On the analyticity of the Dirichlet–Neumann operator and Stokes waves

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Massimiliano Berti, Alberto Maspero, Paolo Ventura

To the memory of Antonio Ambrosetti

Abstract: We prove an analyticity result for the Dirichlet-Neumann operator under space periodic boundary conditions in any dimension in an unbounded domain with infinite depth. We derive an analytic bifurcation result of analytic Stokes waves –i.e. space periodic traveling solutions– of the water waves equations in deep water.

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Contents

1 Introduction and main results

The Dirichlet-Neumann operator plays an important role in fluid mechanics, for example in the Craig-Sulem-Zhakarov [19, 42] formulation of the water waves equations (cfr. Section 1.2), and in several other branches of analysis, as in the theory of inverse problems. Roughly speaking it is defined as the linear operator which maps the Dirichlet datum of a harmonic function in a given domain into the normal derivative at its boundary (Neumann datum). The Dirichlet-Neumann operator is nonlinear with respect to the boundary of the domain. In view of many applications it is important to determine its regularity in different function spaces.

Several results about the analyticity of the Dirichlet-Neumann operator acting between Sobolev spaces, with respect to the variation of the boundary, have been proved, starting with the pioneering works of Coifmann-Meyer [16], Craig-Schanz-Sulem [18], Craig-Nicholls [17] and Lannes [26] where we refer for an extended bibliography. We also mention the deep analysis of the Dirichlet-Neumann operator in [5, 4, 3, 40], on which we will comment later.

The major aim of this paper is to prove a further analyticity result for the Dirichlet-Neumann operator $G(\eta)$ defined in (1.3) on the unbounded domain $\mathbb{T}^d \times \{y \leq \eta(x)\}$, where $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ is the standard d-dimensional flat torus, in any space dimension $d \geq 1$. Assuming that $\eta(x)$ is analytic, we prove in Theorem 1.2 the analyticity of the map $\eta \mapsto G(\eta)$ acting between suitable spaces of analytic periodic functions. The delicate point of this result is that η and ψ are assumed to have the *same* regularity (if η is more regular than ψ the result is simpler). Following Lannes [25, 26] and Alazard-Burq-Zuily [1] we make use of a regularizing diffeomorphism to flatten the domain to the half cylinder, in which the transformed harmonic function solves a perturbed elliptic equation. Then the proof relies on a perturbative approach to invert the transformed Laplacian over suitable spaces of functions $u(x, y)$ which are analytic in x, with Sobolev regularity in y and decay to zero as $y \to -\infty$, cfr. (2.7). The key step is obtain linear elliptic regularity estimates for the Poisson equation in these spaces, see Lemma 2.10. Then the elliptic estimates for the modified problem are obtained by a perturbative argument differently from [1].

As a consequence of Theorem 1.2, we derive an analytic bifurcation result of analytic Stokes waves –i.e. space periodic traveling solutions, which look stationary in a moving frame with constant speed– of the pure gravity water waves equations in infinite depth, see Theorem 1.3. Existence of traveling waves which are constant in one space dimension, i.e. are 1-dimensional waves, dates back to classical works of Levi-Civita [27], Nekrasov [31] and Struik [37], in the twenties of the last century. Then Lewy [28] proved that a traveling wave which is at least C^1 is actually analytic. Theorem 1.3 proves in addition that small amplitude Stokes waves depend analytically on the amplitude taking values in a space of analytic functions. In finite depth and with surface tension, a result of this kind is proved in Nicholls-Reitich [33] by a power series expansion approach.

In this paper we deduce Theorem 1.3 by the analytic Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue, as presented in the book of Ambrosetti-Prodi [6], thanks to the analytic estimates of the Dirichlet-Neumann operator obtained in Theorem 1.2.

In addition to their interests per se –traveling waves have fundamental importance in fluid mechanics–, these results have been used in the study of the Benjiamin-Feir instability of the Stokes waves in [13].

We now state precisely our results. Along this paper we use the following notation. We denote the spatial variables by $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, $d \geq 1$, where $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ is the standard flat torus. The symbol ∇ denotes the gradient

$$\nabla := (\partial_{x_j})_{j=1,\ldots,d} \qquad \text{and} \qquad \Delta := \sum_{j=1}^{d} \partial_{x_j}^2 \,, \quad \Delta_{x,y} := \Delta + \partial_y^2 \,.$$

A dot will denote the standard scalar product in $\mathbb{R}^d$. Moreover $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

1.1 Dirichlet-Neumann operator

We consider the cylindrical domain

$$\mathcal{D}_\eta := \left\{ (x, y) \in \mathbb{T}^d \times \mathbb{R} \ : \ y < \eta(x) \right\}, \quad d \geq 1 \,, \tag{1.1}$$

delimited by the graph $\partial\mathcal{D}_\eta = \{y = \eta(x)\}$ of a periodic function $\eta(x)$, and, given a periodic Dirichlet datum $\psi(x)$, we consider the unique harmonic function $\Phi(x, y)$ solving the system

$$\begin{cases} \Delta_{x,y}\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \Phi(x, y) = \psi(x) & \text{at } y = \eta(x) \\ \partial_y \Phi(x, y) \to 0 & \text{as } y \to -\infty \,. \end{cases} \tag{1.2}$$

The Dirichlet-Neumann operator $G(\eta)$ is then defined as the linear operator

$$\begin{aligned} [G(\eta)\psi](x) &:= \sqrt{1 + |\nabla\eta|^2}\, \partial_n \Phi_{|y=\eta(x)} \\ &= (\partial_y \Phi)(x, \eta(x)) - \nabla\eta(x) \cdot (\nabla\Phi)(x, \eta(x)) \end{aligned} \tag{1.3}$$

where n denotes the exterior normal

$$n := \frac{1}{\sqrt{1 + |\nabla\eta|^2}} \begin{bmatrix} -\nabla\eta \\ 1 \end{bmatrix}, \quad \partial_n := \frac{1}{\sqrt{1 + |\nabla\eta|^2}} (\partial_y - \nabla\eta \cdot \nabla) \,.$$

The reason of the name "Dirichlet-Neumann" is that the operator $G(\eta)$ maps the Dirichlet datum $\psi(x)$ of the harmonic function $\Phi(x, y)$ into the (normalized) normal derivative $\partial_n \Phi$ at the boundary $\partial\mathcal{D}_\eta = \{y = \eta(x)\}$ (Neumann datum).

Remark 1.1. In (1.2) it is equivalent to require the boundary condition $\nabla\Phi(x, y) \to 0$ as $y \to -\infty$, see Remark 2.8. Actually $\nabla\Phi(x, y)$ decays to zero exponentially fast as $y \to -\infty$.

Simple algebraic properties of the Dirichlet-Neumann operator are recalled in Appendix A.

Since Calderon it is known that the Dirichlet-Neumann operator $G(\eta)$ is, if η is a C^∞ function, a classical pseudo-differential operator, elliptic of order 1, with an asymptotic expansion in classical decreasing symbols. For the flat surface $\eta(x) = 0$, the Dirichlet-Neumann operator is the Fourier multiplier

$$G(0) = |D| = (-\Delta)^{\frac{1}{2}}$$

as follows by the elementary calculus (2.25). In space dimension $d = 1$ the Dirichlet-Neumann operator is equal to $|D|$ up to infinitely many times regularizing operators, see e.g. [14, 8]. If

$\eta(x)$ has a finite smoothness, Lannes [25, 26] proved an analogous expansion in symbols with finite smothness.

The Dirichlet-Neumann operator is a nonlinear map with respect to the boundary of the domain $\partial\mathcal{D}_\eta$. The analytic dependence with respect to η of the Dirichlet-Neumann operator $\eta \mapsto G(\eta)$ has been first established in the two dimensional setting by Coifman-Meyer [16], and in the three dimensional setting by Craig, Schanz and Sulem [18], showing that, if $\eta \in C^{k+1}$, $\psi \in H^{k+1}$, $k \in \mathbb{N}$, then $G(\eta)[\psi] \in H^k$ is analytic in $C^{k+1} \cap \{\|\eta\|_{C^1} < r\}$ for r sufficiently small.

In view of of application to water waves Craig-Nicholls [17], Wu [40, 41], and Lannes [25, 26] proved, with different approaches, that if η, ψ have the same Sobolev regularity H^s then $G(\eta)[\psi] \in H^{s-1}$. In particular Lannes proved tame estimates using *regularizing* diffeomorphisms to straighten the domain.

The paralinearization of $G(\eta)\psi$, which enables to prove optimal estimates for the action of the Dirichlet-Neumann operator, has been obtained in Alazard-Metivier [5], Alazard-Delort [4], and Alazard-Burq-Zuily [2, 3] in rough domains, using a variational analysis to construct the solution and applying elliptic regularity theory. The paralinearization of the Dirichlet-Neumann operator in $d = 1$ with a multilinear expansion in η is proved in Berti-Delort [10], by using a paradifferential parametrix à la Boutet de Monvel.

Finally we mention the work of Alazard-Burq-Zuily [1] for the study of the Dirichlet-Neumann operator acting in analytic function spaces, making use of a regularizing diffeomorphism as in [25], variational methods and elliptic regularity analysis.

In this paper we prove an analyticity result (Theorem 1.2) for the Dirichlet-Neumann map $\eta \mapsto G(\eta)\psi$ in the cylindrical domain $\mathcal{D}_\eta$ defined in (1.1), acting between spaces of periodic analytic functions defined in (1.4) below. We suppose that the functions η and ψ belong to the spaces of periodic functions

$$H^{\sigma,s} := H^{\sigma,s}(\mathbb{T}^d) := \Big\{ u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}\, k\cdot x} \ : \ \|u\|^2_{H^{\sigma,s}} := \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k\rangle^{2s} |u_k|^2 < \infty \Big\} \tag{1.4}$$

where, for any $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, we set

$$|k|_1 := |k_1| + \cdots + |k_d| , \quad \langle k\rangle := \max(1, |k|) , \quad |k| := \Big(\sum_{j=1}^d k_j^2 \Big)^{1/2} .$$

Clearly, if the dimension $d = 1$ then $|k| = |k|_1$.

If $\sigma = 0$ the space $H^{0,s}$ is the usual Sobolev space H^s. If $\sigma > 0$, a periodic function $u(x)$ belongs to $H^{\sigma,s}(\mathbb{T}^d)$, if and only if it admits an analytic extension in the strip $|y|_\infty := \max\{|y_1|, \dots, |y_d|\} < \sigma$ and the traces at the boundaries $u(\cdot + \mathrm{i}\, y)$, $|y|_\infty = \sigma$, belong to the Sobolev space $H^s := H^s(\mathbb{T}^d)$. In Appendix B.1 we prove this characterization, together with the property that the spaces $H^{\sigma,s}$ form, for $s > d/2$, an algebra with respect to the product of functions and satisfy tame estimates.

The main result of this section is the following theorem.

Let $B^{\sigma,s}(r)$ denote the open ball in $H^{\sigma,s}$ of center 0 and radius $r > 0$.

Theorem 1.2. ***(Dirichlet-Neumann operator)*** *Let $\sigma \geq 0$ and s, s_0 such that $s + \frac{1}{2}$, $s_0 \in \mathbb{N}$, and $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. Then there exists $\epsilon_0 := \epsilon_0(s) > 0$ such that the Dirichlet-Neumann operator map*[1]

$$\eta \mapsto G(\eta) , \quad H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, H^{\sigma,s-1}) ,$$

is analytic and fulfills the tame estimate

$$\|G(\eta)\psi\|_{H^{\sigma,s-1}} \leq C(s)\big(\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}}\big) . \tag{1.5}$$

We remark that, in Theorem 1.2, the functions η, ψ have the same analytic regularity. The proof of such result, given in Section 2, relies on a regularizing flattening method (following [25, 1]) together with a perturbative argument in suitable functional spaces.

1.2 Stokes waves

As an application of Theorem 1.2 we prove that 1-dimensional Stokes waves solutions of the pure gravity water waves equations in deep water are analytic functions belonging to the spaces $H^{\sigma,s}$, and moreover depend analytically with respect to the amplitude parameter. Clearly 1-dimensional traveling waves are also 2d-traveling waves which are constant in one space direction, so it extends to higher dimensional Stokes waves. We first present the water waves equations.

The pure gravity water waves equations. We consider the Euler equations for a bi-dimensional incompressible, inviscid, irrotational fluid under the action of gravity, filling the region $\mathcal{D}_\eta$ defined in (1.1) with $d = 1$,

$$\begin{cases} \partial_t\Phi + \frac{1}{2}\big((\partial_x\Phi)^2 + (\partial_y\Phi)^2\big) + g\eta = 0 & \text{at } y = \eta(x) \\ \partial_t\eta = \partial_y\Phi - (\partial_x\eta)\,(\partial_x\Phi) & \text{at } y = \eta(x) \\ \Delta_{x,y}\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \partial_y\Phi \to 0 & \text{as } y \to -\infty , \end{cases} \tag{1.6}$$

where $g > 0$ is the acceleration of gravity. The irrotational velocity field is the gradient of the harmonic scalar potential $\Phi = \Phi(t, x, y)$, determined by its trace $\psi(t, x) = \Phi(t, x, \eta(t, x))$ at the free surface $y = \eta(t, x)$. Actually $\Phi(t, \cdot)$ is the unique solution of the elliptic equation (1.2). The time evolution of the fluid is determined by the first two boundary conditions in (1.6) at the free surface. The first states that the pressure of the fluid is equal, at the free surface, to the constant atmospheric pressure (dynamic boundary condition) and the second one that the fluid particles remain, along the evolution, on the free surface (kinematic boundary condition).

As shown by Zakharov [42] and Craig-Sulem [19], the evolutionary system (1.6) amounts to the following equations for the unknowns $(\eta(t, x), \psi(t, x))$,

$$\eta_t = G(\eta)\psi , \quad \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 , \tag{1.7}$$

[1] $H^{\sigma,s} \cap B^{\sigma,s_0}(\epsilon_0)$ is an open set in the $H^{\sigma,s}$ topology.

where $G(\eta)$ is the Dirichlet-Neumann operator in (1.3). In addition the equations (1.7) are the Hamiltonian system

$$\partial_t \eta = \nabla_\psi \mathcal{H}\,, \quad \partial_t \psi = -\nabla_\eta \mathcal{H}\,, \tag{1.8}$$

where ∇_η, ∇_ψ denote the L^2-gradients of the Hamiltonian

$$\mathcal{H}(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \big(\psi\, G(\eta)\psi + g\eta^2\big)\,\mathrm{d}x\,,$$

which is the sum of the kinetic energy (cfr. (A.1)) and potential gravitational energy of the fluid. Actually, as proved in [19, 42], the L^2-gradient with respect to η of the kinetic energy

$$K(\eta,\psi) := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2} \overset{(A.1)}{=} \frac{1}{2}\int_{\mathcal{D}_\eta} |\nabla\Phi|^2\,\mathrm{d}x\,, \tag{1.9}$$

is equal to

$$\nabla_\eta K(\eta,\psi) = -\frac{1}{2}\psi_x^2 + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2\,, \tag{1.10}$$

yielding the equivalence between (1.8) and (1.7).

We also remark that the water waves equations (1.7) are invariant under space translations namely, by (A.2),

$$\mathcal{H}\circ\tau_\theta = \mathcal{H}\,, \quad \forall\theta\in\mathbb{R}^d\,.$$

In addition, the water waves equations are reversible with respect to the involution

$$\rho\begin{bmatrix}\eta(x)\\ \psi(x)\end{bmatrix} := \begin{bmatrix}\eta(-x)\\ -\psi(-x)\end{bmatrix}, \quad \text{i.e. } \mathcal{H}\circ\rho = \mathcal{H}\,,$$

as a consequence of (A.3).

The Stokes waves. Noteworthy solutions of (1.7) are the so-called Stokes waves, namely traveling solutions of the form

$$\eta(t,x) = \breve{\eta}(x-ct)\,, \quad \psi(t,x) = \breve{\psi}(x-ct)\,, \tag{1.11}$$

for some real c (the speed) and 2π-periodic functions $(\breve{\eta}(x), \breve{\psi}(x))$ (the profiles). In a reference frame in translational motion with constant speed c, the water waves equations (1.7) then become, by using the translation invariance property (A.2),

$$\eta_t = c\eta_x + G(\eta)\psi\,, \quad \psi_t = c\psi_x - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2\,. \tag{1.12}$$

The Stokes waves profiles $(\breve{\eta}, \breve{\psi})$ in (1.11) are then equilibrium steady solutions of (1.12), namely solve the system

$$c\eta_x + G(\eta)\psi = 0\,, \quad c\psi_x - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 = 0\,. \tag{1.13}$$

The next theorem is the main bifurcation result of small amplitude Stokes waves proved in this paper. We denote by $B(r) := \{x\in\mathbb{R}\colon |x| < r\}$ the real ball with center 0 and radius r.

Theorem 1.3. ***(Stokes waves)*** *For any $\sigma \geq 0$, $s > 5/2$ and $k \in \mathbb{N}$, there exists $\epsilon_0 := \epsilon_0(\sigma, s, k) > 0$ and a unique family of solutions*

$$(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon) \in H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$$

of the system (1.13)*, parameterized by $|\epsilon| \leq \epsilon_0$, such that*

1. *the map $\epsilon \mapsto (\eta_\epsilon, \psi_\epsilon, c_\epsilon)$, $B(\epsilon_0) \to H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$ is analytic;*

2. *$\eta_\epsilon(x)$ is even, $\eta_\epsilon(x)$ has zero average, $\psi_\epsilon(x)$ is odd;*

3. *the solutions $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$ have the expansion*

$$(\eta_\epsilon(x), \psi_\epsilon(x)) = \epsilon(\sqrt{k}\cos(kx), \sqrt{g}\sin(kx)) + O(\epsilon^2)\,, \quad c_\epsilon \to \sqrt{\frac{g}{k}} \quad as \quad \epsilon \to 0\,. \tag{1.14}$$

Theorem 1.3 is proved in Section 3. Let us make some comments on the result.

1. As already mentioned in the introduction, the first rigorous bifurcation proof of small amplitude Stokes waves for pure gravity water waves goes back to Levi-Civita [27] and Nekrasov [31] in deep water, and Struik [37] in finite depth. We refer to the monographs of Ambrosetti-Prodi [6] and Buffoni-Toland [15] for a complete presentation. Concerning regularity, it is known since Lewy [28] that a Stokes wave which is at least C^1 is actually analytic. Theorem 1.3 proves in addition that the Stokes waves $(\eta_\epsilon(x), \psi_\epsilon(x))$ depend analytically on the amplitude ϵ taking values in a space of analytic functions $H^{\sigma,s} \times H^{\sigma,s}$. In finite depth and in presence of surface tension, an analyticity result of this kind is proved in Nicholls-Reitich [33], by a power series expansion. We also mention Plotnikov-Toland [35] for related results about analytic continuation of Stokes waves.

Existence of traveling water waves has been also proved by Zeidler [43] under the effect of capillary forces and Martin [30], Walhén [39] also for constant vorticity flows. We expect that, thanks to Theorem 1.2, an analyticity result for the Stokes waves, analogous to Theorem 1.3, holds also in these cases.

2. Higher order Taylor expansions of the Stokes waves in ϵ are known, see e.g. [20], [32], [33]. We remark that Theorem 1.3 proves the convergence of the Taylor series of the Stokes waves in ϵ, taking values in spaces of analytic periodic functions.

3. *Quasi-periodic traveling waves.* More general 1d time quasi-periodic traveling Stokes waves have been recently obtained in Berti-Franzoi-Maspero [11, 12], with or without surface tension, and Feola-Giuliani [21], by means of a Nash-Moser implicit function iterative scheme. We remark that these solutions are not steady in any moving frame. This implies a small divisor problem.

4. *Higher space dimension: existence.* For three dimensional fluids, in addition to Stokes waves, also traveling wave solutions which are nontrivially periodic in both spatial directions are known, for example forming hexagonal patterns. Their existence was first proved in Craig-Nicholls [17] for gravity-capillary water waves, by applying variational bifurcation arguments à la Weinstein-Moser, exploiting the Hamiltonian nature (1.8) of the water waves equations. The surface tension allows to apply, in the bifurcation analysis, the standard implicit function theorem. On the other

hand the existence of 2d pure gravity doubly-periodic traveling wave solutions is a small divisor problem. In this case, solutions with Sobolev regularity were constructed by Iooss-Plotinkov [22, 23] by means of a Nash-Moser implicit function theorem, requiring suitable Diophantine conditions on the speed vector.

5. *Regularity.* In higher space dimensions a regularity result à la Lewy [28], i.e. a traveling wave surface which is at least C^1 is actually analytic, has been proved for gravity-capillary water waves by Craig-Matei [16]. For pure gravity waves, a result of this kind is false, because the system is no more elliptic. This feature is the counterpart of the small divisor problem arising in the existence proof of Iooss-Plotinkov [22, 23]. Assuming Diophantine conditions on the speed vector, Alazard-Metivier [5] proved that the periodic traveling waves constructed in [22, 23], which have Sobolev regularity, are indeed C^∞.

6. We finally note that, for larger values of the amplitude ϵ, the regularity of the traveling wave solutions may break down. Indeed it is well known that large traveling waves have cusps, as proved in the celebrated works about the Stokes conjecture of Amick, Fraenkel, Toland [7] and Plotinkov [34].

2 Analyticity of the Dirichlet-Neumann operator

In this section we prove Theorem 1.2 concerning the analyticity of the Dirichlet-Neumann operator. The first step is to straighten the free surface.

Regularizing diffeomorphism. Following [25, 1] we apply the regularizing change of variables

$$x = x' , \qquad y = \rho(x', y') , \quad \rho(x', y') := y' + e^{y'|D|}\eta(x') , \tag{2.1}$$

where $e^{y|D|}$ is the Fourier multiplier

$$\left(e^{y|D|}g\right)(x) := \sum_{k\in\mathbb{Z}^d} g_k \, e^{y|k|} \, e^{\mathrm{i}\, k\cdot x} , \quad \forall \; g(x) = \sum_{k\in\mathbb{Z}^d} g_k \, e^{\mathrm{i}\, k\cdot x} .$$

Note that

$$\rho(x', 0) = \eta(x') , \quad \lim_{y'\to-\infty} \rho(x', y') - y' = \eta_0 ,$$

and, since

$$\partial_{y'}\rho(x', y') = 1 + e^{y|'D|}|D|\eta ,$$

if $\sup_{y'<0} \|e^{y'|D|}|D|\eta\|_{L^\infty(\mathbb{T}^d)} < 1$ the change of coordinates (2.1) is a diffeomorphism between the domain $\mathcal{D}_\eta = \{(x, y) \colon y \le \eta(x)\}$ and the flat half-cylinder $\{(x', y') \colon y' \le 0\} = \mathbb{T}^d \times \mathbb{R}_{\le 0}$ where $\mathbb{R}_{\le 0} := (-\infty, 0]$. By the change of variables (2.1) the derivatives ∂_y and ∇_x become respectively

$$\Lambda_1 = \frac{1}{\partial_{y'}\rho}\partial_{y'} , \qquad \Lambda_2 = \nabla_{x'} - \frac{\nabla_{x'}\rho}{\partial_{y'}\rho}\partial_{y'} ,$$

and the transformed harmonic function

$$\varphi(x', y') := \Phi(x', y' + \rho(x', y'))$$

solves the elliptic problem

$$\begin{cases} (\Lambda_1^2 + \Lambda_2^2)\varphi = 0 \\ \varphi(x,0) = \psi(x) \\ \partial_y \varphi(x,y) \to 0 \qquad \text{as } y \to -\infty\,. \end{cases} \tag{2.2}$$

By means of chain rule, system (2.2) is rewritten (cfr. [1]) as the perturbed elliptic problem (we rename the variables x', y' as x, y)

$$\begin{cases} \Delta_{x,y}\varphi = F(\eta)[\varphi] \\ \varphi(x,0) = \psi(x) \\ \partial_y \varphi(x,y) \to 0 \qquad \text{as } y \to -\infty\,, \end{cases} \tag{2.3}$$

where

$$F(\eta)[\varphi] := \big(\alpha(\eta)\partial_y^2 + \beta(\eta)\Delta + \gamma(\eta)\cdot\nabla\partial_y + \delta(\eta)\partial_y\big)\,\varphi \tag{2.4}$$

with, since $\nabla\rho(x,y) = e^{y|D|}\nabla\eta$ and $\partial_y\rho(x,y) = 1 + e^{y|D|}|D|\eta$,

$$\begin{aligned} \alpha(\eta) &:= 1 - \frac{1+|\nabla\rho|^2}{\partial_y\rho} = \frac{e^{y|D|}|D|\eta - |e^{y|D|}\nabla\eta|^2}{1+e^{y|D|}|D|\eta}\,, \\ \beta(\eta) &:= 1 - \partial_y\rho = -e^{y|D|}|D|\eta\,, \\ \gamma(\eta) &:= 2\nabla\rho = 2e^{y|D|}\nabla\eta\,, \\ \delta(\eta) &:= \frac{1}{\partial_y\rho}\Big(-2\nabla\rho\cdot\nabla\partial_y\rho + \partial_y\rho\Delta\rho + \frac{1+|\nabla\rho|^2}{\partial_y\rho}\partial_y^2\rho\Big)\,. \end{aligned} \tag{2.5}$$

In the new variables (2.1), the Dirichlet-Neumann operator defined in (1.3) becomes

$$[G(\eta)\psi](\cdot) = -\nabla\eta\cdot\nabla\varphi(\cdot,0) + \frac{1+|\nabla\eta|^2(\cdot)}{1+(|D|\eta)(\cdot)}\,(\partial_y\varphi)(\cdot,0)\,. \tag{2.6}$$

Function spaces. In order to state our main existence result for the solutions of (2.3), we introduce some function spaces. Given $s\in\mathbb{N}_0$, $\sigma, a \geq 0$, we define

$$\mathcal{H}^{\sigma,s,a} := \Big\{ u(x,y) = \sum_{k\in\mathbb{Z}^d} u_k(y)e^{\mathrm{i}\,k\cdot x} : \mathbb{T}^d\times(-\infty,0] \to \mathbb{C} \ : \ \|u\|_{\sigma,s,a} < \infty \Big\} \tag{2.7}$$

endowed with the norm

$$\begin{aligned} \|u\|^2_{\sigma,s,a} &:= \sum_{j=0}^{s} \|\partial_y^j u\|^2_{L^{2,a}(\mathbb{R}_{\leq 0}, H^{\sigma,s-j})} && (2.8) \\ &= \sum_{j=0}^{s} \int_{-\infty}^{0} \|\partial_y^j u(\cdot,y)\|^2_{H^{\sigma,s-j}}\, e^{-2ay}\mathrm{d}y \\ &= \sum_{j=0}^{s} \int_{-\infty}^{0} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\,\langle k\rangle^{2(s-j)}\,|\partial_y^j u_k(y)|^2 e^{2a|y|}\mathrm{d}y \\ &= \sum_{j=0}^{s}\sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\,\langle k\rangle^{2(s-j)}\,\|\partial_y^j u_k\|^2_{L^{2,a}} && (2.9) \end{aligned}$$

where, given a Hilbert space X, we have used the notation

$$\|u\|^2_{L^{2,a}(\mathbb{R}_{\le 0},X)} := \int_{-\infty}^0 \|u(y)\|_X^2 e^{-2ay}\,\mathrm{d}y = \int_{-\infty}^0 \|u(y)\|_X^2 e^{2a|y|}\,\mathrm{d}y\,. \tag{2.10}$$

Remark 2.1. For $\sigma = a = 0$, the space $H^{0,s,0}$ coincides with the Sobolev space $H^s(\mathbb{T}^d \times \mathbb{R}_{\le 0})$ of L^2 functions $u : \mathbb{T}^d \times \mathbb{R}_{\le 0} \to \mathbb{C}$ possessing weak derivatives $\partial^\alpha u$ in L^2, for any multiindex $\alpha \in \mathbb{N}^{d+1}$ with modulus $|\alpha| \le s$, with equivalent norm $\|u\|_s^2 = \sum_{\alpha\in\mathbb{N}^{d+1},|\alpha|\le s} \|\partial^\alpha u\|^2_{L^2}$.

We point out that, for any $s \in \mathbb{N}$,

$$\|u\|^2_{\sigma,s,a} = \|u\|^2_{L^{2,a}(\mathbb{R}_{\le 0},H^{\sigma,s})} + \|\partial_y u\|^2_{\sigma,s-1,a}\,,$$

and, by (2.8) and $\|\partial_{x_i} v\|_{H^{\sigma,s-1}} \le \|v\|_{H^{\sigma,s}}$, we directly get the following simple lemma.

Lemma 2.2. *Let $s \in \mathbb{N}$, $\sigma \ge 0$, $a \ge 0$. The linear maps*

$$\partial_{x_i} : \mathcal{H}^{\sigma,s,a} \mapsto \mathcal{H}^{\sigma,s-1,a}\,,\ \ \forall i = 1,\dots,d, \quad \partial_y : \mathcal{H}^{\sigma,s,a} \mapsto \mathcal{H}^{\sigma,s-1,a}\,,$$

are continuous.

We also denote

$$\mathbb{C} \oplus \mathcal{H}^{\sigma,s,a} := \left\{c + u(x,y)\,,\ c \in \mathbb{C}\,,\ u \in \mathcal{H}^{\sigma,s,a}\right\}, \quad \Pi : \mathbb{C} \oplus \mathcal{H}^{\sigma,s,a} \to \mathcal{H}^{\sigma,s,a},\ \Pi[c+u] = u\,, \tag{2.11}$$

and, with a small abuse of notation, given a function $g \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s,a}$, we denote its norm by $\|g\|_{\sigma,s,a} := \|\Pi g\|_{\sigma,s,a} + |g - \Pi g|$. The function spaces $\mathcal{H}^{\sigma,s,a}$ and $\mathbb{C} \oplus \mathcal{H}^{\sigma,s,a}$ are modeled to mimic the decay of the harmonic function $\underline{\varphi}$ in (2.25) as $y \to -\infty$, cfr. Lemma 2.5.

We now list a series of properties of the spaces $\mathcal{H}^{\sigma,s,a}$ used in the sequel; we defer their proofs in Appendix B.2.

Lemma 2.3 (Trace). *Let $\sigma \ge 0$, $s \in \mathbb{R}$. Then one has*

$$\|u\|_{C^0(\mathbb{R}_{\le 0},H^{\sigma,s})} \le \|u\|_{L^2(\mathbb{R}_{\le 0},H^{\sigma,s+\frac12})} + \|\partial_y u\|_{L^2(\mathbb{R}_{\le 0},H^{\sigma,s-\frac12})}. \tag{2.12}$$

In particular, the trace operator

$$\Gamma(u) := u(\cdot,0) := u|_{y=0} \tag{2.13}$$

is, for any $s \in \mathbb{N}_0$, $a \ge 0$, a linear bounded map between $\mathcal{H}^{\sigma,s+1,a} \to H^{\sigma,s+\frac12}$, satisfying

$$\|\Gamma(u)\|_{H^{\sigma,s+\frac12}} \le \|u\|_{\sigma,s+1,0} \le \|u\|_{\sigma,s+1,a}\,. \tag{2.14}$$

If $s > \frac{d+1}{2}$, the space $\mathcal{H}^{\sigma,s,a}$ is an algebra with respect to the product of functions and the following tame estimates hold.

Proposition 2.4 (Tame). *Let $\sigma, a \ge 0$ and $s \ge s_0 > \frac{d+1}{2}$, $s, s_0 \in \mathbb{N}$. Then there exist positive constants $C_s \ge 1$ (non-decreasing in s) such that, for any $u \in \mathcal{H}^{\sigma,s,0}$ and $v \in \mathcal{H}^{\sigma,s,a}$,*

$$\|uv\|_{\sigma,s,a} \le C_s\big(\|u\|_{\sigma,s,0}\,\|v\|_{\sigma,s_0,a} + \|u\|_{\sigma,s_0,0}\,\|v\|_{\sigma,s,a}\big)\,. \tag{2.15}$$

In particular one has

$$\|u^j\|_{\sigma,s,a} \le \big(2C_s\|u\|_{\sigma,s_0,a}\big)^{j-1}\|u\|_{\sigma,s,a}\,, \quad \forall j \ge 1\,. \tag{2.16}$$

The next lemma proves the continuity of the harmonic function $e^{y|D|}g$, which solves the Dirichlet-Neumann elliptic problem (2.25), with respect to the Dirichlet datum g at $y = 0$.

Lemma 2.5 (Harmonic propagator). *Let $\sigma \geq 0$ and $s + \frac{1}{2} \in \mathbb{N}$. Then, for any $g \in H^{\sigma,s}$, the function*

$$(e^{y|D|}g)(x) := \sum_{k \in \mathbb{Z}^d} g_k\, e^{y|k|}\, e^{\mathrm{i}\, k \cdot x}$$

belongs to $\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$, $a \in (0,1)$, and the linear map

$$H^{\sigma,s} \to \mathcal{H}^{\sigma,s+\frac{1}{2},a}\,, \quad g \mapsto \Pi[e^{y|D|}g] = e^{y|D|}g - g_0\,,$$

is continuous.

We now come back to Theorem 1.2. The key result of its proof is the following proposition regarding the solution of the elliptic problem (2.3).

The parameter $a \in (0,1)$ plays a technical role in studying the decay as $y \to -\infty$ of the solution of the elliptic problem (2.2) (see in particular Lemma C.1). *In the sequel we fix $a = \frac{1}{2}$.*

Proposition 2.6. *Let $\sigma \geq 0$ and s, s_0 such that $s + \frac{1}{2}$, $s_0 \in \mathbb{N}$ and $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. Then there exist $\epsilon_0 := \epsilon_0(s) > 0$ and, for any $\eta \in H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ and $\psi \in H^{\sigma,s}$, a unique solution $\varphi \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$ of the elliptic problem (2.3), satisfying*

$$\|\Pi\varphi\|_{\sigma,s+\frac{1}{2},a} \leq C(s)\big(\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}}\big)\,. \tag{2.17}$$

Moreover $\varphi = \Psi(\eta)[\psi]$, where Ψ is an analytic map $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})$, and $\Psi(0)\psi = e^{y|D|}\psi$.

Postponing the proof of this proposition, we first use it to deduce Theorem 1.2.

Proof of Theorem 1.2. By Proposition 2.6, for any $\eta \in H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ and $\psi \in H^{\sigma,s}$, there exists a unique solution $\varphi \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$ of (2.3). The Dirichlet-Neumann operator is computed in (2.6). Since $\varphi(x,0) = \psi(x)$, using the trace operator $\Gamma(u) = u(\cdot,0)$ in (2.13), and recalling the definition of Π in (2.11), we rewrite (2.6) as

$$\begin{aligned} G(\eta)\psi &= -\nabla\eta\cdot\nabla\psi + \frac{1+|\nabla\eta|^2}{1+(|D|\eta)}\,\Gamma[\partial_y\varphi] \\ &= \underbrace{-\nabla\eta\cdot\nabla\psi}_{=:G_1(\eta)\psi} + \underbrace{\Gamma[\partial_y\Pi\varphi]}_{=:G_2(\eta)\psi} + \underbrace{\frac{|\nabla\eta|^2-(|D|\eta)}{1+(|D|\eta)}\,\Gamma[\partial_y\Pi\varphi]}_{=:G_3(\eta)\psi}\,. \end{aligned} \tag{2.18}$$

We prove that each map

$$G_i : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, H^{\sigma,s-1})\,, \quad i = 1,2,3\,, \quad \text{is analytic}\,, \tag{2.19}$$

and fulfills the tame estimate (1.5). Regarding $G_1(\eta)\psi$, it suffices to note that it is linear in η and by (B.3),

$$\|\nabla\eta\cdot\nabla\psi\|_{H^{\sigma,s-1}} \lesssim_s \|\eta\|_{H^{\sigma,s_0+\frac{1}{2}}}\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{1}{2}}}\,. \tag{2.20}$$

Next we consider $G_2(\eta)\psi = \Gamma[\partial_y \Pi \Psi(\eta)\psi]$. By Lemma 2.2 and 2.3, the map $\varphi \mapsto \Gamma[\partial_y \Pi\varphi] \in \mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}, H^{\sigma,s-1})$ which, together with the analyticity of $\eta \mapsto \Psi(\eta)$ stated in Proposition 2.6, implies the analyticity of $\eta \mapsto G_2(\eta)$ as in (2.19). Moreover by (2.14), Lemma 2.2 and (2.17), we have

$$\|\Gamma[\partial_y \Pi\varphi]\|_{H^{\sigma,s-1}} \le \|\partial_y \Pi\varphi\|_{\sigma,s-\frac{1}{2},0} \lesssim_s \|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}} \|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}} . \tag{2.21}$$

Finally consider $G_3(\eta)\psi = f(\eta)G_2(\eta)\psi$, where $f(\eta)$ is the multiplication operator by the function

$$f(\eta) = \frac{|\nabla\eta|^2 - (|D|\eta)}{1 + (|D|\eta)} = \left(|\nabla\eta|^2 - (|D|\eta)\right) \sum_{j=0}^{\infty} (-|D|\eta)^j . \tag{2.22}$$

By Lemma B.2 we have that $\|(|D|\eta)^j\|_{H^{\sigma,s-1}} \le (C(s)\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}})^j \|\eta\|_{H^{\sigma,s}}$ for any $j \in \mathbb{N}$, and therefore $f(\eta)$ in (2.22) is bounded, on the domain $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, by

$$\|f(\eta)\|_{H^{\sigma,s-1}} \lesssim_s \|\eta\|_{H^{\sigma,s}} . \tag{2.23}$$

Moreover $f(\eta)$ in (2.22) is a series of analytic functions uniformly convergent on the sets $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, $\forall R > 0$. Thus, by Weierstrass theorem, $\eta \mapsto f(\eta)$ is analytic on $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, and, by the arbitrariness of R, on the whole open set $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to H^{\sigma,s-1}$. We conclude that also $G_3(\eta)$ is analytic as stated in (2.19). Finally, (2.23) and (2.21) imply that $G_3(\eta)$ satisfies the tame estimate (1.5). □

Remark 2.7. It follows from the proof that $G(0)\psi = G_2(0)\psi = \Gamma[\partial_y \Pi \Psi(0)\psi]$, which, together with $\Psi(0)\psi = e^{y|D|}\psi$, recovers the identity $G(0)\psi = |D|\psi$.

The final paragraph is devoted to the proof of Proposition 2.6.

Proof of Proposition 2.6: the perturbative argument. We look for a solution φ of (2.3) of the form

$$\varphi(x,y) = \underline{\varphi}(x,y) + u(x,y) \tag{2.24}$$

where $\underline{\varphi}$ is the harmonic solution of

$$\begin{cases} \Delta_{x,y}\underline{\varphi} = 0 \\ \underline{\varphi}(x,0) = \psi(x) \\ \partial_y \underline{\varphi}(x,y) \to 0 \qquad \text{as } y \to -\infty , \end{cases} \qquad \text{i.e.} \quad \underline{\varphi}(x,y) := e^{y|D|}\psi(x) , \tag{2.25}$$

whereas u solves the elliptic problem

$$\begin{cases} \Delta_{x,y} u = F(\eta)[\phi + u] , \\ u(x,0) = 0 \\ \partial_y u(x,y) \to 0 \qquad \text{as } y \to -\infty , \end{cases} \tag{2.26}$$

with $\phi := \underline{\varphi}$. The harmonic function $\underline{\varphi} = e^{y|D|}\psi$ ia estimated by Lemma 2.5.

Remark 2.8. Also the derivative $\partial_x \underline{\varphi}(x,y) \to 0$ as $y \to -\infty$. Actually any solution of (1.2) satisfies $\nabla\Phi(x,y) \to 0$ as $y \to -\infty$. Indeed let a such that $\mathbb{T} \times \{y = -a\} \subset \mathcal{D}_\eta$. Since the harmonic function $\Phi(x,y)$ is analytic then $\vartheta(x) := \Phi(x,-a)$ is analytic as well. Thus (cfr. (2.25)) we can represent $\Phi(x,y) = \sum_{k\in\mathbb{Z}^d} \vartheta_k \, e^{|k|(y+a)} \, e^{\mathrm{i}\,k\cdot x}$, which proves that

$$\nabla\Phi(x,y) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \mathrm{i}\,k\,\vartheta_k\, e^{|k|(y+a)}\, e^{\mathrm{i}\,k\cdot x} \to 0 \quad \text{as} \quad y \to -\infty\,,$$

actually exponentially fast.

The solution of system (2.26) is given by the following lemma.

Lemma 2.9. *Let $\sigma \geq 0$ and s, s_0 such that $s + \frac{1}{2}$, $s_0 \in \mathbb{N}$ and $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. Then there exist $\epsilon_0 := \epsilon_0(s) > 0$ and a unique analytic map*

$$\eta \mapsto U(\eta)\,, \quad U : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \longrightarrow \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}\big)\,,$$

such that $u = U(\eta)[\phi] = U(\eta)[\Pi\phi]$, with Π in (2.11)*, solves* (2.26)*, satisfying*

$$\|\Pi U(\eta)[\phi]\|_{\sigma,s+\frac{1}{2},a} \leq C(s)\big(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}\big)\,. \tag{2.27}$$

The proof of Lemma 2.9 relies on Lemmata 2.10 and 2.11 below.

Given a function $g(x,y)$ defined in $\mathbb{T}^d \times (-\infty, 0)$, we first consider the linear elliptic problem

$$\begin{cases} \Delta_{x,y} u = g \\ u(x,0) = 0 \\ \partial_y u(x,y) \to 0 \quad \text{as } y \to -\infty\,. \end{cases} \tag{2.28}$$

The following key lemma is proved in Appendix C.

Lemma 2.10 (**Elliptic regularity**)**.** *Fix $\sigma \geq 0$, $s \in \mathbb{N}_0$ and $a \in (0,1)$. For any $g \in \mathcal{H}^{\sigma,s,a}$, the elliptic problem* (2.28) *has a unique solution $u := L(g) \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+2,a}$. The linear map*

$$L : \mathcal{H}^{\sigma,s,a} \to \mathbb{C} \oplus \mathcal{H}^{\sigma,s+2,a}\,, \quad g \mapsto L(g)\,,$$

is continuous, i.e. there exists $C_a > 0$ such that $\|Lg\|_{\sigma,s+2,a} \leq C_a \|g\|_{\sigma,s,a}$.

Thanks to Lemma 2.10, we recast the nonlinear elliptic problem (2.26) into the equation

$$\big(\mathrm{Id} - L \circ F(\eta)\big)[u] = L \circ F(\eta)[\phi]\,. \tag{2.29}$$

Note that the linear operator $\mathrm{Id} - L \circ F(\eta)$ depends non-linearly on η and that, recalling (2.4), $F(\eta)[\phi] = F(\eta)[\Pi\phi]$ depends only on the component $\Pi\phi \in \mathcal{H}^{\sigma,s,a}$ of ϕ defined in (2.11), for the presence of the derivatives $\partial_y, \partial_{yy}, \nabla\partial_y$. In the next lemma we study the regularity of the nonlinear map $\eta \mapsto F(\eta)$.

Lemma 2.11. *Let $\sigma \geq 0$, $s + \frac{1}{2}, s_0 \in \mathbb{N}$ with $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. There exists $\epsilon_0 := \epsilon_0(s) > 0$ such that the nonlinear map*

$$\begin{aligned} F : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) &\to \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}, \mathcal{H}^{\sigma,s-\frac{3}{2},a}\big)\,, \\ \eta &\mapsto \big\{\phi \mapsto F(\eta)[\phi]\big\}\,, \end{aligned}$$

defined in (2.4) *is analytic and satisfies the tame estimate*

$$\|F(\eta)[\phi]\|_{\sigma,s-\frac{3}{2},a} \leq C(s)\big(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}}\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \|\eta\|_{H^{\sigma,s}}\|\Pi\phi\|_{\sigma,s_0+2,a}\big)\,. \tag{2.30}$$

Proof. We write $F(\eta)[\phi]$ in (2.4) as

$$F(\eta)[\phi] = \mathcal{F}_1[\alpha(\eta),\phi] + \mathcal{F}_2[\beta(\eta),\phi] + \sum_{j=1}^{d}\mathcal{F}_{3j}[\gamma_j(\eta),\phi] + \mathcal{F}_4[\delta(\eta),\phi]$$

with bilinear maps

$$\mathcal{F}_1[g,\phi] := g\partial_y^2\phi\,, \quad \mathcal{F}_2[g,\phi] := g\Delta\phi\,, \quad \mathcal{F}_{3j}[g,\phi] := g\partial_{x_j}\partial_y\phi\,, \quad \mathcal{F}_4[g,\phi] := g\partial_y\phi\,.$$

In view of (2.15), Lemma 2.2 and (2.11), each of these maps is bounded $\mathcal{H}^{\sigma,s-\frac{3}{2},a}\times(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac{1}{2},a}) \to \mathcal{H}^{\sigma,s-\frac{3}{2},a}$ and any $\mathcal{F} \in \{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_{3j},\mathcal{F}_4\}$ fulfills the tame estimates

$$\|\mathcal{F}[g,\phi]\|_{\sigma,s-\frac{3}{2},a} \lesssim_s \|g\|_{\sigma,s-\frac{3}{2},a}\|\Pi\phi\|_{\sigma,s_0+2,a} + \|g\|_{\sigma,s_0,a}\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a}\,. \tag{2.31}$$

We claim that the maps

$$\begin{aligned} &H^{\sigma,s} \cap B^{\sigma,s_0+\frac{1}{2}}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{1}{2},a}\,, \quad \eta \mapsto \alpha(\eta)\,,\beta(\eta)\,,\gamma_j(\eta)\,, \quad j = 1,\dots,d\,, \\ &H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{3}{2},a}\,, \quad \eta \mapsto \delta(\eta)\,, \end{aligned} \tag{2.32}$$

are analytic and, for any $s \geq s_0 + \frac{3}{2}$, $j = 1,\dots,d$,

$$\|\alpha(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\beta(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\gamma_j(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\delta(\eta)\|_{\sigma,s-\frac{3}{2},a} \leq C(s)\|\eta\|_{H^{\sigma,s}}\,. \tag{2.33}$$

It is clear that these properties, together with (2.31), imply the Lemma.

Let us consider first $\alpha(\eta)$, defined in (2.5), which we rewrite as

$$\alpha(\eta) = \Big(1 - \frac{1}{\partial_y\rho(\eta)}\Big) + \Big(1 - \frac{1}{\partial_y\rho(\eta)}\Big)|\nabla\rho(\eta)|^2 - |\nabla\rho(\eta)|^2\,.$$

We first prove that $\eta \mapsto 1 - \frac{1}{\partial_y\rho(\eta)}$ is analytic as a map $H^{\sigma,s} \cap B^{\sigma,s_0}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{1}{2},a}$. We first note that Lemma 2.5 implies

$$\|\partial_y\rho(\eta) - 1\|_{\sigma,s-\frac{1}{2},a} = \|e^{y|D|}|D|\eta\|_{\sigma,s-\frac{1}{2},a} \lesssim_s \|\eta\|_{H^{\sigma,s}}\,. \tag{2.34}$$

Then by (2.16) and (2.34) the series

$$1 - \frac{1}{\partial_y\rho(\eta)} = -\sum_{j\geq 1}(1 - \partial_y\rho(\eta))^j \tag{2.35}$$

is bounded by

$$\Big\|\frac{1}{\partial_y\rho(\eta)}-1\Big\|_{\sigma,s-\frac12,a}\le\|\partial_y\rho(\eta)-1\|_{\sigma,s-\frac12,a}\sum_{j\ge1}\big(2C_s\|\partial_y\rho(\eta)-1\|_{\sigma,s_0,a}\big)^{j-1}\le C(s)\|\eta\|_{H^{\sigma,s}}\qquad(2.36)$$

provided $\|\eta\|_{H^{\sigma,s_0+\frac12}}<\epsilon_0(s)$ is small enough. The series (2.35) of analytic functions in uniformly convergent in $\mathcal{H}^{\sigma,s-\frac12,a}$ on the domain $\eta\in B^{\sigma,s}(R)\cap B^{\sigma,s_0}(\epsilon_0)$, $\forall R>0$, thus it defines an analytic map on $H^{\sigma,s}\cap B^{\sigma,s_0}(\epsilon_0)$. Moreover the linear map $\eta\mapsto\nabla\rho(\eta)=e^{y|D|}\nabla\eta$ is, by Lemma 2.5, bounded between $H^{\sigma,s}\to\mathcal{H}^{\sigma,s-\frac12,a}$. Therefore $\alpha(\eta)$ is the product of analytic functions $H^{\sigma,s}\cap B^{\sigma,s_0+\frac12}(\epsilon_0)\to\mathcal{H}^{\sigma,s-\frac12,a}$, and using the tame estimate (2.15) we get (2.33).

The analyticity and the estimates of the functions $\eta\mapsto\beta(\eta),\gamma_j(\eta)$, $j=1,\dots,d$ stated in (2.32) follow similarly. Finally consider $\delta(\eta)$ in (2.5). The biggest loss of derivatives follows from the linear maps $\eta\mapsto\Delta\rho(\eta)$, $\partial_y^2\rho(\eta)$, $\nabla\partial_y\rho(\eta)$ which, by Lemmata 2.2 and 2.5, are bounded between $H^{\sigma,s}\to\mathcal{H}^{\sigma,s-\frac32,a}$. Moreover $\delta(\eta)$ satisfies the estimate $\|\delta(\eta)\|_{H^{\sigma,s-\frac32,a}}\le C(s,\|\eta\|_{H^{\sigma,s_0+\frac32}})\|\eta\|_{H^{\sigma,s}}$ for any $s-\frac32\ge s_0$. □

Proof of Lemma 2.9. For any $s\ge s_0+\frac32$ such that $s+\frac12\in\mathbb{N}$, by Lemmata 2.10 and 2.11, the map

$$\eta\mapsto P(\eta):=L\circ F(\eta)\,,\quad H^{\sigma,s}\cap B^{\sigma,s_0+\frac12}(\epsilon_0)\to\mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac12,a})\,,$$

is analytic and, for positive constants $C(s)\ge C'(s_0)>0$, in view of (2.11),

$$\begin{aligned}&\|P(\eta)[\phi]\|_{\sigma,s+\frac12,a}\le C(s)\big(\|\eta\|_{H^{\sigma,s_0+\frac32}}\|\Pi\phi\|_{\sigma,s+\frac12,a}+\|\eta\|_{H^{\sigma,s}}\|\Pi\phi\|_{\sigma,s_0+2,a}\big)\\&\|P(\eta)[\phi]\|_{\sigma,s_0+2,a}\le C'(s_0)\,\|\eta\|_{H^{\sigma,s_0+\frac32}}\|\Pi\phi\|_{\sigma,s_0+2,a}\,.\end{aligned}\qquad(2.37)$$

We claim that, for any $\eta\in H^{\sigma,s}\cap B^{\sigma,s_0+\frac32}(\epsilon_0)$ and $\epsilon_0(s)>0$ small enough, the operator $\mathrm{Id}-P(\eta)$ is invertible in $\mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac12,a})$ and the inverse map

$$\eta\mapsto(\mathrm{Id}-P(\eta))^{-1}=\sum_{j=0}^{\infty}P(\eta)^j[\phi]\,,\quad H^{\sigma,s}\cap B^{\sigma,s_0+\frac32}(\epsilon_0)\to\mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac12,a})\,,\qquad(2.38)$$

is analytic. As each $\eta\mapsto P(\eta)^j$ is analytic $H^{\sigma,s}\cap B^{\sigma,s_0+\frac12}(\epsilon_0)\to\mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac12,a})$, the claim follows by proving that the series (2.38) converges uniformly in $\mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac12,a})$ for $\eta\in B^{\sigma,s}(R)\cap B^{\sigma,s_0+\frac32}(\epsilon_0)$ for any $R>0$. By (2.37) we have, for any $j\in\mathbb{N}$,

$$\|P(\eta)^j[\phi]\|_{\sigma,s_0+2,a}\le\big(C'(s_0)\|\eta\|_{H^{\sigma,s_0+\frac32}}\big)^j\|\Pi\phi\|_{\sigma,s_0+2,a}\,,\qquad(2.39)$$

and, by induction, we prove that

$$\|P(\eta)^j[\phi]\|_{\sigma,s+\frac12,a}\le C(s)^j\|\eta\|^{j-1}_{H^{\sigma,s_0+\frac32}}\big(\|\eta\|_{H^{\sigma,s_0+\frac32}}\|\Pi\phi\|_{\sigma,s+\frac12,a}+j\|\eta\|_{H^{\sigma,s}}\|\Pi\phi\|_{\sigma,s_0+2,a}\big)\,.\qquad(2.40)$$

Indeed, for $j=1$ this is (2.37). Then assuming that (2.40) holds for j, we get

$$\|P(\eta)^{j+1}[\phi]\|_{\sigma,s+\frac12,a}\overset{(2.37)}{\le}C(s)\big(\underbrace{\|\eta\|_{H^{\sigma,s_0+\frac32}}\|P(\eta)^j[\phi]\|_{\sigma,s+\frac12,a}}_{=:A}+\underbrace{\|\eta\|_{H^{\sigma,s}}\|P(\eta)^j[\phi]\|_{\sigma,s_0+2,a}}_{=:B}\big).$$

By the inductive hypothesis the first term is bounded by

$$A \le C(s)^j \|\eta\|^j_{H^{\sigma,s_0+\frac{3}{2}}} \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + j\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \right),$$

whereas, by (2.39),

$$B \le (C'(s_0)\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}})^j \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a},$$

and we deduce, as $C'(s_0) \le C(s)$, that

$$\|P(\eta)^{j+1}[\phi]\|_{\sigma,s+\frac{1}{2},a} \le C(s)^{j+1} \|\eta\|^j_{H^{\sigma,s_0+\frac{3}{2}}} \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + (j+1)\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \right),$$

which proves (2.40) at the step $j+1$.

By (2.40), the series in (2.38) is bounded by

$$\begin{aligned} \|(\mathrm{Id} - P(\eta))^{-1}[\phi]\|_{\sigma,s+\frac{1}{2},a} &\le \sum_{j\ge0} \|P(\eta)^j[\phi]\|_{\sigma,s+\frac{1}{2},a} \\ &\le \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \sum_{j\ge1} C(s)^j \|\eta\|^{j-1}_{H^{\sigma,s_0+\frac{3}{2}}} \|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} \\ &\qquad + \sum_{j\ge1} C(s)^j \|\eta\|^{j-1}_{H^{\sigma,s_0+\frac{3}{2}}} \, j \, \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \\ &\le 2\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + C\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \end{aligned} \tag{2.41}$$

provided $\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} < \epsilon_0(s)$ is sufficiently small. In particular this shows the claim on the uniform convergence of the series on $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ for any $R > 0$.

The analytic map

$$U : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \longrightarrow \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}\big), \quad U(\eta)[\phi] := (\mathrm{Id} - L \circ F(\eta))^{-1}\big[L \circ F(\eta)[\phi]\big],$$

defines the unique solution $u = U(\eta)[\phi]$ of (2.29) and, consequently, of system (2.26). By (2.41) and (2.37) we deduce (2.27). This proves Lemma 2.9. □

Proof of Proposition 2.6. It follows with $\varphi = \Psi(\eta)[\psi] = e^{y|D|}\psi + U(\eta)[e^{y|D|}\psi]$, see (2.24), (2.25) and Lemma 2.9. □

3 Analyticity of the Stokes wave

In this section we prove Theorem 1.3. With the aid of the analyticity result of Theorem 1.2, the bifurcation proof is classical. We report it for completeness. It is based on the application of the analytic Crandall-Rabinowitz Theorem 3.1 below. For the proof we refer e.g to [15], and Theorem 4.1 in Chap. 5 of [6] for its smooth version.

Theorem 3.1 (Crandall-Rabinowitz bifurcation Theorem)**.** *Let X, Y be Banach spaces and $U \subset X$ be an open neighbourhood of 0. Let $F : U \times \mathbb{R} \to Y$, $F(u, c)$, be an analytic map satisfying $F(0, c) = 0$ for any $c \in \mathbb{R}$. Let c^* be such that $L := \mathrm{d}_u F(0, c^*) \in \mathcal{L}(X, Y)$ is not invertible and*

1. *Ker$(L) = \mathrm{span}\langle u^* \rangle$, $u^* \in X$, is 1-dimensional;*

2. the range $R := Rng(L)$ *is closed and* codim $R = 1$*;*

3. (transversality) $\partial_c \mathrm{d}_u F(0,c^*)[u^*] \notin R$.

Then there exist $\epsilon_* > 0$ *and an analytic function*

$$(-\epsilon_*,\epsilon_*) \to U \times \mathbb{R}, \quad \epsilon \mapsto (u_\epsilon, c_\epsilon), \qquad u_\epsilon = \epsilon u^* + O(\epsilon^2), \quad c_\epsilon = c^* + O(\epsilon),$$

such that $F(u_\epsilon, c_\epsilon) = 0$ *for any* $|\epsilon| < \epsilon_*$.

Theorem 1.3 is proved by applying Theorem 3.1 to the nonlinear operator

$$\begin{aligned} &F : \big(H^{\sigma,s}_{\mathtt{ev}_0} \cap B^{\sigma,s_0}(\epsilon_0)\big) \times H^{\sigma,s}_{\mathtt{odd}} \times \mathbb{R} \longrightarrow H^{\sigma,s-1}_{\mathtt{odd}} \times H^{\sigma,s-1}_{\mathtt{ev}_0}, \ \sigma \geq 0, \ s > 5/2, \\ &F(\eta,\psi,c) := \Big(c\eta_x + G(\eta)\psi \ , \ c\psi_x - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2\Big) \end{aligned} \tag{3.1}$$

where $H^{\sigma,s}_{\mathtt{ev}_0}$, respectively $H^{\sigma,s}_{\mathtt{odd}}$, denote the space of even, respectively odd, and average-free real valued functions in $H^{\sigma,s}$ defined in (1.4), and $\epsilon_0 := \epsilon_0(\sigma,s,s_0) > 0$ is provided by Theorem 1.2. Note that a real function $(\eta,\psi) \in H^{\sigma,s}_{\mathtt{ev}_0} \times H^{\sigma,s}_{\mathtt{odd}}$ admits a Fourier series expansion

$$\begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} = \sum_{k\geq 1} \begin{bmatrix} \eta_k \cos(kx) \\ \psi_k \sin(kx) \end{bmatrix} \quad \text{with norm} \quad \|(\eta,\psi)\|^2_{H^{\sigma,s}} \simeq \sum_{k\geq 1} e^{2\sigma|k|}\langle k\rangle^{2s}(\eta_k^2 + \psi_k^2). \tag{3.2}$$

The fact that the nonlinear operator F in (3.1) maps a pair of functions (η,ψ) which are odd/even in x into a pair of functions which are even/odd in x is verified thanks to the reversibility property (A.3). Moreover, the second component of F has zero average thanks to the following lemma.

Lemma 3.2. *Let* $G(\eta)$ *be the Dirichlet-Neumann operator defined in* (1.3)*. Then*

$$\int_{\mathbb{T}} -\frac{1}{2}\psi_x^2 + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \,\mathrm{d}x = 0. \tag{3.3}$$

Proof. By (A.4), the kinetic energy $K(\eta,\psi) = \frac{1}{2}(\psi, G(\eta)\psi)_{L^2}$ in (1.9) satisfies $K(\eta+m,\psi) = K(\eta,\psi)$ for any $m \in \mathbb{R}$. Thus

$$0 = \frac{\mathrm{d}}{\mathrm{d}m}K(\eta+m,\psi) = \mathrm{d}_\eta K(\eta,\psi)[1] = (\nabla_\eta K(\eta,\psi),1)_{L^2} = \int_{\mathbb{T}} \nabla_\eta K(\eta,\psi)\,\mathrm{d}x.$$

In view of (1.10), the identity (3.3) is proved. ☐

We now start verifying the assumptions of the Crandall-Rabinowitz Theorem 3.1. First, by Theorem 1.2, the nonlinear operator F defined in (3.1) is *analytic*. Moreover, by inspection,

$$F(0,0,c) = 0, \quad \forall c \in \mathbb{R}.$$

The possible bifurcation values of non-trivial solutions of $F(\eta,\psi,c) = 0$ are those speeds c such that the linearized operator

$$\mathrm{d}_{(\eta,\psi)}F(0,0,c) : H^{\sigma,s}_{\mathtt{ev}_0} \times H^{\sigma,s}_{\mathtt{odd}} \to H^{\sigma,s-1}_{\mathtt{odd}} \times H^{\sigma,s-1}_{\mathtt{ev}_0}, \quad \begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix} \mapsto \begin{bmatrix} c\partial_x & |D| \\ -g & c\partial_x \end{bmatrix} \begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix}, \tag{3.4}$$

has a nontrivial kernel. In the next lemma we characterize such values.

Lemma 3.3. ***(Bifurcation speeds)*** *The kernel of* $\mathrm{d}_{(\eta,\psi)}F(0,0,c)$ *in* (3.4) *is nontrivial if and only if*

$$c = \pm\sqrt{\frac{g}{k}} \qquad \textit{for some } k \in \mathbb{N}\,. \tag{3.5}$$

For any $k \in \mathbb{N}$*, the Kernel of* $L := \mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$*, where we set* $c_k^* := \sqrt{\frac{g}{k}}$*, is one dimensional and*

$$\mathit{Ker}(L) = \langle u^* \rangle \quad \textit{with} \quad u^* := \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix}. \tag{3.6}$$

Proof. By the Fourier expansion (3.2), it results that the kernel of $\mathrm{d}_{(\eta,\psi)}F(0,0,c)$ is nontrivial if and only if at least one of the matrices $\begin{bmatrix} -ck & k \\ -g & ck \end{bmatrix}$, $k \in \mathbb{N}$, has zero determinant. This is verified provided $c^2 k = g$ for some $k \in \mathbb{N}$, i.e. (3.5) holds. In addition, a vector $\begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} = \sum_{j\geq 1} \begin{bmatrix} \eta_j \cos(jx) \\ \psi_j \sin(jx) \end{bmatrix}$ belongs to the Kernel of $\mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$ if and only if

$$\begin{bmatrix} -c_k^* j & j \\ -g & c_k^* j \end{bmatrix} \begin{bmatrix} \eta_j \\ \psi_j \end{bmatrix} = 0\,, \quad \forall j \geq 1\,. \tag{3.7}$$

If $j \neq k$ then

$$\det \begin{bmatrix} -c_k^* j & j \\ -g & c_k^* j \end{bmatrix} = -(c_k^*)^2 j^2 + gj = j^2\big((c_j^*)^2 - (c_k^*)^2\big) \neq 0\,, \tag{3.8}$$

since the map $k \mapsto (c_k^*)^2 = g/k$ is injective on $\mathbb{N}$. Hence $\eta_j = \psi_j = 0$ for any $j \neq k$. On the other hand, if $j = k$ then (3.7) is solved provided $\sqrt{g}\eta_k = \sqrt{k}\psi_k$, proving (3.6). ☐

We apply Theorem 3.1 with $c_k^* := \sqrt{\frac{g}{k}}$. By Lemma 3.3 assumption 1 holds. The next lemma verifies the assumptions 2)-3).

Lemma 3.4. *The range* $R := \mathit{Rng}L$*,* $L = \mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$*, is*

$$R = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^{\sigma,s-1}_{\mathtt{odd}}(\mathbb{T}) \times H^{\sigma,s-1}_{\mathtt{ev}_0}(\mathbb{T}) : \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} f_k \sin(kx) \\ c_k^* f_k \cos(kx) \end{bmatrix} + \sum_{j\geq 1, j\neq k} \begin{bmatrix} f_j \sin(jx) \\ g_j \cos(jx) \end{bmatrix} \right\}. \tag{3.9}$$

In particular R *is closed and* $\operatorname{codim} R = 1$.

The vector $(\partial_c \mathrm{d}_{(\eta,\psi)}F)(0,0,c_k^*) \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix}$ *does not belong to* R*.*

Proof. A vector $\begin{bmatrix} f \\ g \end{bmatrix} \in H^{\sigma,s-1}_{\mathtt{odd}}(\mathbb{T}) \times H^{\sigma,s-1}_{\mathtt{ev}_0}(\mathbb{T})$ belongs to R if and only if there is $\begin{bmatrix} \eta \\ \psi \end{bmatrix} \in H^{\sigma,s}_{\mathtt{ev}_0} \times H^{\sigma,s}_{\mathtt{odd}}$ such that, recalling (3.4) and (3.2),

$$\begin{bmatrix} -c_k^* j & j \\ -g & c_k^* j \end{bmatrix} \begin{bmatrix} \eta_j \\ \psi_j \end{bmatrix} = \begin{bmatrix} f_j \\ g_j \end{bmatrix} \quad \forall j \geq 1 \quad \text{where} \quad \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \sum_{j\geq 1} \begin{bmatrix} f_j \sin(jx) \\ g_j \cos(jx) \end{bmatrix}. \tag{3.10}$$

For any $j \neq k$, by (3.8), system (3.10) has the unique solution

$$\eta_j = \frac{1}{g}\frac{\sqrt{k}}{k-j}\big(\sqrt{g}f_j - \sqrt{k}g_j\big)\,, \quad \psi_j = \frac{1}{j\sqrt{g}}\frac{\sqrt{k}}{k-j}\big(\sqrt{kg}f_j - jg_j\big)\,. \tag{3.11}$$

If $j = k$, the system (3.10) is solvable if and only if

$$\sqrt{g} f_k = \sqrt{k} g_k \tag{3.12}$$

and a solution is $\eta_k = -\frac{1}{\sqrt{kg}} f_k$, $\psi_k = 0$. By (3.11) we deduce that $|\eta_j|, |\psi_j| \leq \frac{C_k}{j}(|f_j| + |g_j|)$, for any $j \in \mathbb{N} \setminus \{k\}$, implying that $(\eta, \psi) \in H^{\sigma,s}$, actually

$$\|\eta\|_{H^{\sigma,s}}, \ \|\psi\|_{H^{\sigma,s}} \leq C_k(\|f\|_{H^{\sigma,s-1}} + \|g\|_{H^{\sigma,s-1}}) .$$

In conclusion, the range R of L has the form (3.9), by (3.12) and $c_k^* = \sqrt{g/k}$.

Finally differentiating (3.4) one computes

$$(\partial_c \mathrm{d}_{(\eta,\psi)} F)(0,0,c_k^*) \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix} \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix} = \begin{bmatrix} -k^{\frac{3}{2}}\sin(kx) \\ k\sqrt{g}\cos(kx) \end{bmatrix}$$

which does not belong to the range R in (3.9). □

All the assumptions of the Crandall-Rabinowitz Theorem are verified, proving Theorem 1.3.

A Basic properties of the Dirichlet-Neumann operator

The linear Dirichlet-Neumann operator $G(\eta)$ defined in (1.3) is self-adjoint with respect to the L^2 scalar product,

$$\big(G(\eta)\psi_1, \psi_2\big)_{L^2} = \int_{\mathcal{D}_\eta} \nabla\Phi_1 \cdot \nabla\Phi_2 \, \mathrm{d}x = \big(G(\eta)\psi_2, \psi_1\big)_{L^2} ,$$

where Φ_1 and Φ_2 are the harmonic functions associated to ψ_1, ψ_2 as in (1.2). Thus $G(\eta)$ is semi-positive definite

$$\big(G(\eta)\psi, \psi\big)_{L^2} = \int_{\mathcal{D}_\eta} |\nabla\Phi|^2 \mathrm{d}x \geq 0 , \tag{A.1}$$

and its kernel contains only the constant functions, $G(\eta)[1] = 0$. In particular (A.1) implies also the unicity of the solutions of (1.2).

We list other classical algebraic properties of the Dirichlet-Neumann used in the paper.

Lemma A.1. *The Dirichlet-Neumann $G(\eta)$ in* (1.3) *is:*

(i) invariant under space translations

$$\tau_\theta G(\eta)\psi = G(\tau_\theta\eta)[\tau_\theta\psi] , \quad \tau_\theta u(x) := u(x+\theta) , \quad \forall \theta \in \mathbb{R}^d ; \tag{A.2}$$

(ii) invariant under the reflection at the origin, namely

$$G(\eta^\vee)[\psi^\vee] = (G(\eta)[\psi])^\vee \quad \textit{where} \quad f^\vee(x) := f(-x) ; \tag{A.3}$$

(iii) constant along vertical translations, i.e.

$$G(\eta + m) = G(\eta) , \quad \forall m \in \mathbb{R} . \tag{A.4}$$

Proof. Let us prove (A.2). Let Φ be the solution of (1.2). For any $\theta \in \mathbb{R}^d$ the harmonic function

$$\Phi_\theta(x,y) := \Phi(x+\theta, y) \quad \forall (x,y) \in \mathcal{D}_{\tau_\theta \eta} = \{y < \eta(x+\theta)\}$$

solves

$$\Delta_{x,y}\Phi_\theta = 0 \text{ in } \mathcal{D}_{\tau_\theta\eta}\,, \quad \Phi_\theta(x, \tau_\theta\eta(x)) = \tau_\theta\psi(x)\,, \quad \partial_y \Phi_\theta(x,y) \to 0 \text{ as } y \to -\infty\,.$$

Therefore, by (1.3),

$$\begin{aligned} G(\tau_\theta\eta)[\tau_\theta\psi] &= (\partial_y\Phi_\theta)(x,\tau_\theta\eta) - (\nabla\tau_\theta\eta)(x)\cdot(\nabla\Phi_\theta)(x,\tau_\theta\eta) \\ &= (\partial_y\Phi)(x+\theta,\eta(x+\theta)) - (\nabla\eta)(x+\theta)\cdot(\nabla\Phi)(x+\theta,\eta(x+\theta)) = \tau_\theta G(\eta)[\psi] \end{aligned}$$

proving (A.2). To prove (A.3), consider the harmonic function

$$\Phi^\vee(x,y) := \Phi(-x,y) \quad \forall (x,y) \in \mathcal{D}_{\eta^\vee} = \{y < \eta^\vee(x)\}$$

which solves

$$\Delta_{x,y}\Phi^\vee = 0 \text{ in } \mathcal{D}_\eta^\vee\,, \quad \Phi^\vee(x,\eta^\vee(x)) = \psi^\vee(x)\,, \quad \partial_y\Phi^\vee(x,y) \to 0 \text{ as } y \to -\infty\,.$$

Therefore (A.3) follows by

$$\begin{aligned} G(\eta^\vee)[\psi^\vee] &= (\partial_y\Phi^\vee)(x,\eta^\vee(x)) - (\nabla\eta^\vee)(x)\cdot(\nabla\Phi^\vee)(x,\eta^\vee(x)) \\ &= (\partial_y\Phi)(-x,\eta(-x)) - (\nabla\eta)(-x)\cdot(\nabla\Phi)(-x,\eta(-x)) = G(\eta)[\psi](-x)\,. \end{aligned}$$

For any $m \in \mathbb{R}$ the harmonic function

$$\Phi_m(x,y) := \Phi(x, y-m) \quad \forall (x,y) \in \mathcal{D}_{\eta+m} = \{y < \eta(x) + m\}$$

solves

$$\Delta_{x,y}\Phi_m = 0 \text{ in } \mathcal{D}_{\eta+m}\,, \quad \Phi_m(x,\eta(x)+m) = \psi(x)\,, \quad \partial_y\Phi_m(x,y) \to 0 \text{ as } y \to -\infty\,.$$

Therefore, by (1.3),

$$\begin{aligned} G(\eta+m)[\psi] &= (\partial_y\Phi_m)(x,\eta(x)+m) - (\nabla\eta)(x)\cdot(\nabla\Phi_m)(x,\eta(x)+m) \\ &= (\partial_y\Phi)(x,\eta(x)) - (\nabla\eta)(x)\cdot(\nabla\Phi)(x,\eta(x)) = G(\eta)[\psi] \end{aligned}$$

proving (A.4). □

B Functional spaces

We collect in this Appendix some properties of the function spaces $H^{\sigma,s}$ and $\mathcal{H}^{\sigma,s,a}$.

B.1 The spaces $H^{\sigma,s}$

We first note the following characterization of the spaces $H^{\sigma,s}$.

Lemma B.1. ***(Characterization of*** $H^{\sigma,s}$***)*** *The space* $H^{\sigma,s}(\mathbb{T}^d)$*,* $\sigma > 0$*, coincides with the periodic functions* $u(x)$ *which admit an extension* $u(z)$ *in the complex strip*

$$\mathbb{T}^d_\sigma := \mathbb{T}^d + \mathrm{i}\,[-\sigma,\sigma]^d = \Big\{ z = x + \mathrm{i}\,y \colon\ x \in \mathbb{T}^d\,,\ y \in \mathbb{R}^d,\ \ |y|_\infty := \max\{|y_1|,\dots,|y_d|\} \le \sigma \Big\}\,,$$

which is analytic in $|y|_\infty < \sigma$*, and whose traces at the boundaries* $u(\cdot + \mathrm{i}\,y)$*,* $|y|_\infty = \sigma$*, belong to the Sobolev space* $H^s := H^s(\mathbb{T}^d)$*, with equivalence of the norms*

$$\|u\|_{H^{\sigma,s}} \simeq_d \sup_{|y|_\infty \le \sigma} \Big\{ \|u(\cdot + \mathrm{i}\,y)\|_{H^s} \Big\}\,. \tag{B.1}$$

Proof. Let $u(x)$ be a function in $H^{\sigma,s}(\mathbb{T}^d)$. For any $z \in \mathbb{C}^d$, $z = x + \mathrm{i}\,y$, $|y|_\infty \le \sigma$, we define its extension

$$u(z) := \sum_{k\in\mathbb{Z}^d} u_k\, e^{\mathrm{i}\,k\cdot z}$$

which is analytic for $|y|_\infty < \sigma$. For any $y \in \mathbb{R}^d$ with $|y|_\infty \le \sigma$, the Sobolev norm $\|\ \|_{H^s}$ of the periodic function

$$x \mapsto u^{(y)}(x) := u(x + \mathrm{i}\,y) = \sum_{k\in\mathbb{Z}^d} u_k\, e^{-k\cdot y} e^{\mathrm{i}\,k\cdot x} \tag{B.2}$$

is bounded by

$$\begin{aligned}\|u^{(y)}\|^2_{H^s} = \sum_{k\in\mathbb{Z}^d} |u_k|^2 e^{-2k\cdot y}\langle k\rangle^{2s} &\le \sum_{k\in\mathbb{Z}^d} |u_k|^2 e^{2|y|_\infty |k|_1}\langle k\rangle^{2s}\\ &\le \sum_{k\in\mathbb{Z}^d} |u_k|^2 e^{2\sigma|k|_1}\langle k\rangle^{2s} = \|u\|^2_{H^{\sigma,s}}\,.\end{aligned}$$

Thus $u^{(y)}(\cdot)$ belongs to H^s and $\|u^{(y)}\|_{H^s} \le \|u\|_{H^{\sigma,s}}$.
In order to prove the equivalence (B.1), consider the partition of $\mathbb{Z}^d$,

$$\mathbb{Z}^d = \bigcup_{\vec\epsilon\in\{\pm1\}^d} \mathbb{Z}^d_{\vec\epsilon}\,,\quad \mathbb{Z}^d_{\vec\epsilon} := \left\{ k = (k_1,\dots,k_d) \in \mathbb{Z}^d\ :\ \begin{cases} k_j > 0\,, & \text{if } \epsilon_j = -1\,,\\ k_j \le 0\,, & \text{if } \epsilon_j = 1\,,\end{cases}\quad \forall j = 1,\dots,d \right\}.$$

For any $\vec\epsilon = (\epsilon_1,\dots,\epsilon_d) \in \{\pm1\}^d$, the function $u^{(\sigma\vec\epsilon)}$ defined as in (B.2) satisfies

$$\|u^{(\sigma\vec\epsilon)}\|^2_{H^s} = \sum_{k\in\mathbb{Z}^d} |u_k|^2 e^{-2\sigma k\cdot\vec\epsilon}\langle k\rangle^{2s} \ge \sum_{k\in\mathbb{Z}^d_{\vec\epsilon}} |u_k|^2 \langle k\rangle^{2s} e^{2\sigma(|k_1|+\ldots+|k_d|)}$$

and therefore

$$\|u\|^2_{H^{\sigma,s}} = \sum_{\vec\epsilon\in\{\pm1\}^d}\ \sum_{k\in\mathbb{Z}^d_{\vec\epsilon}} |u_k|^2\langle k\rangle^{2s} e^{2\sigma|k|_1} \le 2^d \sup_{|y|_\infty = \sigma} \|u^{(y)}\|^2_{H^s}\,.$$

The equivalence (B.1) is proved. □

The spaces $H^{\sigma,s}$, $s > d/2$, form an algebra with respect to the product of functions, and the following more general tame estimates hold.

Lemma B.2. ***(Tame)*** *Let $\sigma \geq 0$ and $s \geq \mathfrak{s}_0 > d/2$. There exist positive constants $C_{s,\mathfrak{s}_0} \geq 1$ (non decreasing in s) such that, for any $f, g \in H^{\sigma,s}$, one has*

$$\|fg\|_{H^{\sigma,s}} \leq C_{s,\mathfrak{s}_0}\big(\|f\|_{H^{\sigma,s}}\|g\|_{H^{\sigma,\mathfrak{s}_0}} + \|f\|_{H^{\sigma,\mathfrak{s}_0}}\|g\|_{H^{\sigma,s}}\big). \tag{B.3}$$

In particular, for any $j \geq 1$,

$$\|f^j\|_{H^{\sigma,s}} \leq (2C_{s,\mathfrak{s}_0}\|f\|_{H^{\sigma,\mathfrak{s}_0}})^{j-1}\|f\|_{H^{\sigma,s}}. \tag{B.4}$$

Proof. The classical proof follows adapting the proof of Lemma 4.5.1 in [9] and it is quite similar to that of Lemma B.5. So we omit it. Estimate (B.4) follows by induction from (B.3) in the same way (2.16) descends from (2.15). □

B.2 The spaces $\mathcal{H}^{\sigma,s,a}$

Proof of Lemma 2.3. For any $u \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}_{\leq 0})$, any $y_0 \leq 0$, we have the inequality

$$|u_k(y_0)|^2 \leq 2\int_{-\infty}^0 |\partial_y u_k(y)|\,|u_k(y)|\,\mathrm{d}y\,.$$

Multiplying by $\langle k\rangle^{2s}$ and using the elementary inequality $2\langle k\rangle^{2s}ab \leq \langle k\rangle^{2s-1}a^2 + \langle k\rangle^{2s+1}b^2$, for any $a, b \geq 0$, we get that

$$\begin{aligned}
\|u(\cdot,y_0)\|^2_{H^{\sigma,s}} &= \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\langle k\rangle^{2s}|u_k(y_0)|^2 \\
&\leq \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\int_{-\infty}^0 2\langle k\rangle^{2s}|\partial_y u_k(y)|\,|u_k(y)|\,\mathrm{d}y \\
&\leq \int_{-\infty}^0 \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\langle k\rangle^{2s-1}|\partial_y u_k(y)|^2\mathrm{d}y + \int_{-\infty}^0 \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\langle k\rangle^{2s+1}|u_k(y)|^2\mathrm{d}y \\
&= \|u\|^2_{L^2(\mathbb{R}_{\leq 0},H^{\sigma,s+\frac{1}{2}})} + \|\partial_y u\|^2_{L^2(\mathbb{R}_{\leq 0},H^{\sigma,s-\frac{1}{2}})}
\end{aligned}$$

which proves (2.12) for smooth functions with compact support and then by density for all functions. Finally, recalling the definition of the norm $\|\ \|_{\sigma,s,a}$ in (2.8), we deduce (2.14).

Proof of Lemma 2.5. In view of (2.8) we have that

$$\begin{aligned}
\|e^{y|D|}g - g_0\|^2_{\sigma,s+\frac{1}{2},a} &= \sum_{j=0}^{s+\frac{1}{2}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{2\sigma|k|_1}\langle k\rangle^{2(s+\frac{1}{2}-j)}|g_k|^2\int_{-\infty}^0 |\partial_y^j e^{|k|y}|^2 e^{-2ay}\mathrm{d}y \\
&= \sum_{j=0}^{s+\frac{1}{2}}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{2\sigma|k|_1}\langle k\rangle^{2s+1}\langle k\rangle^{-2j}|g_k|^2|k|^{2j}\int_{-\infty}^0 e^{2(|k|-a)y}\mathrm{d}y \\
&= (s+\frac{1}{2})\sum_{k\neq 0}\frac{\langle k\rangle^{2s+1}}{2(|k|-a)}e^{2\sigma|k|_1}|g_k|^2 \leq C_{a,s}\|g\|^2_{H^{\sigma,s}}
\end{aligned}$$

proving the lemma.

Proof of Proposition 2.4. We define

$$\mathcal{H}^{\sigma,s,a}_{\mathbb{R}} := \big\{u : \mathbb{T}^d \times \mathbb{R} \to \mathbb{C} : \quad \|u\|_{\sigma,s,a,\mathbb{R}} < \infty\big\}$$

endowed with the norm

$$\|u\|^2_{\sigma,s,a,\mathbb{R}} := \sum_{j=0}^{s} \|\partial_y^j u\|^2_{L^{2,a}(\mathbb{R},H^{\sigma,s-j})}$$
$$= \sum_{j=0}^{s} \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k\rangle^{2(s-j)} |\partial_y^j u_k(y)|^2 e^{2a|y|} \mathrm{d}y$$
$$= \sum_{j=0}^{s} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k\rangle^{2(s-j)} \|\partial_y^j u_k\|^2_{L^{2,a}(\mathbb{R})} \tag{B.5}$$

where, given a Hilbert space X, we have used the notation

$$\|u\|^2_{L^{2,a}(\mathbb{R},X)} := \int_{\mathbb{R}} \|u(y)\|^2_X e^{2a|y|} \mathrm{d}y \,.$$

By adapting the method in [29] of "extension by reflection" we have the following lemma.

Lemma B.3. ***(Extension operator)*** *There exists a linear bounded extension operator* $\mathcal{E}_s : \mathcal{H}^{\sigma,s,a} \to \mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$ *such that* $\mathcal{E}_s u = u$ *a.e. on* $(-\infty, 0)$*. Thus*

$$\|u\|_{\sigma,s,a} \le \|\mathcal{E}_s u\|_{\sigma,s,a,\mathbb{R}} \lesssim_s \|u\|_{\sigma,s,a} \,. \tag{B.6}$$

Proof. We follow [29]. For any $u \in \mathcal{C}^\infty_c(\mathbb{R}, H^{\sigma,s})$ we define

$$(\mathcal{E}_s u)(y) := \begin{cases} u(y), & y \le 0\,, \\ \alpha_0^{(s)} u(-y) + \cdots + \alpha_s^{(s)} u(-(s+1)y)\,, & y > 0\,, \end{cases} \tag{B.7}$$

where the coefficients $\alpha_j^{(s)}$, $j = 0, \dots, s$ are to be chosen in order to have

$$\partial_y^j (\mathcal{E}_s u)(0) = (\partial_y^j u)(0), \quad \forall j = 0, \dots, s\,,$$

namely solve the linear system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ -1 & -2 & \dots & -s-1 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^s & (-2)^s & \dots & (-s-1)^s \end{pmatrix} \begin{pmatrix} \alpha_0^{(s)} \\ \vdots \\ \vdots \\ \vdots \\ \alpha_s^{(s)} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

The above Vandermonde matrix is invertible and thus the coefficients $\alpha_0^{(s)}, \dots, \alpha_s^{(s)}$ are uniquely well-defined. Then by (B.7)

$$\|\mathcal{E}_s u\|^2_{\sigma,s,a,\mathbb{R}} \le \|u\|^2_{\sigma,s,a} + C_s \sum_{j=0}^{s} \sum_{i=0}^{s} \int_0^\infty \|\partial_y^j u(-(i+1)y)\|^2_{H^{\sigma,s-j}} e^{2a|y|} \mathrm{d}y$$
$$\le C'_s \sum_{i,j=0}^{s} \int_{-\infty}^0 \|\partial_y^j u(z)\|^2_{H^{\sigma,s-j}} e^{\frac{2a|z|}{i+1}} \mathrm{d}z \lesssim_s \|u\|^2_{\sigma,s,a} \,.$$

By density the operator $u \mapsto \mathcal{E}_s u$ admits a bounded linear extension to $\mathcal{E}_s : \mathcal{H}^{\sigma,s,a} \to \mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$ and the lemma follows. □

In order to analyze the space $\mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$ we can use the Fourier transform in the variable y. Given a function $y \mapsto u(y)$ in $L^2(\mathbb{R}, X)$, where X is a Hilbert space, we denote its Fourier transform

$$\widehat{u}(\xi) := (\mathcal{F}u)(\xi) := \int_{\mathbb{R}} \widehat{u}(y) e^{-\mathrm{i}\,\xi\, y} \mathrm{d}y$$

and, by the inverse Fourier transform formula,

$$u(y) = \int_{\mathbb{R}} \widehat{u}(\xi) e^{\mathrm{i}\,\xi\, y} \text{đ}\xi\,, \quad \text{đ} := \frac{1}{2\pi}\mathrm{d}\,.$$

When $X = H^{\sigma,s}(\mathbb{T}^d)$ we may also Fourier expand in x writing

$$u(y,x) = \sum_{k\in\mathbb{Z}^d} u_k(y) e^{\mathrm{i}\,k\cdot x} = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} \widehat{u}_k(\xi) e^{\mathrm{i}\,(\xi y + k\cdot x)} \text{đ}\xi\,. \tag{B.8}$$

In the sequel we shall also denote $\widehat{u}_k(\xi) = \widehat{u}(k,\xi)$.

The following lemma characterizes the space $\mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$.

Lemma B.4. ***(Characterization of*** $\mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$***)*** *The space* $\mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$, $a > 0$, *coincides with the functions* $y \mapsto u(y) \in L^2(\mathbb{R}, H^{\sigma,s})$ *whose Fourier transform* $\xi \mapsto \widehat{u}(\xi)$ *admits an extension* $\widehat{u}(\zeta)$ *in the complex strip* $\{\zeta = \xi + \mathrm{i}\,\eta \colon\, \xi \in \mathbb{R}\,,\ \eta \in \mathbb{R},\ |\eta| \leq a\}$, *analytic in* $|\eta| < a$, *whose traces at the boundaries* $\widehat{u}(\cdot + \varsigma\mathrm{i}\,a)$, *belong to* $L^2(\mathbb{R}, H^{\sigma,s})$ *and* $\langle\xi\rangle^j \widehat{u}(\xi + \varsigma\mathrm{i}\,a) \in L^2(\mathbb{R}, H^{\sigma,s-j})$, *for any* $j = 0, \ldots, s$, $\varsigma = \pm 1$, *with equivalence of the norms*

$$\|u\|^2_{\sigma,s,a,\mathbb{R}} \simeq_{s,a} \max_{\substack{j=0,\ldots,s\\ \varsigma=\pm1}} \big\|\langle\xi\rangle^j \widehat{u}(\xi + \varsigma\mathrm{i}\,a)\big\|^2_{L^2(\mathbb{R}, H^{\sigma,s-j})} \tag{B.9}$$

$$\simeq_{s,a} \max_{\substack{j=0,\ldots,s\\ \varsigma=\pm1}} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} \langle\xi\rangle^{2j} \langle k\rangle^{2(s-j)} e^{2\sigma|k|_1} |\widehat{u}_k(\xi + \varsigma\mathrm{i}\,a)|^2 \mathrm{d}\xi$$

$$\simeq_{s,a} \max_{\varsigma=\pm1} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} \langle k,\xi\rangle^{2s} e^{2\sigma|k|_1} |\widehat{u}_k(\xi + \varsigma\mathrm{i}\,a)|^2 \mathrm{d}\xi \tag{B.10}$$

where $\langle k,\xi\rangle := \sqrt{1 + |k|^2 + |\xi|^2}$.

Proof. For any $|\eta| \leq a$, integrating by parts,

$$(-\mathrm{i}\,\xi)^j \widehat{u}(\xi \pm \mathrm{i}\,\eta) = \int_{\mathbb{R}} u(y) e^{\mp\eta y} (-\mathrm{i}\,\xi)^j e^{-\mathrm{i}\,\xi y} \mathrm{d}y = \int_{\mathbb{R}} u(y) e^{\mp\eta y} \partial_y^j (e^{-\mathrm{i}\,\xi y}) \mathrm{d}y$$
$$= (-1)^j \int_{\mathbb{R}} \partial_y^j \big(u(y) e^{\mp\eta y}\big) e^{-\mathrm{i}\,\xi y} \mathrm{d}y = \int_{\mathbb{R}} \Big(\sum_{p=0}^{j} \binom{j}{p} (\mp\eta)^{j-p} e^{\mp\eta y} \partial_y^p u(y)\Big) e^{-\mathrm{i}\,\xi y} \mathrm{d}y\,.$$

By Plancherel theorem we have

$$\|\xi^j \widehat{u}(\xi \pm \mathrm{i}\,\eta)\|^2_{L^2_\xi(\mathbb{R}, H^{\sigma,s-j})} \lesssim \sum_{p=0}^{j} \int_{\mathbb{R}} \|\partial_y^p u(y)\|^2_{H^{\sigma,s-j}} e^{2|y|a} \mathrm{d}y \lesssim \|u\|^2_{\sigma,s,a,\mathbb{R}}\,. \tag{B.11}$$

Conversely, for any $0 \leq j \leq s$, we have

$$\|e^{a|y|} \partial_y^j u\|^2_{L^2(\mathbb{R}, H^{\sigma,s-j})} = \int_0^\infty e^{2ay} \|\partial_y^j u(y)\|^2_{H^{\sigma,s-j}} \mathrm{d}y + \int_{-\infty}^0 e^{-2ay} \|\partial_y^j u(y)\|^2_{H^{\sigma,s-j}} \mathrm{d}y$$

$$\leq \|e^{ay}\partial_y^j u(y)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})} + \|e^{-ay}\partial_y^j u(y)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})}\,. \tag{B.12}$$

Now

$$\begin{aligned}\mathcal{F}\big(e^{\pm ay}\partial_y^j u(y)\big)(\xi) &= \int_{\mathbb{R}} e^{-\mathrm{i}\,\xi y} e^{\pm ay}\partial_y^j u(y)\mathrm{d}y = (-1)^j \sum_{p=0}^{j}\binom{j}{p}(\pm a)^{j-p}(-\mathrm{i}\,\xi)^p \int_{\mathbb{R}} e^{-\mathrm{i}\,\xi y} e^{\pm ay} u(y)\mathrm{d}y \\ &= \sum_{p=0}^{j}\binom{j}{p}(\mp a)^{j-p}(\mathrm{i}\,\xi)^p \widehat{u}(\xi \mp \mathrm{i}\,a)\,,\end{aligned}$$

and thus, by Plancharel theorem,

$$\begin{aligned}\|e^{\pm ay}\partial_y^j u(y)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})} &\simeq \|\mathcal{F}\big(e^{\pm ay}\partial_y^j u(y)\big)(\xi)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})} \\ &\lesssim_{a,j} \|\langle\xi\rangle^j \widehat{u}(\xi \mp \mathrm{i}\,a)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})}\,. \end{aligned}\tag{B.13}$$

We deduce by (B.12) and (B.13) that, for any $j = 0, \ldots, s$,

$$\|e^{a|y|}\partial_y^j u\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})} \lesssim_{a,s} \max_{\varsigma=\pm} \|\langle\xi\rangle^j \widehat{u}(\xi + \varsigma\mathrm{i}\,a)\|^2_{L^2(\mathbb{R},H^{\sigma,s-j})}\,. \tag{B.14}$$

The estimates (B.11) and (B.14) prove (B.9).

The last equivalence in (B.10) follows by Young's inequality $\langle\xi\rangle^{2j}\langle k\rangle^{2(s-j)} \leq \langle\xi\rangle^{2s} + \langle k\rangle^{2s}$. ☐

We now prove the tame estimate for the product of two functions in $\mathcal{H}^{\sigma,s,a}_{\mathbb{R}}$.

Lemma B.5 (**Tame**). *Let $\sigma, a \geq 0$, $s, s_0 \in \mathbb{N}$ such that $s \geq s_0 > \frac{d+1}{2}$. Then*

$$\|uv\|_{\sigma,s,a,\mathbb{R}} \leq C_s \|u\|_{\sigma,s,0,\mathbb{R}}\|v\|_{\sigma,s_0,a,\mathbb{R}} + C_s \|u\|_{\sigma,s_0,0,\mathbb{R}}\|v\|_{\sigma,s,a,\mathbb{R}}\,. \tag{B.15}$$

Proof. The product of the functions (cfr. (B.8))

$$u(x,y) = \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} \widehat{u}(k,\xi)e^{\mathrm{i}\,(\xi y + k\cdot x)}\text{đ}\xi\,, \quad v(x,y) = \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} \widehat{v}(k,\xi)e^{\mathrm{i}\,(\xi y + k\cdot x)}\text{đ}\xi\,,$$

is $uv = \sum_{m\in\mathbb{Z}^d}\int_{\mathbb{R}} \widehat{uv}(m,\eta)e^{\mathrm{i}\,\eta y}\, e^{\mathrm{i}\,mx}\text{đ}\eta$ with

$$\widehat{uv}(m,\eta) = \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} \widehat{u}(k,\xi)\,\widehat{v}(m-k,\eta-\xi)\text{đ}\xi\,. \tag{B.16}$$

By (B.10), (B.16) we have that

$$\begin{aligned}\|uv\|^2_{\sigma,s,a,\mathbb{R}} &\simeq \max_{\varsigma\in\{\pm1\}}\sum_{m\in\mathbb{Z}^d}\int_{\mathbb{R}} |\widehat{uv}(m,\eta+\varsigma\mathrm{i}\,a)|^2 e^{2\sigma|m|_1}\langle m,\eta\rangle^{2s}\text{đ}\eta \qquad\qquad (B.17)\\ &\leq \max_{\varsigma\in\{\pm1\}}\sum_{m\in\mathbb{Z}^d}\int_{\mathbb{R}}\Big(\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}} |\widehat{u}(k,\xi)|\,|\widehat{v}(m-k,\eta-\xi+\varsigma\mathrm{i}\,a)|e^{\sigma|m|_1}\langle m,\eta\rangle^s\text{đ}\xi\Big)^2\text{đ}\eta.\end{aligned}$$

We split the frequency space into

$$\begin{aligned}A &:= \big\{(m,\eta,k,\xi) \in \mathbb{Z}^d\times\mathbb{R}\times\mathbb{Z}^d\times\mathbb{R} \;:\; \langle m,\eta\rangle \leq 2\langle k,\xi\rangle\big\},\\ B &:= \big\{(m,\eta,k,\xi) \in \mathbb{Z}^d\times\mathbb{R}\times\mathbb{Z}^d\times\mathbb{R} \;:\; \langle m,\eta\rangle > 2\langle k,\xi\rangle\big\}\end{aligned}$$

and, since $e^{\sigma|m|_1} \le e^{\sigma|k|_1} e^{\sigma|m-k|_1}$, we estimate (B.17) as

$$\|uv\|^2_{\sigma,s,a,\mathbb{R}} \le I_1 + I_2 \tag{B.18}$$

where

$$I_1 := \sum_{m\in\mathbb{Z}^d} \int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}_A} e^{\sigma|k|_1} |\widehat{u}(k,\xi)| \langle k,\xi\rangle^s e^{\sigma|m-k|_1} |\widehat{v}(m-k,\eta-\xi+\varsigma \mathrm{i}\, a)| \langle m-k,\eta-\xi\rangle^{s_0} \frac{\langle m,\eta\rangle^s}{\langle k,\xi\rangle^s \langle m-k,\eta-\xi\rangle^{s_0}} đ\xi\Big)^2 đ\eta,$$

$$I_2 := \sum_{m\in\mathbb{Z}^d} \int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}_B} e^{\sigma|k|_1} |\widehat{u}(k,\xi)| \langle k,\xi\rangle^{s_0} e^{\sigma|m-k|_1} |\widehat{v}(m-k,\eta-\xi+\varsigma \mathrm{i}\, a)| \langle m-k,\eta-\xi\rangle^{s} \frac{\langle m,\eta\rangle^s}{\langle k,\xi\rangle^{s_0} \langle m-k,\eta-\xi\rangle^{s}} đ\xi\Big)^2 đ\eta,$$

where, given $m, k \in \mathbb{Z}^d$ and $\eta \in \mathbb{R}$, we denoted

$$\mathbb{R}_A := \Big\{\xi \in \mathbb{R} \,:\, (m,\eta,k,\xi) \in A\Big\}, \qquad \mathbb{R}_B := \Big\{\xi \in \mathbb{R} \,:\, (m,\eta,k,\xi) \in B\Big\}.$$

Using that, if $(m,\eta,k,\xi) \in A$ then $\langle k,\xi\rangle > \frac{1}{2}\langle m,\eta\rangle$, the Cauchy-Schwarz inequality and exchanging the order of integration we get

$$\begin{aligned}
I_1 &\lesssim_s \sum_{m\in\mathbb{Z}^d} \int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\sigma|k|_1} |\widehat{u}(k,\xi)|^2 \langle k,\xi\rangle^{2s} e^{2\sigma|m-k|_1} |\widehat{v}(m-k,\eta-\xi+\varsigma \mathrm{i}\, a)|^2 \langle m-k,\eta-\xi\rangle^{2s_0} đ\xi\Big) \\
&\quad \times \Big(\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}_A} \frac{1}{\langle m-k,\eta-\xi\rangle^{2s_0}} đ\xi\Big) đ\eta \\
&\lesssim_s \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}} e^{2\sigma|k|_1} |\widehat{u}(k,\xi)|^2 \langle k,\xi\rangle^{2s} \Big(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}^d} e^{2\sigma|m-k|_1} |\widehat{v}(m-k,\eta-\xi+\varsigma \mathrm{i}\, a)|^2 \langle m-k,\eta-\xi\rangle^{2s_0} đ\eta\Big) đ\xi \\
&\overset{(B.10)}{\lesssim_s} \|u\|^2_{\sigma,s,0,\mathbb{R}} \|v\|^2_{\sigma,s_0,a,\mathbb{R}}\,.
\end{aligned} \tag{B.19}$$

Note that since $s_0 \in \mathbb{N}$, $s_0 > (d+1)/2$ then $s_0 \ge \frac{d+1}{2} + \frac{1}{2}$. If $(m,\eta,k,\xi) \in B$, i.e. $\langle k,\xi\rangle < \frac{1}{2}\langle m,\eta\rangle$, then $\langle m-k,\eta-\xi\rangle > \frac{1}{2}\langle m,\eta\rangle$, and one deduces similarly that

$$I_2 \lesssim_{s,s_0} \|u\|^2_{\sigma,s_0,0,\mathbb{R}} \|v\|^2_{\sigma,s,a,\mathbb{R}}\,,$$

proving, in view of (B.18), the tame estimate (B.15). ☐

We now prove (2.15). Given $u \in \mathcal{H}^{\sigma,s,0}$ and $v \in \mathcal{H}^{\sigma,s,a}$, we consider their extensions $\mathcal{E}_s u$ and $\mathcal{E}_s v$ obtained by Lemma B.3. Since the product $\mathcal{E}_s u \mathcal{E}_s v$ is an extension of uv we have that $\|uv\|_{\sigma,s,a} \le \|\mathcal{E}_s u \mathcal{E}_s v\|_{\sigma,s,a,\mathbb{R}}$. Thus the tame estimate (B.15) and the equivalence of the norms in (B.6), imply (2.15).

The proof of (2.16) follows by induction on j. For $j = 1$ it is trivial and if it holds for j then

$$\begin{aligned}
\|u^{j+1}\|_{\sigma,s,a} &\overset{(2.15)}{\le} C_s\big(\|u\|_{\sigma,s,a}\|u^j\|_{\sigma,s_0,a} + \|u\|_{\sigma,s_0,a}\|u^j\|_{\sigma,s,a}\big) \\
&\overset{(2.16)_j}{\le} C_s\big(\|u\|_{\sigma,s,a}(2C_{s_0})^{j-1}\|u\|^j_{\sigma,s_0,a} + \|u\|_{\sigma,s_0,a}(2C_s\|u\|_{\sigma,s_0,a})^{j-1}\|u\|_{\sigma,s,a}\big) \\
&\le (2C_s)^j \|u\|^j_{\sigma,s_0,a}\|u\|_{\sigma,s,a}
\end{aligned}$$

proving (2.16) at the step $j+1$.

C Proof of the elliptic regularity Lemma 2.10

We define the weighted L^2-space of functions (cfr. (2.10))

$$L^{2,a} := \left\{ p : \mathbb{R}_{\le 0} \to \mathbb{C} \quad : \quad \|p\|_{L^{2,a}}^2 := \int_{-\infty}^0 |p(y)|^2 e^{-2ay} \mathrm{d}y < \infty \right\} . \tag{C.1}$$

For any $\lambda \ge 0$, define also the integral operators

$$(T_\lambda p)(y) := \int_{-\infty}^y e^{\lambda(z-y)} p(z) \mathrm{d}z , \quad (\widetilde{T}_\lambda p)(y) := \int_y^0 e^{\lambda(y-z)} p(z) \mathrm{d}z , \quad \forall y \le 0 . \tag{C.2}$$

The next technical lemma shows that the operators $T_\lambda, \widetilde{T}_\lambda$ extend to bounded operators on $L^{2,a}$.

Lemma C.1. *Let $s \ge 0, a > 0$. For any $\lambda \ge 0$ one has*

$$\|T_\lambda p\|_{L^{2,a}} \le \frac{1}{\lambda + a} \|p\|_{L^{2,a}} , \tag{C.3}$$

$$|(T_\lambda p)(y)| \le \frac{e^{ay}}{\sqrt{2(\lambda + a)}} \|p\|_{L^{2,a}} , \quad \forall y \le 0 , \tag{C.4}$$

$$\|\partial_y^j (T_\lambda p)\|_{L^{2,a}} \le C_a \sum_{i=0}^{j-1} \langle \lambda \rangle^{j-i-1} \|\partial_y^i p\|_{L^{2,a}} , \quad \forall j \ge 1 , \tag{C.5}$$

where $\langle \lambda \rangle = \max\{1, |\lambda|\}$. For any $\lambda > a$, one has

$$\|\widetilde{T}_\lambda p\|_{L^{2,a}} \le \frac{1}{\lambda - a} \|p\|_{L^{2,a}} , \tag{C.6}$$

$$|(\widetilde{T}_\lambda p)(y)| \le \left(\frac{e^{2ay} - e^{2\lambda y}}{2(\lambda - a)} \right)^{\frac{1}{2}} \|p\|_{L^{2,a}} , \quad \forall y \le 0 , \tag{C.7}$$

$$\|\partial_y^j (\widetilde{T}_\lambda p)\|_{L^{2,a}} \le C_a \sum_{i=0}^{j-1} \langle \lambda \rangle^{j-i-1} \|\partial_y^i p\|_{L^{2,a}} , \quad \forall j \ge 1 . \tag{C.8}$$

Proof. We consider first the operator T_λ. Recalling (C.2) we have

$$\|T_\lambda p\|_{L^{2,a}}^2 = \int_{-\infty}^0 \left| \int_{-\infty}^y e^{\lambda(z-y)} p(z) \mathrm{d}z \right|^2 e^{-2ay} \,\mathrm{d}y \le \int_{-\infty}^0 \left(\int_{-\infty}^y e^{(\lambda+a)(z-y)} |p(z)| e^{-az} \mathrm{d}z \right)^2 \mathrm{d}y .$$

Since $\int_{-\infty}^y e^{(\lambda+a)(z-y)} \mathrm{d}z = \frac{1}{\lambda+a}$, the measure $(\lambda + a) e^{(\lambda+a)(z-y)} \mathrm{d}z$ is normalized on the domain $(-\infty, y)$, and Jensen inequality and exchanging the order of integration implies

$$\begin{aligned}
\|T_\lambda p\|_{L^{2,a}}^2 &\le \frac{1}{(\lambda + a)^2} \int_{-\infty}^0 \int_{-\infty}^y (\lambda + a) e^{(\lambda+a)(z-y)} |p(z)|^2 e^{-2az} \mathrm{d}z \mathrm{d}y \\
&= \frac{1}{\lambda + a} \int_{-\infty}^0 e^{-2az} |p(z)|^2 e^{(\lambda+a)z} \left(\int_z^0 e^{-(\lambda+a)y} \mathrm{d}y \right) \mathrm{d}z \\
&= \frac{1}{(\lambda + a)^2} \int_{-\infty}^0 e^{-2az} |p(z)|^2 \left(1 - e^{(\lambda+a)z} \right) \mathrm{d}z \le \frac{1}{(\lambda + a)^2} \|p\|_{L^{2,a}}^2
\end{aligned}$$

as $1 - e^{(\lambda+a)z} \le 1$ for any $z \le 0$. This proves (C.3). Estimate (C.4) descends, recalling (C.2), (C.1), and applying Cauchy-Schwarz inequality,

$$\begin{aligned}
|(T_\lambda p)(y)| &\le e^{-\lambda y} \int_{-\infty}^y e^{\lambda z} |p(z)| \mathrm{d}z \\
&\le e^{-\lambda y} \left(\int_{-\infty}^y e^{2(\lambda+a)z} \mathrm{d}z \right)^{\frac{1}{2}} \left(\int_{-\infty}^0 |p(z)|^2 e^{-2az} \mathrm{d}z \right)^{\frac{1}{2}} = \frac{e^{ay}}{\sqrt{2(\lambda + a)}} \|p\|_{L^{2,a}} .
\end{aligned}$$

In order to prove the estimate (C.5) for $\partial_y^j(T_\lambda p)$, we first note the following inductive formula

$$\partial_y^j\,(T_\lambda p) = \sum_{i=0}^{j-1}(-\lambda)^{j-i-1}\,\partial_y^i p + (-\lambda)^j T_\lambda p\,, \quad \forall j \geq 1\,.$$

Then (C.5) follows estimating $\|T_\lambda p\|_{L^{2,a}}$ by (C.3).

Now we consider the operator $\widetilde{T}_\lambda$ in (C.2). Since $\int_y^0 e^{(\lambda-a)(y-z)}\mathrm{d}z = \frac{1-e^{(\lambda-a)y}}{\lambda-a}$, then, for any $\lambda > a$, the measure $\frac{\lambda-a}{1-e^{(\lambda-a)y}}e^{(\lambda-a)(y-z)}\mathrm{d}z$ is normalized on the domain $(y,0)$, and, by Jensen inequality and exchanging the order of integration,

$$\begin{aligned}
\|\widetilde{T}_\lambda p\|_{L^{2,a}}^2 &\leq \int_{-\infty}^0 \frac{1-e^{(\lambda-a)y}}{\lambda-a}\int_y^0 |p(z)|^2 e^{-2az}e^{(\lambda-a)(y-z)}\mathrm{d}z\mathrm{d}y \\
&= \frac{1}{\lambda-a}\int_{-\infty}^0 |p(z)|^2 e^{-2az}e^{-(\lambda-a)z}\int_{-\infty}^z \big(e^{(\lambda-a)y}-e^{2(\lambda-a)y}\big)\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{(\lambda-a)^2}\int_{-\infty}^0 |p(z)|^2 e^{-2az}\big(1-\frac{1}{2}e^{(\lambda-a)z}\big)\mathrm{d}y \leq \frac{1}{(\lambda-a)^2}\|p\|_{L^{2,a}}^2
\end{aligned}$$

as $1-\frac{1}{2}e^{(\lambda-a)z} \leq 1$ for any $z \leq 0$. This proves (C.6). The estimate (C.7) follows similarly to (C.4). Finally (C.8) descends from the identity

$$\partial_y^j\,(\widetilde{T}_\lambda p) = -\sum_{i=0}^{j-1}\lambda^{j-i-1}\,\partial_y^i p + \lambda^j \widetilde{T}_\lambda p\,, \quad \forall j \geq 1\,,$$

together with the estimate for $\widetilde{T}_\lambda p$ in (C.6). □

Proof of Lemma 2.10. Writing $u(x,y) = \sum_{k\in\mathbb{Z}^d} u_k(y)e^{\mathrm{i}\,k\cdot x}$, we expand (2.28) in Fourier in the x variables, obtaining for any $k \in \mathbb{Z}^d$ the second order system for $u_k(y)$,

$$\begin{cases} -|k|^2 u_k(y) + \partial_y^2 u_k(y) = g_k(y) \\ u_k(0) = 0, \quad \partial_y u_k(y) \to 0 \text{ as } y \to -\infty\,. \end{cases} \tag{C.9}$$

<u>Case $k = 0$:</u> The solution of (C.9) is, for $k = 0$,

$$u_0(y) = \int_{-\infty}^y \int_{-\infty}^{y'} g_0(z)\mathrm{d}z\mathrm{d}y' - \int_{-\infty}^0 \int_{-\infty}^{y'} g_0(z)\mathrm{d}z\mathrm{d}y' \overset{(C.2)}{=} \underbrace{(T_0^2 g_0)(y)}_{=\Pi u_0}\underbrace{-(T_0^2 g_0)(0)}_{=u_0-\Pi u_0}\,. \tag{C.10}$$

First note that, since $g \in \mathcal{H}^{\sigma,s,a}$, then, by (2.9),

$$\|\partial_y^j g_0\|_{L^{2,a}} \leq \|g\|_{\sigma,s,a}\,, \quad \forall j = 0,\ldots,s\,. \tag{C.11}$$

By (C.3), (C.4), the function $\Pi u_0 = T_0^2 g_0$ and the constant $u_0 - \Pi u_0 = -(T_0^2 g_0)(0)$ satisfy

$$\|\Pi u_0\|_{L^{2,a}} \leq a^{-2}\|g_0\|_{L^{2,a}} \leq a^{-2}\|g\|_{\sigma,s,a}\,,\ |u_0 - \Pi u_0| \leq \frac{1}{\sqrt{2}a^{3/2}}\|g_0\|_{L^{2,a}} \leq \frac{1}{\sqrt{2}\,a^{3/2}}\|g\|_{\sigma,s,a}\,. \tag{C.12}$$

Thus $u_0 \in \mathbb{C} \oplus L^{2,a}$. Moreover $\partial_y u_0 = T_0 g_0$ and (C.4), (C.3) imply that

$$|(\partial_y u_0)(y)| \leq \frac{e^{ay}}{\sqrt{2a}}\|g_0\|_{L^{2,a}}\,, \quad \|\partial_y u_0\|_{L^{2,a}} \leq a^{-1}\|g_0\|_{L^{2,a}} \leq a^{-1}\|g\|_{\sigma,s,a}\,. \tag{C.13}$$

In addition, since $\partial_y^2 u_0 = g_0$, we get $\partial_y^j u_0 = \partial_y^{j-2} g_0$, for any $j \geq 2$, and then, by (C.11),

$$\|\partial_y^j u_0\|_{L^{2,a}} = \|\partial_y^{j-2} g_0\|_{L^{2,a}} \leq \|g\|_{\sigma,s,a}, \quad \forall 2 \leq j \leq s+2. \tag{C.14}$$

The bounds (C.12), (C.13), (C.14), and recalling (2.9), imply that

$$|u_0 - \Pi u_0|^2 + \|\Pi u_0\|_{L^{2,a}}^2 + \sum_{j=1}^{s+2} \|\partial_y^j u_0\|_{L^{2,a}}^2 \leq C_a \|g\|_{\sigma,s,a}^2. \tag{C.15}$$

Case $k \neq 0$: The solution of the linear equation (C.9) is (by the variation of constants method)

$$\begin{aligned} u_k(y) &= -\frac{1}{2|k|} \int_{-\infty}^{y} e^{|k|(z-y)} g_k(z)\,\mathrm{d}z - \frac{1}{2|k|} \int_{y}^{0} e^{|k|(y-z)} g_k(z)\,\mathrm{d}z + \frac{e^{|k|y}}{2|k|} \int_{-\infty}^{0} g_k(z) e^{|k|z} \mathrm{d}z \\ &\overset{(C.2)}{=} -\frac{1}{2|k|} (T_{|k|} g_k)(y) - \frac{1}{2|k|} (\widetilde{T}_{|k|} g_k)(y) + \frac{e^{|k|y}}{2|k|} (T_{|k|} g_k)(0). \end{aligned} \tag{C.16}$$

By (2.9), each $g_k \in L^{2,a}$ and $\|g_k\|_{L^{2,a}} \leq \|g\|_{\sigma,s,a}$. Thus by $\|e^{|k|y}\|_{L^{2,a}} = 1/\sqrt{2(|k|-a)}$, Lemma C.1, and recalling that $a \in (0,1)$, we bound (C.16) for any $|k| \geq 1$, as

$$\|u_k\|_{L^{2,a}} \leq \frac{1}{2|k|} \|T_{|k|} g_k\|_{L^{2,a}} + \frac{1}{2|k|} \|\widetilde{T}_{|k|} g_k\|_{L^{2,a}} + \frac{|(T_{|k|} g_k)(0)|}{2|k|\sqrt{2(|k|-a)}}| \lesssim_a \frac{1}{|k|^2} \|g_k\|_{L^{2,a}}. \tag{C.17}$$

Thus each $u_k \in L^{2,a}$, $k \neq 0$. Note also that $\partial_y u_k(y) = \frac{1}{2}(T_{|k|} g_k)(y) - \frac{1}{2}(\widetilde{T}_{|k|} g_k)(y) + \frac{e^{|k|y}}{2}(T_{|k|} g_k)(0)$ satisfies, by (C.4) and (C.6),

$$\begin{aligned} |\partial_y u_k(y)| &\leq \frac{1}{2} |(T_{|k|} g_k)(y)| + \frac{1}{2} |(\widetilde{T}_{|k|} g_k)(y)| + \frac{e^{|k|y}}{2} |(T_{|k|} g_k)(0)| \\ &\leq \frac{1}{2} \frac{e^{ay}}{\sqrt{2(|k|+a)}} \|g_k\|_{L^{2,a}} + \frac{1}{2} \Big(\frac{e^{2ay} - e^{2|k|y}}{2(|k|-a)} \Big)^{\frac{1}{2}} \|g_k\|_{L^{2,a}} + \frac{e^{|k|y}}{2} \frac{1}{\sqrt{2(|k|+a)}} \|g_k\|_{L^{2,a}} \end{aligned}$$

thus tends to 0 as $y \to -\infty$.

By (C.17) and recalling (2.9) we deduce that

$$\sum_{k \neq 0} e^{2\sigma|k|_1} \langle k \rangle^{2(s+2)} \|u_k\|_{L^{2,a}}^2 \lesssim_a \|g\|_{\sigma,s,a}^2 \tag{C.18}$$

and we conclude that $u = u_0(y) + \sum_{k \neq 0} u_k(y) e^{\mathrm{i}\, k \cdot x}$ is in $\mathbb{C} \oplus L^{2,a}(\mathbb{R}_{\leq 0}, H^{\sigma,s+2})$ with

$$\|\Pi u\|_{L^{2,a}(\mathbb{R}_{\leq 0}, H^{\sigma,s+2})}^2 = \|\Pi u_0\|_{L^{2,a}}^2 + \sum_{k \neq 0} e^{2\sigma|k|_1} \langle k \rangle^{2(s+2)} \|u_k\|_{L^{2,a}}^2 \overset{(C.12),(C.18)}{\leq} C_a \|g\|_{\sigma,s,a}^2. \tag{C.19}$$

Now we estimate the derivatives $\partial_y^j u$, $j \geq 1$. Differentiating (C.16) we get, for any $j \geq 1$,

$$\partial_y^j u_k(y) = -\frac{1}{2|k|} \partial_y^j (T_{|k|} g_k)(y) - \frac{1}{2|k|} \partial_y^j (\widetilde{T}_{|k|} g_k) + \frac{1}{2} |k|^{j-1} e^{|k|y} (T_{|k|} g_k)(0)$$

and, using $\|e^{|k|y}\|_{L^{2,a}} = 1/\sqrt{2(|k|-a)}$, Lemma C.1, $a \in (0,1)$, we get, for any $|k| \geq 1$,

$$\|\partial_y^j u_k\|_{L^{2,a}} \leq \frac{1}{2|k|} \|\partial_y^j T_{|k|} g_k\|_{L^{2,a}} + \frac{1}{2|k|} \|\partial_y^j \widetilde{T}_{|k|} g_k\|_{L^{2,a}} + C_a |k|^{j-\frac{3}{2}} |(T_{|k|} g_k)(0)|$$

$$\overset{(C.5),(C.8),(C.4)}{\lesssim_a} \sum_{i=0}^{j-1} \langle k \rangle^{j-i-2} \, \|\partial_y^i g_k\|_{L^{2,a}} \, . \tag{C.20}$$

By (C.20) we conclude that, for any $1 \le j \le s+1$,

$$\begin{aligned} \sum_{k\neq 0} e^{2\sigma|k|_1} \langle k \rangle^{2(s+2-j)} \|\partial_y^j u_k\|^2_{L^{2,a}} &\lesssim_a \sum_{i=0}^{j-1} \sum_{k\neq 0} e^{2\sigma|k|_1} \, \langle k \rangle^{2(s+2-j)} \langle k \rangle^{2(j-i-2)} \|\partial_y^i g_k\|^2_{L^{2,a}} \\ &\lesssim_a \sum_{i=0}^{j-1} \sum_{k\neq 0} e^{2\sigma|k|_1} \, \langle k \rangle^{2(s-i)} \|\partial_y^i g_k\|^2_{L^{2,a}} \overset{(2.9)}{\le} C_a \|g\|^2_{\sigma,s,a} \, . \end{aligned} \tag{C.21}$$

We finally estimate the last derivative $\partial_y^{s+2} u_k$. Differentiating (C.9) with respect to ∂_y^s, we get

$$\partial_y^{s+2} u_k(y) = \partial_y^s g_k(y) + |k|^2 \partial_y^s u_k(y)$$

and then

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \|\partial_y^{s+2} u_k\|^2_{L^{2,a}} &\lesssim \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \Big(\|\partial_y^s g_k\|^2_{L^{2,a}} + |k|^4 \|\partial_y^s u_k\|^2_{L^{2,a}} \Big) \\ &\overset{(2.9),(C.20)}{\lesssim_a} \|g\|^2_{\sigma,s,a} + \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k \rangle^4 \sum_{i=0}^{s-1} \langle k \rangle^{2(s-i-2)} \, \|\partial_y^i g_k\|^2_{L^{2,a}} \\ &\lesssim_a \|g\|^2_{\sigma,s,a} + \sum_{i=0}^{s-1} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k \rangle^{2(s-i)} \, \|\partial_y^i g_k\|^2_{L^{2,a}} \overset{(2.9)}{\le} C_a \|g\|^2_{\sigma,s,a} \, . \end{aligned} \tag{C.22}$$

Recalling (2.8), summing the estimates (C.15), (C.19), (C.21) and (C.22), we deduce that $u \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+2,a}$ and $\|u\|_{\sigma,s+2,a} = |u - \Pi u| + \|\Pi u\|_{\sigma,s+2,a} \le C_{s,a} \|g\|_{\sigma,s,a}$. Lemma 2.10 is proved. ☐

References

[1] Alazard T., Burq N., Zuily C. *Cauchy theory for the water waves system in an analytic framework*, to appear in Tokyo Journal of Mathematics.

[2] Alazard T., Burq N., Zuily C. *On the water-wave equations with surface tension.* Duke Math. J., 158, 413-499, 2011.

[3] Alazard T., Burq N., Zuily C. *On the Cauchy problem for gravity water waves.* Invent. Math., 198, 71–163, 2014.

[4] Alazard T., Delort J-M. *Sobolev estimates for two dimensional gravity water waves. Astérisque*, 374, viii + 241, 2015.

[5] Alazard T., Métivier G. *Paralinearization of the Dirichlet to Neumann operator, and regularity of the three dimensional water waves.* Comm. Partial Differential Equations 34, no. 10-12, 1632-1704, 2009.

[6] Ambrosetti A., Prodi G. *A Primer of Nonlinear Analysis.* Cambridge Studies in Advanced Math. 34, 1993.

[7] Amick C. J., Fraenkel L. E. , Toland J. F. *On the Stokes conjecture for the wave of extreme form.* Acta Matematica 148, 193 - 214, 1982.

[8] Baldi P., Berti M., Haus E., Montalto R. *Time quasi-periodic gravity water waves in finite depth.* Inv. Math. 214 (2): 739–911, 2018.

[9] Berti M., Bolle P. *Quasi-Periodic Solutions of Nonlinear Wave Equations on the d-Dimensional Torus*, EMS Monographs in Mathematics, xv - 358 pp. ISBN: 978-3-03719-211-5, 2020.

[10] Berti M., Delort J.-M. *Almost Global Solutions of Capillary-gravity Water Waves Equations on the Circle.* UMI Lecture Notes 2018, ISBN 978-3-319-99486-4.

[11] Berti M., Franzoi L., Maspero A. *Traveling quasi-periodic water waves with constant vorticity*, Archive for Rational Mechanics, 240: 99–202, 2021.

[12] Berti M., Franzoi L., Maspero A. *Pure gravity traveling quasi-periodic water waves with constant vorticity*, arXiv:2101.12006, 2021.

[13] Berti M., Maspero A., Ventura P. *Full description of Benjamin-Feir instability of Stokes waves in deep water*, arXiv:2109.11852 .

[14] Berti M., Montalto R. *Quasi-periodic standing wave solutions of gravity-capillary water waves*, Volume 263, MEMO 1273, Memoires AMS, ISSN 0065-9266, 2020.

[15] Buffoni B., Toland J. *Analytic Theory of Global Bifurcation.* Princeton University Press, 2016.

[16] Coifman R., Meyer Y. *Nonlinear harmonic analysis and analytic dependence.* Proc. Symp. Pure Math., Vol. 43, pp. 71-78, 1985.

[17] Craig W., Nicholls D. *Travelling two and three dimensional capillary gravity water waves.* SIAM J. Math. Anal. 32, no. 2, 323-359, 2000.

[18] Craig W., Schanz U., Sulem C. *The modulational regime of three-dimensional water waves and the Davey-Stewartson system.* Annales de l'I.H.P. Analyse non linèaire 14(5): 615–667, 1997.

[19] Craig W., Sulem C. *Numerical simulation of gravity waves.* J. Comput. Phys., 108(1): 73–83, 1993.

[20] Fenton, J.D. *Nonlinear wave theories*, Ocean Engineering Science, The Sea, 9A, pp. 3-25, ISBN 9780674017399, 1990.

[21] Feola R., Giuliani F. *Quasi-periodic traveling waves on an infinitely deep fluid under gravity.* arXiv:2005.08280, to appear Memoires AMS.

[22] Iooss G., Plotnikov P. *Small divisor problem in the theory of three-dimensional water gravity waves*, Mem. Amer. Math. Soc., 200(940):viii+128, 2009.

[23] Iooss G., Plotnikov P. *Asymmetrical tridimensional traveling gravity waves*, Arch. Rat. Mech. Anal., 200(3):789–880, 2011.

[24] Iooss G., Plotnikov P., Toland J. *Standing waves on an infinitely deep perfect fluid under gravity.* Arch. Ration. Mech. Anal., 177(3):367–478, 2005.

[25] Lannes D. *Well-posedness of the water-waves equations*, J. Amer. Math. Soc., 3, 605–654, 18, 2005.

[26] Lannes D. *The water waves problem: mathematical analysis and asymptotics.* Mathematical Surveys and Monographs, 188, 2013.

[27] Levi-Civita T. *Détermination rigoureuse des ondes permanentes d' ampleur finie,* Math. Ann. 93 , pp. 264-314, 1925.

[28] Lewy H. *A note on harmonic functions and a hydrodynamical application,* Proc. Amer. Math. Soc., 3: 111–113, 1952.

[29] Lions J.L., Magenes E., *Probléms aux limites non homogénes at applications*, vol. 1, 1968.

[30] Martin C. *Local bifurcation and regularity for steady periodic capillary-gravity water waves with constant vorticity*, Nonlinear Anal. Real World Appl. 14, no. 1, 131-149, 2013.

[31] Nekrasov A. *On steady waves.* Izv. Ivanovo-Voznesenk. Politekhn. 3, 1921.

[32] Nguyen H., Strauss W. *Proof of modulational instability of Stokes waves in deep water.* To appear in Comm. Pure Appl. Math., 2020.

[33] Nicholls D., Reitich F. *On analyticity of travelling water waves*, Proc. R. Soc. A, 461, 1283-130, 2005.

[34] Plotnikov P. *A proof of the Stokes conjecture in the theory of surface waves*, Dinamika Splosh. Sredy, 57: 41-76, 1982.

[35] Plotnikov P.I., Toland J.F. *The Fourier Coefficients of Stokes' Waves.* Nonlinear Problems in Mathematical Physics and Related Topics I. International Mathematical Series, vol 1. Springer, Boston, MA, 2002.

[36] Stokes. G. *On the theory of oscillatory waves.* Trans. Cambridge Phil. Soc. 8: 441–455, 1847.

[37] Struik D. *Détermination rigoureuse des ondes irrotationelles périodiques dans un canal á profondeur finie.* Math. Ann. 95: 595–634, 1926.

[38] Toland J. F. *On the existence of a wave of greatest height and Stokes conjecture*, Proc. Roy. Soc. London Ser. A 363, 1715, 469-485, 1978.

[39] Wahlén E. *Steady water waves with a critical layer*, J. Diff. eq., 246, 2468-2483, 2009.

[40] Wu S. *Well-posedness in Sobolev spaces of the full water wave problem in 2D*, Inv. Math., 130 (1): 39-72, 1997.

[41] Wu S. *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc., 12 (2): 445-495, 1999.

[42] Zakharov V. *Stability of periodic waves of finite amplitude on the surface of a deep fluid.* Zhurnal Prikladnoi Mekhaniki i Teckhnicheskoi Fiziki 9(2): 86–94, 1969.

[43] Zeidler E. *Existenzbeweis für cnoidal waves unter Berücksichtigung der Oberflächen spannung*, Arch. Rational Mech. Anal, 41, pp. 81-107, 1971.

International School for Advanced Studies (SISSA), Via Bonomea 265, 34136, Trieste, Italy.
Emails: berti@sissa.it, alberto.maspero@sissa.it, paolo.ventura@sissa.it