SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Isomonodromic Deformations Along the Caustic of a Dubrovin-Frobenius Manifold

Candidate:
Luis Felipe López Reyes

Advisor:
Davide Guzzetti

Academic Year 2021-2022

MA PER SEGUIR VIRTUTE E CANOSCENZA • SISSA •

Contents

Acknowledegments

During the wonderful journey of the PhD I had the pleasure of being surrounded by amazing people that helped me grow as a mathematician but most importantly as a human being. I would like to dedicate some words to them.

First of all I would like to thank all of my collegues that worked with me in SISSA. Special thanks to Sohaib who was always willing to share his thoughts in mathematics and was always really kind in listening my own disordered ideas of mathematics. I would also like to thank Umar for all the beautiful discussions (and grills) that we shared during the two hard years of the pandemic. Also I have a lot of good memories from the Kirilovers study group, Leo, Mario, the Micheles (tall and small) and Stefano; I will always remember the great moments we shared studying together for more than one year, I can only say that I am really thankful for all the things I learned from you and with you. I would also like to thank my advisor Davide Guzzetti for taking me as his student.

Vorrei anche ringraziare a tutti gli italiani che mi hanno fatto sentire a casa. Tra di voi, ricorderò per sempre a Serena, che dal momento che sono arrivato senza visto in Italia, mi ha auitato in tanti sensi che non trovo le parole giuste per ringraziarle. Devo ringraziare anche a Michele e Ciccio (grandi sporchi!), non dimenticherò mai tutte le sere che abbiamo condiviso, siete delle persone bravissime e spero di trovarvi ancora in futuro. Certo che non potevo dimenticare a Guido, ricordo (con tanta voglia di tornare) le bellisime due settimane che abbiamo fatto a Serluca-Calabaia, sei un grande Guido! Ringraziamenti speciali a Teresa che è stata sempre bravissima, sopratutto perché ha saputo soppravivere a tutta la matematica che Michele e io gli sparavamo, davvero grazie di tutto Tere! Tante grazie a Dan e Pepe, siete veramente dei grandi amici e delle grande persone. Con tanto affetto mi ricordo delle serate passate al bar di Sofia, grazie Sofi per accògliercì nel tuo bar, grazie Elisabetta per tutti i belli momenti, vi porterò sempre vicino al cuore! Non mi resta che ringraziare a tutti ragazzi con chi ho arrampicato in questo ultimo anno, tra di voi devo veramente ringraziare a Sara per portarci tante volte e a tante persone sulle belle vie che Trieste ci offre; ma sopratutto devo ringraziare a Massimo, Chef! sei delle persone più belle, brave e forte che ho mai conosciuto in vita mia, ho una grande ammirazione e rispetto per te, grazie per portarmi in montagna, per portarmi ad arrampicare ma sopratutto *per aver condiviso la tua amicizia. A tutti voi ragazzi, vi voglio tanto bene e mi mancherete.*

Por supuesto que también debo agradecer a todas las personas de México que, a pesar de la distancia, me han ayudado y apoyado a lo largo de mi vida. Primero que nada quisiera agradecer a Laura Ortiz Bobadilla, gracias por introducirme al bellísimo mundo de las ecuaciones diferenciales y sobre todo a la manera geométrica de pensar en ellas. Espero en esta tesis se vean un poco reflejadas tus enseñanzas. También quisiera agradecer a Alejandro, Ana, Carmen y Niels, a pesar de que nos hemos visto poco en estos últimos años siempre pienso en ustedes y todas las cosas que hemos vivido juntos. Emiliano, a pesar de que tomamos caminos distintos (tu la filosofía y yo las matemáticas) quiero que sepas que siempre has sido una persona muy importanto en mi vida, gracias por todo. Por último debo agradecer a mi papá y a mi mamá, sin ustedes nada de lo que he logrado en mi vida habría sido posible, los extraño y los amo con todo mi ser. Gracias mamá por todos tus cuidados, apoyos y amor, sin duda eres la persona más magnífica que conozco y eres una grande fuente de inspiración en mi vida.

Introduction

Dubrovin-Frobenius manifolds were invented by Boris Dubrovin to geometrize the study of certain $2D$ Topological Field Theories ([11],[12]). The primary free energy F of a family of such theories satisfies the so called $WDVV$ equations. Given a quasi-homogeneous solution to these equations one constructs a Dubrovin-Frobenius manifold structure on the domain of definition M of the solution.

The first condition a Dubrovin-Frobenius manifold must satisfy is that the tangent sheaf $\mathcal{T}_M$ carries $\mathcal{O}_M$-bilinear multiplication $\circ\colon \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$, this multiplication is required to be unital, associative and commutative. The multiplication is required to satisfy an integrability condition (equation 1.1.1); a manifold satisfying this conditions is called an *F-manifold*. This integrability condition ensures that, the decomposition of each tangent space T_pM into irreducible subalgebras extends to a local decomposition of M into irreducible F-manifolds. Next up in the definition comes the Euler vector field E, a global vector field required to satisfy $\mathcal{L}_E \circ = \circ$. Lastly one requires the existence of a flat metric η compatible with the multiplication in the sense that for any vector fields u, v, w one has $\eta(u \circ v, w) = \eta(u, v \circ w)$ and such that $\mathcal{L}_E \eta = (2-d)\eta$ for some complex number d called the charge of the Dubrovin-Frobenius manifold.

As vector spaces, each tangent space T_pM of a manifold is isomorphic to $\mathbb{C}^n$, on a Dubrovin-Frobenius manifold each tangent space is a $\mathbb{C}$-algebra and as such it is no longer necessarily isomorphic to $\mathbb{C}^n$. If as algebras $T_pM \cong \mathbb{C}^n$ then the point p is called semisimple. In this case there exists a neighborhood V of p such that all points in V are semisimple; in V there exists n-linearly independent vector fields π_i such that $\pi_i \circ \pi_j = \delta_{ij}\pi_i$, these vectors are called *orthogonal idempotents*. The points which are not semisimple form an hypersurface K (which can be empty) called the *caustic*. It is the purpose of this work to study the structure of a Dubrovin-Frobenius manifold in a neighborhood of a non-semisimple point $p \in K$. In particular we are interested in the restriction to the caustic of a family of differential equations associated to the Dubrovin-Frobenius manifold.

Let us briefly recall some sources of examples of Dubrovin-Frobenius manifolds. We start by some Dubrovin-Frobenius manifolds coming from isolated hypersurface singularities. Historically these where of the first examples where the Dubrovin-Frobenius manifold

was found but was not yet called that way (see [17], [18] and [19]). In these manifolds the multiplication is easy to define but the flat metric η is more involved. These examples are also useful for this work because the caustic is always non-empty.
Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be the germ of a holomorphic function and suppose that $df|_0 = 0$. Then the origin is said to be a singularity of f. If $df|_0 \neq 0$ then at least one of the partial derivatives $\frac{\partial f}{\partial x_i}$ does not vanish at $x = 0$ and as such, it is invertible in the local ring $\mathbb{C}\{x_0,\dots,x_n\}$. Hence the $\mathbb{C}$-algebra $\mathbb{C}\{x_0,\dots,x_n\}$ modulo the ideal

$$J_f = \left(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_n}\right)$$

is the zero ring and as such has dimension zero. If the origin is a singularity of f then this is no longer the case and the dimension μ of the algebra $A_0 = \mathbb{C}\{x\}/J_f$ is called the *Milnor number* of f. We can choose representatives $a_i(x) \in \mathbb{C}\{x\}, i = 1,\dots,\mu$ of a basis of A_0 and construct a new function $F\colon (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu},0) \to (\mathbb{C},0)$ by setting

$$F(x,t) := f(x) + \sum_{k=1}^{\mu} a_k(x)t_k.$$

This function is called a “semiuniversal unfolding” of f and using it one can construct a Dubrovin-Frobenius manifold structure on $(M,0) = (\mathbb{C}^{\mu},0)$. To define the multiplication one considers first the *critical space* $(C,0)$ of the semiuniversal unfolding F which is defined by

$$(C,0) := \{\, (x,t) \in (\mathbb{C}^d \times M,0) \mid \frac{\partial F}{\partial x_i}(x,t) = 0, i = 1,\dots,d \,\}.$$

Using the projection $\pi_C\colon C \to M$ one gets an isomorphism $\mathcal{T}_{M,0} \cong (\pi_C)_*\mathcal{O}_{C,0}$ via $u \mapsto \tilde{u}(F)|_{(C,0)}$ where $\tilde{u} \in \mathcal{T}_{\mathbb{C}^d\times M,0}$ is any lift of $u \in \mathcal{T}_{M,0}$. This isomorphism induces an associative, commutative and unital multiplication $\circ$ on $\mathcal{T}_{M,0}$ and the inverse image of $F|_{(C,0)}$ under this isomorphism turns out to be an Euler vector field E for the multiplication. The above construction makes $(M,\circ,e,E)$ into an F-manifold with Euler vector field. To get the metric one looks for another sheaf $\mathcal{F}$ with a non-degenerate bilinear form and an isomorphism $\mathcal{T}_M \cong \mathcal{F}$. It turns out that this sheaf $\mathcal{F}$ is the pushforward by the projection $\pi\colon (\mathbb{C}^{n+1} \times M,0) \to M$ of the sheaf of relative differentials of the map $\varphi\colon (\mathbb{C}^{n+1} \times M,0) \to (\mathbb{C} \times M,0)$. Indeed, this sheaf is equipped with the Grothendieck residue pairing and is a locally free $\mathcal{O}_M$-module of rank $\mu = dimM$. It turns out that it is also a free $(\pi_C)_*\mathcal{O}_C \cong \mathcal{T}_M$ module of rank one. The choice of a generator induces an isomorphism $\mathcal{T}_M \cong \mathcal{F}$. But in order that the induced metric is flat one needs to choose special generators, these are the so called *primitive forms* (see [19]).

Another source of examples comes from the quantum cohomology of certain symplectic manifolds X with $H^{2k+1}(X;\mathbb{C}) = 0$. In these examples the flat structure is easy to define, the metric is just the Poincaré pairing in the ordinary cohomology ring of X and as such it is trivially flat. The multiplication is much more involved and to obtain it one must consider the Gromov-Witten invariants of X. Using the Gromov-Witten invariants it is possible to deform the cup-product of $H^\bullet(X;\mathbb{C})$ and obtain an associative, commutative and unital multiplication. The first Chern class of X is an Euler vector field for this multiplication.

As mentioned before we are interested in a family of differential equations associated to any Dubrovin-Frobenius manifolds. Let us explain how this family is constructed and why the monodromy data of this family are so important for the Dubrovin-Frobenius manifold. By definition, the Levi-Civita connection ∇ of the metric η of a Dubrovin-Frobenius manifold is flat. Flatness of ∇, the associativity and commutativity of the multiplication $\circ$ and a potentiality condition on $\circ$ (see section 2.2) allows us to define a 1-parameter family of flat connections

$$\nabla^z := \nabla + z\,\circ, \quad z \in \mathbb{C}$$

on the tangent sheaf $\mathcal{T}_M$.

Furthermore, the conformal condition $\mathcal{L}_E\eta = (2-d)\eta$ implies that the endomorphism of $\mathcal{T}_M$ given by

$$\mu := \frac{2-d}{2} Id - \nabla E$$

is η antisymmetric. Let $\pi_M \colon \mathbb{P}^1 \times M \to M$ denote the projection. Using the endomorphism μ and multiplication by the Euler vector field E, one can extend the 1-parameter family of connections ∇^z to a flat connection $\bar{\nabla}$ on the vector bundle $\pi_M^*\mathcal{T}_M$ over $\mathbb{P}^1 \times M$. The covariant derivatives in the direction of vectors tangent to M are the same as the covariant derivatives of ∇^z. If z is a global coordinate on $\mathbb{C}$ then the covariant derivative in the direction of ∂_z is given by

$$\bar{\nabla}_{\partial_z} v := \frac{\partial v}{\partial z} + E \circ v - \frac{1}{z}\mu v.$$

Once again the potentiality and the condition $\mathcal{L}_E \circ = \circ$ ensures that $\bar{\nabla}$ is a flat connection.

When looking for flat sections of $\bar{\nabla}$ we need to solve the overdetermined system of partial differential equations

$$\bar{\nabla} v = 0.$$

While doing this we can start by solving the differential equation on the z-variable. This ordinary differential equation reads

$$\frac{dY}{dz} = \left(\frac{1}{z}\mu - E\circ\right) Y. \tag{$\star$}$$

In this way every Dubrovin-Frobenius manifold parametrizes a family of meromorphic ordinary differential equations on $\mathbb{P}^1$. All members of this family have a regular singularity at $z = 0$ and a Poincaré rank one singularity at $z = \infty$.

Let us describe the monodromy data one associates to such differential equation. At the regular singularity $z = 0$ one can find a local holomorphic Gauge transformation taking the differential equation ($\star$) to its "normal form" (see section 4.1). Then one is able to write explicitly a fundamental matrix solution Y_{Lev} in Levelt form in a neighborhood of $z = 0$. The monodromy transformation takes the form

$$Y_{Lev}(z) \mapsto Y_{Lev}(e^{2\pi i}z) = Y_{Lev}(z)e^{2\pi i\mu}e^{2\pi iR}$$

and is completely determined by the matrices μ and R. This two matrices are called the *monodromy data at* $z = 0$.

In general, at the irregular singularity $z = \infty$ one can only find a formal Gauge transformation which takes the differential equation ($\star$) to its "normal form" (see section 4.2). Using this one can write a formal fundamental matrix solution Y_F and as before compute a formal monodromy

$$Y_F(z) \mapsto Y_F(e^{2\pi i}z) = Y_F(z)e^{2\pi iB}.$$

The matrix B is called the *exponent of formal monodromy* and is part of the monodromy data at $z = \infty$.
Since the solutions obtained by the formal procedure are in general not convergent, one can ask if they in some sense approximate actual holomorphic solutions. This is indeed the case and using a result from Sibuya one can get holomorphic fundamental matrix solutions $Y_\nu, \nu \in \mathbb{Z}$ such that the asymptotic expansion of Y_ν as $z \to \infty$ in certain sectors $\mathcal{S}_\nu$ is precisely the formal fundamental matrix solution Y_F. As we leave the sector $\mathcal{S}_\nu$ and enter the next sector $\mathcal{S}_{\nu+1}$, the asymptotic expansion of Y_ν will no longer be given by Y_F. But on the overlap $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+1}$ the two solutions will be related by a *Stokes matrix* S_ν defined by the relation:

$$Y_{\nu+1} = Y_\nu S_\nu.$$

The Stokes matrices are also part of the monodromy data at $z = \infty$.

Finally the solutions Y_{Lev} and Y_0 will be related by a *central connection matrix*

$$Y_{Lev} = Y_0 C.$$

The matrices μ, R, B, S_ν, C are called the *monodromy data* of the Dubrovin-Frobenius manifold M. In the works [11] and [12] Dubrovin showed that on a sufficiently small neighborhood W of a semisimple point $p \in M$ such that the eigenvalues of the endomorphism

$$E_p\circ\colon T_pM \to T_pM$$

are distinct; the monodromy data are constant and moreover, starting from the monodromy data one can reconstruct the structure of a Dubrovin-Frobenius manifold on the neighborhood W.
Recently on [7] Cotti, Dubrovin and Guzzetti extended this result to a small neighborhood W of a semisimple point p but without any restrictions on the eigenvalues of the endomorphism $E_p\circ$. Their result is based on the extension of the theory of isomonodromy deformations when the eigenvaluse at the irregular singularity coalesce [6], but the multiplication remains semisimple in a neighbourhood of the coalescence locus.

After the extension result by Cotti, Dubrovin and Guzzetti, two problems remained open. The first one is to extend the isomonodromy deformation theory at a locus where the eigenvalues of $E\circ$ coalesce and the multiplication is no longer semisimple. That is, extend the theory of isomonodromic deformations to the caustic. The second problem, which is even more difficult, is to describe in a neighbourhood of a point belonging to the caustic the transition between semisimple points of this neigbourhood and points of the caustic. In this thesis, we make use of the geometry of a Dubrovin-Frobenius manifold to obtain insight and possible solutions to these problems.

Main results: The most important results of this work are theorem 4.3.2 and proposition 6.1.2. The first of these theorems says that under some assumptions, after restricting the family $(\star)$ to certain submanifolds $L \subset K$ of the caustic, one is able to find isomonodromic fundamental matrix solutions of equation $(\star)$. The reason why we restrict ourselves to this submanifolds is because the Jordan form of the endomorphism $E\circ$ changes as we approach the caustic. As a consequence, the fundamental matrix solutions computed outside the caustic become singular at the caustic. Proposition 6.1.2 says that after a suitable renoramlization some of the columns of the formal fundamental matrix solution outside the caustic have a well defined limit at the caustic and moreover, these columns coincide with some of the columns of the formal fundamental matrix computed inside the caustic.

It is worth mentioning that while this work was under development, the theory of isomondromic deformations of differential equations of the same type as $(\star)$ was stablished independently by Guzzetti in [13]. Basicaly what it is shown in this work is that, under some assumptions, by the restricting equation $(\star)$ to certain submanifolds of a Dubrovin-Frobenius manifold, the hypothesis of [13] are realized.

The reason to start searching for isomonodromic fundamental matrix solutions of equation $(\star)$ restricted to certain submanifolds $L \subset K$ is that, in a really over simplistic way, one could say that the fact of the monodromy data are constant depends only on two facts

1. Being able to define a "normal form" near the singular points of the differential equation $(\star)$ and compute the monodromy data of the corresponding solutions. In

order to be able to compute the monodromy data for an open set $W \subset M$ this "normal" form should vary holomorphically as we move in W.

2. In order to get isomonodromic fundamental matrix solutions of differential equation ($\star$), one exploits the fact that our differential equation is part of an over determined system of integrable partial differential equations.

More concretely, we will show that under some assumptions the restriction of the family of differential equations ($\star$) to certain submanifolds of the caustic is isomonodromic. For our purpose, of the two steps mentioned above, only the first step represents a problem. Indeed, flatness of the connection $\bar{\nabla}$ doesn't depend on a point $p \in M$ being semisimple whereas the "normal form" of the differential equation ($\star$) does depend on the point p being semisimple or not. In our case the "normal form" will depend on the Jordan form of the endomorphism $E\circ$. Let us try to explain why the normal form will change at a non-semisimple point.

On a neighborhood of a semisimple point there exists m-linearly independent vector fields $\pi_1, \dots, \pi_m$. We can write any other vector field as $v = v_1\pi_1 + \cdots + v_m\pi_m$. Thus we immediately obtain $v \circ \pi_i = v_i\pi_i$. As such, all the operators $v\circ$ are diagonalizable in a neighborhood of a semisimple point.
The set of non-semisimple points, the caustic, is an hypersurface K or the empty set (proposition 1.1.4). At a point $p \in K$ the endomorphism $v_p\circ$ might or might not be diagonalizable. In any case, since any neighborhood of p intersects the semisimple loci; the basis that diagonalizes $v\circ$ outside the caustic cannot be extended to the basis that puts $v\circ$ in Jordan form inside the caustic. Indeed, outside the caustic the idempotents are a basis of eigenvectors and by definition this vectors no longer exist on the caustic.

To get around this problem first we describe some multiplication invariant submanifolds $L \subset K$ (proposition 1.2.1). Along this submanifolds it is easy to describe the vector fields v such that $v\circ$ is diagonalizable along L; the endomorphism $v\circ$ will be diagonalizable along L if and only if v is tangent to L (proposition 1.2.2). This brings us to the first assumption we will use throughout this work.

Assumption 1: The Euler vector field is tangent to the multiplication invariant submanifolds L described in proposition 1.2.1.

By restricting ourselves to the submanifolds L we will almost get our desired "normal form". Note that on the semisimple case, thanks to the compatibility of the metric and the multiplication, the idempotents are orthogonal. Since the endomorphism $E\circ$ is η-symmetric and μ is η-antisymmetric, a convenient basis for writing the differential equation ($\star$) is the basis consisting of the normalized orthogonal idempotents. In our case we make the following assumption.

Assumption 2: The restriction of the metric η to the multiplication invariant submanifolds is non-degenerate.

A word of comment is due; on the complex numbers any symmetric bilinear form η has null-vectors ($i.e.$ vectors v such that $\eta(v,v)=0$). Hence, even if η is non-degenerate, the restriction of η to some subspaces might be degenerate and thus the reason for the assumption we make. What we really want is that the normal space of the submanifolds L is transversal to their tangent space.

With assumptions 1 and 2 one is able to get nice "normal forms" of the differential equation ($\star$) restricted to L and compute monodromy data for the corresponding fundamental matrix solutions. In order to get isomonodromic fundamental matrix solutions we will need to make another assumption which is probably best to leave for later, let us just mention that it is related with the exponent of formal monodromy B. We might just add that, if at the caustic we only loose one idempotent then this third assumption is not needed, it is always satisfied. The reason for this is that in this case the exponent of formal monodromy totally determines the underlying F-manifold structure of M in a small neighborhood of $p \in K$, and vice versa.

Let us at least explain how does the structure of the underlying structure of F-manifold appears in the exponent of formal monodromy B. We will use two deep results by Hertling that can be found in [14]. First is the fact that (see [14] theorem 2.11 or theorem 1.1.1 of this work) if at $p \in M$ the tangent space decomposes as

$$T_pM = \bigoplus_{k=1}^{l}(T_pM)_k$$

where each piece $(T_pM)_k$ is an irreducible $\mathbb{C}$-algebra then the germ (M,p) of the F-manifold M at p decomposes as

$$(M,p) \cong \Pi_{k=1}^{l}(M_k,p)$$

where each (M_k,p) is an irreducible F-manifold.
If at the caustic K we only loose one idempotent then, since the only irreducible 2-dimensional algebra is $\mathbb{C}[z]/z^2$, for $p \in K$ we will have

$$T_pM \cong \mathbb{C}[z]/(z^2) \oplus \mathbb{C}^{m-2}.$$

Correspondingly, the germ of M at p will decompose as

$$(M,p) \cong F^2 \times \Pi_{k=1}^{m-2}A_1$$

where F^2 is the germ of a two-dimensional F-manifolds and A_1 is the only germ of a one-dimensional F-manifold. In this case the caustic will be a multiplication invariant submanifold and the multiplication will be generically semisimple, we will often refer to this case as the semisimple caustic case. Now we use the second result by Hertling which is a classification of the germs of two-dimensional F-manifolds ([14] theorem 4.7). Essentially it says that they are classified by a natural number $n \in \mathbb{N}_{\geq 2}$ (see example 1.1.5). In this case, all the entries of the exponent of formal monodromy B will be zero except two of them which will be (theorem 5.1.2)

$$\pm i\frac{n-2}{n}.$$

In the semisimple case all the entries of the exponent of formal monodromy are zero. Correspondingly the two dimensional germ of F-manifold appearing in the decomposition of M will be $I_2(2)$ which is isomorphic two $A_1 \times A_1$. Note that the above formula still works in this case. Therefore, on the semisimple case and on the semisimple caustic case the exponent of formal monodromy "knows" the underlying structure of F-manifold. We expect that this is still true when we loose more idempotents when arriving at the caustic. To study this it would be useful to have a classification of irreducible germs of F-manifolds, but already on dimension 3 the classification is vast, recent and still incomplete (see [3]).

Let us describe the organization of this work. In chapter 1 we start studying F-manifolds. The most important results are proposition 1.1.1 and theorem 1.1.1. The proposition states that the product of two F-manifolds is again an F-manifold. The theorem says that the decomposition of T_pM into irreducible algebras gives a local decomposition of M into a product irreducible F-manifolds. Both of these results aren't new and can be found on [14]. Of crucial importance for this work are propositions 1.2.1 and 1.2.2. The first of these propositions describes certain multiplication invariant submanifolds L to which we will restrict when studying the monodromy data of the differential equation $(\star)$. Proposition 1.2.2 says that the endomorphism $v_p\circ\colon T_pM \to \mathcal{T}_pM$ for $p \in L$ is diagonalizable if and only if $v_p \in T_pL \leq T_pM$. This proposition is the reason for assumption 1.

In chapter 2 we start studying Dubrovin-Frobenius manifolds. The definition we give is not the original one but it is convenient for the purpose of this work. Here we also show that under assumptions 1 and 2, the multiplication invariant submanifolds of proposition 1.2.1, with the induced structures, satisfy all the axioms of a Dubrovin-Frobenius manifold except for the flatness of the metric. On a three-dimensional Dubrovin-Frobenius manifold with a semisimple caustic, the caustic is always a Dubrovin-Frobenius manifold. This results were previously noted by Strachan [21] but we provide different and coordinate independent proofs.

Section 2.2 shows that the definition we use always gives a Dubrovin-Frobenius manifold as were defined by Dubrovin. At the end of the chapter we introduce the most important object of this work, the deformed connection $\bar{\nabla}$. It is also shown that this connection is flat. Nothing from this chapter is new and all of its contents can be found in [11],[12] and [14].

In chapter 3 the real work of this thesis begins. Here, under assumptions 1 and 2, we study the deformed connection when pulled back to the multiplication invariant submanifolds L of proposition 1.2.1. To the best of our knowledge this was only done previously on a neighborhood of a semisimple point. In this case assumptions 1 and 2 are always satisfied. Under assumptions 1 and 2 the connection $\bar{\nabla}$ when restricted to L satisfies analogous properties to that of $\bar{\nabla}$ in the semisimple case (see equations (3.1.1), (3.1.2) and (3.2.2)). In the semisimple case, using the endomorphism μ one can recover the connection matrices of the Levi-Civita connection of the metric η (equation (3.1.2)). On a point of the caustic we cannot do this anymore because the kernel of $ad_{E\circ}$ are no longer only the diagonal matrices. In spite of this, the parts of the Levi-Civita connection that we cannot recover from μ give new flat connections $\pi_k \circ \nabla$ on certain subbundles over L which are determined by the irreducible algebra decomposition of T_pM with $p \in L$. This facts are established in propositions 3.2.1, 3.2.2 and theorem 3.2.1. Some other properties of the flat connections $\pi_k \circ \nabla$, analogous to the properties of ∇, are summarized in propositions 3.2.3 and 3.2.4. Finally on proposition 3.2.5 we use these new flat connections to get a basis from which the desired "normal form" of the differential equation $(\star)$ will be computed.

Chapter 4 is a central one: we establish the isomonodromy deformation theory along the multiplication invariant submanifolds L. We identify the relevant monodromy data, the flat sections at infinity, and prove isomonodromy. We start by studying the monodromy data at $z = 0$. It was previously known (see [11] and [12]) that these monodromy data are constant in a neighborhood of any point $p \in M$. For completeness we write a proof of this fact (theorem 4.1.1). Then in section 4.2 we go on to study the monodromy data at $z = \infty$. Here the results of the previous chapter become crucial. First we make our third assumption, it concerns the diagonal blocks of the matrix V of proposition 3.2.5.

Assumption 3: The eigenvalues b_{k_i} of any diagonal block of the matrix V of proposition 3.2.5 don't differ by a non-zero integer. That is $b_{k_i} - b_{k_j} \notin \mathbb{Z} \setminus \{0\}$ for any $i, j \in \{1, \ldots, dim(T_pM)_k\}$.

With this assumption we obtain a "normal form" for equation $(\star)$ from which we can compute a formal solution Y_F. Proposition 3.2.5 immediately implies that the exponent of formal monodromy of this solution is constant. In theorems 4.2.2 and 4.3.1, assumption 3 will also allow us to show that the corresponding Stokes matrices and central connection matrix are constant. Thus under assumptions 1,2 and 3 we get that the monodromy data of the differential equation $(\star)$ are constant.

On chapter 5 we apply the previous results to the case of a semisimple caustic. In particular we show that assumption 3 is not needed in this case and we compute explicitly the exponent of formal monodromy in terms of the irreducible F-manifold decomposition (theorem 5.1.2). In section 5.2 we compute the monodromy data along the semisimple caustic of a large class of three dimensional Dubrovin-Frobenius manifolds.

Having established the constancy of the monodromy data one natural question arises.

Question: How does the monodromy data for different multiplication invariant submanifolds $L, \tilde{L}$ are related? In particular, if $\tilde{L} = \bar{L} \setminus L$ is the (topological) boundary of L, what can we say about solutions of equation $(\star)$ restricted to L as we approach $\tilde{L}$?

On chapter 6 we address this question and give a partial answer with proposition 6.1.2: After an adequate Gauge transformation, the columns of the formal fundamental matrix solution corresponding to blocks that don't coalesce with other blocks as we move from L to $\tilde{L}$ remain holomorphic on $\tilde{L}$.

Chapter 1

F-Manifolds

In this chapter we start preparing for the definition of a Dubrovin-Frobenius manifold. Every Dubrovin-Frobenius manifold underlies an F-manifold structure and this chapter is dedicated to studying their basic properties. Much of this material can be found in [14].
The tangent sheaf $\mathcal{T}_M$ of any F-manifold carries a multiplication $\circ$. This multiplication endows each tangent space T_pM with the structure of a $\mathbb{C}$-algebra. One of the most important results in F-manifold theory is that, the decomposition of T_pM into irreducible algebras induces a local decomposition of M into "simpler" F-manifolds (theorem 1.1.1).
Most of the results known in Dubrovin Frobenius manifolds are done in neighborhoods where the algebras T_pM contain no nilpotents ([11],[12]). For points p in such neighborhoods T_pM is a direct sum of 1-dimensional algebras and multiplication by any vector $v_p \circ T_pM \to \mathcal{T}_pM$ is diagonalizable.

In this work we will assume that generically the multiplication has no nilpotents but, we will mainly focus on neighborhoods of points $p \in M$ such that T_pM contains nilpotent elements. In this case the set of nilpotent elements is an hypersurface K called the caustic (proposition 1.1.4). Correspondingly, for a point $p \in K$ not all of the operators $v_p \circ T_pM \to T_pM$ will be diagonalizable.
On a neighborhood of a point $p \in K$ the Jordan form of the endomorphism $v\circ \colon \mathcal{T}_M \to \mathcal{T}_M$ might change from diagonalizable to non-diagonalizable. Proposition 1.2.2 tells us which endomorphisms $v\circ$ remain diagonalizable when we arrive at the caustic.
But even if $v\circ$ is still diagonalizable at $p \in K$, the basis that diagonalizes it outside of K cannot be extended to K. Hence the basis that diagonalizes $v\circ$ outside of K is different from the one that diagonalizes it inside K. For the purpose of the last chapter of this work, this fact causes a lot of trouble and therefore we will restrict ourselves to certain multiplication invariant submanifolds $L \subset K$ in which the basis that diagonalizes the operators $v\circ$ does not change as we move on L. This submanifolds are described in proposition 1.2.1.

1.1 General Theory

Let M be a connected complex manifold of dimension m then, each point p of M has a vector space associated to it, namely its tangent space T_pM. This collection of vector spaces parametrized by the points of M varies holomorphically and; on any open set U, multiplication by any local holomorphic function $f \in \mathcal{O}_M(U)$ by a local vector field $v \in \mathcal{T}_M(U) := \Gamma(U, TM)$ is $\mathcal{O}_M(U)$-bilinear. One of the crucial ingredients is that on an F-manifold the tangent spaces carry a multiplication $\circ \colon \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ (the notation means that we have such a map for any open set $U \subset M$), this multiplication is required to be commutative, associative, with unit and $\mathcal{O}_M$-bilinear. As such, each tangent space becomes a commutative, associative, unital finite-dimensional algebra over $\mathbb{C}$.
Just for being a vector space over $\mathbb{C}$ each tangent space T_pM is isomorphic to $\mathbb{C}^n$; as $\mathbb{C}$-algebras this is no longer the case.

Example 1.1.1. Consider $\mathbb{C}^2$ with its canonical basis e_1, e_2, the formulas $e_i \circ e_j := \delta_{ij} e_i$ define a structure of a commutative, associative algebra over $\mathbb{C}$ (the unit is $e = e_1 + e_2$). This algebra is the direct sum of two copies of the the only one-dimensional $\mathbb{C}$-algebra. Note that this algebra doesn't posses nilpotent elements. The algebra $\mathbb{C}[z]/(z^2)$ is two-dimensional but z is a nilpotent element. Hence these two algebras cannot be isomorphic. It can be shown that up to isomorphism this is the only irreducible two-dimensional $\mathbb{C}$-algebra.

On a complex manifold any local chart induces a basis of T_pM for any $p \in M$, hence we obtain a vector space isomorphism $T_pM \cong \mathbb{C}^n$. On an F-manifold we wish that the decomposition of T_pM into irreducible $\mathbb{C}$-algebras (an algebra is *irreducible* if it is not isomorphic to a direct sum of other two algebras) extends to a decomposition of the germ (M, p) of M around p into "irreducible" F-manifolds (see proposition 1.1.1 and definition 1.1.3). This will be achieved with the following definition.

Definition 1.1.1. Let M be a complex manifold. Let $\circ \colon \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ be a commutative, associative, unital and $\mathcal{O}_M$-bilinear multiplication on $\mathcal{T}_M$. Denote by $e \in \mathcal{T}_M(M)$ the global vector field corresponding to the unit of the multiplication $\circ$. The triple $(M, \circ, e)$ is called an *F-manifold* if for any local vector fields $u, v \in \mathcal{T}_M(U)$ we have that

$$\mathcal{L}_{u \circ v}(\circ) = u \circ (\mathcal{L}_v \circ) + v \circ (\mathcal{L}_u \circ). \tag{1.1.1}$$

Remark 1.1.1. Note that the expression

$$\mathcal{L}_{u \circ v}(\circ)(w, z) - u \circ \mathcal{L}_v(\circ)(w, z) - v \circ \mathcal{L}_u(\circ)(w, z) \tag{1.1.1}$$

is $\mathcal{O}_M$-bilinear in its four arguments and hence it defines a tensor. Condition (1.1.1) is equivalent to the vanishing of the above tensor and by $\mathcal{O}_M$-bilinearity this only has to be checked on a basis of $\mathcal{T}_M$.

Example 1.1.2. Consider $\mathbb{C}$ with a local coordinate t. Define a multiplication by $f(t)\partial_t \circ g(t)\partial_t := f(t)g(t)\partial_t$. Then $\partial_t \circ f(t)\partial_t = f(t)\partial_t$ and so ∂_t is a unit for $\circ$. Condition (1.1.1) is immediately true and thus we have defined a one dimensional F-manifold. We will denote this F-manifold as A_1

Example 1.1.3. Consider $\mathbb{C}^2$ with local coordinates (t_1, t_2) and let $n \in \mathbb{N}_{\geq 2}$. Define a multiplication by $\partial_{t_1} = e, \partial_{t_2} \circ \partial_{t_2} = t_2^{n-2}\partial_{t_1}$ and then extend linearly. A simple computation shows that $\mathcal{L}_{\partial_{t_1}} \circ = 0$. With this we get

$$\mathcal{L}_{\partial_{t_1} \circ \partial_{t_2}}(\circ) = \mathcal{L}_{\partial_{t_2}} \circ = e \circ (\mathcal{L}_{\partial_{t_2}} \circ) + \partial_{t_2} \circ (\mathcal{L}_{\partial_{t_1}} \circ).$$

To show that the above define an F-manifold we just need to show that

$$\mathcal{L}_{t_2^{n-2}\partial_{t_1}}(\circ) = 2\partial_{t_2} \circ (\mathcal{L}_{\partial_{t_2}} \circ).$$

But $(\mathcal{L}_{t_2^{n-2}\partial_{t_1}}(\circ))(\partial_{t_2}, \partial_{t_2}) = -2[t_2^{n-2}\partial_{t_1}, \partial_{t_2}] \circ \partial_{t_2}$, where as $(\mathcal{L}_{\partial_{t_2}} \circ)(\partial_{t_2}, \partial_{t_2}) = 2[\partial_{t_2}, t_2^{n-2}\partial_{t_1}]$. The remaining equalities are satisfied trivially. This two dimensional F-manifolds are denoted by $I_2(n)$ and correspond to the orbit spaces of the finite Coxeter groups $I_2(n)$ (see [10]).

Fix a point $p \in M$. The family of operators $X_p \circ \colon T_pM \to T_pM$ with $X_p \in T_pM$ is commutative. Hence there is a common eigenspace decomposition $T_pM = \oplus_{k=1}^{l}(T_pM)_k$.

Lemma 1.1.1. *Let $(A, \circ, e)$ be a commutative, associative, unital $\mathbb{C}$-algebra and let $A = \sum_{k=1}^{l} A_k$ be the common eigenspace decomposition of all operators $a\circ, a \in A$. Then each A_k is a irreducible commutative, associative, unital $\mathbb{C}$-algebra.*

Proof. Since $A_i \circ A_j \subset A_j$ and $A_j \circ A_i \subset A_i$ then $A_i \circ A_j = 0$ for $i \neq j$ and $A_i \circ A_i \subset A_i$. This shows that $\oplus_{i \in I} A_i$ is an ideal for any $I \subset \{1, \ldots, l\}$. Hence $A_i \cong A / \oplus_{j=1, j \neq i} A_j$ is a commutative associative and unital $\mathbb{C}$-algebra. Reducibility of any A_i would give a finer eigenspace decomposition. □

Remark 1.1.2. The proposition implies that each irreducible algebra A_i has an identity $\pi_i \in A_i$ and $e = \sum_{i=1}^{l} \pi_i$. The vectors π_i will be called *idempotents*.

Now we define another global vector field which will be of crucial importance in the future.

Definition 1.1.2. Let $(M, \circ, e)$ be an F-manifold. A vector field E will be called *Euler vector field* if one has

$$\mathcal{L}_E(\circ) = \circ. \tag{1.1.2}$$

Example 1.1.4. In the A_1 F-manifold the vector field $E := t\partial_t$ is an Euler vector field. Indeed,

$$(\mathcal{L}_E \circ)(\partial_t, \partial_t) = -2[t\partial_t, \partial_t] = \partial_t = \partial_t \circ \partial_t.$$

For the F-manifolds $I_2(n)$ an Euler vector field is

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{2}{m} t_2 \frac{\partial}{\partial t_2}.$$

We have $[E, \partial_{t_1}] = -\partial_{t_1}$ and $[E, \partial_{t_2}] = -\frac{2}{n}\partial_{t_2}$ so

$$\begin{aligned}(\mathcal{L}_E\circ)(\partial_{t_2}, \partial_{t_2}) &= [E, t_2^{n-2}\partial_{t_2}] - 2[E, \partial_{t_2}] \circ \partial_{t_2} \\ &= E(t_2^{n-2})\partial_{t_1} + t_2^{n-2}[E, \partial_{t_1}] + \frac{4}{m}\partial_{t_2} \circ \partial_{t_2} = (2n-4-n+4)\frac{t_2^{n-2}}{n}\partial_{t_1} \\ &= t_2^{n-2}\partial_{t_1} = \partial_{t_2} \circ \partial_{t_2}.\end{aligned}$$

Equation (1.1.2) for the pairs $(\partial_{t_1}, \partial_{t_1})$ and $(\partial_{t_1}, \partial_{t_2})$ are easy to check.

As we mentioned before, condition (1.1.1) will allow us to decompose an F-manifold into simpler F-manifolds. First we show that the product of two F-manifolds with Euler vector fields is again an F-manifold with an Euler vector field.

Proposition 1.1.1. *Let $(M_1, \circ_1, e_1)$ and $(M_2, \circ_2, e_2)$ be two F-manifolds. Then $(M, \circ, e) = (M_1 \times M_2, \circ_1 \oplus \circ_2, e_1 + e_2)$ is an F-manifold. Moreover, if E_1 and E_2 are Euler vector fields on M_1 and M_2 then $E_1 + E_2$ is an Euler vector field on M.*

Proof. The fact that $\circ$ is commutative, associative, $\mathcal{O}_M$-bilinear and that e is a unit is immediate. We only need to check that condition (1.1.1) holds and that the vector field E satisfies (1.1.2). Let $p_i \colon M \to M_i$ denote the projections. Recall that

$$\mathcal{T}_M = \mathcal{O}_M \otimes_{\mathcal{O}_{M_1}} p_1^{-1}\mathcal{T}_{M_1} \oplus \mathcal{O}_M \otimes_{\mathcal{O}_{M_2}} p_2^{-1}\mathcal{T}_{M_2}.$$

Condition (1.1.1) is equivalent to

$$\mathcal{L}_{u \circ v}(\circ)(x \circ y) - u \circ \mathcal{L}_v(\circ)(x, y) - v \circ \mathcal{L}_u(\circ)(x, y) = 0.$$

This expression is $\mathcal{O}_M$-linear in all of its arguments and therefore we only need to verify it for vectors in $p_1^{-1}\mathcal{T}_{M_1} \cup p_2^{-1}\mathcal{T}_{M_2}$. Take $a, b \in p_1^{-1}\mathcal{T}_{M_1} \cup p_2^{-1}\mathcal{T}_{M_2}$ and write them as $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_i, b_i \in p_i^{-1}\mathcal{T}_{M_i}$. Since $[a_1, b_2] = [a_2, b_2] = 0$ we get $[a, b] = [a_1, b_1] + [a_2, b_2]$. For $u, v, x, y \in p_1^{-1}\mathcal{T}_{M_1} \cup p_2^{-1}\mathcal{T}_{M_2}$ we have

$$\begin{aligned}\mathcal{L}_{u \circ v}(\circ)(x, y) &= [u_1 \circ_1 v_1, x_1 \circ_1 y_1] - [u_1 \circ_1 v_1, x_1] \circ y - x \circ [u_1 \circ_1 v_1, y_1] \\ &\qquad [u_2 \circ_2 v_2, x_2 \circ_2 y_2] - [u_2 \circ_2 v_2, x_2] \circ y - x \circ [u_2 \circ_2 v_2, y_2]\end{aligned}$$

But $[p_i^{-1}\mathcal{T}_{M_i}, p_i^{-1}\mathcal{T}_{M_i}] \subset p_i^{-1}\mathcal{T}_{M_i}$ and $p_i^{-1}\mathcal{T}_{M_i} \circ p_j^{-1}\mathcal{T}_{M_j} = 0$ for $i \neq j$ so we can write

$$\begin{aligned}\mathcal{L}_{u \circ v}(\circ)(x, y) &= [u_1 \circ_1 v_1, x_1 \circ_1 y_1] - [u_1 \circ_1 v_1, x_1] \circ_1 y_1 - x_1 \circ [u_1 \circ_1 v_1, y_1] \\ &\quad + [u_2 \circ_2 v_2, x_2 \circ_2 y_2] - [u_2 \circ_2 v_2, x_2] \circ_2 y_2 - x_2 \circ_2 [u_2 \circ_2 v_2, y_2] \\ &= \mathcal{L}_{u_1 \circ_1 v_1}(\circ_1)(x_1, y_1) + \mathcal{L}_{u_2 \circ_2 v_2}(\circ_2)(x_2, y_2). \quad (1.1.3)\end{aligned}$$

On the other hand

$$u \circ \mathcal{L}_v(\circ)(x,y) = (u_1 + u_2)(\mathcal{L}_{v_1}(\circ)(x,y) + \mathcal{L}_{v_2}(\circ)(x,y))$$

but for $i \neq j$ one has

$$\mathcal{L}_{v_i}(\circ)(x_i, y_j) = 0 \qquad\qquad \mathcal{L}_{v_i}(\circ)(x_j, y_j) = 0$$

so that

$$\begin{aligned} u \circ \mathcal{L}_v(\circ)(x,y) + v \circ \mathcal{L}_u(\circ)(x,y) &= u_1 \circ_1 \mathcal{L}_{v_1}(\circ_1)(x_1,y_1) + u_2 \circ_2 \mathcal{L}_{v_2}(\circ_2)(x_2,y_2) \\ &+ v_1 \circ_1 \mathcal{L}_{u_1}(\circ_1)(x_1,y_1) + v_2 \circ_2 \mathcal{L}_{u_2}(\circ_2)(x_2,y_2) \\ &= \mathcal{L}_{u_1 \circ v_1}(\circ_1)(x_1,y_1) + \mathcal{L}_{u_2 \circ v_2}(\circ_2)(x_2,y_2). \end{aligned}$$

Comparing this expression with (1.1.3) we get that $\circ$ satisfies (1.1.1).
Now suppose the F-manifolds M_1 and M_2 have Euler vector fields E_1, E_2. Condition (1.1.2) is $\mathcal{O}_M$-bilinear so again we only need to verify it for $x, y \in p_1^{-1}\mathcal{T}_{M_1} \cup p_2^{-1}\mathcal{T}_{M_2}$. Me have

$$\mathcal{L}_E(\circ)(x,y) = \mathcal{L}_{E_1}(\circ)(x,y) + \mathcal{L}_{E_2}(\circ)(x,y)$$

But as before

$$\mathcal{L}_{E_i}(\circ)(x,y) = \mathcal{L}_{E_i}(\circ_i)(x_i,y_i) = x_i \circ_i y_i$$

so that

$$\mathcal{L}_E(\circ)(x,y) = x_1 \circ_1 y_1 + x_2 \circ_2 y_2 = x \circ y.$$

$$\begin{aligned} \mathcal{L}_E(\circ)(x,y) = \mathcal{L}_{E_1}(\circ)(x,y) + \mathcal{L}_{E_2}(\circ)(x,y) &= \mathcal{L}_{E_1}(\circ_1)(x_1,y_1) + \mathcal{L}_{E_2}(\circ_2)(x_2,y_2) \\ &= x_1 \circ_1 y_1 + x_2 \circ y_2 = x \circ y. \end{aligned}$$

$\square$

Definition 1.1.3. We will say that the F-manifold $(M, \circ, e)$ is *irreducible* if it is not isomorphic to a product of two F-manifolds of smaller dimension.

Example 1.1.5. The F-manifold $I_2(2)$ is reducible because it is isomorphic to the product $A_1 \times A_1$. Using theorem 1.1.1 one can see that the germ of the F-manifold $I_2(n)$ is reducible for any (t_1, t_2) with $t_2 \neq 0$. On the other hand, when $n > 2$ the germ of $I_2(n)$ at points of the form $(t_1, 0)$ are irreducible. Indeed, on those points we have $\partial_{t_2} \circ \partial_{t_2} = 0$ so $T_{(t_1,0)}I_2(n)$ is isomorphic as an algebra to $\mathbb{C}[z]/z^2$; if the germ were reducible we would get that an isomorphism of algebras between $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C}[z]/z^2$. But this cannot happen because $\mathbb{C} \times \mathbb{C}$ has no nilpotents.

Take $p \in M$. Let us exhibit some irreducible $\mathcal{O}_{M,p}$-algebras that are induced by the decompostion of T_pM into irreducible $\mathbb{C}$-algebras. In this work this algebras will be of the utmost importance. The $\mathbb{C}$-algebra structure of T_pM comes from the $\mathcal{O}_{M,p}$-algebra structure of $\mathcal{T}_{M,p}$ which in turn is obtained by the $\mathcal{O}_M$-algebra structure of the tangent sheaf $\mathcal{T}_M$. The kernel of the evaluation map at p, $ev_p \colon \mathcal{T}_{M,p} \to T_pM$ is the $\mathcal{O}_{M,p}$-submodule $\mathfrak{m}_{M,p}\mathcal{T}_{M,p}$ of vector fields vanishing at $p \in M$. The $\mathcal{O}_{M,p}$-linearity of the multiplication implies that the kernel is also an ideal and this gives the natural algebra structure of T_pM. The irreducible $\mathbb{C}$-algebra decomposition of T_pM induces an irreducible $\mathcal{O}_{M,p}$-algebra decomposition of $\mathcal{T}_{M,p} = \oplus_{i=1}^{l}(\mathcal{T}_{M,p})_i$ with $(\mathcal{T}_{M,p})_i := ev_p^{-1}(T_pM)_i$. Therefore the irreducible $\mathbb{C}$-algebra decomposition of T_pM gives a irreducible $\mathcal{O}_{M,p}$-algebra decomposition of $\mathcal{T}_{M,p} = \oplus_{i=1}^{l}(\mathcal{T}_{M,p})_i$. The integrability condition (1.1.1) gives nice properties of these subalgebras.

Lemma 1.1.2. *Let $(M, \circ, e)$ be an F-manifold and suppose that for $p \in M$ we have $\mathcal{T}_{M,p} = \oplus_{i=1}^{l}(\mathcal{T}_{M,p})_i$ with each $(\mathcal{T}_{M,p})_i$ an irreducible $\mathcal{O}_{M,p}$-algebra with unit π_i. Then*

1. $\mathcal{L}_{\pi_i}(\circ) = 0$.

2. $[\pi_i, \pi_j] = 0$.

3. $[\pi_i, (\mathcal{T}_{M,p})_j] \subset (\mathcal{T}_{M,p})_j$.

4. $[(\mathcal{T}_{M,p})_i, (\mathcal{T}_{M,p})_j] \subset (\mathcal{T}_{M,p})_i \oplus (\mathcal{T}_{M,p})_j$.

Proof. Evaluating the integrability condition (1.1.1) on π_i we get

$$\mathcal{L}_{\pi_i}(\circ) = 2\pi_i \circ \mathcal{L}_{\pi_i}(\circ).$$

Multiplying by π_i we get $\pi_i \circ \mathcal{L}_{\pi_i}(\circ) = 2\pi_i \circ \mathcal{L}_{\pi_i}(\circ)$ so that $\mathcal{L}_{\pi_i}(\circ) = 2\pi_i \circ \mathcal{L}_{\pi_i}(\circ) = 0$. For 2; we have

$$0 = \mathcal{L}_{\pi_i}(\circ)(\pi_j, \pi_j) = \mathcal{L}_{\pi_i}(\pi_j) - 2\pi_j \circ \mathcal{L}_{\pi_i}\pi_j.$$

Again multiplying by π_j gives the result. For 3; fix i and j and take $v_j \in (\mathcal{T}_{M,p})_j$, we have

$$0 = \mathcal{L}_{\pi_i}(\circ)(\pi_j, v_j) = \mathcal{L}_{\pi_i}v_j - \pi_j \circ \mathcal{L}_{\pi_i}v_j.$$

Therefore $\mathcal{L}_{\pi_i}v_j = \pi_j \circ \mathcal{L}_{\pi_i}v_j \in (\mathcal{T}_{M,p})_j$. For the last part take $u \in (\mathcal{T}_{M,p})_i$ and $v \in (\mathcal{T}_{M,p})_j$ then if $i \neq k \neq j$

$$\begin{aligned} 0 &= \mathcal{L}_{\pi_k \circ u}(\circ)(\pi_k, v) \\ &= \pi_k \circ ([u, \pi_k \circ v] - [u, \pi_k] \circ v - \pi_i \circ [u, v]) \\ &= -\pi_k \circ [u, v]. \end{aligned}$$

This means that the projection of $[u, v]$ on $(\mathcal{T}_{M,p})_k$ is zero unless $k = i$ or $k = j$. □

Finally we have the desired decomposition theorem.

Theorem 1.1.1. *Let (M,p) be the germ at $p \in M$ of an F-manifold $(M, \circ, e)$. Then the common eigenspace decomposition $T_pM = \oplus_{k=1}^{l}(T_pM)_k$ into irreducible algebras extends to a decomposition*

$$(M,p) \cong \Pi_{k=1}^{l}(M_k, p) \tag{1.1.4}$$

of the germ (M,p) into irreducible germs of F-manifolds (M_k, p). Moreover, an Euler vector field E on M decomposes into a sum of Euler vector fields on the germs (M_k, p).

Proof. Consider the multiplication invariant subsheaves $(\mathcal{T}_{M,p})_k, k = 1, \ldots, l$. By the last item of the preceding lemma for any j we have that the subbundles

$$\bigoplus_{\substack{k=1\\k\neq j}}^{l}(\mathcal{T}_{M,p})_k$$

are integrable. Indeed,

$$\begin{aligned}\left[\bigoplus_{\substack{i=1\\i\neq j}}^{l}(\mathcal{T}_{M,p})_i, \bigoplus_{\substack{k=1\\k\neq j}}^{l}(\mathcal{T}_{M,p})_k\right] &= \sum_{\substack{i=1\\i\neq j}}^{l}\sum_{\substack{k=1\\k\neq j}}^{l}[(\mathcal{T}_{M,p})_i, (\mathcal{T}_{M,p})_k]\\ &\subset \sum_{\substack{i=1\\i\neq j}}^{l}\sum_{\substack{k=1\\k\neq j}}^{l}(\mathcal{T}_{M,p})_i \oplus (\mathcal{T}_{M,p})_k = \bigoplus_{\substack{i=1\\i\neq j}}^{l}(\mathcal{T}_{M,p})_i.\end{aligned}$$

By the Frobenius integrability theorem we get a submersion $f_j \colon (M,p) \to (\mathbb{C}^{dim(T_pM)_j}, 0)$ such that the fibers are the integral submanifolds of this subbundle. Since the image of a direct sum of linear maps is the direct sum of the images of each map we get that the map $f := \oplus_{j=1}^{l} f_j \colon (M,p) \to (\mathbb{C}^{dim M}, 0)$ is an isomorphism.
Consider the submanifolds

$$(M_k, p) := ((\bigoplus_{\substack{j=1\\j\neq k}} f_j)^{-1}(0), p).$$

Since the kernel of a direct sum of linear maps is the intersection of the kernels of all the maps we have $\mathcal{T}_{M_k,p} = \iota^*(\mathcal{T}_{M,p})_k$ where $\iota \colon (M_k, p) \to (M,p)$ is the inclusion. Under the isomorphism f the germ manifolds (M_k, p) get mapped to germs of transversal linear subspaces of $(\mathbb{C}^{dim M}, 0)$ and thus we get $(M,p) \cong \Pi_{k=1}^{l}(M_k, p)$. Let us show that the manifolds (M_k, p) are F-manifolds. Consider the projections $p_k \colon (M,p) \to (M_k, p)$. The projections are open maps so for any open set $U \subset M$ we have $p_k^{-1}\mathcal{T}_{M_k,p}(U) = \mathcal{T}_{M_k,p}(p_k(U))$;

therefore to define a multiplication on $\mathcal{T}_{M_k,p}$ it is enough to show that for $u, v \in p_k^{-1}\mathcal{T}_{M_k,p}$ we have $u \circ v \in p_k^{-1}\mathcal{T}_{M_k,p}$. We have

$$\mathcal{T}_{M,p} \cong \bigoplus_{k=1}^{l} \mathcal{O}_{M,p} \otimes_{\mathcal{O}_{M_k,p}} p_k^{-1}\mathcal{T}_{M_k,p} = \bigoplus_{k=1}^{l} (\mathcal{T}_{M,p})_k.$$

Since the sheaves $(\mathcal{T}_{M,p})_k$ are multiplication invariant we have that $u \circ v \in (\mathcal{T}_{M,p})_k$ whenever $u, v \in p_k^{-1}\mathcal{T}_{M_k,p}$. Now $u \circ v \in p_k^{-1}\mathcal{T}_{M_k,p}$ if and only if for any j and any $w \in (\mathcal{T}_{M,p})_j$ one has $[w, u \circ v] \in (\mathcal{T}_{M,p})_j$. Condition (1.1.1) gives

$$[w, u \circ v] - [w, u] \circ v - u \circ [w, v] = \mathcal{L}_{\pi_j \circ w}(\circ)(u, v)$$
$$\pi_j \circ ([w, u \circ v] - [w, u] \circ v - u \circ [w, v]).$$

But $[w, u], [w, v] \in (\mathcal{T}_{M,p})_j$ and therefore

$$[w, u \circ v] = \pi_j \circ [w, u \circ v] \in (\mathcal{T}_{M,p})_j.$$

Condition (1.1.1) holds for all vector fields on M and therefore it follows for vector fields in $p_k^{-1}\mathcal{T}_{M_k,p}$.

Finally suppose that $(M, \circ, e)$ has an Euler vector field E. We will show that $\pi_k \circ E \in p_k^{-1}\mathcal{T}_{M_k,p}$ and that $\pi_k \circ E$ satisfies (1.1.2) for all $u, v \in (\mathcal{T}_{M,p})_k$. For any $w \in (\mathcal{T}_{M,p})$ we have

$$\begin{aligned}
[\pi_k \circ E, w] &= \mathcal{L}_{\pi_k \circ E}(\circ)(\pi_j, w) + [\pi_k \circ E, \pi_j] \circ w + \pi_j \circ [\pi_k \circ E, w] \\
&= \pi_k \circ \pi_j \circ w + [\pi_k \circ E, \pi_j] \circ w + \pi_j \circ [\pi_k \circ E, w] \\
&= \pi_j \circ (\pi_k \circ w + [\pi_k \circ E, \pi_j] \circ w + [\pi_k \circ E, w]) \in (\mathcal{T}_{M,p})_j
\end{aligned}$$

because $w \circ \pi_j = w$. For the last part if $u, v \in (\mathcal{T}_{M,p})_k$ then

$$\mathcal{L}_{\pi_k \circ E}(\circ)(u, v) = \pi_k \circ \mathcal{L}_E(\circ)(u, v) = \pi_k \circ u \circ v = u \circ v.$$

□

Definition 1.1.4. A point $p \in M$ will be called *semisimple* if as algebras T_pM is isomorphic to $\mathbb{C}^m$. An F-manifold $(M, \circ, e)$ is called *massive* if it is generically semisimple. The set of non-semisimple points is called the *caustic* and will be denoted by K.

Remark 1.1.3. If the point $p \in M$ is semisimple then by theorem 1.1.1 a small neighborhood around p consist of semisimple points. The same does not hold true if the point p is not semisimple. The decomposition (1.1.4) holds true in a neighborhood of p but for a point $q \neq p$ the decomposition of the germ (M, q) into irreducible F-manifolds may be finer. For example on the $I_2(n)$ F-manifolds the whole t_1 axis consists of non-semisimple points. Hence any neighborhood of a non-semisimple point contains semisimple points. More generally, if at $p \in M$ the tangent space T_pM decomposes as a direct sum of l_p irreducible algebras then, in a small neighborhood W of p, for all points $q \in M$ we have that T_qM decomposes as a direct sum of l_q algebras with $l_q \geq l_p$. Indeed, by theorem 1.1.1 for any point $q \in W$ we have that T_qM is isomorphic to a direct sum of l_p algebras. This algebras may or may not be irreducible.

For the purpose of this work it will be important to recognize if a given point $p \in M$ is semisimple or not. In this direction we have

Proposition 1.1.2. *Let $(M, \circ, e)$ be an F-manifold. Suppose that at a point $p \in M$ there exists a vector v such that the operator $v\circ$ has different eigenvalues. Then p is a semisimple point*

Proof. Let $v \circ e_i = u_i e_i$ then $v \circ (e_i \circ e_i) = e_i \circ v \circ e_i = u_i e_i \circ e_i$ so that $e_i \circ e_i$ is an eigenvector of $v\circ$ with eigenvalue u_i. Since all eigenvalues are different we obtain $e_i \circ e_i = \lambda_i e_i$ and $\pi_i := \frac{e_i}{\lambda_i}$ satisfies $\pi_i \circ \pi_i = \pi_i$. Now $u_i(\pi_i \circ \pi_j) = v \circ (\pi_i \circ \pi_j) = u_j(\pi_i \circ \pi_j)$ but since $u_i \neq u_j$ we obtain $\pi_i \circ \pi_j = 0$. □

It may well happen that the operator $v_p\circ \colon T_pM \to T_pM$ has repeated eigenvalues and nevertheless the point p might still be semisimple. A trivial example of this is any scalar multiple of the identity, multiplication by this element will have only one eigenvalue but it will be diagonalizable for any point $p \in M$. Later we will give less trivial examples of this phenomena.
To identify a point in the caustic we use the following proposition.

Proposition 1.1.3. *Let $(M, e, \circ)$ be an F-manifold. If the point $p \in M$ is semisimple then for all $v \in T_pM$ the operator $v \circ T_pM \to T_pM$ is diagonalizable.*

Proof. Write $v = v_1\pi_1 + \cdots + v_m\pi_m$, then $v \circ \pi_i = v_i\pi_i$ so the basis $\pi_1, \ldots, \pi_m$ is a basis of eigenvectors. □

Example 1.1.6. On the F-manifold $I_2(n)$ on the basis $\partial_{t_1}, \partial_{t_2}$, multiplication by the vector $a\partial_{t_1} + b\partial_{t_2}$ has matrix

$$\begin{pmatrix} a & bt_2^{n-2} \\ b & a \end{pmatrix}.$$

The eigenvalues of this matrix are $a \pm b t_2^{\frac{n-2}{2}}$ so at the caustic $\{\, t_2 = 0 \,\}$, multiplication by vector fields with $b \neq 0$ will not be diagonalizable. Later we will be more precise about the vector fields whose multiplication operator is diagonalizable at a point $p \in M$ (see proposition 1.2.2)

We finish this section with a proposition about the non-semisimple points on a massive F-manifold.

Proposition 1.1.4. *Let $(M, \circ, e)$ be a massive F-manifold. Then the caustic is either empty or an hypersurface.*

Proof. First we show that the caustic is an analytic set so that its dimension is well defined. Let $W \subset M$ be an open set. For any vector field $v \in \mathcal{T}_M(W)$ let $p_{v\circ} := det(v \circ -\lambda)$ be the characteristic polynomial of the endomorphism $v\circ$. This defines a function $p_{v\circ} \colon W \to \mathbb{C}^m$. Now $K_v := \{ q \in W \,|\, p_{v_q\circ} \text{ has a repeated root} \}$ is the inverse image under $p_{v\circ}$ of the set of polynomials with a repeated root and as such is an analytic set. Let us show that

$$K \cap W = \cap_{v \in \mathcal{T}_M(W)} K_v.$$

If $p \in K$ then by proposition1.1.2 the operators $v_p\circ$ all have repeated eigenvalues so we get the inclusion $\subset$. For the other inclusion note that at a semisimple point p there is always a vector field v such that $v_p\circ$ has different eigenvalues so if $v_p\circ$ has repeated a repeated eigenvalue for all $v \in \mathcal{T}_M(U)$ then p cannot be semisimple.
Now take a semisimple point $p \in M$ and consider the $m = dimM$ idempotent vector fields $\pi_1, \dots, \pi_m$. Suppose that $dimK < dimM - 2$ then for a sufficiently small open set W such that $W \cap K \neq \emptyset$ the set $W \setminus K$ is simply connected. Since M is massive we can take a semisimple point $p \in W \setminus K$ and consider the $m = dimM$ idempotent vector fields $\pi_1, \dots, \pi_m$. The simply-connectedness of $W \setminus K$ implies that the vector fields π_i have no monodromy and therefore they extend to the whole neighborhood W. Now $e - (\pi_1 + \dots + \pi_m) = 0, \pi_i \circ \pi_j = 0$ and $\pi_i \circ \pi_i - \pi_i = 0$ all hold true on $W \setminus K$ so by continuity this relations hold in the whole neighborhood W. Therefore on the whole neighborhood W we have $m = dimM$ idempotents and hence $K \cap W = \emptyset$. $\square$

1.2 The Caustic and its Multiplication Invariant Submanifolds

Suppose that at a point $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ with each $(T_pM)_k$ an irreducible algebra. By lemma 1.1.1 T_pM has l idempotent vectors $\pi_1, \dots, \pi_l$. Theorem 1.1.1 says that this vectors extend to idempotent vector fields $\pi_1 \dots, \pi_l$ on a neighborhood of p. Thanks to lemma 1.1.2 we get that $[\pi_i, \pi_j] = 0$. Therefore there exists a submanifold $L \subset M$ such that $p \in L$ and the tangent space of L is generated by the idempotents $\pi_1, \dots, \pi_l$. We will refer to L as the *integral submanifold* of the idempotents $\pi_1, \dots, \pi_l$ passing through p. In this section we will study some properties of these submanifolds.

Proposition 1.2.1. *Let $(M, \circ, e)$ be an F-manifold. Take $p \in M$ and suppose that $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ with each $(T_pM)_k$ an irreducible algebra. Let L be the integral submanifold of the idempotent vector fields $\pi_1, \dots, \pi_l$ passing through p. Then, with the induced structures $(L, \circ, e)$ is a massive F-manifold.*

Proof. We have $\pi_i \circ \pi_j = \delta_{ij}\pi_i \in \mathcal{T}_L$. This means that L is multiplication invariant. Since $e = \pi_1 + \cdots \pi_l$ the unit e is tangent to L. The condition (1.1.1) holds for all vector fields on M in particular it holds for the vector fields tangent to L. $\square$

Example 1.2.1. Take a semisimple point $p \in M$ then the integral submanifold is an open set $L \subset M$. If the caustic is empty then $L = M$ if it is non-empty then since M is connected and K is an hypersurface $L = M \setminus K$.
Let $p \in K$ be a regular point (non-singular point) of the caustic and suppose that T_pM has $m-1$ idempotents, then L is the regular part of an irreducible component $\tilde{K} \subset K$ of the caustic. Note that in this case the irreducible component of the caustic $\tilde{K}$ is a massive F-manifold. We will refer to this case as the semisimple caustic case. Later we will see examples of F-manifolds such that the tangent spaces of one irreducible component of the caustic have less than $m-1$ idempotents (see example 2.2.3).

Suppose that $(M, \circ, e)$ is a semisimple F-manifold (all points of M are semisimple) then proposition 1.1.3 says the operator of multiplication by any vector field $v\circ\colon \mathcal{T}_M \to \mathcal{T}_M$ is diagonalizable. Note that in this case any vector field is trivially tangent to the integral submanifold through any point $p \in M$. Now suppose that the caustic is non-empty. By definition, the basis $\pi_1, \ldots, \pi_m$ that diagonalizes the operator of multiplication of any vector field $v \in \mathcal{T}_M$ ceases to exist on the caustic. Nevertheless it might happen that $v \circ \mathcal{T}_M \to \mathcal{T}_M$ is still diagonalizable for a point $p \in K$.
To eliminate the complications arising from the fact that the basis diagonalizing $v\circ$ outside the caustic cannot be extended holomorphically to the caustic, we restrict ourselves to the integral submanifold of the idempotents passing through p. This is achieved by pulling back the tangent bundle $\mathcal{T}_M$ via the inclusion map $i\colon L \to M$. Recall that the fiber of $i^*\mathcal{T}_M$ at a point p is equal to the fiber of $\mathcal{T}_M$ at the point $i(p)$. As such the multiplication $\circ$ on $\mathcal{T}_M$ induces a multiplication on $i^*\mathcal{T}_M$, by abuse of notation we will denote this two multiplications with the same symbol $\circ$. Just as in the semisimple case, the next proposition says that the vector fields v such that $v\circ$ is diagonalizable along L are the ones that are tangent to L.

Proposition 1.2.2. *Suppose that $(M, p) \cong \Pi_{k=1}^{l}(M_k, p)$ with each M_k irreducible. Let L be the integral submanifold of the idempotents $\pi_1, \ldots, \pi_l$. Let $i\colon L \to M$ denote the inclusion. Then for any $v \in i^*\mathcal{T}_M$ the operator of multiplication $v\circ\colon i^*\mathcal{T}_M \to i^*\mathcal{T}_M$ is diagonalizable if and only if $v \in \mathcal{T}_L$.*

Proof. If $v \in \mathcal{T}_L$ we can write $v = v_1\pi_1 + \cdots + v_l\pi_l$. Take $w \in (\mathcal{T}_M)_k$ then $v \circ w = v_k w$. This means that on each irreducible algebra $(\mathcal{T}_{M,p})_k$ the operator $v\circ$ acts by multiplication by v_k and therefore is diagonalizable. Suppose now that $v\circ\colon i^*\mathcal{T}_M \to i^*\mathcal{T}_M$ is diagonalizable. Since the irreducible algebras $(\mathcal{T}_{M,p})_k$ are multiplication invariant the operator $v \circ|_{(\mathcal{T}_{M,p})_k}$ is also diagonalizable. On each of this algebras $v\circ$ can have only one eigenvalue (otherwise we would have more than l idempotents) say v_k. This tells us that $v \circ|_{(\mathcal{T}_{M,p})_k} = v_k\pi_k \circ|_{(\mathcal{T}_{M,p})_k}$ and therefore $v = v_1\pi_1 + \cdots + v_l\pi_l \in \mathcal{T}_L$. ☐

We now obtain a special local coordinate system for the submanifolds L which will be useful for later computations. Since the idempotent vector fields $\pi_1, \ldots, \pi_l \in \mathcal{T}_L$ commute there exists a coordinate system $u_1, \dots, u_l$ such that

$$\pi_i = \frac{\partial}{\partial u_i}.$$

Definition 1.2.1. Let $L \subset M$ be the integral submanifold of the idempotent vector fields $\pi_1, \dots, \pi_l$. The local coordinates $(u_1 \dots, u_l)$ on L such that $\partial_{u_i} = \pi_i$ are called *canonical coordinates.*

Proposition 1.2.3. *Let L be the integral submanifold of the idempotents $\pi_1, \dots, \pi_l$ and let $(u_1, \dots, u_m)$ be canonical coordinates on L. Suppose that the Euler vector field E is tangent to L. Then*

$$E = \sum_{s=1}^{l} (u_s + c_s)\pi_s$$

for some constants c_s. In particular, the eigenvalues of the endomorphism $E\circ\colon i^*T_L \to i^*\mathcal{T}_L$ are $(u_s + c_s)$ $s = 1 \dots, l$.

Proof. Write $E = \sum_{s=1}^{l} E^s \pi_s$. For $i \neq j$ the integrability condition (1.1.1) gives

$$0 = (\mathcal{L}_E \circ)(\pi_i, \pi_j) = (\pi_i E^j)\pi_j + (\pi_j E^i)\pi_i.$$

Hence $E^i = E^i(u_i)$. But

$$\pi_i = (\mathcal{L}_E \circ)(\pi_i, \pi_i) = (\pi_i E^i)\pi_i$$

so that $E^i = u_i + c_i$. $\square$

Chapter 2

Dubrovin Frobenius Manifolds

In this chapter we start by defining a Dubrovin-Frobenius manifold. This definition is not the original one. In this work we want to study Dubrovin-Frobenius manifolds near points that are not semisimple. Theorem 1.1.1 gives a good starting point for studying non-semisimple points and that is why we choose this alternative definition. In section 2.2 we show that this definition is equivalent to the original definition by Dubrovin. In the last section we introduce the most important object of this work, namely the deformed connection $\bar{\nabla}$. This connection is constructed by first defining a 1-parameter family of flat connections on a Dubrovin-Frobenius manifold M and then extending it to a flat connection on a certain vector bundle over $\mathbb{P}^1 \times M$. The study of the deformed connection on a neighborhood of a non-semisimple point is the topic of next chapter. The deformed connection also induces a family of ordinary meromorphic differential equations on $\mathbb{P}^1$. The monodromy data of this family is the topic of subsequent chapters of this work. All the material of this chapter can be found in [14],[11] and [12].

2.1 General Theory

We start with the most important definition.

Definition 2.1.1. A *Dubrovin-Frobenius* manifold is a quintuple $(M, \circ, e, E, \eta)$ where $(M, \circ, e, E)$ is an F-manifold with Euler vector field E and $\eta \in Sym^2\mathcal{T}_M^*$ is a metric such that

1. For all vector fields u, v, w we have $\eta(u \circ v, w) = \eta(u, v \circ w)$ (Compatibility of the multiplication and the metric).

2. The Euler vector field satisfies $\mathcal{L}_E\eta = (2-d)\eta$ for some $d \in \mathbb{C}$ (Conformality).

3. The unit is e flat, namely $\nabla e = 0$ where ∇ is the Levi-Civita connection of η.

4. The metric η is flat.

Example 2.1.1. On the F-manifold A_1 we can define a Dubrovin-Frobenius manifold structure of charge $d = 0$ by setting $\eta(\partial_t, \partial_t) = 1$. All the axioms of a Dubrovin-Frobenius manifold are verified immediately.
Consider $I_2(n)$ F-manifold with the Euler vector field

$$E = t_1\partial_{t_1} + \tfrac{2}{n}t_2\partial_{t_2}.$$

Define a metric η by $\eta(\partial_{t_1}, \partial_{t_1}) = \eta(\partial_{t_2}, \partial_{t_2}) = 0$ and $\eta(\partial_{t_1}, \partial_{t_2}) = 1$. For the compatibility of the multiplication and the metric, the only non-trivial thing we need to check is

$$\eta(\partial_{t_1} \circ \partial_{t_2}, \partial_{t_2}) = 0 = t_2^{n-2}\eta(\partial_{t_1}, \partial_{t_1}) = \eta(\partial_{t_1}, \partial_{t_2} \circ \partial_{t_2}).$$

To check conformality note that $(\mathcal{L}_E\eta)(\partial_{t_1}, \partial_{t_1}) = (\mathcal{L}_E\eta)(\partial_{t_2}, \partial_{t_2}) = 0$ and

$$(\mathcal{L}_E\eta)(\partial_{t_1}, \partial_{t_2}) = -\eta([E, \partial_{t_1}], \partial_{t_2}) - \eta(\partial_{t_1}, [E, \partial_{t_2}]) = 1 + \frac{2}{n}.$$

Thus we see that $\mathcal{L}_E\eta = (2 - d)\eta$ with $d = \frac{n-2}{n}$. Since on the coordinate vector fields $\partial_{t_1}, \partial_{t_2}$ the metric is constant we get that η is flat and $\partial_{t_1} = e$ is ∇-flat.
More generally, on $\mathbb{C}^2$ given $d \in \mathbb{C} \setminus \{ 1 \}$ we can define an associative multiplication by

$$\begin{aligned} \partial_{t_1} &= e \\ \partial_{t_2} \circ \partial_{t_2} &= t_2^{2d/(1-d)}. \end{aligned}$$

If we let $E = t_1\partial_{t_1} + (1 - d)t_2\partial_{t_2}$ and η the same as before, then we get a two dimensional Dubrovin-Frobenius manifold of charge d. Note that the multiplication is not defined on $\{ t_2 = 0 \}$ unless $\frac{2d}{1-d} \in \mathbb{N}$ in which case we recover the Dubrovin-Frobenius manifolds $I_2(n)$ defined above. In other words, if $\frac{2d}{1-d} \notin \mathbb{N}$ then the caustic of the above Dubrovin-Frobenius manifolds is empty.

The following lemma will be useful.

Lemma 2.1.1. *Let M be a manifold with metric η and Levi-Civita connection ∇. A vector field e is flat if and only if $\mathcal{L}_e\eta = 0$ and the 1-form $\eta(e, -)$ is closed.*

Proof. For any vector fields u, v we have

$$(\mathcal{L}_e\eta)(u, v) = e(\eta(u, v)) - \eta([e, u], v) - \eta(u, [e, v]).$$

By the compatibility of the metric $\nabla\eta = 0$, we get $e(\eta(u, v)) = \eta(\nabla_e u, v) + \eta(u, \nabla_e v)$ so that

$$(\mathcal{L}_e\eta)(u, v) = \eta(\nabla_e u, v) + \eta(u, \nabla_e v) - \eta([e, u], v) - \eta(u, [e, v]).$$

Since the connection ∇ is torsionless we get $\nabla_e = \nabla e + [e, -]$ so

$$(\mathcal{L}_e \eta)(u, v) = \eta(\nabla_u e, v) + \eta(u, \nabla_v e). \tag{2.1.1}$$

We also have

$$d(\eta(e, -))(u, v) = u(\eta(e, v)) - v(\eta(e, u)) - \eta(e, [u, v]),$$

again by the compatibility and torsionless of ∇ we get

$$d(\eta(e, -))(u, v) = \eta(\nabla_u e, v) - \eta(u, \nabla_v u). \tag{2.1.2}$$

Now suppose $\nabla e = 0$, then equalities (2.1.1) and (2.1.2) imply $\mathcal{L}_e \eta = 0$ and $d(\eta(e, -)) = 0$. Conversely adding up (2.1.1) and (2.1.2) we get $\eta(\nabla_u e, v) = 0$ for all vector fields u, v. Since η is non-degenerate we conclude $\nabla_u e = 0$ for all vector fields u so that $\nabla e = 0$ and e is flat. □

Proposition 1.2.1 says that the integral submanifolds of the idempotents are massive F-manifolds. Note that if the Euler vector field E is tangent to L and the metric η when restricted to L is non-degenerate, then $(L, \circ, e, E, \eta|_L)$ satisfies all the axioms of a Dubrovin-Frobenius manifold but the last one.

Corollary 2.1.1. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold and suppose that $p \in M$ we have $T_pM = \oplus_{k=1}^{l} (T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotents $\pi_1, \dots, \pi_l$ passing through p. Then $(L, \circ, e, E, \eta|_L)$ satisfies all the axioms of Dubrovin-Frobenius manifold except possibly for the flatness of $i^*\eta$. Moreover, if M is 3-dimensional and the caustic K is generically semisimple then $(K, \circ, e, E, \eta|_K)$ is a Dubrovin-Frobenius manifold.*

Proof. The only thing that needs to be proven is the statement about the 3-dimensional Dubrovin-Frobenius manifold. Let $g = \eta|_K$ and let $\tilde{\nabla}$ denote the Levi-Civita connection of g. Since $\tilde{\nabla} e = 0$ by the previous lemma we get $\mathcal{L}_e g = \tilde{\nabla}_e g = 0$. Call $\partial_1 = e$ and pick a vector field ∂_2 such that $[\partial_1, \partial_2] = 0$. Then $\mathcal{L}_e g = 0$ implies $\partial_1 g_{ij} = 0$ so that the components of the metric in this basis are constant in the direction of the unit vector field. Since the Christoffel symbols are functions of the metric and its derivatives they are also constant along the unit vector field. Now $[\tilde{\nabla}_{\partial_1}, \tilde{\nabla}_{\partial_2}] = 0$ because $\tilde{\nabla}_e = 0$. Finally

$$[\tilde{\nabla}_{\partial_1}, \tilde{\nabla}_{\partial_2}] = \partial_1 \Gamma^1_{22} \partial_1 + \partial_1 \Gamma^2_{22} \partial_2 = 0.$$

□

Example 2.1.2. Let us consider the Dubrovin-Frobenius manifold M associated with the singularity A_n. This manifold consists of the polynomials of the form

$$F(a; z) = z^{n+1} + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a = (a_0, \dots, a_{n-1}) \in \mathbb{C}^n$. This manifold is an affine space modeleled on the vector space of polynomials of degree at most $n-1$. This means that we can identify the tangent space to any point $a \in M$ with the space of polynomials of degree at most $n-1$. Given two polynomials $f, g \in T_aM$ the multiplication is defined by

$$f \circ g := fg \bmod \left.\frac{\partial F}{\partial z}\right|_a.$$

If we write $\frac{\partial F}{\partial z} = (n+1)\Pi_{i=1}^n (z - \alpha_i)$ then one can easily check that the polynomials

$$e_i := \frac{1}{z - \alpha_i} \frac{\partial F}{\partial z}$$

satisfy $e_i \circ e_j = \delta_{ij}\lambda_i e_i$ and therefore they are multiples of the orthogonal idempotents. Therefore the caustic K consist of the points a such that the polynomial $\frac{\partial F}{\partial z}$ has a double root. The set of points where $\frac{\partial F}{\partial z}$ has only a double root and all other simple is an open set inside the caustic. In this open set the polynomials e_i, with α_i a simple root, still are multiples of orthogonal idempotents π_i; we have $n-2$ of them, say $\pi_3, \dots, \pi_n$. But we have another orthognal idempotent given by $e - \pi_3 - \dots - \pi_n$. By proposition **??**, the caustic is a massive F-manifold. Note that we can apply the proposition again, indeed, the caustic contains the locus of points $\tilde{K}$ such that the polynomial $\frac{\partial F}{\partial z}$ has a triple root and all the other roots simple. The same argument as before shows that along $\tilde{K}$ we have $n-2$ orthogonal idempotents. Continuing in this way we arrive at a 2-dimensional F-manifold, the locus of points where $\frac{\partial F}{\partial z}$ has a root of multiplicity $n-1$ and a simple root. By the corollary this surface is a Dubrovin-Frobenius manifold.

2.2 Potentiality and Dubrovin's Definition

In this section we show that the definition of a Dubrovin-Frobenius manifold always gives a Dubrovin-Frobenius manifold as Dubrovin defined them. The converse is also true (see [14]).

Lemma 2.2.1. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold and consider the tensor $A(u, v, w) = \eta(u \circ v, w)$. Then ∇A is symmetric in its four arguments.*

Proof. Let us look back at condition (1.1.1)

$$\mathcal{L}_{u \circ v} \circ = u \circ (\mathcal{L}_v) \circ + v \circ (\mathcal{L}_u \circ).$$

Using the torsion freeness of the connection ∇ we can write

$$\begin{aligned}(\mathcal{L}_{u\circ v}\circ)(w,z) &= \mathcal{L}_{u\circ v}(w\circ z) - (\mathcal{L}_{u\circ v}w)\circ z - w\circ(\mathcal{L}_{u\circ v}z)\\ &= \nabla_{u\circ v}(w\circ z) - \nabla_{z\circ w}(u\circ v) - (\nabla_{u\circ v}w - \nabla_w u\circ v)\circ z\\ &= -w\circ(\nabla_{u\circ v}z - \nabla_z u\circ v)\\ &= \nabla_{u\circ v}(w\circ z) - (\nabla_{u\circ v}w)\circ z - w\circ(\nabla_{u\circ v}z)\\ &- \nabla_{z\circ w}(u\circ v) + (\nabla_w u\circ v)\circ z + w\circ(\nabla_z u\circ v)\\ &= \nabla\circ(u\circ v,w,z) - \nabla_{z\circ w}(u\circ v) + (\nabla_w u\circ v)\circ z + w\circ(\nabla_z u\circ v)\,.\end{aligned}$$

Similarly

$$\begin{aligned}(u\circ(\mathcal{L}_v))(w,z) = u\circ[\nabla_v(w\circ z) - \nabla_{w\circ z}v\\ - (\nabla_v w - \nabla_w v)\circ z - w\circ(\nabla_v z - \nabla_z v)]\end{aligned} \quad (2.2.1)$$

$$\begin{aligned}(v\circ(\mathcal{L}_u))(w,z) = v\circ[\nabla_u(w\circ z) - \nabla_{w\circ z}u\\ - (\nabla_u w - \nabla_w u)\circ z - w\circ(\nabla_u z - \nabla_z u)]\end{aligned} \quad (2.2.2)$$

Hence the torsion freeness of ∇ allows us to write the integrability condition (1.1.1) as

$$\begin{aligned}0 = (\mathcal{L}_{u\circ v}\circ - u\circ(\mathcal{L}_v\circ) - v\circ(\mathcal{L}_u\circ))(w,z) =\\ \nabla\circ(u\circ v,w,z) - u\circ\nabla\circ(v,w,z) - v\circ\nabla\circ(u,w,z)\\ - \nabla\circ(w\circ z,u,v) + w\circ\nabla\circ(z,u,v) + z\circ\nabla\circ(w,u,v).\end{aligned} \quad (2.2.3)$$

Compatibility of the metric gives

$$\begin{aligned}\nabla A(u,v,w,z) &= u\eta(v\circ w,z) - \eta(\nabla_u v\circ w,z) - \eta(v\circ\nabla_u w,z) - \eta(v\circ w,\nabla_u z)\\ &= \eta(\nabla_u(v\circ w) - \nabla_u v\circ w - v\circ\nabla_u w,z) = \eta(\nabla\circ(u,v,w),z).\end{aligned}$$

Using this and taking the inner product of (2.2.3) with the unit vector field e gives

$$\begin{aligned}0 = \nabla A(u\circ v,w,z,e) - \nabla A(v,w,,z,u) - \nabla A(u,w,z,v)\\ \nabla A(w\circ z,u,v,e) + \nabla A(z,u,v,w) + \nabla A(w,u,v,z).\end{aligned} \quad (2.2.4)$$

But compatibility of ∇ and flatness of e give

$$\nabla A(x,y,z,e) = x\eta(y,z) - \eta(\nabla_x y,z) - \eta(y,\nabla_x z) = 0.$$

Therefore we conclude

$$\nabla A(z,u,v,w) - \nabla A(v,w,z,u) = \nabla A(w,u,v,z) - \nabla A(u,w,z,v).$$

The tensor A is symmetric so that ∇A is symmetric in the last three entries. Hence the left hand side is symmetric in u and w. On the other hand

$$\begin{aligned}\nabla A(w,u,v,z) - \nabla A(u,w,z,v) &= \eta(\nabla \circ (w,u,z) - \nabla \circ (u,w,z), v)\\ &= -\eta(\nabla \circ (u,w,z) - \nabla \circ (w,u,z), v)\\ &= -(\nabla A(u,w,v,z) - \nabla A(w,u,z,v)).\end{aligned}$$

So that the left hand side is antisymmetric in u, w. Thus both sides must vanish and we have

$$\nabla A(u,w,v,z) = \nabla A(w,u,z,v) = \nabla A(w,u,v,z).$$

□

Remark 2.2.1. Consider some flat coordinates $(t^1, \dots, t^m)$ on an open set $u \subset M$ and let $\partial_{t^i} \circ \partial_{t^j} = \sum_k c_{ij}^k \partial_{t^k}$ and define $c_{ijk} := \sum_s c_{ij}^s \eta_{sk}$. Then

$$\nabla A(\partial_{t^l}, \partial_{t^i}, \partial_{t^j}, \partial_{t^k}) = \frac{\partial c_{ijk}}{\partial t^l}.$$

By the symmetry of ∇A we get

$$\frac{\partial c_{ijk}}{\partial t^l} = \frac{\partial c_{ijl}}{\partial t^k} = \frac{\partial c_{ilk}}{\partial t^j}.$$

Moreover, if $\eta^{\alpha\beta}$ denote the components of the inverse matrix of η we have $c_{ij}^k = \sum_s c_{ijs}\eta^{sk}$ but then

$$\frac{\partial c_{ij}^k}{\partial t^l} = \sum_{s=1}^m \frac{\partial c_{ijs}}{\partial t^l}\eta^{sk} = \sum_{s=1}^m \frac{\partial c_{ils}}{\partial t^j}\eta^{sk} = \frac{\partial c_{il}^k}{\partial t^j}.$$

Now

$$\nabla \circ (\partial_{t^l}, \partial_{t^i}, \partial_{t^j}) = \sum_{k=1}^m \frac{\partial c_{ij}^k}{\partial t^l}\partial_{t^k}$$

so we conclude that the tensor $\nabla\circ$ is symmetric in its three arguments.

Lemma 2.2.2. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold of charge d. For any vector fields u, v, w let $A(u,v,w) = \eta(u \circ v, w)$ then*

$$\mathcal{L}_E A = (3-d)A.$$

Proof. We have

$$\begin{aligned}(\mathcal{L}_E A)(u,v,w) &= E\eta(u\circ v, w) - \eta([E,u]\circ v, w) - \eta(u \circ [E,v], w) - \eta(u\circ v, [E,w])\\ &= (\mathcal{L}_E\eta)(u\circ v, w) + \eta([E, u\circ v] - [E,u]\circ v - u\circ[E,v], w)\\ &= (2-d)\eta(u\circ v, w) + \eta((\mathcal{L}_E\circ)(u,v), w) = (3-d)\eta(u\circ v, w)\\ &= (3-d)A(u,v,w).\end{aligned}$$

□

Proposition 2.2.1. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold then*

$$\nabla\nabla E = 0.$$

Proof. On flat coordinates $(t^1, \cdots, t^m)$ the components η_{ij} of the metric are constant. Writing $E = \sum_s E^s \partial_{t^s}$ the conformal condition $\mathcal{L}_E \eta = (2-d)\eta$ reads

$$(\mathcal{L}_E \eta)(\partial_{t^i}, \partial_{t^j}) = \sum_{s=1}^{m} \partial_{t^i} E^s \eta_{sj} + \partial_{t^i} E^s \eta_{si} = (2-d)\eta_{ij}.$$

Setting $E_\alpha := \sum_s E^s \eta_{s\alpha}$ (so that $E^\beta = \sum_s E_s \eta^{s\beta}$ where η^{rs} are the components of the inverse matrix of η) and taking the derivative with respect to t^k gives

$$\partial_{t^k}\partial_{t^i} E_j = -\partial_{t^k}\partial_{t^j} E_i.$$

This equation holds for any indices i, j, k, hence

$$\partial_{t^k}\partial_{t^j} E_i = -\partial_{t^k}\partial_{t^i} E_j = -\partial_{t^i}\partial_{t^k} E_j = \partial_{t^i}\partial_{t^j} E_k = \partial_{t^j}\partial_{t^i} E_k = -\partial_{t^j}\partial_{t^k} E_i.$$

Therefore $\partial_{t^k}\partial_{t^j} E_i = 0$ and since E^i is a $\mathbb{C}$-linear combination of the E_j we get $\partial_{t^k}\partial_{t^j} E^i = 0$. In flat coordinates we have

$$\nabla\nabla E = \sum_{p,r,s} \partial_{t^p}\partial_{t^r} E^s dt^p \otimes dt^r \otimes \partial_{t^s} = 0.$$

$\square$

Theorem 2.2.1. *Let $(M, \circ, e, E, \eta)$ be a Frobenius manifold of charge d and let $(t^1, \ldots, t^m)$ be flat coordinates on a simply connected open set $U \subset M$ with $\partial_{t^1} = e$. Let $\partial_{t^i} \circ \partial_{t^j} = \sum_k c_{ij}^k \partial_{t^k}$ and $c_{ijk} = \sum_s c_{ij}^s \eta_{sk}$. Then there exists a holomorphic function $F\colon U \to \mathbb{C}$ such that*

1. $c_{ijk} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}$
2. $\eta_{ij} = c_{1ij}$.
3. $c_{ij}^k = \sum_{s=1}^m c_{ijs}\eta^{sk}$ *where $\eta^{\alpha\beta}$ are the components of the inverse matrix of η.*
4. $E(F) - (3-d)F$ *is a quadratic polynomial.*

Proof. By the preceding remark

$$\frac{\partial c_{ijk}}{\partial t^l} = \frac{\partial c_{ijl}}{\partial t^k} = \frac{\partial c_{ilk}}{\partial t^j},$$

since U is simply connected there exists functions b_{ij} such that

$$\frac{\partial b_{ij}}{\partial t^k} = c_{ijk}.$$

But

$$\frac{\partial b_{ij}}{\partial t^k} = c_{ijk} = c_{jik} = \frac{\partial b_{ji}}{\partial t^k}$$

and hence we can choose functions b_{ij} such that $b_{ij} = b_{ji}$. Now

$$\frac{\partial b_{ij}}{\partial t^k} = c_{ijk} = c_{ikj} = \frac{\partial b_{ik}}{\partial t^j}$$

so we can find functions a_i such that

$$\frac{\partial a_i}{\partial t^j} = b_{ij}.$$

By the above choice

$$\frac{\partial a_i}{\partial t^j} = b_{ij} = b_{ji} = \frac{\partial a_j}{\partial t^i}$$

so we can find a function F such that

$$\frac{\partial F}{\partial t^i} = a_i.$$

By construction

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k} = \frac{\partial^2 a_i}{\partial t^j \partial t^k} = \frac{\partial b_{ij}}{\partial t^k} = c_{ijk}.$$

This proves the first item. For the second one note that

$$\partial_{t^1} \circ \partial_{t^i} = \sum_{s=1}^{m} c_{ij}^s \partial_{t_s} = \partial_{t^i}$$

and therefore $c_{1i}^s = \delta_i^s$. Hence

$$c_{1ij} = \sum_{s=1}^{m} c_{1i}^s \eta_{sj} = \eta_{ij}.$$

For the third part we have

$$\sum_{s=1}^{m} c_{ijs}\eta^{sk} = \sum_{s=1}^{m}\sum_{r=1}^{m} c_{ij}^r \eta_{rs}\eta^{sk} = \sum_{r=1}^{m} c_{ij}^r \delta_r^k = c_{ij}^k.$$

For the last part write $E = \sum_s E^s \partial_{t^s}$. In flat coordinates the condition $\nabla\nabla E = 0$ is simply $\partial_{t^i}\partial_{t^j} E^s = 0$. Using this one gets

$$\partial_{t^i}\partial_{t^j}\partial_{t^k} E(F) = \sum_{s=1}^{m} E^s \partial_{t^s} c_{ijk} + (\partial_{t^i} E^s) c_{jks} + (\partial_{t^j} E^s) c_{iks} + (\partial_{t^k} E^s) c_{ijs}.$$

Computing $(\mathcal{L}_E A)(\partial_{t^i}, \partial_{t^j}, \partial_{t^k})$ we get $\partial_{t^i}\partial_{t^j}\partial_{t^k} E(F) = (\mathcal{L}_E A)(\partial_{t^i}, \partial_{t^j}, \partial_{t^k})$. We also have $A(\partial_{t^i}, \partial_{t^j}, \partial_{t^k}) = c_{ijk}$. Using lemma 2.2.2 we get

$$\partial_{t^i}\partial_{t^j}\partial_{t^k} E(F) = (3-d) A(\partial_{t^i}, \partial_{t^j}, \partial_{t^k}) = (3-d)\partial_{t^i}\partial_{t^j}\partial_{t^k} F.$$

Therefore

$$\partial_{t^i}\partial_{t^j}\partial_{t^k}(E(F) - (3-d)F) = 0.$$

□

The function F will be called *potential* of the Dubrovin-Frobenius manifold.

Example 2.2.1. Let us write down the potential for the two-dimensional Dubrovin-Frobenius manifolds of example 2.1.1 when $d \neq -1, 1, 3$. We have that

$$\begin{array}{lll} c_{11}^1 = 1 & c_{12}^1 = 0 & c_{22}^1 = t_2^{2d/(1-d)} \\ c_{11}^2 = 0 & c_{12}^2 = 1 & c_{22}^2 = 0. \end{array}$$

From the equality $\frac{\partial^3 F}{\partial_i \partial_j \partial_k} = c_{ijk} = \sum_s \eta_{is} c_{jk}^s$ we get

$$\begin{array}{ll} \dfrac{\partial^3 F}{\partial t_1 \partial t_1 \partial t_1} = 0 & \dfrac{\partial^3 F}{\partial t_1 \partial t_1 \partial t_2} = 1 \\ \dfrac{\partial^3 F}{\partial t_1 \partial t_2 \partial t_2} = 0 & \dfrac{\partial^3 F}{\partial t_2 \partial t_2 \partial t_2} = t_2^{2d/(1-d)}. \end{array}$$

From the two equalities of the first line we get

$$\frac{\partial F}{\partial t_1} = t_1 f(t_2) + g(t_2)$$

for some functions f, g. From the two equalities of the second line we get

$$\frac{\partial F}{\partial t_2} = \frac{(1-d)^2}{2(1+d)} t_2^{2/(1-d)} + h(t_1).$$

Comparing the last two equations we conclude $f(t_2) = t_2$, $h(t_1) = \frac{t_1^2}{2}$ and we can put $g = 0$. Putting this together we arrive at

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + \frac{(1-d)^3}{2(1+d)(3-d)} t_2^{(3-d)/(1-d)}.$$

In particular for the $I_2(n)$ Dubrovin-Frobenius manifolds we get

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+\frac{1}{(n+1)n(n-1)}t_2^{n+1}.$$

When $d=3$ an identical procedure yields

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+\frac{1}{2}\log(t_2).$$

For $d=-1$ we obtain

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+\frac{t_2^2}{2}\log(t_2)-t_2^2,$$

but note that the functions c_{ijk} and therefore c_{ij}^k don't chance if we add a polynomial of degree at most two. Hence, when $d=-1$ we can take

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+\frac{t_2^2}{2}\log(t_2)$$

as the potential. When $d=1$ the potential

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+e^{t_2}$$

defines a Dubrovin-Frobenius manifold. Indeed, the metric η is the same as before, the multiplication is given by $\partial_{t_2}\circ\partial_{t_2}=e^{t_2}\partial_{t_1}$. The Euler vector field must be of the form $E=t_1\partial_{t_1}+f(t_1,t_2)\partial_{t_2}$. Imposing that $E(F)-F$ must be a polynomial of degree at most two we get $f(t_1,t_2)=2$. With this one gets

$$(\mathcal{L}_E\eta)(\partial_{t_1},\partial_{t_2})=\eta(\partial_{t_1},\partial_{t_2})=(2-1)\eta(\partial_{t_1},\partial_{t_2}).$$

We now find a normal form for the potential F similar to the one found in the previous example. We suppose that the eigenvalues of the operator ∇E are simple.

Proposition 2.2.2. *Let $(M,\circ,e,E,\eta)$ be a Dubrovin-Frobenius manifold of charge d and suppose that the eigenvalues of ∇E are simple. Then there is a flat coordinate system $(t_1,\dots,t_m)$ with $e=\partial_{t_1}$ such that the potential F is written*

$$F=\frac{1}{2}t_1^2t_m+\frac{t_1}{2}\sum_{k=2}^{m-1}t_kt_{m+1-k}+f(t_2,\dots,t_m)$$

if $d\neq 0$ or $\eta(e,e)=0$ and

$$F=\frac{c}{6}t_1^3+\frac{t_1}{2}\sum_{k=2}^{m-1}t_kt_{m+1-k}+f(t_2,\dots,t_m)$$

if $d=0$ and $\eta(e,e)=c\neq 0$.

Proof. Since $\nabla\nabla E = 0$ the eigenvalues and eigenvectors of the endomorphism ∇E are ∇-flat. For the eigenvalues this just mean that they are constant. Let $\partial_{t_1}, \ldots, \partial_{t_m}$ be a basis of eigenvectors with $\nabla_{\partial_{t_i}} E = d_i \partial_{t_i}$. On this basis we must have

$$E = \sum_{s=1}^{m} (d_i t_i + r_i)\partial_{t_i},$$

and $[E, \partial_{t_i}] = -d_i \partial_{t_i}$. From conformality we obtain

$$(2-d)\eta_{ij} = (\mathcal{L}_E \eta)(\partial_{t_i}, \partial_{t_j}) = (d_i + d_j)\eta_{ij}.$$

Therefore, if $d_i + d_j \neq \eta_{ij}$ then $\eta_{ij} = 0$.
The metric η is non-degenerate so given an eigenvector ∂_{t_i} of ∇E there must exist at least another eigenvector ∂_{t_j} such that $\eta_{ij} \neq 0$; since the eigenvalues of ∇E are simple there exists at most one of them, namely the eigenvector with eigenvalue $2 - d - d_i$.
Since $e = (\mathcal{L}_E \circ)(e, e) = -[E, e]$ and $\nabla_e E = [e, E]$ we get that the unit vector field is always an eigenvector of ∇E. We set $e = \partial_{t_1}$.
Now suppose $d \neq 0$. Conformality gives

$$(2-d)\eta(e,e) = 2\eta(e,e)$$

and hence we conclude that $\eta(e, e) = 0$. We can now order and normalize the eigenvectors ∂_{t_i} in such a way that the matrix of the metric η is antidiagonal (there exists at most one eigenvector ∂_{t_i} such that $\eta(\partial_{t_i}, \partial_{t_i}) \neq 0$) *i.e.*

$$\eta(\partial_{t_i}, \partial_{t_j}) = \delta_{i,m+1-j}.$$

We have that

$$1 = \eta_{1m} = \partial_{t_1}\partial_{t_1}\partial_m F$$

so $\partial_{t_1}\partial_{t_1} F = t_m + h(t_1, \ldots, t_{m-1})$. For $i \neq m$ we have

$$0 = \eta_{1i} = \partial_{t_1}\partial_{t_1}\partial_{t_i} F = \partial_i h$$

and hence $h(t_1, \ldots, t_{m-1})$ is a constant which we take to be zero. Thus, $\partial_{t_1}\partial_{t_1} F = t_m$ and $\partial_{t_1} F = t_1 t_m + f_1(t_2, \ldots, t_m)$. Now for all $i > 1$ we have

$$0 = \eta_{jm} = \partial_{t_1}\partial_{t_j}\partial_{t_m} F_{1jm} = \partial_{t_j}\partial_{t_m} f_1.$$

Hence the function $\partial_{t_m} f_1$ is a constant which we take to be zero and thus $f_1 = f_1(t_2, \ldots, t_{m-1})$ and

$$F_1 = t_1 t_m + f_1(t_2, \ldots, t_{m-2}).$$

We now have

$$1 = \eta_{2m-1} = \partial_{t_1}\partial_{t_2}\partial_{t_{m-1}} F = \partial_{t_2}\partial_{t_{m-1}} f_1$$

so $\partial_{t_2} f_1 = t_{m-1} + f_2(t_2, \ldots, t_{m-2})$. But for $1 < i < m-1$ we have

$$0 = \eta_{2i} = \partial_{t_1}\partial_{t_2}\partial_{t_i} = \partial_{t_i}\partial_{t_2} f_2$$

so $\partial_{t_2} f_2$ is a constant which we again take to be zero. Hence $\partial_{t_2} f_1 = t_{m-1}$, $f_1 = t_2 t_{m-1} + f_2(t_3, \ldots, t_{m-2})$ and $\partial_{t_1} F = t_1 t_m + t_2 t_{m-1} = f_2(t_3, \ldots, t_{m-1})$. For $2 < i$ we have

$$0 = \eta_{im-1} = \partial_{t_1}\partial_{t_i}\partial_{t_{m-1}} F = \partial_{t_i}\partial_{t_{m-1}} f_2$$

so just as before we can take $f_2 = f_2(t_3, \ldots, t_{m-2})$ and

$$\partial_{t_1} F = t_1 t_m + t_2 t_{m-1} + f_2(t_3, \ldots, t_{m-2}).$$

Continuing in the same way we get

$$F_1 = t_1 t_m + \frac{1}{2}\sum_{k=2}^{m-1} t_k t_{m+1-k}$$

so that

$$F = \frac{1}{2} t_1^2 t_m + \frac{t_1}{2}\sum_{k=2}^{m-1} t_k t_{m+1-k} + f(t_2, \ldots, t_m).$$

For the other case we would have $c = \partial_{t_1}\partial_{t_1}\partial_{t_1} F$ so $\partial_{t_1}\partial_{t_1} F = ct_1 + g(t_2, \ldots, t_m)$ but since $\eta_{1i} = 0$ for $i > 1$ we get $\partial_{t_1}\partial_{t_1} F = \frac{c}{2} t_1^2 + f_1(t_2, \ldots, t_m)$. Reasoning as before we get the desired result. □

On dimension 3 and when the charge $d \neq 0$ the potential is

$$F = \frac{1}{2} t_1^2 t_2 + \frac{1}{2} t_1 t_2^2 + f(t_2, t_3).$$

The multiplication table is given by

$$\begin{aligned}
\partial_{t_2} \circ \partial_{t_2} &= f_{,223}\partial_{t_1} + f_{,222}\partial_{t_2} + \partial_{t_3} \\
\partial_{t_2} \circ \partial_{t_3} &= f_{,233}\partial_{t_1} + f_{,223}\partial_{t_2} \\
\partial_{t_3} \circ \partial_{t_3} &= f_{,333}\partial_{t_1} + f_{,233}\partial_{t_2},
\end{aligned}$$

where $f_{,ijk}$ denotes the third partial derivative of f with respect to the variables t_i, t_j, t_k. For associativity to hold the equations

$$\begin{aligned}
(\partial_{t_2} \circ \partial_{t_2}) \circ \partial_{t_3} &= \partial_{t_2} \circ (\partial_{t_2} \circ \partial_{t_3}) \\
\partial_{t_2} \circ \partial_{t_3} \circ \partial_{t_3} &= \partial_{t_2} \circ \partial_{t_3} \circ \partial_{t_3}
\end{aligned}$$

must hold. Both of these equalities lead to the partial differential equation

$$\left(\frac{\partial^3 f}{\partial_{t_2}\partial_{t_2}\partial_{t_3}}\right)^2 = \frac{\partial^3 f}{\partial_{t_2}\partial_{t_2}\partial_{t_2}}\frac{\partial^3 f}{\partial_{t_2}\partial_{t_3}\partial_{t_3}} + \frac{\partial^3 f}{\partial_{t_3}\partial_{t_3}\partial_{t_3}}.$$

Example 2.2.2. The potentials

$$
\begin{aligned}
F_A &= \frac{1}{2}x^2z + \frac{1}{2}xy^2 - \frac{1}{16}y^2z^2 + \frac{1}{960}z^5 \\
F_B &= \frac{1}{2}x^2z + \frac{1}{2}xy^2 + \frac{1}{6}y^3z + \frac{1}{6}y^2z^3 + \frac{1}{210}z^7 \\
F_H &= \frac{1}{2}x^2z + \frac{1}{2}xy^2 + \frac{1}{6}y^3z^2 + \frac{1}{20}y^2z^5 + \frac{1}{3960}z^{11}
\end{aligned} \tag{2.2.5}
$$

define massive Dubrovin-Frobenius manifolds. These Dubrovin-Frobenius manifolds come from the orbit spaces of the Coxeter groups A_3, B_3 and H_3 (see [10]). The corresponding Euler vector fields are

$$
\begin{aligned}
E_A &= x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z} \\
E_B &= x\frac{\partial}{\partial x} + \frac{2}{3}y\frac{\partial}{\partial y} + \frac{1}{3}z\frac{\partial}{\partial z} \\
E_H &= x\frac{\partial}{\partial x} + \frac{3}{5}y\frac{\partial}{\partial y} + \frac{1}{5}z\frac{\partial}{\partial z}.
\end{aligned} \tag{2.2.6}
$$

Later we will analyze the caustic of these Dubrovin-Frobenius manifolds.

Example 2.2.3. The following family of examples exhibit three-dimensional Dubrovin-Frobenius manifolds such that at one irreducible component of the caustic K, the tangent space consists of only one irreducible algebra. That is, along this component we loose two idempotents and therefore the caustic is not semisimple. Instead the caustic is foliated by the integral curves of the unit vector field e which is the only idempotent along this component of the caustic. The potential is the function

$$F(x, y, z) = \frac{1}{2}x^2z + \frac{1}{2}xy^2 + y^4f(z)$$

for some function f of one variable. The Euler vector field is

$$E = x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}.$$

The multiplication table is given by

$$
\begin{aligned}
\partial_y \circ \partial_y &= 12y^2f'(z)\partial_x + 24yf(z)\partial_y + \partial_z \\
\partial_y \circ \partial_z &= 4y^3f''(z)\partial_x + 12y^2f'(z) \\
\partial_z \circ \partial_z &= y^4f'''(z)\partial_x + 4y^3f''(z)
\end{aligned}
$$

Notice that along $y = 0$ we have $\partial_z \circ \partial_z = 0$ so $y = 0$ is contained on the caustic. Along this component $E = x\partial_x$ and so the Euler vector field is tangent to $\{ y = 0 \}$. The vector ∂_z is also tangent to this component of the caustic and so this component is a globally nilpotennt F-manifold. Note that the Euler vector field is tangent to the integral submanifolds of the only idempotent $\pi_1 = e$. But since $\eta(e, e) = 0$, the metric restricted to this submanifolds is degenerate. More about this examples can be found on [11] appendix C or [4].

Remark 2.2.2. The associativity of the multiplication $\circ$ implies that the function F must satisfy a highly overdetermined system of partial differential equations known as the *WDVV equations* (Witten-Dijkgraaf-Verlinde-Verinde). Indeed, for all vector fields u, v, w we must have

$$\eta((u \circ v) \circ w, -) = \eta(u \circ (v \circ w), -).$$

In terms of the flat vector fields $\partial_{t^i}, i = 1, \dots, m$ and the function F the above equality becomes

$$\sum_{r,s} \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^r} \eta^{rs} \frac{\partial^3 F}{\partial t^s \partial t^k \partial t^l} = \sum_{r,s} \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^r} \eta^{rs} \frac{\partial^3 F}{\partial t^s \partial t^i \partial t^l}$$

2.3 The Deformed Connection

In this section we introduce one of the most important objects one can associate to a Dubrovin-Frobenius manifold, the deformed connection. In this section we merely state and proof its most important property, namely its flatness. In the next chapter we will study in detail the consecuences of its flatness. This was already done by Dubrovin on the semisimple loci, the novelty of the next chapter is that we drop the semisimple hypothesis.

Definition 2.3.1. Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold of dimension m and Levi-Civita connection ∇. We define a 1-parameter family of connections ∇^z in the following way: for each $z \in \mathbb{C}$ set and any vector fields u, v set

$$\nabla^z_u v := \nabla_u v + z(u \circ v). \tag{2.3.1}$$

Proposition 2.3.1. *For any $z \in \mathbb{C}$ the connection ∇^z is flat.*

Proof. Fix a coordinate system $(x^1, \dots, x^m)$. The connection matrices of the deformed connection are

$$\omega^z_i = \omega_i + z\partial_{x^i} \circ$$

where the matrices ω_i are the connection matrices of the flat connection ∇. We have

$$\frac{\partial \omega^z_i}{\partial x^j} - \frac{\partial \omega^z_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} + z\left(\partial_{x^j}(\partial_{x^i}\circ) - \partial_{x^i}(\partial_{x^j}\circ)\right)$$

and

$$\begin{aligned}\left[\omega^z_i, \omega^z_j\right] &= [\omega_i, \omega_j] + z([\omega_i, \partial_{x^j}\circ] + [\partial_{x^i}, \omega_j]) + z^2[\partial_{x^i}, \partial_{x^j}] \\ &= [\omega_i, \omega_j] + z([\omega_i, \partial_{x^j}\circ] + [\partial_{x^i}, \omega_j])\end{aligned}$$

because the multiplication $\circ$ is associative and commutative. Since the connection ∇ is flat, flatness of the connection ∇^z is equivalent to the equations (see appendix A):

$$\partial_{x^j}(\partial_{x^i}\circ) - \partial_{x^i}(\partial_{x^j}\circ) = [\omega_i, \partial_{x^j}\circ] + [\partial_{x^i}, \omega_j].$$

On flat coordinates $(t^1, \dots, t^n)$ the connection matrices ω_i vanish and therefore we only need to verify
$$\partial_{t^j}(\partial_{t^i}\circ) - \partial_{t^i}(\partial_{t^j}\circ) = 0.$$
We have
$$\partial_{t^i}\circ = c_{i\alpha}^{\beta} dt^{\alpha} \otimes \partial_{t^\beta} \qquad \text{and} \qquad \partial_{t^j}\circ = c_{j\alpha}^{\beta} dt^{\alpha} \otimes \partial_{t^\beta}$$
Be we just need to verify
$$\frac{\partial c_{i\alpha}^{\beta}}{\partial t^j} = \frac{\partial c_{j\alpha}^{\beta}}{\partial t^i} = 0.$$
But as lemma 2.2.1 and remark 2.2.1 show, this is always true on a Dubrovin-Frobenius manifold. □

We now extend the 1-parameter family of connections ∇^z to a meromorphic flat connection on a certain vector bundle over $\mathbb{P}^1 \times M$. We start by defining an endomorphism $\mu\colon \mathcal{T}_M \to \mathcal{T}_M$ which will be of crucial importance.
Since the Euler vector field is conformal, $\mathcal{L}_E\eta = (2-d)\eta$, we get that the endomorphism of the tangent sheaf
$$\mu := \frac{2-d}{2} Id - \nabla E \tag{2.3.2}$$
is η-antisymmetric. Indeed, compatibility and torsion freeness of ∇ imply (see equation (2.1.1))
$$(2-d)\eta(u,v) = (\mathcal{L}_E\eta)(u,v) = \eta(\nabla_u E, v) + \eta(u, \nabla_v E),$$
hence
$$\begin{aligned} 0 &= \eta\left(\frac{2-d}{2}u - \nabla_u E -, v\right) + \eta\left(u, \frac{2-d}{2}v - \nabla_v E\right) \\ &= \eta(\mu u, v) + \eta(u, \mu v). \end{aligned}$$
Proposition 2.2.1 says that $\nabla\nabla E = 0$ thus we get:

Proposition 2.3.2. *The endomorphism μ is flat i.e. $\nabla\mu = 0$.*

Consider the projection $\pi_M\colon \mathbb{P}^1 \times M \to M$. The importance of the endomorphism μ is that it allows us to define a flat connection $\bar{\nabla}$ on the bundle $\pi_M^* TM \to \mathbb{P}^1 \times M$. This connection extends the family of flat connections (2.3.1) on $TM \to M$.
The multiplication $\circ$, the endomorphism μ and the connection ∇ induce the same kind of objects on $\pi_M^*\mathcal{T}_M$ which should be denoted by $\pi_M^*\circ, \pi_M^*\mu$ and $\pi_M^*\nabla$. Abusing notation we will denote them by $\circ, \mu$ and ∇. Recall that $\mathcal{T}_{\mathbb{P}^1\times M} \cong \pi_{\mathbb{P}^1}^*\mathcal{T}_{\mathbb{P}^1} \oplus \pi_M^*\mathcal{T}_M$ and $\pi_M^*\mathcal{T}_M \cong \mathcal{O}_{\mathbb{P}^1\times M} \otimes_{\mathcal{O}_M} \pi_M^{-1}\mathcal{T}_M$. Let z be a global coordinate on $\mathbb{C}$.

Definition 2.3.2. Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold with Levi-Civita connection ∇. The *deformed connection* is defined in the following way: for $u \in \pi_M^* \mathcal{T}_M \leq \mathcal{T}_{\mathbb{P}^1 \times M}$ and $v \in \pi_M^* \mathcal{T}_M$ set

$$\bar{\nabla}_u v := \nabla_u v + z u \circ v.$$

The covariant derivative in the ∂_z direction is defined as

$$\bar{\nabla}_{\partial_z} v := \partial_z v + E \circ v - \frac{1}{z} \mu v.$$

Proposition 2.3.3. *The deformed connection $\bar{\nabla}$ on the vector bundle $\pi_M^* \mathcal{T}_M$ over $\mathbb{P}^1 \times M$ is flat.*

Proof. Let $(t^1, \ldots, t^m)$ be flat coordinates on a neighborhood $U \subset M$. The connection matrix $\bar{\omega}$ of $\bar{\nabla}$ is

$$\bar{\omega} = \sum_{i=1}^{m} \bar{\omega}_i dt^i + \bar{\omega}_z dz,$$

where

$$\bar{\omega}_i = z \partial_{t^i} \circ$$
$$\bar{\omega}_z = E \circ - \frac{1}{z} \mu.$$

Thanks to proposition 2.3.1 we only need to check that

$$\frac{\partial \bar{\omega}_i}{\partial z} - \frac{\partial \bar{\omega}_z}{\partial t^i} = [\bar{\omega}_i, \bar{\omega}_z].$$

Proposition 2.3.2 says that on flat coordinates $\partial_{t^i} \mu = 0$ and since $\mu = \frac{2-d}{2} Id - \nabla E$ we need to verify that

$$\partial_{t^i} \circ = \partial_{t^i}(E \circ) + [\partial_{t^i}, \nabla E].$$

Evaluating $\mathcal{L}_E \circ = \circ$ on ∂_{t^i} we get

$$\partial_{t^i} \circ = \sum_{\beta,\gamma} \left(\sum_{\alpha} E^\alpha \frac{\partial c_{i\gamma}^\beta}{\partial t^\alpha} - c_{i\gamma}^\alpha \frac{\partial E^\beta}{\partial t^\alpha} + c_{\alpha\gamma}^\beta \frac{\partial E^\alpha}{\partial t^i} + c_{i\alpha}^\beta \frac{\partial E^\alpha}{\partial t^\gamma} \right) dt^\gamma \otimes \partial_{t_\beta}.$$

We also have

$$\partial_{t^i}(E \circ) = \sum_{\beta,\gamma} \left(\sum_{\alpha} E^\alpha \frac{\partial c_{\alpha\gamma}^\beta}{\partial t^i} + c_{\alpha\gamma}^\beta \frac{\partial E^\alpha}{\partial t^i} \right) dt^\gamma \otimes \partial_{t^\beta}$$
$$[\partial_{t^i} \circ, \nabla E] = \sum_{\beta,\gamma} \left(\sum_{\alpha} c_{i\alpha}^\beta \frac{\partial E^\alpha}{\partial t^\gamma} - c_{i\gamma}^\alpha \frac{\partial E^\beta}{\partial t^\alpha} \right) dt^\gamma \otimes p_\beta.$$

On flat coordinates we have

$$\frac{\partial c_{\alpha\gamma}^\beta}{\partial t^i} = \frac{\partial c_{i\gamma}^\beta}{\partial t^\alpha}$$

(see lemma 2.2.1 and remark 2.2.1). Using this and comparing the previous equations we get the result. □

The flat connection $\bar{\nabla}$ is of fundamental importance. As shown by Dubrovin, on a small enough semisimple neighborhood W, the monodromy data (which will be defined in chapter 4) of the connection $\bar{\nabla}$ are constant and they determine the Dubrovin-Frobenius manifold structure on W. In the next chapter we will study the restriction of $\bar{\nabla}$ to the submanifolds L described in proposition 1.2.1. Except for the semisimple case we won't be able to recover the Dubrovin-Frobenius manifold structure on L but we will show that under some mild conditions the monodromy data are still constant.

Chapter 3

The Deformed Connection

In this chapter we describe some geometric properties of the deformed connection $\bar{\nabla}$ when restricted to the multiplication invariant submanifolds L of the caustic K (see proposition 1.2.1). We will always assume that the Euler vector field is tangent to this submanifolds and that the metric η, when restricted to these submanifolds is non-degenerate. In this case proposition 1.2.2 tells us that the endomorphism of multiplication by the Euler vector field is diagonalizable along these submanifolds. To warm up we start with the semisimple loci, this is the case that was originally studied by Dubrovin (see [11], [12]). Then we start studying the deformed connection when restricted to the submanifolds $L \subset K$. Many of the properties of $\bar{\nabla}$ remain the same but now the deformed connection induces flat connections on the irreducible algebras $\iota^*(\mathcal{T}_{M,p})_k$. These flat connections will play an important role in the next chapter.

3.1 The Semisimple locus

Suppose that the point $p \in M$ is semisimple. Since the caustic (the non-semisimple locus) is an hypersurface (see proposition 1.1.4) there is an open neighborhood $W \subset M$ which consists only of semisimple points. Therefore, in W there exists $\pi_1, \ldots, \pi_m$ vector fields which satisfy

$$\pi_i \circ \pi_j = \delta_{ij} \pi_i.$$

Since the metric satisfies $\eta(u \circ v, w) = \eta(u, v \circ w)$ we obtain that the idempotents $\pi_1, \ldots, \pi_m$ are orthogonal. We also have that the metric is non-degenerate so that $|\pi_i| = \eta(\pi_i, \pi_i) \neq 0$ and we can define the *normalized orthogonal idempotents* as

$$f_i := \frac{\pi_i}{|\pi_i|}.$$

Let $(u_1, \ldots, u_m)$ be canonical coordinates around $p \in M$ so that $\partial_{u_i} = \pi_i$. Using the basis $f_1, \ldots, f_m$ of $\pi_M^* \mathcal{T}_M$ and the coordinate system $(z, u_1, \ldots, u_m)$ of $\mathbb{P}^1 \times M$, let us write down the equations of flatness of the deformed connection $\bar{\nabla}$. First we compute the connection matrices,

$$\sum_{s=1}^{m}(\bar{\omega}_i)_j^s f_s = \bar{\nabla}_{\pi_i} f_j = \nabla_{\pi_i} f_j + z\pi_i \circ f_j = \sum_{s=1}^{m}(\omega_i)_j^s f_s + z\delta_{ij} f_i$$

and

$$\sum_{s=1}^{m}(\bar{\omega}_z)_j^s f_s = \bar{\nabla}_{\partial_z} f_j = E \circ f_j - \frac{1}{z}\mu f_j.$$

We have that $\pi_i \circ f_j = \delta_{ij} f_i$ so that the matrix of $\pi_i\circ$ on the basis $f_1, \ldots, f_m$ is $(E_i)_\beta^\alpha = \delta_i^\alpha \delta_\beta^i$. The condition $\eta(E \circ v, w) = \eta(v, E \circ w)$ says that the endomorphism $E\circ\colon \mathcal{T}_M \to \mathcal{T}_M$ is η-symmetric and as we have seen the endomorphism μ is η-antisymmetric. Therefore on the basis $f_1 \ldots, f_m$ the matrix U representing the endomorphism $E\circ$ is symmetric (the matrix of $E\circ$ is actually diagonal $U = diag(u_1, \ldots, u_m)$ where u_i are the canonical coordinates) and the matrix V representing the endomorphism μ is antisymmetric. Moreover, compatibility of the Levi-Civita connection ∇ of η says that in this basis the connection matrices ω_i of ∇ are antisymmetric. Hence we have

$$\begin{aligned}\bar{\omega}_i &= \omega_i + zE_i\\ \bar{\omega}_z &= U - \frac{1}{z}V.\end{aligned}$$

The equations of flatness

$$\frac{\partial \bar{\omega}_i}{\partial u_j} - \frac{\partial \bar{\omega}_j}{\partial u_i} = [\bar{\omega}_i, \bar{\omega}_j]$$

gives

$$[E_i, \omega_j] = [E_j, \omega_i] \tag{3.1.1}$$

because the connection ∇ is flat and $[E_i, E_j] = 0$. Since $[E_i, U] = 0$ the equation

$$\frac{\partial \bar{\omega}_i}{\partial z} - \frac{\partial \bar{\omega}_z}{\partial u_i} = [\bar{\omega}_i, \bar{\omega}_z]$$

gives

$$\begin{aligned}\frac{\partial V}{\partial u_i} &= [V, E_i]\\ [U, \omega_i] &= -[E_i, V].\end{aligned} \tag{3.1.2}$$

In the next section we will see that on the non-semisimple loci we also have equalities analogous to the ones above.

3.2 The Caustic

For a point $p \in K$ on the caustic the algebra $\mathcal{T}_{M,p}$ has less than $m = dimM$ idempotents. Correspondingly, the decomposition of $T_pM = \oplus_{k=1}^{l}(T_pM)_k$ into irreducible algebras has at least one algebra of dimension at least two. Let us introduce some notation that will be useful in the following. First we order the algebras $(T_pM)_k$ in some way. Then we define numbers k_i with $k \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, dim(T_pM)_k\}$ by

$$k_i = \sum_{j=1}^{k-1} dim(T_pM)_j + i.$$

We also define subsets $(k) \subset \{1, \ldots, m\}$ by

$$(k) = \left\{ k_i \in \mathbb{N} \;\Big|\; \sum_{j=1}^{k-1} dim(T_pM)_j < k_i \leq \sum_{j=1}^{k} dim(T_pM)_j \right\}.$$

Let L be the integral submanifold of the idempotent vector fields $\pi_1, \ldots, \pi_l$ passing through p and let $\iota\colon L \to M$ denote the inclusion. As proposition 1.2.1 shows this manifolds are multiplication invariant. We will assume that the metric $\iota^*\eta|_L$ is non-degenerate. This means that the normal bundle $\mathcal{N}_L$ of the submanifold L is transverse to $\mathcal{T}_L$ and $\iota^*\mathcal{T}_M = \mathcal{T}_L \oplus \mathcal{N}_L$. Since the bundles $\iota^*(\mathcal{T}_{M,p})_k$ are orthogonal between each other we can choose a unitary basis n_{k_i} of $\mathcal{N}_L$ $(i > 1)$ such that

$$n_{k_i} \in \iota^*(\mathcal{T}_{M,p})_k \cap \mathcal{N}_L.$$

Introducing the normalized idempotents

$$f_k := \frac{\pi_k}{|\pi_k|}, \quad k = 1 \ldots, l,$$

we obtain an orthonormal basis of $\iota^*\mathcal{T}_M$. We order this orthonormal basis in the following way:

$$f_{k_1} := f_k \quad f_{k_i} := n_{k_i} \text{ for } i > 1.$$

For example, in this orthonormal basis multiplication by $\pi_k\circ$ has a matrix E_k which on the diagonal block with indices belonging to (k) has an identity matrix of size $\#(k) = dim(T_pM)_k$ and all other entries are zero.

We will also assume that the Euler vector field E is tangent to the manifold L. In this case proposition 1.2.2 guarantees that the endomorphism $E\circ\colon \iota^*\mathcal{T}_M \to \iota^*\mathcal{T}_M$ is diagonalizable. Let $u_1 \ldots, u_l$ be canonical coordinates of L around p. Then, after a translation (see proposition 1.2.3)

$$E = \sum_{k=1}^{l} u_k \pi_k.$$

We have that $E \circ n_{i_k} = u_k n_{i_k}$ so that the matrix U representing the endomorphism $E\circ\colon \iota^*\mathcal{T}_M \to \iota^*\mathcal{T}_M$ on this orthonormal basis is diagonal.
By η-antisymmetry, the matrix V representing the endomorphism $\mu\colon \iota^*\mathcal{T}_M \to \iota^*\mathcal{T}_M$ is antisymmetric.

Using the inclusion $(id \times \iota)\colon \mathbb{P}^1 \times L \to \mathbb{P}^1 \times M$, we will now pullback the deformed connection $\bar\nabla$ on $\pi_M^*\mathcal{T}_M$ to $(\pi_M \circ (id \times \iota))^*\mathcal{T}_M$. Since the connection $\bar\nabla$ is flat the connection $(id\times\iota)^*\bar\nabla$ is also flat. The connection matrices in the orthonormal basis of $(\pi_M\circ(id\times\iota))^*\mathcal{T}_M$ we just constructed and the local coordinates $(z, u_1, \ldots, u_l)$ of $\mathbb{P}^1 \times L$ are

$$\begin{aligned} \bar\omega_i &= \omega_i + zE_i \\ \bar\omega_z &= U - \frac{1}{z}V. \end{aligned} \tag{3.2.1}$$

Flatness of $(id \times \iota)^*\bar\nabla$ gives

$$\begin{aligned} [U, \omega_i] &= -[E_i, V] \\ [E_i, \omega_j] &= [E_j, \omega_i] \\ \frac{\partial V}{\partial u_i} &= [V, \omega_i]. \end{aligned} \tag{3.2.2}$$

Lemma 3.2.1. *We have*

$$\begin{aligned} (\omega_i)^{p_\alpha}_{q_\beta} &= 0 && \text{if } (p) \neq (q) \text{ and } (p) \neq (i) \neq (q) \\ (u_i - u_p)(\omega_i)^{i_\alpha}_{p_\beta} &= V^{i_\alpha}_{p_\beta} && \text{if } (p) \neq (i) \\ (\omega_i)^{i_\alpha}_{j_\beta} &= -(\omega_j)^{i_\alpha}_{j_\beta}. \end{aligned} \tag{3.2.3}$$

Proof. The first two equations follow from the first equation of (3.2.2) and the last one follows from the second one. □

As we can see from equations (3.2.3) the flatness of the connection $\bar\nabla$ does not give any information about the diagonal blocks $(\omega_i)^{k_\alpha}_{k_\beta}$ of the connection matrices of the connection ∇. In the following proposition we show that the partial differential equation satisfied by the matrix V consists of $l + 1$ uncoupled systems of partial differential equations. Of these $l + 1$ systems, l consists of the l diagonal blocks of V and the last one consists of the off-diagonal entries of V. To write down the system of partial differential equations for the off-diagonal blocks first we need to solve the diagonal blocks. Theorem 3.2.1 gives a more geometrical interpretation of this fact.

Proposition 3.2.1. *For any $i,k=1,\dots,l$ and any $\alpha,\beta=1,\dots dim(T_pM)_k$ we have*

$$\frac{\partial V^{k_\alpha}_{k_\beta}}{\partial u_i}=\sum_{\substack{k_s\in(k)\\ k_s\neq k_\alpha,k_\beta}} V^{k_\alpha}_{k_s}(\omega_i)^{k_s}_{k_\beta}-(\omega_i)^{k_\alpha}_{k_s}V^{k_s}_{k_\beta}. \tag{3.2.4}$$

Let $i,j,k\in\{1,\dots,l\}$ with $i\neq j\neq k\neq i$. Then for any $\alpha\in\{1\dots,dim(T_pM)_i\}$ and $\beta\in\{1,\dots,dim(T_pM)_j\}$ we have

$$\begin{aligned}\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_k}&=\sum_{\substack{j_s\in(j)\\ j_s\neq j_\beta}} V^{i_\alpha}_{j_s}(\omega_k)^{j_s}_{j_\beta}-\sum_{\substack{i_s\in(i)\\ i_s\neq i_\alpha}}(\omega_k)^{i_\alpha}_{i_s}V^{i_s}_{j_\beta}\\ &\quad+\sum_{s\in(k)}\frac{u_i-u_j}{(u_i-u_k)(u_j-u_k)}V^{i_\alpha}_sV^s_{j_\beta}.\\ \frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_i}&=\sum_{\substack{i_s\in(i)\\ i_s\neq i_\alpha}}\left(\frac{V^{i_\alpha}_{i_s}}{u_j-u_i}-(\omega_i)^{i_\alpha}_{i_s}\right)V^{i_s}_{j_\beta}+\sum_{\substack{j_s\in(j)\\ j_s\neq j_\beta}}\left((\omega_i)^{j_s}_{j_\beta}-\frac{V^{j_s}_{j_\beta}}{u_j-u_i}\right)V^{i_\alpha}_{j_s}\\ &\quad-\sum_{r_s\notin(i)\cup(j)}\frac{V^{i_\alpha}_{r_s}V^{r_s}_{j_\beta}}{u_r-u_i}.\end{aligned}\tag{3.2.5}$$

Proof. The last equation of (3.2.2) says

$$\frac{\partial V^{k_\alpha}_{k_\beta}}{\partial u_i}=\sum_{s=1}^m V^{k_\alpha}_s(\omega_i)^s_{k_\beta}-(\omega_i)^{k_\alpha}_sV^s_{k_\beta}.$$

Suppose $(k)\neq(i)$, then we can split the sum in the indexes $s\in(k)$ and $s\notin(k)$. When $s\notin(k)$ the first equation of (3.2.3) tell us that the functions $(\omega_i)^s_{k_\beta}$ and $(\omega_i)^{k_\alpha}_s$ are zero unless $s\in(i)$. So when $(k)\neq(i)$ then

$$\frac{\partial V^{k_\alpha}_{k_\beta}}{\partial u_i}=\sum_{s\in(i)\cup(k)}^m V^{k_\alpha}_s(\omega_i)^s_{k_\beta}-(\omega_i)^{k_\alpha}_sV^s_{k_\beta}.$$

The second equation of (3.2.3) gives

$$V^{k_\alpha}_{i_s}(\omega_i)^{i_s}_{k_\beta}-(\omega_i)^{k_\alpha}_{i_s}V^{i_s}_{k_\beta}=(u_k-u_i)(\omega_i)^{k_\alpha}_{i_s}(\omega_i)^{i_s}_{k_\beta}+(u_i-u_k)(\omega_i)^{k_\alpha}_{i_s}(\omega_i)^{i_s}_{k_\beta}=0$$

therefore only the summands with $s\in(k)$ survive. The restriction $k_s\neq k_\alpha,k_\beta$ comes from antisymmetry of the matrices V and (ω_i). The other case $i=k$ is similar.
For the second equation we have

$$\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_k}=\sum_{s=1}^m V^{i_\alpha}_s(\omega_k)^s_{j_\beta}-(\omega_k)^{i_\alpha}_sV^s_{j_\beta}.$$

But $s \notin (j) \cup (k)$ implies $(\omega_k)^s_{j_\beta} = 0$ and $s \notin (i) \cup (k)$ gives $(\omega_k)^{i_\alpha}_s = 0$ so

$$\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_k} = \sum_{\substack{j_s \in (j) \\ j_s \neq j_\beta}} V^{i_\alpha}_{j_s} (\omega_k)^{j_s}_{j_\beta} - \sum_{\substack{i_s \in (i) \\ i_s \neq i_\alpha}} (\omega_k)^{i_\alpha}_{i_s} V^{i_s}_{j_\beta} + \sum_{k_s \in (k)} V^{i_\alpha}_{k_s} (\omega_k)^{k_s}_{j_\beta} - (\omega_k)^{i_\alpha}_{k_s} V^{k_s}_{j_\beta}.$$

The first equation of (3.2.3) gives

$$(\omega_k)^{k_s}_{j_\beta} = \frac{V^{k_s}_{j_\beta}}{u_j - u_k} \qquad\qquad (\omega_k)^{i_\alpha}_{k_s} = \frac{V^{i_\alpha}_{k_s}}{u_i - u_k}$$

and this implies

$$\sum_{k_s \in (k)} V^{i_\alpha}_{k_s} (\omega_k)^{k_s}_{j_\beta} - (\omega_k)^{i_\alpha}_{k_s} V^{k_s}_{j_\beta} = \sum_{k_s \in (k)} \frac{u_i - u_j}{(u_i - u_k)(u_j - u_k)} V^{i_\alpha}_{k_s} V^{k_s}_{j_\beta}.$$

Finally, for the last equation we have

$$\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_i} = \sum_{s=1}^{m} V^{i_\alpha}_s (\omega_i)^s_{j_\beta} - (\omega_i)^{i_\alpha}_s V^s_{j_\beta}.$$

If $s \notin (i) \cup (j)$ then $(\omega_i)^{i_\alpha}_s = 0$ so we can write the previous equation as

$$\begin{aligned}
\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_i} &= \sum_{i_s \in (i)} V^{i_\alpha}_{i_s} (\omega_i)^{i_s}_{j_\beta} - (\omega_i)^{i_\alpha}_{i_s} V^{i_s}_{j_\beta} + \sum_{j_s \in (j)} V^{i_\alpha}_{j_s} (\omega_i)^{j_s}_{j_\beta} - (\omega_i)^{i_\alpha}_{j_s} V^{j_s}_{j_\beta} \\
&= \sum_{r_s \notin (i) \cup (j)} (\omega_i)^{i_\alpha}_{r_s} V^{r_s}_{j_\beta}.
\end{aligned}$$

But

$$(\omega_i)^{i_s}_{j_\beta} = \frac{V^{i_s}_{j_\beta}}{u_j - u_i} \qquad\qquad (\omega_i)^{i_\alpha}_{j_s} = \frac{V^{i_\alpha}_{j_s}}{u_j - u_i}$$

and

$$(\omega_i)^{i_\alpha}_{r_s} = \frac{V^{i_\alpha}_{r_s}}{u_r - u_i}$$

so we get

$$\begin{aligned}
\frac{\partial V^{i_\alpha}_{j_\beta}}{\partial u_i} = &\sum_{\substack{i_s \in (i) \\ i_s \neq i_\alpha}} \left(\frac{V^{i_\alpha}_{i_s}}{u_j - u_i} - (\omega_i)^{i_\alpha}_{i_s} \right) V^{i_s}_{j_\beta} + \sum_{\substack{j_s \in (j) \\ j_s \neq j_\beta}} \left((\omega_i)^{j_s}_{j_\beta} - \frac{V^{j_s}_{j_\beta}}{u_j - u_i} \right) V^{i_\alpha}_{j_s} \\
&- \sum_{r_s \notin (i) \cup (j)} \frac{V^{i_\alpha}_{r_s} V^{r_s}_{j_\beta}}{u_r - u_r}.
\end{aligned}$$

□

Corollary 3.2.1. *Suppose $\#(k) = 2$. Then $V_{k_2}^{k_1} = -V_{k_1}^{k_2}$ is a constant.*

If we have a two dimensional irreducible algebra $\iota^*(\mathcal{T}_{M,p})_k$ in the decomposition of $\iota^* T_{M,p}$ then there is essentially one choice for the unitary normal n_{k_2} to L that lie on $\mathcal{N}_L \cap \iota^*(\mathcal{T}_{M,p})_k$. Correspondingly there is not too many freedom on the constant $V_{k_2}^{k_1}$. On theorem 5.1.2 (with a slight change of notation) we will compute explicitly this constant. We now show that each diagonal block of the curvature form Ω of the connection ∇ only depends on the entries of the same diagonal block of the connection matrices ω_i of the connection ∇.

Proposition 3.2.2. *Let $(u_1, \dots, u_l)$ be canonical coordinates around $p \in L$ and consider the orthonormal basis $f_{k_1} = f_k, f_{k_i} = n_{k_i}$ of $\iota^* \mathcal{T}_M$. Let Ω be the curvature form of the connection $\iota^* \nabla$ written down on these coordinates and this basis. Then for any $i, j, k \in \{1, \dots, l\}$ and $\alpha, \beta \in \{1, \dots, dim(T_p M)_k\}$ we have*

$$(\Omega_{ij})_{k_\beta}^{k_\alpha} = \frac{\partial (\omega_i)_{k_\beta}^{k_\alpha}}{\partial u_j} - \frac{\partial (\omega_j)_{k_\beta}^{k_\alpha}}{\partial u_i} - \sum_{k_s \in (k)} (\omega_i)_{k_s}^{k_\alpha} (\omega_j)_{k_\beta}^{k_s} - (\omega_j)_{k_s}^{k_\alpha} (\omega_i)_{k_\beta}^{k_s}. \tag{3.2.6}$$

Proof. The block diagonal entries of the curvature form Ω are

$$(\Omega_{ij})_{k_\beta}^{k_\alpha} = \frac{\partial (\omega_i)_{k_\beta}^{k_\alpha}}{\partial u_j} - \frac{\partial (\omega_j)_{k_\beta}^{k_\alpha}}{\partial u_i} - \sum_{s=1}^{m} (\omega_i)_{s}^{k_\alpha} (\omega_j)_{k_\beta}^{s} - (\omega_j)_{s}^{k_\alpha} (\omega_i)_{k_\beta}^{s}.$$

Suppose $(i) \neq (k) \neq (j)$ and divide the last sum in $s \in (k)$ and $s \notin (k)$. When $s \notin (k)$ and $(\omega_i)_s^{k_\alpha}, (\omega_i)_{k_\beta}^s \neq 0$ then $s \in (i) \neq (j)$ but then $(\omega_j)_{k_\beta}^s, (\omega_j)_s^{k_\alpha} = 0$ and the result follows. Now suppose $(i) = (k) \neq (j)$, then

$$(\Omega_{ij})_{i_\beta}^{i_\alpha} = \frac{\partial (\omega_i)_{i_\beta}^{i_\alpha}}{\partial u_j} - \frac{\partial (\omega_j)_{i_\beta}^{i_\alpha}}{\partial u_i} - \sum_{s=1}^{m} (\omega_i)_{s}^{i_\alpha} (\omega_j)_{i_\beta}^{s} - (\omega_j)_{s}^{i_\alpha} (\omega_i)_{i_\beta}^{s}.$$

In the last sum let us look at the indices $s \notin (i)$. Then $(\omega_j)_{i_\beta}^s, (\omega_j)_s^{i_\alpha} = 0$ unless $s \in (j)$. But then the last equation of (3.2.3) gives

$$(\omega_i)_s^{i_\alpha} (\omega_j)_{i_\beta}^s - (\omega_j)_s^{i_\alpha} (\omega_i)_{i_\beta}^s = -(\omega_j)_s^{i_\alpha} (\omega_j)_{i_\beta}^s + (\omega_j)_s^{i_\alpha} (\omega_j)_{i_\beta}^s = 0$$

so again the result follows. $\square$

Equation (3.2.6) has an incredible geometric consequence. First note that the function $\pi_k \circ \nabla \colon \iota^*(\mathcal{T}_{M,p})_k \to \Omega_L^1 \otimes \iota^*(\mathcal{T}_{M,p})_k$ (here $\pi_k \circ \colon \Omega_L^1 \otimes \iota^*(\mathcal{T}_{M,p})_k \to \Omega_L^1 \otimes \iota^*(\mathcal{T}_{M,p})_k$ is defined as $id \otimes \pi_k \circ$) define connections on the vector bundles $\iota^*(\mathcal{T}_{M,p})_k$. Indeed, for $v \in \iota^*(\mathcal{T}_{M,p})_k$ and $f \in \mathcal{O}_L$ we have

$$\begin{aligned}(\pi_k \circ \nabla) f v &= \pi_k \circ (df \otimes v + f \nabla v) \\ &= df \otimes \pi_k \circ v + f \pi_k \circ \nabla v \\ &= df \otimes v + f \pi_k \circ \nabla v\end{aligned}$$

because $\pi_k \circ v = v$ for $v \in \iota^*(\mathcal{T}_{M,p})_k$. Equation (3.2.6) says that these connections are flat.

Theorem 3.2.1. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold with Levi-Civita connection ∇. Assume that at $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotents $\pi_1, \ldots, \pi_l$ passing through p and denote $\iota\colon L \to M$ the inclusion. Suppose that*

1. *The Euler vector field E is tangent to L.*
2. *The inner product $\iota^*\eta|_{\mathcal{T}_L} \in Sym^2\mathcal{T}_L^*$ is non-degenerate.*

Then for each $k = 1, \ldots, l$ the connections $\pi_k \circ \nabla\colon \iota^*(\mathcal{T}_{M,p})_k \to \Omega_L^1 \otimes \iota^*(\mathcal{T}_{M,p})_k$ are flat.

Proof. Consider the orthonormal basis $f_{k_1} = \frac{\pi_k}{|\pi_k|}, f_{k_\alpha} = n_{k_\alpha}$. We have

$$\nabla f_{k_\alpha} = \sum_{s=1}^{m}\sum_{i=1}^{l}(\omega_i)^s_{k_\alpha} du_i \otimes f_s$$

so that

$$\pi_k \circ \nabla f_{k_\alpha} = \sum_{k_s \in (k)}\sum_{i=1}^{l}(\omega_i)^{k_s}_{k_\alpha} du_i \otimes f_{k_s}.$$

Let ${}_k\omega_i$ be the connection matrices of $\pi_k \circ \nabla$ then the above computation gives

$$({}_k\omega_i)^{k_\alpha}_{k_\beta} = (\omega_i)^{k_\alpha}_{k_\beta}.$$

Let Ω^k be the curvature form of $\pi_k \circ \nabla$ and Ω the one of ∇. Proposition (3.2.2) and flatness of ∇ give

$$\begin{aligned}
(\Omega^k_{ij})^{k_\alpha}_{k_\beta} &= \frac{\partial({}_k\omega_i)^{k_\alpha}_{k_\beta}}{\partial u_j} - \frac{\partial({}_k\omega_j)^{k_\alpha}_{k_\beta}}{\partial u_i} - \sum_{k_s \in (k)} ({}_k\omega_i)^{k_\alpha}_{k_s}({}_k\omega_j)^{k_s}_{k_\beta} - ({}_k\omega_j)^{k_\alpha}_{k_s}({}_k\omega_i)^{k_s}_{k_\beta} \\
&= \frac{\partial(\omega_i)^{k_\alpha}_{k_\beta}}{\partial u_j} - \frac{\partial(\omega_j)^{k_\alpha}_{k_\beta}}{\partial u_i} - \sum_{k_s \in (k)} (\omega_i)^{k_\alpha}_{k_s}(\omega_j)^{k_s}_{k_\beta} - (\omega_j)^{k_\alpha}_{k_s}(\omega_i)^{k_s}_{k_\beta} \\
&= (\Omega_{ij})^{k_\alpha}_{k_\beta} = 0.
\end{aligned}$$

□

Consider now the projection $\pi_L\colon \mathbb{P}^1 \times L \to L$. We now define a 1-parameter family of flat connections on the bundles $\iota^*(\mathcal{T}_{M,p})_k$ and flat connections on the bundles $(\iota \circ \pi_L)^*(\mathcal{T}_{M,p})_k$. This connections are analogous to the connections ∇^z and $\bar{\nabla}$ defined on the previous chapter.

Proposition 3.2.3. *For every $z \in \mathbb{C}$ the connection $(\pi_k \circ \nabla)^z := \pi_k \circ \nabla + z\circ$ on the bundle $\iota^*(\mathcal{T}_{M,p})_k$ is flat.*

Proof. First note that since the bundle $\iota^*(\mathcal{T}_{M,p})_k$ is multiplication invariant, the expression $(\pi_k \circ \nabla)^z$ does indeed define a connection. Let ${}_k\omega_i$ denote the connection matrices of the connection $\pi_k \circ \nabla$. Then the connection matrices ${}_k\omega_i^z$ of the connection $(\pi_k \circ \nabla)^z$ are given by

$$ {}_k\omega_i^z = \begin{cases} {}_k\omega_i & \text{if } i \neq k \\ {}_k\omega_k + zId & \text{if } i = k. \end{cases} $$

where Id is an $dim(T_pM)_k \times dim(T_pM)_k$ identity matrix. Since the partial derivatives of the identity matrix are zero and the identity commutes with any matrix, flatness of $(\pi_k \circ \nabla)^z$ follows from flatness of $\pi_k \circ \nabla$. □

Definition 3.2.1. Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold and suppose that at $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ with each $(T_pM)_k$ an irreducible algebra. Let $\pi_L \colon \mathbb{P}^1 \times L \to L$ be the projection and let z be a global coordinate on $\mathbb{C}$. We define a connection $\overline{\pi_k \circ \nabla}$ on $(\iota \circ \pi_L)^*(\mathcal{T}_{M,p})_k$ in the following way: for $u \in \pi_L^*\mathcal{T}_L \leq \mathcal{T}_{\mathbb{P}^1 \times L}$ and $v \in (\iota \circ \pi_L)^*\mathcal{T}_M$ set

$$ \overline{\pi_k \circ \nabla}_u v := (\pi_k \circ \nabla)_u v + zu \circ v. $$

The covariant derivative in the ∂_z direction is defined as

$$ \overline{\pi_k \circ \nabla}_{\partial_z} v := \partial_z v - E \circ v - \frac{1}{z}\pi_k \circ \mu v. $$

Proposition 3.2.4. *The connections $\overline{\pi_k \circ \nabla}$ are flat.*

Proof. Let $e_{k_i}, i = 1, \ldots, dim(T_pM)_k$ be a flat basis of the connection $\pi_k \circ \nabla$ and let ${}_k\bar{\omega}_j, j = 1 \ldots, l$ and ${}_k\bar{\omega}_z$ denote the connection matrices of the connection $\overline{\pi_k \circ \nabla}$. We have

$$ {}_k\bar{\omega}_j = \begin{cases} 0 & \text{if } j \neq k \\ zId & \text{if } j = k \end{cases} $$

and since on $(\mathcal{T}_{M,p})_k$ multiplication by $E\circ$ is just scalar multiplication by u_k,

$$ {}_k\bar{\omega}_z = u_k Id - \frac{1}{z} V_k $$

where V_k is the k-th diagonal block of the matrix V. Since e_{k_i} is $\pi_k \circ \nabla$-flat the previous proposition says that V_k is a constant matrix. Hence

$$ \frac{\partial_k \bar{\omega}_z}{\partial u_j} = \begin{cases} 0 & \text{if } j \neq k \\ Id & \text{if } j = k \end{cases} $$

and

$$\frac{\partial_k \bar{\omega}_j}{\partial z} = \begin{cases} 0 & \text{if } j \neq k \\ Id & \text{if } j = k. \end{cases}$$

In all cases we get

$$\frac{\partial_k \bar{\omega}_z}{\partial u_j} - \frac{\partial_k \bar{\omega}_j}{\partial z} = 0.$$

On the other hand since scalar multiplication commutes with all matrices

$$v[_k\bar{\omega}_z,_k \bar{\omega}_j] = 0.$$

□

We finish this chapter with a proposition that will help us find "normal forms" for a certain family of differential equations associated to a Dubrovin-Frobenius manifold. Consider a $\pi_k \circ \nabla$-flat basis e_{k_i} for each algebra $(\mathcal{T}_{M,p})_k$. The metric $\iota^*\eta|_{(\mathcal{T}_{M,p})_k}$ is compatible with the connection $\pi_k \circ \nabla$ and therefore the components of the metric in this basis are constant. Without loss of generality we can suppose that the basis e_{i_k} of $\iota^*\mathcal{T}_M$ is orthonormal. Since $E\circ|_{(\mathcal{T}_{M,p})_k}$ is just multiplication by u_k the matrix of $E\circ$ in this basis is diagonal. The operators of multiplication by π_k have the same matrix E_k. Therefore equations (3.2.3) remain valid. But now since $0 = (\omega_i)^{k_\alpha}_{k_\beta} = (\omega_i)^{k_\alpha}_{k_\beta}$ equation (3.2.4) says that for all $i = 1, \ldots, l$

$$\frac{\partial V^{k_\alpha}_{k_\beta}}{\partial u_i} = 0.$$

Summarizing we have the following

Proposition 3.2.5. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold with Levi-Civita connection ∇. Assume that at $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotents $\pi_1, \ldots, \pi_l$ passing through p and denote $\iota\colon L \to M$ the inclusion. Suppose that*

1. *The Euler vector field E is tangent to L.*
2. *The inner product $\iota^*\eta|_{\mathcal{T}_L} \in Sym^2\mathcal{T}^*_L$ is non-degenerate.*

Then there exists a η-orthonormal frame e_{k_i} of $\iota^*\mathcal{T}_{M,p}$ such that

1. *$e_{k_i} \in \iota^*(\mathcal{T}_{M,p})_k$.*
2. *The matrix U of the endomorphism $E\circ\colon \iota^*\mathcal{T}_M \to \iota^*\mathcal{T}_M$ is diagonal and $E\circ e_{k_i} = u_k e_{k_i}$.*
3. *The matrix V of the endomorphism $\mu\colon \iota^*\mathcal{T}_M \to \iota^*\mathcal{T}_M$ is antisymmetric and the block diagonal entries $V^{k_\alpha}_{k_\beta}$ are constant.*

Chapter 4

Isomonodromic Deformations Inside the Caustic

In the previous chapters we saw that from any Dubrovin-Frobenius manifold $(M, \circ, e, E, \eta)$ we can construct a meromorphic flat connection $\bar{\nabla}$ on the vector bundle $\pi_M^* \mathcal{T}_M$ over $\mathbb{P}^1 \times M$ ($\pi_M \colon \mathbb{P}^1 \times M \to M$). Take a point $p \in M$ and consider the inclusion $\iota_p \colon \mathbb{P}^1 \times \{ p \} \to \mathbb{P}^1 \times M$. Since the composition $\iota_p \circ \pi_M$ maps $\mathbb{P}^1$ to the point $p \in M$ we get that the vector bundle $(\iota_p \circ \pi_M)^* \mathcal{T}_M$ is trivial. So by pulling back $\bar{\nabla}$ via ι_p we obtain a trivial vector bundle with flat meromorphic connection over $\mathbb{P}^1$. But this kind of object is nothing more than a meromorphic differential equation on $\mathbb{P}^1$ (see [8]). Indeed, for each point $p \in M$ the corresponding differential equation reads

$$\frac{dY}{dz} = \left(\frac{1}{z} \mu_p - E_p \circ \right) Y. \tag{4.0.1}$$

Hence, from any Dubrovin-Frobenius manifold we get a family of meromorphic ordinary differential equations on $\mathbb{P}^1$. This family is parametrized by the points of the Dubrovin-Frobenius manifold and reads

$$\frac{dY(z,p)}{dz} = \left(\frac{1}{z} \mu(p) - E(p) \circ \right) Y(z,p). \tag{4.0.1}$$

It has a Fuchsian singularity at $z = 0$ and a Poincaré rank 1 singularity at $z = \infty$. The existence and unicity theorem for ordinary differential equations says that, on a neighborhood of any non-singular point, the solution to equation (4.0.1) exists and once we fix an initial condition it is unique. Moreover this theorem also asserts that if the differential equation depends holomorphically on additional parameters, then the solutions will also be holomorphic in these additional parameters.
In this chapter we study the monodromy data of this family. This monodromy data consists of the exponents μ, R of the monodromy transformation $Y(z) \mapsto Y(e^{2\pi i} z)$ at $z = 0$ of

a particular class of fundamental matrix solutions Y_{Lev}, an exponent of formal monodromy associated to the monodromy transformation at $z=\infty$ of a formal fundamental matrix solution Y_F near $z=\infty$, Stokes matrices S_ν which codify how the asymptotics when $z\to\infty$ of certain fundamental matrix solutions $Y_\nu, \nu\in\mathbb{Z}$ changes as we change certain sectors $\mathcal{S}_\nu$ and a central connection matrix C relating the fundamental matrix solutions Y_{Lev} and Y_0.

The principal part at $z=0$ of equation (4.0.1) (the endomorphism μ) is ∇-flat so that its eigenvalues and Jordan form are constant along the Dubrovin-Frobenius manifold. This no longer is true for the principal part at $z=\infty$ (the endomorphism $E\circ$) so that the Jordan form of $E\circ$ may change as we move on the Dubrovin-Frobenius manifold. This fact makes it difficult to find formal solutions in neighborhoods $W\subset M$ in which the Jordan form of $E\circ$ changes. Therefore we will restrict ourselves to the integral submanifolds L of the idempotents $\pi_1,\dots,\pi_l$ passing through a point $p\in M$ (see proposition 1.2.1). We will also suppose that the metric η when restricted to these submanifolds is non-degenerate and that the Euler vector field is tangent to them. Proposition 1.2.2 says that the Euler vector field E is tangent to L if and only if $E\circ$ is diagonalizable along L.

4.1 Monodromy Data at $z=0$

At $z=0$ the differential equation (4.0.1) has a Fuchsian singularity. In this section we study the monodromy transformation $Y(z,p)\mapsto Y(e^{2\pi i}z,p)$ on a neighborhood W of any point $p\in M$. We will show that we can find a solution $Y(z,p)$ such that the monodromy transformation is independent of $q\in W$.

Recall that on a Dubrovin-Frobenius manifold we have a 1-parameter family of flat connections (proposition 2.3.1)

$$\nabla^z=\nabla+z\circ\,.$$

Note that this connection depends holomorphically on z and therefore the solutions will also be holomorphic in z. By flatness, the space of ∇^z-flat sections is m-dimensional.
Let $(t^1,\dots,t^m)$ be flat coordinates on a neighborhood W of p and choose any basis of $\mathcal{T}_M(W)$ and let

$$\bar\omega_t:=\sum_{i=1}^m\bar\omega_i dt^i$$

be the connection matrix of ∇^z. Let $\Phi=\Phi(z,t)\colon\mathbb{C}\times W\to GL(m,\mathbb{C})$ be a fundamental matrix solution of $\nabla^z v=0$. This means that the columns of Φ are ∇^z-flat sections of $\mathcal{T}_M(W)$ and as such, Φ satisfies the partial differential equation

$$d_t\Phi=-\bar\omega_t\Phi$$

where d_t denotes the differential with respect the variables t^i. Consider now the deformed flat connection $\bar{\nabla}$ on the vector bundle $\pi_M^* \mathcal{T}_M$. Let z be a global coordinate on $\mathbb{C}$; on the chosen coordinates and basis for M and $\mathcal{T}_M$ the connection matrix of $\bar{\nabla}$ is

$$\bar{\omega} = \bar{\omega}_t + z^{-1}(A_0 + A_1 z)dz =: \bar{\omega}_t + \bar{\omega}_z dz,$$

where A_0, A_1 are the matrices representing $-\mu$ and $E\circ$ in the chosen basis. Recall that $\bar{\nabla}$ is also flat (proposition 2.3.3) so that in particular we have

$$\frac{\partial \bar{\omega}_z}{\partial t^i} - \frac{\partial \bar{\omega}_i}{\partial z} - [\bar{\omega}_z, \bar{\omega}_i] = \tag{4.1.1}$$

which we can write compactly as

$$d_t \bar{\omega}_z = d_z \bar{\omega}_t + [\bar{\omega}_t, \bar{\omega}_z]. \tag{4.1.1}$$

Proposition 4.1.1. *Let $(M, \circ, e, E, \eta)$ be Dubrovin-Frobenius manifold. Let Φ be a fundamental matrix solution of $\nabla^z = 0$. Then*

$$d_t(\Phi^{-1}\bar{\omega}_z\Phi + \Phi^{-1}d_z\Phi) = 0. \tag{4.1.2}$$

In particular, after the Gauge transformation $Y = \Phi X$ we have $d_t X = 0$ and $\frac{dX}{dz}$ does not depend on the variables t^i.

Proof. We have

$$\begin{aligned} d_t(\Phi^{-1}\bar{\omega}_z\Phi) = &-\Phi^{-1}(-\bar{\omega}_t\Phi)\Phi^{-1}\bar{\omega}_z\Phi \\ &+ \Phi^{-1}(d_z\bar{\omega}_t + [\bar{\omega}_z, \bar{\omega}_t])\Phi + \Phi^{-1}\bar{\omega}_z(-\bar{\omega}_t u\Phi) = \Phi^{-1}(d_z\bar{\omega}_t)\Phi. \end{aligned}$$

and

$$\begin{aligned} d_t(\Phi^{-1}d_z\Phi) &= -\Phi^{-1}(-\bar{\omega}_t\Phi)\Phi^{-1}d_z\Phi + \Phi^{-1}d_z d_t\Phi \\ &= \Phi^{-1}(\bar{\omega}_t d_z\Phi - (d_z\bar{\omega}_t)\Phi - \bar{\omega}_t d_z\Phi) \\ &= -\Phi^{-1}(d_z\bar{\omega}_t)\Phi. \end{aligned}$$

□

Since $\Phi = \Phi(z, t)$ is holomorphic on the variable z we can expand

$$\Phi(z, t) = \Phi_0 + \sum_{k=1}^{\infty} \Phi_k(t) z^k.$$

Plugging this into $\nabla^z\Phi = 0$ we get

$$\nabla\Phi_0 = 0. \tag{4.1.3}$$

That is, the columns of Φ_0 are ∇-flat vector fields for the Levi-Civita connection of η. Recall that $\nabla\mu = 0$ (proposition 2.3.2) so the basis that puts μ in Jordan form consists of ∇-flat vector fields. In particular we can choose the columns of Φ_0 to be the basis that puts μ in its Jordan form which we will also denote by μ and by simplicity we will assume that it is diagonal.

Let us now show that there is a fundamental matrix solution of $\bar{\nabla}Y(z,t) = 0$ such that the monodromy transformation $Y(z,t) \mapsto Y(e^{2\pi i}z,t)$ is constant.
By the last proposition after the Gauge transformation $Y(z,t) = \Phi(z,t)X(z,t)$ we have

$$\begin{aligned} &d_tX = 0 \\ &\frac{dX}{dz} = z^{-1}\left(\mu + \sum_{k=1}^{\infty}\bar{A}_k z^k\right)X. \end{aligned}$$

Since $\mu + \sum_{k=1}^{\infty}\bar{A}_k z^k = \Phi^{-1}\bar{\omega}_z\Phi + \Phi^{-1}d_z\Phi$, equality (4.1.2) says that the matrices $\bar{A}_k$ are constant. Given that the matrix valued function $\Phi(z,t)$ is holomorphic on $\mathbb{C}\times W$ it is a univalued function of the variable z and as such $\Phi(e^{2\pi i}z,t) = \Phi(z,t)$ so the monodromy transformation of $Y(z,t) = \Phi(z,t)X(z)$ can only come from the matrix $X(z)$. But this matrix does not depend on t so its monodromy transformation is t-independent. Let us be more explicit on this point.

We do a Gauge transformation

$$X = \left(Id + \sum_{k=1}^{\infty}T_k z^k\right)Z$$

where the matrices T_k are to be found. We put

$$\frac{dZ}{dz} = \left(\mu + \sum_{k=1}^{\infty}R_k z^z\right)$$

where the matrices R_k are also to be found. Substituting we get the recursive relations

$$[\mu, T_k] - kT_k = R_k + \sum_{s=1}^{k-1}T_{k-s}R_s - \bar{A}_1T_{k-1}. \tag{4.1.4}$$

We can solve the above equations by putting

$$(T_k)^i_j = \begin{cases} \frac{1}{\mu_i-\mu_j-k}\left(\sum_{s=1}^{k-1}(T_{k-s}R_s)^i_j - \bar{A}_1T_{k-1}\right) & \text{if } \mu_i - \mu_j \neq k \\ 0 & \text{if if } \mu_i - \mu_j = k \end{cases}$$

and

$$(R_k)^i_j = \begin{cases} 0 & \text{if } \mu_i - \mu_j \neq k \\ (\bar{A}_1 T_{k-1})^i_j - \sum_{s=1}^{k-1}(T_{k-s}R_s) & \text{if } \mu_i - \mu_j = k. \end{cases}$$

Since the matrix Φ is holomorphic on $z = 0$ the series $\sum_{k=1} \bar{A}_k z^k$ is holomorphic at $z = 0$, but the equation satisfied by X has a Fuchsian singularity at $z = 0$ and therefore the Gauge transformation just found $T = Id + \sum_{k=1} T_k z^k$ will be convergent on a neighborhood of $z = 0$ (see [5]). In the end we obtain a system

$$\frac{dZ}{dz} = z^{-1}\left(\mu + R_1 z + \cdots + R_p z^p\right) Z$$

where p is the maximum integer difference of the eigenvalues of μ. In particular $(R_k)^i_j$ may not be zero only if $\mu_i - \mu_j = k$. Let

$$R := R_1 + \cdots + R_p \tag{4.1.5}$$

Theorem 4.1.1. *Equation (4.0.1) has a holomorphic fundamental matrix solution Y_{Lev} such that on any compact subset of $\mathbb{C}$ we have*

$$Y_{Lev}(z,t) = \Phi(z,t)T(z,t)z^{\mu}z^{R} \tag{4.1.6}$$

and the monodromy transformation of this solution is constant and equal to

$$Y_{Lev}(e^{2\pi i}z) = Y_{Lev}(z)e^{2\pi i\mu}e^{2\pi i R}. \tag{4.1.7}$$

Moreover if Φ is ∇^z-flat then Y_{Lev} is $\bar{\nabla}$-flat.

Proof. The only things we need to check is that

$$\frac{dz^{\mu}z^{R}}{dz} = z^{-1}(\mu + R_1 z + \cdots + R_p z^p)z^{\mu}z^{R}$$

and that the monodromy transformation of $z^{\mu}z^{R}$ is $e^{2\pi i\mu}e^{2\pi i R}$. We have that

$$\frac{dz^{\mu}z^{R}}{dz} = z^{-1}\mu z^{\mu}z^{R} + z^{-1}z^{\mu}Rz^{R} = z^{-1}(\mu + z^{\mu}Rz^{-\mu})z^{\mu}z^{R}.$$

Since μ is diagonal we have that $(z^{\mu}Rz^{-\mu})^{\alpha}_{\beta} = z^{\mu_\alpha - \mu_\beta}R^{\alpha}_{\beta} = \sum_{k=1}^{p} z^{\mu_\alpha-\mu_\beta}(R_k)^{\alpha}_{\beta}$. But if $(R_k)^{\alpha}_{\beta} \neq 0$ then $\mu_\alpha - \mu_\beta = k$ and therefore $(z^{\mu}Rz^{-\mu})^{\alpha}_{\beta} = \sum_{k=1}^{p}(R_k)^{\alpha}_{\beta}z^k$ and $z^{\mu}Rz^{-\mu} = R_1 z + \cdots + R_p z^p$. Now we compute the monodromy transformation. First we have that

$$[e^{2\pi i\mu}, R]^{\alpha}_{\beta} = \sum_{k=1}^{p}(R_k)^{\alpha}_{\beta}(e^{2\pi i\mu_\alpha} - e^{2\pi i\mu_\beta}) = \sum_{k=1}^{p}(R_k)^{\alpha}_{\beta}(e^{2\pi i(\mu_\alpha-\mu_\beta)} - 1)e^{2\pi i\mu_\beta}.$$

But again $(R_k)^\alpha_\beta \neq 0$ gives $\mu_\alpha - \mu_\beta = k \in \mathbb{Z}$ and therefore $e^{2\pi i(\mu_\alpha - \mu_\beta)} = 1$. Hence $[e^{2\pi i\mu}, R] = 0$ and from this we immediately get $[e^{2\pi i\mu}, z^R] = 0$. Thus,

$$(e^{2\pi i}z)^\mu (e^{2\pi i}z)^R = z^\mu e^{2\pi i\mu} z^R e^{2\pi i R} = z^\mu z^R e^{2\pi i\mu} e^{2\pi i R}.$$

□

The solution (4.1.6) of equation (4.0.1) is said to be in *Levelt form*. Its monodromy transformation is $e^{2\pi i\mu}e^{2\pi i R}$. The matrices μ and R are called *monodromy data* of equation (4.0.1) at $z = 0$.

4.2 Monodromy Data at $z = \infty$

We now study the monodromy data of equation (4.0.1) at $z = \infty$. This data consists of Stokes matrices and an exponent of formal monodromy. Since the connection ∇^z is not holomorphic at $z = \infty$ we cannot use it to argue as in the past section. Instead we will start by finding formal solutions to equation (4.0.1). Then we will use a theorem from Sibuya's to get holomorphic solutions on certain sectors whose asymptotic expansions on these sectors are the formal solutions that we found. There is some freedom on the formal solutions that we will find and we will show that these formal solutions can be chosen in such a way that they are $\bar{\nabla}$-flat. Finally, using $\bar{\nabla}$-flatness we will show that the Stokes matrices are constant and, under some conditions the exponent of formal monodromy will also be constant.

From now on we will use the notation for the indices k_i that was established at the beginning of chapter 3 section 3.2. Suppose that at $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotent vector fields passing through p, let $\iota\colon L \to M$ denote the inclusion and let $u_1 \ldots, u_l$ be canonical coordinates on L around p.
Suppose that the Euler vector field E is tangent to L, then $(L, \circ, e, E)$ is an F-manifold with Euler vector field and the endomorphism $E\circ$ is diagonalizable along E (see propositions 1.2.1 and 1.2.2).
If we further suppose that the restriction of the metric η to L is non-degenerate then proposition 3.2.5 shows that we can find an orthonormal basis e_{k_i} of $\iota^*\mathcal{T}_M$ such that the matrix U representing $E\circ$ is diagonal with $E \circ e_{k_i} = u_k e_{k_i}$, the matrix V representing μ is antisymmetric and the block-diagonal entries $V^{k_\alpha}_{k_\beta}$ are constant. We now write equation (4.0.1) in this basis and start looking for a formal solution.

We start by doing a formal Gauge transformation

$$Y(z,u) = \left(Id + \sum_{k=1}^{\infty} G_k(u) z^{-k}\right) \bar{Y}(z,u) = \left(\sum_{k=0}^{\infty} G_k z^{-k}\right) \bar{Y}. \tag{4.2.1}$$

where the matrices G_k are to be determined. Setting

$$\frac{d\bar{Y}}{dz} = \left(-U + \sum_{k=1}^{\infty} B_k(u) z^{-k}\right) \bar{Y}, \tag{4.2.2}$$

where the matrices B_k are also to be determined, we get the recursive relations

$$-[U, G_k] + (k-1)G_{k-1} + VG_{k-1} - \sum_{s=1}^{k-1} G_{k-s} B_s = B_k \text{ for } k \geq 1. \tag{4.2.3}$$

So if we already now $G_1 \ldots, G_{k-1}$ and $B_1, \ldots, B_{k-1}$ we can try to solve the above equation and obtain G_k and B_k. We do this in the following way. For $k = 1$ we need to solve

$$-[U, G_1] + V = B_1.$$

Taking the entry on the i_α row and the j_β column of the above equation we get

$$-(u_i - u_j)(G_1)^{i_\alpha}_{j_\beta} + (V)^{i_\alpha}_{j_\beta} = (B_1)^{i_\alpha}_{j_\beta}. \tag{4.2.4}$$

So if $i \neq j$ we can put

$$(G_1)^{i_\alpha}_{j_\beta} = \frac{(V)^{i_\alpha}_{j_\beta}}{u_i - u_j}$$
$$(B_1)^{i_\alpha}_{j_\beta} = 0.$$

If $i = j$ then $u_i = u_j$ so we can put $(G_1)^{i_\alpha}_{i_\beta} = 0$ but we are forced to put

$$(B_1)^{i_\alpha}_{i_\beta} = (V)^{i_\alpha}_{i_\beta}.$$

In particular note that the matrix B_1 is constant.
Analogously, for $k > 1$ and $i \neq j$ we can put

$$(G_k)^{i_\alpha}_{j_\beta} = \frac{1}{u_i - u_j}\left((k-1)G_{k-1} + VG_{k-1} - \sum_{s=1}^{k-1} G_{k-s} B_s\right)^{i_\alpha}_{j_\beta}$$
$$(B_k)^{i_\alpha}_{j_\beta} = 0.$$

For $i = j$ we put

$$(G_k)^{i_\alpha}_{i_\beta} = 0$$
$$(B_k)^{i_\alpha}_{i_\beta} = \left((k-1)G_{k-1} + VG_{k-1} - \sum_{s=1}^{k-1} G_{k-s} B_s\right)^{i_\alpha}_{i_\beta}.$$

By construction the matrices B_k are block diagonal $i.e.$ $(B_k)^{i_\alpha}_{j_\beta} = 0$ when $i \neq j$.
We now do the Gauge transformation

$$\bar{Y} = e^{-Uz}\hat{Y},$$

we get

$$\left(-U + \sum_{k=1}^{\infty} B_k z^{-k}\right) e^{-Uz}\hat{Y} = e^{-Uz}\left(-U\hat{Y} + \frac{d\hat{Y}}{dz}\right).$$

But the matrices U and B_k have the same block-diagonal structure and on each block e^{-Uz} acts by multiplication by $e^{-u_i z}$. Therefore we get

$$\frac{d\hat{Y}}{dz} = \left(\sum_{k=1}^{\infty} B_k z^{-k}\right)\hat{Y}. \tag{4.2.5}$$

This equation is a direct sum of l formal local Fuchsian systems of dimension $dim(T_pM)_k$. The matrices H that we will define in the following have the same block structure of this direct sum. We now take $H_0 = H_0(u)$ a matrix diagonalizing the matrix

$$(B_1)^{k_\alpha}_{k_\beta} = V^{k_\alpha}_{k_\beta}.$$

We will call the matrix

$$B := H_0^{-1} B_1 H_0 \tag{4.2.6}$$

the *exponent of formal monodromy*. We will see that if we choose the k-th block of $H_0(u)$ consisting of flat sections of $\pi_k \circ \nabla$ then, a solution having a prescribed asymptotic expansion, depending on H_0, will also be $\bar{\nabla}$-flat i, e by solving recursively the equation for the z-component we get a solution for the whole system. For $k > 1$ let

$$\hat{B}_k = H_0^{-1} B_k H_0. \tag{4.2.7}$$

Then after the Gauge transformation $\hat{Y} = H_0\hat{X}$ we get

$$\frac{d\hat{X}}{dz} = \left(Bz^{-1} + \sum_{k=2}^{\infty} \hat{B}_k z^k\right)\hat{X}.$$

Remark 4.2.1. In the following we make the assumption that each of the blocks of the matrix B are non-resonant. At present, except for the semisimple caustic case, when B is always non-resonant (see section 5.1) , there is no geometrical or algebraic interpretation for the block diagonal entries of V.

We can now find a formal Gauge transformation (it is formal because it depends on the formal series of the $\hat{B}_k$)

$$\hat{X} = \left(Id + \sum_{k=1}^{\infty} H_k z^k\right)\bar{X}.$$

The non-resonance condition implies that by solving recursively the equations (having found $H_0, \ldots, H_{k-1}$ and $B_1, \ldots, B_k$)

$$H_k = -\frac{1}{k}\left(B_{k+1} + \sum_{l=1}^{k-1} B_{k+1-l} H_l\right) \tag{4.2.8}$$

we can write a formal solution of equation (4.0.1) in the following form

$$Y_F = \left(Id + \sum_{k=1}^{\infty} G_k z^{-k}\right) e^{-Uz} H_0 \left(Id + \sum_{l=1}^{\infty} H_l z^{-l}\right) z^B.$$

Since on each block of the H_k the matrix e^{-Uz} is acts by scalar multiplication, we can write the above as

$$Y_F = (H_0 + (H_0 H_1 + G_1 H_0) z^{-1} + O(z^{-2})) e^{-Uz} z^B. \tag{4.2.9}$$

We now discuss the sectors in which certain holomorphic solutions to equation (4.0.1) will have the above formal series as asymptotic expansion.

The gauge transformation (4.2.1) is usually divergent, but there are certain sectors $\mathcal{S}_\nu$ of the z-plane in which this formal power series is the asymptotic expansion of a holomorphic gauge transformation which takes equation (4.0.1) to the block diagonal equation (4.2.2).

Definition 4.2.1. A line ℓ through the origin of the z-plane is called *admissible* for the system (4.0.1) if for all $z \in \ell \setminus \{ 0 \}$ we have that $\mathrm{Re}(z(u_i - u_j)) \neq 0$ whenever $u_1 - u_j \neq 0$. Let ϕ be the oriented angle between the positive real axis and an admissible line ℓ. For ϵ sufficiently small, N sufficiently big and $\nu \in \mathbb{Z}$ we define sectors $\mathcal{S}_\nu$ of opening angle $\pi + 2\epsilon$ by

$$\begin{aligned} \mathcal{S}_0 :=& \{ z \in \mathbb{C} | arg(z) \in (\phi - \pi - \epsilon, \phi + \epsilon), |z| > N \} \\ \mathcal{S}_\nu :=& e^{i\nu\pi} \mathcal{S}_0. \end{aligned}$$

Note that the intersection of two subsequent sectors has opening angle 2ϵ.

On the following u denotes a parameter on a small domain $W \subset \mathbb{C}^l$, for the applications we have in mind $u = (u_1, \cdots, u_l)$ are the coordinates on a neighborhhod of the point $p \in L$.

Theorem 4.2.1. *(Sibuya [20])Let $A(z,u) = \sum_{k=0}^{\infty} A_k(u) z^{-k}$ with $A_k \in Mat_n(\mathcal{O}_{\mathbb{C}^l})$ be holomorphic on $\{ z \geq N_0 > 0 \} \times \{ |u| \leq \epsilon_0 \}$ such that $A_0(u) = \Lambda(u) = \Lambda_1 \oplus \cdots \oplus \Lambda_l$ is diagonal with $l \leq n$ distinct eigenvalues where each matrix $Lambda_k$ is diagonal with only one eigenvalue. Then, for any proper subsector $\bar{S}(\alpha, \beta)$ of $\mathcal{S}_\nu$ there exists positive numbers $N \geq N_0, \epsilon \leq \epsilon_0$ and a matrix $G(z,u)$ with the following properties:*

1. *$G(z,u)$ is holomorphic in (z,u) for $|z| \geq N$, $z \in \bar{S}(\alpha, \beta)$ and $|u| \leq \epsilon$.*

2. *$G(z,u)$ has uniform asymptotic expansion for $|u| \le \epsilon$ with holomorphic coefficients $G_k(u)$,*

$$G(z,u) \sim Id + \sum_{k=1}^{\infty} G_k(u) z^{-k}, \;\; z \to \infty, z \in \bar{S}(\alpha,\beta),$$

where the matrices G_k are computed from (4.2.3)

3. *The gauge transformation $Y(z,u) = G(z,u)\tilde{Y}(z,u)$ reduces the system $\frac{dY}{dz} = AY$ to block diagonal form*

$$\frac{d\tilde{Y}}{dz} = \tilde{B}(z,u)\tilde{Y}, \quad \tilde{B}(z,u) = \tilde{B}_1(z,u) \oplus \cdots \oplus \tilde{B}_s(z,u)$$

and $\tilde{B}$ has uniform asymptotic expansion for $|u| \le \epsilon$ with holomorphic coefficients $B_k(u)$

$$\tilde{B}(z,u) \sim \Lambda(u) + \sum_{k=1}^{\infty} B_k(u) z^{-k}, \;\; z \to \infty, z \in \bar{S}(\alpha,\beta).$$

Now we apply this theorem to the matrix $A = -U + Vz^{-1}$ of system (4.0.1) restricted to the submanifold L. We get that the formal gauge transformation of (4.2.1) is asymptotic, in proper sectors $\mathcal{S}_\nu$, to a holomorphic gauge transformation G_ν that takes system (4.0.1) to the block diagonal form (4.2.2). Hence, on each sector there is a holomorphic matrix G_ν and a fundamental matrix solution of system (4.0.1) of the form

$$Y_\nu = G_\nu H e^{-Uz} z^B = \hat{Y}_\nu e^{-Uz} z^B \tag{4.2.10}$$

with asymptotic expansion on the sector $\mathcal{S}_\nu$

$$Y_\nu \sim Y_F.$$

Stokes matrices are defined in the usual way. On the overlap of two adjacent sectors $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+1}$ we have that

$$Y_{\nu+1}(z;u) = Y_\nu(z;u) S_\nu(u).$$

The matrix S_ν is called *Stokes matrix*. Now we proceed to show that the matrices $S_\nu(u)$ are independent of the parameter u. We need some preliminaries.

So far we have constructed solutions $Y_\nu(z,u)$ of equation (4.0.1) with some prescribed asymptotic expansion on certain sectors $\mathcal{S}_\nu$. Now we come back to flat sections of the connection $\bar{\nabla}$ on the vector bundle $(\pi_M \circ (id \times \iota))^* \mathcal{T}_M$ over $\mathbb{P}^1 \times L$. If Y is a fundamental matrix of $\bar{\nabla}$-flat sections then

$$dY = -\bar{\omega} Y$$

where $\bar{\omega}$ is the connection form of $\bar{\nabla}$. In particular the differential equation (4.0.1) is satisfied for all $p \in W \subset L$. Therefore there exist $GL(n,\mathbb{C})$-valued functions $D_\nu(u)$ such that $Y(z,u) = Y_\nu(z,u)D_\nu(u)$. In the following two lemmas we show that if we choose $H_0(u)$ in an appropriate way we can make D_ν to be independent of $u \in W$. Recall that the connections $\pi_k \circ \nabla$ on $i^*(\mathcal{T}_{M,p})_k$ are flat so that the connection $\oplus_{k=1}^{l} \pi_k \circ \nabla$ on $\iota^* \mathcal{T}_{M,p}$ is flat. Let ω^Δ denote its connection matrix.

Lemma 4.2.1. *There exists a matrix $H_0(u)$ whose columns are a basis for the $\oplus_{k=1}^{l} \pi_k \circ \nabla$-flat sections and such that $H_0^{-1} B_1 H_0$ is the exponent of formal monodromy.*

Proof. Let H_0 be a matrix of $\oplus_{k=1}^{l} \pi_k \circ \nabla$-flat sections of $\iota^* \mathcal{T}_{M,p}$. Then

$$\begin{aligned} d(H_0^{-1} B_1 H_0) &= -H_0^{-1}(-\omega^\Delta) B_1 H_0 + H_0^{-1} dB_1 H_0 + H_0 B_1 (-\omega^\Delta) H_0 \\ &= H_0^{-1}(dB_1 + [\omega^\Delta, B_1]) H_0. \end{aligned}$$

Now recall that B_1 is block diagonal and $(B_1)^{k_\alpha}_{k_\beta} = (V)^{k_\alpha}_{k_\beta}$. But by proposition 3.2.1 $dB_1 = [B_1, \omega^\Delta]$. Therefore the matrix $H_0^{-1} B_1 H_0$ is a constant matrix. Let C be a constant matrix diagonalizing $H_0^{-1} B_1 H_0$. Then $H_0 C$ still consists of $\oplus_{k=1}^{l} \pi_k \circ \nabla$-flat sections and $B = C^{-1} H_0^{-1} B_1 H_0 C$. □

Lemma 4.2.2. *Let $Y(z,u)$ be a fundamental matrix of $\bar{\nabla}$-flat sections and let $Y_\nu(z,u)$ be a solution of equation (4.0.1) with asymptotic behavior (4.2.9) on the sector $\mathcal{S}_\nu$. Then Y_ν is $\bar{\nabla}$-flat if and only if the blocks of H_0 are $\pi_k \circ \nabla$-flat.*

Proof. Let d_u denote the differential with respect to the u_i variables and let $\bar{\omega}_u$ be the part of the connection form of $\bar{\omega}$ disregarding the dz-component (not that since the matrices $G_k.H_k$ and D_ν don't depend on z we have $d_u H_k = dH_k$). First

$$d_u Y \cdot Y^{-1} - d_u Y_\nu \cdot Y_\nu^{-1} = Y_\nu dD_\nu \cdot D_\nu^{-1} Y_\nu^{-1}.$$

On the sector $\mathcal{S}_\nu$ we have

$$\begin{aligned} d_u Y_\nu &\sim (dH_0 + d(H_0 H_1 + G_1 H_0) z^{-1} + O(z^{-2})) e^{-Uz} z^B \\ &\quad - (H_0 dU z + (H_0 H_1 + G_1 H_0) dU + O(z^{-1})) e^{-Uz} z^B \\ Y_\nu^{-1} &\sim z^{-B} e^{Uz} (H_0^{-1} - (H_1 H_0^{-1} + H_0^{-1} G_1) z^{-1} + O(z^{-2})). \end{aligned}$$

Using that the matrix dU which is diagonal with entries du_i commutes with the matrices H_k we have

$$d_u Y_\nu \cdot Y_\nu^{-1} \sim -z dU + dH_0 \cdot H_0^{-1} + [dU, G_1] + O(z^{-1}).$$

Since $d_u Y \cdot Y^{-1} = -\bar{\omega}_u$ on the sector $\mathcal{S}_\nu$ we have

$$-\bar{\omega}_u + z dU - [dU, G_1] - dH_0 \cdot H_0^{-1} + O(z^{-1}) \sim Y_\nu (dD_\nu \cdot D_\nu^{-1}) Y_\nu^{-1}.$$

Let us now compute $-\bar{\omega}_u + zdU - [dU, G_1]$. Recall that (see equation (3.2.1))

$$\begin{aligned}\bar{\omega}_u &= \sum_{k=1}^{l} (\omega_k + zE_k) du_k \\ &= zdU + \sum_{k=1}^{l} \omega_k du_k\end{aligned}$$

where ω_k are the connection matrices of $\iota^* \nabla$. The block diagonal entries of $[dU, G_1]$ are zero. Writing $dU = \sum_k E_k du_k$, then from equation (4.2.4) we get

$$[E_k du_k, G_1]^{k_\alpha}_{j_\beta} = V^{k_\alpha}_{j_\beta} \frac{du_k}{u_k - u_j}.$$

but from equation

$$[U, \omega_k] = -[\bar{E}_k, V] \tag{3.2.2}$$

we obtain

$$(\omega_k)^{k_\alpha}_{j_\beta} du_k = -V^{k_\alpha}_{j_\beta} \frac{du_k}{u_k - u_j}.$$

For $i \neq j \neq k \neq i$ we also have

$$(\omega_k)^{i_\alpha}_{j_\beta} du_k = [E_k du_k, G_1]^{i_\alpha}_{j_\beta} = 0.$$

In the end we obtain that

$$-\bar{\omega}_u + zdU - [dU, G_1]$$

is the connection matrix of $\oplus_k \pi_k \circ \nabla$. Hence, the blocks H_0 are $\pi_k\circ$-flat if and only if on the sector $\mathcal{S}_\nu$

$$Y_\nu (dD_\nu \cdot D_\nu^{-1}) Y_\nu^{-1} \sim O(z^{-1})$$

Let us write

$$Y_\nu dD_\nu \cdot D_\nu^{-1} Y_\nu^{-1} \sim \sum_{k=1}^{\infty} F_k z^{-k} =: F_\nu.$$

Using (4.2.10) on $\mathcal{S}_\nu$ we have

$$e^{-zU} z^B dD_\nu \cdot D_\nu^{-1} z^{-B} e^{Uz} \sim \hat{Y}_\nu^{-1} F_\nu \hat{Y}_\nu. \tag{4.2.11}$$

Note that since the matrix $\hat{Y}_\nu$ is holomorphic on $z = \infty$, the term $\hat{Y}_\nu^{-1} F_\nu \hat{Y}_\nu$ vanishes as z^{-1} when $z \to \infty$.

In the following let us denote by $A_{[i,j]}$ the block of the matrix A consisting of the entries $A^{i_\alpha}_{j_\beta}$ with $i_\alpha \in (i)$ and $j_\beta \in (j)$. The off-diagonal blocks of the left hand side of (4.2.11) are of the form

$$e^{(u_j - u_i)z} z^{B_{[i,i]}} (dD_\nu \cdot D_\nu^{-1})_{[i,j]} z^{B_{[j,j]}}.$$

Since the sector $\mathcal{S}_\nu$ has opening angle bigger than π this sector intersects the line $Re((u_i - u_j)z) = 0$. On one side of this line the function $e^{(u_i-u_j)z}$ diverges when $z \to \infty$. But the above expression must vanish as z^{-1} when $z \to \infty$ so we conclude

$$(dD_\nu \cdot D_\nu^{-1})_{[i,j]} = 0.$$

The block diagonal entries of the left hand side of (4.2.11) are of the form

$$z^{b_{i_\alpha} - b_{i_\beta}} (dD_\nu \cdot D_\nu^{-1})^{i_\alpha}_{i_\beta}.$$

For $\alpha = \beta$ we obtain that $(dD_\nu \cdot D_\nu^{-1})^{i_\alpha}_{i_\alpha}$ vanishes as z^{-1} when $z \to \infty$ and therefore $(dD_\nu \cdot D_\nu^{-1})^{i_\alpha}_{i_\alpha} = 0$. When $\alpha \neq \beta$ then since we supposed that the diagonal blocks of the matrix B are non resonant, the above expression can be $O(z^{-1})$ if and only if $(dD_\nu \cdot D_\nu^{-1})^{i_\alpha}_{i_\beta} = 0$. Hence we conclude that the blocks of H_0 are $\pi_k \circ \nabla$-flat if and only if $dD_\nu = 0$ and this is true if and only if Y_ν is $\bar{\nabla}$-flat. □

Now we can prove

Theorem 4.2.2. *There exists holomorphic solutions $Y_\nu(z,u)$ of equation (4.0.1) such that*

$$Y_\nu(z,u) \sim (H_0 + (H_0 H_1 + G_1 H_0)z^{-1} + O(z^{-2}))e^{-Uz} z^B \text{ for } z \in \mathcal{S}_\nu,$$

and the corresponding Stokes matrices S_ν are u-independent.

Proof. By the last lemma we can choose Y_ν in such a way that

$$d_u Y_\nu \cdot Y_\nu = -\omega_u$$

for all $\nu \in \mathbb{Z}$. On the overlap $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+1}$ we have

$$\begin{aligned} -\omega_u &= d_u Y_{\nu+1} \cdot Y_{\nu+1}^{-1} = d_u Y_\nu \cdot Y_\nu^{-1} + Y_\nu dS_\nu \cdot S_\nu Y_\nu^{-1} \\ &= -\omega_u + Y_\nu dS_\nu \cdot S_\nu Y_\nu^{-1}. \end{aligned}$$

□

4.3 The Central Connection Matrix

Up till now we have seen that on a neighborhood W of $p \in L$, differential equation (4.0.1) admits a solution Y_{Lev} that locally around zero is written as (4.1.6) and whose monodromy data (μ, R) don't depend on $u \in W$. We have also seen that there are certain solutions Y_ν that on sectors $\mathcal{S}_\nu$ near $z = \infty$ have asymptotic expansion (4.2.9) and whose Stokes matrices and exponent of formal monodromy are constant. The last part of the monodromy data that one associates to the meromorphic differential equation (4.0.1) is a *central connection matrix C* defined by

$$Y_{Lev}(z,u) =: Y_0(z,u) C(u). \tag{4.3.1}$$

Theorem 4.3.1. *The central connection matrix is constant.*

Proof. By theorem 4.1.1 and lemma 4.2.2 the fundamental matrix solutions Y_L and Y_0 of equation (4.0.1) can be chosen to be $\bar{\nabla}$-flat. Hence

$$-\omega_u = dY_L \cdot Y_L^{-1} = dY_0 \cdot Y_0^{-1} + Y_0 dC \cdot C^{-1} Y_0^{-1} = -\omega_u + Y_0 dC \cdot C^{-1} Y_0^{-1}.$$

□

Putting together theorems 4.1.1,4.2.2 and 4.3.1 we obtain

Theorem 4.3.2. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold and suppose that at $p \in M$ we have $T_pM = \oplus_{k=1}^{l}(T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotents $\pi_1, \ldots, \pi_l$ passing through p and denote $\iota\colon L \to M$ the inclusion. Suppose that:*

1. *The Euler vector field E is tangent to L.*
2. *The inner product $\iota^*\eta|_{\mathcal{T}_L} \in Sym^2\mathcal{T}_L^*$ is non-degenerate.*

Let e_{k_i} be the orthonormal basis of $\iota^*\mathcal{T}_M$ of proposition 3.2.5 and suppose further that:

3. *The eigenvalues of any of the diagonal blocks of the matrix V representing the endomorphism $\mu\colon \iota^*\mathcal{T}_M \to \mathcal{T}_M$ on the basis e_{k_i} don't differ by a non-zero integer.*

Then there exists holomorphic fundamental matrix solutions $Y_{Lev}, Y_\nu, \nu \in \mathbb{Z}$ and a formal fundamental matrix solution Y_F of equation 4.0.1 such that the corresponding monodromy data are constant.

Let us now obtain a relation between all the monodromy data (μ, R, S_0, S_1, C). By construction, whenever $w \in \mathcal{S}_2$ the fundamental matrix solution Y_2 has asymptotic expansion $Y_F = (H_0 + O(w^{-1}))e^{-Uw}w^B$ (see equation 4.2.9) so when $z \in \mathcal{S}_0$ then

$$Y_2(e^{2\pi i}z) \sim (H_0 + O(z^{-1}))e^{-Uz}z^B e^{2\pi i B}.$$

But then the fundamental matrix solutions $Y_0(z)e^{2\pi i B}$ and $Y_2(e^{2\pi i}z)$ have the same asymptotic expansion on the sector $\mathcal{S}_0$. Arguing as in lemma 4.2.2, since the sector $\mathcal{S}_0$ contains a basic set of Stokes rays, we conclude

$$Y_2(e^{2\pi i}z) = Y_0(z)e^{2\pi i B}.$$

The definition of the Stokes matrices immediately gives

$$Y_2(e^{2\pi i}z) = Y_0(e^{2\pi i}z)S_1S_2.$$

Using this last to equation we get

$$Y_0(e^{2\pi i}z) = Y_0(z)e^{2\pi i B}(S_1S_2)^{-1} \tag{4.3.2}$$

The fact that $E\circ$ is η-symmetric and μ is η-antisymmetric will give us a relation between consecutive Stokes matrices. First we have

Lemma 4.3.1. *Let $Y_\alpha(z,u)$ and $Y_\beta(z,u)$ be two solutions of equation (4.0.1). Then*

$$\frac{d}{dz}\eta\left(Y_\alpha(e^{\pm\pi i}z), Y_\beta(z)\right) = 0$$

Proof. We have

$$\begin{aligned}\eta(d_zY_\alpha(-z), Y_\beta) &= \eta(-(-E\circ -z^{-1}\mu)Y_\alpha(-z), Y_\beta(z)) \\ &= \eta(Y_\alpha(-z), (E\circ -z^{-1}\mu)Y_\beta).\end{aligned}$$

and

$$\eta(Y_\alpha(-z), d_zY_\beta(z)) = \eta\left(Y_\alpha(-z), (-U+z^{-1}\mu)Y_\beta\right).$$

□

Let us now apply this to the three consecutive solutions Y_0, Y_1 and Y_2. By the previous lemma we have

$$\begin{aligned}\eta(Y_1(-z), Y_0) &= P_1 \\ \eta(Y_2(-z), Y_1) &= P_2\end{aligned}$$

for some constant matrices P_i. Using the defining relations of Stokes matrices we have

$$\begin{aligned}P_2 &= \eta(Y_2(-z), Y_1) \\ &= \eta(Y_1(-z)S_1, Y_0(z)S_0) \\ &= S_1^TP_1S_0.\end{aligned}$$

So that

$$S_1^T = P_2S_0^{-1}P_1.$$

Let us now obtain a relation between the matrix H_0 and the exponent of formal monodromy B. If $z\in\mathcal{S}_0$ then $e^{\pi i}z\in\mathcal{S}_1$ so that

$$\begin{aligned}Y_0(z) &\sim (H_0+O(z^{-1}))e^{-Uz}z^B \\ Y_1(e^{\pi i}z) &\sim (H_0+O(z^{-1}))e^{Uz}z^Be^{\pi iB}.\end{aligned}$$

Therefore

$$\begin{aligned}P_1 &= \eta(Y_1(e^{\pi i}z), Y_0(z)) \\ &\sim e^{\pi iB}z^B\eta(H_0,H_0)z^B + e^{\pi iB}z^Be^{Uz}O(z^{-1})e^{-Uz}z^B.\end{aligned}$$

In particular the term $\sim e^{\pi iB}z^B\eta(H_0,H_0)z^B$ must be z independent so that taking its derivative with respect to z we obtain

$$BH_0^TH_0 + H_0^TH_0B = 0.$$

We can summarize the above in the following propostion

Proposition 4.3.1. *Consider the holomorphic fundamental matrix solutions Y_0, Y_1 and Y_2 of equation (4.0.1) then*

$$S_1^T = P_2 S_0^{-1} P_1$$

where

$$\eta(Y_1(-z), Y_0) = P_1$$
$$\eta(Y_2(-z), Y_1) = P_2$$

and

$$P_1 = e^{\pi i B} z^B \eta(H_0, H_0) z^B.$$

Chapter 5

The Semisimple Caustic Case

5.1 The case of a semisimple Caustic

Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold of dimension m. By definition the caustic $K \subset$ is the set of points $p \in M$ such that T_pM has less than m idempotents. Proposition 1.1.4 says that the caustic is an hypersurface which we will suppose non-empty. We will assume that, generically, for $p \in K$ the algebra T_pM has exactly $m-1$ idempotents. In this case the integral submanifold L of the $m-1$ idempotents passing through p will be an irreducible component of the regular points of K. For any point $p \in L$ the germ of the F-manifold M at p will decompose as a product of one 2-dimensional F-manifold and $m-2$ one-dimensional F-manifolds. Up to isomorphism there is only one 1-dimensional germ of F-manifold which we denote by A_1. Germs of 2-dimensional F-manifolds were classified by Hertling. Up to isomorphism they are classified by a natural number $n \geq 2$ and the corresponding germ is denoted by $I_2(n)$. Let V be the matrix of the endomorphism μ restricted to L. In this section we will show that if at $p \in L$ we have

$$(M,p) \cong I_2(n) \times \Pi_{k=1}^{m-2} A_1$$

and $(T_pM)_1$ is the only two-dimensional irreducible subalgebra of T_pM then

$$V_2^1 = -V_1^2 = \pm \frac{i}{2} \frac{n-2}{n}.$$

Correspondingly, the exponent of formal monodromy of Y_F (see equation (4.2.9)) has all diagonal blocks equal to zero except for the first one which is

$$B_{[1,1]} = \begin{pmatrix} \frac{n-2}{2n} & 0 \\ 0 & -\frac{n-2}{2n} \end{pmatrix}.$$

This means that we can read of the structure of F-manifold of the germ (M,p) only from the exponent of formal monodromy. To begin let us state the classification by Hertling of germs of two-dimensional massive F-manifolds

Theorem 5.1.1. *(Hertling [14] theorem 4.7) Up to isomorphism the only germs of two-dimensional massive F-manifolds are the germs $I_2(n) = ((\mathbb{C}^2, 0), \circ, e)$ with $n \in \mathbb{N}_{\geq 2}$. On some coordinates t, u_2 the multiplication is given by*

$$\partial_{u_2} = e \qquad\qquad \partial_t \circ \partial_t = t^{n-2}\partial_{u_2}.$$

An Euler vector field is

$$E = \frac{2}{n} t\partial_t + u_2\partial_{u_2},$$

and the caustic has equation $t = 0$.

This theorem gives us a useful coordinate system on an open neighborhood $W \subset M$ around a point $p \in K$. Indeed, the germ of M at p is a product of one $I_2(n)$ manifold and $m-2$ A_1 manifolds so around p we can use the functions $(t, u_2, \ldots, u_m)$ as a coordinate system. In these coordinates the Euler vector field is

$$E = \frac{2}{n} t\partial_t + \sum_{k=2}^{m} u_i\pi_i.$$

Proposition 5.1.1. *Let $(M, \circ, e, E)$ be an F-manifold with Euler vector field. Suppose that an irreducible component $\tilde{K} \subset K$ of the caustic is semisimple. Then the Euler vector field is tangent to the regular part of $\tilde{K}$.*

Proof. On the coordinates $(t, u_2, \ldots, u_m)$ around a point $p \in \tilde{K}$ we have that $\tilde{K} = \{ t = 0 \}$ and the tangent space to $\tilde{K}$ is generated by the vector fields $\pi_i = \partial_{u_i}$ $i = 2, \ldots, m$. But on $t = 0$ we have

$$E = \sum_{k=2}^{m} u_i\pi_i.$$

□

Since the Euler vector field is tangent to $\tilde{K}$, proposition 1.2.2 tells us that $E\circ$ is diagonalizable along $\tilde{K}$.
Now let us look at the form of the metric η on the basis $\partial_t, \pi_i, i = 2, \ldots, m$. Since the algebras $(T_pM)_k$ are orthogonal between themselves and $\eta(\partial_t, \partial_t) = t^{n-2}\eta(\pi_2, \pi_2)$ we have that (here we assign the index 1 to the variable t)

$$\begin{pmatrix} t^{n-2}\eta_{22} & \eta_{12} & 0 & \ldots & 0 \\ \eta_{21} & \eta_{22} & 0 & \ldots & 0 \\ 0 & 0 & \eta_{33} & & \vdots \\ \vdots & & & \ddots & \\ 0 & 0 & \ldots & & \eta_{mm} \end{pmatrix}$$

In the case we are dealing the component $\tilde{K}$ of the caustic K has a well defined normal direction, so after choosing one side and noting that at the caustic $\eta_{11} = 0$ we get that a unitary normal to the caustic is

$$N = -i\frac{\sqrt{\eta_{22}}}{\eta_{12}}\partial_t + \frac{i}{\sqrt{\eta_{22}}}\pi_2.$$

Now consider the orthonormal basis consisting of the normal vector N and the normalized idempotents

$$f_i := \frac{\pi_i}{|\pi_i|}.$$

Theorem 5.1.2. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold with non-empty caustic K and suppose that for a point $p \in K$ the germ of M at p as an F-manifold is isomorphic to $I_2(n) \times (A_1)^{m-2}$ with $n \geq 3$. Then the only non-zero entries of the exponent of formal monodromy are $\pm\frac{i}{2}\frac{n-2}{n}$.*

Proof. We need to compute

$$V_1^2 = \eta(f_2, \mu N) = -\eta(f_2, \nabla_N E).$$

We have

$$\nabla E = \frac{2}{n}dt \otimes \partial_t + \sum_{s=2}^{m} du_s \otimes \pi_s + \frac{2}{n}t\nabla\partial_t + \sum_{s=2}^{m} u_s \nabla\pi_s.$$

Therefore using the Christoffel symbols Γ^k_{ij} of the basis $\partial_t, \pi_i, i = 2\ldots, m$ gives

$$\begin{aligned}
\nabla_{\partial_t} E &= \left(\frac{2}{n} + \frac{2}{n}t\Gamma^1_{11} + \sum_{s=2}^{m} u_s\Gamma^1_{1s}\right)\partial_t + \left(\frac{2}{n}t\Gamma^2_{11} + \sum_{s=2}^{m} u_s\Gamma^2_{1s}\right)\pi_2 + \cdots,\\
\nabla_{\pi_2} E &= \left(\frac{2}{n}t\Gamma^1_{21} + \sum_{s=2}^{m} u_s\Gamma^1_{2s}\right)\partial_t + \left(1 + \frac{2}{n}t\Gamma^2_{21} + \sum_{s=2}^{m} u_s\Gamma^2_{2s}\right)\pi_2 + \cdots.
\end{aligned}$$

With this we get

$$\begin{aligned}
V_1^2 = i\frac{n-2}{n} &+ \frac{2}{n}t\left[\frac{i}{\eta_{22}}\left(\Gamma^1_{21}\eta_{12} + \Gamma^2_{21}\eta_{22}\right) - \frac{i}{\eta_{12}}\left(\Gamma^1_{11}\eta_{12} + \Gamma^2_{11}\eta_{22}\right)\right]\\
&+ \sum_{s=2}^{m} u_s\left[\frac{i}{\eta_{22}}\left(\Gamma^1_{2s}\eta_{12} + \Gamma^2_{2s}\eta_{22}\right) - \frac{i}{\eta_{12}}\left(\Gamma^1_{1s}\eta_{12} + \Gamma^2_{1s}\eta_{22}\right)\right].
\end{aligned}$$

Now using the form of the metric and the fact that, on the caustic $\{t = 0\}$, we have $\eta_{22,s} = 0$ for $s \geq 2$ ($f_{,s}$ denotes the partial derivative of the function f with respect to the

s-th coordinate) we get

$$\frac{i}{\eta_{22}}\left(\Gamma^1_{22}\eta_{12}+\Gamma^2_{22}\eta_{22}\right)=\frac{i}{2}\frac{\eta_{22,2}}{\eta_{22}}$$
$$-\frac{i}{\eta_{12}}\left(\Gamma^1_{12}\eta_{12}+\Gamma^2_{12}\eta_{22}\right)=-\frac{i}{2}\frac{\eta_{22,1}}{\eta_{12}},$$

and for $s\geq 3$

$$\frac{i}{\eta_{22}}\left(\Gamma^1_{2s}\eta_{12}+\Gamma^2_{2i}\eta_{22}\right)=\frac{i}{2}\frac{\eta_{22,s}}{\eta_{22}}$$
$$-\frac{i}{\eta_{12}}\left(\Gamma^1_{1s}\eta_{12}+\Gamma^2_{1s}\eta_{22}\right)=-\frac{i}{2}\frac{\eta_{12,s}}{\eta_{12}}.$$

So on the caustic

$$\begin{aligned} V_1^2 &= i\left[\frac{n-2}{n}+\frac{1}{2}\left(u_2\left(\frac{\eta_{22,2}}{\eta_{22}}-\frac{\eta_{22,1}}{\eta_{12}}\right)+\sum_{s=3}^{m}u_s\left(\frac{\eta_{22,s}}{\eta_{22}}-\frac{\eta_{12,s}}{\eta_{12}}\right)\right)\right] \\ &= i\left[\frac{n-2}{n}+\frac{1}{2}\left(\sum_{i=s}^{m}u_s\left(\frac{\eta_{22,s}}{\eta_{22}}-\frac{\eta_{12,s}}{\eta_{12}}\right)+\frac{u_2}{\eta_{12}}(\eta_{12,2}-\eta_{22,1})\right)\right]. \end{aligned}$$

Along the caustic we have $E=\sum_{s=2}^m u_i\pi_i$ and the condition $\mathcal{L}_E\eta=(2-d)\eta$ implies $E(\eta_{22})-d\eta_{22}$ and $E(\eta_{12})=(-d+\frac{n-2}{n})\eta_{12}$. This gives

$$V_1^2=\frac{i}{2}\left(\frac{n-2}{n}+\frac{u_2}{\eta_{12}}(\eta_{12,2}-\eta_{22,1})\right).$$

On this coordinates we also have $\eta(e,-)=\eta_{12}dt+\sum_{i=2}^m\eta_{ii}du_i$ but by lemma 2.1.1 this form is closed and therefore $\eta_{12,2}-\eta_{22,1}=0$. □

Remark 5.1.1. The germ of F-manifold $I_2(2)$ is isomorphic to $A_1\times A_1$ and as such is not irreducible. If we suppose that at p the canonical coordinates u_1 and u_2 are equal but p is semisimple, an analogous computation would show that $\eta(f_2,\mu\frac{\pi_1}{|\pi_1|})=0$. On the work [7] it is shown that under this conditions, one can extend the isomonodromic fundamental matrix solutions of equation (4.0.1) on a semisimple point p such that $E_p\circ$ has different eigenvalues, to semisimple points q such that $E_q\circ$ has repeated eigenvalues.

Example 5.1.1. Let us compute the matrix P_1 of proposition 4.3.1. The matrix $(H_0)_{[1,1]}$ diagonalizing the block $B_{[1,1]}$ is of the form

$$(H_0)_{[1,1]}=\begin{pmatrix} r(u) & s(u)\\ ir(u) & -is(u)\end{pmatrix}$$

hence

$$H_0^T H_0 = \begin{pmatrix} 0 & 2rs \\ 2rs & 0 \end{pmatrix}$$

This gives

$$(P_1)_{[1,1]} = e^{\pi i B_{[1,1]}} z^{B_{[1,1]}} H_0^T H_0 z^{B_{[1,1]}} = \begin{pmatrix} 0 & 2rse^{\pi i \frac{n-2}{2n}} \\ 2rse^{-\pi i \frac{n-2}{2n}} & 0 \end{pmatrix}.$$

Let us use theorem 5.1.2 to compute the F-manifold decomposition for the three-dimensional Dubrovin-Frobenius manifolds of example 2.2.2. We will do this explicitly for the F_H potential, the others being analogous and simpler.
From the potential F_H of example 2.2.2 we get that on the basis $\partial_x, \partial_y, \partial_z$ the operator of multiplication by the Euler vector field has the form

$$\begin{pmatrix} x & \frac{7}{10}yz(2y+z^3) & \frac{1}{20}(12y^3 36y^2z^3+z^9) \\ \frac{3}{5}y & x+yz^2+\frac{1}{5}z^5 & \frac{7}{10}yz(2y+z^3) \\ \frac{1}{5}z^5 & \frac{3}{5}y & x \end{pmatrix}$$

Since the canonical coordinates are the eigenvalues of this matrix (proposition 1.2.3) the caustic is contained in the locus where at least two of these canonical coordinates coincide. This locus is described by the discriminant of the characteristic polynomial of this matrix which in this case is a multiple of the polynomial

$$y^2(y-z^3)^5(27y+5z^3)^3.$$

We can divide the zeroes of this polynomial in two components: the semisimple coalescence locus, where the multiplication remains semisimple and the caustic. To identify each of these components we use propositions 1.1.2 and 1.1.3. For example, along the first component of this surface $y=0$ multiplication by ∂_y has three different eigenvalues and thus $y=0$ belongs to the semisimple coalescence locus. Along the components $y=z^3$ and $y=-\frac{5}{27}z^3$ the operator of multiplication by ∂_y is not diagonalizable and therefore the caustic is the union of this two components.

The component $y=z^3$ is parametrized by $x=r, y=s^3, z=s$ and the tangent space to this surface is generated by $\partial_r = e$ and $\partial_s = 3s^2\partial_y + \partial_z$. In this basis multiplication by ∂_s has matrix

$$\begin{pmatrix} 0 & \frac{175}{4}s^8 \\ 1 & 9s^4 \end{pmatrix}.$$

The eigenvectors of this matrix are $e_2 = -\frac{25}{2}s^4\partial_x + 3s^2\partial_y + \partial_z$ and $e_3 = \frac{7}{2}s^4\partial_x + 3s^2\partial_y + \partial_z$. Along the caustic the tangent space decomposes as the direct sum of a two-dimensional and a one-dimensional algebra. To identify the unit in each of this algebras we use the

Euler vector field. In our previous notation, the eigenvalue associated with π_2 must have multiplicity two and that of π_3 has multiplicity one. Thus we obtain $e = \pi_2 + \pi_3 = -\frac{1}{16s^2}e_2 + \frac{1}{16s^2}e_3$ so the square norms of π_2 and π_3 are $-\frac{1}{16s^4}$ and $\frac{1}{16s^4}$ respectively. The unitary normal is the vector $N = -3s^2\partial_x + \partial_y$ and therefore an orthonormal basis along this component of the caustic consists of the vectors

$$\begin{aligned} N &= -3s^2\partial_x + \partial_y \\ f_2 &= i4s^2\pi_2 \\ f_3 &= 4s^2\pi_3. \end{aligned}$$

On the basis $\partial_x, \partial_y, \partial_z$ the endomorphism μ has matrix $diag(-\frac{2}{5}, 0, \frac{2}{5})$ and this gives

$$\mu_{12} = \eta(N, \mu f_2) = i\frac{3}{10}.$$

Therefore, along the component $y = z^3$ we have

$$n = 5.$$

We can parametrize the other component $y = -\frac{5}{27}z^3$ by $x = r, y = -\frac{5}{27}s^3, z = s$. An identical procedure now gives

$$n = 3.$$

The cases of B_3 and A_3 are analogous and simpler. On the B_3 Dubrovin-Frobenius manifold the matrix of the endomorphism μ is $diag(-\frac{1}{3}, 0, \frac{1}{3})$ and the bifurcation diagram has equation

$$y^2(2y - 3z^2)^4(2y + z^2)^3.$$

Again $y = 0$ corresponds to the semisimple coalescence locus and the other two components conform the caustic. On the component $\{\, 2y - 3z^2 = 0 \,\}$ we have $n = 4$ and on the component $\{\, 2y + z^2 = 0 \,\}$ we have $n = 3$. Finally the A_3 manifold has bifurcation diagram

$$y^2(27y^2 + 8z^2)$$

Once again $y = 0$ is the semisimple coalescence locus and on the other component we have $n = 3$.

5.2 Three-dimensional Dubrovin-Frobenius manifolds

In this section we compute the exponent of formal monodromy, Stokes matrices and central connection matrix of system (4.0.1) when restricted to a semisimple component of the caustic K of a three-dimensional Dubrovin-Frobenius manifold M of charge $d \neq 0$.

Let $(M, \circ, e, E, \eta)$ be a three dimensional Dubrovin-Frobenius manifold of charge $d \neq 0$ and suppose that at $p \in M$ we have

$$(M, p) \cong I_2(n) \times A_1.$$

Then p belongs to the caustic K. Let $\pi_2 \in \mathcal{T}_{I_2(n)}, \pi_3 \in \mathcal{T}_{A_1}$ be the unit vectors and suppose that the metric η restricted to K is non-degenerate (for three-dimensional Dubrovin-Frobenius manifolds of dimension 3 and charge $d \neq 0$ this is always true).

Proposition 5.2.1. *Let $(M, \circ, e, E, \eta)$ be a three dimensional Dubrovin-Frobenius manifold of charge $d \neq 0$ with non-empty caustic K. Suppose that for a point p in the regular part of an irreducible component of K we have*

$$(M, p) \cong I_2(n) \times A_1.$$

Then on the basis

$$N, f_2 = \frac{\pi_2}{|\pi_2|}, f_3 = \frac{\pi_3}{|\pi_3|}$$

The matrix V representing the endomorphism μ is

$$V = \begin{pmatrix} 0 & -i\frac{n-2}{2n} & \frac{n-2}{2n} \\ i\frac{n-2}{2n} & 0 & i\frac{d}{2} \\ -\frac{n-2}{2n} & -i\frac{d}{2} & 0 \end{pmatrix}.$$

Proof. Evaluating $\mathcal{L}_E \circ = \circ$ on (e, e) we obtain $[E, e] = -e$. Evaluating $\mathcal{L}_E \eta = (2-d)\eta$ on (e, e) and since $\nabla e = 0$ we get $(2-d)\eta(e, e) = 2\eta(e, e)$. Since $d \neq 0$ we get $\eta(e, e) = 0$. At the caustic we have $e = \pi_2 + \pi_3$ and therefore

$$|\pi_3| = i|\pi_2|.$$

Hence we obtain $e = |\pi_2|e_2 + |\pi_3|e_3 = |\pi_2|(e_2 + ie_3)$. On the other hand we have $\mu e = \frac{2-d}{2}e - \nabla_e E = -\frac{d}{2}e$ so that the vector $(0, 1, i)^T$ is an eigenvector of the matrix V with eigenvalue $\frac{d}{2}$. Thus we obtain

$$-\frac{d}{2}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 & V_2^1 & V_3^1 \\ -V_2^1 & 0 & V_3^2 \\ -V_3^1 & -V_3^2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$$

From this we get the equations $V_3^1 = iV_2^1$ and $V_3^2 = i\frac{d}{2}$. But by theorem 5.1.2 we now that the entry V_2^1 is $-\frac{i}{2}t\frac{n-2}{2n}$. $\square$

We now start the computation of the Stokes matrices and the central connection matrix. Let $a = i\frac{d}{2}$ and $b = V_2^1$. The differential equation (4.0.1) is

$$\frac{dY}{dz} = \left(\begin{pmatrix} -u_2 & 0 & 0 \\ 0 & -u_n & 0 \\ 0 & 0 & -u_3 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 0 & b & ib \\ -b & 0 & a \\ -ib & -a & 0 \end{pmatrix}\right)Y. \tag{5.2.1}$$

Using the Gauge transformation $Y = e^{u_2 z}\tilde{Y}$ we get a new system

$$\frac{d\tilde{Y}}{dz} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_2 - u_3 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 0 & b & ib \\ -b & 0 & a \\ -ib & -a & 0 \end{pmatrix} \right) \tilde{Y}.$$

Next we do the change of variables $z = (u_2 - u_3)w$ and obtain

$$\frac{dY}{dw} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{w} \begin{pmatrix} 0 & b & ib \\ -b & 0 & a \\ -ib & -a & 0 \end{pmatrix} \right) Y. \tag{5.2.2}$$

By doing the formal Gauge transformation (4.2.1) we get a block diagonal system $\frac{d\tilde{Y}}{dw} = (-U + B_1 w^{-1} + B_2 w^{-2} + \cdots)\tilde{Y}$ with

$$B_1 = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} -b^2 & iab & 0 \\ iab & a^2 & 0 \\ 0 & 0 & b^2 - a^2 \end{pmatrix} \qquad G_1 = \frac{1}{u_2 - u_3} \begin{pmatrix} 0 & 0 & ib \\ 0 & 0 & a \\ ib & a & 0 \end{pmatrix}.$$

This matrices are obtained from the equations (4.2.3).We now do the usual transformation $Y = e^{A_0}Y$ with $A_0 = diag(0, 01)$ to cancel the matrix A_0. A matrix that diagonalizes the matrix B_1 is

$$H_0 := \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we perform the Gauge transformation $\tilde{Y} = (Id + \sum_{k=1}^{\infty} H_k w^{-k})\hat{Y}$ to obtain the system $\frac{d\hat{Y}}{dw} = \frac{B}{w}\hat{Y}$ with $B = diag(ib, -ib, 0)$. We have

$$H_1 = \begin{pmatrix} \frac{b^2 - a^2}{2} & -\frac{i(a-b)^2}{4b-2i} & 0 \\ \frac{i(a+b)^2}{4b+2i} & \frac{b^2-a^2}{2} & 0 \\ 0 & 0 & a^2 - b^2 \end{pmatrix}$$

and H_1 is computed from the equation (see equation (4.2.8))

$$[B, H_1] + H_1 + \hat{B}_2 = 0.$$

Putting all together we obtain that a formal fundamental matrix solution of system (5.2.2) is given by

$$Y_F = (H_0 + (H_0 H_1 + G_1 H_0) w^{-1} + O(w^{-2})) e^{A_0 w} w^B$$

For the computation of the Stokes matrix we will need the first row of this matrix which consists of the following series

$$Y_{11} = w^{ib} + O(w^{-1}) \qquad Y_{12} = w^{-ib} + O(w^{-1}) \qquad Y_{13} = ibe^{w} w^{-1} + O(w^{-2}). \tag{5.2.3}$$

The differential system (5.2.2) consists of the equations

$$\begin{aligned}
\dot{y}_1 &= b\frac{y_2 + iy_3}{w} \\
\dot{y}_2 &= -\frac{by_1 - ay_3}{w} \\
\dot{y}_3 &= y_3 - \frac{iby_1 + ay_2}{w}.
\end{aligned} \tag{5.2.4}$$

To compute the Stokes matrix we will use the third order scalar ordinary differential equation satisfied by y_1. One can check that this equation is

$$\dddot{y} + \frac{2-w}{w}\ddot{y} + \frac{a(a-i)-w}{w^2}\dot{y}_1 - \frac{b^2}{w^2}y = 0. \tag{5.2.5}$$

The general solution to this equation is

$$\begin{aligned}
&\alpha_2F_2\left(\begin{smallmatrix} -ib & ib \\ 1+ia & -ia \end{smallmatrix}; w\right) + \beta e^{a\pi} w^{-ia}{}_2F_2\left(\begin{smallmatrix} -i(a+b) & i(b-a) \\ 1-ia & -2ia \end{smallmatrix}; w\right) + \\
&\qquad \gamma e^{-a\pi} w^{1+ia}{}_2F_2\left(\begin{smallmatrix} 1+i(a-b) & 1+i(a+b) \\ 2+ia & 2+2ia \end{smallmatrix}; w\right) =: \alpha F_1 + \beta F_2 + \gamma F_3.
\end{aligned} \tag{5.2.6}$$

The constants α, β and γ are chosen according to the solutions provided by the first row (5.2.3) of a formal solution of the system (5.2.2), we now give more details. First let us write down the asymptotic expansion of the general solution (5.2.6), we have that on the sector $-(2+\epsilon)\frac{\pi}{2} < arg(w) < (2-\epsilon)\frac{\pi}{2}$, with $\epsilon\pm 1$, the function ${}_2F_2$ has asymptotic expansion (see [9])

$$\begin{aligned}
{}_2F_2\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix} | w\right) \sim \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}&[e^w w^{\alpha_1+\alpha_2-\beta_1-\beta_2} + \\
&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2\alpha_1)}(e^{\epsilon\pi i}w)^{-\alpha_1} + \\
&\qquad \frac{\Gamma(\alpha_2)\Gamma(\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_2-\alpha_2)}(e^{\epsilon\pi i}w)^{\alpha_2}](1+O(w^{-1}))
\end{aligned} \tag{5.2.7}$$

which we will write as

$${}_2F_2\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix} | w\right) \sim [R_i e^w w^{\alpha_1+\alpha_2-\beta_1-\beta_2} + S_i(e^{\epsilon\pi i}w)^{-\alpha_1} + T_i(e^{\epsilon\pi i}w)^{-\alpha_2}](1+O(w^{-1}))$$

The subindices of R_i, S_i, T_i will correspond to the three solutions F_1, F_2, F_3 of (5.2.6). In this way we obtain that on the sector $-(2+\epsilon)\frac{\pi}{2} < arg(w) < (2-\epsilon)\frac{\pi}{2}$ the functions F_i have asymptotic expansion

$$\begin{aligned}
F_1 &\sim (R_1 e^w w^{-1} + S_1 e^{-\epsilon\pi b} w^{ib} + T_1 e^{\epsilon\pi b} w^{-ib})(1+O(w^{-1})) \\
F_2 &\sim (R_2 e^{\pi a} e^w w^{-1} + S_2 e^{(1-\epsilon)\pi a} e^{-\epsilon\pi b} w^{ib} + T_2 e^{(1-\epsilon)\pi a} e^{\epsilon\pi b} w^{-ib})(1+O(w^{-1})) \\
F_3 &\sim (R_3 e^{-\pi a} e^w w^{-1} - S_3 e^{(\epsilon-1)\pi a} e^{-\epsilon\pi b} w^{ib} - T_3 e^{(\epsilon-1)\pi a} e^{\epsilon\pi b} w^{-ib})(1+O(w^{-1})).
\end{aligned} \tag{5.2.8}$$

With this, on the sectors $-(2+\epsilon)\frac{\pi}{2} < arg(w) < (2-\epsilon)\frac{\pi}{2}$, the general solution (5.2.6) of equation (5.2.5) has asymptotic expansion

$$[(\alpha R_1 + \beta R_2 e^{\pi a} + \gamma R_3 e^{-\pi a})e^w w^{-1} + \\ (\alpha S_1 e^{-\epsilon\pi b} + \beta S_2 e^{(1-\epsilon)\pi a} e^{\epsilon\pi b} - \gamma S_3 e^{(\epsilon-1)\pi a} e^{-\epsilon\pi b}) w^{ib} + \\ (\alpha T_1 e^{\epsilon\pi b} + \beta T_2 e^{(1-\epsilon)\pi a} e^{\epsilon\pi b} - \gamma T_3 e^{(\epsilon-1)\pi a} e^{\epsilon\pi b}) w^{-ib}](1 + O(w^{-1})). \quad (5.2.9)$$

We now compute the Stokes matrix. According to Sibuya's theorem on the sectors $S^{(\epsilon)} := \{ w \in \mathbb{P}^1 | -(2+\epsilon)\frac{\pi}{2} < arg(w) < (2-\epsilon)\frac{\pi}{2} \} \cap \{ w | R < |w| \}$, with $\epsilon = \pm 1$ and $R >> 0$ there exists solutions $y_1^{(\epsilon)}, y_2^{(\epsilon)}, y_3^{(\epsilon)}$ of equation (5.2.5) which have the asymptotic expansion Y_{11}, Y_{12}, Y_{13} of (5.2.3). To compute the Stokes matrix we need to write the solutions $y_i^{(-1)}$ as linear combinations of the solutions $y_i^{(1)}$. To do this we use the solutions F_i given by (5.2.6) of equation (5.2.5); we will write $(F_1, F_2, F_3) = (y_1^{(1)}, y_2^{(1)}, y_3^{(1)})P$ and $(y_1^{(-1)}, y_2^{(-1)}, y_3^{(-1)}) = (F_1, F_2, F_3)Q$ so that $(y_1^{(-1)}, y_2^{(-1)}, y_3^{(-1)}) = (y_1^{(1)}, y_2^{(1)}, y_3^{(1)})PQ$ and the Stokes matrix is $S = PQ$.
In order to do this note that since the sectors $S^{(\epsilon)}$ contain a complete collection of Stokes rays, the solutions having a prescribed asymptotic expansion on each sector are unique. Comparing (5.2.9) with (5.2.8) we get

$$F_1 = S_1 e^{-\pi b} y_1^{(1)} + T_1 e^{\pi b} y_2^{(1)} - \frac{i}{b} R_1 y_3^{(1)}$$
$$F_2 = S_2 e^{-\pi b} y_1^{(1)} + T_2 e^{\pi b} y_2^{(1)} - \frac{i}{b} R_2 e^{\pi a} y_3^{(1)}$$
$$F_3 = S_3 e^{-\pi b} y_1^{(1)} + T_3 e^{\pi b} y_2^{(1)} - \frac{i}{b} R_3 e^{-\pi a} y_3^{(1)}$$

so the matrix P is

$$\begin{pmatrix} S_1 e^{-\pi b} & S_2 e^{-\pi b} & -S_3 e^{-\pi b} \\ T_1 e^{\pi b} & T_2 e^{\pi b} & -T_3 e^{\pi b} \\ -\frac{i}{b} R_1 & -\frac{i}{b} R_2 e^{\pi a} & -\frac{i}{b} R_3 e^{-\pi a} \end{pmatrix}. \quad (5.2.10)$$

To obtain the matrix Q we need to find the constants α, β, γ such that (5.2.9) has the asymptotic expansion of (5.2.3). That is we need to solve the following systems of linear algebraic equations

$$\begin{pmatrix} R_1 & R_2 e^{\pi a} & R_3 e^{-\pi a} \\ S_1 e^{\pi b} & S_2 e^{2\pi a} e^{\pi b} & -S_3 e^{-2\pi a} e^{\pi b} \\ T_1^{-\pi b} & T_2 e^{2\pi a} e^{-\pi b} & -T_3 e^{-2\pi a} e^{-\pi b} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} ib \\ 0 \\ 0 \end{pmatrix}.$$

The solutions can be obtained with the help of a computer software. The matrix Q has as columns the solutions $(\alpha, \beta, \gamma)^T$ and again, using a computer software, and substituting the values of R_i, S_i, T_i which can be obtained from (5.2.7) we get that the Stokes matrix $S = PQ$ is

$$S = \begin{pmatrix} 1 & 0 & \frac{2b\pi\Gamma(2ib)}{\Gamma(i(b-a))\Gamma(1+ib)\Gamma(1+i(a+b))} \\ 0 & 1 & \frac{2b\pi\Gamma(-2ib)}{\Gamma(1-ib)\Gamma(1+i(a-b))\Gamma(-i(a+b))} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.11}$$

We now compute the central connection matrix relating the solutions $Y_{\mathcal{L}}$ and Y_0. We want to find solutions of system (5.2.2) around $z = 0$. First we diagonalize V via the transformation $Y = T_0\tilde{Y}$ with

$$T_0 = \begin{pmatrix} 0 & a & 2ab \\ 1 & ib & -i(a^2+b^2) \\ i & b & b^2 - a^2 \end{pmatrix}.$$

We obtain a new system

$$\frac{d\tilde{Y}}{dw} = \left(\frac{1}{w}\begin{pmatrix} ia & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ia \end{pmatrix} + \frac{1}{a^2}\begin{pmatrix} \frac{a^2-b^2}{2} & \frac{ib(b^2-a^2)}{2} & \frac{i(a^2-b^2)^2}{2} \\ ib & b^2 & b^3 - a^2 b \\ -\frac{i}{2} & -\frac{b}{2} & \frac{a^2-b^2}{2} \end{pmatrix} \right) \tilde{Y}$$
$$=: \left(\frac{\mu}{w} + A_1 \right) \tilde{Y}$$

For simplicity we now assume $2ia \notin \mathbb{Z}$ and therefore there exists a holomorphic Gauge transformation $\tilde{Y} = (Id + \sum_{k=1}^{\infty} T_k w^k)\hat{Y}$ which takes the previous equation to $\frac{d\hat{Y}}{dw} = \mu z^{-1}\hat{Y}$. The matrices T_k can be computed from equation (4.1.4). In particular we have

$$T_1 = \frac{1}{a^2}\begin{pmatrix} \frac{a^2-b^2}{2} & \frac{b(b^2-a^2)}{2(a+i)} & -\frac{a^2+b^2}{2(2a+i)} \\ \frac{b}{a-1} & b^2 & i\frac{b(b^2-a^2)}{a+i} \\ \frac{1}{2(i-2a)} & \frac{ib}{2(a-i)} & \frac{a^2-b^2}{2} \end{pmatrix}$$

and the fundamental matrix solution in Levelt form of (5.2.2) is $Y = T_0(Id + \sum_{k=1}^{\infty} T_k z^k)z^V$. In particular the first row of this matrix has expansion

$$\begin{aligned} y_1 &= \frac{b}{a(2a-3i)-1} w^{1+ia} + \cdots \\ y_2 &= aw + \frac{b^2}{a-i} w^2 + \cdots \\ y_3 &= w^{-ia}\left(2ab + \frac{b(a^2-b^2)}{a+i} w + \cdots\right) \end{aligned} \tag{5.2.12}$$

Now recall that the first component y_1 of the system (5.2.2) satisfies the third order differential equation (5.2.5). A basis of the solutions consists of the functions F_1, F_2, F_3 of

(5.2.6). To compute the central connection matrix we follow the same procedure as for the Stokes matrix: The matrix P of (5.2.10) expresses the solutions F_i as a linear combination of the solutions $y_i^{(1)}$ having asymptotic expansion (5.2.3) on the sector $S^{(1)}$, now we compute a matrix D expressing the solutions y_i of (5.2.12) as linear combinations of the solutions F_i. The connection matrix will be $C = PD$.
The hypergeometric function ${}_2F_2$ appearing in the solutions F_i is a holomorphic at $z = 0$ so at this point this solutions have the expansion (see [9])

$$
\begin{aligned}
F_1 &= w + \frac{b^2}{a(a-i)} w^2 + \cdots \\
F_2 &= e^{a\pi} w^{-ia} (1 + \frac{a^2 - b^2}{2a(a+i)} w + \cdots) \\
F_3 &= e^{-a\pi} w^{1+ia} (1 + \frac{1 + 2ia - (a^2 - b^2)}{4 + 6ia - 2a^2} w + \cdots).
\end{aligned}
\tag{5.2.13}
$$

Comparing with (5.2.12) we get

$$
D = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 2e^{-a\pi} ab \\ \frac{e^{a\pi} b}{a(2a-3i)-1} & 0 & 0 \end{pmatrix}.
$$

Again, with the help of a computer software we get that the central connection matrix C is

$$
\begin{pmatrix}
-\frac{2a^2 b e^{(a-b)\pi} \Gamma(ia)\Gamma(2ia)\Gamma(2ib)}{\Gamma(1+ib)\Gamma(1+i(a+b))^2} & \frac{i2^{-1+2ib} a e^{-b\pi} \sqrt{\pi} csch(a\pi)\Gamma(\frac{1}{2}+ib)}{\Gamma(-i(a-b))\Gamma(1+i(a+b))} & \frac{2abe^{-(a+b)\pi}\Gamma(1-ia)\Gamma(-2ia)\Gamma(2ib)}{\Gamma(-i(a-b))^2\Gamma(1+ib)} \\
-\frac{2a^2 b e^{(a+b)\pi} \Gamma(ia)\Gamma(2ia)\Gamma(-2ib)}{\Gamma(1-ib)\Gamma(1+ia-ib)^2} & \frac{i2^{-1-2Ib} a e^{b\pi} \sqrt{\pi} csch(a\pi)\Gamma(\frac{1}{2}-ib)}{\Gamma(1+i(a-b))\Gamma(-i(a+b))} & \frac{2abe^{(-a+b)\pi}\Gamma(1-ia)\Gamma(-2ia)\Gamma(-2ib)}{\Gamma(1-ib)\Gamma(-i(a+b))^2} \\
-\frac{2a\Gamma(1+ia)\Gamma(2ia)}{\Gamma(1+i(a-b))\Gamma(1+i(a+b))} & a\, csch(a\pi) sinh(b\pi) & -\frac{2a^2\Gamma(-ia)\Gamma(-2ia)}{\Gamma(-i(a-b))\Gamma(-i(a+b))}
\end{pmatrix}.
$$

Chapter 6

Open Problem: Changing Strata

6.1 Changing Strata

Let $p \in M$ with $T_pM \cong \oplus_{k=1}^{l}(T_pM)_k$ and let $L \subset M$ be the integral manifold of the idempotents $\pi_1, \ldots, \pi_l$ passing through p. We have seen that the points of the Dubrovin-Frobenius manifold parametrize a family of meromorphic ordinary differential equations on $\mathbb{P}^1$. By restricting ourselves to a neighborhood $W \subset L$ of p we have seen that it is possible to construct solutions Y_{Lev}, Y_ν such that the corresponding monodromy data (μ, R, B, S, C) are constant. The natural question now is how this monodromy data changes as we move to further substrata.

For example, if the caustic is non-empty then the boundary of the semisimple locus is the caustic. When writing the solutions of equation (4.0.1) outside the caustic, one uses the basis of idempotents. At the caustic this basis no longer exists and as such, some of these solutions will "diverge" as we approach the caustic. Nevertheless let us argue that there should be a relation between the solutions outside and inside the caustic.
Take $z_0 \in \mathbb{C}^*$ and a point $p_0 \in K$. Since the differential equation (4.0.1) is holomorphic in a small neighborhood of (z_0, p_0), the existence theorem for ordinary differential equations states that there exists a small neighborhood $T \times W \subset \mathbb{P}^1 \times M$ such that equation (4.0.1) has a fundamental matrix solution Y_{NS} and moreover this solution depends holomorphically on $(z, p) \in T \times W$. Since the caustic is an hypersurface and $W \subset M$ is open then $W \setminus K \neq \emptyset$. Applying again the existence theorem to a point (z_0, q_0) with $q_0 \in W \setminus K$ we get a new open set $\tilde{T} \times \tilde{W} \subset \mathbb{P}^1 \times M$ and a new fundamental matrix solution Y_S of equation (4.0.1). Notice that $(z_0, q_0) \in (T \times W) \cap (\tilde{T} \times \tilde{W})$. Since on the open $(T \times W) \cap (\tilde{T} \times \tilde{W})$ set Y_{NS} and Y_L are both fundamental matrix solutions of equation (4.0.1), there exists a matrix $\mathcal{C}$ such that

$$Y_S = Y_{NS}\mathcal{C}.$$

Note that this relation allows to extend the fundamental matrix solution Y_S to points on the caustic. Indeed for $(z,p) \in (\mathbb{P}^1 \times K) \cap (T \times W)$ we can set

$$Y_S(z,p) := Y_{NS}(z,p)\mathcal{C}.$$

We now give a partial answer to the problem of finding the matrix $\mathcal{C}$. Let $\tilde{L} := \bar{L} \setminus L$ be the topological boundary of L. First we have

Proposition 6.1.1. *Suppose $\tilde{L} \neq \emptyset$. If $p \in L$ with $T_pM \cong \oplus_{k=1}^{l_p}(T_pM)_k$ and $q \in \tilde{L}$ with $T_qM \cong \oplus_{k=1}^{l_q}(T_qM)_k$ then $l_q < l_p$. That is, as we move to further substrata the dimensions of the irreducible algebras in which the tangent space decomposes can only grow.*

Proof. If at a point $q \in M$ the tangent space T_qM has l idempotents then theorem 1.1.1 says that in a neighborhood of this point this l we have at least l idempotent vector fields. Since every neighborhood of a point $q \in \tilde{L}$ intersects L we get $l_q \leq l_p$. But equality would imply $q \in L$. □

So as we pass from a point $p \in L$ to a point $q \in \tilde{L}$ the irreducible algebras in which $\mathcal{T}_M$ is decomposed grow in dimension and as a consequence we loose some idempotents and some canonical coordinates u_i of L coalesce.
Let $Y_{Lev}, Y_\nu, \tilde{Y}_{Lev}, \tilde{Y}_\nu$ be the isomonodromic solutions of family (4.0.1) restricted to L and $\tilde{L}$ respectively. The basis used to construct the solution Y_{Lev} of equation (4.0.1) doesn't depend on the multiplication structure and as such we have

$$Y_{Lev}|_{\tilde{L}} = \tilde{Y}_{Lev}.$$

On the other hand, the basis used to construct the solutions Y_ν used the decomposition of T_pM into multiplication invariant subspaces. In particular the formal solution Y_F (see (4.2.9)) of equation (4.0.1) have terms of the form $\frac{1}{u_i - u_j}$ and therefore the asymptotic expansions cease to have meaning for points in $\tilde{L}$. In this section we show that, after a proper Gauge transformation, the columns of Y_F corresponding to blocks whose corresponding canonical coordinate u_k does not collide with any other canonical coordinate, coincide with some of the columns of the formal solution $\tilde{Y}_F$.

Let $q \in \tilde{L}$ and suppose that $T_qM \cong \oplus_{k=1}^{l_q}(T_qM)_k$ where each $(T_qM)_k$ is an irreducible algebra. By Hertling's decomposition (1.1.1) we have that

$$(M,q) \cong \Pi_{k=1}^{l_q}(M_k, q).$$

Now take $p \in L \cap (M,q)$, by the above we have $T_pM \cong \oplus_{k=1}^{l_q} T_pM_k$ where each T_pM_k is an algebra but not necessarily irreducible. Suppose that out of these l_q algebras n of them are reduible. We order the algebras T_pM_k in such a way that for $k \leq n$ the algebras T_pM_k are reducible and for $k > n$ they are irreducible.
The algebras T_pM_k for $k \leq n$ decompose as a direct sum of the l_k irreducible algebras

$(T_pM)_{k_i}$. For each of these irreducible algebras we have a canonical coordinate u_{k_i} on the manifold L. When we move to $\tilde{L}$ the canonical coordinates u_{k_i} coalesce to a single function $\tilde{u}_k$ which is a canonical coordinate on the manifold $\tilde{L}$. On the other hand, for $k > n$ the algebras T_pM_k are irreducible and on the limit $p \to q$ they remain irreducible. Correspondingly the canonical coordinates u_k don't coalesce with any other canonical coordinate and on the limit we have $u_k = \tilde{u}_k$ which again is a canonical coordinate on the manifold $\tilde{L}$.
This ordering induces a block decomposition on the matrices we are about to compute. In the following we will use the partition of the set $\{1, \ldots, m\}$ by the sets (k) given by the decomposition of T_qM into irreducible algebras. The symbol $A_{[i,j]}$ will denote the block consisting the entries with rows in the set (i) and columns in the set (j).
Take a point $q \in \tilde{L}$ and let $W \subset (M, q)$ be a sufficiently small open neighborhood of q. Let $\tilde{e}_{k_i}$ be the basis of $\tilde{\imath}^*\mathcal{T}_M$ of proposition 3.2.5. Since the decomposition of (M, q) into irreducible F-manifolds holds true in the open set W and the metric η is holomorphic, we can extend this basis to an orthonormal basis of $\mathcal{T}_W$.
Since $\tilde{L}$ is the (topological) boundary of L then $W \cap L \neq \emptyset$. Applying again proposition 3.2.5 we get an orthonormal basis e_{k_i} of $\iota^*\mathcal{T}_M$. If the open set W is small enough then we can extend the basis e_{k_i} to an orthonormal basis of $\mathcal{T}_{W\setminus\tilde{L}}$. The reason why we cannot extend the basis e_{k_i} to the whole neighborhood W is that some elements of the basis e_{k_i} no longer exists on $\tilde{L}$. For example, according to our ordering on the first $k \leq n$ algebras $(\mathcal{T}_{M,p})_k$ with $p \in W \cap L$, as we move to $q \in \tilde{L}$ the idempotents $\pi_k \in (\mathcal{T}_{M,p})_k$ no longer exist at q.
Now let Q be the matrix whose columns are the vectors $\tilde{e}_{k_i}$ written as linear combinations of the vectors e_{k_i} (*i.e.* $[\tilde{e}_1, \ldots, \tilde{e}_m] = [e_1, \ldots, e_m]Q$). The matrix Q is holomorphic on $W \setminus \tilde{L}$. By the compatibility of the metric and the multiplication Q is a block diagonal matrix with blocks $Q_i := Q_{[i,i]}$ for $i \leq n$ and $Q_i = Id$ for $i > n$. We consider the familiy of differential equations (4.0.1) restricted to L

$$\frac{dY}{dz} = \left(\frac{1}{z}V - U\right)Y \tag{6.1.1}$$

and to $\tilde{L}$

$$\frac{d\tilde{Y}}{dz} = \left(\frac{1}{z}\tilde{V} - \tilde{U}\right)\tilde{Y}. \tag{6.1.2}$$

The above change of basis induces a Gauge transformation $Y = QX$ from which we obtain

$$\tilde{V}_{[i,j]} = \begin{cases} Q_i^{-1}V_{[i,j]}Q_j & i, j \leq n \\ Q_i^{-1}V_{[i,j]} & i \leq n < j \\ V_{[i,j]}Q_j & j \leq n < i \\ V_{[i,j]} & i, j > n, \end{cases} \tag{6.1.3}$$

where the right hand side is evaluated at $\tilde{L}$. We also have

$$\begin{aligned}\left[\tilde{U},T\right]_{[i,j]} &= \sum_{s=1}^{l_q} \tilde{U}_{[i,s]}T_{[s,j]} - T_{[i,s]}\tilde{U}_{[s,j]} \\ &= \tilde{U}_{[i,i]}T_{[i,j]} - T_{[i,j]}\tilde{U}_{[j,j]} = (\tilde{u}_i - \tilde{u}_j)T_{[i,j]}\end{aligned}$$

because the diagonal blocks $\tilde{U}_{[i,i]}$ have only one eigenvalue. On the other hand, since the diagonal blocks of U may have more than one eigenvalue we only get

$$[U,T]_{[i,j]} = U_{[i,i]}T_{[i,j]} - T_{[i,j]}U_{[j,j]}$$

but when we evaluate at points of $\tilde{L}$ the eigenvalues of $U_{[i,i]}$ become equal and we recover

$$[U,T]_{[i,j]} = (u_i - u_j)T_{[i,j]} = (\tilde{u}_i - \tilde{u}_j)T_{[i,j]}.$$

Recall the recursive equations (4.2.3) used to compute the formal solutions Y_F and $\tilde{Y}_F$. We can now show

Lemma 6.1.1. *With the above notations, for all k we have*

$$(\tilde{G}_k)_{[i,j]} = Q_i^{-1}(G_k)_{[i,j]} \qquad\qquad (\tilde{B}_k)_{[i,j]} = (B_k)_{[i,j]} = 0$$

when $i \le n < j$ and

$$(\tilde{G}_k)_{[i,j]} = (G_k)_{[i,j]} \qquad\qquad (\tilde{B}_k)_{[i,j]} = (B_k)_{[i,j]}$$

when $i, j > n$ and the right hand sides of the equalities is evaluated at $\tilde{L}$.

Proof. We proceed by induction, all the quantities without ˜ are evaluated at $\tilde{L}$. For $k = 1$ and $i \le n < j$ we have

$$(\tilde{u}_i - \tilde{u}_j)(\tilde{G}_1)_{[i,j]} = \tilde{V}_{[i,j]} = Q_i^{-1}V_{[i,j]} = (u_i - u_j)Q^{-1}(G_k)_{[i,j]}.$$

and if $i, j > n$ then

$$(\tilde{u}_i - \tilde{u}_j)(\tilde{G}_1)_{[i,j]} + (\tilde{B}_k)_{[i,j]} = \tilde{V}_{[i,j]} = V_{[i,j]} = (u_i - u_j)(G_1)_{[i,j]} + (B_k)_{[i,j]}$$

so if $i = j$ then $(B_1)_{[i,j]} = (\tilde{B}_1)_{[i,j]} = V_{[i,j]}$ and $(G_1)_{[i,j]} = (\tilde{G}_1)_{[i,j]} = 0$. If $i \ne j$ then $(B_1)_{[i,j]} = (\tilde{B}_1)_{[i,j]} = 0$ and $(G_1)_{[i,j]} = (\tilde{G}_1)_{[i,j]}$. Now we make the induction step, we have that

$$[\tilde{U}, \tilde{G}_k] + \tilde{B}_k = (k-1)\tilde{G}_{k-1} + \tilde{V}\tilde{G}_{k-1} - \sum_{s=1}^{k-1} \tilde{G}_{k-s}\tilde{B}_s.$$

Suppose $i \leq n < j$ then we can write

$$\begin{aligned}
(\tilde{V}\tilde{G}_k)_{[i,j]} &= \sum_{s=1}^{l_q} \tilde{V}_{[i,s]}(\tilde{G}_k)_{[s,j]} \\
&= \sum_{s\leq n} \tilde{V}_{[i,s]}(\tilde{G}_k)_{[s,j]} + \sum_{s>n} \tilde{V}_{[i,s]}(\tilde{G}_k)_{[s,j]} \\
&= \sum_{s\leq n} Q_i^{-1}V_{[i,s]}Q_sQ_s^{-1}(G_k)_{[s,j]} + \sum_{s>n} Q_i^{-1}(V_{[i,s]}G_k)_{[s,j]} \\
&= Q_i^{-1}(VG_k)_{[i,j]}.
\end{aligned}$$

In the same way

$$\begin{aligned}
(\tilde{G}_{k-s}\tilde{B}_s)_{[i,j]} &= \sum_{t=1}^{l_q}(\tilde{G}_{k-s})_{[i,t]}(\tilde{B}_k)_{[t,j]} = \sum_{t>n}(\tilde{G}_{k-s})_{[i,t]}(\tilde{B}_k)_{[t,j]} \\
&= \sum_{t>n} Q_i^{-1}(G_{k-s})_{[i,t]}(B_k)_{[t,j]} \\
&= Q_i^{-1}(G_{k-s}B_s).
\end{aligned}$$

Putting these last two equations we get

$$\begin{aligned}
(\tilde{u}_i - \tilde{u}_j)(\tilde{G}_k)_{[i,j]} &= ((k-1)\tilde{G}_{k-1} + \tilde{V}\tilde{G}_{k-1} - \sum_{s=1}^{k-1} \tilde{G}_{k-s}\tilde{B}_s)_{[i,j]} \\
&= Q_i^{-1}((k-1)G_{k-1} + VG_{k-1} - \sum_{s=1}^{k-1} G_{k-s}B_s)_{[i,j]} \\
&= (u_i - u_j)Q_i^{-1}(G_k)_{[i,j]}.
\end{aligned}$$

The induction step for the blocks $i, j > n$ is done analogously. $\square$

Observe that after applying the Gauge transformation G followed by e^{-Uz} we obtain the block diagonal system (4.2.5). Since $(B_k)_{[i,j]} = (\tilde{B}_k)_{[i,j]}$ for $i, j > n$ we get that $(H_k)_{[i,j]} = (\tilde{H}_k)_{[i,j]}$. This observation and the previous lemma immediately imply

Proposition 6.1.2. *Let $(M, \circ, e, E, \eta)$ be a Dubrovin-Frobenius manifold. Suppose that at $p \in M$ we have $T_pM \cong \oplus_{k=1}^{l_p}(T_pM)_k$ where each $(T_pM)_k$ is an irreducible algebra. Let L be the integral submanifold of the idempotents $\pi_1, \ldots, \pi_l$ passing through p and suppose that $\tilde{L} := \bar{L} \setminus L \neq \emptyset$. Let $q \in \tilde{L}$ and suppose that as germ of F-manifolds*

$$(M, q) \cong \Pi_{k=1}^{l_q}(M_k, q),$$

with each (M_k, q) *irreducible. Order the algebras* T_pM_k *in such a way that for* $k \leq n$ *the algebra* T_pM_k *is reducible and for* $k > n$ *we have* $T_pM_k = (T_pM)_k$. *Let* Q *be the matrix of change of basis on the open neighborhood* $W \subset (W, q)$ *between the basis* e_{k_i} *and* $\tilde{e}_{k_i}$ *of proposition 3.2.5 applied to* L *and* $\tilde{L}$ *respectively. Let* Y_F *and* $\tilde{Y}_Y$ *be the formal solutions of equations (6.1.1) and (6.1.2) respectively.*
Then the last $m - \sum_{k=1}^{n} dim T_pM_k$ *columns of* $\tilde{Y}_F$ *are the last columns of* $Q^{-1}Y_F|_{\tilde{L}}$

Conclusions

The most important result of this work is theorem 4.3.2. In a certain sense in it paves the way for the study of non-generic isomonodromic deformations. To put this result in context, let us briefly recall the history of isomonodromic deformations of meromorphic ordinary differential equations over $\mathbb{P}^1$.

The study of isomonodromic deformations of meromorphic ordinary differential equation goes back to Riemann. In the case of regular singularities he already posed the problem in its full generality: To construct a system of functions with regular singularities that has the prescribed monodromy data. This problem, in the non-resonant case, was solved by Schlesinger, Fuchs and Garnier and can be summarized in the Schlesinger equations. This are a system of non-linear partial differential equations that the matrices defining the family of ordinary differential equations must satisfy in order that the family has constant monodromy data.

For a long time the interest in isomonodromic families of meromorphic ordinary differential equations receded and it was until the work of Jimbo, Miwa and Ueno that the study of isomonodromic deformations regained interest. In the seminal paper [15], Jimbo, Miwa and Ueno studied monodromy deformations of "generic" families of meromorphic ordinary differential equations. In particular they studied systems that not only had regular singularities. Part of the genericness assumption is that the eigenvalues of the leading term at the poles of the differential equation have different eigenvalues. In the case of Dubrovin-Frobenius manifolds this means that we should restrict ourselves to the subset of semisimple loci such that the endomorphism of multiplication by the Euler vector fields has different eigenvalues.

On recent years Cotti, Dubrovin and Guzzetti studied isomonodromic deformations of meromorphic differential equations of the form

$$\frac{dY}{dz} = \left(\Lambda(t) + \frac{A_1(t)}{z}\right) Y \tag{6.1.4}$$

where Λ is a diagonal matrix. In [6] theorem 1.1 they showed that, if t varies in an open domain W which contains a subset Δ where some of the eigenvalues of Λ coalesce and if the entries of the matrix A_1 corresponding to the eigenvalues that coalesce at Δ vanish at Δ (*i.e.* if $\lambda_i(t) - \lambda_j(t)|_\Delta = 0$ then $(A_1)_{ij}|_\Delta = 0$), then certain fundamental matrix solutions (holomorphic and formal) of equation (6.1.4) defined outside the coalescence loci Δ can be holomorphically continued in the parameter t to the coalescence loci. Moreover, if the fundamental matrix solutions were isomonodromic then their extension to Δ remains isomonodromic. In particular this means that one can compute the monodromy data of the family (6.1.4) on a point of the coalescence loci Δ.

As we have seen, on the caustic of a Dubrovin-Frobenius manifold the vanishing condition of theorem 1.1 of [6] is not necessarily satisfied. As a consequence one cannot extend the isomonodromic fundamental matrix solutions outside the caustic to points inside the caustic. Nevertheless if the coefficients of equation (6.1.4) are holomorphic on the coalescence loci Δ then the existence theorem for ordinary differential equations guarantees that fundamental matrix solutions of equation (6.1.4) exist for points $t_\Delta \in \Delta$ and they depend holomorphically on a small open neighborhood $\tilde{W} \subset W$.

Basically our result says that, thanks to the geometric properties of a Dubrovin-Frobenius manifold, the fundamental matrix solutions computed in the coalescence loci (the multiplication invariant submanifolds $L \subset K$) can also be taken to be isomonodromic. The obvious and really hard question is how these two sets of isomonodromic fundamental matrix solutions (outside and inside the coalescence) are related.

Our second important result, proposition 6.1.2 gives a partial answer to this question, namely, after a proper renormalization (the matrix Q) some of the columns of the formal fundamental matrix solutions outside the coalescence loci and in a neighborhood of $z = \infty$, have a well definde limit in the coalescence loci Δ and moreover, when evaluated at Δ they are equal to some of the columns of the formal fundamental matrix solutions inside the coalescence loci in a neighborhood of $z = \infty$.

Notice that again we used the geometric properties of a Dubrovin-Frobenius manifold to cook up the matrix Q which renormalizes some of the columns of the formal fundamental matrix solution outside the coalescence loci. In a more general setting getting the correct renormalization matrix may be much more harder.

Understanding completely (even in the examples provided by Dubrovin-Frobenius manifolds) how the holomorphic fundamental matrices solutions are related remains an open question which deserves further investigation.

Finally let us make some remarks about the flat connections $\pi_k \circ \nabla$ that played a fundamental role in the construction of the isomonodromic fundamental matrix solutions inside the coalescence loci. In recent years interest has grown in studying weaker structures than that of a Dubrovin-Frobenius manifold. In particular lots of interesting results have been found on the so called flat F-manifolds (see [1] and [2]). A flat F-manifold is just an F-manifold M with a flat and torsionless connection ∇ on the tangent sheaf $\mathcal{T}_M$ such that $\nabla e = 0$. Just as in the case of a Dubrovin-Frobenius manifold, one can construct a 1-parameter family of flat connections ∇^z on $\mathcal{T}_M$. Although the bundles $\iota^*(\mathcal{T}_{M,p})_k$ over the multiplication invariant submanifolds L are not the tangent bundle and the unit of these algebras is not necessarily flat; they posses a multiplication and are also equipped with the flat connection $\pi_k \circ \nabla$ and can be extended to a 1-parameter family of flat connections. This connections were of fundamental importance for this work but their 1-parameter extensions were nowhere used. Further study of these families of connections might give more insight as to what parts of the Dubrovin-Frobenius manifold we can recover from family of meromorphic differential equations associated to it.

Appendix A

Flat Connections

A.1 Flat Connections

Let M be a complex manifold of dimension m, let $\mathcal{T}_M$ be the sheaf of holomorphic vector fields and let Ω^1_M be the sheaf of holomorphic 1-forms. Consider a holomorphic vector bundle $\pi\colon V \to M$ and denote its sheaf of holomorphic sections by $\mathcal{V}$.

Definition A.1.1. A connection on the vector bundle $\pi\colon V \to M$ is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{V} \to \Omega^1_M \otimes \mathcal{V}$$

which satisfies the Leibniz rule: For any open set U and any $f \in \mathcal{O}_M(U), v \in \mathcal{V}(U)$ one has

$$\nabla fv = df \otimes v + f\nabla v.$$

Let r be the rank of the vector bundle $\pi\colon V \to M$ and suppose that the local sections $e_1, \ldots, e_r \in \mathcal{V}(U)$ are a local frame. We can write

$$\nabla e_j = \sum_{s=1}^{r} \omega_j^s \otimes e_s$$

for some $\omega_j^s \in \Omega^1_M(U)$. The matrix of 1-forms $\omega = (\omega_\beta^\alpha)$ is called *the connection matrix* associated to the frame e_j.

If $(x^1, \ldots, x^m)$ is a local coordinate system on some open neighborhood U of M then we can write

$$\omega_\beta^\alpha = \sum_{s=1}^{m} (\omega_s)_\beta^\alpha dx^s$$

where $(\omega_s)_\beta^\alpha \in \mathcal{O}_M(U)$. The matrices $\omega_s = (\omega_s)_\beta^\alpha$ are called the *connection matrices* associated to the frame e_i and the local coordinates x^j.

Example A.1.1. Suppose V is the tangent bundle and $e_i = \partial_i$ is the frame associated to some local coordinates $(x_1, \dots, x_m)$. We have

$$\nabla_{\partial_k} \partial_j = \sum_{s=1}^{m} \Gamma_{kj}^{s} \partial_s$$

where Γ_{kj}^{s} are the Christoffel symbols associated to the frame ∂_i. Then

$$\nabla \partial_j = \sum_{s=1}^{m} \left(\sum_{k=1}^{n} \Gamma_{kj}^{s} \, dx^k \right) \otimes \partial_s.$$

Therefore in this example we have

$$\omega_j^s = \sum_{k=1}^{m} \Gamma_{kj}^{s} \, dx^k.$$

Any section $s \in \mathcal{V}(U)$ can be written as a $\mathcal{O}_M(U)$-linear combination of the sections e_i, by the Leibniz rule we obtain

$$\begin{aligned} \nabla s = \nabla \left(\sum_{j=1}^{r} f^j e_j \right) &= \sum_{j=1}^{r} df^j \otimes e_j + f^j \nabla e_j \\ &= \sum_{s=1}^{r} \left(df^s + \sum_{j=1}^{n} f^j \omega_j^s \right) \otimes e_s. \end{aligned}$$

We can write this conveniently as

$$\nabla s = d \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix} + \omega \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix}$$

Given a vector bundle with a connection one could ask about the flat sections $\mathcal{V}_F \leq \mathcal{V}$. By definition this sections satisfy $\nabla s = 0$. If we are given a local frame e_i with connection matrix ω, finding a flat section amounts to solving the system of partial differential equations

$$df = -\omega f \tag{A.1.1}$$

where $f = (f^1, \dots, f^n)^T$. This is a system of $n \times (dimM)$ partial differential equations for the n unknowns f^i, hence unless $dimM = 1$ we don't expect this system to have a solution. Suppose we can find n linearly independent flat sections $s_j = \sum f_j^s e_s$ and let $F := (f_j^s)$ be the matrix whose columns consist of these flat sections. Then from equation (A.1.1) we get

$$0 = ddF = -d(\omega F) = -(d\omega + \omega \wedge \omega)F,$$

(here we used that for 1-forms η we have $d(\eta \wedge \xi) = d\eta \wedge \xi - \eta \wedge d\xi$). Since the sections s_j are linearly independent the matrix F is invertible and therefore

$$\Omega := d\omega + \omega \wedge \omega = 0. \tag{A.1.2}$$

It turns out that this condition is also sufficient for the existence of n linearly independent flat sections of V (see [16]). On a local system of coordinates $(x_1, \dots, x_m)$ equation (A.1.2) becomes

$$\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} = [\omega_i, \omega_j].$$

The $End(\mathcal{V})$-valued two-form Ω is known as the *curvature form* or *curvature tensor* of the connection ∇

A.2 Compatible metrics

Suppose that on our vector bundle we have a metric $g \in Sym^2(\mathcal{V}^*)$. Using the connection we can define the total covariant derivative of g in the usual way:

$$(\nabla g)(u, v) := d(g(u, v)) - g(\nabla u, v) - g(u, \nabla v).$$

We will say that g is compatible with ∇ if $\nabla g = 0$.

Proposition A.2.1. *Let $\pi \colon V \to M$ be a vector bundle with connection ∇. Suppose that the metric g is compatible with ∇. Then for any local coordinate system $(x^1, \dots, x^m)$ on $U \subset M$ and any orthonormal frame $e_1, \dots, e_r \in \mathcal{V}(U)$, the corresponding connection matrices are antisymmetric i.e, $(\omega_i)^\alpha_\beta = -(\omega_i)^\beta_\alpha$ for any $i = 1, \dots, m$ and $\alpha, \beta = 1, \dots, r$.*

Proof. By compatibility and orthonormality we have

$$0 = g(\nabla e_\alpha, e_\beta) + g(e_\alpha, \nabla e_\beta).$$

Evaluating at ∂_{x_i} we get

$$\begin{aligned} 0 &= \sum_{s=1}^{r} g((\omega_i)^s_\alpha e_s, e_\beta) + g(e_\alpha, (\omega_i)^s_\beta) \\ &= (\omega_i)^\beta_\alpha + (\omega_i)^\alpha_\beta. \end{aligned}$$

$\square$

A.3 Pullback Connection

Let $\pi\colon V \to M$ be a vector bundle with a connection ∇ and let $f\colon L \to M$ be a holomorphic function. Using the function f and the vector bundle $\pi\colon V \to M$ we can construct the pullback vector bundle $\bar{\pi}\colon f^*V \to L$. The sheave of sections of $f^*\mathcal{V}$ can be described in the following way. Given an open set $U \subset M$ and a section $s \in \mathcal{V}(U)$ by precomposing with f we get a section $f^*s \in f^*\mathcal{V}(f^{-1}(U))$; sections of $f^*\mathcal{V}$ over $f^{-1}(U)$ are $\mathcal{O}_L(f^{-1}(U))$-linear combinations of sections of the form f^*s with $s \in \mathcal{V}(U)$. For general open set $W \subset L$, to compute $f^*\mathcal{V}(W)$ we first take the direct limit of the vector spaces $\mathcal{V}(U)$ where $U \subset M$ is open and $W \subset f^{-1}(U)$ (the direct limit is computed using the restriction maps) and then take $\mathcal{O}_L$-linear combinations.
We can also pullback the connection ∇ to $f^*\mathcal{V}$. For sections of the form f^*s we set

$$(f^*\nabla)f^*s := f^*(\nabla s)$$

and for general sections we extend using the Leibniz rule.

Proposition A.3.1. *Let $\pi\colon V \to M$ be a vector bundle with flat connection ∇ and let $f\colon L \to M$ be a holomorphic function. Then the connection $f^*\nabla$ on the vector bundle $\bar{\pi}\colon f^*V \to$ is flat.*

Proof. Note that both vector bundles have the same rank r. It ∇ is flat then we can find $s_1, \dots, s_r \in \mathcal{V}$ linearly independent sections such that $\nabla s_i = 0$. But then $(f^*\nabla)f^*s = f^*(\nabla s) = 0$ so that $f^*\mathcal{V}$ has r-linearly independent flat sections and therefore $f^*\nabla$ is flat. ☐

Bibliography

[1] Alessandro Arsie, Alexandr Buryak, Paolo Lorenzoni, and Paolo Rossi. Semisimple flat f-manifolds in higher genus, 2020.

[2] Alessandro Arsie, Alexandr Buryak, Paolo Lorenzoni, and Paolo Rossi. Riemannian f-manifolds, bi-flat f-manifolds, and flat pencils of metrics, 2021.

[3] Alexey Basalaev and Claus Hertling. 3-dimensional f-manifolds. *Letters in Mathematical Physics*, 111(4), jul 2021.

[4] Alexey Basalaev and Atsushi Takahashi. On rational frobenius manifolds of rank three with symmetries. *Journal of Geometry and Physics*, 84:73–86, oct 2014.

[5] Werner Basler. *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*. Springer New York, NY, 2000.

[6] Giordano Cotti, Boris Dubrovin, and Davide Guzzetti. Isomonodromy deformations at an irregular singularity with coalescing eigenvalues. *Duke Mathematical Journal*, 168(6), apr 2019.

[7] Giordano Cotti, Boris Dubrovin, and Davide Guzzetti. Local moduli of semisimple Frobenius coalescent structures. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 040, 105, 2020.

[8] Pierre Deligne. *Equations Différentielles à Points Singuliers Réguliers*. Springer-Verlag, 1970.

[9] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.7 of 2022-10-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[10] Boris Dubrovin. Differential geometry of the space of orbits of a Coxeter group. *Surveys in Differential Geometry*, 2 1993.

[11] Boris Dubrovin. Geometry of 2-D topological field theories. *Lect. Notes Math.*, 1620:120–348, 1996.

[12] Boris Dubrovin. *Painlevé Transcendents in Two-Dimensional Topological Field Theory*, pages 287–412. Springer New York, New York, NY, 1999.

[13] Davide Guzzetti. Isomonodromic deformations along a stratum of the coalescence locus. *Journal of Physics A: Mathematical and Theoretical*, 55(45):455202, nov 2022.

[14] Claus Hertling. *Frobenius Manifolds and Moduli Spaces for Singularities.* Cambridge University Press, 2002.

[15] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. i. general theory and tau-function. *Physica D: Nonlinear Phenomena*, 2(2):306–352, 1981.

[16] Claude Sabbah. *Isomonodromic Deformations and Frobenius Manifolds an Introduction.* Springer London, 2007.

[17] Kyoji Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. *J.Fac.Sci.Univ.Tokyo*, 28:775–792, 1981.

[18] Kyoji Saito. Period mapping associated to a primitive form. *Publications of the Research Institute for Mathematical Sciences*, 19(3):1231–1264, 1983.

[19] Kyoji Saito and Atsushi Takahashi. From primitive forms to frobenius manifolds. In *Proc. Sympos. Pure Math., 78, AMS*, 2008.

[20] Y Sibuya. Simplification of a system of linear ordinary differential equations about a singular point. *Funkcial. Ekvac*, 4:29–56, 1962.

[21] I.A.B. Strachan. Frobenius manifolds: natural submanifolds and induced bi-hamiltonian structures. *Differential Geometry and its Applications*, 20(1):67–99, jan 2004.

Index