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Pure SU (2) gauge theory partition function and generalized Bessel kernel

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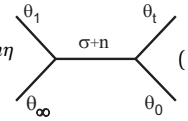
Abstract

We show that the dual partition function of the pure $\mathcal{N}=2$ SU(2) gauge theory in the self-dual Ω-background (a) is given by Fredholm determinant of a generalized Bessel kernel and (b) coincides with the tau function associated to the general solution of the Painlevé III equation of type D_8 (radial sine-Gordon equation). In particular, the principal minor expansion of the Fredholm determinant yields Nekrasov combinatorial sums over pairs of Young diagrams.

1 Introduction

The study of quantitative aspects of the isomonodromy/CFT correspondence [SMJ, K89, M90, T11] has been initiated in the work [GIL12], where the general tau function of the sixth Painlevé equation was conjectured to coincide with the Fourier tranform of the 4-point $c=1$ Virasoro conformal block

$$\tau_{\rm VI}\big(t\,|\,\sigma,\eta,\vec{\theta}\big)=\sum_{n\in\mathbb{Z}}e^{2\pi i n\eta}$$



$$(t). \tag{1.1}$$

This proposal was later proved in [ILTe, BSh1] by CFT methods. The parameters $\vec{\theta}=(\theta_0,\theta_t,\theta_1,\theta_\infty)$ represent local monodromy exponents on the Painlevé side, and are related to external conformal dimensions of primaries in the conformal block by $\Delta_\nu=\theta_\nu^2$. The intermediate dimension is $\Delta=(\sigma+n)^2$.

As is well-known, the AGT correspondence [AGT] relates Virasoro 4-point conformal blocks to partition functions of the $\mathcal{N}=2$ supersymmetric 4D gauge theories with the gauge group SU(2) and $N_f=4$ matter multiplets, regularized by an appropriate deformation (the Ω-background) with two parameters ϵ_1,ϵ_2. The $c=1$ case corresponds to the self-dual Ω-background ($\epsilon_1+\epsilon_2=0$). Expanding conformal blocks around $t=0$ corresponds to the weak coupling expansion in the gauge theory, explicitly computed in [Nek].

The Painlevé VI is the most general equation in the Painlevé family. All the others can be obtained from it by appropriate degeneration limits. In [GIL13], some of these limits have been computed at the level of solutions. This produces explicit formulas for Painlevé V and all three types (D_6, D_7 and D_8) of Painlevé III functions in the form of power series. From the gauge theory point of view, such degenerations correspond to decoupling of the massive fields, which means that Painlevé V and III's are related to $N_f<4$ gauge theories, and explicit formulas for the tau functions are known in their weak coupling regions. On the CFT side, these cases are related to conformal blocks involving Whittaker vectors [G09, BMT, GT]. In contrast to the $N_f=4$ case, there are interesting situations for $N_f<4$ where explicit (asymptotic) series representations of solutions are not known: they correspond to strong coupling regions on the gauge theory side, and to conformal blocks with irregular vertex operators in the CFT framework. The present work is concerned with the most degenerate case of Painlevé III equation of type D_8 corresponding to the pure gauge theory.

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It is interesting to note that an avatar of the Painlevé III (D_8) tau function was already studied by Nekrasov and Okounkov in [NO], although at the time the relevant object had not yet been related to isomonodromy nor to CFT. The equation of interest is of 2nd order and contains no parameters; its tau function is given by

$$\tau_{\rm III}\left(t\,|\,\sigma,\eta\right) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta}\mathcal{Z}_{\rm SU(2)}\left(t\,|\,\sigma+n\right), \tag{1.2}$$

where (σ,η) represent the initial data. The right side of (1.2) was dubbed in [NO] the dual partition function of the pure gauge theory. The first reason to consider it was purely technical: it is convenient to introduce a Lagrange multiplier to control (in the non-ϵ-deformed limit) the value of σ, the vacuum expectation value of the scalar field. A second reason is the existence of a fermionic representation for the dual partition function, presented in [NO] in the special case $\tau_{\rm III}\left(t\,|\,\frac{1}{4},\eta\right)$. Setting in addition $\eta = 0$ or $\eta = \frac{1}{4}$, we obtain elementary solutions of PIII:

$$\tau_{\rm III}\left(t\,\middle|\,\frac{1}{4},\frac{1\pm1}{8}\right) = t^{\frac{1}{16}}e^{\pm4\sqrt{t}}. \tag{1.3}$$

They are related to twisted representations in the intermediate channel [Zam, AZ] generated by the realization of the Virasoro algebra in terms of one Ramond boson [BSh2].

In order to get a physically interesting result, namely the partition function without ϵ-deformation, the dual partition function should be considered in the limit $\eta \to i\infty$. In this case the sum can be computed in a saddle-point approximation[3]. Different quantities scale as follows:

$$\begin{gathered}\eta = \epsilon^{-1}\tilde\eta, \qquad \sigma = \epsilon^{-1}\tilde\sigma, \qquad t = \epsilon^{-4}\tilde t,\\ \mathcal{Z}_{\rm SU(2)}\left(\epsilon^{-4}\tilde t\,|\,\epsilon^{-1}\tilde\sigma\right) \sim \exp\left\{\epsilon^{-2}\mathcal{F}_0(\tilde t\,|\,\tilde\sigma) + \mathcal{F}_1(\tilde t\,|\,\tilde\sigma) + \ldots\right\},\end{gathered} \tag{1.4}$$

which means that the saddle point is defined by the equation $\partial_{\tilde\sigma}\mathcal{F}_0\left(\tilde\sigma\,|\,\tilde t\right) = -4\pi i\tilde\eta$. One of the main results of [NO] is the statement that the Seiberg-Witten prepotential [SW] — the function encoding the low-energy behaviour of the $\mathcal{N}=2$ pure SU (2) gauge theory — coincides with $\mathcal{F}_0\left(\tilde\sigma\,|\,\tilde t\right)$, which confirms the Seiberg-Witten solution at the microscopic level.

A related procedure was used in [BLMST] to identify the Painlevé I–V tau functions also with the dual partition functions of *strongly* coupled gauge theories, including the Argyres-Douglas theories of type H_0, H_1 and H_2. Specifically, it has been checked that the long-distance (irregular type) tau function expansions match various magnetic and dyonic strong coupling expansions on the gauge side. A CFT counterpart of this correspondence has been suggested in [Nag1, Nag2], where some of the long-distance asymptotic series for Painlevé V and IV were conjecturally related to Fourier transforms of conformal blocks with irregular vertex operators.

In a recent paper [GL16], we have developed a method of representing the isomonodromic tau functions of Fuchsian systems as block Fredholm determinants. The construction is based on the Riemann-Hilbert approach. The main input is given by monodromy of a connection $\partial_z - A(z)$ with simple poles together with a pants decomposition of the appropriate punctured Riemann sphere. The relevant integral operators act on vector-valued functions defined on a collection of circles (internal boundary components of pants). Their kernels are expressed in terms of solutions of Fuchsian systems associated to different pairs of pants and having only 3 regular singular points. In rank 2, where the isomonodromy equations are equivalent to the Garnier system containing Painlevé VI as the simplest case, Fredholm determinant representations become completely explicit as the kernels have hypergeometric expressions. Furthermore, the principal minor expansion of the determinant written in the Fourier basis coincides with the combinatorial evaluation [Nek] of the dual partition function of the 4D $\mathcal{N}=2$ linear quiver U (2) gauge theory. This yields in particular a rigorous proof of the series representation of the Painlevé VI tau function, which bypasses the use of the AGT correspondence and does not rely on CFT arguments such as crossing symmetry, null vector decoupling equations, etc.

While it is in principle clear that the approach of [GL16] may be extended to at least some classes of irregular isomonodromic systems, its practical implementation within the Riemann-Hilbert framework is not obvious. Our main goal in this paper is to work out the details for Painlevé III (D_8) equation which exhibits most of the subtleties of the irregular case and at the same time keeps the notational fuss to a minimum. We hope

[3] This is the original proposal from [NO, Eq. (5.5)]. The actual answer for the dual partition function also contains non-perturbative corrections (in ϵ) of crucial importance which we are going to study in a future work.

that the Fredholm determinant representation of $\tau_{\rm III}\left(t\,|\,\sigma,\eta\right)$ obtained here, besides producing a combinatorial series at weak coupling, may also turn out to be useful for the analysis of the strongly coupled regime. Let us mention that a different (?) Fredholm determinant representation for the special tau function $\tau_{\rm III}\left(t\,|\,\sigma,0\right)$ has recently appeared in the proof [BGT1] of a 4D version of the conjecture of [GHM] relating topological strings and spectral theory (see also [BGT2] for higher-rank generalizations). Our results could also provide some insight in this context.

A useful guideline for our work is provided by the geometric Painlevé confluence diagram proposed in [CM, CMR]. In this picture, the monodromy manifolds of different Painlevé equations are interpreted as moduli spaces of Riemann spheres with cusped boundaries. One is then tempted to replace the usual decomposition of the Painlevé VI four-holed sphere into two pairs of pants by cutting the Painlevé III (D_8) decorated cylinder into two, each of them having one regular and one 1-cusped puncture, see Fig. 1.

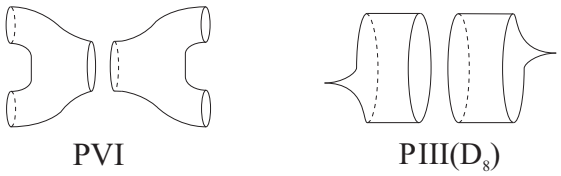


Figure 1: Pants decomposition for Painlevé VI and Painlevé III (D_8).

Furthermore, the number of cusps at a particular hole was heuristically related [CMR, Appendix A] to the number of Stokes rays at the corresponding irregular singular point, and to the pole order of the quadratic differential $\det A(z)\,dz^2$. As we will see, the former interpretation turns out to be the most adapted to our purposes, cf e.g. the Riemann-Hilbert contour in Fig. 3.

The paper is organized as follows. In Section 2, we introduce an irregular linear system leading to Painlevé III (D_8), describe its generalized monodromy, and explain the "decorated pants decomposition" of the associated Riemann-Hilbert problem. In Section 3, it is shown that the PIII (D_8) tau function admits a Fredholm determinant representation with a generalized Bessel kernel, the main result being Theorem 3.2. Section 4 is devoted to derivation of the series over pairs of Maya/Young diagrams and its identification with the dual partition function of the pure gauge theory (Theorems 4.1 and 4.4).

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2 Isomonodromy and Riemann-Hilbert setup

2.1 Associated irregular system

Our starting point is a system of linear differential equations

$$\partial_z Y = A(z)\,Y, \tag{2.1a}$$

where $A(z)$ is a given $N\times N$ matrix with rational dependence on z. The fundamental matrix solution $Y(z)$ in general has branched singularities at the poles of the 1-form $A(z)\,dz$ on the Riemann sphere $\mathbb{P}^1$. It involves no loss of generality to assume that $\mathrm{Tr}\,A(z)=0$; otherwise it suffices to transform $Y\mapsto fY$ with a suitably adjusted scalar factor.

We are going to study a special class of such linear systems in rank $N=2$ characterized by the number of singularities and their type. Specifically, assume that there are only two irregular singular points (e.g. 0 and ∞) of Poincaré rank $\frac{1}{2}$. By this we mean that

$$A(z)=A_{-2}z^{-2}+A_{-1}z^{-1}+A_0,\qquad A_k\in \mathrm{Mat}_{2\times 2}(\mathbb{C}),\quad \mathrm{Tr}\,A_k=0,$$

with non-diagonalizable A_0 and A_{-2}. Using constant gauge transformations $Y \mapsto GY$, $A(z) \mapsto GA(z)G^{-1}$ and rescaling $z \mapsto \lambda z$ if necessary, it may be further assumed that either (i) $A_0 = \sigma_+$, $A_{-2} = \sigma_-$ or (ii) $A_0 = A_{-2} = \sigma_+$, where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the case (ii), the remaining freedom of conjugation by upper triangular matrices with unit diagonals leaves only two nontrivial parameters in A_{-1}. The corresponding linear system does not admit isomonodromic deformations and reduces to a special case of doubly-cofluent Heun equation. We will therefore focus on the case (i) and, after suitable rescalings, parameterize $A(z)$ as

$$A(z) = q\sigma_- z^{-2} + q^{-1}\begin{pmatrix} -p & t \\ -q & p \end{pmatrix} z^{-1} - \sigma_+. \tag{2.1b}$$

The system (2.1a) with $A(z)$ given by (2.1b) is the linear problem associated to Painlevé III (D_8) equation. Among 3 parameters p, q and t, the latter plays the role of time in the associated isomonodromic problem, and the former two are coordinates on the PIII (D_8) phase space.

The system (2.1) can be put to a more convenient form using non-constant gauge transformation. Let us define a new matrix $\tilde{Y}(\xi)$ by

$$\tilde{Y}(\xi) = G(\xi)^{-1} Y\left(\xi^2\right), \qquad G(\xi) = \frac{1}{i\sqrt{2}}\begin{pmatrix} \xi^{\frac{1}{2}} & \xi^{\frac{1}{2}} \\ \xi^{-\frac{1}{2}} & -\xi^{-\frac{1}{2}} \end{pmatrix}.$$

It solves the linear system

$$\partial_\xi \tilde{Y} = \tilde{A}(\xi)\,\tilde{Y}, \tag{2.2a}$$

with $\tilde{A}(\xi) = 2\xi G(\xi)^{-1} A\left(\xi^2\right) G(\xi) - G(\xi)^{-1} G'(\xi)$. Computing the latter matrix explicitly, one may see that the system (2.2a) also has irregular singularities at 0 and ∞:

$$\begin{gathered} \tilde{A}(\xi) = \tilde{A}_{-2}\xi^{-2} + \tilde{A}_{-1}\xi^{-1} + \tilde{A}_0, \\ \tilde{A}_{-2} = \left(q + \frac{t}{q}\right)\sigma_z + \left(q - \frac{t}{q}\right) i\sigma_y, \qquad \tilde{A}_{-1} = -\left(\frac{2p}{q} + \frac{1}{2}\right)\sigma_x, \qquad \tilde{A}_0 = -2\sigma_z, \end{gathered} \tag{2.2b}$$

where $\sigma_{x,y,z}$ denote the Pauli matrices. The above is by no means a generic form of 2×2 systems with 2 irregular singular points of Poincaré rank 1; one of the properties that singles out the class described by (2.2b) is a discrete $\mathbb{Z}_2$-symmetry $\tilde{A}(-\xi) = -\sigma_x \tilde{A}(\xi)\sigma_x$.

2.2 Monodromy

The fact that the transformed coefficients $\tilde{A}_{-2}$ and $\tilde{A}_0$ are diagonalisable, in contrast to their counterparts in (2.1b), allows to write formal fundamental solutions of (2.2a) at 0 and ∞ in the standard form,

$$\tilde{Y}^{(0)}_{\text{form}}(\xi) = \left(-\frac{q}{\sqrt{t}}\right)^{-\frac{\sigma_x}{2}} \left[\mathbb{1} + \sum_{k=1}^{\infty} y_k^{(0)} \xi^k\right] e^{2\sigma_z\sqrt{t}\xi^{-1}}, \qquad \xi \to 0, \tag{2.3a}$$

$$\tilde{Y}^{(\infty)}_{\text{form}}(\xi) = \left[\mathbb{1} + \sum_{k=1}^{\infty} y_k^{(\infty)} \xi^{-k}\right] e^{-2\sigma_z\xi}, \qquad \xi \to \infty. \tag{2.3b}$$

The $\mathbb{Z}_2$-symmetry of $\tilde{A}(\xi)$ implies that formal solutions satisfy $\tilde{Y}^{(\nu)}_{\text{form}}(-\xi) = \sigma_x \tilde{Y}^{(\nu)}_{\text{form}}(\xi)\sigma_x$. The expansion coefficients $y_k^{(\nu)}$ can be computed in a straightforwad way to any finite order using (2.2). In what follows, the only explicit expression we need concerns the first such coefficient in (2.3a), namely,

$$y_1^{(0)} = -\frac{1}{\sqrt{t}}\left[\left(\frac{p^2}{q^2} + \frac{p}{2q} - q - \frac{t}{q} + \frac{1}{16}\right)\sigma_z + \left(\frac{p}{2q} + \frac{1}{8}\right) i\sigma_y\right]. \tag{2.4}$$

The actual solutions of the unfolded system (2.2) can only be asymptotic to $\tilde{Y}^{(\nu)}_{\text{form}}(\xi)$ inside the Stokes sectors $\tilde{\mathscr{S}}^{(\nu)}_k$ ($k=1,2,3$) defined by

$$\tilde{\mathscr{S}}^{(0)}_k = \left\{\xi\in\mathbb{C}\,\middle|\,\frac{\arg t-3\pi}{2}+k\pi<\arg\xi<\frac{\arg t+\pi}{2}+k\pi,\ |\xi|<R\right\},$$
$$\tilde{\mathscr{S}}^{(\infty)}_k = \left\{\xi\in\mathbb{C}\,\middle|\,-\frac{3\pi}{2}+k\pi<\arg\xi<\frac{\pi}{2}+k\pi,\ |\xi|>R\right\}.$$

Furthermore, the requirement that $\tilde{Y}^{(\nu)}_k \simeq \tilde{Y}^{(\nu)}_{\text{form}}$ inside $\mathscr{S}^{(\nu)}_k$ as $\xi\to\nu$ fixes the solutions $\tilde{Y}^{(\nu)}_k$ uniquely. As is well-known, such canonical solutions associated to the same point are related by constant (i.e. independent of ξ) Stokes matrices

$$S^{(\nu)}_k = \tilde{Y}^{(\nu)}_k(\xi)^{-1}\tilde{Y}^{(\nu)}_{k+1}(\xi), \qquad \nu=0,\infty, \qquad k=1,2. \tag{2.5}$$

Constant connection matrix $E=\tilde{Y}^{(0)}_1(\xi)^{-1}\tilde{Y}^{(\infty)}_1(\xi)$ relates the canonical solutions at 0 and ∞. Stokes and connection matrices describe global asymptotic behavior of solutions of (2.1), (2.2) and constitute the relevant set of generalized monodromy data.

The discrete symmetry mentioned above and interlacing structure of dominant/recessive solutions imply that Stokes matrices can be written as

$$S^{(0)}_1=\sigma_x S^{(0)}_2\sigma_x=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}, \qquad S^{(\infty)}_1=\sigma_x S^{(\infty)}_2\sigma_x=\begin{pmatrix}1&0\\\beta&1\end{pmatrix}. \tag{2.6a}$$

Similar constraints are also valid for the connection matrix,

$$\sigma_x E\sigma_x = {S^{(0)}_1}^{-1} E\, S^{(\infty)}_1 = S^{(0)}_2 E\, {S^{(\infty)}_2}^{-1}. \tag{2.6b}$$

The relations (2.6) imply that, in general, the monodromy data $\{S^{(\nu)}_k\}$, E can be parameterized by a pair of complex parameters (σ,η) in the following way:

$$\begin{aligned} E &= \frac{1}{\sin 2\pi\sigma}\begin{pmatrix}\sin 2\pi\eta & -i\sin 2\pi(\eta+\sigma)\\ i\sin 2\pi(\eta-\sigma) & \sin 2\pi\eta\end{pmatrix}, \qquad \sigma\notin\mathbb{Z}/2,\\ S^{(0)}_1 &= {S^{(0)}_2}^T = {S^{(\infty)}_1}^T = S^{(\infty)}_2 \equiv S=\begin{pmatrix}1&-2i\cos 2\pi\sigma\\0&1\end{pmatrix}. \end{aligned} \tag{2.7}$$

It can be furthermore assumed that σ and η belong to the strips $-\frac{1}{2}\le\Re\sigma\le 0$ and $-\frac{1}{2}<\Re\eta\le\frac{1}{2}$. Note that the counterclockwise monodromy matrix $\tilde{M}_0$ of $\tilde{Y}^{(0)}_1(z)$ around 0 can be expressed as

$$\tilde{M}_0^{-1} = SS^T = ES^TSE^{-1}. \tag{2.8}$$

Let us finally comment on how to recover monodromy of the initial system (2.1) from the Stokes data of the unfolded equation (2.2). Introduce the solutions $Y^{(\nu)}(z)=G(\sqrt{z})\,\tilde{Y}^{(\nu)}_1(\sqrt{z})$, uniquely defined by their asymptotic behavior

$$Y^{(\nu)}(z)\simeq G(\sqrt{z})\,\tilde{Y}^{(\nu)}_{\text{form}}(\sqrt{z}),$$

as $z\to\nu$ inside the sectors $\arg t-\pi<\arg z<\arg t+3\pi$ (for $\nu=0$) and $-\pi<\arg z<3\pi$ (for $\nu=\infty$). The monodromy matrix in $Y^{(0)}(ze^{2\pi i})=Y^{(0)}(z)\,M_0$ can be computed using the Stokes matrix connecting unfolded solutions $\tilde{Y}^{(0)}_{1,2}(\xi)$ together with the symmetry properties $G(\xi e^{i\pi})=iG(\xi)\sigma_x$ and $\sigma_x\tilde{Y}^{(0)}_{\text{form}}(-\xi)\sigma_x=\tilde{Y}^{(0)}_{\text{form}}(\xi)$. The result reads

$$M_0 = i\sigma_x S^{-1}=\begin{pmatrix}0&i\\i&-2\cos 2\pi\sigma\end{pmatrix}=U^{-1}e^{2\pi i\mathfrak{S}}U, \tag{2.9a}$$

$$\mathfrak{S}=\left(\sigma+\frac{1}{2}\right)\sigma_z, \qquad U=\frac{1}{\sqrt{2\sin 2\pi\sigma}}\begin{pmatrix}e^{-i\pi(\sigma+\frac{1}{4})}&e^{i\pi(\sigma+\frac{1}{4})}\\e^{i\pi(\sigma+\frac{1}{4})}&-e^{-i\pi(\sigma+\frac{1}{4})}\end{pmatrix}. \tag{2.9b}$$

As expected, the monodromy matrices of $Y^{(0)}$ and $\tilde{Y}^{(0)}_1$ are related by $\tilde{M}_0=-M_0^2$. In the same way, the monodromy of $Y^{(\infty)}(z)$ around 0 is given by $E^{-1}M_0E=\sigma_xM_0\sigma_x$.

2.3 Deformation equations and tau function

The usual construction of isomonodromic family of systems (2.2) involves varying the "time" parameter t appearing in the exponentials in (2.3), while keeping the data $\{S_k^{(\nu)}\}$, E fixed. The latter requirement implies that the matrix $\partial_t \tilde{Y} \cdot \tilde{Y}^{-1}$ is meromorphic on $\mathbb{P}^1$ with poles only possible at 0 and ∞. Analyzing the local behavior of this quantity with the help of expansions of formal solutions and recasting the result in terms of $Y(z)$, one finds that

$$\partial_t Y = B(z)\, Y, \qquad B(z) = \begin{pmatrix} 0 & -q^{-1} \\ -\frac{q}{tz} & 0 \end{pmatrix}.$$

The compatibility of this isomonodromy constraint with the system (2.1) yields the zero-curvature condition $\partial_t A - \partial_z B + [A, B] = 0$, which is equivalent to a pair of scalar equations

$$\begin{cases} t q_t = 2p + q, \\ t p_t = \dfrac{2p^2}{q} + p + q^2 - t, \end{cases} \tag{2.10}$$

or, equivalently, to a single 2nd order ODE:

$$q_{tt} = \frac{q_t^2}{q} - \frac{q_t}{t} + \frac{2q^2}{t^2} - \frac{2}{t}. \tag{2.11}$$

This is the most degenerate Painlevé III equation (of type D_8). In applications, it usually appears in the form of the radial sine-Gordon equation

$$u_{rr} + \frac{u_r}{r} + \sin u = 0, \tag{2.12}$$

which is obtained from (2.11) after the change of variables $q\left(2^{-12} r^4\right) = -2^{-6} r^2 e^{iu(r)}$. The isomonodromic provenance of these equations implies that the quantities (σ, η) introduced above to parameterize the Stokes data provide a pair of conserved quantities for (2.11) and (2.12).

Let us define the tau function $\tau_{\mathrm{III}}(t)$ of PIII (D_8) by the logarithmic derivative

$$\zeta(t) = t\partial_t \ln \tau_{\mathrm{III}} = \frac{\left(t q_t - q\right)^2}{4q^2} - q - \frac{t}{q}. \tag{2.13}$$

Conversely, one can express $q = -t\zeta'$. The function $\zeta(t)$ essentially coincides with the time-dependent Hamiltonian of PIII (D_8) and satisfies the equation

$$\left(t\zeta''\right)^2 = 4\left(\zeta'\right)^2\left(\zeta - t\zeta'\right) - 4\zeta'. \tag{2.14}$$

The tau function plays a crucial role in the rest of this note. We are going to express it in terms of monodromy, thereby providing explicit formulas for the general solution of Painlevé III (D_8).

2.4 Riemann-Hilbert problem

It is convenient to replace the linear system (2.2) by an equivalent Riemann-Hilbert problem (RHP). It will be defined by a pair (Γ, J) where Γ is an oriented contour on $\mathbb{P}^1$ and $J : \Gamma \to \mathrm{SL}(2, \mathbb{C})$ is a jump matrix. The relevant contour $\Gamma = \ell^{[0]} \cup \ell^{[\infty]} \cup C_E$ is represented by solid lines in Fig. 2 where it is assumed for simplicity that $\arg t = 0$. The segments $\ell^{[0]}$ and $\ell^{[\infty]}$ correspond to portions of anti-Stokes rays at 0 and ∞. They are close relatives of the cusps of the Chekhov-Mazzocco-Rubtsov geometric confluence diagram [CM, CMR] in the Riemann-Hilbert setting.

The Riemann sphere is decomposed by Γ into 2 connected open domains $D^{[0]}$ and $D^{[\infty]}$. The relevant RHP is to find a 2×2 matrix $\Psi(z)$ holomorphic and invertible inside each of these domains such that

(i) its boundary values on the positive and negative side of Γ satisfy $\Psi_+ = \Psi_- J$, where the piecewise constant jump matrix is given by

$$J\big|_{\ell^{[0]}} = M_0^{-1}, \qquad J\big|_{\ell^{[\infty]}} = \sigma_x M_0^{-1} \sigma_x, \qquad J\big|_{C_E} = E.$$

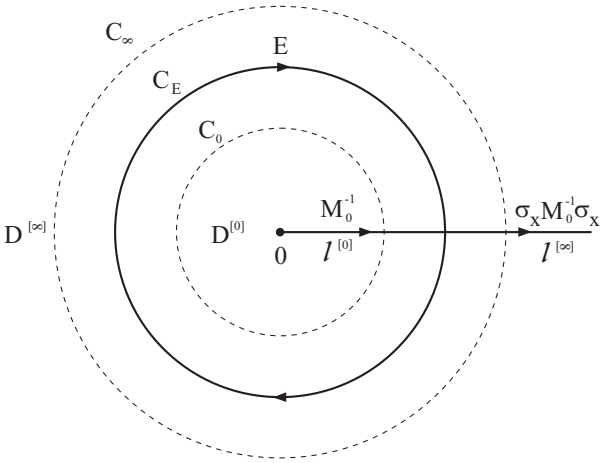


Figure 2: Contour Γ of the PIII (D_8) Riemann-Hilbert problem.

(ii) at 0 and ∞, the function $\Psi(z)$ behaves as

$$\Psi(z) \simeq G\left(\sqrt{z}\right) \begin{cases} Q\left[\mathbb{1} + O\left(\sqrt{z}\right)\right] e^{2\sigma_z \sqrt{t/z}}, & z \to 0, \\ \left[\mathbb{1} + O\left(\frac{1}{\sqrt{z}}\right)\right] e^{-2\sigma_z \sqrt{z}}, & z \to \infty, \end{cases}$$

where $\arg z \in]-\pi, \pi[$ and Q is a constant invertible matrix such that $[Q, \sigma_x] = 0$, cf (2.3).

The unique solution of this RHP is related to the fundamental solution of the irregular system (2.1) by

$$Y^{(0)}(z) = \begin{cases} \Psi(z), & z \in D^{[0]}, \\ \Psi(z) E^{-1}, & z \in D^{[\infty]}. \end{cases}$$

Let $\mathscr{A}$ be the open annulus bounded by two circles C_0 and C_∞ as shown in Fig. 2. Consider a piecewise analytic function $\hat{\Psi}(z)$ defined by

$$\hat{\Psi}(z) = \begin{cases} \Psi(z), & z \notin \bar{\mathscr{A}}, \\ Y^{(0)}(z) U^{-1} z^{-\mathfrak{S}}, & z \in \mathscr{A}, \end{cases} \tag{2.15}$$

where $\mathfrak{S} = \left(\sigma + \frac{1}{2}\right)\sigma_z$ and U are defined by (2.9b). This new function solves a Riemann-Hilbert problem defined by the pair $\left(\hat{\Gamma}, \hat{J}\right)$, where the contour $\hat{\Gamma}$ and the relevant jump matrices are represented in Fig. 3. The transformed RHP is of course equivalent to the initial one. The function $\hat{\Psi}(z)$ has been designed so that it has no jumps inside $\mathscr{A}$ and coincides with $\Psi(z)$ inside C_0 and outside C_∞. Cancellation of the jumps inside $\mathscr{A}$ can only be done at the expense of introducing new jumps on the circles C_0 and C_∞; as we will see in a moment, their choice above models regular singularities at ∞ and 0, respectively.

There is a natural decomposition $\hat{\Gamma} = \hat{\Gamma}^{[0]} \cup \hat{\Gamma}^{[\infty]}$, where $\hat{\Gamma}^{[0]}$ (and $\hat{\Gamma}^{[\infty]}$) consist of C_0 (resp. C_∞) and the part of the positive real axis contained inside C_0 (resp. outside C_∞). Denoting

$$\hat{J}^{[0]} = \hat{J}\Big|_{\hat{\Gamma}^{[0]}}, \qquad \hat{J}^{[\infty]} = \hat{J}\Big|_{\hat{\Gamma}^{[\infty]}},$$

we can assign to the original RHP two simpler RHPs for functions $\hat{\Psi}^{[0]}(z)$ and $\hat{\Psi}^{[\infty]}(z)$ defined by the pairs $\left(\hat{\Gamma}^{[0]}, \hat{J}^{[0]}\right)$ and $\left(\hat{\Gamma}^{[\infty]}, \hat{J}^{[\infty]}\right)$. The latter correspond to two rank 2 Fuchsian systems having one regular singular point and one irregular singular point of Poincaré rank $\frac{1}{2}$ which can be expicitly solved in terms of Bessel functions.

Let us also remark that (i) only $\hat{\Psi}^{[0]}(z)$ depends on PIII (D_8) independent variable t (via the asymptotic condition at $z = 0$); (ii) the initial RHP may also be rewritten as a RHP on a single circle inside $\mathscr{A}$ with the jump $\hat{\Psi}^{[0]}(z)\,\hat{\Psi}^{[\infty]}(z)^{-1}$. The study of an equivalent RHP on a circle is the main tool used in [Nil] for the asymptotic analysis of PIII (D_8).

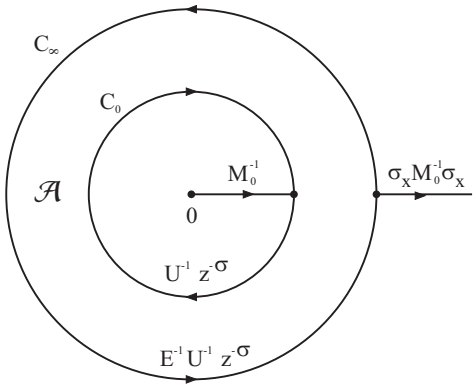


Figure 3: Contour $\hat{\Gamma}$ and associated jumps.

2.5 Building block solutions

Consider a model differential system

$$\partial_z Y^{[\infty]} = A^{[\infty]}(z)\, Y^{[\infty]}, \qquad A^{[\infty]}(z) = -\sigma_+ + \begin{pmatrix} \sigma+\frac{1}{2} & 0 \\ -1 & -\sigma-\frac{1}{2} \end{pmatrix} z^{-1}. \tag{2.16}$$

Such an ansatz is inspired by the following: we would like to have an irregular singularity of Poincaré rank $\frac{1}{2}$ at $z=\infty$ and a regular singularity at $z=0$ with local monodromy exponents given by $\pm\left(\sigma+\frac{1}{2}\right)$, cf (2.9a).

Choose the fundamental matrix solution of (2.16) as

$$Y_\infty^{[\infty]}(z) = \frac{1}{i\sqrt{2\pi}} \begin{pmatrix} 2\sqrt{z}\,K_{2\sigma}\left(2\sqrt{z}\right) & 2\pi\sqrt{z}\,I_{2\sigma}\left(2\sqrt{z}\right) \\ 2K_{2\sigma+1}\left(2\sqrt{z}\right) & -2\pi I_{2\sigma+1}\left(2\sqrt{z}\right) \end{pmatrix} \begin{pmatrix} 1 & -ie^{2\pi i\sigma} \\ 0 & 1 \end{pmatrix}, \tag{2.17}$$

where $I_\nu(x)$, $K_\nu(x)$ denote the modified Bessel functions of the 1st and 2nd kind. This fundamental solution is defined in the domain $\arg z \in]0, 2\pi[$ where it has the asymptotics

$$Y_\infty^{[\infty]}(z) \simeq G\left(\sqrt{z}\right)\left[\mathbb{1} + \sum_{k=1}^{\infty} y_k^{[\infty]} z^{-\frac{k}{2}}\right] e^{-2\sigma_z\sqrt{z}}, \qquad z\to\infty.$$

In the vicinity of $z=0$, it becomes convenient to rewrite it as

$$Y_\infty^{[\infty]}(z) = Y_0^{[\infty]}(z)\, e^{2\pi i\eta\sigma_z} z^{\Theta} UE, \tag{2.18a}$$

$$Y_0^{[\infty]}(z) = i\sqrt{\frac{\pi}{\sin 2\pi\sigma}} \begin{pmatrix} z^{-\sigma} I_{2\sigma}\left(2\sqrt{z}\right) & -z^{\sigma+1} I_{-2\sigma}\left(2\sqrt{z}\right) \\ -z^{-\sigma-\frac{1}{2}} I_{2\sigma+1}\left(2\sqrt{z}\right) & z^{\sigma+\frac{1}{2}} I_{-2\sigma-1}\left(2\sqrt{z}\right) \end{pmatrix} e^{-i\pi\left(\sigma-\frac{1}{4}\right)\sigma_z}. \tag{2.18b}$$

In fact, we used in (2.17) the same σ as in the parameterization of Stokes data precisely to achieve (2.18a). The matrix function $Y_0^{[\infty]}(z)$ is holomorphic and invertible in the entire complex plane, and normalized as to have unit determinant. Therefore, the solution $\hat{\Psi}^{[\infty]}(z)$ of the exterior auxiliary RHP may be written as

$$\hat{\Psi}^{[\infty]}(z) = \begin{cases} Y_\infty^{[\infty]}(z), & z \text{ outside } C_\infty, \\ Y_0^{[\infty]}(z)\, e^{2\pi i\eta\sigma_z}, & z \text{ inside } C_\infty. \end{cases} \tag{2.19}$$

Similarly, the function

$$Y_0^{[0]}(z) = \frac{1}{i\sqrt{2\pi}} \begin{pmatrix} 2\pi I_{-2\sigma-1}\left(\frac{2}{\sqrt{z}}\right) & 2K_{-2\sigma-1}\left(\frac{2}{\sqrt{z}}\right) \\ \frac{2\pi}{\sqrt{z}} I_{-2\sigma}\left(\frac{2}{\sqrt{z}}\right) & -\frac{2}{\sqrt{z}} K_{-2\sigma}\left(\frac{2}{\sqrt{z}}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ie^{2\pi i\sigma} & 1 \end{pmatrix} \tag{2.20}$$

defines a fundamental matrix solution of the linear system

$$\partial_z Y^{[0]} = A^{[0]}(z)\, Y^{[0]}, \qquad A^{[0]}(z) = -\sigma_- z^{-2} + \begin{pmatrix} \sigma+\frac{1}{2} & -1 \\ 0 & -\sigma-\frac{1}{2} \end{pmatrix} z^{-1}. \tag{2.21}$$

It is characterized by the asymptotic behavior

$$Y_0^{[0]}(z) \simeq G\left(\sqrt{z}\right)\left[\mathbb{1} + \sum_{k=1}^{\infty} y_k^{[0]} z^{\frac{k}{2}}\right] e^{2\sigma_z/\sqrt{z}}, \tag{2.22}$$

as $z \to 0$ inside the sector $\arg z \in]0, 2\pi[$. In the neighborhood of $z = \infty$, this model solution $Y_0^{[0]}(z)$ can be suitably rewritten as

$$Y_0^{[0]}(z) = Y_\infty^{[0]}(z)\, z^{\mathfrak{S}} U, \tag{2.23a}$$

$$Y_\infty^{[0]}(z) = i\sqrt{\frac{\pi}{\sin 2\pi\sigma}} \begin{pmatrix} z^{-\sigma-\frac{1}{2}} I_{-2\sigma-1}\left(\frac{2}{\sqrt{z}}\right) & z^{\sigma+\frac{1}{2}} I_{2\sigma+1}\left(\frac{2}{\sqrt{z}}\right) \\ z^{-\sigma-1} I_{-2\sigma}\left(\frac{2}{\sqrt{z}}\right) & z^{\sigma} I_{2\sigma}\left(\frac{2}{\sqrt{z}}\right) \end{pmatrix} e^{-i\pi(\sigma+\frac{1}{4})\sigma_z}, \tag{2.23b}$$

Taking into account that the matrix ratio $G\left(\sqrt{z}\right)^{-1} G\left(\sqrt{\frac{z}{t}}\right) = t^{-\frac{\sigma_x}{4}}$ is independent of z, the solution $\hat{\Psi}^{[0]}(z)$ of the interior auxiliary RHP may now be expressed as

$$\hat{\Psi}^{[0]}(z) = \begin{cases} Y_0^{[0]}\left(\frac{z}{t}\right), & z \text{ inside } C_0, \\ Y_\infty^{[0]}\left(\frac{z}{t}\right) t^{-\mathfrak{S}}, & z \text{ outside } C_0. \end{cases} \tag{2.24}$$

The parameterization of Stokes data introduced in Subsection 2.2 now becomes more transparent. The variable σ encodes the spectrum of the single nontrivial monodromy matrix M_0 whose eigenvalues are given by $-e^{\pm 2\pi i\sigma}$, cf (2.9a). The 2nd parameter η measures a relative twist of local parametrices $Y_0^{[\infty]}(z)$, $Y_\infty^{[0]}\left(\frac{z}{t}\right)$ in the full solution $\hat{\Psi}(z)$.

3 Fredholm determinant representation

3.1 Boundary spaces

Let $V(C)$ be the space of smooth functions on a circle C which will be sometimes identified with the space of holomorphic functions in an annulus containing C. Also, define the space $H(C) = \mathbb{C}^2 \otimes V(C)$ whose elements will be represented as 2-rows of elements of $V(C)$. The subspaces of $V(C)$ and $H(C)$ that consist of functions with only positive or negative Fourier modes will be denoted by $V_\pm(C)$ and $H_\pm(C)$.

In relation with the previously discussed RHP for the function $\hat{\Psi}(\xi)$, introduce the spaces

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-, \qquad \mathscr{H}_\pm = H_\pm(C_0) \oplus H_\mp(C_\infty). \tag{3.1}$$

Observe that each of the subspaces $\mathscr{H}_\pm$ can be identified in a natural way with the space of vector-valued holomorphic functions on the annulus $\mathscr{A}$. We are now going to consider two operators acting on $\mathscr{H}$ *from the right* and generalizing the usual projections on positive and negative modes.

1. The first operator, to be denoted by $\mathscr{P}_\Sigma$, is defined by

$$\left(f\mathscr{P}_\Sigma\right)(z) = \frac{1}{2\pi i}\int_{C_0\cup C_\infty} \frac{f\left(z'\right)\hat{\Psi}_+\left(z'\right)^{-1}\hat{\Psi}_+(z)\,dz'}{z-z'}, \qquad z \in C_0 \cup C_\infty.$$

   The convention used to interpret the singularity at $z' = z$ is to slightly deform the integration contour so that it goes clockwise around this point.

2. The second operator, $\mathscr{P}_\oplus$, is constructed in a similar way with the help of elementary building block solutions $\hat\Psi^{[0]}(z)$ and $\hat\Psi^{[\infty]}(z)$,

$$\left(f\mathscr{P}_\oplus\right)(z)=\frac{1}{2\pi i}\int_{C_k}\frac{f\left(z'\right)\hat\Psi_+^{[k]}\left(z'\right)^{-1}\hat\Psi_+^{[k]}(z)\,dz'}{z-z'},\qquad z\in C_k,\quad k=0,\infty.$$

The absence of jumps of $\hat\Psi(z)\,\hat\Psi_+^{[0]}(z)^{-1}$ (and $\hat\Psi(z)\,\hat\Psi_+^{[\infty]}(z)^{-1}$) inside C_0 (resp. outside C_∞), and systematic application of residue theorem/collapsing the contours imply the following properties:

- $\mathscr{P}_\Sigma^2=\mathscr{P}_\Sigma$ and $\mathscr{P}_\oplus^2=\mathscr{P}_\oplus$, i.e. the operators $\mathscr{P}_\Sigma$, $\mathscr{P}_\oplus$ are projections.
- $\ker\mathscr{P}_\oplus=\mathscr{H}_-$, $\ker\mathscr{P}_\Sigma\supseteq\mathscr{H}_\mathscr{A}$, where $\mathscr{H}_\mathscr{A}$ is the space of boundary values of functions holomorphic on $\mathscr{A}$.
- $\mathscr{P}_\Sigma\mathscr{P}_\oplus=\mathscr{P}_\Sigma$, $\mathscr{P}_\oplus\mathscr{P}_\Sigma=\mathscr{P}_\oplus$; this means that $\mathscr{P}_\Sigma$ and $\mathscr{P}_\oplus$ have the same range, to be denoted by $\mathscr{H}_\mathscr{T}$.

Loosely speaking, the space $\mathscr{H}_\mathscr{T}$ consists of functions on $C_0\cup C_\infty$ whose continuations outside $\mathscr{A}$ share the global monodromy properties of the fundamental matrix solution of (2.2). The operators $\mathscr{P}_\Sigma$, $\mathscr{P}_\oplus$ project on $\mathscr{H}_\mathscr{T}$ along $\mathscr{H}_\mathscr{A}$ and $\mathscr{H}_-$, respectively, which may be denoted as $\mathscr{P}_\Sigma=\mathscr{H}\xrightarrow{\mathscr{H}_\mathscr{A}}\mathscr{H}_\mathscr{T}$, $\mathscr{P}_\oplus=\mathscr{H}\xrightarrow{\mathscr{H}_-}\mathscr{H}_\mathscr{T}$.

According to the decomposition (3.1), write $f\in\mathscr{H}$ as

$$f=\begin{pmatrix}f_+^{[0]} & f_-^{[\infty]}\end{pmatrix}\oplus\begin{pmatrix}f_-^{[0]} & f_+^{[\infty]}\end{pmatrix}.$$

The action of $\mathscr{P}_\oplus$ is then given by

$$f\mathscr{P}_\oplus=\begin{pmatrix}f_+^{[0]} & f_-^{[\infty]}\end{pmatrix}\oplus\begin{pmatrix}f_+^{[0]}\mathsf{d} & f_-^{[\infty]}\mathsf{a}\end{pmatrix},\tag{3.2}$$

where the matrix integral operators $\mathsf{a}:H_-(C_\infty)\to H_+(C_\infty)$ and $\mathsf{d}:H_+(C_0)\to H_-(C_0)$ are expressed in terms of elementary solutions $\hat\Psi^{[0]}(z)$, $\hat\Psi^{[\infty]}(z)$ and have integrable form:

$$\begin{gathered}\left(f\mathsf{a}\right)(z)=\frac{1}{2\pi i}\oint_{C_\infty}f\left(z'\right)\mathsf{a}\left(z',z\right)dz',\qquad \left(f\mathsf{d}\right)(z)=-\frac{1}{2\pi i}\oint_{C_0}f\left(z'\right)\mathsf{d}\left(z',z\right)dz',\\ \mathsf{a}\left(z',z\right)=\frac{\hat\Psi_+^{[\infty]}\left(z'\right)^{-1}\hat\Psi_+^{[\infty]}(z)-\mathbb{1}}{z-z'},\qquad \mathsf{d}\left(z',z\right)=\frac{\mathbb{1}-\hat\Psi_+^{[0]}\left(z'\right)^{-1}\hat\Psi_+^{[0]}(z)}{z-z'}.\end{gathered}\tag{3.3}$$

The minus sign is introduced into the definition of $\mathsf{d}\left(z',z\right)$ to absorb the opposite orientation of the circles $C_{0,\infty}$ in some computations below. Let us note in passing that the action of a and d may be extended from the boundary circles to vector-valued functions holomorphic on $\mathscr{A}$.

The result (3.2) suggests that $f_+^{[0]}$, $f_-^{[\infty]}$ provide convenient coordinates on $\mathscr{H}_\mathscr{T}$. We are going to use this basis to describe the operator $\mathscr{P}_\Sigma$ involving the solution $\hat\Psi(\xi)$. Given $f\in\mathscr{H}$, write $f=g+h$ with $g\in\mathscr{H}_\mathscr{T}$ and $h\in\mathscr{H}_A$. These conditions translate into

$$g=\begin{pmatrix}g_+^{[0]} & g_-^{[\infty]}\end{pmatrix}\oplus\begin{pmatrix}g_+^{[0]}\mathsf{d} & g_-^{[\infty]}\mathsf{a}\end{pmatrix},\qquad h=\begin{pmatrix}h_+^{[0]} & h_-^{[\infty]}\end{pmatrix}\oplus\begin{pmatrix}h_-^{[\infty]} & h_+^{[0]}\end{pmatrix}$$

Expressing $h_+^{[0]}$, $h_-^{[\infty]}$ in terms of $g_+^{[0]}$, $g_-^{[\infty]}$, one obtains the equation

$$\begin{pmatrix}g_+^{[0]} & g_-^{[\infty]}\end{pmatrix}(\mathbb{1}-K)=\begin{pmatrix}f_+^{[0]}-f_+^{[\infty]} & f_-^{[\infty]}-f_-^{[0]}\end{pmatrix},\qquad K=\begin{pmatrix}0 & \mathsf{a}\\ \mathsf{d} & 0\end{pmatrix}.$$

Below we assume invertibility of $\mathbb{1}-K$, which ensures the existence of a unique splitting $\mathscr{H}=\mathscr{H}_\mathscr{T}\oplus\mathscr{H}_\mathscr{A}$. Computing the action of $\mathscr{P}_\Sigma$ on $\mathscr{H}$ (essentially equivalent to solving the original RHP) thereby amounts to finding the inverse $(\mathbb{1}-K)^{-1}$.

Consider the restrictions $\mathscr{P}_{\oplus,+}=\mathscr{P}_\oplus\big|_{\mathscr{H}_+}$, $\mathscr{P}_{\Sigma,+}=\mathscr{P}_\Sigma\big|_{\mathscr{H}_+}$. We have just seen that in the previously described basis $\mathscr{P}_{\oplus,+}$ is given by the identity matrix, whereas $\mathscr{P}_{\Sigma,+}$ coincides with $(\mathbb{1}-K)^{-1}$.

Definition 3.1. *The tau function of the Riemann-Hilbert problem for $\hat\Psi(\xi)$ is defined as*

$$\tau(t)=\det\left(\mathscr{H}_+\xrightarrow{\mathscr{H}_-}\mathscr{H}_\mathscr{T}\xrightarrow{\mathscr{H}_\mathscr{A}}\mathscr{H}_+\right)=\det\left(\mathscr{P}_{\oplus,+}\mathscr{P}_{\Sigma,+}{}^{-1}\right)=\det(\mathbb{1}-K).\tag{3.4}$$

The first two expressions of $\tau(t)$ are "coordinate-free" while the last Fredholm determinant corresponds to the choice of a specific basis. Our next task is to understand the relation between the last definition and the tau function of Painlevé III (D_8) equation introduced in (2.13).

3.2 Relation to $\tau_{\mathrm{III}}(t)$

Let t_0 be a constant parameter close to Painlevé III (D_8) independent variable t and consider the ratio

$$\begin{aligned}\frac{\tau(t)}{\tau(t_0)} &= \det\left(\mathscr{H}_+ \xrightarrow{\mathscr{H}_-} \mathscr{H}_{\mathscr{T}}(t) \xrightarrow{\mathscr{H}_{\mathscr{A}}} \mathscr{H}_+ \xrightarrow{\mathscr{H}_{\mathscr{A}}} \mathscr{H}_{\mathscr{T}}(t_0) \xrightarrow{\mathscr{H}_-} \mathscr{H}_+\right) = \\ &= \det\left(\mathscr{H}_{\mathscr{T}}(t_0) \xrightarrow{\mathscr{H}_-} \mathscr{H}_{\mathscr{T}}(t) \xrightarrow{\mathscr{H}_{\mathscr{A}}} \mathscr{H}_{\mathscr{T}}(t_0)\right) = \\ &= \det\left(\mathscr{P}_\oplus(t)\big|_{\mathscr{H}_{\mathscr{T}}(t_0)} \mathscr{P}_\Sigma(t_0)\big|_{\mathscr{H}_{\mathscr{T}}(t)}\right).\end{aligned}$$

Since for $\nu = \oplus, \Sigma$ we can express the inverses as $\left(\mathscr{P}_\nu(t)\big|_{\mathscr{H}_{\mathscr{T}}(t_0)}\right)^{-1} = \mathscr{P}_\nu(t_0)\big|_{\mathscr{H}_{\mathscr{T}}(t)}$, the logarithmic derivative of $\tau(t)$ may be written as

$$\begin{aligned}\partial_t \ln\tau(t) &= -\operatorname{Tr}_{\mathscr{H}_{\mathscr{T}}(t_0)}\left\{\mathscr{P}_\oplus(t)\big|_{\mathscr{H}_{\mathscr{T}}(t_0)} \mathscr{P}_\Sigma(t_0)\big|_{\mathscr{H}_{\mathscr{T}}(t)} \partial_t\left(\mathscr{P}_\Sigma(t)\big|_{\mathscr{H}_{\mathscr{T}}(t_0)} \mathscr{P}_\oplus(t_0)\big|_{\mathscr{H}_{\mathscr{T}}(t)}\right)\right\} = \\ &= -\operatorname{Tr}_{\mathscr{H}}\left\{\mathscr{P}_\oplus(t)\,\mathscr{P}_\Sigma(t_0)\,\partial_t\mathscr{P}_\Sigma(t)\,\mathscr{P}_\oplus(t_0)\right\} = \\ &= -\operatorname{Tr}_{\mathscr{H}}\left\{\mathscr{P}_\oplus(t)\,\partial_t\mathscr{P}_\Sigma(t)\right\}. \qquad (3.5)\end{aligned}$$

Here the middle line is obtained by using that $\operatorname{ran}\mathscr{P}_\nu(t) = \mathscr{H}_{\mathscr{T}}(t)$. The last line follows from the transversality of $\mathscr{H}_{\mathscr{T}}(t)$ and $\mathscr{H}_{\mathscr{A}}$ (as well as $\mathscr{H}_{\mathscr{T}}(t)$ and $\mathscr{H}_-$) in $\mathscr{H}$, which implies that

$$\left(\mathscr{H} \xrightarrow{\mathscr{H}_{\mathscr{T}}(t_0)} \mathscr{H}_{\mathscr{A}} \xrightarrow{\mathscr{H}_{\mathscr{A}}} \mathscr{H}_{\mathscr{T}}(t)\right) \quad = \quad \left(\mathscr{H} \xrightarrow{\mathscr{H}_{\mathscr{T}}(t_0)} \mathscr{H}_- \xrightarrow{\mathscr{H}_-} H_{\mathscr{T}}(t)\right) \quad = \quad 0,$$

i.e. the corresponding compositions of projections are equal to zero.

The next task is to compute the trace in the right side of (3.5). Collapsing the contours and computing residues as in Step 2 of the proof of Theorem 2.9 in [GL16], we arrive at

$$\operatorname{Tr}_{\mathscr{H}}\left\{\mathscr{P}_\oplus(t)\,\partial_t\mathscr{P}_\Sigma(t)\right\} = \sum_{\nu=0,\infty}\frac{1}{2\pi i}\oint_{C_\nu}\operatorname{Tr}\left\{\partial_z\left(\hat\Psi_+(z)\,\hat\Psi_+^{[\nu]}(z)^{-1}\right)\hat\Psi_+^{[\nu]}(z)\,\partial_t\left(\hat\Psi_+(z)^{-1}\right)\right\}dz.$$

Recall that $\hat\Psi$ has the same jumps as $\hat\Psi^{[0]}$ inside C_0 and as $\hat\Psi^{[\infty]}$ outside C_∞. Therefore the "+"-indices in the above expression are redundant, the contours $C_{0,\infty}$ can be replaced by small circles around 0 and ∞, and the resulting integrals may be computed by residues. On these circles, the integrand may be represented by series involving only integer (but not half-integer) powers of z, which can be shown using once again the symmetry properties such as $G\left(\xi e^{i\pi}\right) = iG(\xi)\sigma_x$ and $\sigma_x\tilde Y^{(0)}_{\mathrm{form}}(-\xi)\sigma_x = \tilde Y^{(0)}_{\mathrm{form}}(\xi)$. Furthermore, the series at ∞ has the form $\sum_{k\geq 0} f_k z^{-k-2}$, hence the corresponding residue vanishes.

On the other hand, the residue at 0 reads (note the negative orientation of C_0)

$$\frac{\left(y_1^{(0)}\right)_{11} - \left(y_1^{(0)}\right)_{12}}{\sqrt{t}} - \frac{\left(y_1^{[0]}\right)_{11} - \left(y_1^{[0]}\right)_{12}}{t},$$

where $y_1^{(0)}$ is the first nontrivial coefficient of the formal solution (2.3a) and $y_1^{[0]}$ is its counterpart in the expansion (2.22) of the model solution $Y_0^{[0]}(z)$. The former quantity is explicitly given by (2.4), while the latter is readily deduced from (2.20):

$$y_1^{[0]} = \left(\sigma + \frac{1}{4}\right)\frac{i\sigma_y}{2} - \left(\sigma + \frac{1}{4}\right)^2\sigma_z.$$

Combining the two results with (3.5) yields

$$\partial_t \ln\tau(t) = -\operatorname{Tr}_{\mathscr{H}}\left\{\mathscr{P}_\oplus(t)\,\partial_t\mathscr{P}_\Sigma(t)\right\} = \frac{1}{t}\left[\frac{p^2}{q^2} - q - \frac{t}{q} - \left(\sigma + \frac{1}{2}\right)^2\right].$$

The Fredholm determinant $\tau(t)$ from the Definition 3.1 may therefore be identified with the usual Painlevé III (D_8) tau function $\tau_{\mathrm{III}}(t)$ defined by (2.13):

$$\tau_{\mathrm{III}}(t) = \mathrm{const}\cdot t^{\left(\sigma+\frac{1}{2}\right)^2}\tau(t). \qquad (3.6)$$

In combination with explicit solutions (2.19), (2.24) of auxiliary RHPs which appear in the definition (3.3) of operators a and d, this yields the following result.

Theorem 3.2. *Let $(\sigma,\eta)\in\mathbb{C}^2$ with $\sigma\notin\mathbb{Z}/2$ be the coordinates on the generic stratum of the space of the Stokes data of the linear system (2.1), introduced in Subsection 2.2. The corresponding Painlevé III (D_8) tau function $\tau_{\rm III}(t)=\tau_{\rm III}(t\,|\,\sigma,\eta)$ can be expressed as Fredholm determinant*

$$\tau_{\rm III}(t)=\text{const}\cdot t^{(\sigma+\frac12)^2}\det(\mathbb{1}-K),\qquad K=\begin{pmatrix}0&\mathsf{a}\\ \mathsf{d}&0\end{pmatrix}. \tag{3.7}$$

Here the operators a, d *act on vector-valued functions $f\in H(C)$ on a circle C centered at the origin and oriented counterclockwise,*

$$(f\mathsf{a})(z)=\frac{1}{2\pi i}\oint_C f(z')\,\mathsf{a}(z',z)\,dz',\qquad (f\mathsf{d})(z)=\frac{1}{2\pi i}\oint_C f(z')\,\mathsf{d}(z',z)\,dz', \tag{3.8}$$

and the integral kernels $\mathsf{a}(z',z)$, $\mathsf{d}(z',z)$ are explicitly given by

$$\mathsf{a}(z',z)=e^{i\pi(\sigma-2\eta)\sigma_z}\,\frac{\mathbb{J}_\sigma(z',z)-\mathbb{1}}{z-z'}\,e^{i\pi(2\eta-\sigma)\sigma_z}, \tag{3.9a}$$

$$\mathsf{d}(z',z)=t^{\mathfrak{S}}e^{i\pi\sigma\sigma_z}\sigma_y\,\frac{\mathbb{1}-\mathbb{J}_\sigma\left(\frac{t}{z'},\frac{t}{z}\right)}{z-z'}\,\sigma_y e^{-i\pi\sigma\sigma_z}t^{-\mathfrak{S}}, \tag{3.9b}$$

$$\mathbb{J}_\sigma(z',z)=\frac{\pi}{\sin2\pi\sigma}\begin{pmatrix} z'j_{\sigma+\frac12}(z)j_{-\sigma}(z')-j_\sigma(z)j_{-\sigma-\frac12}(z') & iz'j_{-\sigma-\frac12}(z)j_{-\sigma}(z')-izj_{-\sigma}(z)j_{-\sigma-\frac12}(z')\\ ij_{\sigma+\frac12}(z)j_\sigma(z')-ij_\sigma(z)j_{\sigma+\frac12}(z') & zj_{-\sigma}(z)j_{\sigma+\frac12}(z')-j_{-\sigma-\frac12}(z)j_\sigma(z')\end{pmatrix}, \tag{3.9c}$$

with $j_\sigma(z)=z^{-\sigma}I_{2\sigma}(2\sqrt{z})=\dfrac{{}_0F_1(2\sigma+1;z)}{\Gamma(2\sigma+1)}$ and $\mathfrak{S}=(\sigma+\frac12)\sigma_z$.

Remark 3.3. The kernels $\mathsf{a}(z',z)$, $\mathsf{d}(z',z)$ are not singular at $z=z'$. That $j_\sigma(z)$ are holomorphic in the entire complex plane is a signature of the fact that $\operatorname{ran}\mathsf{a}\subseteq H_+(C)\subseteq\ker\mathsf{a}$ and $\operatorname{ran}\mathsf{d}\subseteq H_-(C)\subseteq\ker\mathsf{d}$. The Fredholm determinant may therefore be rewritten as

$$\det(\mathbb{1}-K)=\det(\mathbb{1}+\mathsf{a}+\mathsf{d}).$$

The latter form may seem more compact while the integral kernel of $\mathsf{a}+\mathsf{d}$ is still integrable. However it turns out to be beneficial for our purposes to work with the block structure of K in (3.7).

Remark 3.4. Let us note that the tau function (2.13) differs from [GIL13, Eq. (2.14)] or [ILT14, Eq. (2.15)] by a $\mathbb{Z}_2$-Bäcklund transformation. This discrete symmetry becomes most explicit at the level of the sine-Gordon equation (2.12) where it corresponds to the mapping $u\mapsto -u$. The relevant monodromy parameters transform as $\sigma\mapsto\frac12-\sigma$, $\eta\mapsto-\eta$ which should be taken into account before comparing (3.7)–(3.9) with (1.2). An interesting representation-theoretic interpretation of this symmetry has been recently suggested in [BSh2].

Remark 3.5. The monodromy data $\{S_k^{(\nu)}\}$, E in (2.7) are invariant with respect to integer shifts $\sigma\mapsto\sigma+1$. The Painlevé III (D_8) tau function should thus be quasi-periodic in σ, namely,

$$\tau_{\rm III}(t\,|\,\sigma+1,\eta)=\text{const}\cdot\tau_{\rm III}(t\,|\,\sigma,\eta),$$

where the constant expression depends on the choice of normalization of $\tau_{\rm III}(t\,|\,\sigma,\eta)$. This quasi-periodicity is not obvious at all at the level of Fredholm determinant representation (3.7)–(3.9) but will be made manifest in the next section.

Upon truncation of the Taylor expansion of the right side of (3.9b) in t, the operator d becomes finite rank so that the corresponding $t\to0$ asymptotics of $\tau_{\rm III}(t)$ is given by a finite determinant, cf [GL16, Theorem 2.11]. From the point of view of this asymptotic analysis, the most efficient choice of σ is to set $-1<\Re\sigma\le0$. The leading asymptotic terms obtained by such procedure coincide with the known results [Jim, IN, Nov, FIKN, Nil]. It is an instructive exercise to check that the subleading asymptotic terms derived from the Fredholm determinant reproduce [ILT14, Eqs. (3.3)–(3.5)].

Remark 3.6. The Painlevé III (D_8) isomonodromic RHP is usually formulated in the literature for the unfolded system (2.2), see e.g. [FIKN, IP, Nil]. While such formulation has a number of technical advantages, the correspondence with CMR confluence diagram is not manifest therein. Furthermore, the analog of the Fredholm determinant (3.4) does *not* coincide with the tau function (2.13). Instead, it gives the tau function of a special case of PIII (D_6) equation ($N_f = 2$ in the gauge theory language), related to PIII (D_8) by a quadratic transformation.

4 Series over Young diagrams

4.1 Cauchy matrix representations

Let us now express the operators a and d from (3.8) in the basis of Fourier modes, where they are given by semi-infinite matrices. Denoting $\mathbb{Z}' = \mathbb{Z} - \frac{1}{2}$, $\mathbb{Z}'_+ = \mathbb{N} - \frac{1}{2}$, write

$$\mathsf{a}(z',z) = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{a}_{-q}^{\ p} z'^{-\frac{1}{2}+p} z^{-\frac{1}{2}+q}, \tag{4.1a}$$

$$\mathsf{d}(z',z) = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{d}_{\ q}^{-p} z'^{-\frac{1}{2}-p} z^{-\frac{1}{2}-q}, \tag{4.1b}$$

where $z', z \in \mathbb{C}^*$. The mode operators $\mathsf{a}_{-q}^{\ p}, \mathsf{d}_{\ q}^{-p}$ are 2×2 matrices whose elements will be represented as $\mathsf{a}_{-q;s}^{\ p;s'}$, $\mathsf{d}_{\ q;s}^{-p;s'}$, with "color" indices $s', s \in \{+,-\}$. Our convention is that "+" and "−" correspond to the first and second row/column.

In order to compute these matrix elements explicitly, let us return to the original definition (3.3) of a and d. Recall that inside the annulus $\mathscr{A}$ we have $\hat\Psi^{[\infty]}(z) = Y_\infty^{[\infty]}(z)\, E^{-1} U^{-1} z^{-\mathfrak{S}}$ and $\hat\Psi^{[0]}(z) = Y_0^{[0]}\left(\frac{z}{t}\right) U^{-1} z^{-\mathfrak{S}}$, where $Y_\infty^{[\infty]}(z)$, $Y_0^{[0]}(z)$ solve the linear systems (2.16), (2.21). These relations may be used to differentiate the kernels $\mathsf{a}(z',z)$, $\mathsf{d}(z',z)$ with respect to their arguments. In particular, for $z', z \in \mathscr{A}$ one has

$$\begin{aligned}
&\left(z\partial_z + z'\partial_{z'} + 1\right)\mathsf{a}(z',z) - \left[\mathfrak{S}, \mathsf{a}(z',z)\right] = \\
&\quad = \hat\Psi^{[\infty]}(z')^{-1}\, \frac{zA^{[\infty]}(z) - z'A^{[\infty]}(z')}{z-z'}\, \hat\Psi^{[\infty]}(z) = -\hat\Psi^{[\infty]}(z')^{-1} \sigma_+ \hat\Psi^{[\infty]}(z) = \\
&\quad = -e^{-2\pi i\eta\sigma_z} Y_0^{[\infty]}(z')^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} Y_0^{[\infty]}(z)\, e^{2\pi i\eta\sigma_z} = \\
&\quad = -\frac{\pi}{\sin 2\pi\sigma} \begin{pmatrix} f_{-\sigma-\frac{1}{2}}(z') \\ ie^{2\pi i(2\eta-\sigma)} f_{\sigma+\frac{1}{2}}(z') \end{pmatrix} \otimes \begin{pmatrix} f_{\sigma+\frac{1}{2}}(z) & ie^{2\pi i(\sigma-2\eta)} f_{-\sigma-\frac{1}{2}}(z) \end{pmatrix}.
\end{aligned}$$

Substituting into the last equation the Fourier representation (4.1a) and using the factorization of the right hand side, we obtain

$$\left(p+q\right)\mathsf{a}_{-q}^{\ p} - \left[\mathfrak{S}, \mathsf{a}_{-q}^{\ p}\right] = e^{i\pi(\sigma-2\eta)\sigma_z} \psi^p(\nu) \otimes \bar\psi_q(\nu)\, e^{i\pi(2\eta-\sigma)\sigma_z}, \tag{4.2}$$

with

$$\psi^{p;s}(\nu) = \sqrt{\frac{\Gamma(1+2s\nu)}{\Gamma(1-2s\nu)}}\, \frac{e^{-i\pi s/4}}{\left(p-\frac{1}{2}\right)!\,(1-2s\nu)_{p-\frac{1}{2}}}, \tag{4.3a}$$

$$\bar\psi_{p;s}(\nu) = \sqrt{\frac{\Gamma(1-2s\nu)}{\Gamma(1+2s\nu)}}\, \frac{e^{i\pi s/4}}{\left(p-\frac{1}{2}\right)!\,(2s\nu)_{p+\frac{1}{2}}}, \tag{4.3b}$$

where $s = \pm$, $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ denotes the Pochhammer symbol, and we have introduced instead of σ a shifted monodromy parameter $\nu = \sigma + \frac{1}{2}$ to make the resulting expressions more symmetric. Further introducing shifted momenta

$$x_{p;s} = p - s\nu, \qquad p \in \mathbb{Z}',\ s = \pm,$$

the solution of (4.2) can be written as

$$\mathsf{a}_{-q;s}^{\;p;s'} = \frac{\psi^{p;s'}(\nu)\,\bar\psi_{q,s}(\nu)}{x_{p;s'}-x_{-q;s}}\,e^{i\pi(2\eta-\sigma)(s-s')}, \tag{4.4a}$$

where $p,q\in\mathbb{Z}'_+$, $s',s=\pm1$. We thus conclude that in the Fourier basis the operator a is given, up to left and right diagonal factors, by a Cauchy matrix $M_{jk}=\frac{1}{x_j-y_k}$. This allows, *inter alia*, to compute any minor of a in a factorized form.

The matrix elements of d may be computed in a similar fashion, or alternatively deduced by comparison of (3.9a) and (3.9b). The result is again a Cauchy matrix,

$$\mathsf{d}^{-q;s}_{\;p;s'} = \frac{\psi^{q;s}(-\nu)\,\bar\psi_{p,s'}(-\nu)}{x_{p;s'}-x_{-q;s}}\,e^{i\pi\sigma(s-s')}\,t^{(s-s')\nu+p+q}, \tag{4.4b}$$

and its nontrivial part coincides with that of (4.4b) after replacement $\nu\to-\nu$. The dependence on PIII variable t is isolated in the diagonal factors; cf Remark 3.5.

4.2 Maya and Young diagrams

Given a matrix $A\in\mathrm{Mat}_{m\times m}(\mathbb{C})$, the von Koch's formula

$$\det(\mathbb{1}+A)=\sum_{n=0}^{\infty}\sum_{i_1<\ldots<i_n}\det\left(A_{i_j i_k}\right)_{j,k=1}^{n}$$

expresses the determinant $\det(\mathbb{1}+A)$ as the sum of principal minors of A. While this series of course terminates at $n=m$, the formula has a straightforward generalization to infinite matrices. If A is indexed by elements of a discrete set $\mathfrak{X}$ instead of $\{1,\ldots,m\}$, then

$$\det(\mathbb{1}+A)=\sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}}\det A_{\mathfrak{Y}}, \tag{4.5}$$

where the sum is taken over all subsets $\mathfrak{Y}$ of $\mathfrak{X}$ and $A_{\mathfrak{Y}}$ is the principal minor of A obtained by choosing the rows and columns labeled by $\mathfrak{Y}$.

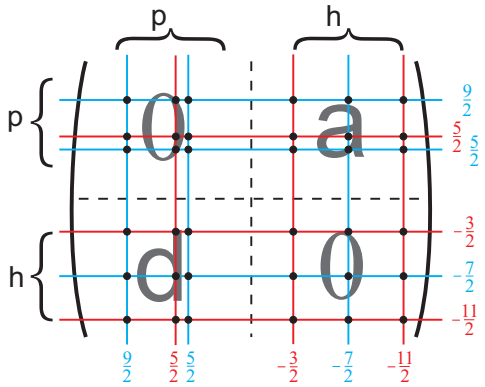


Figure 4: Labeling of principal minors of K by positions (p,h) of particles and holes of color + (red) and − (blue). Here $\mathsf{p}^+=\{\frac{5}{2}\}$, $\mathsf{h}^+=\{-\frac{3}{2},-\frac{11}{2}\}$, $\mathsf{p}^-=\{\frac{9}{2},\frac{5}{2}\}$, $\mathsf{h}^-=\{-\frac{7}{2}\}$, so that $\mathsf{m}^+=\{\frac{5}{2},-\frac{3}{2},-\frac{11}{2}\}$, $\mathsf{m}^-=\{\frac{9}{2},\frac{5}{2},-\frac{7}{2}\}$ and $Q(\mathsf{m}^+)=-Q(\mathsf{m}^-)=-1$.

We are going apply the last formula to the Fredholm determinant (3.7) with K written in the Fourier basis. Represent appropriate subsets as $\mathfrak{Y}=(\mathsf{p},\mathsf{h})$, where p and h correspond, respectively, to the first and second block of K, see Fig. 4. The sum in (4.5) may be restricted to (p,h) with $\sharp(\mathsf{p})=\sharp(\mathsf{h})$, as otherwise the corresponding minors obviously vanish. It follows that

$$\det(\mathbb{1}-K)=\sum_{(\mathsf{p},\mathsf{h}):\,\sharp(\mathsf{p})=\sharp(\mathsf{h})}(-1)^{\sharp(\mathsf{p})}\det\mathsf{a}^{\mathsf{p}}_{\mathsf{h}}\det\mathsf{d}^{\mathsf{h}}_{\mathsf{p}}, \tag{4.6}$$

where e.g. $a^{\mathsf{p}}_{\mathsf{h}}$ is a square $\sharp(\mathsf{p}) \times \sharp(\mathsf{p})$ matrix obtained by restricting a to rows p and columns h. Let us now take a closer look at the structure of subsets (p,h) labeling different contributions to (4.6):

- The set p has the form $\mathsf{p}^+ \sqcup \mathsf{p}^-$, where $\mathsf{p}^+ = \{p_1^+,\dots,p_L^+\}$, $\mathsf{p}^- = \{p_1^-,\dots,p_M^-\}$, and $p_j^\pm \in \mathbb{Z}'_+$ are Fourier indices of elements of p of color $\pm$. Similarly, $\mathsf{h} = \mathsf{h}^+ \sqcup \mathsf{h}^-$, where $\mathsf{h}^+ = \{-q_1^+,\dots,-q_{L'}^+\}$, $\mathsf{h}^- = \{-q_1^-,\dots,-q_{M'}^-\}$ with $q_j^\pm \in \mathbb{Z}'_+$ consist of Fourier indices of elements of h of colors $+$ and $-$. The elements of $\mathsf{p}^\pm$ and $\mathsf{h}^\pm$ are thus distinct positive (resp. negative) half-integers.

- Let us consider the combinations $\mathsf{m}^\pm = \mathsf{p}^\pm \cup \mathsf{h}^\pm$. Both $\mathsf{m}^\pm$ are finite subsets of $\mathbb{Z}'$ and can be represented in the usual way by Maya diagrams; $\mathsf{p}^\pm$ and $\mathsf{h}^\pm$ are positions of particles and holes of color $\pm$ in the Dirac sea, see Fig. 4 and bottom part of Fig. 5. Given a Maya diagram m, the difference $Q(\mathsf{m}) = \sharp(\text{particles}) - \sharp(\text{holes})$ is called the charge of m. The constraint $\sharp(\mathsf{p}) = \sharp(\mathsf{h})$ is then nothing but the neutrality condition $Q(\mathsf{m}^+) + Q(\mathsf{m}^-) = 0$.

- On the other hand, the set $\mathbb{M}$ of Maya diagrams can be bijectively mapped to the set $\mathbb{Y} \times \mathbb{Z}$ of charged Young diagrams/partitions. This correspondence is represented graphically in Fig. 5. The profile of the Young diagram $\mathsf{Y} \in \mathbb{Y}$ associated to a Maya diagram $\mathsf{m} \in \mathbb{M}$ is obtained by starting far away on the NW-axis and going south-east above each filled circle and north-east above each empty circle of m. The charge corresponds to relative position of the bottom boundary of Y and the NE-axis, and coincides with $Q(\mathsf{m})$.

Different contributions to (4.6) may therefore be labeled (i) by positions of particles $\mathsf{p}^\pm \in 2^{\mathbb{Z}'_+}$ and holes $\mathsf{h}^\pm \in 2^{-\mathbb{Z}'_+}$ of two colors $\pm$ satisfying the balance condition $\sharp(\mathsf{p}^+) + \sharp(\mathsf{p}^-) = \sharp(\mathsf{h}^+) + \sharp(\mathsf{h}^-)$; (ii) by pairs $(\mathsf{m}^+,\mathsf{m}^-) \in \mathbb{M}^2$ of Maya diagrams of zero total charge; and also (iii) by pairs $(\mathsf{Y}^+,\mathsf{Y}^-) \in \mathbb{Y}^2$ of Young diagrams corresponding to m^+ and m^-, and an integer $Q \equiv Q(\mathsf{m}^+)$.

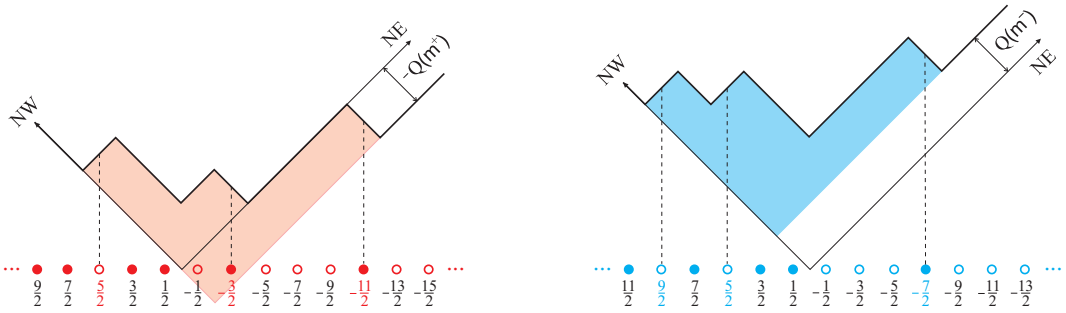


Figure 5: Young diagrams Y^+ and Y^- (shaded regions) associated to $\mathsf{m}^+ = \{\frac{5}{2},-\frac{3}{2},-\frac{11}{2}\}$ and $\mathsf{m}^- = \{\frac{9}{2},\frac{5}{2},-\frac{7}{2}\}$.

The individual contributions can be readily computed using the Cauchy matrix representations (4.4).

Theorem 4.1. *The Painlevé III (D_8) tau function $\tau_{\mathrm{III}}(t) = \tau_{\mathrm{III}}(t|\sigma,\eta)$ from Theorem 3.2 admits the following series representation:*

$$\tau_{\mathrm{III}}(t) = \sum_{(\mathsf{p},\mathsf{h}):\sharp(\mathsf{p})=\sharp(\mathsf{h})} e^{-4\pi i\eta Q}\, \Xi_{\mathsf{p},\mathsf{h}}(v)\, \Delta^2_{\mathsf{p},\mathsf{h}}(v)\, t^{(v-Q)^2+|\mathsf{Y}^+|+|\mathsf{Y}^-|}, \tag{4.7}$$

where $|\mathsf{Y}|$ denotes the total number of boxes in $\mathsf{Y} \in \mathbb{Y}$ and

$$\Xi_{\mathsf{p},\mathsf{h}}(v) = (-1)^Q \frac{\Gamma^{2Q}(1+2v)}{\Gamma^{2Q}(1-2v)} \left[\prod_{(p,s')\in\mathsf{p}} \left(p-\tfrac{1}{2}\right)! \left(1-2s'v\right)_{p-\frac{1}{2}} \prod_{(-q,s)\in\mathsf{h}} \left(q-\tfrac{1}{2}\right)! (2sv)_{q+\frac{1}{2}} \right]^{-2}, \tag{4.8a}$$

$$\Delta_{\mathsf{p},\mathsf{h}}(v) = \frac{\prod\limits_{(p,s')<(\tilde{p},\tilde{s}')\in\mathsf{p}} \left(x_{p;s'} - x_{\tilde{p};\tilde{s}'}\right) \prod\limits_{(-q,s)<(-\tilde{q},\tilde{s})\in\mathsf{h}} \left(x_{-\tilde{q};\tilde{s}} - x_{-q;s}\right)}{\prod\limits_{(p;s')\in\mathsf{p}} \prod\limits_{(-q;s)\in\mathsf{h}} \left(x_{p;s'} - x_{-q;s}\right)}. \tag{4.8b}$$

Proof. From (4.4) it follows that

$$(-1)^{\sharp(\mathsf{p})}\det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}}\det \mathsf{d}_{\mathsf{p}}^{\mathsf{h}}=(-1)^{\sharp(\mathsf{p})}\prod_{(p,s')\in\mathsf{p}}\psi^{p;s'}(\nu)\,\bar{\psi}_{p;s'}(-\nu)\prod_{(-q,s)\in\mathsf{h}}\psi^{q;s}(-\nu)\,\bar{\psi}_{q;s}(\nu)\times \\ \times\left(t^{\nu}e^{2\pi i\eta}\right)^{\sum_{(-q,s)\in\mathsf{h}}s-\sum_{(p,s')\in\mathsf{p}}s'}t^{\sum_{(-q,s)\in\mathsf{h}}q+\sum_{(p,s')\in\mathsf{p}}p}\Delta^2_{\mathsf{p},\mathsf{h}}(\nu)\,. \tag{4.9}$$

The power of $t^{\nu}e^{2\pi i\eta}$ can be further transformed as

$$\sum_{(-q,s)\in\mathsf{h}}s-\sum_{(p,s')\in\mathsf{p}}s'=\sharp\left(\mathsf{h}^{+}\right)-\sharp\left(\mathsf{h}^{-}\right)-\sharp\left(\mathsf{p}^{+}\right)+\sharp\left(\mathsf{p}^{-}\right)=Q\left(\mathsf{m}^{-}\right)-Q\left(\mathsf{m}^{+}\right)=-2Q.$$

It may be also easily shown (see Fig. 13 in [GL16]) that

$$\sum_{(-q,s)\in\mathsf{h}^{\pm}}q+\sum_{(p,s')\in\mathsf{p}^{\pm}}p=\frac{Q^2}{2}+\left|\mathsf{Y}^{\pm}\right|,$$

so that the power of t in the second line of (4.9) becomes $Q^2+\left|\mathsf{Y}^{+}\right|+|\mathsf{Y}^{-}|$. The prefactor $\Xi_{\mathsf{p},\mathsf{h}}(\nu)$ is obtained from the diagonal products in the first line by simple algebra. □

4.3 Nekrasov functions

In this subsection we rewrite the factorized expressions $\Xi_{\mathsf{p},\mathsf{h}}(\nu)\,\Delta^2_{\mathsf{p},\mathsf{h}}(\nu)$ for the coefficients of the tau function expansion in a notation close to gauge theory. The main tool we need is a technical statement that can be found, for example, in [GL16, GM]. In order to formulate it, let $(\mathsf{Y}^s,Q^s)\in\mathbb{Y}\times\mathbb{Z}$ (with $s=\pm$) be two charged Young diagrams, not necessarily the same as above. Denote by $\mathsf{m}^s\in\mathbb{M}$ the associated Maya diagrams. At this point we do not need to assume that $Q^{+}+Q^{-}=0$.

Introduce the following three quantities:

1. An explicit factorized function

$$\tilde{Z}_{\mathsf{bif}}\left(\nu\left|\mathsf{Y}^{+},Q^{+};\mathsf{Y}^{-},Q^{-}\right.\right)=\prod_{-q\in\mathsf{h}^{+}}(-\nu)_{q+\frac{1}{2}}\prod_{-q\in\mathsf{h}^{-}}(\nu+1)_{q-\frac{1}{2}}\prod_{p\in\mathsf{p}^{-}}(-\nu)_{p+\frac{1}{2}}\prod_{p\in\mathsf{p}^{+}}(\nu+1)_{p-\frac{1}{2}}\times \\ \times\frac{\prod\limits_{-q\in\mathsf{h}^{+}}\prod\limits_{p\in\mathsf{p}^{-}}\left(\nu-q-p\right)\prod\limits_{-q\in\mathsf{h}^{-}}\prod\limits_{p\in\mathsf{p}^{+}}\left(\nu+p+q\right)}{\prod\limits_{-q'\in\mathsf{h}^{-}}\prod\limits_{-q\in\mathsf{h}^{+}}\left(\nu-q+q'\right)\prod\limits_{p'\in\mathsf{p}^{-}}\prod\limits_{p\in\mathsf{p}^{+}}\left(\nu+p-p'\right)}. \tag{4.10}$$

   which, as we will see in a moment, constitutes the main building block of $\Xi_{\mathsf{p},\mathsf{h}}(\nu)\,\Delta^2_{\mathsf{p},\mathsf{h}}(\nu)$.

2. Another factorized expression, representing the Nekrasov bifundamental contribution:

$$Z_{\mathsf{bif}}\left(\nu\left|\mathsf{Y}^{+},\mathsf{Y}^{-}\right.\right):=\prod_{\square\in\mathsf{Y}^{+}}\left(\nu+1+a_{\mathsf{Y}^{+}}(\square)+l_{\mathsf{Y}^{-}}(\square)\right)\prod_{\square\in\mathsf{Y}^{-}}\left(\nu-1-a_{\mathsf{Y}^{-}}(\square)-l_{\mathsf{Y}^{+}}(\square)\right), \tag{4.11}$$

   where $\mathsf{Y}^{\pm}\in\mathbb{Y}$ and the notation for Young diagrams follows Fig. 6. The expressions $a_{\mathsf{Y}}(\square)$, $l_{\mathsf{Y}}(\square)$ and $h_{\mathsf{Y}}(\square)$ represent the arm-, leg-, and hook length of the box $\square$ in $\mathsf{Y}\in\mathbb{Y}$. In the case where $\square=(i,j)$ does not belong to Y, the definition of the former two quantities is extended by $a_{\mathsf{Y}}(\square)=\mathsf{Y}_i-j$ and $l_{\mathsf{Y}}(\square)=\mathsf{Y}'_j-i$. In particular, we have

$$Z_{\mathsf{bif}}\left(-\nu\left|\mathsf{Y}^{-},\mathsf{Y}^{+}\right.\right)=(-1)^{\left|\mathsf{Y}^{+}\right|+\left|\mathsf{Y}^{-}\right|}Z_{\mathsf{bif}}\left(\nu\left|\mathsf{Y}^{+},\mathsf{Y}^{-}\right.\right), \\ Z_{\mathsf{bif}}\left(0\left|\mathsf{Y},\mathsf{Y}\right.\right)=(-1)^{|\mathsf{Y}|}\prod_{\square\in\mathsf{Y}}h^2_{\mathsf{Y}}(\square)\,. \tag{4.12}$$

3. For $Q\in\mathbb{Z}$, define the "structure constant" $\Upsilon(\nu\,|\,Q)$ by

$$\Upsilon(\nu\,|\,Q)=\frac{\Gamma^{Q}(1+\nu)\,G(1+\nu)}{G(1+\nu+Q)}. \tag{4.13}$$

   Here $G(z)$ denotes the Barnes G-function satisfying the relation $G(z+1)=\Gamma(z)\,G(z)$. Note that $\Upsilon(\nu\,|\,Q)$ is actually a rational function of ν.

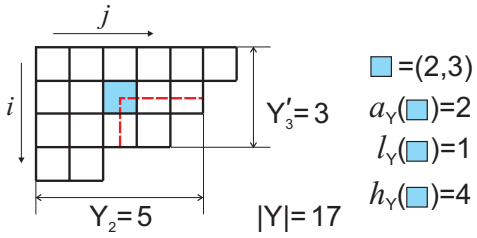


Figure 6: Characteristics of Young diagrams.

Lemma 4.2. *We have*

$$\tilde{Z}_{\mathsf{bif}}\left(\nu\middle|\mathsf{Y}^{+},Q^{+};\mathsf{Y}^{-},Q^{-}\right)=\pm\Upsilon^{-1}\left(\nu\middle|Q^{+}-Q^{-}\right)Z_{\mathsf{bif}}\left(\nu+Q^{+}-Q^{-}\middle|\mathsf{Y}^{+},\mathsf{Y}^{-}\right),\tag{4.14}$$

where $\pm$ *means that the equality holds up to an overall sign.*

Proof. See [GL16, Appendix A]. □

Lemma 4.2 thus relates certain products over boxes of Young diagrams to some explicit functions of particle and hole coordinates in the relevant Maya diagrams. Let us now use it to identify the corresponding Nekrasov functions in (4.1). We first prove

Lemma 4.3. *We have*

$$\Xi_{\mathsf{p},\mathsf{h}}(\nu)\,\Delta_{\mathsf{p},\mathsf{h}}^{2}(\nu)=\frac{\Gamma^{2Q}(1+2\nu)}{\Gamma^{2Q}(1-2\nu)}\frac{\Upsilon(2\nu|-2Q)\,\Upsilon(-2\nu|2Q)}{\prod\limits_{s,s'=\pm1}Z_{\mathsf{bif}}\left((Q-\nu)(s'-s)\,|\,\mathsf{Y}^{s'},\mathsf{Y}^{s}\right)}.\tag{4.15}$$

Proof. First of all, one may further decompose $\Delta_{\mathsf{p},\mathsf{h}}$:

$$\Delta_{\mathsf{p},\mathsf{h}}(\nu)=\Delta_{\mathsf{p},\mathsf{h}}^{++}\Delta_{\mathsf{p},\mathsf{h}}^{+-}(\nu)\Delta_{\mathsf{p},\mathsf{h}}^{--},$$

where

$$\Delta_{\mathsf{p},\mathsf{h}}^{\pm\pm}=\frac{\prod\limits_{p,\bar{p}\in\mathsf{p}^{\pm}:p<\bar{p}}(p-\bar{p})\prod\limits_{-q,-\bar{q}\in\mathsf{h}^{\pm}:q>\bar{q}}(q-\bar{q})}{\prod\limits_{p\in\mathsf{p}^{\pm}}\prod\limits_{-q\in\mathsf{h}^{\pm}}(p+q)},$$

$$\Delta_{\mathsf{p},\mathsf{h}}^{+-}(\nu)=\frac{\prod\limits_{p_+\in\mathsf{p}^{+}}\prod\limits_{p_-\in\mathsf{p}^{-}}(-2\nu+p_+-p_-)\prod\limits_{-q_+\in\mathsf{h}^{+}}\prod\limits_{-q_-\in\mathsf{h}^{-}}(-2\nu-q_++q_-)}{\prod\limits_{p_+\in\mathsf{p}^{+}}\prod\limits_{-q_-\in\mathsf{h}^{-}}(-2\nu+p_++q_-)\prod\limits_{p_-\in\mathsf{p}^{-}}\prod\limits_{-q_+\in\mathsf{h}^{+}}(2\nu+p_-+q_+)}.$$

Comparing (4.10) with (4.8b), we can write

$$\begin{aligned}\prod_{s,s'=\pm1}\tilde{Z}_{\mathsf{bif}}\left(\nu(s-s')\,|\,\mathsf{Y}^{s'},Qs';\mathsf{Y}^{s},Qs\right)&=\pm\left[\Delta_{\mathsf{p},\mathsf{h}}^{++}\Delta_{\mathsf{p},\mathsf{h}}^{+-}(\nu)\Delta_{\mathsf{p},\mathsf{h}}^{--}\right]^{-2}\times\\&\times\prod_{-q\in\mathsf{h}^{-}}(-2\nu)_{q+\frac{1}{2}}\prod_{-q\in\mathsf{h}^{+}}(2\nu+1)_{q-\frac{1}{2}}\prod_{p\in\mathsf{p}^{+}}(-2\nu)_{p+\frac{1}{2}}\prod_{p\in\mathsf{p}^{-}}(2\nu+1)_{p-\frac{1}{2}}\times\\&\times\prod_{-q\in\mathsf{h}^{+}}(2\nu)_{q+\frac{1}{2}}\prod_{-q\in\mathsf{h}^{-}}(-2\nu+1)_{q-\frac{1}{2}}\prod_{p\in\mathsf{p}^{+}}(2\nu)_{p+\frac{1}{2}}\prod_{p\in\mathsf{p}^{+}}(-2\nu+1)_{p-\frac{1}{2}}\times\\&\times\left[\prod_{-q\in\mathsf{h}^{-}}\left(q-\tfrac{1}{2}\right)!\prod_{-q\in\mathsf{h}^{+}}\left(q-\tfrac{1}{2}\right)!\prod_{p\in\mathsf{p}^{-}}\left(p-\tfrac{1}{2}\right)!\prod_{p\in\mathsf{p}^{+}}\left(p-\tfrac{1}{2}\right)!\right]^{2}.\end{aligned}\tag{4.16}$$

Using the identity $(z)_{q+\frac{1}{2}}=z\cdot(z+1)_{q-\frac{1}{2}}$ for the Pochhammer's symbol, the balance condition $\sharp\left(\mathsf{h}^{+}\right)+\sharp\left(\mathsf{h}^{-}\right)=\sharp\left(\mathsf{p}^{+}\right)+\sharp\left(\mathsf{p}^{-}\right)$, and comparing the last three lines with (4.8a), we can rewrite (4.16) as

$$\prod_{s,s'=\pm1}\tilde{Z}_{\mathsf{bif}}\left(\nu(s-s')\,|\,\mathsf{Y}^{s'},Qs';\mathsf{Y}^{s},Qs\right)=\pm\frac{\Gamma^{2Q}(1+2\nu)}{\Gamma^{2Q}(1-2\nu)}\,\Xi_{\mathsf{p},\mathsf{h}}^{-1}(\nu)\,\Delta_{\mathsf{p},\mathsf{h}}^{-2}(\nu).\tag{4.17}$$

Combining this result with (4.14), we immediately obtain (4.15) up to an overall sign. It suffices to check it for real $\nu \notin \mathbb{Z}$. For the left side this sign is obviously equal to $(-1)^Q$. From the identities (4.12) it follows that the right side of (4.15) may be rewritten as

$$\frac{G(1+2\nu)\,G(1-2\nu)}{G(1+2\nu-2Q)\,G(1-2\nu+2Q)}\left[Z_{\mathsf{bif}}\left(2\nu-2Q\,|\,\mathsf{Y}^-,\mathsf{Y}^+\right)\prod_{\square\in\mathsf{Y}^+}h_{\mathsf{Y}^+}(\square)\prod_{\square\in\mathsf{Y}^-}h_{\mathsf{Y}^-}(\square)\right]^{-2}.$$

Its sign is therefore determined by the Barnes function prefactor in the last expression, and can be easily shown to be $(-1)^Q$. □

We can now formulate our final result.

Theorem 4.4. *Let $\mathcal{Z}_{\mathrm{SU}(2)}(t\,|\,\nu)$ be the Nekrasov instanton partition function of the pure gauge theory, defined as a double sum over partitions:*

$$\mathcal{Z}_{\mathrm{SU}(2)}(t\,|\,\nu)=C(\nu)\sum_{\mathsf{Y}^+,\mathsf{Y}^-\in\mathbb{Y}}\frac{t^{\nu^2+|\mathsf{Y}^+|+|\mathsf{Y}^-|}}{\prod_{s,s'=\pm1}Z_{\mathsf{bif}}\left(\nu(s-s')\,|\,\mathsf{Y}^{s'},\mathsf{Y}^{s}\right)},\tag{4.18}$$

with $C(\nu)=\left[\prod_{s=\pm1}G(1+2s\nu)\right]^{-1}$ and $Z_{\mathsf{bif}}\left(\nu\,|\,\mathsf{Y}',\mathsf{Y}\right)$ defined by (4.11). The dual partition function

$$\mathcal{Z}^{\mathrm{dual}}_{\mathrm{SU}(2)}\left(t\,|\,\nu,\eta\right)=\sum_{n\in\mathbb{Z}}e^{4\pi i n\eta}\mathcal{Z}_{\mathrm{SU}(2)}(t\,|\,\nu+n)\tag{4.19}$$

admits Fredholm determinant representation

$$\mathcal{Z}^{\mathrm{dual}}_{\mathrm{SU}(2)}\left(t\,|\,\nu,\eta\right)=C(\nu)\,t^{\nu^2}\det(\mathbb{1}-K),\tag{4.20}$$

where K is the generalized Bessel kernel from Theorem 3.2 (with $\sigma=\nu-\frac{1}{2}$), and thereby coincides with the general tau function of the Painlevé III (D_8) equation.

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