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Existence and properties of minimal surfaces and varifolds with contact angle conditions

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Chapter 1

Introduction

1.1 Background

This thesis is devoted to the study of minimal surfaces with prescribed angle conditions and to their generalizations in the sense of Geometric Measure Theory and is part of the work carried out by the author during his PhD studies under the supervision of Professor Guido De Philippis.

The study of minimal surfaces whose area is stationary under a certain class of variations (the so called *minimal surfaces*) has been one of the most active fields in Mathematics, both for its own intrinsic interest and for its connections to problems in physics and engineering.

One of the most studied topics is the existence and regularity of minimal surfaces under various boundary conditions. For a smooth minimal hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the stationarity condition can be translated in the well-known equation:

$$H_\Sigma = 0, \tag{1.1}$$

where H_Σ is the mean curvature vector of the surface Σ. If $U \subset \mathbb{R}^n$ is open and Σ is the graph of a function $u\colon U \to \mathbb{R}$, then (1.1) is equivalent to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{in } U,$$

which is a (degenerate) elliptic equation. As said, it is natural to look for solutions of the above equations under various boundary conditions: for example, one can look for minimizers of the area when the boundary is a fixed closed $(n-1)$-submanifold (the so called *Plateau problem*); or one can look for a stationary surface $\Sigma \subset \mathcal{M}$, where $\mathcal{M} \subset \mathbb{R}^{n+1}$ is a fixed container, $\partial\Sigma \subset \partial\mathcal{M}$ and $\partial\Sigma$ is free to move inside $\partial\mathcal{M}$ (called a *free boundary problem*). Each of these problems arises from physical situations, where the free energy of a membrane is proportional to its area; thus hypersurfaces that are stationary with respect to these energies are equilibrium configurations of the given physical system.

While solutions of Plateau problems model situations where the boundary of the surface is fixed, free boundary problems give rise to minimal surfaces meeting the boundary of the container with a prescribed $\pi/2$ angle.

These type of angle conditions originate in various problem, the most famous one being the modelling of *capillarity* surfaces, but see also [Li20] for an application to the *positive mass theorem*. In this case, one studies surfaces which are boundaries of an open subset $\Omega \subseteq \mathcal{M}$ of the fixed container $\mathcal{M}$, (modeling for instance a drop of liquid in a glass), and its energy is a weighted sum of the area of its interfaces with the other fluid contained in $\mathcal{M}$ and with the glass itself:

$$F_a(\Omega) = \mathcal{H}^n(\partial\Omega \cap \mathcal{M}^\circ) + a\mathcal{H}^n(\partial\Omega \cap \partial\mathcal{M}), \tag{1.2}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure and $a \in [0, 1)$. It is easy to see that a smooth stationary point Ω for F_a is such that $\partial\Omega \cap \mathcal{M}^o$ is a minimal hypersurface that meets $\partial\mathcal{M}$ with a fixed angle θ such that $\theta = \arccos a$.

Solutions of the above mentioned problems are usually found via various techniques in global analysis, for instance stationary surfaces with prescribed boundary can be obtained via minimization techniques. However it is easy to see that the unique minimizer of F_a is the empty set. In this case it is natural to look for the existence of a stationary point through the use of *min-max techniques* which would, at least in principle, select a saddle point for the energy. Application of min-max techniques to geometric problems has a long story and goes back to the seminal work of Birkhoff [Bir17] and his extension to the area functional by Almgren-Pitts [Pit14].

In this thesis, the main point we want to investigate is the study of existence and regularity of minimal surfaces $\Sigma \subset \mathcal{M}$ meeting $\partial\mathcal{M}$ with a fixed angle $\theta \in (0, \pi/2)$; as said before this will be obtained by performing a min-max procedure in the spirit of Almgren-Pitts to the capillarity functional F_a defined in (1.2). Referring to the second part of this introduction for a more detailed history of the problem, we would like to point out here a few features of the problem, which are also motivating the study of the questions in the first part of the thesis.

As for minimizing methods, the desired minimal surface is obtained as a limit of a suitable sequence of approximating objects (called *min-max sequence*). The usual problem is that the surfaces which form a min-max (or minimizing) sequence have just uniformly bounded area and the space of smooth surfaces does not enjoy good compactness properties under area bounds. Thus, in order to get a candidate limit for the approximating sequence, one has to enlarge the class of objects where we look for a limit and to weaken the notion of convergence. In other words, one is forced to introduce weaker notions of surfaces in order to obtain the existence of a solution. Regularity theory would later allow to show that this weak solution is indeed a classical one.

During the last decades, many of these weak notions of surfaces have been developed and each of them is suitable for a certain class of problems. When dealing with min-max, one of the best suited weak notions of surfaces are varifolds. A k-varifold on $\mathbb{R}^n$ is a positive Radon measure, and a k-surface Σ can be seen in a natural way as a k-varifold V_Σ for which the total mass corresponds to the k-dimensional area of the surface Σ. Thus, since a sequence of Radon measures with uniformly bounded masses is pre-compact, any sequence of k-surfaces Σ_k with uniformly bounded k-area has a subsequence which converges in the sense of varifolds to a k-varifold V.

For a varifold one can define also a notion of *first variation* of the mass, thus defining the concept of *stationary varifold*, which is the weak counterpart of the notion of minimal surface and allows us to define a weak notion of solution to our problem. Moreover, by first variation, one can define a weak notion of *contact angle* condition for a varifold V in a container $\mathcal{M}$, see [KT17] and Definition 2.20.

Varifolds, unlike other weak notions of surfaces like currents, lack of orientation; this, while makes the notion of boundary a bit problematic, implies that the mass is *continuous* under varifold convergence, whereas it is just lower semi-continuous for convergence of currents. This feature makes varifold a well-suited tool in solving min-max problems, where one has to prove convergence of the values of functional to the value attained on the saddle point.

Due their utility in this and other kind of problems, the study of the properties of varifolds which satisfy suitable conditions on their first variation is an interesting topic of research. The author studied some of these problems during his PhD: in particular, some boundary properties of varifolds with contact angle conditions at $\partial\mathcal{M}$ in a given container $\mathcal{M}$; since these conditions are usually given in a very weak way, it is natural to ask if they hold in some stronger sense and if varifolds which satisfy any of them enjoy some of the properties satisfied by regular surfaces which meet $\partial\mathcal{M}$ with a prescribed contact angle.

We now focus with some more details on the main results of the thesis.

1.2 Varifolds with free boundary

1.2.1 Bounded first variation

In the first part of the thesis, taken by the work [DeM21], we study boundary properties of varifolds that satisfy a contact angle condition.

Varifolds are a generalization of the notion of surface in the sense of Geometric Measure Theory: it is possible to see a k-surface $\Sigma \subset \mathbb{R}^n$ as a positive Radon measure on the product space $G_k(\mathbb{R}^n) \coloneqq \mathbb{R}^n \times G(k,n)$, where $G(k,n)$ is the Grassmannian of unoriented k-planes in $\mathbb{R}^n$. Any positive Radon measures on this space is called a k-varifold. Roughly speaking, a k-varifold is a measure which distributes the mass, at every point, among all possible k-planes. To give an example, a smooth k-surface $\Sigma \subset \mathbb{R}^n$ induces a k-varifold V_Σ in a natural way:

$$V_\Sigma = \mathcal{H}^k \llcorner \Sigma \otimes \delta_{T_x\Sigma},$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure; in other words, V_Σ concentrates the mass of Σ only on the tangent planes to Σ. To a varifold it is associated its *mass* $\|V\|$, which is defined as $\|V\|(A) = V\big(A \times G_k(\mathbb{R}^n)\big)$ and measures the equivalent of the "area" of V charged on A; indeed, it holds $\|V_\Sigma\| = \mathcal{H}^k \llcorner \Sigma$.

A general k-varifold can be a wild object, which is k-dimensional just in the sense that it is a measure on $G_k(\mathbb{R}^n)$. This generalization of the notion of surface is used in several frameworks, in order to show existence of (weak) solutions to geometric problems involving surfaces, in the same spirit for which one uses Sobolev spaces instead of regular functions in the solutions of problems involving functions.

As in the case of Sobolev spaces, one can ask whether (and in what sense) these weak notions of surfaces enjoy some of the properties of their regular counterparts. Besides their practical usefulness, these questions are of their own interest and their investigation is an active field of research.

This part of the thesis deals with the study of some "boundary properties" of a k-varifold contained in a compact domain $\mathcal{M} \subset \mathbb{R}^n$ which satisfies a contact angle condition. This contact angle condition is given using the idea of first variation of the mass of V: let us consider a vector field $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ and consider the quantity

$$\delta V[X] \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|(\psi_t)_\sharp V\|(\mathcal{M})\Big)\Big|_{t=0} = \int_{G_k(\mathbb{R}^n)} \operatorname{div}_S(x)\,\mathrm{d}V(x,S), \tag{1.3}$$

where $(\psi_t)_\sharp V$ is the push-forward of V through the flow map ψ_t of X. $\delta V[X]$ is called the *first variation* of V applied to X and measures the derivative at $t=0$ of the mass $\|V\|$ of V when we let V "evolve" by the action of the flow ψ_t of X.
If $V = V_\Sigma$ for a smooth surface Σ with smooth boundary $\partial\Sigma$, then further computations show that

$$\delta V_\Sigma[X] = \int_{G_k(\mathbb{R}^n)} \operatorname{div}_S(x)\,\mathrm{d}V_\Sigma(x,S) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k + \int_{\partial\Sigma} \langle X, \eta_\Sigma\rangle\,\mathrm{d}\mathcal{H}^{k-1}, \tag{1.4}$$

where H is the mean curvature vector of Σ and $\eta_\Sigma(x)$ is the exterior unit conormal vector to $\partial\Sigma$, that is the unique unit vector which lies on $T_x\Sigma$, is orthogonal to $T_x\partial\Sigma$ and points outside Σ.

The key observation about (1.4) is that it can be used to describe the possibility that Σ meets $\partial\mathcal{M}$ orthogonally. In fact, the more natural geometrical meaning of this condition is that $\partial\Sigma \subset \partial\mathcal{M}$ and that η_Σ is orthogonal to $\partial\mathcal{M}$. This is equivalent to say that

$$\eta_\Sigma(x) \cdot X(x) = 0 \qquad \forall x \in \partial\Sigma,\ \forall X \in \mathfrak{X}_t(\mathcal{M}),$$

where $\mathfrak{X}_t(\mathcal{M})$ is the space of vector fields which are tangent to $\partial\mathcal{M}$ on $\partial\mathcal{M}$. Putting this in (1.4), we obtain the condition

$$\int_{G_k(\mathcal{M})} \mathrm{div}_S(x)\,\mathrm{d}V_\Sigma(x,S) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \tag{1.5}$$

We can see that the formulation of (1.5) does not require any regularity of $\partial\Sigma$; thus it can be intended as an othogonality condition in a weak sense: if V is a general k-varifold in $\mathcal{M}$, one can say that V has *free boundary* at $\partial\mathcal{M}$ *if there exists* a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ such that it holds

$$\int_{G_k(\mathcal{M})} \mathrm{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_\mathcal{M} \langle X, H\rangle\,\mathrm{d}\|V\| \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \tag{1.6}$$

By the similarity with (1.5), H is called the *generalized mean curvature* of V. If a smooth surface Σ intersects $\partial\mathcal{M}$ orthogonally, that is if Σ satisfies (1.5), simply by testing it with a vector field $\bar{X}$ which coincides with the exterior unit normal to $\partial\mathcal{M}$ on $\partial\mathcal{M}$ and satisfies $\|\bar{X}\|_\infty = 1$, one obtains a-priori upper bounds on $\mathcal{H}^{k-1}(\partial\Sigma)$ in terms of $\mathcal{H}^k(\Sigma)$, of H and of the second fundamental form of $\partial\mathcal{M}$.

It is a natural question if similar properties hold for a general varifold with free boundary as well. More precisely, it is natural to ask if, when testing the first variation δV of V also with vector fields which are not tangent to $\partial\mathcal{M}$, this first variation $\delta V[X]$ (which is in principle just a first-order distribution by (1.3)) can be bounded in terms of $\|X\|_{C^0}$, that is if V has *bounded first variation*; if this is the case, δV is vector-valued Radon measure and it is also natural to ask whether there are bounds on the mass of this Radon measure similar to the ones which hold in the case of smooth surfaces. We summarize these problems.

Question 1.1. *Is it true that a varifold V with free boundary at $\partial\mathcal{M}$ has bounded first variation δV? Can we find bounds on $|\delta V|$ in terms of H, $\|V\|$ and the second fundamental form of $\partial\mathcal{M}$?*

These question are studied in Chapter 3, where we give affirmative answers, see Theorem 3.3 and Corollary 3.13 which generalize results of Grüter and Jost [GJ86a] and [Ede18]; we show that there exists a Radon measure $\tilde{H}\|V\| + N\sigma_V$ supported on $\partial\mathcal{M}$ which, together with $H\|V\|$, make up the first variation δV of V.

These results can be seen as a sort of varifold-versions of the trace theorem for Sobolev functions. Indeed, the definition of varifold with free boundary (1.6) is the analogous of the one of Sobolev functions: $f \in W^{1,p}(\mathcal{M})$ if and only if there exists a measurable vector field $Df \in L^p(\mathcal{M}, \mathbb{R}^n)$ such that

$$\int_\mathcal{M} f \,\mathrm{div}\, X(x)\,\mathrm{d}x = -\int_\mathcal{M} X(x)\cdot Df(x)\,\mathrm{d}x \qquad \forall X \in \mathfrak{X}_c(\mathcal{M}),$$

where $\mathfrak{X}_c(\mathcal{M})$ is the class of C^1 vector fields compactly supported in $\mathcal{M}^\circ$. The trace theorem essentially says that it can be possible to test the first variation of f with a larger class of vector fields, up to adding a measure supported on $\partial\mathcal{M}$ (the trace of f), which is exactly the analogous of Theorem 3.3 and Corollary 3.13.

1.2.2 Density set and Haudorff measures

One of the other parallels between the theory of Sobolev functions and the theory of varifolds is the existence of the density depending on the integrability of the "weak derivative" H of V.

The classical result [Sim84, Corollary 17.8], based on monotonicity identity, states that if a k-varifold on $\mathcal{M}$ has generalized mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the

k-density $\Theta^k(\|V\|, x)$ of $\|V\|$ at x exists and is finite *for every* $x \in \mathcal{M}$. Here the k-density is defined as

$$\Theta^k(\|V\|, x) = \lim_{r \to 0} \frac{\|V\|\big(B_r(x)\big)}{r^k}.$$

On the other hand, if H has lower integrability or if V has simply bounded first variation, then the classical theory states that $\Theta^k(\|V\|, x)$ exists and is finite $\|V\|$-a.e. , see [Sim84, Lemma 40.5].

This is similar to the classification of the set $\mathrm{Leb}(f)$ of Lebesgue points for $f \in W^{1,p}(\mathcal{M})$. Indeed, if $p > n$, then $\mathrm{Leb}(f) = \mathcal{M}$ and the density of the function exists for every $x \in \mathcal{M}$ (in fact it turns out that f has a Hölder representative); on the other hand, by Lebesgue differentiation theorem, in general one can say that $\mathcal{L}^n$-a.e. $x \in \mathcal{M}$ belongs to $\mathrm{Leb}(f)$ and the density of f exists $\mathcal{L}^n$-a.e. .

In the case of Sobolev function with lower integrability assumptions on the gradient, there exists a more precise estimate on the size of $\mathcal{M} \setminus \mathrm{Leb}(f)$ given by Federer and Ziemer in [FZ72]: if $f \in W^{1,p}(\mathcal{M})$ for some $p \in [1, n]$, then

$$\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Leb}(f)\big) = 0 \qquad \forall s > n - p,$$

that is $\mathcal{M} \setminus \mathrm{Leb}(f)$ has Hausdorff dimension at most $n - p$. This in some sense fills the gap between the case $p > n$ and the full Lebesgue measure of $\mathrm{Leb}(f)$, which holds also without requiring any differentiability property of f, that is even for $f \in L^1(\mathcal{M})$. It is natural to ask if a similar "bridge" can be build also for the theory of varifolds.

Question 1.2. *Let V be a k-varifold in $\mathcal{M}$ with free boundary at $\partial\mathcal{M}$ with generalized mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in [1, k]$. What can be said about the Hausdorff dimension of the set where $\Theta^k(\|V\|, \cdot)$ is infinite or does not exist?*

The answer is given by Theorem 3.5, where we show that there exists a set $\mathrm{Dens}(V)$ such that

$$\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0 \qquad \forall s > k - p,$$

where the density exists and is finite, which is the analogue of the result of Federer and Ziemer.

1.2.3 Rectifiability of the free boundary

If a smooth k-surface Σ intersects $\partial\mathcal{M}$ orthogonally, then Sard's theorem yields that $\partial\Sigma$ is a regular $(k-1)$-submanifold. Looking at (1.4), this means that the first variation δV_Σ of V_Σ is given by

$$\delta V_\Sigma = H\|V_\Sigma\| + \eta\mathcal{H}^{k-1} \llcorner \partial\Sigma,$$

with $\partial\Sigma$ regular $(k-1)$-submanifold. In other words, the part of the first variation which is singular with respect to $\|V_\Sigma\| = \mathcal{H}^k \llcorner \Sigma$ is given by $\mathcal{H}^{k-1} \llcorner \partial\Sigma$, where $\partial\Sigma$ is a regular $(k-1)$-submanifold. Since a varifold with free boundary at $\partial\mathcal{M}$ meets $\partial\mathcal{M}$ orthogonally in a weak sense, it seems natural to study the following problem.

Question 1.3. *Let V be a k-varifold with free boundary at $\partial\mathcal{M}$ and let σ_V be the Radon measure such that $\delta V = (H + \tilde{H})\|V\| + N\sigma_V$ as stated by Corollary 3.13. Under what conditions on V we can say that σ_V is singular with respect to $\|V\|$? Under what conditions on V it can be said that σ_V is $(k-1)$-rectifiable?*

These questions are studied and partially answered in Chapter 4 by Theorem 4.1, which appeared in [DeM21]; there we show that, under suitable conditions, a rectifiability property holds for σ_V.

1.3 Min-max construction of minimal surfaces with fixed contact angle

The second part of the thesis is taken by the work [DD21] in collaboration with Guido De Philippis and is covered by Chapter 5. It debates the existence of minimal surfaces in a given container $\mathcal{M} \subset \mathbb{R}^3$ which meet $\partial\mathcal{M}$ with a fixed contact angle $\theta \in (0, \pi/2)$. This is done via a min-max procedure based on the tecniques introduced by Almgren and Pitts in [Pit14] applied to the *capillarity energy*: for a relatively open set $\Omega \subset \mathcal{M}$ with finite perimeter, it is defined as

$$F_a(\Omega) = \mathcal{H}^2(\partial_i\Omega) + a\mathcal{H}^2(\partial_b\Omega),$$

where $\partial_i\Omega = \partial^*\Omega \cap \mathcal{M}^\circ$, $\partial_b\Omega = \partial^*\Omega \cap \partial\mathcal{M}$ (we recall that $\partial^*\Omega$ denotes the reduced boundary of Ω) and $a = \cos\theta$. The first variation δ_{F_a} of F_a of a smooth Ω under the action of $X \in \mathfrak{X}_t(\mathcal{M})$ (that is a vector field tangent to $\partial\mathcal{M}$ on $\partial\mathcal{M}$) is given by the formula

$$\delta_{F_a}\Omega[X] = -\int_{\partial_i\Omega} X \cdot H \, \mathrm{d}\mathcal{H}^2 + \int_{\gamma(\Omega)} \left(\eta_{\partial_i\Omega} + a\,\eta_{\partial_b\Omega}\right) \cdot X \, \mathrm{d}\mathcal{H}^1,$$

where H is the mean curvature vector of $\partial_i\Omega$, $\gamma(\Omega)$ is the common boundary of $\partial_i\Omega$ and $\partial_b\Omega$ and $\eta_{\partial_i\Omega}$, $\eta_{\partial_b\Omega}$ are respectively the exterior unit conormal vectors of $\partial_i\Omega$ and of $\partial_b\Omega$.

The key observation here is that a smooth stationary point Ω of F_a (that is a open set Ω such that $\delta_{F_a}\Omega[X] = 0$ for every $X \in \mathfrak{X}_t(\mathcal{M})$) has the property that $\partial_i\Omega$ is a minimal surface in $\mathcal{M}$ (that is it has vanishing mean curvature H) which meets $\partial\mathcal{M}$ with a fixed angle θ; i.e. that its exterior unit conormal satisfies

$$\eta_{\partial_i\Omega} \cdot N \equiv \sin\theta,$$

where N is the exterior unit normal to the container $\mathcal{M}$. Thus, in order to obtain a minimal surface in $\mathcal{M}$ which meets $\partial\mathcal{M}$ with a fixed angle θ one can try to find a relatively open set with regular boundary that is stationary for F_a with respect to vector fields in $\mathfrak{X}_t(\mathcal{M})$.

As usually happens in this kind of variational problems, one has first to look for a solution in a larger class of competitors. The choice clearly depends on the particular problem and on the used techniques. When dealing with the minimization of a functional involving the perimeter of a open set, the framework of sets of finite perimeter is the ideal space to find a solution, since on a minimizing sequence the perimeter is bounded and it is lower semicontinuous under weak convergence of open sets. Unfortunately, in this case the problem cannot be solved in this way, simply because any minimizing sequence for F_a of open subsets of $\mathcal{M}$ converges to the empty set. Hence we have to rely on a different strategy in order to solve the problem. The use of min-max techniques for this kind of variational problem has a long story and goes back to the work of Birkhoff [Bir17] which used min-max to prove existence of closed geodesics on any 2-Riemanniam manifold which is topologically a 2-sphere. Almgren and Pitts extended in the 80's this approach to the case of area functional, strongly using tools of Geometric Measure Theory, [Pit14]. More precisely they proved existence and regularity of minimal hypersurfaces in closed (that is compact without boundary) Riemannian n-manifolds, for $n \leq 6$. Since their work, several versions of their techniques have been adapted to different problems and they allowed for the solutions of a series of long standing problems in Geometric Analysis; without the aim of being exhaustive we can summarize some of the history as follows:

- In 1981, Schoen and Simon extended in [SS81] the min-max approach to closed Riemannian n-manifolds, for any dimension $n \in \mathbb{N}$; as one can expect by the well-known regularity theory for surfaces which minimize the area, for $n \geq 8$, one has to allow the existence of a singular set of Hausdorff dimension at most $n - 8$ on the obtained hypersurface;
- In 2003, Colding and De Lellis simplified in [CD03] the original approach of Pitts in the case of closed Riemannian 3-manifolds;

- In 2013, De Lellis and Ramic extended in [DT13] the simplified version of min-max approach to closed Riemannian n-manifolds for any dimension $n \in \mathbb{N}$;
- In 2014, Marques and Neves applied in [MN14] the min-max theory to prove the longstanding Willmore conjecture;
- In 2017, Marques and Neves used in [MN17] the min-max machinery to prove existence of infinitely many minimal hypersurfaces in closed Riemannian n-manifolds with positive Ricci curvature and $3 \leq n \leq 7$, proving the validity of Yau's conjecture in this particular case; afterwards, Irie, Marques and Neves showed in [IMN18] the Yau's conjecture in the case of a generic metric and next Song proved in [Son18] existence of infinitely many minimal hypersurfaces in Riemannian manifolds without any conditions on the metric;
- In 2018, De Lellis and Ramic applied in [DR18] the min-max techniques to the case of compact Riemannian n-manifolds $\mathcal{M}$ with boundary, dealing with both the Plateau problem and the free boundary problem for the area functional; in the latter, they proved existence and regularity of minimal hypersurfaces which meet $\partial\mathcal{M}$ orthogonally;
- In 2019 [ZZ19] used min-max to prove existence of hypersurfaces in Riemannian manifolds with constant mean curvature.

The main result of this second part of the thesis concerns existence and regularity of 2-dimensional surfaces meeting the boundary of a convex container $\mathcal{M} \subset \mathbb{R}^3$ at a prescribed angle $\theta \in (0, \pi/2)$, see Theorem 5.1 below.

Let us also point out that during the completion of this thesis Chao Li, X. Zhou and J.J. Zhu [LZZ21] obtained independently the same result with related techniques.

As explained above one of the key difficulties in this setting is that the capillarity functional is naturally defined on boundary of sets, while the natural framework for setting the min-max procedure is the class of varifolds.

In order to overcome this difficulty, the key idea is to define the capillarity functional F_a on *pairs of* 2*-surfaces* instead of open sets: we perform the min-max procedure on pairs (Σ, Γ) where $\Sigma \subset \mathcal{M}$ and $\Gamma \subset \partial\mathcal{M}$; in other words here Σ plays the role of $\partial_i\Omega$ and Γ plays the role of $\partial_b\Omega$ and this is actually true when we choose the min-max sequence of competitors, but the two components Σ, Γ are now somehow treated as independent.

The limit of a min-max sequence of pairs of surfaces is a pair of varifolds (V, W), where V is a 2-varifold in $\mathcal{M}$ and W is a 2-varifold in $\partial\mathcal{M}$. Since the definition of F_a can be easily extended to such pairs of varifols, we are able to show that the limits of particular min-max sequences are pairs of varifolds (V, W) which are stationary for F_a; This means that V is stationary in $\mathcal{M}^o$ and satisfies a contact angle condition at $\partial\mathcal{M}$ which is a slight variant of the one proposed by Kagaya and Tonegawa in [KT17]. Thus this V is the candidate to be the minimal surface which meets $\partial\mathcal{M}$ with the fixed angle.

Once we get a weak solution, we have to prove its regularity. Since a limit of a min-max sequence is just stationary, the regularity theory is harder than the one for minimizing or stable objects. The idea to prove regularity, which dates back to the work of Pitts, is to show that, even if the limit is not globally stable, it enjoys some stability property when we deform it *locally*. This sort of stability (called *almost minimizing* property) allows us to use the regularity theory for stable minimal surfaces, thanks to which we can conclude the proof of regularity and it is in this step that we need to restrict to 2-dimensional minimal surfaces, since the desired regualrity theory is not yet known in higher dimension (see however [LZZ21, Theorem C.1] where good estimates in higher dimensions are obtained for a certain range of contact angles). The final step is to remove possible point singularities which are left in the procedure. In order to do so we exploit the peculiarity of the contact angle condition which allows to show that each connected component of the surface is, locally, a graph, together with some classical regularity theory for free boundary problems.

1.4 Plan of the work

We just sketch the principal results of each chapter and we refer to the introductions of the chapters for more detailed accounts.

- In Chapter 2 we define all the tools which are needed in the following;
    - In section 2.1 we set all the notations and definitions used throughout the thesis;
    - In section 2.2 we define the contact angle conditions for surfaces and varifolds and we analyze their basic properties;
- In Chapter 3 we study the boundedness of the first variation of a varifold with contact angle conditions, that is we answer to Question 1.1 and Question 1.2 essentially by proving Theorem 3.3 and Theorem 3.5.
- In Chapter 4 we study the rectifiability properties of the free boundary of a varifold, partially answering to Question 1.3 by Theorem 4.1;
- In Chapter 5 we prove Theorem 5.1 by showing existence and regularity of a minimal surface $\Sigma \subset \mathcal{M}$ which meets $\partial\mathcal{M}$ with a fixed angle $\theta \in (0, \pi/2)$.

Ringraziamenti

Un percorso come quello del dottorato coinvolge talmente tanti aspetti di crescita e cambiamento che diventa impossibile tracciarne i confini. Sono soprattutto le persone che ci accompagnano e che incontriamo a determinarne gli esiti. Diventa quindi ugualmente impossibile ringraziare tutti coloro che hanno partecipato a questo viaggio così ricco di esperienze e di evoluzioni.
Ciononostante, desidero provare a menzionare alcune delle figure il cui ruolo è stato sicuramente fondamentale in questi anni.

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Chapter 2

Preliminary facts

2.1 Basic notations and definitions

2.1.1 Euclidean space, submanifolds

We work in an euclidean space $\mathbb{R}^n$ for some positive integer $n \geq 3$. For a fixed orthonormal system of coordinates, we denote by e_i the i-th coordinate unit vector. If $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the closed ball with center x and radius r and by ω_k the Lebesgue measure of the unit ball in $\mathbb{R}^k$. If the center of the ball is the origin we do not write it, that is we set $B_r := B_r(0)$. If $v \in \mathbb{R}^n$ we call L_v the translation

$$L_v : x \mapsto x + v.$$

For each $A \subset \mathbb{R}^n$, we denote by $\mathbf{1}_A$ the indicator function of A, by $\overline{A}$ and A^{o} respectively the closure and the interior of A in the euclidean topology. If $A \subset \mathbb{R}^n$, and $r > 0$ we write $U_r(A)$ for the tubular neighborhood of A, that is

$$U_r(A) = \bigcup_{x \in A} B_r(x)^{\mathrm{o}}.$$

We denote by S a generic k-dimensional linear subspace (or k-plane) of $\mathbb{R}^n$ and we write $S^\perp$ for the orthogonal complement of S in $\mathbb{R}^n$. We denote by P_S the orthogonal projection on S. If X is a C^1 vector field, we call $\operatorname{div}_S X$ the scalar product $P_S \cdot DX$. If $\tau_1, \dots, \tau_k$ is an orthonormal basis of S, by simple computations one has

$$\operatorname{div}_S X(x) = \sum_{i=1}^{k} \langle D_{\tau_i} X(x), \tau_i \rangle. \tag{2.1}$$

Throughout the paper, $\gamma \in C^\infty\big([0,\infty)\big)$ denotes a cut-off function such that

- $\gamma(t) = 1$ for each $t \in [0, \frac{1}{2}]$;
- $\gamma(t) = 0$ for each $t \geq 1$;
- $\gamma'(t) \leq 0$ and $|\gamma'(t)| \leq 3$ for every $t \in \mathbb{R}$.

For each $r > 0$ and $x \in \mathbb{R}^n$ we consider the dilation map

$$\tau_{x,r}(y) = \frac{1}{r}(y - x). \tag{2.2}$$

If Γ is a C^1 k-dimensional submanifold in $\mathbb{R}^n$ and $x \in \Gamma$, we write $T_x\Gamma$ for the tangent space to Γ at x. We see $T_x\Gamma$ as the blow-up of Γ at the point x. If Γ has non-empty boundary $\partial\Gamma$ of class C^1 and if $x \in \partial\Gamma$, we see $T_x\Gamma$ as containing $T_x\partial\Gamma$, which divides $T_x\Gamma$ into two half-spaces. We call these two parts $T_x^+\Gamma$ and $T_x^-\Gamma$ and we set $T_x^+\Gamma$ to be the blow-up at x of the interior part of Γ.

2.1.2 Ambient domain $\mathcal{M}$

We work on a compact domain $\mathcal{M} \subset \mathbb{R}^n$ with $C^{2,\alpha}$ boundary $\partial\mathcal{M}$. We write $N(x)$ for the exterior unit normal vector to $\partial\mathcal{M}$ at x. In the following, d denotes the signed distance function from $\partial\mathcal{M}$ such that $d > 0$ in $\mathcal{M}^{\mathrm{o}}$, that is

$$d(x) = \begin{cases} \inf\{|x-y| \mid y \in \partial\mathcal{M}\} & \text{if } x \in \mathcal{M} \\ -\inf\{|x-y| \mid y \in \partial\mathcal{M}\} & \text{if } x \in \mathbb{R}^n \setminus \mathcal{M}. \end{cases} \tag{2.3}$$

Unless otherwise specified, we denote by $R = R(\mathcal{M}) > 0$ a number such that d is C^2 in $\overline{U_R(\partial\mathcal{M})}$; such a R exists by [GT01, Lemma 14.16]. Thus ∇d exists in $\overline{U_R(\partial\mathcal{M})}$ and points inside $\mathcal{M}$.

We work with several classes of vector fields on $\mathcal{M}$, which we denote with the letter $\mathfrak{X}$ with subscripts based on their behavior on $\partial\mathcal{M}$:

$$\mathfrak{X}(\mathcal{M}) = C^1(\mathcal{M},\mathbb{R}^n), \qquad \mathfrak{X}_t(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) \in T_x\partial\mathcal{M},\ \forall x \in \partial\mathcal{M}\},$$
$$\mathfrak{X}_\perp(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) \in (T_x\partial\mathcal{M})^\perp,\ \forall x \in \partial\mathcal{M}\},$$
$$\mathfrak{X}_0(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) = 0,\ \forall x \in \partial\mathcal{M}\}, \quad \mathfrak{X}_c(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid \operatorname{supp} X \subset\subset \mathcal{M}^{\mathrm{o}}\}.$$

If Γ is a C^2 submanifold of $\mathbb{R}^n$, by slight abuse of notation we write $\mathfrak{X}_t(\Gamma)$ (respectively $\mathfrak{X}_0(\Gamma)$) for the set of compactly supported C^1 vector fields on $\mathbb{R}^n$ that are tangent to Γ (respectively that vanish on Γ).

2.1.3 Measures, rectifiable sets

If $A \subset \mathbb{R}^n$ is open, $\mathscr{M}(A,\mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued Radon measures on A and $\mathscr{M}^+(A)$ is the space of positive Radon measures on A. If $\mu \in \mathscr{M}(A,\mathbb{R}^m)$ we denote by $|\mu|$ the total variation measure of μ. If $B \subset \mathbb{R}^n$ is Borel, we write $\mu \llcorner B$ for the restriction of the measure μ to B. We endow $\mathscr{M}(A,\mathbb{R}^m)$ with the weak*-topology: i.e. we say that a sequence of Radon measures $\{\mu_j\}_j$ converges to μ ($\mu_j \overset{*}{\rightharpoonup} \mu$) if

$$\lim_{j\to\infty} \int f \,\mathrm{d}\mu_j = \int f \,\mathrm{d}\mu \qquad \forall f \in C_c(A,\mathbb{R}^m).$$

If $C \subset \mathbb{R}^n$ is closed, we define the space $\mathscr{M}(C,\mathbb{R}^m)$ of Radon measures on C as

$$\mathscr{M}(C,\mathbb{R}^m) = \{\mu \in \mathscr{M}(\mathbb{R}^n,\mathbb{R}^m) \mid \operatorname{supp}\mu \subset C\}.$$

We say that a sequence $\{\mu_j\}_j \subset \mathscr{M}(C,\mathbb{R}^m)$ converges to $\mu \in \mathscr{M}(C,\mathbb{R}^m)$ if $\mu_j \overset{*}{\rightharpoonup} \mu$ as Radon measures in $\mathbb{R}^n$.

If $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, we define the upper and the lower k-densities of μ at x:

$$\Theta^{*k}(\mu,x) = \limsup_{r\to 0} \frac{\mu\big(B_r(x)\big)}{r^k}, \qquad \Theta^k_*(\mu,x) = \liminf_{r\to 0} \frac{\mu\big(B_r(x)\big)}{r^k}.$$

If the above limits coincide, then we define the k-density of μ at x as their common value, which we denote by $\Theta^k(\mu,x)$.

If $\mu \in \mathscr{M}(A,\mathbb{R}^m)$ and $f\colon A \to \mathbb{R}^N$ is continuous and *proper*[1], we define the push-forward $f_\#\mu$ of μ through f as the Radon measure in $\mathscr{M}(\mathbb{R}^N,\mathbb{R}^m)$ defined by

$$f_\#\mu(B) = \mu\big(f^{-1}(B)\big) \qquad \forall B \subset \mathbb{R}^N \text{ Borel}.$$

---

[1] $f\colon A \to \mathbb{R}^N$ is said to be proper if for every compact set $K \subset \mathbb{R}^N$, $f^{-1}(K)$ is compact.

If $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, we say that ν is a k-blow-up of μ at x or a k-tangent measure to μ in x if there exists a sequence $r_j \downarrow 0$ such that

$$\mu_j := \frac{1}{r_j^k}\big(\tau_{x,r_j}\big)_{\#}\mu \overset{*}{\rightharpoonup} \nu. \tag{2.4}$$

We denote by $\operatorname{Tan}^k(\mu, x)$ the (possibly empty) set of k-blow-ups of μ at the point x. If $\Theta^{*k}(\mu, x) < \infty$ then, by Banach-Alaoglu Theorem, $\operatorname{Tan}^k(\mu, x)$ is non-empty. If $\Theta^k_*(\mu, x) > 0$, then every k-blow-up of μ at x is non-trivial; indeed, if $\nu \in \operatorname{Tan}^k(\mu, x)$ and $\mu_j \overset{*}{\rightharpoonup} \nu$ as in (2.4), then

$$\nu(B_1) \geq \limsup_j \mu_j(B_1) = \limsup_k \frac{\mu(B_{r_j}(x))}{r_j^k} \geq \Theta^k_*(\mu, x) > 0.$$

For each $s > 0$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and, if $A \subset \mathbb{R}^n$, $\mathcal{H}_{\dim}(A)$ denotes the Hausdorff dimension of A. We say that a Borel set $M \subset \mathbb{R}^n$ is k-rectifiable if there exist $M_0 \subset \mathbb{R}^n$ with $\mathcal{H}^k(M_0) = 0$ and a countable family of C^1 k-submanifolds $\{M_j\}_{j=1}^{\infty}$ such that

$$M \subset M_0 \cup \bigcup_{i=0}^{\infty} M_j.$$

We say that a measure $\mu \in \mathscr{M}^+(\mathbb{R}^n)$ is k-rectifiable if there exist a k-rectifiable set M and a positive function $\theta \in L^1_{\mathrm{loc}}(M, \mathcal{H}^k)$ such that $\mu = \theta \mathcal{H}^k \llcorner M$.

2.1.4 Surfaces

If $\Sigma \subset \mathbb{R}^n$ is a k-surface of class C^2, then it has a well defined its *second fundamental form*, that we denote by A_Σ. Its trace H is the *mean curvature* of Σ. If Σ has sufficiently regular boundary $\partial\Sigma$ we can define its exterior unit conormal.

Definition 2.1 (conormal)**.** Let $\Sigma \subset \mathbb{R}^n$ be a k-surface of class C^1 with boundary $\partial\Sigma$ of class C^1 and let $x \in \partial\Sigma$. The *exterior unit conormal* $\eta_\Sigma(x)$ is defined as the unique unit vector that satisfies

$$\eta_\Sigma(x) \in T_x\Sigma \cap (T_x\partial\Sigma)^\perp$$

and points outward Σ.

If $\Sigma \subset \mathbb{R}^n$ is a k-surface with locally bounded $\mathcal{H}^k$-measure, then we can study the variation of its k-dimensional area under the action of a vector field $X \in \mathfrak{X}(\mathcal{M})$, which is called the *first variation* of the area of Σ under the action of X.

In order to define it, let us consider a vector field $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$; it is associated to X a flow map $\psi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ at the time $t \in (-\varepsilon, \varepsilon)$. It is well-known that $(\psi_t)_{t\in(-\varepsilon,\varepsilon)}$ is a 1-parameter family of diffeomorphisms of $\mathbb{R}^n$ on $\mathbb{R}^n$. Moreover, since X is compactly supported, there exists a ball B_r such that

$$\psi_t(x) = x \qquad \forall t \in (-\varepsilon, \varepsilon),\ \forall x \in \mathbb{R}^n \setminus B_r.$$

Now we can define the first variation of the area of Σ.

Definition 2.2 (first variation of the area)**.** Let $\Sigma \subset \mathbb{R}^n$ be a k-surface with locally bounded $\mathcal{H}^k$-measure, let $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field such that $\operatorname{supp} X \subset B_r$ and let ψ_t be its flow map. The *first variation* of the area of Σ under the action of X is defined as

$$\delta_{\mathcal{H}^k}(\Sigma)[X] = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{H}^k\big(\psi_t(\Sigma) \cap B_r\big)\Big)\Big|_{t=0}.$$

The above quantity is well-defined, as shown by the following result ([Sim84, (9.3)]).

Proposition 2.3. *Let $\Sigma \subset \mathbb{R}^n$ be a k-surface with locally bounded $\mathcal{H}^k$-measure and let $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$; the following formula holds true:*

$$\delta_{\mathcal{H}^k}(\Sigma)[X] = \int_\Sigma \operatorname{div}_{T_x\Sigma} X(x) \, \mathrm{d}\mathcal{H}^k(x).$$

If Σ is assumed to be of class C^2 with boundary $\partial\Sigma$ of class C^1, then the first variation can be written in terms of the mean curvature H of Σ and its unit conormal η_Σ.

Proposition 2.4. *Let $\Sigma \subset \mathbb{R}^n$ be a k-surface of class C^2 with (possibly empty) boundary $\partial\Sigma$ of class C^1 and let $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$; then it holds*

$$\int_\Sigma \operatorname{div}_{T_x\Sigma} X(x) \, \mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma \langle X, H\rangle \, \mathrm{d}\mathcal{H}^k + \int_{\partial\Sigma} \langle X, \eta_\Sigma\rangle \, \mathrm{d}\mathcal{H}^{k-1}, \tag{2.5}$$

where H is the mean curvature vector of Σ and η_Σ is the exterior unit conormal of Σ.

We now particularize these ideas for surfaces contained in the domain $\mathcal{M}$. It is of particular importance the case of a surface $\Sigma \subset \mathcal{M}$ such that the first variation of the area of Σ vanishes for every $X \in \mathfrak{X}_c(\mathcal{M})$; such a Σ is called *minimal*.

Definition 2.5 (Minimal surface)**.** We say that a k-surface $\Sigma \subset \mathcal{M}$ is *minimal* if it satisfies

$$\delta_{\mathcal{H}^k}(\Sigma)[X] = 0 \qquad \forall X \in \mathfrak{X}_c(\mathcal{M}).$$

Remark 2.6. If $\Sigma \subset \mathcal{M}$ is minimal and of class C^2 with boundary $\partial\Sigma$ of class C^1, then by (2.5) it is straightforward to prove that it satisfies:

- $\partial\Sigma \subset \partial\mathcal{M}$;
- $H \equiv 0$ in $\mathcal{M}^\circ$.

2.1.5 Varifolds

If $1 \le k \le n$ we call $G(k,n)$ the Grassmannian of the un-oriented k-dimensional linear subspaces (or k-planes) of $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ we denote by $G_k(A) := A \times G(k,n)$ the trivial Grassmannian bundle over A.

A *k-varifold* on A is a positive Radon measure on $G_k(A)$. We denote by $\mathcal{V}_k(A)$ the set of all k-varifolds on A and we endow $\mathcal{V}_k(A)$ with the topology of the weak*-convergence of Radon measures, i.e. we say that $V_j \overset{*}{\rightharpoonup} V$ if

$$\lim_{j\to\infty} \int_{G_k(A)} \varphi(x,S) \, \mathrm{d}V_j(x,S) = \int_{G_k(A)} \varphi(x,S) \, \mathrm{d}V(x,S) \qquad \forall \varphi \in C_c(G_k(A)).$$

A k-rectifiable measure $\mu = \theta\mathcal{H}^k \llcorner M$ in $\mathbb{R}^n$ induces the k-varifold

$$V = \theta\mathcal{H}^k \llcorner M \otimes \delta_{T_xM},$$

where θ is called the *multiplicity function* and T_xM is the approximate tangent space of M at x. A varifold that is induced by a rectifiable measure is called a *rectifiable varifold*. If the multiplicity function θ assumes only integer values, we say that the varifold is *integer rectifiable*. If V is a k-varifold on Ω, the *mass* $\|V\|$ (or *total variation*) of V is the positive Radon measure defined as

$$\|V\|(A) = V(G_k(A)) \qquad \forall A \subset \Omega \text{ Borel}.$$

If V is the k-varifold induced by the rectifiable measure $\mu = \theta\mathcal{H}^k\llcorner M$, then

$$\|V\|(B) = \int_B \theta(x)\,\mathrm{d}\mathcal{H}^k(x).$$

By slight abuse of notation, we often denote $\operatorname{supp}\|V\|$ by $\operatorname{supp} V$. If $\Omega \subset \mathbb{R}^n$ is a domain, $V \in \mathcal{V}_k(\Omega)$ and if $\psi : \Omega \to \mathbb{R}^n$ is a diffeomorphism, the *push forward* $\psi_\sharp V$ of V through ψ is the varifold in $\mathcal{V}_k\big(\psi(\Omega)\big)$ such that, $\forall \varphi \in C_c\big(G_k(\psi(\Omega))\big)$,

$$\int_{G_k(\psi(\Omega))} \varphi(y,T)\,\mathrm{d}\psi_\sharp V(y,T) = \int_{G_k(\Omega)} J_S\psi(x)\varphi\big(\psi(x), \mathrm{d}\psi_x(S)\big)\,\mathrm{d}V(x,S), \tag{2.6}$$

where $J_S\psi(x)$ is the Jacobian of ψ relative to the k-plane S, i.e.

$$J_S\psi(x) = \sqrt{\det\big((\mathrm{d}\psi_x)^*_{|S} \circ \big((\mathrm{d}\psi_x)_{|S}\big)}.$$

We notice that this *is not* the push forward of measures previously defined (which is denoted by the different symbol $f_\#\mu$). In fact, the push forward of varifolds is defined in this way in order to ensure the validity of the area formula: indeed if V is induced by a rectifiable set M, then $\psi_\sharp V$ is induced by $\psi(M)$. If $V \in \mathcal{V}_k(\mathbb{R}^n)$, we say that $C \in \mathcal{V}_k(\mathbb{R}^n)$ is a *blow-up* of V at x or a *tangent varifold* to V at x if there exists a sequence of radii $r_j \downarrow 0$ such that

$$(\tau_{x,r_j})_\sharp V \overset{*}{\rightharpoonup} C.$$

We write $\operatorname{Tan}(V,x)$ for the set of tangent varifold to V at the point x.

If $V \in \mathcal{V}_k(\mathcal{M})$ and if $X \in \mathfrak{X}(\mathcal{M})$ the *first variation* $\delta V[X]$ of V with respect to X is

$$\delta V[X] = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|(\psi_t)_\sharp V\|(\mathcal{M})\Big)\Big|_{t=0}$$

where ψ_t is the flow map of X at the time t. The following *first variation formula* holds:

$$\delta V[X] = \int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}V(x,S). \tag{2.7}$$

We now define the class of varifolds with *bounded first variation*:

Definition 2.7. We say that a varifold V has *bounded first variation* in $\mathcal{M}$ if

$$\sup\{|\delta V[X]| \mid X \in \mathfrak{X}(\mathcal{M}), \max|X| \le 1\} < +\infty. \tag{2.8}$$

If (2.8) holds with a proper subset of $\mathfrak{X}(\mathcal{M})$(e.g. $\mathfrak{X}_c(\mathcal{M})$, $\mathfrak{X}_0(\mathcal{M})$...) in place of $\mathfrak{X}(\mathcal{M})$, we say that V has bounded first variation with respect to this subset.

Therefore V has bounded first variation if there exists $\delta V \in \mathscr{M}(\mathcal{M},\mathbb{R}^n)$ such that, for any $X \in \mathfrak{X}(\mathcal{M})$,

$$\delta V[X] = \int_{\mathcal{M}} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{\mathcal{M}} \langle X(x), \zeta(x)\rangle\,\mathrm{d}|\delta V|(x),$$

where ζ is the polar vector of δV with respect to $|\delta V|$.

If V has bounded first variation, then by Lebesgue decomposition there exist $|\delta^s V| \in \mathscr{M}^+(\mathcal{M})$, a $|\delta^s V|$-measurable function $\eta_V : \mathcal{M} \to \mathbb{R}^n$ and a $\|V\|$-measurable function $H : \mathcal{M} \to \mathbb{R}^n$ such that

$$\delta V[X] = -\int_{\mathcal{M}} \langle H, X\rangle\ \mathrm{d}\|V\| + \int_{\mathcal{M}} \langle X, \eta_V\rangle\,\mathrm{d}|\delta^s V| \qquad \forall X \in \mathfrak{X}(\mathcal{M}), \tag{2.9}$$

where $|\delta^s V|$ is the singular part of $|\delta V|$ with respect to $\|V\|$ which satisfies the following property, as stated in [Sim84, Theorem 4.7]:

$$|\delta^s V| = |\delta V| \llcorner Z, \qquad Z = \Big\{ x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{|\delta V|(B_r(x))}{\|V\|(B_r(x))} = +\infty \Big\}.$$

Since (2.9) is similar to the corresponding one for smooth surfaces, we call H the *generalized mean curvature* of V, $|\delta^s V|$ the *boundary measure* of V, the set Z is the *boundary* of V and η_V is the *unit co-normal* of V.

We now define the classes of varifolds with *generalized mean curvature*:

Definition 2.8. We say that $V \in \mathcal{V}_k(\mathcal{M})$ has *generalized mean curvature* with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$) if there exists a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ such that $H(x) = 0$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ and for any $X \in \mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$) the following formula holds:

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x) \ \mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle H, X \rangle \ \mathrm{d}\|V\|. \tag{2.10}$$

Remark 2.9. The assumption that $H(x) = 0$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ is important: without this hypothesis, the generalized mean curvature is not uniquely defined, that is any combination $H+K$ with $\operatorname{supp} K \subset \partial\mathcal{M}$ satisfies (2.10). We have to exclude this ambiguity in order to make true the results of Chapter 3, as we shall see later.

Thus V has *generalized mean curvature* with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$) if it has bounded first variation with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$, $\mathfrak{X}_t(\mathcal{M})$) and δV has no singular part with respect to $\|V\|$ when we test with vector fields in $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$).

2.1.6 Capillarity free energy

If $\Omega \subset \mathcal{M} \subset \mathbb{R}^{n+1}$ is open in the relative topology of $\mathcal{M}$ and has finite perimeter, we call $\partial_i\Omega$ and $\partial_b\Omega$ respectively the internal and the external reduced boundary of Ω, that is

$$\partial_i\Omega = \partial^*\Omega \cap \mathcal{M}^{\circ}, \qquad \partial_b\Omega = \partial^*\Omega \cap \partial\mathcal{M}, \tag{2.11}$$

where $\partial^*\Omega$ denotes the reduced boundary of Ω. If $\partial_i\Omega$ and $\partial_b\Omega$ are regular n-submanifolds of $\mathbb{R}^{n+1}$, we denote by $\gamma(\Omega)$ their common boundary (as manifolds).

Definition 2.10 (Capillarity free energy)**.** If $\Omega \subset \mathcal{M}$ is relatively open with finite perimeter and if $a \in [0,1)$, the *a-capillarity energy* F_a applied to Ω is defined as

$$F_a(\Omega) = \mathcal{H}^n(\partial_i\Omega) + a\mathcal{H}^n(\partial_b\Omega). \tag{2.12}$$

If we look at $\partial_i\Omega$ and $\partial_b\Omega$ as rectifiable n-varifolds (the reduced boundary $\partial^*\Omega$ is n-rectifiable), (2.7) shows that the first variation of F_a can be computed by the following formula.

Proposition 2.11. *Let $\Omega \subseteq \mathcal{M}$ be relatively open with finite perimeter and let $a \in [0,1)$. Then*

$$\begin{aligned} \delta_{F_a}\Omega[X] &:= \frac{\mathrm{d}}{\mathrm{d}t}\Big(F_a\big(\psi_t(\Omega)\big)\Big)\Big|_{t=0} \\ &= \int_{\partial_i\Omega} \operatorname{div}_{T_x\partial_i\Omega} X(x) \ \mathrm{d}\mathcal{H}^n(x) + a\int_{\partial_b\Omega} \operatorname{div}_{T_x\partial\mathcal{M}} X(x) \ \mathrm{d}\mathcal{H}^n(x) \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \end{aligned}$$

We now define $\mathscr{O}(\mathcal{M})$ as the class of all subsets of $\mathcal{M}$ which are open in the relative topology of $\mathcal{M}$ and whose internal and external boundary are n-dimensional submanifolds of class C^2 with C^1 boundary, that is

$$\mathscr{O}(\mathcal{M}) = \{\Omega \subset \mathcal{M} \mid \Omega \text{ is open in } \mathcal{M},\ \partial_i(\Omega) \text{ and } \partial_b(\Omega) \text{ are of class } C^2 \text{ with } C^1 \text{ boundary } \gamma(\Omega)\}, \tag{2.13}$$

If $\Omega \in \mathscr{O}(\mathcal{M})$, Propositions 2.11 and 2.4 yield the first variation of F_a with respect to $X \in \mathfrak{X}_t(\mathcal{M})$:

$$\delta_{F_a}\Omega[X] = -\int_{\partial_i\Omega} X \cdot H \,\mathrm{d}\mathcal{H}^n + \int_{\gamma(\Omega)} \big(\eta_{\partial_i\Omega} + a\,\eta_{\partial_b\Omega}\big) \cdot X \,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.14}$$

where H is the mean curvature vector of $\partial_i\Omega$, $\eta_{\partial_i\Omega}$ is the exterior unit conormal vector to $\partial_i\Omega$ and $\eta_{\partial_b\Omega}$ is the exterior unit conormal vector to $\partial_b\Omega$.

In the following we need the following extension of the capillarity functional to the set of pairs of varifolds $\mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$.

Definition 2.12 (Capillarity functional for pairs of varifolds)**.** If $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ and if $a \in [0, 1)$, the *a-capillarity functional* F_a applied to (V, W) is defined as

$$F_a(V, W) = \|V\|(\mathcal{M}) + a\|W\|(\partial\mathcal{M}).$$

Definition 2.13 (First variation of F_a)**.** If $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$, the first variation of F_a with respect a vector field $X \in \mathfrak{X}_t(\mathcal{M})$ is given by

$$\begin{aligned}\delta_{F_a}(V, W)[X] &:= \frac{\mathrm{d}}{\mathrm{d}t}\Big(F_a\big((\psi_t)_\sharp V, (\psi_t)_\sharp W\big)\Big)\Big|_{t=0} \\ &= \int_{G_n(\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}V(x, S) + a\int_{G_n(\partial\mathcal{M})} \operatorname{div}_{T_x\partial\mathcal{M}} X(x)\ \mathrm{d}W(x, T_x\partial\mathcal{M}).\end{aligned}$$

2.2 Contact angle conditions

We are going to describe contact angle conditions for surfaces and varifolds in a compact domain $\mathcal{M}$ of an euclidean space. We define these conditions in a variational way: we first look at the natural definition of contact angle in the case of sufficiently regular surfaces; next we observe that these conditions imply some relations on the first variation of a suitable functional; since no regularity of the surface is actually necessary in order to state these relations, we use them as weak definitions of the corresponding contact angle conditions.

The definition is slightly different from $\frac{\pi}{2}$ to general angle: in the former case one has to consider the area functional associated to the surface Σ; in the latter (well-defined only in the codimension-1 case), one has to assume that Σ is the part of boundary of a relatively open set $\Omega \subset \mathcal{M}$ which lies in $\mathcal{M}^\circ$ (that is $\Sigma = \partial_i\Omega$ as in (2.11)). This is due to the fact that also $\partial_b\Omega$ (that is the boundary part of Ω) appears in the expression of the capillarity energy F_a defined in (2.12), which is the functional whose first variation gives the weak contact angle condition.

In the first part of the discussion we talk about the orthogonal angle condition, while in the second part we deal with general angles.

2.2.1 Surfaces and varifolds with free boundary

We have seen in Proposition 2.4 that the first variation of the area of a k-surface Σ of class C^2 with C^1 boundary $\partial\Sigma$ can be written in terms of the mean curvature H and the unit conormal η_Σ of Σ:

$$\int_\Sigma \operatorname{div}_{T_x\Sigma} X(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k + \int_{\partial\Sigma} \langle X, \eta_\Sigma\rangle\,\mathrm{d}\mathcal{H}^{k-1}. \tag{2.15}$$

Now let us suppose that $\Sigma \subset \mathcal{M}$. We want to use (2.15) to define a orthogonality condition for Σ at $\partial\mathcal{M}$. In order to do this, we notice that a surface Σ which meets $\partial\mathcal{M}$ orthogonally has to satisfy

$$\partial\Sigma \subset \partial\mathcal{M} \qquad \text{and} \qquad \eta_\Sigma(x) = N(x) \quad \forall x \in \partial\Sigma,$$

where $N(x)$ is the exterior unit normal vector to $\partial\mathcal{M}$ at x. Thus, if Σ meets $\partial\mathcal{M}$ orthogonally and we test (2.15) with any vector field $X \in \mathfrak{X}_t(\mathcal{M})$ (that is if X is tangent to $\partial\mathcal{M}$, see (2.1.2)), then the boundary term disappears; in other words, if Σ meets $\partial\mathcal{M}$ orthogonally, it satisfies

$$\int_\Sigma \operatorname{div}_{T_x\Sigma} X(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \tag{2.16}$$

Conversely, if Σ is a k-surface of class C^2 with C^1 boundary $\partial\Sigma$ and Σ satisfies (2.16), then (2.16) implies that $\partial\Sigma \subset \partial\mathcal{M}$ and that the unit conormal η_Σ to $\partial\Sigma$ coincides with the exterior unit normal vector N to $\partial\mathcal{M}$. In conclusion, for a sufficiently regular k-surface $\Sigma \subset \mathcal{M}$, (2.16) is equivalent to the fact that Σ meets $\partial\mathcal{M}$ orthogonally.

By the way, we can see that (2.16) has meaning also if $\partial\Sigma \subset \partial\mathcal{M}$ is not regular, even if $\partial\Sigma$ and η_Σ are not well-defined. Thus it can be used to *define* a weak notion of surfaces which meet $\partial\mathcal{M}$ orthogonally, that is to generalize in a weak sense the idea of regular surfaces that meet $\partial\mathcal{M}$ orthogonally.

Definition 2.14 (Surface with free boundary)**.** We say that a surface $\Sigma \subset \mathcal{M}$ of class C^2 has *free boundary* at $\partial\mathcal{M}$ if it satisfies

$$\int_\Sigma \operatorname{div}_{T_x\Sigma} X(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \tag{2.17}$$

As we can see, (2.17) is similar to the formula for varifolds with generalized mean curvature in Definition 2.8. This suggests that a notion of orthogonality condition for k-varifolds can be defined in a way which is similar to the above one for surfaces.

Definition 2.15 (Varifold with free boundary)**.** We say that $V \in \mathcal{V}_k(\mathcal{M})$ has *free boundary* at $\partial\mathcal{M}$ if there exists a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ which is tangent to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ and such that

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}V(x,S) = -\int_\mathcal{M} \langle H, X\rangle\ \mathrm{d}\|V\|. \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}). \tag{2.18}$$

Remark 2.16. As in Remark 2.9, the assumption that H is tangent to $\partial\mathcal{M}$ $\|V\|$-a.e. is important because otherwise H is not uniquely defined: every sum $H + K$, with K orthogonal to $\partial\mathcal{M}$, satisfies (2.18) for any $X \in \mathfrak{X}_t(\mathcal{M})$, whereas when we test with non-tangent vector fields the presence of K is relevant. In the following we need to test the first variation of a varifold with free boundary also with vector fields which are not tangent to $\partial\mathcal{M}$ to study the first variation of V and to derive for istance (3.28), which is no longer true if H is not tangent to $\partial\mathcal{M}$.

By comparing Definition 2.14 and Definition 2.15, we see that a varifold with free bondary meets $\partial\mathcal{M}$ orthogonally in a weak sense.

2.2.2 General angle

In this section we define a notion of contact angle at the boundary for n-varifolds in $\mathcal{M} \subset \mathbb{R}^{n+1}$; in order to justify the definitions, we first look at the case of smooth surfaces. Throughout this section, we assume that $\mathcal{M} \subset \mathbb{R}^{n+1}$.

Remark 2.17. We notice that our weak definition of contact angle works only in the codimension-1 case, that is for n-varifolds in $\mathcal{M} \subset \mathbb{R}^{n+1}$, whereas Definition 2.15 of k-varifold with free boundary is valid for every $1 \leq k \leq n+1$.

If $\Sigma \subset \mathcal{M}$ is regular with regular boundary $\partial\Sigma \subset \partial\mathcal{M}$, the contact angle at $\partial\mathcal{M}$ can be defined using the exterior unit conormal η_Σ to $\partial\Sigma$.

Definition 2.18. Let $\Sigma \subset \mathcal{M}$ be a surface of class C^1 with boundary $\partial\Sigma \subset \partial\mathcal{M}$ of class C^1 and let $\theta \in (0, \pi/2]$. Σ is said to have *contact angle* θ at $\partial\mathcal{M}$ if

$$N(x) \cdot \eta_\Sigma = \sin\theta \qquad \forall x \in \partial\Sigma.$$

This definition works for every codimension, but requires the regularity of Σ and $\partial\Sigma$, since it relies on the the existence of the conormal η_Σ.

In order to obtain a definition of contact angle in a weak sense for hypersurfaces, we use the capillarity energy for open sets defined in (2.12). This definition works only for n-varifolds, that is when the codimension is 1.

Proposition 2.19. *Suppose* $\theta \in (0, \pi/2)$*,* $a = \cos\theta$ *and let us assume* $\Omega \in \mathscr{O}(\mathcal{M})$*, where* $\mathscr{O}(\mathcal{M})$ *is defined in* (2.13)*, with internal boundary* $\partial_i\Omega$*, external boundary* $\partial_b\Omega$ *and* $\gamma(\Omega)$ *as their common boundary. If*

$$\delta_{F_a}\Omega[X] = -\int_{\partial_i\Omega} X \cdot H \, \mathrm{d}\mathcal{H}^n \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}), \tag{2.19}$$

where H *is the mean curvature vector of* $\partial_i\Omega$*, then* $\partial_i\Omega$ *meets* $\partial\mathcal{M}$ *with angle* θ*.*

Proof. For $\Omega \in \mathscr{O}(\mathcal{M})$, by comparing with (2.14), (2.19) implies that

$$\eta_{\partial_i\Omega} + a\,\eta_{\partial_b\Omega} \perp T_x\partial\mathcal{M} \qquad \forall x \in \gamma(\Omega).$$

This implies that

$$P_{T_x\partial\mathcal{M}}\big(\eta_{\partial_i\Omega}(x)\big) = -a\,\eta_{\partial_b\Omega}(x) \qquad \forall x \in \gamma(\Omega),$$

which implies that $N(x) \cdot \eta_{\partial_i\Omega} = \sin\theta$ for every $x \in \gamma(\Omega)$, that is $\partial_i\Omega$ meets $\partial\mathcal{M}$ with angle θ. ☐

Since (2.19) has meaning also if Ω has just finite perimeter, where $\partial_i\Omega$ and $\partial_b\Omega$ are n-rectifiable varifolds, Proposition 2.19 suggests the following generalization of the contact angle condition in a weak sense for codimension n-varifolds in $\mathcal{M} \subset \mathbb{R}^{n+1}$.

Definition 2.20 (Contact angle condition for pairs of varifolds)**.** Let $\mathcal{M} \subset \mathbb{R}^{n+1}$, let $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ and suppose $\theta \in (0, \pi/2]$. We say that V *meets* $\partial\mathcal{M}$ *with contact angle* θ if there exists a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ that is tangent to $\partial\mathcal{M}$ at $\|V\|$-a.e. $x \in \partial\mathcal{M}$ and is such that, for every $X \in \mathfrak{X}_t(\mathcal{M})$, it holds

$$\begin{aligned}\delta_{F_a}(V, W)[X] &= \int_{G_n(\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}V(x, S) + a \int_{G_n(\partial\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}W(x, S) \\ &= -\int X \cdot H \, \mathrm{d}\|V\|,\end{aligned}$$

where $a = \cos\theta$. We say that (V, W) is *stationary* for F_a if it meets $\partial\mathcal{M}$ with angle θ with $H \equiv 0$, that is if

$$\delta_{F_a}(V, W)[X] = 0 \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}).$$

Remark 2.21. The vector field H is assumed to be tangent to $\partial\mathcal{M}$ on $\partial\mathcal{M}$ for the same reasons explained in Remark 2.16.

Remark 2.22. Definition 2.20 is a slight generalization of the one given by Kagaya and Tonegawa in [KT17]. The present definition differs from the other for the fact that we allow the boundary part W to be a n-varifold on $\partial\mathcal{M}$, that is a positive Radon measure on $\partial\mathcal{M}$, whereas the one given in [KT17] prescribes that the boundary part is a subset of $\partial\mathcal{M}$. We need our generalization in order to have compactness properties which are not satisfied by sets.

In order to define the class of stable pairs for F_a, we define the admissible variations for stability.

Definition 2.23 (Admissible variations)**.** A family $\psi_t \colon (-\varepsilon,\varepsilon)\times\mathcal{M}\to\mathcal{M}$ of maps is said to be an admissible variation if it satisfies the following conditions:

1. there exists a compact set $K\subseteq\mathcal{M}$ such that

$$\psi_t(x)=x \qquad \forall x\in\mathcal{M}\setminus K,\ \forall t\in(-\varepsilon,\varepsilon)$$

2. $\psi_t(\partial\mathcal{M})\subset\partial\mathcal{M}$ for every $t\in(-\varepsilon,\varepsilon)$;
3. $\psi_0=\mathrm{Id}_{\mathcal{M}}$;
4. there exists $X\in\mathfrak{X}_t(\mathcal{M})$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x)\big|_{t=0}=X(x)$ for any $x\in\mathcal{M}$.

In particular, the flow maps of vector fields in $\mathfrak{X}_t(\mathcal{M})$ are admissible. We now give the definition of stability for pairs with respect to F_a.

Definition 2.24 (Stable pairs)**.** If $\theta\in[0,\frac{\pi}{2})$ and $a=\cos\theta$, we say that the pair $(V,W)\in\mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ is *stable* for F_a if it satisfies

$$\delta^2_{F_a}(V,W)[X]\coloneqq\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(F_a\big((\psi_t)_\sharp V,(\psi_t)_\sharp W\big)\Big)\bigg|_{t=0}\geq 0 \qquad \forall\psi_t \text{ admissible,} \tag{2.20}$$

where $\delta^2_{F_a}$ is the second variation of F_a. If $U\subset\mathcal{M}$ is relatively open in $\mathcal{M}$, we say that (V,W) is stable in U if it satisfies (2.20) for every 1-parameter family of admissible variations such that $\psi_t(x)=x$ for every $x\notin U$.

We refer to Section 3.6 for the statements of some boundedness and monotonicity results that that are needed in Chapter 5.

Part I

Varifolds with contact angle conditions

Chapter 3

Bounded variation and consequences

3.1 Introduction

3.1.1 Motivations

In this chapter we study the phenomenon for which a varifold $V \in \mathcal{V}_k(\mathcal{M})$ that has generalized mean curvature with respect to a class of vector fields with some condition at $\partial\mathcal{M}$, has in fact bounded first variation with respect to a larger class of vector fields and this first variation is obtained by adding to $H\|V\|$ an extra term supported on $\partial\mathcal{M}$.

This fact can be seen as a sort of trace theorem for Sobolev functions: if f has an L^1 function as weak derivative in the interior of $\mathcal{M}$, then we can test the first variation of f with a larger class of vector fields which do not vanish on $\partial\mathcal{M}$, by adding a measure supported on $\partial\mathcal{M}$. In order to further motivate the main result, let us consider the following example.

If $\Sigma \subset \mathcal{M}$ is a smooth k-surface with smooth boundary $\partial\Sigma$ and which meets $\partial\mathcal{M}$ orthogonally, we can obtain an a-priori bound on the $(k-1)$-dimensional measure of $\partial\Sigma$ by testing the first variation formula

$$\int_\Sigma \operatorname{div}_{T_x\Sigma} X(x)\,\mathrm{d}\mathcal{H}^k(x) = -\int_\Sigma \langle X, H\rangle\,\mathrm{d}\mathcal{H}^k + \int_{\partial\Sigma} \langle X, N\rangle\,\mathrm{d}\mathcal{H}^{k-1} \tag{3.1}$$

with a smooth vector field X such that $X(x) = N(x)$ on $\partial\mathcal{M}$; moreover we can choose X such that its C^1-norm of X is bounded from above (up to a constant) by the C^2-norm of $\partial\mathcal{M}$, that is by $R(\mathcal{M})^{-1}$, where $R(\mathcal{M})$ is the minimum radius of curvature of $\partial\mathcal{M}$. This yields the estimate

$$\mathcal{H}^{k-1}(\partial\Sigma) \le c\frac{\mathcal{H}^k(\Sigma)}{R(\mathcal{M})} + \int_\Sigma |H|\,\mathrm{d}\mathcal{H}^k,$$

where $c = c(k,\mathcal{M})$. This bound can be easily localized to any ball $B_r(x)$ where $x \in \partial\mathcal{M}$: if we cut-off X with a function $\varphi \in C_c^\infty\big(B_r(x)\big)$ such that $\varphi(y) = 1$ for $y \in B_{r/2}(x)$, then we get

$$\mathcal{H}^{k-1}\big(\partial\Sigma \cap B_{r/2}(x)\big) \le c\frac{\mathcal{H}^k\big(\Sigma \cap B_r(x)\big)}{r} + \int_{\Sigma\cap B_r(x)} |H|\,\mathrm{d}\mathcal{H}^k,$$

where again $c = c(k,\mathcal{M})$.

Remark 3.1. In the particular case where $\mathcal{M} = B_1$ and $\Sigma \subset B_1$ is a minimal k-surface that meets ∂B_1 orthogonally, we can choose $X = x$ as test vector field in (3.1); since for any k-plane $S \in G(k,n)$ it holds $\operatorname{div}_S X = k$, we obtain the nice identity

$$\mathcal{H}^{k-1}(\partial\Sigma) = k\mathcal{H}^k(\Sigma).$$

The simple proofs of these a-priori bounds strongly rely on the fact that Σ and $\partial\Sigma$ are assumed to be smooth (which in particular implies that the first variation of Σ is bounded) and on the assumption that the conormal η_Σ of $\partial\Sigma$ points outside $\mathcal{M}$. It is natural to ask if similar estimates hold also for a general varifold with free boundary V.

Question 3.2. *Is it true that a varifold V with free boundary at $\partial\mathcal{M}$ has bounded first variation? In this case, is it true that its unit conormal on $\partial\mathcal{M}$ is orthogonal to $\partial\mathcal{M}$ and points outside $\mathcal{M}$?*

3.1.2 Main result

Some of above questions have already been partially studied by Grüter and Jost in [GJ86a, 4.11(ii)], where they prove that a varifold with free boundary with $\|V\|(\partial\mathcal{M}) = 0$ has bounded first variation δV; More recently, Edelen proved the same result in [Ede18, Proposition 3.2] removing the hypothesis that $\|V\|(\partial\mathcal{M}) = 0$, but assuming that V is rectifiable.

In [DeM21] the author refines these boundedness results, extending them to general varifolds and removing the assumption $\|V\|(\partial\mathcal{M}) = 0$. We state it in a slightly more general setting: if V has generalized mean curvature with respect to vector fields that vanish on $\partial\mathcal{M}$, then it has bounded first variation with respect to vector fields that are orthogonal to $\partial\mathcal{M}$. As we shall see later, this implies a varifold V with free boundary at $\partial\mathcal{M}$ has bounded first variation.

Theorem 3.3. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a k-varifold with generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$, with $H \in L^1(\mathcal{M}, \|V\|)$. Then there exist a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial\mathcal{M}$ such that, for any $X \in \mathfrak{X}_\perp(\mathcal{M})$, it holds*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\, \mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H}\rangle \,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle \,\mathrm{d}\sigma_V \tag{3.2}$$

where $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends only on the second fundamental form of $\partial\mathcal{M}$. In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$. Moreover, the following global and local estimates hold:

$$\sigma_V(\partial\mathcal{M}) \le c\|V\|(\mathcal{M}) + \int_{\mathcal{M}} |H| \,\mathrm{d}\|V\|; \tag{3.3}$$

$$\sigma_V\big(B_{r/2}(x_0)\big) \le \frac{c}{r}\|V\|\big(B_r(x_0)\big) + \int_{B_r(x_0)} |H| \,\mathrm{d}\|V\| \qquad \forall x_0 \in \partial\mathcal{M},\ \forall r \le R(\mathcal{M}), \tag{3.4}$$

where the constant $c = c(\mathcal{M})$ depends only on the second fundamental form of $\partial\mathcal{M}$ and $R(\mathcal{M})$ is such that the signed distance function from $\partial\mathcal{M}$ is of class C^2 in $U_R(\partial\mathcal{M})$.

Let us comment the result: when we test with vector fields which are orthogonal to $\partial\mathcal{M}$, we detect two new components of the first variation of V:

- An absolutely continuous part with respect to $\|V\|$ given by $\tilde{H}\|V\|$, which takes into account the fact that V can "lean tangentially" on $\partial\mathcal{M}$. To better understand this fact, let us assume that V is induced by a smooth surface: then the mean curvature of Σ is orthogonal to $\partial\mathcal{M}$ on the set of tangential contact; this part of the mean curvature, which is not seen when testing with vector fields which vanish on $\partial\mathcal{M}$, is relevant if we test with vector fields orthogonal to $\partial\mathcal{M}$. Moreover, we expect that for $\mathcal{H}^k$-a.e. point on this contact set, the mean curvature of Σ is bounded by the second fundamental form of $\partial\mathcal{M}$. Theorem 3.3 states that these properties are valid also for a general varifold, see (3.18).

- A part given by $N\sigma_V$, which "points outward $\mathcal{M}$" and is bounded. If V is induced by a surface Σ with C^1 boundary $\partial\Sigma$, then this component of the first variation corresponds to $\eta_\Sigma\mathcal{H}^{k-1}$; thus, roughly speaking, we expect that this is the "transversal boundary" of V at $\partial\mathcal{M}$, that is where V meets $\partial\mathcal{M}$ orthogonally.

Remark 3.4. We notice that Theorem 3.3 does not say anything about the tangential part of the first variation of V; in fact, let $\mathcal{M} \subset \mathbb{R}^n$ and consider $W \subset \partial\mathcal{M}$ relatively open in $\partial\mathcal{M}$ with infinite perimeter in $\partial\mathcal{M}$. Since any closed set is the 0-level set of a smooth function, there exists a smooth $(n-1)$-surface $\Sigma \subset \mathcal{M}$, which is locally a graph on W near ∂W and meets $\partial\mathcal{M}$ tangentially in ∂W. Let us consider the varifold V induced by Σ. Since $\|V\|(\partial\mathcal{M}) = 0$, by Corollary 3.10 we have

$$\delta V[X] = \int_\Sigma H \cdot X \,\mathrm{d}\mathcal{H}^{n-1} \qquad \forall X \in \mathfrak{X}_0(\mathcal{M}),$$

where H is the mean curvature vector of Σ. But the tangent part of δV is unbounded, due to fact that it holds

$$\delta V[X] = \int_\Sigma H \cdot X \,\mathrm{d}\mathcal{H}^{n-1} + \langle D_{\partial\mathcal{M}} W, X\rangle \qquad \forall X \in \mathfrak{X}_t(\mathcal{M}),$$

where $D_{\partial\mathcal{M}} W$ is the distributional gradient of $\mathbf{1}_W$ in $\partial\mathcal{M}$, which is not a Radon measure. This means that V can have unbounded first variation on $\partial\mathcal{M}$ where "V meets $\partial\mathcal{M}$ tangentially".

Looking at the case of varifolds with free boundary, since the tangent part to $\partial\mathcal{M}$ of the first variation of such a varifold is controlled by definition, Theorem 3.3 easily implies that varifolds with free boundary have bounded first variation (Corollary 3.13).

As we said before, Corollary 3.13 was already proved by Grüter and Jost when $\|V\|(\partial\mathcal{M}) = 0$, and Edelen extended the result to $\|V\|(\partial\mathcal{M}) > 0$ but assuming that V is rectifiable. In Edelen's proof, the rectifiability is used to show that for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ the only planes charged by V are those included in $T_x\partial\mathcal{M}$.
In order to remove the rectifiability assumption, we use Lemma 3.6 which asserts that on $\partial\mathcal{M}$, V charges only planes that are included in $T_x\partial\mathcal{M}$, also in the case where V is not rectifiabile. The tangential component of the first variation of a varifold with free boundary, in the case of $\|V\|(\partial\mathcal{M}) > 0$ also arises in the work of Mizuno and Tonegawa [MT15].

We can see Lemma 3.6 as a form of the Constancy Theorem (see [Sim84, Theorem 41.1]), which states that a stationary m-varifold contained in a m-surface of class C^2 coincides with the surface up to a multiplicative constant.

3.1.3 Consequences

Theorem 3.3 implies similar boundedness results for other classes of varifolds. These results are collected in section 3.4.

Another interesting consequence of the result is a monotonicity formula (Corollaries 3.15 and 3.16) at boundary points for a k-varifold V with free boundary, which roughly speaking states that, if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the map

$$\rho \mapsto \frac{\|V\|\big(B_\rho(x)\big)}{\rho^k}$$

is increasing up to a small correction. This implies that, if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the density $\Theta^k(\|V\|, x)$ exists and is finite *for every* $x \in \partial\mathcal{M}$.

For $x \in \mathcal{M}^\circ$, this is known by the celebrated work of Allard [All72]. Grüter and Jost proved in [GJ86a] the equivalent version of this result for varifolds with free boundaries, but involving reflections of the balls through $\partial\mathcal{M}$. Using Theorem 3.3, the author proved in [DeM21] a monotonicity formula at the boundary without reflecting the balls in the case of varifolds with free boundary (section 3.5); a monotonicity formula for varifolds which satisfy the contact angle condition is reported section 3.6.1 and is a variant of the one obtained by Kagaya and Tonegawa in [KT17]. For more details, we refer to the introduction of section 3.5.

3.1.4 A result on Hausdorff dimension of the set where density exists

The above mentioned results study the question of the monotonicity of density ratios (and, consequently, the existence of the density) when the mean curvature of the k-varifold satisfy $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$.

One can see that, in general, the density ratios are increasing for $\|V\|$-a.e. point $x \in \mathcal{M}$. This is true also if no absolute continuity of the first variation of V with respect to $\|V\|$ is assumed, that is even if V has just bounded first variation, see e.g. [Sim84, Lemma 40.5].

In particular, if V has free boundary at $\partial\mathcal{M}$, by Corollary 3.13 and these considerations we infer that the density ratios are increasing at $\|V\|$-a.e. $x \in \mathcal{M}$. This fact essentially follows by the Lebesgue differentiation theorem applied to the measures δV and $\|V\|$.

This statement can be rather unsatisfactory, especially at boundary points: since one expects that generally we have $\|V\|(\partial\mathcal{M}) = 0$, the set of points on $\partial\mathcal{M}$ for which the monotonicity holds (and the density exists) may be in principle empty. Since in Chapter 4 we need a more precise estimate for the size of the set where the density of $\|V\|$ does not exists or is infinite, we have to study the question more carefully.

We notice that the threshold $p > k$ is similar to the one for continuity of Sobolev functions: if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p > n$, then every point is a lebesgue point for f and f has a continuous representative. If $p \leq n$, then by Lebesgue differentiation theorem it follows $\mathcal{L}^n(\mathcal{M} \setminus \mathrm{Leb}(f)) = 0$, where $\mathrm{Leb}(f)$ is the set of Lebesgue points of f. In fact, by the work of Federer and Ziemer [FZ72], it can be said more: if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \leq n$, then

$$\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Leb}(f)\big) = 0 \qquad \forall s > n - p.$$

In other words, the Hausdorff dimension of the set $\mathrm{Leb}(f)$ of Lebesgue points for f is at most $n - p$.

Thus one may guess that a similar statement holds for varifolds with generalized mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \leq k$[1]; in fact this is true, as shown by the author in [DeM21]. More precisely, we define the *density set* of V (Definition 3.22), denoted by $\mathrm{Dens}(V)$, and we prove the following result, which can be seen as the natural counterpart of the results of Federer and Ziemer.

Theorem 3.5. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a varifold with free boundary at $\partial\mathcal{M}$ and $H \in L^p(\mathcal{M}, \|V\|)$ for some $1 \leq p \leq \infty$. Then*

$$\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0 \qquad \forall s > k - p \tag{3.5}$$

and, for every $x_0 \in \mathrm{Dens}(V)$ there exists an increasing function $\varphi_{x_0} \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{\|V\|\big(B_r(x_0)\big)}{r^k} \leq \frac{\|V\|\big(B_t(x_0)\big)}{t^k} + \varphi_{x_0}(t) \quad \forall\, 0 < r < t, \qquad \lim_{t \to 0} \varphi_{x_0}(t) = 0. \tag{3.6}$$

In particular, the k-density $\Theta^k(\|V\|, x_0)$ exists and is finite for every $x_0 \in \mathrm{Dens}(V)$; moreover, the restrictions of $\Theta^k(\|V\|, \cdot)$ to $\mathrm{Dens}(V) \cap \mathcal{M}^\circ$ and to $\mathrm{Dens}(V) \cap \partial\mathcal{M}$ are upper semi-continuous.

Thus the set of points where $\Theta^k(\|V\|, \cdot)$ is infinte or does not exist has Hausdorff dimension at most $k - p$. As said before, $\mathrm{Dens}(V)$ is the "good set" where we are able to study the pointwise properties of V which are essential to prove the results of Chapter 4 in the desired generality.

---

[1] Since the mean curvature represents a sort of "second derivative" of the function which generates the varifold, it would be better to compare the class of varifolds with mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ with the Sobolev space $W^{2,p}(\mathcal{M})$; nevertheless, we do the comparison with $W^{1,p}$ because in our opinion this makes more clear the intuitions behind the results stated.

We remark that, if $p > k$, then the theorem states the well-known existence of the density at *every* point and is a straightforward consequence of Corollary 3.15. Thus Theorem 3.5 can be seen as a generalization of Corollaries 3.15 and 3.16.

The behavior of the density for points in $\mathcal{M}^{\circ}$ with respect to lower dimensional Hausdorff measures has already been studied; see for example the work of Menne [Men09, pp. 2.9–2.11]: in that paper the author was interested in a lower bound for the lower density of the varifold with respect to Hausdorff measures; in particular he shows that, if the density has a lower bound $\|V\|$-a.e. and $H \in L^p(\mathcal{M}, \|V\|)$, then at $\mathcal{H}^{k-p}$-a.e. point, either the lower density still satisfies the lower bound, or it is equal to 0.

We finally remark that a version of Theorem 3.5 holds for varifolds with contact angle: we refer the reader to section 3.7.1.

3.1.5 Plan of the chapter

This chapter is organized as follows:

- In section 3.2 we prove the constancy type lemma which is used in the proof of Theorem 3.3;
- In section 3.3 we prove Theorem 3.3;
- In section 3.4 we state some consequences of Theorem 3.3, adapting it to varifolds with generalized mean curvature with respect to other classes of vector fields. In particular, in Corollary 3.13 we prove that varifolds with free boundary have bounded first variation;
- In section 3.5 we establish some monotonicity formulae at boundary points for varifolds with free boundary;
- In section 3.6 we adapt some of the above results to pair of varifolds which satisfies a contact angle condition for a general angle θ; the key remark in order to obtain such extensions is the fact that, if $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition with $a = \cos\theta$, then $V + aW$ has free boundary at $\partial\mathcal{M}$. This is stated in Proposition 3.17.
- In section 3.7 we define the density set $\mathrm{Dens}(V)$ of V and we prove Theorem 3.5 and in section 3.7.1 we state its generalization to varifolds which satisfy the contact angle condition.

3.2 Constancy lemma

In this section we prove Lemma 3.6, which is a variant of the Constancy Theorem [Sim84, Theorem 41.1]. There are some differences: the classical Constancy Theorem asserts that a stationary k-varifold V whose support is contained in a k-manifold Γ of class C^2 satisfies $V = \theta_0 \Gamma$ where θ_0 is a non-negative constant. On the contrary, Lemma 3.6 has much weaker hypotheses (and conclusion): if V has bounded first variation with respect to vector fields that vanish on a C^2 hypersurface Γ, then V charges on Γ only k-planes which are included in $T_x\Gamma$.

This result plays a key role in the proof of Theorem 3.3, in order to deal with the case $\|V\|(\partial\mathcal{M}) > 0$.

Lemma 3.6. *Let $\Gamma \subset \mathbb{R}^n$ be a C^2-hypersurface without boundary, let $V \in \mathcal{V}_k(\mathbb{R}^n)$ have bounded first variation δV with respect to $\mathfrak{X}_0(\Gamma)$ (that is vector fields that vanish on Γ). Then*

$$V\big(\{(x,S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma, S \not\subset T_x\Gamma\}\big) = 0.$$

Proof. By a simple covering argument it is enough to prove the result locally: that is that, for each $x_0 \in \Gamma$, there exist $r = r(x_0) > 0$ and a ball $B_r(x_0)$ such that

$$V\big(\{(x,S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma \cap B_r(x_0), S \not\subset T_x\Gamma\}\big) = 0.$$

We fix $x_0 \in \Gamma$ and without loss of generality we can assume that $x_0 = 0$.

Since Γ is locally-orientable, there exists $r' > 0$ and a ball $B_{r'}$ such that $B_{r'} \setminus \Gamma$ is made of two connected components D^+ and D^- separated by Γ. Only in this proof, d denotes a fixed one of the two signed distance functions from Γ in $B_{r'}$, that is

$$d(x) = \begin{cases} \inf\{|x-y| \mid y \in \Gamma\} & \text{if } x \in D^+ \\ -\inf\{|x-y| \mid y \in \Gamma\} & \text{if } x \in D^-. \end{cases}$$

Since Γ is of class C^2, there exists $r \leq r'$ such that $d \in C^2\big(B_r\big)$. We set $r(x_0) = r$.

We recall that γ is the cut-off function defined in section 2.1.1. Since $\nabla d(x)$ is orthogonal to Γ for every $x \in \Gamma$, we have $|P_S \nabla d(x)|^2 = 0$ if and only if $S \subset T_x\Gamma$. Therefore, to get the conclusion, it is enough to prove that

$$\int_{G_k(\Gamma)} \gamma\Big(\frac{|x|}{r}\Big) |P_S \nabla d(x)|^2 \,\mathrm{d}V(x,S) = 0. \tag{3.7}$$

To do so, we test (2.10) with a suitable vector field $X \in \mathfrak{X}_0(\mathcal{M})$. If $\rho < r$, we choose $X(x) = d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\nabla d(x)$. We clearly have $X \in \mathfrak{X}_0(\mathcal{M})$ and

$$\begin{aligned} \operatorname{div}_S X(x) = & \gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 + \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle \frac{x}{|x|}, P_S\nabla d(x)\rangle \\ & + \frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 + d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x). \end{aligned}$$

Since V has bounded first variation $\delta V = \zeta|\delta V|$ with respect to $\mathfrak{X}_0(\Gamma)$ (where ζ is the polar vector of δV with respect to its total variation $|\delta V|$), testing (2.10) with X we obtain

$$\begin{aligned}\Big|\int_{G_k(\mathbb{R}^n)} &\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S)\Big| \\ &\le \Big|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle\frac{x}{|x|}, P_S\nabla d(x)\rangle\,\mathrm{d}V(x,S)\Big| \\ &\quad+\Big|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S)\Big| \\ &\quad+\Big|\int_{G_k(\mathbb{R}^n)} d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)\Big| \\ &\quad+\Big|\int_{\mathbb{R}^n}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta V|(x)\Big|.\end{aligned} \tag{3.8}$$

For the left-hand side of the above inequality, by dominated convergence theorem we have

$$\lim_{\rho\to 0}\int_{G_k(\mathbb{R}^n)}\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S) = \int_{G_k(\Gamma)}\gamma\Big(\frac{|x|}{r}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S).$$

Therefore, to show (3.7), we have to prove that the terms on the right-hand side of (3.8) go to 0 as $\rho \to 0$:

1. Since γ' is bounded and $d(x) \le \rho$ by cut-off, for the first term we have

$$\lim_{\rho\to 0}\Big|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle\frac{x}{|x|}, P_S\nabla d(x)\rangle\,\mathrm{d}V(x,S)\Big| \le c\lim_{\rho\to 0}\frac{\rho}{r}\|V\|\big(U_\rho(\Gamma)\cap B_r\big) = 0.$$

2. Since $\big|\frac{d}{\rho}\big| \le 1$ and $\gamma'(s) \ne 0$ only if $s \in (1/2, 1)$, for the second term we have

$$\begin{aligned}\lim_{\rho\to 0}\Big|\int_{G_k(\mathbb{R}^n)} &\frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S)\Big| \\ &\le 3\lim_{\rho\to 0}\|V\|\Big(\big(U_\rho(\Gamma)\setminus U_{\rho/2}(\Gamma)\big)\cap B_r\Big) = 0.\end{aligned}$$

3. By the choice of r we have that $|\operatorname{div}_S\nabla d(x)| \le c$ in B_r; thus

$$\lim_{\rho\to 0}\Big|\int_{G_k(\mathbb{R}^n)} d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)\Big| \le c\lim_{\rho\to 0}\rho\|V\|\big(U_\rho(\Gamma)\cap B_r\big) = 0.$$

4. For the last term, since δV is a Radon measure, it holds

$$\lim_{\rho\to 0}\Big|\int_{\mathbb{R}^n}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta_0 V|(x)\Big| \le \lim_{\rho\to 0}\rho|\delta V|\big(U_\rho(\Gamma)\cap B_r\big) = 0.$$

This completes the proof. □

3.3 Proof of Theorem 3.3

We can now prove Theorem 3.3. The idea of the proof is to decompose $X \in \mathfrak{X}_\perp(\mathcal{M})$ into its tangent and orthogonal parts X^T, $X^\perp$ and to cut-off in a tubular neighborhood U_ρ of $\partial\mathcal{M}$; we then prove that, if we let $\rho \to 0$, the contribution of the X^T vanishes (since $X \in \mathfrak{X}_\perp(\mathcal{M})$) and the contribution of $X^\perp$ splits into two components:

- the absolutely continuous part $\tilde{H}\|V\|$, which depends on the second fundamental form of $\partial\mathcal{M}$;
- a part which defines a signed distribution, thus it is a Radon measure whose polar vector is $N(x)$; this gives rise to the measure $N\sigma_V$.

The estimates (3.3) and (3.4) are then obtained by testing (3.2) with suitable vector fields, as in the smooth case.

Proof of Theorem 3.3. Let us fix $R > 0$ (as in section 2.1) so that the distance function d from $\partial\mathcal{M}$ defined in (2.3) is of class C^2 in $\overline{U_R(\partial\mathcal{M})}$.

In what follows we are going to repeatedly use the decomposition of a vector field $X \in \mathfrak{X}(\mathcal{M})$ we now present; within $U_R(\partial\mathcal{M})$, we can decompose X in its normal and tangent component: there exists a scalar function $\chi(x)$ such that $X = X^\perp + X^T$ with $X^\perp(x) = \chi(x)\nabla d(x)$ and $\langle X^T(x), \nabla d(x)\rangle = 0$ for all $x \in U_R(\partial\mathcal{M})$.

Step 1: We begin by cut-offing a vector field in its "interior" and "boundary" part. For every $X \in \mathfrak{X}_\perp(\mathcal{M})$ and $\rho < R$ one has

$$X(x) = \gamma\Big(\frac{d(x)}{\rho}\Big)X(x) + \Big(1 - \gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x).$$

Therefore

$$\begin{aligned}\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = &\int_{G_k(\mathcal{M})} \operatorname{div}_S \Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S)\\ &+ \int_{G_k(\mathcal{M})} \operatorname{div}_S \Big[\Big(1 - \gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x)\Big]\,\mathrm{d}V(x,S).\end{aligned} \tag{3.9}$$

Thus we have splitted X in the "interior" and the "boundary part" by cut-offing with $\gamma(d/\rho)$ and the idea is to send $\rho \to 0$.

Step 2: For the interior part, since $\Big(1 - \gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x) \in \mathfrak{X}_c(\mathcal{M})$ (i.e. it is compactly supported in the interior of $\mathcal{M}$), by (2.10) and dominated convergence theorem we have

$$\begin{aligned}&\lim_{\rho\to 0}\int_{G_k(\mathcal{M})} \operatorname{div}_S \Big[\Big(1 - \gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x)\Big]\,\mathrm{d}V(x,S)\\ &\quad= -\lim_{\rho\to 0}\int_{\mathcal{M}} \Big(1 - \gamma\Big(\frac{d(x)}{\rho}\Big)\Big)\langle X(x), H(x)\rangle\,\mathrm{d}\|V\|(x)\\ &\quad= -\int_{\mathcal{M}^\circ} \langle X(x), H(x)\rangle\,\mathrm{d}\|V\|(x)\\ &\quad= -\int_{\mathcal{M}} \langle X(x), H(x)\rangle\,\mathrm{d}\|V\|(x),\end{aligned} \tag{3.10}$$

where the last equality follows by the fact that H is assumed to be equal to 0 on $\partial\mathcal{M}$ (see Remark 2.9 for more details).

Step 3: Since the limit in (3.10) exists, also for the "boundary part" of X the limit

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})} \operatorname{div}_S \Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S) \tag{3.11}$$

exists. We want now to compute (3.11) and to show the existence of $\tilde{H}$ and σ_V on $\partial\mathcal{M}$. We begin by writing

$$\begin{aligned}\int \operatorname{div}_S\Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S) = &\int \gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\\ &+\int \frac{1}{\rho}\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x),X(x)\rangle\,\mathrm{d}V(x,S).\end{aligned}\tag{3.12}$$

At this point, we want to study separately the limit as $\rho\to 0$ of each term in the right-hand side of the above equation.

- For the first term of the right-hand side in (3.12), we expect that, as $\rho\to 0$, it give a sort of mean curvature of $\partial\mathcal{M}$. This expectation is justified by the fact that, on $\partial\mathcal{M}$, V charges only planes that are tangent to $\partial\mathcal{M}$ by Lemma 3.6 and because X is orthogonal to $\partial\mathcal{M}$ on $\partial\mathcal{M}$. More precisely, we are going to prove that there exists a $\|V\|$-measurable vector field $\tilde{H}$ such that

  $$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\partial\mathcal{M}}\langle\tilde{H}(x),X(x)\rangle\,\mathrm{d}\|V\|(x),\tag{3.13}$$

  which is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x\in\partial\mathcal{M}$. To this aim, we first observe that by dominated convergence theorem we have

  $$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S).\tag{3.14}$$

  We now have to compute the right-hand side of (3.14). To do this, we decompose $\operatorname{div}_S X(x) = \operatorname{div}_S X^\perp(x)+\operatorname{div}_S X^T(x)$. For the tangent part, we claim that

  $$\int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X^T(x)\,\mathrm{d}V(x,S)=0.\tag{3.15}$$

  In fact $\operatorname{div}_S X^T(x)=0$ for any $x\in\partial\mathcal{M}$, and $\forall S\subset T_x\partial\mathcal{M}$. This is true because $X^T\equiv 0$ on $\partial\mathcal{M}$, therefore $D_\tau\langle X(x),\tau\rangle=0$ for any $\tau\in T_x\partial\mathcal{M}$. Thus (3.15) follows by definition of tangential divergence (2.1) and by Lemma 3.6.
  For the orthogonal component we get

  $$\operatorname{div}_S X^\perp(x)=\langle P_S\nabla\chi,\nabla d(x)\rangle+\chi(x)\operatorname{div}_S\nabla d(x).$$

  Since $\nabla d(x)$ is orthogonal to $\partial\mathcal{M}$, by Lemma 3.6 again we have $\langle P_S\nabla\chi(x),\nabla d(x)\rangle=0$ for V-a.e. $(x,S)\in G_k(\partial\mathcal{M})$. Hence

  $$\int_{G_k(\partial\mathcal{M})}\langle P_S\nabla\chi(x),\nabla d(x)\rangle\,\mathrm{d}V(x,S)=0.$$

  Since $\nabla d(x)=-N(x)$ for every $x\in\partial\mathcal{M}$, where $N(x)$ is the unit normal vector to $\partial\mathcal{M}$ at x, we obtain

  $$\int_{G_k(\partial\mathcal{M})}\chi(x)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)=\int_{G_k(\partial\mathcal{M})}\langle X(x),N(x)\rangle\operatorname{div}_S N(x)\,\mathrm{d}V(x,S).\tag{3.16}$$

  Thus, combining (3.15)-(3.16), we obtain

  $$\int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)=\int_{G_k(\partial\mathcal{M})}\langle X(x),N(x)\rangle\operatorname{div}_S N(x)\,\mathrm{d}V(x,S).\tag{3.17}$$

We are now going to define $\tilde{H}$ and write the last integral in terms of it. To do so, by disintegration of V we can write

$$V = \|V\| \otimes \nu_x$$

where for $\|V\|$-a.e. x, ν_x is a probability measure on $G(k,n)$. Hence the right-hand side of (3.17) can be written as

$$\begin{aligned}\int_{G_k(\partial\mathcal{M})} &\langle X(x), N(x)\rangle \operatorname{div}_S N(x)\,\mathrm{d}V(x,S)\\ &= \int_{\partial\mathcal{M}} \langle X(x), N(x)\rangle \Big(\int_{G(k,n)} \operatorname{div}_S N(x)\,\mathrm{d}\nu_x(S)\Big)\,\mathrm{d}\|V\|(x).\end{aligned}$$

If we define

$$\tilde{H}(x) := -N(x) \int_{G(k,n)} \operatorname{div}_S N(x)\,\mathrm{d}\nu_x(S) \qquad \text{for } \|V\|\text{-a.e. } x \in \partial\mathcal{M}, \tag{3.18}$$

we can write (3.17) as

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\partial\mathcal{M}} \langle \tilde{H}(x), X(x)\rangle\,\mathrm{d}\|V\|(x). \tag{3.19}$$

Loosely speaking, $\tilde{H}(x)$ can be interpreted as the "mean curvature of $\partial\mathcal{M}$ weighted according to the planes charged by V at x"; in fact, in co-dimension 1, $\tilde{H}$ turns out to be precisely the mean curvature of $\partial\mathcal{M}$ (see Corollary 3.8). By its definition, it is clear that $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$, that $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and that $\|\tilde{H}\|_{L^\infty(\partial\mathcal{M},\|V\|)}$ depends only on the second fundamental form of $\partial\mathcal{M}$. Gathering (3.14) and (3.19) we finally get (3.13).

- We now have to study the second term in the right-hand side of (3.12). Roughly speaking, we can see it as "the mean orthogonal part to $\partial\mathcal{M}$ of V" on the tubular neighborhood $U_\rho(\partial\mathcal{M})$. When $\rho \to 0$, we expect that this term takes into account the "transversal boundary" of V at $\partial\mathcal{M}$, that is the singular part of the first variation of V on $\partial\mathcal{M}$.
  More precisely, we are going to show the existence of a positive Radon measure σ_V (as expressed in the statement of the theorem), such that

$$\lim_{\rho\to 0} \frac{1}{\rho} \int \gamma'\Big(\frac{d(x)}{\rho}\Big) \langle P_S \nabla d(x), X(x)\rangle\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} \langle X(x), N(x)\rangle\,\mathrm{d}\sigma_V(x), \tag{3.20}$$

  where $N(x)$ is the exterior unit normal vector to $\partial\mathcal{M}$. We first of all remark that the above limit exists by the existence of limits of the other two terms in (3.12).
  To compute the limit in (3.20), we use again the decomposition $X = X^T + X^\perp = X^T + \chi\nabla d$. As $\rho \to 0$, we expect that the contribution of X^T is zero. Indeed, since $X \in \mathfrak{X}_\perp(\mathcal{M})$ and X is of class C^1, there exists a constant $c > 0$ such that $|X^T(x)| \leq c d(x)$. Therefore, since $\gamma'(s) \neq 0$ only for $s \in (1/2, 1)$, we obtain

$$\begin{aligned}\lim_{\rho\to 0} \frac{1}{\rho} \int_{G_k(\mathcal{M})} &\Big|\gamma'\Big(\frac{d(x)}{\rho}\Big) \langle P_S \nabla d(x), X^T(x)\rangle\Big|\,\mathrm{d}V(x,S)\\ &\leq \lim_{\rho\to 0} 3c\|V\|\big(U_\rho(\partial\mathcal{M}) \setminus U_{\rho/2}(\partial\mathcal{M})\big) = 0.\end{aligned} \tag{3.21}$$

  This proves the existence of the limit for the orthogonal part

$$\begin{aligned}\lim_{\rho\to 0}\frac{1}{\rho} \int_{G_k(\mathcal{M})} &\gamma'\Big(\frac{d(x)}{\rho}\Big) \langle P_S \nabla d(x), X^\perp(x)\rangle\,\mathrm{d}V(x,S)\\ &= \lim_{\rho\to 0} \frac{1}{\rho} \int_{G_k(\mathcal{M})} \chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big) |P_S \nabla d(x)|^2\,\mathrm{d}V(x,S).\end{aligned} \tag{3.22}$$

(3.21) and (3.22) yield

$$\begin{aligned}\lim_{\rho\to 0}\int \frac{1}{\rho}\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X(x)\rangle \,\mathrm{d}V(x,S)\\ = \lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})}\chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S).\end{aligned}$$

We now observe that the map

$$T\colon X\in\mathfrak{X}_\perp(\mathcal{M})\longmapsto \lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})}\chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S)$$

is a well-defined distribution and, by its definition, $\operatorname{supp}T\subseteq\partial\mathcal{M}$. Again by definition, if $\chi(x)\le 0$ for every $x\in\partial\mathcal{M}$ (i.e. if X point outward $\partial\mathcal{M}$), then $T(X)\ge 0$. Therefore T is a signed distribution and by Riesz Representation Theorem there exists a positive Radon measure σ_V such that

$$\langle T, X\rangle = \int_{\partial\mathcal{M}}\langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X\in\mathfrak{X}_\perp(\mathcal{M}).$$

This completes the proof of (3.20).

Step 4: We now gather the previous computations to get (3.2).

By (3.12), (3.13) and (3.20), we can rewrite (3.11) as

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S) = -\int_{\partial\mathcal{M}}\langle\tilde{H}(x),X(x)\rangle\,\mathrm{d}\|V\|(x) + \int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V. \tag{3.23}$$

Going back to (3.9), by (3.10) and (3.23) we finally get

$$\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}}\langle X(x), H(x)+\tilde{H}(x)\rangle\,\mathrm{d}\|V\|(x) + \int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V,$$

which completes the proof of (3.2).

Step 5: We are left with the proof of (3.3) and (3.4). We begin with (3.4). Let us fix $x_0\in\partial\mathcal{M}$ and $r\le R$. Without loss of generality we can assume that $x_0=0$. To prove the estimate, we test (3.2) with $X(x) = -\gamma\big(\frac{|x|}{r}\big)\nabla d(x)$ which clearly belongs to $\mathfrak{X}_\perp(\mathcal{M})$. We have

$$\operatorname{div}_S X(x) = -\gamma'\Big(\frac{|x|}{r}\Big)\frac{1}{r}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle - \gamma\Big(\frac{|x|}{r}\Big)\operatorname{div}_S\nabla d(x).$$

Therefore

$$\begin{aligned}\sigma_V\big(B_{r/2}\big) &\le \int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{r}\Big)\,\mathrm{d}\sigma_V\\ &= \int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V\\ &= \int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) + \int_{\mathcal{M}}\langle X, H+\tilde{H}\rangle\,\mathrm{d}\|V\|\\ &= \int_{G_k(\mathcal{M})}\Big[-\gamma'\Big(\frac{|x|}{r}\Big)\frac{1}{r}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle - \gamma\Big(\frac{|x|}{r}\Big)\operatorname{div}_S\nabla d(x)\Big]\,\mathrm{d}V(x,S)\\ &\quad - \int_{\mathcal{M}}\gamma\Big(\frac{|x|}{r}\Big)\langle\nabla d(x), H(x)+\tilde{H}(x)\rangle\,\mathrm{d}\|V\|(x).\end{aligned} \tag{3.24}$$

We want to estimate the last member of the above inequality. To do so, we choose a constant $c = c(\partial\mathcal{M}, R)$ such that

$$|\operatorname{div}_S \nabla d(x)| \le c \qquad \forall x \in U_R(\partial\mathcal{M}).$$

By (3.18), the choice of c provides also $|\tilde{H}(x)| \le c$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$. Thus

$$-\int_{G_k(\mathcal{M})} \gamma\Big(\frac{|x|}{r}\Big) \operatorname{div}_S \nabla d(x)\, \mathrm{d}V(x,S) - \int_{\mathcal{M}} \gamma\Big(\frac{|x|}{r}\Big) \langle \nabla d(x), H(x) + \tilde{H}(x)\rangle\, \mathrm{d}\|V\|(x)$$
$$\le \int_{B_r} \big(c + |H(x)|\big)\, \mathrm{d}\|V\|(x).$$

Substituting in (3.24) and since γ can be taken so that $\|\gamma'\|_\infty \approx 2$, we get

$$\sigma_V\big(B_{r/2}\big) \le \frac{c}{r}\|V\|\big(B_r\big) + \int_{B_r} |H(x)|\, \mathrm{d}\|V\|(x).$$

The proof of (3.3) is similar to the previous one and is in fact easier: it is enough to take a vector field $X \in \mathfrak{X}_\perp(\mathcal{M})$ such that $X(x) = -\gamma\Big(\frac{d(x)}{\rho}\Big)\nabla d(x)$ for some ρ sufficiently small, so that $X = N$ on $\partial\mathcal{M}$ and use (3.2).

□

3.4 Consequences of Theorem 3.3

In this section we clarify some consequences of Theorem 3.3.

- In section 3.4.1 we extend Theorem 3.3 to the case of varifolds with bounded first variation with respect to $\mathfrak{X}_0(\mathcal{M})$;
- In section 3.4.2 we study the codimension 1 case: we prove that if $k = n-1$, then the vector field $\tilde{H}$ given by Theorem 3.3 is the mean curvature vector of $\partial\mathcal{M}$ (Corollary 3.8); next we state a refined version of Lemma 3.6 for varifolds with free boundary: if $k = n-1$, then the restriction of V to $\partial\mathcal{M}$ is $(n-1)$-rectifiable (Corollary 3.9);
- In section 3.4.3 we extend Theorem 3.3 to varifolds with generalized mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, i.e. vector fields compactly supported in the interior of $\mathcal{M}$ (see (2.1.2)), assuming that $\|V\|(\partial\mathcal{M}) = 0$ (Corollary 3.10); As a consequence, such varifolds have generalized mean curvature with respect to the larger class of vector fields $\mathfrak{X}_0(\mathcal{M})$;
- In section 3.4.4 we show that varifolds with free boundary have bounded first variation (Corollary 3.13);

3.4.1 Varifolds with bounded first variation with respect to $\mathfrak{X}_0(\mathcal{M})$

The fact that the first variation of V with respect to $\mathfrak{X}_0(\mathcal{M})$ is absolutely continuous with respect to $\|V\|$ is not essential to prove Theorem 3.3. In fact the following slight modification holds true.

Theorem 3.7. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a k-varifold with bounded first variation δV with respect to $\mathfrak{X}_0(\mathcal{M})$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial\mathcal{M}$ such that, for any $X \in \mathfrak{X}_\perp(\mathcal{M})$, it holds*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\, \mathrm{d}V(x,S) = \int_{\mathcal{M}} \langle X, \zeta\rangle\, \mathrm{d}|\delta V| - \int_{\mathcal{M}} \langle X, \tilde{H}\rangle\, \mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\, \mathrm{d}\sigma_V$$

where $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends on the second fundamental form of $\partial\mathcal{M}$ and ζ is the polar vector of δV with respect to $|\delta V|$. In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$. Moreover, the following estimates hold:

$$\sigma_V(\partial\mathcal{M}) \le c\|V\|(\mathcal{M}) + |\delta V|(\mathcal{M}); \tag{3.25}$$

$$\sigma_V\big(B_{r/2}(x_0)\big) \le \frac{c}{r}\|V\|\big(B_r(x_0)\big) + |\delta V|\big(B_r(x_0)\big) \qquad \forall x_0 \in \partial\mathcal{M}, \forall r \le R(\mathcal{M}) \tag{3.26}$$

where $R(\mathcal{M})$ is such that the distance function from $\partial\mathcal{M}$ is of class C^2 in $U_R(\partial\mathcal{M})$ and the constant $c = c(\mathcal{M})$ depends on the second fundamental form of $\partial\mathcal{M}$.

Proof. The proof follows the one of Theorem 3.3. The only part of the proof of (3.2) where we use the hypothesis $\delta V \ll \|V\|$ is in (3.10). If V has bounded first variation δV with respect to $\mathfrak{X}_0(\mathcal{M})$, then one easily obtains

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{G_k(\mathcal{M})} \operatorname{div}_S\left[\left(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\right)X(x)\right]\mathrm{d}V(x,S)\\
&\quad= \lim_{\rho\to 0}\int_{\mathcal{M}}\left(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\right)\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta V|(x)\\
&\quad= \int_{\mathcal{M}^\circ}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta V|(x)\\
&\quad= \int_{\mathcal{M}}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta V|(x)
\end{aligned}$$

where ζ is the polar vector of δV with respect to $|\delta V|$ and the last equality is due to the assumption $|\delta V|(\partial\mathcal{M}) = 0$ (see Remark 2.9).

The modifications to the proofs of (3.3) and (3.4) to obtain (3.25) and (3.26) are obvious. □

3.4.2 The codimension 1 case

If $k = n-1$, we can characterize $\tilde{H}$ in a simpler way: if $x \in \partial\mathcal{M}$, then $\tilde{H}(x)$ is the mean curvature vector of $\partial\mathcal{M}$.

Corollary 3.8. *Let $V \in \mathcal{V}_{n-1}(\mathcal{M})$ have generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$ with $H \in L^1(\mathcal{M}, \|V\|)$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ such that*

$$\int_{G_{n-1}(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}}\langle X, H+\tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}}\langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X \in \mathfrak{X}_\perp(\mathcal{M})$$

where $\tilde{H}$ is the mean curvature vector of $\partial\mathcal{M}$, that is $\tilde{H}(x) := -N(x)\big(\operatorname{div}_{T_x\partial\mathcal{M}} N(x)\big)$ for $x \in \partial\mathcal{M}$. Moreover, (3.4) *holds true.*

Proof. If S is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, then $S \subset T_x\partial\mathcal{M}$ if and only if $S = T_x\partial\mathcal{M}$. Therefore, if $k = n-1$ and $V \in \mathcal{V}_{n-1}(\mathcal{M})$ with generalized mean curvature with respect to $\mathfrak{X}_0(\mathcal{M})$, Lemma 3.6 yields

$$V\left(\{(x,S) \in G_{n-1}(\partial\mathcal{M}) \mid S \ne T_x\Gamma\}\right) = 0. \tag{3.27}$$

Hence, for $X \in \mathfrak{X}_\perp(\mathcal{M})$,

$$\int_{G_{n-1}(\partial\mathcal{M})} X(x)\operatorname{div}_S N(x)\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} X(x)\operatorname{div}_{T_x\partial\mathcal{M}} N(x)\,\mathrm{d}\|V\|,$$

thus (3.17) becomes

$$\int_{G_{n-1}(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} \langle N(x), X(x)\rangle \operatorname{div}_{T_x\partial\mathcal{M}} N(x)\,\mathrm{d}\|V\|(x).$$

Thus, defining

$$\tilde{H}(x) := -N(x)\operatorname{div}_{T_x\partial\mathcal{M}} N(x),$$

we get (3.19). □

If $k = n-1$ and V has bounded first variation with respect to $\mathfrak{X}_t(\mathcal{M})$ instead of $\mathfrak{X}_0(\mathcal{M})$ (which is included in $\mathfrak{X}_t(\mathcal{M})$), we can strengthen the conclusion of Lemma 3.6: $V \llcorner G_{n-1}(\partial\mathcal{M})$ is $(n-1)$-rectifiable.

Corollary 3.9. *Let $V \in \mathcal{V}_{n-1}(\mathcal{M})$ with bounded first variation with respect to $\mathfrak{X}_t(\mathcal{M})$. Then $V \llcorner G_{n-1}(\partial\mathcal{M})$ is an $(n-1)$-rectifiable varifold. More precisely, if $\varphi \in C_c\big(G_{n-1}(\mathcal{M})\big)$, then*

$$\int_{G_{n-1}(\mathcal{M})} \varphi(x,S)\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} \varphi(x, T_x\partial\mathcal{M})\theta(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{G_{n-1}(\mathcal{M}^\circ)} \varphi(x,S)\,\mathrm{d}V(x,S),$$

where $\theta(x) = (\omega_{n-1})^{-1}\Theta^{n-1}(\|V\|, x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\mathcal{M}$.

Proof. By Lemma 3.6 and in particular by (3.27), for any $\varphi \in C_c\big(G_{n-1}(\mathcal{M})\big)$ we have

$$\int_{G_{n-1}(\mathcal{M})} \varphi(x,S)\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} \varphi(x, T_x\partial\mathcal{M})\,\mathrm{d}\|V\|(x) + \int_{G_{n-1}(\mathcal{M}^\circ)} \varphi(x,S)\,\mathrm{d}V(x,S).$$

By [Sim84, Lemma 40.5]), for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ there exists the density $\Theta^{n-1}(\|V\|, x) < \infty$. Hence, since $\partial\mathcal{M}$ is of class C^2, the quantity

$$\theta(x) := \lim_{\rho\to 0} \frac{\|V\|\big(B_\rho(x)\big)}{\mathcal{H}^{n-1}\llcorner\partial\mathcal{M}\big(B_\rho(x)\big)} = \frac{\Theta^{n-1}(\|V\|, x)}{\omega_{n-1}}$$

exists and is finite for $\|V\|$-a.e. $x \in \partial\mathcal{M}$. By Radon-Nikodym Theorem [Sim84, Theorem 4.7], since the singular set $\{x \in \mathcal{M} \mid \theta(x) = +\infty\}$ of $\|V\|\llcorner\partial\mathcal{M}$ is $\|V\|$-negligible, we have $\|V\|\llcorner\partial\mathcal{M} \ll \mathcal{H}^{n-1}\llcorner\partial\mathcal{M}$ and the conclusion follows. □

3.4.3 Varifolds with mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$

The analogue of Theorem 3.3 holds, adding the extra hypothesis $\|V\|(\partial\mathcal{M}) = 0$, even if V has generalized mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, i.e. the vector fields with compact support in the interior of $\mathcal{M}$. In fact, if we analyze the proof of Theorem 3.3, we can see that the only point where we used the existence of generalized mean curvature with respect $\mathfrak{X}_0(\mathcal{M})$ is to obtain (3.17), that is to obtain the identity

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{G_k(\partial\mathcal{M})} \langle X(x), N(x)\rangle \operatorname{div}_S N(x)\,\mathrm{d}V(x,S)$$

by the use of Lemma 3.6, whereas if $\|V\|(\partial\mathcal{M}) = 0$ then obviously we have

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = 0.$$

Since the remaining arguments remain valid also if V has mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, we have proved the following corollary.

Corollary 3.10. *Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature H with respect to $\mathfrak{X}_c(\mathcal{M})$ with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial\mathcal{M}) = 0$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ such that*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X \in \mathfrak{X}_\perp(\mathcal{M}).$$

In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$ and the estimates (3.3), (3.4) *on σ_V hold true.*

Remark 3.11. If we remove the hypothesis $\|V\|(\partial\mathcal{M}) = 0$ nothing can be said about the behavior of V on $\partial\mathcal{M}$, because any vector field in $\mathfrak{X}_c(\mathcal{M})$ has first derivatives compactly supported in the interior of $\mathcal{M}$. So we have a lack of test vector fields to establish any property of V on $\partial\mathcal{M}$: e.g. take a smooth surface in $\mathcal{M}^\circ$ and add any varifold $W \in \mathcal{V}_k(\partial\mathcal{M})$ with unbounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$.

Since $\mathfrak{X}_0(\mathcal{M}) \subset \mathfrak{X}_\perp(\mathcal{M})$, the following result follows.

Corollary 3.12. *Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature H with respect to $\mathfrak{X}_c(\mathcal{M})$, with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial\mathcal{M}) = 0$. Then V has generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$.*

3.4.4 Varifolds with free boundaries

As an immediate corollary of Theorem 3.3, varifolds with free boundaries have bounded first variation.

Corollary 3.13. *Let $V \in \mathcal{V}_k(\mathcal{M})$ have free boundary at $\partial\mathcal{M}$ with $H \in L^1(\mathcal{M}, \|V\|)$. Then V has bounded first variation. More precisely, there exist a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X \in \mathfrak{X}(\mathcal{M}), \tag{3.28}$$

where $\tilde{H}$ is defined as in (3.18)*. In particular $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends only on the second fundamental form of $\partial\mathcal{M}$. Moreover,* (3.3) (3.4) *hold true.*

Proof. If $X \in \mathfrak{X}(\mathcal{M})$, there exist $X^T, X^\perp$ such that $X = X^T + X^\perp$, with $X^T \in \mathfrak{X}_t(\mathcal{M})$ and $X^\perp \in \mathfrak{X}_\perp(\mathcal{M})$ (see the decomposition at the beginning of the proof of Theorem 3.3). We have

$$\operatorname{div}_S X(x) = \operatorname{div}_S X^T(x) + \operatorname{div}_S X^\perp(x).$$

Since the mean curvature of V with respect to $\mathfrak{X}_0(\mathcal{M})$ is given by $H\mathbf{1}_{\mathcal{M}^\circ}$, by Theorem 3.3 there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that

$$\begin{aligned}\int_{G_k(\mathcal{M})} \operatorname{div}_S X^\perp(x)\,\mathrm{d}V(x,S) &= -\int_{\mathcal{M}} \langle X^\perp, H\mathbf{1}_{\mathcal{M}^\circ} + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \\ &= -\int_{\mathcal{M}} \langle X^\perp, H + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V,\end{aligned}$$

where the last equality follows because H is assumed to be tangent to $\partial\mathcal{M}$ on $\partial\mathcal{M}$ (see Remark 2.16). Moreover we have the same estimates (3.3) and (3.4) on σ_V. For what concerns the tangent part X^T, the definition of varifold with free boundary yields

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X^T(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X^T, H\rangle\,\mathrm{d}\|V\|.$$

This shows the conclusion. □

3.5 Monotonicity formulae

A very useful tool in the study of the properties of varifolds is the so-called *monotonicity formulae.* These results state that for a k-varifold V the map

$$\rho \in (0,+\infty) \mapsto \frac{\|V\|\big(B_\rho(x)\big)}{\rho^k}$$

is monotone increasing (up to a small correction which is negligible for small radii), under suitable integrability assumption on the generalized mean curvature H of V in a neighborhood of x. This shows, in particular, that the density $\Theta^k(\|V\|, x)$ exists and is finite and is moreover upper semi-continuous (which can be seen as a first regularity property of V).

The existence of the density has a lot of remarkable consequences, one of the most important is the existence of *tangent varifolds* to V; the study of these tangent varifolds (also called *blow-ups*) is often useful in establishing structure properties of varifolds and in the regularity theory for variational problems.

Monotonicity formulae were firstly proved by Allard in his seminal work [All72] (see [Sim84, §17 and §40] for a detailed account). A Boundary version of these monotonicity results is proved in [All75] and, under weaker regularity assumption, in [Bou15].

Grüter and Jost established in [GJ86a] some versions of monotonicity formulae for varifolds with free boundary at points $x \in \partial\mathcal{M}$: [GJ86a, Theorem 3.1]. The monotonicity results are obtained by reflecting the balls across $\partial\mathcal{M}$, that is they prove monotonicity (up to the usual correction) of the map

$$\rho \in (0,+\infty) \mapsto \frac{\|V\|\big(B_\rho(x)\big) + \|V\|\big(\tilde{B}_\rho(x)\big)}{\rho^k},$$

where $\tilde{B}_\rho(x)$ is the reflected ball across $\partial\mathcal{M}$.

Using Corollary 3.13 and the estimates on σ_V, it is possible to obtain the monotonicity of the mass in $B_r(x)$, without reflecting the balls, as obtained in [DeM21]. The first step in order to prove the monotonicity formula at boundary points is the following monotonicity inequality, which we state in this slightly unusual form, since it is used also in the proof of Theorem 3.5.

Lemma 3.14 (Monotonicity inequality)**.** *Suppose $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$, with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in [1,+\infty)$. Then there exists a constant $c > 0$ that depends only on n, k, p and on the second fundamental form of $\partial\mathcal{M}$ such that, for all $x_0 \in \partial\mathcal{M}$ and $s \in \mathbb{R}$ the following inequality holds:*

$$\begin{aligned}(1+c\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} \geq &-\rho^{-\frac{k-s}{p}}\Big(\frac{1}{p}+c\rho\Big)\left(\frac{1}{\rho^s}\int_{B_\rho(x_0)}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\\ &-c(1+\rho)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}.\end{aligned} \tag{3.29}$$

Proof. Without loss of generality we can suppose $x_0 = 0$. Since for large ρ the statement is obvious, we have to prove it only for $0 < \rho < R(\mathcal{M})$, where $R(\mathcal{M})$ is defined in section 2.1. We want to bound from below the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1-p}{p}}. \tag{3.30}$$

To do so, we want to bound from below the derivative in the right-hand side to get a differential inequality. We have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) = -\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\left(k\gamma\Big(\frac{|x|}{\rho}\Big) + \frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\right)\mathrm{d}V(x,S).$$

Let us choose $X(x) = \gamma\Big(\frac{|x|}{\rho}\Big)x$. Then

$$\operatorname{div}_S X(x) = k\gamma\Big(\frac{|x|}{\rho}\Big) + \frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_S\frac{x}{|x|}\right|^2.$$

We use Corollary 3.13: by testing (3.28) with X we get the following *monotonicity identity*:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) =& -\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\\ &-\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_{S^\perp}\frac{x}{|x|}\right|^2\mathrm{d}V(x,S)\\ =&\frac{1}{\rho^{k+1}}\int_{\mathcal{M}}\langle X, H+\tilde{H}\rangle\,\mathrm{d}\|V\| - \frac{1}{\rho^{k+1}}\int_{\partial\mathcal{M}}\langle X, N\rangle\,\mathrm{d}\sigma_V\\ &-\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_{S^\perp}\frac{x}{|x|}\right|^2\mathrm{d}V(x,S).\end{aligned} \tag{3.31}$$

We have to estimate from below the last member of the above identity.

Since $\gamma' \le 0$, we can neglect the last integral and, since $|x| \le \rho$ by cut-off, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) \ge& -\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\|\\ &-\frac{1}{\rho^k}\int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\Big|\langle\frac{x}{|x|}, N\rangle\Big|\,\mathrm{d}\sigma_V.\end{aligned} \tag{3.32}$$

We have now to bound the two terms in the right-hand side of (3.32).

- For the first one, since $H \in L^p(\mathcal{M}, \|V\|)$ and $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$, also $H+\tilde{H} \in L^p(\mathcal{M}, \|V\|)$. Therefore, by Hölder inequality we get

$$\begin{aligned}\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\| \le&\frac{1}{\rho^k}\left(\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\int_{\mathcal{M}}\gamma^{\frac{p}{p-1}}\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}\\ \le&\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\rho^{-\frac{k-s}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned} \tag{3.33}$$

- We now move on the estimate of the second integral in the right-hand side of (3.32). Since $\partial\mathcal{M}$ is of class C^2 and since $0 \in \partial\mathcal{M}$, there exists a constant c such that

$$|\langle\frac{x}{|x|}, N(x)\rangle| \le c|x| \qquad \forall x \in \partial\mathcal{M}.$$

  This yields

$$\frac{1}{\rho^k}\int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|\langle\frac{x}{|x|}, N\rangle|\,\mathrm{d}\sigma_V \le \frac{c}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V. \tag{3.34}$$

We have to further estimate the righ-hand side of this inequality. This is done by testing (3.28) with $X(x) = -\gamma\Big(\frac{|x|}{\rho}\Big)\nabla d(x)$; as in (3.24) we get

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V =& -\frac{1}{\rho^{k-1}}\int_{G_k(\mathcal{M})}\gamma'\Big(\frac{|x|}{\rho}\Big)\frac{1}{\rho}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle\,\mathrm{d}V(x,S)\\ &-\frac{1}{\rho^{k-1}}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{|x|}{\rho}\Big)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)\\ &-\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\langle\nabla d(x),H+\tilde H\rangle\,\mathrm{d}\|V\|(x).\end{aligned}$$

Since $\gamma'(s)=0$ if $s\in(0,1/2)$, we have $\frac{|x|}{\rho}\geq\frac{1}{2}$; Moreover, using $|\operatorname{div}_S\nabla d|\leq c$ (because $\rho<R$ and d is of class C^2 in $\overline{U_R(\partial\mathcal{M})}$) and (3.33), we obtain

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq& -\frac{2}{\rho^{k-1}}\int_{\mathcal{M}}\gamma'\Big(\frac{|x|}{\rho}\Big)\frac{|x|}{\rho^2}\,\mathrm{d}\|V\|(x)+\frac{c}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ &+\rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}\\ =&\frac{2}{\rho^{k-1}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)-\frac{2(k-1)}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ &+\frac{2(k-1)+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ &+\rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned}\tag{3.35}$$

Concerning the last member, we now observe that

$$\begin{aligned}\frac{2}{\rho^{k-1}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)&-\frac{2(k-1)}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ &=2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right).\end{aligned}$$

Substituting in (3.35) we get

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq& 2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ &+\frac{2(k-1)+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ &+\rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned}$$

Taking into account that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ &\qquad=\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)+\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq{}& \frac{2k+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)+2\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ &+\rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned} \tag{3.36}$$

This complete the estimate of the second term in the right-hand side of (3.32).

Gathering (3.33), (3.34) and (3.36) in (3.32), we can estimate (3.30) as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} ={}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1-p}{p}}\\ \geq{}&-\rho^{-\frac{k-s}{p}}\Big(\frac{1}{p}+c\rho\Big)\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\\ &-c(1+\rho)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}\\ &-c\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}},\end{aligned}$$

which is the desired inequality. □

We now use the monotonicity inequality to prove the monotonicity formulae at boundary points.

Corollary 3.15 (Monotonicity formula for $p\in(k,+\infty)$)**.** *Let $V\in\mathcal{V}_k(\mathcal{M})$ have free boundary at $\partial\mathcal{M}$, with $H\in L^p(\mathcal{M},\|V\|)$ for some $p\in(k,+\infty)$. Then there exists $\Lambda=\Lambda(k,p,\mathcal{M},\|H\|_{L^p})>0$ such that, for all $x_0\in\partial\mathcal{M}$ the function*

$$\rho\longmapsto e^{\Lambda\rho}\left(\frac{\|V\|\big(B_\rho(x_0)\big)}{\rho^k}\right)^{\frac{1}{p}}+\Lambda e^{\Lambda\rho}\rho^{1-\frac{k}{p}}$$

is monotone increasing.

Proof. Without loss of generality we can assume $x_0=0\in\partial\mathcal{M}$. We first notice that we have to prove the result only for small ρ, since it is clearly true for $\rho>\operatorname{diam}(\mathcal{M})$.

We choose $s=0$ in (3.29); therefore $\frac{k}{p}\in(0,1)$ since $p>k$. Rearranging we obtain the existence of $\Lambda>0$, which depends on the constant c of Lemma 3.14, on $\|H\|_{L^p(\mathcal{M})}$ and on the second fundamental form of $\partial\mathcal{M}$, such that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}e^{\Lambda\rho}\left[\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}+\Lambda\rho^{1-\frac{k}{p}}\right]\geq 0.$$

Since the constant c of Lemma 3.14 does not depend on the choice of γ (the estimates are independent on the choice of γ unless that $\gamma'(s)=0$ for $s\in(0,1/2)$), letting γ increase to $\mathbf{1}_{[0,1)}$ we have that the function

$$\rho\longmapsto e^{\Lambda\rho}\left[\left(\frac{\|V\|(B_\rho)}{\rho^k}\right)^{\frac{1}{p}}+\Lambda\rho^{1-\frac{k}{p}}\right]$$

is monotone increasing. □

If $H \in L^\infty(\mathcal{M}, \|V\|)$, in (3.33) we simply estimate

$$\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\| \leq \frac{\|H+\tilde{H}\|_\infty}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|.$$

By repeating the previous arguments we have the following result:

Corollary 3.16. *Let $V \in \mathcal{V}_k(\mathcal{M})$ have free boundary at $\partial\mathcal{M}$ with $H \in L^\infty(\mathcal{M}, \|V\|)$. Then there exists $\Lambda = \Lambda(k, \mathcal{M}, \|H\|_\infty)$ such that, for all $x \in \partial\mathcal{M}$ the function*

$$\rho \longmapsto e^{\Lambda\rho}\frac{\|V\|\big(B_\rho(x)\big)}{\rho^k}$$

is monotone increasing.

3.6 Varifolds with contact angle

In this section, we adapt some of the previous results to the case of pairs of varifolds which satisfies the contact angle condition given in Definition 2.20. These results are needed in Chapter 5. All the properties stated are direct consequences of the following fact.

Proposition 3.17. *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ and let $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ have fixed angle condition θ for some $\theta \in (0, \pi/2]$ and let us call $a = \cos\theta$. Then $V + aW$ has free boundary at $\partial\mathcal{M}$ with mean curvature $\hat{H}$ satisfying*

$$\int_A H\,\mathrm{d}\|V\| = \int_A \hat{H}\,\mathrm{d}(\|V\| + a\|W\|) \qquad \forall A \subset \mathcal{M} \text{ Borel.} \tag{3.37}$$

Moreover the expression of $\hat{H}$ is the following:

$$\hat{H}(x) := H(x)\mathbf{1}_B(x)D_{\|V\|+a\|W\|_{ac}}\|V\|(x) \qquad \text{for } (\|V\| + a\|W\|)\text{-a.e. } x \in \mathcal{M},$$

where B is a Borel set such that $\|V\|(\mathbb{R}^{n+1} \setminus B) = 0$, $D_{\|V\|+a\|W\|_{ac}}\|V\|(x)$ is the Radon-Nikodym derivative of $\|V\|$ with respect to $\|V\| + a\|W\|_{ac}$ at x, and $\|W\|_{ac}$ is the absolute continuous part of $\|W\|$ with respect to $\|V\|$.

Proof. First of all, up to changing H on a set of $\|V\|$-measure 0, we can assume that H is a Borel function. Next, for simplicity we call $\mu := \|V\|$ and $\nu := a\|W\|$. By Lebesgue decomposition [EG15, Theorem 1.31], there exists a Borel set $B \subset \mathcal{M}$ with $\mu(\mathcal{M} \setminus B) = 0$ such that

$$\nu = \nu_{ac} + \nu_s,$$

where

$$\nu_{ac} \ll \mu, \qquad \nu_s = \nu \llcorner (\mathcal{M} \setminus B).$$

Since

$$\mu + \nu_{ac} \ll \mu \ll \mu + \nu_{ac},$$

by Lebesgue differentiation theorem [EG15, Theorem 1.30], for any $X \in \mathfrak{X}_t(\mathcal{M})$ we have

$$\begin{aligned}\int X(x) \cdot H(x)\,\mathrm{d}\mu(x) &= \int X(x) \cdot H(x)D_{\mu+\nu_{ac}}\mu(x)\,\mathrm{d}(\mu + \nu_{ac})(x)\\ &= \int X(x) \cdot H(x)\mathbf{1}_B(x)D_{\mu+\nu_{ac}}\mu(x)\,\mathrm{d}(\mu + \nu_{ac})(x)\\ &= \int X(x) \cdot H(x)\mathbf{1}_B(x)D_{\mu+\nu_{ac}}\mu(x)\,\mathrm{d}(\mu + \nu)(x),\end{aligned} \tag{3.38}$$

where $D_{\mu+\nu_{ac}}\mu(x)$ is the Radon-Nikodym derivative of μ with respect to $\mu+\nu_{ac}$ at the point $x\in\mathcal{M}$. Recalling that (V,W) has fixed angle and that $\mu:=\|V\|$ and $\nu:=a\|W\|$, we have

$$\begin{aligned}\int \operatorname{div}_S X(x)\,\mathrm{d}(V+aW)(x,S) &= -\int X\cdot H\,\mathrm{d}\|V\| \\ &= \int X\cdot\Big(H\mathbf{1}_B D_{\|V\|+a\|W\|}\|V\|\Big)\,\mathrm{d}(\|V\|+a\|W\|) \quad \forall X\in\mathfrak{X}_t(\mathcal{M}).\end{aligned}$$

This means that $V+aW$ has free boundary at $\partial\mathcal{M}$, according to Definition 2.15, with mean curvature $\hat{H}$ defined by

$$\hat{H}:=H\mathbf{1}_B D_{\|V\|+a\|W\|}\|V\|.$$

(3.37) follows by (3.38). □

3.6.1 Consequences of Proposition 3.17

We now state some of the direct consequences of Proposition 3.17: in fact, since a pair $(V,W)\in\mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ that satisfies Definition 2.20 is such that $V+aW$ has free boundary at $\partial\mathcal{M}$, all the results for varifolds with free boundary directly apply to $V+aW$. We recall some of them for the reader convenience and for future reference.

We first state that, if $(V,W)\in\mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ has contact angle θ, then $V+aW$ has bounded first variation in $\mathcal{M}$.

Corollary 3.18. *Let $(V,W)\in\mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ have contact angle θ. Then $V+aW$ has bounded first variation. More precisely, there exist a positive Radon measure σ_V on $\partial\mathcal{M}$ such that*

$$\begin{aligned}\int_{G_2(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}\big(V(x,S)+aW(x,S)\big) &= -\int_{\mathcal{M}} X\cdot H\,\mathrm{d}\|V\| \\ -\int_{\mathcal{M}} X\cdot\tilde{H}\,\mathrm{d}(\|V\|+a\|W\|) &+ \int_{\partial\mathcal{M}} X\cdot N\,\mathrm{d}\sigma_V \qquad \forall X\in\mathfrak{X}(\mathcal{M}),\end{aligned}\tag{3.39}$$

where $\tilde{H}$ is the mean curvature of $\partial\mathcal{M}$. In particular $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$, $\tilde{H}$ is bounded and $\|\tilde{H}\|_\infty$ depends only on the second fundamental form of $\partial\mathcal{M}$. Moreover, (3.3) (3.4) *hold true.*

Proof. The fact that $V+aW$ has bounded first variation immediately follows by Proposition 3.17 and by Corollary 3.13. By Corollary 3.13 and (3.37) one obtains (3.39). The fact that $\tilde{H}$ is the mean curvature of $\partial\mathcal{M}$ is a consequence of the definition of $\tilde{H}$ given in (3.18): we have

$$\tilde{H}=-N(x)\operatorname{div}_{\partial\mathcal{M}}N(x),$$

since on $\partial\mathcal{M}$, V charges only planes which are tangent to $\partial\mathcal{M}$, by Lemma 3.6 and the only n-plane contained in $T_x\partial\mathcal{M}$ is $T_x\partial\mathcal{M}$ itself. This is precisely the definition of the mean curvature of $\partial\mathcal{M}$. □

As in [KT17], a monotonicity formula at points on $\partial\mathcal{M}$ holds for a pair (V,W) that satisfies the contact angle condition. The following monotonicity is slightly different from the one in [KT17], since it does not involve reflections over $\partial\mathcal{M}$. It is obtained immediately by Proposition 3.17 and Corollary 3.15.

Corollary 3.19. *Suppose $(V,W)\in\mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition with $H\in L^p(\mathcal{M},\|V\|)$ for some $p\in(n,+\infty)$. Then there exists $\Lambda>0$ which depends only on $p,\mathcal{M},(\|V\|+a\|W\|)(\mathcal{M}),\|H\|_{L^p}$ such that, for all $x_0\in\partial\mathcal{M}$ the function*

$$\rho\longmapsto e^{\Lambda\rho}\left[\left(\frac{\|V\|\big(B_\rho(x_0)\big)+a\|W\|\big(B_\rho(x_0)\big)}{\rho^n}\right)^{\frac{1}{p}}+\Lambda\rho^{1-\frac{n}{p}}\right]$$

is monotone increasing.

If $H \in L^\infty(\mathcal{M}, \|V\|)$, we have the following result, which again is a straightforward consequence of Proposition 3.17 and Corollary 3.16.

Corollary 3.20. *Suppose $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition with $H \in L^\infty(\mathcal{M}, \|V\|)$. Then there exists $\Lambda = \Lambda(\mathcal{M}, \|H\|_{L^\infty}) > 0$ such that, for all $x_0 \in \partial\mathcal{M}$ the function*

$$\rho \longmapsto e^{\Lambda\rho} \frac{\|V\|\big(B_\rho(x)\big) + a\|W\|\big(B_\rho(x_0)\big)}{\rho^n}$$

is monotone increasing.

Remark 3.21. By Corollaries 3.19 and 3.20, it follows that if $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in (n, \infty]$, then the density

$$\Theta_n(\|V\| + a\|W\|, x) = \lim_{\rho\to 0} \frac{\|V\|\big(B_\rho(x)\big) + a\|W\|\big(B_\rho(x_0)\big)}{\rho^n}$$

exists and is finite for every $x \in \partial\mathcal{M}$.

3.7 Density set and proof of Theorem 3.5

We introduce the definition of *density set* for a k-varifold V.

Definition 3.22 (Density set)**.** Let $V \in V_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ and $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$. Then the density set for V is defined as

$$\mathrm{Dens}(V) = \bigcup_{s\in(k-p,k]} \Big\{x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{1}{r^s} \int_{B_r(x)} |H + \tilde{H}|^p \,\mathrm{d}\|V\| = 0\Big\}.$$

Remark 3.23. We notice that, if $p > k$, then $\mathrm{Dens}(V) = \mathcal{M}$ (since $s = 0 \in (k-p, k]$) and Theorem 3.5 is an easy consequence of the monotonicity of density ratios, Corollaries 3.15 and 3.16.

Proof of Theorem 3.5. We first show that $\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0$ for all $s > k - p$. We have

$$\mathrm{Dens}(V) = \bigcup_{s\in(k-p,k]} A_s; \qquad A_s = \Big\{x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{1}{r^s} \int_{B_r(x)} |H + \tilde{H}|^p \,\mathrm{d}\|V\| = 0\Big\}.$$

If μ is absolutely continuous with respect to $\|V\|$ and has density $|H + \tilde{H}|^p$, that is $\mu = |H + \tilde{H}|^p\|V\|$, then for any $s \in (k - p, k]$ we have

$$\mathcal{M} \setminus A_s = \bigcup_{i\in\mathbb{N}} \Big\{x \in \mathcal{M} \mid \Theta^{*s}(\mu, x) \geq \frac{1}{i}\Big\}.$$

Then, by [Mat95, Theorem 6.9], for every $i \in \mathbb{N}$

$$\mathcal{H}^s\Big(\Big\{x \mid \Theta^{*s}(\mu, x) \geq \frac{1}{i}\Big\}\Big) \leq i\mu\Big(\Big\{x \mid \Theta^{*s}(\mu, x) \geq \frac{1}{i}\Big\}\Big).$$

We want to show that for every $i \in \mathbb{N}$ the right-hand side is equal to 0, which would prove (3.5). To see this, we first recall that for a varifold with bounded first variation in $\mathcal{M}$ the $\Theta^k(\|V\|, x)$ exists and is finite for $\|V\|$-a.e. $x \in \mathcal{M}$ by [Sim84, Lemma 40.5]; since

$$\lim_{r\to 0} \frac{1}{\|V\|(B_r(x))} \int_{B_r(x)} |H + \tilde{H}|^p \,\mathrm{d}\|V\| < \infty \qquad \text{for } \|V\|\text{-a.e. } x \in \mathcal{M},$$

we have

$$\mu\Big(\{x \mid \Theta^{*s}(\mu,x) \geq \frac{1}{i}\}\Big) = 0.$$

Thus $\mathcal{H}^s\big(\mathcal{M}\setminus \mathrm{Dens}(V)\big) \leq \mathcal{H}^s\big(\mathcal{M}\setminus A_s\big) = 0$. This shows (3.5).

In order to prove (3.6), we fix $x_0 \in \mathrm{Dens}(V)\cap\partial\mathcal{M}$ and $s \in (k-p,k]$ such that $x_0 \in A_s$. Without loss of generality we can assume $x_0 = 0$. It is enough to show the result only for small radii, because it is clearly valid if $\mathrm{diam}(\mathcal{M}) \leq r < t$. By the choice of s, we have $\frac{k-s}{p} \in [0,1)$. Applying Lemma 3.14 and taking into account $0 \in A_s$, we obtain that there exists $\Lambda > 0$ (which depends on the point x_0 chosen) such that

$$\frac{\mathrm{d}}{\mathrm{d}\rho} e^{\Lambda\rho}\left[\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}} + \Lambda\rho^{1-\frac{k-s}{p}}\right] \geq 0.$$

Since this is independent on the choice of γ, letting γ to increase to $\mathbf{1}_{[0,1)}$ we obtain that the function

$$\rho \mapsto e^{\Lambda\rho}\left[\left(\frac{\|V\|(B_\rho)}{\rho^k}\right)^{\frac{1}{p}} + \Lambda\rho^{1-\frac{k-s}{p}}\right]$$

is monotone increasing. This shows that the function

$$\varphi_0(t) := \sup_{0<r<t}\left(\frac{\|V\|(B_r)}{r^k} - \frac{\|V\|(B_t)}{t^k}\right)$$

is well-defined and satisfies (3.6).

If $x_0 \in \mathrm{Dens}(V)\cap\mathcal{M}^o$ the proof of (3.6) is the same, except for the fact that we have to use the classical monotonicity inequality for interior points, which indeed is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} \geq &-\frac{\rho^{-\frac{k-s}{p}}}{p}\left(\frac{1}{\rho^s}\int_{B_\rho(x_0)} |H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} \\ &-\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}.\end{aligned}$$

The existence of $\Theta^k(\|V\|,\cdot)$ on $\mathrm{Dens}(V)$ and the upper semi-continuity of the restrictions of $\Theta^k(\|V\|,\cdot)$ to $\mathrm{Dens}(V)\cap\mathcal{M}^o$ and to $\mathrm{Dens}(V)\cap\partial\mathcal{M}$ are easy consequences of (3.6). ☐

Remark 3.24. We point out that, for every $t > 0$, the function $x \mapsto \varphi_x(t)$ is Borel-measurable; this follows from the fact that for every $t, r > 0$ the function $x \mapsto \|V\|(B_r)/r^k - \|V\|(B_t)/t^k$ is upper semi-continuous and by

$$\varphi_x(t) = \sup_{\substack{0<r<t\\ r\in\mathbb{Q}}}\left(\frac{\|V\|(B_r)}{r^k} - \frac{\|V\|(B_t)}{t^k}\right).$$

Since the supremum of a countable family of measurable function is measurable, we have that every $x \mapsto \varphi_x(t)$ is Borel-measurable.

3.7.1 Varifolds with contact angle

The analogous of Theorem 3.5 holds for $V + aW$, if $(V,W) \in \mathcal{V}_n(\mathcal{M})\times\mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition at $\partial\mathcal{M}$ (Definition 2.20). We record it explicitly for future reference.

Corollary 3.25. *Let* $\mathcal{M} \subset \mathbb{R}^{n+1}$ *and let* $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ *be a pair of varifolds that satisfies the contact angle condition given by Definition 2.20, with* $H \in L^p(\mathcal{M}, \|V\|)$ *for some* $1 \leq p \leq \infty$*. Then*

$$\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Dens}(V + aW)\big) = 0 \qquad \forall s > k - p$$

and, for every $x_0 \in \mathrm{Dens}(V)$ *there exists an increasing function* $\varphi_{x_0} \colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_{t\to 0} \varphi_{x_0}(t) = 0$ *and*

$$\frac{\|V\|\big(B_r(x_0)\big) + a\|W\|\big(B_r(x_0)\big)}{r^k} \leq \frac{\|V\|\big(B_t(x_0)\big) + a\|W\|\big(B_t(x_0)\big)}{t^k} + \varphi_{x_0}(t) \qquad \forall\, 0 < r < t;$$

In particular, the k*-density* $\Theta^k(\|V\|+a\|W\|, x_0)$ *exists and is finite for every* $x_0 \in \mathrm{Dens}(V+aW)$*; moreover, the restrictions of* $\Theta^k(\|V\|+a\|W\|, \cdot)$ *to* $\mathrm{Dens}(V+aW)\cap\mathcal{M}^o$ *and to* $\mathrm{Dens}(V+aW)\cap\partial\mathcal{M}$ *are upper semi-continuous.*

The result is a straightforward consequence of Proposition 3.17 and Theorem 3.5.

Chapter 4

Rectifiability of the free boundary

4.1 Introduction and main results

In this chapter we want to study more carefully the properties of the measure σ_V, whose existence is stated by Theorem 3.3, in the case of varifolds with free boundary at $\partial\mathcal{M}$, see Corollary 3.13.

Indeed, if the varifold is induced by a smooth k-surface $\Sigma \subset \mathcal{M}$ with smooth $\partial\Sigma$ and with free boundary at $\partial\mathcal{M}$, then

$$\sigma_V = \mathcal{H}^{k-1} \llcorner \partial\Sigma,$$

as one can see by (2.15). In particular, in this case σ_V is $(k-1)$-rectifiable. It is then natural to ask if σ_V is $(k-1)$-rectifiable in a more general case as well.

In order to state precisely our main result concerning σ_V, we define the $(k-1)$-*dimensional part* of σ_V, written σ_V^*, as the restriction of σ_V to those points with strictly positive lower $(k-1)$-density and finite upper $(k-1)$-density :

$$\sigma_V^* = \sigma_V \llcorner E, \qquad E = \{x \mid 0 < \Theta_*^{k-1}(\sigma_V, x) \le \Theta^{*(k-1)}(\sigma_V, x) < +\infty\}. \tag{4.1}$$

Our main result on σ_V^* is the following:

Theorem 4.1. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$ and $\Theta^k(\|V\|, x) \ge 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$. Then σ_V^* is $(k-1)$-rectifiable.*

In section 4.2 we prove that, if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the condition $\Theta^{*(k-1)}(\sigma_V, x) < +\infty$ is not restrictive, since it holds for every point x, basically by (3.4) and by the monotonicity formula for $\|V\|$ that we prove in Corollary 3.15. Thus in this case, Theorem 4.1 reads as follows.

Theorem 4.2. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$ and $\Theta^k(\|V\|, x) \ge 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$. Then the restriction $\sigma_V \llcorner \{x \mid \Theta_*^{k-1}(\sigma_V, x) > 0\}$ is $(k-1)$-rectifiable.*

Since we deal with the same conditions on V, throughout this chapter we assume the following hypotheses, if not otherwise specified.

Assumption 4.3. $V \in \mathcal{V}_k(\mathcal{M})$ is a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$, with generalized mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \ge 1$ and $\Theta^k(\|V\|, x) \ge 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$.

We remark that if V satisfies Assumption 4.3, Corollary 3.13 establishes the existence of the measure σ_V and that V has bounded first variation.

4.1.1 Varifolds with contact angle condition

If $(V,W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ satisfies the contact angle condition given by Definition 2.20, then by Proposition 3.17 the varifold $V + aW$ has free boundary according to Definition 2.15. Then one can state the following version of Theorem 4.1 for varifolds with contact angle condition.

Corollary 4.4. *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ and let $(V,W)\mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ be a pair of varifolds such that V meets $\partial\mathcal{M}$ with contact angle $\theta \in (0,\pi/2]$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$; let us call $a = \cos\theta$ and let us assume that $\Theta^k(\|V\| + a\|W\|, x) \geq a$ for $(\|V\| + a\|W\|)$-a.e. $x \in \mathcal{M}$. Then σ^*_{V+aW} is $(n-1)$-rectifiable.*

We remark that, since by Proposition 3.17 $V + aW$ has free boundary at $\partial\mathcal{M}$, Corollary 3.13 state the existence of the measure σ_{V+aW}.

4.1.2 Strategy of proof

The main idea behind the proof of the above result is to use the Marstrand-Mattila Rectifiability Criterion (Theorem 4.5) combined with an analysis of the blow-ups of V on $\partial\mathcal{M}$ and of their stratification: we prove that, for σ^*_V-a.e. $x \in \partial\mathcal{M}$, every $(k-1)$-blow-up of σ^*_V at x is of the form $\beta\mathcal{H}^{k-1} \llcorner S$ for some $(k-1)$-dimensional plane S and $\beta > 0$. This allows us to apply the Marstrand-Mattila Rectifiability Criterion to show that σ^*_V is $(k-1)$-rectifiable.

The Marstrand-Mattila Rectifiability Criterion was firs proved by Marstrand in [Mar61] for $m = 2, n = 3$ and next extended by Mattila in [Mat75] to any $m, n \in \mathbb{N}$. See also [DeL08, Theorem 5.1]. We state explicitly the criterion for the reader convenience

Theorem 4.5 (Marstrand-Mattila Rectifiability Criterion)**.** *Let $m \leq n$ be a natural number and let μ be a positive Radon measure on $\mathbb{R}^n$ such that, for μ-a.e. $x \in \mathbb{R}^n$, we have*

1. $0 < \Theta^m_*(\mu, x) \leq \Theta^{*m}(\mu, x) < \infty$;
2. *Every m-tangent measure of μ at x is of the form $\beta\mathcal{H}^m \llcorner S$ for some m-dimensional linear subspace of $\mathbb{R}^n$ and some $\beta > 0$.*

Then μ is m-rectifiable.

Since by definition of σ^*_V it is straightforward to prove that

$$0 < \Theta^{k-1}_*(\sigma^*_V, x) \leq \Theta^{*(k-1)}(\sigma^*_V, x) < +\infty \qquad \text{for } \sigma^*_V\text{-a.e. } x \in \partial\mathcal{M},$$

in order to apply Marstrand-Mattila Rectifiability Criterion to σ^*_V it remains to show that for σ^*_V-a.e. $x_0 \in \partial\mathcal{M}$, every tangent measure to σ_V is a $(k-1)$-dimensional plane. Section 4.4 is devoted to prove this, which is achieved in several steps.

1. We first prove that every tangent varifold to V is a cone C in a half-space which has free boundary. Thus there exists the measure σ_C; moreover we prove that, for σ^*_V-a.e., $\mathrm{Tan}^{k-1}(\sigma^*_V, x_0)$ is made of these measures σ_C;
2. Then we study these tangent measures σ_C. We show that each σ_C has an invariant subspace D_C, which we expect to be the intersection between the invariant subspace of the cone C with $T_{x_0}\partial\mathcal{M}$.

   Our idea is to prove that, at points of approximate continuity of $\Theta^k(\|V\|, \cdot)$ with respect to σ^*_V (thus σ^*_V-a.e.), D_C coincides with the support of σ_C. Roughly speaking, at points of approximate continuity of $\Theta^k(\|V\|, \cdot)$ with respect to σ^*_V, we expect that the invariant subspace of σ_C has full dimension, because of the fact that the density $\Theta^k(\|V\|, \cdot)$ "has no jumps".

3. We prove that, at those points, every σ_C coincides with a multiple of the $(k-1)$-dimensional Hausdorff measure restricted to D_C;

Gathering these facts, we conclude the proof of Theorem 4.1.

4.1.3 Plan of the chapter

The plan of the chapter is the following:

- In section 4.2 we prove that Theorem 4.2 can be obtained by Theorem 4.1 using the estimate (3.4), that is in the case $H \in L^p(\mathcal{M}, \|V\|)$ for $p > k$ the hypothesis $\Theta^{*k}(\sigma_V, \cdot) < +\infty$ is redundant.
- In section 4.3 we adapt some well-known facts to σ_V^*: we prove that
    - for σ_V^*-a.e. $x \in \partial\mathcal{M}$, σ_V^* has the same tangent measures of σ_V;
    - if $p > 1$, then $\sigma_V^*\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0$
- Section 4.4 is the core of this chapter: here we prove that σ_V^* satisfies the second condition of Marstrand-Mattila Rectifiability Criterion;
- In section 4.5 we gather all these facts in order to conclude the proof of Theorem 4.1.

4.1.4 Some comments on the hypotheses of Theorem 4.1

We now want briefly comment the hypotheses of Theorem 4.1.

- We used the hypothesis $p > 1$ in the proof of the second condition of Marstrand-Mattila Rectifiability Criterion for σ_V^*, that is to prove that at σ_V^*-a.e. point, every blow-up of σ_V^* is a $(k-1)$-plane. The proof is based on the study of the blow-ups of V, which exist on $\mathrm{Dens}(V)$ by Theorem 3.5. Thus we had to prove that

$$\sigma_V^*\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0. \tag{4.2}$$

    - if $p > k$, then (4.2) is clearly true, since in this case $\mathrm{Dens}(V) = \mathcal{M}$;
    - the situation is more delicate if $p \leq k$. In this case, using classical results on desities, we have only bounds on the Hausdorff dimension of $\mathcal{M} \setminus \mathrm{Dens}(V)$; thus, in order to obtain (4.2), we have to compare σ_V^* with the Hausdorff measures. The condition $\Theta^{*(k-1)}(\sigma_V, x) < \infty$ (Lemma 4.10) implies that

$$\sigma_V^* \ll \mathcal{H}^{k-1}.$$

Since if $p > 1$, then $\mathcal{M} \setminus \mathrm{Dens}(V)$ is $\mathcal{H}^{k-1}$-negligible by (3.5), it follows

$$\sigma_V^*\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0 \qquad \text{if } p > 1.$$

Incidentally, proving Theorem 4.1 in the case $p \leq k$ was precisely the early reason for which Theorem 3.5 was proved. In fact, if $p \leq k$, then the Lebesgue-Besicovitch differentation Theorem applied to the classical monotonicity identity for varifolds with bounded first variation guarantees (see e.g. [Sim84, Lemma 40.5]) the existence and finiteness of the k-density and of tangent cones just $\|V\|$-a.e.. Since $\|V\|(\partial\mathcal{M}) = 0$ may hold true, it was in principle possible that $\Theta^k(\|V\|, x) = +\infty$ for every point in $\partial\mathcal{M} \cap \mathrm{supp}\|V\|$, and this would have stopped our analysis. On the contrary, Theorem 3.5 implies that the blow-ups of V exist $\mathcal{H}^{k-1}$-a.e. if $p > 1$.

It is reasonable to expect that, using another type of argument, it may be possible to avoid this comparison between σ_V^* and $\mathcal{H}^{k-1}$, in order to prove (4.2) in the case $p = 1$ as well.

- The rectifiability assumption on V is redundant since it follows by the lower bound on the density $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$, by Corollary 3.13 and by Allard's Rectifiability Theorem [Sim84, Theorem 42.4].
- If V is integer rectifiable, since σ_V represents the singular part of the first variation of V, one could guess that σ_V do not sees the set of points where $\Theta_*^{k-1}(\sigma_V, \cdot) = 0$. If this is the case, then Theorem 4.1 remains true also removing the restriction $\sigma_V \llcorner \{\Theta_*^{k-1}(\sigma_V, \cdot) > 0\}$.
- The hypothesis $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$ is used to prove that every tangent varifold to V at points in $\mathrm{Dens}(V)$ are (rectifiable) cones (see Lemma 4.13) and the value 1 can be replaced by any strictly positive lower bound. Indeed, in the version of the rectifiability theorem for n-varifolds with contact angle θ at $\partial\mathcal{M}$ (Corollary 4.4), we use the bound $a = \cos\theta$ on the density $\Theta^n(\|V\| + a\|W\|, \cdot)$, since it is the natural lower bound on the density of $\|V\| + a\|W\|$ if V and W are assumed to be integer rectifiable.

4.1.5 Comments on the integer rectifiability of σ_V^*

Theorem 4.1 state the rectifiability of σ_V^* when V has free boundary and is rectifiable. A natural question concerning the structure of σ_V^* is the following.

Question 4.6. *Let V satisfy the hypotheses of the rectifiability theorem 4.1 and let us assume in addition that V is integer rectifiable. Is σ_V^* $(k-1)$-integer rectifiable?*

The answer is negative, even if we assume that V has multiplicity 1 $\|V\|$-a.e. and it is induced by smooth surfaces, as shown by the following example.

Example 4.7. Let us consider

$$\begin{gathered}\mathcal{M} \coloneqq \{x_1 \geq 0\} \subset \mathbb{R}^3,\\ \Sigma_1 = \{(x_1, x_2, x_3) \subset \mathbb{R}^3 \colon x_3 \geq 0, x_1 = (1 + \sin x_2)x_3\},\\ \Sigma_2 = \{(x_1, x_2, x_3) \subset \mathbb{R}^3 \colon x_3 \leq 0, x_1 = -(1 + \sin x_2)x_3\}.\end{gathered}$$

Clearly $\Sigma_1, \Sigma_2 \subset \mathcal{M}$ are two smooth surfaces with boundary

$$B \coloneqq \partial\Sigma_1 = \partial\Sigma_2 = \{(0, x_2, 0) \colon x_2 \in \mathbb{R}\}.$$

Moreover, they are symmetric with respect to the hyperplane $\{x_3 = 0\}$ and for every $x_2 \in \mathbb{R}$ their unit conormals satisfy

$$\begin{aligned}\eta_{\Sigma_1}(0, x_2, 0) &= \Big(- \sin\big(\arctan(1 + \sin x_2)\big), 0, - \cos\big(\arctan(1 + \sin x_2)\big)\Big),\\ \eta_{\Sigma_2}(0, x_2, 0) &= \Big(- \sin\big(\arctan(1 + \sin x_2)\big), 0, \cos\big(\arctan(1 + \sin x_2)\big)\Big).\end{aligned}$$

Hence the varifold V induced by $\Sigma_1 \cup \Sigma_2$ has free boundary at $\partial\mathcal{M}$ and satisfy

$$\delta V = \big(H_{\Sigma_1}\mathcal{H}^2 \llcorner \Sigma_1 + H_{\Sigma_2}\mathcal{H}^2 \llcorner \Sigma_2\big) - e_1\big(\theta\mathcal{H}^1 \llcorner B\big),$$

where H_{Σ_i} is the mean curvature vector of Σ_i and

$$\theta(0, x_2, 0) = 2\sin\big(\arctan(1 + \sin x_2)\big) \qquad \forall x_2 \in \mathbb{R}.$$

Since $\sigma_V = \theta\mathcal{H}^1 \llcorner B$, it is not integer rectifiable.

Remark 4.8. The previous example shows that the integer k-rectifiability of V does not imply the integer $(k-1)$-rectifiability of σ_V^*. Nevertheless, if V satisfies the hypotheses of Theorem 4.2 and $x \in \partial\mathcal{M}$ is such that $\Theta^k(\|V\|, x) = \frac{1}{2}\omega_k$, then one can apply the Allard-Type regularity theorem proved in [GJ86a], obtaining that V is in fact induced by a graph of a function of class $C^{1,\alpha}$, which implies that σ_V is induced by a regular $(k-1)$-submanifold, proving the integer $(k-1)$-rectifiability of σ_V.

4.1.6 Notations

Since in the proof of Theorem 4.1 we have to deal with the blow-ups of V at points on $\partial\mathcal{M}$, we introduce a notation for the scalings of V at a point $x_0 \in \partial\mathcal{M}$ that we use throughout this chapter.

Notation 4.9 (Scalings)**.** If $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$, if $x_0 \in \partial\mathcal{M}$ is a fixed point and $r_j \downarrow 0$ is a fixed sequence, we use the following notations:

$$\mathcal{M}_j := \tau_{x_0,r_j}(\mathcal{M}) \qquad V_j := (\tau_{x_0,r_j})_\sharp V \in \mathcal{V}_k(\mathcal{M}_j) \qquad \sigma_j := \sigma_{V_j};$$

where τ_{x_0,r_j} is the dilation function defined in (2.2) and $f_\sharp V$ denotes the push-forward of V through f defined in (2.6). In Lemma 4.13 we show that each V_j has free boundary at $\partial\mathcal{M}_j$, thus Corollary 3.13 states the existence of σ_{V_j}, which we call σ_j. Moreover, we denote by H_j the generalized mean curvature of V_j and $\tilde{H}_j$ the vector field provided by Corollary 3.13 relative to V_j. If $x \in \partial\mathcal{M}_j$, we denote by $N_j(x)$ the exterior unit normal to $\mathcal{M}_j$ at x. As $j \to \infty$, according to the definition of tangent space given in section 2.1, we say that $\mathcal{M}_j \to T^+_{x_0}\mathcal{M}$ and that $\partial\mathcal{M}_j \to T_{x_0}\partial\mathcal{M}$.

4.2 Proof of Theorem 4.2

In this section we show that Theorem 4.2 is a corollary of Theorem 4.1.

Proof of Theorem 4.2. We have to prove that, if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the condition $\Theta^{*(k-1)}(\sigma_V, x) < +\infty$ is not restrictive, since it holds *for every* point $x \in \partial\mathcal{M}$; indeed (3.4) and Hölder inequality yield

$$\begin{aligned}\frac{\sigma_V(B_{r/2}(x))}{r^{k-1}} \leq& c\frac{\|V\|\big(B_r(x)\big)}{r^k} + \frac{1}{r^{k-1}}\int_{B_r(x)} |H| \,\mathrm{d}\|V\| \\ \leq& c\frac{\|V\|\big(B_r(x)\big)}{r^k} + r^{1-\frac{k}{p}}\Big(\int_{B_r(x)} |H|^p \,\mathrm{d}\|V\|\Big)^{\frac{1}{p}}\Big(\frac{\|V\|\big(B_r(x)\big)}{r^k}\Big)^{1-\frac{1}{p}};\end{aligned}$$

By Lemma 3.15 and $p > k$, the last member of the above inequality is bounded as $r \to 0$, therefore it follows $\Theta^{*(k-1)}(\sigma_V, x) < +\infty$. □

4.3 Some well-known facts

In this section we adapt to σ_V^* two well-known facts about measures:

- in Lemma 4.10 we show that $\sigma_V^*\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0$; this allows us to check the conditions of the Marstrand-Mattila criterion just on $\mathrm{Dens}(V)$.
- In Lemma 4.11 we prove that for σ_V^*-a.e. $x \in \partial\mathcal{M}$, σ_V and σ_V^* have the same $(k-1)$-tangent measures.

Lemma 4.10. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$. Then $\sigma_V^* \ll \mathcal{H}^{k-1}$. In particular, if $p > 1$, we have*

$$\sigma_V^*\big(\partial\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0. \tag{4.3}$$

Proof. The proof is an easy consequence of [Mat95, Theorem 6.9] which states that if $\mu \in \mathscr{M}^+(\mathbb{R}^n)$ is a positive Radon measure and if

$$A \subset \{x \in \mathbb{R}^n \mid \Theta^{*(k-1)}(\mu, x) \leq \lambda\},$$

then $\mu(A) \leq 2^{k-1}\lambda\mathcal{H}^{k-1}(A)$. Let us assume $\mathcal{H}^{k-1}(A) = 0$; since by definition $\Theta^{*(k-1)}(\sigma_V^*, x) < \infty$ for σ_V^*-a.e. $x \in \mathcal{M}$ we have that

$$\sigma_V^*(A) = \sigma_V^*\Big(\bigcup_{i\in\mathbb{N}} A \cap \{\Theta^{*(k-1)}(\sigma_V^*, x) < i\}\Big) \leq \sum_{i\in\mathbb{N}} i2^{k-1}\mathcal{H}^{k-1}(A) = 0.$$

(4.3) clearly follows by $\sigma_V^* \ll \mathcal{H}^{k-1}$ and (3.5). ☐

Since σ_V^* is concentrated on E, we expect that $\mathrm{Tan}^{k-1}(\sigma_V^*, x) = \mathrm{Tan}^{k-1}(\sigma_V, x)$ for every point of E that has density 1 with respect to σ_V. This is a well-known fact, see e.g. [DeL08, Remark 3.13], but we recall the proof for the reader convenience.

Lemma 4.11. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$, let σ_V be the measure provided by Corollary 3.13 and let E be defined as in* (4.1)*. Then*

$$\mathrm{Tan}^{k-1}(\sigma_V^*, x_0) = \mathrm{Tan}^{k-1}(\sigma_V, x_0) \qquad \forall x_0 \in E \text{ such that } \lim_{r\to 0} \frac{\sigma_V\big(E \cap B_r(x_0)\big)}{\sigma_V\big(B_r(x_0)\big)} = 1. \tag{4.4}$$

In particular, the first equality in (4.4) *holds σ_V^*-a.e..*

Proof. Let us fix $x_0 \in \partial\mathcal{M}$ such that

$$\lim_{r\to 0} \frac{\sigma_V\big(E \cap B_r(x_0)\big)}{\sigma_V\big(B_r(x_0)\big)} = \lim_{r\to 0} \frac{\sigma_V^*\big(B_r(x_0)\big)}{\sigma_V\big(B_r(x_0)\big)} = 1,$$

let $\nu \in \mathrm{Tan}^{k-1}(\sigma_V, x_0)$ and $r_j \downarrow 0$ such that

$$\sigma_j := \frac{1}{r_j^{k-1}}\big(\tau_{x_0,r_j}\big)_\#\sigma_V \overset{*}{\rightharpoonup} \nu.$$

We are going to prove that

$$\sigma_j^* := \frac{1}{r_j^{k-1}}\big(\tau_{x_0,r_j}\big)_\#\sigma_V^* \overset{*}{\rightharpoonup} \nu.$$

Without loss of generality we can assume $x_0 = 0$. Let us consider $f \in C_c(\mathbb{R}^n)$. We have

$$\begin{aligned}\int f(x)\,\mathrm{d}\nu(x) &= \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \int f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x)\\ &= \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \bigg(\int_E f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x) + \int_{\mathbb{R}^n\setminus E} f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x)\bigg).\end{aligned}$$

For the last term there exists $c > 0$ such that

$$\begin{aligned}\frac{1}{r_j^{k-1}} \int_{\mathbb{R}^n\setminus E} \Big|f\Big(\frac{x}{r_j}\Big)\Big|\,\mathrm{d}\sigma_V(x) &\leq \frac{c}{r_j^{k-1}}\sigma_V\big(B_{cr_j} \setminus E\big)\\ &\leq c\frac{\sigma_V\big(B_{cr_j}\big)}{r_j^{k-1}} \frac{\sigma_V\big(B_{cr_j} \setminus E\big)}{\sigma_V\big(B_{cr_j}\big)} \xrightarrow[j\to\infty]{} 0\end{aligned}$$

since E has density 1 with respect to σ_V and $\sigma_V(B_{cr_j})/r_j^{k-1}$ remains bounded by $\sigma_j \overset{*}{\rightharpoonup} \nu$. Hence

$$\int f(x)\,\mathrm{d}\nu(x) = \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \int f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V^*(x),$$

thus

$$\sigma_j^* = \frac{1}{r_j^{k-1}} \big(\tau_{0,r_j}\big)_{\#}\sigma_V^* \overset{*}{\rightharpoonup} \nu.$$

This proves $\mathrm{Tan}^{k-1}(\sigma_V, x_0) \subset \mathrm{Tan}^{k-1}(\sigma_V^*, x_0)$; the reverse inclusion can be proved in a similar way. $\square$

4.4 Tangent measures to σ_V^*

In this section we prove that, for σ_V^*-a.e. $x_0 \in \partial\mathcal{M}$, each tangent measure to σ_V^* coincides to $\mathcal{H}^{k-1} \llcorner S$ for some $(k-1)$-dimensional plane S.

More precisely, the main result of this section is the following proposition.

Proposition 4.12. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$ and $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$. Then there exists a σ_V^*-measurable set $G \subset \partial\mathcal{M}$ which satisfies*

$$\sigma_V^*(\partial\mathcal{M} \setminus G) = 0,$$
$$\mathrm{Tan}^{k-1}(\sigma_V^*, x_0) \subset \{\alpha\mathcal{H}^{k-1} \llcorner S \mid \alpha > 0,\ S \text{ is a } (k-1)\text{-plane}\} \qquad \forall x_0 \in G.$$

The proof is obtained as consequence of some lemmata. The structure is the following:

1. In Lemma 4.13 we first prove that at every $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$, tangent varifolds to V are cones in the half-space $T_{x_0}^+\mathcal{M}$ that are stationary with respect to $\mathfrak{X}_t(T_{x_0}^+\mathcal{M})$. Thus, if $C \in \mathrm{Tan}(V, x_0)$ for some $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$, Corollary 3.13 provides the existence of the measure σ_C. We also prove that the measures σ_j relative to the scalings of V converge weakly to σ_C; This imply that (Corollary 4.14)
$$\mathrm{Tan}^{k-1}(\sigma_V^*, x_0) = \{\sigma_C \mid C \in \mathrm{Tan}(V, x_0)\} \qquad \text{for } \sigma_V^*\text{-a.e. } x_0 \in \partial\mathcal{M}.$$
2. In Lemma 4.15 we prove that each tangent cone C has an invariant linear subspace D_C that coincides with the set of points where the k-density $\Theta^k(\|C\|, \cdot)$ attains its maximum;
3. In Lemma 4.19 we show that for σ_V^*-a.e. $x_0 \in \partial\mathcal{M}$, if $C \in \mathrm{Tan}(V, x_0)$, then σ_C is concentrated on D_C;
4. In Lemma 4.20 we prove that for σ_V^*-a.e. $x_0 \in \partial\mathcal{M}$, if $C \in \mathrm{Tan}(V, x_0)$, then D_C is $(k-1)$-dimensional and $\sigma_C = \alpha\mathcal{H}^{k-1} \llcorner D_C$ for some constant $\alpha = \alpha(x_0, C)$;
5. In subsection 4.5 we summarize all these facts to conclude the proof of Proposition 4.12: for σ_V^*-a.e. $x_0 \in \partial\mathcal{M}$, every $(k-1)$-tangent measure to σ_V at x_0 is a $(k-1)$-plane.

4.4.1 Intermediate lemmata

We begin by studying tangent varifolds to V at points on $\mathrm{Dens}(V)$.

Lemma 4.13. *Let $V \in \mathcal{V}_k(\mathcal{M})$ have free boundary at $\partial\mathcal{M}$, let us assume that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$, that $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$ and $r_j \downarrow 0$ and let us use the notations for the scalings defined in section 4.1; then the following statements hold:*

1. *every V_j has free boundary at $\partial\mathcal{M}_j$ with $\tilde{H}_j(x) = r_j\tilde{H}(r_jx)$ for every $x \in \partial\mathcal{M}_j$ and*
$$\sigma_{V_j} = \frac{1}{r_j^{k-1}}(\tau_{0,r_j})_\#\sigma_V; \tag{4.5}$$
2. *there exist a subsequence of r_j, not relabeled, and a k-varifold C with $\operatorname{supp} C \subset T^+_{x_0}\mathcal{M}$ such that*
$$V_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} C.$$
In particular, $\operatorname{Tan}(V, x_0) \neq \emptyset$.
3. *C is stationary with respect to $\mathfrak{X}_t(T^+_{x_0}\mathcal{M})$;*
4. *if σ_C is the measure given by Corollary 3.13 relative to C, we have*
$$\sigma_j := \sigma_{V_j} \underset{j\to\infty}{\overset{*}{\rightharpoonup}} \sigma_C;$$
5. *If in addition $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$ (that is if V satisfies Assumption 4.3), then C is a rectifiable cone and σ_C is scaling invariant, that is*
$$\frac{1}{r^{k-1}}(\tau_{0,r})_\#\sigma_C = \sigma_C \qquad \forall r > 0.$$

Proof. To simplify the notations, we assume without loss of generality that $x_0 = 0 \in \partial\mathcal{M} \cap \operatorname{Dens}(V)$ and $T^+_0\mathcal{M} = \{x_n \geq 0\}$.

Step 1: We first prove that every V_j has free boundary at $\partial\mathcal{M}_j$. For every $X \in \mathfrak{X}_t(\mathcal{M}_j)$ the vector field $x \mapsto X(\frac{x}{r_j})$ belongs to $\mathfrak{X}_t(\mathcal{M})$, thus we can apply (2.18) to V and get

$$\begin{aligned}\int_{G_k(\mathcal{M}_j)} \operatorname{div}_S X(x)\,\mathrm{d}V_j(x,S) =& \frac{1}{r_j^{k-1}}\int_{G_k(\mathcal{M})} \operatorname{div}_S X\big(\frac{x}{r_j}\big)\,\mathrm{d}V(x,S)\\ =& -\frac{1}{r_j^{k-1}}\int_{\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), H(x)\rangle\,\mathrm{d}\|V\|(x).\end{aligned}$$

Thus, if we define $H_j(x) = r_jH(r_jx)$ for all $x \in \mathcal{M}_j$, by changing again variables we get

$$\int_{G_k(\mathcal{M}_j)} \operatorname{div}_S X(x)\,\mathrm{d}V_j(x,S) = -\int_{\mathcal{M}_j} \langle X(x), H_j(x)\rangle\,\mathrm{d}\|V_j\|(x),$$

that is V_j has free boundary at $\partial\mathcal{M}_j$, with $H_j(x) = r_jH(r_jx)$. By Corollary 3.13, V_j has bounded first variation and there exists $\sigma_j = \sigma_{V_j}$ and $\tilde{H}_j$. By a similar argument and by definition of $\tilde{H}$ it is easily seen that

$$\tilde{H}_j(x) = r_j\tilde{H}(r_jx) \qquad \text{for } \|V_j\|\text{-a.e. } x \in \partial\mathcal{M}_j.$$

If $X \in \mathfrak{X}(\mathcal{M}_j)$, then

$$\begin{aligned}\int_{G_k(\mathcal{M}_j)} \operatorname{div}_S X(x)\,\mathrm{d}V_j(x,S) =& \frac{1}{r_j^{k-1}}\int_{G_k(\mathcal{M})} \operatorname{div}_S X\big(\frac{x}{r_j}\big)\,\mathrm{d}V(x,S)\\ =& -\frac{1}{r_j^{k-1}}\int_{\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), H(x) + \tilde{H}(x)\rangle\,\mathrm{d}\|V\|(x)\\ &+ \frac{1}{r_j^{k-1}}\int_{\partial\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), N(x)\rangle\,\mathrm{d}\sigma_V(x)\\ =& -\int_{\mathcal{M}_j} \langle X, H_j + \tilde{H}_j\rangle\,\mathrm{d}\|V_j\| + \frac{1}{r_j^{k-1}}\int_{\partial\mathcal{M}_j} \langle X, N_j\rangle\,\mathrm{d}(\tau_{0,r_j})_\#\sigma_V.\end{aligned}$$

If we compare this equality with (3.28) for V_j tested with X, we get

$$\sigma_j := \sigma_{V_j} = \frac{1}{r_j^{k-1}}(\tau_{0,r_j})_{\#}\sigma_V,$$

that is σ_j is obtained by scaling σ_V.

Step 2: We now want to study the limit of the sequence V_j. Since $0 \in \mathrm{Dens}(V)$, by Theorem 3.5 there exists a constant $c > 0$ such that

$$\|V_j\|(B_1) = \|(\tau_{0,r_j})_\sharp V\|\big(B_1\big) = \frac{\|V\|\big(B_{r_j}\big)}{r_j^k} \leq c \qquad \forall j \in \mathbb{N}. \tag{4.6}$$

Thus by compactness of Radon measures, there exist $C \in \mathcal{V}_k(\mathbb{R}^n)$ and a subsequence of r_j, not relabeled, such that

$$V_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} C. \tag{4.7}$$

Step 3: Clearly $\operatorname{supp} C \subset T_0^+\mathcal{M}$. To show that C is stationary with respect to $\mathfrak{X}_t(T_0^+\mathcal{M})$, we test with a vector field $X \in \mathfrak{X}_t(T_0^+\mathcal{M})$. We first need the following estimate on H_j (exactly the same relation holds for $\tilde{H}_j$):

$$\|H_j\|_{L^p(B_1)} = \left(\int_{B_1} |H_j|^p \,\mathrm{d}\|V_j\|\right)^{\frac{1}{p}} = \left(\frac{1}{r_j^{k-p}}\int_{B_{r_j}} |H|^p \,\mathrm{d}\|V\|\right)^{\frac{1}{p}} \xrightarrow[j\to\infty]{} 0, \tag{4.8}$$

where in the second equality we have used $H_j(x) = r_j H(r_j x)$ and the limit follows by $0 \in \mathrm{Dens}(V)$ and $|H| \leq |H + \tilde{H}|$ because $H(x) \in T_x\partial\mathcal{M}$ (Remark 2.16) while $\tilde{H} \in (T_x\partial\mathcal{M})^\perp$.

To prove that C is stationary, let us pick $X \in \mathfrak{X}_t(T_0^+\mathcal{M})$ with compact support; there exists a sequence of vector fields $X_j \in \mathfrak{X}_t(\mathcal{M}_j)$ with compact support such that $X_j \to X$ in the C^1 topology. Hence, using (4.7) and (4.8) we get

$$\begin{aligned}\int_{G_k(T_0^+\mathcal{M})} \mathrm{div}_S\, X(x)\,\mathrm{d}C(x,S) &= \lim_{j\to\infty}\int_{G_k(\mathcal{M}_j)} \mathrm{div}_S\, X_j(x)\,\mathrm{d}V_j(x,S)\\ &= -\lim_{j\to\infty}\int_{\mathcal{M}_j} \langle X_j(x), H_j(x)\rangle \,\mathrm{d}\|V_j\|(x)\\ &= 0\end{aligned}$$

that is, by definition, C is stationary with respect to $\mathfrak{X}_t(T_0^+\mathcal{M})$.

Step 4: Corollary 3.13 provides that C has bounded first variation and the existence of σ_C. We want to prove that

$$\sigma_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} \sigma_C. \tag{4.9}$$

To this aim, we first remark that $\operatorname{supp}\sigma_C \subset T_0\partial\mathcal{M}$ and, since $T_0\partial\mathcal{M}$ is flat, we have that $\tilde{H}_C = 0$. We next need an uniform bound on $\sigma_j(B_1)$ and we use (3.4): since the second fundamental form of $\partial\mathcal{M}_j$ go to 0 as $r_j \to 0$, when we apply (3.4) to V_j in B_1, the constant c in (3.4) is bounded uniformly in j. This implies that there exists an uniform constant c such that for each $j \in \mathbb{N}$

$$\begin{aligned}\sigma_j(B_1) =& \frac{\sigma_V(B_{r_j})}{r_j^{k-1}}\\ \leq& \frac{c\|V\|(B_{2r_j})}{r_j^k} + \frac{1}{r_j^{k-1}}\int_{B_{2r_j}} |H|\,\mathrm{d}\|V\| \\ \leq& \frac{c\|V\|(B_{2r_j})}{r_j^k} + r_j^{1-\frac{k-s}{p}}\left(\frac{1}{r_j^s}\int_{B_{2r_j}} |H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{\|V\|(B_{2r_j})}{r_j^k}\right)^{1-\frac{1}{p}}\end{aligned} \tag{4.10}$$

where $s \in (k-p, k]$ is such that 0 satisfies the s-density condition in the definition of $\mathrm{Dens}(V)$. The last member of (4.10) is uniformly bounded in j since $0 \in \mathrm{Dens}(V)$, Theorem 3.5 and by $\frac{k-s}{p} \leq 1$. This proves the uniform bound on $\sigma_j(B_1)$.

To complete the proof of (4.9), let $X \in \mathfrak{X}(\mathbb{R}^n)$ be a vector field with compact support. If e_n is the n-th coordinate unit vector (that is the interior unit normal vector to $\partial T_0^+\mathcal{M}$), we have

$$\begin{aligned}
-\int_{T_0\partial\mathcal{M}} \langle X, e_n\rangle \,\mathrm{d}\sigma_C &= \int_{G_k(T_0^+\mathcal{M})} \mathrm{div}_S\, X(x) \,\mathrm{d}C(x,S) \\
&= \lim_{j\to\infty} \int_{G_k(\mathcal{M}_j)} \mathrm{div}_S\, X(x) \,\mathrm{d}V_j(x,S) \\
&= -\lim_{j\to\infty} \left(\int_{\mathcal{M}_j} \langle X(x), H_j(x) + \tilde{H}_j(x)\rangle \,\mathrm{d}\|V_j\|(x) \right. \\
&\qquad\qquad \left. + \int_{\partial\mathcal{M}_j} \langle X(x), N_j(x)\rangle \,\mathrm{d}\sigma_j(x) \right) \\
&= \lim_{j\to\infty} \int_{\partial\mathcal{M}_j} \langle X(x), N_j(x)\rangle \,\mathrm{d}\sigma_j(x),
\end{aligned} \tag{4.11}$$

where the first identity follows by $H_C = \tilde{H}_C = 0$ and the last one follows by (4.8). Thus $N_j\sigma_j \overset{*}{\rightharpoonup} -e_n\sigma_C$ is proved, since $\mathfrak{X}(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n, \mathbb{R}^n)$, by (4.11) and the uniform bound on $\sigma_j(B_1)$ (4.10). To prove (4.9), it is enough to observe that, if $f \in C_c(\mathbb{R}^n, \mathbb{R})$, then

$$\int f \,\mathrm{d}\sigma_C = \int \langle f e_n, e_n\rangle \,\mathrm{d}\sigma_C = -\lim_{j\to\infty} \int \langle f e_n, N_j\rangle \,\mathrm{d}\sigma_j = \lim_{j\to\infty} \int f \,\mathrm{d}\sigma_j,$$

by the uniform bound on $\sigma_j(B_1)$ and since $\partial\mathcal{M}_j \to \partial T_0^+\mathcal{M}$ in the C^1 topology.

Step 5: We are left to prove that, if $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$, then C is a rectifiable cone. The scaling invariance of σ_C will be an easy consequence of this. We first claim that C is rectifiable. In fact, we have:

- $\Theta^k(\|V_j\|, x) \geq 1$ for $\|V_j\|$-a.e. $x \in \mathcal{M}_j$, since $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$;
- $\sup_{j\in\mathbb{N}} \|V_j\|(B_1) < \infty$ by (4.6);
- the V_j's have locally uniformly bounded first variations because $0 \in \mathrm{Dens}(V)$, by (4.8) and the uniform bound on (4.10).

Thus we can apply to the sequence V_j the compactnes theorem for rectifiable varifolds [Sim84, Theorem 42.7], which proves that C is rectifiable and that $\Theta^k(\|C\|, x) \geq 1$ for $\|C\|$-a.e. $x \in \mathcal{M}$. In particular, there exists a k-rectifiable set $\Gamma \subset T_0^+\mathcal{M}$ such that

$$C = \theta(x)\mathcal{H}^k \llcorner \Gamma \otimes T_x\Gamma,$$

with $\theta(x) = \Theta^k(\|C\|, x) \geq 1$ for $\mathcal{H}^k$-a.e. $x \in \Gamma$ and $T_x\Gamma$ is the approximate tangent space of Γ at x.

It remains to show that C is a cone. The argument is exactly the same as [Sim84, Theorem 19.3], but we recall it for the reader convenience. Since C is rectifiable, to prove

$$(\tau_{0,\lambda})_\sharp C = C \qquad \forall \lambda > 0, \tag{4.12}$$

(that is the fact that C is a cone), it is enough to show that

$$\Theta^k(\|C\|, \lambda x) = \Theta^k(\|C\|, x) \qquad \forall x \in \mathbb{R}^n, \forall \lambda > 0,$$

that is θ is homogeneous of degree 0. This is clearly implied by

$$\|C\|(\lambda A) = \lambda^k \|C\|(A) \qquad \forall A \subset \mathbb{R}^n \text{ Borel}, \forall \lambda > 0.$$

By approximation, it is enough to prove that, for every 0-homogeneous function $h \in C^1(\mathbb{R}^n \setminus \{0\})$, one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\lambda^k}\int \gamma\Big(\frac{|x|}{\lambda}\Big) h(x)\,\mathrm{d}\|C\|(x)\right) = 0 \qquad \forall \lambda > 0, \tag{4.13}$$

where γ is the cut-off function defined in Section 2.1. To this aim, if $\lambda > 0$ is such that $\|C\|(\partial B_\lambda) = 0$, we observe that

$$\frac{\|C\|(B_\lambda)}{\lambda^k} = \lim_{j\to\infty} \frac{\|V_j\|(B_\lambda)}{\lambda^k} = \lim_{j\to\infty} \frac{\|V\|(B_{\lambda r_j})}{\lambda^k r_j^k} = \Theta^k(\|V\|, 0).$$

Since there exists at most a countable number of radii λ_j such that $\|C\|(\partial B_{\lambda_j}) > 0$, by approximation it follows that

$$\frac{\|C\|(B_\lambda)}{\lambda^k} = \Theta^k(\|V\|, 0) \qquad \forall \lambda > 0. \tag{4.14}$$

If we write the monotonicity identity (3.31) for C, since $H_C = \tilde{H_C} = 0$ and $\partial(T_0^+\mathcal{M})$ is flat, integrating between σ, λ we obtain

$$\begin{aligned}
\frac{1}{\lambda^k}\int \gamma\Big(\frac{|x|}{\lambda}\Big)\,\mathrm{d}\|C\|(x) - \frac{1}{\sigma^k}\int \gamma\Big(\frac{|x|}{\sigma}\Big)\,\mathrm{d}\|C\|(x)& \\
= \frac{1}{\lambda^k}\int \gamma\Big(\frac{|x|}{\lambda}\Big)\big|P_{S^\perp}\nabla|x|\big|^2\,\mathrm{d}C(x,S)& \\
- \frac{1}{\sigma^k}\int \gamma\Big(\frac{|x|}{\sigma}\Big)\big|P_{S^\perp}\nabla|x|\big|^2\,\mathrm{d}C(x,S)& \\
+ \int \big|P_{S^\perp}\nabla|x|\big|^2\left(\int_\sigma^\lambda \frac{k}{\rho^{k+1}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\rho\right)\mathrm{d}C(x,S).&
\end{aligned} \tag{4.15}$$

Letting γ increase to $\mathbf{1}_{[0,1)}$ in (4.15), by dominated convergence theorem and (4.14) we get

$$0 = \frac{\|C\|(B_\lambda)}{\lambda^k} - \frac{\|C\|(B_\sigma)}{\sigma^k} = \int_{G_k(B_\lambda\setminus B_\sigma)} \frac{\big|P_{S^\perp}\nabla|x|\big|^2}{|x|^k}\,\mathrm{d}C(x,S).$$

Since the last term is non-negative, we have

$$\big|P_{S^\perp}\nabla|x|\big|^2 = \left|P_{S^\perp}\frac{x}{|x|}\right|^2 = 0 \qquad \text{for } C\text{-a.e. } (x,S) \in G_k(T_0^+\mathcal{M}),$$

that is

$$C\Big(\big\{(x,S) \mid P_S(x) \neq x\big\}\Big) = C\Big(\big\{(x,S) \mid x \notin S\big\}\Big) = 0. \tag{4.16}$$

If h is 0-homogeneous, then $\langle \nabla h(x), x\rangle = 0$ and (4.16) implies

$$\langle \nabla h(x), P_S\, x\rangle = \langle \nabla h(x), x\rangle = 0 \qquad \text{for } C\text{-a.e. } (x,S) \in G_k(T_0^+\mathcal{M}). \tag{4.17}$$

Since C has free boundary at $\partial T_0^+\mathcal{M}$, by testing (2.10) for C with $X(x) = h(x)\gamma\Big(\frac{|x|}{\lambda}\Big)x \in \mathfrak{X}_t(T_0^+\mathcal{M})$, one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\lambda^k}\int \gamma\Big(\frac{|x|}{\lambda}\Big)h(x)\,\mathrm{d}\|C\|(x)\right) &= -\frac{1}{\lambda^{k+1}}\int_{G_k(T_0^+\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}C(x,S) \\
&\quad + \frac{1}{\lambda^{k+1}}\int_{G_k(T_0^+\mathcal{M})} \gamma\Big(\frac{|x|}{\lambda}\Big)\langle \nabla h(x), P_S\, x\rangle\,\mathrm{d}C(x,S) \\
&= 0,
\end{aligned}$$

where the last equality follows by $H_C = 0$ and (4.17). This proves (4.13), thus C is a cone. The scaling invariance of σ_C is a trivial consequence of (3.28) applied to C and (4.12).

□

Before going on, we highlight an easy consequence of Lemma 4.10, Lemma 4.11 and Lemma 4.13.

Corollary 4.14. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$ and let σ_V be the measure provided by Corollary 3.13. Then*

$$\mathrm{Tan}^{k-1}(\sigma_V^*, x_0) = \{\sigma_C \mid C \in \mathrm{Tan}(V, x_0)\} \qquad \text{for } \sigma_V^*\text{-a.e. } x_0 \in \partial\mathcal{M}.$$

Proof. By Lemma 4.13 it clearly follows that for each $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$,

$$\mathrm{Tan}^{k-1}(\sigma_V, x_0) = \{\sigma_C \mid C \in \mathrm{Tan}(V, x_0)\}.$$

By Lemma 4.10 and Lemma 4.11 we have the conclusion. □

It is well-known that, for a cone, points with maximal density form a linear subspace and that the cone is invariant by translation with respect these points (see e.g. [Sim96, Section 3.3] and [Whi97, Theorem 3.1, Example (4) of Section 4]). We report here the simple proof of this fact for the sake of completeness.

Lemma 4.15. *Let V satisfy Assumption 4.3, let $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$ be fixed and let C be a tangent cone to V at x_0. Then the set*

$$D_C = \{y \in T_{x_0}\partial\mathcal{M} \mid \Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0)\}$$

is a linear subspace of $\mathbb{R}^n$. Moreover the translated cone $(L_y)_\sharp C$ coincides with C for all $y \in D_C$. In particular, if σ_C is the measure given by Corollary 3.13 relative to C, then $(L_y)_\# \sigma_C = \sigma_C$.

Definition 4.16. We say that D_C is the *invariant subspace* of the cone C.

Proof of Lemma 4.15. Without loss of generality we can assume that $x_0 = 0$. Let us call $\theta_0 = \Theta^k(\|V\|, 0)$. By Lemma 4.13 C is rectifiable and it holds $\Theta^k(\|V\|, 0) = \Theta^k(\|C\|, 0)$. Since C is a cone, for $y \in T_0\partial\mathcal{M}$ we have

$$\theta_0 \overset{(4.14)}{=} \lim_{r\to+\infty} \frac{\|C\|\big(B_r\big)}{r^k} \geq \lim_{r\to+\infty} \frac{\|C\|\big(B_{r-|y|}(y)\big)}{(r-|y|)^k} \frac{(r-|y|)^k}{r^k} = \lim_{r\to+\infty} \frac{\|C\|\big(B_r(y)\big)}{r^k} \geq \Theta^k(\|C\|, y). \tag{4.18}$$

The last inequality is given by the monotonicity identity for C: in the last member of (3.31), the first two integrals disapper (since $H + \tilde{H} = 0$ and $\langle N(x), x\rangle = 0$) and the last term is non-negative. (4.18) shows that

$$\Theta^k(\|C\|, 0) \geq \Theta^k(\|C\|, y) \qquad \forall y \in T_0\partial\mathcal{M}. \tag{4.19}$$

If $y \in D_C$, then (4.18) yields

$$\frac{\|C\|\big(B_r(y)\big)}{r^k} = \theta_0 \qquad \forall r > 0.$$

By the same arguments of Lemma 4.13, it follows that C is a rectifiable cone also with respect to y. By rectifiability of C, in order to show that $(L_y)_\sharp C = C$, it is enough to prove that

$$\Theta^k(\|C\|, z) = \Theta^k(\|C\|, y + z) \qquad \forall z \in T_0^+\mathcal{M}. \tag{4.20}$$

To this aim, let $z \in T_0^+\mathcal{M}$ be an arbitrary point. Since C is a cone with respect to y, we have that

$$\Theta^k(\|C\|, z) = \Theta^k\Big(\|C\|, y + \frac{1}{2}(z - y)\Big) = \Theta^k\Big(\|C\|, \frac{1}{2}(y + z)\Big).$$

On the other hand, since C is a cone we have

$$\Theta^k(\|C\|, y+z) = \Theta^k\Big(\|C\|, \frac{1}{2}(y+z)\Big).$$

This shows (4.20) and hence $(L_y)_\sharp C = C$. The translation invariance of σ_C is a trivial consequence of (3.28) applied to C and of $(L_y)_\sharp C = C$.

It remains to show that D_C is a linear subspace of $\mathbb{R}^n$. Since C is a cone, if $y \in D_C$, then $\lambda y \in D_C$ for each $\lambda > 0$. Since C is a cone also with respect to y, then $\lambda y \in D_C$ also if $\lambda < 0$. If $y, z \in D_C$, it follows from the previous discussion that also $y + z \in D_C$ and this proves that D_C is a linear subspace. ☐

Before going on, we first recall the definition of *approximate continuity*:

Definition 4.17 (Approximate continuity)**.** If μ is a positive Radon measure and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Borel function, we say that f is *approximately continuous* at $x \in \mathbb{R}^n$ with respect to μ if

$$\lim_{r\to 0} \frac{\mu\big(\{z \in B_r(x) \mid |f(z) - f(x)| > \varepsilon\}\big)}{\mu(B_r(x))} = 0 \qquad \forall \varepsilon > 0.$$

Remark 4.18. It is well-known that, if μ is a Radon measure, then every μ-measurable function is approximate continuous at μ-a.e. point (see e.g. [EG15, Theorem 1.37] where the proof is done for the Lebesgue measure, but the same arguments can be applied to any Radon measure).

The following lemma states that there exists a set F of full σ_V-measure with respect to $\operatorname{Dens}(V)$ such that for every $x \in F$ the invariant subspace D_C of any cone $C \in \operatorname{Tan}(V, x)$ coincides with $\operatorname{supp} \sigma_C$.

Lemma 4.19. *Let V satisfy Assumption 4.3. Then there exists a set $F \subset \operatorname{Dens}(V) \cap \partial\mathcal{M}$ that satisfies*

$$\sigma_V(\operatorname{Dens}(V) \setminus F) = 0$$

and has the following property: for every $x_0 \in F$ and for every $C \in \operatorname{Tan}(V, x_0)$ we have either $\sigma_C = 0$ or $\sigma_C \neq 0$ and

$$\Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0) \qquad \forall y \in \operatorname{supp} \sigma_C.$$

In particular, either $\sigma_C = 0$ or $\operatorname{supp} \sigma_C = D_C$.

Proof. The idea of the proof is the following: we first define the "good" set F of full σ_V-measure with respect to $\operatorname{Dens}(V)$ where $\Theta^k(\|V\|, \cdot)$ exists and is approximate continuous with respect to σ_V. Next, we fix $x_0 \in F$, $r_j \to 0$ and, using the notations for the scalings,

$$V_j \overset{*}{\rightharpoonup} C \in \operatorname{Tan}(V, x_0).$$

We fix $y \in \operatorname{supp} \sigma_C$ and we assume by contradiction that the statement is false. We find a tiny ball $B_r(y)$ where $\Theta^k(\|V_j\|, \cdot)$ is close to $\Theta^k(\|C\|, y)$ for j sufficiently large. Since $\Theta^k(\|C\|, y)$ is close to $\|C\|(B_\rho(y))/\rho^k$ for small ρ, this is achieved by weak convergence $V_j \overset{*}{\rightharpoonup} C$ and using the properties of φ_{x_0} stated in Theorem 3.5. Since there exists a constant β such that $\sigma_j(B_r(y)) > \beta > 0$ for large j, this contradicts the approximate continuity of $\Theta^k(\|V\|, \cdot)$ at x_0 with respect to σ_V.

Step 1: We are going first to define the set F of full σ_V-measure with respect to $\operatorname{Dens}(V)$ and next we will prove that the conclusion of the theorem holds for every $x \in F$.

We call A the set of points $x \in \partial\mathcal{M}$ that satisfy all the following conditions:

1. $x \in \mathrm{Dens}(V) \cap \operatorname{supp} \sigma_V$ and is of density 1 for $\mathrm{Dens}(V)$ with respect to σ_V, that is
$$\lim_{\rho \to 0} \frac{\sigma_V\big(\mathrm{Dens}(V) \cap B_\rho(x)\big)}{\sigma_V\big(B_\rho(x)\big)} = 1; \tag{4.21}$$
2. x is a point of approximate continuity for $\Theta^k(\|V\|, \cdot)$ with respect to σ_V (which is a well-defined Borel function in $\mathrm{Dens}(V)$ and can be extended, for instance, to be 0 outside $\mathrm{Dens}(V)$; the chosen extension does not influence the approximate continuity, because by (4.21) x has density 1 in $\mathrm{Dens}(V)$, where $\Theta^k(\|V\|, \cdot)$ is well-defined).

By Lebesgue differentiation theorem and by Remark 4.18, we have
$$\sigma_V(\mathrm{Dens}(V) \setminus A) = 0. \tag{4.22}$$
In addition, by Theorem 3.5, for every $x \in A$ the map $\rho \mapsto \varphi_x(\rho)$ is monotone increasing and converge pointwise to 0 as $\rho \downarrow 0$, that is
$$\lim_{\rho \to 0} \varphi_x(\rho) = 0 \qquad \forall x \in A,$$
and for every $\rho > 0$ the map $x \mapsto \varphi_x(t)$ is Borel-measurable by Remark 3.24.

Thus, by Egoroff's Theorem, for every $h \in \mathbb{N}$ there exists a set $F_h \subset A$ such that
$$\sigma_V(A \setminus F_h) \leq 1/h, \qquad \varphi_x(\rho) \underset{\rho \to 0}{\overset{\text{unif}}{\longrightarrow}} 0 \text{ on } F_h. \tag{4.23}$$
Up to removing sets of σ_V-measure 0, we can assume that every $x \in F_h$ is a point of density 1 with respect to σ_V, that is
$$\lim_{\rho \to 0} \frac{\sigma_V\big(F_h \cap B_\rho(x)\big)}{\sigma_V\big(B_\rho(x)\big)} = 1 \qquad \forall x \in F_h.$$
We now define
$$F = A \cap \Big(\bigcup_{h \in \mathbb{N}} F_h\Big).$$
By (4.22) and (4.23) it follows
$$\sigma_V^*(\mathrm{Dens}(V) \setminus F) = 0.$$

Let us fix $x_0 \in F$ and let us consider $h \in \mathbb{N}$ such that $x_0 \in F_h$. Without loss of generality we can assume $x_0 = 0$.

Now let us fix $r_j \downarrow 0$. Using the notations for the scalings, since $F \subset \mathrm{Dens}(V)$, by Lemma 4.13 there exists a subsequence, not relabeled, such that $V_j \overset{*}{\rightharpoonup} C$ with $C \in \mathrm{Tan}(V, 0)$ and C is a rectifiable cone with $\Theta^k(\|C\|, y) \geq 1$ for $\|C\|$-a.e. $y \in \mathbb{R}^n$.

We now need a technical remark that is useful in the rest of the proof: recalling that $H_j(y) = r_j H(r_j y)$, by a simple change of variables one obtains
$$r_j x \in \mathrm{Dens}(V) \quad \Leftrightarrow \quad x \in \mathrm{Dens}(V_j).$$
More precisely, if we call φ_x^j the function for V_j defined in Theorem 3.5, we get
$$\varphi_x^j(\rho) = \varphi_{r_j x}(r_j \rho). \tag{4.24}$$

Step 2: We can now begin with the proof. If $\sigma_C = 0$ there is nothing to prove. Thus we can assume $\sigma_C \neq 0$ and $\emptyset \neq \operatorname{supp} \sigma_C \subset \partial T_0^+ \mathcal{M}$. Since C is a cone, by (4.19) we have

$$\Theta^k(\|C\|, y) \leq \Theta^k(\|C\|, 0) \qquad \forall y \in \partial T_0^+ \mathcal{M}.$$

By contradiction, let us assume that there exists $y \in \operatorname{supp} \sigma_C$ and $\varepsilon > 0$ such that

$$\Theta^k(\|C\|, y) < \Theta^k(\|C\|, 0) - \varepsilon. \tag{4.25}$$

Since σ_C is scaling invariant by Lemma 4.13, without loss of generality we can assume that $y \in B_{1/2}$. By the uniform convergence (4.23) and by definition of density, there exists $\rho \in (0, 1/2)$ such that

$$\frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} + \varphi_z(\rho) \leq \Theta^k(\|C\|, y) + \frac{\varepsilon}{8} \qquad \forall z \in F_h. \tag{4.26}$$

Since $V_j \overset{*}{\rightharpoonup} C$, without loss of generality we can choose ρ so that there exists $J \in \mathbb{N}$ and a small $0 < r < \rho$ for which

$$\left| \frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} - \frac{\|V_j\|\big(B_\rho(y)\big)}{(\rho - r)^k} \right| < \frac{\varepsilon}{8} \qquad \forall j > J. \tag{4.27}$$

Let us choose $j > J$; for every $z \in B_r(y)$ such that $r_j z \in F_h$ we have

$$\begin{aligned}
\Theta^k(\|V_j\|, z) &\leq \frac{\|V_j\|\big(B_{\rho-r}(z)\big)}{(\rho - r)^k} + \varphi_z^j(\rho - r) \\
&\leq \frac{\|V_j\|\big(B_\rho(y)\big)}{(\rho - r)^k} + \varphi_{r_j z}(r_j \rho) \\
&\overset{(4.27)}{\leq} \frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} + \varphi_{r_j z}(r_j \rho) + \frac{\varepsilon}{8} \\
&\overset{(4.26)}{\leq} \Theta^k(\|C\|, y) + \frac{\varepsilon}{4} \\
&\overset{(4.25)}{\leq} \Theta^k(\|C\|, 0) - \frac{3\varepsilon}{4} \\
&= \Theta^k(\|V_j\|, 0) - \frac{3\varepsilon}{4},
\end{aligned}$$

where we used the fact that every φ_x is increasing and (4.24). This shows that, for $j > J$,

$$B_r(y) \cap \frac{1}{r_j} F_h \subseteq \left\{ z \in B_1 \mid \big|\Theta^k(\|V_j\|, z) - \Theta^k(\|V_j\|, 0)\big| > \frac{\varepsilon}{2} \right\}.$$

Step 3: We now want to estimate from below the measure of this set to get a contradiction with the approximate continuity of $\Theta^k(\|V\|, \cdot)$ at 0.

By approximate continuity of the $\Theta^k(\|V\|, \cdot)$ at 0 with respect to σ_V we have

$$\begin{aligned}
0 &= \limsup_{j \to \infty} \frac{\sigma_V\left(\left\{ z \in B_{r_j} \mid \big|\Theta^k(\|V\|, z) - \Theta^k(\|V\|, 0)\big| > \frac{\varepsilon}{2} \right\}\right)}{\sigma_V(B_{r_j})} \\
&\geq \limsup_{j \to \infty} \frac{\sigma_V\big(B_{r r_j}(r_j y) \cap F_h\big)}{\sigma_V(B_{r_j})} \\
&= \limsup_{j \to \infty} \left(\frac{\sigma_V\big(B_{r r_j}(r_j y)\big)}{\sigma_V(B_{r_j})} - \frac{\sigma_V\big(B_{r r_j}(r_j y) \setminus F_h\big)}{\sigma_V(B_{r_j})} \right) \\
&= \limsup_{j \to \infty} \frac{\sigma_V\big(B_{r r_j}(r_j y)\big)}{\sigma_V(B_{r_j})} \\
&= \limsup_{j \to \infty} \frac{\sigma_j\big(B_r(y)\big)}{\sigma_j(B_1)},
\end{aligned} \tag{4.28}$$

where the equality in the fourth line is given by

$$0 \le \limsup_{j\to\infty} \frac{\sigma_V\big(B_{rr_j}(r_j y)\setminus F_h\big)}{\sigma_V(B_{r_j})} \le \lim_{j\to\infty} \frac{\sigma_V\big(B_{r_j}\setminus F_h\big)}{\sigma_V(B_{r_j})} = 0,$$

since $y \in B_{1/2}$, $r \in (0, 1/2)$ and the fact that 0 is of density 1 for F_h with respect to σ_V; the last equality in (4.28) follows by (4.5).

We want to estimate from below the last term in (4.28) to get a contradiction. To do so, let us notice that, by $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$ and $y \in \operatorname{supp}\sigma_C$, there exist two constants $c, \beta > 0$ such that, for j sufficiently large,

$$\sigma_j\big(B_1\big) \le c, \qquad \sigma_j\big(B_r(y)\big) \ge \beta,$$

which contradicts (4.28).

This also prove the inclusion $\operatorname{supp}\sigma_C \subset D_C$. To prove the other inclusion, let us notice that the scaling invariance of σ_C and $\operatorname{supp}\sigma_C \neq \emptyset$ imply $0 \in \operatorname{supp}\sigma_C$. Since σ_C is invariant by translations along D_C by Lemma 4.15, we have the opposite inclusion and $\operatorname{supp}\sigma_C = D_C$.

□

We next prove that, for every $x \in F$ (where F is the set defined in the previous Lemma) such that $\Theta_*^{k-1}(\sigma_V, x) > 0$ and for every $C \in \operatorname{Tan}(V, x)$, σ_C is the surface measure of a $(k-1)$-plane, which coincides with D_C.

Lemma 4.20. *Let V satisfy Assumption 4.3 and let F be the set defined in Lemma 4.19. For every $x_0 \in F$ such that $\Theta_*^{k-1}(\sigma_V, x_0) > 0$ and for every $C \in \operatorname{Tan}(V, x_0)$, the invariant subspace D_C of C is $(k-1)$-dimensional; moreover there exists $\alpha_0 > 0$ such that*

$$\sigma_C = \alpha_0 \mathcal{H}^{k-1} \llcorner D_C.$$

Proof. Let $x_0 \in F$ be a fixed point and let us assume $C \in \operatorname{Tan}(V, x_0)$. Without loss of generality we can assume that $x_0 = 0$ and that $T_0\partial\mathcal{M}$ is the subspace $\{x_n = 0\}$. Since $C \in \operatorname{Tan}(V, 0)$ there exists $r_j \downarrow 0$ such that, using the notations for the scalings, $V_j \overset{*}{\rightharpoonup} C$. Lemma 4.13 asserts C is a rectifiable cone and that $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$, where σ_C is the measure relative to C given by Corollary 3.13.

We first recall that the condition

$$\Theta_*^{k-1}(\sigma_V, 0) = \liminf_{r\to 0} \frac{\sigma_V\big(B_r(x_0)\big)}{r^{k-1}} > 0, \tag{4.29}$$

together with $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$, implies that $\sigma_C \neq 0$. Thus, Lemma 4.19 provides $\operatorname{supp}\sigma_C = D_C$, where D_C is the invariant subspace D_C of C, given by Lemma 4.15. D_C is a linear subspace of $\mathbb{R}^n$ and throughout this proof we call $m = \dim D_C$ its dimension. By definition of D_C, we clearly have $D_C \subset \{x_n = 0\}$. Thus $m \le n-1$. After a suitable change of coordinates, we can assume that $D_C = \{x_{m+1} = \cdots = x_n = 0\}$.

For any $y \in D_C$ and any $r > 0$, we denote by $Q_{D_C}(y, r)$ the closed cube included in D_C with center y, side of length r and faces parallel to the coordinate vectors $e_1, \dots, e_m$. If we set

$$\alpha_0 = \liminf_{j\to\infty} \frac{\sigma_V\big(B_{r_j}\big)}{\omega_{k-1} r_j^{k-1}}.$$

we have $\alpha_0 > 0$, by (4.29). Since σ_C is invariant by scalings (by Lemma 4.13) and by translations in D_C (by Lemma 4.15), there exists a fixed $\beta_0 > 0$ such that

$$\sigma_C\big(Q_{D_C}(y, r)\big) = \beta_0 r^{k-1} \qquad \forall y \in D_C \quad \forall r > 0. \tag{4.30}$$

We are going to show that (4.30) implies $m = k-1$. We argue by contradiction and by cases:

- Let us assume, by contradiction, that $m < k-1$. For each $l \in \mathbb{N}$, there exists a covering $\{Q_i^l\}_{i=1}^{2^{lm}}$ of $Q_{D_C}(0,1)$ such that each Q_i^l is a cube included in D_C and of side length 2^{-l}. Therefore
$$\sigma_C\big(Q_{D_C}(0,1)\big) \leq \sum_{i=1}^{2^{lm}} \sigma_C\big(Q_i^l\big) = 2^{lm}\beta_0 2^{-l(k-1)} \leq \beta_0 2^{-l} \xrightarrow[l\to\infty]{} 0.$$
Thus $\sigma_C\big(Q_{D_C}(0,1)\big) = 0$. By translation invariance of σ_C, it follows that $\sigma_C = 0$, which is a contradiction.

- Let us assume now that $m \geq k$. We first observe that by approximation, (4.30) holds also for cubes that are open in D_C. Hence, taking for every $l \in \mathbb{N}$ a covering $\{Q_i^l\}_{i=1}^{2^{lm}}$ of $Q_{D_C}(0,1)$ of cubes included in D_C with disjoint interiors and of side length 2^{-l}, we have
$$\sigma_C\big(Q_{D_C}(0,1)\big) \geq \sum_{i=1}^{2^{lm}} \sigma_C\Big(\big(Q_i^l\big)^{\text{o}}\Big) = 2^{lm}\beta_0 2^{-l(k-1)} \geq \beta_0 2^{l} \xrightarrow[l\to\infty]{} +\infty,$$
(where $\big(Q_i^l\big)^{\text{o}}$ is intended in the topology of D_C) which is a contradiction.

This shows that $\dim D_C = k-1$. Since σ_C is invariant by scalings and translations in D_C, we have
$$\frac{\sigma_C\big(B_r(y)\big)}{\mathcal{H}^{k-1}\big(D_C \cap B_r(y)\big)} = \alpha_0 \qquad \forall y \in D_C,\ \forall r > 0.$$
By Radon-Nikodym Theorem [Sim84, Theorem 4.7], we obtain that $\sigma_C = \alpha_0 \mathcal{H}^{k-1} \llcorner D_C$. □

4.4.2 Proof of Proposition 4.12

We now have all the ingredients to prove Proposition 4.12.

Proof of Proposition 4.12. Let E be the set defined in (4.1) and let F be the set defined in the Lemma 4.19. Since $\Theta_*^{k-1}(\sigma_V, \cdot) > 0$ on E, by Lemma 4.20 it follows that for every $x_0 \in E \cap F$ and for every $C \in \operatorname{Tan}(V, x_0)$, there exists $\alpha > 0$ and a $(k-1)$-dimensional plane S such that
$$\sigma_C = \alpha \mathcal{H}^{k-1} \llcorner S,$$
where σ_C is the measure given by Corollary 3.13 applied to C, see Lemma 4.13. Hence Corollary 4.14 implies that there exists a set $G \subset E \cap F$ which satisfies
$$\sigma_V^*\big((E \cap F) \setminus G\big) = 0,$$
$$\operatorname{Tan}^{k-1}(\sigma_V^*, x_0) \subset \{\alpha \mathcal{H}^{k-1} \llcorner S \mid \alpha > 0,\ S \text{ is a } (k-1)\text{-plane}\} \qquad \forall x_0 \in G.$$
It remains to prove that G is of full σ_V^*-measure. To do so, it is enough to show that $E \cap F$ is of full σ_V^*-measure. We estimate
$$\begin{aligned}\sigma_V^*\big(\mathcal{M} \setminus (E \cap F)\big) \leq& \sigma_V^*(\mathcal{M} \setminus E) + \sigma_V^*(\mathcal{M} \setminus F)\\ =& \sigma_V^*(\mathcal{M} \setminus E) + \sigma_V^*\big(\mathcal{M} \setminus \operatorname{Dens}(V)\big) + \sigma_V^*\big(\operatorname{Dens}(V) \setminus F\big)\\ =& 0.\end{aligned}$$
The last equality is a consequence of the following facts:

- $\sigma_V^*\big(\mathcal{M} \setminus E\big) = 0$ by definition of σ_V^*;
- $\sigma_V^*\big(\mathcal{M} \setminus \operatorname{Dens}(V)\big) = 0$ by Lemma 4.10 and $p > 1$;
- $\sigma_V^*\big(\operatorname{Dens}(V) \setminus F\big) = 0$ by Lemma 4.19 and $\sigma_V^* \ll \sigma_V$.

□

4.5 Conclusion of the proof of Theorem 4.1

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let E be the set defined in (4.1). We define $\tilde{E}$ to be the set of points of E which are of density 1 with respect to σ_V, that is

$$\tilde{E} = \left\{ x \in E \mid \lim_{r \to 0} \frac{\sigma_V\big(E \cap B_r(x)\big)}{\sigma_V\big(B_r(x)\big)} = 1 \right\}.$$

Notice that the above ratios are well-defined by the fact that $\Theta_*^{k-1}(\sigma_V, \cdot) > 0$ on E. In other words, $\tilde{E}$ is the set of points of density 1 of σ_V^* with respect to σ_V. For every $x \in \tilde{E}$ we have

$$\Theta_*^{k-1}(\sigma_V^*, x) = \limsup_{r \to 0} \frac{\sigma_V^*\big(B_r(x)\big)}{r^{k-1}} = \limsup_{r \to 0} \frac{\sigma_V\big(B_r(x)\big)}{r^{k-1}} \frac{\sigma_V^*\big(B_r(x)\big)}{\sigma_V\big(B_r(x)\big)} = \Theta_*^{k-1}(\sigma_V, x),$$

and the same equality holds for $(k-1)$-upper densities. Since by Lebesgue-Besicovitch Differentiation Theorem we have $\sigma_V^*(\mathcal{M} \setminus \tilde{E}) = 0$, by definition of E it follows

$$0 < \Theta_*^{k-1}(\sigma_V^*, x) \leq \Theta^{*(k-1)}(\sigma_V^*, x) < +\infty \qquad \text{for } \sigma_V^*\text{-a.e. } x \in \partial\mathcal{M},$$

thus σ_V^* satisfies the first condition of the Marstrand-Mattila Rectifiability Criterion (Theorem 4.5).

By Proposition 4.12, σ_V^* satisfies also the second condition of the Marstrand-Mattila Rectifiability Criterion, thus it is $(k-1)$ rectifiable. □

Part II

Existence and regularity of minimal surfaces with prescribed angle

Chapter 5

Min-max minimal surfaces with prescribed angle condition

5.1 Introduction

Throughout this chapter, if not otherwise specified, we assume that $\mathcal{M} \subset \mathbb{R}^3$ and that every surface is 2-dimensional.

As we stated in Definition 2.5, a surface $\Sigma \subset \mathcal{M}$ that is stationary with respect to vector fields which are compactly supported in $\mathcal{M}^o$ is called *minimal.* It is then natural to ask whether such surfaces exists. Clearly an example is $S \cap \mathcal{M}$, where S is any 2-plane in $\mathbb{R}^3$. Thus the question is of course trivial if we do not require any other condition on Σ on $\partial\mathcal{M}$. Some of the possible choices are to fix $\partial\Sigma$ to be a given fixed curve (which is a version of the well-known *Plateau problem*) or to give contact angle conditions on Σ at $\partial\mathcal{M}$.

In this chapter we study the existence and regularity of a minimal surface $\Sigma \subset \mathcal{M}$ which meets $\partial\mathcal{M}$ with a fixed angle θ. This is based on the work [DD21].

5.1.1 Background and main result

Minimal surfaces, i.e. critical points of the area functional, are a fundamental object in Geometric Analysis and the problem of finding a minimal surface (possibly with given side constraints) in a given Riemannian manifold $\mathcal{M}$ has been a fundamental topic of research in the last 50 years.

When the topology of the ambient manifold $\mathcal{M}$ is non-trivial, existence results can be often obtained via minimization of the area functional in suitable homology classes. On the other hand, when the topology is trivial, it is easy to see that non-trivial minimizers do not exist and one has to rely on different methods.

Since the seminal work of Birkhoff about the existence of closed geodesic in Riemannian manifolds, the use of min-max techniques has been a successful tool to show existence of critical points for a variety of functionals.

The implementation of min-max techniques for the construction of minimal surfaces has been started in the seminal work of Almgren and Pitts and deeply relies on techniques of Geometric Measure Theory. One of the fundamental outcomes of the theory has been to show the existence of a d dimensional minimal surface in every $(d+1)$-dimensional manifold, [Pit14; CD03; DT13] (to be precise the minimal surface is allowed to have a “small” singular set when $d \geq 7$). Since their seminal work, the strategy has been suitably modified to show the existence of surfaces with prescribed curvature [ZZ19], given boundary [DR18] as well as free boundary, [DR18]. Furthermore these techniques have been as well used to show the existence of *infinitely many* minimal surfaces in a given Riemannian manifold [MN17; IMN18; Son18]

The main result of this chapter (appeared in the work [DD21] and obtained independently by Li, Zhou and Zhu in [LZZ21]) is to show how Almgren-Pitts techniques (and more precisely

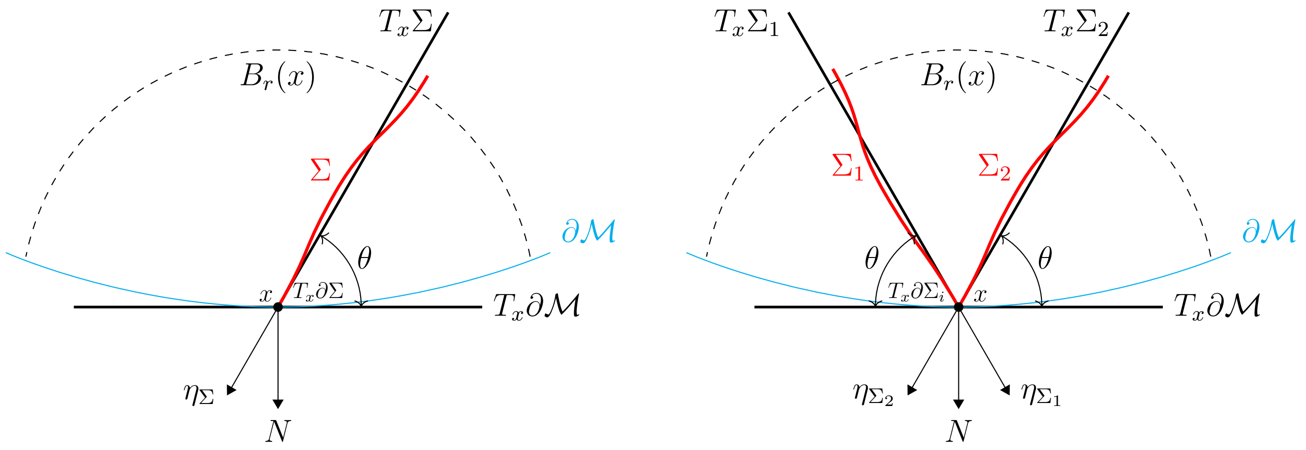

Figure 5.1: The two situations described by Theorem 5.1

their revised version by Colding-De Lellis, [CD03]) allows the existence of minimal surfaces with *prescribed contact angle at the boundary* in convex subsets of $\mathbb{R}^3$.

Theorem 5.1. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a convex bounded open set with $C^{2,\alpha}$ boundary and assume that $\theta \in (0, \pi/2)$. Then there exists a minimal surface $\Sigma \subset \mathcal{M}$ with $\partial\Sigma \subset \partial\mathcal{M}$ with the property that, for every $x \in \partial\Sigma$, there exists $r > 0$ such that one of the following is satisfied:*

1. *$\Sigma \cap B_r(x)$ is made of a single smooth surface Σ such that*
$$\eta_\Sigma \cdot N = \sin\theta \qquad \text{on } \partial\Sigma;$$
2. *$\Sigma \cap B_r(x) = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are smooth minimal surfaces such that*
$$\eta_{\Sigma_i} \cdot N = \sin\theta \qquad \text{on } \partial\Sigma_i, \qquad \text{for } i = 1, 2 \qquad \text{and}$$
$$T_x\Sigma_1 \cap T_x\Sigma_2 = T_x\Sigma_1 \cap T_x\partial\mathcal{M} = T_x\Sigma_2 \cap T_x\partial\mathcal{M}. \tag{5.1}$$

Here η_Σ denotes the exterior co-normal field to Σ at $\partial\Sigma$ and N is the exterior unit normal to $\partial\mathcal{M}$.

We remark that the condition (5.1) means that $T_x\Sigma_1$ and $T_x\Sigma_2$ are *symmetrical* with respect to $N(x)$, as shown in the Figure 5.1.

5.1.2 Sketch of the argument

The main idea of the proof, which has already appeared in the Master Thesis of the first author [DeM18], where some preliminary steps of this program were carried out, is to apply Almgren-Pitts/Colding-De Lellis techniques to the *capillarity energy* defined in section 2.1.6:

$$F_a(\Omega) = \mathcal{H}^2(\partial_i\Omega) + \cos\theta\,\mathcal{H}^2(\partial_b\Omega), \tag{5.2}$$

where $\Omega \subset \mathcal{M}$ has finite perimeter and $\partial_i\Omega = \partial^*\Omega \cap \mathcal{M}^o$, $\partial_b\Omega = \partial^*\Omega \cap \partial\mathcal{M}$. Note indeed that a smooth stationary point of (5.2) with respect to $\mathfrak{X}_t(\mathcal{M})$ (see also Proposition 2.19) satisfies

$$H_\Sigma = 0 \text{ in } \mathcal{M}^o, \qquad \eta_\Sigma \cdot N = \sin\theta \quad \text{on } \partial\Sigma \cap \partial\mathcal{M},$$

where $\Sigma = \partial_i\Omega$. With respect to the case of the pure area functional, or to the free boundary one (when $\theta = \pi/2$), two main new difficulties arise. First, a priori, the capillarity functional is only defined for surfaces which are boundary of a set: we need to extend it to class of rectifiable varifold, a technical step which is instrumental to the proof of Theorem 5.1. The key idea here is to "decouple" the functional and initially look at the interior and boundary part as independent. We use Definition 2.12, where we have defined the capillarity functional F_a for pairs of varifolds $(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$. Once this has been done, the pull-tight procedure can be carried on and one can show the existence of a stationary pair for F_a (see Definition 2.12 again), that is the existence of a stationary varifold which satisfies a suitable notion of "angle condition" which is a slight modification on the one proposed by Kagaya and Tonegawa in [KT17] (see Section 2.2.2). This will be done in Section 5.3.

The second key step is to show regularity of the obtained varifold; here we start by following the stategy of [Pit14; CD03; DT13], and showing that the stationary varifold found in the previous step is ε-minimizing in annuli and thus it admits replacements there, see Section 5.4 and 5.7. Existence of replacements then follows by the the regularity theory of [DM15; DM17] together with the curvature estimates proved for 2-dimensional stable critical point of the capillarity functional. These estimates, recalled in Section 5.6 can be easily obtained if one assumes an a priori area growth on the surface (a condition which is met on our construction) and have been recently established in full generality in [HS21]. Existence of replacements implies that the obtained varifold is induced by a smooth surface outside a finite set of points and the proof follows the one in [DR18]. In order to remove the final singular points we instead rely on a new argument which strongly uses that $\theta < \pi/2$. Under this assumption we can show that stable surface is indeed graphical and satisfies a suitable free boundary problem, from which we can deduce the desired regularity. This is done in Section 5.8.

5.1.3 Comments on the hypotheses

We conclude the introduction by commenting some of the hypotheses of Theorem 5.1.

- The convexity of $\mathcal{M}$ is used in showing that the "interior" part of Σ does not touch the boundary of $\mathcal{M}$ and it looks plausible that by following the argument of [Li15], the assumption can be dispensed. Moreover, it will be clear from the proof that we can endow $\mathcal{M}$ with any (smooth) Riemaniann metric for which the boundary is still convex and obtain the same result.

- The restriction to dimension 3 appears instead only in one (crucial) point of our proof, namely in showing curvature estimates for stable capillarity surfaces in Section 5.6. We conjecture that similar estimates should hold true in higher dimension as well and if this is the case, the same arguments of this chapter will show the existence of n-dimensional surfaces into convex sets of $(n+1)$-dimensional manifolds.

  We remark that Li, Zhou and Zhu have obtained in [LZZ21, Theorem C.1] these higher dimensional curvature estimates for a certain range of contact angles.

  In the general case, we record in section 5.4.4 to what extent one can conclude on the existence of min-max minimal hypersurfaces with contact angle condition for general dimensions for a future reference.

5.1.4 Plan of the chapter

This chapter is divided as follows:

- In Section 5.2 we define the basic elements of min-max procedure: the idea of parametrized family of surfaces and of *sweepout*; next we present the notion of minimizing sequence of sweepouts and of min-max sequence of pairs of surfaces, which are the approximating sequences for the desired minimal surface, whose existence is stated in Theorem 5.1.

- In Section 5.3 we first prove that the limit of any min-max sequence of pairs is non-trivial, that is its first component does not vanish at the limit. Next we prove the *pull-tight* procedure, which shows the existence of a particular minimizing sequence of sweepouts with the property that, every min-max sequence of pairs converges to a stationary pair for the capillarity functional.

- In section 5.4 we begin to develop the tools in order to obtain the regularity for the limit pair: we show that there exists a particular min-max sequence that has the *almost-minimizing* property in annuli: this means that, in a sufficiently large class of annuli, one cannot deform the pairs of the min-max sequence without passing through a pair for which F_a assumes a larger value of a fixed amount. The existence of this almost-minimizing min-max sequence is obtained by a combinatorial argument originally due to Almgren-Pitts. This almost-minimizing property is the prototype of a partial stability for the min-max sequence, which is the key for obtaining regularity. At the end, we record in subsection 5.4.4 what we can conclude for the problem of finding a min-max hypersurface in $\mathcal{M} \subset \mathbb{R}^{n+1}$ with fixed angle at the boundary for $n \geq 3$.

- In section 5.5 we prove the stability inequality for capillarity functional (Proposition 5.25), which is the starting point for the regulairty theory for the limit pair (V, W) of the almost-minimizing min-max sequence.

- In section 5.6, we recall the curvature estimates (Theorem 5.34) for pairs of surfaces which are stationary and stable for F_a in a half-space. These curvature estimates imply a compactness result (Theorem 5.35) for stable pairs which is crucial to obtain existence of the replacements for the min-max limit pair.

- In section 5.7 we collect all the ingredients to get existence of replacements for the limit pair in annuli. A replacement for a pair (V, W) in a annulus An is a new stationary pair (V', W') that coincides with (V, W) outside An and is induced in An by a pair of smooth surfaces which is stationary for F_a. This is obtained by considering a constrained minimizing problem, called the *homotopic Plateau problem*: we fix an annulus An and, for each pair (Σ_k, Γ_k) of the min-max sequence we consider all deformations of (Σ_k, Γ_k) supported in An which decrease the value of F_a without passing through a pair with too higher F_a. Thanks to the almost-minimizing property of the min-max sequence, the solutions of these problems turn out to form another min-max sequence, which converges to a replacement (V', W') for the initial limit pair (V, W).

  Existence of replacements is a key step in the regularity theory: the next step is to prove that the limit pair coincides with its replacements in annuli, obtaining regularity.

- In Section 5.8 we show that the replacement property implies the regularity of the limit pair (V, W) of the min-max sequence. We first prove that the V is integer rectifiable in the interior and that the tangent varifolds to V are half-planes. Next we, establish the regularity of V in any punctured ball $B_{r(x)}(x) \setminus \{x\}$ with suitable conditions on $r(x)$ and this implies that V is regular up to a finite set of possibly singular points. After this step, we remove the final singular points by showing that the V can be written as a graph on $\partial\mathcal{M}$; using the regularity for solutions of a suitable free boundary problem, we obtain the full regularity of V.

5.2 Elements of min-max procedure

In this section we define the elements of the min-max procedure and report some of their basic properties.

Definition 5.2. A family $\{\Sigma_t\}_{t\in[0,1]}$ is a *parametrized smooth family of surfaces* if it satisfies the following properties:

1. For every $t \in [0,1]$ Σ_t is a closed subset of $\mathcal{M}$ of finite $\mathcal{H}^2$-measure;
2. For every $t \in [0,1]$ there exists a finite set S_t such that $\Sigma_t \setminus S_t$ is a smooth surface with boundary $\partial\Sigma_t \subset \partial\mathcal{M}$ which is smooth outside of S_t;
3. if $t \to s$, then $\sup_{x\in\Sigma_t} \mathrm{dist}(x, \Sigma_s) \to 0$;
4. if $U \subset\subset \mathcal{M} \setminus S_s$, then Σ_t converges in the smooth sense to Σ_s in U as $t \to s$;
5. the function $t \mapsto \mathcal{H}^2(\Sigma_t)$ is continuous.

A first elementary consequence of the definition of parametrized family is that the area of the surfaces converge also locally. This is useful in the following, then we record it in the following lemma.

Lemma 5.3. *Let $\{\Sigma_t\}_{t\in[0,1]}$ be a smooth parametrized family of surfaces. Then, for every open set $U \subset \mathcal{M}$, it holds*
$$\lim_{t\to s} \mathcal{H}^2(\Sigma_t \cap U) = \mathcal{H}^2(\Sigma_s \cap U).$$

Proof. For every $\varepsilon > 0$, there exists $W \subset\subset \mathcal{M} \setminus S_s$ such that $\mathcal{H}^2(\Sigma_s \cap W) > \mathcal{H}^2(\Sigma_s) - \varepsilon$. By smooth convergence in W, we have that

$$\lim_{t\to s} \mathcal{H}^2(\Sigma_t \cap U \cap W) = \mathcal{H}^2(\Sigma_s \cap U \cap W), \qquad \lim_{t\to s} \mathcal{H}^2(\Sigma_t \cap (W \setminus U)) = \mathcal{H}^2(\Sigma_s \cap (W \setminus U))$$

Since $\mathcal{H}^2(\Sigma_t) \to \mathcal{H}^2(\Sigma_s)$ by the choice of W we have also $|\mathcal{H}^2(\Sigma_t \cap U \setminus W) - \mathcal{H}^2(\Sigma_s \cap U \setminus W)| < \varepsilon$ for $|t - s|$ sufficiently small. This implies that $|\mathcal{H}^2(\Sigma_t \cap U) - \mathcal{H}^2(\Sigma_s \cap U)| < \varepsilon$ for $|t - s|$ sufficiently small. □

We often refer to a parametrized smooth family of surfaces simply as a family of surfaces.

Definition 5.4 (Sweepout)**.** A pair $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ of parametrized family of surfaces with (finite) singular sets S_t and G_t is called a *sweepout* if there exists a family of open sets $\{\Omega_t\}_{t\in[0,1]}$ in the relative topology of $\mathcal{M}$ that satisfy the following properties:

1. for every $t \in [0,1]$ $\Sigma_t \subset \mathcal{M}$ and $\Gamma_t \subset \partial\mathcal{M}$;
2. for every $t \in [0,1]$ $\Sigma_t \cap \Gamma_t = \partial\Sigma_t = \partial\Gamma_t$ (here the boundary is intended in the sense of differential topology);
3. for every $t \in [0,1]$, $\partial_i\Omega_t \setminus \Sigma_t \subset S_t$;
4. for every $t \in [0,1]$, $\big(\partial_b\Omega_t \setminus (\Sigma_t \cup \Gamma_t)\big) \cup \big((\Sigma_t \cap \partial\mathcal{M}) \setminus \partial_b\Omega_t\big) \cup \big(\Gamma_t \setminus \partial_b\Omega_t\big) \subset S_t \cup G_t$;
5. $\Omega_0 = \emptyset$ and $\Omega_1 = \mathcal{M}$;
6. if $t \to s$, then $\mathrm{Vol}(\Omega_t \setminus \Omega_s) + \mathrm{Vol}(\Omega_s \setminus \Omega_t) \to 0$.

Remark 5.5. Roughly speaking, a sweepout (Σ_t, Γ_t) is made of the internal and external boundaries of a family of open sets which are empty at $t = 0$ and grow until they fill all the container $\mathcal{M}$ at $t = 1$. Notice that Σ_t may have a non-trivial part on $\partial\mathcal{M}$.

As it is stated in [DT13, Proposition 0.4] sweepouts always exist.

Definition 5.6 (Pair compatible with open set)**.** A pair of surfaces (Σ, Γ) and an open set Ω are said to be *compatible* if the properties 1, 2, 3, 4 and 6 of Definition 5.4 holds with (Σ, Γ), Ω, S, G in place of (Σ_t, Γ_t), Ω_t, S_t, G_t.

Definition 5.7. Two sweepouts $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ are *homotopic* if there exists a parametrized family of surfaces $(\Sigma_{(t,s)}, \Gamma_{(t,s)})_{(t,s)\in[0,1]^2}$ such that:

- $\Sigma_{(t,0)} = \Sigma_t$, $\Gamma_{(t,0)} = \Gamma_t$ for every $t \in [0, 1]$;
- $\Sigma_{(t,1)} = \tilde{\Sigma}_t$ and $\Gamma_{(t,1)} = \tilde{\Gamma}_t$ for every $t \in [0, 1]$;
- for every $s \in [0, 1]$ the family $\{(\Sigma_{(t,s)}, \Gamma_{(t,s)})\}_{t\in[0,1]}$ is a sweepout.

Remark 5.8. Roughly speaking, two sweepouts are homotopic if one can be deformed into the other in a continuous way by a family of sweepouts.

Definition 5.9. A non-empty set of sweepouts $\mathscr{S}$ is *homotopically closed* if it contains the homotopy class of all its elements.

From now on, with $\mathscr{S}$ we always denote a homotopically closed set of sweepouts.

Definition 5.10. If $\mathscr{S}$ is an homotopically closed set of sweepouts, the *min-max value of* $\mathscr{S}$ is

$$m_0(\mathscr{S}) = \inf\{\max_{t\in[0,1]} F_a(\Sigma_t, \Gamma_t) \mid (\Sigma_t, \Gamma_t)_{t\in[0,1]} \in \mathscr{S}\}.$$

Definition 5.11 (minimizing sequence of sweepouts)**.** A sequence $\{(\Sigma^i_t, \Gamma^i_t)_{t\in[0,1]}\}^{i\in\mathbb{N}}$ of sweepouts in $\mathscr{S}$ is a *minimizing sequence* if

$$\lim_{i\to\infty} \max_{t\in[0,1]} F_a(\Sigma^i_t, \Gamma^i_t) = m_0(\mathscr{S}).$$

Definition 5.12 (min-max sequence)**.** If $(\Sigma^i_t, \Gamma^i_t)^{i\in\mathbb{N}}_{t\in[0,1]}$ is a minimizing sequence of sweepouts, a sequence $(\Sigma^i, \Gamma^i) := (\Sigma^i_{t_i}, \Gamma^i_{t_i})$ (where $t_i \in [0, 1]$ for every $i \in \mathbb{N}$) is a *min-max sequence* if

$$\lim_{i\to\infty} F_a\big(\Sigma^i, \Gamma^i\big) = m_0(\mathscr{S}).$$

5.3 Existence of non-trivial stationary pairs

In this section we prove that, given a homotopically closed set of sweepouts $\mathscr{S}$, there exists a minimizing sequence of sweepouts such that every min-max sequence converges, up to subsequences, to a pair $(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$ which is stationary with respect to F_a and which is non-trivial, that is $\|V\|(\mathcal{M}) > 0$. This is achieved in two steps:

- In section 5.3.1 we prove that, for every sweepout, the maximum of F_a is strictly greater of $a\mathcal{H}^2(\partial\mathcal{M})$ (Proposition 5.13); this implies that any limit (V, W) of a min-max sequence satisfies $\|V\|(\mathcal{M}) > 0$.
- In section 5.3.2 we perform the pull-tight procedure, which proves the existence of a particular miniminizing sequence of sweepouts, such that any min-max sequence converge to a stationary pair of varifolds.

5.3.1 Limits of min-max sequences are non-trivial

In order to prove that the limit pair of every min-max sequence is non-trivial we need the following proposition.

Proposition 5.13. *There exists* $\eta > 0$ *such that, for every sweepout* $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ *it holds*

$$\max_{t\in[0,1]} F_a(\Sigma_t, \Gamma_t) \geq a\mathcal{H}^2(\partial\mathcal{M}) + \eta.$$

Proof. We are going to prove that there exists $\varepsilon > 0$ and a differentiable function $f\colon [0,\varepsilon) \to \mathbb{R}^+$ such that, for every sweepout (Σ_t, Γ_t), $f(0) = 0$, $f'(s) > 0$ for $s \in (0,\varepsilon)$ and

$$F_a(\Sigma_t, \Gamma_t) \geq a\mathcal{H}^2(\partial\mathcal{M}) + f(|\mathcal{M} \setminus \Omega_t|) \qquad \forall t \in [0,1] \text{ s.t. } |\mathcal{M} \setminus \Omega_t| \leq \varepsilon.$$

This clearly implies the result. To prove the existence of such f, we write F_a in a different way, and next we use the relative isoperimetric inequality to obtain a lower bound on $F_a(\Sigma_t, \Gamma_t)$.

We fix a vector field $X \in \mathfrak{X}(\mathcal{M})$ such that $|X(x)| \leq 1$ for all $x \in \mathcal{M}$ and $X(x) = N(x)$ (the exterior unit normal to $\mathcal{M}$) for all $x \in \partial\mathcal{M}$; the C^1-norm of X depends only on $\mathcal{M}$. Using X we can write $F_a(\Sigma_t, \Gamma_t)$ as

$$\begin{aligned} F_a(\Sigma_t, \Gamma_t) =& \mathcal{H}^2(\Sigma_t) + a\mathcal{H}^2(\Gamma_t) \\ =& \mathcal{H}^2(\Sigma_t) + a\int_{\Sigma_t\cup\Gamma_t} X \cdot \nu_\Sigma \,\mathrm{d}\mathcal{H}^2 - a\int_{\Sigma_t} X \cdot \nu_\Sigma \,\mathrm{d}\mathcal{H}^2 \\ =& \int_{\Sigma_t} (1 - aX \cdot \nu_\Sigma)\,\mathrm{d}\mathcal{H}^2 + a\int_{\Omega_t} \operatorname{div} X(x)\,\mathrm{d}x. \end{aligned} \tag{5.3}$$

Since $\Sigma_t \supset \partial\Omega_t \cap \mathcal{M}^\circ$ holds up to sets of $\mathcal{H}^2$-measure 0, by the relative isoperimetric inequality there exists a constant $c(\mathcal{M}) > 0$ such that

$$\mathcal{H}^2(\Sigma_t) \geq \mathcal{H}^2(\partial\Omega_t \cap \mathcal{M}^\circ) \geq c(\mathcal{M}) \min\{|\Omega_t|, |\mathcal{M} \setminus \Omega_t|\}^{\frac{2}{3}}.$$

Thus, by (5.3) and $1 - aX \cdot \nu_\Sigma \geq 1 - a$, for every $t \in [0,1]$ we have

$$\begin{aligned} F_a(\Sigma_t, \Gamma_t) \geq& c(\mathcal{M})(1-a) \min\{|\Omega_t|, |\mathcal{M} \setminus \Omega_t|\}^{\frac{2}{3}} + a\int_{\mathcal{M}} \operatorname{div} X(x)\,\mathrm{d}x - a\int_{\mathcal{M}\setminus\Omega_t} \operatorname{div} X(x)\,\mathrm{d}x \\ =& c(\mathcal{M})(1-a) \min\{|\Omega_t|, |\mathcal{M} \setminus \Omega_t|\}^{\frac{2}{3}} + a\mathcal{H}^2(\partial\mathcal{M}) - a\int_{\mathcal{M}\setminus\Omega_t} \operatorname{div} X(x)\,\mathrm{d}x. \end{aligned}$$

If $\varepsilon > 0$ is sufficently small, for all $t \in [0,1]$ such that $|\mathcal{M} \setminus \Omega_t| \leq \varepsilon$, the previous inequality becomes

$$\begin{aligned} F_a(\Sigma_t, \Gamma_t) \geq& a\mathcal{H}^2(\partial\mathcal{M}) + c_1|\mathcal{M} \setminus \Omega_t|^{\frac{2}{3}} - c_2|\mathcal{M} \setminus \Omega_t| \\ =& a\mathcal{H}^2(\partial\mathcal{M}) + f(|\mathcal{M} \setminus \Omega_t|), \end{aligned}$$

where c_1, c_2 are positive constants depending on $\mathcal{M}, a$. Clearly $f \in C^1\big((0,\varepsilon)\big)$, $f(0) = 0$ and, up to choosing $\varepsilon > 0$ even smaller, we have and $f'(s) > 0$ for all $s \in (0,\varepsilon)$. $\square$

Proposition 5.13 implies the following corollary, which states that the first component of the limit of every min-max sequence is non-trivial.

Corollary 5.14. *For any homotopically closed set of sweepouts* $\mathscr{S}$, *we have* $m_0(\mathscr{S}) > a\mathcal{H}^2(\partial\mathcal{M})$. *In particular, if* $(\Sigma^k_{t_k}, \Gamma^k_{t_k})_k$ *is a min-max sequence in* $\mathscr{S}$ *which converges to a pair* $(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$, *then it holds*

$$\|V\|(\mathcal{M}) > 0.$$

5.3.2 Pull-tight procedure

We now want to prove that, given a homotopically closed set of sweepouts $\mathscr{S}$, there exists a minimizing sequence of sweepouts such that every min-max sequence converges to a pair which is stationary for the capillarity functional. From now on, we consider fixed the homotopically closed set of sweepouts $\mathscr{S}$ and we call $m_0 = m_0(\mathscr{S})$.

The strategy of the proof is based on the construction of a deformation of a fixed minimizing sequence; this deformation reduce the mass of the surfaces of a quantity that depends on how much the pair is "far" from being stationary. We next prove that this deformed minimizing sequence has the desired property. More precisely, this is sketched in the following steps:

- We construct a continuous map $(V, W) \in A^{2m_0} \mapsto Y_{(V,W)} \in \mathfrak{X}_t(\mathcal{M})$, where A^{2m_0} is the set of pairs of varifolds with a fixed bound on the total mass. The vector field $Y_{(V,W)} \in \mathfrak{X}_t(\mathcal{M})$ associated to a pair (V, W), if applied to (V, W) lower its capillarity functional F_a of an amount which depends on how far is (V, W) from being a stationary pair for F_a; moreover this deformation reduces the distance from (V, W) from the set of ; in other words the map $(V, W) \mapsto Y_{(V,W)}$ induces a sort of gradient flow on A^{2m_0}.
- We fix a minimizing sequence of sweepouts $(\Sigma_t^k, \Gamma_t^k)^{k\in\mathbb{N}}$ and we "deform it" by applying the "gradient flow" constructed above; since this lower F_a, we obtain another minimizing sequence of sweepouts $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)_{t\in[0,1]}^{k\in\mathbb{N}}$;
- If from $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)_{t\in[0,1]}^{k\in\mathbb{N}}$ we could extract a min-max sequence $(\tilde{\Sigma}_{t_k}^k, \tilde{\Gamma}_{t_k}^k)^k$ which has positive distance from the set of stationary pair, this would imply that the sequence $(\Sigma_{t_k}^k, \Gamma_{t_k}^k)^k$ extracted from $(\Sigma_t^k, \Gamma_t^k)_{t\in[0,1]}^{k\in\mathbb{N}}$ satisfies
$$\liminf_{k\to\infty} F_a(\Sigma_{t_k}^k, \Gamma_{t_k}^k) > m_0,$$
which would contradict the fact that the sequence of sweepouts $(\Sigma_t^k, \Gamma_t^k)_{t\in[0,1]}^{k\in\mathbb{N}}$ is minimizing.

Theorem 5.15 (Pull-tight procedure)**.** *There exists a minimizing sequence $\{(\Sigma_t^k, \Gamma_t^k)\}^k$ of sweepouts such that every min-max sequence converges, up to subsequences, to a pair $(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$ which is stationary for the capillarity functional.*

Proof. We follow the construction in [CD03]. Let us fix a minimizing sequence of sweepouts $\{(\Sigma_t^k, \Gamma_t^k)\}^{k\in\mathbb{N}}$.

Let us begin by calling
$$A^{2m_0} = \{(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M}) \mid \|V\|(\mathcal{M}) \le 2m_0, \|W\|(\partial\mathcal{M}) \le \mathcal{H}^2(\partial\mathcal{M})\}$$
and
$$A_\infty = \{(V, W) \in A^{2m_0} \mid (V, W) \text{ is stationary for } F_a\}.$$
Since the weak*-topology is metrizable on bounded sets, we choose two distances which induces it on $\{V \in \mathcal{V}_2(\mathcal{M}) \mid \|V\|(\mathcal{M}) \le 2m_0\}$ and $\{W \in \mathcal{V}_2(\partial\mathcal{M}) \mid \|W\|(\partial\mathcal{M}) \le \mathcal{H}^2(\partial\mathcal{M})\}$ and we call $\mathcal{D}$ the distance on the product space A^{2m_0} which induces the product topology. Up to scaling $\mathcal{D}$, we can assume that $\operatorname{diam} A^{2m_0} = 1$. Moreover A^{2m_0} is compact since it is a product of compact spaces. Since the property of being stationary is preserved under weak*-convergence of varifolds, it follows that A_∞ is also a compact subset of A^{2m_0}. We call $\mathcal{B}_r(V, W)$ the ball of radius r and center (V, W) with respect the distance $\mathcal{D}$.

For every $i \in \mathbb{N}$, we call
$$A_i = \Big\{(V, W) \in A^{2m_0} \mid \frac{1}{2^{i+1}} \le \mathcal{D}\big((V, W), A_\infty\big) \le \frac{1}{2^i}\Big\}.$$
Since every A_i is closed, it is compact.

Step 1: We claim that for every $i \in \mathbb{N}$ there exists a constant $c_i > 0$ such that, for every $(V, W) \in A_i$, there exists $X_{(V,W)} \in \mathfrak{X}_t(\mathcal{M})$ such that $|X_{(V,W)}(x)| \leq 1$ for every $x \in \mathcal{M}$ and

$$\begin{aligned}\delta_{F_a}(V, W)[X_{(V,W)}] = &\int_{G_2(\mathcal{M})} \operatorname{div}_S X_{(V,W)}(x) \,\mathrm{d}V(x, S) + a \int_{G_2(\partial\mathcal{M})} \operatorname{div}_{\partial\mathcal{M}} X_{(V,W)}(x) \,\mathrm{d}W(x, T_x\partial\mathcal{M}) \\ \leq &- c_i.\end{aligned} \tag{5.4}$$

Indeed, suppose by contradiction that the claim is false for some $i \in \mathbb{N}$; thus there exists a sequence of pairs $(V_j, W_j)_j$ in A_i such that for every j and for every $X \in \mathfrak{X}_t(\mathcal{M})$ with $|X| \leq 1$ we have

$$\Big| \int_{G_2(\mathcal{M})} \operatorname{div}_S X(x) \,\mathrm{d}V_j(x, S) + a \int_{G_2(\partial\mathcal{M})} \operatorname{div}_{\partial\mathcal{M}} X(x) \,\mathrm{d}W_j(x, T_x\partial\mathcal{M}) \Big| \leq \frac{1}{j}.$$

By compactness of A_i, (V_j, W_j) converges, up to subsequences, to $(V, W) \in A_i$. By the previous equation, it follows that (V, W) is stationary for F_a, which contradicts $(V, W) \in A_i$. This proves the claim.

Step 2: The map $(V, W) \in A^{2m_0} \setminus A_\infty \mapsto X_{(V,W)} \in \mathfrak{X}_t(\mathcal{M})$ is not necessarily continuous. We want now to use it to construct a continuous one with respect the C^1-topology in $\mathfrak{X}_t(\mathcal{M})$ that satisfies an inequality on the first variation of F_a similar to (5.4) in a neighborhood of (V, W).

We first observe that, by continuity of the first variation under varifold convergence, for every $i \in \mathbb{N}$ and every $(V, W) \in A_i$ there exists $r_{(V,W)} \in (0, \frac{1}{2^{i+2}})$ such that

$$\int_{G_2(\mathcal{M})} \operatorname{div}_S X_{(V,W)}(x) \,\mathrm{d}\tilde{V}(x, S) + a \int_{G_2(\partial\mathcal{M})} \operatorname{div}_{\partial\mathcal{M}} X_{(V,W)}(x) \,\mathrm{d}\tilde{W}(x, T_x\partial\mathcal{M}) \leq -\frac{c_i}{2} \tag{5.5}$$

whenever $\mathcal{D}\big((V, W), (\tilde{V}, \tilde{W})\big) \leq r_{(V,W)}$. Thus the family

$$\mathcal{F}_i = \{\mathcal{B}_{r_{(V,W)}}(V, W) \mid (V, W) \in A_i\}$$

(we recall that $\mathcal{B}_r(V, W)$ is the ball of center (V, W) and radius r with respect the distance $\mathcal{D}$) is an open cover of A_i and by compactness there exists a finite number J_i of pairs $\{(V_i^j, W_i^j)\}^{j=1,\dots,J_i}$ such that

$$A_i \subset \bigcup_{j=1}^{J_i} \mathcal{B}_{\frac{r_{i,j}}{2}}(V_i^j, W_i^j)$$

(where we call $r_{i,j} := r_{(V_i^j, W_i^j)}$). For simplicity of notation, we call $\mathcal{B}_{i,j} = \mathcal{B}_{r_{i,j}}(V_i^j, W_i^j)$ and $\tilde{\mathcal{B}}_{i,j} = \mathcal{B}_{\frac{r_{i,j}}{2}}(V_i^j, W_i^j)$. We underline that by the fact that $r_{(V,W)} \in (0, \frac{1}{2^{i+2}})$ it follows

$$A_h \cap \mathcal{B}_{i,j} = \emptyset \qquad \text{whenever } |i - h| \geq 2.$$

By the assumptions, the family

$$\mathcal{F} = \{\tilde{\mathcal{B}}_{i,j} \mid i \in \mathbb{N}, j = 1, \dots, J_i\}$$

is a locally finite covering of $A^{2m_0} \setminus A_\infty$. We can construct a partition of the unity $\psi_{i,j}$ with respect to this covering and define

$$Y_{(V,W)} = \mathcal{D}\big((V, W), A_\infty\big) \sum_{\substack{i \in \mathbb{N} \\ j=1,\dots,J_i}} \psi_{i,j}(V, W) X_{(V_i^j, W_i^j)} \qquad \forall (V, W) \in A^{2m_0}.$$

The map $(V,W) \mapsto Y_{(V,W)}$ is continuous with respect the C^1 topology of $\mathfrak{X}_t(\mathcal{M})$ and $Y_{(V,W)} = 0$ if $(V,W) \in A_\infty$. If we define for every $(V,W) \in A^{2m_0}$

$$\rho(V,W) = \min\Big\{\frac{r_{i,j}}{2} \mid (V,W) \in \tilde{\mathcal{B}}_{i,j}\Big\},$$

by definition we have $\rho(V,W) \le \frac{1}{2^{i+3}}$ whenever $(V,W) \in A_i$. Hence, if $(V,W) \in A_h$, for every $(\tilde{V},\tilde{W}) \in \mathcal{B}_{\rho(V,W)}(V,W)$ we have that

$$\begin{aligned}\delta_{F_a}(\tilde{V},\tilde{W})(Y_{(V,W)}) &= \mathcal{D}\big((\tilde{V},\tilde{W}),A_\infty\big) \sum_{\substack{|i-h|\le 1\\ j=1,\dots,J_i}} \psi_{i,j}(\tilde{V},\tilde{W})\delta_{F_a}(\tilde{V},\tilde{W})(X_{(V_i^j,W_i^j)})\\ &\le -\frac{1}{2^{h+2}}\min\{c_{h-1},c_h,c_{h+1}\},\end{aligned}$$

by (5.5), since $\mathcal{B}_{\rho(V,W)}(V,W) \subset \mathcal{B}_{i,j}$ for every i,j such that $(V,W) \in \tilde{\mathcal{B}}_{i,j}$.

By compactness again, there exist two continuous functions $d\colon [0,1] \to (0,1]$ and $f\colon (0,1] \to (0,+\infty)$ such that

$$d(s) \le 2^{-(i+3)} \qquad \forall s \in [2^{-(i+1)}, 2^{-i}], \tag{5.6}$$

$$\delta_{F_a}(\tilde{V},\tilde{W})(Y_{(V,W)}) \le -f\big(\mathcal{D}((V,W),A_\infty)\big) \quad \text{whenever} \quad \mathcal{D}\big((V,W),(\tilde{V},\tilde{W})\big) < d\big(\mathcal{D}((V,W),A_\infty)\big). \tag{5.7}$$

Step 3: We now want to use the vector field-valued map constructed to deform the minimizing sequence of sweepouts $(\Sigma_t^k,\Gamma_t^k)^{k\in\mathbb{N}}$ in a new minimizing sequence $(\tilde{\Sigma}_t^k,\tilde{\Gamma}_t^k)^{k\in\mathbb{N}}$ in $\mathscr{S}$.

If $(V,W) \in A^{2m_0}$, we define $(V_t,W_t) = \big((\psi_{(V,W),t})_\sharp V, (\psi_{(V,W),t})_\sharp W\big)$, where $\psi_{(V,W),t}$ is the flow map of $Y_{(V,W)}$ evaluated at the time t. Thus (5.7) yields the existence of a maximum $t_0(V,W) > 0$ such that, for any $t \in [0,t_0(V,W)]$ we have

$$\delta_{F_a}(V_t,W_t)(Y_{(V,W)}) \le -f\big(\mathcal{D}((V,W),A_\infty)\big) \quad \text{and} \quad \mathcal{D}((V_t,W_t),(V,W)) \le d\big(\mathcal{D}((V,W),A_\infty)\big).$$

Again by compactness of the sets A_i there exists a continuous function $T\colon (0,1] \to (0,1]$ such that, for every $(V,W) \in A^{2m_0}$ and for every $t \in [0,T\big(\mathcal{D}((V,W),A_\infty)\big)]$, it holds

$$\begin{aligned}\delta_{F_a}(V_t,W_t)(Y_{(V,W)}) &\le -f\big(\mathcal{D}((V,W),A_\infty)\big),\\ \mathcal{D}((V_t,W_t),(V,W)) &\le d\big(\mathcal{D}((V,W),A_\infty)\big) \overset{(5.6)}{<} \frac{\mathcal{D}((V,W),A_\infty)}{4}.\end{aligned}$$

Hence, setting $\beta = \mathcal{D}((V,W),A_\infty)$

$$F_a(V_{T(\beta)},W_{T(\beta)}) - F_a(V,W) \le \int_0^{T(\beta)} \delta_{F_a}(V_t,W_t)(Y_{(V,W)})\,\mathrm{d}t \le -T(\beta)f\big(\mathcal{D}((V,W),A_\infty)\big).$$

We can re-normalize the flows $\psi_{(V,W),t}$ by defining $\Psi_{(V,W),t}(x) = \psi_{(V,W),T(\beta)t}$, where again $\beta = \mathcal{D}((V,W),A_\infty)$. Thus, re-defining $(V_t,W_t) = \big((\Psi_{(V,W),t})_\sharp V, \Psi_{(V,W),t})_\sharp W\big)$ we have that there exists a continuous function $g\colon (0,1] \to (0,+\infty)$ such that $\lim_{s\to 0} g(s) = 0$ and

$$F_a(V_1,W_1) - F_a(V,W) \le -g\big(\mathcal{D}((V,W),A_\infty)\big), \qquad \mathcal{D}\big((V_1,W_1),(V,W)\big) < \frac{\mathcal{D}((V,W),A_\infty)}{4}.$$

Now we choose a minimizing sequence of sweepouts $\{(\Sigma_t^k,\Gamma_t^k)\}^{k\in\mathbb{N}}$ in $\mathscr{S}$ such that

$$\max_{t\in[0,1]} F_a(\Sigma_t^k,\Gamma_t^k) \le m_0(\mathscr{S}) + \frac{1}{k} \qquad \forall k \in \mathbb{N}.$$

Let us fix $k \in \mathbb{N}$. We would like to deform every (Σ_t^k, Γ_t^k) by the flow $\Psi_{(\Sigma_t^k,\Gamma_t^k),1}$, but since the map

$$Y_k \colon t \in [0,1] \mapsto Y_{(\Sigma_t^k,\Gamma_t^k)} \in \mathfrak{X}_t(\mathcal{M})$$

is only continuous, the family of pairs $t \mapsto (\Psi_{(\Sigma_t^k,\Gamma_t^k),1})_\sharp(\Sigma_t^k, \Gamma_t^k)$ may not belong to $\mathscr{S}$. Therefore we have to smooth the map Y_k. To this aim let us consider a smooth map $Z_k \colon [0,1] \to \mathfrak{X}_t(\mathcal{M})$. If $\sup_{t\in[0,1]} \|Y_k - Z_k\|_{C^1}$ is sufficiently small and if $Z_k(t) = 0$ when $Y_k(t) = 0$, then for every $t \in [0,1]$ we have

$$F_a(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k) - F_a(\Sigma_t^k, \Gamma_t^k) \leq -\frac{g\big(\mathcal{D}((\Sigma_t^k, \Gamma_t^k), A_\infty)\big)}{2}, \qquad \mathcal{D}\big((\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k), (\Sigma_t^k, \Gamma_t^k)\big) \leq \frac{\mathcal{D}\big((\Sigma_t^k, \Gamma_t^k), A_\infty\big)}{4}. \tag{5.8}$$

where $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k) = \Phi_1(t)_\sharp(\Sigma_t^k, \Gamma_t^k)$ and $\Phi_1(t)$ is the flow map of $Z_k(t)$ evaluated at the time 1. Hence $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)_{t\in I} \in \mathscr{S}$. By repeating the same procedure for every $k \in \mathbb{N}$ we obtain a new sequence of sweepouts $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)^{k\in\mathbb{N}}$. By (5.8) we have that this sequence is again a minimizing sequence.

Step 4: We claim that $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)^{k\in\mathbb{N}}$ has the desired property, that is, every min-max sequence converges to a stationary pair for F_a. To prove this, suppose by contradiction that the claim is false and let $(\tilde{\Sigma}_{t_k}^k, \tilde{\Gamma}_{t_k}^k)^{k\in\mathbb{N}}$ a min-max sequence such that

$$\limsup_{k\to\infty} \mathcal{D}((\tilde{\Sigma}_{t_k}^k, \tilde{\Gamma}_{t_k}^k), A_\infty) = \ell > 0.$$

Up to passing to a subsequence we can assume that the above upper limit is in fact a limit. By the second relation in (5.8), we have that, for k sufficiently large,

$$\mathcal{D}\big((\Sigma_{t_k}^k, \Gamma_{t_k}^k), A_\infty\big) \geq \frac{\ell}{2}.$$

By the first inequality in (5.8) we get

$$F_a(\Sigma_{t_k}^k, \Gamma_{t_k}^k) \geq F_a(\tilde{\Sigma}_{t_k}^k, \tilde{\Gamma}_{t_k}^k) + \frac{1}{2} g\Big(\frac{\ell}{2}\Big) \xrightarrow{k\to\infty} m_0 + \frac{1}{2} g\Big(\frac{\ell}{2}\Big) > m_0.$$

Thus

$$\liminf_{k\to\infty} \max_{t\in[0,1]} F_a(\Sigma_t^k, \Gamma_t^k) \geq \liminf_{k\to\infty} F_a(\Sigma_{t_k}^k, \Gamma_{t_k}^k) \geq m_0 + \frac{1}{2} g\Big(\frac{\ell}{2}\Big)$$

which contradicts the fact that the sequence (Σ_t^k, Γ_t^k) was minimizing in $\mathscr{S}$.

□

5.4 Existence of almost minimizing min-max sequences

We now introduce the notion of *almost minimizing* pair (Σ, Γ).

Definition 5.16. Let $\varepsilon > 0$, $U \subset \mathcal{M}$ be open and let (Σ, Γ) be a pair of surfaces that is compatible with an open set $\Omega \subset \mathcal{M}$ (in the sense of Definition 5.6). We say that (Σ, Γ) is *ε-almost minimizing* in U if there exists no parametrized pair of surfaces $(\Phi_t, \Psi_t)_{t\in[0,1]}$ and no family of open sets $\{\Omega_t\}_{t\in[0,1]}$ such that:

1. $(\Phi_t, \Psi_t)_{t\in[0,1]}$ and $\{\Omega_t\}_{t\in[0,1]}$ satisfy properties 1, 2, 3, 4 and 6 of Definition 5.4;
2. $\Omega_0 = \Omega$ and $\Omega_t \setminus U = \Omega \setminus U$ for every $t \in [0,1]$;
3. $\Phi_0 = \Sigma$ and $\Phi_t \setminus U = \Sigma \setminus U$ for every $t \in [0,1]$;

4. $\Psi_0 = \Gamma$ and $\Psi_t \setminus U = \Gamma \setminus U$ for every $t \in [0,1]$;

5. $F_a(\Phi_t, \Psi_t) \leq F_a(\Sigma, \Gamma) + \frac{\varepsilon}{8}$ for every $t \in [0,1]$;

6. $F_a(\Phi_1, \Psi_1) \leq F_a(\Sigma, \Gamma) - \varepsilon$.

A sequence of pairs $\{(\Sigma^k, \Gamma^k)\}^k$ is *almost minimizing* in U if there exists a sequence $\varepsilon_k \downarrow 0$ such that each (Σ^k, Γ^k) is ε_k-almost minimizing in U.

Roughly speaking, a pair is almost minimizing in U if it cannot be continuously deformed in U into one other pair with F_a smaller than ε without passing through a pair with F_a larger than $\varepsilon/8$. The main result of this section is the following.

Proposition 5.17. *Let $\mathscr{S}$ a homotopically closed set of sweepouts; then there exist a min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}} = (\Sigma^k_{t_k}, \Gamma^k_{t_k})^{k\in\mathbb{N}}$ and a function $r\colon \mathcal{M} \to (0, +\infty)$ that satisfy:*

1. *(Σ^k, Γ^k) converges, up to subsequences, to a pair $(V, W) \in \mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$ which is stationary for F_a;*

2. *There exists $\varepsilon_k \downarrow 0$ such that for every $x \in \mathcal{M}$ and for every annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$, there exists $N \in \mathbb{N}$ such that the sequence $(\Sigma^k, \Gamma^k)^{k>N}$ is ε_k-almost minimizing in An.*

The strategy of the proof is as follows: we have an homotopically closed set of sweepouts and a minimizing sequence $\{(\Sigma^k_t, \Gamma^k_t)\}^{k\in\mathbb{N}}$ such that any min-max sequence converges to a stationary varifold, which is given by Theorem 5.15. We prove that, if there were no almost minimizing min-max sequence, then the minimizing sequence can be deformed in another one $(\tilde{\Sigma}^k_t, \tilde{\Gamma}^k_t)^{k\in\mathbb{N}}$ such that $\lim_k \max_{t\in[0,1]} F_a(\tilde{\Sigma}^k_t, \tilde{\Gamma}^k_t) < m_0$ and this would be a contradiction.

5.4.1 Deforming a sweepout near a non-almost minimizing slice

If a slice $(\Sigma_{t_0}, \Gamma_{t_0})$ of a sweepout is not ε-almost minimizing in some open set U, it can be deformed in another pair $(\tilde{\Sigma}, \tilde{\Gamma})$, with $F_a(\tilde{\Sigma}, \tilde{\Gamma}) \leq F_a(\Sigma_{t_0}, \Gamma_{t_0}) - \varepsilon$ by a parametrized family of pairs $(\Phi_t, \Psi_t)_{t\in[0,1]}$ such that $(\Phi_0, \Psi_0) = (\Sigma_{t_0}, \Gamma_{t_0})$ and $(\Phi_1, \Psi_1) = (\tilde{\Sigma}, \tilde{\Gamma})$.

We cannot simply replace $(\Sigma_{t_0}, \Gamma_{t_0})$ with $(\tilde{\Sigma}, \tilde{\Gamma})$, first of all because in this way we no longer have a smooth parametrized family of pairs and because we need to deform also the slices for times near t_0 to build the contradiction argument stated above. Hence we need the following lemma, which is similar to [DT13, Lemma 3.1] and [DR18, Lemma 5.1].

Lemma 5.18. *Let $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ be a sweepout and suppose $t_0 \in (0,1)$ is such that $(\Sigma_{t_0}, \Gamma_{t_0})$ is not ε-almost minimizing in $U \subset \mathcal{M}$ for some $\varepsilon > 0$ and some open set U. Then for every open set $V \subset \mathcal{M}$ with $U \subset\subset V$, there exists $\bar{\eta} > 0$ such that, whenever we choose $\eta \leq \bar{\eta}$ and $0 \leq t_0 - \eta < a < a'' < b'' < b < t_0 + \eta \leq 1$, there exists a sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ that is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and satisfies the following properties:*

1. *$(\tilde{\Sigma}_t, \tilde{\Gamma}_t) = (\Sigma_t, \Gamma_t)$ for every $t \in [0,a] \cup [b,1]$;*

2. *$(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \setminus V = (\Sigma_t, \Gamma_t) \setminus V$ for every $t \in [0,1]$;*

3. *$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) + \frac{\varepsilon}{4}$ for every $t \in [0,1]$;*

4. *$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) - \frac{\varepsilon}{2}$ for every $t \in [a'', b'']$.*

Proof. The number $\bar{\eta}$ will be chosen at the end of the proof.

Step 1: The first step in the proof is to build the new sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$; this is achieved first of all by "deforming the time inside U", to have that the slice is frozen on $(\Sigma_{t_0}, \Gamma_{t_0})$ in an open neighborhood of t_0, whereas in $\mathcal{M} \setminus V$ nothing changes. While $(\Sigma_{t_0}, \Gamma_{t_0})$ remains frozen in U, we deform it therein by the family of (Φ_t, Ψ_t) given by Definition 5.16 which contradicts the ε-almost minimality of $(\Sigma_{t_0}, \Gamma_{t_0})$. Of course, this requires some technicalities.

Let $\{\Omega_t\}_{t\in[0,1]}$ be the family of open sets relative to (Σ_t, Γ_t). Let us fix V and choose two open sets A, B such that

$$U \subset\subset A \subset\subset B \subset\subset V$$

and such that both Σ_{t_0} and Γ_{t_0} are smooth surfaces (with eventually non-empty smooth boundary) in $C := B \setminus A$ (this is possibile since the sets of singularities S_{t_0}, G_{t_0} of Σ_{t_0} and Γ_{t_0} are finite). Let us fix two non-negative smooth functions χ_A, χ_B such that

- $\chi_A \in C^\infty_c(B)$, $\chi_B \in C^\infty_c(\mathcal{M} \setminus \overline{A})$;
- $\chi_A(x) + \chi_B(x) \equiv 1$ for all $x \in \mathcal{M}$.

Since $\partial\mathcal{M}$ is $C^{2,\alpha}$, there exists $r > 0$ such that the distance function $d(\cdot) := \mathrm{dist}(\cdot, \partial\mathcal{M})$ belongs to $C^{2,\alpha}\big(U_{2r}(\partial\mathcal{M})\big)$, where $U_{2r}(\partial\mathcal{M})$ is the tubular neighborhood of $\partial\mathcal{M}$ of radius $2r$. By the regularity of $\partial\mathcal{M}$ and d, there exists a smooth function $F\colon x \in \mathcal{M} \mapsto \varphi_x \in C^{2,\alpha}(B_r, B_r)$ such that each ϕ_x is a $C^{2,\alpha}$-diffeomorphism and

$$\varphi_x\big(B_r \cap \tau_{x,1}(\mathcal{M})\big) = B_r \cap \{x_1 \geq -d(x)\} \qquad \forall x \in \mathcal{M},$$

where $\tau_{x,\lambda}$ is the dilation map defined in (2.2). Essentially, ϕ_x is a local chart that depends smoothly on x and maps $\partial\mathcal{M}$ in $\{x_1 = -d(x)\}$.

By a suitable choice of $A, B, \bar{\eta}$ and by smooth convergence of $\Sigma_t \to \Sigma_{t_0}$ in C, for every $t \in (t_0 - \bar{\eta}, t_0 + \bar{\eta})$ $\Sigma_t \cap C$ can be written as $\psi_t(\Sigma_{t_0})$ for a smooth 1-parameter family of diffeomorphisms such that $\psi_{t_0} = \mathrm{Id}\llcorner\Sigma_{t_0}$. Moreover, if $\bar{\eta}$ is sufficiently small, then $\psi_t(x) \in B_r(x)$ for all $t \in (t_0 - \bar{\eta}, t_0 + \bar{\eta})$ and for all $x \in \mathcal{M}$.

All the previous arguments show that, for all $s, t \in (t_0 - \bar{\eta}, t_0 + \bar{\eta})$, it is well defined

$$f_{t,s}(x) = \phi_x^{-1}\Big(\chi_B(x)\phi_x(\psi_t(x)) + \chi_A(x)\big(\phi_x(\psi_s(x))\big)\Big) \quad \forall x \in \mathcal{M}.$$

We remark these properties:

- $f_{t,s}(\Sigma_{t_0} \cap C)$ coincides with Σ_t in a neighborhood of $\mathcal{M} \setminus B$ and changes in a $C^{2,\alpha}$ way in a convex combination of Σ_t and Σ_s in a neighborhood of A.
- this definition ensures that $\partial[f_{t,s}(\Sigma_{t_0} \cap C)] \subset \partial\mathcal{M}$ for all $\Sigma, t \in (t_0 - \bar{\eta}, t_0 + \bar{\eta})$. (To keep the boundary of surfaces on $\partial\mathcal{M}$ is the main reason of the complicate definition of $f_{t,s}$)
- Since $f_{t,s}(\Sigma_{t_0})$ is essentially a convex combination of Σ_t and Σ_s and by smooth convergence of the Σ_t in C, $\bar{\eta}$ can be chosen so that every $f_{t,s}(\Sigma_{t_0} \cap C)$ is diffeomorphic to $\Sigma_{t_0} \cap C$ and
$$\mathcal{H}^2(f_{t,s}(\Sigma_{t_0} \cap C)) < \mathcal{H}^2(\Sigma_t \cap C) + \frac{\varepsilon}{32} \qquad \forall t, s, \in (t_0 - \bar{\eta}, t_0 + \bar{\eta}). \tag{5.9}$$

In a similar way we can define $g_{t,s}$ which satisfies the same properties of $f_{t,s}$ with Γ_t in place of Σ_t, in particular

$$\mathcal{H}^2(g_{t,s}(\Gamma_{t_0} \cap C)) < \mathcal{H}^2(\Gamma_t \cap C) + \frac{\varepsilon}{32} \qquad \forall t, s, \in (t_0 - \bar{\eta}, t_0 + \bar{\eta}). \tag{5.10}$$

Moreover we have that $g_{t,s}(\Gamma_{t_0} \cap C) \subset \partial\mathcal{M}$ for all $s, t \in (t_0 - \bar{\eta}, t_0 + \bar{\eta})$. We underline that by the construction of $f_{t,s}(\Sigma_{t_0} \cap C)$ and $g_{t,s}(\Gamma_{t_0} \cap C)$, it follows that their boundaries coincide and are contained in $\partial\mathcal{M}$. Again in a similar way, we can define the family of open sets $\Xi_{t,s}$ relative to $f_{t,s}(\Sigma_{t_0})$ and $g_{t,s}(\Sigma_{t_0})$.

We now fix $a, a', a'', b'', b, b' \in (0,1)$ such that $t_0 - \bar{\eta} < a < a' < a'' < b'' < b' < b < t_0 + \bar{\eta}$ and a smooth function $\gamma\colon [0,1] \to [0,1]$ such that $\gamma(t) = t$ for every $t \in [0,a] \cup [b,1]$, $\gamma(t) = t_0$ for every $t \in [a', b']$ and γ is monotone non-decreasing.

We now use the hypotheses that $(\Sigma_{t_0}, \Gamma_{t_0})$ is not ε-almost minimizing in U, that is there exists a parametrized pair of surfaces $(\Phi_t, \Psi_t)_{t\in[0,1]}$ which satisfies the properties of Definition 5.16. Let us consider another smooth function $\omega\colon [0,1] \to [0,1]$ such that $\omega(t) = 0$ for every $t \in [0,a'] \cup [b',1]$ and $\omega(t) = 1$ for every $t \in [a'', b'']$. We now define the new sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ as follows:

- $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) = (\Sigma_t, \Gamma_t)$ for every $t \in [0,a] \cup [b,1]$;
- $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \setminus B = (\Sigma_t, \Gamma_t) \setminus B$ for every $t \in [0,1]$;
- $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \cap C = (f_{t,\gamma(t)}(\Sigma_{t_0} \cap C), g_{t,\gamma(t)}(\Gamma_{t_0} \cap C))$ for every $t \in (a,b)$;
- $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \cap A = (\Sigma_{\gamma(t)}, \Gamma_{\gamma(t)})$ for every $t \in [a,a'] \cup [b',b]$;
- $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \cap A = (\Phi_{\omega(t)}, \Psi_{\omega(t)})$ for every $t \in (a', b')$.

In a similar way are defined the open sets $\tilde{\Omega}_t$ relative to $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$. By definition it is clear that $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and satisfies the first two conditions of the lemma.

Step 2: The next step is to estimate $F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ to complete the proof of the lemma. Since $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) = (\Sigma_t, \Gamma_t)$ for every $t \in [0,a] \cup [b,1]$, we have to estimate $F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ only for $t \in (a,b)$.

- Let us consider $t \in (a,a'] \cup [b',b)$; We have that $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \setminus B = (\Sigma_t, \Gamma_t) \setminus B$. In C we have
$$\mathcal{H}^2(\tilde{\Sigma}_t \cap C) = \mathcal{H}^2(f_{t,\gamma(t),\beta(t)}(\Sigma_{t_0} \cap C)) < \mathcal{H}^2(\Sigma_t \cap C) + \frac{\varepsilon}{32},$$
$$\mathcal{H}^2(\tilde{\Gamma}_t \cap C) = \mathcal{H}^2(g_{t,\gamma(t),\beta(t)}(\Gamma_{t_0} \cap C)) < \mathcal{H}^2(\Gamma_t \cap C) + \frac{\varepsilon}{32},$$
where the inequalities follow by (5.9) and (5.10) and by $\gamma(t) \in (a,b)$ for $t \in (a,b)$. In A we have
$$|\mathcal{H}^2(\tilde{\Sigma}_t \cap A) - \mathcal{H}^2(\Sigma_t \cap A)| = |\mathcal{H}^2(\Sigma_{\gamma(t)} \cap A) - \mathcal{H}^2(\Sigma_t \cap A)| < \frac{\varepsilon}{32}$$
$$|\mathcal{H}^2(\tilde{\Gamma}_t \cap A) - \mathcal{H}^2(\Gamma_t \cap A)| = |\mathcal{H}^2(\Gamma_{\gamma(t)} \cap A) - \mathcal{H}^2(\Gamma_t \cap A)| < \frac{\varepsilon}{32}$$
by Lemma 5.3 if $\bar{\eta}$ is chosen sufficiently small. Gathering these estimates we get
$$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) + \frac{\varepsilon}{8} \qquad \forall t \in (a,a'] \cup [b',b).$$
- Now we consider $t \in (a', a'') \cup (b'', b')$. The estimates in $\mathcal{M} \setminus B$ and in C are the same of the case $t \in (a,a'] \cup [b',b)$, the only difference being in A. Thus we have to estimate only in A. Therein we have
$$\begin{aligned}\mathcal{H}^2(\tilde{\Sigma}_t \cap A) + a\mathcal{H}^2(\tilde{\Gamma}_t \cap A) &= \mathcal{H}^2(\Phi_{\omega(t)} \cap A) + a\mathcal{H}^2(\Psi_{\omega(t)} \cap A)\\ &< \mathcal{H}^2(\Sigma_{t_0} \cap A) + a\mathcal{H}^2(\Gamma_{t_0} \cap A) + \frac{\varepsilon}{8}\\ &< \mathcal{H}^2(\Sigma_t \cap A) + a\mathcal{H}^2(\Sigma_t \cap A) + \frac{\varepsilon}{8} + \frac{\varepsilon}{16}\end{aligned}$$
where the first inequality follows by the fact that (Φ_s, Ψ_s) contradicts the ε-almost minimality of $(\Sigma_{t_0}, \Gamma_{t_0})$ in U (recall that $(\Phi_s, \Psi_s) \setminus U = (\Sigma_{t_0}, \Gamma_{t_0}) \setminus U$), whereas the second inequality is true by Lemma 5.3 if we choose $\bar{\eta}$ sufficiently small.

  By this and the estimate in C we have
$$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) + \frac{\varepsilon}{4} \qquad \forall t \in (a', a'') \cup (b'', b').$$

- Now consider the case $t \in [a'', b'']$. Again, the estimates in $\mathcal{M} \setminus B$ and in C as the same as the other cases. We racall that, by definition, $\tilde{\Sigma}_t \cap A = \Phi_1 \cap A$ and $\tilde{\Gamma}_t \cap A = \Psi_1 \cap A$. Thus

$$\begin{aligned}\mathcal{H}^2(\tilde{\Sigma}_t \cap A) + a\mathcal{H}^2(\tilde{\Gamma}_t \cap A) &= \mathcal{H}^2(\Phi_1 \cap A) + a\mathcal{H}^2(\Psi_1 \cap A)\\ &< \mathcal{H}^2(\Sigma_{t_0} \cap A) + a\mathcal{H}^2(\Gamma_{t_0} \cap A) - \frac{\varepsilon}{2}\\ &< \mathcal{H}^2(\Sigma_t \cap A) + a\mathcal{H}^2(\Gamma_t \cap A) - \frac{3\varepsilon}{8}.\end{aligned}$$

where the first inequality follows by the definition of (Φ_t, Ψ_t) and the second is true if we choose $\bar{\eta}$ sufficiently small. Gathering this estimate with the one in C, we get

$$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) - \frac{\varepsilon}{4} \qquad \forall t \in [a'', b''].$$

If we choose $\bar{\eta} > 0$ so small so that all the previous estimate are valid, $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ satisfies all the conclusion of the lemma. It is clear that all the previous arguments remain true whenever $0 < \eta \leq \bar{\eta}$.

□

5.4.2 Almgren-Pitts combinatorial argument

The aim of this section is to show that there exists a min-max sequence of pairs which is almost minimizing in at least one of any *admissible pair* of open sets, that is a pair of open sets which are at a sufficiently large distance compared to their diameter. This is used in section 5.4.3 to complete the proof of Proposition 5.17.

Definition 5.19 (Admissible pair of sets). If $U_1, U_2 \subset \mathcal{M}$ are relatively open and non-empty, we say that (U_1, U_2) is an *admissible pair of open sets* if

$$\operatorname{dist}(U_1, U_2) \geq 4 \min\{\operatorname{diam} U_1, \operatorname{diam} U_2\}.$$

We call $\mathcal{C}$ the family of admissible pairs of open sets.

Definition 5.20. For any $\varepsilon > 0$ and $(U_1, U_2) \in \mathcal{C}$, we say that (Σ, Γ) is ε-almost minimizing in (U_1, U_2) if it is ε-a.m. in at least one of U_1, U_2.

We can now state the fundamental result of this subsection.

Lemma 5.21 (Almgren-Pitts combinatorial lemma). *Let $\mathscr{S}$ be a homotopically closed family of sweepouts; then there exists a minimizing sequence of sweepouts $(\Sigma_t^k, \Gamma_t^k)_{t\in[0,1]}^{k\in\mathbb{N}}$ and a min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}} := (\Sigma_{t_k}^k, \Gamma_{t_k}^k)^{k\in\mathbb{N}}$ that satisfies the following properties:*

1. $F_a(\Sigma^k, \Gamma^k) \geq m_0 - \frac{1}{k}$ *for every* $k \in \mathbb{N}$;
2. $F_a(\Sigma^k, \Gamma^k)$ *is* $\frac{1}{k}$*-a.m. in every* $(U_1, U_2) \in \mathcal{C}$ *for every* $k \in \mathbb{N}$;
3. (Σ^k, Γ^k) *converges to a pair in* $\mathcal{V}_2(\mathcal{M}) \times \mathcal{V}_2(\partial\mathcal{M})$ *which is stationary for* F_a.

Before the proof of Lemma 5.21 we state the following simple and useful property.

Lemma 5.22. *If $(U_1, U_2), (V_1, V_2)$ are pairs of non-empty open sets such that*

$$\begin{gathered}\operatorname{dist}(U_1, U_2) \geq 2 \min\{\operatorname{diam} U_1, \operatorname{diam} U_2\}\\ \operatorname{dist}(V_1, V_2) \geq 2 \min\{\operatorname{diam} V_1, \operatorname{diam} V_2\},\end{gathered}$$

then there exist two indices $i, j \in \{1, 2\}$ such that $U_i \cap V_j = \emptyset$

Proof. Without loss of generality we can suppose that $\operatorname{diam} U_1 \leq \operatorname{diam} U_2$ and $\operatorname{diam} V_1 \leq \operatorname{diam} V_2$. Moreover we can suppose $\operatorname{diam} U_1 \leq \operatorname{diam} V_1$.

If $U_1 \cap V_1 = \emptyset$ there is nothing to prove. Otherwise, there exists $\xi \in U_1 \cap V_1$ and we have $U_1 \subset U_{\operatorname{diam} U_1}(V_1)$, where $U_{\operatorname{diam} U_1}(V_1)$ is the open neighborhood of V_1 of radius $\operatorname{diam} U_1$. Since $\operatorname{dist}(V_1, V_2) \geq 2 \operatorname{diam} V_1 \geq 2 \operatorname{diam} U_1$, it follows that $U_{\operatorname{diam} U_1}(V_1) \cap V_2 = \emptyset$, thus $U_1 \cap V_2 = \emptyset$. $\square$

We can now pass to the proof of Lemma 5.21.

Proof of Lemma 5.21. Let $\{(\Sigma_t^k, \Gamma_t^k)\}^k$ be the minimizing sequence of sweepouts given by Theorem 5.15; Up to passing to a subsequence, we can assume that for every $k \in \mathbb{N}$

$$\max_{t\in[0,1]} F_a(\Sigma_t^k, \Gamma_t^k) \leq m_0 + \frac{1}{8k}.$$

This suitable subsequence (not relabeled) is clearly still minimizing and we are going to prove that is the minimizing sequence stated in the theorem. Since by Theorem 5.15 every min-max sequence converges to a stationary pair up to subsequences, we have only to find a min-max sequence with the almost minimizing property. For any $k \in \mathbb{N}$ let us define the set of times

$$K_k = \Big\{ t \in [0,1] \mid F_a(\Sigma_t^k, \Gamma_t^k) \geq m_0 - \frac{1}{k} \Big\}.$$

K_k is clearly compact. We are going to prove that, for every $k \in \mathbb{N}$ there exists $t_k \in K_k$ such that $(\Sigma_{t_k}^k, \Gamma_{t_k}^k)$ is $1/k$-almost minimizing in every $(U_{1,}, U_{2,}) \in \mathcal{C}$. This is the min-max sequence stated in the Lemma. We argue by contradiction: if there exists $k \in \mathbb{N}$ such that for every $t \in K_k$ (Σ_t^k, Γ_t^k) is neither $\frac{1}{k}$-almost in $U_{1,t}$ nor in $U_{2,t}$ for some $(U_{1,t}, U_{2,t}) \in \mathcal{C}$, we construct a sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ that is homotopic to $(\tilde{\Sigma}_t^k, \tilde{\Gamma}_t^k)_{t\in[0,1]}$ and satisfies $\max_{t\in[0,1]} F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < m_0$, which is absurd. Thus we fix such a $k \in \mathbb{N}$. Since k is fixed, throughout the rest of the proof we drop the subscripts and superscripts k.

Step 1: As first step, we want to show that there exists a covering $\{J_j\}_{j=1}^M$ of K made of closed intervals with the following properties:

- For every $j = 1, \dots, M$, there exist $(a_j, b_j) \subset J_j$, an open set $W_j \subset \mathcal{M}$ and a sweepout $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)_{t\in[0,1]}$ that is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and satisfies:
  1. $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) = (\Sigma_t, \Gamma_t)$ for every $t \in [0,1] \setminus J_j$;
  2. $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) \setminus W_j = (\Sigma_t, \Gamma_t) \setminus W_j$ for every $t \in [0,1]$;
  3. $F_a(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) < F_a(\Sigma_t, \Gamma_t) + \frac{1}{4k}$ for every $t \in [0,1]$;
  4. $F_a(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) < F_a(\Sigma_t, \Gamma_t) - \frac{1}{2k}$ for every $t \in [a_j, b_j]$.
- $$K \subset \bigcup_{j=1}^M (a_j, b_j)$$
- For every $j = 1, \dots, M$, J_j intersects at most J_{j-1} and J_{j+1};
- If $i \neq j$ and $J_i \cap J_j \neq \emptyset$, then $W_i \cap W_j = \emptyset$.

We recall that we assumed by contradiction that for any $t \in K$ there exists $(U_{1,t}, U_{2,t}) \in \mathcal{C}$ such that (Σ_t, Γ_t) is neither $\frac{1}{k}$-almost in $U_{1,t}$ nor in $U_{2,t}$.

Hence, for every $t \in K$, $i = 1, 2$ and for any open set $V_{i,t} \supset\supset U_{i,t}$, let $\bar{\eta}_t$ be the number given by Lemma 5.18 and for every $0 < \eta_t \leq \bar{\eta}_t$ let us define $I_{t,\eta_t} = [t - \eta_t, t + \eta_t]$. We choose the sets $V_{i,t}$ such that

$$\operatorname{dist}(V_{1,t}, V_{2,t}) \geq 2 \min\{\operatorname{diam} V_{1,t}, \operatorname{diam} V_{2,t}\}.$$

By compactness of K, there exists a finite set of points $t_1 < \cdots < t_m$ such that

$$K \subset \bigcup_{j=1}^{m} (I_{t_j,\bar{\eta}_{t_j}})^{\mathrm{o}}.$$

Since for every t_j we can choose $\eta_j \leq \bar{\eta}_{t_j}$, we can obtain that each I_{t_j,η_j} intersects at most $I_{t_{j-1},\eta_{j-1}}$ and $I_{t_{j+1},\eta_{j+1}}$ and still

$$K \subset \bigcup_{j=1}^{m} (I_{t_j,\eta_j})^{\mathrm{o}}. \tag{5.11}$$

For every $j = 1, \dots, m$ and $i = 1, 2$, we denote $V_j^i := V_{i,t_j}$ and $I_j := I_{t_j,\eta_j}$.

Starting from this covering, we want to construct the covering $\{J_j\}_{j=1}^M$ of K. We begin by defining J_1 and W_1:

- If $I_1 \cap I_2 = \emptyset$, then we set $J_1 = I_1$, $W_1 = V_1^1$;
- If $I_1 \cap I_2 \neq \emptyset$, up to relabeling the sets V_j^i, by Lemma 5.22 we can suppose that $V_1^1 \cap V_2^1 = \emptyset$. Then we set $J_1 = I_1$ and $W_1 = V_1^1$.

Now, in any of the two above cases, by Lemma 5.22, we can assume that $V_1^1 \cap V_2^1 = \emptyset$; to choose J_2, W_2, we have to consider several cases:

1. If $I_2 \cap I_3 = \emptyset$, then we set $J_2 = I_2$ and $W_2 = V_2^1$;
2. If $I_2 \cap I_3 \neq \emptyset$, by Lemma 5.22 there exists two indices i, ℓ such that $V_2^i \cap V_3^\ell = \emptyset$. Without loss of generality we can suppose $\ell = 1$.
   (a) If $i = 1$, we set $J_2 = I_2$, $W_2 = V_2^1$;
   (b) If $i = 2$, we can find two closed intervals L, H such that

   $$L^{\mathrm{o}} \cup H^{\mathrm{o}} = {I_2}^{\mathrm{o}} \qquad L \cap I_3 = \emptyset \qquad H \cap I_1 = \emptyset$$

   (the first condition implies that L, H cover I_2 and that their interiors overlap only in the middle of I_2). We then set $J_2 = L$, $W_2 = V_2^1$, $J_3 = H$, $W_3 = V_2^2$.

To choose the next interval and open set, we proceed as follows:

- In the case 1, we choose J_3, W_3 as we have done for J_1, W_1 (since $I_3 \cap I_2 = \emptyset$);
- In the case 2a we can choose J_3, W_3 as we have done for J_2, W_2;
- In the case 2b we can choose J_4, W_4 as we have done for J_2, W_2.

Thus we can continue this procedure until we complete the choice of all the intervals J_j and the open sets W_j; it is clear that, by construction, each J_j intersects at most J_{j-1} and J_{j+1} and that, if $J_j \cap J_\ell \neq \emptyset$, then $W_j \cap W_\ell = \emptyset$. Moreover, by (5.11) and the above construction, we have $K \subset \bigcup_{j=1}^M {J_j}^{\mathrm{o}}$. Hence for every $j = 1, \dots, M$ we can choose, $a_j, b_j \in {J_j}^{\mathrm{o}}$ such that

$$K \subset \bigcup_{j=1}^{M} (a_j, b_j).$$

This complete the definition of the covering $\{J_j\}_{j=1}^M$ and the open sets W_j; we have now to define the sweepouts $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)_{t\in[0,1]}$ for $j = 1, \dots, M$.

For every $j = 1, \dots, M$ there exists $\ell \in \{1, \dots, m\}$ and $i \in \{0, 1\}$ such that $(a_j, b_j) \subset {J_j}^{\mathrm{o}} \subset I_\ell = I_{t_\ell,\eta_\ell}$, $W_j = V_\ell^i$ and $(\Sigma_{t_\ell}, \Gamma_{t_\ell})$ is not $\frac{1}{k}$-almost minimizing in W_j. Thus, by Lemma 5.18 there exists a sweepout $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)_{t\in[0,1]}$ that is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and such that

1. $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) = (\Sigma_t, \Gamma_t)$ for every $t \in [0,1] \setminus J_j$;
2. $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) \setminus W_j = (\Sigma_t, \Gamma_t) \setminus W_j$ for every $t \in [0,1]$;
3. $F_a(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) < F_a(\Sigma_t, \Gamma_t) + \frac{1}{4k}$ for every $t \in [0,1]$;
4. $F_a(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j) < F_a(\Sigma_t, \Gamma_t) - \frac{1}{2k}$ for every $t \in [a_j, b_j]$,

which are the required conditions to complete the proof of the first step.

Step 2: We now want to "glue together" the deformations $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)$ given by the first step in an unique sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ that is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and satisfies $\max_{t\in[0,1]} F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < m_0$, which is a contradiction.

We define the competitor sweeoput $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ as follows:

- If $t \notin \bigcup_{j=1}^M J_j$, then $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) = (\Sigma_t, \Gamma_t)$;
- If $t \in J_j \setminus (J_{j+1} \cup J_{j-1})$ then $(\tilde{\Sigma}_t, \tilde{\Gamma}_t) = (\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)$;
- If $t \in J_j \cap J_{j+1}$, then $\tilde{\Sigma}_t$ coincides with $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)$ in W_j and with $(\tilde{\Sigma}_t^{j+1}, \tilde{\Gamma}_t^{j+1})$ in W_{j+1}; more precisely:

$$\tilde{\Sigma}_t \cap W_j = \tilde{\Sigma}_t^j \cap W_j, \quad \tilde{\Sigma}_t \cap W_{j+1} = \tilde{\Sigma}_t^{j+1} \cap W_{j+1}, \quad \tilde{\Sigma}_t \setminus (W_j \cup W_{j+1}) = \Sigma_t \setminus (W_j \cup W_{j+1});$$

$$\tilde{\Gamma}_t \cap W_j = \tilde{\Gamma}_t^j \cap W_j, \quad \tilde{\Gamma}_t \cap W_{j+1} = \tilde{\Gamma}_t^{j+1} \cap W_{j+1}, \quad \tilde{\Gamma}_t \setminus (W_j \cup W_{j+1}) = \Gamma_t \setminus (W_j \cup W_{j+1});$$

These are all the possible cases, since t can belong to at most two consecutive intervals. Clearly $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ is well-defined, because by construction of the intervals J_j and open sets W_j, if $J_j \cap J_\ell \neq \emptyset$, then $W_j \cap W_\ell = \emptyset$; Since every $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)_{t\in[0,1]}$ is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$, also the sweepout $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$. The open sets $\tilde{\Omega}_t$ relative to the pairs $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ are defined in the same way, starting from the open sets $\tilde{\Omega}_t^j$ relative to $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)$.

We have now to estimate $F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ to get the contradiction:

1. If $t \notin K$, then t can be in at most two of the intervals J_j, say J_j, J_{j+1}; thus $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)$ coincides with $(\tilde{\Sigma}_t^j, \tilde{\Gamma}_t^j)$ and $(\tilde{\Sigma}_t^{j+1}, \tilde{\Gamma}_t^{j+1})$ respectively in the two open sets W_j, W_{j+1} and coincides with (Σ_t, Γ_t) in $\mathcal{M} \setminus (W_j \cup W_{j+1})$. Since for every j it holds $F_a(\tilde{\Sigma}_s^j, \tilde{\Gamma}_s^j) < F_a(\Sigma_s, \Gamma_s) + \frac{1}{4k}$ for every $s \in [0,1]$, we have
$$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < F_a(\Sigma_t, \Gamma_t) + \frac{2}{4k} < m_0 - \frac{1}{k} + \frac{1}{2k} = m_0 - \frac{1}{2k}, \tag{5.12}$$
where the second inequality is given by $t \notin K$ (thus $F_a(\Sigma_t, \Gamma_t) < m_0 - \frac{1}{k}$).
2. If $t \in K$, then there exists j such that t belongs to at least one (a_j, b_j) and at most in another J_ℓ for $\ell = j-1$ or $\ell = j+1$. Hence, by the properties of the sweepouts $(\tilde{\Sigma}_s^h, \tilde{\Gamma}_s^h)_{s\in[0,1]}$ previously defined, one has
$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}_t^j \cap W_j) + a\mathcal{H}^2(\tilde{\Gamma}_t^j \cap W_j) &< \mathcal{H}^2(\Sigma_t \cap W_j) + a\mathcal{H}^2(\Gamma_t \cap W_j) - \frac{1}{2k} \\ \mathcal{H}^2(\tilde{\Sigma}_t^\ell \cap W_\ell) + a\mathcal{H}^2(\tilde{\Gamma}_t^\ell \cap W_\ell) &< \mathcal{H}^2(\Sigma_t \cap W_\ell) + a\mathcal{H}^2(\Gamma_t \cap W_\ell) + \frac{1}{4k}. \end{aligned}$$
Thus
$$F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) \le F_a(\Sigma_t, \Gamma_t) - \frac{1}{4k} \le m_0 + \frac{1}{8k} - \frac{1}{4k} = m_0 - \frac{1}{8k}, \tag{5.13}$$
where we used $F_a(\Sigma_t, \Gamma_t) \le m_0 + \frac{1}{8k}$.

The estimates (5.12) and (5.13) show that

$$\max_{t\in[0,1]} F_a(\tilde{\Sigma}_t, \tilde{\Gamma}_t) < m_0,$$

which is absurd since $(\tilde{\Sigma}_t, \tilde{\Gamma}_t)_{t\in[0,1]}$ is homotopic to $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and m_0 is the min-max value of the homotopically closed set of sweepouts $\mathscr{S}$.

□

5.4.3 Proof of Proposition 5.17

We use the min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}}$ given by Lemma 5.21 to prove Proposition 5.17.

Proof of Proposition 5.17. Let us consider the min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}}$ given by Lemma 5.21. It converges to a stationary pair and, for every $k \in \mathbb{N}$, it is $\frac{1}{k}$-almost minimizing in any pair of open sets $(U_1, U_2) \in \mathcal{C}$.

We begin with a simple useful remark: if a surface is ε-a.m. in a set U and $V \subset U$, then it is ε-a.m. also in V. For every $x \in \mathcal{M}$ and $r > 0$, we have $\big(B_r(x), \mathcal{M} \setminus B_{9r}(x)\big) \in \mathcal{C}$, provided that r is small enough to have $\mathcal{M} \setminus B_{9r}(x) \neq \emptyset$.

- If for every $x \in \mathcal{M}$ there exists $\rho(x) > 0$ and $N(x) \in \mathbb{N}$ such that (Σ^k, Γ^k) is $\frac{1}{k}$-almost minimizing in $B_{\rho(x)}(x)$ for every $k \geq N(x)$, then the proof is complete if we set $\varepsilon_k = \frac{1}{k}$, $r(x) = \rho(x)$, since (Σ^k, Γ^k) is $\frac{1}{k}$-a.m. in every $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ for $k \geq N(x)$.
- If there exists $x \in \mathcal{M}$ and a subsequence $(\Sigma^{k_j}, \Gamma^{k_j})_{j\in\mathbb{N}}$ such that every $(\Sigma^{k_j}, \Gamma^{k_j})$ is not $\frac{1}{k_j}$-a.m. in $B_{1/j}(x)$, then for every $j \in \mathbb{N}$ the pair $(\Sigma^{k_j}, \Gamma^{k_j})$ is $\frac{1}{k_j}$-a.m. in $\mathcal{M} \setminus B_{9/j}(x)$. For every $j \in \mathbb{N}$ we define

  $$\varepsilon_j = \frac{1}{k_j}; \quad (\tilde{\Sigma}^j, \tilde{\Gamma}^j) = (\Sigma^{k_j}, \Gamma^{k_j}); \quad r(y) = \begin{cases} \operatorname{diam} \mathcal{M} & \text{if } y = x, \\ \operatorname{dist}(y, x) & \text{if } y \neq x. \end{cases}$$

  We claim that the sequence $(\tilde{\Sigma}^j, \tilde{\Gamma}^j)_{j\in\mathbb{N}}$ and the function r satisfy the conclusion. In fact for every $\mathrm{An}(x, s, t) \in \mathcal{AN}_{r(x)}(x)$ there exists $N \in \mathbb{N}$ such that if $j > N$ then $\mathrm{An}(x, s, t) \subset \mathcal{M} \setminus B_{1/9j}(x)$ and by the initial remark $(\tilde{\Sigma}^j, \tilde{\Gamma}^j)$ is $\frac{1}{k_j}$-a.m. in $\mathrm{An}(x, s, t)$. If $y \neq x$ and $\mathrm{An}(y, s, t) \in \mathcal{AN}_{r(y)}(y)$, then $t < \operatorname{dist}(y, x)$ and there exists $N \in \mathbb{N}$ such that if $j > N$ then $\operatorname{dist}(y, x) - \frac{1}{9j} < t$. Thus $\mathrm{An}(y, s, t) \subset \mathcal{M} \setminus B_{1/9j}(x)$ and again by the initial remark $(\tilde{\Sigma}^j, \tilde{\Gamma}^j)$ is $\frac{1}{k_j}$-a.m. in $\mathrm{An}(x, s, t)$.

□

5.4.4 Comments on the role of the dimension

Up to Proposition 5.17, the dimension 3 of $\mathcal{M}$ is irrelevant and all the arguments apply as well to the case of pairs of n-varifolds in a domain $\mathcal{M} \subset \mathbb{R}^{n+1}$ for any $n \in \mathbb{N}$. Hence we have in fact proved the following generalization of Proposition 5.17.

Theorem 5.23. *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a domain with boundary of class $C^{2,\alpha}$ and let $a \in [0, 1)$. Let $\mathscr{S}$ be a homotopically closed set of sweepouts; then there exist a min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}} = (\Sigma^k_{t_k}, \Gamma^k_{t_k})^{k\in\mathbb{N}}$ of pairs of hypersurfaces and a function $r\colon \mathcal{M} \to (0, +\infty)$ that satisfy:*

1. *(Σ^k, Γ^k) converges, up to subsequences, to a pair $(V, W) \in \mathcal{V}_n(\mathcal{M}) \times \mathcal{V}_n(\partial\mathcal{M})$ which is stationary for F_a;*

2. *There exists* $\varepsilon_k \downarrow 0$ *such that for every* $x \in \mathcal{M}$ *and for every annulus* $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$*, there exists* $N \in \mathbb{N}$ *such that the sequence* $(\Sigma^k, \Gamma^k)^{k>N}$ *is* ε_k*-almost minimizing in* An.

3. $V \llcorner G_n(\mathcal{M}^o)$ *is induced by a minimal hypersurface* $\Sigma \subset \mathcal{M}^o$ *which is smooth up to a closed set* $\operatorname{Sing}\Sigma$ *of Hausdorff dimension at most* $n-7$.

Proof. The existence of the min-max sequence $(\Sigma^k, \Gamma^k)^{k\in\mathbb{N}} = (\Sigma^k_{t_k}, \Gamma^k_{t_k})^{k\in\mathbb{N}}$ of pairs and of the function $r\colon \mathcal{M} \to (0, +\infty)$ and the validity of points 1 and 2 are obtained by repeating the same arguments as for the case $n = 2$.

Point 3 follows by the min-max theory for manifolds without boundary carried out in [DT13] □

Remark 5.24. The remaining part of this chapter is devoted to obtain the regularity of V up to $\partial\mathcal{M}$. As stated in the introduction, while all the arguments made in this chapter easily adapt to any dimension n, our restriction to the case of 2-surfaces in $\mathcal{M} \subset \mathbb{R}^3$ comes from the availability of the curvature estimates obtained in section 5.6.

5.5 Stability inequality

The construction of the almost minimizing min-max sequence is needed to prove the existence of replacements in the annuli for the limit pair of varifolds; the important point is that these replacements are in fact locally *stable*, that is their second variation is non-negative on small balls when tested with admissible variations. This allows us to use the regularity theory for minimizers, to show that they are regular on these small balls. To achieve this, we need to prove curvature estimates for stable stationary pairs of surfaces. In the interior these estimates are established by the work of Schoen, Simon and Yau [SSY75]; At the boundary, we cite the extension to the case of free boundaries for the area functional discussed by Grüter and Jost in [GJ86b] and the work of De Lellis and Ramic [DR18, Theorem 6.14] concerning the case of Dirichlet boundary conditions. We would like to prove these curvature estimates for the capillarity functional in the free boundary case.

The starting point for the regularity theory for stationary and stable pairs is the *stability inequality*, which yields an a-priori bound for the norm squared of the second fundamental form of a smooth stable hypersurface in terms of the gradient of a cut-off function, see [Sim84, Remark 9.8 and (B.1) in Appendix B].

The main result of this section is the following stability inequality for the capillarity functional.

Proposition 5.25 (Stability inequality for capillarity functional)**.** *Let* $\mathcal{N} \subset \mathbb{R}^3$ *be a half-space. Then there exists an universal constant* $c(a) > 0$ *with the following property: for every pair* (Σ, Γ) *which is compatible with the open set* Ω *and is stationary and stable for* F_a*, and for every* $\zeta \in C^\infty_c(\mathbb{R}^n)$*, it holds*

$$\int_\Sigma \zeta^2 |A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \le c \int_\Sigma |\nabla\zeta|^2 \,\mathrm{d}\mathcal{H}^2, \tag{5.14}$$

where $|A_\Sigma|^2$ *is the norm squared of the second fundamental form of* Σ.

Without loss of generality, throughout this section we assume that $\mathcal{N} = \{x_1 \ge 0\}$.

5.5.1 Rewriting the capillarity functional

In order to prove Proposition 5.25, it is convenient to write $F_a(\Sigma, \Gamma)$ in terms of an anisotropic functional depending only on Σ.

Lemma 5.26. *Let us assume that $\Omega \subset \mathcal{N}$ is relatively open and that the pair (Σ, Γ) is compatible with Ω in the sense of Definition 5.6. Then it holds*

$$F_a(\Sigma, \Gamma) = \int_\Sigma \Psi(\nu_\Sigma)\,\mathrm{d}\mathcal{H}^2$$

where ν_Σ is the unit normal to Σ which points outside Ω and

$$\Psi(\nu) := |\nu| + a\nu \cdot e_1$$

Proof. Let us choose the vector field $v \equiv -ae_1$. Clearly we have

$$\operatorname{div} v = 0, \qquad v(x) = aN(x) \quad \forall x \in \partial\mathcal{N},$$

where $N(x)$ is exterior unit normal to $\mathcal{N}$; applying the divergence theorem to v and Ω we obtain

$$\begin{aligned} 0 = \int_\Omega \operatorname{div} v\,\mathrm{d}x &= \int_\Sigma \langle v, \nu_\Sigma\rangle\,\mathrm{d}\mathcal{H}^2 + \int_\Gamma \langle v, N\rangle\,\mathrm{d}\mathcal{H}^2 \\ &= \int_\Sigma \langle v, \nu_\Sigma\rangle\,\mathrm{d}\mathcal{H}^2 + a\mathcal{H}^2(\Gamma). \end{aligned}$$

Thus we can write

$$\begin{aligned} F_a(\Sigma, \Gamma) =& \mathcal{H}^2(\Sigma) + a\mathcal{H}^2(\Gamma) \\ =& \mathcal{H}^2(\Sigma) - \int_\Sigma \langle v, \nu_\Sigma\rangle\,\mathrm{d}\mathcal{H}^2 \\ =& \int_{\Sigma_t} (1 - \langle v, \nu_\Sigma\rangle)\,\mathrm{d}\mathcal{H}^2 \\ =& \int_\Sigma \Psi(\nu_\Sigma)\,\mathrm{d}\mathcal{H}^2, \end{aligned}$$

where

$$\Psi(\nu) := |\nu| + a\nu \cdot e_1 \geq (1 - a) > 0 \qquad \forall \nu \in \mathbb{S}^2 := \{\nu \in \mathbb{R}^3 \mid |\nu| = 1\}.$$

□

Remark 5.27. In particular, a pair (Σ, Γ) which is compatible with Ω is stationary and/or stable for F_a if and only if it is stationary and/or stable for the functional

$$\mathcal{F}(\Sigma) := \int_\Sigma \Psi(\nu_\Sigma(x))\,\mathrm{d}\mathcal{H}^2(x).$$

5.5.2 Second variation formula for $\mathcal{F}$

All the results of this subsection hold for any container $\mathcal{M}$ and any *anisotropic functional*

$$\mathcal{F}(\Sigma) := \int_\Sigma \Psi(\nu_\Sigma)\,\mathrm{d}\mathcal{H}^2,$$

where $\Psi \colon \mathbb{R}^3 \to [0, +\infty)$ is 1-homogeneous. We want let evolve the surface Σ by the action of an admissible variation f_t (in the sense of Definition 2.23) and compute the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(f_t(\Sigma))\bigg|_{t=0}.$$

In order to perform this computation, it is useful to write $\mathcal{F}\big(f_t(\Sigma)\big)$ in terms of an integral over Σ.

Lemma 5.28. *Let $\Sigma \subset \mathcal{M}$ be a surface of class C^2, let $\Psi\colon \mathbb{R}^3 \to [0,+\infty)$ be 1-homogeneous and assume that f_t is an admissible variation on $\mathcal{M}$; then*

$$\mathcal{F}\big(f_t(\Sigma)\big) = \int_\Sigma \Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\,\mathrm{d}\mathcal{H}^2(x)$$

where for any $A \in \mathbb{R}^{m\times m}$, $\operatorname{cof} A := (\det A)(A^{-1})^*$.

Proof. Let us fix $t \in (-\varepsilon,\varepsilon)$; for simplicity of notation, we drop the subscript t and write f instead of f_t. We begin by computing

$$\begin{aligned}\mathcal{F}\big(f(\Sigma)\big) &= \int_{f(\Sigma)} \Psi(\nu_\Sigma(y))\,\mathrm{d}\mathcal{H}^2(y)\\ &= \int_\Sigma \Psi\big(\nu_{f(\Sigma)}(f(x))\big)J_\Sigma f(x)\,\mathrm{d}\mathcal{H}^2(x),\end{aligned}$$

where the second equality follows by the area formula for surfaces. We have

$$\big((\nabla f)^{-1}\big)^*[\nu_\Sigma] = \big\langle(\nabla f)^{-1}[\nu_{f(\Sigma)}],\nu_\Sigma\big\rangle\nu_{f(\Sigma)},$$

(where we dropped the dependence on the points x and $f(x)$) which implies

$$\nu_{f(\Sigma)} = \frac{\big((\nabla f)^{-1}\big)^*[\nu_\Sigma]}{|\big((\nabla f)^{-1}\big)^*[\nu_\Sigma]|} = \frac{\big((\nabla f)^{-1}\big)^*[\nu_\Sigma]}{|\big\langle(\nabla f)^{-1}[\nu_{f(\Sigma)}],\nu_\Sigma\big\rangle|}.$$

Since Ψ is 1-homogeneous, we get

$$\mathcal{F}\big(f_t(\Sigma)\big) = \int_\Sigma \Psi\big(\big((\nabla f_t(x))^{-1}\big)^*[\nu_\Sigma(x)]\big)\frac{J_\Sigma f_t(x)}{|\big\langle(\nabla f)^{-1}[\nu_{f(\Sigma)}],\nu_\Sigma\big\rangle|}\,\mathrm{d}\mathcal{H}^2(x).$$

Since, at least for small t we have $\det\nabla f_t > 0$, we write the last fraction as

$$\frac{J_\Sigma f}{|\big\langle(\nabla f)^{-1}[\nu_{f(\Sigma)}],\nu_\Sigma\big\rangle|} = \frac{1}{J_{f(\Sigma)}f^{-1}}\frac{1}{|\big\langle(\nabla f)^{-1}[\nu_{f(\Sigma)}],\nu_\Sigma\big\rangle|} = \frac{1}{\det(\nabla f)^{-1}} = \det\nabla f.$$

Thus, defining $\operatorname{cof} A = (\det A)(A^{-1})^*$ for $A \in \mathbb{R}^{m\times m}$, we have

$$\mathcal{F}\big(f_t(\Sigma)\big) = \int_\Sigma \Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\,\mathrm{d}\mathcal{H}^2(x).$$

□

We now state a linear algebra lemma, which we use in the following to compute the second variation of the functional. For the proof we refer to [Mag12, Lemma 17.4]

Lemma 5.29. *Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices; then*

$$\begin{aligned}\det\Big(\operatorname{Id} + tA + \frac{t^2}{2}B + o(t^2)\Big) &= 1 + t\operatorname{Tr}A + \frac{t^2}{2}\Big((\operatorname{Tr}A)^2 - \operatorname{Tr}A^2 + \operatorname{Tr}B\Big) + o(t^2),\\ \Big(\operatorname{Id} + tA + \frac{t^2}{2}B + o(t^2)\Big)^{-1} &= \operatorname{Id} - tA + \frac{t^2}{2}\Big(2A^2 - B\Big) + o(t^2).\end{aligned}$$

In particular

$$\begin{aligned}\operatorname{cof}\Big(\operatorname{Id} + tA + \frac{t^2}{2}B + o(t^2)\Big) &= \operatorname{Id} + t\Big(\operatorname{Tr}A\operatorname{Id} - A^*\Big)\\ &\quad+\frac{t^2}{2}\Big[\Big((\operatorname{Tr}A)^2 - \operatorname{Tr}A^2 + \operatorname{Tr}B\Big)\operatorname{Id} + 2(A^*)^2 - 2(\operatorname{Tr}A)A^* - B^*\Big] + o(t^2).\end{aligned} \tag{5.15}$$

As a corollary we compute the time derivatives of $\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]$.

Corollary 5.30. *Let f_t be an admissible variation on $\mathcal{M}$; then we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)] = \Big(\operatorname{div} X \operatorname{Id} - DX^*\Big)\nu_\Sigma(x)\\ &\quad + t\Big[\Big((\operatorname{div} X)^2 - \operatorname{div}(DX^2) + \operatorname{div} Z\Big)\operatorname{Id} + 2(DX^*)^2 - 2(\operatorname{div} X)DX^* - DZ^*\Big]\nu_\Sigma(x) + o(t);\end{aligned} \tag{5.16}$$

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)] =\\ &\quad \Big[\Big((\operatorname{div} X)^2 - \operatorname{Tr}(DX^2) + \operatorname{div} Z\Big)\operatorname{Id} + 2(DX^*)^2 - 2(\operatorname{div} X)DX^* - DZ^*\Big]\nu_\Sigma(x) + o(1),\end{aligned} \tag{5.17}$$

where

$$X(x) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}f_t(x)\Big|_{t=0}, \qquad Z(x) \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}t^2}f_t(x)\Big|_{t=0} \equiv 0.$$

Proof. We have

$$f_t(x) = x + tX(x) + \frac{t^2}{2}Z(x) + o(t^2)$$

Thus

$$\nabla f_t(x) = \operatorname{Id} + tDX(x) + \frac{t^2}{2}DZ(x) + o(t^2),$$

and the result follows by (5.15). □

Lemma 5.31 (Second variation formula for $\mathcal{F}$)**.** *Let $\Sigma \subset \mathcal{M}$ be a surface of class C^2, assume that $\Psi\colon \mathbb{R}^3 \to [0,+\infty)$ is 1-homogeneous and that f_t is an admissible variation on $\mathcal{M}$; then*

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(f_t(\Sigma))\Big|_{t=0} = \int_\Sigma \Big[&\nabla^2\Psi\big[DX^*[\nu_\Sigma]\big]\cdot DX^*[\nu_\Sigma] + \Big((\operatorname{div} X)^2 - \operatorname{Tr}(DX^2) + \operatorname{div} Z\Big)\Psi\\ &+ \nabla\Psi\cdot\Big(2(DX^*)^2[\nu_\Sigma] - 2(\operatorname{div} X)DX^*[\nu_\Sigma] - DZ^*[\nu_\Sigma]\Big)\Big]\,\mathrm{d}\mathcal{H}^2.\end{aligned} \tag{5.18}$$

Proof. By Lemma 5.28 and differentiating under integral, in order to obtain the second variation of Ψ we have to compute

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\Big|_{t=0}.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big) = \nabla\Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)].$$

Hence

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\Big|_{t=0} =\\ &\quad+\nabla^2\Psi\big(\nu_\Sigma(x)\big)\Big[\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\Big|_{t=0}\Big]\cdot\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\Big|_{t=0}\\ &\quad+\nabla\Psi\big(\nu_\Sigma(x)]\big)\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\Big|_{t=0}.\end{aligned}$$

Substituting (5.16) and (5.17), we get

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi\big(\operatorname{cof}\nabla f_t(x)[\nu_\Sigma(x)]\big)\Big|_{t=0} =\\ &\quad+(\operatorname{div} X)^2\nabla^2\Psi[\nu_\Sigma]\cdot\nu_\Sigma - 2(\operatorname{div} X)\nabla^2\Psi[\nu_\Sigma]\cdot DX^*[\nu_\Sigma] + \nabla^2\Psi\big[DX^*[\nu_\Sigma]\big]\cdot DX^*[\nu_\Sigma]\\ &\quad+\Big((\operatorname{div} X)^2 - \operatorname{Tr}(DX^2) + \operatorname{div} Z\Big)\nabla\Psi\cdot\nu_\Sigma\\ &\quad+\nabla\Psi\cdot\Big(2(DX^*)^2[\nu_\Sigma] - 2(\operatorname{div} X)DX^*[\nu_\Sigma] - DZ^*[\nu_\Sigma]\Big),\end{aligned} \tag{5.19}$$

where we dropped the dependence on the argument $\nu_\Sigma(x)$. Since Ψ is 1-homogeneous, it satisfies

$$\nabla\Psi(\nu)\cdot\nu = \Psi(x,\nu);$$
$$\nabla^2\Psi(\nu)[\nu] = 0,$$

and substituting in (5.19) we obtain (5.18). $\square$

5.5.3 Proof of Proposition 5.25

We can now prove Proposition 5.25.

Proof of Proposition 5.25. Since Σ and $\partial\Sigma$ are of class C^2, there exists a C^2-extension of ν_Σ to the whole $\mathcal{N}$: more precisely $Y \in \mathfrak{X}(\mathcal{N})$ satisfies

$$Y(x) = \nu_\Sigma(x) \qquad \forall x \in \Sigma.$$

Since $\nabla\Psi(\nu) = \nu/|\nu| - ae_1$ and Σ meets $\partial\mathcal{N}$ with angle $\theta = \arccos a$, it follows

$$\nabla\Psi(Y(x))\cdot N(x) = 0 \qquad \forall x \in \partial\Sigma, \tag{5.20}$$

where $N(x)$ is the exterior unit normal to $\mathcal{N}$, that is $N(x) = -e_1$. Of course there exists a particular choice of Y which satisfies (5.20) for every $x \in \partial\mathcal{N}$ and which is supported in a sufficiently small neighborhood of Σ. Let us fix such a Y, a cut-off function $\zeta \in C_c^\infty(\mathcal{N})$ and define the vector field

$$X(x) := \zeta(x)\nabla\Psi(Y(x)).$$

By the above discussion, $X \in \mathfrak{X}_t(\mathcal{N})$; since it holds

$$\nabla\Psi(\nu) = \frac{\nu}{|\nu|} - ae_1, \qquad \nabla^2\Psi = \mathrm{Id} - \nu_\Sigma\otimes\nu_\Sigma, \qquad H_\Sigma = 0,$$

simple computations show that X satisfies the following properties for every $x \in \Sigma$:

$$DX(x) = \nabla\Psi\otimes\nabla\zeta + \zeta A_\Sigma + \zeta\nabla^2\Psi[\omega]\otimes\nu_\Sigma, \tag{5.21}$$
$$\operatorname{div} X(x) = \nabla\zeta\cdot\nabla\Psi, \tag{5.22}$$
$$DX^*[\nu_\Sigma(x)] = \Psi\nabla\zeta, \tag{5.23}$$

where A_Σ is the second fundamental form of Σ and is extended to be 0 when computed on vectors which are orthogonal to Σ and ω is a suitable vector field of class C^1 which satisfies

$$DY(x) = A_\Sigma(x) + \omega(x)\otimes\nu_\Sigma(x) \qquad \forall x \in \Sigma.$$

Now we consider the variation $f_t(x) = x + tX(x)$; since $X \in \mathfrak{X}_t(\mathcal{N})$, f_t is admissible and for every $x \in \Sigma$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t(x)\bigg|_{t=0} = X(x), \qquad Z(x) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}f_t(x)\bigg|_{t=0} \equiv 0.$$

Applying Lemma 5.31, one has

$$0 \le \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(f_t(\Sigma))\bigg|_{t=0} = \int_\Sigma (T_1+T_2+T_3)\,\mathrm{d}\mathcal{H}^2, \tag{5.24}$$

where the inequality follows by the stability of Σ and

$$T_1 = \Big((\operatorname{div} X)^2 - \operatorname{Tr}(DX^2)\Big)\Psi$$
$$T_2 = 2\nabla\Psi\cdot\big[(DX^*)^2[\nu_\Sigma] - (\operatorname{div} X)DX^*[\nu_\Sigma]\big]$$
$$T_3 = \nabla^2\Psi\big[DX^*[\nu_\Sigma]\big]\cdot DX^*[\nu_\Sigma].$$

- For the first term in T_1, by (5.22) it follows

  $$(\operatorname{div} X)^2 = (\nabla\zeta \cdot \nabla_\nu \Psi)^2.$$

  For what concerns the second term, we have

  $$DX = S + T,$$
  $$S := \nabla\Psi \otimes \nabla\zeta + \zeta A^\Psi_\Sigma, \qquad T := \zeta\nabla^2\Psi[\omega] \otimes \nu_\Sigma.$$

  By 1-homogeneity of Ψ one has $\nabla^2\Psi(\nu)[\nu] = 0$, thus $T^2 = 0$ and

  $$(DX)^2 = S^2 + ST + TS,$$

  hence

  $$\operatorname{Tr}(DX^2) = \operatorname{Tr}(S^2) + 2\operatorname{Tr}(ST).$$

  We have

  $$\begin{aligned}\operatorname{Tr}(S^2) =& \operatorname{Tr}\left[(\nabla\Psi \otimes \nabla\zeta)^2 + \zeta^2(A^\Psi_\Sigma)^2 + 2\zeta(\nabla\Psi \otimes \nabla\zeta)A^\Psi_\Sigma\right]\\ =&(\nabla\Psi \cdot \nabla\zeta)^2 + \zeta^2|A^\Psi_\Sigma|^2 + 2\zeta A^\Psi_\Sigma[\nabla\Psi] \cdot \nabla\zeta\end{aligned}$$

  For the other term it holds

  $$\operatorname{Tr}(ST) = \operatorname{Tr}(TS) = \zeta\Psi\nabla^2\Psi[\omega] \cdot \nabla\zeta.$$

  Thus

  $$\operatorname{Tr}(DX^2) = (\nabla\zeta \cdot \nabla\Psi)^2 + \zeta^2|A_\Sigma|^2 + 2\zeta A_\Sigma[\nabla\Psi] \cdot \nabla\zeta + 2\zeta\Psi\nabla^2\Psi[\omega] \cdot \nabla\zeta,$$

  and

  $$T_1 = -\zeta^2\Psi|A_\Sigma|^2 - 2\zeta\Psi A_\Sigma[\nabla\Psi] \cdot \nabla\zeta - 2\zeta\Psi^2\nabla^2\Psi[\omega] \cdot \nabla\zeta.$$

- Using (5.21), (5.22) and (5.23), elementary computations yield

  $$2\nabla\Psi \cdot (DX^*)^2[\nu_\Sigma] = 2\Psi(\nabla\Psi \cdot \nabla\zeta)^2 + 2\zeta\Psi A_\Sigma[\nabla\Psi] \cdot \nabla\zeta + 2\zeta\Psi^2\nabla^2\Psi[\omega] \cdot \nabla\zeta$$

  and

  $$2\nabla\Psi \cdot (\operatorname{div} X)DX^*[\nu_\Sigma] = 2\Psi(\nabla\Psi \cdot \nabla\zeta)^2.$$

  Thus

  $$T_2 = 2\zeta\Psi A_\Sigma[\nabla\Psi] \cdot \nabla\zeta + 2\zeta\Psi^2\nabla^2\Psi[\omega] \cdot \nabla\zeta.$$

- Finally

  $$T_3 = \nabla^2\Psi\big[DX^*[\nu_\Sigma]\big] \cdot DX^*[\nu_\Sigma] = \Psi^2\nabla^2\Psi[\nabla\zeta] \cdot \nabla\zeta.$$

Summarizing we have

$$T_1 + T_2 + T_3 = -\zeta^2\Psi|A_\Sigma|^2 + \Psi^2\nabla^2\Psi[\nabla\zeta] \cdot \nabla\zeta.$$

Thus (5.24) becomes

$$\int_\Sigma \zeta^2\Psi|A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \le \int_\Sigma \Psi^2\nabla^2\Psi[\nabla\zeta] \cdot \nabla\zeta \,\mathrm{d}\mathcal{H}^2.$$

Using

$$0 < 1 - a \le |\nu_\Sigma| - a\nu_\Sigma \cdot e_1 = \Psi(\nu_\Sigma) \le 2, \qquad \nabla^2\Psi = \operatorname{Id} - \nu \otimes \nu,$$

we obtain

$$(1-a)\int_\Sigma \zeta^2|A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \le \zeta^2\Psi|A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \le \int_\Sigma \Psi^2\nabla^2\Psi[\nabla\zeta] \cdot \nabla\zeta \,\mathrm{d}\mathcal{H}^2 \le 2\int_\Sigma |\nabla\zeta|^2 \,\mathrm{d}\mathcal{H}^2,$$

hence there exists a constant $c(a) > 0$ such that

$$\int_\Sigma \zeta^2|A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \le c\int_\Sigma |\nabla\zeta|^2 \,\mathrm{d}\mathcal{H}^2.$$

$\square$

5.6 Bernstein's Theorem, curvature estimates and compactness

The stability inequality is used to obtain a Bernstein-type theorem for stationary and stable surfaces with respect to the capillarity energy in a three-dimensional half space.

Theorem 5.32 (Bernstein Theorem). *Let us assume that* (Σ,Γ) *is a pair of connected smooth surfaces which is stationary and stable for* F_a *in* $\mathcal{N}\setminus\{0\}$*, where* $\mathcal{N} := \{x_1 \geq 0\} \subset \mathbb{R}^3$*; moreover let us assume that there exists a constant* $M > 0$ *such that*

$$\sup_{R\in(0,\infty)} \frac{\mathcal{H}^2(\Sigma\cap B_R(0))}{R^2} \leq M; \tag{5.25}$$

Then Σ *and* Γ *are planar and fall in one of the following cases:*

1. Σ,Γ *are half planes and* Σ *forms an angle* θ *with* $\partial\mathcal{N}$ *such that* $a = \cos\theta$*;*
2. $\Sigma = h\{x_1 = c\}$ *and* $\Gamma = k\partial\mathcal{N}$ *for some* $c \geq 0$ *and a choice of* $h, k \in \{0,1\}$*;*
3. Σ *is a half-plane orthogonal to* $\partial\mathcal{N}$ *and* $\Gamma = k\partial\mathcal{N}$ *for a choice of* $k \in \{0,1\}$*.*

Remark 5.33. By the definition of stable and stationary pair, we cannot exclude cases 2 and 3 in Theorem 5.32; nevertheless, they do not occour in our work, as we shall see in sections 5.7 and 5.8.

Proof. The proof is classical and we just point out that it is enough to test (5.14) with the following logarithmic cut-off function, for $N \in \mathbb{N}$:

$$\zeta_N(x) = \begin{cases} 0 & \text{if } |x| \leq e^{-2N} \\ 2 + \frac{\log(|x|)}{N} & \text{if } e^{-2N} \leq |x| \leq e^{-N} \\ 1 & \text{if } e^{-N} \leq |x| \leq e^{N} \\ 2 - \frac{\log(|x|)}{N} & \text{if } e^{N} \leq |x| \leq e^{2N} \\ 0 & \text{if } |x| \geq e^{2N}. \end{cases}$$

This, together with the area bound (5.25), gives

$$\int_{\Sigma\cap\left(B_{e^N}\setminus B_{e^{-N}}\right)} |A_\Sigma|^2 \,\mathrm{d}\mathcal{H}^2 \leq \frac{2cMe^2}{N}.$$

By sending $N \to \infty$, we obtain $A_\Sigma \equiv 0$, hence Σ is contained in a plane. Thus we have the conclusion. ☐

As it is well known, a Bernstein's type theorem as above implies local curvature estimates.

Theorem 5.34 (Curvature estimates for stable surfaces). *For every* $M > 0$*, there exists a constant* $\Lambda = \Lambda(a, M) > 0$ *such that, for every pair* (Σ,Γ) *of* C^2 *surfaces which is stationary and stable for* F_a *in* $\mathcal{M}$ *with with* $\mathcal{H}^2(\Sigma) \leq M$*, it holds*

$$\sup_{x\in B_1} |A_\Sigma|(x) \leq \Lambda.$$

Proof. Let us assume, by contradiction, that for every $k \in \mathbb{N}$ there exists a pair $(\tilde\Sigma_k, \tilde\Gamma_k)$ that is stationary and stable for F_a in $\mathcal{M}$ and satisfies

$$\mathcal{H}^2(\tilde\Sigma_k) \leq M \qquad \text{and} \qquad \sup_{x\in B_1} |A_{\tilde\Sigma_k}(x)| \geq k. \tag{5.26}$$

Let us define for every $k \in \mathbb{N}$ the function

$$f_k \colon x \in B_2 \mapsto \operatorname{dist}(x, \partial B_2)|A_{\tilde{\Sigma}_k}(x)|.$$

Since every f_k is continuous, positive and is equal to 0 on ∂B_2, for every $k \in \mathbb{N}$ there exists $x_k \in {B_2}^{\mathrm{o}}$ such that

$$\operatorname{dist}(x_k, \partial B_2)|A_{\tilde{\Sigma}_k}(x_k)| = f_k(x_k) = \max_{B_2} f_k \geq k, \tag{5.27}$$

where the last inequality follows by the curvature estimate in (5.26). We now define, for every $k \in \mathbb{N}$, the dilation maps

$$g_k \colon x \mapsto |A_{\tilde{\Sigma}_k}(x_k)|(x - x_k),$$

and we define the rescaled and translated pairs (Σ_k, Γ_k) and domains $\mathcal{M}_k$:

$$\Sigma_k = g_k(\tilde{\Sigma}_k), \qquad \Gamma_k = g_k(\tilde{\Gamma}_k), \qquad \mathcal{M}_k = g_k(\mathcal{M}).$$

Clearly for every $k \in \mathbb{N}$ (Σ_k, Γ_k) is stationary and stable for F_a in $\mathcal{M}_k$ and

$$g_k\big(B_{\operatorname{dist}(x_k, \partial B_2)}(x_k)\big) = B_{\operatorname{dist}(x_k, \partial B_2)|A_{\tilde{\Sigma}_k}(x_k)|}(0) \supset B_k,$$

where the last inclusion follows by $f_k(x_k) \geq k$ in (5.27). Thus, for every $y \in B_{k/2} \cap \Sigma_k$ we have

$$g_k^{-1}(y) = x_k + \frac{y}{|A_{\tilde{\Sigma}_k}(x_k)|} \in B_2, \qquad \operatorname{dist}\left(x_k + \frac{y}{|A_{\tilde{\Sigma}_k}(x_k)|}, \partial B_2\right) \geq \frac{1}{2} \operatorname{dist}(x_k, \partial B_2).$$

Hence, by scaling and (5.27), we have

$$\begin{aligned} |A_{\Sigma_k}(y)| =& \frac{\left|A_{\tilde{\Sigma}_k}\left(x_k + \frac{y}{|A_{\tilde{\Sigma}_k}(x_k)|}\right)\right|}{|A_{\tilde{\Sigma}_k}(x_k)|} \\ \leq& \frac{\operatorname{dist}(x_k, \partial B_2)}{\operatorname{dist}\left(x_k + \frac{y}{|A_{\tilde{\Sigma}_k}(x_k)|}, \partial B_2\right)} \\ \leq& 2. \end{aligned}$$

This proves that

$$\sup_{y \in B_{k/2}} |A_{\Sigma_k}(y)| \leq 2 \qquad \forall k \in \mathbb{N}, \tag{5.28}$$

that is the C^2 norms of Σ_k are uniformly bounded in a sequence of enlarging balls. Again by scaling, it is easy to see that

$$|A_{\Sigma_k}(0)| = \frac{|A_{\tilde{\Sigma}_k}(x_k)|}{|A_{\tilde{\Sigma}_k}(x_k)|} = 1 \qquad \forall k \in \mathbb{N}. \tag{5.29}$$

We now distinguish two cases:

1. Let us assume that $\liminf_{k \to \infty} \operatorname{dist}(0, \partial M_k) = L < \infty$.

   For every $k \in \mathbb{N}$ we choose $y_k \in \partial \mathcal{M}_k$ such that $|y_k| = \operatorname{dist}(0, \partial \mathcal{M}_k)$. Then there exists a subsequence (not relabeled) such that

   $$y_k \to y \quad \text{with } |y| = L, \qquad \mathcal{M}_k \to \mathcal{N} \quad \text{in } C^{2,\alpha}, \qquad \Sigma_k \to \Sigma,\ \Gamma_k \to \Gamma \quad \text{in } C^{1,\beta},$$

   for some $\beta \in (0,1)$. Up to rotating the coordinates, we can assume that $\mathcal{N} = \{x_1 \geq -L\}$ and $y = (-L, 0, \ldots, 0) \in \mathcal{N}$. Since stationarity and stability for F_a pass to the limit, the pair (Σ, Γ) is stationary and stable for F_a in $\mathcal{N}$.

Each Σ_k satisfy an elliptic equation with Neumann condition in $\mathcal{M}_k$ and both the equations and the Neumann conditions converge to the ones satisfied by Σ, hence by classical Schauder estimates in fact $\Sigma_k \to \Sigma$ in the C^2-topology. This implies, by (5.29), that

$$|A_\Sigma(0)| = 1.$$

Let us fix $R > 0$ such that $\mathcal{H}^2\big(\Sigma \cap \partial B_R(y)\big) + a\mathcal{H}^2\big(\Gamma \cap \partial B_R(y)\big) = 0$. By the C^2-convergence $\Sigma_k \to \Sigma$, we have

$$\frac{\mathcal{H}^2\big(\Sigma \cap B_R(y)\big) + a\mathcal{H}^2\big(\Gamma \cap B_R(y)\big)}{R^2} = \lim_{k\to\infty} \frac{\mathcal{H}^2\big(\Sigma_k \cap B_R(y_k)\big) + a\mathcal{H}^2\big(\Gamma_k \cap B_R(y_k)\big)}{R^2}.$$

Since the mean curvatures H_{Σ_k} of Σ_k vanish and the second fundamental forms of $\mathcal{M}_j$ are converging locally uniformly to 0, by Corollary 3.20 there exists an universal constant $c > 0$ such that for every k it holds

$$\begin{aligned}
\frac{\mathcal{H}^2\big(\Sigma_k \cap B_R(y_k)\big) + a\mathcal{H}^2\big(\Gamma_k \cap B_R(y_k)\big)}{R^2} &\leq \\
&\leq c\,\frac{\mathcal{H}^2\big(\Sigma_k \cap B_{|A_{\tilde{\Sigma}_k}(x_k)|}(y_k)\big) + a\mathcal{H}^2\big(\Gamma_k \cap B_{|A_{\tilde{\Sigma}_k}(x_k)|}(y_k)\big)}{|A_{\tilde{\Sigma}_k}(x_k)|^2} \\
&= c\,\mathcal{H}^2\big(\tilde{\Sigma}_k \cap B_1(g^{-1}(y_k))\big) + a\mathcal{H}^2\big(\tilde{\Gamma}_k \cap B_1(g^{-1}(y_k))\big) \\
&\leq c\big(M + a\mathcal{H}^2(\partial\mathcal{M})\big),
\end{aligned}$$

where the first inequality follows, for k sufficiently large, by Corollary 3.20; the last inequality follows by the uniform bound (5.26). Since R is arbitrary on a subset of full measure in $(0, \infty)$, there exists a constant $c > 0$ such that

$$\sup_{R\in(0,+\infty)} \frac{\mathcal{H}^2\big(\Sigma \cap B_R(y)\big)}{R^2} \leq cM. \tag{5.30}$$

Summarizing, Σ is a surface which is stationary and stable for F_a in $\mathcal{N}$ which satisfies (5.30) and $|A_\Sigma(0)| = 1$; but this contradicts Theorem 5.32.

2. If $\liminf_{k\to\infty} \operatorname{dist}(0, \partial M_k) = +\infty$, we choose a subsequence (not relabeled) Σ_k such that

$$\lim_{k\to\infty} \operatorname{dist}(0, \partial M_k) = +\infty.$$

By (5.28), there exists a further subsequence (not relabeled) of Σ_k such that

$$\Sigma_k \to \Sigma \qquad \text{in } C^{1,\beta}$$

for some $\beta \in (0, 1)$, where Σ is a surface without boundary in the whole $\mathbb{R}^3$ which is stationary and stable for the area functional. By the classical Schauder estimates, the convergence $\Sigma_k \to \Sigma$ is in fact in the C^2 topology, thus by (5.29) one has

$$|A_\Sigma(0)| = 1.$$

Arguing as above, we find a contradiction with the classical Bernstein Theorem for stable entire minimal surfaces.

□

A straightforward consequence of the above curvature estimates is the following compactness result.

Theorem 5.35 (Compactness for stable pairs)**.** *Let* $U \subset \mathcal{M}$ *be open and let* $(\Sigma^k, \Gamma^k)^k$ *be a sequence of pairs in* U *satisfying:*

1. *For every* $k \in \mathbb{N}$, Σ_k *and* Γ_k *are connected and smooth up to the boundary;*
2. *every* (Σ^k, Γ^k) *is stationary and stable for* F_a *in* U*;*
3. $\sup_k F_a(\Sigma^k, \Gamma^k) < +\infty$*;*

Then, up to subsequences, (Σ^k, Γ^k) *converges in the sense of varifolds to a stationary stable pair* (V, W) *of varifolds for* F_a *such that:*

1. W *is equal to* Γ *where* $\Gamma \subset \partial\mathcal{M}$ *is a surface with smooth boundary;*
2. V *is a smooth stable minimal surface* Σ *in* U *which either does not meet* $\partial\mathcal{M}$ *or meets* $\partial\mathcal{M}$ *orthogonally, tangentially or with angle* θ*;*
3. *For every open set* $U' \subset\subset U$*, the convergences* $\Sigma_k \to \Sigma$ *and* $\Gamma_k \to \Gamma$ *in* U' *are smooth.*

5.7 Replacements

To prove regularity for (V, W) we follow Almgren and Pitts and introduce the notion of replacement.

Definition 5.36 (Replacements)**.** Let (V, W) be stationary for F_a in $\mathcal{M}$ and let $U \subset \mathcal{M}$ be a relatively open set; a pair $(\tilde{V}, \tilde{W})$ is a *replacement* of (V, W) in U if it satisfies the following conditions:

1. $(\tilde{V}, \tilde{W})$ is stationary for F_a in $\mathcal{M}$ and $\tilde{V} = V$ and $\tilde{W} = W$ in $\mathcal{M} \setminus U$;
2. $F_a(\tilde{V}, \tilde{W}) = F_a(V, W)$;
3. $\tilde{V}$ and $\tilde{W}$ are induced in U by smooth surfaces $\tilde{\Sigma}, \tilde{\Gamma}$ with boundary, such that the pair $(\tilde{\Sigma}, \tilde{\Gamma})$ is stationary and stable for F_a in U;
4. $\tilde{\Sigma}$ has integer multiplicity and $\tilde{\Gamma}$ has either multiplicity 0 or 1;
5. Any connected component of $\tilde{\Sigma}$ either does not intersect $\partial\mathcal{M}$ or meets $\partial\mathcal{M}$ with angle θ; that is no connected component of $\tilde{\Sigma}$ touches $\partial\mathcal{M}$ tangentially, nor it meets $\partial\mathcal{M}$ orthogonally.

The key result of this section is the existence of replacements in every annulus An in $\mathcal{AN}_{r(x)}(x)$.

Proposition 5.37 (Existence of replacements)**.** *Let* $(\Sigma^k, \Gamma^k)^k, (V, W), r$ *be respectively the min-max sequence, the pair of varifolds and the function* r *given by Proposition 5.17. Let* $x \in \mathcal{M}$ *and let* $\mathrm{An} := \mathrm{An}(x, r_0, s_0) \in \mathcal{AN}_{r(x)}(x)$*; then there exists a min-max sequence* $(\hat{\Sigma}^k, \hat{\Gamma}^k)^k$ *and a function* $\tilde{r} : \mathcal{M} \to (0, +\infty)$ *such that*

- $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ *converges, up to subsequences, to a pair* $(\tilde{V}, \tilde{W})$ *which is a replacement for* (V, W) *in* $\mathrm{An} = \mathrm{An}(x, r_0, s_0)$*;*
- *For every* $y \in \mathcal{M}$*,* $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ *is almost minimizing in every annulus in* $\mathcal{AN}_{\tilde{r}(y)}(y)$*;*
- $\tilde{r}(x) = r(x)$*.*

Since the proposition is already proved for points $x \in \mathcal{M}^o$ in [DT13, Proposition 2.6], throughout the section we assume that $x \in \partial\mathcal{M}$.

The idea to prove the existence of replacements is the following: we fix an annulus $\mathrm{An} = \mathrm{An}(x, r_0, s_0) \in \mathcal{AN}_{r(x)}(x)$; since every (Σ^k, Γ^k) is, for k large enough, ε_k-a.m. in An for the sequence $\varepsilon_k \downarrow 0$ stated by Proposition 5.17, we fix such a k and consider all homotopic deformations $(\tilde{\Sigma}^k_t, \tilde{\Gamma}^k_t)_{t\in[0,1]}$ of (Σ^k, Γ^k) supported in An which do not "pass" through pairs for which F_a is larger than $F_a(\Sigma^k, \Gamma^k) + \frac{\varepsilon_k}{8}$, that is

$$F_a(\tilde{\Sigma}^k_t, \tilde{\Gamma}^k_t) \leq F_a(\Sigma^k, \Gamma^k) + \frac{\varepsilon_k}{8} \qquad \forall t \in [0,1]. \tag{5.31}$$

We can consider a minimizing sequence $(\Sigma^k_j, \Gamma^k_j)_j$ for this class of deformations that converges, up to subsequences, to a pair $(\tilde{V}_k, \tilde{W}_k)$, which is stationary for F_a in An and coincides with (Σ^k, Γ^k) in $\mathcal{M} \setminus \mathrm{An}$, since all deformations are supported in An. The main step is to prove that $(\tilde{V}_k, \tilde{W}_k)$ is induced by regular surfaces in An. This is based on the fact that if we consider a sufficiently small ball $B \subset \mathrm{An}$, the sequence $(\Sigma^k_j, \Gamma^k_j)_j$ is actually minimizing F_a in B *without* the restriction (5.31). This allows us to use the regularity theory for sets which minimize the capillarity functional F_a, see [DM15, Theorem 1.2] and [DM17, Theorem 1.5], to obtain that $(\tilde{V}_k, \tilde{W}_k)$ is regular inside An.

Up to subsequences, $(\tilde{V}_k, \tilde{W}_k)$ converges as $k \to \infty$, to a pair of varifolds $(\tilde{V}, \tilde{W})$. $(\tilde{V}, \tilde{W})$ coincides with (V, W) in $\mathcal{M} \setminus \mathrm{An}$ and, By Theorem 5.35, coincides in An with a regular pair $(\tilde{\Sigma}, \tilde{\Gamma})$ which is stationary and stable for F_a. Thus $(\tilde{V}, \tilde{W})$ is the desired replacement.

We fix $x \in \partial\mathcal{M}$ and an annulus $\mathrm{An} = \mathrm{An}(x, r_0, s_0) \in \mathcal{AN}_{r(x)}(x)$. These are fixed throughout the remainder of the section, and when we simply write An we refer to $\mathrm{An}(x, r_0, s_0)$.

5.7.1 $\frac{\varepsilon_k}{8}$-homotopic Plateau problem

Let $(\Sigma^k, \Gamma^k)^k$ be the almost minimizing min-max sequence given by Proposition 5.17 and for every $k \in \mathbb{N}$ let Ω_k be the open subset of $\mathcal{M}$ which is compatible with (Σ^k, Γ^k). We fix $k \in \mathbb{N}$ such that (Σ^k, Γ^k) is ε_k-a.m. in An and we define the class of pairs $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$.

Definition 5.38. A pair (Σ, Γ) belongs to $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$ if there exists a parametrized family of pairs of surfaces $(\Sigma_t, \Gamma_t)_{t\in[0,1]}$ and a family of open sets $(\Omega_t)_{t\in[0,1]}$ compatible with (Σ_t, Γ_t) that satisfy the following properties:

1. $(\Sigma_0, \Gamma_0) = (\Sigma^k, \Gamma^k)$ and $(\Sigma_1, \Gamma_1) = (\Sigma, \Gamma)$;
2. $(\Sigma_t, \Gamma_t) = (\Sigma^k, \Gamma^k)$ in $\mathcal{M} \setminus \mathrm{An}$ for every $t \in [0,1]$;
3. $F_a(\Sigma_t, \Gamma_t) \leq F_a(\Sigma^k, \Gamma^k) + \frac{\varepsilon_k}{8}$ for every $t \in [0,1]$.

Roughly speaking, (Σ, Γ) belongs to $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$ if it can be obtained deforming (Σ^k, Γ^k) in An without passing through a pair for which F_a is larger than $F_a(\Sigma^k, \Gamma^k) + \frac{\varepsilon_k}{8}$.

Remark 5.39. Since for k sufficiently large (Σ^k, Γ^k) is ε_k-almost minimizing in An, for such $k \in \mathbb{N}$ one has

$$F_a((\Sigma^k, \Gamma^k)) - \varepsilon_k \leq F_a((\Sigma, \Gamma)) \leq F_a((\Sigma^k, \Gamma^k)) + \frac{\varepsilon_k}{8} \qquad \forall (\Sigma, \Gamma) \in \mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big).$$

We now consider the minimizing problem for F_a in the class $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$.

Proposition 5.40 ($\frac{\varepsilon_k}{8}$-Homotopic Plateau problem)**.** *Let $(\Sigma^k_j, \Gamma^k_j)_j$ be a minimizing sequence of pairs in $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$, that is*

$$\lim_{j\to\infty} F_a(\Sigma^k_j, \Gamma^k_j) = \inf \Big\{ F_a(\Sigma, \Gamma) \mid (\Sigma, \Gamma) \in \mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big) \Big\}$$

and let Ω^k_j be open sets compatible with (Σ^k_j, Γ^k_j). Then the following conclusions hold:

1. *(Σ_j^k, Γ_j^k) converge as $j \to \infty$, up to subsequences, to a pair (V^k, W^k) which is stationary and stable for F_a in* An*;*

2. *(V^k, W^k) is induced, in* An*, by a pair of surfaces $(\tilde{\Sigma}^k, \tilde{\Gamma}^k)$ which are smooth and minimize F_a on small balls contained in* An*;*

3. *Ω_j^k converge, as $j \to \infty$, to a set $\tilde{\Omega}^k$ of finite perimeter which is compatible with $(\tilde{\Sigma}^k, \tilde{\Gamma}^k)$ in* An.

5.7.2 Proof of proposition 5.40

The proof is quite long and we divide it in several steps. Since k is fixed, throughout the proof we drop superscripts k, thus we write (Σ, Γ) for (Σ^k, Γ^k), (Σ_j, Γ_j) for (Σ_j^k, Γ_j^k), Ω_j for Ω_j^k, (V, W) for (V^k, W^k), $(\tilde{\Sigma}, \tilde{\Gamma})$ for $(\tilde{\Sigma}^k, \tilde{\Gamma}^k)$ and $\tilde{\Omega}$ for $\tilde{\Omega}^k$.

First of all we notice that of course there exists a subsequence, not relabeled, (Σ_j, Γ_j) which converges to a pair (V, W) in the sense of varifolds, since masses are bounded; moreover the open sets Ω_j compatible with (Σ_j, Γ_j) converge to $\tilde{\Omega}$, which is of finite perimeter since the masses of the pairs (Σ_j, Γ_j) are uniformly bounded. We then have to prove that these objects have the properties stated in the proposition.

Step 1: (V^k, W^k) is stationary and stable for F_a in An

We first prove that (V, W) is stationary in An for F_a; indeed, let us assume by contradiction the existence of a vector field $X \in \mathfrak{X}_t(\mathcal{M})$ with compact support in An such that

$$\delta_{F_a}(V, W)[X] = -\beta < 0.$$

Since (Σ_j, Γ_j) converge to (V, W), by continuity of first variation there exists $J \in \mathbb{N}$ such that

$$\delta_{F_a}(\Sigma_j, \Gamma_j)[X] \leq -\frac{\beta}{2} \qquad \forall j \geq J.$$

Since the first variation with respect to X is continuous with respect to the varifold convergence, there exists $T > 0$ such that

$$\delta_{F_a}(\psi_t)_\sharp(\Sigma_j, \Gamma_j)[X] \leq -\frac{\beta}{4} \qquad \forall t \in [0, T], \forall j \geq J.$$

Thus, for every $j \geq J$ we have

$$F_a(\psi_T)_\sharp(\Sigma_j, \Gamma_j) \leq F_a(\Sigma_j, \Gamma_j) + \int_0^T \delta_{F_a}(\psi_t)_\sharp(\Sigma_j, \Gamma_j)[X]\, \mathrm{d}t \leq F_a(\Sigma_j, \Gamma_j) - \frac{T\beta}{4},$$

which contradicts the minimality of the sequence $(\Sigma_j, \Gamma_j)_j$ in $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$.

By a similar argument, (V, W) is also stable for F_a in An.

Step 2: In small balls pairs with small F_a are in $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$

We show that, if we consider a sufficiently small ball B in An and a sufficiently large j, if $(\tilde{\Sigma}, \tilde{\Gamma})$ coincides with (Σ_j, Γ_j) outside of B and it holds $F_a(\tilde{\Sigma}, \tilde{\Gamma}) \leq F_a(\Sigma_j, \Gamma_j)$, then $(\tilde{\Sigma}, \tilde{\Gamma})$ belongs to $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$.

Lemma 5.41. *Let us assume $y \in$ An; there exists $j_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that $B_{\rho_0}(y) \subset$ An and with the following property: if $j > j_0$, $\rho < \rho_0$ and if $(\tilde{\Sigma}, \tilde{\Gamma})$ is a smooth pair of surfaces such that*

1. $(\tilde{\Sigma}, \tilde{\Gamma}) \setminus B_\rho(y) = (\Sigma_j, \Gamma_j) \setminus B_\rho(y)$*;*

2. $F_a(\tilde{\Sigma}, \tilde{\Gamma}) < F_a(\Sigma_j, \Gamma_j)$*,*

then $(\tilde{\Sigma}, \tilde{\Gamma}) \in \mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$*.*

Proof. Since the case $y \in \mathcal{M}^o$ is covered in [DT13, Lemma 4.1], we fix $y \in \mathrm{An} \cap \partial\mathcal{M}$. Up to translating, we can assume that y coincides with the origin 0.

We first assume that there exists $R > 0$ such that $B_R \cap \partial\mathcal{M}$ is flat, that is, up to rotating the coordinates, we assume that $B_R \cap \partial\mathcal{M} = \{(z_1, z_2, z_3) \in B_R \mid z_3 \geq 0\}$.

1. We begin by considering $j_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that $4\rho_0 \leq R$ and $B_{4\rho_0} \subset \mathrm{An}$. The exact values of j_0 and ρ_0 will be chosen at the end of the proof.

   We choose $\rho < \rho_0$, $j > j_0$; since the sets of singularities S_j, G_j of (Σ_j, Γ_j) are finite, by Sard's Lemma there exists $\tau \in (\rho, 2\rho)$ such that each surface of the pair (Σ_j, Γ_j) intersects ∂B_τ transversally and smoothly.

   We now choose $\delta > 0$ so that $2\delta < \min\{\tau - \rho, 2\rho - \tau\}$, and we consider a monotone non-decreasing smooth function $\gamma : [0, +\infty) \to [0, +\infty)$ such that

   - $\gamma(t) = t$ if $t \in [0, \tau - 2\delta] \cup [\tau + 2\delta, +\infty)$;
   - $\gamma(t) = \tau$ if $t \in [\tau - \delta, \tau + \delta]$.

   We recall, by (2.2), that
   $$\tau_{0,\kappa} : z \in \mathbb{R}^3 \longmapsto \frac{1}{\kappa} z \in \mathbb{R}^3,$$
   and, for every $t \in [0,1]$ and every $\lambda \geq 0$, we define $(\Sigma_{j,t}, \Gamma_{j,t})$ as follows:
   $$(\Sigma_{j,t}, \Gamma_{j,t}) \cap \partial B_\lambda = \tau_{0,[t\gamma(\lambda)+(1-t)\lambda]/\lambda}\Big((\Sigma_j, \Gamma_j) \cap \partial B_{t\gamma(\lambda)+(1-t)\lambda}\Big).$$
   Roughly speaking, we are stretching (Σ_j, Γ_j) in a neighbourhood of ∂B_τ and deforming it in a cone.

   Whatever is $\varepsilon > 0$, if δ is chosen sufficiently small, we have that
   $$F_a(\Sigma_{j,t}, \Gamma_{j,t}) \leq F_a(\Sigma_j, \Gamma_j) + \varepsilon \qquad \forall t \in [0,1].$$
   Since $(\tilde{\Sigma}, \tilde{\Gamma}) = (\Sigma_j, \Gamma_j)$ outside B_ρ, the same modification can be done on $(\tilde{\Sigma}, \tilde{\Gamma})$ with the same estimates on F_a. This allows us to assume, without loss of generality, that (Σ_j, Γ_j) (and thus $(\tilde{\Sigma}, \tilde{\Gamma})$, since it coincides with (Σ_j, Γ_j) outside B_ρ) is a cone in $\mathrm{An}(0, \tau - \delta, \tau + \delta)$.

2. We now define C, K as the cones with base $\Sigma_j \cap \partial B_\tau$ and $\Gamma_j \cap \partial B_\tau$ and vertex 0, that is:
   $$C \cap \partial B_\lambda = \tau_{0,\tau/\lambda}\Big(\Sigma_j \cap \partial B_\tau\Big) \qquad \forall \lambda > 0,$$
   $$K \cap \partial B_\lambda = \tau_{0,\tau/\lambda}\Big(\Gamma_j \cap \partial B_\tau\Big) \qquad \forall \lambda > 0.$$

   We can now construct the homotopy beetween (Σ_j, Γ_j) and $(\tilde{\Sigma}, \tilde{\Gamma})$, that is the family of pairs $\{(\Sigma_{j,t}, \Gamma_{j,t})\}_{t \in [0,1]}$ such that $(\Sigma_{j,0}, \Gamma_{j,0}) = (\Sigma_j, \Gamma_j)$ and $(\Sigma_{j,1}, \Gamma_{j,1}) = (\tilde{\Sigma}, \tilde{\Gamma})$; we define it in the following way:

   - $(\Sigma_{j,t}, \Gamma_{j,t}) \setminus B_\tau = (\Sigma_j, \Gamma_j) \setminus B_\tau$ for every $t \in [0,1]$;
   - $(\Sigma_{j,t}, \Gamma_{j,t}) \cap \mathrm{An}(0, |1-2t|\tau, \tau) = (C, K) \cap \mathrm{An}(0, |1-2t|\tau, \tau)$ for every $t \in [0,1]$;

- $(\Sigma_{j,t},\Gamma_{j,t})\cap B_{(1-2t)\tau}=\tau_{0,1/(1-2t)}\Big((\Sigma_j,\Gamma_j)\cap B_\tau\Big)$ for $t\in[0,\frac{1}{2}]$;
- $(\Sigma_{j,t},\Gamma_{j,t})\cap B_{(2t-1)\tau}=\tau_{0,1/(2t-1)}\Big((\tilde{\Sigma},\tilde{\Gamma})\cap B_\tau\Big)$ for $t\in[\frac{1}{2},1]$.

The family of open sets compatible with $(\Sigma_{j,t},\Gamma_{j,t})$ is defined in the same way.

It is clear that this family of pair of surfaces satisfies conditions 1 and 2 of definition 5.38; we have only to check that condition 3 is satisfied.

3. By coarea formula, we have

$$\begin{aligned}\mathcal{H}^2(\Sigma_j\cap\mathrm{An}(0,\rho,2\rho))&=\int_\rho^{2\rho}\mathcal{H}^1\big(\Sigma_j\cap\partial B_\lambda\big)\,\mathrm{d}\lambda,\\ \mathcal{H}^2(\Gamma_j\cap\mathrm{An}(0,\rho,2\rho))&=\int_\rho^{2\rho}\mathcal{H}^1\big(\Gamma_j\cap\partial B_\lambda\big)\,\mathrm{d}\lambda.\end{aligned}$$

By Chebyshev inequality we have

$$\begin{aligned}\mathcal{H}^1\Big(\Big\{\lambda\in[\rho,2\rho]\mid\mathcal{H}^1(\Sigma_j\cap\partial B_\lambda)\geq\frac{3\mathcal{H}^2(\Sigma_j\cap\mathrm{An}(0,\rho,2\rho))}{\rho}\Big\}\Big)&\leq\frac{\rho}{3},\\ \mathcal{H}^1\Big(\Big\{\lambda\in[\rho,2\rho]\mid\mathcal{H}^1(\Gamma_j\cap\partial B_\lambda)\geq\frac{3\mathcal{H}^2(\Gamma_j\cap\mathrm{An}(0,\rho,2\rho))}{\rho}\Big\}\Big)&\leq\frac{\rho}{3}.\end{aligned}$$

Since by Sard's Lemma the set of $\lambda\in[\rho,2\rho]$ such that Σ_j and Γ_j intersect ∂B_λ transversally has full measure, we can choose $\tau\in[\rho,2\rho]$ such that:

- Σ_j and Γ_j intersect ∂B_λ transversally; in particular $\Sigma_j\cap\partial B_\tau$ and $\Gamma_j\cap\partial B_\tau$ are smooth 1-manifolds;
- $$\begin{aligned}\mathcal{H}^1(\Sigma_j\cap\partial B_\tau)&\leq\frac{3\mathcal{H}^2(\Sigma_j\cap\mathrm{An}(0,\rho,2\rho))}{\rho},\\ \mathcal{H}^1(\Gamma_j\cap\partial B_\tau)&\leq\frac{3\mathcal{H}^2(\Gamma_j\cap\mathrm{An}(0,\rho,2\rho))}{\rho}.\end{aligned}$$

This gives an estimate for the area of $C\cap\mathrm{An}(y,|1-2t|\tau,\tau)$ and $K\cap\mathrm{An}(y,|1-2t|\tau,\tau)$:

$$\begin{aligned}\mathcal{H}^2(C\cap\mathrm{An}(y,|1-2t|\tau,\tau))&=\int_{|1-2t|\tau}^{\tau}\mathcal{H}^1(K\cap\partial B_\lambda)\,\mathrm{d}\lambda\\ &=\mathcal{H}^1(\Sigma_j\cap\partial B_\tau)\int_{|1-2t|\tau}^{\tau}\Big(\frac{\lambda}{\tau}\Big)\,\mathrm{d}\lambda\\ &\leq6\mathcal{H}^2(\Sigma_j\cap B_{2\rho}).\end{aligned}\tag{5.32}$$

In a similar way we get

$$\mathcal{H}^2(K\cap\mathrm{An}(y,|1-2t|\tau,\tau))\leq6\mathcal{H}^2(\Gamma_j\cap B_{2\rho}).$$

If $t\in[0,\frac{1}{2}]$ we have

$$\begin{aligned}\mathcal{H}^2(\Sigma_{j,t}\cap B(0,|1-2t|\tau))=|1-2t|^2\mathcal{H}^2(\Sigma_j\cap B_\tau)\leq\mathcal{H}^2(\Sigma_j\cap B_{2\rho})\\ \mathcal{H}^2(\Gamma_{j,t}\cap B(0,|1-2t|\tau))=|1-2t|^2\mathcal{H}^2(\Gamma_j\cap B_\tau)\leq\mathcal{H}^2(\Gamma_j\cap B_{2\rho})\end{aligned}\tag{5.33}$$

If $t\in[\frac{1}{2},1]$ we similarly have

$$\begin{aligned}\mathcal{H}^2(\Sigma_{j,t}\cap B(0,|1-2t|\tau))=|1-2t|^2\mathcal{H}^2(\tilde{\Sigma}\cap B_\tau)\leq\mathcal{H}^2(\tilde{\Sigma}\cap B_{2\rho}),\\ \mathcal{H}^2(\Gamma_{j,t}\cap B(0,|1-2t|\tau))=|1-2t|^2\mathcal{H}^2(\tilde{\Gamma}\cap B_\tau)\leq\mathcal{H}^2(\tilde{\Gamma}\cap B_{2\rho}).\end{aligned}$$

4. Gathering the estimates (5.32), (5.33) and $F_a(\tilde{\Sigma}, \tilde{\Gamma}) < F_a(\Sigma_j, \Gamma_j)$, we obtain

$$\mathcal{H}^2(\Sigma_{j,t} \cap B_\tau) + a\mathcal{H}^2(\Gamma_{j,t} \cap B_\tau) \leq 7\Big[\mathcal{H}^2(\Sigma_j \cap B_{2\rho}) + a\mathcal{H}^2(\Gamma_j \cap B_{2\rho})\Big].$$

We now recall that (Σ_j, Γ_j) converges to (V, W) $(= (V^k, W^k))$ which is stationary in An. By Corollary 3.20, we have that, for a suitable constant $c > 0$, it holds

$$\|V\|(B_\lambda) + a\|W\|(B_\lambda) \leq c\lambda^2.$$

By varifold convergence $(\Sigma_j, \Gamma_j) \to (V, W)$, up to choosing the radius ρ_0 such that

$$\|V\|(\partial B_{2\rho_0}) + a\|W\|(\partial B_{2\rho_0}) = 0,$$

there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and $\rho < \rho_0$

$$\begin{aligned}\mathcal{H}^2(\Sigma_j \cap B_{2\rho}) + a\mathcal{H}^2(\Gamma_j \cap B_{2\rho}) &\leq \mathcal{H}^2(\Sigma_j \cap B_{2\rho_0}) + a\mathcal{H}^2(\Gamma_j \cap B_{2\rho_0}) \\ &\leq 2\Big[\|V\|(B_{2\rho_0}) + a\|W\|(B_{2\rho_0})\Big] \\ &\leq 2^3 c\rho_0^2.\end{aligned}$$

If ρ_0 is chosen such that $7 \cdot 2^3 c\rho_0^2 < \frac{\varepsilon_k}{8}$, we obtain

$$F_a(\Sigma_{j,t}, \Gamma_{j,t}) \leq F_a(\Sigma_j, \Gamma_j) + \frac{\varepsilon_k}{8} \qquad \forall t \in [0,1],\ \forall j \geq j_0.$$

Thus, with these choices of j_0 and ρ_0, we obtain $(\tilde{\Sigma}, \tilde{\Gamma}) \in \mathcal{HP}\big((\Sigma^k, \Gamma^k), \mathrm{An}\big)$.

5. If $\partial\mathcal{M}$ is not flat in a neighborhood of y, there exists a ball $B := B_R(y)$, a neighborhood D of the origin and a diffeomorphism $\Phi : B \to D$ such that

$$\Phi(y) = 0 \qquad \Phi(B \cap \mathcal{M}) = \{(z_1, \dots, z_3) \in D \mid z_3 \geq 0\}.$$

By regularity of Φ, there exists a constant $b > 0$ such that

$$\begin{aligned}\frac{1}{b}\mathcal{H}^k(A) \leq \mathcal{H}^k(\Phi(A)) \leq b\mathcal{H}^k(A) \qquad &\forall A \subset B_R(y),\ k \in \{1, \dots, 3\} \\ \frac{1}{b}|z - w| \leq |\Phi(z) - \Phi(w)| \leq b|z - w| \qquad &\forall z, w \in B_R(y).\end{aligned}$$

We can repeat all the arguments made above and, up to adjusting the constants, obtain the thesis.

□

Step 3: $\tilde{\Omega}^k$ locally minimizes F_a in An

Lemma 5.42. *Let us assume $y \in \mathcal{M}$ and $0 < \rho < \rho_0$, where ρ_0 is given by Lemma 5.41; then, for every open set Ξ of finite perimeter such that $\Xi \setminus B_{\rho/2}(y) = \tilde{\Omega} \setminus B_{\rho/2}(y)$, we have*

$$F_a(\tilde{\Omega}) \leq F_a(\Xi).$$

Proof. If $y \in \mathcal{M}^\circ$, then the lemma is proved by showing that $\tilde{\Omega}$ locally minimizes the perimeter in a small ball with center y; the proof is the same as the one of [DT13, Lemma 4.2], then we can assume that $y \in \partial\mathcal{M}$.

Let us assume by contradiction that there exists $\eta > 0$ and an open set Ξ of finite perimeter such that $\Xi \setminus B_{\rho/2}(y) = \tilde{\Omega} \setminus B_{\rho/2}(y)$ and

$$F_a(\Xi) < F_a(\tilde{\Omega}) - \eta. \tag{5.34}$$

We want to use this assumption to construct a sequence of competitors for F_a which is better than Ω_j.

Since $\mathbf{1}_{\Omega_j} \to \mathbf{1}_{\tilde{\Omega}}$ in L^1, by coarea formula the sequence of function

$$\psi_j \colon t \mapsto \mathcal{H}^2\big((\Omega_j \Delta \tilde{\Omega}) \cap \partial B_t\big)$$

converge to 0 in $L^1([0,\rho])$ as $j \to \infty$. Hence, up to subsequences, we have

$$\lim_{j\to\infty} \mathcal{H}^2\big((\Omega_j \Delta \tilde{\Omega}) \cap \partial B_t\big) = 0 \qquad \text{for a.e. } t \in [0,\rho]. \tag{5.35}$$

We define, for each $j \in \mathbb{N}$,

$$\Xi_j = \big(\Omega_j \setminus B_\tau\big) \cup \big(\Xi \cap B_\tau\big),$$

where τ has to be chosen in $\left(\frac{\rho}{2}, \rho\right)$. We have

$$\begin{aligned} F_a(\Xi_j) \leq & \mathcal{H}^2\big((\partial\Omega_j \setminus B_\tau) \cap \mathcal{M}^\circ\big) + \operatorname{Per}(\Xi, {B_\tau}^\circ \cap \mathcal{M}^\circ) + \mathcal{H}^2\big((\Omega_j \Delta \tilde{\Omega}) \cap \partial B_\tau\big) \\ & + a\mathcal{H}^2\big((\partial\Omega_j \setminus B_\tau) \cap \partial\mathcal{M}\big) + a\mathcal{H}^2\big(\partial^*\Xi \cap B_\tau \cap \partial\mathcal{M}\big) \end{aligned}$$

If we choose τ such that (5.35) holds true for $t = \tau$ and such that the perimeter measure of Ω_j and $\tilde{\Omega}$ do not charge ∂B_τ, we have that

$$\begin{aligned} \limsup_{j\to\infty} \big(F_a(\Xi_j) - F_a(\Omega_j)\big) \leq & \limsup_{j\to\infty} \Big[\operatorname{Per}(\Xi, {B_\tau}^\circ \cap \mathcal{M}^\circ) + \mathcal{H}^2\big((\Omega_j \Delta \tilde{\Omega}) \cap \partial B_\tau\big) \\ & \quad + a\mathcal{H}^2\big(\partial^*\Xi \cap B_\tau \cap \partial\mathcal{M}\big) \\ & \quad - \operatorname{Per}(\Omega_j, {B_\tau}^\circ \cap \mathcal{M}^\circ) - a\mathcal{H}^2\big(\partial^*\Omega_j \cap B_\tau \cap \partial\mathcal{M}\big)\Big] \\ \leq & \operatorname{Per}(\Xi, {B_\tau}^\circ \cap \mathcal{M}^\circ) + a\mathcal{H}^2\big(\partial^*\Xi \cap B_\tau \cap \partial\mathcal{M}\big) \\ & \quad - \liminf_{j\to\infty} \Big[\operatorname{Per}(\Omega_j, {B_\tau}^\circ \cap \mathcal{M}^\circ) + a\mathcal{H}^2\big(\partial^*\Omega_j \cap B_\tau \cap \partial\mathcal{M}\big)\Big] \\ \leq & \operatorname{Per}(\Xi, {B_\tau}^\circ \cap \mathcal{M}^\circ) + a\mathcal{H}^2\big(\partial^*\Xi \cap B_\tau \cap \partial\mathcal{M}\big) \\ & \quad - \Big[\operatorname{Per}(\tilde{\Omega}, {B_\tau}^\circ \cap \mathcal{M}^\circ) + a\mathcal{H}^2\big(\partial^*\tilde{\Omega} \cap B_\tau \cap \partial\mathcal{M}\big)\Big] \\ \leq & -\eta, \end{aligned} \tag{5.36}$$

where the second inequality holds by (5.35), the third by lower semicontinuity of the capillarity functional under L^1 convergence of sets, see [Mag12, Proposition 19.1] and the last inequality is given by the assumption (5.34).

Ξ_j is not an admissible candidate, since it is not necessarily smooth in $B_{\rho/2}$, hence we have to smooth up these set without increasing F_a too much.

By minimality of the sequence $\{\partial\Omega_j\}_j$ for the homotopic Plateau problem, we choose $j_0 \in \mathbb{N}$ such that Lemma 5.41 holds and so that

$$F_a(\Omega_j) \leq \inf\{F_a(\Sigma,\Gamma) \mid (\Sigma,\Gamma) \in \mathcal{HP}\big((\Sigma^k,\Gamma^k), \operatorname{An}\big)\} + \frac{\eta}{8} \qquad \forall j \geq j_0.$$

We choose an interval $(a,b) \subset\subset (\tau,\rho)$ such that $\partial_i\Omega_j$ and $\partial_b\Omega_j$ are smooth in $\operatorname{An}(y,a,b)$ and have a smooth common boundary $\gamma_j \subset \partial\mathcal{M} \cap \operatorname{An}(y,a,b)$. We can choose $D \subset\subset \operatorname{An}(y,a,b)$ which has smooth boundary ∂D and so that there exist $d > 0$ and (a',b') such that

$$D \supseteq T := \{(\xi,\zeta) \mid |\xi| \in (a',b'), \zeta \in [0,d]\},$$

where (ξ, ζ) are normal coordinates in a tubular neighborhood of $\partial\mathcal{M}$; roughly speacking, in the normal coordinates, D contains a “tube” T with base on $\partial\mathcal{M}$.

With a small deformation supported in $\mathrm{An}(y, a, b)$ we can “push-in” γ_j in $\mathrm{Int}(\mathcal{M})$ in D, that is we can assume that, in the normal coordinates ξ, ζ,

$$\partial_i\Omega_j \cap T = \gamma_j \times (0, d].$$

Moreover d and $|a' - b'|$ can be chosen small enough so that the possible gain of F_a is as small as we want, and then we can assume, without loss of generality, that Ω_j is of this form.

From this we can consider D', T' obtained by reflecting D across $\partial\mathcal{M}$ and extend Ω_j in D' obtaining Ω'_j so that, in the in the normal coordinates (ξ, ζ) we have

$$\partial_i\Omega'_j \cap T' = \gamma_j \times (-d, d).$$

We now extend every $\mathbf{1}_{\Xi_j}$ to a function $f_j \in BV(\mathbb{R}^3)$ such that $|Df_j|(\partial\mathcal{M}) = 0$ and such that $f_j \llcorner D' = \mathbf{1}_{\Omega'_j}$.

Roughly speacking, we enlarged Ω_j a bit outside $\mathcal{M}$ in correspondence of $\mathrm{An}(y, a, b)$ and next we considered the BV-extension to $\mathbb{R}^3$ of the enlargement.

Then, if φ_ε is the standard mollifier, we define

$$g_{j,\varepsilon} := f_j * \varphi_\varepsilon.$$

For every $j > j_0$, by standard approximation procedure [Mag12, Theorem 13.8], for almost every $t \in (0, 1)$ we have that the set $\Delta_{j,\varepsilon} := \{g_{j,\varepsilon} > t\} \cap \mathcal{M}$ is an open subset of $\mathcal{M}$ which is smooth in $\mathcal{M}$ and satisfies

$$\Delta_{j,\varepsilon} \xrightarrow[\varepsilon\to 0]{L^1_{\mathrm{loc}}(\mathcal{M})} \mathbf{1}_{\Xi_j}, \qquad \mathrm{Per}\left(\Delta_{j,\varepsilon}, \mathcal{M}^o\right) \xrightarrow[\varepsilon\to 0]{} \mathrm{Per}\left(\Xi_j, \mathcal{M}^o\right).$$

Since F_a is continuous under strict convergence of BV-functions, we obtain

$$\lim_{\varepsilon\to 0} F_a(\Delta_{j,\varepsilon}) = F_a(\Xi_j).$$

Hence for every $j > j_0$ there exists $\bar{\varepsilon}_j > 0$ such that the set Δ_{j,ε_j} satisfies

$$F_a(\Delta_{j,\varepsilon}) \le F_a(\Xi_j) + \frac{\eta}{4} \qquad \forall j > j_0,\ \forall \varepsilon \in (0, \bar{\varepsilon}_j).$$

Since $\Delta_{j,\varepsilon}$ does not coincide with Ω_j outside $B_a(y)$, we have to construct a new sequence of sets that “connect” $\Delta_{j,\varepsilon}$ inside $B_a(y)$ with Ω_j outside $B_b(y)$ with a small possible increase of F_a. This is possible since both Ω_j and $\Delta_{j,\varepsilon}$ are smooth within $\mathrm{An}(y, a, b)$.

We fix $j > j_0$ a tubular neighbourhood U of $\partial_i\Omega_j \cap \mathrm{An}(y, a', b')$ with normal coordinates (w, s); since by mollification $\partial_i\Delta_{j,\varepsilon}$ and $\partial_b\Delta_{j,\varepsilon}$ converge smoothly respectively to $\partial_i\Omega_j$ and $\partial_b\Omega_j$ in $\mathrm{An}(y, a', b')$ as $\varepsilon \to 0$ (here the enlargement of Ω_j in D' plays an important role), there exist functions $\psi_{j,\varepsilon}$ such that

$$\Omega_j \cap U = \{(w, s) \mid s < 0\}, \qquad \Delta_{j,\varepsilon} \cap U = \{(w, s) \mid s < \psi_{j,\varepsilon}(w)\} \quad \text{in } \mathrm{An}(y, a', b')$$

and $\psi_{j,\varepsilon} \to 0$ smoothly as $\varepsilon \to 0$.

To join smoothly the two sets we choose a function $\chi \in C^\infty_c(B_{b'}(y))$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ in a neighbourhood of $B_{a'}(y)$ and we construct a new family of open sets $\Theta_{j,\varepsilon}$:

- $\Theta_{j,\varepsilon} \cap B_{a'}(y) = \Delta_{j,\varepsilon} \cap B_{a'}(y)$;
- $\Theta_{j,\varepsilon} \cap \mathrm{An}(y, a', b') = \{(w, s) \mid s < \chi(w)\psi_{j,\varepsilon}(w)\}$;
- $\Theta_{j,\varepsilon} \setminus B_{b'} = \Omega_j \setminus B_{b'}$.

By construction $\partial_i\Theta_{j,\varepsilon}, \partial_b\Theta_{j,\varepsilon}$ are smooth outside of a finite set.

By smooth convergence $\psi_{j,\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$, for every $j > j_0$ there exists $\varepsilon_j \in (0, \bar{\varepsilon}_j)$ such that

$$F_a(\Theta_{j,\varepsilon_j}) \le F_a(\Xi_j) + \frac{\eta}{2} \le F_a(\Omega_j) - \frac{\eta}{4}$$

if j is sufficiently large, by (5.36). Since by Lemma 5.41 $\Theta_{j,\varepsilon_j} \in \mathcal{HP}\big((\Sigma^k,\Gamma^k),\mathrm{An}\big)$, this is a contradiction with the minimality of the sequence Ω_j in $\mathcal{HP}\big((\Sigma^k,\Gamma^k),\mathrm{An}\big)$ for F_a.

□

Step 4: (V^k, W^k) is compatible with $\tilde{\Omega}^k$

We have now to show that the pair (V, W) is induced by $\partial_i\tilde{\Omega}$ and $\partial_b\tilde{\Omega}$. By [DT13, Proposition A.1], we have only to prove that

$$\lim_{j\to\infty} \mathcal{H}^2(\Sigma_j) = \mathcal{H}^2(\partial_i\tilde{\Omega}), \qquad \lim_{j\to\infty} \mathcal{H}^2(\Gamma_j) = \mathcal{H}^2(\partial_b\tilde{\Omega}).$$

Since F_a is lower semicontinuous under L^1-convergence of open sets, it holds

$$F_a(\tilde{\Omega}) \le \lim_{j\to\infty} F_a(\Sigma_j, \Gamma_j).$$

The above inequality cannot be strict, because otherwise there would exist a sufficiently small ball B such that the same inequality holds and we can argue as the proof of Lemma 5.42, defining new competitors $\Xi_j = \big(\Omega_j \setminus B\big) \cup \big(\tilde{\Omega} \cap B\big)$ which would contradict the minimality of sequence Ω_j. Hence

$$F_a(\tilde{\Omega}) = \lim_{j\to\infty} F_a(\Sigma_j, \Gamma_j). \tag{5.37}$$

If we prove $\lim_j \mathcal{H}^2(\Gamma_j) = \mathcal{H}^2(\partial_b\tilde{\Omega})$, (5.37) would imply the conclusion. In order to do this, since $\mathbf{1}_{\Omega_j} \to \mathbf{1}_{\tilde{\Omega}}$ in $L^1(\mathbb{R}^n)$, we have

$$\mathrm{Per}(\tilde{\Omega}) \le \liminf_{j\to\infty} \mathrm{Per}(\Omega_j);$$

Thus

$$\begin{aligned}(1-a)\mathcal{H}^2(\partial_b\tilde{\Omega}) =& \mathrm{Per}(\tilde{\Omega}) - F_a(\tilde{\Omega})\\ \le& \liminf_{j\to\infty} \Big(\mathrm{Per}(\Omega_j) - F_a(\Sigma_j,\Gamma_j)\Big)\\ =&(1-a)\liminf_{j\to\infty} \mathcal{H}^2(\Gamma_j).\end{aligned}$$

On the other hand, again by (5.37) and $\mathbf{1}_{\Omega_j} \to \mathbf{1}_{\tilde{\Omega}}$ in $L^1(\mathbb{R}^n)$, one has

$$\left.\begin{aligned}\mathcal{H}^2(\partial_i\tilde{\Omega}) = \mathrm{Per}(\tilde{\Omega}, \mathcal{M}^\circ) \le \liminf_{j\to\infty}\mathrm{Per}(\Omega_j,\mathcal{M}^\circ) = \liminf_{j\to\infty}\mathcal{H}^2(\Sigma_j)\\ \mathcal{H}^2(\partial_i\tilde{\Omega}) + a\mathcal{H}^2(\partial_b\tilde{\Omega}) = \lim_{j\to\infty}\big(\mathcal{H}^2(\Sigma_j) + a\mathcal{H}^2(\Gamma_j)\big)\end{aligned}\right\} \Longrightarrow \mathcal{H}^2(\partial_b\tilde{\Omega}) \ge \limsup_{j\to\infty}\mathcal{H}^2(\Gamma_j).$$

This implies that

$$\lim_{j\to\infty} \mathcal{H}^2(\Gamma_j) = \mathcal{H}^2(\partial_b\tilde{\Omega}),$$

which, together with (5.37), yields

$$\lim_{j\to\infty} \mathcal{H}^2(\Sigma_j) = \mathcal{H}^2(\partial_i\tilde{\Omega}).$$

Step 5: $\partial_i\tilde{\Omega}^k$ and $\partial_b\tilde{\Omega}^k$ are regular surfaces in An

By the minimality of $\tilde{\Omega}$ for F_a in small balls, we use use the regularity theorems for minimizers of capillarity functional [DM15, Theorem 1.2 and Corollary 1.4] to obtain that V and W, which are induced by $\partial_i\tilde{\Omega}$ and $\partial_b\tilde{\Omega}$, coincide in An with surfaces $\tilde{\Sigma}, \tilde{\Gamma}$ that are smooth and $\tilde{\Sigma}$ meets $\partial\mathcal{M}$ on $\partial\tilde{\Gamma}$ with angle θ.

5.7.3 Construction of replacements: proof of theorem 5.37

Proof of the theorem 5.37.

- *(V^k, W^k) converges to $(\tilde{V}, \tilde{W})$, which is made of smooth surfaces $\tilde{\Sigma}, \tilde{\Gamma}$ in* An.
  Let us consider for every $k \in \mathbb{N}$ the pair (V^k, W^k) given by Proposition 5.40; (V^k, W^k) is a stationary stable pair made of smooth surfaces in An, by Theorem 5.35 the sequence of pairs (V^k, W^k) converges, up to subsequences, to a pair $(\tilde{V}, \tilde{W})$ which is stationary and stable for F_a in An and is induced by a pair of regular surfaces $\tilde{\Sigma}, \tilde{\Gamma}$ in An.

- *$\tilde{\Sigma}$ has integer multiplicity, $\tilde{\Gamma}$ has either multiplicity* 0 *or* 1*. Σ has no connected components which meet $\partial\mathcal{M}$ orthogonally or tangentially.*
  Since the pair $(\tilde{\Sigma}^k, \tilde{\Gamma}^k)$ given by Proposition 5.40 in An is induced by $(\partial_i\tilde{\Omega}^k, \partial_b\tilde{\Omega}^k)$, $\tilde{\Sigma}^k, \tilde{\Gamma}^k$ have multiplicity 1. By Theorem 5.35 $(\tilde{\Sigma}, \tilde{\Gamma})$ is the smooth limit of the sequence $(\tilde{\Sigma}^k, \tilde{\Gamma}^k)$, thus $\tilde{\Sigma}, \tilde{\Gamma}$ have integer multiplicity. For the same reason, $\tilde{\Gamma}$ has either multiplicity 0 or 1 (because it is the smooth limit of the surfaces $\tilde{\Gamma}^k$, which are contained in $\partial\mathcal{M}$).

  Concerning the last assertion, we notice that $\tilde{\Sigma}^k = \partial_b\tilde{\Omega}^k$, and $\tilde{\Omega}^k$ is a set of finite perimeter stationary for F_a in An. Thus $\tilde{\Sigma}^k$ does not have connected components which meet $\partial\mathcal{M}$ orthogonally. Since $\tilde{\Sigma}$ is a smooth limit of $\tilde{\Sigma}^k$, it does not have connected components which meet $\partial\mathcal{M}$ orthogonally.

  The sets $\tilde{\Omega}^k$ are, locally on small balls contained in An, minimizers for F_a. By the regularity theory for minimizers, see section 5.7.2, $\partial_i\tilde{\Omega}^k \cap \text{An}$ is made of surfaces which either do not meet $\partial\mathcal{M}$ or meet $\partial\mathcal{M}$ with angle θ. Since $\tilde{\Sigma}$ is the smooth limit of $\partial_i\tilde{\Omega}^k$ in An, if a connected component $\tilde{\Sigma}_c$ of $\tilde{\Sigma}$ meets $\partial\mathcal{M}$ tangentially, then it is a limit of connected components of $\partial_i\tilde{\Omega}^k$ that do not intersect $\partial\mathcal{M}$. Thus $\tilde{\Sigma}_c$ would be a minimal surface which touches $\partial\mathcal{M}$ in a point $\bar{y} \in \partial\mathcal{M} \cap \text{An}$ and is contained in $\mathcal{M}$. This would give a positive sign on the second fundamental form of $\tilde{\Sigma}_c$ which contradicts the fact that $\tilde{\Sigma}_c$ is the smooth limit of surfaces with 0 mean curvature.

- *Definition of the min-max sequence $(\hat{\Sigma}^k, \hat{\Gamma}^k)^k$*
  Let us consider, for every $k \in \mathbb{N}$, a minimizing sequence $\{\Omega_j^k\}_j$ for the $\frac{\varepsilon_k}{8}$-homotopic Plateau problem $\mathcal{HP}\big((\Sigma^k, \Gamma^k), \text{An}\big)$. Since by Proposition 5.40 for every $k \in \mathbb{N}$ the sequence $(\partial_i\Omega_j^k, \partial_b\Omega_j^k)_j$ converges, up to subsequences, to (V^k, W^k), we can select a diagonal sequence $(\hat{\Sigma}^k, \hat{\Gamma}^k)^k = (\partial_i\Omega_{j(k)}^k, \partial_b\Omega_{j(k)}^k)$ which satisfies
  $$F_a(\hat{\Sigma}^k, \hat{\Gamma}^k) \leq F_a(\Sigma^k, \Gamma^k) \qquad \forall k \in \mathbb{N}$$
  and such that $(\hat{\Sigma}^k, \hat{\Gamma}^k) \to (\tilde{V}, \tilde{W})$. Since every $(\hat{\Sigma}^k, \hat{\Gamma}^k) \in \mathcal{HP}\big((\Sigma^k, \Gamma^k), \text{An}\big)$, by Remark 5.39 one has
  $$F_a(\Sigma^k, \Gamma^k) - \varepsilon_k \leq F_a(\hat{\Sigma}^k, \hat{\Gamma}^k) \leq F_a(\Sigma^k, \Gamma^k),$$
  thus $(\hat{\Sigma}^k, \hat{\Gamma}^k)^k$ is a new min-max sequence for F_a.

- *Definition of $\tilde{r}$*
  We fix $\eta > 0$ and $\text{An}' = \text{An}(x, r_0 - \eta, s_0 + \eta)$ such that $\text{An}' \in \mathcal{AN}_{r(x)}(x)$.

- We set $\tilde{r}(x) = r(x)$; We claim that for every annulus in $\mathcal{AN}_{r(x)}(x)$, the sequence $(\hat{\Sigma}^k, \hat{\Gamma}^k)_k$ is ε_k-almost minimizing in every annulus for k sufficiently large; indeed let us consider $\mathrm{An}'' \in \mathcal{AN}_{r(x)}(x)$ and $\mathrm{An}''' \in \mathcal{AN}_{r(x)}(x)$ such that $\mathrm{An}''' \supset \mathrm{An} \cup \mathrm{An}''$ and let us fix $k \in \mathbb{N}$ such that (Σ^k, Γ^k) is ε_k-almost minimizing in An'''.

  For every k $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ is obtained by deforming (Σ^k, Γ^k) in An without passing through a pair with F_a larger than $F_a(\Sigma^k, \Gamma^k) + \varepsilon_k/8$, thus $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ cannot be deformed in An''' in another pair (Σ, Γ) with

  $$F_a(\Sigma, \Gamma) \leq F_a(\hat{\Sigma}^k, \hat{\Gamma}^k) - \varepsilon_k \leq F_a(\Sigma^k, \Gamma^k) - \varepsilon_k$$

  without passing through a pair with F_a larger than

  $$F_a(\Sigma^k, \Gamma^k) + \varepsilon_k/8 \geq F_a(\hat{\Sigma}^k, \hat{\Gamma}^k) + \varepsilon_k/8,$$

  otherwise this would contradicts the ε_k-almost minimality of (Σ^k, Γ^k) in An'''.
- if $y \in \mathrm{An}'$, we set $\tilde{r}(y) = \min\{r(y), \mathrm{dist}(y, \partial \mathrm{An}')\}$; In this case $(\hat{\Sigma}^k, \hat{\Gamma}^k)_k$ is almost minimizing in every annulus belonging to $\mathcal{AN}_{\tilde{r}(y)}(y)$ because such an annulus is contained in An' and $(\hat{\Sigma}^k, \hat{\Gamma}^k)_k$ is almost minimizing in An'.
- if $y \notin \mathrm{An}'$, we set $\tilde{r}(y) = \min\{r(y), \mathrm{dist}(y, \mathrm{An})\}$. If $\mathrm{An}'' \in \mathcal{AN}_{\tilde{r}(y)}(y)$ then $\mathrm{An}'' \subset \mathcal{M} \setminus \mathrm{An}$; since $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ and (Σ^k, Γ^k) coincide outside An and since $(\Sigma^k, \Gamma^k)_k$ is almost minimizing in every annulus in $\mathcal{AN}_{r(y)}(y)$, we obtain that $(\hat{\Sigma}^k, \hat{\Gamma}^k)_k$ is almost minimizing in An''.

- $F_a(\tilde{V}, \tilde{W}) = F_a(V, W)$

  We are left to prove that $(\tilde{V}, \tilde{W})$ is a replacement for (V, W) in $\mathrm{An} = \mathrm{An}(x, r_0, s_0)$. Since $(\hat{\Sigma}^k, \hat{\Gamma}^k)^k$ is a min-max sequence, we have

  $$F_a(\tilde{V}, \tilde{W}) = F_a(V, W).$$

- *$(\tilde{V}, \tilde{W})$ is stationary in $\mathcal{M}$*

  By construction, it follows that $(\tilde{V}, \tilde{W})$ is stationary in An and in $\mathcal{M} \setminus \mathrm{An}$. We have only to prove that it is stationary in $\mathcal{M}$. Let us assume, by contradiction, that $(\tilde{V}, \tilde{W})$ is not stationary. We fix $\eta > 0$, a vector field $X \in \mathfrak{X}_t(\mathcal{M})$ such that

  $$\delta_{F_a}(\tilde{V}, \tilde{W})[X] \leq -\eta$$

  and we choose a partition of unity for the cover $\{\mathrm{An}', \mathcal{M} \setminus \mathrm{An}\}$ of $\mathcal{M}$: $\varphi \in C_c^\infty(\mathrm{An}')$ and $\psi \in C_c^\infty(\mathcal{M} \setminus \mathrm{An})$. Hence

  $$\delta_{F_a}(\tilde{V}, \tilde{W})[X] = \delta_{F_a}(\tilde{V}, \tilde{W})[\varphi X] + \delta_{F_a}(\tilde{V}, \tilde{W})[\psi X].$$

  Since $(\tilde{V}, \tilde{W})$ is stationary in $\mathcal{M} \setminus \mathrm{An}$, it follows that $\delta_{F_a}(\tilde{V}, \tilde{W})[\psi X] = 0$. Thus

  $$\delta_{F_a}(\tilde{V}, \tilde{W})[\varphi X] \leq -\eta.$$

  By continuity of first variation there exists $T > 0$ such that, if Φ_t is the flow of φX, then

  $$\delta_{F_a}(\Phi_t)_\sharp(\tilde{V}, \tilde{W})[\varphi X] \leq -\frac{\eta}{2} \qquad \forall t \in [0, T].$$

Since $(\hat{\Sigma}^k, \hat{\Gamma}^k) \underset{k}{\rightarrow} (\tilde{V}, \tilde{W})$, by continuity of first variation again, there exists $\bar{k} \in \mathbb{N}$ such that

$$\delta_{F_a}(\Phi_t)_\sharp(\hat{\Sigma}^k, \hat{\Gamma}^k)[\varphi X] \leq -\frac{\eta}{4} \qquad \forall t \in [0,T], \forall k > \bar{k}.$$

Hence

$$F_a\big((\Phi_t)_\sharp(\hat{\Sigma}^k, \hat{\Gamma}^k)\big) \leq F_a(\hat{\Sigma}^k, \hat{\Gamma}^k) - t\,\frac{\eta}{4} \qquad \forall t \in [0,T], \ \ \forall k > \bar{k}.$$

Thus we have a smooth deformation of $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ with support in An' which lowers F_a of a fixed amount. But this contradicts, for k large enough, the almost minimizing property of $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ in An'.

□

Remark 5.43. Since $(\hat{\Sigma}^k, \hat{\Gamma}^k)$ is a new min-max sequence which is almost minimizing in each annulus in $\mathcal{AN}_{\tilde{r}(y)}(y)$, we can repeat the previous arguments to construct a new replacement of $(\tilde{V}, \tilde{W})$ in each annulus in $\mathcal{AN}_{\tilde{r}(y)}(y)$. In particular, since $\tilde{r}(x) = r(x)$, we can construct a again a new min-max sequence and a new replacement of $(\tilde{V}, \tilde{W})$ in every annulus in $\mathcal{AN}_{r(x)}(x)$. This procedure can be in fact repeated an infinite number of times.

5.8 Regularity of V

In this final section we prove that the pair (V, W) obtained by Proposition 5.17 is such that V is indeed induced in $\mathcal{M}^o$ by a smooth surface Σ which meets $\partial\mathcal{M}$ with angle θ, thus completing the proof of Theorem 5.1

5.8.1 Integer rectifiability of V

We begin by studying the set $\mathrm{Tan}\,\big((V,W),x\big)$ of tangent pairs to (V,W), where (V,W) is the limit of the almost-minimizing min-max sequence stated in Proposition 5.17. We call $\mathrm{Tan}\,\big((V,W),x\big)$ the set of pairs (C,K) where $C \in \mathrm{Tan}(V,x)$ and $K \in \mathrm{Tan}(W,x)$.

Proposition 5.44. *Let (V,W) be the pair of varifolds given by Proposition 5.17; Then*

1. $V' := V \llcorner G_2(\mathcal{M}^o)$ *is integer rectifiable;*
2. V *and* W *are rectifiable;*
3. *For every* $x \in \partial\mathcal{M} \cap \mathrm{supp}\|V\| + a\|W\|$, *every* $(C,K) \in \mathrm{Tan}((V,W),x)$ *is such that* $C + aK = \tilde{\Sigma} + a\tilde{\Gamma}$ *where* $\mathrm{supp}\|\tilde{\Sigma}\| \subset T_x\mathcal{M}$ *and* $\mathrm{supp}\|\tilde{\Gamma}\| \subset T_x\partial\mathcal{M}$*; moreover they satisfy:*
    - $\tilde{\Gamma} = 0$, $\tilde{\Gamma}$ *is a half plane or* $\tilde{\Gamma} = T_x\partial\mathcal{M}$*;*
    - $\tilde{\Sigma} = h\tilde{\Sigma}_\theta^+ + k\tilde{\Sigma}_\theta^- + \ell T_x\partial\mathcal{M}$ *where* $h,k \in \{0,1\}$, $\ell \in \mathbb{N}$ *and* $\tilde{\Sigma}_\theta^+, \tilde{\Sigma}_\theta^-$ *are half planes that meet* $T_x\partial\mathcal{M}$ *with angle* θ *and* $\partial\tilde{\Sigma}_\theta^+ = \partial\tilde{\Sigma}_\theta^-$.

    More precisely we have the following cases:

    (a) *If* $\tilde{\Gamma} = 0$, *then* $h = k$ *and* $h + k + \ell > 0$*; moreover* $h = k > 0$ *if and only if* $\ell = 0$*;*

    (b) *If* $\tilde{\Gamma}$ *is a half plane, then* $h + k = 1$, $\ell = 0$ *and* $\partial\tilde{\Gamma} = \partial\tilde{\Sigma}_\theta^+$*;*

    (c) *If* $\tilde{\Gamma} = T_x\partial\mathcal{M}$, *then* $h = k = 0$.

Proof. We divide the proof in few steps; the integer rectifiability of V' easily follows by the interior theory carried out in [DT13], while the assertions on V, W and on tangent cones are less straightforward and in order to obtain it, we have to study tangent varifolds to $V + aW$.

- *V' is integer rectifiable*

Let us consider $x \in \mathcal{M}^\circ$. Then by the classical min-max theory in the interior, see [DT13, Lemma 5.2], there exists a neighborhood $U \subset \mathcal{M}^\circ$ of x such that V is integer rectifiable in U, thus V' is integer rectifiable.

- *V and W are rectifiable*

Let us fix $x \in \partial\mathcal{M} \cap \operatorname{supp}\big(\|V\| + a\|W\|\big)$. Our aim is to prove a uniform lower bound on the density $\Theta_2(\|V\| + a\|W\|, x)$; this, together with the fact that $V + aW$ has bounded first first variation, allows us to apply Allard's Rectifiability Theorem [Sim84, Theorem 42.4] to conclude that V and W are rectifiable.

First of all we recall that the density $\Theta_2(\|V\| + a\|W\|, x)$ exists and is finite for every $x \in \partial\mathcal{M}$ by Remark 3.21. If we choose a sequence $r_j \downarrow 0$ the rescaled pairs $(V_j, W_j) := (\tau_{x,r_j})_\sharp(V, W)$ converge, up to subsequences, to a pair (C, K) such that:

1. $\operatorname{supp} C \subset T_x\mathcal{M}$ and $\operatorname{supp} K \subset T_x\partial\mathcal{M}$;
2. $\delta_{F_a}(C, K)[X] = 0$ for every $X \in \mathfrak{X}_t(T_x\mathcal{M})$;
3. $\rho^{-2}\big(\|C\|(B_\rho) + a\|K\|(B_\rho)\big) = \sigma^{-2}\big(\|C\|(B_\sigma) + a\|K\|(B_\sigma)\big) = \Theta_2(\|V\| + a\|W\|, x)$ for every $\rho, \sigma > 0$.

All these statements follow by the fact that, since (V, W) is stationary for F_a, then by definition the varifold $V + aW$ has free boundary (Definition 2.15) with $H \equiv 0$, that is it is stationary for the mass with respect to all variations tangent to $T_x\partial\mathcal{M}$. Hence the above statements follows by the same arguments of the proof of [DeM21, Lemma 5.4]

Since (V, W) has replacement $(\tilde{V}_j, \tilde{W}_j)$ in every annulus $\operatorname{An}(x, r_j, 2r_j)$ for j sufficiently large, by Theorem 5.35 the rescaled pairs $(\tau_{x,r_j})_\sharp(\tilde{V}_j, \tilde{W}_j)$ converge to a pair $(\tilde{C}, \tilde{K})$ which is a replacement for (C, K) in $\operatorname{An}(0, 1, 2)$. By the properties of replacements, we have

$$\begin{aligned}\frac{\|\tilde{C}\|(B_{1/2}) + a\|\tilde{K}\|(B_{1/2})}{(1/2)^2} &= \frac{\|C\|(B_{1/2}) + a\|K\|(B_{1/2})}{(1/2)^2}\\ &= \frac{\|C\|(B_{5/2}) + a\|K\|(B_{5/2})}{(5/2)^2}\\ &= \frac{\|\tilde{C}\|(B_{5/2}) + a\|\tilde{K}\|(B_{5/2})}{(5/2)^2}.\end{aligned}$$

Since $\|\tilde{C}\| + a\|\tilde{K}\|$ is a stationary varifold in $T_x\mathcal{M}$ with respect to variations in $\mathfrak{X}_t(T_x\partial\mathcal{M})$, by the monotonicity identity for varifolds with free boundary (see e.g. Step 5 in the proof of [DeM21, Lemma 5.4]) one gets

$$\frac{\|\tilde{C}\|(B_\rho) + a\|\tilde{K}\|(B_\rho)}{\rho^2} = \frac{\|\tilde{C}\|(B_\sigma) + a\|\tilde{K}\|(B_\sigma)}{\sigma^2} = \Theta_2(\|V\| + a\|W\|, x) \qquad \forall \rho, \sigma > 0. \tag{5.38}$$

This means that for every $\rho > 0$ we have $\operatorname{supp}(\|\tilde{C}\| + a\|\tilde{K}\|) \cap \partial B_\rho \neq \emptyset$, because otherwise (5.38) does not hold for $\rho - \varepsilon, \rho + \varepsilon$ for some small $\varepsilon > 0$.

For any $y \in \partial B_{3/2} \cap \operatorname{supp}(\|\tilde{C}\| + a\|\tilde{K}\|)$, since $\tilde{C}$ and $\tilde{K}$ are regular in $\operatorname{An}(0, 1, 2)$, it follows that $\Theta_2(\|\tilde{C}\| + a\|\tilde{K}\|, y) \geq 1/2$. By monotonicity of density ratios for $\tilde{C} + a\tilde{K}$ given by Corollary 3.20, we have that there exists an universal constant $c > 0$ such that

$$\|\tilde{C}\|(B_{1/2}(y)) + a\|\tilde{K}\|(B_{1/2}(y)) \geq c.$$

(5.38) and since $B_{1/2}(y) \subset B_2$, we obtain the existence of an universal constant $c' > 0$ such that

$$\Theta_2(\|V\| + a\|W\|, x) > c' \qquad \forall x \in \operatorname{supp}(\|V\| + a\|W\|).$$

Since $V + aW$ is stationary with respect to $\mathfrak{X}_t(\mathcal{M})$,[DeM21, Corollary 4.7, (1.6)] implies that it has bounded first variation in the whole space $\mathbb{R}^n$ with a uniform upper bound on the first variation given by the mass $(\|V\| + a\|W\|)(\mathcal{M})$. This, together with the above uniform lower bound on the density and the Allard Rectifiability Theorem [Sim84, Theorem 42.4], proves that $V + aW$ is a rectifiable varifold. Since W is rectifiable because it is the weak*-limit of a sequence in $L^\infty(\partial\mathcal{M}, \mathcal{H}^2)$, we have that V is rectifiable.

- *$C + aK$ is a rectifiable cone; C, K are rectifiable*

We claim that every tangent pair (C, K) to (V, W) is such that $C + aK$ is a rectifiable cone. Indeed $C + aK$ is obtained as limit of rescalings $(\tau_{x,r_j})_\sharp(V + aW)$ which are stationary for $\mathfrak{X}_t\big(\tau_{x,r_j}(\mathcal{M})\big)$; by [DeM21, Theorem 1.1 and (1.6)], we have that $(\tau_{x,r_j})_\sharp(V + aW)$ have uniformly bounded first variation, then we can apply the Compactness Theorem for rectifiable varifolds [Sim84, Theorem 42.7], to get the rectifiability of $C + aK$. The fact that $C + aK$ is a cone follows by the same arguments in Step 5 in the proof of [DeM21, Lemma 5.4].

K is rectifiable because, as above, it is the weak*-limit of a bounded sequence in $L^\infty(\partial\mathcal{M}, \mathcal{H}^2)$; thus also C is rectifiable.

- *$C + aK = \tilde{\Sigma} + a\tilde{\Gamma}$, where $\tilde{\Sigma} \subset T_x\mathcal{M}, \tilde{\Gamma} \subset T_x\partial\mathcal{M}$ are half-planes and $\tilde{\Sigma}$ intersects $T_x\partial\mathcal{M}$ with angle θ.*

For any $\lambda \in (0, 1)$ and for any j sufficiently large, (V, W) has a replacement $(\tilde{V}_j, \tilde{W}_j)$ in the annulus $\mathrm{An}(x, \lambda r_j, \lambda^{-1} r_j)$ and the rescaled pairs $(\tau_{x,r_j})_\sharp(\tilde{V}_j, \tilde{W}_j)$ converge to a pair $(\tilde{C}, \tilde{K})$ which is a replacement for (C, K) in $\mathrm{An}(0, \lambda, \lambda^{-1})$. Moreover, by the same arguments of the above discussion, $\tilde{C} + a\tilde{K}$ is a rectifiable varifold.

As above, for any $\sigma \in (0, \lambda)$ and for any $\rho > \lambda^{-1}$, we have

$$\begin{aligned}\frac{\|\tilde{C}\|(B_\sigma) + a\|\tilde{K}\|(B_\sigma)}{\sigma^2} &= \frac{\|C\|(B_\sigma) + a\|K\|(B_\sigma)}{\sigma^2}\\ &= \frac{\|C\|(B_\rho) + a\|K\|(B_\rho)}{\rho^2}\\ &= \frac{\|\tilde{C}\|(B_\rho) + a\|\tilde{K}\|(B_\rho)}{\rho^2}.\end{aligned}$$

Again this and the same arguments in Step 5 in the proof of [DeM21, Lemma 5.4] prove that $\tilde{C} + a\tilde{K}$ is a rectifiable cone.

Since $(\tilde{C}, \tilde{K})$ is a replacement of (C, K) in $\mathrm{An}(0, \lambda, \lambda^{-1})$ and $\tilde{C}$ and $\tilde{K}$ are cones in the annulus $\mathrm{An}(0, \lambda, \lambda^{-1})$, there exist two surfaces of constant multiplicity c which induce $(\tilde{C}, \tilde{K})$ in $\mathrm{An}(0, \lambda, \lambda^{-1})$. If we extend them in a conical way up to the origin and the infinity, we obtain a pair of surfaces $(\tilde{\Sigma}, \tilde{\Gamma})$ which is stationary and stable in $\mathbb{R}^n \setminus \{0\}$, thus we can apply the Bernstein Theorem (Theorem 5.32) to obtain that $\tilde{\Sigma} + a\tilde{\Gamma}$ is a linear combination of the three cases 1,2,3.

Since replacements do not have components which meet $\partial\mathcal{M}$ orthogonally, we have to exclude case 3.

On the other hand, since the density of $\tilde{\Gamma}$ belongs to $\{0, 1\}$ (because it is a smooth limit of smooth surfaces in $\tau_{x,r_j}(\partial\mathcal{M})$) it is equal to 0, to a half plane or to $T_x\partial\mathcal{M}$. This, together with the fact that $\tilde{\Sigma} + a\tilde{\Gamma}$ is induced by the smooth limit of open sets which are stationary for F_a, implies the classification of $\tilde{\Sigma}$ and $\tilde{\Gamma}$ stated.

□

5.8.2 Regularity of V

We first note that V is regular and stable outside a finite number of point

Proposition 5.45. *Let (V, W) be the pair of varifolds given by Proposition 5.17, let us define $V' := V\llcorner G_2(\mathcal{M}^\circ)$ and let us fix $x \in \partial\mathcal{M}$. Then there exists $\rho > 0$ such that V' is induced in $B_\rho(x) \setminus \{x\}$ by the union Σ of smooth minimal surfaces Σ_i that satisfy the following conditions:*

1. *The surfaces Σ_i are disjoint in $\mathcal{M}^\circ$;*
2. *Each Σ_i either meet $\partial\mathcal{M}$ with angle θ or do not intersect $\partial\mathcal{M}$;*
3. *For every $y \in \partial\mathcal{M} \cap B_\rho(x) \setminus \{x\}$ there exists $\rho' > 0$ such that $\Sigma \cap B_{\rho'}(y)$ fall in one of the two cases stated by Theorem 5.1.*

Moreover Σ is stable for the capillarity energy.

Proof. The proof is very similar to the one of [DR18, p. 10.3] but we sketch here the argument for the reader convenience.

Let us consider $x \in \partial\mathcal{M}$ and $r(x)$ so that (V, W) has replacements in any annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$. Let us fix $\rho > 0$ such that $4\rho < r(x)$. By the results in [DT13], we have that V is regular in $\mathcal{M}^\circ$ and it is stable in annuli. Let us call $\Sigma_1, \Sigma_2, \dots$ the connected components of V in $\mathrm{An}(x, \rho, 4\rho) \cap \mathcal{M}^\circ$ and let us fix one of them Σ_i with a regular point $y \in \Sigma_i$. Since y is regular for Σ_i, if $s = |x - y|$ we can find $\bar{\sigma} > 0$ such that

- Σ_i is regular in $B_{\bar{\sigma}}(y)$;
- $\Sigma_i \cap \partial B_{s+\sigma}(x) \neq \emptyset$ (by maximum principle [DT13, Theorem 5.1 (*ii*)]) for every $\sigma \in (0, \bar{\sigma})$;
- the intersection between Σ_i and $\partial B_{s+\sigma}(x)$ is transversal (by Sard's Lemma) for almost every $\sigma \in (0, \bar{\sigma})$.

We consider such a σ, call $t = s + \sigma$ and fix a point $z \in \Sigma_i \cap \partial B_t(x) \cap \mathcal{M}^\circ$. We consider a replacement $(\tilde{V}, \tilde{W})$ for (V, W) in $\mathrm{An}(x, \rho, t)$. Again by the interior theory in [DT13], $\tilde{V}$ is regular in $\mathcal{M}^\circ$. Since V and $\tilde{V}$ coincide outside $B_t(x)$, any tangent cone C to $\tilde{V}$ at z must contain a half-plane π (since z is regular for Σ_i); since C is stationary and stable (because it is a tangent cone of a varifold with the replacement property which, by the interior theory in [DT13], is stable in annuli), it is a plane, hence C coincides with the whole plane π. This implies that z is regular for $\tilde{V}$.

If we call $\tilde{\Sigma}$ the connected component of $\tilde{V}$ that contains z, by unique continuation, $\tilde{\Sigma}$ and Σ_i coincide and, since $\tilde{\Sigma}$ is regular up to $\partial\mathcal{M}$ in $\mathrm{An}(x, \rho, t)$ (because $(\tilde{V}, \tilde{W})$ is a replacement for (V, W)), also Σ_i is regular up to $\partial\mathcal{M}$ in $\mathrm{An}(x, \rho, t)$. Since the choice of y is arbitrary, we have that Σ_i is regular up to $\partial\mathcal{M}$ in the whole annulus $\mathrm{An}(x, \rho, 4\rho)$ and either meets $\partial\mathcal{M}$ with angle θ or does not meet $\partial\mathcal{M}$.

By monotonicity formula, it follows that there are only finitely many connected components of V' in $\mathrm{An}(x, 2\rho, 3\rho)$: $\overline{\Sigma}_1, \dots, \overline{\Sigma}_N$ and each of them is a regular minimal surface up to $\partial\mathcal{M}$; we have only to verify that $\bigcup_{i=1}^N \overline{\Sigma}_i$ is regular.

In order to do this, let us consider a point $y \in \overline{\Sigma}_i \cap \mathrm{An}(x, 2\rho, 3\rho)$. We claim that y is a regular point for V', that is y belongs to one of the two cases in the statement of Theorem 5.1.

If y does not belong to any other $\overline{\Sigma}_j$, then it is of course regular for V'; if y belongs to a $\overline{\Sigma}_j$, then y is regular also for $\overline{\Sigma}_j$ and by the properties of replacements $\overline{\Sigma}_j$ meets $\partial\mathcal{M}$ with angle θ.

Since $\Sigma_i \cap \Sigma_j \cap \mathcal{M}^\circ = \emptyset$, then $T_y\overline{\Sigma}_i \cap T_y\partial\mathcal{M} = T_y\overline{\Sigma}_j \cap T_y\partial\mathcal{M}$. Thus $\overline{\Sigma}_i$ and $\overline{\Sigma}_j$ meet $\partial\mathcal{M}$ with angle θ. By Hopf Lemma, we have that $T_y\overline{\Sigma}_i$ and $T_y\overline{\Sigma}_j$ have to be symmetric with respect to the unit normal $N(y)$ to $\partial\mathcal{M}$. Again by the same arguments and Hopf Lemma, there exists no other surface $\overline{\Sigma}_k$ such that $y \in \overline{\Sigma}_k$. Thus $\Sigma \cap B_{\rho'}(y)$ falls in one of the two cases stated by Theorem 5.1.

We now observe that, by definition, $\|V'\|(\partial\mathcal{M}) = 0$ and any point $y \in \operatorname{supp}\|V\| \cap \operatorname{An}(x, 2\rho, 3\rho)$ belongs to some $\overline{\Sigma}_i$. This shows that V' is regular up to $\partial\mathcal{M}$ in $\operatorname{An}(x, \rho, 4\rho)$; since the radius ρ is arbitrary (except for the fact that $4\rho < r(x)$), this argument proves that V' is regular in $B_{r(x)/4}(x) \setminus \{x\}$.

Since the above arguments show that V' coincides in any $\operatorname{An} \mathcal{AN}_{r(x)}(x)$ with the interior components of replacements of (V, W), this implies that V' is stable for the capillarity energy. $\square$

In order to conclude the proof we only need to remove the final (possible) singular point x. Here, the stability of V in a in a (small) punctuerd ball plays a crucial role. The following Proposition is indeed a generalization to capillarity surfaces of the "removable" singularity theorem for two dimensional stable minimal surfaces first established by Lawson and Gulliver, [GB86], see also [Sim82] and [Mee07]. A removable singularity theorem is proved by Li, Zhou and Zhu [LZZ21, Theorem 1.15] in order to remove the final possible singular point x. Their argument uses this removable singularity theorem combined with a blow-up analysis at x in order to show the regularity of V' up to x.

We present here a different route to obtain the regularity of V' up to x. To explain the idea, let us assume that $\mathcal{M} = \{x_1 \geq 0\}$ and that Σ is globally stable. Then it is well known that

$$v = \nu_\Sigma \cdot e_1$$

is a Jacobi field which is negative at the boundary due to the contact angle condition. By the stability inequality v should be negative also inside since otherwise its positive part would induce a negative variation. This implies that Σ can be expressed as a union of graph with respect to $\{x_1 = 0\}$. Each of this graph will be a stationary point of the functional

$$\mathcal{F}(g) = \int \sqrt{1 + |\nabla g|^2} + a|\{g > 0\}|$$

and the regularity will follow by the regularity theory of [ACF84] (see also [CF85]) and more precisely the implementation in [DeS11]). In order to make the proof rigorous we shall first localize the above argument and then precise in which sense g is solving a free boundary problem.

Proposition 5.46. *Let (V, W) be the pair of varifolds given by Proposition 5.17, let us fix $x \in \partial\mathcal{M}$ and let $\rho > 0$ be the radius given by Proposition 5.45. Then V is regular in the whole $B_\rho(x)$, that is there exists $0 < \rho' \leq \rho$ and a minimal surface Σ in $B_\rho(x)$ such that*

1. *either $\Sigma \cap B_{\rho'}(x) = \emptyset$;*
2. *or $\Sigma \cap B_{\rho'}(x)$ fall in one of the two cases stated by Theorem 5.1, that is it is made of at most two smooth minimal surfaces which meet $\partial\mathcal{M}$ with angle θ.*

Proof. The proof is divided in few steps:

Step 1: First by arguing exactly as in [DR18, p. 10.4], we note that $V \llcorner B_\rho(x)$ consists by finitely many connected components Σ_i with integer multiplicities h_i which are homeomorphic to 2-semidisks without the center x.

Step 2: We now fix such a surface $\Sigma^i = \Sigma$, remove the multiplicity h_i and we aim to show that it can be extended smoothly through x. We first claim that for $y \in \Sigma \cap \partial\mathcal{M} \cap B_\rho(x)$ the sequence $\Sigma^y_j := (\tau_{y,r_j})_\sharp \Sigma$ converges (up to subsequences) to a plane π_y which forms the correct angle with $T_y\partial\mathcal{M}$. Note that this is obvious if $y \neq x$ (since the surface is regular there) and we only need to show it for $y = x$. For, we note that the curvature estimates (Theorem 5.34) yield the existence of a constant $c > 0$ such that

$$|A_\Sigma(y)| \leq \frac{c}{|y - x|} \qquad \forall y \in B_\rho(x).$$

Thus Σ^x_j converges locally smoothly in $\mathbb{R}^n \setminus \{0\}$ to an entire surface Ξ which is smooth and stable outside the origin. We want to show that the only possibilities for Ξ are the following:

- Ξ is empty;
- Ξ is a half plane π which intersects $T_x\partial\mathcal{M}$ with angle θ;

In order to prove the above classification for $\mathfrak{X}$, we have to distinguish two cases.

- If there exists $j \in \mathbb{N}$ such that Σ_j^x does not intersect $\tau_{x,r_j}(\partial\mathcal{M}) \cap B_1 \setminus \{0\}$, then we claim that $x \notin \operatorname{supp}\Sigma$ (this clearly implies that $\Xi = \emptyset$). Indeed, let us consider such a j and let us assume by contradiction that $x \in \operatorname{supp}\Sigma$. Since Σ does not intersect $\partial\mathcal{M} \cap B_{r_j} \setminus \{0\}$, if we call Z the varifold induced by Σ_j^x, we have

$$\|Z\|(\partial\mathcal{M}) = 0, \qquad \delta Z[X] = 0 \quad \forall X \in \mathfrak{X}_c(\mathcal{M})$$

where $\mathfrak{X}_c(\mathcal{M})$ denotes the class of vector fields with compact support in the interior of $\mathcal{M}$. Thus, by Corollary 3.10 and by Theorem 3.3, Z has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$, that is the class of vector fields which are orthogonal to $\partial\mathcal{M}$ on $\partial\mathcal{M}$. The first variation measure σ_Z measure of Z with respect to $\mathfrak{X}_\perp(\mathcal{M})$ satisfies

$$\operatorname{supp}\sigma_Z \subset \{x\}, \qquad \Theta^{1*}(\sigma_Z, x) < \infty$$

by (3.4), where $\Theta^{1*}(\sigma_Z, x)$ is the upper 1-density of σ_Z at x. Hence $\sigma_Z = 0$ and

$$\delta Z[X] = 0 \qquad \forall X \in \mathfrak{X}_\perp(\mathcal{M}),$$

that is Z is stationary with respect to $\mathfrak{X}_\perp(\mathcal{M})$.

Since $\partial\mathcal{M}$ is strictly convex, we can apply a version of the maximum principle to Z in order to obtain that $x \notin \operatorname{supp} Z$. Indeed, let us consider the vector field

$$X = \gamma(|x|)\gamma\Big(\frac{d(x)}{\rho}\Big)\nabla d(x),$$

where γ is the cut-off function defined in section 2.1 and $\rho > 0$ is choses so small so that

$$\operatorname{supp} Z \cap \operatorname{An}(0, 1/2, 1) \cap U_\rho(\partial\mathcal{M}_j) = \emptyset. \tag{5.39}$$

This is possible since Σ_j^x is smooth in $B_1(0)\setminus\{0\}$. Now we compute the tangential divergence of X with respect a 2-plane S:

$$\begin{aligned}\operatorname{div}_S X(x) =& \gamma'(|x|)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle\frac{x}{|x|}, P_S\nabla d(x)\rangle \\ &+ \frac{1}{\rho}\gamma(|x|)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 + \gamma(|x|)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x).\end{aligned}$$

Since Z is stationary with respect to $\mathfrak{X}_\perp(\mathcal{M})$, it holds

$$0 = \int_{G_2(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}Z(x, S). \tag{5.40}$$

We are now going to show that in fact the last integral is strictly negative. In fact, by (5.39), it follows that $\gamma'(|x|)\gamma\Big(\frac{d(x)}{\rho}\Big) = 0$ on $\operatorname{supp} Z$, thus we have

$$\begin{aligned}\int_{G_2(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}Z(x, S) = \int_{G_2(\mathcal{M})} \Big(&\frac{1}{\rho}\gamma(|x|)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 \\ &+ \gamma(|x|)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x)\Big)\,\mathrm{d}Z(x, S).\end{aligned}$$

By the strict convexity assumption on $\partial\mathcal{M}$, the last integrand is strictly negative, thus $\delta Z[X] < 0$, which contradicts (5.40).

- On the other hand, if for every $j \in \mathbb{N}$, Σ_j^x intersects $\tau_{x,r_j}(\partial\mathcal{M}) \cap B_1 \setminus \{0\}$, by Proposition 5.45 Σ_j^x meets $\tau_{x,r_j}(\partial\mathcal{M})$ with angle θ. Since the convergence of Σ_j^x to Ξ is locally smooth in $\mathbb{R}^n \setminus \{0\}$, then Ξ is is a connected, smooth, stationary and stable point in $\mathbb{R}^n \setminus \{0\}$ for F_a, meets $T_x\mathcal{M}$ with angle θ and is such that $0 \in \operatorname{supp}\Xi$. Thus, by Theorem 5.32, Ξ is a half-plane π that meets $T_x\partial\mathcal{M}$ with angle θ.

Step 3: We now show that Σ is a Lipschitz graph with respect to $T_x\partial\mathcal{M}$. To this end we can assume that

$$N_{T_x\partial\mathcal{M}} = -e_1.$$

We now let r_j be a sequence such that $\Sigma_j \setminus \{0\}$ (where $\Sigma_j := (\tau_{x,r_j})_\sharp\Sigma$) converges to a half plane Ξ locally smoothly in $B_1(0) \setminus \{0\}$. Note that the plane might depend on the subsequence, but in any case we can rotate the coordinates so that

$$\nu_\pi = -ae_1 + \sqrt{1-a^2}e_2.$$

Since the convergence of $\Sigma_j \to \pi$ is in the C^1-topology on annuli, for every $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that

$$\sup_{y \in B_{r_j}(x)\setminus B_{r_j/4}(x)} |\nu_\Sigma(y) - \nu_\pi| < \varepsilon \qquad \forall j > J,$$

where ν_Σ is the unit normal vector to Σ. Hence, for any $k > j > J$, we have

$$\sup_{y \in \Sigma\cap(\partial B_{r_j}(x)\cup\partial B_{r_k}(x))} |\nu_\Sigma(y) - \nu_\pi| < \varepsilon. \tag{5.41}$$

Let us consider for $e \in \mathbb{S}^2$ and $|e - e_1| \leq \delta$ to be fixed ,the function

$$v(y) = \nu_\Sigma(y)\cdot e \quad \forall y \in \Sigma.$$

By (5.41), it follows that for every $\varepsilon, \delta > 0$ there exists $J \in \mathbb{N}$ such that

$$\sup_{y \in \Sigma\cap(\partial B_{r_j}(x)\cup\partial B_{r_k}(x))} v(y) \leq -a + \varepsilon + \delta \qquad \forall k > j \geq J.$$

Moreover, by the contact angle condition and the regularity of Σ up to $\partial\mathcal{M}$, we get $v(y) \leq -a + 2\delta$ for every on $y \in \Sigma \cap \partial\mathcal{M} \cap \operatorname{An}(x, r_k, r_j)$. Hence

$$\sup_{y \in \Sigma\cap\left(\partial\mathcal{M}\cup\partial B_{r_j}(x)\cup\partial B_{r_k}(x)\right)} v(y) \leq -a + \varepsilon + \delta < 0 \qquad \forall k > j \geq J.$$

provided δ and ε are chosen sufficiently small but depending only on a.

It is a classical fact that v is a Jacobi field, that is

$$\Delta_\Sigma v + |A_\Sigma|^2 v = 0.$$

We now claim that v has constant non-positive sign also in the interior of the annulus $\operatorname{An}(x, r_k, r_j)$. Indeed, the positive part $v_+ := \max\{v, 0\}$ is a function with compact support in the interior of $\operatorname{An}(x, r_k, r_j)$ and we can test the weak form of the Jacobi equation with the function v_+ as test function, that is

$$\begin{aligned} 0 &= \int_\Sigma \nabla v_+ \cdot \nabla v - |A_\Sigma|^2 v v_+ \, d\mathcal{H}^2 \\ &= \int_\Sigma |\nabla v_+|^2 - |A_\Sigma|^2 (v_+)^2 \, d\mathcal{H}^2 \\ &= Q(v_+) \end{aligned}$$

where Q is the stability operator for Σ. Since $Q \geq 0$ (by stability of Σ), it follows that v_+ also is a non negative solution of the Jacobi equation, a contradiction with the strong minimum principle. Hence for each e sufficiently close to e_1,

$$\nu_\Sigma \cdot e < 0 \qquad \text{on } \Sigma \cap \mathrm{An}(x, r_k, r_j)$$

which is easily seen to imply that $\Sigma \cap \mathrm{An}(x, r_k, r_j)$ can be represented as the graph of a Lipschitz function $g : \mathbb{R}^2 \to \mathbb{R}$

$$\Sigma \cap \mathrm{An}(x, r_k, r_j) = \{x_1 = g(x_2, x_3)\} \cap \mathrm{An}(x, r_k, r_j)$$

and $|\nabla g| \leq 1/\delta$ for $\delta > 0$ depending only on a. Letting $k \to \infty$ we deduce that $\Sigma \cap B_{r_j}$ is the graph of a Lipschitz function.

Step 4: We now claim that the function g in the above step is a solution of a free boundary problem. To this end note first that, up to taking j large and a scaling (and translation) of the coordinates that we can assume that

$$\Sigma \cap B_1 = \{x_1 = g\} \cap B_1, \qquad \partial\mathcal{M} \cap B_1 = \{x_1 = h\} \cap B_1$$

for a suitable function h with $h(0) = |\nabla h(0)| = 0$. Note that $g \geq h$. It is now easy to check that g is a viscosity solution of the equation

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla g}{\sqrt{1+|\nabla g|^2}}\Big) = 0 & \text{on } \{g > h\} \\ -\nabla g \cdot \nabla h + 1 = a\sqrt{1+|\nabla g|^2}\sqrt{1+|\nabla h|^2} & \text{on } \{g = h\}. \end{cases}$$

Indeed the first equation is clearly satisfied classically. As for the boundary condition we first note that for $x_0 \in \{g = h\}$, the point $(g(x_0), x_0) \in \Sigma \cap \partial\mathcal{M}$ and thus, thanks to Step 1, the functions

$$g_r(x) = \frac{g(x_0 + rx) - g(x_0)}{r}$$

converges (up to subsequences) to a linear function ℓ with the property that

$$-\nabla \ell \cdot \nabla h(x_0) + 1 = a\sqrt{1+|\nabla \ell|^2}\sqrt{1+|\nabla h(x_0)|^2}.$$

It is easy to verify that this implies that the condition is satisfied in the viscosity sense, see for instance the computations in [DSV21]. Note that this also implies that, up to choose r_j smaller, that

$$\|g - \ell\|_{L^\infty(B_1'(0))} < \varepsilon,$$

where $\ell(x) = x_2\sqrt{(1-a^2)/a^2}$ (recall the normalizations at x). One can then apply some straightforward modifications of the arguments in [DeS11] to show that the free boundary a C^1 curve. More precisely one can either argue directly or observe that $g - h$ is satisfying a free boundary problem with variable coefficients, note also that the Lipschitz assumption on g ensures the uniform ellipticity of the minimal surface operator.

Step 5: The previous step implies that each of the connected components Σ^j of Σ are smooth graphs near x_0 and that they satisfy the minimal surface equation with constant capillarity conditions. Moreover, the components Σ^j cannot intersect transversally outside x. Hence their tangents at x have to be

- either a half-plane π which intersects $T_x\partial\mathcal{M}$ with angle θ
- or two half-planes π_1, π_2 which intersect $T_x\partial\mathcal{M}$ with angle θ and

$$\pi_1 \cap \pi_2 = \pi_1 \cap T_x\partial\mathcal{M} = \pi_2 \cap T_x\partial\mathcal{M}$$

(the last condition means that π_1 and π_2 are symmetrical with respect to exterior unit normal N to $\partial\mathcal{M}$, as in the second case in the statement of Theorem 5.1.). By Hopf Lemma, the graphs that have the same tangent at $\partial\mathcal{M}$ have to coincide; hence x is a regular point as well, that is there exists $\rho' > 0$ such that $\Sigma \cap B_{\rho'}(x)$ falls in one of the two cases stated by Theorem 5.1. $\square$

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