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EXACT INTEGRABILITY CONDITIONS FOR COTANGENT VECTOR FIELDS

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Abstract. In Quantum Hydro-Dynamics the following problem is relevant: let $(\sqrt{\rho}, \Lambda) \in W^{1,2}(\mathbb{R}^d, \mathcal{L}^d, \mathbb{R}^+) \times L^2(\mathbb{R}^d, \mathcal{L}^d, \mathbb{R}^d)$ be a finite energy hydrodynamics state, i.e. $\Lambda = 0$ when $\rho = 0$ and

$$E = \int_{\mathbb{R}^d} \frac{1}{2} \big|\nabla \sqrt{\rho}\big|^2 + \frac{1}{2} \Lambda^2 \mathcal{L}^d < \infty.$$

The question is under which conditions there exists a wave function $\psi \in W^{1,2}(\mathbb{R}^d, \mathcal{L}^d, \mathbb{C})$ such that

$$\sqrt{\rho} = |\psi|, \quad J = \sqrt{\rho}\Lambda = \Im\big(\bar{\psi} \nabla \psi\big).$$

The second equation gives for $\psi = \sqrt{\rho} w$ smooth, $|w| = 1$, that $i\Lambda = \sqrt{\rho} \bar{w} \nabla w$.

Interpreting $\rho \mathcal{L}^d$ as a measure in the metric space $\mathbb{R}^d$, this question can be stated in generality as follows: given metric measure space (X, d, μ) and a cotangent vector field $v \in L^2(T^*X)$, is there a function $w \in W^{1,2}(X, \mu, \mathbb{S}^1)$ such that

$$dw = iwv.$$

Under some assumptions on the metric measure space (X, d, μ) (conditions which are verified on Riemann manifolds with the measure $\mu = \rho \mathrm{Vol}$ or more generally on non-branching $\mathrm{MCP}(K, N)$), we show that the necessary and sufficient conditions for the existence of w is that (in the case of differentiable manifold)

$$\int v(\gamma(t)) \cdot \dot{\gamma}(t) dt \in 2\pi\mathbb{Z}$$

for π-a.e. γ, where π is a test plan supported on closed curves. This condition generalizes the condition that the vorticity is quantized. We also give a representation of every possible solution.

In particular, we deduce that the wave function $\psi = \sqrt{\rho} w$ is in $W^{1,2}(X, \mu, \mathbb{C})$ whenever $\sqrt{\rho} \in W^{1,2}(X, \mu, \mathbb{R}^+)$.

Contents

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1. Introduction

This paper is devoted to the following question arising in Quantum Hydro-Dynamics (QHD): assume that $(\sqrt{\rho},\Lambda) \in W^{1,2}(\mathbb{R}^d,\mathcal{L}^d,\mathbb{R}^+) \times L^2(\mathbb{R}^d,\mathcal{L}^d,\mathbb{R}^d)$ are such that $\Lambda = 0$ when $\rho = 0$. The question is under which conditions there exists a wave function $\psi \in W^{1,2}(\mathbb{R}^d,\mathcal{L}^d,\mathbb{C})$ such that

$$\sqrt{\rho} = |\psi|, \quad J = \sqrt{\rho}\Lambda = \Im\big(\bar{\psi}\nabla\psi\big). \tag{1.1}$$

Here and in the following $\Im(\cdot)$ is the imaginary part of a complex number.
The second equation in (1.1) gives for $\psi = \sqrt{\rho}w$ smooth, $|w| = 1$, that $i\Lambda = \sqrt{\rho}\bar{w}\nabla w$. The interest in the above question arises because ρ, J are obtained as solutions to the QHD system

$$\begin{cases} \partial_t\rho + \operatorname{div} J = 0, \\ \partial_t J + \operatorname{div}\big(\Lambda\otimes\Lambda + p(\rho)\mathrm{id}\big) = \operatorname{div}\left(\dfrac{\nabla^2\rho}{4} - \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\right), \end{cases} \tag{1.2}$$

which is the system of PDE satisfied by the hydrodynamics quantities in (1.1) if $\psi \in C(\mathbb{R}, W^{1,2}(\mathbb{R}^d,\mathcal{L}^d,\mathbb{C}))$ solves the Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + f'(|\psi|^2)\psi, \tag{1.3}$$

where

$$f(\rho) = \rho\int_{\rho*}^{\rho}\frac{p(s)-p(\rho*)}{s^2}ds, \quad \rho^* \text{ reference density.}$$

Apart from the 1d case, where a weak solution to (1.2) can be constructed and then the wave function ψ is recovered (see [AMZ21]), in the general case one first obtains the solution to (1.3) and then shows that the associated hydrodynamics quantities (1.1) satisfy (1.2); for an overview of the mathematical theory for QHD we refer to [AM].

An implication that (ρ, J) are derived from a wave function ψ is that

$$\operatorname{curl} J = 2\nabla(\sqrt{\rho})\wedge\Lambda, \tag{1.4}$$

which mimics the fact that $\operatorname{curl}\nabla\theta = 0$, where $\psi = \sqrt{\rho}e^{i\theta}$: here and in the following $\wedge$ is the outer product in $\mathbb{R}^d$. Being this condition a local condition, it is not sufficient for the existence of a wave function ψ, as one can see with the elementary example in $\mathbb{R}^2$

$$\rho(x) \in C^1_c(\mathbb{R}^2), \rho(0) = 0, \qquad J(x) = \rho(x)\frac{\alpha x^\perp}{|x|^2},\ \alpha\notin\mathbb{Z}. \tag{1.5}$$

Indeed (1.4) is verified and being $v = \alpha x^\perp/|x|^2$, $\operatorname{curl} v = 2\pi\alpha\delta_0$, for $\alpha\notin\mathbb{N}$ it is impossible to solve

$$Dw = iwv,$$

where Dw is the differential of the function w.

In [AMZ] the authors present some results which extend the 1d case reconstruction of the wave function ψ to the case when $x \in \mathbb{R}^d$ and ρ, J are radially symmetric, and more interestingly when $x\in\mathbb{R}^2$, the mass ρ is continuous and the vacuum set $\{\rho = 0\}$ is given by isolated points $\{x_1, x_2, \dots\}$, and the velocity field $v = J/\rho$ is a distribution such that

$$\operatorname{curl} v = 2\pi\sum_j k_j\delta_{x_j}, \qquad k_j\in\mathbb{Z}. \tag{1.6}$$

In this case the local function $w(x)$ given by $Dw = ivw$ (which is obtained by the above curl-free condition (1.4)) can be extended to a global one because of the *integer circuitation condition* (1.6): indeed, considering a single singular point $x_1 = 0$

$$\operatorname{curl} v = 2\pi k\delta_0,$$

one deduces that for all test function ϕ

$$-\int\nabla^\perp\phi\cdot v\mathcal{L}^2 = 2\pi k\phi(0).$$

In particular (by considering radial function ψ, for example) one obtains that for $\mathcal{L}^1$-a.e. r it holds

$$\int_0^1 v(\gamma_r(t))\cdot\dot{\gamma}_r(t)dt = 2\pi k, \quad \gamma_r(t) = r(\cos(2\pi t),\sin(2\pi t)). \tag{1.7}$$

Hence, the local functions w can be extended globally, since after a rotation around x_1 the value

$$w(\gamma_r(1)) = w(\gamma_r(0))e^{i\int_0^1 v(\gamma_r(t))\cdot\dot\gamma_r(t)dt} = w(\gamma_r(0)) \tag{1.8}$$

is the same, and indeed the counterexample (1.5) does not satisfy exactly (1.6) about the point $x = 0$.

More in general, one can see that the condition (1.6) is equivalent to the following. Let $\Gamma(\mathbb{R}^2)$ be the metric space of Lipschitz curves $\gamma : [0, 1] \mapsto \mathbb{R}^2$ with the uniform topology C_0, and define a *test plan* π as a probability measure such that its evaluation $e(t)_\sharp\pi$ at time t satisfies

$$\int \phi(x)(e(t)_\sharp\pi)(dx) = \int \phi(\gamma(t))\pi(d\gamma) \leq C \int \phi(x)dx,$$

for all $\phi \geq 0$ continuous function. The formula (1.8) is then rewritten as follows: for π-a.e. γ such that $\gamma(0) = \gamma(1)$ (i.e. it is a closed path) it holds

$$\int_0^1 v(\gamma(t)) \cdot \dot\gamma(t)dt = 2\pi \sum_j k_j \mathrm{Wi}(\gamma, x_j) \in 2\pi\mathbb{Z}, \tag{1.9}$$

where $\mathrm{Wi}(\gamma, x)$ is the *winding number* of γ around $z \notin \gamma([0, 1])$. Being the set $\{\rho = 0\}$ locally finite (or more generally of $\mathcal{H}^1$-measure 0), one can prove that π-a.e. γ is not passing through $\{\rho = 0\}$, so that for π-a.e. γ the winding number $\mathrm{Wi}(\gamma, x_i)$ is defined for all x_i. The above formula is actually equivalent to the weak formulation (1.7) because for sufficiently regular functions the boundary of almost all level sets is the union of closed curves.

Formula (1.9) is meaningful in any dimension, and actually even in metric measure spaces, and gives rise to the following questions.

Problem. *Let (X, d, μ) be a metric measure space (a Polish space with $\mu \in \mathcal{P}(X)$ Borel probability measure) and let $v \in L^2(T^*X)$ be a cotangent vector field such that, if π is a test plan supported on the set of closed curves, then*

$$\int_0^1 v(\gamma(t)) \cdot \dot\gamma(t)dt \in 2\pi\mathbb{Z} \qquad \textit{for } \pi\textit{-a.e. } \gamma. \tag{1.10}$$

(1) Is there a function $w \in L^\infty(X, \mathbb{S}^1)$ such that its differential satisfies

$$Dw = iwv\,? \tag{1.11}$$

(2) Assuming moreover that m is the ambient measure in (X, d) and

$$\mu = \frac{|f|^2 m}{\|f\|_2^2} \quad \textit{with } f \in W^{1,2}(X, m, \mathbb{R}), \tag{1.12}$$

does the function fw belong to $W^{1,2}(X, m, \mathbb{C})$?

Above and in the following we will use the identification $\{z \in \mathbb{C}, |z| = 1\} = \mathbb{S}^1$, this should not generate confusion. In the case $X = \mathbb{R}^d$ (or a differentiable manifold) the objects considered in the previous statement are standard, while in Section 3 we recall their definitions for generic metric measure spaces following [GP20].

The assumption that $v \in L^2(T^*X)$ is the analog of the requirement $\Lambda = \sqrt{\rho}v \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ in the Euclidean setting. We will call the condition (1.10) the *integer circuitation condition*. It is obvious that the opposite is trivial, i.e. $v = -iDw/w$ has integer circuitation.

It turns out that in general metric spaces the answer is no, simply because the space may have too few curves and the measure μ may be too singular (Example 6.3). But in more regular spaces the answer is positive:

Theorem 1.1. *Assume that (X, d, m) has the measure contraction property* $\mathrm{MCP}(K, N)$ *(and if branching the ambient measure m is the Hausdorff measure $\mathcal{H}^N$). Then, for every $f \in W^{1,2}(X, m, \mathbb{R})$ and cotangent vector field $v \in L^2(T^*X)$ w.r.t. the measure $\mu = |f|^2 m/\|f\|_2^2$, there exists $w \in L^\infty(X, m, \mathbb{S}^1)$ such that*

$$Dw = iwv \quad \textit{and} \quad fw \in W^{1,2}(X, m, \mathbb{C}). \tag{1.13}$$

Moreover, there exists a partition of $\{|f| > 0\}$ into at most countably many sets Ω_i, $m(\Omega_i) > 0$,

$$\{|f| > 0\} = \bigcup_{i\in\mathbb{N}} \Omega_i,$$

such that for every function w' satisfying $Dw = iwv$ it holds

$$\frac{w'(x)}{w(x)} = c_i \in \mathbb{S}^1 \quad \text{constant for } m\text{-a.e. } x \in \Omega_i.$$

The second part of the statement is the same as the 1d case, where in every interval such that $\{|f| > 0\}$ the function w is determined up to a constant. Moreover, every Riemann manifold satisfies the assumptions of the theorem.

Before giving an overview of the proof, some comments are in order.

The actual result proved here is more general: the idea is to decompose the space X into regions $\{\Omega_\alpha\}_\alpha$ in which a.e. points $x, y \in \Omega_\alpha$ can be connected by test plans, and in these regions to construct the function w by integrating along curves. The technical part is to remove curves γ which are not "good" for v: either the integral along γ is not defined or the circuitation (1.13) is not integer if γ is closed. The non technical part is that such a decomposition does not need to be sufficiently regular to decompose the metric measure space (X, d, m) into metric measure spaces $(\Omega_\alpha, d, m_\alpha)$. Example 6.3 indeed uses the well known construction of the Vitali set to show that the decomposition above does not exist in general. From a measure theoretical point of view, one needs to assume that the partition $\{\Omega_\alpha\}_\alpha$ admits a *strongly consistent disintegration*, i.e. one can write

$$\mu = \int \mu_\alpha \omega(d\alpha) \quad \text{with} \quad \mu_\alpha \in \mathcal{P}(\Omega_\alpha). \tag{1.14}$$

This condition is certainly fullfilled if the partition $\{\Omega_\alpha\}_\alpha$ is countable, and this countability property is true under the assumptions of Theorem 1.1. It is not clear to us if such a decomposition into these measure-theoretic indecomposable components Ω_α has some interests on its own.

Within these setting, the function v is not locally integrable, so $\operatorname{curl} v$ is meaningless even distributionally. On the other hand, as in the example of (1.5), the study of $\operatorname{curl}\Lambda$ does not yield any condition into the existence of the function w. It is not clear if there exists some curl-like functional which allows to write the integer circuitation condition in a more PDE point of view, for example identifying the quantum vortex structure on which the rotational part of v is concentrated and has $2\pi\mathbb{Z}$-density, as in (1.6).

Another question is whether the integer circuitation assumption (1.10) is preserved by the flow of (1.2). A tautological argument is that since a solution to the QHD system is constructed by means of a wave function ψ solving Shcrödinger equation, a posteriori the integer circuitation condition must be compatible with the QHD formulation. Similarly for the study of the identity (1.4), it can be shown that the integer circuitation condition is compatible for smooth solutions of (1.2). Indeed one rewrites the system (1.2) as

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \partial_t v + v \cdot \nabla v = \nabla\Big(\dfrac{\Delta\sqrt{\rho}}{2\sqrt{\rho}} - f'(\rho)\Big), \end{cases} \tag{1.15}$$

so that both ρ and v are transported by the same vector field v. Hence any curve γ at time t is transported back to a curve γ' at time 0 (being the flow $\dot X = v(t, X)$ locally smooth), and defining

$$\gamma_t(\alpha) = X(t, \gamma(\alpha))$$

one has

$$\begin{aligned} \frac{d}{dt}\int v(t,\gamma_t(\alpha)) \cdot \dot\gamma_t(\alpha) d\alpha\bigg|_{t=0} &= \int \nabla\Big(-\frac{|v_0|^2}{2} + \frac{\Delta\sqrt{\rho_0}}{2\sqrt{\rho_0}} - f'(\rho_0)\Big) \cdot \dot\gamma(\alpha) d\alpha \\ &\quad + \int v_0(\gamma(\alpha)) \cdot \nabla v_0(\gamma(\alpha)) \dot\gamma(\alpha) d\alpha \\ &= \int \nabla\Big(\frac{\Delta\sqrt{\rho_0}}{2\sqrt{\rho_0}} - f'(\rho_0)\Big) \cdot \dot\gamma(\alpha) d\alpha = 0. \end{aligned}$$

Hence the fact that the r.h.s. of the equation for v in (1.15) is a gradient yields that the integer circuitation condition is preserved.

The last two remarks (i.e. if one can give a meaning to $\operatorname{curl} v$ and if the computation above can be performed for nonsmooth solutions) are related to the fundamental question of regularity of the QHD solution, which is echoed by the regularity of the flow of the transport equation with vector field v. As far as we know, these questions are completely open in the general case and they are deeply connected to the proof of existence of a solution by purely fluid dynamics techniques.

As a last observation, a completely similar analysis can be done for the question:

given $v \in L^1(T^*X)$, there exists a function w such that $Dw = v$, i.e. the form is exact?

The results and the proof are exactly the same, by replacing the integer circuitation condition with the *circuitation free condition*: if π is a test plan supported on the set of closed curves, then

$$\int_0^1 v(\gamma(t)) \cdot \dot{\gamma}(t) dt = 0 \qquad \text{for } \pi\text{-a.e. } \gamma. \tag{1.16}$$

1.1. **Sketch of the proof and structure of the paper.** In a nutshell, the idea of constructing w is the same as for the smooth case: given a value $w(\bar{x})$ as some point $\bar{x}$, find $w(x)$ by solving the ODE

$$\frac{d}{dt} w(\gamma(t)) = i w(\gamma(t)) v(\gamma(t)) \cdot \dot{\gamma}(t), \qquad w(\gamma(0)) = w(\bar{x}),\ \gamma(0) = \bar{x}, \gamma(1) = x. \tag{1.17}$$

The integer circuitation condition should take care of the possibility of having multiple curves connecting $\bar{x}$ to x.

The problem is that all definitions are given in terms of measures, and then one has to consider only the good path, i.e. the ones for which the circuitation condition holds. This difficulty occurs also for $X = \mathbb{R}^d$, because of the density $f \in W^{1,2}(\mathbb{R}^d, \mathcal{L}^d, \mathbb{R})$. Given a closed curve γ such that $\gamma(t) = \gamma(s)$ for some $0 < s < t < 1$, it is not even true in general that the restricted closed curve $R_{s,t}\gamma = \gamma \llcorner_{[s,t]}$ is of integer circuitation, even if γ is. The first results (Proposition 3.11 and Theorem 3.13) state that under the integer circuitation condition (1.10) every test plan is concentrated on a set of curves A_π such that every closed curve obtained by patching pieces of finitely many curves of A_π has integer circuitation. In particular, restrictions of curves satisfy (1.10). We call the curves obtained by piecing together parts of other curves *subcurve* or *subloops* if closed.

The argument in $\mathbb{R}^d$ (and in differentiable manifolds) is much easier than the general case in MMS: the case $X = \mathbb{R}^2$ is particularly straightforward (Section 4). Indeed, one can use test plans to move mass along the i-axis, $i = 1, 2$, and then construct a finite dimensional map from $\mathbb{R}^4$ into the set of curves Γ such that every point in some Cantor-like neighborhood is connected with another one by a specific curve moving first along the 1-direction and then along the 2-direction (Lemma 4.1). The integer circuitation condition and the result of the previous part imply that we can remove a set of measure 0 in $\mathbb{R}^4$ and the remaining curves have integer circuitation together with the subcurves (Corollary 4.2). Since by $W^{1,2}$-regularity of the function f one can locally assume that $f > 1/2$ on sufficiently many of these segments, the function w can be constructed on this set (which is of positive $\mathcal{L}^2$-measure, Lemma 4.3). This procedure can be repeated in every Lebesgue point of f and Df, and the different w (which are defined up to a constant) can be related with each other by a test plan (if it exists) concentrated on curves with initial points in one set and end points in the other set, making them compatible (Lemma 4.5). The same analysis can be done in $\mathbb{R}^d$, $d \geq 3$, at the expense of a more complicated selection procedure for the curves. Since we will address the problem in general spaces, we will leave this extension to the reader which wants to avoid some measure theoretic arguments used in the remaining parts.

The observation above is that locally one can construct suitable paths which have integer circuitation, where $f > 0$. In particular these sets cannot be further divided by just removing a set of measure 0 trajectories of the above plan. The idea is then to find this minimal partition (see Section 5 for the abstract theory, and Section 5.1 for the application to our case).

Given a set of curves $\{\gamma_\alpha\}_\alpha$, define the minimal (by set inclusion) equivalence relation E by requiring that $\gamma_\alpha(t) E \gamma_\alpha(s)$ for all $s, t \in [0,1]$, γ_α (Definition 5.7). If we have a test plan π, for every set of curves A_π, $\pi(A_\pi) = 1$, one has clearly different equivalence relations E^{A_π}. However the space X is separable, and we use this fact to deduce that there is a set $\bar{A}_\pi$ such that the equivalence relation $E^{\bar{A}_\pi}$ is minimal: this means that if $\mathcal{M}^{\bar{A}_\pi}$ is the family of μ-measurable sets which are saturated w.r.t. $E^{\bar{A}_\pi}$, then for all other sets A_π where π is concentrated there exists a μ-conegligible set where the σ-algebra generated by E^{A_π} is a subalgebra of $\mathcal{M}^{E^{\bar{A}_\pi}}$ (Proposition 5.8). This does not mean that every atom of $\mathcal{M}^{\bar{A}_\pi}$ coincides with an equivalence class of $E^{\bar{A}_\pi}$: the situation is completely equivalent to the Vitali set of $[0,1]$, where $E = [0,1]/\mathbb{Q}$ and then the E-saturated $\mathcal{L}^1$-measurable sets have Lebesgue measure 0 or 1 (see Example 6.3).

Notwithstanding the counterexample right above, the next step we perform is to find a test plan $\bar{\pi}$ such that the maximal measure algebra $\mathcal{M}^{\bar{\pi}}$ is the minimal among all measure algebras $\mathcal{M}^{\pi}$ (Proposition 5.9). The reason for this analysis is the following: first, being $\mathcal{M}^{\bar{\pi}}$ maximal, up to a negligible set the space X cannot be further divided into smaller atoms, i.e. the atoms of $\mathcal{M}^{\bar{\pi}}$ are the minimal components used by the test plan $\bar{\pi}$. Being $\mathcal{M}^{\bar{\pi}}$ minimal among the maximal measure algebras, we deduce that the all others test plans use curves which are contained into atoms of $\mathcal{M}^{\bar{\pi}}$ (up to a negligible set as usual, see Corollary 5.11). These components are in some sense the minimal connected-by-curves components of (X, μ), in the sense that one cannot find a further decomposition such that all test plans have trajectories concentrated on the atoms of this new refined decomposition.

Unfortunately finding the minimal-maximal σ-algebra is not sufficient: indeed the problem is that in general the components of the minimal-maximal decomposition do not coincide with the equivalence classes of the generating relation (as observed above, and as Example 6.3 shows). This condition seems essential and it is assumed in the construction of w (Assumption 6.1). In the case of sufficiently regular MMS (in particular MCP(K,N)-spaces where the ambient measure is $f\mathcal{H}^N$ with $f \in W^{1,2}(X, \mathcal{H}^N, \mathbb{R}^+)$), it is satisfied (Example 7.2). Under the assumption that the equivalence classes coincide with the atoms of the measure algebra $\mathcal{M}^{\bar{\pi}}$, the construction is now fairly simple, and follows the line for the $\mathbb{R}^2$ case: for the test plan $\bar{\pi}$ take the set $A^{\bar{\pi}}$ such that every subloop is of integer circuitation, choose a Borel section and a Borel function ω on the quotient space and for every x compute w by solving the ODE (1.17) with initial data $\omega(y)$ along a path connecting y to x, where y is the point in Borel section in the equivalence class of x (Proposition 6.4). It is important to note that the existence of such a path follows because we assume that the atoms of the measure algebra coincide with the equivalence classes, or equivalently that the disintegration is strongly consistent (1.14). The fact that $\bar{A}^{\bar{\pi}}$ is of free circuitation gives that the value $w(x)$ is independent of the particular path chosen. The same argument yields that any other solution w' is such that w'/w is constant on the equivalence class (Theorem 6.6).

The final analysis of the case when $\mu = fm$, with $f \in W^{1,2}(X, m, \mathbb{R}^+)$ is at this point a corollary, once it is assumed that the space (X, d, m) has a test plan connecting all points at lest locally (a *Democratic test plan*, as defined in [LV07], see Assumption 7.1): indeed in this case the equivalence classes have positive measure, and then the disintegration is strongly consistent. Metric measure spaces enough regular satisfy this assumption (Example 7.2). For the existence of w solving (1.11), the only thing to verify is that there exists a test plan $\bar{\pi}$ satisfying the assumptions of the previous sections, and this is true in these spaces (Lemma 7.3). Finally, to check that $fw \in W^{1,2}(X, m, \mathbb{C})$ it is enough to prove that that it holds along test plans for m, and this is fairly easy once one have a Borel way of passing from test plans for m to test plans for μ (Lemma 7.6 and Proposition 7.7).

2. Notation

The real numbers are denoted by $\mathbb{R}$, the rational numbers by $\mathbb{Q}$, the natural numbers by $\mathbb{N}$ and the integers by $\mathbb{Z}$. The unitary circle in $\mathbb{R}^2$ is denoted by $\mathbb{S}^1$. The cardinality of a set A is denoted by $\sharp A$.

A generic constant (which may vary from line to line) is denoted with const. or C, sometimes with an index or accent.

Let X be a separable complete metric space with distance d. A point in X will be denoted by x. The balls of radius r centered at x is denoted as $B_r^X(x)$, and more generally the r neighborhood of a set A as $B_r^X(A)$. The family of compact subsets of a metric space X is denoted by $\mathcal{K}(X)$, and we will use the Hausdorff distance $d_H(K_1, K_2)$. Countable intersection of open sets are G_δ-sets, while countable union of closed sets are F_σ-sets. The projection onto X is written as $\mathtt{p}_X$, or sometimes $\mathtt{p}_i$ if it is the i-component of $\prod_j X_j$. The Borel σ-algebra is $\mathcal{B}(X)$, the projection of closed sets are the analytic sets Σ_1^1, and the σ-algebra generated by them is $\sigma(\Sigma_1^1)$.

In case there is no ambiguity, the geodesic connecting x to y is $\overrightarrow{[x,y]}$: this notation will be used mainly in $\mathbb{R}^d$.

A generic function from X to Y will be denoted by $f : X \to Y$, and we will call f a Borel function when it is a Borel function whose domain of definition $\mathrm{Dom}(f)$ is a Borel set. The graph of a function $f : X \to Y$ is denoted by

$$\mathrm{Graph}(f) = \big\{(x, f(x)), x \in X\big\} \subset X \times Y,$$

and its range by

$$f(X) = \{f(x), x \in X\}.$$

For a (multi-valued) function $f : X \to Y$ we will denote $f(x)$ both as the image of $f(x)$ or as the section $\mathrm{Graph}\, f \cap \{x\} \times Y$. In general this will not generate confusion.

The space of Lipschitz functions $\mathrm{Lip}([0,1]; X)$ from $[0,1]$ into X is denoted by Γ: we will refer to it as the set of Lipschitz curves on X. Its subset of closed Lipschitz curves (or closed loops or loops for shortness) will be written as Γ_c. An element of both sets will be written as γ. Explicitly

$$\Gamma = \mathrm{Lip}([0,1]; X), \quad \Gamma_c = \Big\{\gamma \in \mathrm{Lip}([0,1]; X), \gamma(0) = \gamma(1)\Big\}.$$

It is known that Γ is a separable metric space with the C^0-topology [Sri98, Theorem 2.4.3].

The evaluation map $e(t) = e(t,\gamma)$ is the map

$$[0,1]\times\Gamma \ni \gamma \mapsto e(t,\gamma) = \gamma(t) \in X.$$

The restriction operator $R_{s,t}(\gamma)$ is defined as the restriction of the curve γ to the interval $[s,t] \subset [0,1]$,

$$R_{s,t}(\gamma)(\tau) = \gamma\big((1-\tau)s + \tau t\big), \quad 0 \leq s \leq t \leq 1.$$

The time inversion of a curve is denoted by $R(\gamma)(t) = \gamma(1-t)$.

The metric derivative of γ is denoted by $|\dot{\gamma}|$. The length of a curve $\gamma \in \Gamma$ is defined as

$$L(\gamma) = \int_0^1 |\dot{\gamma}|(t)dt.$$

Given a set $A \subset \Gamma$, we say that $\gamma \in \Gamma$ is a sub-curve of A if there are finitely many curves $\gamma_i \in A$ such that

$$\mathrm{Graph}\,\gamma \subset \bigcup_i \mathrm{Graph}\,\gamma_i.$$

The set of all sub-curves of A is denoted by Γ_A.

The Lebesgue measure in $\mathbb{R}^d$ is denoted with $\mathcal{L}^d$ and the d-dimensional Hausdorff measure by $\mathcal{H}^d$. Let $\mathcal{P}(X)$ be the space of probability measures over X and m,μ elements of $\mathcal{P}(X)$: in the second part of our analysis $\mu = fm$ (and m will be used as ambient probability for the space $W^{1,2}(X)$), but in the first part we will consider (X, d, μ) as a given metric measure space. The topology on $\mathcal{P}(X)$ is the narrow topology. We denote the standard integral spaces by $L^p(\mu)$. The push forward of a measure μ by a map f is written by $f_\sharp\mu$. The disintegration of a probability according to a map $f : X \to Y$ will be denoted as

$$\mu = \int \mu_y \varpi, \quad \varpi = f_\sharp\mu.$$

We will also write $\varpi = \mu \llcorner \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra made of μ-measurable sets saturated w.r.t. the equivalence relation $Ex = f^{-1}(f(x))$.

If E is an equivalence relation, we will write xEx' if x, x' are equivalent, and the equivalence class for x as Ex or $E(x)$. We will not distinguish between E and its graph in the product space. We will often use the letters $\alpha, \beta, \ldots$ for the elements of the quotient space X/E. The quotient map is considered either as a section of X (i.e. a set S containing a single point for every equivalence class) or as a map $\mathsf{e} : X \to [0,1]$ whose level sets are the equivalence classes.

3. Settings and basic computations

In this section we introduce the setting for our problem: it is based on the analysis on metric measure space of [Gig, GP20], with elementary slight variations.

3.1. Test plans.

Definition 3.1. A measure $\pi \in \mathcal{P}(\Gamma)$ is a *plan with barycenter in* $L^\infty(\mu)$ [AMS] (or *admissible plan* when the reference measure μ is clear from the context) if for some constant C it holds

$$\int_0^1 e(t)_\sharp(|\dot{\gamma}|\pi)dt \leq C\mu. \tag{3.1}$$

The set of μ-admissible plans is denoted by $\mathrm{Adm}(\mu)$, and C is called the *compressibility constant* of π.

The above formula (3.1) is equivalent to

$$\int \bigg[\int_0^1 \mathbb{1}_{\gamma(t)\in A}(t,\gamma)|\dot{\gamma}|(t)dt \bigg] \pi(d\gamma) \leq C\mu(A).$$

It is clear that the above definition is invariant if we reparameterize the curves γ. In the following we will often consider the parametrization of γ by length, so that $|\dot{\gamma}| = 1$: this is to avoid to write the factor $|\dot{\gamma}|$ in many equations. In that case the interval of integration is $[0, L(\gamma)]$, and we will often not specify the interval of integration if clear from the context.

Remark 3.2. For the analysis in this work we do not require the condition

$$e(t)_\sharp \pi \leq C\mu, \tag{3.2}$$

see for example [ACM]. These assumptions will be analyzed in Proposition 3.10: here we can just say that since the integer circuitation condition (1.10) is required for π-a.e. γ and is independent on the parametrization of γ, it is natural to consider admissible plans as in Definition 3.1.

Remark 3.3. An admissible admissible plan π is given by the constant curves: define the map

$$X \ni x \mapsto \gamma_x \in \Gamma_c, \qquad \gamma_x(t) = x \quad \forall t \in [0,1],$$

and set

$$\pi = (\gamma_x)_\sharp (f(x)\mu(dx)) \tag{3.3}$$

for every $f \in L^\infty(\mu)$, $\|f\|_1 = 1$. It is immediate from the definition to see that

$$\pi(\Gamma) = 1, \quad e(t)_\sharp \pi = f\mu,$$

so that it is admissible. In particular the measure defined in (3.3) when $f \equiv 1$ is denoted with $\hat{\pi}$.

It may happens that these are the only admissible plans: for example, if (X, d) is totally disconnected (or zero dimensional [Sri98, Section 2.2]), the only Lipschitz curves $\gamma : [0,1] \to X$ are the constant ones.

It is immediate to see that if $\pi \in \mathsf{Adm}(\mu)$ and $\gamma \mapsto (s(\gamma), t(\gamma))$ is a Borel function, then $(R_{s,t})_\sharp \pi, R_\sharp \pi \in \mathsf{Adm}(\mu)$ because

$$e(\tau)_\sharp \big((R_{s,t})_\sharp \pi \big) = \gamma((1-\tau)s(\gamma) + \tau t(\gamma))_\sharp \pi, \quad e(\tau)_\sharp \big(R_\sharp \pi \big) = e(1-\tau)_\sharp \pi.$$

3.2. **Tangent and cotangent modules.** Here we recall the definition of tangent and cotangent modules in a metric measure space X, following the approach of [Gig, GP20], see also [ACM].

Let $S^1(X)$ be the set of functions $f : X \to \mathbb{R}$ with the following property: there exists $g \in L^1(X)$ such that for all admissible plan π

$$f \circ \gamma \text{ is a.c. } \pi\text{-a.e. } \gamma \quad \wedge \quad e_\sharp \bigg(\bigg| \frac{d}{dt} f(\gamma(t)) \bigg| \mathcal{L}^1 \llcorner_{[0,1]} \times \pi \bigg) \leq C g \mu,$$

where C is the compressibility constant of π (see (3.1)) and $e : [0,1] \times \Gamma \to X$ is the evaluation map. The essential infimum of all g's is called the *minimal upper gradient* $|Df|$. In what follows, we will consider also complex valued functions: we will say that $f = \Re f + i \Im f$ belongs to $S^1(X)$ (or other regularity classes) if both $\Re f, \Im f \in S^1(X)$.

Remark 3.4. The standard definition of the pre-Sobolev class $S^1(X)$ requires that the test plan satisfies also

$$e(t)_\sharp \tilde{\pi} \leq C\mu.$$

Following the proof of Proposition 3.10, one can see that the definitions are equivalent.

Let v be an element of the cotangent module $L^1(T^*X)$, which can be defined as the $L^1(\mu)$-normed $L^\infty(\mu)$-module closure of the set

$$\bigcup_{n \in \mathbb{N}} \Big\{ \{(\Omega_i, f_i)\}_{i=1}^n, \{\Omega_i\}_{i=1}^n \text{ partition of } X, \{f_i\}_{i=1}^n \subset S^1(X) \Big\} / \sim$$

w.r.t. the norm

$$\big\| \{(\Omega_i, f_i)\}_{i=1}^n \big\| = \sum_i \int_{\Omega_i} |Df_i| \mu,$$

where the equivalence relation $\sim$ is w.r.t. the above norm, and $|Df|$ is the minimal upper gradient (see [GP20, Theorem 4.1.1]). We can associate to a function $f \in S^1(X)$ the cotangent vector field $Df = (X, f)$, so that

$$\{(\Omega_i, f_i)\}_{i=1}^n = \sum_{i=1}^n \mathbb{1}_{\Omega_i} Df_i. \tag{3.4}$$

The tangent module $L^\infty(TX)$ is defined as the dual $L^\infty(\mu)$-normed $L^\infty(\mu)$-module, i.e. the space of linear functional $\mathscr{L} : L^1(T^*X) \to L^1(\mu)$ such that for $f \in L^\infty(\mu)$, $v \in L^1(T^*X)$

$$\mathscr{L}(fv) = f\mathscr{L}(v), \quad \|\mathscr{L}\| = \sup_{\|v\|\leq 1} \int |\mathscr{L}(v)|\mu < \infty.$$

The above norm definition gives that the pointwise norm is

$$|\mathscr{L}| = \text{ess-sup}\big\{\mathscr{L}(v), \|v\| \leq 1\big\} \in L^\infty(\mu).$$

If π is an admissible plan, then for every simple element $v = \sum_i \mathbb{1}_{\Omega_i} Df_i$ we can define

$$\Lambda_\gamma(t, v) = \sum_i \mathbb{1}_{\Omega_i}(\gamma(t)) \frac{d}{dt} f_i(\gamma)(t), \tag{3.5}$$

which satisfies by (3.1)

$$\int \bigg[\int_0^{L(\gamma)} |\Lambda_\gamma(t, v)| dt \bigg] \pi(d\gamma) \leq \sum_i C \int_{\Omega_i} |Df_i|\mu = C\|v\|_{L^1(T^*X)}.$$

Let now $v \in L^1(T^*X)$: being the elements of the form (3.4) dense, we can consider a sequence of simple elements $w_n = \sum_i \mathbb{1}_{\Omega_{i,n}} Df_{i,n}$ such that

$$\|w_n\| = \sum_i \int_{\Omega_{i,n}} |Df_{i,n}|\mu < 2^{-n}, \quad v = \sum_n w_n.$$

For every n, the function $\Lambda_\gamma(t, w_n)$ is defined by (3.5) on a set of curves $G_n \subset \Gamma$: by redefining $\Lambda_\gamma(t, w_n)$ on a negligible set of times, we can assume that $t, \gamma \mapsto \Lambda_\gamma(t, v)$ is Borel.
Let

$$G = \bigg\{ \gamma \in \Gamma : \sum_n \int_0^{L(\gamma)} |\Lambda_\gamma(t, w_n)| dt < \infty \bigg\},$$

and define

$$\Lambda_\gamma(t, v) = \begin{cases} \sum_n \Lambda_\gamma(t, w_n) & \gamma \in G, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Proposition 3.5. *The function $\Lambda_\gamma(t, v)$ is Borel and for every admissible plan π it holds*

$$\|\Lambda_\cdot(\cdot, v)\|_{L^1(\mathcal{L}^1 \otimes \pi)} \leq C\|v\|_{L^1(T^*(X))}, \quad C \text{ compressibility constant.} \tag{3.7}$$

Moreover, if $v_\ell \to v \in L^1(T^*X)$, v_ℓ simple elements, and π is an admissible plan, then

$$\lim_{\ell \to \infty} \int \bigg[\int_0^{L(\gamma)} \big|\Lambda_\gamma(t, v) - \Lambda_\gamma(t, v_\ell)\big| dt \bigg] \pi(d\gamma) = 0.$$

Finally, if $v = \alpha_1 v_1 + \alpha_2 v_2$, then for every admissible plan

$$\int \bigg[\int_0^{L(\gamma)} \big|\Lambda_\gamma(t, v) - \alpha_1 \Lambda_\gamma(t, v_1) - \alpha_2 \Lambda_\gamma(t, v_2)\big| dt \bigg] \pi(d\gamma) = 0.$$

Proof. The conditions defining $\Lambda_\gamma(t, v)$ and G are Borel, so that $\Lambda_\gamma(t, v)$ is Borel. The second part follows from the definition of norm in the cotangent module. Finally, the linearity as in the third part of the statement is easily verified for simple functions, and then one can pass to the limit. $\square$

We will need the following property of $\Lambda_\gamma(t, v)$: we write the proof for completeness.

Lemma 3.6. *There exists a $\mathcal{L}^1 \times \pi$-negligible set N such that*
$$\gamma(t) = \gamma(s) \quad \Rightarrow \quad \Lambda_\gamma(t,v) = \Lambda_\gamma(s,v)$$
for all $s,t \in [0,L(\gamma)] \setminus N(\gamma)$.

In particular, the function Λ_γ can be defined as an L^1-function of $\mathcal{H}^1 \llcorner_{\gamma(0,L(\gamma))}$.

Proof. By (3.7) we have that if $v_n \to v$ in $L^1(T^*X)$ then $\Lambda_\gamma(t,v_n) \to \Lambda_\gamma(t,v)$ in $L^1(\mathcal{L}^1 \times \pi)$, so that up to subsequences it converges $\mathcal{L}^1 \times \pi$-a.e.. In particular, let $v_n \to v$ be a sequence of cotangent vector fields in $L^1(T^*X)$ such that the statement is true for each v_n, and let $N_n \subset [0,1] \times \Gamma$ be the corresponding $\mathcal{L}^1 \times \pi$-negligible set such that
$$\forall s,t \notin N_n(\gamma) \ \Big(\gamma(t) = \gamma(s) \ \Rightarrow \ \Lambda_\gamma(s,v_n) = \Lambda_\gamma(t,v_n) \Big).$$
Clearly the set $N = \cup_n N_n$ is negligible. Since $\Lambda_\gamma(t,v_n) \to \Lambda_\gamma(t,v)$ up to a subsequence in a $\mathcal{L}^1 \times \pi$-conegligble set $A \subset \mathbb{R} \times \Gamma$, we obtain that for all $(t,\gamma),(s,\gamma) \in A \setminus N$ it holds
$$\gamma(t) = \gamma(s) \ \Rightarrow \ \Lambda_\gamma(s,v) = \Lambda_\gamma(t,v).$$
It is thus sufficient to prove the statement is true for v of the form (3.4), and in particular for a single f.

Define the set of transversal self-intersections
$$\begin{aligned} N' &= \bigcup_n N'_n \\ &= \bigcup_{n\in\mathbb{N}} \bigg\{ (s,\gamma) : \exists t \in [0,L(\gamma)] \ \Big(\gamma(s+2^{-n}(-1,1)) \cap \gamma(t+2^{-n}(-1,1)) = \{\gamma(s)\} = \{\gamma(t)\} \Big) \bigg\}. \end{aligned}$$
It is fairly easy to see that for every γ $N'_n(\gamma)$ is finite for every γ: indeed if $s_n,t_n \in [0,L(\gamma)]$ are a sequence of points in the definition of $N'_n(\gamma)$, then it is clear that $|s_n - s_m| \vee |t_n - t_m| \geq 2^{-n}$. Hence $N'(\gamma)$ is countable and N' is $\mathcal{L}^1 \times \pi$-negligible.

For a given $f \in S^1(X)$, consider the conegligible set
$$M = \Big\{ (t,\gamma) \notin N' : |\dot{\gamma}|(t) \text{ exists and } f \circ \gamma \text{ differentiable in } t \Big\}.$$
Note that if $\gamma(s) = \gamma(t)$, $s,t \in M(\gamma)$, then there are sequences $s_n \to s, t_n \to t$, $s_n,t_n \neq s,t$ respectively such that $\gamma(s_n) = \gamma(t_n)$: hence by the formula (3.5) we deduce that
$$\frac{d}{dt} f \circ \gamma(t) = \frac{d}{dt} f \circ \gamma(s).$$
The complement $N = M^c$ satisfies the statement. □

In particular we deduce that

Corollary 3.7. *For π-a.e. γ the function Λ_γ depends only on $\gamma(t)$ for $\mathcal{L}^1$-a.e. t, i.e. can be written as a function on $\gamma([0,1])$ defined $\mathcal{H}^1$-a.e. or equivalently it is a measurable function for the pull-back σ-algebra $e^{-1}(\gamma)(\mathcal{B}(X))$.*

3.3. The integer circuitation condition.

Let $v \in L^1(T^*X)$ and $\Lambda_\gamma(t) = \Lambda(t,v) \in L^1(\mathcal{L}^1 \times \pi)$ be the corresponding L^1-function given by (3.5): being v fixed from now on, for shortness we will not write the dependence w.r.t. v. We restate the assumption of integer circuitation (1.10) here below.

Assumption 3.8. For all admissible plans π concentrated on Γ_c it holds
$$\int_0^1 \Lambda_\gamma(t)|\dot{\gamma}|(t)dt \in 2\pi\mathbb{Z} \quad \pi\text{-a.e. } \gamma, \tag{3.8}$$
i.e. v (or Λ) has *integer circuitation*.

We define
$$\mathcal{I}_c = \bigg\{ \gamma \in \Gamma_c, \int_0^1 \Lambda_\gamma(t)|\dot{\gamma}|(t)dt \in 2\pi\mathbb{Z} \bigg\},$$
so that we can rephrase the above assumption as $\mathcal{I}_c$ has full π-measure. Note that as far as we know $\mathcal{I}_c$ depends on π, but this should not create confusion being π fixed in the construction of the function w satisfying (1.11) (see Section 6).

An elementary example of an integer circuitation cotangent vector field is given by

$$v = -i \frac{Df}{f},$$

where $f \in L^0(\mu, \mathbb{S}^1)$ with $|Df| \in L^1$: in this case

$$\Lambda_\gamma(t) = -\frac{i}{f} \frac{d}{dt} f \circ \gamma(t) \quad (|\dot{\gamma}| = 1),$$

with π-a.e. γ, with π admissible plan.

Remark 3.9. Another possible condition is to ask that

$$\int_0^1 \Lambda_\gamma(t) |\dot{\gamma}|(t) dt = 0 \quad \pi\text{-a.e. } \gamma.$$

In this case we are looking for functions whose differential along γ is Λ_γ, i.e. $v = Df$ is exact: we will say that every admissible π has 0 *circuitation* or it is *circuitation free*. The analysis of this situation is completely similar to the case we are considering in the following, so the same constructions apply.

For completeness we show that we can restrict the set of admissible plans to a more tame one.

Proposition 3.10. *Assume that Assumption 3.8 holds only for admissible plans which satisfy*

$$L(\gamma) \in L^\infty(\pi), \quad e(t)_\sharp \pi \leq C\mu,$$

Then Assumption 3.8 holds for all admissible plans.

Proof. Let $\pi \in \mathsf{Adm}(\mu)$ be an admissible plan supported on Γ_c as in Definition 3.1.

Since Assumption 3.8 is invariant for countable unions of admissible plans, we can assume that π is concentrated on curves with length $L(\gamma) \in (\bar{L}, \bar{L} + 1)$, $\bar{L}$ fixed, and we thus consider the pameterization

$$L(\gamma \llcorner_{[0,t]}) = L(\gamma) t.$$

In this case Condition (3.1) implies

$$\int_0^1 e(t)_\sharp \pi \leq \frac{C}{\bar{L}} \mu = C' \mu.$$

Hence for all $\delta t > 0$ the measure π' defined by

$$\pi' = \fint_0^\delta \big[(\gamma^\epsilon)_\sharp \pi \big] d\epsilon, \quad \gamma^\epsilon(t) = \gamma(\epsilon + t \mod L(\gamma)),$$

is supported on Γ_c and satisfies $\bar{L} \leq L(\gamma) \leq \bar{L} + 1$ and

$$e(t)_\sharp \pi' = \fint_t^{\delta + t} \big[e(s)_\sharp \pi \big] ds \leq \frac{C'}{\delta} \mu,$$

where for simplicity we assume that $0 \leq t \leq \bar{L} - \delta$. The assumptions in the proposition thus imply that Assumption 3.8 holds for π'-a.e. γ, and by Corollary 3.7

$$\Lambda_{\gamma^\epsilon}(t) = \Lambda_\gamma(t + \epsilon \mod L(\gamma)),$$

so that we conclude that π satisfies Assumption 3.8. □

3.4. **Intersection properties of curves.** By definition, any admissible plan π supported on Γ_c is concentrated on the set of curves such that

$$\int_0^{L(\gamma)} \Lambda_\gamma(t) dt \in 2\pi\mathbb{Z}.$$

However a priori nothing is said about properties of intersecting curve: for example, given two curves γ, γ' such that

$$\gamma(s) = \gamma'(s'), \ \gamma(t) = \gamma'(t'), \quad s \leq t, s' \leq t', \tag{3.9}$$

one can construct a closed curve γ'' by setting

$$\gamma''(\tau) = \begin{cases} R_{s,t}(\gamma)(2\tau) & \tau \in [0, 1/2], \\ R_{1-t', 1-s'}(R(\gamma'))(2\tau - 1) & \tau \in (1/2, 1], \end{cases} \tag{3.10}$$

i.e. moving from $\gamma(s)$ to $\gamma(t) = \gamma'(t')$ along $\gamma(\tau)$ and then returning to $\gamma'(s') = \gamma(s)$ along $\gamma(-\tau)$. We will use the above construction (i.e. a new closed curve obtained by piecing together parts of other curves) to show that we can also require

$$\int_0^1 \Lambda(\gamma'')(\tau)|\dot{\gamma''}|(\tau)d\tau \in 2\pi\mathbb{Z}$$

by removing a π-negligible set of trajectories, where γ'' is the curve constructed above.

We first start with self-intersecting curves.

Proposition 3.11. *Let Λ be of integer circuitation. Every $\pi \in \mathsf{Adm}(\mu)$ supported on Γ_c is concentrated on a set of closed curves A_π such that if $\gamma(s) = \gamma(t)$ then $R_{s,t}(\gamma)$ has integer circuitation.*

Proof. The proof is given in 3 steps.

Step 1: construction of a countable Borel set of loops. The following argument is the same as in [Sri98, Proposition 5.2.7]: we give it for completeness. The set

$$F = \big\{(\gamma, s, t) : \gamma(s) = \gamma(t)\big\} \subset \Gamma_c \times [0,1]^2$$

is closed, and moreover

$$\Big\{\gamma : \gamma(s) = \gamma(t) \text{ for some } (s,t) \in \bar{B}^2_r((\bar{s},\bar{t}))\Big\}$$

is closed in Γ. Hence $F : \Gamma_c \to [0,1]^2$ is a Borel measurable multifunction with $F(\gamma)$ closed. For every $\bar{B}^2_{2^{-k}}((\bar{s}_m,\bar{t}_m))$, $s_m, t_m \in \mathbb{Q}$ and $k \in \mathbb{N}$, there is a Borel selection [Sri98, Theorem 5.2.1]

$$L_{k,m} : A_{k,m} = F^{-1}(\bar{B}^2_{2^{-k}}((\bar{s}_m,\bar{t}_m))) \to [0,1]^2, \quad L_{k,m}(\gamma) = (s_{k,m}, t_{k,m}) \in F(\gamma) \cap \bar{B}^2_{2^{-k}}((\bar{s}_m,\bar{t}_m)),$$

By construction, the graphs of the functions $L_{k,m}$ are dense in $F(\gamma)$ for each γ, i.e.

$$F(\gamma) = \operatorname{clos}\big\{L_{k,m}(\gamma), \gamma \in A_{k,m}\big\}.$$

By reordering, we will denote the functions $L_{m,k}$ and domains $A_{m,k}$ with L_n, A_n respectively.

Step 2: negligibility of the set of bad loops. If $\pi \in \mathcal{P}(\Gamma_c)$ is an admissible plan, for each map $L_n(\gamma) = (s_n, t_n)$ define the transport plan

$$\pi_n = (R_{s_n,t_n})_\sharp(\pi \llcorner_{A_n}), \quad A_n = \mathrm{Dom}(L_n).$$

Being a restriction, clearly $\pi_n \in \mathsf{Adm}(\mu)$. In particular, it follows by Assumption 3.8 that

$$\pi_n\big(\{\gamma \notin \mathcal{I}_c\}\big) = \pi\big(\{\gamma \in A_n : R_{s_n,t_n}(\gamma) \notin \mathcal{I}_c\}\big) = 0.$$

By σ-additivity

$$\pi\Big(\big\{\gamma \in A_n : \exists n\big(R_{s_n,t_n}(\gamma) \notin \mathcal{I}_c\big)\big\}\Big) = 0.$$

Step 3: conclusion. The previous point gives that $\pi \in \mathsf{Adm}(\mu)$ is concentrated on a set of curves A_π such that for all $n \in \mathbb{N}$ it holds $R_{s_n,t_n}(\gamma) \in \mathcal{I}_c$. Let now $(s_n,t_n), (\bar{s},\bar{t}) \in F(\gamma)$, with $(s_n,t_n) \to (\bar{s},\bar{t})$. Then

$$2\pi\mathbb{Z} \ni \int_{s_n}^{t_n} \Lambda_\gamma(\tau)|\dot{\gamma}|(\tau)d\tau \to \int_{\bar{s}}^{\bar{t}} \Lambda_\gamma(\tau)|\dot{\gamma}|(\tau)d\tau,$$

which gives that every subcurve of $\gamma \in A_\pi$ belongs to $\mathcal{I}_c$, i.e. it is of integer circuitation. □

As a consequence of the previous proposition, the next lemma shows that we can construct the function w along π-a.e. curve.

Lemma 3.12. *Let γ be a Lipschitz curve such that if $\gamma(s) = \gamma(t)$ then $R_{s,t}\gamma \in \mathcal{I}_c$. Then there exists a function $w_\gamma : \gamma([0,1]) \to \mathbb{S}^1$ such that*

$$\frac{d}{dt} w_\gamma(\gamma(t)) = i\Lambda_\gamma(t)|\dot{\gamma}|(t) w_\gamma(\gamma(t)).$$

Proof. Define the function

$$w_\gamma(t) = e^{i\int_0^t \Lambda_\gamma |\dot{\gamma}| \mathcal{L}^1}.$$

If $\gamma(t) = \gamma(s)$, then by the assumption $R_{s,t}\gamma \in \mathcal{I}_c$ it holds

$$w_\gamma(t) = w_\gamma(s) e^{i\int_s^t \Lambda_\gamma |\dot{\gamma}| \mathcal{L}^1} = w_\gamma(s),$$

so that the function w_γ can be actually defined on $\mathrm{Graph}(\gamma)$ by

$$w_\gamma(x) = w_\gamma(t), \quad x = \gamma(t).$$

□

We next generalize Proposition 3.11 to closed subcurves of the set of curves where π is concentrated: given a set $A \subset \Gamma_c$ define

$$A(n) = \left\{ \gamma \in \Gamma_c : \exists \gamma_1, \dots, \gamma_n \in A \left(\mathrm{Graph}\,\gamma \subset \bigcup_i \mathrm{Graph}\,\gamma_i \right) \right\}.$$

Theorem 3.13. *Let Λ be of integer circuitation. Every $\pi \in \mathsf{Adm}(\mu)$ is concentrated on a set of curves A_π such that $A_\pi(n) \subset \mathcal{I}_c$ for all n, i.e. every subloop of A_π belongs to $\mathcal{I}_c$.*

Proof. The proof is given in two steps. The first one illustrates the procedure in a simple case, while the second is the general case.

Step 1: the case of two curves. Consider the set $(\Gamma_c)^2 = \Gamma_c \times \Gamma_c$ and its closed subset

$$\mathcal{C}(2) = \left\{ (\gamma_1, \gamma_2) \in (\Gamma_c)^2 : \gamma_1([0,1]) \cap \gamma_2([0,1]) \neq \emptyset \right\}.$$

In particular the multifunction $F(\gamma_1, \gamma_2)$ with graph

$$F = \left\{ (\gamma_1, \gamma_2, t_1, t_2) \in \mathcal{C}(2) \times [0,1]^2 : \gamma_1(t_1) = \gamma_2(t_2) \right\}$$

is Borel measurable and $F(\gamma_1, \gamma_2)$ is closed, so that by [Sri98, Theorem 5.2.1] there is a Borel selection

$$\mathfrak{L} : \mathcal{C}(2) \to [0,1]^2, \quad \mathfrak{L}(\gamma_1, \gamma_2) = (t_1, t_2) \text{ such that } \gamma_1(t_1) = \gamma_2(t_2).$$

We thus can define the curve

$$\gamma'(\gamma_1, \gamma_2; t) = \begin{cases} \gamma_1(2t) & t \in [0, t_1/2] \\ \gamma_2(t_2 + 2t - t_1 \mod 1) & t \in (t_1/2, t_1/2 + 1/2), \\ \gamma_1(2t-1) & t \in [t_1/2 + 1/2, 1]. \end{cases} \tag{3.11}$$

Clearly the map $(\gamma_1, \gamma_2) \to \gamma'(\gamma_1, \gamma_2)$ is Borel. Up to reparametrization, the curve $\gamma'(\gamma_1, \gamma_2)$ is the closed curve obtained by moving along the first curve and then along the second one.

Consider now the set of positive measures

$$\Pi^{\leq}(\pi, \pi) = \left\{ \xi \in \mathcal{M}^+(\mathcal{C}(2)) : (\mathtt{p}_1)_\sharp \xi \leq \pi, (\mathtt{p}_2)_\sharp \xi \leq \pi \right\},$$

where $\pi \in \mathsf{Adm}(\mu)$ is the admissible plan in the statement. We can define the new admissible plan

$$\pi' = (\gamma')_\sharp \xi, \quad \xi \in \Pi^{\leq}(\pi, \pi).$$

It is fairly easy to see that $\pi' \in \mathsf{Adm}(\mu)$ because

$$\begin{aligned} \int \left[\int_0^1 \phi(\gamma(t)) |\dot{\gamma}|(t) dt \right] \pi'(d\gamma) &= \sum_{i=1,2} \int \left[\int_0^1 \phi(\gamma(t)) |\dot{\gamma}|(t) dt \right] (\mathtt{p}_i)_\sharp \xi(d\gamma) \\ &\leq 2 \int \left[\int_0^1 \phi(\gamma(t)) |\dot{\gamma}|(t) dt \right] \pi(d\gamma), \end{aligned}$$

and since $\gamma' \in \Gamma_c$ then π' is supported on Γ_c.

Proposition 3.11 implies that the set

$$\tilde{\mathcal{C}}(2) = \left\{ (\gamma_1, \gamma_2) \in \mathcal{C}(2) : \gamma'(\gamma_1, \gamma_2) \text{ has a self-intersection with non integer circuitation} \right\}$$

is negligible for every $\xi \in \Pi^{\leq}(\pi, \pi)$. We recall now the following result [Kel84, Theorem 2.19 and Proposition 3.5]: if B is Borel (or analytic)

$$\sup_{\xi \in \Pi^{\leq}(\pi,\pi)} \xi(B) = \min \left\{ \pi(B_1) + \pi(B_2) : \, B \subset (B_1 \times \Gamma) \cup (\Gamma \times B_2) \right\}.$$

In our case, we have observed that the l.h.s. is 0 when $B = \tilde{\mathcal{C}}(2)$, and then there are set B_1, B_2 negligible w.r.t. π such that for every $\gamma_1, \gamma_2 \in \Gamma_c \setminus (B_1 \cup B_2)$ every subloop of the curve $\gamma'(\gamma_1, \gamma_2)$ has integer circuitation.

This proves the theorem for $n = 2$.

Step 2: the general case. Define the closed set

$$\mathcal{C}(n) = \left\{ (\gamma_1, \dots, \gamma_n) \in (\Gamma_c)^n : \bigcup_i \mathrm{Graph}\,\gamma_i \text{ is connected} \right\}.$$

Starting with γ_1, as in the previous step we can join it to some curve γ_i, $i \neq 1$, by formula (3.11): call this curve $\tilde{\gamma}_2$. Using the assumption that $\cup_i \operatorname{Graph} \gamma_i$ is connected, we can repeat the process by adding to $\tilde{\gamma}_2$ a new curve γ_j, $j \neq 1, i$, as in (3.11), obtaining a new curve $\tilde{\gamma}_3$. Proceeding by adding curves, after n steps we obtain a single curve $\tilde{\gamma}$: it is fairly easy to see that since each merging operation is Borel, so the map $(\gamma_1, \dots, \gamma_n) \to \tilde{\gamma}(\gamma_1, \dots, \gamma_n)$ is Borel too.

As in the previous step, for every

$$\xi \in \Pi^{\leq}\big(\underbrace{\pi, \dots, \pi}_{n \text{ times}}\big) = \big\{\xi \in \mathcal{M}^+(\mathcal{C}(n)), (\mathtt{p}_i)_\sharp \xi \leq \pi\big\}$$

we can construct the admissible plan

$$\pi' = \big(\tilde{\gamma}(\gamma_1, \dots, \gamma_n)\big)_\sharp \xi,$$

so that we deduce that the set

$$\tilde{\mathcal{C}}(n) = \Big\{(\gamma_1, \dots, \gamma_n) \in \mathcal{C}(n) : \gamma' \text{ has a self intersection with non integer circuitation}\Big\}$$

is negligible for every $\xi \in \Pi^{\leq}(\pi, \dots, \pi)$. We now use [Kel84, Lemma 1.8, Theorems 2.14 and 2.21]: they state that if B is Borel (or analytic) then

$$\sup_{\xi \in \Pi^{\leq}(\pi, \dots, \pi)} \xi(B) = \min \bigg\{ \sum_{i=1}^n \int f_i \pi : \mathbb{I}_B(\gamma_1, \dots, \gamma_n) \leq \sum_{i=1}^n f_i(\gamma_i), 0 \leq f_i \leq 1 \bigg\}.$$

In our case the left side is 0, so that we can take the sets $B_i = \{f_i > 0\}$ to deduce that there are π-negligible sets B_i such that

$$\tilde{\mathcal{C}}(n) \subset \bigcup_i (\mathtt{p}_i)^{-1}(B_i), \quad \text{where } \mathtt{p}_i(\gamma, \dots, \gamma_n) = \gamma_i.$$

In particular in the complement of $\tilde{B}_n = \cup_i B_i$ it holds that every subloop of $\tilde{\gamma}(\gamma_1, \dots, \gamma_n)$ belongs to $\mathcal{I}_c$. Removing the negligible set $\cup_n \tilde{B}_n$ we obtain the set A_π of the statement. □

Given a sequence of admissible plan π_n, $n \in \mathbb{N}$, it is immediate to see that

$$\frac{1}{\sum_n \frac{2^{-n}}{C_n}} \sum_n \frac{2^{-n}}{C_n} \pi_n \in \mathsf{Adm}(\mu), \tag{3.12}$$

where C_n is the constant for π_n in (3.1) (the constant in front the sum is for normalization). Hence the previous theorem gives the following statement.

Corollary 3.14. *If $\{\pi_n\}_{n \in \mathbb{N}} \subset \mathsf{Adm}(\mu)$ is a countable family of μ-admissible plans, then there exist Borel sets A_{π_n} such that $\pi_n(A_{\pi_n}) = 1$ and it holds*

$$\bigg\{\gamma \in \Gamma_c : \exists \gamma_1 \in A_{\pi_1}, \dots, \gamma_n \in A_{\pi_n} \bigg(\operatorname{Graph} \gamma \subset \bigcup_i \operatorname{Graph} \gamma_i \bigg) \bigg\} \subset \mathcal{I}_c.$$

4. A simple case: $x \in \mathbb{R}^2$, $\mu = f^2 \mathcal{L}^2 / \|f\|_2^2$ with $f \in W^{1,2}(\mathbb{R}^2, \mathcal{L}^2, \mathbb{R}+)$

As a preliminary analysis, we perform the construction of a function $w : \mathbb{R}^2 \mapsto \mathbb{S}^1$ such that

$$D(fw) = Df w + f v,$$

where $v \in \mathbb{R}^d$ is a (cotangent) vector field defined f-a.e. having integer circuitation, and $f \in W^{1,2}(\mathbb{R}^2, \mathcal{L}^d, \mathbb{R}^+)$: the integer circuitation assumption is

$$\int_0^1 v(\gamma(t)) \cdot \dot{\gamma}(t) dt \in 2\pi \mathbb{Z}$$

for π-a.e. γ, with π-admissible plan concentrated on $\Gamma_c(\mathbb{R}^2)$ for the measure $\mu = f^2 \mathcal{L}^2 / \|f\|_2^2$. With some additional technicalities, a similar analysis can be adapted to the d-dimensional case: the advantage in the $2d$-case is that the construction of the admissible plans for the measure $\mu = f^2 \mathcal{L}^2$ is particularly easy.

Consider a Lebesgue point for f and Df: let us assume for definiteness

$$⨍_{B_r^2(0)} |f - 1|^2 \mathcal{L}^d, ⨍_{B_r^2(0)} |Df - 1|^2 < \epsilon^2, \quad \epsilon = o(r). \tag{4.1}$$

By standard density argument, for the directions $\mathbf{e}_1, \mathbf{e}_2$ along the coordinate axis there are segments

$$\ell_{k,y} = \big\{x \in \mathbb{R}^2 : \mathbf{e}_k \cdot x \in [-r,r], \mathtt{p}_{\mathbf{e}_k^\perp} x = y\big\}, \quad y \in K_k \subset [-r,r] \text{ compact},$$

such that

$$\mathcal{L}^1(K_k) \geq r, \quad \int_{\ell_{k,y}} |f-1|^2 \mathcal{H}^1, \int_{\ell_{k,y}} |\partial_{x_k} f - 1|^2 \mathcal{H}^1 < C\epsilon^2 r.$$

In particular it follows that $|f| \geq 1/2$ $\mathcal{H}^1$-a.e. on all the segments $\{\ell_{k,y}, y \in K_k, k = 1,2\}$.

Consider the map $\mathbb{R}^4 \ni (y,y') = (y_1,y_2,y_1',y_2') \mapsto \gamma_{(y,y')}(t) \in \Gamma_c$ where

$$\gamma_{(y,y')} = \overrightarrow{[y,(y_1',y_2)]} \cup \overrightarrow{[(y_1',y_2),y']} \cup \overrightarrow{[y',(y_1,y_2')]} \cup \overrightarrow{[(y_1,y_2'),y]}, \tag{4.2}$$

where $\overrightarrow{[x,y]}$ is the geodesic connecting x, y: $\gamma_{(y,y')}$ is the oriented boundary of the rectangle with opposite corners y, y'.

Lemma 4.1. *For all $g \in L^\infty(\mathbb{R}^4, \mathcal{L}^4)$, $\|g\|_1 = 1$ with bounded support, the measure $\pi = (\gamma_{(y,y')})_\sharp(g\mathcal{L}^4)$ is an admissible plan in $(\mathbb{R}^2, \mathcal{L}^2)$.*

Proof. Since $\gamma_{(y,y')}$ is the union of 4 curves, it is enough to prove the statement for a single part, for example $\overrightarrow{[y,(y_1',y_2)]}$: in this case

$$\begin{aligned}\int \phi\big((1-t)y_1 + ty_1', y_2\big) g(y,y') dy dy' &= \int \phi(z_1,y_2) g(z,y')((1-t)^2+t^2) dz dy' \\ &\leq C\|g\|_\infty \big(\operatorname{diam}(\operatorname{supp} g)\big)^2 \int \phi(z_1,y_2) dz_1 dy_2,\end{aligned}$$

where

$$\begin{pmatrix} z_1 \\ z_1' \end{pmatrix} = \begin{bmatrix} 1-t & t \\ -t & 1-t \end{bmatrix} \begin{pmatrix} y_1 \\ y_1' \end{pmatrix},$$

which shows that the evaluation of π is $\leq C(g)\mathcal{L}^2$. □

Using the same ideas of the first part of the proof of Theorem 3.13 applied to the measures

$$\xi \in \Pi^{\leq}\big(\mathcal{L}^2 \llcorner_{K_1 \times K_2}, \mathcal{L}^2 \llcorner_{K_1 \times K_2}\big),$$

we deduce the following corollary.

Corollary 4.2. *There is a $\mathcal{L}^2$-conegligible set $I_0 \subset K_1 \times K_2$ such that for every $y, y' \in I_0$ the curve $\gamma_{(y,y')}$ has integer circuitation.*

Using Fubini, we obtain that for $\mathcal{L}^2$-a.e. $y = (y_1,y_2) \in I_0$ the set

$$\big\{y' \in I_0 : p_2(y') = y_2\big\} \subset K_2, \tag{4.3}$$

has full measure. Define for one of these y

$$w(y,y') = e^{i\int_{y_1}^{y_1'} v_1(t,y_2)dt + i\int_{y_2}^{y_2'} v_2(y_1',t)dt}, \quad y' \in K_1 \times K_2,$$

i.e. the integral along the curve $\overrightarrow{[y,(y_1',y_2)]} \cup \overrightarrow{[(y_1',y_2),y']}$. The function $w(y,y')$ is defined on $\mathcal{L}^2$-a.e. point of $K_1 \times K_2$.

Lemma 4.3. *For every $(y',y'') \in (K_1 \times K_2)^2 \setminus N$, $\mathcal{L}^4(N) = 0$, it holds*

$$w(y,y'') = w(y,y') e^{i\int_{y_1'}^{y_1''} v_1(t,y_2')dt + i\int_{y_2'}^{y_2''} v_2(y_1'',t)dt}.$$

Proof. The proof is a consequence that the statement holds for $y'' \in I_0$ such that $(y_1'',y_2) \in I_0$ because setting $y''' = (y_1'',y_2)$

$$\begin{aligned}&\int_{y_1}^{y_1'} v_1(t,y_2)dt + \int_{y_2}^{y_2'} v_2(y_1',t)dt + \int_{y_1'}^{y_1''} v_1(t,y_2')dt + \int_{y_2'}^{y_2''} v_2(y_1'',t)dt \\ &\quad = \int_{y_1}^{y_1''} v_1(t,y_2)dt + \int_{y_2}^{y_2''} v_2(y_1'',t)dt + \int_{\gamma_{(y',y''')}} v_1 \cdot \dot\gamma_{(y',y''')} \\ &\quad \in \int_{y_1}^{y_1''} v_1(t,y_2)dt + \int_{y_2}^{y_2''} v_2(y_1'',t)dt + 2\pi\mathbb{Z},\end{aligned}$$

and the set

$$\big\{y' \in I_0, (y_1', y_2) \in I_0\big\} \times \big\{y'' \in I_0, (y_1'', y_2) \in I_0\big\}$$

has full measure due to the assumption on (4.3) and on I_0. □

By repeating the construction, we obtain the following corollary.

Corollary 4.4. *The is a countable family of sets $K_{1,n} \times K_{2,n}$ and of functions w_n such that*

$$\bigcup_{n \in \mathbb{N}} K_{1,n} \times K_{2,n} \quad \text{has full } |f|^2 \mathcal{L}^2\text{-measure}$$

and w_n satisfies Lemma 4.3 in every $K_{1,n} \times K_{2,n}$.

Note that we do not require the sets $K_{1,n} \times K_{2,n}$ to be disjoint.

Proof. Since by restricting the balls we can require that

$$\frac{\mathcal{L}^2(K_{1,n} \times K_{2,n})}{(2r)^2} \to 1,$$

a standard covering argument as in the proof of [Bog, Theorem 5.5.1] gives the statement. □

The above lemma defines the function w in a set of positive Lebesgue measure, with the property that every two points are connected with a path where $|f| \geq 1/2$. This is the hardest step, since the remaining part needs only to show that this w is compatible with all admissible plans and it can be extended to the whole $\{f \neq 0\}$.

Consider an admissible plan π such that

$$e(0)_\sharp \pi \leq C \mathcal{L}^2 \llcorner_{K_{1,i} \times K_{2,i}}, \quad e(1)_\sharp \pi \leq C \mathcal{L}^2 \llcorner_{K_{1,j} \times K_{2,j}}.$$

For every γ, γ' such that

$$\gamma(0), \gamma'(0) \in K_{1,i} \times K_{2,i}, \quad \gamma(1), \gamma'(1) \in K_{1,j} \times K_{2,j},$$

let $\hat{\gamma}_{\gamma,\gamma'}$ be the closed curve obtained by joining $\gamma(0), \gamma'(0)$ and $\gamma(1), \gamma'(1)$ with the curves used in Lemma 4.3 to compute w. Define

$$\tilde{\pi} = (\hat{\gamma}_{\gamma,\gamma})_\sharp \big(\pi(d\gamma) \times \pi(d\gamma')\big),$$

which is a transference plan due to Lemma 4.1. The condition of integer circuitation yields that by removing a $\tilde{\pi}$-negligible set of trajectories the curves $\hat{\gamma}_{\gamma,\gamma}$ have integer circuitation. For all these curves it follows that

$$\frac{w_j(\gamma(1))}{w_i(\gamma(0)) e^{i \int_0^1 v \cdot \dot{\gamma} \mathcal{L}^1}} = \frac{w_j(\gamma'(1)) e^{i \int v \cdot \overrightarrow{[\gamma'(1), \gamma(1)]}^{\,\cdot}}}{w_i(\gamma'(0)) e^{i \int v \cdot \overrightarrow{[\gamma'(0), \gamma(0)]}^{\,\cdot}) + i \int_0^1 v \cdot \dot{\gamma} \mathcal{L}^1}} = \frac{w_j(\gamma'(1))}{w_i(\gamma'(0)) e^{i \int_0^1 v \cdot \dot{\gamma}' \mathcal{L}^1}}. \tag{4.4}$$

The above computation yields the following lemma.

Lemma 4.5. *There exists a constant c_{ij} such that for every admissible plan π with*

$$e(0)_\sharp \pi \leq C \mathcal{L}^2 \llcorner_{K_{1,i} \times K_{2,i}}, \quad e(1)_\sharp \pi \leq C \mathcal{L}^2 \llcorner_{K_{1,j} \times K_{2,j}},$$

it holds for π-a.e. γ

$$c_{ij} w_j(\gamma(1)) = w_i(\gamma(0)) e^{i \int v \cdot \dot{\gamma}}.$$

Proof. For π-a.e. γ the set γ' such that $\hat{\gamma}_{\gamma,\gamma'}$ is of integer circuitation has full π-measure. For every such a γ, define c_{ij} as the inverse of the ratio of (4.4):

$$c_{ij} = \frac{w_i(\gamma(0)) e^{i \int_0^1 v \cdot \dot{\gamma} \mathcal{L}^1}}{w_j(\gamma(1))}.$$

Since by (4.4) the ratio is the same for π-a.e. γ, the proof is complete. □

Starting from the function w_1 defined in $K_{1,1} \times K_{2,1}$, we can then find constants c_{1j} such that the above lemma holds for all sets $K_{1,j} \times K_{2,j}$ for which there is an admissible plan connecting $K_{1,1} \times K_{2,1}$ to $K_{1,j} \times K_{2,j}$. If at this step we cover $\{|f| > 0\}$ up to a $\mathcal{L}^2$-negligible set, we are done, otherwise we take the next $K_{1,i} \times K_{2,i}$ and connect to some others subsets $K_{1,j'} \times K_{2,j'}$: clearly these sets cannot belong to the previous family, otherwise there is an admissible plan from $K_{1,1} \times K_{2,1}$ to $K_{1,j'} \times K_{2,j'}$ and then to $K_{1,i} \times K_{2,i}$.

We this conclude with the following statement.

Proposition 4.6. *There exists a function w such that for all admissible plans π w.r.t. the measure $|f|^2\mathcal{L}^2$ it holds*

$$w(\gamma(1)) = w(\gamma(0))e^{i\int v\cdot\dot\gamma} \quad \pi\text{-a.e. } \gamma.$$

Moreover all these functions depend at most on countably many multiplicative constants c_j.

Proof. We are left in proving the second part, which follows from the following observations:

(1) every solution $\tilde w$ satisfies

$$\frac{\tilde w}{w_j} = \text{ constant in every } K_{1,j} \times K_{2,j} :$$

indeed it is constant on the curves used in Lemma 4.3;

(2) the relation R

$$(K_{1,j} \times K_{2,j})R(K_{1,i} \times K_{2,i}) \quad \iff \quad \exists\pi \text{ plan connecting them as in Lemma 4.5}$$

is an equivalence relation;

(3) the construction of w presented before the statement of this proposition is defined up to a multiplicative constant in each equivalence class.

This concludes the proof. □

Once the function w is constructed, it follows from the chain rule in one dimension that for all admissible plans w.r.t. $|f|^2\mathcal{L}^2$ it holds

$$\frac{d}{dt}(fw)\circ\gamma = \frac{df\circ\gamma}{dt}w + f\frac{dw\circ\gamma}{dt} = wDf\cdot\dot\gamma + iwfv\cdot\gamma,$$

for π-a.e. γ. In particular, by Corollary 4.4 and Proposition 4.6 we can consider the $\mathcal{L}^1$-conegligible set of $y_2 \in \mathbb{R}$ such that $y_1 \mapsto f(y_1, y_2)$ is in $H^1(\mathbb{R})$, and the function $y_1 \mapsto w(y_1, y_2)$ satisfies

$$|w(y_1,y_2)| = 1, \quad |\partial_{y_1}w(y_1,y_2)| = |v_1(y_1,y_2)|$$

for all intervals of the open set $\{y_1 : |f(y_1,y_2)| > 0\} \subset \mathbb{R}$. Since $fw = 0$ when $f = 0$ because $|w| = 1$, for these y_2's it holds $y_1 \mapsto (fw)(y_1,y_2)$ is a.c. and has derivative $w\partial_{x_1}f + ifwv_1$, and this means that

$$\|D(fw)\|_2 \leq \|Df\|_2 + \int |v|^2|f|^2\mathcal{L}^2 < \infty,$$

We thus conclude with the following result.

Proposition 4.7. *For every w given by Proposition 4.6 the function fw is in $W^{1,2}(\mathbb{R}^2, \mathcal{L}^2, \mathbb{C})$ and has differential $D(fw) = wDf + ifwv$.*

5. Construction of the minimal equivalence relation

In this section we construct a decomposition $X = \cup_\alpha X_\alpha$ into disjoint sets with the following properties:

- each admissible plan is concentrated on a set of curves γ such that $\gamma([0,1])$ belongs to a singe elements Z_α up to a $\mathcal{H}^1$-negligible est;
- there exists an admissible plan $\bar\pi$ generating the partition $\{X_\alpha\}_\alpha$ up to a μ-negligible set.

The first property means that every admissible plan moves mass only inside the equivalence classes X_α: this can be interpreted by saying that the X_α are the connected components from the point of view of the MMS (X,μ), in the sense that no mass transfer can occur between different equivalence classes. For example, the MMS

$$B_1((-2,0)) \cup B_1((2,0)) \cup [-1,1]\times\{0\} \subset \mathbb{R}^2 \tag{5.1}$$

is such that the segment connecting $[-1,1]$ has $\mathcal{L}^2$-measure 0, so the no admissible plans π can charge mass on it. Hence it is not seen by any Sobolev function $f \in W^{1,2}(X,\mu,\mathbb{R})$: from their point of view the two balls are completely disjoint.
The second property instead says that no smaller partitions can be constructed if they need to satisfy the first property: even by changing the set where the plan $\bar{\pi}$ is concentrated, the partition $\{X_\alpha\}_\alpha$ remains the same in a set of full μ-measure. Returning to the example (5.1), it is elementary to see that the two balls cannot be further subdivided if we require the above first point to hold (recall Section 4 and see also Example 7.2).

In the first part of this section we take an abstract point of view as follow.

(1) The (complete) σ-algebra $\mathcal{M}$ of μ is separable, i.e. countably generated: identifying elements of $\mathcal{M}$ up to negligible sets, we obtain a separable measure algebra which we still denote by $(\mathcal{M},\mu)$ or $\mathcal{M}$ since the measure μ is fixed here. The function
$$d(\Omega,\Omega') = \mu(\Omega\Delta\Omega') \tag{5.2}$$
is a distance on $\mathcal{M}$, which makes $\mathcal{M}$ a complete and separable metric space.

(2) Each equivalence relation E on X generates the sub σ-algebra $\mathcal{M}^E$ made of all saturated measurable sets,
$$\mathcal{M}^E = \big\{\Omega \in \mathcal{M} : E\Omega = \Omega\big\}. \tag{5.3}$$
The corresponding measure algebra $\mathcal{M}^E$ is clearly a measure sub-algebra of $\mathcal{M}$. In particular it is separable and closed.

The statements below can be deduced easily from the properties of measure algebras, see [Fre02a, Chapter 3].

Consider a family of measure sub-algebras $\{\mathcal{M}^\alpha\}_\alpha$, $\mathcal{M}^\alpha \subset \mathcal{M}$, closed under countable union: for every sequence $\mathcal{M}^{\alpha_i}$ there exists $\mathcal{M}^{\bar{\alpha}}$ such that
$$\bigcup_i \mathcal{M}^{\alpha_i} \subset \mathcal{M}^{\bar{\alpha}}.$$
The following result shows that there is a maximal element $\mathcal{M}^{\bar{\alpha}}$.

Proposition 5.1. *There exists $\mathcal{M}^{\bar{\alpha}} \in \{\mathcal{M}^\alpha\}_\alpha$ such that $\mathcal{M}^\alpha \subset \mathcal{M}^{\bar{\alpha}}$ for all α.*

The proof is just the observation that every subset of a separable metric space has a countable dense subset.

Proof. Consider the measure sub-algebra generated by the union of all $\mathcal{M}^\alpha$: it corresponds to the closure of the algebra generated by the algebra $\cup_\alpha \mathcal{M}^\alpha$ w.r.t. the distance (5.2). In particular, being countably generated, it coincides with the measure sub-algebra generated by countably many elements $\Omega_n \in \cup_\alpha \mathcal{M}^\alpha$.

Let the sub-algebras $\mathcal{M}^{\alpha_n}$ be such that $\Omega_n \in \mathcal{M}^{\alpha_n}$, and let $\mathcal{M}^{\bar{\alpha}}$ be measure sub-algebra containing all of them. This is the measure sub-algebra of the statement. □

A symmetric result considers the case where the family $\{\mathcal{M}^\alpha\}_\alpha$ is closed under countable intersection: for every sequence $\{\mathcal{M}^{\alpha_n}\}_n$ there exists $\mathcal{M}^{\bar{\alpha}}$ such that
$$\mathcal{M}^{\bar{\alpha}} \subset \bigcap_n \mathcal{M}^{\alpha_n}. \tag{5.4}$$

Proposition 5.2. *There exists $\mathcal{M}^{\bar{\alpha}} \in \{\mathcal{M}^\alpha\}_\alpha$ such that $\mathcal{M}^{\bar{\alpha}} \subset \mathcal{M}^\alpha$ for all α.*

The proof follows by observing that every union of open balls is obtained by considering only countably many balls.

Proof. First of all, for all
$$\Omega \notin \bigcap_\alpha \mathcal{M}^\alpha$$
there is α_Ω such that $\Omega \notin \mathcal{M}^{\alpha_\Omega}$, and then there exists $r_\Omega > 0$ such that the set
$$B^{\mathcal{M}}_{R_\Omega}(\Omega) = \big\{\Omega' : d(\Omega,\Omega') < r_\Omega\big\}$$
has empty intersection with $\mathcal{M}^{\alpha_\Omega}$.

By separability, we need only a countable family of balls $\{B^{\mathcal{M}}_{R_{\Omega_n}}(\Omega_n)\}_n$ to cover

$$\mathcal{M} \setminus \bigcap_\alpha \mathcal{M}^\alpha.$$

Hence we deduce that if

$$B^{\mathcal{M}}_{R_{\Omega_n}}(\Omega_n) \cap \mathcal{M}^{\alpha_n} = \emptyset,$$

then

$$\bigcap_\alpha \mathcal{M}^\alpha = \bigcap_n \mathcal{M}^{\alpha_n},$$

which is the statement by (5.4). □

In terms of σ-algebras, the two statements above can be rewritten as follows. Consider a family of μ-complete σ-algebras $\mathcal{M}^\alpha \subset \mathcal{M}$, where $\mathcal{M}$ is the μ-completion of a separable σ-algebra.

Corollary 5.3. *The following holds:*

(1) *if $\{\mathcal{M}^\alpha\}_\alpha$ is closed under countable union, then there exists an element $\mathcal{M}^{\bar\alpha}$ such that*
$$\forall \alpha, \Omega \in \mathcal{M}^\alpha \, \exists \Omega' \in \mathcal{M}^{\bar\alpha} \big(\mu(\Omega \Delta \Omega') = 0 \big);$$

(2) *if $\{\mathcal{M}^\alpha\}_\alpha$ is closed under countable intersection, then there exists an element $\mathcal{M}^{\bar\alpha}$ such that*
$$\forall \alpha, \Omega \in \mathcal{M}^{\bar\alpha} \, \exists \Omega' \in \mathcal{M}^{\alpha} \big(\mu(\Omega \Delta \Omega') = 0 \big).$$

The same statement can be rewritten in terms of equivalence relations, by reversing the intersection and union.

Corollary 5.4. *The following holds:*

(1) *if $\{E^\alpha\}_\alpha$ is closed under countable intersection, then there exists an element $E^{\bar\alpha}$ such that if $\mathcal{M}^\alpha$ is the complete σ-algebra generated by E^α according to (5.3) then*
$$\forall \alpha, \Omega \in \mathcal{M}^\alpha \, \exists \Omega' \in \mathcal{M}^{\bar\alpha} \big(\mu(\Omega \Delta \Omega') = 0 \big);$$

(2) *if $\{E^\alpha\}_\alpha$ is closed under countable union, then there exists an element $E^{\bar\alpha}$ such that if $\mathcal{M}^\alpha$ is the complete σ-algebra generated by E^α according to (5.3) then*
$$\forall \alpha, \Omega \in \mathcal{M}^{\bar\alpha} \, \exists \Omega' \in \mathcal{M}^{\alpha} \big(\mu(\Omega \Delta \Omega') = 0 \big).$$

5.1. **Application to admissible plans.** The statement of Proposition 5.1 in our setting will be used as follows.

Definition 5.5. Let $\{\Omega_\alpha\}_\alpha$ be a family of subsets of a set X. The equivalence relation generated by $\{\Omega_\alpha\}_\alpha$ is the smallest equivalence relation E such that

$$\bigcup_\alpha \Omega_\alpha \times \Omega_\alpha \subset E.$$

Lemma 5.6. *The equivalence relation generated by $\{\Omega_\alpha\}_\alpha$ is the following:*

$$xEx' \quad \Longleftrightarrow \quad \exists n \in \mathbb{N}, \{\Omega_{\alpha_i}\}_{i=1}^n, x_i, x_{i+1} \in \Omega_{\alpha_i} \big(x_1 = x \ \wedge \ x_{n+1} = x' \big).$$

Proof. Immediate from the definition. □

We apply the above construction to our case, the sets Ω_α being the range of the curves $\{\gamma \in A_\pi\}$.

Definition 5.7. We will say that two points x, x' in X are equivalent according to the set $A_\pi \subset \Gamma$ and we will write $x E^{A_\pi} x'$ if they are equivalent according to Definition 5.5 where the sets Ω_α are the family of sets $\{\gamma([0,1]), \gamma \in A_\pi\}$.

Equivalently, $x E^{A_\pi} x'$ if there exist a finite set $\{\gamma_i\}_i \subset A_\pi$ of curves and a path $\gamma \in \Gamma$ such that

$$\mathrm{Graph}\, \gamma \subset \bigcup_i \mathrm{Graph}\, \gamma_i \quad \text{and} \quad \gamma(0) = x, \ \gamma(1) = x'.$$

Indeed, considering the sets $\Omega_1 = \{\gamma_1([0,1])\}, \dots, \Omega_n = \{\gamma_n([0,1])\}$, $\gamma_1, \dots, \gamma_n \in A_\pi$, and the points $x_i, x_{i+1} \in \Omega_i$, $\gamma_i(t_i^-) = x_i, \gamma_i(t_i^+) = x_{i+1}$, of Lemma 5.6, the curve γ can be obtained by

$$\gamma(t) = \gamma_i\big(t_i^- + (nt - (i-1))(t_{i+1}^+ - t_i^-)\big), \qquad nt \in [i-1, i],$$

up to a reparametrization.

For every admissible plan $\pi \in \mathsf{Adm}(\mu)$ consider the family of sets A_π satisfying Theorem 3.13 (i.e. every closed subloop of A_π is of integer cicuitation), and let E^{A_π}, $\mathcal{M}^{E^{A_\pi}}$ be the corresponding equivalence relation and σ-algebra, respectively (see (5.3)).

Proposition 5.8. *There exists $\bar{A}_\pi$ such that $\pi(\bar{A}_\pi) = 1$ and $\mathcal{M}^{E^{\bar{A}_\pi}}$ is maximal.*

In particular, for every subset $A_\pi \subset \bar{A}_\pi$ such that $\pi(A_\pi) = 1$ it holds

$$\mathcal{M}^{E^{\bar{A}_\pi}} = \mathcal{M}^{E^{A_\pi}} \quad \text{as measure algebras.} \tag{5.5}$$

We will denote $\mathcal{M}^{E^{\bar{A}_\pi}}$ as $\mathcal{M}^\pi$, and the corresponding equivalence relation by $E^\pi = E^{\bar{A}_\pi}$. We stress that the only unique object is $\mathcal{M}^\pi$ considered as a complete σ-subalgebra of $\mathcal{M}$ or as a measure algebra.

Proof. We need only to verify that the family $\{E^{A_\pi}\}_{A_\pi}$ is closed under countable intersection: this follows from the implications

$$\forall i \big(\pi(A^i_\pi) = 1\big) \quad \Rightarrow \quad \pi\bigg(\bigcap_i A^i_\pi\bigg) = 1,$$

and

$$E^{\cap_i A^i_\pi} \subset \bigcap_i E^{A^i_\pi}. \qquad \square$$

We next apply Proposition 5.2 to sequences of admissible plans.

Proposition 5.9. *There exists $\bar{\pi} \in \mathsf{Adm}(\mu)$ for every $\pi \in \mathsf{Adm}(\mu)$ it holds*

$$\mathcal{M}^{\bar{\pi}} \subset \mathcal{M}^\pi \quad \textit{as measure algebras.}$$

Proof. Consider a sequence of minimal equivalence relations $\{E^{\pi_i}\}_i$, and define the admissible plan

$$\hat{\pi} = \frac{1}{\sum_i \frac{2^{-i}}{C_i}} \sum_i \frac{2^{-i}}{C_i} \pi_i,$$

where C_i is the compressibility constant (3.1) (as before we normalized the sum of measures in order to be a probability). Let $\bar{A}_{\hat{\pi}}, \bar{A}_{\pi_i}$ be a set of curves satisfying Proposition 5.8 above for $\hat{\pi}, \pi_i$ respectively.

If we consider the sets

$$\hat{A}_{\pi_i} = \bar{A}_{\hat{\pi}} \cap \bar{A}_{\pi_i},$$

then $\pi_i(\hat{A}_{\pi_i}) = 1$ because $\pi_i \ll \hat{\pi}$, and then by maximality of $\mathcal{M}^{E^{\bar{A}_{\pi_i}}}$ we conclude that by (5.5) that

$$\mathcal{M}^{E^{\bar{A}_{\pi_i}}} = \mathcal{M}^{E^{\hat{A}_{\pi_i}}} = \mathcal{M}^{\pi_i} \quad \text{as measure algebras.}$$

We thus deduce from $E^{\hat{A}_{\pi_i}} \subset E^{\bar{A}_{\hat{\pi}}}$ that $\mathcal{M}^{\hat{\pi}} \subset \mathcal{M}^{\pi_i}$ for all i, which shows that we are in the situation of Point (2) of Corollary 5.3: the minimal measure algebra $\mathcal{M}^{\bar{\pi}}$ satisfies the statement. $\square$

Let $\bar{A}_{\bar{\pi}}$ be a σ-compact set where $\bar{\pi}$ is concentrated such that $E^{\bar{A}_{\bar{\pi}}}$ generates the maximal measure algebra $\mathcal{M}^{\bar{\pi}}$, which by the previous proposition is the smallest among all maximal measure algebras $\mathcal{M}^\pi$.

Let $\{\Omega_n\}_n$ be a countable family of $E^{\bar{A}_{\bar{\pi}}}$-saturated sets generating $\mathcal{M}^{\bar{\pi}}$ up to negligible sets, and define the equivalence relation

$$\bar{E} = \bigcap_n \big(\Omega_n \times \Omega_n\big) \cup \big((X \setminus \Omega_n) \times (X \setminus \Omega_n)\big). \tag{5.6}$$

By construction

$$E^{\bar{A}_{\bar{\pi}}} \subset \bar{E},$$

and in general the inclusion is strict. The equivalence relation $\bar{E}$ is made of the atoms of the σ-algebra generated by the $\{\Omega_n\}_n$, whose completion coincides with $\mathcal{M}^{\bar{\pi}}$.

Remark 5.10. Consider another set $\bar{A}'_{\bar{\pi}}$ generating $\mathcal{M}^{\bar{\pi}}$, and let $\{\Omega'_n\}_n$ be $E^{\bar{A}'_{\bar{\pi}}}$-saturated sets generating $\mathcal{M}^{\bar{\pi}}$. Since $\mathcal{M}^{\bar{\pi}}$ is unique and it is the completion of the σ-algebra generated by $\bar{E}$, then there are $\bar{E}$-saturated sets $\{\Omega''_n\}_n$ such that

$$\mu(\Omega''_n \Delta \Omega'_n) = 0.$$

Defining

$$N = \bigcup_n \Omega''_n \Delta \Omega'_n, \quad \mu(N) = 0,$$

we deduce that the equivalence relation obtained by (5.6) using the family $\{\Omega'_n\}_n$ is equal to the one constructed through the sets $\{\Omega''_n\}_n$ up to a μ-negligible set. Indeed the equivalence relation generated by $\{\Omega''_n \setminus N\}_n$ contains $\bar{E}$ since the sets Ω''_n are $\bar{E}$-saturated. Hence

$$E^{\bar{A}'_{\bar{\pi}}} \setminus N \times N \subset \bar{E}.$$

Reversing the argument we obtain that there exists a μ-negligible set N' such that

$$\bar{E} \setminus N' \times N' \subset E^{\bar{A}'_{\bar{\pi}}},$$

and then $E^{\bar{A}'_{\bar{\pi}}}$ is equal to $\bar{E}$ up to a μ-negligible set on X.

Let $\pi \in \mathsf{Adm}(\mu)$ and $\bar{A}_\pi$ generating the maximal $\mathcal{M}^\pi$ as in Proposition 5.8. By minimality of $\mathcal{M}^{\bar{\pi}}$, for every Ω_n there is a set $\Omega_n^\pi \in \mathcal{M}^\pi$ such that

$$\mu\big(\Omega_n \Delta \Omega_n^\pi\big) = 0. \tag{5.7}$$

In particular, following the reasoning of the previous remark, there exists a μ-negligible set $N \subset X$ such that if $\bar{E}$ is given by (5.6) then

$$E^{\bar{A}_\pi} \cap (X \setminus N \times X \setminus N) \subset \bar{E} \cap (X \setminus N \times X \setminus N). \tag{5.8}$$

Since $\gamma \in \bar{A}_\pi$ takes values in a single class of $E^{\bar{A}_\pi}$, we conclude that

Corollary 5.11. *Every $\pi \in \mathsf{Adm}(\mu)$ is concentrated on a family of trajectories A_π such that*

$$\forall \gamma \in A_\pi \ \big(\gamma(s)\bar{E}\gamma(t) \text{ up to a } \mathcal{L}^1\text{-negligible subset of } [0, L(\gamma)]\big).$$

Proof. Since we know that $\gamma(t)E^{\bar{A}_\pi}\gamma(s)$ for $s, t \in [0, L(\gamma)]$ by construction, then we deduce from (5.8) that

$$\gamma(s)\bar{E}\gamma(t) \quad \forall s, t \in [0, L(\gamma)] \setminus \gamma^{-1}(N).$$

Being N μ-negligible,

$$\int \bigg[\int_0^{L(\gamma)} \mathbf{I}_N(\gamma(t))dt\bigg]\pi(d\gamma) = 0,$$

so that we need only to remove the negligible set of curves γ such that $\mathcal{L}^1(\gamma^{-1}(N)) > 0$. □

Remark 5.12. The presence of the negligible set N is natural: take for example the space $X = \mathbb{S}^1$, $\mu = \mathcal{L}^1$, and the equivalence relation

$$E = \{(0,0)\} \cup (\mathbb{S}^1 \setminus \{0\})^2.$$

The measure algebra is the trivial one, but any curve passing through 0 sees $N = \{0\}$.

6. Construction of the function w

In the previous section we have constructed an equivalence relation $E^{\bar{A}_{\bar{\pi}}}$ for the admissible plan $\bar{\pi}$ such that the measure algebra $\mathcal{M}^{\bar{\pi}}$ generated by the saturated sets $\mathcal{M}^{\bar{\pi}}$ satisfies the following properties:

(1) $\bar{A}_{\bar{\pi}}$ satisfies Theorem 3.13,
(2) it is the maximal measure algebra among all measure algebra generated by sets where $\bar{\pi}$ is concentrated,
(3) all others admissible plans π are concentrated on a set of curves $\bar{A}_\pi$ generating a measure algebra which contains $\mathcal{M}^{\bar{\pi}}$.

Nothing is said about the strong consistency of the disintegration w.r.t. $E^{\bar{A}_{\bar{\pi}}}$: if

$$\mu = \int \mu_\alpha \zeta(d\alpha), \quad \zeta = \mu \llcorner_{\mathcal{M}^{\bar{\pi}}}, \tag{6.1}$$

then in general the conditional probability μ_α is not concentrated on the corresponding equivalence class $E_\alpha^{\bar{A}_{\bar{\pi}}}$: this implies that $\bar{E} \supseteq E^{\bar{A}_{\bar{\pi}}}$.

Assumption 6.1. We assume that the minimal disintegration w.r.t. $E^{\bar{A}_{\bar{\pi}}}$ is strongly consistent for every $\bar{A}_{\bar{\pi}}$ generating $\mathcal{M}^{\bar{\pi}}$, i.e.

$$\mu_\alpha(E^{\bar{A}_{\bar{\pi}}}) = 1 \quad \text{for } \zeta\text{-a.e. } \alpha.$$

We recall that this is equivalent to each of the following conditions ([BC09] or [Fre02b, Section 452]):

(1) the equivalence relation $E^{\bar{A}_{\bar{\pi}}}$ is essentially countably generated, and in particular we can assume that $\bar{E} = E^{\bar{A}_{\bar{\pi}}}$ up to a μ-negligible set, where $\bar{E}$ is given by (5.6);
(2) there exists a μ-measurable map $f : X \to [0,1]$ whole level sets are the equivalence classes:

$$\big\{(x,x') : f(x) = f(x')\big\} = E^\pi. \tag{6.2}$$

Moreover, strong consistency implies that, up to a μ-negligible set of equivalence classes of $E^{\bar{A}_{\bar{\pi}}}$, there exists a Borel section S, i.e. a Borel set in X such that it contains exactly one point of each equivalence class of $E^{\bar{A}_{\bar{\pi}}}$.

We collect the previous observations in the following statement.

Lemma 6.2. *Under Assumption 6.1, there exists an admissible plan $\bar{\pi}$ concentrated on a set of trajectories $\bar{A}_{\bar{\pi}}$ satisfying Theorem 3.13 such that:*

(1) the equivalence relation $\bar{E} = \bar{E}^{\bar{A}_{\bar{\pi}}}$ generated by $\bar{A}_{\bar{\pi}}$ admits a Borel section $\bar{S}$ and has the following equivalent properties (up to a μ-negligible saturated set):
  (a) the disintegration (6.1) is consistent,
  (b) the atoms of the separable σ-algebra $\mathcal{M}^{\bar{\pi}}$ are the equivalence classes of $\bar{E}$;
(2) every two points of an equivalence class are connected by finitely many curves in $\bar{A}_{\bar{\pi}}$.

Assumption 6.1 is necessary to prove that the integer circuitation condition of Λ implies the existence of the function w: indeed if the disintegration is not consistent, then there are counterexamples as the following one.

Example 6.3. Consider the Cantor 1/3-set

$$K = [0,1] \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{s\in\{0,2\}^{n-1}} \sum_{i=1}^{n-1} s_i 3^{-i} + 3^{-n}(1,2),$$

and let $V : [0,1] \to [0,1]$ be the Vitali function

$$V \in C^0([0,1]), \qquad V\bigg(\sum_{i=1}^{n-1} s_i 3^{-i} + 3^{-n}(1,2)\bigg) = \sum_{i=1}^{n-1} s_i 2^{-1-i} + \{2^{-n}\}, \quad s_i \in \{0,2\}.$$

Define for $\alpha \in [0,1] \setminus \mathbb{Q}$

$$f : K \to K, \quad f(x) = V^{-1} \circ (\mathrm{id} + \alpha \mod 1) \circ V(x),$$

which is defined up to the countably many points belonging to the boundary Z of the intervals $\sum_{i=1}^{n-1} s_i 3^{-i} + 3^{-n}(1,2)$, $n \in \mathbb{N}, s_i \in \{0,2\}$. For definiteness we can assume that those points are removed from K, in which case the set $K \setminus Z$ is a G_δ-set, i.e. still a Polish space.

It is fairly easy to construct a family of totally disconnected curves $\gamma : [0,1] \times K \to \mathbb{R}^3$ of length 1 such that

$$\gamma(0,x) = (0,0,x), \quad \gamma(1,x) = (0,0,f(x)).$$

The MMS (X,d,μ) is the the union of all these trajectories with the Euclidean distance in $\mathbb{R}^3$, and with the measure $\mathcal{L}^1 \times \mathcal{H}^{1/3}$ in the coordinates $(t,x) \in [0,1) \times K$ see Figure 1.

Let $o(\cdot,x) : \mathbb{R} \to \mathbb{R}^3$ be the orbit starting from $x \in K$ and defined by

$$o(t,x) = \gamma(t \mod 1, f^{\lfloor t \rfloor}(x)), \quad \lfloor t \rfloor \text{ integer part of } t.$$

The fact that the segments γ are totally disconnected implies that the only admissible plans π are concentrated on curves which are subsets of $o(\cdot,x)$, and in particular any measurable function $v : [0,1] \times K \to \mathbb{R}$ gives an integer (or free) circuitation cotangent vector field, being the tangent space one dimensional in each point.

Setting $v = 2\pi\beta$ constant with $z \cdot (\alpha,\beta) \notin \mathbb{Z}$ for all $z \in \mathbb{Z}^2 \setminus 0$, we obtain that any candidate w such that $\dot{w} = iwv$ should satisfy

$$w(0,0,f(x)) = w(0,0,x)e^{2\pi i\beta},$$

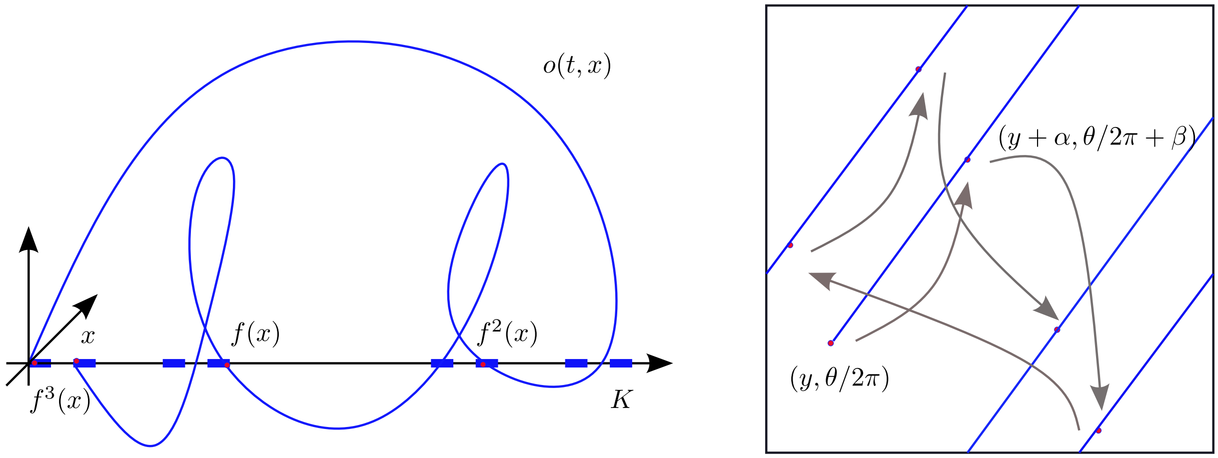

Figure 1. The Cantor set K together with a piece of the orbit o, and the evolution of the point (y,θ) in the angular coordinates.

which, calling

$$w'(y) = w(0,0,V^{-1}(y)),$$

by definition of f gives

$$w'(y+\alpha \mod 1) = e^{i\theta(y+\alpha \mod 1)} = w'(y)e^{2\pi i\beta} = e^{i\theta(y)+2\pi i\beta},$$

where the function $\theta : [0,1] \to \mathbb{S}^1$ is the angle.

If we consider the torus $\mathbb{T}^2$, then the values of the angles $\theta(y)$ on a single orbit corresponds to the trajectory of the points

$$n \mapsto \left(y + n\alpha \mod 1, \frac{\theta(y)}{2\pi} + n\beta \mod 1\right).$$

We are looking for the invariant graph $\theta = \theta(y)$. Consider the measure $\varpi = \theta_\sharp \mathcal{L}^1$ and its Fourier transform

$$\hat{\varpi} = \sum_{z\in\mathbb{Z}^2} c_z e^{2\pi i z\cdot(y,\theta)}.$$

The invariance implies

$$\sum_{z\in\mathbb{Z}^2} c_z e^{2\pi i z\cdot(y,\theta)} = \sum_{z\in\mathbb{Z}^2} c_z e^{2\pi i z\cdot(y,\theta)+2\pi i z\cdot(\alpha,\beta)},$$

i.e.

$$\forall c_z \neq 0 \ \big(z\cdot(\alpha,\beta) \in \mathbb{Z}^2\big).$$

This is impossible for $z \neq 0$ by the assumptions on α, β, hence there are no integrals w to $Dw = iwv$.

We can slightly change the space X and the admissible plan $\bar\pi$ in order to see that the assumption that the disintegration is strongly consistent for all $\bar A_{\bar\pi}$ is essential. Set

$$\tilde X = X \cup [0,1] \times \{0\},$$

i.e. we add the segment containing K. This segment is negligible for all admissible plans, since we still keep the measure supported on X. To have a set of curves with integer circuitation, just remove a point to every orbit $o(t,x)$, for example assume that

$$\bar A_{\bar\pi} = \big\{(t,0,0), t\in[0,1]\big\} \cup \bigcup_{x\in K, n\in\mathbb{N}} \big\{o(t,x), t \in [0,1/2-2^{-n}]\big\} \cup \big\{o(-t,x), t\in[0,1/2-2^{-n}]\big\}.$$

It is fairly easy to see that $\bar A_{\bar\pi}$ satisfies Theorem 3.13, and the equivalence class is $\tilde X \setminus \cup_x\{o(1/2,x)\}$. However removing the negligible curve $(t,0,0)$ the measure algebra $\mathcal{M}$ is not the minimal one $\mathcal{M}^{\bar\pi}$.

The fact that the last possibility is ruled out by Assumption 6.1 is essential in the proof of Theorem 6.6.

We now prove the existence of a function w using the curves of $\bar A_{\bar\pi}$.

Proposition 6.4. *There exists a Borel function $\breve{w}$ such that for all sub-curves γ of $\bar{A}_{\bar{\pi}}$ it holds*

$$\breve{w}(\gamma(1)) = \breve{w}(\gamma(0)) e^{i \int_0^1 \Lambda_\gamma(t) |\dot{\gamma}|(t) dt}.$$

Proof. Let $\bar{S}$ be a Borel section of $\bar{E}$, and let $\Gamma_{\bar{A}_{\bar{\pi}},\bar{S}}$ be the family of subcurves of $\bar{A}_{\bar{\pi}}$ passing through $\bar{S}$. We will assume that $\bar{S}$ has a point in each equivalence class, by removing a μ-negligible set of equivalence classes.

For every curve $\gamma \in \Gamma_{\bar{A}_{\bar{\pi}},\bar{S}}$ let

$$s(\gamma) = \max \big\{ t : \gamma(t) \in \bar{S} \big\}.$$

Notice that by the assumption that $\bar{A}_{\bar{\pi}}$ generates $\bar{E}$, every point $x \in X$ is connected to the unique point $\bar{S} \cap Ex$ by a subcurve $\gamma \in \Gamma_{\bar{A}_{\bar{\pi}},\bar{S}}$. Being $\{(t,\gamma), \gamma(t) \in \bar{S}\}$ a Borel set, it follows that s is Borel.

Define the function

$$\breve{w} : [0,1] \times \Gamma_{\bar{A}_{\bar{\pi}},\bar{S}} \to L^\infty((0,1)), \quad \breve{w}(t,\gamma) = e^{i \int_{s(\gamma)}^t \Lambda(\gamma)(\tau) |\dot{\gamma}|(\tau) d\tau}. \tag{6.3}$$

It is easy to see that this function is Borel, being $s(\gamma)$ and Λ Borel. Moreover, from the integer circuitation assumption on $\bar{A}_{\bar{\pi}}$, if $\gamma(t) = \gamma'(t')$, $\gamma, \gamma' \in \Gamma_{\bar{A}_{\bar{\pi}},\bar{S}}$, then

$$\frac{\breve{w}(\gamma'(t'))}{\breve{w}(\gamma(t))} = e^{i \big(\int_{s(\gamma')}^{t'} \Lambda(\gamma')(\tau) |\dot{\gamma}|(\tau) d\tau + \int_t^{s(\gamma)} \Lambda(\gamma)(\tau) |\dot{\gamma}|(\tau) d\tau \big)} = 1,$$

because the curve

$$\gamma(\tau) = \begin{cases} R_{s(\gamma'),t'}(\gamma')(2\tau) & \tau \in [0,1/2], \\ R_{t,s(\gamma)}(\gamma)(2(1-\tau)) & \tau \in (1/2,1], \end{cases}$$

is a sub-loop of $\bar{A}_{\bar{\pi}}$. This shows that $\breve{w}(t,\gamma)$ is the graph of a Borel function $\breve{w}$ defined on X.

Finally, let $\gamma \in \Gamma_{\bar{A}_{\bar{\pi}}}$ be a subcurve of $\bar{A}_{\bar{\pi}}$. Let $\gamma_0, \gamma_1 \in \Gamma_{\bar{A}_{\bar{\pi}},\bar{S}}$ be subcurves connecting $\gamma(0), \gamma(1)$ to a point $\bar{S}$, respectively: such curves exists because γ takes values in a single equivalence class, and each point in an equivalence class is connected to $\bar{S}$ by a subcurve. The subloop made by these 3 curves has integer circuitation, and then

$$\begin{aligned} \breve{w}(\gamma(1)) &= \breve{w}(\gamma(0)) e^{-i \int_1^{-s(\gamma_0)} \Lambda(\gamma_0) |\dot{\gamma}_0| \mathcal{L}^1 + \int_{s(\gamma_1)}^1 \Lambda(\gamma_1) |\dot{\gamma}_1| \mathcal{L}^1} \\ &= \breve{w}(\gamma(0)) e^{i \int_0^1 \Lambda(\gamma) |\dot{\gamma}| \mathcal{L}^1} e^{-i \int_0^1 \Lambda(\gamma) |\dot{\gamma}| \mathcal{L}^1 - i \int_1^{-s(\gamma_0)} \Lambda(\gamma_0) |\dot{\gamma}_0| \mathcal{L}^1 + \int_{s(\gamma_1)}^1 \Lambda(\gamma_1) |\dot{\gamma}_1| \mathcal{L}^1} \\ &= \breve{w}(\gamma(0)) e^{i \int_0^1 \Lambda(\gamma) |\dot{\gamma}| \mathcal{L}^1}. \end{aligned} \tag{6.4}$$

where we have used that the curve obtained by joining $\gamma, \gamma_0, \gamma_1$ is a subloop of $\bar{A}_{\bar{\pi}}$. □

The next proposition shows the importance that $\mathcal{M}^{\bar{\pi}}$ is the maximal measure algebra generated by $\bar{\pi}$.

Proposition 6.5. *If w satisfies*

$$\frac{d}{dt} w(\gamma(t)) = 0, \quad \pi\text{-a.e. } \gamma, \tag{6.5}$$

for every admissible plan π, then up to a μ-negligible set w is constant in each equivalence class.

Observe that in this prof we do not need the assumption on the strong consistency of the disintegration, Assumption 6.1: this is due to the fact that the function w is μ-measurable.

Proof. Consider the admissible plan $\bar{\pi}$ generating the minimal equivalence relation $\bar{E}$.

Assumption (6.5) implies that $t \mapsto w(\gamma(t))$ is equal to a constant $h(\gamma)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$ for $\bar{\pi}$-a.e. γ. Define the set

$$W = \Big\{ (\gamma,\gamma') : \gamma([0,1]) \cap \gamma'([0,1]) \neq \emptyset, h(\gamma) \neq h(\gamma') \Big\} \subset \Gamma \times \Gamma.$$

The same analysis of Step 1 of Theorem 3.13 implies that there is a $\bar{\pi}$-negligible set $N \subset \Gamma$ such that

$$\forall \gamma, \gamma' \in \Gamma \setminus N \Big(\gamma([0,1]) \cap \gamma'([0,1]) \ \Rightarrow \ h(\gamma) = h(\gamma') \Big).$$

Indeed, for every measure $0 < \xi \in \Pi^{\leq}(\bar{\pi},\bar{\pi})$ concentrated on W the admissible plan obtained by (3.11) would not satisfy the assumption of the proposition.

Hence, $t \mapsto w(\gamma(t))$ is constant in each subcurve γ of $\Gamma \setminus N$ for $\mathcal{L}^1$-a.e. t. In particular, it is constant in each equivalence class of the equivalence relation $E^{\bar{A}_{\bar{\pi}} \setminus N}$ generated by $\bar{A}_{\bar{\pi}} \setminus N$.

This equivalence relation $E^{\bar{A}_{\bar{\pi}} \setminus N}$ is smaller than $\bar{E}$, but by the maximality of $\mathcal{M}^{\bar{E}}$ it generates the same measure algebra $\mathcal{M}^{\bar{E}} = \mathcal{M}^{\bar{\pi}}$. If w is not constant in each equivalence class of $\bar{E}$ up to a μ-negligible set $N' \subset X$, then it follows that taking f as in (6.2) for the equivalence relation $\bar{E}$ and defining

$$f'(x) = \big(f(x), w(x)\big),$$

we obtain a Borel map such that its equivalence relation $\bar{E}' = (f', f')^{-1}(\mathrm{id})$ is smaller than $\bar{E}$ and the measure algebra $\mathcal{M}^{\bar{E}'}$ strictly contains $\mathcal{M}^{\bar{\pi}}$, which contradicts the maximality of the latter. $\square$

The last result gives the general formula for every solution to the differential equation $Dw = iwv$ or more precisely

$$\frac{d}{dt} w \circ \gamma = i w(\gamma) \Lambda_\gamma(t) |\dot{\gamma}|(t),$$

rewritten in (6.6) below.

Theorem 6.6. *Given a function $\omega \in L^\infty(\zeta, \mathbb{S}^1)$ (ζ given by (6.1)), there exists a function $w \in L^\infty(\mu, \mathbb{S}^1)$ solving the equation for every admissible plan π*

$$w(\gamma(t)) = w(\gamma(s)) e^{i \int_s^t \Lambda_\gamma |\dot{\gamma}| \mathcal{L}^1}, \quad \pi\text{-a.e. } \gamma,\ s,t \in [0,1] \setminus N_\gamma, \mathcal{L}^1(N_\gamma) = 0, \tag{6.6}$$

and such that

$$w(x) = \omega(y) \breve{w}(x) \quad \text{for } \zeta\text{-a.e. } y \in \bar{S} \cap \bar{E}x,$$

where $\breve{w}$ is the function constructed in Proposition 6.4.

Viceversa, if w is a solution to the equation (6.6) above, then the ratio $w/\breve{w}$ is constant on each equivalence class $\bar{E}_y$ for ζ-a.e. y and defines a function $\omega \in L^\infty(\zeta, \mathbb{S}^1)$ on the quotient space $X/\bar{E}$.

In particular the map

$$L^\infty(\zeta, \mathbb{S}^1) \ni \omega(y) \longleftrightarrow \omega(\bar{E}x) \breve{w}(x) \in L^\infty(\mu, \mathbb{S}^1)$$

is bijective on the set of solutions.

Proof. Given a function $\omega \in L^\infty(\zeta, \mathbb{S}^1)$, the following extension of formula (6.3) constructs the candidate function for the first part of the statement:

$$w(x; \omega) = \omega(\bar{E}x) e^{i \int_{s(\gamma)}^1 \Lambda(\gamma) |\dot{\gamma}| \mathcal{L}^1} = \omega(Ex) \breve{w}, \tag{6.7}$$

where $\gamma \in \Gamma_{\bar{A}_{\bar{\pi}}, \bar{S}}$ is a subcurve of $\bar{A}_{\bar{\pi}}$ connecting y to x.

We need to prove that the integral formula (6.6) holds for all π. Let π be an admissible plan: by removing a negligible set of curves, we can assume that $(\pi + \bar{\pi})/2$ is concentrated on a set of integer circuitation curves, together with its subloops (Corollary 3.14). Being the equivalence relation $\bar{E}$ maximal in the sense of Proposition 5.9, we can apply Corollary 5.11: the curves γ on which π is concentrated are such that

(1) $\gamma(t)$ belongs to the same equivalence class of $\bar{E}$ up to a $\mathcal{L}^1$-negligible subset of $[0,1]$;
(2) the function $w(\gamma(t); \omega)$ given by formula (6.7) is defined $\mathcal{L}^1$-a.e.

Since the disintegration is strongly consistent, for every γ as above there is a subcurve curve γ' of the set where $\bar{\pi}$ is concentrated connecting $\gamma(s)$ to $\gamma(t)$ (by Point (1) above and Assumption 6.1). Being $\pi + \bar{\pi}$ of integer circuitation, we deduce as in (6.4) that (6.6) holds.

Finally we show that every solution w is computed in this way, i.e. the second part of the statement. By construction, along $\bar{\pi}$-a.e. γ

$$\frac{d}{dt} \frac{w(\gamma(t))}{\breve{w}(\gamma(t))} = \frac{w(\gamma(t))}{\breve{w}(\gamma(t))} \big(\Lambda(\gamma)(t) |\dot{\gamma}|(t) - \Lambda(\gamma)(t) |\dot{\gamma}|(t)\big) = 0,$$

so that we can apply Proposition 6.5. $\square$

7. The $W^{1,2}(X,m,\mathbb{R})$-case

In this section we apply the above analysis to the case $\mu = |f|^2 m/\|f\|_2^2$, where m is now the ambient measure on X and $f \in W^{1,2}(X,m,\mathbb{R}^+)$. We still need some assumptions on the structure of the MMS (X,d,m), as explained in the introduction.

Assumption 7.1. For m-a.e. point $x \in X$ there exist two sequence of numbers r_n, R_n, $r_n \leq R_n$, converging to 0 and an admissible plan $\pi_n \in \mathcal{P}(\Gamma)$ such that for some constant C_x

$$\int_0^1 e(t)_\sharp(|\dot{\gamma}|\pi_n)dt \leq \frac{C_x}{m(B_{r_n}(x))} m\llcorner_{B_{R_n}(x)}, \quad L(\gamma) \leq 2R_n,\ m(B_{R_n}(x)) \leq C_x m(B_{r_n}(x)),$$

and

$$(e(0),e(1))_\sharp \pi_n = \frac{1}{m(B_{r_n}(x))^2} m \times m\llcorner_{B_{r_n}(x)\times B_{r_n}(x)}.$$

Up to the restriction that the curves γ used by π_n are localized inside the ball $B_{R_n}(x)$, the admissible plans π_n are the Democratic Plans of [LV07].

Example 7.2. The above condition holds for example for $[0,1]^d$, $d \in \mathbb{N}$, with $m = \prod_i \mathcal{L}^1\llcorner_{[0,1]}$: indeed one considers the image measure

$$((1-t)x + ty)_\sharp(m \times m)$$

which is equivalent to m by coarea formula, being

$$\sqrt{\sum_i (\det M_i)^2} = \sqrt{((1-t)^2 + t^2)^d} \simeq 1,$$

where M_i are the $d \times d$-minors of the matrix $[(1-t)\mathrm{id}\ t\mathrm{id}]$. In particular, it holds in a Riemann manifold by a slight variation of the above computation.

More generally, it holds in non-branching metric measure space under the Measure Contraction Property $\mathrm{MCP}(K,N)$ [Oht07, Definition 2.1], just by adapting the proof of [LV07, Theorem 3.1]. We briefly recall the definition of $\mathrm{MCP}(K,N)$. Define the function

$$s_K(t) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & K > 0, \\ t & K + 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t) & K < 0, \end{cases}$$

A MMS (X,d,m) satisfies the (K,N)-measure contraction property if for all $x \in X$ and $A \in \mathcal{B}(X)$, $m(A) > 0$, there is a plan $\pi \in \mathcal{P}(\Gamma)$ concentrated on geodesics such that $e(t)_\sharp \pi$ has bounded second order moments,

$$e(0)_\sharp \pi = \delta_x, \quad e(1)_\sharp \pi = \frac{m\llcorner_A}{m(A)},$$

and the density of $e(t)_\sharp \pi$ w.r.t. m satisfies

$$e(t)_\sharp \bigg(t \bigg(\frac{s_K(tL(\gamma)/\sqrt{N-1})}{s_K(L(\gamma)/\sqrt{N-1})} \bigg)^{N-1} \pi(d\gamma) \bigg) \leq \frac{m}{m(A)}.$$

In particular, since

$$s_K(t) \sim t \quad \text{for } t \leq 1,$$

it follows that if $A \subset B_1(x)$ then the density is controlled by

$$e(t)_\sharp \pi \leq \frac{Cm}{t^N m(A)}.$$

Now fixing the ball $B_r(x)$, and considering the plans π_x as in the definition of $\mathrm{MCP}(K,N)$, the non branching condition gives that the plan

$$\pi = \int_{B_r(x)} \pi_x dx$$

is unique [LV07, Example 3.1], it is concentrated on $B_{3r}(x)$ (being concentrated on geodesics), and

$$e(t)_\sharp \pi \leq \frac{C}{t^N m(B_r(x))} m\llcorner_{B3r(x)} \leq \frac{2^N C}{m(B_r(x))} m\llcorner_{B3r(x)}, \quad t \in [1/2,1].$$

By the nonbranching conditions, one deduces that $R_\sharp\pi = \pi$ as in [LV07, Theorem 3.1], where we recall that $R(\gamma)(t) = \gamma(1-t)$, so that the above estimate holds also for $t \in [0,1/2]$. Using again MCP(K,N) one obtains

$$m(B_{rt}(x)) \geq ct^N m(B_r(x)),$$

which gives the last condition of Assumption 7.1.

In the case of branching spaces, the same condition holds if (locally, being all computations done in a small ball)

$$\frac{1}{C} \leq \frac{m(B_r(x))}{r^N} \leq C \tag{7.1}$$

and for every $x \in X$, $0 < r < 1$, there is a transport plan $\pi_{x,r} \in \mathrm{Geo}(X)$ concentrated on geodesics such that

$$e(0)_\sharp\pi_{x,r} = \delta_x,\ e(1)_\sharp\pi_{x,r} = \frac{m\llcorner_{B_r(x)}}{m(B_r(x))}, \quad e(t)_\sharp\pi_{x,r} \leq \frac{C}{t^N}\frac{m\llcorner_{B_{rt}(x)}}{m(B_r(x))}. \tag{7.2}$$

In particular, it holds for branching MCP(K,N)-spaces with the Hausdorff measure $\mathcal{H}^N$ as ambient measure m [Raj].

To prove this fact, consider any ball $B_{\bar r}(\bar x)$, $0 < \bar r < 1/2$, and define

$$\bar\pi = \fint_{B_{\bar r}(\bar x)} \tilde\pi_{y,\bar x,\bar r} m(dy), \quad \tilde\pi_{y,\bar x,\bar r} = \frac{m(B_{2\bar r}(y))}{m(B_{\bar r}(\bar x))}\pi_{y,2\bar r}\llcorner_{\{\gamma(1)\in B_{\bar r}(\bar x)\}}.$$

Being $\pi_{x,\bar r} \in \mathcal{P}(\mathrm{Geo}(X))$, i.e. it is concentrated on the set of geodesic, it follows that $L(\gamma) \leq 2\bar r$, and thus $e(t)_\sharp\pi_{y,\bar x,\bar r}$ is concentrated on $B_{2\bar r}(\bar x)$ for $y \in B_{\bar r}(\bar x)$. The measure contraction property (the last estimate of (7.2)) gives that $m(B_{2\bar r}(\bar x)) \leq C2^N m(B_{\bar r}(\bar x))$. Moreover by construction

$$(e(0),e(1))_\sharp m = \frac{1}{m(B_{\bar r}(\bar x))^2} m \times m\llcorner_{B_{\bar r}(\bar x)\times B_{\bar r}(\bar x)}.$$

Finally, consider a ball $B_r(x)$ and compute

$$e(t)_\sharp\bar\pi(B_r(x)) = \fint e(t)_\sharp\pi_{y,\bar x,\bar r}(B_r(x))m(dy) \leq \min\left\{1, \frac{Cm(B_r(x))}{t^N m(B_{\bar r}(\bar x))}\right\}\frac{m(B_{r+2t\bar r}(x))}{m(B_{\bar r}(\bar x))},$$

where we have used the last estimate in (7.2) and the simple observation

$$e(t)_\sharp\bar\pi_{y,\bar x,\bar r}(B_r(x)) \leq \begin{cases} 1 & y \in B_{r+2t\bar r}(x), \\ 0 & \text{otherwise.}\end{cases}$$

Using again the measure contraction property one obtains

$$m(B_{r+t\bar r}(x)) \leq C\left(1 + \frac{t\bar r}{r}\right)^N m(B_r(x)),$$

and then for $t \geq r/\bar r$ and using (7.1)

$$\begin{aligned}\frac{Cm(B_r(x))}{t^N m(B_{\bar r}(\bar x))}\frac{m(B_{r+2t\bar r}(x))}{m(B_{\bar r}(\bar x))} &\leq C^2\left(\frac{1}{t} + \frac{2\bar r}{r}\right)^N \frac{m(B_r(x))^2}{m(B_{\bar r}(\bar x))^2} \leq \frac{C^2(3\bar r)^N}{r^N}\frac{m(B_r(x))^2}{m(B_{\bar r}(\bar x))^2} \\ &\leq C^4\frac{m(B_r(x))}{m(B_{\bar r}(\bar x))}.\end{aligned}$$

For $t \leq r/\bar r$ we have $m(B_{r+2t\bar r}(x)) \leq C4^N m(B_r(x))$, and thus we obtain

$$e(t)_\sharp\bar\pi(B_r(x)) \leq \frac{C''}{m(B_{\bar r}(x))}m(B_r(x))$$

for some constant C''. This shows that $e(t)_\sharp\bar\pi \leq C''m/m(B_{\bar r}(\bar x))$.

Lemma 7.3. *There exists an admissible plan π satisfying Assumption 7.1 and such that the equivalence relation $\bar E$ given by (5.6) is made of countably many equivalence classes of positive measure.*

Proof. Take a point x as in Assumption 7.1, fix n and let A_{π_n} be a set of curves where π_n is concentrated: by assumption m-a.e. point $x_1 \in B_{r_n}(x)$ is connected by a curve to m-a.e. point $x_2 \in B_{r_n}(x)$. By the transitivity of the equivalence relation, all these points are in the same equivalence class, i.e. $B_{r_n}(x)$ is in a single equivalence class up to a m-negligible set. It is clear that by removing a negligible set of trajectories we changes the equivalence class only in a m-negligible set.

Let $\{\Omega_n\}_n = \{B_{r_n}(x_n)\}_n$ be the family of balls as above covering m-a.e. $x \in X$. Let $\pi_{nn'}$ be an admissible plan (if it exists) such that

$$e(0)_\sharp \pi_{nn'} \ll m \llcorner_{B_{r_n}(x_n)}, \quad e(1)_\sharp \pi_{nn'} \ll m \llcorner_{B_{r_{n'}}(x_{n'})}.$$

The admissible plan of the statement is

$$\bar\pi = \sum_n c_n \pi_n + \sum_{nn'} c_{nn'} \pi_{nn'},$$

with $c_n, c_{nn'}$ chosen as in (3.12) to have a probability measure. $\square$

We next consider a function $f \in W^{1,2}(X, m, \mathbb{R}^+)$ and define probability measure

$$\mu = \frac{1}{\|f\|_2^2} |f|^2 m.$$

We assume that we are given the cotangent vector field $v \in L^2(T^*X)$ on the space (X, μ), and we denote with $\Lambda_\gamma(t)$ the corresponding function as in (3.6).

Proposition 7.4. *There exists an admissible plan $\bar\pi$ (w.r.t. the measure μ) satisfying Assumption 6.1 such that its equivalence classes are of positive measure.*

Proof. As in the previous proof, it is enough to prove the statement locally by showing that nearby to a Lebesgue point of $|f|, |Df| > 0$ there exists a plan $\bar\pi$ generating an equivalence class of positive measure essentially invariant w.r.t. the choice of the sets $A_{\bar\pi}$.

Let $\bar x \in X$ be a Lebesgue point for $|f|$ and $|Df|$ (being $|Df|$ the upper gradient), i.e. for definiteness

$$⨍_{B_{\bar r}(\bar x)} |f - 1|^2 m, ⨍_{B_{\bar r}(\bar x)} ||Df| - 1|^2 m < \epsilon^2. \tag{7.3}$$

Consider the ball $B_{r_n}(\bar x)$, with $r_n \leq R_n \leq \bar r$, and the admissible plan π_n defined in Assumption 7.1: we have

$$\begin{aligned}
&\int \bigg[\int_0^1 |Df|(\gamma(t)) |\dot\gamma|(t) dt \bigg] \pi_n(d\gamma) = \int L(\gamma) \pi_n(d\gamma) + \int \bigg[\int_0^1 \big[|Df|(\gamma(t)) - 1 \big] |\dot\gamma|(t) dt \bigg] \pi_n(d\gamma) \\
&\bigg[L(\gamma) \leq 2R_n, e(t)_\sharp(|\dot\gamma| \pi_n dt) \leq \frac{C_{\bar x} m}{m(B_{r_n}(\bar x))} \bigg] \quad \leq 2R_n + C_{\bar x} \int_{B_{R_n}(\bar x)} ||Df| - 1| \frac{m}{m(B_{r_n}(\bar x))} \\
&\qquad \leq 2R_n + C_{\bar x} \epsilon \frac{m(B_{R_n}(\bar x))}{m(B_{r_n}(\bar x))} \leq 2R_n + C_{\bar x}^2 \epsilon.
\end{aligned}$$

In particular, using Chebyshev inequality, the π-measure of the set $\mathcal{W}$ of trajectories for which

$$\int_0^1 |Df|(\gamma(t)) |\dot\gamma|(t) dt > \frac{1}{2}$$

is bounded by

$$\pi(\mathcal{W}) = \pi\bigg(\bigg\{ \gamma : \int_0^1 |Df|(\gamma(t)) |\dot\gamma|(t) dt > \frac{1}{2} \bigg\} \bigg) < 4R_n + 2C_{\bar x}^2 \epsilon < \frac{1}{64}$$

for $n \gg 1$.

By the disintegration w.r.t. the map $e(0) : \gamma \mapsto \gamma(0)$,

$$\pi_n = ⨍_{B_{r_n}(\bar x)} \pi_{n,x} m(dx),$$

it follows that

$$\frac{1}{m(B_{r_n}(\bar x))} m\bigg(\bigg\{ x : \pi_{n,x}(\mathcal{W}) \geq \frac{1}{8} \bigg\} \bigg) < \frac{1}{8}.$$

For all other points, there exists a family of geodesics of $\pi_{n,x}$-measure greater than $7/8$ such that

$$\int_0^1 ||Df(\gamma(t))| - 1|^2 |\dot{\gamma}| dt \leq \frac{1}{2}. \tag{7.4}$$

By Assumption 7.1,

$$e(1)_\sharp \pi_{n,x} = \frac{m}{m(B_{r_n}(\bar{x}))},$$

so that the end points of the previous trajectories cover a subset of $B_\delta(\bar{x})$ with m-measure larger than $7m(B_{r_n}(\bar{x}))/8$.

The set of x such that $|f(x) - 1| > 1/4$ has measure smaller than $16\epsilon^2 m(B_{r_n}(\bar{x}))$, so that there is a set of positive m-measure such that $|f(x) - 1| > 1/4$ and the trajectories starting from x verifying (7.4) have end points in a set with measure $> 7m(B_{r_n}(\bar{x}))/8$. In particular the trajectories starting from each of these points must intersect, and since

$$|f(\gamma(t))| \geq |f(\gamma(0))| - \int_0^1 |Df(\gamma(s))||\dot{\gamma}|(s) ds \geq \frac{1}{4},$$

then they live in an equivalence class of positive measure. □

Corollary 7.5. *The disintegration w.r.t. the equivalence classes of $E^{\bar{A}_{\bar{\pi}}}$ for the measure μ is strongly consistent for every $\bar{A}_\pi$, hence that there exists a solution w.*

This corollary, together with Example 7.2, proves the existence part of (1.13) of Theorem 1.1.

Proof. We just observe that the disintegration with equivalence classes of positive measure is certainly strongly consistent, corresponding to a countable partition of the ambient space up to a negligible set. □

Finally, we consider the case $f \in W^{1,2}(X, m, \mathbb{R}^+)$ and $v \in L^2(T^*X)$ cotangent vector field in (X, μ). The above computations can be applied to the equivalence classes of $\mu = f^2 m$, and then by Theorem 6.6 we can construct a function $w \in L^\infty(\mu, \mathbb{S}^1)$ such that

$$w(\gamma(1)) - w(\gamma(0)) = \int_0^1 \Lambda(\gamma)|\dot{\gamma}|\mathcal{L}^1, \quad \pi\text{-a.e. } \gamma, \ \pi \text{ admissible plan of } (X, \mu).$$

Let now π' be an admissible plan for (X, m): we can assume that π is concentrated on a family of curves A_π such that $f \circ \gamma \in W^{1,2}([0, L(\gamma)], \mathcal{L}^2, \mathbb{R})$ for all $\gamma \in A_\pi$. In particular, $f \circ \gamma$ is continuous and for every $n \in \mathbb{N}$ define

$$\big\{|f \circ \gamma| > 2^{-n}\big\} = \bigcup_k \big(t^-_{k,n}(\gamma), t^+_{k,n}(\gamma)\big). \tag{7.5}$$

The next lemma assures that the dependence w.r.t. k, γ of $t^\pm_{k,n}$ is Borel.

Lemma 7.6. *There exists a family of countable maps of Borel regularity*

$$\gamma \mapsto t^-_{k,n}(\gamma), t^+_{k,n}(\gamma)$$

such that (7.5) *holds.*

Proof. Being $(t, \gamma) \mapsto e(t, \gamma) = \gamma(t)$ continuous, by selecting a Borel representative of f we obtain that

$$\tilde{f}(t, \gamma) = \big[|f(\gamma(t))| - 2^{-n}\big]^+$$

is Borel, and by the choice of A_π it is continuous w.r.t. t for every $\gamma \in \Gamma$ by setting $\tilde{f}(t, \gamma) = 0$ if $\gamma \notin A_\pi$. We will prove the statement for the function $\tilde{f}$.

Let $s_\ell \in [0, 1] \cap \mathbb{Q}$ be a countable dense sequence of times, set

$$A_{\pi,k} = \{\gamma \in A_\pi : \tilde{f}(s_k, \gamma) > 0\}$$

and define the function $\tilde{t}^+_k : A_{\pi,k} \to [0, 1]$ as the function whose graph is

$$\big\{\gamma : \tilde{t}^+_k(\gamma) \geq t\big\} = \bigcap_{s_k \leq s_\ell \leq t} A_{\pi,\ell}.$$

In the same way the function $\tilde{t}_k^- : A_{\pi,k} \to [0,1]$ is given by

$$\bigl\{\gamma : \tilde{t}_k^-(\gamma) \leq t\bigr\} = \bigcap_{t \leq s_\ell \leq s_k} A_{\pi,\ell}.$$

Begin $A_{\pi,k}$ Borel, the functions $\tilde{t}_k^\pm$ are Borel with domain $A_{\pi,k}$. It is clear that with this procedure we have

$$\{\tilde{f} > 0\} = \bigcup_\gamma \bigcup_k \bigl(t_k^-(\gamma), t_k^+(\gamma)\bigr).$$

The only step left is to have that every interval of $\{\tilde{f}(\cdot,\gamma) > 0\}$ is counted once. Set $t_{1,n}^\pm(\gamma) = \tilde{t}_1^\pm(\gamma)$, and in general

$$t_{k,n}^\pm(\gamma) = \tilde{t}_k^\pm(\gamma) \quad \text{if } k = \min\bigl\{j : s_j \in (\tilde{t}_k^-(\gamma), \tilde{t}_k^+(\gamma))\bigr\}.$$

Being the last set a Borel set, the functions $t_k^\pm$ are again Borel. □

By defining

$$\gamma_{k,n} = \gamma\llcorner_{(t_{k,n}^-(\gamma), t_{k,n}^+(\gamma))}, \qquad \pi_{k,n} = (\gamma_{k,n})_\sharp \pi,$$

we have that $\pi_{k,n}$ is an admissible plan for the measure $\mu = f^2 m / \|f\|_2^2$, with compressibility constant $2^{2n} C / \|f\|_2^2$, C being the compressibility constant of π. Along $\pi_{k,n}$-a.e. curve, by elementary 1d computations we have that

$$\frac{d}{dt}(fw)(\gamma(t)) = \frac{d}{dt} f(\gamma(t)) w(\gamma(t)) + f(\gamma(t)) \frac{d}{dt} w(\gamma(t)), \tag{7.6}$$

so that, recalling $w \in \mathbb{S}^1$, for $\pi_{n,k}$-a.e. γ we deduce

$$\left|\frac{d}{dt}(fw) \circ \gamma(t)\right| \leq |Df|(\gamma(t)) + |f(\gamma(t))||Dw(\gamma(t))| = |Df|(\gamma(t)) + |f(\gamma(t))||v(\gamma(t))|. \tag{7.7}$$

The last function belongs to $L^2(X,m)$, because $f \in W^{1,2}(X,m,\mathbb{R}^+)$ and $|Dw| \in L^2(X,\mu)$, i.e.

$$\int |Dw|^2 |f|^2 \mu < \infty.$$

By letting $n \to \infty$ and extending $t \mapsto (fw)(\gamma(t))$ as a continuous function, we deduce that (7.6), (7.7) holds for π-a.e. γ, and since trivially $|fw| = |f| \in L^2(X,m)$ we conclude with the following proposition.

Proposition 7.7. *Under the assumptions $f \in W^{1,2}(X,m,\mathbb{R}^+)$, $v \in L^2(X,|f|^2 m,\mathbb{C})$, there exists a solution w such that $fw \in W^{1,2}(X,m,\mathbb{C})$ and*

$$D(fw) = wDf + fDw. \tag{7.8}$$

Moreover any other solution w' satisfies

$$\frac{w(x)}{w'(x)} = \omega(Ex),$$

where E is the equivalence relation constructed in Section 5 in the space (X,μ) with $\mu = f^2 m / \|f\|_2^2$.

In particular there are at most $(\mathbb{S}^1)^{\mathbb{N}}$-different solutions. Together with Proposition 7.4, the above statement completes the proof of Theorem 1.1.

Proof. Because of Theorem 6.6, we are left to prove (7.8): this follows from (7.6) and the observation that every cotangent vector field is uniquely defined by its behavior on admissible plans as in (3.5). □

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