, from $\mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar y})$ to $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$.

Moreover, the so defined isometry ι respects the operations on the tangent cones, i.e.

$$|\mathsf{v}|^2_u = \int_\Omega |\iota(\mathsf{v})_x|^2_{u(x)}\,\mathrm{d}\mathfrak{m}(x), \tag{3.3.3a}$$

$$\langle \mathsf{v}_1, \mathsf{v}_2 \rangle_u = \int_\Omega \langle \iota(\mathsf{v}_1)_x, \iota(\mathsf{v}_2)_x \rangle_{u(x)}\,\mathrm{d}\mathfrak{m}(x), \tag{3.3.3b}$$

$$\mathsf{d}^2_u(\mathsf{v}_1, \mathsf{v}_2) = \int_\Omega \mathsf{d}^2_{u(x)}(\iota(\mathsf{v}_1)_x, \iota(\mathsf{v}_2)_x)\,\mathrm{d}\mathfrak{m}(x), \tag{3.3.3c}$$

$$\iota(\lambda \mathsf{v}) = \lambda \iota(\mathsf{v}), \tag{3.3.3d}$$

$$\iota(\mathsf{v}_1 \oplus \mathsf{v}_2) = \iota(\mathsf{v}_1) \oplus \iota(\mathsf{v}_2), \tag{3.3.3e}$$

for any $\mathsf{v}, \mathsf{v}_1, \mathsf{v}_2 \in \mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar y})$ and $\lambda \in \mathbb{R}^+$.

Proof. Let $v^1, v^2 \in L^2(\Omega, \mathrm{Y}_{\bar y})$, $\alpha_1, \alpha_2 \geq 0$, consider the sections $\mathsf{S}^1, \mathsf{S}^2 \in L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ given by $\mathsf{S}^i := \iota(\alpha_i(\mathsf{G}_u^{v^i})'_0)$. Notice that

$$\begin{aligned}
\mathsf{d}_u^2\big(\alpha_1(\mathsf{G}_u^{v_1})'_0, \alpha_2(\mathsf{G}_u^{v_2})'_0\big) &= \lim_{t\downarrow 0} \frac{\mathsf{d}_{L^2}^2\big((\mathsf{G}_u^{v_1})_{\alpha_1 t}, (\mathsf{G}_u^{v_2})_{\alpha_2 t}\big)}{t^2} \\
&= \lim_{t\downarrow 0} \int_\Omega \frac{\mathsf{d}_{\mathrm{Y}}^2\big((\mathsf{G}_{u(x)}^{v_1(x)})_{\alpha_1 t}, (\mathsf{G}_{u(x)}^{v_2(x)})_{\alpha_2 t}\big)}{t^2}\, \mathrm{d}\mathfrak{m}(x) \\
&= \int_\Omega \lim_{t\downarrow 0} \frac{\mathsf{d}_{\mathrm{Y}}^2\big((\mathsf{G}_{u(x)}^{v_1(x)})_{\alpha_1 t}, (\mathsf{G}_{u(x)}^{v_2(x)})_{\alpha_2 t}\big)}{t^2}\, \mathrm{d}\mathfrak{m}(x) \\
&= \int_\Omega \mathsf{d}_{u(x)}^2\big(\mathsf{S}_x^1, \mathsf{S}_x^2\big)\, \mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

where, in passing the limit inside the integral, we used the dominate convergence theorem and the fact that the integrand is non-negative and non-decreasing in t (recall (2.1.3)). This proves at once that ι passes to the quotient to a map on $\mathsf{Geo}_u L^2(\Omega, \mathrm{Y}_{\bar y})/\sim$ and that the so induced map is an isometry which therefore can be extended to a map from $\mathrm{T}_u L^2(\Omega, \mathrm{Y}_{\bar y})$ to $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$. The fact that such extension is surjective follows from an approximation argument based on the requirement ii) in Definition 3.3.2.

Now observe that (3.3.3c) has already been proved by the fact that ι is an isometry. Then (3.3.3a) and (3.3.3b) follow as well. Also, (3.3.3d) is obvious by definition and then (3.3.3e) follows from (3.3.3c), (3.3.3d) and the metric characterization of the midpoints of x, y as the point m such that $\mathsf{d}^2(x,m) + \mathsf{d}^2(m,y) = \mathsf{d}^2(x,y)/2$. $\square$

The separable setting

In this section we assume instead that Y is a separable and locally $\mathsf{CAT}(\kappa)$-space and we study the Borel structure of the pullback $u^*\mathrm{T}_G\mathrm{Y}$ of the geometric tangent bundle of Y. We shall then see in the space case of Y being separable and $\mathsf{CAT}(0)$ how such Borel structure relates to the space $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ studied in the previous section.

Thus let Y be separable and locally $\mathsf{CAT}(\kappa)$, (X, d) be a complete and separable metric space and $\Omega \subset \mathrm{X}$ be open.

As before, for a given Borel map $u\colon \Omega \to \mathrm{Y}$, the pullback geometric tangent bundle $u^*\mathrm{T}_G\mathrm{Y}$ is defined as

$$u^*\mathrm{T}_G\mathrm{Y} := \big\{(x, y, v)\colon x \in \Omega,\ y = u(x),\ v \in \mathrm{T}_y\mathrm{Y}\big\} \subset \mathrm{X} \times \mathrm{T}_G\mathrm{Y}$$

and a section of $u^*\mathrm{T}_G\mathrm{Y}$ is a map $\mathsf{S}\colon \Omega \to u^*\mathrm{T}_G\mathrm{Y}$ such that $\mathsf{S}_x \in \mathrm{T}_{u(x)}\mathrm{Y}$ for every $x \in \Omega$.

Now equip $u^*\mathrm{T}_G\mathrm{Y} \subset \mathrm{X} \times \mathrm{T}_G\mathrm{Y}$ with the restriction of the product σ-algebra $\mathcal{B}(\mathrm{X}) \otimes \mathcal{B}(\mathrm{T}_G\mathrm{Y})$, which, with abuse of terminology, we shall call Borel σ-algebra on $u^*\mathrm{T}_G\mathrm{Y}$ and denote $\mathcal{B}(u^*\mathrm{T}_G\mathrm{Y})$. In particular, we shall say that a section is Borel if it is measurable w.r.t. $\mathcal{B}(\mathrm{X})$ and $\mathcal{B}(u^*\mathrm{T}_G\mathrm{Y})$.

A section is *simple* provided there are a Borel partition (E_n) of Ω, $(\alpha_n) \subset \mathbb{R}^+$ and points $(y_n) \subset \mathrm{Y}$ s.t. $y_n \in B_{r_{u(x)}}(u(x))$, for every $x \in E_n$ and $\mathsf{S}_{|E_n} = \alpha_n(\mathsf{G}_{u(\cdot)}^{y_n})'_0$. We shall formally denote such section by $\sum_n \chi_{E_n}\alpha_n u^*(\mathsf{G}_\cdot^{y_n})'_0$. Notice that the restriction of such section to E_n coincides with the (graph of) the composition of u with the simple section of $\mathrm{T}_G\mathrm{Y}$ given by $y \mapsto (y, \alpha_n(\mathsf{G}_y^{y_n})'_0)$. In particular, recalling Proposition 2.1.10 we see that simple sections of $u^*\mathrm{T}_G\mathrm{Y}$ are Borel.

Moreover, they are dense in the space of Borel sections:

Lemma 3.3.6 (Density of simple sections)**.** *Let* (X, d) *be a metric space,* $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$ *be separable and locally* $\mathsf{CAT}(\kappa)$*-space,* $\Omega \subset \mathrm{X}$ *an open subset and* $u\colon \Omega \to \mathrm{Y}$ *Borel. Let* $\mathsf{S}\colon \Omega \to u^*\mathrm{T}_G\mathrm{Y}$ *be a Borel section of* $u^*\mathrm{T}_G\mathrm{Y}$ *and* $\varepsilon > 0$*.*

Then, there is a simple section T *such that* $\mathsf{d}_{u(x)}(\mathsf{S}_x, \mathsf{T}_x) < \varepsilon$ *for every* $x \in \Omega$*.*

Proof. We can reduce the proof to the case of Y being $\mathsf{CAT}(\kappa)$ by using the Lindelöf property of Y and the coverings made by $B_{\mathsf{r}_y/2}(y)$. Doing so, we achieve uniqueness of geodesics between any couple of points. Let $D \subset \mathrm{Y}$ be countable and dense (y_n, α_n) be an enumeration of $D \times \mathbb{Q}^+$. Then for every $n \in \mathbb{N}$ consider the function $F_n \colon \mathrm{T}_G\mathrm{Y} \to \mathbb{R}$ given by

$$F_n(y, v) := \mathsf{d}_y(v, \alpha_n(\mathsf{G}_y^{y_n})'_0) = \sqrt{|v|_y^2 + |\alpha_n|^2 \mathsf{d}^2(y, y_n) - 2\big\langle v, \alpha_n(\mathsf{G}_y^{y_n})'_0 \big\rangle_y}.$$

The defining requirements of $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$ and the property (2.1.8) ensure that F_n is Borel. Hence so is the map $\tilde{F}_n \colon u^*\mathrm{T}_G\mathrm{Y} \to \mathbb{R}$ defined as $\tilde{F}_n := F_n \circ \pi_{\mathrm{T}_G\mathrm{Y}}$, where $\pi_{\mathrm{T}_G\mathrm{Y}} \colon u^*\mathrm{T}_G\mathrm{Y} \subset \mathrm{X} \times \mathrm{T}_G\mathrm{Y} \to \mathrm{T}_G\mathrm{Y}$ is the canonical projection.

Hence given a Borel section S of $u^*\mathrm{T}_G\mathrm{Y}$ the map $\tilde{F}_n \circ \mathsf{S} \colon \mathrm{X} \to \mathbb{R}$ is Borel and thus, for given $\varepsilon > 0$, so is the set $\tilde{E}_n := (\tilde{F}_n \circ \mathsf{S})^{-1}([0, \varepsilon))$. We then put $E_n := \tilde{E}_n \setminus \cup_{i<n} \tilde{E}_i$ and notice that the property (2.1.4) ensures that the E_n's form a partition of X, thus giving the conclusion. □

Thanks to such density result we can show that the operations on the tangent cones preserve Borel regularity. The statement below is similar in spirit to (part of) the statement of Proposition 3.3.4, but here no measure is fixed on Ω and that the sections are defined for every $x \in \Omega$, not for $\mathfrak{m}$-a.e. x.

Proposition 3.3.7. *Let (X, d) be a metric space, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ be separable and locally $\mathsf{CAT}(\kappa)$, $\Omega \subset \mathrm{X}$ an open subset and $u \colon \Omega \to \mathrm{Y}$ a Borel map. Let $\mathsf{S}^1, \mathsf{S}^2$ be Borel sections of $u^*\mathrm{T}_G\mathrm{Y}$ and $f \colon \mathrm{X} \to \mathbb{R}^+$ be a Borel map.*

Then the functions sending $x \in \mathrm{X}$ to $|\mathsf{S}_x^2|_{u(x)}, \langle \mathsf{S}_x^1, \mathsf{S}_x^2 \rangle_{u(x)}, \mathsf{d}_{u(x)}(\mathsf{S}_x^1, \mathsf{S}_x^2)$ are Borel and the sections $x \mapsto f(x)\mathsf{S}_x^1, \mathsf{S}_x^1 \oplus \mathsf{S}_x^2$ are Borel as well.

Proof. Let $\mathsf{S}^1, \mathsf{S}^2$ be simple of the form: $\mathsf{S}^1 = \chi_{E_1} \alpha\, u^*(\mathsf{G}_\cdot^{y_1})'_0$ and $\mathsf{S}^2 = \chi_{E_2} \beta\, u^*(\mathsf{G}_\cdot^{y_2})'_0$, with $E_i := u^{-1}(A_i)$ and $A_i \in \mathcal{B}(\mathrm{Y})$ such that $y_i \in B_{\mathsf{r}_y}(y)$ for every $y \in A_i, i = 1, 2$. Then they are the (graph of the) composition of u with the simple sections of $\mathrm{T}_G\mathrm{Y}$ given by $\chi_{A_1} \alpha\, (\mathsf{G}_\cdot^{y_1})'_0$ and $\chi_{A_2} \beta\, (\mathsf{G}_\cdot^{y_2})'_0$ respectively, hence in this case the conclusion comes from Proposition 2.1.11.

Then the conclusion comes from the 'fiberwise' continuity of all the expressions considered (granted by Proposition 2.1.5) and the density of simple sections established in Lemma 3.3.6 above. □

We now come to the relation between the space of (equivalence classes up to $\mathfrak{m}$-a.e. equality of) Borel sections of $u^*\mathrm{T}_G\mathrm{Y}$ and the space $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|_\Omega})$ in the case where Y is separable and $\mathsf{CAT}(0)$. As expected, these spaces coincide when the right integrability of the first ones is in place:

Proposition 3.3.8. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \bar{y})$ be a pointed separable CAT(0)-space, $\Omega \subset \mathrm{X}$ an open subset and $u \colon \Omega \to \mathrm{Y}$ be a Borel map.*

Then, $\mathsf{S} \in L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|_\Omega})$ if and only if it is the equivalence class up to $\mathfrak{m}$-a.e. equality of a Borel section T of $u^*\mathrm{T}_G\mathrm{Y}$ with $\int_\Omega |\mathsf{T}|_{u(x)}^2 \,\mathrm{d}\mathfrak{m}(x) < \infty$.

Proof. Assume at first that $\mathsf{S} \in L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|_\Omega})$. Then the fact that $\int_\Omega |\mathsf{S}_x|_{u(x)}^2 \,\mathrm{d}\mathfrak{m}(x) < \infty$ is a direct consequence of the definition and of Proposition 3.3.5 above, thus we only need to prove that S is the equivalence class up to $\mathfrak{m}$-a.e. equality of a Borel section of $u^*\mathrm{T}_G\mathrm{Y}$. To see this we need to prove that, letting $\pi_\mathrm{X}, \pi_{\mathrm{T}_G\mathrm{Y}}$ be the projections of $u^*\mathrm{T}_G\mathrm{Y} \subset \mathrm{X} \times \mathrm{T}_G\mathrm{Y}$ to $\mathrm{X}, \mathrm{T}_G\mathrm{Y}$ respectively, the maps $\pi_\mathrm{X} \circ \mathsf{S}$ and $\pi_{\mathrm{T}_G\mathrm{Y}} \circ \mathsf{S}$ are equivalence classes up to $\mathfrak{m}$-a.e. equality of Borel maps. For the first one this is obvious, because it is the identity on X. For the second one we recall the definition of $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$ to see that we need to prove that $\pi_\mathrm{Y} \circ \pi_{\mathrm{T}_G\mathrm{Y}} \circ \mathsf{S}$ is Borel (which it is, because it coincides with u) and that $x \mapsto \langle \mathsf{S}_x, (\mathsf{G}_{u(x)}^z)'_0 \rangle$ is Borel for every $z \in \mathrm{Y}$ (which is easily seen to be the case from the requirement i) in Definition 3.3.2).

We pass to the converse implication and start observing that Lemma 3.3.1 and the definition of $\mathcal{B}(\mathrm{T}_G\mathrm{Y})$ just recalled ensure that for any $v \in L^2(\Omega, \mathrm{Y}_{\bar{y}})$ the section given by $(\mathsf{G}_{u(x)}^{v(x)})'_0$ is the equivalence class up to $\mathfrak{m}$-a.e. equality of a Borel section. It follows that if T is a Borel section as in the statement, then it satisfies the requirement i) in Definition 3.3.2. We now claim that if T is also simple, then it also satisfies the requirement ii). To see this write $\mathsf{T} = \sum_n \chi_{E_n} \alpha_n\, u^*(\mathsf{G}_\cdot^{y_n})'_0$ and put $\mathsf{T}^i := \sum_{n \leq i} \chi_{E_n} \alpha_n\, u^*(\mathsf{G}_\cdot^{y_n})'_0$ where it is intended that for $x \notin \cup_{n \leq i} E_n$ we have $\mathsf{T}_x^i = 0_{u(x)} \in$

$\mathrm{T}_{u(x)}\mathrm{Y}$. Then putting $\beta_i := \max_{n\leq i}\alpha_n$, $y_{n,i} := (\mathsf{G}^{y_n}_{u(x)})_{\alpha_n/\beta_i}$ and defining $v^i \in L^2(\Omega,\mathrm{Y}_{\bar y})$ as $v^i|_{E_n} := y_{n,i}$ for $n\leq i$ and $v^i|_{\Omega\setminus\cup_{n\leq i}E_n} \equiv u$ we see that $\mathsf{T}^i = \iota(\beta_i(\mathsf{G}^{v^i}_u)'_0)$, so that (the equivalence class up to $\mathfrak{m}$-a.e. equality of) T^i belongs to $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|\Omega})$. It is then clear that $\mathsf{d}_{L^2}(\mathsf{T}^i,\mathsf{T})\to 0$, proving that the equivalence class of T belongs to $L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|\Omega})$.

Then the conclusion for a generic section T as in the statement can be easily obtained by an approximation argument starting from the density result in Lemma 3.3.6. □

3.3.2 The Laplacian of a $\mathsf{CAT}(0)$-valued map

Let us start by giving the general definition of Laplacian of a $\mathsf{CAT}(0)$-valued Sobolev map:

Definition 3.3.9 (Tension field/Laplacian)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(K,N)$ space, $\Omega\subset\mathrm{X}$ an open subset, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\bar y)$ a pointed $\mathsf{CAT}(0)$-space and $\bar u\in\mathsf{KS}^{1,2}(\Omega,\mathrm{Y}_{\bar y})$.*

Then the domain of the Laplacian $D(\Delta_{\bar u})\subset\mathsf{KS}^{1,2}_{\bar u}(\Omega,\mathrm{Y}_{\bar y})$ is defined as $D(\Delta_{\bar u}) := D(|\partial^-\mathsf{E}^{\mathsf{KS}}_{\bar u}|)$ and for $u\in D(\Delta_{\bar u})$ we put

$$\Delta_{\bar u}u := \iota(\mathsf{v})\in L^2(u^*\mathrm{T}_G\mathrm{Y},\mathfrak{m}_{|\Omega}),\qquad \textit{where } \mathsf{v} \textit{ is the element of minimal norm in } -\partial^-\mathsf{E}^{\mathsf{KS}}_{\bar u}(u).$$

Similarly, for maps u from X to Y we say that u is in the domain of the Laplacian $D(\Delta)$ if $|\partial^-\mathsf{E}^{\mathsf{KS}}|(u)<\infty$ and in this case $\Delta u := \iota(\mathsf{v})$, where $\iota(\mathsf{v})$ is the element of minimal norm in $-\partial^-\mathsf{E}^{\mathsf{KS}}(u)$.

Proposition 3.3.10 (Laplacian and variation of the energy)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(K,N)$ space, $\Omega\subset\mathrm{X}$ an open subset, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\bar y)$ a pointed $\mathsf{CAT}(0)$-space and $\bar u\in\mathsf{KS}^{1,2}(\Omega,\mathrm{Y}_{\bar y})$. Also, let $u\in D(\Delta_{\bar u})$. Then, for every $v\in L^2(\Omega,\mathrm{Y}_{\bar y})$, we have*

$$-\int_\Omega \big\langle \Delta_{\bar u}u(x),\big(\mathsf{G}^{v(x)}_{u(x)}\big)'_0\big\rangle_{u(x)}\,\mathrm{d}\mathfrak{m}(x)\leq \lim_{t\downarrow 0}\frac{\mathsf{E}^{\mathsf{KS}}_{\bar u}((\mathsf{G}^v_u)_t)-\mathsf{E}^{\mathsf{KS}}_{\bar u}(u)}{t}. \tag{3.3.4}$$

Moreover, u is harmonic with $\bar u$ as boundary value if and only if $u\in D(\Delta_{\bar u})$ with $\Delta_{\bar u}u=0$.

Proof. Inequality (3.3.4) follows applying (3.2.6), the definition of $\Delta_{\bar u}u$ and recalling Proposition 3.3.5. The second claim is a restatement of (3.2.7) in this setting. □

Remark 3.3.11. This last proposition shows that our definition is compatible with the classical one valid in the smooth category. Indeed, if X,Y are smooth Riemannian manifold, $\bar u,u\colon\bar\Omega\subset\mathrm{X}\to\mathrm{Y}$ are smooth maps with the same boundary values, v is a smooth section of $u^*T\mathrm{Y}$ (in the smooth setting $\mathrm{T}_G\mathrm{Y}$ is canonically equivalent to the standard tangent bundle $T\mathrm{Y}$) which is 0 on $\partial\Omega$, then we can produce a smooth perturbation of u by putting $u_t(x) := \exp_{u(x)}(t\mathsf{v}_x)$. A direct computation then shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathsf{E}^{\mathsf{KS}}_{\bar u}(u_t) = -\int_\Omega\big\langle\tau(u)_x,\mathsf{v}_x\big\rangle_{u(x)}\,\mathrm{d}\mathfrak{m}(x),$$

where $\tau(u)$ is the *tension field of* u, see for instance [125, Section 9.2]. This formula is the smooth version of (3.3.4). Notice indeed that $u_t=(\mathsf{G}^{u_1}_u)_t$ for $t\in[0,1]$ (and similarly $u_t=(\mathsf{G}^{u_{-1}}_u)_{-t}$ for $t\in[-1,0]$) and that if everything is smooth, then $t\mapsto\mathsf{E}^{\mathsf{KS}}_{\bar u}(u_t)$ is C^1, hence differentiable in 0, so that the one-sided bound in (3.3.4) becomes an equality in the smooth case.

It is worth to underline that in our framework the lack of equality in (3.3.4) is not only related to the lack of smoothness of $t\mapsto\mathsf{E}^{\mathsf{KS}}_{\bar u}(u_t)$, which a priori could produce different left and right derivatives in 0, but also to the fact that tangent *cones* are not really tangent *spaces*: the opposite of a vector field does not necessarily exist and thus we are forced to take one-sided perturbations only. ■

A direct consequence of Proposition 3.3.10 above is the following:

Corollary 3.3.12. *With the same assumptions and notation as in Proposition 3.3.10 we have*

$$\mathsf{E}^{\mathsf{KS}}_{\bar u}(u) - \int_{\mathrm{X}} \big\langle \Delta_{\bar u} u(x), \big(\mathsf{G}^{v(x)}_{u(x)}\big)'_0 \big\rangle_{u(x)} \, \mathrm{d}\mathfrak{m}(x) + \mathsf{E}^{\mathsf{KS}}(d) \le \mathsf{E}^{\mathsf{KS}}_{\bar u}(v),$$

where $d := \mathsf{d}(u,v)$*.*

Proof. Couple (3.3.4) with (2.3.4). □

In the next discussion, we are interested in properties of the composition $f \circ u$, whenever u is a harmonic map and f is λ-convex functional. Observe that, in a smooth framework, the chain rule $\Delta(f \circ u) = \mathrm{Hess} f(\nabla u, \nabla u) + \mathrm{d}f(\Delta u)$ immediately implies that

$$\Delta(f \circ u) \ge \lambda |\mathrm{d}u|^2_{\mathsf{HS}} \qquad \text{if } f \text{ is } \lambda\text{-convex and } u \text{ is harmonic.} \tag{3.3.5}$$

A nonsmooth version of (3.3.5) has already been addressed in [148] (see Theorem 1.2 there) for maps with *Euclidean* source domain and $\mathsf{CAT}(0)$-target. Nevertheless, as we are going to show in Theorem 3.3.15, the discussion generalizes to our framework: the main stumbling block to overcome being the absence of *Lipschitz* vector field on a RCD-space. In the next, we shall need the following property of Sobolev functions and, specifically, of their directional derivatives (for the definition of test vector field see [97] and for the concept of Regular Lagrangian Flow see [28]):

Proposition 3.3.13. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(K,N)$ *space,* $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \bar y)$ *a pointed complete metric space,* $\Omega \subset \mathrm{X}$ *open,* $v \in L^2(T\mathrm{X})$ *a test vector field and* (Fl^v_s) *the associated Regular Lagrangian Flow. Also, let* $u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar y})$*.*

Then, for every $K \subset \Omega$ *compact, we have that*

$$\lim_{s \to 0} \frac{\mathsf{d}_{\mathrm{Y}}(u \circ \mathsf{Fl}^v_s, u)}{s} = |\mathrm{d}_2 u(v)| \qquad \text{in } L^2(K). \tag{3.3.6}$$

(notice that for $|s|$ *small the map* $u \circ \mathsf{Fl}^v_s$ *is well defined from* K *to* Y*).*

Similarly, for a real valued Sobolev function $g \in W^{1,2}(\Omega)$ *we have*

$$\lim_{s \to 0} \frac{g \circ \mathsf{Fl}^v_s - g}{s} = \mathrm{d}g(v) \qquad \text{in } L^2(K). \tag{3.3.7}$$

Proof. Property (3.3.7) is (an equivalent version of) the definition of Regular Lagrangian Flow, see for instance [107, Proposition 2.7]. For (3.3.6) recall first [109, Remark 4.15] to get that functions in $\mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar y})$ also belong to the 'direction' Korevaar-Schoen space as defined in [108], then recall [108, Theorem 4.5]. □

The next Lemma deals with variations of a map u, suitably obtained through gradient flows trajectories in the target space, and the rate of change at the level of Korevaar-Schoen energy (see (3.3.8)-(3.3.9) below). In the following statement, notice that $f \circ u$ belongs to $W^{1,2}(\Omega)$ - and thus $\mathrm{d}(f \circ u)$ is well defined - because f is Lipschitz, Ω has finite measure and by ii) in Theorem 2.3.4. Also, for the very same reason, we shall drop the subscript $\bar y$ from Y when Ω is bounded as the L^2-integrability depends no more on the particular chosen point $\bar y \in \mathrm{Y}$. Compare the proof with [148, Lemma 3.1].

Lemma 3.3.14. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(K,N)$ *space,* Y CAT*(0)-space and* $\Omega \subset \mathrm{X}$ *open and bounded. Also, let* $f \in \mathrm{Lip}(\mathrm{Y})$ *be* λ*-convex,* $\lambda \in \mathbb{R}$*, and* $u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y})$*. For* $g \in \mathrm{Lip}_{bs}(\mathrm{X})^+$*, define the (equivalence class of the) variation map*

$$u_t(x) = \mathsf{GF}^f_{tg(x)}(u(x)) \qquad \forall t > 0, \ x \in \Omega.$$

Then, $u_t \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y})$ *for every* $t > 0$ *and there is a constant* $C > 0$ *depending on* f, g *such that*

$$|\mathrm{d}u_t|^2_{\mathsf{HS}} \le e^{-2\lambda t g}\big(|\mathrm{d}u|^2_{\mathsf{HS}} - 2t \left\langle \mathrm{d}g, \mathrm{d}(f \circ u) \right\rangle + Ct^2\big) \qquad \mathfrak{m}\text{-a.e. in } \Omega, \tag{3.3.8}$$

holds for every $t \in [0,1]$*. In particular*

$$\varlimsup_{t \downarrow 0} \frac{\mathsf{E}^{\mathsf{KS}}(u_t) - \mathsf{E}^{\mathsf{KS}}(u)}{t} \le -\int_\Omega \frac{\lambda}{d+2} g |\mathrm{d}u|^2_{\mathsf{HS}} + \left\langle \mathrm{d}(f \circ u), \mathrm{d}g \right\rangle \mathrm{d}\mathfrak{m}. \tag{3.3.9}$$

Proof. The map $x \mapsto (tg(x), u(x))$ is Borel and essentially separably valued and the map $(t, y) \mapsto \mathsf{GF}^f_t(y)$ is continuous, hence $x \mapsto u_t(x)$ is Borel and essentially separably valued. Also, the identity (3.2.1) and the trivial estimate $|\partial^- f| \le \mathrm{Lip}(f)$ show that $t \mapsto \mathsf{GF}^f_t(y)$ is $\mathrm{Lip}(f)$-Lipschitz for every $y \in \mathrm{Y}$, thus $\mathsf{d}_\mathrm{Y}(u_t(x), \bar{y}) \le t \sup(g)\mathrm{Lip}(f) + \mathsf{d}_\mathrm{Y}(u(x), \bar{y})$, for every $\bar{y} \in \mathrm{Y}$, from which it easy follows that $u_t \in L^2(\Omega, \mathrm{Y})$. Taking also into account the contraction property (3.2.3) we obtain that

$$
\begin{aligned}
\mathsf{d}_\mathrm{Y}(u_t(x), u_t(y)) &\le e^{\lambda^- t(g(x)+g(y))}\mathsf{d}_\mathrm{Y}\big(u(x), \mathsf{GF}^f_{t|g(y)-g(x)|}(u(y))\big) \\
&\le e^{2\lambda^- t \sup g}\big(\mathsf{d}_\mathrm{Y}(u(x), u(y)) + t\mathrm{Lip}(g)\mathrm{Lip}(f)\mathsf{d}(x, y)\big)
\end{aligned}
$$

and thus

$$
\mathsf{ks}^2_{2,r}[u_t, \Omega](x) \le 2e^{4\lambda^- t \sup g}\big(\mathsf{ks}^2_{2,r}[u, \Omega](x) + t^2\mathrm{Lip}^2(g)\mathrm{Lip}^2(f)\big).
$$

Integrating and using the fact that $\mathfrak{m}(\Omega) < \infty$ we conclude that $u_t \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y})$.

In order to obtain (3.3.8) we need to be more careful in our estimates and to this aim we shall use Lemma 3.2.3 and Proposition 3.3.13 above. Let $\gamma \colon [0, S] \to \Omega$ be a Lipschitz curve: for any $s \in [0, S]$ the bound (3.2.5) gives (here we are fixing a Borel representative of u and thus of u_t, but notice that the estimate (3.3.12) does not depend on such choice):

$$
\begin{aligned}
&\mathsf{d}^2_\mathrm{Y}(u_t(\gamma_0), u_t(\gamma_s)) \\
&\le e^{-2\lambda t(g(\gamma_0) \pm |g(\gamma_0)-g(\gamma_s)|)}\Big(\mathsf{d}^2_\mathrm{Y}(u(\gamma_0), u(\gamma_s)) + 2t(g(\gamma_s) - g(\gamma_0))(f(u(\gamma_0)) - f(u(\gamma_s))) \\
&\quad + \int_0^{|t(g(\gamma_0)-g(\gamma_s))|} 2\mathrm{Lip}^2(f)\theta_\lambda(r) + \lambda^-\big(\mathsf{d}^2_\mathrm{Y}(\mathsf{GF}^f_r(u(\gamma_0)), u(\gamma_s)) + \mathsf{d}^2_\mathrm{Y}(\mathsf{GF}^f_r(u(\gamma_s)), u(\gamma_0))\big)\,\mathrm{d}r\Big),
\end{aligned}
$$

where the sign in $\pm|g(\gamma_0) - g(\gamma_s)|$ depends on the sign of λ. Now use again the fact that $r \mapsto \mathsf{GF}^f_r(u(\gamma_0))$ is $\mathrm{Lip}(f)$-Lipschitz to get that

$$
\mathsf{d}^2_\mathrm{Y}(\mathsf{GF}^f_r(u(\gamma_0)), u(\gamma_s)) \le 2r^2\mathrm{Lip}^2(f) + 2\mathsf{d}^2_\mathrm{Y}(u(\gamma_0), u(\gamma_s)),
$$

notice that the same bounds holds for $\mathsf{d}^2_\mathrm{Y}(\mathsf{GF}^f_r(u(\gamma_s)), u(\gamma_0))$, that

$$
|t(g(\gamma_0) - g(\gamma_s))| \le ts\mathrm{Lip}(g)\mathrm{Lip}(\gamma)
$$

and that $\theta_\lambda(t) \le te^{2\lambda^- t}$ to conclude that, for some constant C depending only on $f, g, \mathrm{Lip}(\gamma), T$ and every $t \in [0, T]$, we have

$$
\begin{aligned}
\mathsf{d}^2_\mathrm{Y}(u_t(\gamma_0), u_t(\gamma_s)) &\le e^{-2\lambda t g(\gamma_0)+Cs}\Big(\mathsf{d}^2_\mathrm{Y}(u(\gamma_0), u(\gamma_s)) \\
&\quad + 2t(g(\gamma_s) - g(\gamma_0))(f(u(\gamma_0)) - f(u(\gamma_s))) + Ct^2s^2 + Cts\mathsf{d}^2_\mathrm{Y}(u(\gamma_0), u(\gamma_s))\Big).
\end{aligned} \tag{3.3.10}
$$

Now let v be a test vector field on X and Fl^v_s its Regular Lagrangian Flow and recall that since $g, f \circ u \in W^{1,2}(\Omega)$, by (3.3.7) we know that for any $K \subset \Omega$ compact we have

$$
\frac{g \circ \mathsf{Fl}^v_s - g}{s} \to \mathrm{d}g(v) \qquad \text{and} \qquad \frac{f \circ u \circ \mathsf{Fl}^v_s - f \circ u}{s} \to \mathrm{d}(f \circ u)(v) \tag{3.3.11}
$$

in $L^2(K)$ as $s \downarrow 0$. Thus writing (3.3.10) for $\gamma_s := \mathsf{Fl}^v_s(x)$ for $\mathfrak{m}$-a.e. $x \in \Omega$, dividing by s^2, letting $s \downarrow 0$ and recalling (3.3.6) and (3.3.11) we conclude that

$$
|\mathrm{d}u_t(v)|^2 \le e^{-2\lambda t g}\Big(|\mathrm{d}u(v)|^2 - 2t\,\mathrm{d}g(v)\,\mathrm{d}(f \circ u)(v) + Ct^2\Big) \qquad \mathfrak{m}\text{-}a.e. \text{ in } \Omega, \tag{3.3.12}
$$

having also used the arbitrariness of $K \subset \Omega$ compact and the fact that the Lipschitz constant of $t \mapsto \mathsf{Fl}^v_s(x)$ is bounded by $\|v\|_{L^\infty}$. We have established (3.3.12) for v regular, but both sides of the inequality are continuous w.r.t. L^0-convergence of uniformly bounded vectors v with values in $L^0_2(T\mathrm{X})$, thus by density we deduce that (3.3.12) is valid for any $v \in L^\infty(T\mathrm{X})$. Hence writing (3.3.12) for v varying in a local Hilbert base of $L^2(T\mathrm{X})$ (recall Theorem 2.3.3) and adding up we deduce (3.3.8). Then (3.3.9) also follows. □

In order to state the analogue of (3.3.5), we consider the notion of measured value Laplacian as defined in Section 1.4.2.

Theorem 3.3.15. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(K, N)$ *space,* Y *be* $\mathsf{CAT}(0)$ *and* $\Omega \subset \mathrm{X}$ *open and bounded. Also, let* $f \in \mathrm{Lip}(\mathrm{Y})$ *be* λ*-convex,* $\lambda \in \mathbb{R}$ *and* $u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y})$ *be harmonic.*

Then, $f \circ u \in D(\mathbf{\Delta}, \Omega)$ *and* $\mathbf{\Delta}(f \circ u)_{|\Omega}$ *is a Radon measure satisfying*

$$\mathbf{\Delta}(f \circ u)_{|\Omega} \geq \frac{\lambda}{d+2} |\mathrm{d}u|^2_{\mathsf{HS}} \mathfrak{m}. \tag{3.3.13}$$

Proof. As noticed before Lemma 3.3.14, under the stated assumptions we have $f \circ u \in W^{1,2}(\Omega)$. Now let $g \in \mathrm{Lip}_c(\Omega)^+$ be arbitrary and apply Lemma 3.3.14 with these functions f, g, u and define $u_t \in \mathsf{KS}^{1,2}_{\bar{u}}(\Omega, \mathrm{Y})$ accordingly. Notice that since $\mathrm{supp}(g) \subset \Omega$, we have that u_t and u agree on a neighbourhood of $\partial\Omega$ and thus have the same boundary value.

Therefore from the fact that u is harmonic and (3.3.9) we deduce

$$-\int_\Omega \langle \mathrm{d}(f \circ u), \mathrm{d}g \rangle \, \mathrm{d}\mathfrak{m} \geq \frac{\lambda}{d+2} \int_\Omega g |\mathrm{d}u|^2_{\mathsf{HS}} \, \mathrm{d}\mathfrak{m} \qquad \forall g \in \mathrm{Lip}_c(\Omega)^+$$

and the conclusion comes from (1.4.4). □

Corollary 3.3.16. *Let* $\Omega \subset \mathrm{X}$ *be open and bounded,* Y *be* $\mathsf{CAT}(0)$*,* $\bar{u} \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y})$*,* u *harmonic map with* $\bar{u}$ *as boundary values and* $f \in \mathrm{Lip}(\mathrm{Y})$ *be* 2*-convex. If* $f \circ u$ *is constant then* u *itself is constant map.*

Proof. Apply Theorem 3.3.15, then $|\mathrm{d}u|_{\mathsf{HS}}$ vanishes and conclude. □

Let us now discuss a simple and explicit example of Laplacian of a map.

Example 3.3.17. Let $\mathrm{Y} := \mathbb{R}^2$, $\mathrm{X} := \mathbb{R}/\mathrm{Z}$ equipped with the standard distances and measure, and $\Omega = \mathrm{X}$. Then a direct application of the definitions in Theorem 2.3.4 show that $u = (u_1, u_2) \colon \mathrm{X} \to \mathrm{Y}$ is in $\mathsf{KS}^{1,2}(\mathrm{X}, \mathrm{Y})$ if and only if $u_1 \circ \mathrm{p}, u_2 \circ \mathrm{p} \colon \mathbb{R} \to \mathbb{R}$ are in $W^{1,2}_{loc}(\mathbb{R})$, where $\mathrm{p} \colon \mathbb{R} \to \mathbb{R}/\mathrm{Z} = \mathrm{X}$ is the natural projection, with

$$\mathsf{E}^{\mathsf{KS}}(u) = \tfrac{c}{2}\Big(\int_{\mathrm{X}} |u_1'|^2(\theta) + |u_2'|^2(\theta) \, \mathrm{d}\theta\Big),$$

for some universal constant $c > 0$. Then it is clear that $u \in D(\Delta)$ if and only if $(u_1 \circ \mathrm{p})'', (u_2 \circ \mathrm{p})'' \in L^2_{loc}(\mathbb{R})$ and that in this case

$$\Delta u = c(u_1'', u_2'').$$

Now let $u(\theta) := (\cos(2\pi\theta), \sin(2\pi\theta))$ be the canonical embedding of X in Y. Then $\Delta u = -u$ and in particular for any $\theta \in \mathrm{X}$ we have that $\Delta u(\theta) \in \mathrm{T}_{u(\theta)}\mathbb{R}^2 \sim \mathbb{R}^2$ is orthogonal to the tangent space of X seen as a subset of $\mathbb{R}^2 = \mathrm{Y}$.

This is interesting because one can define the differential $\mathrm{d}u$ of u, even in very abstract situations [106], by means related to Sobolev calculus on the metric measure space $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mu := u_\sharp(|\mathrm{d}u|^2_{\mathsf{HS}}\mathfrak{m}))$ and tangent vector fields in this metric measure space only see directions which are tangent to the graph of u (this is rather obvious in this example, but see for instance [81] for a discussion of this phenomenon in more general cases). This means that, curiously, Δu cannot be computed starting from $\mathrm{d}u$ and using Sobolev calculus in the spirit developed in [97], [96], simply because Δu does not belong to the tangent module $L^2_\mu(T\mathrm{Y})$ ■

We conclude pointing out that while in the Definition 3.3.9 of Laplacian of a map we called into play the space $L^2(u^*\mathrm{T}_G\mathrm{Y}, \mathfrak{m}_{|\Omega})$ as introduced in Definition 3.3.3, in some circumstances it might be useful to deal with a notion of Laplacian related to the Borel σ-algebra $\mathcal{B}(u^*\mathrm{T}_G\mathrm{Y})$ - and thus to the characterization given in Proposition 3.3.8 -, which however is only available for separable spaces Y.

In this direction it is worth to underline that one can always reduce to such case thanks to the following two simple results: the first says that given $u \in L^2(\Omega, \mathrm{Y}_{\bar y})$ we can always find a separable $\mathsf{CAT}(0)$ subspace $\tilde{\mathrm{Y}}$ of Y containing the gradient flow trajectory of $\mathsf{E}^{\mathsf{KS}}_{\bar u}$ starting from u, the second ensures that this restriction does not affect the notion of minus-subdifferential.

Proposition 3.3.18. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K,N)$ space, $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \bar y)$ a pointed $\mathsf{CAT}(0)$-space, $\Omega \subset \mathrm{X}$ open, $\bar u \in \mathsf{KS}^{1,2}(\Omega, \mathrm{Y}_{\bar y})$ and $u \in L^2(\Omega, \mathrm{Y}_{\bar y})$. Also, let (u_t) be the gradient flow trajectory for $\mathsf{E}^{\mathsf{KS}}_{\bar u}$ starting from u.*

Then, there exists a separable $\mathsf{CAT}(0)$ subspace $\tilde{\mathrm{Y}} \subset \mathrm{Y}$ such that $\mathfrak{m}(u_t^{-1}(\mathrm{Y} \setminus \tilde{\mathrm{Y}})) = 0$ for every $t \geq 0$. Similarly for the functional E^{KS}.

Proof. From the fact that geodesics on Y are unique and vary continuously with the endpoint it is easy to see that the closed convex hull of a separable set (i.e. the smallest closed and convex set containing the given set) is also separable. Use this and the fact that maps in $L^2(\Omega, \mathrm{Y}_{\bar y})$ are by definition essentially separably valued to find $\tilde{\mathrm{Y}} \subset \mathrm{Y}$ which is $\mathsf{CAT}(0)$ with the induced metric and such that $\mathfrak{m}(u_t^{-1}(\mathrm{Y} \setminus \tilde{\mathrm{Y}})) = 0$ for every $t \in \mathbb{Q}^+$. We claim that $\tilde{\mathrm{Y}}$ satisfies the conclusion. To see this, pick $t \geq 0$, let $(t_n) \subset \mathbb{Q}^+$ be converging to t and up to pass to a non-relabeled subsequence assume that $\sum_n \mathsf{d}_{L^2}(u_{t_{n+1}}, u_{t_n}) < \infty$. Then from the triangle inequality in $L^2(\mathfrak{m}_{|\Omega})$ and the monotone convergence we see that $\|\sum_n \mathsf{d}_{\mathrm{Y}}(u_{t_{n+1}}, u_{t_n})\|_{L^2} \leq \sum_n \mathsf{d}_{L^2}(u_{t_{n+1}}, u_{t_n}) < \infty$ so that in particular for $\mathfrak{m}$-a.e. $x \in \Omega$ we have $\sum_n \mathsf{d}_{\mathrm{Y}}(u_{t_{n+1}}, u_{t_n})(x) < \infty$ which in turn implies that $(u_{t_n}(x)) \subset \tilde{\mathrm{Y}}$ is a Cauchy sequence, so that its limit $v(x)$ also belongs to $\tilde{\mathrm{Y}}$. The same kind of argument also shows that (u_{t_n}) converges to v in $L^2(\Omega, \mathrm{Y}_{\bar y})$ and since we know, by the continuity of (u_t) as $L^2(\Omega, \mathrm{Y}_{\bar y})$-valued curve, that $u_{t_n} \to u_t$ in $L^2(\Omega, \mathrm{Y}_{\bar y})$ we conclude that $u_t = v$, which proves our claim. □

To present our final result we need a bit of notation. Let Y be a $\mathsf{CAT}(0)$-space and $\tilde{\mathrm{Y}}$ a subspace which is also $\mathsf{CAT}(0)$ with the induced metric. Call $\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}} \colon \tilde{\mathrm{Y}} \to \mathrm{Y}$ the inclusion map. Then for every $y \in \tilde{\mathrm{Y}}$ the tangent space $\mathrm{T}_y\tilde{\mathrm{Y}}$ embeds isometrically into $\mathrm{T}_y\mathrm{Y}$ via the continuous extension of the map which sends $\alpha(\mathsf{G}_y^z)'_0 \in \mathrm{T}_y\tilde{\mathrm{Y}}$ to $\alpha(\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}(\mathsf{G}_y^z))'_0 \in \mathrm{T}_y\mathrm{Y}$. In other words, we can regard a geodesic in $\tilde{\mathrm{Y}}$ also as a geodesic in Y and this provides a canonical immersion of $\mathrm{T}_y\tilde{\mathrm{Y}}$ in $\mathrm{T}_y\mathrm{Y}$ which for trivial reasons is an isometry. Abusing a bit the notation we shall denote such isometry by $\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}$.

Proposition 3.3.19. *Let Y be a $\mathsf{CAT}(0)$-space, $\mathsf{E} \colon \mathrm{Y} \to \mathbb{R} \cup \{+\infty\}$ a λ-convex and lower semicontinuous functional, (y_t) a gradient flow trajectory for E starting from $y_0 \in \mathrm{Y}$ and $\tilde{\mathrm{Y}} \subset \mathrm{Y}$ a subset which is also a $\mathsf{CAT}(0)$-space with the induced metric and such that $(y_t) \subset \tilde{\mathrm{Y}}$. Denote by $\tilde{\mathsf{E}}$ the restriction of E to $\tilde{\mathrm{Y}}$*

Then, $-\partial^-\mathsf{E}(y_0) \neq \emptyset$ if and only if $-\partial^-\tilde{\mathsf{E}}(y_0) \neq \emptyset$ and letting $v, \tilde v$ be the respective elements of minimal norm we have $\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}(\tilde v) = v$. Moreover, (y_t) is also a gradient flow trajectory for $\tilde{E}$.

Proof. Assume that $-\partial^-\tilde{\mathsf{E}}(y_0) \neq \emptyset$. Then we know from Theorem 3.2.9 that $\frac{1}{h}(\mathsf{G}_{y_0}^{y_h})'_0 \to \tilde v$ as $h \downarrow 0$. Then clearly $\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}(\frac{1}{h}(\mathsf{G}_{y_0}^{y_h})'_0) \to \mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}(\tilde v)$ and thus by Theorem 3.2.9 to conclude it is sufficient to prove that $|\partial^-\mathsf{E}|(y_0) < \infty$, because in that case we have that $\mathcal{I}^{\mathrm{Y}}_{\tilde{\mathrm{Y}}}(\frac{1}{h}(\mathsf{G}_{y_0}^{y_h})'_0)$ converges to the element of minimal norm in $-\partial^-\mathsf{E}(y_0) \neq \emptyset$ (which in particular is not empty) as $h \downarrow 0$.

Since $\frac{1}{h}(\mathsf{G}_{y_0}^{y_h})'_0 \to \tilde v$ we have in particular that $\frac{\mathsf{d}_{\mathrm{Y}}(y_0, y_h)}{h} = |\frac{1}{h}(\mathsf{G}_{y_0}^{y_h})'_0|_{y_0} \to |v|_{y_0}$ and thus $S := \sup_{h \in (0,1)} \frac{\mathsf{d}_{\mathrm{Y}}(y_0, y_h)}{h} < \infty$. By the contractivity property (3.2.3) we deduce that

$$\sup_{t,h \in (0,1)} \frac{\mathsf{d}_{\mathrm{Y}}(y_t, y_{t+h})}{h} < (e^{\lambda} \vee 1) S =: S'$$

and thus letting $h \downarrow 0$ we deduce that $|\dot y_t^+| \leq S'$ for every $t \in (0,1)$. Taking into account (3.2.1) and the lower semicontinuity of the slope recalled in Lemma 1.1.8 we conclude.

Viceversa, assume that $-\partial^-\mathsf{E}(y_0) \neq \emptyset$. Then by Theorem 3.2.9 we know that $|\partial^-\mathsf{E}|(y_0) < \infty$ and since trivially we have $|\partial^-\tilde{\mathsf{E}}| \leq |\partial^-\mathsf{E}|_{|_{\tilde{\mathrm{Y}}}}$ we also have $|\partial^-\tilde{\mathsf{E}}|(y_0) < \infty$. Hence by Theorem 3.2.9 we deduce $-\partial^-\tilde{\mathsf{E}}(y_0) \neq \emptyset$ and the first part of the proof applies.

The last statement is a consequence of the first applied to y_t for every $t > 0$ and of Corollary 3.2.10. $\square$

4 Progress on the independence on p of weak gradients: a first order regularity condition

Contents

4.1 Introduction

There are two different reasons for which p-minimal weak upper gradients might depend on p:

i) in [82], a detailed study on weighted Euclidean spaces has been performed to build a family of metric measure spaces for which $|Df|_{p_1} < |Df|_{p_2}$ may occur for $p_1 \neq p_2$;

ii) in [20], an example due to P. Koskela has been reported showing that we may have:

$$f \in W^{1,p_1}(\mathrm{X}),\ f, |Df|_{p_1} \in L^{p_2}(\mathfrak{m}) \quad \Rightarrow \quad f \notin W^{1,p_2}(\mathrm{X}), \quad \forall p_2 > p_1.$$

On the other hand, we saw in Theorem 4.2.2 that Doubling & Poincaré ensures that i) never occurs but still, it is not clear if ii) may still happen under this circumstances. However, this it not the case on $\mathsf{RCD}(K,\infty)$ spaces as proved in Theorem 2.3.6, especially suggesting that two different kinds of independence are achievable.

In this Chapter, we pursue these two goals:

a) to propose an axiomatization of spaces having p-independent weak upper gradients in Definition 4.2.1, especially distinguishing between a *weak* and *strong* kind of independence;

b) to single out a regularity property at first order shared by a large class of spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ yielding p-independent weak upper gradients in a strong sense.

More in details, we are going to address b) by proposing a condition that we call Bounded Interpolation Property (BIP). Roughly, we require the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to have the following *first-order* constraint on Wasserstein geodesics: given a diameter $D > 0$, there exists a positive constant $C(D) > 0$ such that for any two probability measures $\rho_0\mathfrak{m}, \rho_1\mathfrak{m}$ whose supports are enclosed in a ball of diameter D and whose densitities are bounded, there exists a Wasserstein geodesic $t \mapsto \mu_t$ interpolating between the two, that is absolutely continuous with respect to $\mathfrak{m}$ and whose density is bounded at any time t in terms of $C(D), \|\rho_1\|_{L^\infty}, \|\rho_0\|_{L^\infty}$ (see Definition 4.2.3 below for the precise formulation). By means of a well known superposition principle in Optimal Transport, there are dynamical plans $\pi \in \mathscr{P}(C([0,1], \mathrm{X}))$ concentrated on geodesics of the space representing these interpolants. What is special about the class of plans given by the (BIP) is that they are test plans able to detect the Sobolev space: by this we mean that, to prove that a function f belong to $W^{1,p}(\mathrm{X})$, it suffices to check the Sobolev property only against this special collection of plans (for the precise statement see Proposition 4.3.4). Once we have this tool at hand, it can be easily seen that the *strong* independence on p of p-weak upper gradients will follow easily out of this analysis (see Theorem 4.3.5 below). The reason is that we have reduced significantly -to only geodesics- the number of curves entering into play in the definition of the object $|Df|_p$.

Finally, we enrich this analysis with:

i) the unification, under the (BIP), of the nonsmooth p-differential structure in the language of $L^0(\mathfrak{m})$-normed module in Theorem 4.4.1. We provide a construction of an *universal cotangent module*, together with a *universal differential*, that are independent on p;

ii) the proof in Theorem 4.3.8 that the (BIP) is stable under pmGH-convergence;

iii) the proof that a single plan can detect the Sobolev property. Indeed, we push the recent technique developed in [169] to produce a single plan $\boldsymbol{\pi}_q$, called master q-test plan, which is able to detect both the Sobolev space $W^{1,p}(\mathrm{X})$ and the minimal p-weak upper gradient (see Theorem 4.6.2). This provides a resolution of [169, Problem 2.7]. On BIP-spaces, this plan can be also taken independent on q and concentrated on geodesics (see Theorem 4.6.4);

iv) the proof in Section 4.5 of some spaces satisfying the (BIP). Among them, we can find a broad class of spaces satisfying synthetic curvature dimension conditions. We do this by reviewing in Appendix A the interpolation estimates of [174, 57] obtained in the 2-Wasserstein space, to arbitrary exponent $q \in (1, \infty)$ in the context of the CD and MCP spaces.

Structure of the Chapter. This Chapter is organized as follows:

In Section 4.2, we propose the main definitions of this Chapter, namely the axiomatization of weak and strong p-independent weak upper gradients with respect to p and a definition of a first order regularity condition of a metric measure space that we call Bounded Interpolation Property (BIP).

In Section 4.3, we then start our analysis concerning the (BIP) and prove that the stronger kind of independence is assured by this property. Finally, we conclude by showing that this condition is stable with respect to the pointed measure Gromov Hausdorff convergence.

In Section 4.4, we show that the p-differential calculus (in the language of normed modules [97]) in independent on p under the (BIP). In particular, we prove here that there is a unique notion of $L^0(\mathfrak{m})$ cotangent module where differential of Sobolev function lives independently of their integrability exponent.

In Section 4.5, we prove the sort of 'known spaces' verifying different kind of curvature dimension condition verify the (BIP).

We end this Chapter, in Section 4.6, with the construction of a master test plan for the Sobolev space, first on arbitrary metric measure space and then specifically under the (BIP).

4.2 Main definitions

In this Chapter, we will deal frequently with the Hölder duality between integrable exponent as they naturally enters into play when working with the definition of p-Sobolev functions and q-test plan. Hence, to lighten the presentation, we make once and for all the following clarification.

> **Notation**. Throughout this Chapter, it comes without saying that, whenever we fix $p \in [1, \infty)$, the letter q, even if not introduced, is automatically defined as the *conjugate exponent* by the relation
> $$\tfrac{1}{p} + \tfrac{1}{q} = 1$$
> and thus ranging in $(1, \infty]$ (with the usual extreme case $p = 1$ and $q = \infty$). The converse convention is also kept, i.e. if we start fixing a number q, then automatically p is defined as above. Typical recurrent exponents will be $p_1, p_2, \bar{p}$, thus giving rise to the conjugate numbers $q_1, q_2, \bar{q}$.

4.2.1 Weak and strong p-independent weak upper gradients

As already highlighted in (1.2.9), we shall only expect one inequality between minimal weak upper gradients with different p's. In light of the two different pathological situations described in the Introduction, we give the following definition.

Definition 4.2.1 (p-independent weak upper gradients)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. We say that it has p-independent weak upper gradients in the weak sense, provided for any* $p_1, p_2 \in (1, \infty)$*:*

a) $W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$ *is dense in both* $W^{1,p_1}(\mathrm{X})$ *and* $W^{1,p_2}(\mathrm{X})$*;*

b) for any $f \in W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$ *it holds* $|Df|_{p_1} = |Df|_{p_2}$ $\mathfrak{m}$*-a.e.;*

Moreover, we say that X *has p-independent weak upper gradients in the strong sense if we require* a)-b) *and*

c) any $f \in W^{1,p_1}(\mathrm{X})$ *with* $f, |Df|_{p_1} \in L^{p_2}(\mathfrak{m})$ *belongs to* $W^{1,p_2}(\mathrm{X})$*.*

A remarkable example of space with p-independent weak upper gradients in the weak sense has been already faced during this Thesis.

Theorem 4.2.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a doubling metric measure space supporting a weak local* $(1, 1)$*-Poincaré inequality. Then it has p-independent weak upper gradients in the weak sense.*

Proof. This is just a reformulation of Proposition 1.3.8 according the above axiomatization. □

4.2.2 The Bounded Interpolation Property

Here we present our first order regularity constraint over a metric measure space within its implication concerning Sobolev spaces. This will be done by imposing a special behavior of the spreading of mass along the geodesic of the space according to the next definition.

Definition 4.2.3 (Bounded interpolation property)**.** *We say that a complete and separable metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *has the* bounded interpolation property*, provided:*

for every $q \in (1, \infty)$ *there exists a* profile function $\mathbb{R}^+ \ni D \mapsto C_q(D) \in [1, \infty)$ *so that for every* $\mu_0, \mu_1 \in \mathscr{P}(\mathrm{X})$ *absolutely continuous with bounded densities and* $\mathrm{diam}(\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1)) < D$*, there exists* $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ *satisfying*

$$(\mathsf{e}_t)_\sharp \pi = \rho_t \mathfrak{m}, \quad \|\rho_t\|_{L^\infty} \leq C_q(D) \|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0, 1], \tag{BIP}$$

where $\rho_i := \frac{\mathrm{d}\mu_i}{\mathrm{d}\mathfrak{m}}$*.*

When this holds, we say for brevity that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a BIP-space, or that it has the (BIP) with profile function* $D \mapsto C_q(D)$*.*

This axiomatization is inspired by the results of [174], where the very same special behavior of mass transportation has been investigated under synthetic lower Ricci bounds. Such analysis was carried for the exponent $q = 2$, but we will see in Appendix A that it can be actually be performed for all $q \in (1, \infty)$. In this direction, [174, Theorem 4.1] ensures that the (BIP) yields the following:

A BIP-space supports a weak local $(1, 1)$-Poincaré inequality.

Thus we already know from Theorem 4.2.2 that if a BIP space is also doubling, then it has p-independent weak upper gradients in the weak sense. Our goal in this section is to show that, regardless of the doubling assumption, a BIP space has p-independent weak upper gradients in the strong sense. We then postpone to Section 4.5 the study of which sort of 'known' spaces satisfy (BIP).

Notice that, for every q, it is not restrictive to suppose the profile function $D \mapsto C_q(D)$ to be nondecreasing and continuous, thus we shall implicitly use these facts sometime. Moreover, when the profile function is independent on q, as it will be in all the cases faced in Section 4.5, we shall omit the subscript and simply write $D \mapsto C(D)$.

4.3 Independence of $|Df|_p$ under the (BIP)

For the sake of conciseness, we collect, for every $q \in (1, \infty)$ all the relevant interpolants in the class

$$\mathsf{Geod}_q(\mathrm{X}) := \left\{ \pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m}, \rho_1\mathfrak{m}) \colon \begin{array}{l} D > 0, \rho_0, \rho_1 \in L^\infty(\mathfrak{m}) \text{ probabilities} \\ \mathrm{diam}(\mathrm{supp}(\rho_0) \cup \mathrm{supp}(\rho_1)) \le D \\ (\mathsf{e}_t)_\sharp \pi \le C_q(D) \|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})} \mathfrak{m} \end{array} \right\}, \tag{4.3.1}$$

Notice the important fact that, no matter of the fixed exponent q, the defining property of this class ensures that

$$\text{any } \pi \in \mathsf{Geod}_q(\mathrm{X}) \text{ is an } \infty\text{-test plan, and thus also a } q'\text{-test plan for any } q' \in (1, \infty). \tag{4.3.2}$$

Indeed, every plan is concentrated on geodesics of the space whose lengths are controlled by above by some diameter. We shall also work with the 'polygonal' version $\mathsf{PolGeo}_q(\mathrm{X})$ of the above, defined as the set of plans $\pi \in \mathscr{P}(C([0, 1], \mathrm{X}))$ for which there are a finite Borel partition $(A_i)_{i=1,\ldots,N}$ of $C([0, 1], \mathrm{X})$ with $\alpha_i := \pi(A_i) > 0$ and, for every $j = 0, \ldots, m-1, m \in \mathbb{N}$ and $i = 1, \ldots, N$, we have $\alpha_i^{-1}(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp(\pi|_{A_i}) \in \mathsf{Geod}_q(\mathrm{X})$.

Lemma 4.3.1 (Approximation with polygonal plans)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a BIP-space,* $q \in (1, \infty)$ *and* π *a* q*-test plan. Then there are* $(\pi_n) \subset \mathscr{P}(C([0, 1], \mathrm{X}))$ *and* $(\pi_{n,m}) \subset \mathsf{PolGeo}_q(\mathrm{X})$*,* $n, m \in \mathbb{N}$*, such that:*

i) for every $n \in \mathbb{N}$*, we have*

- *a)* $\pi_{n,m} \rightharpoonup \pi_n$ *as* $m \to \infty$*;*
- *b)* $\overline{\lim}_{m\to\infty} \mathrm{Ke}_q(\pi_{n,m}) \le \mathrm{Ke}_q(\pi_n)$*;*
- *c) for some* $C(q, n) > 0$ *we have* $(\mathsf{e}_t)_\sharp \pi_{n,m} \le C(q, n)\mathfrak{m}$ *for every* $m \in \mathbb{N}$*, and* $t \in [0, 1]$*;*

ii) and moreover

- *a)* $\pi_n \rightharpoonup \pi$ *as* $n \to \infty$*;*
- *b)* $\overline{\lim}_{n\to\infty} \mathrm{Ke}_q(\pi_n) \le \mathrm{Ke}_q(\pi)$*;*
- *c) for some* $C(q) > 0$ *we have* $(\mathsf{e}_t)_\sharp \pi_n \le C(q)\mathfrak{m}$ *for every* $n \in \mathbb{N}$*, and* $t \in [0, 1]$*.*

Proof. Let us set for brevity $\mathrm{Y} := C([0,1],\mathrm{X})$ and assume at first that π has compact support so that its trace (recall (1.2.1)) $[\pi] = \{\gamma_t \colon \gamma \in \mathrm{supp}(\pi), t \in [0,1]\} \subset \mathrm{X}$ is also compact. We put $D := \mathrm{diam}([\pi]) < \infty$.

Case i). Let $m, n \in \mathbb{N}$ be fixed. Using the compactness of the support of π, find a finite Borel partition $(A_i)_{i=1,\dots,N_n}$ of its support made of sets with positive π-measure and diameter $\leq \frac{1}{n}$. For $i \in \{1,\dots,N_n\}$ put $\alpha^i := \pi(A_i)^{-1}\pi_{|_{A_i}}$ and then for $j \in \{0,\dots,m-1\}$ let $\beta^{i,j} \in \mathrm{OptGeo}_q((\mathsf{e}_{\frac{j}{m}})_\sharp\alpha^i, (\mathsf{e}_{\frac{j+1}{m}})_\sharp\alpha^i) \cap \mathsf{Geod}_q(\mathrm{X})$ given by the (BIP). With a gluing argument we can then find a plan $\boldsymbol{\beta}^i$ such that $(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp\boldsymbol{\beta}^i = \beta^{i,j}$ for every $j \in \{0,\dots,m-1\}$. We put $\pi_{n,m} := \sum_{i=1}^{N_n}\pi(A_i)\boldsymbol{\beta}^i$ and we notice that the construction and the BIP assumption easily grant that property (i-c) holds. Moreover, we claim

$$\mathrm{Ke}_q(\pi_{n,m}) \leq \mathrm{Ke}_q(\pi) \tag{4.3.3}$$

and to this aim we notice that

$$\begin{aligned}
\iint_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\boldsymbol{\beta}^i &= \sum_{j=0}^{m-1} \iint_{\frac{j}{m}}^{\frac{j+1}{m}} |\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\boldsymbol{\beta}^i \\
(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp\boldsymbol{\beta}^i = \beta^{i,j} \qquad &= \sum_{j=0}^{m-1} \iint_0^1 m^{q-1}|\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\beta^{i,j} \\
(1.1.11a) \qquad &= \sum_{j=0}^{m-1} m^{q-1} W_q^q\big((\mathsf{e}_{\frac{j}{m}})_\sharp\alpha^i, (\mathsf{e}_{\frac{j+1}{m}})_\sharp\alpha^i\big) \\
&\leq \sum_{j=0}^{m-1} m^{q-1} \int \mathsf{d}^q(\gamma_{\frac{j}{m}}, \gamma_{\frac{j+1}{m}}) \,\mathrm{d}\alpha^i \\
&\leq \sum_{j=0}^{m-1} m^{q-1} \int \Big(\int_{\frac{j}{m}}^{\frac{j+1}{m}} |\dot\gamma_t| \,\mathrm{d}t\Big)^q \,\mathrm{d}\alpha^i \\
\text{Jensen} \qquad &\leq \sum_{j=0}^{m-1} m^{q-1-\frac{q}{p}} \iint_{\frac{j}{m}}^{\frac{j+1}{m}} |\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\alpha^i = \iint_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\mathrm{d}\alpha^i,
\end{aligned}$$

for all $i = 0, \dots, N_n$. This in particular grants that $(\pi_{n,m})_m$ is a sequence of q-test plans with uniformly bounded q-kinetic energy and compression. We are going now to produce a weak limit π_n, arguing by tightness.

Fix t, let $j := j(t,m)$ so that $t \in [j/m, (j+1)/m]$ and, using that $\pi_{n,m} \in \mathsf{PolGeo}_q(\mathrm{X})$, we estimate

$$\begin{aligned}
W_q((\mathsf{e}_t)_\sharp\pi_{n,m}, (\mathsf{e}_t)_\sharp\pi) &\leq W_q((\mathsf{e}_t)_\sharp\pi_{n,m}, (\mathsf{e}_{\frac{j}{m}})_\sharp\pi_{n,m}) + W_q((\mathsf{e}_{\frac{j}{m}})_\sharp\pi, (\mathsf{e}_t)_\sharp\pi) \\
&\leq \Big(\int \mathsf{d}^q(\gamma_{\frac{j}{m}}, \gamma_t)\,\mathrm{d}\pi_{n,m}\Big)^{1/q} + \Big(\int \mathsf{d}^q(\gamma_{\frac{j}{m}}, \gamma_t)\,\mathrm{d}\pi\Big)^{1/q} \\
AC^q\text{-supported} \qquad &\leq \Big(\int \Big(\int_{\frac{j}{m}}^t |\dot\gamma_t|\,\mathrm{d}t\Big)^q \mathrm{d}\pi_{n,m}\Big)^{1/q} + \Big(\int \Big(\int_{\frac{j}{m}}^t |\dot\gamma_t|\,\mathrm{d}t\Big)^q \mathrm{d}\pi\Big)^{1/q} \\
\text{Hölder and (4.3.3)} \qquad &\leq 2m^{\frac{1}{pq}}\mathrm{Ke}_q^{1/q}(\pi).
\end{aligned} \tag{4.3.4}$$

Taking into account that $\{(\mathsf{e}_t)_\sharp\pi : t \in [0,1]\}$ is W_q-compact (because π has finite q-energy and thus $t \mapsto (\mathsf{e}_t)_\sharp\pi \in (\mathscr{P}_q(\mathrm{X}), W_q)$ is continuous), this last estimate ensures that $\{(\mathsf{e}_t)_\sharp\pi_{n,m} : t \in [0,1], m \in \mathbb{N}\}$ is W_q-precompact for every $n \in \mathbb{N}$. In particular such set is tight, and thus by Prokhorov's Theorem 1.1.1 there exists a function $\psi : \mathrm{X} \to \mathbb{R}$ with compact sublevels such that

$$\sup_{t \in [0,1], m \in \mathbb{N}} \int \psi \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_{n,m} < \infty. \tag{4.3.5}$$

Now, consider the functional $\Psi : C([0,1],\mathrm{X}) \to \mathrm{X}$ defined by

$$\Psi(\gamma) := \int_0^1 \psi(\gamma_t) + |\dot\gamma_t|^q \,\mathrm{d}t, \quad \text{if } \gamma \in AC^q([0,1],\mathrm{X}), \quad +\infty \text{ otherwise}$$

and notice that it has compact sublevels as well thanks to 1.1.4. By construction we have

$$\sup_m \int \Psi \,\mathrm{d}\pi_{n,m} \le \sup_{t\in[0,1],m\in\mathbb{N}} \Big(\int \psi \,\mathrm{d}(\mathsf{e}_t)_\sharp \pi_{n,m} + \mathrm{Ke}_q(\pi_{n,m}) \Big) \overset{(4.3.3),(4.3.5)}{<} \infty.$$

Again, by Prokhrov's Theorem, we conclude that $(\pi_{n,m})_m$ is tight family and, up to a not relabeled subsequences, we get the existence of a weak limit π_n as m goes to infinity. We thus obtained (i-a). Also, from (4.3.4) we get

$$(\mathsf{e}_t)_\sharp \pi_n = (\mathsf{e}_t)_\sharp \pi, \qquad t \in [0,1]. \tag{4.3.6}$$

Now notice that (4.3.6) ensures *a posteriori* $(\pi_{n,m})_m$ to be also a *polygonal interpolation* of π_n (recall that $\pi_{n,m}$ was built freezing marginals of $(\mathsf{e}_t)_\sharp\pi$ on a uniform time grid) whence (4.3.3) here reads $\mathrm{Ke}_q(\pi_{n,m}) \le \mathrm{Ke}_q(\pi_n)$ for every $m \in \mathbb{N}$. Taking now the limsup yields (i-b).
Case ii). We immediately notice that (4.3.6) ensures (ii-c) with $C = \mathrm{Comp}(\pi)$. Next, we show (ii-a) and, to this aim, we remark that $\Delta^n := \sum_{i=0}^{N_n} \pi_n|_{A_i} \otimes \pi|_{A_i} \in \mathrm{Adm}(\pi_n,\pi)$ by construction. Then, we can estimate

$$\mathcal{W}_q^q(\pi^n,\pi) \le \int_{\mathrm{Y}\times\mathrm{Y}} \mathsf{d}^q_{\sup}(\gamma,\theta)\,\mathrm{d}\Delta^n(\gamma,\theta) = \sum_{i=0}^{N_n} \int_{A_i\times A_i} \mathsf{d}^q_{\sup}(\gamma,\theta)\,\mathrm{d}\pi_n(\gamma)\pi(\theta) \le \tfrac{1}{n^q},$$

where, evidently, we used that for $\pi_n\otimes\pi$-a.e. $(\gamma,\theta) \in A_i \times A_i$ we have $\mathsf{d}_{\sup}(\gamma,\theta) \le \frac{1}{n}$ due to the uniform bound of the diameter of A_i. This clearly implies (ii-a). But now, arguing again by weak lower semicontinuity (1.2.2), we conclude recalling (i-b) and (4.3.3) that

$$\varlimsup_{n\to\infty} \mathrm{Ke}_q(\pi_n) \le \varlimsup_{n\to\infty} \varliminf_{m\to\infty} \mathrm{Ke}_q(\pi_{n,m}) \le \mathrm{Ke}_q(\pi),$$

that is (ii-b).
Reduction step. In this final step, we relieve the proof of the Lemma of assumption π supported on a compact set. Being π a probability measure on the complete and separable space Y, it is concentrated on a sigma-compact set. Let then $\Gamma_k \subset \mathrm{Y}$ be compact so that $\pi(\cup_k\Gamma_k) = 1$, and consider, for every $k \in \mathbb{N}$, the plans $\pi^k := \pi(\cup_{i\le k}\Gamma_i)^{-1}\pi|_{\cup_{i\le k}\Gamma_i}$. They are clearly of compact support, so that we can apply i)-ii) to produce the sequences $\pi^k_{n,m}$ and π^k_n satisfying all the listed properties. Now, a diagonalization argument in k and n gives the conclusion.

□

Before passing to the next Lemma, let us comment the strategy behind the above proof. The idea of approximation via a polygonal argument is due to [142]. However, differently from there, on BIP-spaces the *tightness* step, i.e. sending n to infinity, is not a delicate issue here thanks to the defining compression properties of the plans $\mathsf{Geod}_q(\mathrm{X})$.

Lemma 4.3.2. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space,* $q \in (1,\infty)$ *and* $(\pi_n) \subset \mathscr{P}(C([0,1],\mathrm{X}))$ *be a sequence such that* $\pi_n \rightharpoonup \pi$ *as* n *goes to infinity for some* q*-test plan* $\pi \in \mathscr{P}(C([0,1],\mathrm{X}))$*. Assume that*

$$(\mathsf{e}_t)_\sharp\pi_n \le C\mathfrak{m}, \qquad \forall n \in \mathbb{N},\ t \in [0,1], \tag{4.3.7}$$

and that

$$\varlimsup_{n\to\infty} \mathrm{Ke}_q(\pi_n) \le \mathrm{Ke}_q(\pi). \tag{4.3.8}$$

Then for every $G \in L^p(\mathfrak{m})$ *we have*

$$\lim_{n\to\infty} \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi_n = \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi. \tag{4.3.9}$$

Proof. Let $\mathsf{d}' := \mathsf{d}_{\sup} \vee 1$ and $\mathcal{W}_q$ be the q-Wasserstein distance induced by d'. Thus $\mathcal{W}_q(\pi, \pi_n) \to 0$ as n goes to infinity because d' is a bounded distance equivalnt to the original one.

We write again for brevity $\mathrm{Y} := C([0,1], \mathrm{X})$ and consider, for every $n \in \mathbb{N}$, first the plans $\beta^n \in \mathrm{Opt}_q(\pi, \pi_n)$ and then, using repeatedly a gluing argument, the plan $\boldsymbol{\beta}^n \in \mathscr{P}\mathrm{Y} \times \mathrm{Y}^n$ so that

$$(P^{0,n})_\sharp \boldsymbol{\beta}^n = \beta^n \quad \text{and} \quad (P^{0,1,\dots,n-1})_\sharp \boldsymbol{\beta}^n = \boldsymbol{\beta}^{n-1}.$$

Kolmogorov's Theorem ensures the existence of $\boldsymbol{\beta} \in \mathscr{P}(\mathrm{Y} \times \mathrm{Y}^{\mathbb{N}})$ so that $(P^{0,1,\dots,n})_\sharp \boldsymbol{\beta} = \boldsymbol{\beta}^n$ for all $n \in \mathbb{N}$. Thanks to the assumptions, we can write

$$0 = \lim_{n\to\infty} \mathcal{W}_q^q(\pi, \pi_n) = \lim_n \int_{\mathrm{Y}\times\mathrm{Y}} (\mathsf{d}')^q(\gamma^0, \gamma^n)\, \mathrm{d}\beta^n(\gamma^0, \gamma^n) = \lim_n \int_{\mathrm{Y}\times\mathrm{Y}^{\mathbb{N}}} (\mathsf{d}')^q(P^0(\boldsymbol{\gamma}), P^n(\boldsymbol{\gamma}))\, \mathrm{d}\boldsymbol{\beta}(\boldsymbol{\gamma}).$$

Therefore, one gets that, up to a not relabeled subsequence, $P^n(\boldsymbol{\gamma}) \to P^0(\boldsymbol{\gamma})$ uniformly for $\boldsymbol{\beta}$-a.e. $\boldsymbol{\gamma}$. Now let $f_n(\boldsymbol{\gamma}, t) := |\dot\gamma_t^n|$ and $g_n(\boldsymbol{\gamma}) := \int_0^1 f_n^q(\boldsymbol{\gamma}, t)\, \mathrm{d}t = \mathsf{Ke}_q(\gamma^n)$ and similarly f, g. Notice that (4.3.8) reads as $\overline{\lim} \int g_n \,\mathrm{d}\boldsymbol{\beta} \le \int g \,\mathrm{d}\boldsymbol{\beta}$ and the lower semicontinuity of the q-kinetic energy ensures that $\underline{\lim}\, g_n(\boldsymbol{\gamma}) \ge g(\boldsymbol{\gamma})$ for $\boldsymbol{\beta}$-a.e. $\boldsymbol{\gamma}$. Hence the simple Lemma 4.3.3 below ensures that $g_n \to g$ in $L^1(\boldsymbol{\beta})$ and thus, up to a non-relabeled subsequence, also $\boldsymbol{\beta}$-a.e.. Thus by Lemma 1.1.3 we deduce that for $\boldsymbol{\beta}$-a.e. $\boldsymbol{\gamma}$ we have $f_n(\boldsymbol{\gamma}, \cdot) \to f(\boldsymbol{\gamma}, \cdot)$ in $L^q(0,1)$ and thus also in measure. By Fubini's theorem we then see that $f_n \to f$ in measure (w.r.t. $\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]}$). Now observe that (4.3.8) (and the identity $\|f_n\|_{L^q} = \mathsf{Ke}_q(\pi_n)$) grants that (f_n) is bounded in $L^q(\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]})$ and what we just proved shows that any weak limit must coincide a.e. with f, i.e. $f_n \rightharpoonup f$ in $L^q(\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]})$. Using again (4.3.8) and the uniform convexity of L^q we conclude that

$$f_n \to f \quad \text{in } L^q(\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]}). \tag{4.3.10}$$

Putting $\hat G_n(\boldsymbol{\gamma}, t) := G(\gamma_t^n)$ and analogously $\hat G(\boldsymbol{\gamma}, t) := G(\gamma_t)$, we then see that to conclude it is sufficient to show that $\hat G_n \to \hat G$ in $L^p(\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]})$. This is obvious by dominated convergence if $G \in C_b(\mathrm{X})$, thus the conclusion will follow if we show that the linear maps $L^p(\mathrm{X}, \mathfrak{m}) \ni G \mapsto \hat G_n, \hat G \in L^p(\boldsymbol{\beta} \times \mathscr{L}^1|_{[0,1]})$ are uniformly continuous. This follows from (4.3.7), which give

$$\iint_0^1 |\hat G_n|^p \,\mathrm{d}t\, \mathrm{d}\boldsymbol{\beta} = \int_0^1 \int |G|^p(\cdot, t)\, \mathrm{d}\pi_n\, \mathrm{d}t = \int_0^1 \int |G|^p \,\mathrm{d}(\mathsf{e}_t)_\sharp \pi_n \,\mathrm{d}t \le C \int |G|^p \,\mathrm{d}\mathfrak{m},$$

and the analogous estimates for π. $\square$

Lemma 4.3.3. *Let μ be a Borel probability measure on a Polish space* Y*, and $f_n, f : \mathrm{Y} \to [0,\infty]$, $n \in \mathbb{N}$, Borel such that*

$$f(y) \le \underline{\lim}_{n\to\infty} f_n(y), \qquad \textit{and} \qquad \overline{\lim}_{n\to\infty} \int f_n \,\mathrm{d}\mu \le \int f \,\mathrm{d}\mu < \infty.$$

Then $f_n \to f$ in $L^1(\mu)$.

Proof. Let $g := \underline{\lim}_{n\to\infty} f_n$ and $g_n := \inf_{k\ge n} f_k$, so that the monotone convergence theorem and the assumptions give $\|g_n - g\|_{L^1(\mu)} \to 0$. Also, we have

$$\overline{\lim}_{n\to\infty} \|f_n - g_n\|_{L^1(\mu)} = \overline{\lim}_{n\to\infty} \int f_n - g_n \,\mathrm{d}\mu \le \int f - g \,\mathrm{d}\mu \le 0,$$

forcing in particular $f = g$ μ-a.e.. The conclusion follows. $\square$

Thanks to these approximation result we get the following:

Proposition 4.3.4. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a BIP-space, $p \in (1, \infty)$, $f \colon \mathrm{X} \to \mathbb{R}$ Borel and $G \in L^p(\mathrm{X})$ positive. Then, the following are equivalent:*

i) $f \in S^p(\mathrm{X})$ *and* G *is a* p*-weak upper gradient;*

ii) *the inequality*

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\pi \tag{4.3.11}$$

holds for any $\pi \in \mathsf{Geod}_q(\mathrm{X})$.

Proof. The implication i) $\Rightarrow$ ii) is obvious, so we are left to show the converse. We start noticing that (4.3.11) holds for any $\pi \in \mathsf{PolGeo}_q(\mathrm{X})$. Indeed, for A_i, α_i, N, m as in the definition of $\mathsf{PolGeo}_q(\mathrm{X})$ we have

$$\begin{aligned}
\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi &\leq \sum_{i=1}^{N}\sum_{j=0}^{m-1} \int |f(\gamma_{\frac{j+1}{m}}) - f(\gamma_{\frac{j}{m}})| \,\mathrm{d}\pi|_{A_i} \\
&= \sum_{i=1}^{N}\sum_{j=0}^{m-1} \alpha_i \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\big(\alpha_i^{-1}(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp(\pi|_{A_i})\big) \\
&\overset{*}{\leq} \sum_{i=1}^{N}\sum_{j=0}^{m-1} \alpha_i \iint_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\big(\alpha_i^{-1}(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp(\pi|_{A_i})\big) \\
&= \iint_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\pi
\end{aligned}$$

having used the fact that $\alpha_i^{-1}(\mathsf{Restr}_{\frac{j}{m}}^{\frac{j+1}{m}})_\sharp(\pi|_{A_i})$ is in $\mathsf{Geod}_q(\mathrm{X})$ and the assumption ii) in the starred inequality.

The conclusion now comes by approximation. Let π be an arbitrary q-test plan and assume for the moment that $f(\gamma_1) - f(\gamma_0)$ has the same sign for π-a.e. γ, say non-negative (otherwise replace π with $(\mathsf{Restr}_1^0)_\sharp\pi$ and notice that (4.3.11) is unaffected). Let $(\pi_{n,m}), (\pi_n)$ be given by Lemma 4.3.1, put $f^k := (-k) \vee f \wedge k$ for $k \in \mathbb{N}$ and notice that $f^k(\gamma_1) - f^k(\gamma_0) \geq 0$ for π-a.e. γ. The fact that $f^k \in L^\infty(\mathfrak{m})$ and the compression bounds given by Lemma 4.3.1 give that

$$\lim_{n\to\infty}\lim_{m\to\infty} \int f^k \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_{n,m} = \lim_{n\to\infty} \int f^k \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_n = \int f^k \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi, \qquad \forall t \in [0,1],$$

therefore by monotone convergence we get

$$\begin{aligned}
\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi &= \lim_{k\to\infty} \int |f^k(\gamma_1) - f^k(\gamma_0)| \,\mathrm{d}\pi \\
&= \lim_{k\to\infty} \int f^k(\gamma_1) - f^k(\gamma_0) \,\mathrm{d}\pi \\
&= \lim_{k\to\infty} \Big(\int f^k \,\mathrm{d}(\mathsf{e}_1)_\sharp\pi - \int f^k \,\mathrm{d}(\mathsf{e}_0)_\sharp\pi \Big) \\
&= \lim_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} \Big(\int f^k \,\mathrm{d}(\mathsf{e}_1)_\sharp\pi_{n,m} - \int f^k \,\mathrm{d}(\mathsf{e}_0)_\sharp\pi_{n,m} \Big) \\
&\leq \varliminf_{k\to\infty}\varliminf_{n\to\infty}\varliminf_{m\to\infty} \int |f^k(\gamma_1) - f^k(\gamma_0)| \,\mathrm{d}\pi_{n,m} \\
&\leq \varliminf_{n\to\infty}\varliminf_{m\to\infty} \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi_{n,m} \\
&\leq \varliminf_{n\to\infty}\varliminf_{m\to\infty} \iint_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\pi_{n,m},
\end{aligned}$$

where in the last step we used the fact that $\pi_{n,m} \in \mathsf{PolGeo}_q(\mathrm{X})$ and what previously proved. To conclude we apply Lemma 4.3.2 first as $m \to \infty$ and then as $n \to \infty$.

If both $A^+ := \{\gamma : f(\gamma_1) - f(\gamma_0) \geq 0\}$ and $A^- := \{\gamma : f(\gamma_1) - f(\gamma_0) < 0\}$ have positive π-measure, we apply the above to the q-test plans $\pi^\pm := \pi(A^\pm)^{-1}\pi_{|_{A^\pm}}$ observing that

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\pi = \pi(A^+) \int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\pi^+ + \pi(A^-) \int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\pi^-.$$

The conclusion follows. □

We now come to the main result of the section, which is also the main reason behind the definition of BIP spaces:

Theorem 4.3.5. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a BIP-space and let $p_1, p_2 \in (1, \infty)$. Suppose $f \in S^{p_1}_{loc}(\mathrm{X})$ is such that $|Df|_{p_1} \in L^{p_2}_{loc}(\mathfrak{m})$.*

Then, $f \in S^{p_2}_{loc}(\mathrm{X})$ and

$$|Df|_{p_1} = |Df|_{p_2}, \qquad \mathfrak{m}\text{-a.e.}.$$

Proof. Assume for a moment that $f \in S^{p_1}(\mathrm{X})$ and $|Df|_{p_1} \in L^{p_2}(\mathfrak{m})$. Then we know that (4.3.11) holds for every q_1-test plan with $G := |Df|_{p_1}$, and thus, recalling (4.3.2), also for every plan in $\mathsf{Geod}_{q_2}(\mathrm{X})$. Hence Proposition 4.3.4 tells that $f \in S^{p_2}(\mathrm{X})$ with $|Df|_{p_2} \leq |Df|_{p_1}$ $\mathfrak{m}$-a.e..

In the general case we pick $k \in \mathbb{N}$ and $\eta \in \mathrm{Lip}_{bs}(\mathrm{X})$, define the truncated function $f^k := (-k) \vee f \wedge k$ and then consider ηf^k. The Leibniz rule (1.2.10) (which is trivially valid also for locally Sobolev functions) gives

$$|D(\eta f^k)|_{p_1} \leq |\eta||Df|_{p_1} + |D\eta|k \quad \in L^{p_1} \cap L^{p_2}(\mathfrak{m}). \tag{4.3.12}$$

Thus $\eta f^k \in S^{p_1}(\mathrm{X})$ with $|D(\eta f^k)|_{p_1} \in L^{p_2}(\mathfrak{m})$ and the previous argument applies to conclude that $\eta f^k \in S^{p_2}(\mathrm{X})$ with $|D(\eta f^k)|_{p_2}$ bounded by the right hand side of (4.3.12). By the very Definition 1.2.12 this means that $f \in S^{p_2}_{loc}(\mathrm{X})$ with $|Df|_{p_2} \leq |Df|_{p_1}$ $\mathfrak{m}$-a.e..

Now we can swap p_1 and p_2 to get that equality, and thus the conclusion, holds. □

As a direct implication of the above, we can treat analogously the case of minimal weak upper gradient for Sobolev maps.

Proposition 4.3.6. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a BIP-space, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ be a complete metric space and let $p_1, p_2 \in (1, \infty)$. Then, for every $u \in S^{p_1}(\mathrm{X}, \mathrm{Y})$ with $|Du|_{p_1} \in L^{p_2}(\mathfrak{m})$, we have*

$$u \in S^{p_2}(\mathrm{X}, \mathrm{Y}) \quad \text{and} \quad |Du|_{p_1} = |Du|_{p_2} \qquad \mathfrak{m}\text{-a.e.}.$$

Proof. Simply notice that, the assumption and Theorem 4.3.5 yields that for every $\phi \in \mathrm{Lip}(\mathrm{Y})$, $\varphi \circ u \in S^{p_2}(\mathrm{X})$ and $|D(\varphi \circ u)|_{p_1} = |D(\varphi \circ u)|_{p_2}$ $\mathfrak{m}$-a.e.. Finally,

$$|Du|_{p_1} \overset{(1.2.13)}{=} \operatorname{ess\,sup} |D(\varphi \circ u)|_{p_1} = \operatorname{ess\,sup} |D(\varphi \circ u)|_{p_1} \overset{(1.2.13)}{=} |Du|_{p_1}, \quad \mathfrak{m}\text{-a.e.}$$

as φ varies in $\mathrm{Lip}(\mathrm{Y})$ with $\mathrm{Lip}(\varphi) \leq 1$, and this concludes the proof. □

4.3.1 Stability of (BIP)

We aim at proving the stability of the (BIP) under pointed measured Gromov Hausdorff convergence as defined in Definition 1.1.11

The stability of the (BIP) is a consequence of the following simple compactness property of dynamical optimal test plans under pmGH-convergence:

Lemma 4.3.7. *Let $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$ as in Definition 1.1.11, $R > 0$ and $q \in (1, \infty)$. For every $n \in \mathbb{N}$, and $i = 0, 1$, let $\mu_i^n \in \mathscr{P}_q(\mathrm{X}_n)$ be with $\mathrm{supp}(\mu_i^n) \subseteq B_R(x_n)$ and $\pi^n \in \mathrm{OptGeo}_q(\mu_0^n, \mu_1^n)$. Assume that $\overline{\lim}_{n\to\infty} \mathrm{Comp}(\pi^n) < \infty$.*

Then $(\pi^n) \subset \mathscr{P}(C([0,1], \mathrm{Z}))$ is tight and for any weak limit π along some subsequence $n_k \uparrow +\infty$ we have

i) $\mu_i^{n_k} \rightharpoonup \mu_i$, $i = 0, 1$, *for some* $\mu_i \in \mathscr{P}(\mathrm{X}_\infty) \subset \mathscr{P}(\mathrm{Z})$ *with support contained in* $\bar{B}_R(x_\infty)$*,*

ii) $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ *and* $\lim_{n\to\infty} \mathrm{Ke}_q(\pi^n) = \mathrm{Ke}_q(\pi)$*;*

iii) $\mathrm{Comp}(\pi) \le \overline{\lim}_{n\to\infty} \mathrm{Comp}(\pi^n)$*.*

Proof. From (1.1.11a) and the fact that the measures μ_i^n have uniformly bounded supports in $\mathscr{P}(\mathrm{Z})$ and arguing as in the proof of Lemma 4.3.1 we see that tightness will follow if we show that $\{(\mathsf{e}_t)_\sharp \pi^n : t \in [0,1], n \in \mathbb{N}\}$ is tight. To see this, let $\eta : \mathrm{Z} \to [0,1]$ be Lipschitz with bounded support and identically 1 on $B_{R+1}(x_\infty) \subset \mathrm{Z}$. Since $x_n \to x_\infty$ we see that $\int \eta \, \mathrm{d}\mathfrak{m}_n > 0$ for every n sufficiently big, hence the definition

$$\tilde{\mathfrak{m}}_n := \tfrac{1}{z_n} \eta \mathfrak{m}_n \in \mathscr{P}(\mathrm{Z}), \qquad z_n := \int \eta \, \mathrm{d}\mathfrak{m}_n,$$

is well posed for every $n \in \mathbb{N}$ sufficiently big and we have $\tilde{\mathfrak{m}}_n \rightharpoonup \tilde{\mathfrak{m}}_\infty$ in duality with $C_b(\mathrm{Z})$. In particular, $(\tilde{\mathfrak{m}}_n)$ is a tight sequence. Now fix $\varepsilon > 0$ and find $K \subset \mathrm{Z}$ compact such that $\overline{\lim}_n \tilde{\mathfrak{m}}_n(\mathrm{Z} \setminus K) < \varepsilon$. Then observe that for every $n \in \mathbb{N}$ big enough we have $\mathrm{supp}((\mathsf{e}_t)_\sharp \pi^n) \subset \{\eta = 1\}$ and thus

$$(\mathsf{e}_t)_\sharp \pi^n \le \mathrm{Comp}(\pi^n) \tilde{\mathfrak{m}}_n, \qquad \text{for every } t \in [0,1] \text{ and } n \text{ big enough.} \tag{4.3.13}$$

Hence for any $S > \overline{\lim}_n \mathrm{Comp}(\pi^n)$ we have

$$\overline{\lim_{n\to\infty}} \sup_{t\in[0,1]} (\mathsf{e}_t)_\sharp \pi^n(\mathrm{Z} \setminus K) \le \overline{\lim_{n\to\infty}} \mathrm{Comp}(\pi^n) \tilde{\mathfrak{m}}_n(\mathrm{Z} \setminus K) \le S\varepsilon,$$

proving the desired tightness.

Now say that $\pi^{n_k} \rightharpoonup \pi \in \mathscr{P}(C([0,1], \mathrm{Z}))$. Then $(\mathsf{e}_t)_\sharp \pi^n \rightharpoonup (\mathsf{e}_t)_\sharp \pi$ for every $t \in [0,1]$ (in particular i) holds), thus passing to the limit in (4.3.13) we obtain

$$(\mathsf{e}_t)_\sharp \pi \le S \tilde{\mathfrak{m}}_\infty \le S \mathfrak{m}_\infty, \qquad \forall t \in [0,1] \text{ and } S > \overline{\lim_n} \mathrm{Comp}(\pi^n),$$

thus iii) holds. To see ii) notice that since the measures μ_i^n have uniformly bounded support, the weak convergence $\mu_i^{n_k} \rightharpoonup (\mathsf{e}_i)_\sharp \pi$ implies W_q-convergence, thus recalling the characterization (1.1.11a) of optimal geodesic plans we have

$$\mathsf{Ke}_q(\pi) \le \underline{\lim_{k\to\infty}} \mathsf{Ke}_q(\pi^{n_k}) = \underline{\lim_{k\to\infty}} W_q^q(\mu_0^{n_k}, \mu_1^{n_k}) = W_q^q(\mu_0, \mu_1)$$

and the conclusion follows. □

We come to the actual stability result:

Theorem 4.3.8 (pmGH-stability of (BIP))**.** *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$*,* $n \in \bar{\mathbb{N}}$*, be a sequence of pointed metric measure spaces with* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n) \overset{pmGH}{\to} (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$*. Suppose* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ *satisfies the (BIP) with profile function* $D \mapsto C_q^n(D)$ *for all* $n \in \mathbb{N}$ *and there exist non increasing assignments* $D \mapsto C_q(D)$ *so that* $\overline{\lim}_n C_q^n(D) \le C_q(D) < \infty$ *for every* $D > 0, q \in (1, \infty)$*.*

Then $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ *has the (BIP) with profile function* $D \mapsto C_q(D)$*.*

Proof. We subdivide the proof in two steps.
Step 1. Let $q \in (1, \infty)$, $\mu_\infty = \rho_\infty \mathfrak{m}_\infty \in \mathscr{P}(\mathrm{X}_\infty) \subset \mathscr{P}(\mathrm{Z})$ be with bounded support and $A \subset \mathrm{Z}$ open bounded with $\mathsf{d}(\mathrm{supp}(\mu_\infty), \mathrm{Z} \setminus A) > 0$. We claim that there is a sequence $n \mapsto \mu_n = \rho_n \mathfrak{m}_n \in \mathscr{P}(\mathrm{X}_n) \subset \mathscr{P}(\mathrm{Z})$ W_q-converging to μ_∞ with

$$\overline{\lim_n} \|\rho_n\|_{L^\infty(\mathfrak{m}_n)} \le \|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)} \tag{4.3.14}$$

such that $\mathrm{supp}(\mu_n) \subset A$ for every n sufficiently big.

To see this, let $\eta : \mathrm{Z} \to [0,1]$ be continuous, identically 1 on $\mathrm{supp}(\mu)$ and with support contained in A. Put

$$\tilde{\mathfrak{m}}_n := \frac{1}{z_n}\eta\mathfrak{m}_n, \qquad \text{where} \qquad z_n := \int \eta\,\mathrm{d}\mu_n$$

and similarly $\mathfrak{m}_\infty$. Notice that the assumptions on μ_∞ ensure that $z_\infty > 0$, so that $\tilde{\mathfrak{m}}_\infty$ is well defined, and thus the pmGH-convergence grants that $z_n > 0$ for every n sufficiently big, so that for these n's the probability measures $\tilde{\mathfrak{m}}_n \in \mathscr{P}(\mathrm{Z})$ are well defined and weakly converge to $\tilde{\mathfrak{m}}_\infty$ in duality with $C_b(\mathrm{Z})$. In the forthcoming discussion we will neglect the small n's and think the $\tilde{\mathfrak{m}}_n$'s to be defined for every $n \in \mathbb{N}$.

By construction $\mathrm{supp}(\tilde{\mathfrak{m}}_n) \subset A$ for every $n \in \mathbb{N}$ and since A is bounded we deduce that $W_q(\tilde{\mathfrak{m}}_n, \tilde{\mathfrak{m}}_\infty) \to 0$ as $n \to \infty$. Let $\alpha^n \in \mathrm{Opt}_q(\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n)$, and define

$$\mu_n := P^2_\sharp \beta^n, \qquad \text{where} \qquad \mathrm{d}\beta^n(x,y) := \frac{\mathrm{d}\mu_\infty}{\mathrm{d}\tilde{\mathfrak{m}}_\infty}(x)\,\mathrm{d}\alpha^n(x,y) \in \mathscr{P}(\mathrm{Z}\times\mathrm{Z}).$$

Notice that $P^1_\sharp\beta^n = \mu_\infty$, and thus $\beta^n \in \mathrm{Adm}(\tilde{\mu}_\infty, \tilde{\mu}_n)$. Also, from $\frac{\mathrm{d}\mu_\infty}{\mathrm{d}\tilde{\mathfrak{m}}_\infty} = z_\infty \frac{\mathrm{d}\mu_\infty}{\mathrm{d}\mathfrak{m}_\infty} = z_\infty\rho_\infty$ we get $\beta^n \le z_\infty \|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)}\alpha^n$ and thus

$$\mu_n \le z_\infty\|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)} P^2_\sharp\alpha^n = z_\infty\|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)}\tilde{\mathfrak{m}}_n \le \frac{z_\infty}{z_n}\|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)}\mathfrak{m}_n.$$

Since clearly $z_n \to z_\infty$, (4.3.14) holds. Moreover, we have

$$\begin{aligned} W_q^q(\mu_\infty, \mu_n) &\le \int \mathsf{d}^q(x,y)\,\mathrm{d}\beta^n(y_1,y_2) \\ &\le z_\infty\|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)}\int \mathsf{d}^q(x,y)\,\mathrm{d}\alpha^n(y_1,y_2) \le z_\infty\|\rho_\infty\|_{L^\infty(\mathfrak{m}_\infty)} W_q^q(\tilde{\mathfrak{m}}_\infty, \tilde{\mathfrak{m}}_n) \to 0, \end{aligned} \tag{4.3.15}$$

and the claim is proved.

Step 2. Let $D > 0$, $\mu_0, \mu_1 \in \mathscr{P}(\mathrm{X}_\infty)$ be absolutely continuous with bounded densities and $\mathrm{diam}(\mathrm{supp}(\mu_0)\cup\mathrm{supp}(\mu_1)) < D$. Let $A \subset \mathrm{Z}$ be open with $\mathrm{diam}(A) < D$ and $\mathsf{d}((\mathrm{supp}(\mu_0)\cup\mathrm{supp}(\mu_1)), \mathrm{Z}\setminus A) > 0$: apply the previous step with A and the measures μ_0, μ_1 to find corresponding sequences (μ_i^n) as above. Since X_n is a BIP space we can find $\pi^n \in \mathrm{OptGeo}_q(\mu_0^n, \mu_1^n)$ with

$$\mathrm{Comp}(\pi^n) \le C_q^n(D)\big(\|\rho_0^n\|_{L^\infty(\mathfrak{m}_n)} \vee \|\rho_1^n\|_{L^\infty(\mathfrak{m}_n)}\big), \tag{4.3.16}$$

where $\rho_i^n := \frac{\mathrm{d}\mu_i^n}{\mathrm{d}\mathfrak{m}}$. By Lemma 4.3.7 above, the sequence (π^n) has a subsequence weakly converging to some $\pi \in \mathrm{OptGeo}_q(\mu_0,\mu_1)$, so that taking into account iii) of Lemma 4.3.7 and the previous step, by taking the $\overline{\lim}$ in (4.3.16) we conclude that

$$\mathrm{Comp}(\pi) \le \varlimsup_{n\to\infty} C_q^n(D)\big(\|\rho_0\|_{L^\infty(\mathfrak{m}_\infty)} \vee \|\rho_1\|_{L^\infty(\mathfrak{m}_\infty)}\big)$$

and the conclusion follows. □

4.4 p-independent differential calculus

4.4.1 Unification of p-differential calculus

In this section we study the effect of p-independence of weak upper gradients in terms of differential calculus. Informally, the idea is that on this sort of spaces we should have a concept of differential (and thus of cotangent module) which is independent on the chosen Sobolev exponent p.

Theorem 4.4.1 (Universal cotangent module - weak version)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space with p-independent weak upper gradients in weak sense.*

Then there is a unique couple $(L^0(T^*\mathrm{X}), \mathrm{d})$ where $L^0(T^*\mathrm{X})$ is a $L^0(\mathfrak{m})$-normed module and $\mathrm{d}\colon \cup_{p\in(1,\infty)} S^p_{loc}(\mathrm{X}) \to L^0(T^*\mathrm{X})$ is such that for any $p \in (1,\infty)$ it holds

i) *The restriction of* d *on* $S^p_{loc}(\mathrm{X})$ *is linear;*

ii) *For any* $f \in S^p_{loc}(\mathrm{X})$*, it holds* $|Df|_p = |\mathrm{d}f|$ $\mathfrak{m}$*-a.e.;*

iii) *The space* $\{\mathrm{d}f \colon f \in W^{1,p}(\mathrm{X})\}$ *generates* $L^0(T^*\mathrm{X})$ *as a module.*

Here, uniqueness is intended up to unique isomorphism, i.e. if $(\mathscr{M}, L)$ *is another couple with the same properties, there there is a unique module isomorphism* $\Phi\colon \mathscr{M} \to L^0(T^*\mathrm{X})$ *so that* $\Phi \circ L = \mathrm{d}$.

Moreover, the identification $I_p\colon L^0_p(T^*\mathrm{X}) \to L^0(T^*\mathrm{X})$ *sending* $\mathrm{d}_p f \mapsto \mathrm{d}f$ *induces the module isomorphism* $J_q\colon L^0_q(T\mathrm{X}) \to (L^0(T\mathrm{X}))^*$ *for any* $p, q \in (1, \infty)$ *conjugate exponents.*

Proof.

Uniqueness. For any $p \in (1,\infty)$, the couple $(L^0(T^*\mathrm{X}), \mathrm{d}_{|S^p_{loc}})$ satisfies the same properties of $(L^0_p(T^*\mathrm{X}), \mathrm{d}_p)$ in Theorem 1.2.27. Therefore, uniqueness is a direct consequence of the uniqueness part such Theorem.

Existence. Fix $p_1, p_2 \in (1,\infty)$, $f, g \in W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$ and $E \subset \mathrm{X}$ Borel. We observe that locality of differentials together with the assumption on the metric measure space yield

$$\mathrm{d}_{p_1} f = \mathrm{d}_{p_1} g \quad \mathfrak{m}\text{-a.e. on } E \quad \Leftrightarrow \quad \mathrm{d}_{p_2} f = \mathrm{d}_{p_2} g \quad \mathfrak{m}\text{-a.e. on } E \tag{4.4.1}$$

Indeed,

$$\begin{aligned} \mathrm{d}_{p_1} f = \mathrm{d}_{p_1} g \quad \mathfrak{m}\text{-a.e. on } E \quad &\Leftrightarrow \quad |\mathrm{d}_{p_1}(f-g)| = 0 \quad \mathfrak{m}\text{-a.e. on } E \\ &\Leftrightarrow \quad |D(f-g)|_{p_1} = 0 \quad \mathfrak{m}\text{-a.e. on } E \\ &\Leftrightarrow \quad |D(f-g)|_{p_2} = 0 \quad \mathfrak{m}\text{-a.e. on } E \\ &\Leftrightarrow \quad |\mathrm{d}_{p_2}(f-g)| = 0 \quad \mathfrak{m}\text{-a.e. on } E \\ &\Leftrightarrow \quad \mathrm{d}_{p_2} f = \mathrm{d}_{p_2} g \qquad \mathfrak{m}\text{-a.e. on } E \end{aligned}$$

Building up on property (4.4.1) we are going to construct an isomorphism $I^{p_2}_{p_1} : L^0_{p_1}(T^*\mathrm{X}) \to L^0_{p_2}(T^*\mathrm{X})$ sending $\mathrm{d}_{p_1} f$ to $\mathrm{d}_{p_2} f$. We start by defining its action on simple 1-forms. Denote by $V_{p_j} \subset L^0_{p_j}(T^*\mathrm{X})$, $j = 1,2$, the space of covector fields of type $\sum_{i=1}^n \chi_{E_i} \mathrm{d}_{p_j} f_i$, where (E_i) is a finite Borel partition of X, and $(f_i) \subset W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$. Then define $I^{p_2}_{p_1}\colon V_{p_1} \to V_{p_2}$ by the formula

$$I^{p_2}_{p_1}\Big(\sum_{i=1}^n \chi_{E_i} \mathrm{d}_{p_1} f_i\Big) := \sum_{i=1}^n \chi_{E_i} \mathrm{d}_{p_2} f_i.$$

It can be readily checked that (4.4.1) ensures the well posedness of such map. Moreover, $I^{p_2}_{p_1}$ is linear and, due to the independence of weak upper gradients, it is a pointwise isometry, since

$$\Big| I^{p_2}_{p_1}\Big(\sum_{i=1}^n \chi_{E_i} \mathrm{d}_{p_1} f_i\Big)\Big| = \sum_{i=1}^n \chi_{E_i} |\mathrm{d}_{p_2} f_i| = \sum_{i=1}^n \chi_{E_i} |\mathrm{d}_{p_1} f_i| = \Big|\Big(\sum_{i=1}^n \chi_{E_i} \mathrm{d}_{p_1} f_i\Big)\Big| \quad \mathfrak{m}\text{-a.e.}$$

Therefore, it is continuous and, with a little abuse of notation, it uniquely extend to a pointwise isometry from the closure of V_{p_1} with values in $L^0_{p_2}(T^*\mathrm{X})$. It is clear that, thanks to a) of Definition 4.2.1 and ii) of Theorem 1.2.27, the closure of V_{p_j} coincides with $L^0_{p_j}(T^*\mathrm{X})$ itself, $j = 1,2$. We thus built a module isomorphism $I^{p_2}_{p_1}$ such that

$$I^{p_2}_{p_1}(\mathrm{d}_{p_1} f) = \mathrm{d}_{p_2} f, \tag{4.4.2}$$

holds for every $f \in W^{1,p_1}(\mathrm{X}) \cap W^{1,p_2}(\mathrm{X})$ and it is then clear from definition of $S^{p_j}_{loc}(\mathrm{X})$ and the locality of the differential that (4.4.2) holds for any $f \in S^{p_1}_{loc}(\mathrm{X}) \cap S^{p_2}_{loc}(\mathrm{X})$.

To conclude, fix $\bar{p} \in (1,\infty)$, set $I_p := I^{\bar{p}}_p$, define the module $L^0(T^*\mathrm{X}) := L^0_{\bar{p}}(T^*\mathrm{X})$, and the differential d as $\mathrm{d}_{|S^p_{loc}(\mathrm{X})} := I_p \circ \mathrm{d}_p$. Notice that (4.4.2) ensures that the definition of d is well posed.

Then property i) follows from the linearity of d_p and $I^{\bar{p}}_p$. Property ii) follows from the fact that for every $p \in (1,\infty)$, I_p is a pointwise isometry, as $|\mathrm{d}f| = |I_p(\mathrm{d}_p f)| = |\mathrm{d}_p f| = |Df|_p$, $\mathfrak{m}$-a.e.. Finally, we know that $\{\mathrm{d}_p f \colon f \in W^{1,p}(\mathrm{X})\}$ generates $L^0_p(T^*\mathrm{X})$, thus iii) follows from the fact that I_p is an isomorphism. Finally, it is standard to define J_q as the dual isomorphism of $\mathrm{d}_p f \mapsto \mathrm{d}f$, for every p, q conjugate exponents. $\square$

Definition 4.4.2 (Universal differential structures)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with p-independent weak upper gradients in weak sense. We call the* universal cotangent module *the module* $L^0(T^*\mathrm{X})$ *and* d *the associated* universal differential *given by Theorem 1.2.27. Moreover, we call the* universal tangent module*, and denote it by* $L^0(T\mathrm{X})$*, the dual module* $(L^0(T^*\mathrm{X})^*$*.*

In spaces with p-independent weak gradients in the strong sense, the following stronger result holds. Basically, it says that in this case the situation is closer to the standard Euclidean one, where one first has a distributional differential and then, by investigating its integrability, deduces whether the function is Sobolev:

Theorem 4.4.3 (Universal cotangent module - strong version)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with p-independent weak upper gradients in strong sense.*

Then in addition to the results in Theorem 4.4.1, the following holds. Let $f \in \cup_{p\in(1,\infty)} S^p_{loc}(\mathrm{X})$ *be such that for some* $\bar p \in (1,\infty)$ *we have* $|\mathrm{d}f| \in L^{\bar p}_{loc}(\mathfrak{m})$*.*

Then $f \in S^{\bar p}_{loc}(\mathrm{X})$*.*

Proof. By assumption we know that for some $p \in (1,\infty)$ we have $f \in S^p_{loc}(\mathrm{X})$. For $k, R > 0$ let $f^k := (-k) \vee f \wedge k$, $\eta_R : \mathrm{X} \to [0,1]$ given by $\eta_R := (1 - \mathsf{d}(\cdot, B_R(\bar x)))^+$, where $\bar x \in \mathrm{X}$ is some fixed chosen point, and $f^k_R := \eta_R f^k$. Then f^k_R is bounded with bounded support, hence it belongs to $L^p \cap L^{\bar p}(\mathfrak{m})$. Moreover, we have

$$|Df^k_R|_p \leq \eta_R |Df|_{p_1} + \chi_{\mathrm{supp}(\eta_R)} k = \eta_R |\mathrm{d}f| + \chi_{\mathrm{supp}(\eta_R)} k$$

and the assumption $|\mathrm{d}f| \in L^{\bar p}_{loc}(\mathfrak{m})$ gives that the rightmost side in the above is in $L^{\bar p}(\mathfrak{m})$. Then the fact that X has p-independent weak gradients in the strong sense implies that $f^k_R \in W^{1,\bar p}(\mathrm{X})$ with $|Df^k_R|_{\bar p} \leq |\mathrm{d}f| + \chi_{\mathrm{supp}(\eta_R)} k$. By the very definition of $S^{\bar p}_{loc}(\mathrm{X})$ we just proved that $f \in S^{\bar p}_{loc}(\mathrm{X})$, which is the conclusion. ☐

4.4.2 Infinitesimal Hilbertianity and universal gradient

Intuitively, when a metric measure is infinitesimal Hilbertian, there is a hidden scalar product between tangent directions at small scales. Such geometric concept has in principle nothing to do with the particular choice of the exponent $p = 2$. Nevertheless, due to the dependence of weak upper gradients on p, this hidden geometry remains unseen by all other $W^{1,p}(\mathrm{X})$.

In this section we investigate some consequences of having both infinitesimal Hilbertianity and p-independent weak upper gradients. We start with the following proposition, whose proof essentially boils down into verifying that the classical Clarkson inequalities are valid also for elements of a generic Hilbert module:

Proposition 4.4.4. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimal Hilbertian metric measure space with p-independent gradients in weak sense. Then,* $L^p(T^*\mathrm{X})$ *and* $W^{1,p}(\mathrm{X})$ *are uniformly convex and consequently also reflexive for every* $p \in (1,\infty)$*.*

Proof. Reflexivity is a consequence of uniform convexity, so we focus on this latter property. Also, the map

$$W^{1,p}(\mathrm{X}) \ni f \quad \mapsto \quad (f, \mathrm{d}f) \in L^p(\mathrm{X}) \times_p L^p(T^*\mathrm{X})$$

is an isometry and since $L^p(\mathrm{X})$ is uniformly convex and so is the L^p-norm on $\mathbb{R}^2$ used to define the product norm, the uniform convexity of $W^{1,p}(\mathrm{X})$ will follow if we show the one of $L^p(T^*\mathrm{X})$. We thus concentrate on this latter space and observe that it is sufficient to show that the Clarkson inequalities hold:

$$\begin{aligned} p \in [2,\infty) \quad &\Rightarrow \quad \Big\|\frac{\omega+\eta}{2}\Big\|^p_{L^p(T^*\mathrm{X})} + \Big\|\frac{\omega-\eta}{2}\Big\|^p_{L^p(T^*\mathrm{X})} \leq \tfrac12 \|\eta\|^p_{L^p(T^*\mathrm{X})} + \tfrac12 \|\omega\|^p_{L^p(T^*\mathrm{X})}, \\ p \in (1,2] \quad &\Rightarrow \quad \Big\|\frac{\omega+\eta}{2}\Big\|^q_{L^p(T^*\mathrm{X})} + \Big\|\frac{\omega-\eta}{2}\Big\|^q_{L^p(T^*\mathrm{X})} \leq \Big(\tfrac12 \|\omega\|^p_{L^p(T^*\mathrm{X})} + \tfrac12 \|\eta\|^p_{L^p(T^*\mathrm{X})}\Big)^{\frac{q}{p}}, \end{aligned} \tag{4.4.3}$$

where $q \in (1,\infty)$ is the conjugate exponent of p and ω, η are arbitrary elements in $L^0(T^*\mathrm{X})$.

To see these, we start noticing that the assumption of infinitesimal Hilbertianity (and Theorem 4.4.1 and its proof) gives that

$$2|\eta|^2 + 2|\omega|^2 = |\eta+\omega|^2 + |\eta-\omega|^2, \qquad \mathfrak{m}\text{-a.e.}, \qquad \forall \eta,\omega \in L^0(T^*\mathrm{X}). \tag{4.4.4}$$

Case $p \geq 2$. From the inequality $\|x\|_p \leq \|x\|_2$ valid for any $x \in \mathbb{R}^2$ we see that for any $\eta, \omega \in L^0(T^*\mathrm{X})$ we have

$$\Big|\frac{\omega+\eta}{2}\Big|^p + \Big|\frac{\omega-\eta}{2}\Big|^p \leq \Big(\Big|\frac{\omega+\eta}{2}\Big|^2 + \Big|\frac{\omega-\eta}{2}\Big|^2\Big)^{p/2} \overset{(4.4.4)}{=} \Big(\frac{|\eta|^2}{2} + \frac{|\omega|^2}{2}\Big)^{p/2} \leq \frac{|\omega^n|^p}{2} + \frac{|\eta^n|^p}{2}, \tag{4.4.5}$$

having used the fact that $\mathbb{R}^+ \ni t \mapsto t^{\frac{p}{2}}$ is convex in the last step. Integrating we deduce the first in (4.4.3).

Case $p \in (1,2]$. Obviously $p \leq 2 \leq q$ and thus for any $\eta, \omega \in L^0(T^*\mathrm{X})$ we have

$$\Big|\frac{\omega+\eta}{2}\Big|^q + \Big|\frac{\omega-\eta}{2}\Big|^q \overset{(4.4.5)}{\leq} \Big(\frac{|\eta|^2}{2} + \frac{|\omega|^2}{2}\Big)^{q/2} \overset{*}{\leq} \Big(\frac{|\eta|^p}{2} + \frac{|\omega|^p}{2}\Big)^{q/p}, \tag{4.4.6}$$

where in the starred inequality we used the fact that $\|x\|_2 \leq \|x\|_p$ for any $x \in \mathbb{R}^2$.

Now suppose we already know the reverse triangle inequality in $L^r(\mathfrak{m})$ spaces for $r \in (0,1)$, i.e. that for f, g Borel non-negative it holds

$$\Big(\int f^r \,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{r}} + \Big(\int g^r \,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{r}} \leq \Big(\int (f+g)^r \,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{r}} \tag{4.4.7}$$

and apply it with $r := \frac{p}{q}$, $f := \frac{1}{2}|\omega+\eta|^p$ and $g := \frac{1}{2}|\omega+\eta|^p$ to obtain

$$\Big(\Big\|\frac{\omega+\eta}{2}\Big\|^q_{L^p(T^*\mathrm{X})} + \Big\|\frac{\omega-\eta}{2}\Big\|^q_{L^p(T^*\mathrm{X})}\Big)^{p/q} \leq \int \Big(\Big|\frac{\omega+\eta}{2}\Big|^q + \Big|\frac{\omega-\eta}{2}\Big|^q\Big)^{p/q} \mathrm{d}\mathfrak{m}.$$

This and (4.4.6) give the second in (4.4.3), thus it remains to prove (4.4.7). Putting $\varphi(x) := (f^r(x), g^r(x))$ and $\Psi(a,b) := (a^{\frac{1}{r}} + b^{\frac{1}{r}})^r$, (4.4.7) takes the form

$$\Psi\Big(\int \varphi \,\mathrm{d}\mathfrak{m}\Big) \leq \int \Psi \circ \varphi \,\mathrm{d}\mathfrak{m}.$$

Now observe that since $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is convex and positively 1-homogeneous we have

$$\Psi = \sup_{\ell \leq \Psi,\ \ell \text{ linear}} \ell,$$

therefore

$$\Psi\Big(\int \varphi \,\mathrm{d}\mathfrak{m}\Big) = \sup_\ell \ell\Big(\int \varphi \,\mathrm{d}\mathfrak{m}\Big) = \sup_\ell \int \ell \circ \varphi \,\mathrm{d}\mathfrak{m} \leq \int \Psi \circ \varphi \,\mathrm{d}\mathfrak{m}$$

and the conclusion follows. □

In presence of infinitesimal Hilbertianity and independence of p-upper gradients in the weak sense, we can naturally define a linear notion of gradient:

Theorem 4.4.5 (Universal gradient). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a infinitesimal Hilbertian metric measure space with p-independent weak upper gradients in weak sense. Then there is a unique map $\nabla \colon \cup_{p\in(1,\infty)} S^p_{loc}(\mathrm{X}) \to L^0(T\mathrm{X})$, called* universal gradient, *such that for any $p \in (1,\infty)$ it holds*

i) *The restriction of ∇ on $S^p_{loc}(\mathrm{X})$ is linear;*

ii) *For any $f \in S^p_{loc}(\mathrm{X})$, it holds $\mathrm{d}f(\nabla f) = |\nabla f|^2_* = |\mathrm{d}f|^2$ $\mathfrak{m}$-a.e.;*

iii) *The space $\{\nabla f \colon f \in W^{1,p}(\mathrm{X})\}$ generates $L^0(T\mathrm{X})$ as a module.*

Proof. The assumption of infinitesimal Hilbertianity (together with Theorem 4.4.1 and its proof) ensures that $L^0(T^*\mathrm{X})$ is a Hilbert module: let $\mathcal{R} : L^0(T^*\mathrm{X}) \to L^0(T\mathrm{X})$ be the Riesz isomorphism.

Then from Theorem 4.4.1 above it is clear that $\nabla f := \mathcal{R}(\mathrm{d}f)$ satisfies the requirements. □

4.5 Spaces with the (BIP) and consequences

In this section we list some spaces and conditions that we can relate to the bounded interpolation property and we analyze the consequences.

4.5.1 Finite dimensional spaces with Ricci lowerbounds

We begin with measure contraction property.

Theorem 4.5.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a non branching* $\mathsf{MCP}(K, N)$*-space for some* $K \in \mathbb{R}, N \in [1, \infty)$. *Then, it is a BIP-space with profile function* $D \mapsto 2^N e^{D\sqrt{(N-1)K^-}}$ *and consequently:*

i) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *has p-independent weak upper gradients in the strong sense;*

ii) *there exists a unique couple* $(L^0(T^*\mathrm{X}), \mathrm{d})$ *of universal cotangent module and differential;*

iii) *there exists an* ∞*-plan* $\boldsymbol{\pi}_{\mathsf{master}}$ *concentrated on geodesics so that:*
for every $p \in (1, \infty)$, f *Borel and* $G \in L^p(\mathfrak{m})$ *with* $G \geq 0$, *the following are equivalent*

- $f \in S^p(\mathrm{X})$ *and* G *is a p-weak upper gradient;*
- *it holds*
$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t, \qquad \boldsymbol{\pi}_{\mathsf{master}}\text{-}a.e.\ \gamma.$$

Proof. We subdivide the proof in two steps.
Step 1. Let $q \in (1, \infty), D > 0$ and $\mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X})$ as in the definition of the (BIP). Then, Theorem A.1.1 ensures that we can find $\pi^+ \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ so that

$$\|\rho_t^+\|_{L^\infty(\mathfrak{m})} \leq 2^N e^{D\sqrt{(N-1)K^-}} \|\rho_0\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0, \tfrac{1}{2}],$$

and $\pi^- \in \mathrm{OptGeo}_q(\mu_1, \mu_0)$ so that

$$\|\rho_t^-\|_{L^\infty(\mathfrak{m})} \leq 2^N e^{D\sqrt{(N-1)K^-}} \|\rho_1\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0, \tfrac{1}{2}],$$

having denoted $\rho_t^\pm := \frac{\mathrm{d}(\mathsf{e}_t)_\sharp \pi^\pm}{\mathrm{d}\mathfrak{m}}$ for every $t \in [0, \frac{1}{2}]$. Hence, by non branching, the set $\mathrm{OptGeo}_q(\mu_0, \mu_1)$ is a singleton (1.1.12) and therefore we can glue the *forward* and *backward* estimates to verify the (BIP).
Step 2. Finally, recalling that MCP-spaces are doubling, we have thanks to Theorem 1.3.5 that $W^{1,p}(\mathrm{X})$ is reflexive and therefore $\mathrm{Lip}_{bs}(\mathrm{X})$ is dense in every $W^{1,p}(\mathrm{X})$ (see Corollary 1.3.6). Consequently, $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ have p-independent weak upper gradient in the strong sense by appealing to Theorem 4.3.5. The existence of a universal cotangent module $L^0(T^*\mathrm{X})$ is granted by Theorem 4.4.1. Finally, iii) is the content of Theorem 4.6.4. □

As the reader may have noticed, the fact that $\mathsf{MCP}(K, N)$-spaces have p-independent weak upper gradients in weak sense directly follows by the deep results contained in [64] (see the proof of Proposition 4.2.2, recalling that MCP spaces are doubling and supports a Poincaré inequality). Nevertheless, the novelties that follows from our investigation, is their independence in the stronger sense of Definition 4.2.1 as well as properties ii)-iii).

Finally, as noticed in (2.5.1), since non branching CD_q spaces are MCP spaces, we have the simple corollary.

Corollary 4.5.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a non branching* $\mathsf{CD}(K, N)$ *space for some* $K \in \mathbb{R}, N \in [1, \infty)$. *Then, it is a BIP-space and the results of Theorem 4.5.1 hold.*

4.5.2 Infinite dimensional spaces with Ricci lowerbounds

In Appendix A, we prove in Theorem A.2.1 that the interpolation estimates of [174] stated in Theorem 2.2.9 extend to all exponent $q \neq 2$. This will be achieved also in the infinite dimensional setting of $\mathsf{CD}_q(K,\infty)$ spaces, hence the (BIP) and its consequences have the rights to hold also in this framework. Unfortunately, the $\mathsf{CD}_q(K,\infty)$-class is not known at present to be independent on q as this class is not covered by the analysis of [1]. A simple hypothesis then ensures:

Theorem 4.5.3. *Let $K \in \mathbb{R}$ and suppose that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}_q(K,\infty)$-space for every $q \in (1,\infty)$. Then, it has the (BIP) with profile function $D \mapsto e^{K^- D^2/12}$ and, consequently, Theorem 4.3.5 holds true.*

However, when restricting the attention to $\mathsf{RCD}(K,\infty)$-spaces, we have the stronger result.

Proposition 4.5.4. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(K,\infty)$-space for some $K \in \mathbb{R}, N \in [1,\infty]$. Then, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ has p-independent weak upper gradient in the strong sense. Moreover, there are unique couples $(L^0(T^*\mathrm{X}),\mathsf{d})$ and $(L^0(T\mathrm{X}),\nabla)$ of universal cotangent and tangent module, with associated linear universal differential and gradient, respectively.*

Proof. From [99] we know that $\mathsf{RCD}(K,\infty)$-spaces do satisfy *b)* and *c)* of Definition 4.2.1. It remains to show a). Call now $B := \overline{\mathrm{Lip}_{bs}(\mathrm{X})}^{W^{1,p}}$ and observe that, since $\mathrm{Lip}_{bs}(\mathrm{X}) \subset W^{1,p}(\mathrm{X}) \cap W^{1,2}(\mathrm{X})$ and

$$2|Df|_p^2 + 2|Dg|_p^2 = |D(f+g)|_p^2 + |D(f-g)|_p^2, \qquad \mathfrak{m}\text{-a.e.}, \forall f,g \in \mathrm{Lip}_{bs}(\mathrm{X}),$$

one can argue as in Proposition 4.4.4 in order to show that B is uniformly convex. Then, for every $f \in W^{1,p}(\mathrm{X})$, we consider thanks to [20] and a truncation argument, a sequence $(f_n) \subset \mathrm{Lip}_{bs}(\mathrm{X})$ so that

$$f_n \to f, \quad |Df_n|_p \to |Df|_p, \quad \text{in } L^p(\mathfrak{m}). \tag{4.5.1}$$

Then, for any G weak limit of $\frac{|D(f_n+f_m)|_p}{2}$, by lower semicontinuity of the $W^{1,p}$-norm, we have

$$\varliminf_{n,m} \left\| \frac{f_n + f_m}{2} \right\|_{W^{1,p}(\mathrm{X})} \geq \left(\|f\|^p_{L^p(\mathfrak{m})} + \|G\|^p_{L^p(\mathfrak{m})} \right)^{1/p} \geq \|f\|_{W^{1,p}(\mathrm{X})}.$$

Suppose, by contradiction, that (f_n) is not Cauchy. Then, $\exists \epsilon > 0$ so that $\|f_n - f_m\| \geq \epsilon$ for countably many n,m. Observe that, uniform convexity of B ensures that $\exists \delta > 0$ such that

$$\|f\|_{W^{1,p}(\mathrm{X})} - \delta \geq \left\| \frac{f_n + f_m}{2} \right\|_{W^{1,p}(\mathrm{X})},$$

for countably many n,m, which clearly is absurd in light of (4.5.1). Therefore, (f_n) is Cauchy and, its B-limit must be f. In other words, we showed that $\mathrm{Lip}_{bs}(\mathrm{X})$ is a dense collection in $W^{1,p}(\mathrm{X})$ for every $p \in (1,\infty)$, which is a stronger statement implying a). Finally, for the last claim, simply invoke Theorem 4.4.5 and Theorem 4.4.1. □

Remark 4.5.5. The reason for which the arguments in [99], contrary to the present note, does not involve other exponents than $p = 2$ is that, using heat-flow regularization techniques (which are well understood and at hand in this class, see e.g. [21]), the problem of independence is reduced to the Lipschitz class which is *large* enough in the RCD-setting in every Sobolev space $W^{1,p}(\mathrm{X})$ and capable to lead to the conclusion of p-independent weak upper gradient even in the infinite-dimensional setting of $\mathsf{RCD}(K,\infty)$-spaces. ■

4.5.3 Curvature dimension condition with negative dimension

In this section, we consider a notion of the curvature dimension condition for metric measure spaces with generalized *negative* dimension $N < 0$. This was first introduced in [165] and recently studied in [150, 149] where the authors considered a larger class of metric measure spaces equipped with a *quasi Radon* reference measures. In the present note we are not interested in working in full generality and recall the definition of this CD-class sticking to our notion of $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Our goal is to show that very naturally, the results of [174] and Appendix A extends also to this class.

First, we need to modify the *distortion coefficient* for the case $N \in (-\infty, 0)$ and $K \in \mathbb{R}, t \in [0,1]$:

$$
{}^{-}\sigma_{K,N}^{(t)}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \le N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})}, & \text{if } N\pi^2 < K\theta^2 < 0, \\ t, & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}, & \text{if } 0 < K\theta^2. \end{cases}
$$

Set also ${}^{-}\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}}\sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}}$. Finally, for $\mu \in \mathscr{P}(\mathrm{X})$, we define the *$N$-Rényi relative entropy* with respect to $\mathfrak{m}$ with Negative N by

$$
{}^{-}\mathcal{U}_N(\mu|\mathfrak{m}) := \int_{\mathrm{X}} \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}, \qquad \text{if } \mu = \rho\mathfrak{m}, \quad \infty \text{ otherwise.}
$$

Definition 4.5.6 ($\mathsf{CD}_q(K,N)$ with $N<0$)**.** *Let $q \in (1,\infty)$, $K \in \mathbb{R}$ and $N < 0$. We say that a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the* curvature dimension condition $\mathsf{CD}_q(K,N)$ *if any pair of probabilities $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_q(\mathrm{X})$ admits a plan $\pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m},\rho_1\mathfrak{m})$ so that*

$$
{}^{-}\mathcal{U}_{N'}(\mu_t|\mathfrak{m}) \le \int {}^{-}\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_0(\gamma_0)^{-\frac{1}{N}} +{}^{-}\tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_1(\gamma_1))^{-\frac{1}{N}}\,\mathrm{d}\pi(\gamma), \qquad (4.5.2)
$$

for every $t \in [0,1], N' \in [N,0)$ and having denoted $\mu_t := (\mathsf{e}_t)_\sharp\pi$.

It is rather obvious, see e.g. [150, Proposition 2.6] for $q = 2$, that, if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}_q(K,N)$-condition $K \in \mathbb{R}, N < 0$, then it satisfies the $\mathsf{CD}_q(K',N')$-condition for every $K' \le K, N' \in [N,0)$.

Remark 4.5.7. In order to avoid confusion with the standard definition, we considered writing ${}^{-}\sigma, {}^{-}\tau, {}^{-}\mathcal{U}$ intentionally to distinguish the notation when $N < 0$. Moreover, we point out that it is natural to consider defining the entropy as $\int_{\mathrm{X}} \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}$ without a minus sign in front of the integral, as $h(s) := s^{1-\frac{1}{N}}$ is a convex function for $N < 0$.

■

Following [174] and Appendix A, we show that also in this case, the curvature dimension condition spreads the support of the measure.

Lemma 4.5.8. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space that is a $\mathsf{CD}_q(K,N)$-space for some $K \in R$, $N < 0$ and $q \in (1,\infty)$. Then, for any $\rho_0, \rho_1 \in L^\infty(\mathfrak{m})$ probability densities with $D := diam(supp(\rho_0) \cup supp(\rho_1))) < \infty$, we have:*

i) *If $K \ge 0$, there exists $\pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m},\rho_1\mathfrak{m})$ so that*

$$
\mathfrak{m}(\{\rho_{\frac{1}{2}} > 0\}) \ge \frac{1}{\|\rho_0\|_{L^\infty} \vee \|\rho_1\|_{L^\infty}} e^{-\frac{1}{2}\sqrt{(1-N)K}D};
$$

ii) *If $K < 0$ and $D < \pi\sqrt{\frac{N-1}{K}}$, there exists $\pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m},\rho_1\mathfrak{m})$ so that*

$$
\mathfrak{m}(\{\rho_{\frac{1}{2}} > 0\}) \ge \frac{1}{\|\rho_0\|_{L^\infty} \vee \|\rho_1\|_{L^\infty}} \cos^{1-N}(\tfrac{1}{2}D\sqrt{K/N-1});
$$

where $(\mathsf{e}_{\frac{1}{2}})_\sharp\pi = \rho_{\frac{1}{2}}\mathfrak{m} + \mu_{\frac{1}{2}}^s$ with $\mu_{\frac{1}{2}}^s \perp \mathfrak{m}$.

Proof. Fix $\pi \in \mathrm{OpGeo}(\rho_0\mathfrak{m}, \rho_1\mathfrak{m})$ satisfying (4.5.2), denote $E := \{\rho_{\frac{1}{2}} > 0\}$ and notice that $\mathsf{d}(\gamma_0, \gamma_1) < D$ π-a.e. γ.

Suppose first that $K \geq 0$ and estimate

$$-\sigma_{K,N}^{(\frac{1}{2})}(\mathsf{d}(\gamma_0,\gamma_1)) = \frac{1}{e^{-\frac{1}{2}\sqrt{-K/N}\mathsf{d}(\gamma_0,\gamma_1)} + e^{\frac{1}{2}\sqrt{-K/N}\mathsf{d}(\gamma_0,\gamma_1)}} \leq \frac{1}{2}e^{\frac{1}{2}\sqrt{-K/N}D}.$$

so that $^{-}\tau_{K,N}^{(\frac{1}{2})}(\mathsf{d}(\gamma_0,\gamma_1)) \leq \frac{1}{2}\big(e^{\frac{1}{2}\sqrt{(1-N)KD}}\big)^{-\frac{1}{N}}$ π-a.e. γ and lastly

$$\mathfrak{U}_N(\mu_{\frac{1}{2}}|\mathfrak{m}) \leq (\|\rho_0\|_{L^\infty} \vee \|\rho_1\|_{L^\infty})^{-\frac{1}{N}}\big(e^{\frac{1}{2}\sqrt{K(1-N)}D}\big)^{-\frac{1}{N}}. \tag{4.5.3}$$

Moreover, begin $1 - 1/N \geq 1$, an application of Jensen inequality yields the following estimate from below

$$\mathfrak{U}_N(\mu_{\frac{1}{2}}|\mathfrak{m}) = \mathfrak{m}(E)\fint_E \rho_{\frac{1}{2}}^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} \geq \mathfrak{m}(E)\big(\frac{1}{\mathfrak{m}(E)}\big)^{1-\frac{1}{N}} = \mathfrak{m}(E)^{\frac{1}{N}}. \tag{4.5.4}$$

We can now combine (4.5.4) with (4.5.3), raise to the $-N$ power and rearrange to conclude in the case $K \geq 0$.

Suppose now instead $K < 0$ and $D < \pi\sqrt{N - 1/K}$. Then, π-a.e. γ we have that $\frac{1}{2}\mathsf{d}(\gamma_0,\gamma_1)\sqrt{K/N-1}) < \pi/2$ and, since the cosine is monotone decreasing and strictly positive on $[0, \pi/2)$, we estimate

$$^{-}\tau_{K,N}^{(\frac{1}{2})}(\mathsf{d}(\gamma_0,\gamma_1)) = \frac{1}{2}\Big(\frac{1}{\cos(\frac{1}{2}\mathsf{d}(\gamma_0,\gamma_1)\sqrt{K/N-1})}\Big)^{1-\frac{1}{N}} \leq \frac{1}{2}\Big(\frac{1}{\cos(\frac{1}{2}D\sqrt{K/N-1})}\Big)^{1-\frac{1}{N}}.$$

Then, combining with (4.5.4), we achieve

$$\mathfrak{m}(E)^{\frac{1}{N}} \leq (\|\rho_0\|_{L^\infty} \vee \|\rho_1\|_{L^\infty})^{-\frac{1}{N}}\big(\cos(\tfrac{1}{2}D\sqrt{K/N-1})\big)^{\frac{1-N}{N}},$$

that easily implies the conclusion. □

From this, it follows:

Theorem 4.5.9. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space that is a* $\mathsf{CD}_q(K,N)$*-space for some* $K \in \mathbb{R}, N \in (-\infty, 0)$*, and* $q \in (1,\infty)$*. Then, for any* $D > 0$ *and* $\rho_0, \rho_1 \in L^\infty(\mathfrak{m})$ *probability densities with diam(supp(*ρ_0*)* $\cup$ *supp(*ρ_1*))* $< D$*, it holds*

i) *if* $K \geq 0$*, there exists* $\pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m}, \rho_1\mathfrak{m})$ *with* $\mu_t := (\mathsf{e}_t)_\sharp\pi \ll \mathfrak{m}$ *and*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})} \leq \|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})};$$

ii) *If* $K < 0$ *and* $D \leq \mathrm{diam}(\mathrm{X}) < \pi\sqrt{\frac{N-1}{K}}$*, there exists* $\pi \in \mathrm{OptGeo}_q(\rho_0\mathfrak{m}, \rho_1\mathfrak{m})$ *with* $\mu_t := (\mathsf{e}_t)_\sharp \ll \mathfrak{m}$ *and*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})} \leq \frac{\big(\frac{D}{4}\sqrt{\frac{K}{N-1}}\big)^{1-N}}{\sin^{1-N}\big(\frac{D}{4}\sqrt{\frac{K}{N-1}}\big)}\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})};$$

for all $t \in [0,1]$ *and having set* $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$.

Proof. The proof follows directly from the proof of Theorem A.2.1 in Appendix A, by replacing Step 1 there with Lemma 4.5.8 and repeating *verbatim* Step 2 - Step 3 - Step 4. For the case i), we simplified the estimate directly working in the larger $\mathsf{CD}_q(0,N)$-class. For the case ii), when $K < 0$, we shall also make use (to get the L^∞-bound by completion and induction) of the identity:

$$\lim_{n\to\infty}\prod_{i=1}^n \cos(\theta 2^{-i}) = \frac{\sin(\theta)}{\theta}, \qquad \text{for } \theta = \tfrac{D}{4}\sqrt{\tfrac{K}{N-1}},$$

proven in Lemma 4.5.12 below. □

We finish this section by proving two straightforward corollaries.

Corollary 4.5.10. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space that is a* $\mathsf{CD}_q(K, N)$*-space for every* $q \in (1, \infty)$ *and for some* $K \in \mathbb{R}, N \in (-\infty, 0)$*. If* $K \geq 0$ *or* $K < 0$ *and diam*$(\mathrm{X}) < \pi\sqrt{\frac{N-1}{K}}$*, then* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a BIP-space and, consequently, Theorem 4.3.5 holds true.*

Proof. Observe that the hypotheses with Theorem 4.5.9 gives the conclusion. $\square$

Corollary 4.5.11. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space that is a* $\mathsf{CD}(K, N)$*-space for some* $N \in (-\infty, 0)$ *and* $K \in \mathbb{R}$*. Then, if* $K \geq 0$ *or* $K < 0$ *and diam*$(\mathrm{X}) < \pi\sqrt{\frac{N-1}{K}}$*,* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *supports a weak local* $(1,1)$*-Poincaré inequality.*

Proof. This is a direct consequence of [174, Theorem 4.1] recalling Theorem 4.5.9. $\square$

Lemma 4.5.12. *It holds*

$$\lim_{n\to\infty} \prod_{i=1}^{n} \cos(2^{-i}\theta) = \frac{\sin(\theta)}{\theta}, \qquad \textit{pointwise.}$$

Proof. Recalling the identity $\cos(2^{-i}\theta) = \frac{1}{2}\sin(2^{f}-i+1\theta)\sin^{-1}(2^{-i}\theta)$, we have for every $n \in \mathbb{N}$:

$$\prod_{i=1}^{n} \cos(2^{-n}\theta) = \frac{1}{2^n} \prod_{i=1}^{n} \frac{\sin(2^{-i+1}\theta)}{\sin(2^{-1}\theta)} = \frac{2^{-n}\theta}{\theta} \frac{\sin(\theta)}{\sin(2^{-n}\theta)}.$$

The claim follows simply by taking the limit as n goes to infinity. $\square$

4.6 Master test plan for the Sobolev space

This section is devoted to the study of master test plans on metric measure spaces, i.e. test plans that are capable to detect the Sobolev space and weak upper gradients. This notion has been the main object of the study in [169], where the author asked whether this special object exists in [169, Problem 2.7]. We will conduct this analysis first on arbitrary metric measure spaces to provide a positive answer to this problem and then move to BIP-spaces where we can actually achieve a more sophisticated result.

4.6.1 Master test plans on arbitrary metric measure spaces

Let us start by defining the main object of this part.

Definition 4.6.1 (Master q-test plan)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $q \in (1, \infty)$*. A* master q-test plan $\boldsymbol{\pi}_q$ *is a* q*-test plan so that:*

if $f\colon \mathrm{X} \to \mathbb{R}$ *Borel and* $G \in L^p(\mathfrak{m})$ *with* $G \geq 0$ *are so that*

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t, \qquad \boldsymbol{\pi}_q\text{-}a.e.\ \gamma,$$

then $f \in S^p(\mathrm{X})$ *and* G *is a* p*-weak upper gradient.*

We point out that the definition is given differently from the original one in [169, Definition 2.5] where the function f is assumed to be Sobolev. The main reason is that, differently from there, we are going to prove that master test plans are also able to detect the full Sobolev space and not only the minimal weak upper gradients.

We now come to the first main result of this part giving a positive answer to [169, Problem 2.7].

Theorem 4.6.2. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then, for every $q \in (1, \infty)$, there exists a master q-test plan $\boldsymbol{\pi}_q$.*

Proof. We fix an arbitrary $q \in (1, \infty)$ and subdivide the proof in different steps.
STEP 1. Let us start defining for every $\alpha, \beta > 0$ the set

$$\Pi_{\alpha,\beta} := \{\pi \in \mathscr{P}(C([0,1], \mathrm{X}))\colon \mathrm{Comp}(\pi) \leq \alpha, \mathrm{Ke}_q(\pi) \leq \beta\}.$$

Recalling that $C([0,1], \mathrm{X})$ with $\mathsf{d}_{\sup}$ is a complete and separable metric space, we have that $\mathscr{P}(C([0,1], \mathrm{X}))$ is weakly separable. Therefore, we can consider for every $\alpha, \beta \in \mathbb{Q}$ countable and dense sets $\mathcal{D}_{\alpha,\beta} \subset \Pi_{\alpha,\beta}$ and finally set

$$\mathcal{D} := \bigcup_{\alpha,\beta \in \mathbb{Q}} \mathcal{D}_{\alpha,\beta}.$$

By construction, $\mathcal{D}$ is countable.
STEP 2. Let us then fix arbitrary f Borel and $G \in L^p(\mathfrak{m})$ with $G \geq 0$. We claim that $f \in S^p(\mathrm{X})$ and G is a p-weak upper gradient if and only if

$$\int |f(\gamma_0) - f(\gamma_1)| \,\mathrm{d}\pi \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t\mathrm{d}\pi, \qquad \forall \pi \in \mathcal{D}. \tag{4.6.1}$$

Obviously, we shall only prove the if-implication, as the converse is straightforward. To this aim, we fix an arbitrary q-test plan π and consider sequences $(\alpha_n), (\beta_n) \subset \mathbb{Q}$ so that $\alpha_n \downarrow \mathrm{Comp}(\pi)$ and $\beta_n \downarrow \mathrm{Ke}_q(\pi)$ as n goes to infinity. Being $\pi \in \Pi_{\alpha_n,\beta_n}$ for all $n \in \mathbb{N}$, thanks to a diagonalization argument, we can find a sequence $\pi_n \in \mathcal{D}$ so that

$$\pi_n \rightharpoonup \pi, \qquad \text{and} \qquad \begin{array}{l} \overline{\lim}_{n\to\infty} \mathrm{Ke}_q(\pi_n) \leq \mathrm{Ke}_q(\pi), \\ \overline{\lim}_{n\to\infty} \mathrm{Comp}(\pi_n) \leq \mathrm{Comp}(\pi). \end{array}$$

We observe that, passing the limit in (4.6.1) with $\pi = \pi_n$ would give, given the arbitrariness of π, that $f \in S^p(\mathrm{X})$ and G is a p-weak upper gradient. To this aim, we invoke Lemma 4.3.2 to get that

$$\lim_{n\to\infty} \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t\mathrm{d}\pi_n = \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t\mathrm{d}\pi.$$

Now, arguing as in the proof of Proposition 4.3.4, it suffices to take to the limit the term $\int f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi_n$ without the absolute value inside to conclude. This can be done as the uniformly bounded compression of (π_n) ensures that $\int f(\gamma_t) \,\mathrm{d}\pi_n \to \int f(\gamma_t) \,\mathrm{d}\pi$ as n goes to infinity for every $t \in [0,1]$ when f is bounded. For the general case, we argue with a truncation argument, again as in the proof of Proposition 4.3.4.
STEP 3. We now follows closely the strategy of [169] to reduce the countable collection $\mathcal{D}$ to a single plan. We include all the details for completeness. Let then $(\pi_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathcal{D}$ and set

$$\eta := \sum_{k\in\mathbb{N}} \frac{\pi_k}{2^k \max\{\mathrm{Comp}(\pi^k), \mathrm{Ke}_q(\pi_k), 1\}}, \qquad \boldsymbol{\pi}_q := \frac{\eta}{\eta(C([0,1], \mathrm{X})}.$$

The definition is well posed as $\eta(C([0,1], \mathrm{X})) \leq \sum_{n,l,i} 2^{-k} = 1$. We claim that the so-defined plan is q-test plan. Indeed, it is by definition a probability measure on $C([0,1], \mathrm{X})$ and, given any $t \in [0,1]$, the estimate

$$(\mathsf{e}_t)_\sharp \eta \leq \sum_{k\in\mathbb{N}} \frac{(\mathsf{e}_t)_\sharp \pi^k}{2^k \mathrm{Comp}(\pi^k)} \leq \sum_{n\in\mathbb{N}} 2^{-k} \mathfrak{m} = \mathfrak{m},$$

ensures that $\boldsymbol{\pi}_q$ has bounded compression. Moreover, we can estimate

$$\iint_0^1 |\dot{\gamma}_t|^q \,\mathrm{d}t\mathrm{d}\eta \leq \sum_{k\in\mathbb{N}} \frac{1}{2^k \mathrm{Ke}_q(\pi_k)} \iint_0^1 |\dot{\gamma}_t|^q \,\mathrm{d}t\mathrm{d}\pi_k = \sum_{k\in\mathbb{N}} 2^{-k} = 1,$$

to show that $\mathrm{Ke}_q(\boldsymbol{\pi}) < \infty$ and consequently that $\boldsymbol{\pi}_q$ is a q-test plan.
Step 4. We conclude the proof by proving that $\boldsymbol{\pi}_q$ is a master q-test plan. By construction, for every f Borel and $G \in L^p(\mathfrak{m})$ with $G \geq 0$ we have that $|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t$ holds $\boldsymbol{\pi}_q$-a.e. if and only if it holds π-a.e. for every $\pi \in \mathcal{D}$. Integrating then yields (4.6.1) which in turn implies that $f \in S^p(\mathrm{X})$ and G is a p-weak upper gradient. The proof is now concluded. □

4.6.2 Master test plan on BIP-spaces

Here, we specialize the previous analysis to the context of BIP-spaces. In this case, we will also achieve that there exists a unique master test plan independent on q that is also concentrated on geodesics.

A special role here will be played by the set

$$\mathsf{Geod}_{(1,\infty)}(\mathrm{X}) := \cup_{q\in(1,\infty)}\mathsf{Geod}_q(\mathrm{X})$$

which, recalling (4.3.1), is a set of ∞-test plans thanks to the (BIP). Our first task is to reduce the class $\mathsf{Geod}_{(1,\infty)}(\mathrm{X})$ given by the (BIP) to a countable number of plans, yet taking care that they are still capable of detecting the Sobolev space as in Proposition 4.3.4.

Proposition 4.6.3. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a BIP-space. Then, there exists a countable family $\mathcal{D} \subset \mathsf{Geod}_{(1,\infty)}(\mathrm{X})$ of ∞-test plans concentrated on geodesics such that:*

for every $p \in (1,\infty)$ and $f\colon \mathrm{X} \to \mathbb{R}$ Borel and $G \in L^p(\mathrm{X})$ with $G \geq 0$, the following are equivalent

i) *$f \in S^p(\mathrm{X})$ and G is a p-weak upper gradient;*

ii) *it holds*

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi, \qquad \forall \pi \in \mathcal{D}.$$

Proof. We subdivide the proof in a reduction step and afterwards, we prove the equivalence.
Reduction. Let $\bar x \in \mathrm{X}$ be a point and consider, for every $k \in \mathbb{N}$, the set of plans

$$\Pi_k := \big\{\pi \in \mathsf{Geod}_{(1,\infty)}(\mathrm{X})\colon \mathrm{Comp}(\pi) \leq k,\ \mathrm{supp}((\mathsf{e}_i)_\sharp\pi) \subseteq B_k(\bar x),\ i = 0, 1\big\}.$$

Fix $k \in \mathbb{N}$, any $\pi \in \Pi_k$ is concentrated on geodesics lying in $B_k(\bar x)$ and $(\mathsf{e}_t)_\sharp\pi \leq k\mathfrak{m}|_{B_k(\bar x)}$ for every $t \in [0,1]$. Hence, the family $\{(\mathsf{e}_t)_\sharp\pi\colon t \in [0,1], \pi \in \Pi_k\}$ is tight and, by Prokhorov's Theorem 1.1.1, there exists a functional $\psi\colon \mathrm{X} \to \mathbb{R}$ with compact sublevels so that

$$\sup_{\pi\in\Pi_k, t\in[0,1]} \int \psi\,\mathrm{d}(\mathsf{e}_t)_\sharp\pi < \infty.$$

Then, by Lemma 1.1.4 and recalling that only geodesics with uniformly bounded length are to be considered, we can consider lifting ψ to the functional $\Psi\colon C([0,1],\mathrm{X}) \to \mathbb{R}$, defined via $\Psi(\gamma) := \int \psi(\gamma_t)\,\mathrm{d}t + \mathsf{d}(\gamma_0,\gamma_1)$ if $\gamma \in \mathsf{Geo}(\mathrm{X})$ and $+\infty$ otherwise, that has compact sublevels and satisfies

$$\sup_{\pi\in\Pi_k} \int \Psi(\gamma)\,\mathrm{d}\pi < \infty.$$

Using again Prokhorov's Theorem 1.1.1, we get that Π_k is relative compact in the weak topology of $\mathscr{P}(C([0,1],\mathrm{X}))$. Now, for every $k \in \mathbb{N}$, consider a countable and dense collection $\mathcal{D}_k \subset \Pi_k$ and lastly define

$$\mathcal{D} := \bigcup_{k\in\mathbb{N}} \mathcal{D}_k \subseteq \mathsf{Geod}_{(1,\infty)}(\mathrm{X}). \tag{4.6.2}$$

It is then obvious by construction that the class $\mathcal{D}$ is a countable collection of ∞-test plans concentrated on geodesics.

Equivalence. The implication $(i) \Rightarrow (ii)$ is obvious. For the converse $(ii) \Rightarrow (i)$, we remark that it is sufficient to show

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi \le \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi, \qquad \forall \pi \in \mathsf{Geod}_{(1,\infty)}(\mathrm{X}), \tag{4.6.3}$$

as the conclusion will then follows invoking Proposition 4.3.4 and arguing as in the proof of Proposition 4.3.4 to improve from the above inequality to the one having $\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi$ at the left hand side. Then, we pick $\pi \in \mathsf{Geod}_{(1,\infty)}(\mathrm{X})$ and observe that there exists $k \in \mathbb{N}$ so that $\pi \in \Pi_k$. Then, consider a sequence $(\pi_n) \subseteq \mathcal{D}$ so that $\pi_n \rightharpoonup \pi$ as n goes to infinity and, by construction, we can take in $\mathcal{D}_k$ for a suitable k). Then, the hypotheses ensures that

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi^n(\gamma) \le \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\mathrm{d}\pi^n(\gamma) = \iint_0^1 G(\gamma_t)\mathsf{d}(\gamma_0,\gamma_1)\,\mathrm{d}t\mathrm{d}\pi^n(\gamma), \qquad \forall n \in \mathbb{N}, \tag{4.6.4}$$

having used the fact that π_n is concentrated on geodesics in the last step. Since the function $\gamma \mapsto \mathsf{d}(\gamma_0,\gamma_1)$ is continuous and bounded on bounded sets, the plans $(\mathsf{d}(\gamma_0,\gamma_1)\,\mathrm{d}\pi_n(\gamma))$ weakly converge to $\mathsf{d}(\gamma_0,\gamma_1)\,\mathrm{d}\pi(\gamma)$. Since clearly they have uniformly bounded compression, by arguing as in the proof of Proposition 4.3.4 we see that

$$\iint_0^1 G(\gamma_t)\mathsf{d}(\gamma_0,\gamma_1)\,\mathrm{d}t\mathrm{d}\pi_n(\gamma) \quad \to \quad \iint_0^1 G(\gamma_t)\mathsf{d}(\gamma_0,\gamma_1)\,\mathrm{d}t\mathrm{d}\pi(\gamma)$$

To pass to the limit in the left hand side of (4.6.4) we can argue e.g. as in the proof of Proposition 4.3.4, again using the assumption of uniformly bounded compression. Finally, we achieved (4.6.3) and the conclusion. □

Mimicking an argument in [169], we can pass from a countable collection of plans detecting the minimal p-weak upper gradient to just one.

Theorem 4.6.4. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a BIP-space. Then, there exists a ∞-test plan $\boldsymbol{\pi}_{\mathsf{master}}$ concentrated on geodesics so that*

$$\boldsymbol{\pi}_{\mathsf{master}} \textit{ is a master } q\textit{-test plan}, \qquad \forall q \in (1,\infty).$$

Proof. Let $\mathcal{D}$ be given by Proposition 4.6.3 and (π^n) an enumeration of the countable collection

$$\mathcal{C} := \Big\{ \big(\mathsf{Restr}_{\frac{i-1}{k}}^{\frac{i}{k}}\big)_\sharp \pi \ : \ k \ge \|\mathrm{Lip}(\gamma)\|_{L^\infty(\pi)},\ k \in \mathbb{N},\ i = 1,...,k,\ \pi \in \mathcal{D} \Big\} \subset \mathscr{P}(C([0,1],\mathrm{X}))$$

and define

$$\eta := \sum_{n\in\mathbb{N}} \frac{\pi^n}{2^n \max\{\mathrm{Comp}(\pi^n),1\}}, \qquad\qquad \boldsymbol{\pi}_{\mathsf{master}} := \frac{\eta}{\eta(C([0,1],\mathrm{X})}.$$

The definition is well posed as $\eta(C([0,1],\mathrm{X})) \le \sum_n 2^n < \infty$. We claim that $\boldsymbol{\pi}_{\mathsf{master}} \in \mathscr{P}(C([0,1],\mathrm{X}))$ satisfies the requirements and we start checking that it is a ∞-test plan.

For $t \in [0,1]$ we have

$$(\mathsf{e}_t)_\sharp \pi \le \frac{1}{\eta(C([0,1],\mathrm{X})} \sum_{n\in\mathbb{N}} \frac{(\mathsf{e}_t)_\sharp \pi^n}{2^n \mathrm{Comp}(\pi^n)} \le \frac{1}{\eta(C([0,1],\mathrm{X})} \sum_{n\in\mathbb{N}} \frac{\mathfrak{m}}{2^n} = \frac{\mathfrak{m}}{\eta(C([0,1],\mathrm{X})},$$

and thus $\boldsymbol{\pi}_{\mathsf{master}}$ has bounded compression. Also, since every element π^n of $\mathcal{C}$ is such that $\|\mathrm{Lip}(\gamma)\|_{L^\infty(\pi^n)} \le 2$, we have that $\|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi}_{\mathsf{master}})} \le 2$ as well. We thus proved that $\boldsymbol{\pi}_{\mathsf{master}}$ is a ∞-test plan.

Now let $p \in (1,\infty)$,f Borel and $G \in L^p(\mathfrak{m})$ with $G \ge 0$, be such that

$$|f(\gamma_1) - f(\gamma_0)| \le \int_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t, \qquad \boldsymbol{\pi}_{\mathsf{master}}\text{-a.e. } \gamma,$$

and notice that since by construction $\boldsymbol{\pi}_{\mathsf{master}}$-negligible sets are also π-negligible for any $\pi \in \mathcal{C}$, we have

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \le \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\pi(\gamma)$$

for every $\pi \in \mathcal{C}$. By definition of $\mathcal{C}$ and a simple gluing argument it is then clear that this last inequality holds for any $\pi \in \mathcal{D}$, hence the conclusion follows from Proposition 4.6.3. □

Remark 4.6.5. We point out a key remark from [169] which is worth to notice also in this note. The results contained in Theorem 4.6.2 and Theorem 4.6.4 are not just technical. Indeed, the existence of master test plans as in Definition 4.6.1 make it possible to identify which are the exceptional curves for which the weak upper gradient inequality (0.0.1) fails. This could be previously done by appealing to the notion of q-Modulus from [134] and further employed in [178] for a systematic definition of Sobolev space. The key difference is that the q-Modulus is only an outer measure, while master q-test plans $\boldsymbol{\pi}_q$ (or, on BIP-spaces, the more powerful $\boldsymbol{\pi}_{\mathsf{master}}$ plan) are Borel probability measures.

■

5 | Master test plans for the space of BV functions

Contents

5.1 Introduction

In this Chapter, we push the ideas of the previous analysis developed in Chapter 4 to cover the case of the BV space. Recalling the characterizations of the space $BV(\mathrm{X})$ found in [13] and collected in Theorem 1.2.23, we can consider a (sub)collection of ∞-test plans (cf. Definition 1.2.1) and consider the (possibly larger) collection of functions of bounded Π-variations: $f \in BV^*_\Pi(\mathrm{X})$ provided $f \in L^1(\mathfrak{m})$ and there exists a constant $\mathsf{C}_\Pi > 0$ so that

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi \leq \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mathsf{C}_\Pi, \qquad \text{for all } \infty\text{-test plan } \pi. \tag{5.1.1}$$

This choice of terminology supports the goal of this Chapter: roughly speaking, we aim at reducing the number of ∞-test plans needed in (5.1.1) to detect both a BV function. More specifically, our two main results are the following ones:

i) On any metric measure space, there exists a *master family*, i.e. a countable family $\mathcal{D}$ of ∞-test plans such that $BV^*_{\mathcal{D}}(\mathrm{X}) = BV(\mathrm{X})$ and that is capable of detecting the total variation $|\boldsymbol{D}f|$ of any $f \in BV(\mathrm{X})$.

ii) On $\mathsf{CD}(K,N)$ metric measure spaces (whose reference measure has finite mass) that are also *non-branching*, the family of those ∞-test plans that are concentrated on geodesics constitutes a master family for $BV(\mathrm{X})$.

As in Chapter 4, we shall work with a suitable polygonal interpolation of ∞-test plans mainly working with the condition expressed by the $BV^*_\Pi(\mathrm{X})$ space. The first main difference with respect to the Sobolev case, is that the BV^* condition is 'integrated' and presents a 'lack of linearity' in the right hand side of (5.1.1) that needs to be dealt with. However, this is not an issue as the main ideas carry to this settings with minor modifications. The second main difference is that a master family is required to detect also the measure $|\boldsymbol{D}f|$. To this aim, we can suitable localize (5.1.1) on open sets $\Omega \subset \mathrm{X}$ and define

$$|\boldsymbol{D}f|^*_\Pi(\Omega) := \sup \frac{\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi}{\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)},$$

the sup taken over $\pi \in \Pi$ concentrated on curves living on a compactly supported subset of Ω. Here, we are not claiming that $|\boldsymbol{D}f|^*_\Pi$ is the restriction to open set of a Borel measure but, when Π it the totality of the ∞-test plan, this is actually the case as proved in [78] and agrees with $|\boldsymbol{D}f|$. Directly from the definition (recall the chacarterization a) of the BV space in Theorem 1.2.23) we have $|\boldsymbol{D}f|^*_\Pi(\Omega) \le |\boldsymbol{D}f|(\Omega)$ for all $\Omega \subset \mathrm{X}$ open hence, to detect $|\boldsymbol{D}f|$ is sufficient to exhibit a countable collection Π of ∞-test plans so that

$$|\boldsymbol{D}f|^*_\Pi(\mathrm{X}) = |\boldsymbol{D}f|(\mathrm{X}).$$

Finally, recalling that the underlying goal is also to produce a single plan out of a countable master family, we mention other two achievements:

iii) We introduce yet another notion of BV space, which we call the *curvewise BV space*. We prove that (on arbitrary metric measure spaces) this new notion is equivalent to the others, and i) yields existence of a single master test plan in the curvewise sense.

iv) In the setting of $\mathsf{RCD}(K,N)$ metric measure spaces, building upon the results of [99], we construct a unique test plan concentrated on geodesics that is able to detect the $W^{1,1}$ space.

Structure of the Chapter. This Chapter is organized as follows:

In Section 5.2 we face the main definitions of this chapter, namely two auxiliary notions of BV space defined in duality with an arbitrary collection of plans. Then, we move to important semicontinuity and compactness property of the collection of ∞-test plans.

In Section 5.3, we achieve a countable collection of ∞-test plan, called master family, that are able to detect the BV space. We settle this investigation first in the class of non branching CD space and then on arbitrary metric measure space.

In Section 5.4, we address the problem of reducing a master family to a single plan, called master test plan, detecting the BV space. To this aim, we consider a define a further notion of BV^{cw} space in the curvewise sense and prove its equivalence with the other notions of BV space. We prove that a master test plan in the curvewise sense is achievable.

Finally, in Section 5.5, we restrict the attention to the RCD class and combine the previous analysis to detect $W^{1,1}$ functions.

5.2 Main definitions and properties

5.2.1 Auxiliary BV spaces

Recalling Theorem 1.2.23, we can define two auxiliary (and possibly larger) set of functions. Recall (1.2.1) for the definition of $[\pi] \subset \mathrm{X}$ the trace of a plan.

Definition 5.2.1. *We say that $f \in BV^*_\Pi(\mathrm{X})$, if $f \in L^1(\mathfrak{m})$ and there exists $\mathsf{C}_\Pi > 0$ so that*

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi \le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mathsf{C}_\Pi, \qquad \forall \pi \in \Pi.$$

We denote $|\boldsymbol{D}f|^*_\Pi(\mathrm{X})$ the least constant satisfying the above and, for every $\Omega \subset \mathrm{X}$ open, we define also

$$|\boldsymbol{D}f|^*_\Pi(\Omega) := \sup_{\pi \in \Pi(\Omega)} \frac{\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi}{\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)},$$

where $\Pi(\Omega) := \{\pi \in \Pi \colon [\pi] \subset \Omega$ with $\mathsf{d}([\pi], \mathrm{X} \setminus \Omega) > 0\}$.

It can be certainly possible to extend $|\boldsymbol{D}f|^*_\Pi$ via a Charathéodory construction (with common abuse of notation)

$$|\boldsymbol{D}f|^*_\Pi(B) := \inf\{|\boldsymbol{D}f|^*_\Pi(\Omega)\colon B \subset \Omega \text{ open}\}, \qquad \forall B \subset \mathrm{X} \text{ Borel}.$$

Here, we are not claiming that the above is a finite Borel measure and, for this reason, we shall also need the second auxiliary definition.

Definition 5.2.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $\Pi \neq \emptyset$ *be a collection of* ∞*-test plans. We say that* $f \in L^1(\mathfrak{m})$ *is a function of bounded* Π*-variation, and we write* $f \in BV_\Pi(\mathrm{X})$*, provided*

i) *the composition* $f \circ \gamma \in BV(0,1)$ π*-a.e.* γ*, for every* $\pi \in \Pi$*;*

ii) *there exists a finite Borel measure* $\mu \geq 0$ *so that*

$$\int \gamma_\sharp |D(f\circ\gamma)|(B)\,\mathrm{d}\pi \leq \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\mu(B), \qquad \forall B \in \mathcal{B}(\mathrm{X}), \pi \in \Pi.$$

We denote $|\boldsymbol{D}f|_\Pi$ *the least* μ *satisfying the above.*

This one, differently from the first, gives us in turn a measure $|\boldsymbol{D}f|_\Pi$ which has the right to be called Π-total variation measure. We start with few remarks on the above sets.

Remark 5.2.3. Let us start observing that the definition of the space $BV_\Pi(\mathrm{X})$ is well posed by the very same considerations of Remark 1.2.24. While, for the space $BV^*_\Pi(\mathrm{X})$, well posedness is obvious since we are not claiming that the set value map $|\boldsymbol{D}f|^*_\Pi(\Omega)$ defined on open set is a (restriction of a Borel) measure. However, when Π is the totality of ∞-test plans, this is infact true (see [78, Section 4.4.4], where the associated space is called Beppo-Levi space and denoted $BV_{BL}(\mathrm{X})$).

Finally, we point out the following inclusions: since Π is in general smaller (or equal) to the totality of ∞-plans, we should always expect that

$$BV(\mathrm{X}) \subset BV_\Pi(\mathrm{X}) \subset BV^*_\Pi(\mathrm{X}),$$

and consequently, for all $\Omega \subset \mathrm{X}$ open, that

$$|\boldsymbol{D}f|^*_\Pi(\Omega) \leq |\boldsymbol{D}f|_\Pi(\Omega) \leq |\boldsymbol{D}f|(\Omega).$$

■

It is worth to clarify the link with the previously defined BV space:

> **Notation**. When Π is the totality of the ∞-test plans, we shall stick to the notation
>
> $$\begin{array}{lcl} BV(\mathrm{X}) := BV_\Pi(\mathrm{X}) & & BV(\mathrm{X}) := BV^*_\Pi(\mathrm{X}) \\ |\boldsymbol{D}f| := |\boldsymbol{D}f|_\Pi & \text{and} & |\boldsymbol{D}f|^*_\Pi := |\boldsymbol{D}f| \end{array}$$
>
> as, in light of Theorem 1.2.23 and the above Remark, the associated notions coincides with the space $BV(\mathrm{X})$.

In light of this, we can finally give one of the main definition of this Chapter.

Definition 5.2.4 (Master family for BV))**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then a given family* Π *of* ∞*-test plans on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be a master family for* $BV(\mathrm{X})$ *provided*

$$BV_\Pi(\mathrm{X}) = BV(\mathrm{X}), \quad |\boldsymbol{D}f|_\Pi = |\boldsymbol{D}f| \ \textit{for every}\ f \in BV(\mathrm{X}).$$

It is worth pointing out that that the totality ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is, by definition, a master family for $BV(\mathrm{X})$. However, it is not clear *a priori* whether any strictly smaller family of ∞-test plans can be a master family for BV and, as said before, this will be the main investigation of this Chapter. Finally, let us explain the importance of the above auxiliary definitions. Even if Definition 5.2.2 gives automatically a Borel measure associated to $f \in BV_\Pi(\mathrm{X})$ so that a (direct and not intricate) definition of a master family is possible, we will see that from an operative point of view, it is not the suitable one for our purposes. Indeed, we shall work in the sequel with polygonal approximations of ∞-test plans and Definition 5.2.1 is much more easier to handle in this situation.

5.2.2 Semicontinuity and compactness results of ∞-test plans

The underlying motivation of this part is that, under appropriate condition fulfilled by the metric measure space, we would like to prove the existence of W_∞ geodesics by approximation with W_q-geodesics. This will be later done on CD spaces thanks to the investigation of this Section about general semicontinuity and compactness results for ∞-test plan.

We start by referring to the definitions of the functionals $\mathrm{Ke}_q, \mathrm{Lip}\colon \mathscr{P}(C([0,1],\mathrm{X}) \to [0,\infty]$ in Section 1.2.1 and we derive the following Mosco-type convergence result that provides a precise relation between the concept of Kinetic and Lipschitz energy of a plan.

Proposition 5.2.5. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $(q_n)_{n\in\mathbb{N}} \subseteq (1,\infty)$ be any sequence satisfying $q_n \uparrow \infty$ as $n \to \infty$. Then the following properties are verified:*

i) *If $(\pi_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq \mathscr{P}\big(C([0,1],\mathrm{X})\big)$ and $\pi_n \rightharpoonup \pi_\infty$ as $n \to \infty$, then*

$$\mathrm{Lip}(\pi_\infty) \leq \varliminf_{n\to\infty} \mathrm{Ke}_{q_n}^{1/q_n}(\pi_n). \tag{5.2.1}$$

In particular, it holds that $\mathrm{Lip}\colon \mathscr{P}\big(C([0,1],\mathrm{X})\big) \to [0,+\infty]$ is lower semicontinuous.

ii) *Given any $\pi \in \mathscr{P}\big(C([0,1],\mathrm{X})\big)$, it holds that*

$$\mathrm{Lip}(\pi) \geq \varlimsup_{n\to\infty} \mathrm{Ke}_{q_n}^{1/q_n}(\pi).$$

In particular, it holds that $\mathrm{Ke}_{q_n}^{1/q_n}(\pi) \to \mathrm{Lip}(\pi)$ as $n \to \infty$.

Proof. By applying (1.1.3) with $\mu := \pi$ and $f := \mathsf{ms}$, we get $\mathrm{Lip}(\pi) = \lim_{n\to\infty} \mathrm{Ke}_{q_n}^{1/q_n}(\pi)$ for every $\pi \in \mathscr{P}\big(C([0,1],\mathrm{X})\big)$, thus proving ii). To prove i), fix $(\pi_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq \mathscr{P}\big(C([0,1],\mathrm{X})\big)$ such that $\pi_n \rightharpoonup \pi_\infty$ as $n \to \infty$. Given any $k,n \in \mathbb{N}$ with $n \geq k$, we have that $q_n \geq q_k$ and thus $\mathrm{Ke}_{q_k}^{1/q_k}(\pi_n) \leq \mathrm{Ke}_{q_n}^{1/q_n}(\pi_n)$ by Hölder inequality. Therefore, the lower semicontinuity (5.2.1) of each $\mathrm{Ke}_{q_k}^{1/q_k}$ gives

$$\mathrm{Lip}(\pi_\infty) = \lim_{k\to\infty} \mathrm{Ke}_{q_k}^{1/q_k}(\pi_\infty) \leq \lim_{k\to\infty} \varliminf_{n\to\infty} \mathrm{Ke}_{q_k}^{1/q_k}(\pi_n) \leq \varliminf_{n\to\infty} \mathrm{Ke}_{q_n}^{1/q_n}(\pi_n),$$

which proves i). $\square$

Next we consider the following compactness result for collection of ∞-test plans.

Proposition 5.2.6 (Compactness result for ∞-test plans)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{supp}\big((\mathsf{e}_0)_\sharp\pi_n\big)$ is bounded, $\sup_{n\in\mathbb{N}} \mathrm{Lip}(\pi_n) < +\infty$, and $\sup_{n\in\mathbb{N}} \mathrm{Comp}(\pi_n) < +\infty$. Then there exist a subsequence $(\pi_{n_i})_{i\in\mathbb{N}}$ of $(\pi_n)_{n\in\mathbb{N}}$ and a ∞-test plan π on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ such that $\pi_{n_i} \rightharpoonup \pi$ as $i \to \infty$.*

Proof. Setting $S := \bigcup_{n\in\mathbb{N}} \mathrm{supp}\big((\mathsf{e}_0)_\sharp\pi_n\big)$ and $L := \sup_{n\in\mathbb{N}} \mathrm{Lip}(\pi_n)$, we have that the closed L-neighbourhood B of S is a bounded set containing $\bigcup_{n\in\mathbb{N}} \mathrm{supp}(\pi_n)$. Indeed, for every $n \in \mathbb{N}$ and π_n-a.e. γ we have that $\gamma_0 \in S$ and $\mathsf{d}(\gamma_t,\gamma_0) \leq \mathrm{Lip}(\pi_n) \leq L$ hold for all $t \in [0,1]$. Recall that $\mathfrak{m}(B)$

is finite and thus the measure $\mathfrak{m}_{|B}$ is tight. Since $(\mathsf{e}_t)_\sharp\pi_n \leq \mathrm{C}\mathfrak{m}_{|B}$ for every $n \in \mathbb{N}$ and $t \in [0,1]$, where we set $\mathrm{C} := \sup_{n\in\mathbb{N}} \mathrm{Comp}(\pi_n)$, we deduce that the family

$$\big\{(\mathsf{e}_t)_\sharp\pi_n \,:\, n \in \mathbb{N},\, t \in [0,1]\big\} \subseteq \mathscr{P}(\mathrm{X}) \quad \text{ is tight.}$$

Therefore, by Prokhorov's theorem, there exists a function $\psi\colon \mathrm{X} \to [0,+\infty]$ having compact sublevels such that

$$\sup\left\{\int \psi \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_n \,:\, n \in \mathbb{N},\, t \in [0,1]\right\} < +\infty. \tag{5.2.2}$$

Now let us define $\Psi\colon C([0,1],\mathrm{X}) \to [0,+\infty]$ as $\Psi(\gamma) := \int_0^1 \psi(\gamma_t)\,\mathrm{d}t + \mathrm{Lip}(\gamma)$ if $\gamma \in \mathrm{Lip}([0,1],\mathrm{X})$ and $\Psi(\gamma) := +\infty$ otherwise. We claim that Ψ has compact sublevels. To prove it amounts to showing that the set $\mathcal{K}_\lambda := \big\{\gamma \in C([0,1],\mathrm{X}) \,:\, \Psi(\gamma) \leq \lambda\big\}$ is compact for any given $\lambda > 0$. First, $\mathcal{K}_\lambda$ is closed: if $(\gamma^n)_{n\in\mathbb{N}} \subseteq \mathcal{K}_\lambda$ and $\gamma \in C([0,1],\mathrm{X})$ satisfy $\lim_{n\to\infty} \mathsf{d}_{\sup}(\gamma^n,\gamma) = 0$, then

$$\begin{aligned}
\Psi(\gamma) &\leq \int_0^1 \varliminf_{n\to\infty} \psi(\gamma^n_t)\,\mathrm{d}t + \varliminf_{n\to\infty} \mathrm{Lip}(\gamma^n) \leq \varliminf_{n\to\infty} \int_0^1 \psi(\gamma^n_t)\,\mathrm{d}t + \varliminf_{n\to\infty} \mathrm{Lip}(\gamma^n)\\
&\leq \varliminf_{n\to\infty} \Psi(\gamma^n) \leq \lambda,
\end{aligned}$$

where we used the lower semicontinuity of ψ and Fatou's lemma. This shows that $\gamma \in \mathcal{K}_\lambda$ and thus $\mathcal{K}_\lambda$ is closed. In order to prove that $\mathcal{K}_\lambda$ is compact, we want to use Theorem 1.1.5. The set $\mathcal{K}_\lambda$ verifies item i) of Theorem 1.1.5, since $\mathrm{Lip}(\gamma) \leq \lambda$ for all $\gamma \in \mathcal{K}_\lambda$. About item ii), fix $\varepsilon > 0$ and pick the compact set $K_{\lambda,\varepsilon} := \big\{x \in \mathrm{X} \,:\, \psi(x) \leq \lambda/\varepsilon\big\}$. For any $\gamma \in \mathcal{K}_\lambda$, one has

$$\mathscr{L}^1\big(\big\{t \in [0,1] \,:\, \gamma_t \notin K_{\lambda,\varepsilon}\big\}\big) = \mathscr{L}^1\big(\big\{t \in [0,1] \,:\, \psi(\gamma_t) > \lambda/\varepsilon\big\}\big) \leq \frac{\varepsilon}{\lambda}\int_0^1 \psi(\gamma_t)\,\mathrm{d}t \leq \varepsilon,$$

where we used Chebyshev's inequality. This shows that $\mathcal{K}_\lambda$ verifies item ii) of Theorem 1.1.5 and thus it is compact. All in all, we proved that Ψ has compact sublevels. Finally, note that

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \int \Psi \,\mathrm{d}\pi_n &\leq \sup_{n\in\mathbb{N}} \iint_0^1 \psi(\gamma_t)\,\mathrm{d}t\,\mathrm{d}\pi_n + \sup_{n\in\mathbb{N}} \int \mathrm{Lip}(\gamma)\,\mathrm{d}\pi_n(\gamma)\\
&\leq \sup_{n\in\mathbb{N}} \int_0^1\!\!\int \psi \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_n\,\mathrm{d}t + \sup_{n\in\mathbb{N}} \mathrm{Lip}(\pi_n)\\
&\leq \sup_{\substack{n\in\mathbb{N},\\ t\in[0,1]}} \int \psi \,\mathrm{d}(\mathsf{e}_t)_\sharp\pi_n + L \overset{(5.2.2)}{<} +\infty.
\end{aligned}$$

Therefore, a second application of Prokhorov's theorem yields the statement. $\square$

Finally, we can combine the two above statements to produce a toolbox capable to show in the sequel the existence of ∞-optimal geodesic plans.

Proposition 5.2.7. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $q_n \nearrow \infty$ be a given sequence. Fix any $\mu_0,\mu_1 \in \mathscr{P}_\infty(\mathrm{X})$ and suppose there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ of ∞-test plans $\pi_n \in \mathrm{OptGeo}_{q_n}(\mu_0,\mu_1)$ such that $\sup_{n\in\mathbb{N}} \mathrm{Comp}(\pi_n) < +\infty$. Then there exists a ∞-test plan $\pi \in \mathrm{OptGeo}_\infty(\mu_0,\mu_1)$ such that $\pi_{n_i} \rightharpoonup \pi$ as $i \to \infty$ for some subsequence $(\pi_{n_i})_{i\in\mathbb{N}}$.*

Proof. First, notice that $\mathrm{supp}(\mu_0) = \bigcup_{n\in\mathbb{N}} \mathrm{supp}\big((\mathsf{e}_0)_\sharp\pi_n\big)$ is bounded. Moreover, calling D the diameter of $\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1)$, we claim that $\sup_{n\in\mathbb{N}} \mathrm{Lip}(\pi_n) \leq D$. Indeed, given any $n \in \mathbb{N}$, we know that π_n-a.e. curve γ is a geodesic satisfying $\gamma_0 \in \mathrm{supp}(\mu_0)$ and $\gamma_1 \in \mathrm{supp}(\mu_1)$, so that accordingly $\mathrm{Lip}(\gamma) = \mathsf{d}(\gamma_0,\gamma_1) \leq D$. Therefore, since $\sup_{n\in\mathbb{N}} \mathrm{Comp}(\pi_n) < +\infty$ by assumption, we are in a position to apply Proposition 5.2.6, thus obtaining that there exists a ∞-test plan π on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ such that $\pi_n \rightharpoonup \pi$, up to a not relabelled subsequence in n. By weak continuity (see Remark 1.2.3), we see that $(\mathsf{e}_0)_\sharp\pi = \mu_0$ and $(\mathsf{e}_1)_\sharp\pi = \mu_1$. Being $\mathsf{Geo}(\mathrm{X})$ a closed subset of

$C([0,1],\mathrm{X})$, one has $\pi(\mathsf{Geo}(\mathrm{X})^c) \leq \underline{\lim}_n \pi_n(\mathsf{Geo}(\mathrm{X})^c) = 0$, which shows that π is concentrated on geodesics. In order to prove that $\pi \in \mathrm{OptGeo}_\infty(\mu_0,\mu_1)$, it only remains to show that $\alpha := (\mathsf{e}_0,\mathsf{e}_1)_\sharp\pi \in \mathrm{Opt}_\infty(\mu_0,\mu_1)$. This readily follows from the estimates

$$\begin{aligned}\operatorname*{ess\,sup}_{\alpha\text{-a.e. }(x,y)} \mathsf{d}(x,y) &= \operatorname*{ess\,sup}_{\pi\text{-a.e. }\gamma} \mathsf{d}(\gamma_0,\gamma_1) = \mathrm{Lip}(\pi) \overset{(5.2.1)}{\leq} \lim_{n\to\infty} \mathrm{Ke}_{q_n}^{1/q_n}(\pi_n) \\ &\overset{(1.1.11a)}{=} \lim_{n\to\infty} W_{q_n}(\mu_0,\mu_1) \overset{(1.1.9)}{=} W_\infty(\mu_0,\mu_1).\end{aligned}$$

Therefore, the statement is achieved. □

5.3 Existence of master families of test plans

5.3.1 Master geodesic plans on non-branching CD spaces

Here we prove that on non-branching $\mathsf{CD}(K,N)$ spaces (with finite reference measure) the ∞-test plans concentrated on geodesics form a master family for BV; in fact, we prove a stronger result, namely that those ∞-test plans which are ∞-optimal dynamical plans between their marginals form a master family for BV. As described in the Introduction, the first step is to prove existence of interpolating ∞-optimal geodesic plans having 'well-behaved' compression constants.

Theorem 5.3.1 (Existence of 'good' ∞-optimal dynamical plans)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a non-branching* $\mathsf{CD}(K,N)$ *space, with* $K\in\mathbb{R}$ *and* $N<\infty$. *Suppose the measure* $\mathfrak{m}$ *is finite. Then there exists a nondecreasing function* $C_\infty\colon(0,+\infty)\to(0,+\infty)$, *depending on* K,N *and called the profile function of* $(\mathrm{X},\mathsf{d},\mathfrak{m})$, *such that* $C_\infty(L)\to1$ *as* $L\to0$ *and with the following property: given any* ∞*-test plan* $\boldsymbol{\eta}$ *with bounded support, there exists a* ∞*-optimal dynamical plan* $\pi\in\mathrm{OptGeo}_\infty\big((\mathsf{e}_0)_\sharp\eta,(\mathsf{e}_1)_\sharp\eta\big)$ *such that*

$$\mathrm{Comp}(\pi) \leq C_\infty\big(\mathrm{Lip}(\eta)\big)\mathrm{Comp}(\eta). \tag{5.3.1}$$

Moreover, in the case $K\geq0$, *the profile function can be additionally required to satisfy* $C\leq1$ *on a right neighbourhood of* 0.

Proof. Fix any boundedly-supported ∞-test plan η on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Call $\mu_0=\rho_0\mathfrak{m}:=(\mathsf{e}_0)_\sharp\eta$ and $\mu_1=\rho_1\mathfrak{m}:=(\mathsf{e}_1)_\sharp\eta$. We know from Theorem 2.2.3 that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a (non-branching) $\mathsf{CD}_q(K,N)$ space for every $q\in(1,\infty)$, thus by [181, Proposition 4.2 iv] (see also [185, Theorem 30.32], in both cases the arguments extends for $q\neq2$) ensures that for any $n\in\mathbb{N}$ the unique (by non branching (1.1.12)) element π_n of $\mathrm{OptGeo}_n(\mu_0,\mu_1)$ satisfies

$$\rho_t^n(\gamma_t)^{-\frac{1}{N}} \geq \rho_0(\gamma_0)^{-\frac{1}{N}}\tau_{K,N}^{(1-t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)+\rho_1(\gamma_1)^{-\frac{1}{N}}\tau_{K,N}^{(t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big), \tag{5.3.2}$$

for π_n-a.e. γ and every $t\in[0,1]$, where we write $(\mathsf{e}_t)_\sharp\pi_n=\rho_t^n\mathfrak{m}$. Observe that

$$\lim_{\theta\searrow0}\frac{\tau_{K,N}^{(t)}(\theta)}{t}=1,\qquad\text{uniformly in } t\in(0,1).$$

In particular, the non-decreasing, continuous function $C_\infty\colon(0,+\infty)\to(0,+\infty)$, given by

$$C_\infty(L):=\left(\sup_{\theta\in(0,L]}\sup_{t\in(0,1)}\frac{t}{\tau_{K,N}^{(t)}(\theta)}\right)^N,\qquad\text{for every } L>0,$$

converges to 1 as $L\to0$. Notice that if $K\geq0$, then $C_\infty\leq1$ on a right neighbourhood of 0. Now define $\varepsilon_n:=\mathrm{Lip}(\eta)(\sqrt[n]{n}-1)$ for every $n\in\mathbb{N}$ and observe that $\varepsilon_n\to0$ as $n\to\infty$. Denote

$$\Gamma_n:=\left\{\gamma\in\mathrm{Lip}([0,1],\mathrm{X})\;:\;\int_0^1|\dot\gamma_t|^n\,\mathrm{d}t\leq\big(\mathrm{Lip}(\eta)+\varepsilon_n\big)^n\right\}.$$

Standard verifications show that Γ_n is a Borel subset of $C([0,1],\mathrm{X})$. Given that

$$\begin{aligned}\mathrm{Ke}_n(\pi_n) &\overset{(1.1.11a)}{=} W_n^n(\mu_0,\mu_1) \leq \int \mathsf{d}^n(x,y)\,\mathrm{d}(\mathsf{e}_0,\mathsf{e}_1)_\sharp\eta(x,y) = \int \mathsf{d}^n(\gamma_0,\gamma_1)\,\mathrm{d}\eta(\gamma)\\ &\leq \int \bigg(\int_0^1 |\dot\gamma_t|\,\mathrm{d}t\bigg)^n \mathrm{d}\eta(\gamma) \leq \iint_0^1 |\dot\gamma_t|^n\,\mathrm{d}t\,\mathrm{d}\eta(\gamma) = \mathrm{Ke}_n(\eta) \leq \mathrm{Lip}(\eta)^n,\end{aligned}$$

an application of Chebyshev's inequality yields

$$\pi_n(\Gamma_n^c) \leq \frac{1}{(\mathrm{Lip}(\eta)+\varepsilon_n)^n}\iint_0^1 |\dot\gamma_t|^n\,\mathrm{d}t\,\mathrm{d}\pi_n(\gamma) = \frac{\mathrm{Ke}_n(\pi_n)}{(\mathrm{Lip}(\eta)+\varepsilon_n)^n} \leq \bigg(\frac{\mathrm{Lip}(\eta)}{\mathrm{Lip}(\eta)+\varepsilon_n}\bigg)^n = \frac{1}{n},$$

so that $\pi_n(\Gamma_n) \to 1$ as $n \to \infty$. Calling $M := \max\big\{\|\rho_0\|_{L^\infty(\mathfrak{m})}, \|\rho_1\|_{L^\infty(\mathfrak{m})}\big\} \leq \mathrm{Comp}(\eta)$, we deduce from (5.3.2) that

$$\begin{aligned}\rho_t^n(\gamma_t)^{\frac{1}{N}} &\leq \bigg(\rho_0(\gamma_0)^{-\frac{1}{N}}(1-t)\,C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{-\frac{1}{N}} + \rho_1(\gamma_1)^{-\frac{1}{N}}\,t\,C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{-\frac{1}{N}}\bigg)^{-1}\\ &\leq C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{\frac{1}{N}} M^{\frac{1}{N}}, \qquad \text{for } \pi_n\text{-a.e. } \gamma \text{ and for every } t\in[0,1].\end{aligned} \tag{5.3.3}$$

If we denote by D the diameter of $\mathrm{supp}(\mu_0)\cup\mathrm{supp}(\mu_1)$, then π_n-a.e. γ satisfies $\mathrm{Lip}(\gamma) \leq D$. Indeed, π_n-a.e. γ is a geodesic joining $\gamma_0 \in \mathrm{supp}(\mu_0)$ to $\gamma_1 \in \mathrm{supp}(\mu_1)$, so $\mathrm{Lip}(\gamma) = \mathsf{d}(\gamma_0,\gamma_1) \leq D$. In particular, (5.3.3) implies that (π_n) is a sequence of ∞-test plan with $\sup_{n\in\mathbb{N}}\mathrm{Comp}(\pi_n) \leq C_\infty(D)M$. Hence, an application of Proposition 5.2.7 yields the existence of a ∞-test plan $\pi \in \mathrm{OptGeo}_\infty(\mu_0,\mu_1)$ such that $\pi_n \rightharpoonup \pi$ as $n\to\infty$, up to a not relabelled subsequence. Now let us define the sequence

$$\tilde\pi_n := \frac{\pi_n|_{\Gamma_n}}{\pi_n(\Gamma_n)}, \qquad \text{for every } n\in\mathbb{N}.$$

Observe that $\tilde\pi_n \rightharpoonup \pi$ as $n\to\infty$ and that, writing $(\mathsf{e}_t)_\sharp\tilde\pi_n = \tilde\rho_t^n\mathfrak{m}$, it holds $\tilde\rho_t^n \leq \rho_t^n/\pi_n(\Gamma_n)$. For $\tilde\pi_n$-a.e. γ one has $\mathsf{d}(\gamma_0,\gamma_1) \leq \int_0^1|\dot\gamma_t|\,\mathrm{d}t \leq \big(\int_0^1|\dot\gamma_t|^n\,\mathrm{d}t\big)^{1/n} \leq \mathrm{Lip}(\eta)+\varepsilon_n$, so (5.3.3) yields

$$\tilde\rho_t^n(\gamma_t) \leq \frac{C_\infty\big(\mathrm{Lip}(\eta)+\varepsilon_n\big)M}{\pi_n(\Gamma_n)}, \qquad \text{for } \tilde\pi_n\text{-a.e. } \gamma \text{ and for every } t\in[0,1].$$

This implies that $\mathrm{Comp}(\tilde\pi_n) \leq C_\infty\big(\mathrm{Lip}(\eta)+\varepsilon_n\big)M/\pi_n(\Gamma_n)$, thus it follows from Remark 1.2.3 that $\mathrm{Comp}(\pi) \leq C_\infty\big(\mathrm{Lip}(\eta)\big)M$, as desired. Therefore, the statement is achieved. □

Remark 5.3.2 (Sobolev vs BV)**.** In Chapter 4, we defined a condition that we called Bounded Interpolation Property and it certainly applies to the context of non branching CD spaces (recall Corollary 4.5.2). However, there are two fundamental differences between the above result and a BIP-space that it is worth to notice now before going further.

- ▷ Given the profile function $D \mapsto C_q(D)$ of a BIP-space, we did not require $C_q(D) \to 1$ as $D \downarrow 0$, assumption that is actually present in Theorem 5.3.1. Avoiding this assumption let us work in Chapter 4 with possibly σ-finite reference measure on non branching CD space. Here instead, finiteness of $\mathfrak{m}$ is a technical yet crucial assumption as we heavily rely on the independence of the CD_q-condition on q [1] (see Theorem 2.2.3).
- ▷ The profile functions $D \mapsto C_q(D)$ for $q < \infty$ given by the BIP-condition are morally different from the above profiles $L \mapsto C_\infty(L)$. Indeed, the parameter D is a rough uniform estimate of the mutual distances $\mathsf{d}(x,y) \leq D$ for $(x,y) \in \mathrm{supp}((\mathsf{e}_0)_\sharp\pi)\times\mathrm{supp}((\mathsf{e}_1)_\sharp\pi)$ while, L is a more precise uniform estimate of the quantity $\mathrm{Lip}(\gamma)$, for $\gamma\in\mathrm{supp}(\pi)$ and π a given plan.

The above discussion point towards the following unifying definition: $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a BIP-space, provided there are profile functions $t \mapsto C_q(t)$ with $C_q(t) \downarrow 0$ a $t \downarrow 0$ for every $q \in (1,\infty]$ so that: for every $\eta \in \mathscr{P}([0,1],\mathrm{X})$ with bounded support and bounded compression, and for every $q \in (1,\infty]$, there exists $\pi \in \mathrm{OptGeo}_q((\mathsf{e}_0)_\sharp\eta, (\mathsf{e}_1)_\sharp\eta)$ so that

$$\begin{aligned}
&\mathrm{Comp}(\pi) \leq C_q(\mathrm{diam}([\eta]))\mathrm{Comp}(\eta), &&\text{if } \eta \text{ is a } q\text{-test plan and } q < \infty,\\
&\mathrm{Comp}(\pi) \leq C_\infty(\mathrm{Lip}(\eta))\mathrm{Comp}(\eta), &&\text{if } \eta \text{ is a } \infty\text{-test plan.}
\end{aligned}$$

Finally, given that the results of [1] do not currently extend to arbitrary reference measure, we do not pursue in this Chapter the BIP-axiomatization. ■

Next we introduce and study the concept of a polygonal interpolation of a given test plan by piecewise ∞-optimal dynamical plans.

Definition 5.3.3. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then we define*

$$\Pi_{bs}(\mathrm{X}) := \big\{\pi \ \infty\text{-test plan} : \mathrm{supp}(\pi) \text{ is bounded}\big\},$$
$$\mathsf{Geod}_\infty(\mathrm{X}) := \Big\{\pi \in \Pi_{bs}(\mathrm{X}) \,:\, \pi \in \mathrm{OptGeo}_\infty\big((\mathsf{e}_0)_\sharp\pi, (\mathsf{e}_1)_\sharp\pi\big)\Big\}.$$

Moreover, we define $\mathsf{PolGeod}_\infty(\mathrm{X})$ as the family of all those ∞-test plans π on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ for which there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $(\mathsf{Restr}_{t_{i-1}}^{t_i})_\sharp\pi \in \mathsf{Geod}_\infty(\mathrm{X})$ for every $i = 1,\ldots,n$.

Finally, recall that for an open subset $\Omega \subset \mathrm{X}$, the objects $\Pi_{bs}(\Omega), \mathsf{Geod}_\infty(\Omega)$ and $\mathsf{PolGeo}_\infty(\Omega)$ are defined as the respectively subcollection of plans supported in curves compactly supported in Ω (recall in Definition 5.2.1.

We will say that $\pi \in \mathsf{PolGeo}_\infty(\mathrm{X})$ is a polygonal interpolation of a given $\eta \in \Pi_{bs}(\mathrm{X})$ provided $(\mathsf{e}_{t_i})_\sharp\pi = (\mathsf{e}_{t_i})_\sharp\eta$ for every $i = 1,\ldots,n$, where $t_0,\ldots,t_n$ are chosen as in Definition 5.3.3.

Remark 5.3.4. We claim that

$$\mathrm{Lip}(\pi) \leq \mathrm{Lip}(\eta), \qquad \text{whenever } \pi \in \mathsf{PolGeo}_\infty(\mathrm{X}) \text{ is a polygonal interpolation of } \eta \in \Pi_{bs}(\mathrm{X}).$$

To prove it, call $\pi_i := (\mathsf{Restr}_{t_{i-1}}^{t_i})_\sharp\pi$ and $\eta_i := (\mathsf{Restr}_{t_{i-1}}^{t_i})_\sharp\eta$ for every $i = 1,\ldots,n$, where $t_0,\ldots,t_n$ are as in Definition 5.3.3. Since $(\mathsf{e}_0,\mathsf{e}_1)_\sharp\eta_i \in \mathrm{Adm}\big((\mathsf{e}_0)_\sharp\pi_i, (\mathsf{e}_1)_\sharp\pi_i\big)$, one has that

$$\begin{aligned}
\mathrm{Lip}(\pi_i) = W_\infty\big((\mathsf{e}_0)_\sharp\pi_i, (\mathsf{e}_1)_\sharp\pi_i\big) &\leq \operatorname*{ess\,sup}_{\boldsymbol{\eta}_i\text{-a.e. }\gamma} \mathsf{d}(\gamma_0,\gamma_1) \leq \operatorname*{ess\,sup}_{\boldsymbol{\eta}_i\text{-a.e. }\gamma} \mathrm{Lip}(\gamma) = \mathrm{Lip}(\eta_i)\\
&\leq (t_i - t_{i-1})\mathrm{Lip}(\eta).
\end{aligned}$$

Hence, we conclude that $\mathrm{Lip}(\pi) = \max_{i=1,\ldots,n} \mathrm{Lip}(\pi_i)/(t_i - t_{i-1}) \leq \mathrm{Lip}(\eta)$, as claimed. ■

Lemma 5.3.5. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a non-branching $\mathsf{CD}(K,N)$ space, with $K \in \mathbb{R}$, $N \in (1,\infty)$, and $\mathfrak{m}(\mathrm{X}) < +\infty$. Let $\eta \in \Pi_{bs}(\mathrm{X})$ be given. Then there exists a sequence $(\pi_n)_{n\in\mathbb{N}} \subseteq \mathsf{PolGeo}_\infty(\mathrm{X})$ of polygonal interpolations of η such that*

$$\mathrm{Lip}(\pi_n) \leq \mathrm{Lip}(\eta), \quad \mathrm{Comp}(\pi_n) \leq C_\infty\big(\mathrm{Lip}(\eta)/n\big)\mathrm{Comp}(\eta), \qquad \text{for every } n \in \mathbb{N},$$

where $L \mapsto C_\infty(L)$ stands for the profile function of $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Moreover, we can additionally require that each trace $[\pi_n]$ is contained in the closed $\frac{\mathrm{Lip}(\boldsymbol{\eta})}{n}$-neighbourhood of $[\eta]$.

Proof. Let $n \in \mathbb{N}$ be fixed. Given any $i = 1,\ldots,n$, choose any test plan $\pi_n^i \in \mathsf{Geod}(\mathrm{X})$ such that $(\mathsf{e}_0)_\sharp\pi_n^i = (\mathsf{e}_{(i-1)/n})_\sharp\eta$, $(\mathsf{e}_1)_\sharp\pi_n^i = (\mathsf{e}_{i/n})_\sharp\eta$, and

$$\mathrm{Comp}(\pi_n^i) \leq C_\infty\big(\mathrm{Lip}(\eta)/n\big)\mathrm{Comp}(\eta), \tag{5.3.4}$$

whose existence is guaranteed by Theorem 5.3.1. Thanks to a glueing argument, we find a plan $\pi_n \in \mathsf{PolGeo}_\infty(\mathrm{X})$ such that $(\mathsf{Restr}_{(i-1)/n}^{i/n})_\sharp\pi_n = \pi_n^i$ for every $i = 1,\ldots,n$. Note that (5.3.4) yields $\mathrm{Comp}(\pi_n) \leq C_\infty\big(\mathrm{Lip}(\eta)/n\big)\mathrm{Comp}(\eta)$. Also, π_n is a polygonal interpolation of η by construction, thus $\mathrm{Lip}(\pi_n) \leq \mathrm{Lip}(\eta)$ by Remark 5.3.4. Finally, since each ∞-test plan π_n^i satisfies $\mathrm{Lip}(\pi_n^i) \leq \mathrm{Lip}(\eta)/n$, we deduce that $[\pi_n^i]$ lies inside the closed $\frac{\mathrm{Lip}(\boldsymbol{\eta})}{n}$-neighbourhood of $\mathrm{supp}\big((\mathsf{e}_{i/n})_\sharp\eta\big)$, and thus accordingly $[\pi_n]$ lies in the $\frac{\mathrm{Lip}(\boldsymbol{\eta})}{n}$-neighbourhood of $[\eta]$. □

Finally, recalling the notation and definitions of the auxiliary space in Section 5.2 we can prove:

Theorem 5.3.6 (Master geodesic plans on non-branching CD spaces)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a non-branching* $\mathsf{CD}(K,N)$ *space, for some* $K\in\mathbb{R}$ *and* $N\in(1,\infty)$*. Suppose the measure* $\mathfrak{m}$ *is finite. Then* $\mathsf{Geod}:=\mathsf{Geod}_\infty(\mathrm{X})$ *is a master family for* $BV(\mathrm{X})$*. More generally,* $BV(\mathrm{X})=BV^\star_{\mathsf{Geod}}(\mathrm{X})$ *and*

$$|\boldsymbol{D}f|(B)=|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(B),\qquad \textit{for every } f\in BV(\mathrm{X}) \textit{ and } B\subseteq\mathrm{X} \textit{ Borel.}$$

Proof. For the sake of brevity, we will write Π_∞ for the collection of all ∞ test plans and again Geod in place of $\mathsf{Geod}_\infty(\mathrm{X})$. It follows from the very definitions of the involved objects that $BV^\star_{\Pi_\infty}(\mathrm{X})=BV(\mathrm{X})\subseteq BV_{\mathsf{Geod}}(\mathrm{X})\subseteq BV^\star_{\mathsf{Geod}}(\mathrm{X})$ and

$$|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(B)\le|\boldsymbol{D}f|_{\mathsf{Geod}}(B)\le|\boldsymbol{D}f|(B)=|\boldsymbol{D}f|^\star_{\Pi_\infty}(B),\qquad\text{for all } f\in BV(\mathrm{X})\text{ and } B\subseteq\mathrm{X}\text{ Borel.}$$

Hence, in order to achieve the statement it suffices to prove that $BV^\star_{\mathsf{Geod}}(\mathrm{X})\subseteq BV^\star_{\Pi_\infty}(\mathrm{X})$ and

$$|\boldsymbol{D}f|^\star_{\Pi_\infty}(\Omega)\le|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\Omega)\quad\text{for every } f\in BV^\star_{\mathsf{Geod}}(\mathrm{X})\text{ and }\Omega\subseteq\mathrm{X}\text{ open.}\tag{5.3.5}$$

Step 1. Let $\Omega\subseteq\mathrm{X}$ be a given open set. Calling $\mathsf{PolGeo}:=\mathsf{PolGeo}_\infty(\mathrm{X})$ for brevity, we claim that

$$BV^\star_{\mathsf{Geod}}(\mathrm{X})\subseteq BV^\star_{\mathsf{PolGeo}}(\mathrm{X}),\qquad|\boldsymbol{D}f|^\star_{\mathsf{PolGeo}}(\Omega)\le|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\Omega)\quad\text{for every } f\in BV^\star_{\mathsf{Geod}}(\mathrm{X}).\tag{5.3.6}$$

In order to prove it, fix $f\in BV^\star_{\mathsf{Geod}}(\mathrm{X})$ and $\pi\in\mathsf{PolGeo}(\Omega)$. Choose $0=t_0<t_1<\ldots<t_n=1$ such that $\pi_i:=(\mathsf{Restr}_{t_{i-1}}^{t_i})_\sharp\pi\in\mathsf{Geod}(\Omega)$ for all $i=1,\ldots,n$. Since $\mathrm{Lip}(\pi_i)\le(t_i-t_{i-1})\mathrm{Lip}(\pi)$ and $\mathrm{Comp}(\pi_i)\le\mathrm{Comp}(\pi)$ for every $i=1,\ldots,n$, we may estimate

$$\begin{aligned}\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi(\gamma)\mathsf{e}\sum_{i=1}^n\int f(\gamma_{t_i})-f(\gamma_{t_{i-1}})\,\mathrm{d}\pi(\gamma)&=\sum_{i=1}^n\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi_i(\gamma)\\&\le\sum_{i=1}^n\mathrm{Comp}(\pi_i)\mathrm{Lip}(\pi_i)|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\Omega)\le\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\Omega),\end{aligned}$$

whence (5.3.6) follows thanks to the arbitrariness of $\pi\in\mathsf{PolGeo}(\Omega)$.
Step 2. Next we claim that, calling $\Pi_{bs}:=\Pi_{bs}(\mathrm{X})$ for brevity, it holds that

$$BV^\star_{\mathsf{PolGeo}}(\mathrm{X})\subseteq BV^\star_{\Pi_{bs}}(\mathrm{X}),\qquad|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\Omega)\le|\boldsymbol{D}f|^\star_{\mathsf{PolGeo}}(\Omega)\quad\text{for every } f\in BV^\star_{\mathsf{PolGeo}}(\mathrm{X}).\tag{5.3.7}$$

In order to prove it, fix $f\in BV^\star_{\mathsf{PolGeo}}(\mathrm{X})$ and $\pi\in\Pi_{bs}(\Omega)$. Suppose that $f(\gamma_1)-f(\gamma_0)$ does not change sign and it is positive for π-a.e. γ. Lemma 5.3.5 yields the existence of polygonal interpolations $(\pi_n)_{n\in\mathbb{N}}\subseteq\mathsf{PolGeo}$ of π with $\mathrm{Comp}(\pi_n)\le C_\infty\big(\mathrm{Lip}(\pi)/n\big)\mathrm{Comp}(\pi)$ and $\mathrm{Lip}(\pi_n)\le\mathrm{Lip}(\pi)$ for every $n\in\mathbb{N}$, and such that $[\pi_n]$ lies in the closed $\frac{\mathrm{Lip}(\pi)}{n}$-neighbourhood of $[\pi]$. Chosen $\bar n\in\mathbb{N}$ so that $\mathrm{Lip}(\pi)/\bar n<\mathsf{d}\big([\pi],\mathrm{X}\setminus\Omega\big)$, we thus have that $\pi_n\in\mathsf{PolGeo}(\Omega)$ for every $n\ge\bar n$. Given that $C(L)\to1$ as $L\to0$, by letting $n\to\infty$ in

$$\begin{aligned}\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi(\gamma)&=\int f\,\mathrm{d}(\mathsf{e}_1)_\sharp\pi-\int f\,\mathrm{d}(\mathsf{e}_0)_\sharp\pi=\int f\,\mathrm{d}(\mathsf{e}_1)_\sharp\pi_n-\int f\,\mathrm{d}(\mathsf{e}_0)_\sharp\pi_n\\&=\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi_n(\gamma)\le\mathrm{Comp}(\pi_n)\mathrm{Lip}(\pi_n)|\boldsymbol{D}f|^\star_{\mathsf{PolGeo}}(\Omega)\\&\le C_\infty\big(\mathrm{Lip}(\pi)/n\big)\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\mathsf{PolGeo}}(\Omega)\end{aligned}$$

we conclude that $\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi(\gamma)\le\mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\mathsf{PolGeo}}(\Omega)$ obtaining (5.3.7).
Step 3. Finally, we claim that

$$BV^\star_{\Pi_{bs}}(\mathrm{X})\subseteq BV^\star_{\Pi_\infty}(\mathrm{X}),\qquad|\boldsymbol{D}f|^\star_{\Pi_\infty}(\Omega)\le|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\Omega)\quad\text{for every } f\in BV^\star_{\Pi_{bs}}(\mathrm{X}).\tag{5.3.8}$$

In order to prove it, fix any $f\in BV^\star_{\Pi_{bs}}(\mathrm{X})$ and a ∞-test plan $\pi\in\Pi_\infty(\Omega)$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Fix a curve $\bar\gamma\in\mathrm{supp}(\pi)$ and define $\Gamma_n:=\big\{\gamma\in C([0,1],\mathrm{X})\,:\,\mathsf{d}_{\sup}(\gamma,\bar\gamma)\le n\big\}$ for every $n\in\mathbb{N}$. Define

$$\pi_n:=\frac{\pi|_{\Gamma_n}}{\pi(\Gamma_n)}\in\Pi_{bs}(\Omega),\qquad\text{for every } n\in\mathbb{N}.$$

Observe that $\mathrm{Comp}(\pi_n) \leq \mathrm{Comp}(\pi)/\pi(\Gamma_n)$ and $\mathrm{Lip}(\pi_n) \leq \mathrm{Lip}(\pi)$ for every $n \in \mathbb{N}$. Also, it holds $\pi(\Gamma_n) \nearrow 1$ as $n \to \infty$. Therefore, by using the dominated convergence theorem we get

$$\begin{aligned}\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi(\gamma) &= \lim_{n\to\infty} \int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi_n(\gamma)\\ &\leq \varliminf_{n\to\infty} \mathrm{Comp}(\pi_n)\mathrm{Lip}(\pi_n)|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\Omega)\\ &\leq \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\Omega) \lim_{n\to\infty} \frac{1}{\pi(\Gamma_n)}\\ &= \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\Omega),\end{aligned}$$

thus proving (5.3.8). By combining (5.3.6), (5.3.7), and (5.3.8), we eventually obtain (5.3.5). □

Remark 5.3.7. We point out that if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a non-branching $\mathsf{CD}(K,N)$ space with $K \geq 0$, then the proof of Theorem 5.3.6 can be significantly simplified. Indeed, in this case one can prove that $BV^\star_{\mathsf{Geod}}(\mathrm{X}) \subseteq BV^\star_{\Pi_{bs}}(\mathrm{X})$ and $|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\mathrm{X}) \leq |\boldsymbol{D}f|^\star_{\Pi_{\mathrm{G}}}(\mathrm{X})$ for every $f \in BV^\star_{\mathsf{Geod}}(\mathrm{X})$ in the following way. Given any $f \in BV^\star_{\mathsf{Geod}}(\mathrm{X})$ and $\eta \in \Pi_{bs}$, let us just take the ∞-test plan $\pi \in \mathrm{OptGeo}_\infty\big((\mathsf{e}_0)_\sharp\eta, (\mathsf{e}_1)_\sharp\eta\big)$ satisfying both $\mathrm{Comp}(\pi) \leq \mathrm{Comp}(\eta)$ and $\mathrm{Lip}(\pi) \leq \mathrm{Lip}(\eta)$, whose existence is granted by Theorem 5.3.1. Then,

$$\begin{aligned}\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\eta(\gamma) = \int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi(\gamma) &\leq \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\mathrm{X})\\ &\leq \mathrm{Comp}(\eta)\mathrm{Lip}(\eta)|\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\mathrm{X}).\end{aligned}$$

thus proving that $f \in BV^\star_{\Pi_{bs}}(\mathrm{X})$ and $|\boldsymbol{D}f|^\star_{\Pi_{bs}}(\mathrm{X}) \leq |\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\mathrm{X})$. ■

In the more general setting of non-branching MCP spaces, we can prove a weaker statement:

Remark 5.3.8 (A weaker form of master geodesic plans on MCP spaces)**.** Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a non-branching $\mathsf{MCP}(K,N)$ metric measure space with $N < \infty$ (but without finiteness assumptions on $\mathfrak{m}$) and recall that it is weaker than $\mathsf{CD}_q(K,N)$ for any $q \in (1,\infty)$. By adapting our previous arguments, we can get $BV_{\mathsf{Geod}}(\mathrm{X}) = BV(\mathrm{X})$ and

$$|\boldsymbol{D}f|_{\mathsf{Geod}}(B) \leq |\boldsymbol{D}f|(B) \leq 2^N|\boldsymbol{D}f|_{\mathsf{Geod}}(B), \quad \text{for every } f \in BV(\mathrm{X}) \text{ and } B \subseteq \mathrm{X} \text{ Borel.} \tag{5.3.9}$$

The first inequality is always verified. Below we sketch the proof of the second inequality.

Fix a boundedly-supported ∞-test plan η and $q \in (1,\infty)$. Call $\mu_i := (\mathsf{e}_i)_\sharp\eta$ for $i = 0,1$. We know from [56, Proposition 9.1] that the unique element π_q of $\mathrm{OptGeo}_q(\mu_0,\mu_1)$ satisfies

$$\rho^q_t(\gamma_t)^{-\frac{1}{N}} \geq \rho_0(\gamma_0)^{-\frac{1}{N}}\tau^{(1-t)}_{K,N}\big(\mathsf{d}(\gamma_0,\gamma_1)\big), \quad \text{for } \pi_q\text{-a.e. } \gamma \text{ and every } t \in [0,1), \tag{5.3.10}$$

where we set $\rho^q_t\mathfrak{m} := (\mathsf{e}_t)_\sharp\pi_q$. We point out that [56, Proposition 9.1] concerns the case $q = 2$, but the proof argument works for $q \in (1,\infty)$ arbitrary recalling that the MCP condition is by definition independent of q (see Remark 2.5.2). By applying (5.3.10) to the 'reversed-in-time' plan $(\mathsf{Restr}^0_1)_\sharp\pi_q$, which is the unique element of $\mathrm{OptGeo}_q(\mu_1,\mu_0)$, we obtain the symmetric estimate

$$\rho^q_t(\gamma_t)^{-\frac{1}{N}} \geq \rho_1(\gamma_1)^{-\frac{1}{N}}\tau^{(t)}_{K,N}\big(\mathsf{d}(\gamma_0,\gamma_1)\big), \quad \text{for } \pi_q\text{-a.e. } \gamma \text{ and every } t \in (0,1]. \tag{5.3.11}$$

Define the function $C_\infty \colon (0,+\infty) \to (0,+\infty)$ as in the proof of Theorem 5.3.1. Then we have

$$\rho^q_t(\gamma_t)^{\frac{1}{N}} \overset{(5.3.10)}{\leq} \frac{1}{1-t}\,C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{\frac{1}{N}}\rho_0(\gamma_0)^{\frac{1}{N}} \leq 2\,C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{\frac{1}{N}}\mathrm{Comp}(\eta)^{\frac{1}{N}},$$

for π_q-a.e. γ and every $t \in [0,1/2]$. Similarly, we get $\rho^q_t(\gamma_t)^{\frac{1}{N}} \leq 2\,C_\infty\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^{\frac{1}{N}}\mathrm{Comp}(\eta)^{\frac{1}{N}}$ for π_q-a.e. γ and every $t \in [1/2,1]$ by using (5.3.11) in place of (5.3.10). By arguing as in the last part of the proof of Theorem 5.3.1 (letting $q \to \infty$ and using the compactness result in Proposition 5.2.6), we thus obtain a plan $\pi \in \mathrm{OptGeo}_\infty(\mu_0, \mu_1)$ such that

$$\mathrm{Comp}(\pi) \leq 2^N C_\infty\big(\mathrm{Lip}(\eta)\big)\mathrm{Comp}(\eta). \tag{5.3.12}$$

Finally, by arguing as in the proof of Theorem 5.3.6, but using (5.3.12) in place of (5.3.1), we conclude that $|\boldsymbol{D}f|(\Omega) = |\boldsymbol{D}f|^\star_{\Pi_\infty}(\Omega) \leq 2^N |\boldsymbol{D}f|^\star_{\mathsf{Geod}}(\Omega) \leq 2^N |\boldsymbol{D}f|_{\mathsf{Geod}}(\Omega)$ holds for every choice of $f \in BV_{\mathsf{Geod}}(\mathrm{X})$ and $\Omega \subseteq \mathrm{X}$ open, whence (5.3.9) follows by outer regularity. ■

5.3.2 Master sequences of test plans on metric measure spaces

Aim of this section is to prove that on any metric measure space one can find a countable master family for BV. On non-branching CD spaces, the master sequence can be required to consist of geodesic plans.

Theorem 5.3.9. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then there exists an (at most) countable family $\mathcal{D}$ of ∞-test plans on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ that is a master family for $BV(\mathrm{X})$.*

If $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a non-branching $\mathsf{CD}(K,N)$ space with $K \in \mathbb{R}$, $N \in (1,\infty)$, and $\mathfrak{m}(\mathrm{X}) < +\infty$, then we can additionally require that $\mathcal{D} \subseteq \mathsf{Geod}_\infty(\mathrm{X})$.

Proof. First, fix any master family Π for $BV(\mathrm{X})$. In particular, one can take Π the totality of ∞-test plans or, in setting of non-branching $\mathsf{CD}(K,N)$ spaces with $N, \mathfrak{m}$ finite, we know from Theorem 5.3.6 that also the choice $\Pi = \mathsf{Geod}_\infty(\mathrm{X})$ is allowed. Now define

$$\Pi_{\alpha,\beta} := \Big\{\pi \in \Pi \,:\, \mathrm{Comp}(\pi) \leq \alpha,\, \mathrm{Lip}(\pi) \leq \beta\Big\}, \quad \text{for every } \alpha, \beta \in \mathbb{Q}^+.$$

Note that $\Pi = \bigcup_{\alpha,\beta\in\mathbb{Q}^+} \Pi_{\alpha,\beta}$. Given any $\alpha, \beta \in \mathbb{Q}^+$, we select a countable set $\mathcal{D}_{\alpha,\beta} \subseteq \Pi_{\alpha,\beta}$ that is dense in $\Pi_{\alpha,\beta}$ with respect to the weak topology. Consider the countable family

$$\mathcal{D} := \bigcup_{\alpha,\beta\in\mathbb{Q}^+} \mathcal{D}_{\alpha,\beta} \subseteq \Pi.$$

We aim to show that $\mathcal{D}$ fulfills the requirements. Given that $BV^\star_\Pi(\mathrm{X}) = BV(\mathrm{X}) \subseteq BV_{\mathcal{D}}(\mathrm{X})$ and $|\boldsymbol{D}f|_{\mathcal{D}} \leq |\boldsymbol{D}f|$ for every $f \in BV(\mathrm{X})$, it is sufficient to prove that $BV_{\mathcal{D}}(\mathrm{X}) \subseteq BV^\star_\Pi(\mathrm{X})$ and $|\boldsymbol{D}f|^\star_\Pi(\mathrm{X}) \leq |\boldsymbol{D}f|_{\mathcal{D}}(\mathrm{X})$ for every $f \in BV_{\mathcal{D}}(\mathrm{X})$, which amounts to showing that

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi(\gamma) \leq \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|_{\mathcal{D}}(\mathrm{X}), \quad \text{for every } f \in BV_{\mathcal{D}}(\mathrm{X}) \text{ and } \pi \in \Pi. \tag{5.3.13}$$

To this aim, let $f \in BV_{\mathcal{D}}(\mathrm{X})$ and $\pi \in \Pi$ be fixed. Pick two sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}^+$ satisfying $\alpha_n \searrow \mathrm{Comp}(\pi)$ and $\beta_n \searrow \mathrm{Lip}(\pi)$. For any $n \in \mathbb{N}$ we have that $\pi \in \Pi_{\alpha_n,\beta_n}$, so that we can find a plan $\pi_n \in \mathcal{D}_{\alpha_n,\beta_n} \subseteq \mathcal{D}$ such that $\mathsf{d}_{\mathscr{P}}(\pi_n, \pi) \leq 1/n$. This means that $\pi_n \rightharpoonup \pi$ as $n \to \infty$, so that in particular $(\mathsf{e}_0)_\sharp \pi_n \rightharpoonup (\mathsf{e}_0)_\sharp \pi$ and $(\mathsf{e}_1)_\sharp \pi_n \rightharpoonup (\mathsf{e}_1)_\sharp \pi$ by Remark 1.2.3. Given any $\varepsilon > 0$, we can find a function $f_\varepsilon \in C_{bs}(\mathrm{X})$ such that $\|f - f_\varepsilon\|_{L^1(\mathfrak{m})} \leq \varepsilon$. Using the fact that $\sup_{n\in\mathbb{N}} \mathrm{Comp}(\pi_n) \leq \alpha_1$, we can thus estimate

$$\begin{aligned}
&\left|\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi(\gamma) - \int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi_n(\gamma)\right| \\
\leq &\left|\int f_\varepsilon(\gamma_1) - f_\varepsilon(\gamma_0)\,\mathrm{d}\pi(\gamma) - \int f_\varepsilon(\gamma_1) - f_\varepsilon(\gamma_0)\,\mathrm{d}\pi_n(\gamma)\right| + 4\alpha_1\varepsilon \\
\leq &\left|\int f_\varepsilon\,\mathrm{d}(\mathsf{e}_1)_\sharp\pi - \int f_\varepsilon\,\mathrm{d}(\mathsf{e}_1)_\sharp\pi_n\right| + \left|\int f_\varepsilon\,\mathrm{d}(\mathsf{e}_0)_\sharp\pi - \int f_\varepsilon\,\mathrm{d}(\mathsf{e}_0)_\sharp\pi_n\right| + 4\alpha_1\varepsilon.
\end{aligned}$$

By first letting $n \to \infty$ and then $\varepsilon \searrow 0$, we deduce that

$$\int f(\gamma_1) - f(\gamma_0)\,\mathrm{d}\pi_n(\gamma) \to \int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi(\gamma), \quad \text{as } n \to \infty.$$

Therefore, we can eventually conclude that

$$\begin{aligned}\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi(\gamma) &= \lim_{n\to\infty}\int f(\gamma_1)-f(\gamma_0)\,\mathrm{d}\pi_n(\gamma)\\ &\leq \varliminf_{n\to\infty}\int |D(f\circ\gamma)|([0,1])\,\mathrm{d}\pi_n(\gamma)\\ &= \varliminf_{n\to\infty}\int \gamma_\sharp|D(f\circ\gamma)|(\mathrm{X})\,\mathrm{d}\pi_n(\gamma)\\ &\leq \varliminf_{n\to\infty}\mathrm{Comp}(\pi_n)\mathrm{Lip}(\pi_n)|\boldsymbol{D}f|_{\mathfrak{D}}(\mathrm{X})\\ &\leq \lim_{n\to\infty}\alpha_n\beta_n|\boldsymbol{D}f|_{\mathfrak{D}}(\mathrm{X})\\ &= \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)|\boldsymbol{D}f|_{\mathfrak{D}}(\mathrm{X}).\end{aligned}$$

This gives the desired inequality in (5.3.13), thus yielding the sought conclusion. □

Remark 5.3.10. Actually, the proof of Theorem 5.3.9 shows also the following statement: from any master family Π for $BV(\mathrm{X})$, one can extract a countable subfamily $\tilde{\Pi}\subseteq\Pi$ that is still a master family for $BV(\mathrm{X})$. ■

We expect that, since the defining properties in Definition (5.2.2) and Definition 5.2.1 of $BV_\Pi(\mathrm{X})$ space is in 'integral form', it is not suitable to improve Theorem 5.3.9 and obtain a single master plan for $BV(\mathrm{X})$, i.e., a master family that is a singleton. However, in Section 5.4 we will deal with a 'curvewise' definition of $BV(\mathrm{X})$, which will allow us to build a single test plan (concentrated on geodesics in the non-branching CD case) that is a master plan for $BV(\mathrm{X})$ in the curvewise sense.

5.4 Curvewise BV space

As we already saw during this chapter, a smorgasbord of different (and mostly equivalent) notions of BV space over metric measure spaces have been thoroughly studied in the literature. In this section we propose yet another definition of BV space, which we will refer to as the curvewise BV space. In Subsection 5.4.1 we introduce it and prove its equivalence with $BV(\mathrm{X})$. In Subsection 5.4.2 we show that Theorem 5.3.9 implies the existence of a single ∞-test plan which is a master test plan in the curvewise BV sense.

5.4.1 Definition of $BV^{\mathsf{cw}}(\mathrm{X})$ and its main properties

Here we introduce a new notion of function of bounded variation on a metric measure space: the *curvewise BV space*, which we shall denote by $BV^{\mathsf{cw}}(\mathrm{X})$. Our definition is heavily inspired by the so-called *AM-BV space*, which we are going to recall briefly. The potential-theoretic notion of *approximation modulus*, AM-modulus for short, has been recently introduced by O. Martio in [151]. By building on top of it, he constructed in [152] a 'Newtonian-type' version of BV space, denoted by $BV_{AM}(\mathrm{X})$.

Much like an L^p-function belongs to the Newtonian–Sobolev space $N^{1,p}(\mathrm{X})$ provided it satisfies the weak upper gradient inequality along Mod_p-a.e. path (where Mod_p stands for the p-modulus), an L^1-function is declared to be in $BV_{AM}(\mathrm{X})$ provided it satisfies the BV_{AM} *upper bound* inequality (cf. [152, Eq. (2.6)]) along AM-a.e. path. Our strategy is the following: to replace the quantification 'along AM-a.e. path' by 'along π-a.e. curve, for every ∞-test plan π'. Technically speaking, to do so we first need to slightly adapt the concept of BV_{AM} upper bound, thus introducing that of *curvewise bound*, see Definition 5.4.1. The resulting function space $BV^{\mathsf{cw}}(\mathrm{X})$ (cf. Definition 5.4.2) is a priori larger than $BV_{AM}(\mathrm{X})$, the reason being that AM-null sets are π-null for every ∞-test plan π. We will prove in Theorem 5.4.3 (see also Corollary 5.4.4) that $BV^{\mathsf{cw}}(\mathrm{X})=BV(\mathrm{X})$ on every metric measure space. In this Thesis, we shall not provide the proof that that $BV_{AM}(\mathrm{X})=BV(\mathrm{X})$, and thus $BV_{AM}(\mathrm{X})=BV^{\mathsf{cw}}(\mathrm{X})$, are verified on every metric measure space since it is outside of our main goals. However, these claims are true as verified in [162].

Definition 5.4.1 (Curvewise Π-bound)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix $f \in L^1(\mathfrak{m})$ and a family Π of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Let $\Omega \subseteq \mathrm{X}$ be an open set. Then a given sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^1(\mathfrak{m}_{|\Omega})$ of non-negative functions is said to be a curvewise Π-bound for f on Ω provided for any $\pi \in \Pi$ the following property is verified: for π-a.e. curve γ, it holds that*

$$|D(f\circ\gamma)|((a,b)) \leq \varliminf_{n\to\infty} \int_a^b g_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t, \qquad \textit{for every } 0<a<b<1 \textit{ with } \gamma((a,b)) \subseteq \Omega. \tag{5.4.1}$$

When Π is the totality of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we just speak about curvewise bounds.

The bounded compressibility assumption on the plans π ensures that (5.4.1) is well-posed, since it is independent of the chosen Borel representatives of f and $(g_n)_{n\in\mathbb{N}}$. By building on top of the above notion of curvewise Π-bound, we propose a new space of functions having bounded Π-variation, which we call the curvewise BV_Π space and we denote it by $BV_\Pi^{\mathsf{cw}}(\mathrm{X})$.

Definition 5.4.2 (Curvewise Π-BV space)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let Π be a family of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then we denote by $BV_\Pi^{\mathsf{cw}}(\mathrm{X})$ the space of all those functions $f \in L^1(\mathfrak{m})$ which admit a curvewise Π-bound $(g_n)_n$ on X with $\sup_n \|g_n\|_{L^1(\mathfrak{m})} < \infty$. Given any $f \in BV_\Pi^{\mathsf{cw}}(\mathrm{X})$ and $\Omega \subseteq \mathrm{X}$ open, we define*

$$|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) := \inf_{(g_n)_n} \varliminf_{n\to\infty} \int_\Omega g_n\,\mathrm{d}\mathfrak{m}, \tag{5.4.2}$$

where the infimum is taken among all curvewise Π-bounds $(g_n)_n$ for f on Ω. When Π is the totality of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we use the shorthand notation $BV^{\mathsf{cw}}(\mathrm{X})$ and $|\boldsymbol{D}f|^{\mathsf{cw}}$. When $\Pi = \{\boldsymbol{\pi}\}$ is a singleton, we use the shorthand notation $BV_{\boldsymbol{\pi}}^{\mathsf{cw}}(\mathrm{X})$ and $|\boldsymbol{D}f|_{\boldsymbol{\pi}}^{\mathsf{cw}}$.

Observe that $BV_\Pi^{\mathsf{cw}}(\mathrm{X})$ can be equivalently characterised as the set of all $f \in L^1(\mathfrak{m})$ for which the quantity in the right-hand side of (5.4.2) (with $\Omega = \mathrm{X}$) is finite. Moreover, given any $f \in BV_\Pi^{\mathsf{cw}}(\mathrm{X})$, we can extend the function $\Omega \mapsto |\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega)$ introduced in (5.4.2) to a set-function defined on all Borel sets via Carathéodory construction, in the following way:

$$|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(B) := \inf\big\{|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) \,:\, \Omega \subseteq \mathrm{X} \text{ open}, \, B \subseteq \Omega\big\}, \qquad \text{for every } B \subseteq \mathrm{X} \text{ Borel}. \tag{5.4.3}$$

We will not check whether the set-function $|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}$ in (5.4.3) actually defines a Borel measure on (X,d) when Π is an arbitrary family of ∞-test plans. However, this is the case in the specific situation where Π is a master family for $BV(\mathrm{X})$, as granted by the following result.

Theorem 5.4.3. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let Π be a family of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then $BV(\mathrm{X}) \subseteq BV_\Pi^{\mathsf{cw}}(\mathrm{X}) \subseteq BV_\Pi(\mathrm{X})$ and*

$$|\boldsymbol{D}f|_\Pi(\Omega) \leq |\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) \leq |\boldsymbol{D}f|(\Omega), \qquad \textit{for every } f \in BV(\mathrm{X}) \textit{ and } \Omega \subseteq \mathrm{X} \textit{ open}. \tag{5.4.4}$$

Moreover, if Π is a master family for $BV(\mathrm{X})$, then $BV_\Pi^{\mathsf{cw}}(\mathrm{X}) = BV(\mathrm{X})$ and $|\boldsymbol{D}f|_\Pi^{\mathsf{cw}} = |\boldsymbol{D}f|$ holds for every $f \in BV(\mathrm{X})$, thus in particular $|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}$ is a finite Borel measure on (X,d).

Proof.
Step 1. First of all, we aim to show that $BV(\mathrm{X}) \subseteq BV_\Pi^{\mathsf{cw}}(\mathrm{X})$ and

$$|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) \leq |\boldsymbol{D}f|(\Omega), \qquad \text{for every } f \in BV(\mathrm{X}) \text{ and } \Omega \subseteq \mathrm{X} \text{ open}. \tag{5.4.5}$$

To prove it, pick any sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{Lip}_{loc}(\Omega) \cap L^1(\mathfrak{m}_{|\Omega})$ such that $f_n \to f$ in $L^1(\mathfrak{m}_{|\Omega})$ and $\int_\Omega \mathrm{lip}\, f_n\,\mathrm{d}\mathfrak{m} \to |\boldsymbol{D}f|(\Omega)$, whose existence is granted by (1.2.14). Fix any $\pi \in \Pi$. Choose Borel functions $\bar f\colon \mathrm{X} \to \mathbb{R}$ and $\bar f_n\colon \mathrm{X} \to \mathbb{R}$, $n \in \mathbb{N}$, having the following properties: $\bar f = f$ holds $\mathfrak{m}$-a.e.

on Ω, $\bar{f}_n = f_n$ on Ω for every $n \in \mathbb{N}$, $\bar{f}_n = \bar{f}$ on $\mathrm{X} \setminus \Omega$ for every $n \in \mathbb{N}$, and $\int_{\mathrm{X}\setminus\Omega} |\bar{f}| \,\mathrm{d}\mathfrak{m} < +\infty$. In particular, it holds that $\bar{f}_n \to \bar{f}$ in $L^1(\mathfrak{m})$. Now observe that

$$\bar{f}_n \circ \gamma \to \bar{f} \circ \gamma \text{ in } L^1(0,1), \qquad \text{for } \pi\text{-a.e. } \gamma, \tag{5.4.6}$$

possibly after passing to a (not relabelled) subsequence in n. Indeed, we can estimate

$$\begin{aligned}\iint_0^1 \big|(\bar{f}_n \circ \gamma)(t) - (\bar{f} \circ \gamma)(t)\big| \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) &= \iint_0^1 |\bar{f}_n - \bar{f}| \circ \mathsf{e}_t \,\mathrm{d}t \,\mathrm{d}\pi \\ &\leq \mathrm{Comp}(\pi) \int |\bar{f}_n - \bar{f}| \,\mathrm{d}\mathfrak{m} \to 0, \qquad \text{as } n \to \infty,\end{aligned}$$

thus up to a not relabelled subsequence we have that $\int_0^1 \big|(\bar{f}_n \circ \gamma)(t) - (\bar{f} \circ \gamma)(t)\big| \,\mathrm{d}t \to 0$ as $n \to \infty$ for π-a.e. γ, so that accordingly the claimed property (5.4.6) is proven.

Now pick a π-null Borel set N of curves where the property in (5.4.6) fails. Fix any $\gamma \notin N$ and $0 < a < b < 1$ such that $\gamma((a,b)) \subseteq \Omega$. Thanks to the lower semicontinuity and the locality of the total variation measures, we obtain that

$$\begin{aligned}|D(f \circ \gamma)|((a,b)) = |D(\bar{f} \circ \gamma)|((a,b)) &\leq \varliminf_{n\to\infty} |D(\bar{f}_n \circ \gamma)|((a,b)) = \varliminf_{n\to\infty} \int_a^b |(\bar{f}_n \circ \gamma)'_t| \,\mathrm{d}t \\ &\leq \varliminf_{n\to\infty} \int_a^b (\mathrm{lip}\ \bar{f}_n)(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \leq \varliminf_{n\to\infty} \int_a^b \big(\chi_\Omega \,\mathrm{lip}\ f_n\big)(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t.\end{aligned}$$

This shows that $\big(\chi_\Omega \,\mathrm{lip}\ f_n\big)_n$ is a curvewise Π-bound for f on Ω. When considering $\Omega = \mathrm{X}$, we get that $f \in BV_\Pi^{\mathsf{cw}}(\mathrm{X})$. The same estimates also give

$$|\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) \leq \varliminf_{n\to\infty} \int \chi_\Omega \,\mathrm{lip}\ f_n \,\mathrm{d}\mathfrak{m} = |\boldsymbol{D}f|(\Omega),$$

whence (5.4.5) follows.
Step 2. Next we claim that $BV_\Pi^{\mathsf{cw}}(\mathrm{X}) \subseteq BV_\Pi(\mathrm{X})$ and

$$|\boldsymbol{D}f|_\Pi(\Omega) \leq |\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega), \qquad \text{for every } f \in BV_\Pi^{\mathsf{cw}}(\mathrm{X}) \text{ and } \Omega \subseteq \mathrm{X} \text{ open.} \tag{5.4.7}$$

In order to prove it, let $\pi \in \Pi$ and $\varepsilon > 0$ be fixed. We can find a curvewise Π-bound $(g_n)_n$ for f on Ω satisfying $\varliminf_n \int_\Omega g_n \,\mathrm{d}\mathfrak{m} \leq |\boldsymbol{D}f|_\Pi^{\mathsf{cw}}(\Omega) + \varepsilon$. Thanks to Lemma 1.2.26, there exists a π-null Borel set N of curves such that $f \circ \gamma \in BV(0,1)$ and $|D(f \circ \gamma)|(\{0,1\}) = 0$ for all $\gamma \notin N$. Given any $\gamma \notin N$, we can find $N_\gamma \in \mathbb{N} \cup \{\infty\}$ and $\{a_i^\gamma\}_{i<N_\gamma}, \{b_i^\gamma\}_{i<N_\gamma} \subseteq [0,1]$ such that

$$a_i^\gamma < b_i^\gamma < a_{i+1}^\gamma < b_{i+1}^\gamma, \qquad \gamma^{-1}(\Omega) \cap (0,1) = \bigcup_{i<N_\gamma} (a_i^\gamma, b_i^\gamma). \tag{5.4.8}$$

Therefore, for any $\gamma \notin N$ we may estimate

$$\begin{aligned}\gamma_\sharp |D(f \circ \gamma)|(\Omega) = |D(f \circ \gamma)|\big(\gamma^{-1}(\Omega) \cap (0,1)\big) &\overset{(5.4.8)}{=} \sum_{i<N_\gamma} |D(f \circ \gamma)|((a_i^\gamma, b_i^\gamma)) \\ &\leq \sum_{i<N_\gamma} \varliminf_{n\to\infty} \int_{a_i^\gamma}^{b_i^\gamma} g_n(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \overset{\star}{\leq} \varliminf_{n\to\infty} \sum_{i<N_\gamma} \int_{a_i^\gamma}^{b_i^\gamma} g_n(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \\ &= \varliminf_{n\to\infty} \int_{\gamma^{-1}(\Omega)} g_n(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t,\end{aligned}$$

where the starred inequality is granted by Fatou's lemma. By integrating over π, we obtain

$$
\begin{aligned}
\int \gamma_\sharp |D(f\circ\gamma)|(\Omega)\,\mathrm{d}\pi(\gamma) &\le \int \varliminf_{n\to\infty} \int_{\gamma^{-1}(\Omega)} g_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\
&\overset{\star}{\le} \varliminf_{n\to\infty} \iint_{\gamma^{-1}(\Omega)} g_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\
&\le \mathrm{Lip}(\pi)\varliminf_{n\to\infty}\iint (\chi_\Omega\, g_n)\circ \mathsf{e}_t\,\mathrm{d}t\,\mathrm{d}\pi\\
&\le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\varliminf_{n\to\infty}\int_\Omega g_n\,\mathrm{d}\mathfrak{m}\\
&\le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\big(|\boldsymbol{D}f|^{\mathsf{cw}}_\Pi(\Omega)+\varepsilon\big),
\end{aligned}
$$

where the starred inequality is a consequence of Fatou's lemma. By letting $\varepsilon \searrow 0$, and exploiting the arbitrariness of $\pi \in \Pi$ and the open set Ω, we conclude that $f \in BV_\Pi(\mathrm{X})$ and that $|\boldsymbol{D}f|_\Pi(\Omega) \le |\boldsymbol{D}f|^{\mathsf{cw}}_\Pi(\Omega)$ for every $\Omega \subseteq \mathrm{X}$ open, thus proving the claimed property (5.4.7).

Step 3. What is left to prove is only the last part of the statement. If Π is a master family for $BV(\mathrm{X})$, then (5.4.4) forces the identities $BV^{\mathsf{cw}}_\Pi(\mathrm{X}) = BV(\mathrm{X})$, and $|\boldsymbol{D}f|^{\mathsf{cw}}_\Pi(\Omega) = |\boldsymbol{D}f|(\Omega)$ for every $f \in BV(\mathrm{X})$ and $\Omega \subseteq \mathrm{X}$ open. By virtue of the definition (5.4.3) and the outer regularity of the measure $|\boldsymbol{D}f|$, we deduce that $|\boldsymbol{D}f|^{\mathsf{cw}}_\Pi(B) = |\boldsymbol{D}f|(B)$ holds for every Borel set $B \subseteq \mathrm{X}$, thus in particular $|\boldsymbol{D}f|^{\mathsf{cw}}_\Pi$ is a finite Borel measure as well. The statement is achieved. □

For the sake of clarity, we report the following immediate consequence of Theorem 5.4.3.

Corollary 5.4.4. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then $BV^{\mathsf{cw}}(\mathrm{X}) = BV(\mathrm{X})$ and*

$$|\boldsymbol{D}f|^{\mathsf{cw}} = |\boldsymbol{D}f|, \qquad \textit{for every } f \in BV(\mathrm{X}).$$

Proof. Since the totality of ∞-test plans on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a master family for $BV(\mathrm{X})$, the claim directly follows from the last part of the statement of Theorem 5.4.3. □

5.4.2 A master curvewise test plan for BV

Aim of this section is to build a ∞-test plan $\boldsymbol{\pi}_{\mathsf{m}}$ (concentrated on geodesics when the underlying space is non-branching CD) such that

$$BV^{\mathsf{cw}}_{\boldsymbol{\pi}_{\mathsf{m}}}(\mathrm{X}) = BV(\mathrm{X}).$$

The test plan $\boldsymbol{\pi}_{\mathsf{m}}$ will be obtained by suitably combining the elements that constitute the master sequence for $BV(\mathrm{X})$ provided by Theorem 5.3.9. Before passing to the construction of the plan $\boldsymbol{\pi}_{\mathsf{m}}$ in Theorem 5.4.6, we need to prove the following technical lemma, concerning the behaviour of $BV^{\mathsf{cw}}_\Pi(\mathrm{X})$ under different manipulations of the family Π of ∞-test plans.

Lemma 5.4.5. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then the following properties hold:*

i) *Let $(\pi_n)_{n\in\mathbb{N}}$ be a given sequence of ∞-test plans. Let $(\alpha_n)_{n\in\mathbb{N}} \subseteq (0,1)$ be a sequence with $\sum_{n=1}^\infty \alpha_n = 1$. Suppose $\sup_n \mathrm{Lip}(\pi_n) < +\infty$ and $\sum_{n=1}^\infty \alpha_n \mathrm{Comp}(\pi_n) < +\infty$. Then $\pi := \sum_{n=1}^\infty \alpha_n\pi_n$ is a ∞-test plan satisfying $BV^{\mathsf{cw}}_\pi(\mathrm{X}) = BV^{\mathsf{cw}}_{\{\pi_n\}_n}(\mathrm{X})$ and*

$$|\boldsymbol{D}f|^{\mathsf{cw}}_\pi = |\boldsymbol{D}f|^{\mathsf{cw}}_{\{\pi_n\}_n}, \qquad \textit{for every } f \in BV^{\mathsf{cw}}_{\{\pi_n\}_n}(\mathrm{X}).$$

ii) *Let $\Pi = \{\pi_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of ∞-test plans. For any $\lambda \in \Lambda$, fix a number $k(\lambda) \in \mathbb{N}$ and a subdivision $0 = t^0_\lambda < t^1_\lambda \ldots < t^{k(\lambda)}_\lambda = 1$ of $[0,1]$. Define the ∞-test plans*

$$\pi^i_\lambda := \big(\mathsf{Restr}_{t^{i-1}_\lambda}^{t^i_\lambda}\big)_\sharp \pi_\lambda, \qquad \textit{for every } \lambda \in \Lambda \textit{ and } i = 1,\ldots,k(\lambda).$$

Then the family $\hat\Pi := \big\{\pi^i_\lambda \,:\, \lambda \in \Lambda,\, 1 \le i \le k(\lambda)\big\}$ satisfies $BV^{\mathsf{cw}}_{\hat\Pi}(\mathrm{X}) = BV^{\mathsf{cw}}_\Pi(\mathrm{X})$ and

$$|\boldsymbol{D}f|^{\mathsf{cw}}_{\hat\Pi} = |\boldsymbol{D}f|^{\mathsf{cw}}_\Pi, \qquad \textit{for every } f \in BV^{\mathsf{cw}}_\Pi(\mathrm{X}).$$

Proof.
i) Observe that a Borel set $N \subseteq C([0,1],\mathrm{X})$ is π-null if and only if it is π_n-null for all $n \in \mathbb{N}$. This implies that curvewise π-bounds and curvewise $\{\pi_n\}_n$-bounds coincide, yielding i).
ii) Fix any $f \in L^1(\mathfrak{m})$ and an open set $\Omega \subseteq \mathrm{X}$. It is then sufficient to show that a given sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^1(\mathfrak{m}_{|_\Omega})$ is a curvewise Π-bound for f on Ω if and only if it is a curvewise $\hat{\Pi}$-bound for f on Ω. Suppose the former holds. Fix any $\lambda \in \Lambda$ and $i = 1,\dots,k(\lambda)$. Let N be a π_λ-null Borel set of curves such that the property in (5.4.1) holds for every fixed $\gamma \notin N$; here and in the rest of the proof, we are assuming to have fixed a Borel representative of f. In particular, $\sigma := \mathsf{Restr}_{t_\lambda^{i-1}}^{t_\lambda^i}(\gamma)$ has this property: if $0 < a < b < 1$ and $\sigma((a,b)) \subseteq \Omega$, then

$$|D(f\circ\sigma)|((a,b)) = |D(f\circ\gamma)|((a',b')) \leq \varliminf_{n\to\infty}\int_{a'}^{b'} g_n(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t = \varliminf_{n\to\infty}\int_a^b g_n(\sigma_t)|\dot{\sigma}_t|\,\mathrm{d}t,$$

where we set $a' := (1-a)t_\lambda^{i-1} + a\,t_\lambda^i$ and $b' := (1-b)t_\lambda^{i-1} + b\,t_\lambda^i$. Hence, given that the set of such curves σ's is π_λ^i-null, we have proved that $(g_n)_n$ is a curvewise $\hat{\Pi}$-bound for f on Ω.

Conversely, suppose $(g_n)_n$ is a curvewise $\hat{\Pi}$-bound for f on Ω. Fix any $\lambda \in \Lambda$. Given any index $i = 1,\dots,k(\lambda)$, we can find a π_λ^i-null Borel set N_i of curves such that (5.4.1) holds for all $\sigma \notin N_i$. Thanks to Lemma 1.2.26, we can find a π_λ-null Borel set $\tilde{N}$ of curves such that

$$|D(f\circ\gamma)|\big(\{t_\lambda^1,\dots,t_\lambda^{k(\lambda)-1}\}\big) = 0, \qquad \text{for every } \gamma \notin \tilde{N}. \tag{5.4.9}$$

Now let us consider the π_λ-null set N of curves, which is defined as

$$N := \tilde{N} \cup \bigcup_{i=1}^{k(\lambda)} \big(\mathsf{Restr}_{t_\lambda^{i-1}}^{t_\lambda^i}\big)^{-1}(N_i).$$

Fix $\gamma \notin N$ and $0 < a < b < 1$ with $\gamma((a,b)) \subseteq \Omega$. For any $i = 1,\dots,k(\lambda)$, we denote by I_i the open interval $(a,b) \cap (t_\lambda^{i-1}, t_\lambda^i)$. Given that $\sigma^i := \mathsf{Restr}_{t_\lambda^{i-1}}^{t_\lambda^i}(\gamma) \notin N_i$, we may estimate

$$\begin{aligned}|D(f\circ\gamma)|((a,b)) &\overset{(5.4.9)}{=} \sum_{i=1}^{k(\lambda)} |D(f\circ\gamma)|(I_i) = \sum_{i=1}^{k(\lambda)} |D(f\circ\sigma^i)|(I_i') \leq \sum_{i=1}^{k(\lambda)} \varliminf_{n\to\infty}\int_{I_i'} g_n(\sigma_t^i)|\dot{\sigma}_t^i|\,\mathrm{d}t \\ &\leq \varliminf_{n\to\infty}\sum_{i=1}^{k(\lambda)}\int_{I_i'} g_n(\sigma_t^i)|\dot{\sigma}_t^i|\,\mathrm{d}t = \varliminf_{n\to\infty}\int_a^b g_n(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t,\end{aligned}$$

where we set $I_i' := \mathsf{Restr}_{t_\lambda^{i-1}}^{t_\lambda^i}(I_i)$. This shows that $(g_n)_n$ is a curvewise Π-bound for f on Ω. $\square$

We are now in a position to build the master curvewise plan $\boldsymbol{\pi}_\mathsf{m}$ for BV.

Theorem 5.4.6 (Master curvewise plan for BV)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then there exists a* ∞*-test plan* $\boldsymbol{\pi}_\mathsf{m}$ *on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *such that* $BV_{\boldsymbol{\pi}_\mathsf{m}}^{\mathsf{cw}}(\mathrm{X}) = BV(\mathrm{X})$ *and*

$$|\boldsymbol{D}f|_{\boldsymbol{\pi}_\mathsf{m}}^{\mathsf{cw}} = |\boldsymbol{D}f|, \qquad \textit{for every } f \in BV(\mathrm{X}).$$

Moreover, if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a non-branching* $\mathsf{CD}(K,N)$ *space for some* $K \in \mathbb{R}$*,* $N \in (1,\infty)$*, and the measure* $\mathfrak{m}$ *is finite, then we can additionally require that* $\boldsymbol{\pi}_\mathsf{m}$ *is concentrated on geodesics.*

Proof. First, let $\mathcal{D} = \{\pi_i\}_{i\in\mathbb{N}}$ be a master family for $BV(\mathrm{X})$, whose existence is granted by Theorem 5.3.9; the same result ensures that, in the non-branching $\mathsf{CD}(K,N)$ case, each π_i can be additionally chosen to be concentrated on geodesics. Given any $i \in \mathbb{N}$, pick $n_i \in \mathbb{N}$ such that $\mathrm{Lip}(\pi_i) \leq n_i$ and define the ∞-test plans $(\pi_i^j)_{j=1}^{n_i}$ as

$$\pi_i^j := \big(\mathsf{Restr}_{(j-1)/n_i}^{j/n_i}\big)_\sharp \pi_i, \qquad \text{for every } j = 1,\dots,n_i.$$

Notice that $\mathrm{Lip}(\pi_i^j) \leq 1$. Now define $\hat{\mathcal{D}} := \{\pi_i^j : i \in \mathbb{N},\, j = 1, \dots, n_i\}$. Theorem 5.4.3 and item ii) of Lemma 5.4.5 ensure that $BV(\mathrm{X}) = BV^{\mathsf{cw}}_{\hat{\mathcal{D}}}(\mathrm{X})$ and $|\boldsymbol{D}f| = |\boldsymbol{D}f|^{\mathsf{cw}}_{\hat{\mathcal{D}}}$ for all $f \in BV(\mathrm{X})$. Moreover, let us relabel $\hat{\mathcal{D}}$ as $\{\pi^n\}_{n\in\mathbb{N}}$. For any $n \in \mathbb{N}$, we define $\alpha_n \in (0,1)$ as

$$\alpha_n := \frac{1}{2^n \alpha \max\{\mathrm{Comp}(\pi^n), 1\}}, \qquad \text{where } \alpha := \sum_{m\in\mathbb{N}} \frac{1}{2^m \max\{\mathrm{Comp}(\pi^m), 1\}}.$$

Finally, we define the ∞-test plan $\boldsymbol{\pi}_{\mathsf{m}}$ as $\boldsymbol{\pi}_{\mathsf{m}} := \sum_{n\in\mathbb{N}} \alpha_n \pi^n$. Observe that in the non-branching $\mathsf{CD}(K,N)$ case we have that $\boldsymbol{\pi}_{\mathsf{m}}$ is concentrated on geodesics. What previously proved and item i) of Lemma 5.4.5 imply that $BV(\mathrm{X}) = BV^{\mathsf{cw}}_{\boldsymbol{\pi}_{\mathsf{m}}}(\mathrm{X})$ and that $|\boldsymbol{D}f| = |\boldsymbol{D}f|^{\mathsf{cw}}_{\boldsymbol{\pi}_{\mathsf{m}}}$ for every $f \in BV(\mathrm{X})$, thus yielding the statement. □

5.5 Master test plan for $W^{1,1}$ on RCD spaces

While the BV-theory on metric measure spaces is well-established by now, the $W^{1,1}$-theory seems to be much more complex to deal with. In [13, 78] several definitions of $W^{1,1}(\mathrm{X})$ are proposed, but it is shown that some of them are not equivalent. In particular, in the CD-setting, the available identifications [27] are mainly based on Doubling & Poincaré rather than on the curvature hypothesis. However, the situation greatly improves in the RCD setting and we restrict the attention here to this latter class.

We start considering a notion of space $W^{1,1}(\mathrm{X})$ defined in duality with ∞-test plans (which, in [78], is denoted by $w - W^{1,1}(\mathrm{X})$).

Definition 5.5.1 (The space $W^{1,1}(\mathrm{X})$)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We say that $f \in W^{1,1}(\mathrm{X})$, provided $f \in L^1(\mathfrak{m})$ and there exists $G \in L^1(\mathfrak{m})$ non-negative, called 1-weak upper gradient of f, so that*

$$|f(\gamma_1) - f(\gamma_0)| \leq \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t, \qquad \pi\text{-a.e. } \gamma,$$

for every ∞-test plan π.

The $\mathfrak{m}$-a.e. minimal G satisfying the above, denoted $|Df|_1$, is called minimal 1-weak upper gradient.

Notice that the well-posedness of the above definition follows from standard considerations as in Remark 1.2.24. We claim now that $W^{1,1}(\mathrm{X}) \subseteq BV(\mathrm{X})$. Fix any $f \in W^{1,1}(\mathrm{X})$. Given an arbitrary ∞-test plan π, it is standard to see that for π-a.e. γ we have $f \circ \gamma \in W^{1,1}(0,1)$ and $(f\circ\gamma)'_t \leq |Df|_1(\gamma_t)|\dot{\gamma}_t|$ for a.e. $t \in [0,1]$ (note, e.g., in [78, Section 4.6] the inclusion with the Beppo-Levi space $W^{1,1}_{BL}$). Moreover, for every $B \subseteq \mathrm{X}$ Borel and every ∞-test plan π, we can therefore estimate

$$\begin{aligned}
\int \gamma_\sharp |D(f\circ\gamma)|(B)\,\mathrm{d}\pi &= \iint_0^1 \chi_{\gamma^{-1}(B)}(t)(f\circ\gamma)'(t)\,\mathrm{d}t\,\mathrm{d}\pi \\
&\leq \mathrm{Lip}(\pi) \iint_0^1 (\chi_B |Df|_1)\circ \mathsf{e}_t\,\mathrm{d}t\,\mathrm{d}\pi \\
&\leq \mathrm{Lip}(\pi) \int_B |Df|_1\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

All in all, the above shows at the same time that $f \in BV(\mathrm{X})$ and $|\boldsymbol{D}f| \leq |Df|_1\mathfrak{m}$.

Unfortunately, it is not always true that, if $f \in BV(\mathrm{X})$ with $|\boldsymbol{D}f| \ll \mathfrak{m}$, then f belongs to $W^{1,1}(\mathrm{X})$ and $|D^{ac}f| := \frac{\mathrm{d}|\boldsymbol{D}f|}{\mathrm{d}\mathfrak{m}}$ is a 1-weak upper gradient. The reason being (see the discussion at the beginning of Section 4.6 and Example 4.5.4 in [78]), that the BV-condition requires $f\circ\gamma$ to be only $BV(0,1)$ along a.e. curve, while the $W^{1,1}$-condition requires the composition $f\circ\gamma$ to be absolutely continuous. This discrepancy allows in general for the existence of counterexamples.

Nevertheless, as proven in [99, Remark 3.5], this is not the case in the $\mathsf{RCD}(K,N)$ setting where it holds

$$f \in BV(\mathrm{X}) \text{ with } |\boldsymbol{D}f| \ll \mathfrak{m} \qquad \text{if and only if} \qquad f \in W^{1,1}(\mathrm{X}). \tag{5.5.1}$$

Moreover, in this case, $|Df|_1 = |D^{ac}f|$ at $\mathfrak{m}$-a.e. point. Therefore, building on top of [77],[1] and our Theorem 5.3.9, we are then able to prove:

Theorem 5.5.2. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(K,N)$ space with $N<\infty$ and $\mathfrak{m}$ finite. Then, there exists a ∞-test plan, denoted by $\boldsymbol{\pi}_{\mathsf{m}}$ and concentrated on geodesics, so that:*

If $f,G \in L^1(\mathfrak{m})$ are so that $f\circ\gamma \in W^{1,1}(0,1)$ for $\boldsymbol{\pi}_{\mathsf{m}}$-a.e. $\gamma \in AC([0,1],\mathrm{X})$ and

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\Big| \le G(\gamma_t)|\dot\gamma_t| \qquad (\boldsymbol{\pi}_1 \otimes \mathscr{L}^1)\text{-a.e. } (\gamma,t), \tag{5.5.2}$$

then $f \in W^{1,1}(\mathrm{X})$ and G is a 1-weak upper gradient.

Proof. Since $\mathsf{RCD}(K,N)$ spaces are non-branching [77, Theorem 1.3], we can consider from Theorem 5.3.9 a countable master family $\mathcal{D}$ and from Theorem 5.4.6 a master curvewise plan $\boldsymbol{\pi}_{\mathsf{m}}$ (both concentrated on geodesics) for the space $BV(\mathrm{X})$ with the key property:

$$\Gamma \text{ is } \boldsymbol{\pi}_1\text{-negligible} \qquad \Longleftrightarrow \qquad \Gamma \text{ is } \pi\text{-negligible} \quad \forall \pi \in \mathcal{D},$$

for every Borel $\Gamma \subseteq C([0,1],\mathrm{X})$. Finally, $f,G \in L^1(\mathfrak{m})$ satisfy (5.5.2) if and only if

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\Big| \le G(\gamma_t)|\dot\gamma_t| \qquad (\pi \otimes \mathscr{L}^1)\text{-a.e. } (\gamma,t),\ \forall \pi \in \mathcal{D}.$$

This implies that for π-a.e. γ, it holds $f\circ\gamma \in BV(0,1)$ (in fact, it is absolutely continuous) with $|D(f\circ\gamma)|(I) \le \int_I G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t$, for every $I \subseteq [0,1]$ Borel and $\pi \in \mathcal{D}$. Thus, we reach

$$\begin{aligned}
\int \gamma_\sharp |D(f\circ\gamma)|(B)\,\mathrm{d}\pi(\gamma) &\le \iint_0^1 \chi_{\gamma^{-1}(B)}(t) G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\
&\le \mathrm{Lip}(\pi)\iint_0^1 (\chi_B G)\circ \mathsf{e}_t\,\mathrm{d}t\,\mathrm{d}\pi\\
&\le \mathrm{Comp}(\pi)\mathrm{Lip}(\pi)\int_B G\,\mathrm{d}\mathfrak{m},
\end{aligned}$$

for every $B \subseteq \mathrm{X}$ Borel and $\pi \in \mathcal{D}$. This means that $f \in BV_{\mathcal{D}}(\mathrm{X})$ and $|\boldsymbol{D}f|_{\mathcal{D}} \le G\mathfrak{m}$. Finally, since $\mathcal{D}$ is a master family by Theorem 5.3.9, this immediately implies that $f \in BV(\mathrm{X})$ with $|\boldsymbol{D}f| \le G\mathfrak{m}$ and, appealing to (5.5.1), the conclusion. □

6 Rigidity and almost rigidity of Sobolev inequalities

Contents

6.1 Introduction for smooth manifolds

The standard Sobolev inequality in sharp form reads as

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le \mathsf{Eucl}(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in W^{1,p}(\mathbb{R}^n), \tag{6.1.1}$$

where $p \in (1,n)$, $p^* := \frac{pn}{n-p}$ is the Sobolev conjugate exponent and $\mathsf{Eucl}(n,p)$ is the smallest positive constants for which the inequality (6.1.1) is valid. Its precise value (see (6.2.2) below) was computed independently by Aubin [31] and Talenti [182] (see also [72]).

In the setting of compact Riemannian manifolds, the presence of constant functions in the Sobolev space immediately shows that an inequality of the kind of (6.1.1) must fail. Yet, Sobolev embeddings are certainly valid also in this context and they can be expressed by calling into play the full Sobolev norm:

$$\|u\|^p_{L^{p^*}(M)} \le A\|\nabla u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)}, \qquad \forall u \in W^{1,p}(M), \tag{$\star$}$$

where M is a compact n-dimensional Riemannian manifold and $A, B > 0$. From the presence of the two parameters A, B, it is not straightforward which is the notion of best constants in this case. The issue of defining and determining the best constants in ($\star$) has been the central role of the celebrated AB-program, we refer to [115] for a thorough presentation of this topic (see also [85]). The starting point of this program is the definition of the following two different notions of 'best Sobolev constants':

$$\alpha_p(M) := \inf\{A \ :\ (\star) \text{ holds for some } B\}, \qquad \beta_p(M) := \inf\{B \ :\ (\star) \text{ holds for some } A\}.$$

Then the first natural problem is to determine the value of $\alpha_p(M)$ and $\beta_p(M)$. It is rather easy to see that

$$\beta_p(M) = \mathsf{Vol}(M)^{p/p^*-1},$$

indeed constant functions give automatically $\beta_p(M) \ge \mathsf{Vol}(M)^{p/p^*-1}$, while the other inequality follows from the Sobolev-Poincaré inequality (see, e.g. [115, Sec 4.1]). It is instead more subtle to determine whether $\beta_p(M)$ is attained, in the sense that the infimum in its definition is actually a minimum. This is true for $p = 2$ and due to Bakry [34] (see also Proposition 6.5.1), but actually false for $p > 2$ (see e.g. [115, Prop. 4.1]).

Concerning instead the value of $\alpha_p(M)$, it turns out to be precisely the sharp constant in the Euclidean Sobolev inequality (6.1.1). More precisely Aubin in [31] (see also [115]) showed that on any compact n-dimensional Riemannian manifolds M with $n \ge 2$, we have

$$\alpha_p(M) = \mathsf{Eucl}(n,p)^p \qquad \forall p \in (1,n). \tag{6.1.2}$$

We point out that it is hard task to show that $\alpha_p(M)$ is attained, namely that there exists some $B > 0$ for which ($\star$) holds with $A = \alpha_p(M)$ and B. This has been verified for $p = 2$ in [116], answering affirmatively to a conjecture of Aubin.

On the other hand, knowing the value of $\beta_p(M)$ (and that is attained for $p = 2$), we can define a further notion of optimal-constant A, 'relative" to $B = \beta_2(M)$. More precisely we define

$$A^{\rm opt}_{2^*}(M) := \mathrm{Vol}(M)^{1-2/2^*} \cdot \inf\{A \ :\ (\star) \text{ for } p = 2 \text{ holds with } A \text{ and } B = \mathrm{Vol}(M)^{2/2^*-1}\}.$$

For the sake of generality will actually consider $A^{\rm opt}$ also in the so-called *subcritical case*, meaning that we enlarge the class of Sobolev inequalities and consider for every $q \in (2, 2^*]$

$$\|u\|^2_{L^q(M)} \le A\|\nabla u\|^2_{L^2(M)} + \mathrm{Vol}(M)^{2/q-1}\|u\|^2_{L^2(M)}, \qquad \forall u \in W^{1,2}(M), \tag{$\star\star$}$$

for some constant $A \ge 0$. Then we define

$$A^{\rm opt}_q(M) := \mathrm{Vol}(M)^{1-2/q} \cdot \inf\{A \ :\ (\star\star) \text{ holds}\}.$$

Note that the infimum above is always a minimum and that $\mathrm{Vol}(M)^{2/q-1}$ is the 'minimal B' that we can take in ($\star\star$).

Remark 6.1.1. We bring to the attention of the reader the renormalization factor $\mathrm{Vol}(M)^{1-2/q}$ in the definition of $A^{\rm opt}_q(M)$. This is usually not present in the literature concerning the AB-program (see e.g. [115]), however this choice will allow us to to have cleaner inequalities. This also makes $A^{\rm opt}_q$ invariant under rescalings of the volume measure of M. ■

One of the main questions that we will investigate Chapter concerns the value of $A_q^{\rm opt}(M)$. So far $A_q^{\rm opt}(M)$ is known explicitly only in the case of $\mathbb{S}^n$ and was firstly computed by Aubin in [30] in the case of $q = 2^*$ and by Beckner in [40] for a general q:

$$A_q^{\rm opt}(\mathbb{S}^n) = \frac{q-2}{n}, \qquad \forall n \geq 3. \tag{6.1.3}$$

Aubin also exhibited a family of non-constant functions that achieve equality in ($\star\star$) with $A = A_{2^*}^{\rm opt}(\mathbb{S}^n)$. For a general manifold M instead it can be proved that

$$A_q^{\rm opt}(M) \leq C(K, D, N), \tag{6.1.4}$$

where $K \in \mathbb{R}$ is a lower bound on the Ricci curvature of M, N is an upper bound on the dimension and $D \in \mathbb{R}^+$ an upper bound on its diameter. This follows from the Sobolev-Poincaré inequality combined with an inequality by Bakry (see e.g. [85, Theorem 4.4] and also Section 6.5.1). On the other hand, for positive Ricci curvature we have the following celebrated comparison result originally proven in [122] (see also [138, 35] for the case of a general q):

Theorem 6.1.2. *Let M be an n-dimensional Riemannian manifold, $n \geq 3$, with $\mathsf{Ric} \geq n-1$. Then, for every $q \in (2, 2^*]$, it holds*

$$A_q^{\rm opt}(M) \leq A_q^{\rm opt}(\mathbb{S}^n). \tag{6.1.5}$$

One of the main consequence of the results in this Chapter is the characterization of the equality in (6.1.5), in particular we show:

Theorem 6.1.3. *Equality in (6.1.5) holds for some $q \in (2, 2^*]$ if and only if M is isometric to $\mathbb{S}^n$.*

It is important to point out that the novelty of the above result is that it covers the case $q = 2^*$. Indeed, for $q < 2^*$, Theorem 6.1.3 was already established (see e.g. [35, Remark 6.8.5]) and follows from an improvement (only for $q < 2^*$) of (6.1.5) due to [90] involving the spectral gap (see Remark 6.6.9 for more details). On the other hand, up to our knowledge, this is the first time that it appears in the critical case $q = 2^*$.

Structure of the Chapter. This Chapter is organized as follows:

In Section 6.2, we set the AB-program on general CD-spaces to define the constants α_p and $A_q^{\rm opt}$. We then state the main results of this Chapter and also provide the reader with a sketch of proof for the rigidity in Theorem 6.1.3.

Section 6.3 is devoted to show the upper bound of α_p in nonsmooth setting generalizing (6.1.2). This upper bound will be obtained by the combination of local isoperimetric inequalities and a novel concept of Polya-Szego rearrangements of Euclidean type on CD-spaces in the spirit of [159].

Section 6.4 is devoted to achieve the lower bound of α_p generalizing thus the formula (6.1.2) in nonsmooth setting. Here we also derive, as an application, sharp Sobolev inequalities on $\mathsf{CD}(0, N)$ spaces (Section 6.4.2).

In Section 6.5, we consider the constant $A_q^{\rm opt}$ in nonsmooth setting and face two different geometric bounds in terms of the Ricci curvature bounds (Section 6.5.1) and in terms on the first eigenvalue (Section 6.5.2).

In Section 6.6, we prove our main rigidity result on $A_q^{\rm opt}$ on RCD-spaces. To this aim we develop a concentration compactness dichotomy principle under mGH-convergence (Theorem 6.6.1) and a quantitative linearization lemma for the Sobolev inequality in Section 6.6.2.

In Section 6.7, we prove the main almost-rigidity result, namely the characterization of almost equality in Theorem 6.1.3 on compact RCD-space. The main ingredient we develop is the continuity of the constant $A_q^{\rm opt}$ under mGH-convergence (Section 6.7.2).

Finally, in Section 6.8 we conclude this Chapter by studying the so-called generalize Yamabe equation on $\mathsf{RCD}(K, N)$ spaces. We will prove a classical existence result in Section 6.8.1 while in Section 6.8.2 we will show a continuity result for the generalized Yamabe constant under mGH-convergence.

6.2 Main definitions and statements of the main results

Before starting, let us collect once and for all the key constants appearing in this Chapter.

> **Basic notations**. For all $N \in [1,\infty), p \in (1,N)$, we define the generalized[1] unit ball and unit sphere volumes by
>
> $$\omega_N := \frac{\pi^{N/2}}{\Gamma\left(N/2+1\right)}, \qquad \sigma_{N-1} := N\omega_N, \tag{6.2.1}$$
>
> where Γ is the Gamma-function, and the sharp Euclidean Sobolev constant by
>
> $$\mathsf{Eucl}(N,p) := \frac{1}{N}\Big(\frac{N(p-1)}{N-p}\Big)^{\frac{p-1}{p}}\Big(\frac{\Gamma(N+1)}{N\omega_N\Gamma(N/p)\Gamma(N+1-N/p)}\Big)^{\frac{1}{N}}. \tag{6.2.2}$$
>
> For $N>2$ and $p=2$, the above reduces to
>
> $$\mathsf{Eucl}(N,2) = \Big(\frac{4}{N(N-2)\sigma_N^{2/N}}\Big)^{\frac{1}{2}}. \tag{6.2.3}$$
>
> We will sometimes need also the following identity:
>
> $$\int_0^\pi \sin^{N-1}(t)\,\mathrm{d}t = \frac{\sigma_N}{\sigma_{N-1}}, \quad \forall N>1. \tag{6.2.4}$$

6.2.1 Main rigidity and almost rigidity theorems

We will prove Theorem 6.1.3 in the context of RCD metric measure spaces with synthetic Ricci curvature bounds. To this aim, we will consider the AB-program instead in the context of CD-spaces and face classical but also new questions. One of the main reasons to approach these problems in this more general setting is that it will allow us to characterize also the 'almost-equality' in (6.1.5) (see Theorem 6.2.3 below). Indeed, as we will see, in this case we need to compare the manifold M to a class of singular spaces, rather than to the round sphere.

To state our main results for metric measure spaces we need to define first the notion of optimal constant in the Sobolev inequality in the non-smooth setting. Given a (compact) $\mathsf{RCD}(K,N)$ space (or more generally a $\mathsf{CD}(K,N)$ space) $(\mathrm{X},\mathsf{d},\mathfrak{m})$, for some $K\in\mathbb{R}$, $N\in(2,\infty)$, we set $2^*:=2N/(N-2)$ and consider the analogous of ($\star\star$):

$$\|u\|^2_{L^q(\mathfrak{m})} \le A\||Du|\|^2_{L^2(\mathfrak{m})} + \mathfrak{m}(\mathrm{X})^{2/q-1}\|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}), \tag{6.2.5}$$

for $q\in(2,2^*]$ and a constant $A\ge 0$. Then we define

$$A^{\rm opt}_q(\mathrm{X}) := \mathfrak{m}(\mathrm{X})^{1-2/q}\cdot\inf\{A \ :\ (6.2.5) \text{ holds}\},$$

with the convention that $A^{\rm opt}_q(\mathrm{X})=\infty$ when no A exists. Note that $A^{\rm opt}_q(X)$, when is finite, is actually a minimum. Observe also that, as in the smooth case, there is a renormalization factor $\mathfrak{m}(\mathrm{X})^{1-2/q}$ in the definition. However, being not restrictive, we will mainly work asking $\mathfrak{m}(\mathrm{X})=1$ so that the value of $A^{\rm opt}_q(\mathrm{X})$ is equivalent to the non-renormalized one.

Remarkably in this more general framework, a comparison analogous to (6.1.5) holds.

Theorem 6.2.1 ([59]). *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}(N-1,N)$ space, $N\in(2,\infty)$. Then, for every $q\in(2,2^*]$*

$$A^{\rm opt}_q(\mathrm{X}) \le \frac{q-2}{N}. \tag{6.2.6}$$

[1]For an integer N, ω_N is the volume of the unit ball in $\mathbb{R}^N$ and σ_N is the volume of the N-sphere $\mathbb{S}^N$.

See [175] for the definition of the essentially nonbranching condition and the proof that holds in the $\mathsf{RCD}(K,N)$-class. We also mention that Theorem 6.2.1 in the RCD case was previously obtained in [173]. Observe also that, whenever N is an integer and thanks to (6.1.3), for a N-dimensional Riemannian manifolds (6.2.6) is exactly (6.1.5) and in particular Theorem 6.2.1 generalizes Theorem 6.1.2.

We can now state and our main rigidity result in the setting of metric measure spaces.

Theorem 6.2.2 (Rigidity of $A_q^{\rm opt}$)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(N-1,N)$ *space for some* $N\in(2,\infty)$ *and let* $q\in(2,2^*]$*. Then, equality holds in* (6.2.6) *if and only if* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathsf{RCD}(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ *such that* $(\mathrm{X},\mathsf{d},\mathfrak{m})\simeq[0,\pi]\times_{\sin}^{N-1}\mathrm{Z}$.

Let us compare the above result with the rigidity result in [137] for the Sobolev inequality on manifolds with non-negative Ricci curvature (and later improved in [190], see also [37] in the nonsmooth setting). In [137] it is proved that if (6.1.1) is valid on a non-compact manifold with non-negative Ricci curvature, then the manifold must be the Euclidean space. Here instead we consider compact manifolds and the rigidity is obtained in comparison with the Sobolev inequality on the sphere. For this reason our arguments will also be substantially different than the ones in [137, 190]. Moreover, differently from the smooth case, in the more abstract setting of RCD spaces the above result is instead new for all q.

As anticipated above, we can also prove an 'almost-rigidity' statement linked to the almost-equality case in (6.2.6) (see Section 1.1.6 for the notion of measure-Gromov-Hausdorff convergence and distance d_{mGH}.).

Theorem 6.2.3 (Almost-rigidity of $A_q^{\rm opt}$)**.** *For every* $N\in(2,\infty)$*,* $q\in(2,2^*]$ *and every* $\varepsilon>0$*, there exists* $\delta:=\delta(N,\varepsilon,q)>0$ *such that the following holds. Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(N-1,N)$ *space with* $\mathfrak{m}(\mathrm{X})=1$ *and suppose that*

$$A_q^{\rm opt}(\mathrm{X})\geq\frac{(q-2)}{N}-\delta,$$

Then, there exists a spherical suspension $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ *(i.e. there exists an* $\mathsf{RCD}(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ *so that* Y *is isomorphic as a metric measure space to* $[0,\pi]\times_{\sin}^{N-1}\mathrm{Z}$*) such that*

$$\mathsf{d}_{mGH}((\mathrm{X},\mathsf{d},\mathfrak{m}),(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}}))<\varepsilon.$$

Remark 6.2.4. We briefly point out two important facts concerning the two above statements.

- ▷ In the smooth setting, for $q<2^*$, the almost rigidity follows 'directly' from the sharper version of (6.1.5) cited above (see Remark 6.6.9 for the explicit statement) and using the almost-rigidity of the 2-spectral gap [59, 61]. Nevertheless, we are not aware of any such statement in the literature and anyhow, our proof does not rely on any improved version of (6.1.5).
- ▷ The key feature of Theorem 6.2.2 and Theorem 6.2.3 is that they include the 'critical' exponent. Indeed, the difference between the 'subcritical' case $q<2^*$ and $q=2^*$ is not only technical but a major issue linked to the lack of compactness in the Sobolev embedding. As it will be clear in the sequel, the proof of the critical case requires several additional arguments that constitute the heart of our analysis. ■

The almost-rigidity result contained in Theorem 6.2.3 will be actually a consequence of a stronger statement, that is the continuity of $A_q^{\rm opt}$ under measure Gromov-Hausdorff convergence. More precisely we will prove the following:

Theorem 6.2.5 (Continuity of $A_q^{\rm opt}$ under mGH-convergence)**.** *Let* $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$*,* $n\in\bar{\mathbb{N}}:=\mathbb{N}\cup\{\infty\}$*, be a sequence of compact* $\mathsf{RCD}(K,N)$*-spaces with* $\mathfrak{m}_n(\mathrm{X}_n)=1$ *and for some* $K\in\mathbb{R}$*,* $N\in(2,\infty)$ *so that* $\mathrm{X}_n\overset{mGH}{\to}\mathrm{X}_\infty$*. Then,* $A_q^{\rm opt}(\mathrm{X}_\infty)=\lim_n A_q^{\rm opt}(\mathrm{X}_n)$*, for every* $q\in(2,2^*]$.

6.2.2 Best constant in the Sobolev inequality on compact CD spaces

The proof of the rigidity (and almost rigidity) of $A_q^{\rm opt}$ in the case $q = 2^*$, will force us to study also the value of α_p in the context of CD-spaces. The connection of this with the proof of Theorem 6.2.2 will be explained towards the end of Section 6.2.4, where we provide a sketch of the proof yielding the main rigidity theorem.

Let then $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K, N)$ space with $N \in (1, \infty)$. For any $p \in (1, N)$ set $p^* := \frac{Np}{N-p}$ and, in the same fashion of ($\star$), we consider:

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le A\||Du|\|^p_{L^p(\mathfrak{m})} + B\|u\|^p_{L^p(\mathfrak{m})}, \qquad \forall u \in W^{1,p}(\mathrm{X}). \tag{6.2.7}$$

We are then interested in the minimal A for which (6.2.7) holds. In other words we set (with the usual convention that the inf is ∞ when no A exists):

$$\alpha_p(\mathrm{X}) := \inf\{A \ : \ (6.2.7) \text{ holds for some } B\}. \tag{6.2.8}$$

We will be able to compute the value of $\alpha_p(\mathrm{X})$ for every compact $\mathsf{CD}(K, N)$ space X, extending the result of Aubin for Riemannian manifolds (see (6.1.2) above). Before passing to the actual statement, it is useful to explain first the intuition behind it and the geometrical meaning of the constant $\alpha_p(\mathrm{X})$. The rough idea is that its value is tightly linked to the local structure of the space. Indeed, the key observation is that $\alpha_p(\mathrm{X})$ is invariant under rescaling of the form $(\mathrm{X}, \mathsf{d}/r, \mathfrak{m}/r^N)$. For example, since manifolds are locally Euclidean, it is not surprising that in (6.1.2) the optimal Euclidean-Sobolev constant appears. On the other hand, $\mathsf{CD}(K, N)$ spaces have a more singular local behavior and additional parameters must be taken into account. In particular the value of $\alpha_p(\mathrm{X})$ turns out to be related to the Bishop-Gromov density:

$$(0, +\infty] \ni \theta_N(x) := \lim_{r\to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}, \quad x \in \mathrm{X},$$

where ω_N is the volume of the Euclidean unit ball (see (6.2.1) for non integer N). Our result is then the following:

Theorem 6.2.6. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a compact $\mathsf{CD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Then for every $p \in (1, N)$*

$$\alpha_p(\mathrm{X}) = \left(\frac{\mathsf{Eucl}(N, p)}{\min_{x\in\mathrm{X}} \theta_N(x)^{\frac{1}{N}}}\right)^p. \tag{6.2.9}$$

We point out that, since X is compact, $\min_{x\in\mathrm{X}} \theta_N(x)$ always exists because θ_N is lower semicontinuous.

Remark 6.2.7. Note that if X is a n-dimensional Riemannian manifold, $\theta_n(x) = 1$ for every $x \in \mathrm{X}$, hence in this case (6.2.9) (with $N = n$) is exactly Aubin's result in (6.1.2). Recall also that here N needs not to be an integer and thus $\mathsf{Eucl}(N, p)$ has to be defined for arbitrary $N \in (1, \infty)$ (see (6.2.2)). ■

Remark 6.2.8. We are not assuming $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to be renormalized. In particular observe that if we rescale the reference measure $\mathfrak{m}$ as $c \cdot \mathfrak{m}$, then α_p gets multiplied by $c^{-p/N}$, which is in accordance with the scaling in (6.2.9). ■

Remark 6.2.9. Theorem 6.2.6 gives non-trivial information even in the 'collapsed' case, i.e. when $\theta_N = +\infty$ in a set of positive (or even full) measure (see [98] for the notion of collapsed/non-collapsed RCD spaces). Indeed, to have $\alpha_p(\mathrm{X}) > 0$ it is sufficient that $\theta_N(x) < +\infty$ at a single point $x \in \mathrm{X}$. As an example, consider the model space $([0, \pi], |.|, \sin^{N-1} \mathscr{L}^1)$ which is $\mathsf{RCD}(N-1, N)$ with $\theta_N(x) < +\infty$ only for $x \in \{0, \pi\}$. ■

Theorem 6.2.6 will be proved in two steps, by the combination of an upper bound (Theorem 6.3.12), obtained via local Sobolev inequalities (Theorem 6.3.8), and a lower bound (Theorem 6.4.4) derived with a blow-up analysis.

6.2.3 Additional results and application to the Yamabe equation

Euclidean-type Polya-Szego inequality on $\mathsf{CD}(K,N)$ spaces.

We will develop a Polya-Szego inequality (see Section 6.3.1), which is roughly an Euclidean-variant of the Polya-Szego inequality for $\mathsf{CD}(K,N)$ spaces, $K>0$, derived in [159]. The main feature of this inequality is that it holds on arbitrary $\mathsf{CD}(K,N)$ spaces, $K\in\mathbb{R}$, but assumes the validity of an isoperimetric inequality of the type

$$\mathrm{Per}(E)\ge \mathsf{C}_{\mathsf{Isop}}\mathfrak{m}(E)^{\frac{N-1}{N}},\qquad \forall E\subset\Omega \text{ Borel},$$

for some $\Omega\subset \mathrm{X}$ open and where $\mathsf{C}_{\mathsf{Isop}}$ is a positive constant independent of E. For our purposes this Polya-Szego inequality will be used to derive local Sobolev inequalities of Euclidean-type (see Theorem 6.3.8), however it allows us to obtain also sharp Sobolev inequalities under Euclidean-volume growth assumption.

Sharp and rigid Sobolev inequalities under Euclidean-volume growth.

As a by-product of our analysis, we achieve sharp Sobolev inequalities on $\mathsf{CD}(0,N)$ spaces with Euclidean-volume growth. We recall that a $\mathsf{CD}(0,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ has Euclidean-volume growth if

$$\mathsf{AVR}(\mathrm{X}):=\lim_{R\to+\infty}\frac{\mathfrak{m}(B_R(x_0))}{\omega_N R^N}>0,$$

for some (and thus any) $x_0\in\mathrm{X}$. We will prove the following.

Theorem 6.2.10. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}(0,N)$ space for some $N\in(1,\infty)$ and with Euclidean volume growth. Then, for every $p\in(1,N)$, it holds*

$$\|u\|_{L^{p^*}(\mathfrak{m})}\le \mathsf{Eucl}(N,p)\mathsf{AVR}(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})},\qquad \forall u\in \mathrm{Lip}_c(\mathrm{X}). \tag{6.2.10}$$

Moreover (6.2.10) is sharp.

This extends a result recently derived in [37] in the case of Riemannian manifolds and answer positively to a question posed in [37, Sec. 5.2].

Concentration compactness and mGH-convergence.

As often happens for almost-rigidity results in RCD spaces, Theorem 6.2.3 will be proved by compactness. However, in the case $q=2^*$ we have a strong lack of compactness, hence for the proof we will need an additional tool, which is a concentration compactness result under mGH-convergence of compact RCD-spaces. In particular, we will prove a concentration-compactness dichotomy principle (see Lemma 6.6.6 and Theorem 6.6.1 below) in the spirit of [141] (see also the monograph [179]), but under varying underlying measure. As far as we know, this is the first result of this type dealing with varying spaces and we believe it to be interesting on its own.

Existence for the Yamabe equation and mGH-continuity of Yamabe constant on RCD spaces

As an application of Theorem 6.2.6 we show that on a compact $\mathsf{RCD}(K,N)$ space a (non-negative and non-zero) solution to the so-called Yamabe equation

$$-\Delta u+\mathrm{S}\,u=\lambda u^{2^*-1},\qquad \text{for } \lambda\in\mathbb{R},\ \mathrm{S}\in L^p(\mathfrak{m}),\ p>N/2, \tag{6.2.11}$$

exists provided

$$\lambda_{\mathrm{S}}(\mathrm{X}):=\inf_{u\in W^{1,2}(\mathrm{X})\setminus\{0\}}\frac{\int|Du|^2+\mathrm{S}|u|^2\,\mathrm{dVol}}{\|u\|^2_{L^{2^*}(M)}}<\frac{\min\theta_N^{N/2}}{\mathsf{Eucl}(N,2)^2},$$

where λ_{S} is called generalized Yamabe constant (see Theorem 6.8.2). This extends a classical result on smooth Riemannian manifolds (see Section 6.8 for more details and references).

We also show the continuity of the generalized Yamabe constant under measure Gromov-Hausdorff convergence. More precisely for a sequence X_n of compact $\mathsf{RCD}(K,N)$ spaces such that $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$ with X_∞ a compact $\mathsf{RCD}(K,N)$ space, we show that

$$\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) = \lambda_{\mathrm{S}}(\mathrm{X}_\infty),$$

where S_n converges L^p-weak to S for some $p > N/2$. See Theorem 6.8.6 for a precise statement and Section 2.4 for the definition of L^p-weak convergence with varying spaces. This result extends and sharpens an analogous statement proved for Ricci-limits in [121], where an additional boundedness assumption on the sequence $\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$ is required.

6.2.4 Proof-outline of the rigidity

Here we explain the scheme of the proof of rigidity result in Theorem 6.2.2.

We consider only the case $q = 2^*$, since it is the most interesting one and we also restrict to the case of manifolds, which already contains all the main ideas. Suppose that M is a compact n-dimensional manifolds M, with $\mathsf{Ric} \ge n-1$ and $A^{\rm opt}_{2^*}(M) = A^{\rm opt}_{2^*}(\mathbb{S}^n)$, $n \ge 3$. The latter condition is equivalent to the existence of a sequence $(u_i) \subset W^{1,2}(M)$ of non-constant functions satisfying

$$\mathcal{Q}(u_i) := \frac{\|u_i\|^2_{L^{2^*}} - \mathrm{Vol}(M)^{-2/n}\|u_i\|^2_{L^2}}{\mathrm{Vol}(M)^{-2/n}\|\nabla u_i\|^2_{L^2}} \to A^{\rm opt}_{2^*}(\mathbb{S}^n). \tag{6.2.12}$$

Observe also that, by homogeneity, we can and will assume that $\|u_i\|^2_{L^{2^*}(\mathfrak{m})} = 1$. In a nutshell, the strategy of the proof consists in a fine investigation of these sequences.

To clarify the picture, it is effective to look first at the trivial case of $M = \mathbb{S}^n$. In this setting, we can produce three different type of extremal sequences as in (6.2.12):

Case 1. The first is straightforward. Indeed from [30] we know that a family of extremal functions exists. In other words, we have that the functions $u_{\beta,p} := \big(\beta - \cos(d_p)\big)^{1-\frac{n}{2}}$ $p \in \mathbb{S}^n$, $\beta > 1$, (where d_p is the distance from p) satisfy

$$\mathcal{Q}(u_{\beta,x_0}) \equiv A^{\rm opt}_{2^*}(\mathbb{S}^n).$$

Then we can simply take u_i to be constantly equal to a fixed $u_{\beta,p}$.

Case 2. For the second method we need to recall that the spectral gap of $\mathbb{S}^n$ is n, i.e. that there exists $u \in W^{1,2}(\mathbb{S}^n)$ with zero mean and such that

$$\frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^2}} = n.$$

(or equivalently that $\Delta u = -nu$). We define $u_i := 1 + \varepsilon_i u$ for a fixed sequence $\varepsilon_i \downarrow 0$. Then a standard linearization shows that

$$\mathcal{Q}(u_i) \to \frac{(2^*-2)\|u\|^2_{L^2}}{\|\nabla u\|^2_{L^2}} = \frac{2^*-2}{n} = A^{\rm opt}_{2^*}(\mathbb{S}^n), \tag{6.2.13}$$

Case 3. The third way is of local nature and we will only sketch it. It is based on the observation that

$$\mathsf{Eucl}(n,2)^2 = A^{\rm opt}_{2^*}(\mathbb{S}^n)\mathrm{Vol}(\mathbb{S}^n)^{-2/n}, \tag{6.2.14}$$

which comes from the explicit expression of $\mathsf{Eucl}(n,2)$ (see (6.2.2)). Since $\mathbb{S}^n$ is locally Euclidean, if we zoom enough around a point the metric becomes almost flat. In particular for a fixed point $p \in \mathbb{S}^n$ and $\varepsilon > 0$ there exists small enough radius $r > 0$ and a function $u \in \mathrm{Lip}_c(B_r(p))$ such that

$$\|u\|_{L^{2^*}} \ge (\mathsf{Eucl}(n,2) - \varepsilon)\|\nabla u\|_{L^2},$$

(see, for example, [115, pag. 94] or [85, pag. 14] for details). Then we can take a sequence $\varepsilon_i \to 0$ and functions $u_i \in \mathrm{Lip}_c(B_{r_i}(p)))$ as above with $r_i \to 0$ and renormalized so that $\|u_i\|_{L^{2^*}} = 1$. Observe that by the Hölder inequality we must have that $\|u_i\|_{L^2} \to 0$. Therefore from (6.2.14)

$$\mathcal{Q}(u_i) \to \mathsf{Eucl}(n,2)^2 \mathrm{Vol}(M)^{2/n} = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n).$$

Important point. Remarkably, it turns out (Theorem 6.6.1) that the three cases above already describe all the possible behavior of an extremal sequence u_i in (6.2.12) also on a general manifold. Indeed, we will prove that are only three scenarios:

Case 1. Up to a subsequence, u_i converges in L^{2^*} to a *non constant* extremal function u such that $\mathcal{Q}(u) = A_{2^*}^{\mathrm{opt}}(M) = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)$. This is the simplest case and the rigidity follows from the Polya-Szego inequality. Indeed, the identity $\mathcal{Q}(u) = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)$ forces the monotone rearrangement $u^* : \mathbb{S}^n \to \mathbb{R}$ (as defined in Section 2.2.2) to achieve equality in the Polya-Szego inequality. This and the fact that u is not constant is enough to deduce that $M = \mathbb{S}^n$, by the rigidity case of the Polya-Szego inequality (see Theorem 2.3.11).

Case 2. The second situation is when (up to a subsequence) u_i still converges in L^{2^*}, but to a constant function $u \equiv c$. Up to renormalization (of the volume measure), it can be assumed that $\int u_i = 1$ and $u \equiv 1$. In this case we mimic the case ii) for $\mathbb{S}^n$ described above and write $u_i = 1 + v_i$, where $v_i := u_i - 1$ has zero mean. Then, even if v_i is not of the form $\varepsilon_i v$, it turns out that the linearization in (6.2.13) can still be performed to achieve:

$$\frac{2^* - 2}{n} = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n) = \lim_{i\to\infty} \mathcal{Q}(u_i) = \lim_{i\to\infty} \frac{(2^*-2)\|v_i\|_{L^2}^2}{\|\nabla v_i\|_{L^2}^2} \le \frac{2^*-2}{\lambda_1(M)},$$

where $\lambda_1(M)$ is the first non-trivial eigenvalue of M. This however forces $\lambda_1(M) = n$, which by classical Obata's theorem implies that $M = \mathbb{S}^n$.

Case 3. In the third and more delicate case we have that the sequence u_i vanishes, i.e. $\|u_i\|_{L^2} \to 0$ (in fact the following concentration happens: $|u_i|^{2^*} \rightharpoonup \delta_p$ for some point $p \in M$). Here is where the constant $\alpha_2(M)$ defined in Section 6.2.2 enters into play. Indeed, by definition of $\alpha_2(M)$, for every $\varepsilon > 0$ there exists B_ε such that

$$1 = \|u_i\|_{L^{2^*}}^2 \le (\alpha_2(M) + \varepsilon)\|\nabla u_i\|_{L^p}^2 + B_\varepsilon \|u_i\|_{L^2}^2, \qquad \forall\, i \in \mathbb{N}.$$

Moreover, from $\|u_i\|_{L^2} \to 0$ we must have $\varliminf_i \|\nabla u_i\|_{L^p}^2 > 0$. Combining these two observation we obtain that

$$\varlimsup_i \frac{\|u_i\|_{L^{2^*}}^2}{\|\nabla u_i\|_{L^p}^2} \le (\alpha_2(M) + \varepsilon).$$

By assumption $\mathcal{Q}(u_i) \to A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)$, which implies

$$\varliminf_i \frac{\|u_i\|_{L^{2^*}}^2}{\|\nabla u_i\|_{L^p(M)}^2} \ge \mathrm{Vol}(M)^{-2/n} A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n).$$

Therefore $\alpha_2(M) \ge \mathrm{Vol}(M)^{-2/n} A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)$. However combining (6.1.2) with (6.2.14) we have $\alpha_2(M) = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)\mathrm{Vol}(\mathbb{S}^n)^{-2/n}$, that coupled with the previous observation yields

$$\mathrm{Vol}(M) \ge \mathrm{Vol}(\mathbb{S}^n).$$

This and the Bishop-Gromov volume ratio implies that $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n)$, which forces $\mathrm{diam}(M) = \pi$ and the required rigidity follows from Cheng's diameter rigidity theorem.

6.3 The constant α_p: upper bound

To prove an upper bound of α_p we will need to derive a Sobolev inequality of the type (6.2.7) for some explicit A. This will be achieved by proving first a class of local Sobolev-inequalities (see Theorem 6.3.8) and then 'patch' them together (see Theorem 6.2.1) to obtain the desired global inequality. The local-Sobolev inequalities will be achieved through an Euclidean Polya-Szego simmetrization inequality (Theorem 6.3.6).

6.3.1 Polya-Szego inequality of Euclidean-type

The goal of this section is to prove an Euclidean-variant of the Polya-Szego inequality for $\mathsf{CD}(K,N)$ spaces derived in [159] (under essentially nonbranching assumption, see also Section 2.2.2). The main difference is that our inequality holds for arbitrary $K \in \mathbb{R}$ and assumes the a-priori validity of an Euclidean-type isoperimetric inequality, while the one in [159] requires $K > 0$ and it is based on the Lévy-Gromov isoperimetric inequality for the $\mathrm{CD}(K,N)$ condition. As opposed to Section 2.2.2, where the symmetrization has as target the model space for the $\mathsf{CD}(K,N)$ condition with $K > 0$, we will use a notion of symmetrization that lives in the weighted half line $([0,\infty),|.|,t^{N-1}\mathscr{L}^1)$. It should be remarked that, in general, there is not a natural curvature model space to symmetrize functions defined on an arbitrary $\mathsf{CD}(K,N)$-space with $K \leq 0$. This is because there is not a unique model-space for the Lévy-Gromov isoperimetric inequality in the case $K \leq 0$ (see [155]). Therefore, it is unclear in this high-generality where the rearrangements should live. For this reason we will equip the metric measure spaces under consideration with a (possibly local) isoperimetric inequality of Euclidean-type:

$$\mathrm{Per}(E) \geq C\mathfrak{m}(E)^{\frac{N-1}{N}},$$

for $N > 1$ and C a non-negative constant.

We start with the definition of *Euclidean model space* $(I_{0,N},|.|,\mathfrak{m}_{0,N})$, $N \in (1,\infty)$:

$$I_{0,N} := [0,\infty), \qquad \mathfrak{m}_{0,N} := \sigma_{N-1}t^{N-1}\mathscr{L}^1,$$

where $|.|,\mathscr{L}^1$ are the Euclidean distance and Lebesgue measure, respectively. Next, we define the *Euclidean monotone rearrangement.*

Definition 6.3.1 (Euclidean monotone rearrangement)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. For any Borel function $u\colon \Omega \to \mathbb{R}^+$, we define $\Omega^* := [0,r]$ with $\mathfrak{m}_{0,N}([0,r]) = \mathfrak{m}(\Omega)$ (i.e. $r^N = \omega_N^{-1}\mathfrak{m}(\Omega)$) and the monotone rearrangement $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ by*

$$u^*_{0,N}(x) := u^{\#}(\mathfrak{m}_{0,N}([0,x])) = u^{\#}(\omega_N x^N), \qquad \forall x \in \Omega^*,$$

where $u^{\#}$ is the generalized inverse of the distribution function of u, as defined in Section 2.2.2.

In the sequel, whenever we fix Ω and $u\colon \Omega \to [0,\infty)$, the set Ω^* and the rearrangement $u^*_{0,N}$ are automatically defined as above.

Proposition 6.3.2. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $\Omega \subset \mathrm{X}$ be open and bounded with $\mathfrak{m}(\Omega) < +\infty$. Let $u : \Omega \to [0,+\infty)$ be Borel and let $u^*_{0,N} : \Omega^* \to [0,+\infty)$ be its monotone rearrangement.*
Then, u and $u^*_{0,N}$ have the same distribution function. Moreover

$$\|u\|_{L^p(\Omega)} = \|u^*_{0,N}\|_{L^p(\Omega^*)}, \qquad \forall\ 1 \leq p < +\infty, \tag{6.3.1}$$

and the radial decreasing rearrangement operator $L^p(\Omega) \ni u \mapsto u^*_{0,N} \in L^p(\Omega^*)$ is continuous.

The proof of the above proposition is classical, following e.g. [130], with straightforward modification for the metric measure setting (see also [159]). Observe also that, given $u \in L^p(\Omega)$, its monotone rearrangement must be defined by fixing a Borel representative of u. However, this choice does not affect the outcome object $u^*_{0,N}$, as clearly the distribution function $\mu(t)$ of u is independent of the representative.

We now introduce the additional assumption that will make this section meaningful. For some open set $\Omega \subset \mathrm{X}$ and a number $N \in (1,\infty)$, we require the validity of the following local Euclidean-isoperimetric inequality

$$\mathrm{Per}(E) \geq \mathsf{C}_{\mathsf{Isop}}\mathfrak{m}(E)^{\frac{N-1}{N}}, \qquad \forall\, E \subset \Omega \text{ Borel}. \tag{6.3.2}$$

where $\mathsf{C}_{\mathsf{Isop}}$ is a positive constant independent of E.

Remark 6.3.3. There is a rich literature about Euclidean-type isoperimetric inequalities in metric measure spaces. Inequalities as in (6.3.2) have been proven to hold, at least on balls, in the general setting of locally doubling metric measure spaces satisfying a weak local $(1,1)$-Poincaré inequality (see, e.g., [9, 156]). In the context of $\mathrm{CD}(K,N)$ spaces, local almost-Euclidean isoperimetric inequalities have been derived in [60], while in the recent [37], a global version of (6.3.2) is proven to hold in $\mathrm{CD}(0,N)$ spaces with Euclidean-volume growth. In our specific case the validity of (6.3.2) will come from Theorem 6.3.9. ■

Proposition 6.3.4 (Lipschitz to Lipschitz property of the rearrangement)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $\Omega\subset\mathrm{X}$ be open with $\mathfrak{m}(\Omega)<+\infty$. Assume furthermore that, for some $N\in(1,\infty)$ and $\mathsf{C}_{\mathsf{Isop}}>0$, the isoperimetric inequality in (6.3.2) holds in Ω. Finally, let $u\in\mathrm{Lip}_c(\Omega)$ be non-negative with Lipschitz constant $L\geq 0$ and such that $|D^{ac}u|(x)\neq 0$ for $\mathfrak{m}$-a.e. $x\in\{u>0\}$. Then $u^*_{0,N}\in\mathrm{Lip}(\Omega^*)$ with $\mathrm{Lip}(u^*_{0,N})\leq N\omega_N^{\frac{1}{N}}L/\mathsf{C}_{\mathsf{Isop}}$.*

Proof. We closely follow [159]. Let μ be the distribution function associated to u and denote by $M:=\sup u<+\infty$. The assumptions grant that μ is continuous and strictly decreasing. Therefore for any $s,k\geq 0$ such that $s+k\leq\mathfrak{m}(\mathrm{supp}(u))$ we can find $0\leq t-h\leq t\leq M$ in such a way that $\mu(t-h)=s+k$ and $\mu(t)=s$. Then from the coarea formula (1.2.17) and the L-Lipschitzianity of u we get

$$\int_{t-h}^{t}\mathrm{Per}(\{u>r\},\cdot)\,\mathrm{d}r=\int_{\{t-h<u\leq t\}}|Du|_1\,\mathrm{d}\mathfrak{m}\leq L(\mu(t-h)-\mu(t))=kL. \tag{6.3.3}$$

Observe that $\{u>r\}\subset\Omega$ for every $r>0$, therefore we can apply the isoperimetric inequality (6.3.2) and obtain that

$$\mathrm{Per}(\{u>r\})\geq\mathsf{C}_{\mathsf{Isop}}\mu(r)^{\frac{N-1}{N}},\qquad\forall r>0.$$

Therefore from (6.3.3) and the monotonicity of μ we obtain

$$kL\geq\mathsf{C}_{\mathsf{Isop}}\int_{t-h}^{t}\mu(r)^{\frac{N-1}{N}}\,\mathrm{d}r\geq\mathsf{C}_{\mathsf{Isop}}h\mu(t)^{\frac{N-1}{N}},$$

from which, observing that in this case $u^\#$ is the inverse of μ, we reach

$$u^\#(s)-u^\#(s+k)\leq s^{-1+1/N}\mathsf{C}_{\mathsf{Isop}}^{-1}kL.$$

In particular $u^\#$ is Lipschitz in $(\varepsilon,\mathrm{supp}(u)]$ (and thus in $(\varepsilon,\mathfrak{m}(\Omega)]$) for every $\varepsilon>0$ and at every one of its differentiability points $s\in(0,\mathfrak{m}(\Omega))$ it holds that

$$-\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s)\leq s^{1-1/N}\mathsf{C}_{\mathsf{Isop}}^{-1}L.$$

Fix now two arbitrary and distinct points $x,y\in\Omega^*$ and assume without loss of generality that $y>x$. Recalling the definition of $u^*_{0,N}$ we have that $u^*_{0,N}(x)\geq u^*_{0,N}(y)$ and

$$\begin{aligned}u^*_{0,N}(x)-u^*_{0,N}(y)&=u^\#(\omega_Nx^N)-u^\#(\omega_Ny^N)=\int_{\omega_Nx^N}^{\omega_Ny^N}-\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s)\,\mathrm{d}s\\&\leq\int_{\omega_Nx^N}^{\omega_Ny^N}\frac{s^{-1+1/N}}{\mathsf{C}_{\mathsf{Isop}}}L\,\mathrm{d}s=\omega_N^{\frac{1}{N}}\frac{NL}{\mathsf{C}_{\mathsf{Isop}}}|x-y|,\end{aligned}$$

which proves that $u^*_{0,N}:\Omega^*\to[0,\infty)$ is $N\omega_N^{\frac{1}{N}}L/\mathsf{C}_{\mathsf{Isop}}$-Lipschitz. □

The proof of the following result is exactly the same as in Lemma 3.11 of [159], since the only relevant fact for the proof is that $\mathfrak{m}_{0,N}=h_N\mathscr{L}^1$ with a weight h_N which is bounded away from zero out of the origin (recall also (2.2.4)).

Lemma 6.3.5. *Let $p \in (1,\infty)$. Let $u \in W^{1,p}([0,r],|.|,\mathfrak{m}_{0,N})$, with $r \in (0,\infty)$, be monotone. Then $u \in W^{1,1}_{loc}(0,r)$ and it holds that*

$$|D^{ac}u|(t) = |u'|(t) = |Du|(t), \quad \textit{for a.e. } t \in [0,r].$$

Theorem 6.3.6 (Euclidean Polya-Szego inequality)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N')$ space, $K \in \mathbb{R}$ $N' \in (1,\infty)$ and let $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. Assume furthermore that, for some $N \in (1,\infty)$ and $\mathsf{C}_{\mathsf{Isop}} > 0$, the isoperimetric inequality in (6.3.2) holds in Ω. Then the Euclidean-rearrangement maps $W^{1,p}_0(\Omega)$ to $W^{1,p}(\Omega^*,|.|,\mathfrak{m}_{0,N})$ for any $1 < p < +\infty$. Moreover for any $u \in W^{1,p}_0(\Omega)$ it holds*

$$\int_\Omega |Du|^p \mathrm{d}\mathfrak{m} \ge \left(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\right)^p \int_{\Omega^*} |Du^*_{0,N}|^p \mathrm{d}\mathfrak{m}_{0,N}. \tag{6.3.4}$$

Proof. The proof is a minor modification of the arguments in [159], we will however include most of the details. We first prove the result assuming that $u \in \mathrm{Lip}_c(\Omega)$ and $|D^{ac}u|(x) \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u > 0\}$, then the general case will follow by approximation. Set $M := \sup u$ and define the functions $\phi, \psi : [0,M] \to \mathbb{R}^+$ as follows

$$\phi(t) := \int_{\{u>t\}} |D^{ac}u|^p \,\mathrm{d}\mathfrak{m}, \quad \psi(t) := \int_{\{u>t\}} |D^{ac}u| \,\mathrm{d}\mathfrak{m}.$$

An application of the coarea formula (1.2.17) gives at once that both ϕ and ψ are absolutely continuous with

$$\phi'(t) = -\int |D^{ac}u|^{p-1} \,\mathrm{dPer}(\{u > t\},\cdot), \quad \psi'(t) = -\mathrm{Per}(\{u < t\}), \quad \text{for a.e. } t \in [0,M],$$

for any Borel representative of $|D^{ac}u|$, which we assume to be fixed from now until the end of the proof. From the Hölder inequality we have

$$\psi(t-h) - \psi(t) \le (\phi(t-h) - \phi(t))^{1/p}(\mu(t-h) - \mu(t))^{(p-1)/p}, \quad 0 \le t-h \le t < M,$$

where μ denotes the distribution function of u. From Lemma 1.2.22, we know that also μ is absolutely continuous, in particular we have that a.e. $t \in [0,M]$ is at the same time a differentiability point for ϕ, ψ and μ. Choosing one of such t's in the above inequality, dividing by $h > 0$ and passing to the limit as $h \to 0^+$ we obtain

$$-\psi'(t) \le (-\phi'(t))^{1/p}(-\mu'(t))^{(p-1)/p}, \quad \text{for a.e. } t \in [0,M].$$

Moreover, by the validity of (6.3.2), we have that $\mathrm{Per}(\{u > t\}) \ge \mathsf{C}_{\mathsf{Isop}}\mu(t)^{(N-1)/N}$. Therefore

$$-\phi'(t) \ge \frac{\mathsf{C}^p_{\mathsf{Isop}}\mu(t)^{\frac{(N-1)p}{N}}}{(-\mu'(t))^{p-1}}, \quad \text{for a.e. } t \in [0,M].$$

and integrating we reach

$$\int_\Omega |D^{ac}u|^p \,\mathrm{d}\mathfrak{m} = \int_0^M -\phi'(t)\,\mathrm{d}t \ge \int_0^M \frac{\mathsf{C}^p_{\mathsf{Isop}}\mu(t)^{\frac{(N-1)p}{N}}}{(-\mu'(t))^{p-1}}\,\mathrm{d}t. \tag{6.3.5}$$

Recall now from Proposition 6.3.2 that $\mu(t) = \mathfrak{m}(\{u^*_{0,N} > t\})$, where $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ is the Euclidean monotone rearrangement. Moreover, thanks to the non-vanishing assumptions on $|D^{ac}u|$, we have from Proposition 6.3.4 that $u^*_{0,N} \in \mathrm{Lip}(\Omega^*)$. Additionally $u^*_{0,N}$ is strictly decreasing in $(0,\mathfrak{m}(\mathrm{supp}(u)))$ and in particular $\{u^*_{0,N} > t\} = [0,r_t)$ (and $\{u^*_{0,N} = t\} = \{r_t\}$) for some $r_t \in [0,\mathfrak{m}(\Omega)]$, for every $t \in (0,M)$. Note that r_t can be computed explicitly as $r_t = (\omega_N^{-1}\mu(t))^{1/N}$, which also shows that $t \mapsto r_t$ is a locally absolutely continuous map. Combining these observations with Lemma 1.2.22 and recalling also Lemma 6.3.5 we have following expression for the derivative of μ:

$$-\mu'(t) = \int_{\{u^*_{0,N}=t\}} |D^{ac}u^*_{0,N}|^{-1} \mathrm{d}\mathrm{Per}(\{u^*_{0,N} > t\}, \cdot) = \frac{\mathrm{Per}(\{u^*_{0,N} > t\})}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in (0, M),$$

where r_t is as above. It is clear that $\mathrm{Per}([0,r)) = \sigma_{N-1} r^{N-1}$ for every $r \in (0,\infty)$ (where the perimeter is computed in the space $(I_{0,N}, |.|, \mathfrak{m}_{0,N})$, therefore

$$\mu(t)^{\frac{N-1}{N}} = \big(\frac{\sigma_{N-1}}{N}\big)^{\frac{N-1}{N}} r_t^{N-1} = \frac{\mathrm{Per}(\{u^*_{0,N} > t\})}{N\omega_N^{\frac{1}{N}}}, \tag{6.3.6}$$

and thus we can finally obtain that

$$-\mu'(t) = N\omega_N^{\frac{1}{N}} \frac{\mu(t)^{\frac{N-1}{N}}}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in [0, M].$$

Plugging this identity in (6.3.5) and using again (6.3.6) (recalling also Lemma 6.3.5)

$$\begin{aligned}
\int_\Omega |D^{ac}u|^p \,\mathrm{d}\mathfrak{m} &\geq \mathsf{C}^p_{\mathsf{Isop}} (N\omega_N^{1/N})^{1-p} \int_0^M (|(u^*_{0,N})'|(r_t))^{p-1} \mu(t)^{\frac{(N-1)}{N}} \,\mathrm{d}t \\
&= \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p \int_0^M (|(u^*_{0,N})'|(r_t))^{p-1} \mathrm{Per}(\{u^*_{0,N} > t\}, \cdot) \,\mathrm{d}t \\
&= \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p \int_0^M \int (|(u^*_{0,N})'|(r_t))^{p-1} \mathrm{d}\mathrm{Per}(\{u^*_{0,N} > t\}, \cdot) \,\mathrm{d}t = \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p \int_{\Omega^*} |Du^*_{0,N}|^p \,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Recalling (see (1.2.15)) that $|D^{ac}u| \leq \mathrm{lip}\ u$ $\mathfrak{m}$-a.e. for every $u \in \mathrm{Lip}_{bs}(\mathrm{X})$, we obtain (6.3.4). In the case of a general $u \in W^{1,p}_0(\Omega)$ the result follows via approximation exploiting Lemma 2.3.13 exactly as in the proof of Theorem 1.4 in [159]. □

Remark 6.3.7. It follows from its proof, that Theorem 6.3.6 holds with the weaker assumption that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is uniformly locally doubling and supports a weak local $(1,1)$-Poincaré inequality. Recall also from Remark 6.3.3 that under these assumptions an isoperimetric inequality as in (6.3.2) is available. ■

6.3.2 Local Sobolev inequality

The main goal of this section is to prove the following local Sobolev inequality of Euclidean-type.

Theorem 6.3.8 (Local Euclidean-Sobolev inequality)**.** *For every $\varepsilon > 0$, $N \in (1,\infty)$ and $D > 0$ there exists $\delta = \delta(\varepsilon, D, N) > 0$ such that the following holds. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K,N)$ space, $K \in \mathbb{R}$. Let $r, R \in (0, \frac{1}{2}\sqrt{N/K^-})$ and $x \in \mathrm{X}$ be such that $r < \delta R$, $R < \delta\sqrt{N/K^-}$ (with $\sqrt{N/K^-} := +\infty$ if $K \geq 0$) and $\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \leq D(r/R)^N$. Then*

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq (1+\varepsilon)\mathsf{Eucl}(N,p) \left(\frac{\mathfrak{m}(B_R(x))}{R^N\omega_N}\right)^{-\frac{1}{N}} \||Du|\|_{L^p(\mathfrak{m})}, \quad \forall u \in \mathrm{Lip}_c(B_r(x)). \tag{6.3.7}$$

We mention that local ‘almost-Euclidean” Sobolev inequalities as in the above result are well known on Riemannian manifolds, however they usually depend on double sided bounds on the sectional curvature or on Ricci lower bounds coupled with a lower bound on the injectivity radius (see e.g. [32, Lemma 2.24] and [115, Lemma 7.1, Sec. 7.1]). Instead in our case we only need a lower bound on the Ricci curvature and bounds on the measure of small balls, for this reason Theorem 6.3.8 appears interesting also in the smooth setting.

We face now face a necessary step for the proof of Theorem 6.3.8 starting with the following local isoperimetric inequality of Euclidean type to be used in conjunction with Polya-Szego inequality developed in the previous section. The proof relies on the Brunn-Minkowski inequality and it is mainly inspired by [37], where sharp global isoperimetric inequalities for $\mathsf{CD}(0,N)$ spaces have been proved (see also [29] for a refinement and the previous [45] and [89] for the smooth case). It is worth to mention that a class of 'almost-Euclidean' isoperimetric inequalities in essentially nonbranching CD-spaces, similar to the following ones, were proved in [60] via localization-technique. However, the results in [60] present a set of assumptions that are not suitable for our purposes. Moreover our arguments are different and do not assume the space to be essentially non-branching.

Theorem 6.3.9 (Almost-Euclidean isoperimetric inequality)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{CD}(K,N)$ *space for some* $N\in(1,\infty), K\in\mathbb{R}$*. Then for every* $0<r<R<\frac{1}{2}\sqrt{N/K^-}$ *(where* $\sqrt{N/K^-}=+\infty$ *for* $K\geq 0$*) and* $x\in\mathrm{X}$ *we have*

$$\mathrm{Per}(E)\geq \mathfrak{m}(E)^{\frac{N-1}{N}}N\omega_N^{\frac{1}{N}}\theta_{N,R}^{\frac{1}{N}}(x)(1-(2C_{r,R}^{1/N}+1)\delta-2\eta),\quad \forall\, E\subset B_r(x), \tag{6.3.8}$$

where $\delta:=\frac{r}{R}$*,* $\eta:=R\sqrt{K^-/N}$ *and* $C_{r,R}:=\theta_{N,r}(x)/\theta_{N,R}(x)$*.*

Proof. It is sufficient to prove (6.3.8) with the Minkowski content $\mathfrak{m}(E)^+$ instead of the perimeter. Indeed we could then apply the approximation result in Proposition 1.2.20 to deduce that for every $r'\in(r,R)$, (6.3.8) holds with $r=r'$ (this time with $\mathrm{Per}(E)$). Noticing that $\theta_{N,r'}(x)\to\theta_{N,r}(x)$ as $r'\downarrow r$, sending $r'\to r$ would give the conclusion.

Let $r,R\in\mathbb{R}^+$ with $r<R$ and fix $E\subset B_r(x_0)$ with $\mathfrak{m}(E)>0$. We aim to apply the Brunn-Minkowski inequality to the sets $A_0:=E$, $A_1:=B_R(x_0)$. The triangle inequality easily yields that $A_t\subset E^{t(r+R)}$ for every $t\in(0,1)$ (recall that E^ε is the ε-enlargement of the set E, while A_t is the set of t-midpoint between A_0,A_1). We consider first the case $K\geq 0$. From the Brunn-Minkowski applied with $K=0$ we obtain

$$\begin{aligned}
\mathfrak{m}^+(E)&=\lim_{\varepsilon\to 0^+}\frac{\mathfrak{m}(E^\varepsilon)-\mathfrak{m}(E)}{\varepsilon}=\lim_{t\to 0^+}\frac{\mathfrak{m}(E^{t(r+R)})-\mathfrak{m}(E)}{t(r+R)}\\
&\overset{(2.2.8)}{\geq}\lim_{t\to 0^+}\frac{\big(t\mathfrak{m}(B_R(x_0))^{1/N}+(1-t)\mathfrak{m}(E)^{1/N}\big)^N-\mathfrak{m}(E)}{t(r+R)}\\
&=N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{\mathfrak{m}(B_R(x_0))^{1/N}-\mathfrak{m}(E)^{1/N}}{r+R}\\
&\geq N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{\mathfrak{m}(B_R(x_0))^{1/N}-\mathfrak{m}(B_r(x_0))^{1/N}}{r+R},
\end{aligned}$$

where we have used that $E\subset B_r(x_0)$. If instead $K<0$, arguing analogously we obtain

$$\mathfrak{m}^+(E)\geq\frac{N\mathfrak{m}(E)^{\frac{N-1}{N}}}{r+R}\Big(\frac{\theta\sqrt{-K/N}}{\sinh(\theta\sqrt{-K/N})}\mathfrak{m}(B_R(x_0))^{\frac{1}{N}}-\frac{\theta\sqrt{-K/N}\cosh(\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}\mathfrak{m}(B_r(x_0))^{\frac{1}{N}}\Big),$$

where θ denotes the maximal length of geodesics from A_0 to A_1. It is clear that $\theta\leq r+R$. Moreover for $t\leq 1$ we both have $1-t\leq t/\sinh(t)\leq 1$ and $\cosh(t)\leq 1+t$. In particular if $R\leq\frac{1}{2}\sqrt{-N/K}$ we obtain that

$$\mathfrak{m}^+(E)\geq N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{(1-\sqrt{-K/N}(r+R))\mathfrak{m}(B_R(x_0))^{1/N}-(1+\sqrt{-K/N}(r+R))\mathfrak{m}(B_r(x_0))^{1/N}}{r+R}.$$

Going back to the case of a general $K\in\mathbb{R}$, combining the above estimates and rearranging the terms we reach

$$\mathfrak{m}^+(E)\geq\frac{\mathfrak{m}(E)^{\frac{N-1}{N}}N\omega_N^{\frac{1}{N}}\theta_{N,R}(x)^{\frac{1}{N}}}{1+r/R}\Big((1-\sqrt{\tfrac{K^-}{N}}(r+R))-(1+\sqrt{\tfrac{K^-}{N}}(r+R))\frac{r}{R}\Big(\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)}\Big)^{\frac{1}{N}}\Big),$$

provided $R \le \frac{1}{2}\sqrt{N/K^-}$. Setting $\delta := \frac{r}{R}$, $\eta := R\sqrt{K^-/N}$ and $C := \theta_{N,r}(x)/\theta_{N,R}(x)$, the above gives

$$\mathfrak{m}^+(E) \ge \mathfrak{m}(E)^{\frac{N-1}{N}} N\omega_N^{\frac{1}{N}}\theta_{N,R}(x)^{\frac{1}{N}}\frac{1}{1+\delta}\big((1-2\eta)-(1+2\eta)\delta C^{\frac{1}{N}}\big)$$

that easily implies the conclusion. □

Next, we recall the following classical one-dimensional inequality by Bliss [44] (see also [32, 182]).

Lemma 6.3.10 (Bliss inequality)**.** *Let $u\colon [0,\infty)\to\mathbb{R}$ be locally absolutely continuous. Then for any $1<p<N$ it holds*

$$\Big(\sigma_{N-1}\int_0^\infty |u|^{p^*}\,t^{N-1}\,\mathrm{d}t\Big)^{\frac{1}{p^*}} \le \mathrm{Eucl}(N,p)\Big(\sigma_{N-1}\int_0^\infty |u'|^p\,t^{N-1}\,\mathrm{d}t\Big)^{\frac{1}{p}}, \tag{6.3.9}$$

whenever one side is finite and where $p^* := pN/(N-p)$. Moreover the functions $v_b(r) := (1+br^{\frac{p}{p-1}})^{\frac{p-N}{p}}$, $b>0$, satisfy (6.3.9) with equality.

With the above local isoperimetric inequality and the Euclidean Polya-Szego inequality, the strategy is now to symmetrize functions on the space and exploit the Bliss inequality to deduce the desired local-Sobolev inequalities.

Proof of Theorem 6.3.8. We start observing that it is enough to prove (6.3.7) for non-negative functions. Fix $u \in \mathrm{Lip}_c(B_r(x))$ non-negative and consider $u^*_{0,N}\colon B_r(x)^* \to [0,\infty)$ be the Euclidean-rearrangement of u as in Definition 6.3.1, where $B_r(x)^* = [0,t]$ for some $t>0$. The local Euclidean-isoperimetric inequality given by Theorem 6.3.9 implies that the hypothesis of Proposition 6.3.4 and Theorem 6.3.6 are fulfilled with $\Omega = B_r(x)$ and $\mathsf{C}_{\mathsf{Isop}} = (1-(2D^{1/N}+1)\delta' - 2\eta)N\omega_N^{\frac{1}{N}}\theta_{N,R}(x)^{\frac{1}{N}}$, with $\delta' := \frac{r}{R}$, $\eta := R\sqrt{K^-/N}$ and $D := \theta_{N,r}(x)/\theta_{N,R}(x)$. In particular it holds that $u^*_{0,N} \in W^{1,p}([0,t],|.|,\mathfrak{m}_{0,N})$, which implies (recall (2.2.4)) that $u^*_{0,N} \in W^{1,1}_{loc}(0,t)$ with $(u^*_{0,N})' \in L^p(\mathfrak{m}_{0,N})$ and $|Du^*_{0,N}| = |(u^*_{0,N})'|$ a.e.. Moreover, since $\mathfrak{m}_{0,N}$ is bounded away from 0 far from the origin, $u^*_{0,N} \in W^{1,1}(\varepsilon,t]$ for every $\varepsilon>0$ and by definition $u^*_{0,N}(t) = 0$. Therefore $u^*_{0,N}$ (extended by 0 in (t,∞)) satisfies the assumptions for the Bliss inequality. Recall also from Proposition 6.3.2 that $\|u^*_{0,N}\|_{L^p(\mathfrak{m}_{0,N})} = \|u\|_{L^p(\mathfrak{m})}$ for every $p\in[1,\infty)$. Therefore we are in position to apply the Euclidean Polya-Szego inequality given by (6.3.4), that combined with the Bliss-inequality (6.3.9) gives

$$\begin{aligned}\|u\|_{L^{p^*}(\mathfrak{m})} = \|u^*_{0,N}\|_{L^{p^*}(\mathfrak{m}_{0,N})} &\overset{(6.3.9)}{\le} \mathrm{Eucl}(N,p)\||Du^*_{0,N}|\|_{L^p(\mathfrak{m}_{0,N})}\\ &\overset{(6.3.4)}{\le} \frac{\mathsf{Eucl}(N,p)\theta_{N,R}(x)^{-\frac{1}{N}}}{(1-(2D^{1/N}+1)\delta'-2\eta)}\||Du|\|_{L^p(\mathfrak{m})}.\end{aligned}$$

Finally from the above and observing that $\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} = D(r/R)^N$, we immediately see that there exists $\delta := \delta(\varepsilon,D,N)$ so that, provided $\delta',\eta<\delta$, (6.3.7) holds. □

We end this section with another simpler variant of local Sobolev inequality. It will be needed to deal with cases where $\theta_N(x) = +\infty$, where Theorem 6.3.8 does not give the right information.

Proposition 6.3.11 (Local Sobolev embedding)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}(K,N)$ space for some $N\in(1,\infty)$, $K\in\mathbb{R}$. Then, for every $p\in(1,N)$ and every $B_r(x)\subset\mathrm{X}$ with $r\le 1$, it holds*

$$\Big(\int_{B_r(x)}|u|^{p^*}\,\mathrm{d}\mathfrak{m}\Big)^{\frac{p}{p^*}} \le \Big(\frac{C\mathfrak{m}(B_r(x))}{r^N}\Big)^{-\frac{p}{N}}\int_{B_{2r}(x)}|Du|^p\,\mathrm{d}\mathfrak{m} + 2^p\mathfrak{m}(B_r(x))^{-\frac{p}{N}}\int_{B_r(x)}|u|^p\,\mathrm{d}\mathfrak{m}, \tag{6.3.10}$$

for every $u\in\mathrm{Lip}(\mathrm{X})$, where $p^* = pN/(N-p)$ and $C = C(K,N,p)$.

Proof. Applying (2.2.12) and the Bishop-Gromov inequality

$$\Big(\int_{B_r(x)}|u|^{p^*}\,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p^*}} \le C_1 r\frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_{2r}(x))^{1/p}}\Big(\int_{B_{2r}(x)}|Du|^p\Big)^{\frac1p}+\mathfrak{m}(B_r(x))^{1/p^*}|u_{B_r(x)}|$$
$$\le C_2 r\frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_r(x))^{1/p}}\Big(\int_{B_{2r}(x)}|Du|^p\Big)^{\frac1p}+\mathfrak{m}(B_r(x))^{\frac{1}{p^*}-\frac1p}\Big(\int_{B_r(x)}|u|^p\,\mathrm{d}\mathfrak{m}\Big)^{\frac1p},$$

for suitable positive constants C_1, C_2 depending only on K, N, p. The desired conclusion follows raising to the p in the above inequality. □

6.3.3 Proof of the upper bound

The strategy of the proof of the following result is by-now classical and combines local-Sobolev inequalities with a partition of unity argument (see [31],[32, Chp. 2 Sec. 7], [115, Theorem 4.5] and also [3, Prop. 3.3]).

Theorem 6.3.12 (Upper bound on α_p)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathsf{CD}(K,N)$ *space, for some* $N\in(1,\infty)$, $K\in\mathbb{R}$. *Then, for every* $\varepsilon>0$ *and every* $p\in(1,N)$, *there exists a constant* $B=B(\varepsilon,p,\mathrm{X})>0$ *such that*

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\le\Big(\frac{\mathsf{Eucl}(N,p)^p}{\min_{\mathrm{X}}\theta_N(x)^{p/N}}+\varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})}+B\|u\|^p_{L^p(\mathfrak{m})},\qquad \forall u\in \mathrm{Lip}(\mathrm{X}). \tag{6.3.11}$$

Proof. We start claiming that the following local version of (6.3.11) holds: for any $x\in\mathrm{X}$ and every $\varepsilon>0$ there exists $r=r(\varepsilon,x)>0$ and $C=C(\varepsilon,p,x)<+\infty$ such that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\le\Big(\frac{\mathsf{Eucl}(N,p)^p}{\min_{y\in\mathrm{X}}\theta_N(y)^{p/N}}+\varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})}+C\|u\|^p_{L^p(\mathfrak{m})},\qquad \forall u\in \mathrm{Lip}_c(B_r(x)). \tag{6.3.12}$$

To show the above we observe first that in the case that $\theta_N(x)=+\infty$, (6.3.12) follows immediately from (6.3.10) for r small enough. We are left with the case $0<\theta_N(x)<+\infty$. We start by fixing $\varepsilon\in(0,1/2)$. From the definition of $\theta_N(x)$, there exists $r'=r'(x,\varepsilon)$ so that for every $r\in(0,r')$ it holds $\theta_{N,r}(x)\in((1-\varepsilon)\theta_N(x),(1+\varepsilon)\theta_N(x))$. In particular we have that $\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)}\le 4$ for every $r,R\in(0,r')$. We are therefore in position to apply Theorem 6.3.8 and deduce that there exists $\delta=\delta(\varepsilon,N)$ so that for every $r,R\in(0,r'\wedge\delta\sqrt{N/K^-})$, with $r<\delta R$, the following inequality holds for every $u\in\mathrm{Lip}_c(B_r(x))$

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\overset{(6.3.7)}{\le}(1+\varepsilon)^p\frac{\mathsf{Eucl}(N,p)^p}{\theta_{N,R}(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})}\le\frac{(1+\varepsilon)^p}{(1-\varepsilon)^{p/N}}\frac{\mathsf{Eucl}(N,p)^p}{\min_{\mathrm{X}}\theta_N(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})},$$

where in the second inequality we have used $\theta_{N,R}(x)\ge(1-\varepsilon)\theta_N(x)$. Therefore (6.3.12) (with $C=0$) follows from the above provided we choose ε small enough.

Since X is compact we can extract a finite covering of balls $\{B_i\}_{i=1}^M$ from the covering $\cup_{x\in\mathrm{X}}B_{r(\varepsilon,x)/2}(x)$. We also set $C:=\max_i C_i$ and

$$A:=\frac{\mathsf{Eucl}(N,p)^p}{\min_{\mathrm{X}}\theta_N(x)^{p/N}}+\varepsilon.$$

We claim that there exists a partition of unity made of functions $\{\phi_i\}_{i=1}^M$ such that $\phi_i\in\mathrm{Lip}_c(2B_i)$, $0\le\phi_i\le1$ and $\phi_i^{1/p}\in\mathrm{Lip}_c(2B_i)$ for all i, having denoted $2B_i$, the ball of twice the radius. To build such partition of unity we can argue as follows: start considering functions $\psi_i\in\mathrm{Lip}_c(2B_i)$, such that $0\le\psi_i\le1$ and $\psi_i\ge1$ in B_i. Then we fix $\beta>p$ and take

$$\phi_i:=\frac{\psi_i^\beta}{\sum_{j=1}^M\psi_j^\beta}.$$

Since by construction $\sum_{j=1}^{M}\psi_j^{\beta}\geq 1$ everywhere on X, we have that $\phi_i^{1/p}\in \mathrm{Lip}_c(2B_i)$. Finally it is clear that $\sum_{i=1}^{M}\phi_i=1$.

We are now ready to prove (6.3.11). Fix $u\in \mathrm{Lip}(\mathrm{X})$ and observe that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}=\Big\|\sum_i \phi_i|u|^p\Big\|_{L^{p^*/p}(\mathfrak{m})}\leq \sum_i \|\phi_i|u|^p\|_{L^{p^*/p}(\mathfrak{m})}=\sum_i \big\|\phi_i^{1/p}|u|\big\|^p_{L^{p^*}(\mathfrak{m})}. \tag{6.3.13}$$

Since $\phi_i^{1/p}|u|\in \mathrm{Lip}_c(2B_i)$ we can apply (6.3.12) to obtain

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathfrak{m})}&\leq \sum_{i=1}^{M}A\int\Big(|D\phi_i^{1/p}||u|+|Du|\phi_i^{1/p}\Big)^p\,\mathrm{d}\mathfrak{m}+C\int\phi_i|u|^p\,\mathrm{d}\mathfrak{m}\\&\leq \sum_{i=1}^{M}A\int\phi_i|Du|^p+c_1|Du|^{p-1}\phi_i^{\frac{p-1}{p}}|D\phi_i^{1/p}||u|+c_2|D\phi_i^{1/p}|^p|u|^p\,\mathrm{d}\mathfrak{m}+C\int\phi_i|u|^p\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where $c_1,c_2\geq 0$ are such that $(1+t)^p\leq 1+c_1t+c_2t^p$ for all $t\geq 0$. Recalling that the functions $0\leq\phi_i^{1/p}\leq 1$ are Lipschitz we obtain

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\leq A\int|Du|^p\,\mathrm{d}\mathfrak{m}+\tilde C\int|Du|^{p-1}|u|\,\mathrm{d}\mathfrak{m}+\tilde C\int|u|^p\,\mathrm{d}\mathfrak{m},$$

where $\tilde C=\tilde C(p,M,L)$, L begin the maximum of the Lipschitz constants of the functions $\phi_i^{1/p}$. Finally from the Young inequality we have for every $\delta>0$

$$\int|Du|^{p-1}|u|\,\mathrm{d}\mathfrak{m}\leq\frac{p\delta^{\frac{p}{p-1}}}{p-1}\int|Du|^p\,\mathrm{d}\mathfrak{m}+\frac{1}{p\delta^p}\int|u|^p\,\mathrm{d}\mathfrak{m},\quad\forall\delta>0$$

and plugging this estimate above, choosing δ small enough (but independent of u), we obtain that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\leq(A+\varepsilon)\int|Du|^p\,\mathrm{d}\mathfrak{m}+C'\int|u|^p\,\mathrm{d}\mathfrak{m},$$

for some $C'=C'(\varepsilon,L,M,p)$. Since $\varepsilon>0$ and $u\in\mathrm{Lip}(\mathrm{X})$ were arbitrary, this concludes the proof. □

6.4 Lower bound on α_p

To rough idea of the lower bound on α_p is that, when $\theta_N(x)<+\infty$ the space near x has a conical structure, hence the constant in the Sobolev inequality cannot be better than the one of the tangent structures of the underlying space. This will be formalized with a blow-up argument combined with a stability result for the Sobolev constants.

6.4.1 Blow-up analysis of Sobolev constants

For convenience, we introduce the following notation: whenever in a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ it holds that

$$\|u\|^p_{L^q(\mathfrak{m})}\leq A\||Du|_p\|^p_{L^p(\mathfrak{m})}+B\|u\|^p_{L^p(\mathfrak{m})},\qquad\forall u\in W^{1,p}(\mathrm{X}).$$

for some constants $A,B>0$ and exponents $1<p<q$, we will say that X **supports a (q,p)-Sobolev inequality with constants** A,B. This convention will be used often here, and sometimes in the subsequent sections, without further notice.

We make precise the scaling enjoyed by the Sobolev inequalities under consideration. It is immediate to check that if a space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ supports a (p^*,p)-Sobolev for $p\in(1,N)$ and $p^*:=\frac{pN}{N-p}$ with constants A,B, then for every $r>0$ we have

$$(\mathrm{X},\mathsf{d}/r,\mathfrak{m}/r^N)\text{ supports a }(p^*,p)\text{-Sobolev with constants }A,Br^p. \tag{6.4.1}$$

We pass to the stability of Sobolev embeddings under pmGH-convergence (see also [121, Thm. 3.1] for a similar result for Ricci-limits).

Lemma 6.4.1 (pmGH-Stability of Sobolev constants)**.** *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, $n \in \bar{\mathbb{N}}$, *be a sequence of* $\mathsf{CD}(K,N)$ *spaces for some* $K \in \mathbb{R}$, $N \in (1,\infty)$ *with* $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$. *Suppose* X_n *support a* (q,p)*-Sobolev inequality for* $1 < p < q$ *with constants* A, B. *Then also* X_∞ *supports a* (q,p)*-Sobolev inequality with the same constants* A, B.

Proof. Fix $u \in \mathrm{Lip}_c(\mathrm{X}_\infty)$, from the Γ-$\overline{\lim}$ inequality of the Ch_p energy, there exists a sequence $u_n \in W^{1,p}(\mathrm{X}_\infty)$ such that u_n converges in L^p-strong to u and $\overline{\lim}_n \int |Du|^p \,\mathrm{d}\mathfrak{m}_n \le \int |Du|^p \,\mathrm{d}\mathfrak{m}_\infty$. In particular

$$\begin{aligned}\overline{\lim_n} \|u_n\|^p_{L^q(\mathfrak{m}_n)} &\le \overline{\lim_{n\to\infty}} A \||Du_n|\|^p_{L^p(\mathfrak{m}_n)} + B\|u_n\|^p_{L^p(\mathfrak{m}_n)} \\ &\le A\||Du|\|^p_{L^p(\mathfrak{m}_\infty)} + B\|u\|^p_{L^p(\mathfrak{m}_\infty)} < +\infty.\end{aligned}$$

Therefore u_n converge also L^q-weak to u. From the lower semicontinuity of the L^q-norm with respect to L^q-weak convergence and the arbitrariness of $u \in \mathrm{Lip}_c(\mathrm{X}_\infty)$ the conclusion follows. $\square$

The following result is a consequence of the existence of the disintegration and can be found for example in [75, Corollary 3.8].

Lemma 6.4.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{CD}(0,N)$ *space with* $N \in [1,\infty)$. *Suppose that for some* $x_0 \in \mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_N r^N} = 1$ *for every* $r \in (0,\infty)$, *then*

$$\int \phi(\mathsf{d}(x_0,x))\,\mathrm{d}\mathfrak{m} = \sigma_{N-1}\int_0^\infty \phi(r) r^{N-1}\,\mathrm{d}r, \qquad \forall \phi \in C_c([0,\infty]).$$

Lemma 6.4.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{CD}(0,N)$ *space,* $N \in (1,\infty)$, $p \in (1,N)$ *and set* $p^* := \frac{pN}{N-p}$. *Suppose that for some* $x_0 \in \mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_N r^N} = 1$ *for every* $r \in (0,\infty)$. *Then there exists a sequence of non-constant functions* $u_n \in \mathrm{Lip}_c(\mathrm{X})$ *satisfying*

$$\lim_n \frac{\|u_n\|_{L^{p^*}(\mathfrak{m})}}{\||Du_n|\|_{L^p(\mathfrak{m})}} \ge \mathsf{Eucl}(N,p).$$

Proof. Let $v : [0,\infty) \to [0,\infty)$, $v \in C^\infty(0,\infty)$, be an extremal function for the Bliss inequality (6.3.9) as given by Lemma 6.3.10. It can be easily shown that we can approximate v with functions $v_n \in \mathrm{Lip}_c([0,\infty))$ so that $\|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)} \to \|v\|_{L^{p^*}(h_N n\mathscr{L}^1)}$ and $\|v_n'\|_{L^p(h_N\mathscr{L}^1)} \to \|v'\|_{L^p(h_N\mathscr{L}^1)}$, where $h_N\mathscr{L}^1 = \sigma_{N-1}t^{N-1}\mathscr{L}^1$. For example we can take $v_n := \phi_n(u_b)$ with $\phi_n \in \mathrm{Lip}[0,\infty)$, $\phi_n \ge 0$, $\phi_n(t) \le |t|$, $\mathrm{Lip}(\phi_n) \le 2$, $\phi_n(t) = t$ in $[2/n,\infty)$ and $\mathrm{supp}(\phi_n) \subset [1/n,\infty)$. The claimed approximation of the norms then follows immediately from the fact that v is decreasing and vanishing at infinity. Therefore we have

$$\lim_n \frac{\|v_n\|_{L^{p^*}(h_n\mathscr{L}^1)}}{\|v_n'\|_{L^p(h_n\mathscr{L}^1)}} = \mathsf{Eucl}(N,p). \tag{6.4.2}$$

We can now define $u_n := v_n \circ \mathsf{d}_{x_0}$, where $\mathsf{d}_{x_0}(\cdot) := \mathsf{d}(x_0,\cdot)$. We clearly have that $u_n \in \mathrm{Lip}_c(\mathrm{X})$ and from the chain rule also that $|Du_n| = |v_n'|\circ\mathsf{d}_{x_0}|D\mathsf{d}_{x_0}| \le |v_n'|\circ\mathsf{d}_{x_0}$ $\mathfrak{m}$-a.e., since d_{x_0} is 1-Lipschitz. Hence applying Lemma 6.4.2 we obtain $\|u_n\|_{L^{p^*}(\mathfrak{m})} = \|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)}$ and $\||Du_n|\|_{L^p(\mathfrak{m})} \le \|v_n'\|_{L^p(h_N\mathscr{L}^1)}$. This combined with (6.4.2) (up to passing to a subsequence) gives the conclusion. $\square$

Theorem 6.4.4 (Lower bound on the Sobolev constant)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{CD}(K,N)$ *space,* $K \in \mathbb{R}$, $N \in (1,\infty)$ *that supports a* (p^*,p)*-Sobolev inequality for* $p \in (1,N)$ *with constants* A, B, *where* $p^* = pN/(N-p)$. *Then*

$$A \ge \frac{\mathrm{Eucl}(N,p)^p}{\theta_N(x)^{\frac{p}{N}}}, \qquad \forall x \in \mathrm{X}. \tag{6.4.3}$$

Proof. If $\theta_N(x) = \infty$, there is nothing to prove. Hence we can assume that $\theta_N(x) < +\infty$. From the compactness and stability of the $\mathsf{CD}(K, N)$ condition, there exists a sequence $r_i \to 0$ such that $\mathrm{X}_i := (\mathrm{X}, \mathsf{d}/r_i, \mathfrak{m}/{r_i}^N, x)$ pmGH-converge to a $\mathsf{CD}(0, N)$ space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \boldsymbol{o}_\mathrm{Y})$. Moreover, from (6.4.1) we have that X_i supports a (p^*, p)-Sobolev inequality with constants $A, r_i^p B$. This combined with Lemma 6.4.1 shows that $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ supports a (p^*, p)-Sobolev inequality with constants $A, 0$. However we clearly have that $\mathfrak{m}_\mathrm{Y}$ satisfies $\frac{\mathfrak{m}_\mathrm{Y}(B_r(\boldsymbol{o}_\mathrm{Y}))}{\omega_N r^N} = \theta_N(x)$ for every $r > 0$. Therefore Lemma 6.4.3, after a rescaling, ensures that $A \geq \frac{\mathrm{Eucl}(N,p)^p}{\theta_N(x)^{\frac{p}{N}}}$, which is what we wanted. $\square$

The above, together with Theorem 6.3.12, proves our main result Theorem 6.2.6 concerning $\alpha_p(\mathrm{X})$.

6.4.2 Sharp and rigid Sobolev inequalities under Euclidean volume growth

Here we prove the sharp Sobolev inequalities on $\mathsf{CD}(0, N)$ spaces contained Theorem 6.2.10. The validity of the inequality (6.2.10) will be derived as a consequence of the local-Sobolev inequalities in Theorem 6.3.8. The sharpness instead follows from a well known principle for which the validity of an Euclidean-Sobolev inequality implies certain growth on the measure of balls. In particular we have the following result:

Theorem 6.4.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{CD}(0, N)$, $N \in (1, \infty)$ *such that for some* $p \in (1, N)$ *and* $A > 0$

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq A \||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{Lip}_c(\mathrm{X}), \tag{6.4.4}$$

where $p^* := \frac{pN}{N-p}$. *Then* X *has Euclidean volume-growth and*

$$\mathsf{AVR}(\mathrm{X}) \geq \Big(\frac{\mathsf{Eucl}(N, p)}{A}\Big)^N. \tag{6.4.5}$$

On the general setting of CD spaces Theorem 6.4.5 is proved in [135] (see also [136] for the case $p = 2$), extending to non-smooth setting the same results for Riemannian manifolds due to Ledoux [137] and improved by Xia [190]. We mention also [83] and [191] for analogous statements related to different class of inequalities. In all the cited works the arguments depend on rather intricate ODE-comparison (originated in [137] and inspired by the previous [36]) and heavily rely on the explicit knowledge of the extremal functions for the inequalities. However, using the results in Section 6.4 we are able to give a short proof of Theorem 6.4.5, which uses a more direct blow-down procedure, that we believe being interesting on its own. The main advantage of this approach is that we will never need, as opposed to the ODE-comparison approach, the explicit expression of extremals functions in the Euclidean Sobolev inequality (6.1.1).

Proof of Theorem 6.4.5. The fact that $\mathfrak{m}(\mathrm{X}) = +\infty$ can be immediately seen by plugging in the Sobolev inequality functions $u_R \in \mathrm{Lip}_c(\mathrm{X})$ so that $u_R = 1$ in $B_R(x_0)$ $\mathrm{supp}(u_R) \subset B_{2R}(x_0)$ and $\mathrm{Lip}(u_R) \leq 1/R$ and sending $R \to +\infty$. The fact that X has Euclidean volume growth follows by considering instead functions $u_R(\cdot) := (R - \mathsf{d}_{x_0}(\cdot))^+$ as $R \to +\infty$ with fixed $x_0 \in \mathrm{X}$ and using the Bishop-Gromov inequality.

It remains to prove (6.4.5). We argue via blow-down. Let $R_i \to +\infty$. From the Euclidean volume-growth property, up to passing to a non relabeled subsequence, the rescaled spaces $(\mathrm{X}, \mathsf{d}/R_i, \mathfrak{m}/R_i^N, x_0)$, $x_0 \in \mathrm{X}$, pmGH-converge to an $\mathsf{CD}(0, N)$ space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \boldsymbol{o}_\mathrm{Y})$ satisfying $\frac{\mathfrak{m}(B_R(\boldsymbol{o}_\mathrm{Y}))}{\omega_N r^N} = \mathrm{AVR}(\mathrm{X})$. Moreover combining (6.4.4) with Lemma 6.4.1 proves that Y satisfy a (p^*, p)-Sobolev inequality with constants $A, 0$. Then (6.4.5) follows from Lemma 6.4.3. $\square$

We can now move to the proof of the sharp Sobolev inequalities under the Euclidean volume growth assumption.

Proof of Theorem 6.2.10. Fix $x \in \mathrm{X}$. From the definition of $\mathsf{AVR}(\mathrm{X})$, for every r big enough $\theta_{N,r}(x) \leq 2\mathsf{AVR}(\mathrm{X})$. Fix one of such $r > 0$. From the Bishop-Gromov inequality we also have that

$\theta_{N,R}(x) \geq \mathsf{AVR}(\mathrm{X})$ for every $R > 0$. In particular $\theta_{N,r}(x)/\theta_{N,R}(x) \leq 2$ for every $R > 0$. Hence by Theorem 6.3.8 (for $K = 0$) we have that for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ so that for every $R > r/\delta$ the following local Euclidean Sobolev inequality holds:

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq (1+\varepsilon)\mathsf{Eucl}(N,p)\theta_{N,R}(x)^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{Lip}_c(B_r(x)).$$

Taking $R \to \infty$ we achieve

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq (1+\varepsilon)\mathsf{Eucl}(N,p)\mathsf{AVR}(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{Lip}_c(B_r(x)).$$

Since ε was chosen arbitrarily and independent of $r > 0$, we can first send $\varepsilon \to 0^+$ and then $r \to +\infty$ to achieve the first part of the statement.

The sharpness of (6.2.10) instead follows immediately from Theorem 6.4.5. □

6.5 The constant A_q^{opt} in metric measure spaces

In this section we will prove some upper and lower bounds on A_q^{opt} in the case of metric measure spaces. Let us also remark that the results of this part are valid for a general lower bound $K \in \mathbb{R}$.

We start recalling the definition of A_q^{opt}. In this section we assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a metric measure space with $\mathfrak{m}(\mathrm{X}) = 1$. For every $q \in (2, +\infty)$ we define $A_q^{\mathrm{opt}}(\mathrm{X}) \in [0, +\infty]$ as the minimal constant satisfying

$$\|u\|^2_{L^q(\mathfrak{m})} \leq A_q^{\mathrm{opt}}(\mathrm{X})\, \||Du|_2\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}(\mathrm{X}), \tag{6.5.1}$$

with the convention that $A := +\infty$ if no such A exists. Note that, since $\mathfrak{m}(\mathrm{X}) = 1$, this is the same definition given right after (6.2.5). In the following sections we will prove two type of bounds on $A_q^{\mathrm{opt}}(\mathrm{X})$: an upper bound in the case of synthetic Ricci curvature and dimension bounds and a lower bound in terms of the first non-trivial eigenvalue.

6.5.1 Upper bound on A_q^{opt} in terms of Ricci bounds

Here we prove a generalization to the non-smooth setting of a well known estimate on A_q^{opt} valid on manifolds (recall (6.1.4)). The two key ingredients for the proof are the Sobolev-Poincaré inequality and an inequality due to Bakry:

Proposition 6.5.1. *For every $K \in \mathbb{R}$, $N \in (2, \infty)$ and $D > 0$ there exists a constant $A = A(K, N, D) > 0$ such that the following holds. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a compact $\mathsf{CD}(K, N)$ space with $N \in (1, \infty)$, $K \in \mathbb{R}$, $\mathfrak{m}(\mathrm{X}) = 1$ and $\mathrm{diam}(\mathrm{X}) \leq D$. Then for every $q \in (2, 2^*]$ we have*

$$\|u\|^2_{L^q(\mathfrak{m})} \leq A\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}) \tag{6.5.2}$$

and in particular $A_q^{\mathrm{opt}}(\mathrm{X}) \leq A(K, N, D)$.

Proof. The proof is based on the following inequality: for every $q \in (2, \infty)$

$$\Big(\int |u|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} \leq (u_{\mathrm{X}})^2 + (q-1)\Big(\int |u - u_{\mathrm{X}}|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} \qquad \forall u \in L^q(\mathfrak{m}), \tag{6.5.3}$$

where $u_{\mathrm{X}} = \int u \,\mathrm{d}\mathfrak{m}$. See ([34] or [35, Prop. 6.2.2]) for a proof of this fact. Then (6.5.2) follows combining (6.5.3) with (2.2.12) and the Jensen inequality. □

Recall that for $K > 0$ an explicit and sharp upper bound on A_q^{opt} exists and has been proven in [59] (see Theorem 6.2.1). The argument in [59] relies on the powerful *localization technique*. However, it is worth to point out that Theorem 6.2.1 can also be deduced from the Polya-Szego inequality proved in [159] (see Theorem 2.2.8) and the Sobolev inequality on the model space (2.2.13).

6.5.2 Lower bound on $A_q^{\rm opt}$ in terms of the first eigenvalue

It is well known that a 'tight-Sobolev inequality" as in (6.5.1) (i.e. with a constant 1 in front of $\|u\|_{L^2}$ when X is normalized with unit volume) implies a Poincaré-inequality (see e.g. [35, Prop. 6.2.2]). This can be rephrased as a lower bound on $A_q^{\rm opt}$ in terms of the first non-trivial eigenvalue:

Proposition 6.5.2. *Let* $({\rm X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}({\rm X})=1$. *Then for every* $q\in(2,+\infty)$ *it holds*

$$A_q^{\rm opt}({\rm X})\ge\frac{q-2}{\lambda^{1,2}({\rm X})},\tag{6.5.4}$$

(meaning that if $\lambda^{1,2}({\rm X})=0$, *then* $A_q^{\rm opt}({\rm X})=+\infty$*).*

We will give a detailed proof of this result, which amounts to a linearization procedure. Indeed a refinement of the same argument will also play a key role on the rigidity and almost-rigidity results in the sequel (see Section 6.6.2).

We start with an elementary linearization-Lemma.

Lemma 6.5.3. *Let* $({\rm X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}({\rm X})=1$ *and fix* $q\in(2,\infty)$. *Let* $f\in L^2\cap L^q(\mathfrak{m})$ *with* $\int f\,{\rm d}\mathfrak{m}=0$. *Then*

$$\begin{aligned}\Big|\Big(\int|1+f|^q\,{\rm d}\mathfrak{m}\Big)^{2/q}-&\int(1+f)^2\,{\rm d}\mathfrak{m}-(q-2)\int|f|^2\,{\rm d}\mathfrak{m}\Big|\\&\le C_q\Big(\int|f|^{3\wedge q}+|f|^q\,{\rm d}\mathfrak{m}+\Big(\int|f|^q\,{\rm d}\mathfrak{m}\Big)^2+\Big(\int|f|^2\,{\rm d}\mathfrak{m}\Big)^2\Big),\end{aligned}\tag{6.5.5}$$

where C_q *is a constant depending only on* q.

Proof. We start defining $I:=\int|1+f|^q\,{\rm d}\mathfrak{m}-1$ and observe that

$$\Big|\Big(\int|1+f|^q\,{\rm d}\mathfrak{m}\Big)^{2/q}-1-\frac{2}{q}I\Big|\le c_q|I|^2,\tag{6.5.6}$$

which follows from the inequality $||1+t|^{2/q}-1-2t/q|\le c_qt^2$, $t\ge0$. It remains to investigate the behavior of I. Exploiting the inequality $||1+t|^q-1-qt|\le\tilde c_q(|t|^2+|t|^q)$, $t\ge0$, and the fact that f has zero mean we have the following simple bound

$$|I|\le\tilde c_q\int|f|^2+|f|^q\,{\rm d}\mathfrak{m}.\tag{6.5.7}$$

We will also need a more precise estimate of I, which will follow from the following inequality

$$\Big||1+t|^q-1-qt-\frac{q(q-1)}{2}t^2\Big|\le C_q(|t|^{3\wedge q}+|t|^q),\quad\forall t\in\mathbb{R},\tag{6.5.8}$$

that can be seen using Taylor expansion when $|t|\le1/2$ and elementary estimates in the case $|t|\ge1/2$. Using (6.5.8) we obtain that

$$\Big|I-\int qf+\frac{q(q-1)}{2}|f|^2\,{\rm d}\mathfrak{m}\Big|\le C_q\int|f|^{3\wedge q}+|f|^q\,{\rm d}\mathfrak{m}$$

and since we are assuming that f has zero mean, we deduce

$$\Big|I-\frac{q(q-1)}{2}\int|f|^2\,{\rm d}\mathfrak{m}\Big|\le C_q\int|f|^{3\wedge q}+|f|^q\,{\rm d}\mathfrak{m}.\tag{6.5.9}$$

Combining (6.5.6), (6.5.7) and (6.5.9), noting that $\int(1+f)^2\,{\rm d}\mathfrak{m}=1+\int f^2\,{\rm d}\mathfrak{m}$, we deduce (6.5.5). $\square$

Exploiting the above linearization, we can now prove the lower bound on A_q^{opt} in terms of the first eigenvalue.

Proof of Proposition 6.5.2. If $A_q^{\mathrm{opt}}(\mathrm{X}) = +\infty$ there is nothing to prove, hence we assume that $A_q^{\mathrm{opt}}(\mathrm{X}) < +\infty$. Let $f \in \mathrm{Lip}(\mathrm{X}) \cap L^2(\mathfrak{m})$ with $\int f \, \mathrm{d}\mathfrak{m} = 0$ and $\|f\|_{L^2(\mathfrak{m})} = 1$. Observe also that, since $A_q^{\mathrm{opt}}(\mathrm{X}) < +\infty$, $f \in L^q(\mathrm{X})$. Therefore applying (6.5.5) we obtain

$$\Big(\int |1+\varepsilon f|^q \, \mathrm{d}\mathfrak{m}\Big)^{2/q} - \int (1+\varepsilon f)^2 \, \mathrm{d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2 \, \mathrm{d}\mathfrak{m} = o(\varepsilon^2),$$

which combined with (6.5.1) gives

$$A_q^{\mathrm{opt}}(\mathrm{X})\varepsilon^2 \int |Df|_2^2 \, \mathrm{d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2 \, \mathrm{d}\mathfrak{m} \geq o(\varepsilon^2).$$

Dividing by ε^2 and sending $\varepsilon \to 0$ gives that $\lambda^{1,2}(\mathrm{X}) \geq \frac{q-2}{A_q^{\mathrm{opt}}(\mathrm{X})}$, which concludes the proof. $\square$

6.6 Rigidity of A_q^{opt}

6.6.1 Concentration Compactness

In this section we assume that $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ is a sequence of compact $\mathrm{RCD}(K,N)$ spaces, for some fixed $K \in \mathbb{R}, N \in (2,\infty)$, which converges in mGH-topology to a compact $\mathrm{RCD}(K,N)$ space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. We will also adopt the extrinsic approach [100] identifying $\mathrm{X}_n, \mathrm{X}_\infty$ as subset of a common compact metric space $(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$, with $\mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$, $\mathrm{supp}(\mathfrak{m}_\infty) = \mathrm{X}_\infty$, $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_b(\mathrm{Z})$ and $\mathrm{X}_n \to \mathrm{X}_\infty$ in the Hausdorff topology of Z. To lighten the discussion, we shall not recall in the following statements these facts and assume $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and (Z, d) to be fixed as just explained. Also, we will set $2^* := 2N/(N-2)$ without recalling its expression in the statements.

Our main goal then is to prove the following dichotomy for the behavior of extremizing sequence for the Sobolev inequalities, on varying metric measure spaces.

Theorem 6.6.1 (Concentration-compactness for Sobolev-extremals)**.** *Suppose that* $\mathfrak{m}_n(\mathrm{X}_n)$, $\mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$ *and that* X_n *supports a* $(2^*, 2)$*-Sobolev inequality*

$$\|u\|_{L^{2^*}(\mathfrak{m}_n)}^2 \leq A\||Du|\|_{L^2(\mathfrak{m}_n)}^2 + B\|u\|_{L^2(\mathfrak{m}_n)}^2, \qquad \forall u \in W^{1,2}(\mathrm{X}_n),$$

for some constants $A, B > 0$. *Suppose that* $u_n \in W^{1,2}(\mathrm{X}_n)$ *is a sequence of non-zero functions satisfying*

$$\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}^2 \geq A_n\||Du_n|\|_{L^2(\mathfrak{m}_n)}^2 + B_n\|u_n\|_{L^2(\mathfrak{m}_n)}^2,$$

for some sequences $A_n \to A$, $B_n \to B$.

Then, setting $\tilde{u}_n := u_n\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}^{-1}$, *there exists a non relabeled subsequence such that only one of the following holds:*

- I) $\tilde{u}_n$ *converges* L^{2^*}*-strong to* $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$*;*
- II) $\|\tilde{u}_n\|_{L^2(\mathfrak{m}_n)} \to 0$ *and there exists* $x_0 \in \mathrm{X}_\infty$ *so that* $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_0}$ *in duality with* $C_b(\mathrm{Z})$.

The principle behind the concentration compactness technique is very general and was originated in [141, 140]. In our case, since we will work in a compact setting, the lack of compactness is formally due to dilations or rescalings (and not to translations) and the fact that we deal with the critical exponent in the Sobolev embedding. The main idea behind the principle is first to prove that in general the failure of compactness can only be realized by concentration on a countable number of points. The second step is then exploit a strict sub-additivity property of the minimization problem to show that either we have full concentration at a single point or we do not have concentration at all and thus compactness.

We start by proving necessary results towards the proof of Theorem 6.6.1and we begin with a variant of [120, Prop. 3.27]. For the sake of completeness, we provide here a complete proof.

Proposition 6.6.2. *Let $p, q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that u_n converges L^q-strong to u_∞ and that v_n converges L^p-weak to v_∞, then*

$$\lim_{n\to\infty}\int u_n v_n \,\mathrm{d}\mathfrak{m}_n = \int u_\infty v_\infty \,\mathrm{d}\mathfrak{m}_\infty.$$

Proof. It is sufficient to consider the case $u_n \geq 0, u_\infty \geq 0$, then the conclusion will follow recalling that $u_n^+ \to u_\infty^+$, $u_n^- \to u_\infty^-$ strongly in L^q.

The argument is similar to the one for the case $p=2$ (see, e.g., in [24]), except that we need to consider the functions $u_n^{q/p}+tv_n$, $t\in\mathbb{R}$. Observe first that $u_n^{p/q} \to u_\infty^{q/p}$ strongly in L^p (by (vii) of Prop. 2.4.3). In particular $u_n^{q/p}+tv_n$ converges to $u_\infty^{q/p}+tv_\infty$ weakly in L^p and in particular from iii) of Prop. 2.4.3 we have

$$\|u_\infty^{q/p}+tv_\infty\|_{L^p(\mathfrak{m}_\infty)} \leq \varliminf_n \|u_n^{q/p}+tv_n\|_{L^p(\mathfrak{m}_n)}. \tag{6.6.1}$$

The second ingredient is the following inequality

$$\big||a+|b||^p-|b|^p-pa|b|^{p-1}\big| \leq C_p(|a|^{p\wedge 2}|b|^{p-p\wedge 2}+|a|^p), \quad \forall a,b\in\mathbb{R}, \tag{6.6.2}$$

which is easily derived from $\big||1+t|^p-1-pt\big| \leq C_p(|t|^{p\wedge 2}+|t|^p)$, $\forall t\in\mathbb{R}$. Combining (6.6.2) and (6.6.1) we have

$$\begin{aligned}
&\int |u_\infty|^q \,\mathrm{d}\mathfrak{m}_\infty + pt\int u_\infty v_\infty \,\mathrm{d}\mathfrak{m}_\infty - C_p t^{p\wedge 2}\int |v_\infty|^{p\wedge 2}|u_\infty^{q/p}|^{p-p\wedge 2}\,\mathrm{d}\mathfrak{m}_\infty - C_p t^p\int |v_\infty|^p \,\mathrm{d}\mathfrak{m}_\infty\\
&\leq \|u_\infty^{q/p}+tv_\infty\|^p_{L^p(\mathfrak{m}_\infty)} \leq \varliminf_n \|u_n^{q/p}+tv_n\|_{L^p(\mathfrak{m}_n)}\\
&\leq \varliminf_n \int |u_n|^q \,\mathrm{d}\mathfrak{m}_n + pt\int u_n v_n \,\mathrm{d}\mathfrak{m}_n + C_p t^{p\wedge 2}\int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2}\,\mathrm{d}\mathfrak{m}_n + C_p t^p\int |v_n|^p \,\mathrm{d}\mathfrak{m}_n
\end{aligned}$$

Observe that in the case $p<2$ we have

$$\varlimsup_n \int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2} = \varlimsup_n \int |v_n|^p \,\mathrm{d}\mathfrak{m}_n < +\infty,$$

while for $p\geq 2$ using the Hölder inequality

$$\varlimsup_n \int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2} \leq \varlimsup_n \|v_n\|^2_{L^p(\mathfrak{m}_n)}\|u_n\|^{q(p-2)/p}_{L^q(\mathfrak{m}_n)} < +\infty.$$

In particular, recalling that $\int |u_n|^q \mathrm{d}\mathfrak{m}_n \to \int |u_\infty|^q \mathrm{d}\mathfrak{m}_\infty$ and choosing first $t\downarrow 0$ and then $t\uparrow 0$ above we obtain the desired conclusion. □

The following is a version for varying-measure of the famous Brezis-Lieb Lemma [46]. The key difference with the classical version of this result, is that in our setting it does not makes sense to write '$|u_\infty - u_n|$', since u_∞ and u_n will be integrated with respect to different measures. Hence we need to replace this term in (6.6.3) with $|v_n-u_n|$, where v_n is sequence approximating u_∞ in a strong sense.

Lemma 6.6.3 (Brezis-Lieb type Lemma)**.** *Suppose that $\mathfrak{m}_n(\mathrm{X}_n), \mathfrak{m}_\infty(\mathrm{X}_\infty)=1$, let $q\in[2,\infty)$ and $q'\in(1,q)$. Suppose that $u_n\in L^q(\mathfrak{m}_n)$ satisfy $\sup_n\|u_n\|_{L^q(\mathfrak{m}_n)}<+\infty$ and that u_n converges to u_∞ strongly in $L^{q'}$ to some $u_\infty\in L^{q'}\cap L^q(\mathfrak{m}_\infty)$. Then for any sequence $v_n\in L^q(\mathfrak{m}_n)$ such that $v_n\to u_\infty$ strongly both in $L^{q'}$ and L^q, it holds*

$$\lim_{n\to\infty}\int |u_n|^q \,\mathrm{d}\mathfrak{m}_n - \int |u_n-v_n|^q \,\mathrm{d}\mathfrak{m}_n = \int |u_\infty|^q \,\mathrm{d}\mathfrak{m}_\infty. \tag{6.6.3}$$

Proof. The proof is based on the following inequality:

$$\big||a+b|^q - |b|^q - |a|^q\big| \le C_p(|a|^q|b|^{q-1} + |a|^{q-1}|b|^q), \quad \forall a, b \in \mathbb{R}. \tag{6.6.4}$$

Indeed, if $a = v_n - u_n$ and $b = v_n$, we get from the above

$$\int \big||u_n|^q - |v_n - u_n|^q - |v_n|^q\big| \,\mathrm{d}\mathfrak{m}_n \le C_q \int |v_n - u_n||v_n|^{q-1} + |v_n - u_n|^{q-1}|v_n| \,\mathrm{d}\mathfrak{m}_n. \tag{6.6.5}$$

Since $\int |v_n|^q \,\mathrm{d}\mathfrak{m}_n \to \int |u_\infty|^q \,\mathrm{d}\mathfrak{m}_\infty$, to conclude it is sufficient to show that the right hand side of (6.6.5) vanishes as $n \to +\infty$. We wish to apply Proposition 6.6.2. It follows from our assumptions that $|v_n| \to |v_\infty|$ strongly in L^q and $|v_n|^{q-1} \to |v_\infty|^{q-1}$ strongly in L^p, with $p := q/(q-1)$ (recall Prop. 2.4.3). Hence it remains only to show that $|v_n - u_n|, |v_n - u_n|^{q-1}$ converges to 0 weakly in L^q and weakly in L^p respectively. We have that $\sup_n \|u_n - v_n\|_{L^q(\mathfrak{m}_n)} < +\infty$, hence by *iv)* in Prop. 2.4.3 up to a subsequence $|u_n - v_n|$ converge weakly in L^q to a function $w \in L^q(\mathfrak{m})$. However by assumption the sequences $(v_n), (u_n)$ both converge strongly in $L^{q'}$ to u, hence $v_n - u_n \to 0$ strongly in $L^{q'}$ (recall ii) in Prop. 2.4.3) and in particular by from i) of Prop. 2.4.3 we have that $|v_n - u_n| \to 0$ strongly in $L^{q'}$, which implies that $w = 0$. Analogously we also get that up to a subsequence $|u_n - v_n|^{q-1}$ converge weakly in L^p to a non-negative function $w' \in L^p(\mathfrak{m})$. Suppose first that $q' \le q - 1$. taking $t \in [0,1]$ such that $q - 1 = tq' + (1-t)q$ we have

$$\int w' \,\mathrm{d}\mathfrak{m}_\infty = \lim_n \int |u_n - v_n|^{q-1} \,\mathrm{d}\mathfrak{m}_n \le \|v_n - u_n\|^{tq'}_{L^{q'}(\mathfrak{m}_n)} \|v_n - u_n\|^{(1-t)q}_{L^q(\mathfrak{m}_n)} \to 0,$$

where we have used again that $u_n - v_n \to 0$ strongly in $L^{q'}$ and that $u_n - v_n$ is uniformly bounded in L^q. If instead $q' \ge q - 1$ by Hölder inequality we have

$$\int w' \,\mathrm{d}\mathfrak{m}_\infty = \lim_n \int |u_n - v_n|^{q-1} \,\mathrm{d}\mathfrak{m}_n \le \left(\int |u_n - v_n|^{q'} \,\mathrm{d}\mathfrak{m}_n \right)^{(q-1)/q'} \to 0.$$

In both cases we deduce that $w' = 0$, which concludes the proof. □

Lemma 6.6.4. *Let $q \in [2, \infty)$ and let $u_\infty \in W^{1,2}(\mathrm{X}_\infty) \cap L^q(\mathfrak{m}_\infty)$. Then, there exists a sequence $u_n \in W^{1,2}(\mathrm{X}_n) \cap L^q(\mathrm{X}_n)$ that converges both L^q-strong and $W^{1,2}$-strong to u_∞.*

Proof. By truncation and a diagonal argument we can assume that $u_\infty \in L^\infty(\mathfrak{m}_\infty)$. By the Γ-$\overline{\lim}$ inequality of the Ch_2 energy there exists a sequence $v_n \in W^{1,2}(\mathrm{X}_n)$ converging strongly in $W^{1,2}$ to u_∞. Defining $u_n := (v_n \wedge C) \vee -C$, with $C \ge \|u_\infty\|_{L^\infty(\mathfrak{m}_\infty)}$, we have by i) of Proposition 2.4.3 that u_n converges in L^2-strong to u_∞. Moreover $|Du_n| \le |Dv_n|$ $\mathfrak{m}_n$-a.e., therefore $\overline{\lim}_n \int |Du_n|^2 \,\mathrm{d}\mathfrak{m}_n \le \overline{\lim}_n \int |Dv_n|^2 = \int |Du_\infty|^2 \,\mathrm{d}\mathfrak{m}$, which grants that u_n converges also $W^{1,2}$-strongly to u_∞. Finally, the sequence u_n is uniformly bounded in L^∞ and converges to u_∞ in L^2-strong, hence by *(viii)* of Proposition 2.4.3. we have that that u_n is also L^q-strongly convergent to u_∞. □

The following statement is the analogous in metric measure spaces of [141, Lemma I.1]. We shall omit its proof since the arguments presented there in $\mathbb{R}^n$ extend to this setting with obvious modifications (see also Remark I.5 in [141]).

Lemma 6.6.5. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $\mu, \nu \in \mathscr{M}_b^+(\mathrm{X})$. Suppose that*

$$\left(\int |\varphi|^q \,\mathrm{d}\nu \right)^{1/q} \le C \left(\int |\varphi|^p \,\mathrm{d}\mu \right)^{1/p}, \qquad \forall \varphi \in \mathrm{Lip}_b(\mathrm{X}),$$

for some $1 \le p < q < +\infty$ and $C \ge 0$. Then there exists a countable set of indices J, points $(x_j)_{j\in J} \subset \mathrm{X}$ and positive weights $(\nu_j)_{j\in J} \subset \mathbb{R}^+$ so that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge C^{-p} \sum_{j \in J} \nu_j^{p/q} \delta_{x_j}. \tag{6.6.6}$$

Next, we present a generalized Concentration-Compactness principle, with underlying varying ambient space. For the sake of generality and for an application to the Yamabe equation in Section 6.8, we will be working with a slightly more general Sobolev inequality containing an arbitrary L^q-norm (apart from Section 6.8, we will use this statement only with $q=2$).

Lemma 6.6.6 (Concentration-Compactness Lemma)**.** *Suppose that $\mathfrak{m}_n(\mathrm{X}_n), \mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$ and that for some fixed $q \in (1,\infty)$ the spaces X_n satisfy the following Sobolev-type inequalities*

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)} \le A_n \||Du|\|^2_{L^2(\mathfrak{m}_n)} + B_n \|u\|^2_{L^q(\mathfrak{m}_n)}, \qquad \forall u \in W^{1,2}(\mathrm{X}_n), \tag{6.6.7}$$

with uniformly bounded positive constants A_n, B_n. Let also $u_n \in W^{1,2}(\mathrm{X}_n)$ be $W^{1,2}$-weak and both L^2-strong and L^q-strong converging to $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$ and suppose that $|Du_n|^2\mathfrak{m}_n \rightharpoonup \mu$, $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ in duality with $C_b(\mathrm{Z})$ for two given measures $\mu, \nu \in \mathscr{M}^+_b(\mathrm{Z})$.

Then,

i) *there exists a countable set of indices J, points $(x_j)_{j\in J} \subset \mathrm{X}_\infty$ and positive weights $(\nu_j)_{j\in J} \subset \mathbb{R}^+$ so that*
$$\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j \delta_{x_j};$$

ii) *there exist $(\mu_j)_{j\in J} \subset \mathbb{R}^+$ satisfying $\nu_j^{2/2^*} \le (\overline{\lim}_n A_n)\mu_j$ and such that*
$$\mu \ge |Du_\infty|^2\mathfrak{m}_\infty + \sum_{j\in J} \mu_j \delta_{x_j}.$$

In particular, we have $\sum_j \nu_j^{2/2^*} < \infty$.

Proof. We subdivide the proof in two steps.
Step 1. We assume that $u_\infty = 0$. Let $\varphi \in \mathrm{Lip}_b(\mathrm{Z})$ and consider the sequence $(\varphi u_n) \in W^{1,2}(\mathrm{X}_n)$ which plugged in the Sobolev inequality for each X_n gives

$$\left(\int |\varphi|^{2^*}|u_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n\right)^{1/2^*} \le \left(A_n\int |D(\varphi u_n)|^2\,\mathrm{d}\mathfrak{m}_n + B_n\Big(\int |\varphi|^q u_n^q\,\mathrm{d}\mathfrak{m}_n\Big)^{2/q}\right)^{1/2}, \qquad \forall n \in \mathbb{N}.$$

It is clear that, by weak convergence, the left hand side of the inequality tends to $(\int |\varphi|^{2^*}\,\mathrm{d}\nu)^{1/2^*}$. While for the right hand side we discuss the two terms separately. First, by L^q-strong convergence, we have $\int \varphi^q u_n^q\,\mathrm{d}\mathfrak{m}_n \to 0$, while an an application of the Leibniz rule gives $\int |D(\varphi u_n)|\,\mathrm{d}\mathfrak{m}_n \le \int |D\varphi||u_n| + |\phi||Du_n|\,\mathrm{d}\mathfrak{m}_n$. Moreover again by strong convergence $\int |D\varphi|^2|u_n|^2\mathrm{d}\mathfrak{m}_n \to 0$. Combining these observations we reach

$$\left(\int |\varphi|^{2^*}\mathrm{d}\nu\right)^{1/2^*} \le \big(\overline{\lim_n}\, A_n\big)^{1/2}\left(\int |\varphi|^2\,\mathrm{d}\mu\right)^{1/2}, \qquad \forall \varphi \in \mathrm{Lip}_b(\mathrm{Z}).$$

Thus, Lemma 6.6.5 (applied in the space $(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$) gives i)-ii), for the case $u_\infty = 0$, except for the fact that we currently do no know whether the points $(x_j)_{j\in J}$ are in X_∞. This last simple fact can be seen as follows. Fix $j \in J$. From the weak convergence $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$, there must be a sequence $y_n \in \mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$ such that $\mathsf{d}_\mathrm{Z}(y_n, x_j) \to 0$. Then the GH-convergence of X_n to X_∞ ensures that $x_j \in \mathrm{X}_\infty$, which is what we wanted.
Step 2. We now consider the case of a general u_∞. Observe that from Lemma 6.4.1 X_∞ supports a $(2^*,2)$-Sobolev inequality hence, $u_\infty \in L^{2^*}(\mathfrak{m}_\infty)$. From Lemma 6.6.4 there exists a sequence $\tilde u_n \in W^{1,2}(\mathrm{X}_n)$ such that $\tilde u_n$ converges to u_∞ both strongly in $W^{1,2}$ and strongly in L^{2^*}. Consider now the sequence $v_n := u_n - \tilde u_n$. Clearly v_n converges to zero both in L^2-strong and in $W^{1,2}$-weak. Moreover the measures $|v_n|^{2^*}\mathfrak{m}_n$ and $|Dv_n|^2\mathfrak{m}_n$ have uniformly bounded mass. Since (Z,d) is compact, passing to a non-relabeled subsequence we have $|v_n|^{2^*}\mathfrak{m}_n \rightharpoonup \bar\nu$ and $|Dv_n|^2\mathfrak{m}_n \rightharpoonup \bar\mu$ in duality with $C_b(\mathrm{Z})$ for some $\bar\nu, \bar\mu \in \mathscr{M}^+_b(\mathrm{Z})$. Therefore we can apply Step 1 to the sequence v_n to get $\bar\nu = \sum_{j\in J}\nu_j\delta_{x_j}$, $\bar\mu \ge \sum_{j\in J}\mu_j\delta_{x_j}$ for a suitable countable family J, $(x_j) \subset \mathrm{X}_\infty$ and weights

$(\nu_j), (\mu_j)$ satisfying $\nu_j^{2/2^*} \le (\overline{\lim}_n A_n)\mu_j$. To carry the properties of v_n to the sequence u_n we invoke Lemma 6.6.3 (with $q' = 2$ and $q = 2^*$) to deduce that

$$\lim_{n\to\infty} \int |\varphi|^{2^*} |u_n|^{2^*} \, \mathrm{d}\mathfrak{m}_n - \int |\varphi|^{2^*} |v_n|^{2^*} \, \mathrm{d}\mathfrak{m}_n = \int |\varphi|^{2^*} |u_\infty|^{2^*} \, \mathrm{d}\mathfrak{m}_\infty, \tag{6.6.8}$$

and, taking into account the weak convergence, this implies that

$$\int \varphi^{2^*} \, \mathrm{d}\nu - \int \varphi^{2^*} \, \mathrm{d}\bar{\nu} = \int |u_\infty|^{2^*} \varphi^{2^*} \, \mathrm{d}\mathfrak{m}_\infty,$$

for every non-negative $\varphi \in C_b(\mathrm{Z})$. In particular, this is equivalent to say that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \bar{\nu} = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j \delta_{x_j}$, which proves i). Next, we claim that $\mu \ge \sum_{j\in J} \mu_j \delta_{x_j}$ and, to do so, we consider for each $j \in J$ and $\varepsilon > 0$, $\chi_\varepsilon \in \mathrm{Lip}_b(\mathrm{Z})$, $0 \le \chi_\varepsilon \le 1, \chi_\varepsilon(x_j) = 1$ and supported in $B_\varepsilon(x_j)$. The key ingredient is the following estimate

$$\begin{aligned}
\Big| \int \chi_\varepsilon |Du_n|^2 \, \mathrm{d}\mathfrak{m}_n - \int \chi_\varepsilon |Dv_n|^2 \, \mathrm{d}\mathfrak{m}_n \Big| &\le \int \chi_\varepsilon \big| |Du_n| - |Dv_n| \big| \big(|Du_n| + |Dv_n| \big) \, \mathrm{d}\mathfrak{m}_n \\
&\le \int \chi_\varepsilon |D\tilde{u}_n| \big(|Du_n| + |Dv_n| \big) \, \mathrm{d}\mathfrak{m}_n \\
&\le \Big(\int \chi_\varepsilon^2 |D\tilde{u}_n|^2 \, \mathrm{d}\mathfrak{m}_n \Big)^{1/2} \Big(\| |Du_n| \|_{L^2(\mathfrak{m}_n)} + \| |Dv_n| \|_{L^2(\mathfrak{m}_n)} \Big).
\end{aligned}$$

Observe now that from [24, Theorem 5.7] $|D\tilde{u}_n| \to |Du_\infty|$ strongly in L^2 and in particular $\int \chi_\varepsilon^2 |D\tilde{u}_n|^2 \, \mathrm{d}\mathfrak{m}_n \to \int \chi_\varepsilon^2 |Du_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty$. Moreover $\int \chi_\varepsilon^2 |Du_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty \to 0$ as $\varepsilon \to 0^+$ and u_n, v_n are uniformly bounded in $W^{1,2}(\mathrm{X}_n)$. Therefore taking in the above inequality first $n \to +\infty$ and afterwards $\varepsilon \to 0^+$ we ultimately deduce that

$$\mu(\{x_i\}) = \bar{\mu}(\{x_i\}) \ge \mu_j, \quad \forall\, j \in J.$$

In particular, since μ is non-negative, $\mu \ge \sum_{j\in J} \mu_j \delta_{x_j}$, as claimed. Finally, by the weak lower semicontinuity result in [24, Lemma 5.8], we have

$$\int \phi |Du_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty \le \varliminf_n \int \phi |Du_n|^2 \, \mathrm{d}\mathfrak{m}_n = \int \phi \, \mathrm{d}\mu$$

for every $\phi \in C_b(\mathrm{Z})$. Therefore, we get $\mu \ge |Du_\infty|^2 \mathfrak{m}_\infty$ and, by mutual singularity of the two lower bounds, we have ii) and the proof is now concluded. □

We are finally ready to prove the main result of this section.

Proof of Theorem 6.6.1. Set $\tilde{u}_n := u_n \|u_n\|^{-1}_{L^q(\mathfrak{m}_n)}$. By assumption

$$1 \ge A_n \| |D\tilde{u}_n| \|^2_{L^2(\mathfrak{m}_n)} + B_n \|\tilde{u}_n\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall n \in \mathbb{N}. \tag{6.6.9}$$

Moreover again by hypothesis $A_n \to A > 0$, $B_n \to B > 0$, therefore the sequences A_n, B_n are bounded away from zero and thus $\sup_n \|\tilde{u}_n\|_{W^{1,2}(\mathrm{X}_n)} < \infty$. Hence, up to passing to a non relabeled subsequence, Proposition 2.4.4 grants that $\tilde{u}_n$ converges L^2-strongly to a function $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$. Moreover, the measures $|D\tilde{u}_n|^2\mathfrak{m}_n$, $|\tilde{u}_n|^{2^*}\mathfrak{m}_n$ have uniformly bounded mass. In particular up to a further not relabeled subsequence, there exists $\mu, \nu \in \mathscr{M}_b^+(\mathrm{Z})$ so that $|D\tilde{u}_n|^2\mathfrak{m}_n \rightharpoonup \mu$ and $|\tilde{u}_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ in duality with $C_b(\mathrm{Z})$. We are in position to apply Lemma 6.6.5 to get the existence of at most countably many points $(x_j)_{j\in J}$ and weights $(\nu_j)_{j\in J}$, so that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j \delta_{x_j}$ and $\mu \ge |Du_\infty|^2\mathfrak{m}_\infty + \sum_{j\in J} \mu_j \delta_{x_j}$, with $A\mu_j \ge \nu_j^{2/2^*}$ and in particular $\sum_j \nu_j^{2/2^*} < \infty$. Finally from Lemma 6.4.1 we have that X_∞ supports a $(2^*, 2)$-Sobolev inequality with constants A, B.

Therefore we can perform the following estimates

$$
\begin{aligned}
1 = \lim_n \|\tilde u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} &\geq \lim_n A_n \||D\tilde u_n|\|^2_{L^2(\mathfrak{m}_n)} + B\|\tilde u_n\|^2_{L^2(\mathfrak{m}_n)} \\
&= A\mu(\mathrm{X}_\infty) + B\int |u_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty \\
&\geq A\int |Du_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty + B\int |u_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j^{2/2^*} \\
&\geq \Big(\int |u_\infty|^{2^*} \,\mathrm{d}\mathfrak{m}_\infty\Big)^{2/2^*} + \sum_{j\in J}\nu_j^{2/2^*} \\
&\geq \Big(\int |u_\infty|^{2^*} \,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} = \nu(\mathrm{X}_\infty)^{2/2^*} = 1,
\end{aligned}
$$

where in the last inequality we have used the concavity of the function $t^{2/2^*}$. In particular all the inequalities must be equalities and, since $t^{2/2^*}$ is strictly concave, we infer that every term in the sum $\int |u_\infty|^{2^*} \,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j^{2/2^*}$ must vanish except for one that must be equal to 1. If $\int |u_\infty|^{2^*} \,\mathrm{d}\mathfrak{m}_\infty = 1$ then I) must hold. If instead $\nu_j = 1$ for some $j \in J$, then $u_\infty = 0$ and by definition of ν, $|\tilde u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_j}$, which is exactly II). ☐

6.6.2 Quantitative linearization

A key point in our argument for the rigidity, and especially for the almost-rigidity, of $A_q^{\rm opt}$ will be a more 'quantitative' version of the elementary linearization of the Sobolev inequality contained in Lemma 6.5.3. To state our result, given $q \in (2,\infty)$ and $u \in W^{1,2}(\mathrm{X})$ with $\int |Du|_2^2 \,\mathrm{d}\mathfrak{m} > 0$, it is convenient to define the Sobolev ratio associated to u as the quantity

$$
\mathcal{Q}_q^{\mathrm{X}}(u) := \frac{\|u\|^2_{L^q(\mathfrak{m})} - \|u\|^2_{L^2(\mathfrak{m})}}{\||Du|_2\|^2_{L^2(\mathfrak{m})}}. \tag{6.6.10}
$$

Observe that, if $\lambda^{1,2}(\mathrm{X}) > 0$, $\int |Du|_2^2 \mathrm{d}\mathfrak{m} > 0$ as soon as u is not ($\mathfrak{m}$-a.e. equal to a) constant.

Lemma 6.6.7 (Quantitative linearization)**.** *For all numbers $A, B \geq 0$, $q > 2$ and $\lambda > 0$ there exists a constant $C = C(q, A, B, \lambda)$ such that the following holds. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathrm{X}) = 1$, $\lambda^{1,2}(\mathrm{X}) \geq \lambda$ and supporting a $(q,2)$-Sobolev inequality with constants A, B. Then, for every non-constant $f \in W^{1,2}(\mathrm{X})$ satisfying $\|f\|_{L^2(\mathrm{X})} \leq 1/2$, it holds*

$$
\Big|\mathcal{Q}_q^{\mathrm{X}}(1+f) - \frac{(q-2)\int \big(f - \int f \mathrm{d}\mathfrak{m}\big)^2 \,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}}\Big| \leq C\big(\|f\|^{3\wedge q-2}_{W^{1,2}(\mathrm{X})} + \|f\|^{q-2}_{W^{1,2}(\mathrm{X})} + \|f\|^{2q-2}_{W^{1,2}(\mathrm{X})}\big). \tag{6.6.11}
$$

Proof. We claim that it is enough to prove the statement for functions $f \in W^{1,2}(\mathrm{X})$ with zero mean (and arbitrary L^2-norm). Indeed for a generic $f \in W^{1,2}(\mathrm{X})$ satisfying $\|f\|_{L^2(\mathrm{X})} \leq 1/2$, we can take $\tilde f := \frac{f - \int f \,\mathrm{d}\mathfrak{m}}{1 + \int f \,\mathrm{d}\mathfrak{m}}$, which clearly has zero mean. Then the conclusion would follow observing that the left hand side of (6.6.11) computed at $\tilde f$ coincides with the left hand side of (6.6.11) computed at f and from the fact that

$$
\|\tilde f\|_{W^{1,2}(\mathrm{X})} \leq \|f\|_{W^{1,2}(\mathrm{X})}\Big(1 + \int f \,\mathrm{d}\mathfrak{m}\Big)^{-1} \leq \|f\|_{W^{1,2}(\mathrm{X})}(1 - \|f\|_{L^2(\mathrm{X})})^{-1} \leq 2\|f\|_{W^{1,2}(\mathrm{X})}.
$$

Therefore we can now fix $f \in W^{1,2}(\mathrm{X})$ with $\int f \,\mathrm{d}\mathfrak{m} = 0$. We start with a basic estimate of the L^r norm of f for $r \in [1, q]$. Combining the Hölder and the $(q,2)$-Sobolev inequalities we have

$$
\int |f|^r \,\mathrm{d}\mathfrak{m} \leq \Big(\int |f|^q \,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{q}} \leq (A^{r/2} + B^{r/2})\|f\|^r_{W^{1,2}(\mathrm{X})} \tag{6.6.12}
$$

In the case $r \in (2, q]$ the following refined estimate holds:

$$\begin{aligned}\frac{\int |f|^r \, d\mathfrak{m}}{\int |Df|_2^2 \, d\mathfrak{m}} &\le C_q A^{r/2}\Big(\int |Df|_2^2 \, d\mathfrak{m}\Big)^{\frac{r}{2}-1} + C_q B^{r/2}\Big(\int |f|^2 \, d\mathfrak{m}\Big)^{\frac{r}{2}-1} \frac{\int |f|^2 \, d\mathfrak{m}}{\int |Df|_2^2 \, d\mathfrak{m}} \\ &\le C_q(A^{r/2} + B^{r/2}\lambda^{-1})\|f\|^{r-2}_{W^{1,2}(\mathrm{X})}. \end{aligned} \tag{6.6.13}$$

We now apply (6.5.3) to f, which we rewrite here for the convenience of the reader:

$$\begin{aligned}\Big|\Big(\int |1+f|^q \, d\mathfrak{m}\Big)^{2/q} - \int (1+f)^2 \, d\mathfrak{m} - (q-2)\int |f|^2 \, d\mathfrak{m}\Big| \\ \le \tilde{C}_q\Big(\int |f|^{3\wedge q} + |f|^q \, d\mathfrak{m} + \Big(\int |f|^q \, d\mathfrak{m}\Big)^2 + \Big(\int |f|^2 \, d\mathfrak{m}\Big)^2\Big),\end{aligned}$$

where $\tilde{C}_q$ is a constant depending only on q. Dividing by $\int |Df|_2^2 \, d\mathfrak{m}$ the above inequality and rearranging terms, using the definition of $\lambda^{1,2}(\mathrm{X})$ and the estimates (6.6.12), (6.6.13) we obtain (6.6.11). □

6.6.3 Proof of the rigidity

Here we prove Theorem 6.2.2. This result will follow from the following theorem, which characterizes the behavior of extremal sequences for the Sobolev inequality and which combines the tools of concentration compactness and linearization, developed in the previous sections. This result can be summarized as: either there exist non-constant extremals, or we have information on the first eigenvalue $\lambda^{1,2}(\mathrm{X})$, or we have information on the density θ_N.

Theorem 6.6.8 (The Sobolev-alternative)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a compact $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (2, \infty)$ and with $\mathfrak{m}(\mathrm{X}) = 1$. Let $q \in (2, 2^*]$, with $2^* := 2N/(N-2)$. Then at least one of the following holds:*

i) *there exists a non-constant function $u \in W^{1,2}(\mathrm{X})$ satisfying*

$$\|u\|^2_{L^{2^*}(\mathfrak{m})} = A_q^{\mathrm{opt}}(\mathrm{X})\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \tag{6.6.14}$$

ii) $A_q^{\mathrm{opt}}(\mathrm{X}) = \frac{q-2}{\lambda^{1,2}(\mathrm{X})}$,

iii) $q = 2^*$ *and* $A_{2^*}^{\mathrm{opt}}(\mathrm{X}) = \alpha_2(\mathrm{X}) = \frac{\mathsf{Eucl}(N,2)^2}{\min \theta_N^{2/N}}$ *(see the introduction and (6.2.2) for the definition of $\alpha_2(\mathrm{X})$ and $\mathsf{Eucl}(N,2)$).*

Proof. By definition of $A_q^{\mathrm{opt}}(\mathrm{X})$ there exists a sequence of non-constant functions $u_n \in \mathrm{Lip}(\mathrm{X})$ such that $\mathcal{Q}_q^{\mathrm{X}}(u_n) \to A_q^{\mathrm{opt}}(\mathrm{X})$ (recall (6.6.10)). By scaling we can suppose that $\|u_n\|_{L^{2^*}(\mathfrak{m})} \equiv 1$. In particular (u_n) is bounded in $W^{1,2}(\mathrm{X})$. We distinguish two cases.
Subcritical: $q < 2^*$. By compactness (see Proposition 2.4.4), up to passing to a subsequence, $u_n \to u$ strongly in L^q to some function $u \in W^{1,2}(\mathrm{X})$ such that, from the lower semicontinuity of the Cheeger energy, $\mathcal{Q}_q^{\mathrm{X}}(u) = A_q^{\mathrm{opt}}(\mathrm{X})$. If u is non-constant i) holds and we are done, so suppose that u is constant. Then from the renormalization we must have $u \equiv 1$. Moreover, since $\|u_n\|_{L^q(\mathfrak{m})}, \|u_n\|_{L^2(\mathfrak{m})} \to 1$ and $\mathcal{Q}_q^{\mathrm{X}}(u_n) \to A_q^{\mathrm{opt}}(\mathrm{X})$, we deduce that $\||Du|\|^2_{L^2(\mathfrak{m})} \to 0$. Consider now the functions $f_n := u_n - 1 \in \mathrm{Lip}(X)$, which are non-constant and such that $f_n \to 0$ in $W^{1,2}(\mathrm{X})$. We are therefore in position to apply Lemma 6.6.7 and deduce that

$$A_q^{\mathrm{opt}}(\mathrm{X}) = \lim_{n\to\infty} \mathcal{Q}_q^{\mathrm{X}}(u_n) = \lim_n \frac{(q-2)\int \big(f_n - \int f_n d\mathfrak{m}\big)^2 \, d\mathfrak{m}}{\int |Df_n|^2 \, d\mathfrak{m}} \le \frac{q-2}{\lambda^{1,2}(\mathrm{X})}.$$

Combining this with (6.5.4), we get that $A_q^{\mathrm{opt}}(\mathrm{X}) = \frac{q-2}{\lambda^{1,2}(\mathrm{X})}$, i.e. ii) is true and we conclude the proof in this case.

Critical: $q = 2^*$. We apply the concentration-compactness result in Theorem 6.6.1 and deduce that up to a subsequence: either $u_n \to u$ in $L^{2^*}(\mathfrak{m})$ to some $u \in W^{1,2}(\mathrm{X})$ or $\|u_n\|_{L^2(\mathfrak{m})} \to 0$. In the first case we argue exactly as above using Lemma 6.6.7 and deduce that either i) or ii) holds. Hence we are left to deal with the case $\|u_n\|_{L^2(\mathfrak{m})} \to 0$. From the definition of $\alpha_2(\mathrm{X})$, for every ε there exits B_ε so that a $(2^*, 2)$-Sobolev inequality with constants $\alpha_2(\mathrm{X}) + \varepsilon$ and B_ε is valid. Hence we have

$$\mathcal{Q}^{\mathrm{X}}_{2^*}(u_n)\||Du_n|\|^2_{L^2(\mathfrak{m})} + \|u_n\|^2_{L^2(\mathfrak{m})} = \|u_n\|_{L^{2^*}(\mathfrak{m})} \le (\alpha_2(\mathrm{X}) + \varepsilon)\||Du_n|\|^2_{L^2(\mathfrak{m})} + B_\varepsilon \|u_n\|^2_{L^2(\mathfrak{m})},$$

which gives

$$\mathcal{Q}^{\mathrm{X}}_{2^*}(u_n) \le (\alpha_2(\mathrm{X}) + \varepsilon) + B_\varepsilon \|u_n\|^2_{L^2(\mathfrak{m})}(\||Du_n|\|^2_{L^2(\mathfrak{m})})^{-1}.$$

Observing that $\underline{\lim}_n \||Du_n|\|^2_{L^2(\mathfrak{m})} > 0$ (which follows from the Sobolev inequality, $\|u_n\|^2_{L^2(\mathfrak{m})} \to 0$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})} = 1$) and letting $n \to +\infty$ we arrive at $A^{\mathrm{opt}}_{2^*}(\mathrm{X}) \le (\alpha_2(\mathrm{X}) + \varepsilon)$. From the arbitrariness ε we deduce that $A^{\mathrm{opt}}_{2^*}(\mathrm{X}) \le \alpha_2(\mathrm{X})$ and the proof is concluded (indeed by definition $\alpha_2(\mathrm{X}) \ge A^{\mathrm{opt}}_{2^*}(\mathrm{X})$ is always true). □

We can finally come to the proof of the principal result of this Chapter.

Proof of Theorem 6.2.2. The 'if' implication is direct as any N-spherical suspension, X is so that $A^{\mathrm{opt}}_q(\mathrm{X}) = \frac{q-2}{N}$. This can be seen from the lower bound in Proposition 6.5.2 (recall also Theorem 2.3.9) and the upper bound given in Theorem 6.2.1.

For the 'only if' implication, the result will follow from three different rigidity results, one for each of the alternatives in Theorem 6.6.8. Up to scaling the reference measures, we can suppose $\mathfrak{m}(\mathrm{X}) = 1$.

Case 1: i) in Theorem 6.6.8 holds. Let u be the non-constant function satisfying (6.6.14). Observe that we can assume that u is non-negative. We aim to apply the Polya-Szego inequality with the model space I_N as in Section 2.2.2. Let $u^*_N : I_N \to [0, \infty]$ be the monotone-rearrangement of u. From the Polya-Szego inequality in Theorem 2.2.8 we have that $u^*_N \in W^{1,2}(I_N, |.|, \mathfrak{m}_N)$, $\|u\|_{L^p(\mathfrak{m})} = \|u^*_N\|_{L^p(\mathfrak{m}_N)}$ for both $p \in \{q, 2\}$ and that $\||Du^*_N|\|_{L^2(\mathfrak{m}_N)} \le \||Du|\|_{L^2(\mathfrak{m})}$. Combining this with (2.2.13) we have

$$\begin{aligned}\|u\|^2_{L^q(\mathfrak{m})} = \|u^*_N\|^2_{L^q(\mathfrak{m}_N)} &\le \tfrac{q-2}{N}\||Du^*_N|\|^2_{L^2(\mathfrak{m}_N)} + \|u^*_N\|^2_{L^2(\mathfrak{m}_N)}\\ &\le \tfrac{q-2}{N}\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})} = \|u\|^2_{L^q(\mathfrak{m})}.\end{aligned}$$

Therefore $\||Du^*_N|\|_{L^2(\mathfrak{m}_N)} = \||Du|\|_{L^2(\mathfrak{m})}$ and, since u is non-constant, we are in position to apply the rigidity of the Polya-Szego inequality of Theorem 2.3.11 and conclude the proof in this case.

Case 2: ii) in Theorem 6.6.8 holds. We immediately deduce that $\lambda^{1,2}(\mathrm{X}) = N$ and the conclusion follows from the Obata's rigidity (Theorem 2.3.9).

Case 3: iii) in Theorem 6.6.8 holds. From Theorem 6.3.12 and the explicit expression for $\mathsf{Eucl}(N, 2)$ (see (6.2.3)) we have that

$$\frac{2^* - 2}{N} = A^{\mathrm{opt}}_{2^*}(\mathrm{X}) = \alpha_2(\mathrm{X}) = \frac{\mathsf{Eucl}(N,2)^2}{\min_{x\in\mathrm{X}} \theta_N(x)^{2/N}} = \frac{2^* - 2}{N\sigma_N^{2/N} \min_{x\in\mathrm{X}} \theta_N(x)^{2/N}},$$

therefore $\min_{x\in\mathrm{X}} \theta_N = \sigma_N^{-1}$. On the other hand by the Bishop-Gromov inequality and identity (2.2.7)

$$\frac{1}{\sigma_N} = \inf_{\mathrm{X}} \theta_N(x) \ge \frac{\mathfrak{m}(\mathrm{X})}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))} = \frac{1}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))},$$

which, from the definition of $v_{N-1,N}$ and (6.2.4) forces $\mathrm{diam}(\mathrm{X}) = \pi$. The conclusion then follows by the rigidity of the maximal diameter (Theorem 2.3.10). □

Remark 6.6.9. The rigidity result for $A_q^{\rm opt}(M)$ in the subcritical range $q < 2^*$ was already observed in [138] as a consequence of the following sharper estimate due to [90]: for any n-dimensional Riemannian manifolds M, $n \geq 3$, with $\mathsf{Ric} \geq n-1$ it holds

$$A_q^{\rm opt}(M) \leq \frac{(q-2)}{\kappa(\theta)}, \qquad \forall q \in (2, 2^*), \tag{6.6.15}$$

where $\kappa(\theta) := \theta n + (1-\theta)\lambda^{1,2}(M)$, $\lambda^{1,2}(M)$ being the first non trivial eigenvalue and $\theta = \theta(q) \in [0,1]$ is a suitable interpolation parameter. The spectral gap inequality $\lambda^{1,2}(M) \geq n$ grants that the bound (6.6.15) improves the one of (6.1.5). For every $q \in (2, 2^*)$, the condition $A_q^{\rm opt}(M) = A_q^{\rm opt}(\mathbb{S}^n)(= (q-2)/n)$ forces $\kappa(\theta) = n$ which in turn implies $\lambda^{1,2}(M) = n$. By appealing to the classical Obata's Theorem, this argument covers the rigidity of Theorem 6.1.3 for $q < 2^*$. Nevertheless, this does not extend to the critical exponent: more precisely $\theta(q) \to 1$ as $q \to 2^*$, hence the quantity $\kappa(\theta)$ carries no information on the spectral gap in this case. ■

6.7 Almost rigidity of $A^{\rm opt}$

6.7.1 Behavior at concentration points

The following technical result will be needed for the almost-rigidity result and has the role of replacing in the varying-space case, the Sobolev inequality with constants $\alpha_2(\mathrm{X}) + \varepsilon, B_\varepsilon$ which we used in the fixed-space case of the rigidity (see the proof of Theorem 6.6.8). Indeed it is not clear how to control the constant B_ε in a sequence of mGH-converging spaces. Therefore we need a more precise local analysis which fully exploits the local Sobolev inequalities in Theorem 6.3.8 and Proposition 6.3.11.

Lemma 6.7.1 (Behavior at concentration points)**.** *Let $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, $n \in \bar{\mathbb{N}}$, be a sequence of $\mathsf{RCD}(K, N)$ spaces $K \in \mathbb{R}$, $N \in (1, \infty)$, so that $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$. Fix $p \in (1, N)$, set $p^* := pN/(N-p)$ and assume that $u_n \in \mathrm{Lip}_c(\mathrm{X}_n)$ is a sequence satisfying*

$$\|u_n\|^p_{L^{p^*}(\mathfrak{m}_n)} \geq A_n \||Du_n|\|^p_{L^p(\mathfrak{m}_n)} - B_n \|u_n\|^p_{L^s(\mathfrak{m}_n)}, \tag{6.7.1}$$

for some constants $A_n, B_n \geq 0$ uniformly bounded and $s > 0$ so that $s \in [p, p^*)$. Assume furthermore that $u_n \to 0$ strongly in L^p, $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and that $|u_n|^{p^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ for some $y_0 \in \mathrm{X}_\infty$ in duality with $C_{bs}(\mathrm{Z})$ (where $(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$ is a proper space realizing the convergence in the extrinsic approach). Then

$$\theta_N(y_0) \leq \mathsf{Eucl}(N,p)^N (\overline{\lim_n} A_n)^{-N/p}, \tag{6.7.2}$$

meaning that if $\theta_N(y_0) = +\infty$, then $\overline{\lim}_n A_n = 0$.

Proof. We subdivide the proof in two cases.

Case 1: $\theta_N(y_0) < +\infty$.

Fix $\varepsilon < \theta_N(y_0)/4$ arbitrary. Since $\theta_{N,r}(y_0) \to \theta_N(y_0)$ as $r \to 0^+$ there exists $\bar{r} = \bar{r}(\varepsilon)$ such that

$$|\theta_{N,r}(y_0) - \theta_N(y_0)| \leq \varepsilon, \quad \forall r < \bar{r}. \tag{6.7.3}$$

Let $\delta := \delta(2\varepsilon, D, N)$, with $D = 4$, be the constant given by Theorem 6.3.8 and fix two radii $r, R \in (0, \bar{r})$ such that $R < \delta\sqrt{N/K^-}$ and $r < \delta R$. Consider now a sequence $y_n \in \mathrm{X}_n$ such that $y_n \to y_0$. From the convergence of the measures $\mathfrak{m}_n$ to $\mathfrak{m}_\infty$ we have that $\theta_{N,r}(y_n) \to \theta_{N,r}(y_0)$ and $\theta_{N,R}(y_n) \to \theta_{N,R}(y_0)$. In particular by (6.7.3) there exists $\bar{n} = \bar{n}(r, R, \varepsilon)$ such that

$$|\theta_{N,R}(y_n) - \theta_N(y_0)|, |\theta_{N,r}(y_n) - \theta_N(y_0)| \leq 2\varepsilon, \qquad \forall n \geq \bar{n}. \tag{6.7.4}$$

From the initial choice of ε this also implies that $\theta_{N,r}(y_n)/\theta_{N,R}(y_n) \leq 4$ for every $n \geq \bar{n}$. We are in position to apply Theorem 6.3.8 and get that for every $n \geq \bar{n}$

$$\|f\|_{L^{p^*}(\mathfrak{m}_n)} \leq \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}} \||Df|\|_{L^p(\mathfrak{m}_n)}, \qquad \forall f \in \mathrm{Lip}_c(B_r(y_n)). \tag{6.7.5}$$

Choose $\phi \in \mathrm{Lip}(\mathrm{Z})$ such that $\phi = 1$ in $B^{\mathrm{Z}}_{r/8}(y_0)$, $\mathrm{supp}(\phi) \subset B^{\mathrm{Z}}_{r/4}(y_0)$ and $0 \le \phi \le 1$. From the assumptions, we have that $\int \phi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \to 1$, in particular up to increasing $\bar{n}$ it holds that $\int \phi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \ge 1 - \varepsilon$ for all $n \ge \bar{n}$. Moreover, again up to increasing $\bar{n}$, we have that $\mathsf{d}_{\mathrm{Z}}(y_n, y_0) \le r/4$ for all $n \ge \bar{n}$, therefore

$$1 - \varepsilon \le \int_{B_{r/2}(y_n)} |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n, \qquad \forall n \ge \bar{n}. \tag{6.7.6}$$

For every n we choose a cut-off function $\phi_n \in \mathrm{Lip}(\mathrm{X}_n)$ such that $\phi_n = 1$ in $B_{r/2}(y_n)$, $0 \le \phi_n \le 1$, $\mathrm{supp}(\phi_n) \subset \mathrm{Lip}_c(B_r(y_n))$ and $\mathrm{Lip}(\phi_n) \le 2/r$. Plugging the function $u_n \phi_n \in \mathrm{Lip}_c(B_r(y_n))$ in (6.7.5) and using (6.7.6) we obtain

$$(1-\varepsilon)^{\frac{1}{p^*}} \le \|u_n \phi_n\|_{L^{p^*}(\mathfrak{m}_n)} \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}} \big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r} \|u_n\|_{L^p(\mathfrak{m}_n)} \big). \tag{6.7.7}$$

Moreover recalling that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and the assumption (6.7.1), from (6.7.7) we reach

$$(1-\varepsilon)^{\frac{1}{p^*}} \big(A_n^{1/p} \||Du_n|\|_{L^p(\mathfrak{m}_n)} - B_n \|u_n\|^p_{L^s(\mathfrak{m}_n)} \big) \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}} \big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r} \|u_n\|_{L^p(\mathfrak{m}_n)} \big).$$

We also observe that from the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ and the fact that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$, we have by $(viii)$ in Proposition 2.4.3 that $\|u_n\|_{L^s(\mathfrak{m}_n)} \to 0$. Finally by (6.7.7) and the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ it holds that $\underline{\lim}_n \||Du_n|\|_{L^p(\mathfrak{m}_n)} > 0$. In particular for n big enough we can divide by $\||Du_n|\|_{L^p(\mathfrak{m}_n)}$ the above inequality and letting $n \to +\infty$ we get

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(1-\varepsilon)^{1/p^*}(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}}.$$

From the arbitrariness of ε, the conclusions follows.

Case 2: $\theta_N(y_0) = \infty$.

The argument is similar to Case 1, but we will use Proposition 6.3.11 instead of Theorem 6.3.8. Let $M > 0$ be arbitrary. There exists $r \le 1$ such that $\theta_{N,r}(y_0) \ge 2M$. As above we choose a sequence $y_n \to y_0$. For n big enough we have that

$$\theta_{N,r}(y_n) \ge M. \tag{6.7.8}$$

Applying Proposition 6.3.11, from (6.7.8) we get that for every n big enough

$$\|f\|^p_{L^{p^*}(B_r(y_n))} \le \frac{C_{K,N,p}}{M^{\frac{p}{N}}} \||Df|\|^p_{L^p(B_r(y_n))} + \frac{C_{p,N} \|f\|^p_{L^p(B_r(y_n))}}{r^{p/N} M^{\frac{p}{N}}}, \qquad \forall f \in \mathrm{Lip}(\mathrm{X}_n). \tag{6.7.9}$$

Observing that (6.7.6) is still satisfied with $\varepsilon = 1/M$ and n big enough, we can repeat the above argument, using (6.7.1) and plugging $\phi_n u_n$ in (6.7.9), where ϕ_n is as above. This leads us to

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{C_{K,N,p}}{(1-1/M)^{1/p^*} M^{\frac{1}{N}}},$$

which from the arbitrariness M implies the conclusion. $\square$

6.7.2 Continuity of A^{opt} under mGH-convergence

In Lemma 6.4.1, we proved that Sobolev embedding are stable with respect to pmGH-convergence. A much more involved task it to prove that *optimal* constants are also continuous: indeed, if $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$, in general Lemma 6.4.1 ensures only that $A^{\mathrm{opt}}_q(\mathrm{X}_\infty) \le \underline{\lim}_n A^{\mathrm{opt}}_q(\mathrm{X}_n)$. With the concentration compactness tools developed in Section 6.6.1, the 'quantitative-linearization' result in Lemma 6.6.7 and the technical tool developed in the previous section we can now prove the mGH-continuity of $A^{\mathrm{opt}}_q(\mathrm{X}_n)$ as stated in Theorem 6.2.5, that we restate here for convenience of the reader.

Theorem 6.7.2 (Continuity of $A_q^{\rm opt}$ under mGH-convergence)**.** *Let* $({\rm X}_n,{\sf d}_n,\mathfrak m_n)$ *be a sequence,* $n\in\mathbb N\cup\{\infty\}$*, of compact* ${\sf RCD}(K,N)$*-spaces with* $\mathfrak m_n({\rm X}_n)=1$ *and for some* $K\in\mathbb R$*,* $N\in(2,\infty)$ *so that* ${\rm X}_n\overset{mGH}{\to}{\rm X}_\infty$*. Then,* $A_q^{\rm opt}({\rm X}_\infty)=\lim_n A_q^{\rm opt}({\rm X}_n)$*, for every* $q\in(2,2^*]$*.*

Proof. By definition of $A_q^{\rm opt}({\rm X}_n)$, there exists sequence of non-negative and non-constant functions $u_n\in{\rm Lip}({\rm X}_n)$ satisfying

$$\|u_n\|^2_{L^q(\mathfrak m_n)}\ge A_n\||Du_n|\|^2_{L^2(\mathfrak m_n)}+\|u_n\|^2_{L^2(\mathfrak m_n)}, \tag{6.7.10}$$

having set $A_n:=A_q^{\rm opt}({\rm X}_n)-\frac1n$. By scaling invariance, it is not restrictive to suppose $\|u_n\|_{L^q(\mathfrak m_n)}=1$ for every $n\in\mathbb N$. Observe that thanks to Lemma 6.4.1 we already have that $0<A_q^{\rm opt}({\rm X}_\infty)\le\varliminf_n A_q^{\rm opt}({\rm X}_n)$, hence we only need to show that $A_q^{\rm opt}({\rm X})\ge\varlimsup_n A_q^{\rm opt}({\rm X}_n)$. To this aim, we distinguish two cases.

Subcritical: $q<2^*$. It is clear that A_n is uniformly bounded from below whence the sequence u_n has uniformly bounded $W^{1,2}$ norms. Then, by Proposition 2.4.4 and the Γ-$\varliminf$ inequality of the ${\sf Ch}_2$ energy, there exists a (not relabeled) subsequence L^2-strongly converging to some $u_\infty\in W^{1,2}({\rm X}_\infty)$. Moreover, since u_n are bounded in L^{2^*}, they also converge to u_∞ in L^q-strong and in particular $\|u_\infty\|^2_{L^q(\mathfrak m_\infty)}=1$. Suppose first that the function u_∞ is not constant, then we get

$$\begin{aligned} 1=\|u_\infty\|^2_{L^q(\mathfrak m_\infty)}&\ge\varlimsup_{n\to\infty}A_n\||Du_n|\|^2_{L^2(\mathfrak m_n)}+\|u_n\|^2_{L^2(\mathfrak m_n)}\\ \text{(2.4.1)}+\ L^2\text{-strong}\qquad &\ge\varlimsup_{n\to\infty}A_q^{\rm opt}({\rm X}_n)\||Du_\infty|\|^2_{L^2(\mathfrak m_\infty)}+\|u_\infty\|^2_{L^2(\mathfrak m_\infty)}.\end{aligned}$$

Since u_∞ is not constant this in turn yields $\varlimsup_n A_q^{\rm opt}({\rm X}_n)\le A_q^{\rm opt}({\rm X}_\infty)$ which is what we wanted.

Suppose now that u_∞ is constant. Then, necessarily $u_\infty=1$. Define now $f_n:=1-u_n$ and observe that $\|f_n\|_{W^{1,2}({\rm X}_n)}\to0$, which follows from (6.7.10) and the fact that $\|u_n\|_{L^2(\mathfrak m_n)}\to1$. Moreover from (2.4.3) we have that $\lambda^{1,2}({\rm X}_n)$ are uniformly bounded below away from zero. Therefore we can apply Lemma 6.6.7 to deduce (recall (6.6.10) for the def. of $\mathcal Q_q^{\rm X}$)

$$\varlimsup_{n\to\infty}A_q^{\rm opt}({\rm X}_n)=\varlimsup_{n\to\infty}\mathcal Q_q^{{\rm X}_n}(u_n)=\varlimsup_{n\to\infty}\frac{(q-2)\int\left|f_n-\int f_n\,{\rm d}\mathfrak m_n\right|^2{\rm d}\mathfrak m_n}{\int|Df_n|^2\,{\rm d}\mathfrak m_n}\le\varlimsup_{n\to\infty}\frac{(q-2)}{\lambda^{1,2}({\rm X}_n)}=\frac{(q-2)}{\lambda^{1,2}({\rm X}_\infty)},$$

having used, in the last inequality, the continuity of the 2-spectral gap (2.4.3). This combined with (6.5.4) gives that $\varlimsup_n A_q^{\rm opt}({\rm X}_n)\le A_q^{\rm opt}({\rm X}_\infty)$.

Critical exponent: $q=2^*$. Observe that we are now in position to invoke Theorem 6.6.1 and, up to a further not relabeled subsequence, we just need to handle one of the two different situations I),II) occurring in Theorem 6.6.1. If the case I) occurs, we argue exactly as in the Subcritical: $q<2^*$ case, to conclude that $\varlimsup_n A_q^{\rm opt}({\rm X}_n)\le A_q^{\rm opt}({\rm X}_\infty)$. Hence we are left with situation II), where the sequence u_n develops a concentration point $y_0\in{\rm X}_\infty$. Recalling Lemma 6.7.1, either $\theta_N(y_0)=\infty$ and $\varlimsup_n A_{2^*}^{\rm opt}({\rm X}_n)=0$ or $\theta_N(y_0)<\infty$. The first situation cannot happen, since $A_{2^*}^{\rm opt}({\rm X}_\infty)>0$. In the second one rearranging in (6.7.2) we have

$$\varlimsup_{n\to\infty}A_{2^*}^{\rm opt}({\rm X}_n)\overset{(6.7.2)}{\le}\frac{{\sf Eucl}(N,2)^2}{\theta_N(y_0)^{2/N}}\overset{(6.2.8)}{\le}\alpha_2({\rm X}_\infty)\le A_{2^*}^{\rm opt}({\rm X}_\infty).$$

□

6.7.3 Proof of the almost-rigidity

Combining the rigidity result for $A_q^{\rm opt}$ with the continuity result proved in the previous part we can now prove the almost-rigidity result for $A_q^{\rm opt}$.

Proof of Theorem 6.2.3. We argue by contradiction, and suppose that there exists $\varepsilon>0$, $q\in(2,2^*]$ and a sequence $({\rm X}_n,{\sf d}_n,\mathfrak m_n)$ of ${\sf RCD}(N-1,N)$-spaces with $\mathfrak m_n({\rm X}_n)=1$ so that

$${\sf d}_{mGH}(({\rm X}_n,{\sf d}_n,\mathfrak m_n),({\rm Y},{\sf d}_{\rm Y},\mathfrak m_{\rm Y}))>\varepsilon, \tag{6.7.11}$$

for every spherical suspension $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ and $\lim_n A_q^{\mathrm{opt}}(\mathrm{X}_n) = \frac{q-2}{N}$. Theorem 2.4.1 (recall that $\mathfrak{m}_n(\mathrm{X}_n) = 1$) ensures that up to passing to a non-relabeled subsequence we have $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$, for some $\mathsf{RCD}(N-1, N)$-space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ with $\mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$. Hence (6.7.11) implies

$$\mathsf{d}_{mGH}((\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty), (\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})) \geq \varepsilon, \tag{6.7.12}$$

for every spherical suspension $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$. Finally, by Theorem 6.2.5 we deduce

$$A_q^{\mathrm{opt}}(\mathrm{X}_\infty) = \lim_n A_q^{\mathrm{opt}}(\mathrm{X}_n) = \frac{q-2}{N}.$$

Therefore, by invoking the rigidity Theorem 6.2.2, we get that $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is isomorphic to a spherical suspension. This contradicts (6.7.12) and concludes the proof. □

Remark 6.7.3. The results of Theorem 6.2.3 (and therefore of Theorem 6.2.2) extend directly to the class of $\mathsf{RCD}(K, N)$ spaces for some $K > 0$ and $N \geq 2$ with normalized volume. Consider an $\mathsf{RCD}(K, N)$ space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and define $(\mathrm{X}', \mathsf{d}', \mathfrak{m}') := (\mathrm{X}, \sqrt{\frac{K}{N-1}}\mathsf{d}, \mathfrak{m})$ which is $\mathsf{RCD}(N-1, N)$. Then, since $A_q^{\mathrm{opt}}(\mathrm{X}') = \frac{K}{N-1} A_q^{\mathrm{opt}}(\mathrm{X})$, it is straightforward to set $\delta = \delta(K, N, \varepsilon, q) := \frac{N-1}{K}\delta(N, \varepsilon, q)$ and extend the aforementioned results also for arbitrary $K > 0$. ■

6.8 Application: the Yamabe equation on RCD spaces

In this section we apply Theorem 6.2.6 and the concentration compactness results of Section 6.6.1 to study the Yamabe equation to the $\mathsf{RCD}(K, N)$ setting. In particular we prove an existence result for the Yamabe equation and continuity of the generalized Yamabe constants under mGH-convergence, extending and improving some of the results proved in [121] in the case of Ricci limits. For results concerning the Yamabe problem and the Yamabe constant in non-smooth spaces see also [3, 2, 4, 157].

We recall that the Yamabe problem [192] asks if a compact Riemannian manifolds admits a conformal metric with constant scalar curvature. This has been completely solved and shown to be true after the works of Trudinger, Aubin and Schoen [183, 30, 177]. We also refer to [139] for an introduction to this problem and a complete and self-contained proof of this result.

The Yamabe problem turns out to be linked to the so-called Yamabe equation:

$$-\Delta u + \mathrm{S}\, u = \lambda u^{2^*-1}, \qquad \lambda \in \mathbb{R},\, \mathrm{S} \in L^\infty(M), \tag{6.8.1}$$

where $2^* = \frac{2n}{n-2}$. Indeed solving the Yamabe problem is equivalent to find a non-negative and non-zero solution to (6.8.1) for some $\lambda \in \mathbb{R}$ and with $\mathrm{S} = \mathrm{Scal}$, the scalar curvature of M. In this direction, it is relevant to see that the Yamabe equation is the Euler-Lagrange equation of the following functional:

$$Q(u) := \frac{\int |Du|^2 + \mathrm{S}|u|^2 \,\mathrm{dVol}}{\|u\|^2_{L^{2^*}}}, \qquad u \in W^{1,2}(M) \setminus \{0\},$$

where Vol is the volume measure of M. One then defines the Yamabe constant as the infimum of the above functional:

$$\lambda_{\mathrm{S}}(M) := \inf_{u \in W^{1,2}(M)\setminus\{0\}} Q(u).$$

A crucial step in the solution of the Yamabe problem is:

Theorem 6.8.1 ([183, 30, 192])**.** *Let M be a compact n-dimensional Riemannian manifold satisfying $\lambda_{\mathrm{S}}(M) < \mathsf{Eucl}(n,2)^{-2}$. Then there is a non-zero solution to (6.8.1) with $\lambda = \lambda_{\mathrm{S}}(M)$.*

Recall that $\mathsf{Eucl}(n, 2)$ denotes the optimal constant in the sharp Euclidean Sobolev inequality (6.1.1). It has also been proven by Aubin [31] (see also [139]) that

$$\lambda_{\mathrm{S}}(M) \leq \mathsf{Eucl}(n,2)^{-2} \tag{6.8.2}$$

always holds.

The relevant point for our discussion is that Theorem 6.8.1 turns out to be linked to the notion of optimal Sobolev constant $\alpha_2(M)$, in particular it is actually a corollary of the fact that $\alpha_2(M) = \mathsf{Eucl}(n,2)^2$ (recall (6.1.2)). Since we generalized this last result to setting of compact $\mathsf{RCD}(K,N)$-spaces (see Theorem 6.2.6), it is natural to ask if an analogue of Theorem 6.8.1 holds also in this singular framework. We will positively address this in this part of the note.

6.8.1 Existence of solutions to the Yamabe equation

We start by clarifying in which sense (6.8.1) is intended and, to this aim, we fix $(\mathrm{X},\mathsf{d},\mathfrak{m})$ a compact $\mathsf{RCD}(K,N)$ space for some $K\in\mathbb{R}, N\in(2,\infty)$ with $\mathfrak{m}(\mathrm{X})=1$. We will also denote by 2^* the Sobolev-exponent defined as $2^* := 2N/(N-2)$. In the sequel, we fix

$$\mathrm{S} = g\mathfrak{m} \quad \text{for some } g\in L^p(\mathfrak{m}),\ p>N/2, \tag{6.8.3}$$

The reason for this more general choice of S is the fact that on $\mathsf{RCD}(K,N)$ spaces a 'scalar curvature' that is bounded is not natural (recall that to solve the Yamabe problem one would like to take $\mathrm{S}=\mathrm{Scal}\in L^\infty$). Indeed, requiring only a synthetic lower bound on the Ricci curvature, it is more desirable to work only with integrability requirements.

The goal is then to discuss positive solutions $u\in D(\mathbf{\Delta})\cap L^2(|\mathrm{S}|)$ of

$$-\mathbf{\Delta} u = \lambda u^{2^*-1}\mathfrak{m} - u\mathrm{S}, \qquad \lambda\in\mathbb{R}. \tag{6.8.4}$$

Observe that if $u\in D(\mathbf{\Delta})\subset W^{1,2}(\mathrm{X})$, by the Sobolev embedding we have that $u\in L^{2^*}(\mathfrak{m})$ and thus, the right hand side of (6.8.4) is a well defined Radon measure on X. A solution for this equation will be deduced with a variational approach as described above. More precisely we define the functional $Q_\mathrm{S}\colon W^{1,2}(\mathrm{X})\setminus\{0\}\to\mathbb{R}$ defined as

$$u\mapsto Q_\mathrm{S}(u) := \frac{\int |Du|^2\,\mathrm{d}\mathfrak{m} + \int |u|^2\,\mathrm{dS}}{\|u\|^2_{L^{2^*}(\mathfrak{m})}}.$$

Observe that since $\mathrm{S}=g\mathfrak{m}$, with $g\in L^p(\mathfrak{m})$, $p>N/2$, the integral $\int |u|^2\,\mathrm{dS}$ exists, i.e. its value is well defined. We then define

$$\begin{aligned}\lambda_\mathrm{S}(\mathrm{X}) &:= \inf\{Q_\mathrm{S}(u)\ :\ u\in W^{1,2}(\mathrm{X})\setminus\{0\}\}\\ &= \inf\{Q_\mathrm{S}(u)\ :\ u\in W^{1,2}(\mathrm{X}),\ \|u\|_{L^{2^*}(\mathfrak{m})}=1\},\end{aligned} \tag{6.8.5}$$

and claim that

$$\lambda_\mathrm{S}(\mathrm{X})\in(-\infty,+\infty). \tag{6.8.6}$$

Indeed, $\lambda_\mathrm{S}(\mathrm{X})<+\infty$ as can be seen considering constant functions. On the other hand for every $u\in W^{1,2}(\mathrm{X})$ with $\|u\|_{L^{2^*}(\mathfrak{m})}=1$, Hölder inequality yields

$$Q_\mathrm{S}(u)\geq -\|g\|_{L^p(\mathfrak{m})}\|u\|_{L^{2^*}(\mathfrak{m})} = -\|g\|_{L^p(\mathfrak{m})}.$$

The ultimate goal of this section is to prove the following:

Theorem 6.8.2. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact $\mathsf{RCD}(K,N)$ space for some $K\in\mathbb{R}$, $N\in(2,\infty)$ with $\mathfrak{m}(\mathrm{X})=1$ and let S as in (6.8.3). If*

$$\lambda_\mathrm{S}(\mathrm{X}) < \frac{\min_\mathrm{X}\theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2}, \tag{6.8.7}$$

then there exists a non-negative and non-zero $u\in D(\mathbf{\Delta})\cap L^2(|\mathrm{S}|)$ which is a minimum for (6.8.5) and satisfies (6.8.4).

We start by showing that (6.8.4) is the Euler-Lagrange equation for the minimization problem (6.8.5).

Proposition 6.8.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{RCD}(K,N)$*-space for some* $K\in\mathbb{R}, N\in(2,\infty)$ *with* $\mathfrak{m}(\mathrm{X})=1$ *and let* S *be as in* (6.8.3). *Suppose* $u\in W^{1,2}(\mathrm{X})\cap L^2(|\mathrm{S}|)$ *is a minimizer for* (6.8.5) *satisfying* $\|u\|_{L^{2^*}(\mathfrak{m})}=1$. *Then*

$$\int\langle\nabla u,\nabla v\rangle\,\mathrm{d}\mathfrak{m}=-\int uv\,\mathrm{dS}+\lambda_{\mathrm{S}}(\mathrm{X})\int u^{2^*-1}v\,\mathrm{d}\mathfrak{m},\qquad\forall v\in\mathrm{Lip}(\mathrm{X}).\tag{6.8.8}$$

Proof. We consider for every $\varepsilon\in(-1,1)$ and $v\in\mathrm{Lip}(\mathrm{X})$, the function $u^\varepsilon:=\|u+\varepsilon v\|^{-1}_{L^{2^*}(\mathfrak{m})}(u+\varepsilon v)$, whenever $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ is not zero. It can be seen that for a fixed v then u^ε is well defined at least for ε close to zero. Indeed, the fact that $\int|u|^{2^*},\mathrm{d}\mathfrak{m}=1$ grants that $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}\to1$ as $\varepsilon\to0$ (see below) and in particular $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ does not vanish for $|\varepsilon|$ small enough. By minimality we have (recall also (1.4.2))

$$0\le\lim_{\varepsilon\downarrow0}\frac{Q_{\mathrm{S}}(u^\varepsilon)-Q_{\mathrm{S}}(u)}{\varepsilon}=\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\left(\frac{1}{I_\varepsilon^2}-1\right)\lambda_{\mathrm{S}}(\mathrm{X})+\frac{2}{I_\varepsilon^2}\int\langle\nabla u,\nabla v\rangle\,\mathrm{d}\mathfrak{m}+\int uv\,\mathrm{dS},$$

where $I_\varepsilon:=\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$. Furthermore, from the elementary estimate $||a+\varepsilon b|^q-|a|^q|\le q|\varepsilon b|\big||a+\varepsilon b|^{q-1}+|a|^{q-1}\big|$, with $q=2^*$, and the fact that $u,v\in L^{2^*}(\mathfrak{m})$, we have that $\int|u+\varepsilon v|^q\,\mathfrak{m}\to1$ as $\varepsilon\to0$. Thanks to the same estimates, the dominated convergence theorem grants that

$$\lim_{\varepsilon\downarrow0}\frac{1-I_\varepsilon^2}{\varepsilon}=\frac{2}{2^*}\lim_{\varepsilon\downarrow0}\int\frac{|u|^{2^*}-|u+\varepsilon v|^{2^*}}{\varepsilon}\,\mathrm{d}\mathfrak{m}=-2\int u^{2^*-1}v\,\mathrm{d}\mathfrak{m}.$$

Arguing analogously considering $\varepsilon\uparrow0$ gives (6.8.8). □

We can now prove Theorem 6.8.2 which, thanks to the previous proposition, amounts to the existence of a minimizer for (6.8.5). We will do so using the concentration-compactness tools developed in Section 6.6.1, here employed with a fixed space X.

Proof of Theorem 6.8.2. Let $u_n\in W^{1,2}(\mathrm{X})$ be such that $Q_{\mathrm{S}}(u_n)\to\lambda_{\mathrm{S}}(\mathrm{X})$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})}=1$. We claim that u_n are uniformly bounded in $W^{1,2}(\mathrm{X})$. Indeed, this can be seen from the estimate

$$\int|Du_n|^2+|u_n|^2\,\mathrm{d}\mathfrak{m}\le\int|Du_n|^2\,\mathrm{d}\mathfrak{m}+\int|u_n|^2\,\mathrm{dS}+(1+\|g\|_{L^p(\mathfrak{m})})\|u_n\|_{L^{2^*}(\mathfrak{m})}=1+Q_{\mathrm{S}}(u_n)+\|g\|_{L^p(\mathfrak{m})},$$

obtained combining the Hölder inequality with (6.8.3). Hence, by compactness (see Proposition 2.4.4), up to a not relabeled subsequence, we have $u_n\to u$ in $L^2(\mathfrak{m})$ for some $u\in W^{1,2}(\mathrm{X})$. We claim that $u\in L^2(|\mathrm{S}|)$ and

$$\int u^2\,\mathrm{dS}\le\varliminf_n\int u_n^2\,\mathrm{dS}.\tag{6.8.9}$$

Indeed, an application of Hölder inequality (recall that, by Sobolev embedding, $u^2\in L^{p'}(\mathfrak{m})$ with p' being the conjugate exponent of p) reveals that $u^2\in L^2(|sca|)$ and actually continuity occurs in (6.8.9). We now distinguish two cases:

Case 1. $\lambda_{\mathrm{S}}(\mathrm{X})<0$. By lower semicontinuity of the Cheeger-energy and (6.8.9) we have

$$0>\lambda_{\mathrm{S}}(\mathrm{X})=\lim_n Q_{\mathrm{S}}(u_n)\ge\int|Du|^2\,\mathrm{d}\mathfrak{m}+\int u^2\mathrm{dS}.$$

In particular u is not identically zero and by the lower semicontinuity of the $L^{2^*}(\mathfrak{m})$-norm we have $0<\|u\|_{L^{2^*}(\mathfrak{m})}\le1$. Moreover, from the above we have that $\int|Du|^2\,\mathrm{d}\mathfrak{m}+\int u^2\mathrm{dS}$ is negative, hence

$$\lambda_{\mathrm{S}}(\mathrm{X})\ge\|u\|^{-2}_{L^{2^*}(\mathfrak{m})}\left(\int|Du|^2\,\mathrm{d}\mathfrak{m}+\int u^2\mathrm{dS}\right)=Q_{\mathrm{S}}(\|u\|^{-1}_{L^{2^*}(\mathfrak{m})}u).$$

Therefore $\|u\|^{-1}_{L^{2^*}(\mathfrak{m})}u$ is a minimizer for $Q_{\mathrm{S}}(u)$.

Case 2. $\lambda_{\mathrm{S}}(\mathrm{X}) \geq 0$. Recall that the sequence (u_n) is uniformly bounded both in $L^{2^*}(\mathfrak{m})$ and in $W^{1,2}(\mathrm{X})$. Therefore since X is compact, again up to a subsequence, $|Du_n|^2\mathfrak{m} \rightharpoonup \mu$ and $|u_n|^{2^*} \rightharpoonup \nu$ for some $\mu \in \mathscr{M}_b^+(\mathrm{X})$ and $\nu \in \mathscr{P}(\mathrm{X})$ in duality with $C(\mathrm{X})$. By assumption there exists $\varepsilon > 0$ such that $\lambda_{\mathrm{S}}(\mathrm{X}) < \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2+\varepsilon} =: \lambda_\epsilon$. We fix one of such $\varepsilon > 0$ and define $A_\varepsilon = \lambda_\varepsilon^{-1}$. From Theorem 6.2.6 there exists a constant $B_\varepsilon > 0$ so that

$$\|u\|^2_{L^{2^*}(\mathfrak{m})} \leq A_\varepsilon \||Du|\|^2_{L^2(\mathfrak{m})} + B_\varepsilon \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}(\mathrm{X}).$$

Hence we are in position to apply Lemma 6.6.6 (with fixed space X) to deduce that there exists a countable set of indices J, points $(x_j)_{j\in J} \subset \mathrm{X}$ and weights $(\mu_j) \subset \mathbb{R}^+$, $(\nu_j) \subset \mathbb{R}^+$ such that $\mu_j \geq \lambda_\varepsilon \nu_j^{2/2^*}$ for every $j \in J$ and

$$\nu = |u|^{2^*}\mathfrak{m} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \geq |Du|^2\mathfrak{m} + \sum_{j\in J} \mu_j \delta_{x_j}.$$

We now observe that

$$\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int u^2 \,\mathrm{dS} \geq \|u\|^2_{L^{2^*}(\mathfrak{m})} \lambda_{\mathrm{S}}(\mathrm{X}). \tag{6.8.10}$$

Indeed, this is obvious if $u = 0$ $\mathfrak{m}$-a.e., hence we assume that $u \neq 0$ $\mathfrak{m}$-a.e.. In this case, (6.8.10) follows noticing that $\lambda_{\mathrm{S}}(\mathrm{X}) \leq Q_{\mathrm{S}}(u\|u\|^{-1}_{L^{2^*}(\mathfrak{m})}) = \|u\|^{-2}_{L^{2^*}(\mathfrak{m})} \big(\int |Du|^2 \mathrm{d}\mathfrak{m} + \int u^2 \,\mathrm{dS}\big)$. Therefore using again (6.8.9) we have

$$\begin{aligned}
\lambda_{\mathrm{S}}(\mathrm{X}) = \lim_n Q_{\mathrm{S}}(u_n) &\geq \mu(\mathrm{X}) + \int u^2 \,\mathrm{dS} \geq \int |Du|^2 \mathrm{d}\mathfrak{m} + \lambda_\varepsilon \sum_{j\in J} \nu_j^{2/2^*} + \int u^2 \,\mathrm{dS} \\
&\overset{(6.8.10)}{\geq} \|u\|^2_{L^{2^*}(\mathfrak{m})} \lambda_{\mathrm{S}}(\mathrm{X}) + \lambda_\varepsilon \sum_{j\in J} \nu_j^{2/2^*} \geq \lambda_{\mathrm{S}}(\mathrm{X}) (\|u\|^2_{L^{2^*}(\mathfrak{m})} + \sum_{j\in J} \nu_j^{2/2^*}) \\
&\geq \lambda_{\mathrm{S}}(\mathrm{X}) \Big(\int |u|^{2^*} \,\mathrm{d}\mathfrak{m} + \sum_{j\in J} \nu_j \Big)^{2/2^*} = \lambda_{\mathrm{S}}(\mathrm{X})\nu(\mathrm{X}) = \lambda_{\mathrm{S}}(\mathrm{X}),
\end{aligned}$$

where in the last line, we used the concavity of the function $t^{2/2^*}$, the fact that $\nu \in \mathscr{P}(\mathrm{X})$ and finally that $\lambda_{\mathrm{S}}(\mathrm{X}) \geq 0$. Hence all the inequalities are equalities and in particular from the strict concavity of $t^{2/2^*}$ we deduce that either $\int |u|^{2^*} \mathrm{d}\mathfrak{m} = 1$ or $u = 0$ (and the numbers ν_j are all zero except one that is equal to one). In the second case, plugging $u = 0$ in the above chain of inequalities, we infer that $\lambda_\varepsilon = \lambda_{\mathrm{S}}(\mathrm{X})$ which is a contradiction. Hence, we must have $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$ and $u_n \to u$ strongly in $L^{2^*}(\mathfrak{m})$ and in particular u is a minimizer for (6.8.5). This together with Proposition 6.8.3 concludes the proof. □

We conclude by extending the classical upper bound (6.8.2) to the setting of $\mathsf{RCD}(K,N)$ spaces. This in particular shows that (6.8.7) is a reasonable assumption.

Proposition 6.8.4. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathsf{RCD}(K,N)$ *space for some* $K \in \mathbb{R}$, $N \in (2,\infty)$ *and let* S *as in* (6.8.3). *Then*

$$\lambda_{\mathrm{S}}(\mathrm{X}) \leq \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2}.$$

Proof. The argument is almost the same as for Theorem 6.4.4. We start noticing that in the case $\min_{\mathrm{X}} \theta_N = +\infty$, evidently there is nothing to prove. We are left then to deal with the case $0 < \min_{\mathrm{X}} \theta_N < +\infty$. Let $x \in \mathrm{X}$ such that $\theta_N(x) = \min_{\mathrm{X}} \theta_N$. Then there exists a sequence $r_i \to 0$ such that the sequence of metric measure spaces $(\mathrm{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) := (\mathrm{X}, \mathsf{d}/r_i, \mathfrak{m}/r_i^N, x)$ pmGH-converges to an $\mathsf{RCD}(0,N)$ space $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}}, \boldsymbol{o}_{\mathrm{Y}})$ satisfying $\mathfrak{m}_{\mathrm{Y}}(B_r(\boldsymbol{o}_{\mathrm{Y}})) = \omega_N \theta_N(x) r^N$ for every $r > 0$ (this space is actually a cone by [98]). In particular from Lemma 6.4.3 for every $\varepsilon > 0$ there exists a non-zero $u \in \mathrm{Lip}_c(\mathrm{Y})$ such that $\frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_{\mathrm{Y}})}}{\||Du|\|^2_{L^2(\mathfrak{m}_{\mathrm{Y}})}} \geq \frac{\mathsf{Eucl}(N,2)^2 - \varepsilon}{\theta_N(x)^{2/N}}$. Then by the

Γ-convergences of the 2-Cheeger energies there exists a sequence $u_i \in W^{1,2}(\mathrm{X}_i)$ such that $u_i \to u$ strongly in $W^{1,2}$. Moreover, since u_i are uniformly bounded in $W^{1,2}$ (meaning in $W^{1,2}(\mathrm{X}_i)$), by the Sobolev embedding (recall also the scaling property in (6.4.1)) we have $\sup_i \|u_i\|_{L^{2^*}(\mathfrak{m}_i)} < +\infty$. In particular from the lower semicontinuity of the L^{2^*}-norm we get

$$\varliminf_i \frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}{\||Du_i|\|^2_{L^2(\mathfrak{m})}} = \varliminf_i \frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m}_i)}}{\||Du_i|_i\|^2_{L^2(\mathfrak{m}_i)}} \geq \frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_{\mathrm{Y}})}}{\||Du|\|^2_{L^2(\mathfrak{m}_{\mathrm{Y}})}} \geq \frac{\mathsf{Eucl}(N,2)^2 - \varepsilon}{\min_{\mathrm{X}} \theta_N^{2/N}}, \tag{6.8.11}$$

where $|Du_i|_i$ denotes the weak upper gradient computed in the space X_i.

Denote by $p' := p/(p-1)$ the conjugate exponent of p and observe that by hypothesis $2p' < 2^*$. This and the fact that u_i are bounded in L^{2^*}, by Proposition 2.4.3 $(viii)$ imply that u_i converges in $L^{2p'}$-strong to u. Finally using the Hölder inequality we can write

$$\begin{aligned}
\varlimsup_i Q_{\mathrm{S}}(u_i) &\leq \varlimsup_i \frac{\int |Du_i|^2 \,\mathrm{d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}} + \varlimsup_i \frac{\int \mathrm{S}|u_i|^2 \,\mathrm{d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}} \\
&\overset{(6.8.11)}{\leq} \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2 - \varepsilon} + \varlimsup_i \|\mathrm{S}\|_{L^p(\mathfrak{m})} \frac{\left(\int |u_i|^{2p'} \,\mathrm{d}\mathfrak{m}\right)^{1/p'}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}, \\
&= \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2 - \varepsilon} + \varlimsup_i \|\mathrm{S}\|_{L^p(\mathfrak{m})} r_i^{N\left(\frac{1}{p'} - \frac{2}{2^*}\right)} \frac{\|u_i\|^2_{L^{2p'}(\mathfrak{m}_i)}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m}_i)}} = \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{\mathsf{Eucl}(N,2)^2 - \varepsilon}.
\end{aligned}$$

where we have used that $1/p' < 2/2^*$, that $\varliminf_i \|u_i\|_{L^{2^*}(\mathfrak{m}_i)} \geq \|u\|_{L^{2^*}(\mathfrak{m}_{\mathrm{Y}})} > 0$ and as observed above $\|u_i\|_{L^{2p'}(\mathfrak{m}_i)} \to \|u\|_{L^{2p'}(\mathfrak{m}_{\mathrm{Y}})}$. From the arbitrariness of $\varepsilon > 0$ the proof is now concluded. □

6.8.2 Continuity of λ_{S} under mGH-convergence

In [121] it has been proven in the setting of Ricci-limits a result about mGH-continuity of the generalized Yamabe constant, under some additional boundedness assumption on the sequence. In the following result we extend this fact in the setting of RCD-spaces and we remove such extra assumption.

We start proving that λ_{S} is upper semicontinuous under mGH-convergence.

Lemma 6.8.5. *Let $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ be a sequence of compact $\mathsf{RCD}(K,N)$-spaces with $\mathfrak{m}(\mathrm{X}_n) = 1$, $n \in \bar{\mathbb{N}}$, for some $K \in \mathbb{R}, N \in (2,\infty)$ and satisfying $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. Let also $\mathrm{S}_n \in L^p(\mathfrak{m}_n)$ be L^p-weak convergent to S, for some $p > N/2$. Then,*

$$\varlimsup_{n\to\infty} \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \leq \lambda_{\mathrm{S}}(\mathrm{X}_\infty). \tag{6.8.12}$$

Proof. Fix a non-zero $u \in W^{1,2}(\mathrm{X}_\infty)$. By the Sobolev embedding on X_∞ we know that $u \in L^{2^*}(\mathfrak{m}_\infty)$, therefore by Lemma 6.6.4 there exists a sequence $u_n \in W^{1,2}(\mathrm{X}_n)$ that converge $W^{1,2}$-strong and L^{2^*}-strong to u. By definition of $\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$, we have

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \leq \int |Du_n|^2 \,\mathrm{d}\mathfrak{m}_n + \int \mathrm{S}_n |u_n|^2 \,\mathrm{d}\mathfrak{m}_n, \qquad \forall n \in \mathbb{N}.$$

From the assumption that $p > N/2$, we have that its conjugate exponent p' satisfies $2p' < 2^*$, therefore from (vii), $(viii)$ in Proposition 2.4.3 we have that $|u_n|^2$ $L^{p'}$-strongly converges to u^2. Recalling Proposition 6.6.2, we get that all the above quantities pass to the limit and thus we reach

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_\infty)} \varlimsup_{n\to\infty} \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \leq \int |Du|^2 \,\mathrm{d}\mathfrak{m}_\infty + \int \mathrm{S}|u|^2 \,\mathrm{d}\mathfrak{m}_\infty.$$

By arbitrariness of u, we conclude. □

We shall now come to the main continuity result.

Theorem 6.8.6 (mGH-continuity of λ_{S})**.** *Let $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ be a sequence of compact $\mathsf{RCD}(K,N)$-spaces with $\mathfrak{m}(\mathrm{X}_n)=1$, $n\in\bar{\mathbb{N}}$, for some $K\in\mathbb{R}, N\in(2,\infty)$ satisfying $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. Let also $\mathrm{S}_n\in L^p(\mathfrak{m}_n)$ be L^p-weak convergent to $\mathrm{S}\in L^p(\mathfrak{m}_\infty)$, for a given for $p>N/2$. Then,*

$$\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)=\lambda_{\mathrm{S}}(\mathrm{X}_\infty).$$

Proof. In light of Lemma 6.8.5, we only have to prove that

$$\varliminf_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\geq\lambda_{\mathrm{S}}(\mathrm{X}_\infty).$$

It is not restrictive to assume that the $\varliminf$ is actually a limit. For every $n\in\mathbb{N}$, we take $u_n\in W^{1,2}(\mathrm{X}_n)$ non-zero so that $Q_{\mathrm{S}_n}(u_n)-\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\leq n^{-1}$. In other words

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)}\big(\lambda_{\mathrm{S}_n}(\mathrm{X}_n)+\tfrac{1}{n}\big)\geq\int|Du_n|^2\,\mathrm{d}\mathfrak{m}_n+\int\mathrm{S}_n|u_n|^2\,\mathrm{d}\mathfrak{m}_n. \tag{6.8.13}$$

It is also clearly not restrictive to suppose that $u_n\in\mathrm{Lip}_c(\mathrm{X}_n)$ are non-negative and such that $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}\equiv 1$. Hence, arguing as in the proof of Theorem 6.8.2 (using also (6.8.12)), we get that u_n is uniformly bounded in $W^{1,2}$. Then, by compactness (see Proposition 2.4.4), up to a not relabeled subsequence, we have that u_n converge L^2-strong and $W^{1,2}$-weak to some $u_\infty\in W^{1,2}(\mathrm{X}_\infty)$. From $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}\equiv 1$ and the assumption $p>N/2$, Proposition 2.4.3 implies that u_n^2 converges $L^{p/(p-1)}$-strongly to u^2_∞ and that u_n converges $L^{2p/(p-1)}$-strongly to u_∞. From this point we subdivide the proof in three cases to be handled separately.
Case 1: $\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)<0$. In this case, by (6.8.13) we know by lower semicontinuity of the 2-Cheeger energy and Proposition 6.6.2, we have that

$$0>\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\geq\int|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty+\int\mathrm{S}u^2_\infty\,\mathrm{d}\mathfrak{m}_\infty.$$

In particular, u_∞ is not $\mathfrak{m}_\infty$-a.e. equal to zero and by weak-lower semicontinuity, we have that $0<\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\leq 1$. Therefore

$$\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\geq\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\geq\int|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty+\int\mathrm{S}u^2_\infty\,\mathrm{d}\mathfrak{m}_\infty\geq\lambda_S(\mathrm{X}_\infty)\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)},$$

which concludes the proof in this case.
Case 2: $\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)>0$. Before starting, notice that by using the Hölder inequality, for any $n\in\mathbb{N}$ and any $u\in W^{1,2}(\mathrm{X}_n)$ we have by the definition of $\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$ that

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)}\leq\lambda_{\mathrm{S}_n}(\mathrm{X}_n)^{-1}\int|Du|^2\,\mathrm{d}\mathfrak{m}_n+\lambda_{\mathrm{S}_n}(\mathrm{X}_n)^{-1}\|\mathrm{S}_n\|_{L^p(\mathfrak{m}_n)}\|u\|^2_{L^{2p/p-1}(\mathfrak{m}_n)}. \tag{6.8.14}$$

Moreover, since all X_n are compact and renormalized, there are $\mu\in\mathscr{M}^+_b(\mathrm{Z}),\nu\in\mathscr{P}(\mathrm{Z})$ so that, up to a not relabeled subsequence, $|Du_n|^2\mathfrak{m}_n\rightharpoonup\mu$ and $|u_n|^{2^*}\mathfrak{m}_n\rightharpoonup\nu$ in duality with $C(\mathrm{Z})$ as n goes to infinity, where $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}})$ is a (compact) space realizing the convergences via extrinsic approach. Since we are assuming that $\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)>0$, the constant in (6.8.14) are uniformly bounded (for n big enough) and we are in position to apply Lemma 6.6.6. In particular we get the existence of an at most countable set J, points $(x_j)_{j\in\,J}\subset\mathrm{X}_\infty$ and weights $(\mu_j),(\nu_j)\subset\mathbb{R}^+$, so that $\mu_j\geq\lim_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\nu_j^{2/2^*}$ with $j\in J$ and

$$\nu=|u_\infty|^{2^*}\mathfrak{m}+\sum_{j\in J}\nu_j\delta_{x_j},\quad \mu\geq|Du_\infty|^2\mathfrak{m}+\sum_{j\in J}\mu_j\delta_{x_j}.$$

Moreover, recalling Proposition 6.6.2 we have

$$\mu(\mathrm{X})+\int\mathrm{S}u^2_\infty\,\mathrm{d}\mathfrak{m}_\infty=\lim_{n\to\infty}Q_{\mathrm{S}_n}(u_n)\overset{(6.8.13)}{\leq}\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n), \tag{6.8.15}$$

and, arguing as in the proof of (6.8.10), u_∞ is so that $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\lambda_{\mathrm{S}}(\mathrm{X}_\infty) \le \int |Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + \int \mathrm{S}|u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty$. Finally, we can perform the chain of estimates

$$\begin{aligned}\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) &\overset{(6.8.15)}{\ge} \mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty \ge \int |Du_\infty|^2\mathrm{d}\mathfrak{m}_\infty + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\sum_{j\in J}\nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty\\ &\ge \lambda_{\mathrm{S}}(\mathrm{X}_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\sum_{j\in J}\nu_j^{2/2^*}\\ &\overset{(6.8.12)}{\ge} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \sum_{j\in J}\nu_j^{2/2^*}\Big)\\ &\ge \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} \ge \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n),\end{aligned}$$

where in the last line, we used the concavity of $t^{2/2^*}$ and the fact that $\nu \in \mathscr{P}(\mathrm{X})$. In particular, all inequalities must be equalities and by the strict concavity of $t^{2/2^*}$ either $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ and all $\nu_j = 0$, or $u_\infty = 0$ $\mathfrak{m}_\infty$-a.e. and all the weights are zero except one $\nu_j = 1$. The first situation is the easiest one, as in this case the above inequalities which are actually equalities imply that $\lambda_{\mathrm{S}}(\mathrm{X}_\infty) = \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)$, which is what we wanted. Therefore we suppose that we are in the second case, i.e. that there exists a point $y_0 \in \mathrm{X}_\infty$ so that $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ in duality with $C(\mathrm{Z})$ and that u_n converges in L^2-strong to zero. Moreover, from (6.8.13) and Hölder inequality we get

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \ge \big(\lambda_{\mathrm{S}_n}(\mathrm{X}_n) + \tfrac{1}{n}\big)^{-1}\Big(\int |Du_n|^2\,\mathrm{d}\mathfrak{m}_n - \|\mathrm{S}_n\|_{L^p(\mathfrak{m}_n)}\|u_n\|^2_{L^{2p/(p-1)}(\mathfrak{m}_n)}\Big), \qquad \forall n\in\mathbb{N}.$$

We can therefore apply Lemma 6.7.1 to get that $\theta_N(y_0) \le \mathsf{Eucl}(N,2)^N \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)^{N/2}$. Finally, we can rearrange and invoke Proposition 6.8.4 to get

$$\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \ge \frac{\theta_N(y_0)^{2/N}}{\mathsf{Eucl}(N,2)^2} \ge \lambda_{\mathrm{S}}(\mathrm{X}_\infty).$$

Case 3: $\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) = 0$. The argument is the same as in the previous case, only that we replace (6.8.14) with the Sobolev inequality given in Proposition 6.5.1:

$$\|u\|^2_{L^q(\mathfrak{m})} \le A(K,N,D)\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall u\in W^{1,2}(\mathrm{X}_n), \tag{6.8.16}$$

where $D>0$ is constant such that $\mathrm{diam}(\mathrm{X}_n) \le D$. Then we can apply exactly as in the previous case Lemma 6.6.6, except that in this case we obtain $\mu_j \ge A(K,N,D)^{-1}\nu_j^{2/2^*}$ for every $j\in J$. Then the above chain of estimates becomes

$$\begin{aligned}0 = \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) &\overset{(6.8.15)}{\ge} \mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty\\ &\ge \int |Du_\infty|^2\mathrm{d}\mathfrak{m}_\infty + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty\\ &\ge \lambda_{\mathrm{S}}(\mathrm{X}_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*}\\ &\overset{(6.8.12)}{\ge} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} \ge 0.\end{aligned}$$

Therefore we must have that $\nu_j = 0$ for every $j\in J$. This forces $\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ giving in turn that $\lambda_{\mathrm{S}}(\mathrm{X}_\infty) = 0$. Having examined all the three cases, the proof is now concluded. □

A Interpolation density estimates in the q-Wasserstein space

The aim of this appendix is to revisit the 2-Wasserstein interpolation $L^\infty(\mathfrak{m})$-estimates of [174] on CD spaces and [57] on MCP spaces for arbitrary $q \neq 2$.

A.1 The case of MCP spaces

Recall that, if $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{MCP}(K, N)$-space for some $K \in \mathbb{R}, N \in [1, \infty)$ as defined in Definition 2.5.1, we can interpolate between an absolutely continuous measure and a Dirac mass with a plan and 2-optimal dynamical plan π satisfying

$$(\mathsf{e}_t)_\sharp \pi \leq C(t)\mathfrak{m}, \qquad t \in [0, 1),$$

for a suitable profile function that depends on μ_0 and K, N. This has been stated in Theorem 2.5.3 and is due to [57]. We extend now this result for arbitrary $q \in (1, \infty)$.

Theorem A.1.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a non branching* $\mathsf{MCP}(K, N)$*-space for some* $K \in \mathbb{R}, N \in [1, \infty)$*. Then, for every* $q \in (1, \infty)$*,* $D > 0$ *and* $\mu_0, \mu_1 \in \mathscr{P}_q(\mathrm{X})$ *with* $\mu_0 = \rho_0 \mathfrak{m}, \rho_0 \in L^\infty(\mathfrak{m})$*, and* *diam*$(\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1)) < D$*, there exists* $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ *with* $\mu_t := (\mathsf{e}_t)_\sharp \pi \ll \mathfrak{m}$ *and*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})} \leq \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \|\rho_0\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0, 1), \tag{A.1.1}$$

having set $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$ *for* $t < 1$*.*

Proof. For $q = 2$, the statement is proved in [57, Theorem 1.1]. Here, we give some details to handle the general case.

Step 1. We begin by showing that any $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ as in the hypothesis is 'qualitatively non degenerate' according to the axiomatization given in [55, Assumption 1] (actually, under non branching it is equivalent [128, Corollary 5.17]). Indeed, let $K \subset \mathrm{X}$ be compact, $A \subset K$ non negligible and $x \in K$. Then, denoting by $A_{t,x} \subset C([0, 1], \mathrm{X})$ the subset of geodesics linking A to x, we denote $\mu_0 = \mathfrak{m}(A)^{-1}\mathfrak{m}_{|A}$ and appeal to [57, Theorem 1.1] to get that there exists $\pi \in \mathrm{OptGeo}_2(\mu_0, \delta_x)$ so that

$$1 = \mu_t(A_{t,x}) \leq \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \frac{\mathfrak{m}(A_{t,x})}{\mathfrak{m}(A)}, \qquad \forall t \in [0, 1),$$

where $D := \mathrm{diam}\ (K) < \infty$, having used that μ_t is a probability measure concentrated on $A_{t,x}$. In particular, this shows that there exists a profile function $f \colon [0, 1] \to (0, 1]$ (depending on the compact set K) and a positive $\delta < 1$ so that

$$\mathfrak{m}(A_{t,x}) \geq f(t)\mathfrak{m}(A), \qquad \forall t \in [0, \delta].$$

That is, $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ verifies Assumption 1 in [55].

Step 2. Suppose now $\mu_1 = \delta_x$, for $x \in \mathrm{supp}(\mathfrak{m})$. Then, from [57], we know that there exists $\pi \in \mathrm{OptGeo}_2(\mu_0, \delta_x)$ satisfying (A.1) hence, by Remark 2.5.2, we have also that $\pi \in \mathrm{OptGeo}_q(\mu_0, \delta_x)$.

Step 3. Let here $n \in \mathbb{N}$ and suppose μ_1 is a finite convex combination of Dirac masses, namely $\mu_1 := \sum_{j=1}^n \lambda_j \delta_{x_j}$ for $(x_j) \subset \mathrm{X}$ with $x_i \neq x_i$ for $i \neq j$ and $(\lambda_j) \subset [0,1]$ with $\sum_{j=1}^n \lambda_j = 1$. Then, by appealing to Step 1, we are in position to apply [55, Theorem 2.1] (recall X is proper) to deduce that there exists a unique optimal coupling between μ_0 and μ_1 and it is induced by a Borel map $T\colon \mathrm{X} \to \mathrm{X}$, i.e.

$$W_q^q(\mu_0, \mu_1) = \int \mathsf{d}^q(x, T(x))\,\mathrm{d}\mu_0(x).$$

Take this map T and define $\mu_0^j := \mu_0|_{T^{-1}(x_j)}$ for every $j = 1, ..., n$. By Step 2, we know that there are $\pi^j \in \mathrm{OptGeo}_q(\lambda_j^{-1}\mu_0^j, \delta_{x_j})$ verifying

$$\|\rho_t^j\|_{L^\infty(\mathfrak{m})} \leq \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \|\rho_0^j\|_{L^\infty(\mathfrak{m})}, \qquad \forall t \in [0,1),\ j = 1, ..., n,$$

having set $\rho_t^j := \frac{\mathrm{d}\mu_t^j}{\mathrm{d}\mathfrak{m}}$ and $\mu_t^j := (\mathsf{e}_t)_\sharp \pi^j$ for $t < 1$.

We now define $\pi := \sum_{j=1}^n \lambda_j \pi^j$ so that, by construction, we have that $(\mathsf{e}_0, \mathsf{e}_1)_\sharp \pi \in \mathrm{Adm}(\mu_0, \mu_1)$ and $\mu_t := (\mathsf{e}_t)_\sharp \pi \ll \mathfrak{m}$ for every $t < 1$ with $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}} = \sum_{j=1}^n \lambda_j \rho_t^j$. We now claim that ρ_t satisfies (A.1). To this aim, we instead check that

$$\mathfrak{m}(\{\rho_t^i > 0\} \cap \{\rho_t^j > 0\})) = 0, \qquad \forall t \in (0,1), j \neq i,$$

as the latter property implies the claim by construction of μ_t. Suppose the above is not true, namely there exists $\tau \in (0,1)$ so that $\mathfrak{m}(\{\rho_\tau^i > 0\} \cap \{\rho_\tau^j > 0\})) > 0$ for some $j \neq i$. Then there would exists an optimal coupling between the (renormalized) measures $\mathfrak{m}|_{\{\rho_\tau^i>0\}\cap\{\rho_\tau^j>0\}}$ and $\delta_{x_j} + \delta_{x_i}$ that is not induced by a map. This finds a contradiction for what we have previously proved and concludes the step.

Step 4. Since (X, d) is separable, given any μ_1 as in the hypothesis, we can find a sequence of points $(x_j)_{j\in\mathbb{N}} \subset \mathrm{supp}(\mu_1)$ and weights $(\lambda_{n,j})_{j\in\mathbb{N}} \subset [0,1]$ so that

$$\sum_{j=1}^n \lambda_{n,j}\delta_{x_j} =: \mu_1^n \to \mu_1 \qquad \text{in } W_q,$$

as n goes to infinity, recalling that the support of μ_1 is bounded and consequently (1.1.10). By Step 3, we known that there exists a sequence $\pi_n \in \mathrm{OptGeo}_q(\mu_0, \mu_1^n)$ verifying $\mu^n := \rho_t^n \mathfrak{m}$ and (A.1) for every $t < 1, n \in \mathbb{N}$. Finally, arguing as in [57, Lemma 4.4] (the proof is written for $q = 2$ but works for arbitrary q, see also Lemma 4.3.7) we get the existence of a weak limit $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ of π_n verifying all the required properties. $\square$

A.2 The case of CD_q spaces

Aim of this section is to generalize to arbitrary $q \neq 2$ the interpolation estimates of [174]. We stated them in Theorem 2.2.9 that, recall, ensure that on a $\mathsf{CD}_2(K,N)$ space for some $K \in \mathbb{R}, N \in (1,\infty]$, we can interpolate between two absolutely continuous measures bounded with bounded supports with a 2-optimal dynamical plan π satisfying

$$(\mathsf{e}_t)_\sharp \pi \leq C\mathfrak{m}, \qquad t \in [0,1),$$

for a suitable profile function that depends on μ_0, μ_1 and K, N. More precisely, defining

$$C(D,K,N) := \begin{cases} e^{\sqrt{(N-1)K^-}D}, & \text{if } 1 \leq N < \infty \\ e^{K^- D^2/12}, & \text{if } N = \infty, \end{cases}$$

we are going to prove the following:

Theorem A.2.1. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}_q(K,N)$-space for some $K\in\mathbb{R}$, $N\in[1,\infty]$ and $q\in(1,\infty)$. For any $D>0$ and $\rho_0,\rho_1\in L^\infty(\mathfrak{m})$ probability densities with diam(supp(ρ_0) $\cup$ supp(ρ_1)) $<D$, there exists $\pi\in\mathrm{OptGeo}_q(\rho_0\mathfrak{m},\rho_1\mathfrak{m})$ satisfying $\mu_t:=(\mathsf{e}_t)_\sharp\pi\ll\mathfrak{m}$. Moreover, writing $\mu_t:=\rho_t\mathfrak{m}$, we have the following upper bound for the density*

$$\|\rho_t\|_{L^\infty(\mathfrak{m})}\le C(D,K,N)\|\rho_0\|_{L^\infty(\mathfrak{m})}\vee\|\rho_1\|_{L^\infty(\mathfrak{m})},\qquad\forall t\in[0,1].$$

We postpone its proof at the end of this part and start building up all the necessarily material. Since the above result is probably expected to hold by experts of the field, we shall adopt a concise (but complete) style of presentation.

Preparatory Lemmas

Consider for any $q\in(1.\infty)$ two measures $\mu_0,\mu_1\in\mathscr{P}_q(\mathrm{X})$ with $W_q(\mu_0,\mu_1)<\infty$ and denote by

$$\mathcal{I}^q_t(\mu_0,\mu_1):=\{\mu\in\mathscr{P}_q(\mathrm{X})\colon W_q(\mu_0,\mu)=tW_q(\mu_0,\mu_1),\ W_q(\mu,\mu_1)=(1-t)W_q(\mu_0,\mu_1)\},$$

the set of t-intermediate measures between μ_0,μ_1 where $t\in(0,1)$.

Lemma A.2.2. *Let (X,d) be a metric space, $q\in(1,\infty)$ and assume $\mu_0,\mu_1\in\mathscr{P}_q(\mathrm{X})$ have bounded supports. Then, for every $t\in(0,1)$, the set $\mathcal{I}^q_t(\mu_0,\mu_1)$ is closed in $(\mathscr{P}_q(\mathrm{X}),W_q)$.*

Proof. For any $\nu_n\subseteq\mathcal{I}^q_t(\mu_0,\mu_1)$ with $\nu_n\to\nu$ in W_q, the triangular inequality gives

$$\max_{i=0,1}|W_q(\mu_i,\nu)-W_q(\mu_i,\nu_n)|\le W_q(\nu_n,\nu),$$

from which the conclusion follows. □

We now face convexity properties of the set of t-intermediate measures. This statement should be interpreted as a way to redistribute mass on intermediate points of Wasserstein geodesics.

Lemma A.2.3. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space which is also geodesic and $q\in(1,\infty)$. Suppose $\mu_0,\mu_1\in\mathscr{P}(\mathrm{X})$ with $W_q(\mu_0,\mu_1)<\infty$. Then, for any $\pi\in\mathrm{OptGeo}_q(\mu_0,\mu_1)$ and $f\colon\mathsf{Geo}(\mathrm{X})\to[0,1]$ s.t. $c=(f\pi)(\mathsf{Geo}(\mathrm{X}))\in(0,1)$, we have*

$$(\mathsf{e}_t)_\sharp((1-f)\pi)+c\mu\in\mathcal{I}^q_t(\mu_0,\mu_1),$$

for every $\mu\in\mathcal{I}^q_t(\frac{1}{c}(\mathsf{e}_0)_\sharp(f\pi),\frac{1}{c}(\mathsf{e}_1)_\sharp(f\pi),\ t\in(0,1)$.

Proof. We omit the details and refer to [174, Lemma 3.5] for the proof which reads identical for arbitrary q. □

We consider the excess mass functional

$$\mathcal{F}_C(\eta):=\|(\rho-C)^+\|_{L^1(\mathfrak{m})}+\eta^s(\mathrm{X}),$$

and observe that, when it vanishes on a probability measure, it automatically detect both absolutely continuity with respect to the reference measure with corresponding density L^∞-bounded from the constant C. We prove first that it is lower semicontinuous.

Lemma A.2.4. *Let (X,d) be a bounded metric space equipped with a finite Borel measure $\mathfrak{m}$ and $q\in(1,\infty)$. Then, for any $C\ge0$, the functional $\mathcal{F}_C$ is lower semicontinuous on $(\mathscr{P}_q(\mathrm{X}),W_q)$.*

Proof. The proof consists in showing that

$$\mathcal{F}_C(\mu)=\sup\Big\{\int g\,\mathrm{d}\mu-C\int g\,\mathrm{d}\mathfrak{m}\colon g\in C(\mathrm{X}),\ 0\le g\le1\Big\},$$

for every $\mu\in\mathscr{P}_q(\mathrm{X})$. It is proven in [174]) that it is lower semicontinuous in $\mathscr{P}_2(\mathrm{X})$. Then, recalling (1.1.10) and the present hypothesis, $\mathcal{F}_C$ is equivalently lower semicontinuous in the space $(\mathscr{P}_q(\mathrm{X}),W_q)$ independently on $q\in(1,\infty)$. □

Proof of Theorem A.2.1

Proof. We subdivide the proof into several steps. Write for simplicity $\mu_i := \rho_i \mathfrak{m}$ for $i = 0, 1$. First observe that, for every K, we can alternatively prove the statement in the larger (or equal) class of $CD_q(-K^-, N)$-spaces. Hence, for simplicity, we shall consider only $K < 0$ in the proof.

STEP 1. From the $\mathsf{CD}_q(K,N)$ condition, there exits $\pi \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ which is concentrated on geodesics with length at most D. Also, wee denote for simplicity $E := \{\rho_{\frac{1}{2}} > 0\}$. We first analyze the case $N < \infty$: an application of Jensen's inequality yields

$$\mathcal{U}_N(\mu_{\frac{1}{2}}|\mathfrak{m}) = -\mathfrak{m}(E) \fint_E \rho_{\frac{1}{2}}^{1-\frac{1}{N}} \,\mathrm{d}\mathfrak{m} \geq -\mathfrak{m}(E) \Big(\frac{1}{\mathfrak{m}(E)}\Big)^{1-\frac{1}{N}} = -\mathfrak{m}(E)^{\frac{1}{N}}$$

On the other hand, we estimate in (2.2.2) as

$$\mathcal{U}_N(\mu_{\frac{1}{2}}|\mathfrak{m}) \leq -\Big(\frac{1}{\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})}}\Big)^{\frac{1}{N}} \int 2\tau_{K,N}^{(\frac{1}{2})}(\mathsf{d}(\gamma_0,\gamma_1)) \,\mathrm{d}\pi.$$

Because of $K < 0$ and recalling that $\theta := \mathsf{d}(\gamma_0, \gamma_1) < D$ π-a.e. γ, we can also estimate the distortion coefficients as

$$\tau_{K,N}^{(\frac{1}{2})}(\theta) = \frac{1}{2}^{\frac{1}{N}} \Big(\frac{1}{e^{\sqrt{-K/(N-1)}\frac{\theta}{2}} + e^{-\sqrt{-K/(N-1)}\frac{\theta}{2}}}\Big)^{1-\frac{1}{N}} \geq \frac{1}{2}\big(e^{\sqrt{-K(N-1)}\frac{D}{2}}\big)^{-\frac{1}{N}}.$$

While, for the case $N = \infty$, we observe firstly that Jensen inequality with the convexity of $u(x) = x\log x$ grant

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_{\frac{1}{2}}) = \int_E \rho_{\frac{1}{2}} \log \rho_{\frac{1}{2}} \,\mathrm{d}\mathfrak{m} \geq \log\Big(\frac{1}{\mathfrak{m}(E)}\Big),$$

and secondly that

$$\begin{aligned}\mathrm{Ent}_{\mathfrak{m}}(\mu_{\frac{1}{2}}) &\leq \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + \frac{1}{2}\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + \frac{-K}{8}W_q^2(\mu_0,\mu_1)\\ &\leq \log\big(\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})}\big) + \frac{-KD^2}{8}.\end{aligned}$$

We can thus combine in both case the two inequalities to get the following *spreading of mass under curvature dimension condition* principle:

$$\mathfrak{m}(\{\rho_{\frac{1}{2}} > 0\}) \geq \frac{1}{P(D,K,N)\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})}}, \tag{A.2.1}$$

where $P(D,K,N) = e^{\sqrt{(N-1)K^-}D/2}$ if $1 \leq N < \infty$ and $e^{K^-D^2/8}$ if $N = \infty$, for every $K \in \mathbb{R}$. For simplicity, we write from now on $M := P(D,K,N)\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_1\|_{L^\infty(\mathfrak{m})}$.

STEP 2. We now show that for any $C > M$, there exists a minimizer of $\mathcal{F}_C(\cdot)$ in $\mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)$. There is a standard way to achieve minimizers of the functional $\mathcal{F}_C$ even when $\mathcal{I}^q_t(\mu_0,\mu_1)$ need not to be compact[1]. This has been shown in [174] building exactly upon (A.2.1), Lemma A.2.2, Lemma A.2.3 and Lemma A.2.4 for the particular case $q = 2$. Being these results valid also for general $q \in (1,\infty)$ we omit the details of the strategy ensuring that

$$\forall C > M,\ \exists \mu \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1) \quad \text{so that} \quad \mathcal{F}_C(\mu) = \inf_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathcal{F}_C(\eta). \tag{A.2.2}$$

STEP 3. For any $C > M$, we claim that

$$\inf_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathcal{F}_C(\eta) = 0.$$

[1]This typically occurs on proper spaces making possible a 'direct method' argument. As we consider also $N = \infty$, a different argument is needed.

Denote $\mathcal{I}^q_{min} \subset \mathcal{I}^q_{\frac{1}{2}}(\mu_0, \mu_1)$ the set of minimizers of $\mathcal{F}_C$ (which is always nonempty (A.2.2)) and let $\mu \in \mathcal{I}^q_{min}$ be such that

$$\mathfrak{m}(\rho_\mu > C) \geq \Big(\frac{M}{C}\Big)^{\frac{1}{4}} \sup_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathfrak{m}(\{\rho_\eta > C\}), \tag{A.2.3}$$

where $\mu := \rho_\mu \mathfrak{m} + \mu^s$ with $\mu^s \perp \mathfrak{m}$ and $\eta := \rho_\eta \mathfrak{m} + \mu^s$ with $\eta^s \perp \mathfrak{m}$. We argue now by contradiction and suppose $\mathcal{F}_C(\mu) > 0$ whence. If $A := \{\rho_\mu > 0\}$, then this means necessarily that $\mathfrak{m}(A) > 0$ and $\mu^2(\mathrm{X}) > 0$. In the first case, find a $\delta > 0$ so that, denoting $A' := \{\rho_\mu > C + \delta\}$, we have

$$\mathfrak{m}(A') \geq \Big(\frac{M}{C}\Big)^{\frac{1}{2}} \mathfrak{m}(A).$$

Let $\alpha \in \mathrm{Opt}_q(\mu_0, \mu), \beta \in \mathrm{Opt}_q(\mu, \mu_1)$ and consider

$$\tilde{\pi} \in \mathrm{OptGeo}_q\Big((P^0)_\sharp \frac{\alpha|_{\mathrm{X}\times A'}}{\mu(A')}, (P^1)_\sharp \frac{\beta|_{A'\times \mathrm{X}}}{\mu(A')}\Big),$$

given by Step 1. Denote $\Gamma_t := (\mathsf{e}_t)_\sharp \tilde{\pi}$ the corresponding W_q-geodesic and consider its decomposition $\Gamma_{\frac{1}{2}} = \rho_\Gamma \mathfrak{m} + \Gamma^s$. Then, from (A.2.1), it follows that

$$\mathfrak{m}(\{\rho_\Gamma > 0\} \geq \frac{\mu(A')}{M} \geq \frac{C}{M}\mathfrak{m}(A') \geq \Big(\frac{C}{M}\Big)^{\frac{1}{2}} \mathfrak{m}(A). \tag{A.2.4}$$

Now consider redistributing the mass of the measure μ via

$$\tilde{\mu} := \mu|_{X\setminus A'} + \frac{C}{C+\delta}\mu|_{A'} + \frac{\delta}{C+\delta}\mu(A')\Gamma_{\frac{1}{2}}.$$

Arguing then as in [174, Lemma 3.5], the Lemma A.2.3 directly yields $\tilde{\mu} \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0, \mu_1)$. Also, setting $\tilde{\mu} = \rho_{\tilde{\mu}}\mathfrak{m} + \tilde{\mu}^s$ with $\tilde{\mu}^s \perp \mathfrak{m}$, a standard calculation shows that the excess functional decreases computed at $\tilde{\mu}$. We omit here the details to get

$$\mathcal{F}_C(\mu) - \mathcal{F}_C(\tilde{\mu}) = \int_{\{\rho_\mu < C\}} \min\big\{C - \rho_\mu, \frac{\delta}{C+\delta}\mu(A')\rho_\Gamma\big\}\,\mathrm{d}\mathfrak{m}.$$

Notice that, being μ a minimizer, the right hand most side of the above equation is nonpositive whence, necessarily the integral must vanish giving in turn

$$\tilde{\mu} \in \mathcal{I}^q_{min}(\mu_0, \mu_1), \qquad \mathfrak{m}(\{\rho_\mu < C\} \cap \{\rho_\Gamma > 0\}) = 0.$$

Moreover, $\rho_{\tilde{\mu}} > C$ $\mathfrak{m}$-a.e. on the set $\{\rho_\mu \geq C\} \cap \{\rho_\Gamma > 0\}$, hence

$$\mathfrak{m}(\{\rho_{\tilde{\mu}} > C\}) \geq \mathfrak{m}(\{\rho_\Gamma > 0\}) \overset{(A.2.4)}{\geq} \Big(\frac{C}{M}\Big)^{\frac{1}{2}} \mathfrak{m}(\{\rho_\mu > C\}) \overset{(A.2.3)}{\geq} \Big(\frac{C}{M}\Big)^{\frac{1}{4}} \sup_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathfrak{m}(\{\rho_\eta > C\}),$$

yielding a contradiction, since $\tilde{\mu} \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0, \mu_1)$ and $C > M$. Therefore, A is negligible and the first situation does not occur: necessarily μ is purely singular, otherwise there is nothing to prove. But then a similar redistribution of mass for the singular part applies giving a contradiction. Wrapping up, we showed

$$\forall C > M \qquad \exists \min_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathcal{F}_C(\eta) = 0.$$

The extreme case $C = M$ can be obtained with an easy argument as in [174, Corollary 3.12]: being μ_0, μ_1 supported on bounded sets, we can find a bounded set $B \subset \mathrm{X}$ so that every $\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0, \mu_1)$ is supported in B, and hence

$$\min_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathcal{F}_M(\eta) \leq \min_{\eta \in \mathcal{I}^q_{\frac{1}{2}}(\mu_0,\mu_1)} \mathcal{F}_C(\eta) + |C - M|\mathfrak{m}(B) = |C - M|\mathfrak{m}(B),$$

for every $C > M$. Thus, the conclusion follows also for $C = M$ by approximation $C \downarrow M$.
STEP 4. The conclusion of the theorem will be achieved by iterating the above construction from midpoint to a general dyadic partition of $[0, 1]$. Fix $n \in \mathbb{N}$, we now show how to produce from the measures $(\rho_{k2^{-n+1}})$ for $k = 0, ..., 2^{-n+1}$ the successive sequence $(\rho_{k2^{-kn}})$. Consider, for k odd, the midpoints $\mu_{k2^{-n}} \in \mathcal{I}^q_{\frac{1}{2}}(\rho_{k2^{-n+1}}\mathfrak{m}, \rho_{(k+1)2^{-n+1}}\mathfrak{m})$ satisfying

$$\begin{aligned}
&\mu_{k2^{-n}} \ll \mathfrak{m}, \qquad \mu_{k2^{-n}} := \rho_{k2^{-n}}\mathfrak{m} \\
&\|\rho_{k2^{-n}}\|_{L^\infty(\mathfrak{m})} \leq P(K, N, 2^{-n+1}D)\|\rho_{(k-1)2^{-n}}\|_{L^\infty(\mathfrak{m})} \vee \|\rho_{(k+1)2^{-n}}\|_{L^\infty(\mathfrak{m})},
\end{aligned}$$

since $\mathrm{diam}(\mathrm{supp}(\rho_{(k-1)2^{-n}}) \cup \mathrm{supp}(\rho_{(k+1)2^{-n}})) < 2^{-n+1}D$ and Step 1-2-3 apply. By induction, it holds that

$$\|\rho_{k2^{-n}}\|_{L^\infty(\mathfrak{m})} \leq \prod_{i=1}^{n} P(K, N, 2^{-i+1}D)\|\rho_0\|_{L^\infty(\mathfrak{m})} \vee \|\rho_0\|_{L^\infty(\mathfrak{m})}, \qquad \forall n \in \mathbb{N},$$

which, under the assumption $D < \infty$, can be coupled with the fact

$$\lim_{n\to\infty} \prod_{i=1}^{n} P(K, N, 2^{-i+1}D) = C(D, K, N),$$

giving in turn that a geodesic curve μ_t in $\mathrm{OptGeo}_q(\mu_0, \mu_1)$ is well defined by completion. Finally, the sought L^∞-estimate on ρ_t holds by lower semicontinuity of the functional $\mathcal{F}_M$.

$\square$

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