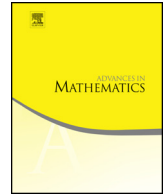




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Airy-kernel determinant on two large intervals



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ABSTRACT

We find the probability of two gaps of the form $(sc, sb) \cup (sa, +\infty)$, $c < b < a < 0$, for large $s > 0$, in the edge scaling limit of the Gaussian Unitary Ensemble, including the multiplicative constant in the asymptotics.

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1. Introduction

For $c < b < a < 0$, set $J = (c, b) \cup (a, +\infty)$. Let $\text{Ai}(x)$, $\text{Ai}'(x)$ be the Airy function and its derivative, respectively. Consider the (trace class) operator K^{Ai} , acting on $L^2(sJ)$, $s > 0$, with kernel

$$K^{\text{Ai}}(z, z') = \frac{\text{Ai}(z)\text{Ai}'(z') - \text{Ai}'(z)\text{Ai}(z')}{z - z'} = \int_0^\infty \text{Ai}(z + \zeta)\text{Ai}(z' + \zeta)d\zeta.$$

We are interested in the large- s behaviour of the corresponding Fredholm determinant

$$P^{\text{Ai}}(sJ) = \det(I - K^{\text{Ai}})_{(sc, sb) \cup (sa, +\infty)}. \tag{1}$$

The determinant $P^{\text{Ai}}(sJ)$ is the probability of 2 gaps (sc, sb) and $(sa, +\infty)$ in the edge scaling limit of the Gaussian Unitary Ensemble (GUE), see, e.g., [23,10].

Clearly, by rescaling s , we can consider the second interval to be $(-s, +\infty)$. In the case of one gap, $(-s, +\infty)$, the determinant

$$P^{\text{Ai}}(-s, +\infty) = \det(I - K^{\text{Ai}})_{(-s, +\infty)}$$

is the Tracy-Widom distribution [25] — the distribution of the largest eigenvalue of the GUE. The same determinant also describes the distribution of the longest increasing subsequence in a random permutation [3]. Its large s asymptotics were first considered by Tracy and Widom [25] in 1994, who observed that

$$P^{\text{Ai}}(-s, +\infty) = \exp \left\{ - \int_{-s}^\infty (x + s)u^2(x)dx \right\}, \tag{2}$$

where $u(x)$ is the Hastings-McLeod solution of the Painlevé II equation

$$u''(x) = xu(x) + 2u^3(x), \tag{3}$$

specified by the following asymptotic condition:

$$u(x) \sim \text{Ai}(x) \quad \text{as } x \rightarrow +\infty. \tag{4}$$

The asymptotics of the logarithmic derivative $(d/ds) \log P^{\text{Ai}}(-s, +\infty)$ follow, up to a constant (which is in fact zero), from (4) and the known asymptotics of the Hastings-McLeod solution at $-\infty$. Integrating, Tracy and Widom obtained

$$\log \det(I - K^{\text{Ai}})_{(-s, +\infty)} = -\frac{1}{12}s^3 - \frac{1}{8} \log s + \chi_{\text{Airy}} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \quad s \rightarrow \infty, \tag{5}$$

up to an undetermined constant χ_{airy} . Tracy and Widom did however, conjecture its value to be

$$\chi_{\text{Airy}} = \frac{1}{24} \log 2 + \zeta'(-1), \tag{6}$$

where ζ denotes Riemann’s ζ -function. The proof of (5) with (6) was given in [12], and, by a different method, in [2].

On the other hand, in the *bulk* of the spectrum of GUE, the probability of a gap $(-s, s)$ is given by the Fredholm determinant $\det(I - K^{\text{sine}})_{(-s,s)}$ of the trace class operator K^{sine} on $L^2(-s, s)$ with the sine kernel

$$K^{\text{sine}}(x, y) = \frac{\sin(x - y)}{\pi(x - y)}.$$

In this case, we have the following large s asymptotics

$$\log \det(I - K^{\text{sine}})_{(-s,s)} = -\frac{s^2}{2} - \frac{1}{4} \log s + c_{\text{sine}} + \mathcal{O}\left(\frac{1}{s}\right), \quad s \rightarrow \infty, \tag{7}$$

where

$$c_{\text{sine}} = \frac{1}{12} \log 2 + 3\zeta'(-1). \tag{8}$$

The leading term $-\frac{s^2}{2}$ in (7) was found by Dyson in 1962 in [16]. The terms $-\frac{s^2}{2} - \frac{1}{4} \log s$ were then computed by des Cloizeaux and Mehta [9] in 1973. The constant (8), known as the Widom-Dyson constant, was identified by Dyson [17] in 1976. The works [16], [9], and [17] are not fully rigorous. The first rigorous confirmation of the main term, i.e. the fact that $\log \det(I - K^{\text{sine}})_{(-s,s)} = -\frac{s^2}{2}(1 + o(1))$, was given by Widom [27] in 1994. The full asymptotics (up to an undetermined value of c_{sine}) was justified by Deift, Its, and Zhou in 1997 in [11]. The value (8) of c_{sine} in (7) was justified in 3 different ways in [21], [14], [18] (see [13] and [19] for more historical details).

As we will see, the present work relates the results (5) and (7) in some sense.

Since J consists of 2 intervals, we expect the appearance of Jacobi θ -functions in the asymptotics. This phenomenon was first observed in [11], where the authors considered the sine-kernel determinant on several large intervals, and found its asymptotics up to a multiplicative constant. The constant in the case of 2 intervals for the sine-kernel determinant was recently found in [19]. In the present work, following many ideas from [19], we establish the asymptotics of (1) including the relevant multiplicative constant. To describe our results, we introduce some notation.

Let

$$p(z) = (z - a)(z - b)(z - c),$$

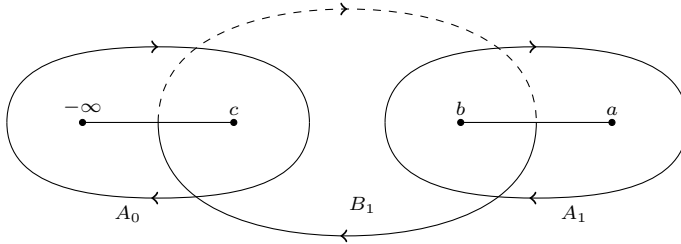


Fig. 1. Cycles on the Riemann surface.

and consider the Riemann surface of the function $p(z)^{1/2}$ with branch cuts on $\mathbb{R} \setminus J$ (see Fig. 1). Fix the first sheet of the surface by the condition $p(z)^{1/2} > 0$ for $z > a$. Define the elliptic integrals around cycles

$$I_k = \frac{1}{2} \oint_{A_1} \frac{\zeta^k d\zeta}{p(\zeta)^{1/2}} = \frac{1}{i} \int_b^a \frac{\zeta^k d\zeta}{|p(\zeta)^{1/2}|}, \tag{9}$$

$$J_k = -\frac{1}{2} \oint_{B_1} \frac{\zeta^k d\zeta}{p(\zeta)^{1/2}} = -\int_c^b \frac{\zeta^k d\zeta}{|p(\zeta)^{1/2}|}, \quad k = 0, 1, 2,$$

where the cycles A_0, A_1, B_1 are depicted in Fig. 1: parts represented by solid lines are on first sheet of the surface, while dotted line is on the second.

Consider the function

$$g(z) = \int_a^z \frac{q(\zeta)}{p(\zeta)^{1/2}} d\zeta, \quad \mathbb{C} \setminus (-\infty, a], \tag{10}$$

on the first sheet. Here

$$q(z) = z^2 + q_1 z + q_0$$

is a polynomial such that

$$\int_a^z \frac{q(\zeta)}{p(\zeta)^{1/2}} d\zeta = \frac{2}{3} z^{3/2} + \mathcal{O}\left(\frac{1}{z^{1/2}}\right), \quad z \rightarrow \infty, \tag{11}$$

and

$$\int_c^b \frac{q(\zeta)}{p(\zeta)^{1/2}} d\zeta = 0. \tag{12}$$

As we verify in Lemma 11 below, these conditions determine the coefficients of $q(z)$:

$$q_1 = -\frac{a + b + c}{2}, \quad q_0 = -\frac{J_2 + q_1 J_1}{J_0} = \frac{1}{3}(ab + ac + bc) + \frac{1}{3}q_1 \frac{J_1}{J_0}. \tag{13}$$

Furthermore, we will see that the function $g(z)$ admits a large- z asymptotic expansion of the form

$$g(z) = \frac{2}{3}z^{3/2} + \frac{\alpha_1}{z^{1/2}} + \frac{\alpha_2}{z^{3/2}} + O\left(z^{-5/2}\right), \quad z \rightarrow \infty, \tag{14}$$

where, in particular,

$$\alpha_2 = -\frac{1}{12} \left(a^3 + b^3 + c^3 - (a + b)(a + c)(b + c) - 8q_0q_1 \right). \tag{15}$$

We show that $\alpha_2 > 0$ in the formula (264) below.

Let

$$\Omega = \frac{g_+(b)}{\pi i} = \frac{2q_1}{3J_0} \in \mathbb{R}, \quad \tau = \frac{I_0}{J_0}, \quad \operatorname{Re} \tau = 0, \quad \operatorname{Im} \tau > 0. \tag{16}$$

The second expression for Ω here follows by Riemann’s period relations, see (138), (151)–(153) below. Recall the third Jacobi θ -function given by

$$\theta(z) = \theta_3(z) = \theta_3(z; \tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i m z + \pi i \tau m^2}. \tag{17}$$

The θ -function satisfies the following periodicity relations, see e.g. [26],

$$\theta_3(z + 1) = \theta_3(z) \quad \text{and} \quad \theta_3(z + \tau) = e^{-2\pi i z - \pi i \tau} \theta_3(z). \tag{18}$$

We now state our result.

Theorem 1. *The following asymptotics hold*

$$\log P^{\text{Ai}}(sJ) = -\alpha_2 s^3 - \frac{1}{2} \log s + \log \frac{\theta_3(s^{3/2}\Omega)}{\theta_3(0)} + \chi + o(1), \quad s \rightarrow +\infty, \tag{19}$$

with

$$\chi = \frac{1}{4} \log(a - c) - \frac{1}{8} \log|2q(a)q(b)q(c)| + c_{\text{sine}} + \chi_{\text{Airy}}, \tag{20}$$

where the constants $c_{\text{sine}}, \chi_{\text{Airy}}$ are given by (8) and (6), respectively.

Remark 2. Alternatively, by the identity for $\theta_3(0)$ in (116) below, (19) can be written as

$$\log P^{\text{Ai}}(sJ) = -\alpha_2 s^3 - \frac{1}{2} \log s + \log \theta_3(s^{3/2}\Omega; \tau) + \chi_1 + o(1), \quad s \rightarrow +\infty, \quad (21)$$

with

$$\chi_1 = -\frac{1}{2} \log \left| \frac{J_0}{\pi} \right| - \frac{1}{8} \log |q(a)q(b)q(c)| + 4\zeta'(-1) \quad (22)$$

This expression for the determinant exhibits a certain *duality* in I_0 and J_0 . By means of the relation

$$\theta_3(z; \tau) = \frac{e^{-i\pi z^2/\tau}}{\sqrt{-i\tau}} \theta_3\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \quad (23)$$

we may write the determinant in a third way. For $\alpha_2^* = \frac{i\pi\Omega^2}{\tau} + \alpha_2$, we have that

$$\log P^{\text{Ai}}(sJ) = -\alpha_2^* s^3 + \log \theta_3\left(s^{3/2} \frac{\Omega}{\tau}; -\frac{1}{\tau}\right) - \frac{1}{2} \log s + \chi_1^* + o(1), \quad s \rightarrow +\infty, \quad (24)$$

with

$$\chi_1^* = -\frac{1}{2} \log \left| \frac{I_0}{\pi} \right| - \frac{1}{8} \log |q(a)q(b)q(c)| + 4\zeta'(-1). \quad (25)$$

In the recent work [4], Blackstone, Charlier and Lenells have simultaneously and independently analyzed the large- s asymptotics of $\log P^{\text{Ai}}(sJ)$. They found the expansion $-\alpha_2 s^3 - \frac{1}{2} \log s + \log \theta_3(s^{3/2}\Omega) + \chi' + \mathcal{O}(1/s)$ with an undetermined constant term $\chi' = \chi'(a, b, c)$. (This analysis was then extended by the authors to the case of n gaps in the bulk of the Airy process in [5], and in the Bessel process in [6].) They followed the approach of [11], and used Riemann-Hilbert analysis to obtain the asymptotics of the derivative $\frac{d}{ds} \log P^{\text{Ai}}(sJ)$. To determine the multiplicative constant in the asymptotics of the determinant, one would need to integrate the logarithmic derivative over s . However, there is no appropriate initial point s_0 where the asymptotics of $P^{\text{Ai}}(s_0J)$ would be independently known. For this reason the problem of determining the constant is different and requires additional ideas. As mentioned above, it was first solved for a single interval in the bulk of the spectrum of GUE. The solution in [21] and also the one in [14] involved representing the sine-kernel determinant as a double-scaling limit of a Toeplitz determinant (whose asymptotics at certain points are either known or can be independently determined) and then integrating a differential identity for Toeplitz determinants starting from this point. The differential identity can be found in the asymptotic form by a Riemann-Hilbert analysis. Similarly, the constant χ_{Airy} in (5) was determined in [12] by integrating a differential identity for a Hankel determinant and using the fact

that the asymptotics of the Hankel determinant can be independently established at a certain point, thus providing an initial point for the integration.

It was observed in [19] that for the sine-kernel determinant on 2 intervals, one does not need to reduce the problem to Toeplitz determinants (although, of course, one still can), but rather it is easier to notice first that if the 2 intervals are far apart from each other in comparison with their width, then the determinant splits (to the main orders in s) into a product of 2 determinants, each on a single interval so that we can use the asymptotics (7). Then to complete the solution, one determines the asymptotic form of a differential identity with respect to the edges of the intervals and performs integration starting from a point where splitting into the product occurs.

Here we follow a similar approach to [19]. First, in Section 2, we establish a separation lemma, which states that if the length of the interval (c, b) is relatively small compared to $b - a$, and a is close to zero from the left then $P^{\text{Ai}}(sJ)$ is written (to main orders in s) as a product of a sine-kernel and Airy-kernel determinant for which we can use the asymptotics (7), (5), respectively. More precisely, we prove

Lemma 3 (*Separation of gaps*). *Set $b = c + \frac{2t_0}{s^{3/2}}$ and $a = -\frac{t_1}{s}$, where $t_0 = t_1 = (\log s)^{1/8}$. Then as $s \rightarrow +\infty$,*

$$\log P^{\text{Ai}}(sJ) = \log \det(I - K^{\text{sine}})_{(-t_0\sqrt{|c|}, t_0\sqrt{|c|})} + \log \det(I - K^{\text{Ai}})_{(-t_1, +\infty)} + o(1) \tag{26}$$

$$= -\frac{|c|}{2}t_0^2 - \frac{1}{4} \log(\sqrt{|c|}t_0) + c_{\text{sine}} - \frac{1}{12}t_1^3 - \frac{1}{8} \log t_1 + \chi_{\text{Airy}} + o(1). \tag{27}$$

Remark 4. We can also choose different values for t_0, t_1 , and slightly larger in s than $(\log s)^{1/8}$.

To solve the problem for arbitrary fixed $c < b < a < 0$, we proceed as follows. In Section 3, we formulate differential identities which express the derivatives with respect to the edges, $\frac{d}{da} \log P^{\text{Ai}}(sJ)$, $\frac{d}{db} \log P^{\text{Ai}}(sJ)$, in terms of a solution of a certain Riemann-Hilbert problem. In Section 4, this problem is solved asymptotically for large s and fixed $c < b < a < 0$, by the Deift-Zhou steepest descent method (as for the sine-kernel on 2 intervals, the solution involves θ -functions). In Section 4.6 we verify that this solution is extendable for variable edges up to the scaling regime of Lemma 3. We then substitute the solution into the differential identities and perform integration: first setting $a = \alpha - x$, $b = \beta + x$, where α is close to zero and β is close to c in the sense of Lemma 3, we integrate $\frac{d}{dx} \log P^{\text{Ai}}(sJ)$ from $x = 0$ to x_0 such that $a = \alpha - x_0$. This fixes the desired value of a . Then we integrate the identity $\frac{d}{db} \log P^{\text{Ai}}(sJ)$ to the general position of b and thus conclude the proof of Theorem 1. Note that unlike the cases of just 1 gap, the integration of the differential identities for 2 gaps is technically involved: we have to use various identities for θ -functions (see Section 4.3) and averaging over fast oscillations. The details of this computation are presented in Section 5.

2. Separation of gaps: proof of Lemma 3

In this section $C_j, j = 1, 2, \dots$, will denote positive constants whose value may change from line to line.

Recall the kernel

$$K^{Ai}(z, z') = \frac{Ai(z)Ai'(z') - Ai'(z)Ai(z')}{z - z'}, \tag{28}$$

and denote

$$K_{\alpha}^{\text{sine}}(z, z') = \frac{\sin(\alpha(z - z'))}{\pi(z - z')}. \tag{29}$$

Thus $K^{\text{sine}}(z, z') = K_1^{\text{sine}}(z, z')$. Let

$$b = c + \frac{2t_0}{s^{3/2}}, \quad t_0/s^{1/2} \rightarrow 0,; \quad a = -\frac{t_1}{s}, \quad t_1/s \rightarrow 0. \tag{30}$$

We first observe the following.

Proposition 5. *For $z, z' \in (sc, sb)$, let*

$$z = cs + \frac{xt_0}{\sqrt{s}}, \quad z' = cs + \frac{yt_0}{\sqrt{s}}, \quad x, y \in (0, 2).$$

Then, uniformly for $x, y \in (0, 2)$,

$$K^{Ai}(z, z') = \frac{\sqrt{s}}{t_0} \left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x, y) + \mathcal{O}\left(\frac{t_0^2}{s^{3/2}}\right) \right), \quad t_0^2/s^{3/2} \rightarrow 0, \quad s \rightarrow +\infty. \tag{31}$$

Proof. We will make use of the following expansions of the Airy function for large, negative argument (see, e.g., [1]):

$$Ai(-z) = \frac{1}{\sqrt{\pi}z^{1/4}} \left[\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{z^{3/2}}\right) \right], \tag{32}$$

and for the derivative ($Ai'(u) = \frac{d}{du}Ai(u)$)

$$Ai'(-z) = -\frac{z^{1/4}}{\sqrt{\pi}} \left[\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{z^{3/2}}\right) \right], \tag{33}$$

as $z \rightarrow +\infty$, where the principal branches of the roots are taken with cuts along $(-\infty, 0)$. Let us define, for $\text{Re}(z) > 0$,

$$\text{Err}_{\text{Ai}}(z) = \sqrt{\pi}z^{1/4}\text{Ai}(-z) - \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right), \tag{34}$$

so that

$$\text{Ai}(-z) = \frac{1}{\sqrt{\pi}z^{1/4}} \left[\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) + \text{Err}_{\text{Ai}}(z) \right]. \tag{35}$$

Since the Airy function is entire it follows, by definition, that $\text{Err}_{\text{Ai}}(z)$ is analytic in $\text{Re}(z) > 0$. Moreover it follows from differentiating Err_{Ai} , making use of the expansions (32) and (33), that

$$\text{Err}_{\text{Ai}}(z) = \mathcal{O}\left(\frac{1}{z^{3/2}}\right), \quad \text{Err}'_{\text{Ai}}(z) = \mathcal{O}\left(\frac{1}{z}\right), \quad \text{Err}''_{\text{Ai}}(z) = \mathcal{O}\left(\frac{1}{z^{1/2}}\right), \tag{36}$$

where for the second derivative we have used, in addition, the Airy equation $\text{Ai}''(z) = z\text{Ai}(z)$.

Let us denote

$$S(z) = \sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right), \quad \text{and} \quad C(z) = \cos\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right).$$

Noting that

$$\frac{(-z')^{1/4}}{(-z)^{1/4}} = 1 + \frac{(-z')^{1/4} - (-z)^{1/4}}{(-z)^{1/4}} = 1 + (z - z')\mathcal{O}(1/s) = 1 + \mathcal{O}\left(\frac{t_0(x - y)}{s^{3/2}}\right),$$

we obtain, by means of (35) and its derivative, that

$$\begin{aligned} \pi(z - z')K^{\text{Ai}}(z, z') &= \pi [\text{Ai}(z)\text{Ai}'(z') - \text{Ai}(z')\text{Ai}'(z)] = \\ &- [S(z)C(z') - S(z')C(z)] + C(z)\text{Err}_{\text{Ai}}(-z') - C(z')\text{Err}_{\text{Ai}}(-z) \\ &+ \frac{1}{(-z')^{1/4}} [S(z') + \text{Err}_{\text{Ai}}(-z')] \text{Err}'_{\text{Ai}}(-z) - \frac{1}{(-z)^{1/4}} [S(z) + \text{Err}_{\text{Ai}}(-z)] \text{Err}'_{\text{Ai}}(-z') \\ &+ \mathcal{O}\left(\frac{t_0(x - y)}{s^{3/2}}\right). \end{aligned} \tag{37}$$

For f, g denoting analytic functions, there are $z_1^*, z_2^* \in (z, z')$ such that

$$\begin{aligned} \frac{f(z)g(z') - f(z')g(z)}{z - z'} &= \frac{f(z)(g(z) + g'(z_1^*)(z - z')) - (f(z) + f'(z_2^*)(z - z'))g(z)}{z - z'} \\ &= f(z)g'(z_1^*) - f'(z_2^*)g(z). \end{aligned} \tag{38}$$

Applying this to the pairs $f(z) = C(z)$, $g(z) = \text{Err}_{\text{Ai}}(-z)$ and $f(z) = \frac{1}{(-z)^{1/4}}[S(z) + \text{Err}_{\text{Ai}}(-z)]$, $g(z) = \text{Err}'_{\text{Ai}}(-z)$, in (37), and using the estimates (36), we obtain (31), uniformly in $x, y \in (0, 2)$. \square

With the assistance of the above result we may now prove Lemma 3.

Consider the kernel

$$\widehat{K}^{\text{Ai}}(z, z') = \begin{cases} K^{\text{Ai}}(z, z'), & z, z' \in (sc, sb) \text{ or } z, z' \in (sa, +\infty) \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

The corresponding determinant splits into the desired product of determinants up to a small error. We have

Proposition 6. *There exist constants $C_0, C_1 > 0$ depending only on c such that*

$$\left| \det(I - \widehat{K}^{\text{Ai}})_{L^2(sJ)} - \det(I - K^{\text{sine}})_{L^2(0,2|c|^{1/2}t_0)} \det(I - K^{\text{Ai}})_{L^2(-t_1, +\infty)} \right| \leq C_0 \frac{e^{C_1 t_0^2}}{s^{3/2}}. \tag{40}$$

Proof. First we note that

$$\begin{aligned} \det(I - \widehat{K}^{\text{Ai}})_{L^2(sJ)} &= 1 + \sum_{m=1}^{\infty} \sum_{k=0}^m \frac{(-1)^m}{(m-k)!k!} \\ &\times \int_{\substack{z_1, \dots, z_k \in (cs, bs) \\ z_{k+1}, \dots, z_m \in (as, +\infty)}} \det(K^{\text{Ai}}(z_i, z_j))_{i,j=1}^k \det(K^{\text{Ai}}(z_i, z_j))_{i,j=k+1}^m dz_1 \cdots dz_m \\ &= \det(I - K^{\text{Ai}})_{L^2(sc, sb)} \det(I - K^{\text{Ai}})_{L^2(-t_1, +\infty)}. \end{aligned} \tag{41}$$

Furthermore, using the change of variables,

$$z_j = sc + \frac{t_0 x_j}{\sqrt{s}}, \quad j = 1, \dots, k, \tag{42}$$

and Proposition 5, we write

$$\begin{aligned} \det(I - K^{\text{Ai}})_{L^2(sc, sb)} &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \\ &\times \int_{x_1, \dots, x_k \in (0,2)} \det \left(K^{\text{sine}}_{t_0 \sqrt{|c|}}(x_i, x_j) + \mathcal{O} \left(\frac{t_0^2}{s^{3/2}} \right) \right)_{i,j=1}^k dx_1 \cdots dx_k. \end{aligned} \tag{43}$$

Let r_j denote the j 'th row of $\left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j)\right)_{i,j=1}^k$ and let e_j denote the j 'th row of the error matrix, so that $e_j = \mathcal{O}\left(\frac{t_0^2}{s^{3/2}}\right)$, $j = 1, \dots, k$. Then we have

$$\begin{aligned} & \det \left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j) + \mathcal{O}\left(\frac{t_0^2}{s^{3/2}}\right) \right)_{i,j=1}^k = \det \begin{pmatrix} r_1 + e_1 \\ r_2 + e_2 \\ \vdots \\ r_k + e_k \end{pmatrix} = \\ & = \det \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} + \sum_{j=1}^k \det \begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ e_j \\ r_{j+1} + e_{j+1} \\ \vdots \\ r_k + e_k \end{pmatrix} = \\ & = \det \left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j) \right)_{i,j=1}^k + \frac{t_0^2}{s^{3/2}} \sum_{j=1}^k \det \begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ \mathcal{O}(1) \\ r_{j+1} + e_{j+1} \\ \vdots \\ r_k + e_k \end{pmatrix}. \end{aligned}$$

Note that the determinants in the sum over j on the right hand side may be estimated by Hadamard's inequality. Let v_ℓ denote the rows of a matrix. If all the matrix elements $\|v_\ell m\| \leq C$,

$$|\det(v_1, v_2, \dots, v_k)^T| \leq \prod_{\ell=1}^k \|v_\ell\| \leq C_1^k k^{k/2} \leq (100 \cdot C_1)^k \sqrt{k!}. \tag{44}$$

Using uniformity of the error term and the estimate

$$\left| K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j) \right| \leq C_1 t_0, \tag{45}$$

we obtain

$$\begin{aligned} & \left| \det \left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j) + \mathcal{O}\left(\frac{t_0^2}{s^{3/2}}\right) \right)_{i,j=1}^k - \det \left(K_{t_0\sqrt{|c|}}^{\text{sine}}(x_i, x_j) \right)_{i,j=1}^k \right| \leq \\ & \leq \frac{t_0^2}{s^{3/2}} k (C_2 t_0)^k \sqrt{k!} \leq \frac{t_0^2}{s^{3/2}} (C_3 t_0)^k \sqrt{k!}. \end{aligned} \tag{46}$$

This estimate and (43) imply (note that $\det(I - K_{t_0\sqrt{|c|}}^{\text{sine}})_{L^2(0,2|c|^{1/2}t_0)} = \det(I - K_{t_0\sqrt{|c|}}^{\text{sine}})_{L^2(0,2)}$)

$$\begin{aligned}
 & \left| \det(I - K^{\text{Ai}})_{L^2(sc, sb)} - \det(I - K^{\text{sine}})_{L^2(0, 2|c|^{1/2}t_0)} \right| \\
 & \leq \frac{t_0^2}{s^{3/2}} \sum_{k=1}^{\infty} \frac{(C_1 t_0)^k}{\sqrt{k!}} \leq \frac{t_0^2}{s^{3/2}} \left(\sum_{k=1}^{\infty} \frac{(C_1 t_0)^{2k} k^2}{k!} \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{1/2} \leq C_0 \frac{e^{C_2 t_0^2}}{s^{3/2}}. \tag{47}
 \end{aligned}$$

Since by (5),

$$\left| \det(I - K^{\text{Ai}})_{L^2(-t_1, +\infty)} \right| \leq C_0,$$

the estimate (47) implies the statement of the proposition. \square

On the other hand, the difference between the determinants corresponding to the kernels K^{Ai} and \widehat{K}^{Ai} is small:

Proposition 7. *There exist constants $C_2, C_3 > 0$ depending only on c such that*

$$\left| \det(I - K^{\text{Ai}})_{L^2(sJ)} - \det(I - \widehat{K}^{\text{Ai}})_{L^2(sJ)} \right| \leq \frac{C_2}{s^{3/4}} e^{C_3 t_1^2 \max\{t_0^2, t_1\}}. \tag{48}$$

Proof. We use another representation of the Airy-kernel,

$$K^{\text{Ai}}(z, z') = \int_0^{\infty} \text{Ai}(z + x) \text{Ai}(z' + x) dx,$$

and the asymptotics of the Airy function at $+\infty$, see, e.g., [1],

$$\text{Ai}(z) = \frac{C_0}{z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right) \left[1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right) \right], \quad z \rightarrow +\infty, \tag{49}$$

and also the arguments similar to those in Proposition 5 to conclude that

$$K^{\text{Ai}}(z, z') = \kappa(z, z') f(z) f(z'), \quad z, z' \in sJ, \tag{50}$$

where

$$|\kappa(z, z')| \leq C_0 t_1^{1/2}, \quad |f(z)| \leq \begin{cases} C_1, & z \in (-t_1, t_1) \\ C_1 e^{-C_2 z^{3/2}}, & z \in (t_1, \infty) \end{cases}, \quad z, z' \in (sa, +\infty). \tag{51}$$

Moreover, on the interval (sc, sb) ,

$$|\kappa(z, z')| \frac{t_0}{s^{1/2}} \leq C_0 t_0, \quad |f(z)| \leq C_1, \quad z, z' \in (sc, sb). \tag{52}$$

We similarly have

$$\widehat{K}^{Ai}(z, z') = \widehat{\kappa}(z, z')f(z)f(z'), \quad z, z' \in sJ, \quad |\widehat{\kappa}(z, z')| \leq \begin{cases} C_1 s^{1/2}, & z, z' \in (sc, sb) \\ C_1 t_1^{1/2}, & z, z' \in (sa, +\infty) \end{cases}. \tag{53}$$

Consider the case when $z \in (sc, sb)$ and $z' \in (sa, +\infty)$. Here we have that $z - z' > (a - b)s$. By representation (28), it follows, from (32), (49), and the asymptotics for the derivatives, that

$$|\kappa(z, z')| \leq \frac{C_1}{s^{3/4}}, \quad z \in (sc, sb), \quad z' \in (sa, +\infty). \tag{54}$$

After making the change of variable (42) in the integrals over (cs, bs) , we can use Proposition 12 in [19] (which is based on Hadamard’s inequality) to estimate the difference $|\det(\kappa(z_j, z_k))_{j,k=1}^{m+1} - \det(\widehat{\kappa}(z_j, z_k))_{j,k=1}^{m+1}|$ below and obtain

$$\begin{aligned} & \left| \det(I - K^{Ai})_{L^2(sJ)} - \det(I - \widehat{K}^{Ai})_{L^2(sJ)} \right| \\ & \leq \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \\ & \times \int_{z_1, \dots, z_{m+1} \in sJ} \left| \det(\kappa(z_i, z_j))_{i,j=1}^{m+1} - \det(\widehat{\kappa}(z_i, z_j))_{i,j=1}^{m+1} \right| f(z_1)^2 dz_1 \cdots f(z_{m+1})^2 dz_{m+1} \\ & = \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} \frac{(-1)^{m+1}}{(m+1-k)!k!} \\ & \times \int_{\substack{z_1, \dots, z_k \in (cs, bs) \\ z_{k+1}, \dots, z_{m+1} \in (sa, +\infty)}} \left| \det(\kappa(z_i, z_j))_{i,j=1}^{m+1} - \det(\widehat{\kappa}(z_i, z_j))_{i,j=1}^{m+1} \right| f(z_1)^2 dz_1 \cdots f(z_{m+1})^2 dz_{m+1} \\ & \leq \frac{C_0}{s^{3/4}} \sum_{m=0}^{\infty} \frac{(C_1 t_1 \max\{t_0, t_1^{1/2}\})^m}{\sqrt{m!}} \leq \frac{C_2}{s^{3/4}} e^{C_3 t_1^2 \max\{t_0^2, t_1\}}. \end{aligned} \tag{55}$$

Indeed, by the estimates (51)–(54) and Proposition 12 in [19], if $z_1, \dots, z_k \in (cs, bs)$,

$$\left| \det(\kappa(z_i, z_j))_{i,j=1}^{m+1} - \det(\widehat{\kappa}(z_i, z_j))_{i,j=1}^{m+1} \right| \left(\frac{t_0}{s^{1/2}} \right)^k \leq \frac{C_0}{s^{3/4}} (C_1 \max\{t_0, t_1^{1/2}\})^m \sqrt{m!}. \quad \square \tag{56}$$

Combining the statements of Propositions 6 and 7 we obtain

$$\left| \det(I - K^{\text{Ai}}) - \det(I - K^{\text{sine}})_{L^2(0,2|c|^{1/2}t_0)} \det(I - K^{\text{Ai}})_{L^2(-t_1,+\infty)} \right| \leq \tag{57}$$

$$\leq \frac{C_1}{s^{3/2}} \exp(C_2 t_0^2) + \frac{C_3}{s^{3/4}} \exp(C_4 t_1^2 \max\{t_0^2, t_1\}).$$

We may thus choose, for example, $t_0 = t_0(s) = t_1 = t_1(s) = \log(s)^{1/8}$, which proves the lemma.

3. Differential identity

In this section we express, for $p = a, b, c$, the derivative $\frac{d}{dp} \log \det(I - K^{\text{Ai}})$ in terms of a solution of a certain Riemann-Hilbert problem (RHP). First let us write the kernel of the operator K^{Ai} in the form

$$K^{\text{Ai}}(z, z') = \frac{\vec{f}(z)^T \vec{g}(z')}{z - z'}, \quad \vec{f}(z) = \begin{pmatrix} \text{Ai}(z) \\ \text{Ai}'(z) \end{pmatrix}, \quad \vec{g}(z) = \begin{pmatrix} \text{Ai}'(z) \\ -\text{Ai}(z) \end{pmatrix}. \tag{58}$$

Note that $\sum_{k=1}^2 f_k(z)g_k(z) = 0$. The operators of this form are known as integrable operators. They possess the following crucial properties (Lemmata 2.8 and 2.12 in [11]).

The resolvent R_s of the operator K^{Ai} is given by

$$(I - K^{\text{Ai}})^{-1} = I + R_s, \tag{59}$$

where R_s can be expressed as

$$R_s(z, z') = \frac{\vec{F}^T(z) \vec{G}(z')}{z - z'}, \quad \vec{F}(z) = (I - K^{\text{Ai}})^{-1} \vec{f}(z), \quad \vec{G}(z) = (I - K^{\text{Ai}})^{-1} \vec{g}(z), \tag{60}$$

and $\sum_{k=1}^2 F_k(z)G_k(z) = 0$. Furthermore,

$$\vec{F}(z) = m_+(z) \vec{f}(z), \quad \vec{G}(z) = (m_+^{-1}(z))^T \vec{g}(z), \tag{61}$$

where $m(z)$ is the 2×2 matrix, the solution to the following RHP:

- (a) $m(z)$ is analytic in $\mathbb{C} \setminus [c, b] \cup [a, \infty]$.
- (b) The function $m(z)$ possesses L^2 boundary values m_+ and m_- as z approaches $(c, b) \cup (a, \infty)$ from above and below, respectively. These boundary values are related by the condition

$$m_+(z) = m_-(z) V^{-1}(sz), \quad z \in (c, b) \cup (a, \infty), \tag{62}$$

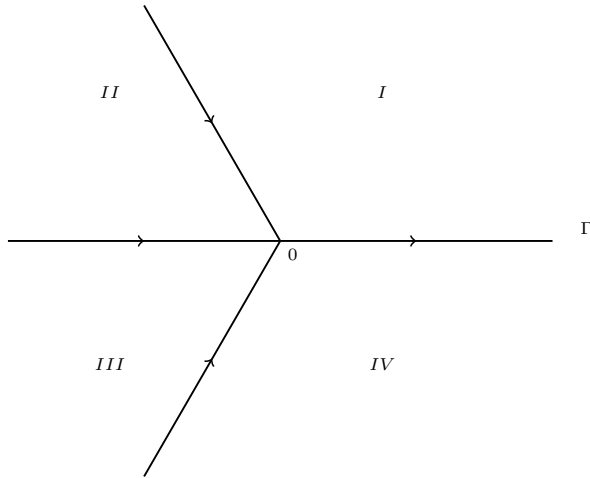


Fig. 2. Jump contour for the Φ -RH problem.

where the jump matrix

$$V^{-1}(z) = I - 2\pi i \vec{f}(z) \vec{g}(z)^T = \begin{pmatrix} 1 - 2\pi i \text{Ai}(z) \text{Ai}'(z) & 2\pi i \text{Ai}(z)^2 \\ -2\pi i \text{Ai}'(z)^2 & 1 + 2\pi i \text{Ai}(z) \text{Ai}'(z) \end{pmatrix}. \tag{63}$$

(c) $m(z)$ satisfies

$$m(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \rightarrow \infty. \tag{64}$$

Before proceeding with the derivation of the differential identity, we will now reduce this RHP to another one, with constant jumps. For that, we need the following model problem.

3.1. Airy model RH problem

Let

$$y_j(z) = \omega^j \text{Ai}(\omega^j z), \quad \omega = e^{\frac{2\pi i}{3}}, \quad j = 0, 1, 2. \tag{65}$$

In each region $I - IV$ as per Fig. 2, consider the function Φ defined by

$$\Phi(z) = \sqrt{2\pi} e^{-\frac{\pi i}{4}} \begin{cases} \begin{pmatrix} y_0 & -y_2 \\ y'_0 & -y'_2 \end{pmatrix}, & \text{for } z \in I, \\ \begin{pmatrix} -y_1 & -y_2 \\ -y'_1 & -y'_2 \end{pmatrix}, & \text{for } z \in II, \\ \begin{pmatrix} -y_2 & y_1 \\ -y'_2 & y'_1 \end{pmatrix}, & \text{for } z \in III, \\ \begin{pmatrix} y_0 & y_1 \\ y'_0 & y'_1 \end{pmatrix}, & \text{for } z \in IV. \end{cases} \tag{66}$$

The pre-factor normalises the determinant to unity at all points. For $\Gamma = \mathbb{R} \cup \mathbb{R}^+ e^{\pm \frac{2\pi i}{3}}$ (see Fig. 2) let

$$v_\Gamma(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \mathbb{R}^+ e^{\pm \frac{2\pi i}{3}}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\infty, 0), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in (0, \infty), \end{cases} \tag{67}$$

it is known, see e.g. [8], [10], that Φ satisfies the following RHP with L^2 boundary values:

$$\begin{aligned} &\Phi(z) \text{ is analytic in } \mathbb{C} \setminus \Gamma, \\ &\Phi_+(z) = \Phi_-(z)v_\Gamma(z), \quad z \in \Gamma \setminus \{0\}, \\ &\Phi(z) = z^{-\frac{1}{4}\sigma_3} N_0 \left(I + \begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix} \frac{1}{48z^{3/2}} + \mathcal{O}\left(\frac{1}{z^3}\right) \right) e^{-\frac{2}{3}z^{\frac{3}{2}}\sigma_3}, \quad z \rightarrow \infty, \\ &\Phi(z) = O(1), \quad z \rightarrow 0, \end{aligned} \tag{68}$$

where N_0 is given by

$$N_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}. \tag{69}$$

The calculation to verify that Φ indeed satisfies (68) rests upon the well-known facts

$$\text{Ai}(z) + e^{\frac{2\pi i}{3}} \text{Ai}(ze^{\frac{2\pi i}{3}}) + e^{-\frac{2\pi i}{3}} \text{Ai}(ze^{-\frac{2\pi i}{3}}) = 0,$$

and

$$\text{Ai}(z) = \frac{z^{-1/4} e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}} \left(1 - \frac{5}{48z^{3/2}} + \mathcal{O}\left(\frac{1}{z^3}\right) \right), \quad z \rightarrow \infty.$$

For further details we refer the reader to p. 216 in [10].

The reason for introducing Φ is because the jump, $V^{-1}(sz)$, of the m -RHP may be factorised in terms of $\Phi(sz)$ as follows:

$$V^{-1}(sz) = \begin{cases} \Phi_+(sz) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \Phi_+^{-1}(sz), & z \in (0, \infty), \\ \Phi_+(sz) \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \Phi_+^{-1}(sz), & z \in (c, b) \cup (a, 0). \end{cases} \tag{70}$$

3.2. RH problem for X

Making use of the Airy model problem, we transform the m -RHP into a form with constant jump matrices. We define the matrix $X(z)$ in each region of \mathbb{C} as per Fig. 3.

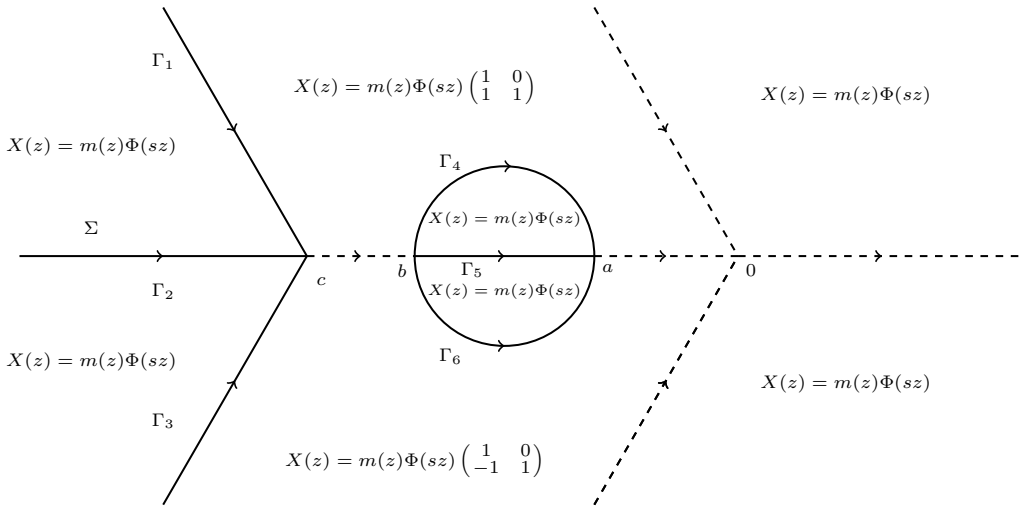


Fig. 3. Definition of X in various regions.

Using (70) and the m and Φ -RH problems, we find that $X(z)$ satisfies the following problem, where Σ denotes the union of bold curves on Fig. 3.

$X(z)$ is analytic in $\mathbb{C} \setminus \Sigma$,

$$X_+(z) = X_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ for } z \in \Gamma_2 \cup \Gamma_5 = (-\infty, c) \cup (b, a),$$

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ for } z \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_6, \tag{71}$$

$$X(z) = m(z)(sz)^{-\frac{1}{4}\sigma_3} N_0 \left(I + \begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix} \frac{1}{48(sz)^{3/2}} + \mathcal{O}\left(\frac{1}{(sz)^3}\right) \right) e^{-\frac{2}{3}(sz)^{\frac{3}{2}}\sigma_3},$$

$z \rightarrow \infty$.

Remark 8. By uniqueness of the solution, the asymptotic condition at infinity can be replaced by the condition

$$X(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) (sz)^{-\frac{1}{4}\sigma_3} N_0 e^{-\frac{2}{3}(sz)^{\frac{3}{2}}\sigma_3}, \quad z \rightarrow \infty.$$

Remark 9. The boundary values of X are L^2 functions. However, more can be said: they are continuous functions at all internal points of the contour with explicitly described logarithmic singularities at the node points. For instance, in a neighbourhood of $z = c$ we have

$$X(z) = \tilde{X}(z) \begin{pmatrix} 1 & \frac{1}{2\pi i} \log(z-c) \\ 0 & 1 \end{pmatrix} \begin{cases} I, & \text{for } -2\pi/3 < \arg(z-c) < 2\pi/3, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } 2\pi/3 < \arg(z-c) < \pi, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z-c) < -2\pi/3, \end{cases}$$

where the branch of the logarithm is fixed by the condition $-\pi < \arg(z-c) < \pi$, and where $\tilde{X}(z)$ is holomorphic in a neighbourhood of $z = c$.

The reader may wish to compare this behaviour of X at $x = c$ with the local parametrix at zero, see, e.g., (221) below.

3.3. The identity

We now proceed with the derivation of the differential identity. Consider the case when $p = b$, the identities at the points a, c are obtained similarly.

$$\frac{d}{db} \log P^{Ai}(sJ) = \text{tr}(I - K^{Ai})^{-1} \frac{dK^{Ai}}{db} = -((I - K^{Ai})^{-1} K^{Ai})(b, b) = -R_s(b, b), \tag{72}$$

where, for the final equality, we used (59). The negative sign at $R_s(b, b)$ comes from the fact that b is the lower limit of integration. (For $p = a, c$, we have the opposite sign.) Now, by (60), we may write

$$R_s(p, p) = \lim_{z, z' \rightarrow p} \left(\frac{F_1(z)G_1(z') + F_2(z)G_2(z')}{z - z'} \right) = - \lim_{z \rightarrow p} (F_1(z)G'_1(z) + F_2(z)G'_2(z)). \tag{73}$$

For definiteness, assume that the limit is taken from outside the lens and with $\text{Im } z > 0$. By (61), the definition of X and that of Φ in sector II,

$$\begin{aligned} \vec{F}(z) &= m_+(z) \vec{f}(z) = X_+(z) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Phi_+^{-1}(sz) \begin{pmatrix} y_0(sz) \\ y'_0(sz) \end{pmatrix} \\ &= \sqrt{2\pi} e^{-i\pi/4} X_+(z) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -y'_2(sz) & y_2(sz) \\ y'_1(sz) & -y_1(sz) \end{pmatrix} \begin{pmatrix} y_0(sz) \\ y'_0(sz) \end{pmatrix}, \end{aligned} \tag{74}$$

where we used the fact that $\det \Phi(z) \equiv 1$. Using it once again, we reduce (74) to

$$\vec{F}(z) = \frac{1}{\sqrt{2\pi} e^{-i\pi/4}} X_+(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2\pi} e^{-i\pi/4}} \begin{pmatrix} X_{11+}(z) \\ X_{21+}(z) \end{pmatrix}. \tag{75}$$

Similarly, we obtain

$$\begin{aligned} \vec{G}(z) &= (m_+^{-1}(z))^T \vec{g}(z) \\ &= \sqrt{2\pi} e^{-i\pi/4} (X_+^{-1}(z))^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -y_1(sz) & -y_1'(sz) \\ -y_2(sz) & -y_2'(sz) \end{pmatrix} \begin{pmatrix} y_0'(sz) \\ -y_0(sz) \end{pmatrix} \\ &= \frac{1}{\sqrt{2\pi} e^{-i\pi/4}} (X_+^{-1}(z))^T \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2\pi} e^{-i\pi/4}} \begin{pmatrix} X_{21+}(z) \\ -X_{11+}(z) \end{pmatrix}. \end{aligned} \tag{76}$$

Substituting (75) and (76) into (73), we finally obtain

$$R_s(b, b) = \frac{1}{2\pi i} \lim_{z \rightarrow b} (X_{11} X'_{21} - X_{21} X'_{11})(z) = \frac{1}{2\pi i} \lim_{z \rightarrow p} (X^{-1}(z) X'(z))_{21}, \tag{77}$$

where the limit is taken from outside the lens with $\text{Im } z > 0$; and therefore by (72)

$$\frac{d}{db} \log P^{Ai}(sJ) = -R_s(b, b) = -\frac{1}{2\pi i} \lim_{z \rightarrow b} (X^{-1}(z) X'(z))_{21}. \tag{78}$$

At the points a and c , we obtain the same result but with the opposite sign.

Thus we have

Lemma 10 (*Differential identity*). *The Fredholm determinant (1) satisfies:*

$$\frac{d}{dp} \log P^{Ai}(sJ) = \pm \frac{1}{2\pi i} \lim_{z \rightarrow p} (X^{-1}(z) X'(z))_{21}, \tag{79}$$

where the $+$ sign is taken if $p = a, c$ and the $-$ sign, if $p = b$. The limit is taken from outside the lens and with $\text{Im } z > 0$.

4. Solution to the Riemann-Hilbert problem for X

To solve the X -RHP for large s we, as usual, apply a series of transformations. The approach, known as the steepest descent method for Riemann-Hilbert problems, was first introduced by Deift and Zhou [15] and used and developed in many subsequent works. The first step is to normalise the exponential behaviour at ∞ by multiplying from the right by a suitable function. This process is set up as follows.

4.1. g -function and the RH problem for S

In the introduction, we defined

$$g(z) = \int_a^z \frac{q(\zeta)}{p(\zeta)^{1/2}} d\zeta, \quad \zeta \in \mathbb{C} \setminus (-\infty, a].$$

Here $p(z) = (z - a)(z - b)(z - c)$, the branch of the root is chosen positive for positive arguments, and the branch cut is on $(-\infty, c) \cup (b, a)$, $c < b < a < 0$; $q(z) = z^2 + q_1z + q_0$ is the second degree polynomial. We require that $g(z)$ satisfy conditions (11) and (12), which as we now show determine the coefficients q_0, q_1 .

We have

Lemma 11. *The function $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, a]$ and satisfies the jump conditions:*

$$g_+(z) + g_-(z) = 0, \quad z \in (-\infty, c) \cup (b, a), \quad g_+(z) = g_-(z) + 2g_+(b), \quad z \in (c, b). \tag{80}$$

The polynomial $q(z) = z^2 + q_1z + q_0$ has coefficients given by (13).

As $z \rightarrow \infty$,

$$g(z) = \frac{2}{3}z^{3/2} + \frac{\alpha_1}{z^{1/2}} + \frac{\alpha_2}{z^{3/2}} + \mathcal{O}(z^{-5/2}), \tag{81}$$

where

$$\alpha_1 = \frac{1}{2}(ab + ac + bc) - \frac{1}{4}(a^2 + b^2 + c^2) - 2q_0,$$

and

$$\alpha_2 = -\frac{1}{12} (a^3 + b^3 + c^2 - (a + b)(a + c)(b + c) + 4(a + b + c)q_0). \tag{82}$$

Proof. Since $\sqrt{p(z)}_+ + \sqrt{p(z)}_- = 0$, on $(-\infty, c) \cup (b, a)$, it follows from (12) that

$$g_+(z) + g_-(z) = 0, \quad \text{for } z \in (-\infty, c) \cup (b, a).$$

On the other hand, for $z \in (c, b)$, we have

$$g_+(z) - g_-(z) = 2 \int_a^b \frac{q(\zeta)}{p(\zeta)_+^{1/2}} = 2g_+(b) = 2g_+(c),$$

where the last equality follows from (12). Thus we have (80).

Write the expansion at infinity

$$\frac{q(z)}{\sqrt{p(z)}} = z^{1/2} \left(1 + \frac{1}{z} \left(q_1 + \frac{a + b + c}{2} \right) + \mathcal{O} \left(\frac{1}{z^2} \right) \right).$$

Setting

$$q_1 = -\frac{a + b + c}{2}$$

and integrating, we obtain

$$g(z) = \frac{2}{3}z^{3/2} + g_0 + \mathcal{O}(z^{-1/2}), \tag{83}$$

for some constant g_0 . It follows from the jump condition $g_+(z) + g_-(z) = 0$ on the half-line $(-\infty, c)$ that $g_0 = 0$. Thus, condition (11) holds with q_1 given in (13). The first expression in (13),

$$q_0 = -\frac{J_2 + q_1 J_1}{J_0}$$

follows immediately from (12). Next, taking the integral over the B_1 cycle (see Fig. 1) on the Riemann surface of $p(z)^{1/2}$, we have

$$0 = -\oint_{B_1} \frac{d}{dz} \sqrt{p(z)} dz = \int_c^b \frac{p'(z) dz}{\sqrt{p(z)}} = 3J_2 + 4q_1 J_1 + (ab + ac + bc)J_0. \tag{84}$$

This allows one to write J_2 in terms of J_0, J_1 , and thus we obtain the second equality in (13).

A straightforward series expansion of $g(z)$ at infinity verifies the values of α_1, α_2 . \square

Now set

$$S(z) = X(z)e^{s^{3/2}g(z)\sigma_3}. \tag{85}$$

It follows from the lemma above and the X -RH problem, in (71), that S satisfies the following problem. For Σ as in Fig. 4, we have

RH problem for S

$S(z)$ is analytic in $\mathbb{C} \setminus \Sigma$,

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, c) \cup (b, a),$$

$$S_+(z) = S_-(z)e^{2s^{3/2}g_+(b)\sigma_3}, \quad z \in (c, b),$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{2s^{3/2}g(z)} & 1 \end{pmatrix}, \quad z \in \Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_6,$$

$$S(z) = s^{-\frac{1}{4}\sigma_3} \left(S_0 + \mathcal{O}\left(\frac{1}{z}\right) \right) z^{-\frac{1}{4}\sigma_3} N_0, \quad z \rightarrow \infty, \quad \text{where } S_0 = \begin{pmatrix} 1 & 0 \\ -\alpha_1 s^{3/2} & 1 \end{pmatrix}. \tag{86}$$

The expansion of $S(z)$ at infinity is found using (81) and the expansions of Φ and m at infinity: it is the same in all sectors.

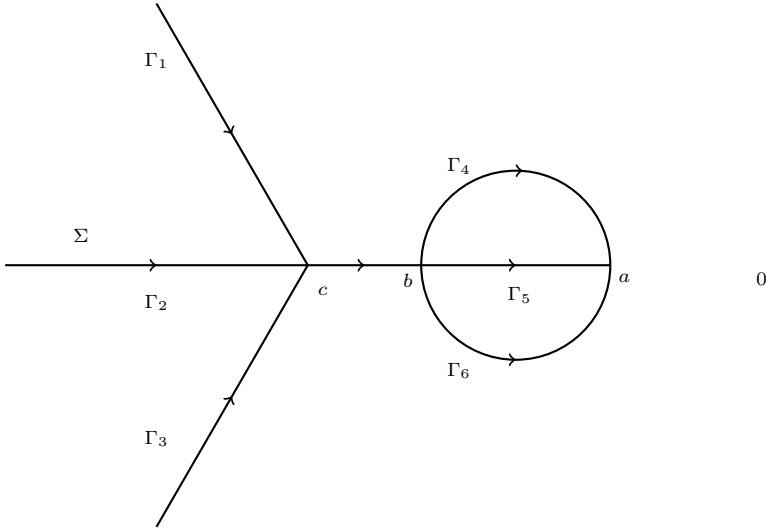


Fig. 4. Jump contour for the S -RH problem.

We now construct approximate solutions (parametrices) for this S -RH problem: outside parametrix away from the points a, b, c , and local parametrices around these points, see Fig. 5. This allows us to construct an asymptotic solution to the S -problem, and therefore to the X -problem.

4.2. Outside parametrix. Jacobi θ -functions

In this section we construct a parametrix for the S -RHP away from the points $p = a, b, c$, which models the behaviour of $S(z)$ at infinity. We proceed as follows.

4.2.1. β model problem

Set

$$\beta(z) = \left(\frac{(z - a)(z - c)}{z - b} \right)^{1/4},$$

with branch cuts along $(-\infty, c) \cup (b, a)$ and such that $\beta(z) > 0$ when $z \rightarrow +\infty$. Consider the function

$$N(z) = N_0^{-1} \beta^{-\sigma_3} N_0 = \begin{pmatrix} \frac{\beta(z) + \beta(z)^{-1}}{2} & \frac{\beta(z) - \beta(z)^{-1}}{2i} \\ -\frac{\beta(z) - \beta(z)^{-1}}{2i} & \frac{\beta(z) + \beta(z)^{-1}}{2} \end{pmatrix}, \tag{87}$$

where N_0 is given by (69). The function $N(z)$ is analytic in $\mathbb{C} \setminus \{(-\infty, c] \cup [b, a]\}$. By definition, it is immediate that

$$N(z) = N_0^{-1} \left(I + \frac{N_1}{z} + \frac{N_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right) z^{-\frac{1}{4}\sigma_3} N_0, \quad z \rightarrow \infty, \tag{88}$$

for some diagonal matrices N_j , which may be written in terms of a, b, c . Moreover, since the boundary values of β satisfy $\beta(z)_+ = i\beta(z)_-$ for $z \in (-\infty, c) \cup (b, a)$, we have that

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } z \in (-\infty, c) \cup (b, a). \tag{89}$$

4.2.2. Abel map

Recall J_0 and τ as defined in (9) and (16) respectively, in the introduction. Consider the map $u(z)$ defined by

$$u(z) = \int_a^z \omega = \frac{1}{2J_0} \int_a^z \frac{dw}{p(w)^{1/2}}, \quad \omega(z) = \frac{dz}{2J_0 p(z)^{1/2}}, \quad \mathbb{C} \setminus (-\infty, a]. \tag{90}$$

We note several relevant properties of $u(z)$ which will be required in defining the outside parametrix.

Lemma 12. *$u(z)$ is analytic in $\mathbb{C} \setminus (-\infty, a]$ and has the following properties.*

$$u(a) = 0, \quad u_+(b) = -\frac{1}{2}\tau, \quad u_+(c) = -\frac{1}{2}(1 + \tau), \tag{91}$$

$$u_+(z) + u_-(z) = -1, \quad z \in (-\infty, c), \tag{92}$$

$$u_+(z) = u_-(z) - \tau, \quad z \in (c, b), \tag{93}$$

$$u_+(z) + u_-(z) = 0, \quad z \in (b, a), \tag{94}$$

and

$$u(z) = -1/2 + \frac{u_1}{z^{1/2}} + \frac{u_2}{z^{3/2}} + \frac{u_3}{z^{5/2}} + \mathcal{O}(z^{-7/2}), \quad z \rightarrow \infty, \tag{95}$$

where the coefficients u_j may be written in terms of a, b, c .

Proof. It is immediate from the definition that on the $+$ side of the cut

$$\int_c^b \omega_+ = \frac{1}{2}, \quad \text{and} \quad \int_b^a \omega_+ = \frac{\tau}{2}.$$

Considering the integration around the A cycles (see Fig. 1), we also note that

$$\int_{-\infty}^c \omega_+ = -\frac{\tau}{2}.$$

From here the statements of the lemma easily follow. \square

Note that we consider $u(z)$ on the first sheet of the Riemann surface of the function $p(z)^{1/2}$ (where we have $p(x)^{1/2} > 0, \beta(x) > 0$, for $x > a$ on the real line). The function $u(z)$ maps the full Riemann surface to a torus (where θ -functions are defined).

4.2.3. *Theta function parametrix*

A θ -function parametrix for the sine-kernel determinant on several intervals was first constructed in [11]. We now modify the construction for our case.

Recall the function $\theta_3(z)$ given by (17) and the constant Ω given by (16). Consider

$$P^\infty(z; s) = N_0 \begin{pmatrix} \frac{\theta_3(0)\theta_3(u(z)+s^{3/2}\Omega+d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z)+d)} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\theta_3(0)\theta_3(u(z)-s^{3/2}\Omega-d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z)-d)} \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ -\frac{\theta_3(0)\theta_3(u(z)+s^{3/2}\Omega-d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z)-d)} \frac{\beta(z)-\beta(z)^{-1}}{2i} & \frac{\theta_3(0)\theta_3(u(z)-s^{3/2}\Omega+d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z)+d)} \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix}, \tag{96}$$

where d a constant to be determined. We will also write $P^\infty(z; s)$ in the short hand notation

$$P^\infty = N_0 \begin{pmatrix} \theta_{11}N_{11} & \theta_{12}N_{12} \\ \theta_{21}N_{21} & \theta_{22}N_{22} \end{pmatrix}, \tag{97}$$

where N_{jk} are the entries of $N(z)$ as defined in (87), and

$$\begin{aligned} \theta_{11}(u(z)) &= \frac{\theta_3(0)\theta_3(u(z) + s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z) + d)}, & \theta_{12}(u(z)) &= \frac{\theta_3(0)\theta_3(u(z) - s^{3/2}\Omega - d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z) - d)}, \\ \theta_{21}(u(z)) &= \frac{\theta_3(0)\theta_3(u(z) + s^{3/2}\Omega - d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z) - d)}, & \theta_{22}(u(z)) &= \frac{\theta_3(0)\theta_3(u(z) - s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u(z) + d)}. \end{aligned} \tag{98}$$

We prove:

Proposition 13. *P^∞ satisfies the following RHP.*

$$\begin{aligned} &P^\infty(z; s) \text{ is analytic in } \mathbb{C} \setminus (-\infty, a], \\ &P_+^\infty(z; s) = P_-^\infty(z; s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\infty, c) \cup (b, a), \\ &P_+^\infty(z; s) = P_-^\infty(z; s)e^{2s^{3/2}g_+(b)\sigma_3}, & z \in (c, b), \\ &P^\infty(z) = N_0(F_0 + \mathcal{O}(1/z))N(z), & z \rightarrow \infty, \end{aligned} \tag{99}$$

where

$$F_0 = \begin{pmatrix} \frac{\theta_3(0)\theta_3(1/2+s^{3/2}\Omega+d)}{\theta_3(s^{3/2}\Omega)\theta_3(1/2+d)} & 0 \\ 0 & \frac{\theta_3(0)\theta_3(1/2+s^{3/2}\Omega-d)}{\theta_3(s^{3/2}\Omega)\theta_3(1/2+d)} \end{pmatrix}. \tag{100}$$

Note that $P^\infty(z)$ has the same jumps and behaviour at infinity (up to left-multiplication by a constant matrix), as $S(z)$.

Proof. To check analyticity in $\mathbb{C} \setminus (-\infty, a]$ one needs only to determine that no poles are introduced from the zeros of the theta functions in the denominators. In fact, as in [11], we determine the constant d so that those which do occur, are precisely cancelled by the zeros of $N_{jk}(z)$. We claim that the function $\beta(z) - \beta(z)^{-1}$ has two zeros

$$\beta_1^* \in (c, b), \quad \beta_2^* \in (a, +\infty). \tag{101}$$

Indeed, we have that

$$\beta(z) \pm \beta(z)^{-1} = 0 \implies \beta(z)^4 = 1 \iff (z - a)(z - c) = (z - b).$$

Let $h(z) = (z - a)(z - c) - (z - b)$. We have

$$h(c) = b - c > 0, \quad h(b) = -(a - b)(b - c) < 0, \quad h(a) = -(a - b) < 0.$$

Since $h(z) \rightarrow +\infty$, if $z \rightarrow \pm\infty$, we deduce that h has a zero, β_1^* in (c, b) and another β_2^* in $(a, +\infty)$. Since $\beta(z) \pm \beta(z)^{-1} = 0 \iff \beta(z)^2 = \pm 1$, the sign of the root (given our choice of branches) implies that β_1^*, β_2^* are the zeros of $\beta(z) - \beta(z)^{-1}$, whereas $\beta(z) + \beta(z)^{-1}$ has no zeros (its zeros are on the second sheet of the Riemann surface).

It is well known (see, e.g., [26]) that $\theta_3(z)$ has only one zero at $z = \frac{1}{2} + \frac{\tau}{2}$ modulo its period lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. By Lemma 12, we easily see that $u(z)$ maps $\mathbb{C} \setminus (-\infty, a]$ onto $(-1/2, 0) \times (-\tau/2, \tau/2)$. Therefore $\theta(u(z)) = 0 \iff z = c$. Define then

$$d = \int_c^{\beta_1^*} \omega. \tag{102}$$

It follows that

$$\theta(u(z) - d) = 0 \iff z = \beta_1^*. \tag{103}$$

This proves that, for our specific choice of d , the zero of $\theta(u(z) - d)$, in $P^\infty(z)$, at β_1^* is cancelled by the zero of $\beta(z) - \beta(z)^{-1}$. It follows that $P^\infty(z)_{12}, P^\infty(z)_{21}$ are analytic functions in $\mathbb{C} \setminus (-\infty, a]$. To see the analyticity of the diagonal terms $P^\infty(z)_{11}, P^\infty(z)_{22}$, we note that since $0 < d < 1/2$ we have, using again the mapping $u(z)$, that $u(z) + d \neq 1/2 + \tau/2 \pmod{\Lambda}$ for all z . Thus $P^\infty(z; s)$ is analytic in $\mathbb{C} \setminus (-\infty, a]$.

We now turn to the jump conditions. The jumps of $P^\infty(z)$ are verified by means of (92)–(94), together with the quasi-periodicity properties of $\theta_3(z)$ in (18) and the jumps for $N(z)$. Indeed for the 11 entry, on the set $(-\infty, c) \cup (b, a)$, we have

$$\begin{aligned} (N_0^{-1}P^\infty(z; s))_{11+} &= \frac{\theta_3(0)\theta_3(u_+(z) + s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u_+(z) + d)}N_{11+} \\ &= -\frac{\theta_3(0)\theta_3(-u_-(z) + s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(-u_-(z) + d)}N_{12-} \\ &= -(N_0^{-1}P^\infty(z; s))_{12-}, \end{aligned}$$

where we have used $\theta_3(z + 1) = \theta_3(z)$, and evenness of $\theta_3(z)$. On the interval (c, b) we have

$$\begin{aligned} (N_0^{-1}P^\infty(z; s))_{11+} &= \frac{\theta_3(0)\theta_3(u_+(z) + s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u_+(z) + d)}N_{11+} \\ &= \frac{\theta_3(0)\theta_3(u_-(z) - \tau + s^{3/2}\Omega + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u_-(z) - \tau + d)}N_{11-} \\ &= \frac{\theta_3(0)\theta_3(u_-(z) + s^{3/2}V + d)}{\theta_3(s^{3/2}\Omega)\theta_3(u_-(z) + d)} \frac{\exp(2\pi i(u_-(z) + s^{3/2}\Omega + d) - \pi i\tau)}{\exp(2\pi i(u_-(z) + d) - \pi i\tau)}N_{11-} \\ &= (N_0^{-1}P^\infty(z; s))_{11-} \exp\left(2s^{3/2}g_+(b)\right). \end{aligned}$$

Repeating this computation for each of the remaining 3 entries yields the jumps of P^∞ .

To obtain the large z expansion of $P^\infty(z)$ we note that $P^\infty(z)N(z)^{-1}$ is analytic at infinity, and use the fact that by (95), $u(z) \rightarrow -1/2$ as $z \rightarrow \infty$. \square

4.2.4. Expansions of $u(z)$, $\beta(z)$, and some properties of θ_{jk}

Below we will use the following easy to obtain expansions:

$$\begin{aligned} u(z) &= u_+(p) + u_{0,p}\sqrt{z-p} \left(1 + u_{1,p}(z-p) + \mathcal{O}((z-p)^2)\right), \\ z \rightarrow p \in \{a, b, c\}, \quad \text{Im } z > 0, \end{aligned} \tag{104}$$

with

$$u_{0,a} = \frac{1}{J_0\sqrt{(a-b)(a-c)}}, \quad u_{1,a} = -\frac{1}{6} \left(\frac{1}{a-b} + \frac{1}{a-c}\right), \tag{105}$$

$$u_{0,b} = \frac{1}{iJ_0\sqrt{(a-b)(b-c)}}, \quad u_{1,b} = \frac{1}{6} \left(\frac{1}{a-b} - \frac{1}{b-c}\right), \tag{106}$$

$$u_{0,c} = -\frac{1}{J_0\sqrt{(a-c)(b-c)}}, \quad u_{1,c} = \frac{1}{6} \left(\frac{1}{a-c} + \frac{1}{b-c}\right), \tag{107}$$

and

$$\begin{aligned} \beta(z) &= \beta_{0,p}(z-p)^{1/4} \left(1 + \beta_{1,p}(z-p) + \mathcal{O}((z-p)^2)\right), \quad z \rightarrow p \in \{a, c\}, \quad \text{Im } z > 0, \\ \beta(z) &= \beta_{0,b}(z-b)^{-1/4} \left(1 + \beta_{1,b}(z-b) + \mathcal{O}((z-b)^2)\right), \quad z \rightarrow b, \quad \text{Im } z > 0, \end{aligned} \tag{108}$$

with

$$\beta_{0,a}^2 = \sqrt{\frac{a-c}{a-b}}, \quad \beta_{0,b}^2 = i\sqrt{(a-b)(b-c)}, \quad \beta_{0,c}^2 = \sqrt{\frac{a-c}{b-c}}. \tag{109}$$

The values of $\beta_{1,p}$ are unimportant in what follows.

By the quasiperiodicity relations (18) and the definition (16) of Ω , we obtain the following connections between the values of θ_{jk} in (98) on the +side of $(-\infty, a)$:

$$\theta_{11}(u_+(b)) = \theta_{12}(u_+(b))e^{2g(b)+s^{3/2}}, \quad \theta_{22}(u_+(b)) = \theta_{21}(u_+(b))e^{-2g(b)+s^{3/2}}. \tag{110}$$

Now considering the derivatives of the second relation in (18), we moreover obtain

$$\theta'_{j1} = -\theta'_{j2}e^{2g(b)+s^{3/2}}, \quad \theta''_{j1} = \theta''_{j2}e^{2g(b)+s^{3/2}}, \quad j = 1, 2, \tag{111}$$

and

$$\theta_{11}\theta''_{22} = \theta_{12}\theta''_{21}, \quad \theta_{22}\theta''_{11} = \theta_{21}\theta''_{12}, \quad \theta'_{11}\theta'_{22} = \theta'_{12}\theta'_{21}, \tag{112}$$

where

$$\theta'_{jk} = \frac{d}{du}\theta_{jk}(u(z)), \quad \theta''_{jk} = \frac{d^2}{du^2}\theta_{jk}(u(z)),$$

and all θ_{jk} and their derivatives are evaluated at $u_+(b)$.

Similarly, at $u(a) = 0$, we obtain

$$\theta_{j1}(0) = \theta_{j2}(0), \quad \theta'_{j1}(0) = -\theta'_{j2}(0), \quad \theta''_{j1}(0) = \theta''_{j2}(0), \quad j = 1, 2. \tag{113}$$

4.3. Identities for θ -functions and elliptic integrals

We now obtain a set of identities involving θ -functions which will be used below in the proof of Theorem 1. First, recall all four Jacobi θ -functions. These may be expressed in terms of $\theta_3(z)$, defined in (17), as follows.

$$\theta_2(z) = \theta_1\left(z + \frac{1}{2}\right) = e^{-\pi iz + \pi i\tau/4}\theta_3\left(z - \frac{\tau}{2}\right), \quad \theta_4(z) = \theta_3\left(z + \frac{1}{2}\right). \tag{114}$$

We note that $\theta_1(z)$ is an odd function of z , whereas $\theta_j(z)$, $j = 2, 3, 4$ are all even functions. θ -functions have the following periodicity properties:

$$\begin{aligned} \theta_1(z+1) &= -\theta_1(z), & \theta_1(z+\tau) &= -e^{-2\pi iz - \pi i\tau}\theta_1(z), \\ \theta_2(z+1) &= -\theta_2(z), & \theta_2(z+\tau) &= e^{-2\pi iz - \pi i\tau}\theta_2(z), \\ \theta_j(z+1) &= \theta_j(z), & \theta_j(z+\tau) &= e^{-2\pi iz - \pi i\tau}\theta_j(z), \quad j = 3, 4. \end{aligned} \tag{115}$$

Denote by θ_k, θ'_k the values of θ -functions and their derivatives at zero (θ -constants), i.e.,

$$\theta_k = \theta_k(0) = \theta_k(0; \tau), \quad \theta'_k = \frac{d}{dz} \theta_k(z)|_{z=0}, \quad k = 1, 2, 3, 4.$$

Lemma 14. Recall $J_0, \tau, \Omega, P^\infty(z; s)$ and d given by (9), (16), (96) and (102), respectively; the polynomial $q(z) = z^2 + q_1z + q_0$ with coefficients given by (13). The following identities hold.

$$i) \quad \theta_3^4 = \frac{J_0^2}{\pi^2}(a - c), \quad ii) \quad \theta_4^4 = \frac{J_0^2}{\pi^2}(a - b), \quad iii) \quad \theta_2^4 = \frac{J_0^2}{\pi^2}(b - c), \quad (116)$$

$$iv) \quad \det P^\infty(z; s) = 1, \quad (117)$$

$$v) \quad \frac{\theta'_3}{\theta_3}(u_+(b) + d) - \frac{\theta'_1}{\theta_1}(u_+(b) + d) = J_0, \quad (118)$$

$$vi) \quad \left(\frac{\theta_1}{\theta_3}\right)'''(u_+(b) + d) = -3J_0 \left(\frac{\theta_1}{\theta_3}\right)''(u_+(b) + d) + \frac{6J_0(2\beta_{1,b} + u_{1,b})}{u_{0,b}^2} \frac{\theta_1}{\theta_3}(u_+(b) + d), \quad (119)$$

$$vii) \quad \frac{d\Omega}{db} = \frac{q(b)}{(a - b)(b - c)J_0}, \quad viii) \quad \frac{d\tau}{db} = -\frac{\pi i}{J_0^2(a - b)(b - c)}, \quad (120)$$

$$ix) \quad \frac{\theta'_3}{\theta_3}(d) - \frac{\theta'_1}{\theta_1}(d) = J_0(a - c), \quad (121)$$

$$x) \quad \left(\frac{\theta_1}{\theta_3}\right)'''(d) = -3J_0(a - c) \left(\frac{\theta_1}{\theta_3}\right)''(d) + \frac{6J_0(a - c)(-2\beta_{1,a} + u_{1,a})}{u_{0,a}^2} \frac{\theta_1}{\theta_3}(d), \quad (122)$$

$$xi) \quad \frac{d\Omega}{da} = -\frac{q(a)}{(a - b)(a - c)J_0}, \quad xii) \quad \frac{d\tau}{da} = \frac{\pi i}{J_0^2(a - b)(a - c)}. \quad (123)$$

Proof. The proof of this lemma is similar to the arguments in [19] wherein the authors derive analogous results in the case of the sine-kernel determinant. For the first three identities we proceed as follows. By the expansion of $u(z)$ at a ,

$$u(z) = \frac{1}{2J_0} \int_a^z \frac{dx}{p(x)^{1/2}} = \frac{1}{J_0} \frac{(z - a)^{1/2}}{\sqrt{(a - b)(a - c)}} (1 + o(1)), \quad z \rightarrow a,$$

we obtain

$$\frac{\theta_3^2(u(z))}{\theta_1^2(u(z))} = J_0^2 \frac{\theta_3^2}{\theta_1^2} \frac{(a - b)(a - c)}{z - a} + \mathcal{O}(1), \quad z \rightarrow a.$$

Since $\theta_1(0) = 0$ is the only zero of $\theta_1(z)$ modulo the lattice, in view of (92)–(94) and (115), the ratio $\frac{\theta_3^2(u(z))}{\theta_1^2(u(z))}$ is a meromorphic function in the whole extended z -plane with only one simple pole at $z = a$, and therefore it follows from Liouville’s theorem that

$$\frac{\theta_3^2(u(z))}{\theta_1^2(u(z))} - J_0^2 \frac{\theta_3^2}{(\theta_1')^2} \frac{(a-b)(a-c)}{z-a} = \text{const}, \quad \forall z \in \mathbb{C}. \tag{124}$$

We evaluate the constant by taking $z \rightarrow \infty$, making use of (114), and obtain

$$\frac{\theta_3^2(u(z))}{\theta_1^2(u(z))} - J_0^2 \frac{\theta_3^2}{(\theta_1')^2} \frac{(a-b)(a-c)}{z-a} = \frac{\theta_4^2}{\theta_2^2}. \tag{125}$$

Substituting here the value $z = b$, and making use of the identities (see, e.g., [26])

$$\theta_1' = \pi\theta_2\theta_3\theta_4, \quad \theta_3^4 = \theta_2^4 + \theta_4^4, \tag{126}$$

we obtain (i). Similarly, substituting $z = c$ into (125), we obtain (ii). The difference of (i) and (ii) gives (iii).

iv) It follows from the RH problem for $P^\infty(z)$ by standard arguments that $\det P^\infty(z)$ is analytic in the complex plane and bounded at infinity. Therefore, it is a constant and, by the condition at infinity,

$$\det P^\infty(z, s) = \det F_0, \quad \forall z \in \mathbb{C}. \tag{127}$$

It remains to check that this value is 1. We prove it as follows. First, by (96),

$$\begin{aligned} \det P^\infty(z; s) &= \frac{\theta_3^2\theta_3(u(z) + s^{3/2}\Omega + d)}{\theta_3^2(s^{3/2}\Omega)\theta_3^2(u(z) + d)}\theta_3(u(z) - s^{3/2}\Omega + d) \left(\frac{\beta(z) + \beta^{-1}(z)}{2}\right)^2 \\ &+ \frac{\theta_3^2\theta(u(z) - s^{3/2}\Omega - d)}{\theta_3^2(s^{3/2}\Omega)\theta_3^2(u(z) - d)}\theta_3(u(z) + s^{3/2}\Omega - d) \left(\frac{\beta(z) - \beta^{-1}(z)}{2i}\right)^2. \end{aligned} \tag{128}$$

Consider the following meromorphic function on the Riemann surface of $p(z)^{1/2}$:

$$\xi(z) = \beta^2(z) - 1 = \frac{(z-a)(z-c)}{p(z)^{1/2}} - 1. \tag{129}$$

By considerations in the previous section, $\xi(z)$ has 2 zeros, β_1^*, β_2^* , see (101). They are located on the first sheet. By Abel's theorem (see, e.g., [20,11])

$$-u(\infty) + u_+(\beta_1^*) + u(\beta_2^*) - u_+(b) = 0 \pmod{\Lambda}. \tag{130}$$

Since by Lemma 12, $u(\infty) = -1/2$, $u_+(b) = -\tau/2$, $u_+(c) = -1/2 - \tau/2$, and furthermore,

$$u_+(\beta_1^*) = u_+(c) + d, \quad d = \int_c^{\beta_1^*} \omega,$$

we have

$$1/2 + u_+(c) + d + u(\beta_2^*) + \tau/2 = u(\beta_2^*) + d = 0 \pmod{\Lambda},$$

or simply

$$u(\beta_2^*) = -d \pmod{\Lambda}. \tag{131}$$

Note, moreover, that since $\beta(\beta_2^*) - \beta^{-1}(\beta_2^*) = 0$ and since by standard arguments $\det N(z) \equiv 1$, we have

$$\left(\frac{\beta(\beta_2^*) + \beta^{-1}(\beta_2^*)}{2}\right)^2 = 1.$$

We now set $z = \beta_2^*$ in the above expression (128) for the determinant and obtain

$$\det P^\infty(z; s) = 1. \tag{132}$$

One may expand the determinant at any point and deduce that all the terms apart from the constant one must be zero, yielding an infinite number of identities linking the derivatives of θ functions. In what follows, we will only need the constant terms in the expansion at $z = b$ and $z = a$. A straightforward computation using expansions (104), (108) gives (in the notation of (97)) for $z \rightarrow b$,

$$\frac{1}{4} \frac{d}{du} [\theta_{11}(u)\theta_{22}(u) - \theta_{12}(u)\theta_{21}(u)]_{u=u_+(b)} \beta_{0,b}^2 u_{0,b} + \theta_{11}(u_+(b))\theta_{22}(u_+(b)) = 1, \tag{133}$$

where $u_{0,b}, \beta_{0,b}$ are defined in (106) and (109), respectively. We note that

$$\beta_{0,b}^2 u_{0,b} = \frac{1}{J_0}. \tag{134}$$

Furthermore, using the identities (110), (111), we can rewrite (133) in the form

$$\frac{1}{2J_0} [\theta'_{11}(u)\theta_{22}(u) + \theta_{11}(u)\theta'_{22}(u)]_{u=u_+(b)} + \theta_{11}(u_+(b))\theta_{22}(u_+(b)) = 1. \tag{135}$$

Similarly, expanding (128) as $z \rightarrow a$, one obtains the identity

$$\frac{1}{2J_0(a-c)} [\theta'_{11}(0)\theta_{22}(0) + \theta_{11}(0)\theta'_{22}(0)] + \theta_{11}(0)\theta_{22}(0) = 1. \tag{136}$$

v) Consider the function $\eta(z)$ defined by

$$\eta(z) = \frac{\theta_1^2(u(z) + d)}{\theta_3^2(u(z) + d)} (\beta(z) + \beta^{-1}(z))^2 - \frac{\theta_1^2(u(z) - d)}{\theta_3^2(u(z) - d)} (\beta(z) - \beta^{-1}(z))^2. \tag{137}$$

Note that this expression is similar to that of $\det P^\infty$ above. Indeed we may employ the same methods and deduce that it is identically equal to 0. On the other hand, by considering the constant coefficient (i.e. the coefficient of $(z - b)^0$) in its expansion for $z \rightarrow b$, we obtain the desired identity (v). Vanishing of the term with $(z - b)^1$ in the expansion yields the identity (vi). Here we expanded the functions $\frac{\theta_1}{\theta_3}$, $\beta(z)$, used (91), (115), the oddness of θ_1 , the evenness of θ_3 , and the fact that

$$\frac{\theta_1}{\theta_3}(u_+(b) - d) = \frac{\theta_1}{\theta_3}(-u_+(b) - d - \tau) = \frac{\theta_1}{\theta_3}(-u_+(b) - d) = -\frac{\theta_1}{\theta_3}(u_+(b) + d).$$

Similarly, expanding the expression at the point $z = a$ instead, one obtains (ix) and (x).

vii) Recall that

$$I_k = \frac{1}{2} \oint_{A_1} \frac{z^k dz}{\sqrt{p(z)}}, \quad J_k = -\frac{1}{2} \oint_{B_1} \frac{z^k dz}{\sqrt{p(z)}}$$

where we integrate along the cycles. Note that

$$\Omega = \frac{g_+(b)}{\pi i} = \frac{i}{\pi} [I_2 + q_1 I_1 + q_0 I_0]. \tag{138}$$

By means of direct differentiation, we obtain

$$\frac{d}{db} I_1 = \frac{1}{2} I_0 + b \frac{d}{db} I_0, \quad \frac{d}{db} J_1 = \frac{1}{2} J_0 + b \frac{d}{db} J_0, \tag{139}$$

and

$$\frac{d}{db} I_2 = \frac{1}{2} I_1 + \frac{b}{2} I_0 + b^2 \frac{d}{db} I_0, \quad \frac{d}{db} J_2 = \frac{1}{2} J_1 + \frac{b}{2} J_0 + b^2 \frac{d}{db} J_0. \tag{140}$$

Furthermore, expanding the integrand of

$$\oint \frac{d}{dz} \left(\frac{(z - a)(z - c)}{z - b} \right)^{1/2} dz = 0 \tag{141}$$

along the cycles, we obtain

$$\frac{d}{db} I_0 = -\frac{I_1 - bI_0}{2(a - b)(b - c)}, \quad \frac{d}{db} J_0 = -\frac{J_1 - bJ_0}{2(a - b)(b - c)}. \tag{142}$$

Differentiating q_0 in (13) and using the above derivatives of J_k , we find

$$\frac{d}{db} q_0 = -\frac{1}{2}(b + q_1) - q(b) \frac{d}{db} \frac{J_0}{J_0}. \tag{143}$$

This identity will be used repeatedly in the proof of the theorem. The final ingredient is obtained by means of Riemann’s period relations. Indeed these relations yield (by similar arguments as in the proof of Lemma 3.45 in [11] and Lemma 27 in [19] using the local variable $\xi = x^{-1/2}$)

$$I_1 J_0 - J_1 I_0 = 2\pi i. \tag{144}$$

Alternatively, to show (144), we can first reduce I_0, I_1, J_0, J_1 to complete elliptic integrals (cf. next section) K, E , and then use the Legendre relation:

$$K(k')E(k) + K(k)E(k') - K(k)K(k') = \frac{\pi}{2}, \quad k' = \sqrt{1 - k^2}.$$

The above facts, taken together, allow us to differentiate Ω directly and obtain (vii). Similarly, we have

$$\frac{d}{da} I_1 = \frac{1}{2} I_0 + a \frac{d}{da} I_0, \quad \frac{d}{da} J_1 = \frac{1}{2} J_0 + a \frac{d}{da} J_0, \tag{145}$$

$$\frac{d}{da} I_2 = \frac{1}{2} I_1 + \frac{a}{2} I_0 + a^2 \frac{d}{da} I_0, \quad \frac{d}{da} J_2 = \frac{1}{2} J_1 + \frac{a}{2} J_0 + a^2 \frac{d}{da} J_0, \tag{146}$$

and

$$\frac{d}{da} I_0 = \frac{I_1 - aI_0}{2(a - b)(a - c)}, \quad \frac{d}{da} J_0 = \frac{J_1 - aJ_0}{2(a - b)(a - c)}. \tag{147}$$

Furthermore,

$$\frac{dq_0}{da} = -\frac{1}{2}(a + q_1) - q(a) \frac{d}{da} \frac{J_0}{J_0}. \tag{148}$$

With the help of these equations, (xi) easily follows.

viii) By definition of τ in (16),

$$\tau = \frac{I_0}{J_0}. \tag{149}$$

Using (142) and (144), we obtain

$$\frac{d\tau}{db} = \frac{I_1 J_0 - J_1 I_0}{2J_0^2(b - a)(b - c)} = -\frac{\pi i}{J_0^2(a - b)(b - c)}, \tag{150}$$

which proves (viii).

Similarly, (xii) follows by (147). \square

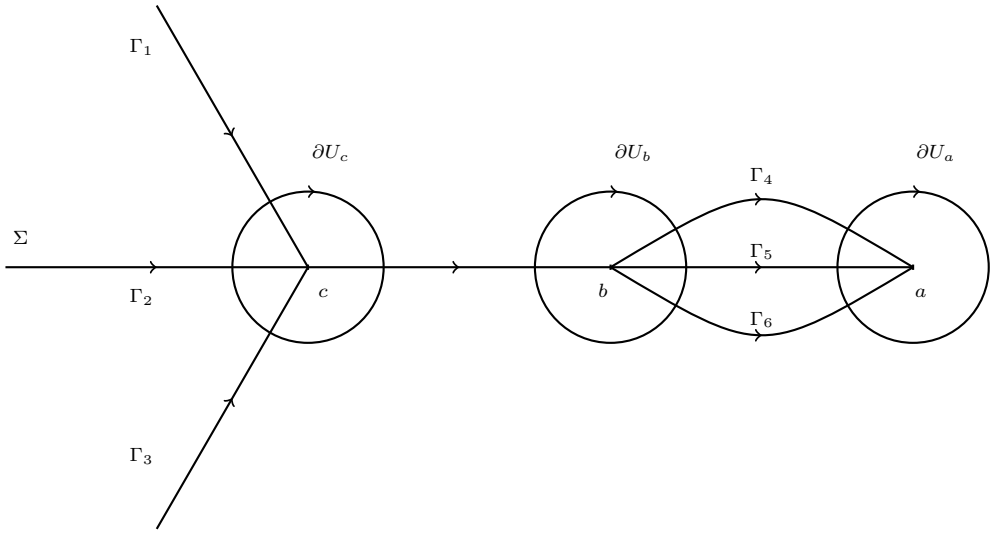


Fig. 5. Jump contour with neighbourhoods U_p .

Remark 15. Substituting the expression (13) for q_0 into (138) yields

$$\Omega = \frac{i}{\pi J_0} ((I_2 + q_1 I_1)J_0 - (J_2 + q_1 J_1)I_0). \tag{151}$$

Now by another application of Riemann’s period relations, we obtain:

$$(I_2 + q_1 I_1)J_0 - (J_2 + q_1 J_1)I_0 = -\frac{2\pi i q_1}{3}, \tag{152}$$

and therefore

$$\Omega = \frac{2q_1}{3J_0}. \tag{153}$$

4.4. Local parametrices

In this section we construct parametrices in small neighbourhoods, U_p , of the points $p = a, b, c$, see Fig. 5 where their boundaries are depicted, which match the exterior parametrix to the main order as $s \rightarrow \infty$ on the boundaries ∂U_p . As in [11], [22], [8] these parametrices involve Bessel functions. We choose U_b, U_c in such a way that the zero $c < \beta_1^* < b$ of $\beta(z) - \beta(z)^{-1}$ (cf. (101)) is outside U_p ’s.

4.4.1. Signs of $q(p)$

Recall the polynomial $q(z) = z^2 + q_1 z + q_0$ with coefficients given by (13). Here we prove that for $0 > a > b > c$,

$$q(a) < 0, \quad q(b) < 0, \quad \text{and} \quad q(c) > 0. \tag{154}$$

This will be used in subsequent analysis.

First note that, since

$$J_k = (-1)^{k+1} \int_c^b \frac{|z|^k dz}{\sqrt{|p(z)|}}, \tag{155}$$

we have that

$$|b|^k |J_0| \leq |J_k| \leq |c|^k |J_0|. \tag{156}$$

Substituting these inequalities into the second form of q_0 in (13), we find that

$$q(b) \leq -\frac{1}{3}(a-b)(b-c) < 0, \tag{157}$$

and

$$q(c) \geq \frac{1}{3}(a-c)(b-c) > 0. \tag{158}$$

To determine the sign of $q(a)$ we require a sharper bound. Let us, first, write

$$J_1 = \int_c^b \frac{a+z-a}{\sqrt{p(z)}} dz = aJ_0 + \int_c^b \sqrt{\frac{a-z}{(b-z)(z-c)}} dz \tag{159}$$

Then note the following standard reductions, with $k = \sqrt{\frac{b-c}{a-c}}$.

$$J_0 = - \int_c^b \frac{dz}{\sqrt{(a-z)(b-z)(z-c)}} = [z = c + (b-c)w^2] = -\frac{2}{\sqrt{a-c}} K(k),$$

$$K(k) = \int_0^1 \frac{dw}{\sqrt{1-k^2w^2}\sqrt{1-w^2}},$$

and, similarly,

$$\int_c^b \sqrt{\frac{a-z}{(b-z)(z-c)}} dz = 2\sqrt{a-c} E(k), \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2w^2}}{\sqrt{1-w^2}} dw,$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively.

Note that $0 < k = \sqrt{\frac{b-c}{a-c}} < 1$. With these expressions for J_0 and J_1 we find that $q(a)$ takes the following form.

$$q(a) = \frac{1}{3}(a - c) \left(a - b - q_1 \frac{E(k)}{K(k)} \right), \quad k = \sqrt{\frac{b - c}{a - c}}. \tag{160}$$

We want to show that $q(a) < 0$, i.e. that

$$\frac{E(k)}{K(k)} > \frac{a - b}{q_1}. \tag{161}$$

As the authors in [4] we will make use of the estimate

$$\frac{E(k)}{K(k)} > \sqrt{1 - k^2}. \tag{162}$$

This reduces the problem to seeing that

$$\sqrt{1 - k^2} = \sqrt{\frac{a - b}{a - c}} > \frac{a - b}{q_1} \quad \text{i.e., that} \quad (a + b + c)^2 > 4(a - b)(a - c). \tag{163}$$

But this is clear since, for $0 > a > b > c$, we have

$$(a + b + c)^2 - 4(a - b)(a - c) > (b + c)^2 - 4bc = (b - c)^2 > 0. \tag{164}$$

The estimate (162) was proven in [24], in a different form, and also follows from log-convexity of the Gauss hypergeometric function [7]. We now provide a proof for the reader’s convenience.

First we have that

$$\frac{\pi^2}{4} = \left(\int_0^1 \frac{dz}{\sqrt{1 - z^2}} \right)^2 < \left(\int_0^1 \frac{\sqrt{1 - k^2 z^2}}{\sqrt{1 - z^2}} dz \right) \left(\int_0^1 \frac{dz}{\sqrt{1 - k^2 z^2} \sqrt{1 - z^2}} \right) = E(k)K(k),$$

and so

$$\frac{E(k)}{K(k)} > \frac{\pi^2}{4K(k)^2}. \tag{165}$$

It, thus, remains to prove the estimate

$$K(k) \leq \frac{\pi}{2} \frac{1}{\sqrt[4]{1 - k^2}}. \tag{166}$$

For $0 < \alpha < \beta$, let us denote by $M(\alpha, \beta)$ the arithmetic-geometric mean. This is defined as the common limit of the following sequences.

$$\alpha_0 = \alpha, \quad \beta_0 = \beta, \quad \alpha_{n+1} = \sqrt{\alpha_n \beta_n}, \quad \alpha_n \leq \alpha_{n+1}; \quad \beta_{n+1} = \frac{\alpha_n + \beta_n}{2}, \quad \beta_{n+1} \leq \beta_n. \tag{167}$$

We then have

Theorem 16 (*Gauss, 1799*).

$$K(k)M(\sqrt{1 - k^2}, 1) = \frac{\pi}{2}. \tag{168}$$

Proof. Define the integral $I(\alpha, \beta)$ by

$$I(\alpha, \beta) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta}}, \quad I(1, \sqrt{1 - k^2}) = I(\sqrt{1 - k^2}, 1) = K(k). \tag{169}$$

Note that

$$I(\alpha, \beta) = \left[\sin \theta = \frac{2\alpha \sin \phi}{(\alpha + \beta) + (\alpha - \beta) \sin^2 \phi} \right] = I\left(\frac{\alpha + \beta}{2}, \sqrt{\alpha\beta}\right). \tag{170}$$

It follows, recursively, that

$$I(\alpha, \beta) = I\left(\frac{\alpha + \beta}{2}, \sqrt{\alpha\beta}\right) = I(\beta_2, \alpha_2) = \dots = I(M(\alpha, \beta), M(\alpha, \beta)) = \frac{\pi}{2M(\alpha, \beta)}, \tag{171}$$

which gives the result upon taking $\alpha = \sqrt{1 - k^2}, \beta = 1$. \square

The required estimate, (166), now follows from (168) by noting that $\alpha_1 = \sqrt{\alpha\beta} \leq M(\alpha, \beta)$ with $\alpha = \sqrt{1 - k^2}, \beta = 1$.

4.4.2. Bessel model RH problem

First recall the Bessel model RH problem of [8], which is a slight modification of the corresponding problem in [22], which in turn, is related to the problem in [11].

Let $I_0, K_0, H_0^{(1)}, H_0^{(2)}$ denote the standard Bessel functions, see e.g. [1], with branch cut along $(-\infty, 0)$. Let, furthermore, $I_0'(x) = \frac{d}{dx}I_0(x)$, etc, and define

$$\Psi_1(\zeta) = e^{-\frac{i\pi}{4}\sigma_3} \pi^{\frac{1}{2}} \sigma_3 \begin{pmatrix} I_0(\zeta^{1/2}) & \frac{i}{\pi} K_0(\zeta^{1/2}) \\ i\pi \zeta^{1/2} I_0'(\zeta^{1/2}) & -\zeta^{1/2} K_0'(\zeta^{1/2}) \end{pmatrix}, \tag{172}$$

$$\Psi_2(\zeta) = \frac{1}{2} e^{-\frac{i\pi}{4}\sigma_3} \pi^{\frac{1}{2}} \sigma_3 \begin{pmatrix} H_0^1(-i\zeta^{1/2}) & H_0^2(-i\zeta^{1/2}) \\ \pi \zeta^{1/2} H_0^{1'}(-i\zeta^{1/2}) & \pi \zeta^{1/2} H_0^{2'}(-i\zeta^{1/2}) \end{pmatrix} \tag{173}$$

and

$$\Psi_3(\zeta) = \frac{1}{2} e^{-\frac{i\pi}{4}\sigma_3} \pi^{\frac{1}{2}} \sigma_3 \begin{pmatrix} H_0^2(i\zeta^{1/2}) & -H_0^1(i\zeta^{1/2}) \\ -\pi \zeta^{1/2} H_0^{2'}(i\zeta^{1/2}) & \pi \zeta^{1/2} H_0^{1'}(i\zeta^{1/2}) \end{pmatrix} \tag{174}$$

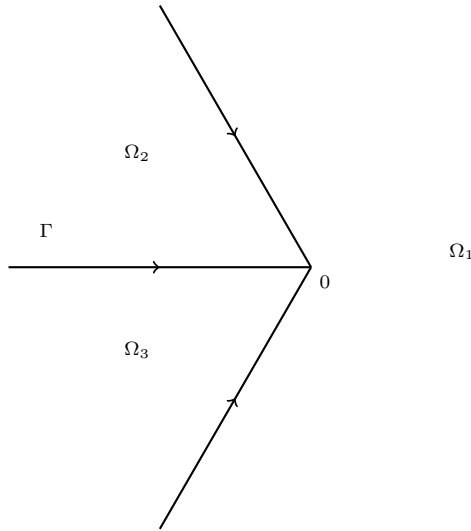


Fig. 6. Jump contour for the Ψ -RHP problem.

Define regions $\Omega_1 = \{\zeta \in \mathbb{C} : -\frac{2\pi}{3} < \arg \zeta < \frac{2\pi}{3}\}$, $\Omega_2 = \{\zeta \in \mathbb{C} : \frac{2\pi}{3} < \arg \zeta < \pi\}$ and $\Omega_3 = \{\zeta \in \mathbb{C} : -\pi < \arg \zeta < -\frac{2\pi}{3}\}$ (see Fig. 6) and consider the function $\Psi(\zeta)$ defined in each region Ω_k by Ψ_k . We have

Lemma 17. [22,8] For $\Gamma = (-\infty, 0) \cup \mathbb{R}_+ \exp\left(\frac{2\pi i}{3}\right) \cup \mathbb{R}_+ \exp\left(-\frac{2\pi i}{3}\right)$, $\Psi(\zeta)$ satisfies the following RHP.

$$\begin{aligned}
 &\Psi(\zeta) \text{ is analytic in } \mathbb{C} \setminus \Gamma, \\
 &\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}_+ \exp\left(\frac{2\pi i}{3}\right) \cup \mathbb{R}_+ \exp\left(-\frac{2\pi i}{3}\right), \\
 &\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in (-\infty, 0), \\
 &\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N_0 \left(I + \frac{1}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta}\right) \right) e^{\zeta^{1/2}\sigma_3}, \quad \zeta \rightarrow \infty.
 \end{aligned} \tag{175}$$

Furthermore, the expansion at infinity holds uniformly as $|\zeta| \rightarrow \infty$.

4.4.3. Construction of the parametrices

Writing

$$g(z) = g_+(p) + \int_p^z \frac{q(\zeta)d\zeta}{p(\zeta)^{1/2}},$$

let

$$f_p(z) = \left(\int_p^z \frac{q(\zeta)d\zeta}{p(\zeta)^{1/2}} \right)^2, \quad f_p(z)^{1/2} = - \int_p^z \frac{q(\zeta)d\zeta}{p(\zeta)^{1/2}} = g_+(p) - g(z), \quad p = a, b, c. \tag{176}$$

It is easy to verify that there exist an open disk U_p centred at p (for each $p = a, b, c$) of a sufficiently small radius ϵ such that $f_p(z)$ is a conformal mapping of U_p onto a neighbourhood of zero. Expanding close to $z = p$, we have, in particular,

$$f_p(z)^{1/2} = \begin{cases} -\frac{2q(a)}{\sqrt{(a-b)(a-c)}}(z-a)^{\frac{1}{2}}(1 + \mathcal{O}(z-a)), & z \rightarrow a, \\ -\frac{2q(b)}{\sqrt{(a-b)(b-c)}}(b-z)^{\frac{1}{2}}(1 + \mathcal{O}(z-b)), & z \rightarrow b, \\ \frac{2q(c)}{\sqrt{(a-c)(b-c)}}(z-c)^{\frac{1}{2}}(1 + \mathcal{O}(z-c)), & z \rightarrow c. \end{cases} \tag{177}$$

Note that the signs of the quantities $q(p)$ in (154) ensure the location of the cuts of $f_p(z)^{1/2}$ in U_p corresponds to the contour of the Ψ -RHP under the mapping

$$\zeta(z) = s^3 f_p(z).$$

We now choose the exact form of the jump contour Σ_S , for the S -RHP problem (86), in U_p so that its image under the mapping $\zeta(z)$ is direct lines. This relates the construction to the Bessel problem.

We look for local parametrices, with $z \in U_p$, in the form

$$\begin{aligned} P_p(z; s) &= E_p(z; s) \Psi (s^3 f_p(z)) \exp(s^{3/2} g(z) \sigma_3), \quad p = a, c \\ P_b(z; s) &= E_b(z; s) \sigma_3 \Psi (s^3 f_b(z)) \sigma_3 \exp(s^{3/2} g(z) \sigma_3), \quad p = b, \end{aligned} \tag{178}$$

where $E_p(z)$ are, analytic in U_p , matrix-valued functions chosen so that $P_p(z; s)$ matches $S(z)$ on ∂U_p to the main order. We will now see that the following functions satisfy these conditions. Set

$$E_p(z; s) = P^\infty(z; s) e^{-s^{3/2} g_\pm(p) \sigma_3} N_0^{-1} f_p(z)^{\frac{1}{4} \sigma_3} s^{\frac{3}{4} \sigma_3}, \quad p = a, c, \tag{179}$$

and

$$E_b(z; s) = P^\infty(z; s) e^{-s^{3/2} g_\pm(b) \sigma_3} \sigma_3 N_0^{-1} \sigma_3 f_b(z)^{\frac{1}{4} \sigma_3} s^{\frac{3}{4} \sigma_3}, \quad p = b, \tag{180}$$

where in \pm we choose $+$, if $\text{Im}(z-p) > 0$ and $-$, if $\text{Im}(z-p) < 0$.

We have

Lemma 18. *The functions $E_p(z) = E_p(z; s)$ defined by (179), (180) are analytic in the respective neighbourhoods U_p .*

Proof. Note that (177) gives

$$f_a^{\frac{1}{4}}(z)_+ = i f_a^{\frac{1}{4}}(z)_-, \quad z \in (a - \epsilon, a); \quad f_c^{\frac{1}{4}}(z)_+ = i f_c^{\frac{1}{4}}(z)_-, \quad z \in (c - \epsilon, c),$$

and

$$f_b^{\frac{1}{4}}(z)_+ = -i f_b^{\frac{1}{4}}(z)_-, \quad z \in (b, b + \epsilon).$$

Consider U_c . First, we verify that $E_c(z)$ has no jumps in U_c . On $(c - \epsilon, c)$, we have using the jump conditions for P^∞ in (99), and the fact that by (80), $g_+(c) = -g_-(c)$,

$$E_c(z)_+ = P_-^\infty(z; s) e^{-s^{3/2} g_-(c)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_0^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} f_c^{\frac{1}{4}\sigma_3}(z) - s^{\frac{3}{4}\sigma_3} = E_c(z)_-.$$

On $(c, c + \epsilon)$, since $g_+(b) = g_+(c)$,

$$E_c(z)_+ = P_-^\infty(z; s) e^{s^{3/2} g_+(c)} N_0^{-1} f_c^{\frac{1}{4}\sigma_3}(z) - s^{\frac{3}{4}\sigma_3} = E_c(z)_-.$$

So that the singularity of $E_c(z)$ at c is not a branch point. Since $f_c^{\frac{1}{4}\sigma_3}(z)$ and the matrix elements $P_{jk}^\infty(z)$ contain at worst a $(z - c)^{-1/4}$ singularity each, it follows that the singularity of $E_c(z)$ at c is removable. Thus $E_c(z)$ is analytic in U_c .

The functions $E_b(z)$ in U_b , and $E_a(z)$ in U_a are considered similarly. \square

Now by the large ζ expansion of $\Psi(\zeta)$ and (176), we find that on the boundaries ∂U_p for large s , uniformly in z ,

$$P_p(z; s) = P^\infty(z; s) e^{-s^{3/2} g_\pm(p)\sigma_3} \left(I + \frac{1}{8s^{3/2} f_p(z)^{1/2}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{s^3}\right) \right) e^{s^{3/2} g_\pm(p)\sigma_3}, \tag{181}$$

for $p = a, c$, and

$$P_b(z; s) = P^\infty(z; s) e^{-s^{3/2} g_\pm(b)\sigma_3} \left(I + \frac{1}{8s^{3/2} f_b(z)^{1/2}} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{s^3}\right) \right) e^{s^{3/2} g_\pm(b)\sigma_3}, \tag{182}$$

where as before, in \pm , we choose $+$, if $\text{Im}(z - p) > 0$ and $-$, if $\text{Im}(z - p) < 0$.

Since $E_p(z)$ are analytic, left-hand multiplication by them does not affect the jump conditions for $P_p(z)$, and it is easy to check that $P_p(z)$ satisfies the same jump conditions as S on $\Sigma_S \cup U_p$.

Thus, we have shown

Lemma 19. *With Σ_S and v_S denoting the contour and jumps, respectively, of the S -RHP (86); the function $P_p(z; s)$ given by (178) in U_p , $p = a, b, c$, has the following properties.*

$$\begin{aligned}
 P_p(z; s) \text{ is analytic in } (\mathbb{C} \setminus \Sigma_S) \cap U_p, \\
 P_p(z; s)_+ = P_p(z; s)_- v_S(z; s), \quad z \in \Sigma_S \cap U_p.
 \end{aligned}
 \tag{183}$$

Moreover, $P_p(z; s)(P^\infty(z; s))^{-1}$ has the following large- s expansion:

$$P_p(z; s)(P^\infty(z; s))^{-1} = I + \Delta_1(z; s) + \mathcal{O}\left(\frac{1}{s^3}\right), \quad \Delta_1(z; s) = \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \quad s \rightarrow \infty,$$

uniformly on ∂U_p . The function $\Delta_1(z; s)$ in z is meromorphic in U_p . For $p = a, c$, we have

$$\Delta_1(z; s) = \frac{1}{8s^{3/2}f_p(z)^{1/2}}P^\infty(z; s)e^{-s^{3/2}g_\pm(p)\sigma_3} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} e^{s^{3/2}g_\pm(p)\sigma_3} (P^\infty(z; s))^{-1},$$

and, for $p = b$,

$$\Delta_1(z; s) = \frac{1}{8s^{3/2}f_b(z)^{1/2}}P^\infty(z; s)e^{-s^{3/2}g_\pm(b)\sigma_3} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} e^{s^{3/2}g_\pm(b)\sigma_3} (P^\infty(z; s))^{-1},$$

where in \pm , we choose $+$, if $\text{Im}(z - p) > 0$ and $-$, if $\text{Im}(z - p) < 0$.

4.5. Small norm RH problem for R

Set

$$R(z) = \left(s^{-\sigma_3/4}S_0N_0F_0^{-1}N_0^{-1}\right)^{-1} S(z) \times \begin{cases} P_p(z)^{-1}, & \text{for } z \in U_p, \\ P^\infty(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus U_p, \end{cases}
 \tag{184}$$

and let Σ_R denote the contour in Fig. 7. Then, R satisfies the following Riemann-Hilbert problem.

R is analytic in $\mathbb{C} \setminus \Sigma_R$

$$\begin{aligned}
 R_+(z) &= R_-(z)P_p(z)(P^\infty(z))^{-1}, \quad z \in \partial U_p, \quad p \in \{a, b, c\} \\
 R_+(z) &= R_-(z)P^\infty(z)v_S(z)(P^\infty(z))^{-1}, \quad z \in \Sigma_R \setminus (\partial U_a \cup \partial U_b \cup \partial U_c), \\
 R(z) &\rightarrow I, \quad z \rightarrow \infty,
 \end{aligned}
 \tag{185}$$

where

$$v_S(z) = \begin{pmatrix} 1 & 0 \\ e^{2s^{3/2}g(z)} & 1 \end{pmatrix}.
 \tag{186}$$

Indeed, the jump conditions easily follow from the RH problems for $S(z)$, $P^\infty(z)$ and $P_p(z)$. The condition at infinity follows from (86), (99), (88).

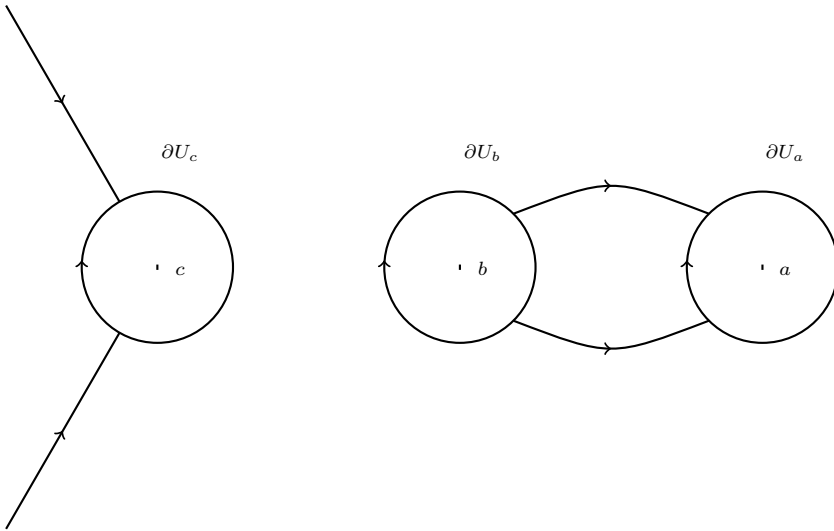


Fig. 7. Jump contour for the R -RH problem.

We will now show that all the jumps for the R -RHP problem are close to the identity for large s both in terms of L^∞ and L^2 norms, which guarantees the solvability of the problem for large s [10].

Let us first consider the jumps of R on $\Sigma_R \setminus (\partial U_a \cup \partial U_b \cup \partial U_c)$. We claim that one may deform the contour $\Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_6$, in the RH problem for $S(z)$ (86), in order to have $\operatorname{Re} g(z) < 0$ on this part of Σ_R . In particular, we show that this implies

$$P^\infty(z)v_S(z)(P^\infty(z))^{-1} = I + \mathcal{O}\left(e^{-c's^{3/2} \max\{1, |z|^{3/2}\}}\right), \quad c' > 0. \tag{187}$$

First, since

$$\operatorname{Re} g(z) = \frac{2}{3}|z|^{3/2} \cos\left(\frac{3}{2} \arg z\right) + \mathcal{O}\left(\frac{1}{\sqrt{z}}\right), \quad z \rightarrow \infty \text{ with } \frac{\pi}{2} < |\arg(z - a)| < \pi, \tag{188}$$

we have that, for sufficiently large $|z|$, $\operatorname{Re}(g) < 0$ in the sector, and hence (187) holds on Σ_R for sufficiently large $|z|$.

On the other hand it follows from (154) that

$$q(z) = (z - x_1)(z - x_2), \quad x_1 \in (c, b), \quad x_2 \in (a, +\infty),$$

and therefore

$$\operatorname{Im} \frac{q(z)}{\sqrt{p(z)}_+} = \frac{\partial}{\partial x} \operatorname{Im} g_+(x) > 0, \quad x \in (-\infty, c) \cup (b, a). \tag{189}$$

Since $\operatorname{Re} g(x) = 0$, for $x \in (-\infty, c) \cup (b, a)$, using the Cauchy-Riemann equations we have, for $0 < y < \delta$, (δ sufficiently small) that

$$\operatorname{Re} g(x + iy) = \int_0^y \frac{\partial \operatorname{Re} g(x + i\psi)}{\partial \psi} d\psi < 0, \quad x \in (-\infty, c) \cup (b, a). \tag{190}$$

Similarly, decreasing δ if need be, we show that

$$\operatorname{Re} g(x + iy) < 0, \quad x \in (-\infty, c) \cup (b, a), \quad -\delta < y < 0. \tag{191}$$

Conditions (190) and (191) together with the boundedness (from infinity and zero) of θ -functions on the contour imply that the jump matrix satisfies

$$P^\infty(z)v_S(z)(P^\infty(z))^{-1} = I + \mathcal{O}\left(e^{-s^{3/2}\epsilon}\right)$$

for some $\epsilon > 0$ uniformly on the lenses of the Σ_R contour around (b, a) , and also for sufficiently small $|z - c|$ on the lenses around $(-\infty, c)$. It remains to obtain an L^∞ estimate on the rest of latter lenses: between a fixed small $|z - c|$ and a fixed large $|z - c|$ (above which we can use the estimate (188)). Consider the part of the lenses with $\operatorname{Im} z > 0$ (for the $\operatorname{Im} z < 0$ part we argue similarly). Clearly, it is sufficient to show that

$$\operatorname{Im} \frac{q(x + iy)}{\sqrt{p(x + iy)}} > 0, \quad x \in (-\infty, c), \quad y > 0. \tag{192}$$

Let $\phi_1 > \phi_2 > \phi_3 > \phi_4 > \phi_5$ denote the acute angles between the real line and the direct lines connecting $x + iy$, $x \in (-\infty, c)$, $y > 0$, and the points $c < x_1 < b < a < x_2$, respectively. Then

$$\frac{q(x + iy)}{\sqrt{p(x + iy)}} = \left| \frac{q(x + iy)}{\sqrt{p(x + iy)}} \right| \exp \left\{ i \left(\frac{\pi}{2} + \frac{\phi_1 + \phi_3 + \phi_4 - 2\phi_2 - 2\phi_5}{2} \right) \right\}.$$

Using the ordering of ϕ_j 's we have that

$$-\pi < \phi_3 - \phi_2 - \phi_5 < \phi_1 + \phi_3 + \phi_4 - 2\phi_2 - 2\phi_5 < \phi_1 - 2\phi_5 < \pi/2,$$

and thus we obtain (192).

To summarize, we established the estimate

$$P^\infty(z)v_S(z)(P^\infty(z))^{-1} = I + \mathcal{O}\left(e^{-s^{3/2}\epsilon' \max\{1, |z|^{3/2}\}}\right), \quad \epsilon' > 0, \tag{193}$$

on the lenses of the contour Σ_R .

It remains to consider the jumps of the R problem on the boundaries ∂U_p . By means of Lemma 19 we have that

$$v_R(z) = P_p(z)(P^\infty(z))^{-1} = I + \Delta_1(z; s) + \mathcal{O}\left(\frac{1}{s^3}\right) = I + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \quad s \rightarrow \infty, \tag{194}$$

uniformly on ∂U_p , $p = a, b, c$.

Thus, by standard theory (see, e.g., [10]), the R -problem is a small-norm problem for large s solvable by means of Neumann series. In what follows, we will need the first 2 terms. We then obtain

$$R(z) = I + R^{(1)}(z) + \mathcal{O}\left(\frac{1}{s^3}\right), \quad R^{(1)}(z) = \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \tag{195}$$

where the estimate for $R^{(1)}(z)$ and the error term are uniform in z . As usual, we substitute these series into the R -RHP, and obtain that $R^{(1)}(z)$ is analytic in $\mathbb{C} \setminus \cup_p \partial U_p$,

$$\begin{aligned} R_+^{(1)}(z) &= R_-^{(1)}(z) + \Delta_1(z), & z \in \cup_p \partial U_p, \\ R_+^{(1)}(z) &\rightarrow 0, & z \rightarrow \infty. \end{aligned} \tag{196}$$

This problem is solved by the Plemelj-Sokhotski formula, so that

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{\Delta_1(\zeta)}{\zeta - z} d\zeta. \tag{197}$$

Note that the integration along the circles ∂U_p is in the negative direction. Below, we will find $R^{(1)}(z)$ by computing the residues. Retracing our transformations $R \rightarrow S \rightarrow X$ yields the asymptotic solution to the RH problem for $X(z)$.

4.6. Extension to the limiting regime of Lemma 3

We now show that the solution to the Riemann-Hilbert problem for fixed $c < b < a < 0$ extends to the regime where $a \rightarrow 0^-$ and $b - c \rightarrow 0^+$, as in Lemma 3. We need to verify that the jumps of the R -RH problem remain small in L^∞ and L^2 norms. Then the R -RH problem is solvable by Neumann series by standard arguments for small norm RH problems on contracting contours.

Let us first consider the jumps on $\partial U_a \cup \partial U_b \cup \partial U_c$. We will now show that in the regime of Lemma 3, by choosing the diameters of U_a, U_b, U_c appropriately, we have uniformly in z

$$P_p(z)(P^\infty(z))^{-1} = I + \begin{cases} \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right), & z \in \partial U_a \\ \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)}\right) & z \in \partial U_b \cup \partial U_c. \end{cases} \tag{198}$$

First, uniformly for $a - b > \varepsilon > 0$ as $b \rightarrow c$ we have

$$J_0 = -\frac{\pi}{\sqrt{a-c}} \left(1 + \frac{1}{2(a-c)}(b-c) + \mathcal{O}((b-c)^2) \right) \tag{199}$$

and

$$J_1 = -\frac{c\pi}{\sqrt{a-c}} \left(1 + \left(\frac{1}{c} + \frac{1}{2(a-c)} \right) (b-c) + \mathcal{O}((b-c)^2) \right), \tag{200}$$

and, therefore,

$$q_0 = \frac{ac}{2} + \frac{a}{4}(b-c) + \mathcal{O}((b-c)^2). \tag{201}$$

With this expression for q_0 we may expand $q(p)$, $p = a, b, c$. Indeed, we find that uniformly for $a - b > \varepsilon > 0$ as $b \rightarrow c$,

$$q(a) = \frac{a(a-c)}{2} + \mathcal{O}(b-c), \tag{202}$$

$$q(b) = -\frac{(a-2c)}{4}(b-c) + \mathcal{O}((b-c)^2), \tag{203}$$

and

$$q(c) = \frac{a-2c}{4}(b-c) + \mathcal{O}((b-c)^2). \tag{204}$$

Note that, for $p = b, c$,

$$\sqrt{z-p} = \mathcal{O}(\sqrt{b-c}), \quad b \rightarrow c, \quad z \in U_p.$$

This is because we have $\text{diam}(U_p) < \min\{|\beta_1^* - b|, |\beta_1^* - c|\}$, where β_j^* , $j = 1, 2$, are the zeros of $\beta(z) - \beta(z)^{-1}$ (they satisfy (101)). We easily derive the expansions

$$\beta_1^* = c + \frac{b-c}{1+a-c} + \mathcal{O}((b-c)^2), \tag{205}$$

$$\beta_2^* = 1 + a - \frac{1}{4} \frac{b-c}{1+a-c} + \mathcal{O}((b-c)^2), \quad b \rightarrow c.$$

We may and do set $\text{diam}(U_p) = \epsilon' \min\{|\beta_1^* - b|, |\beta_1^* - c|\}$, with $0 < \epsilon' < \frac{1}{2}$. Note that the diameter of U_p decreases in this regime. On the other hand, we have that $\sqrt{z-a} = \mathcal{O}(1)$, $a \rightarrow 0^-$, $z \in \overline{U_a}$. The diameter of U_a does not need to decrease, set $\text{diam}(U_a) = \epsilon'$. By (176) and (177) and the above estimates for $q(p)$, we have

$$\begin{aligned} g_+(p) - g(z) &= f_p(z)^{1/2} = \mathcal{O}(b-c), & z \in \overline{U_p}, & \quad p = b, c, \\ g_+(a) - g(z) &= f_a(z)^{1/2} = \mathcal{O}(|a|), & z \in \overline{U_a}. \end{aligned} \tag{206}$$

Recall also that $\operatorname{Re} g_+(p) = 0$, $p = a, b, c$. In view of Lemma 19, to prove (198) it remains to show that $P^\infty(z)_{jk}$, $z \in \partial U_p$, are uniformly bounded in the prescribed regime.

By the first expansion in (205), the functions $\beta(z)$ and $\beta(z)^{-1}$ are uniformly bounded on $\partial U_b, \partial U_c$. Clearly, they are also bounded on ∂U_a .

To investigate the behaviour of θ -functions, first analyze τ , and for that write

$$I_0 = \int_{b+(b-c)^{1/2}}^a \frac{dx}{(x-c)(x-a)^{1/2}} \left(1 + \mathcal{O}(\sqrt{b-c})\right) + \frac{1}{(b-a)^{1/2}} \int_b^{b+(b-c)^{1/2}} \frac{dx}{((x-b)(x-c))^{1/2}} \left(1 + \mathcal{O}(\sqrt{b-c})\right).$$

Then it is easy to compute the integrals and obtain

$$I_0 = \frac{1}{i\sqrt{a-c}} [-\log(b-c) + \log(16(a-c))] \left(1 + \mathcal{O}(\sqrt{b-c})\right), \tag{207}$$

and, by using also (199),

$$\tau = \frac{I_0}{J_0} = \frac{i}{\pi} [-\log(b-c) + \log(16(a-c))] \left(1 + \mathcal{O}(\sqrt{b-c})\right). \tag{208}$$

Thus $i\tau \rightarrow -\infty$ logarithmically as $b \rightarrow c$. By the series representation for $\theta_3(\xi)$, this implies that the θ -functions would converge to unity if their argument was any and fixed or was real. However, since it is in general complex and depends on a, b, c , we need to show that $\theta(u(z) + (-1)^j s^{3/2} \Omega + (-1)^k d)$ in the numerators in P^∞ are bounded, and also that $\theta(u(z) \pm d)$ in the denominators are bounded away from zero.

From the definition of d in (102), we have using (199) and (205) that, as $b \rightarrow c$,

$$d = \frac{1}{2J_0} \int_c^{\beta_1^*} \frac{dz}{((z-a)(z-b)(z-c))^{1/2}} = [z = c + (b-c)w] = \frac{1}{2\pi} (1 + \mathcal{O}(b-c)) \int_0^{\frac{\beta_1^* - c}{b-c}} \frac{dw}{(w(1-w))^{1/2}} = \frac{1}{2\pi} (1 + \mathcal{O}(b-c)) \int_0^{\frac{1}{1+a-c}} \frac{dw}{(w(1-w))^{1/2}} < \frac{1}{2}. \tag{209}$$

Therefore,

$$0 < d < \frac{1}{2}. \tag{210}$$

Using (205), denote

$$m := \frac{1}{b-c} \min\{|\beta_1^* - c|, |b - \beta_1^*|\} = \min\left\{\frac{1}{1+a-c} + \mathcal{O}(b-c), \frac{a-c}{1+a-c} + \mathcal{O}(b-c)\right\}.$$

Thus,

$$\partial U_b = \{z = b + \epsilon' m(b-c)e^{i\phi}, 0 \leq \phi < 2\pi\}, \quad \partial U_c = \{z = c + \epsilon' m(b-c)e^{i\phi}, 0 \leq \phi < 2\pi\}.$$

Arguing as in the case of d above, we obtain

$$u(z) = \frac{\tau}{2} + \frac{1}{2\pi} \int_0^{\epsilon' m e^{i\phi}} \frac{dw}{(w(1-w))^{1/2}} (1 + \mathcal{O}(b-c)), \quad z \in \partial U_b, \tag{211}$$

$$u(z) = \frac{1+\tau}{2} + \frac{1}{2\pi} \int_0^{\epsilon' m e^{i\phi}} \frac{dw}{(w(1-w))^{1/2}} (1 + \mathcal{O}(b-c)), \quad z \in \partial U_c,$$

and $|u(z)| \leq \epsilon$ for $z \in \partial U_a$ for some $\epsilon > 0$. Choosing $\epsilon' > 0$ sufficiently small, we see that these equations and (210) imply in particular that $u(z) \pm d$ is uniformly bounded away from the zero $\frac{1+\tau}{2} \pmod{\Lambda}$ of $\theta_3(\xi)$ on $\partial U_a \cup \partial U_b \cup \partial U_c$. We then obtain for some constants

$$0 < \epsilon < |\theta(u(z) \pm d)| < C < \infty \quad |\theta(u(z) \pm d + r(a, b, c))| < C' < \infty, \tag{212}$$

$$\text{Im } r(a, b, c) = 0, \quad z \in \partial U_a \cup \partial U_b \cup \partial U_c.$$

Thus P^∞ is bounded on $\partial U_a \cup \partial U_b \cup \partial U_c$ and therefore (198) holds.

The analysis of the jumps of R on $\Sigma_R \setminus \partial U_a \cup \partial U_b \cup \partial U_c$ is similar. (Note, in particular, that $0 < \text{Im } u(z) < |\tau|$ on the contour.) We obtain

$$P^\infty(z; s)v_S(z; s)(P^\infty(z; s))^{-1} = I + \mathcal{O}\left(e^{-c' s^{3/2} \min\{|a|, b-c\} \max\{1, |z|^{3/2}\}}\right), \quad c' > 0, \tag{213}$$

on $\Sigma_R \setminus \partial U_a \cup \partial U_b \cup \partial U_c$.

To conclude, we see from (198) and (213) that in the regime of Lemma 3, where $a - b > \varepsilon > 0$,

$$a = -\frac{(\log s)^{1/8}}{s}, \quad b = c + \frac{2(\log s)^{1/8}}{s^{3/2}}, \quad s \rightarrow +\infty, \tag{214}$$

we have

$$\begin{aligned}
 P_p(z)(P^\infty(z))^{-1} &= I + \mathcal{O}\left(\frac{1}{2(\log s)^{1/8}}\right), \quad z \in \partial U_a \cup \partial U_b \cup \partial U_c, \\
 P^\infty(z; s)v_S(z; s)(P^\infty(z; s))^{-1} & \\
 &= I + \mathcal{O}\left(e^{-2c'(\log s)^{1/8} \max\{1, |z|^{3/2}\}}\right), \quad z \in \Sigma_R \setminus \partial U_a \cup \partial U_b \cup \partial U_c.
 \end{aligned}
 \tag{215}$$

Therefore the R -RH problem is solvable in this regime, and we can write

$$\begin{aligned}
 R(z) &= I + R^{(1)}(z) + \mathcal{O}\left(\frac{1}{s^3 \min\{|a|^2, (b-c)^2\}}\right), \\
 R^{(1)}(z) &= \mathcal{O}\left(\frac{1}{s^{3/2} \min\{|a|, b-c\}}\right),
 \end{aligned}
 \tag{216}$$

uniformly in z and uniformly in the range from fixed $c < b < a < 0$ to the regime (214).

5. Proof of Theorem 1

We start with the differential identity (79) for $p = b$, and write $P^{\text{Ai}}(sJ) = P_s^{\text{Ai}}(a, b)$, to indicate the values of a and b ,

$$\frac{d}{db} \log P^{\text{Ai}}(sJ) = \frac{d}{db} \log P_s^{\text{Ai}}(a, b) = - \lim_{z \rightarrow b} \frac{1}{2\pi i} \left(X^{-1}(z) \frac{d}{dz} X(z) \right)_{21}, \tag{217}$$

where the limit is taken from outside the lens and with $\text{Im } z > 0$. We first concentrate on the case of $c < b < a < 0$ fixed, and then provide an extension to the asymptotic regime of Lemma 3.

By the definitions of $S(z)$ and $R(z)$, we have that in the region $U_b \cap \{z : \text{Im } z > 0\}$, and outside the lens,

$$\begin{aligned}
 X^{-1}X' &= e^{s^{3/2}g(z)\sigma_3} P_b^{-1} R^{-1} R' P_b e^{-s^{3/2}g(z)\sigma_3} \\
 &+ e^{s^{3/2}g(z)\sigma_3} P_b^{-1} P_b' e^{-s^{3/2}g(z)\sigma_3} - s^{3/2}g'(z)\sigma_3.
 \end{aligned}
 \tag{218}$$

By definition (178) of $P_b(z)$, in particular in the preimage $\zeta^{-1}(\Omega_1)$ (which is outside the lens),

$$P_b(z) = E_b(z)\sigma_3\Psi(s^3(g(z) - g(b))^2)\sigma_3 e^{s^{3/2}g(z)\sigma_3},$$

and therefore

$$\begin{aligned}
 P_b^{-1}P_b' &= e^{-s^{3/2}g(z)\sigma_3} \sigma_3 \Psi^{-1} \sigma_3 E_b^{-1} E_b' \sigma_3 \Psi \sigma_3 e^{s^{3/2}g(z)\sigma_3} \\
 &+ e^{-s^{3/2}g(z)\sigma_3} \sigma_3 \Psi^{-1} \Psi' \sigma_3 e^{s^{3/2}g(z)\sigma_3} + s^{3/2}g'(z)\sigma_3.
 \end{aligned}$$

Moreover, note that for z in a neighbourhood of b ,

$$R^{-1}(z)R'(z) = \frac{dR^{(1)}}{dz}(z) + \mathcal{O}\left(\frac{1}{s^3}\right), \quad s \rightarrow \infty, \tag{219}$$

where

$$\frac{dR^{(1)}}{dz}(z) = \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{\Delta_1(\zeta; s)}{(\zeta - z)^2} d\zeta. \tag{220}$$

Note that the (uniform) estimate for the error term in (219), follows from the differentiability of the asymptotic expansion of R , which, in turn, follows from the uniform estimate for R , in (195), and Cauchy’s integral formula.

Starting with the (differentiable) expansions, see e.g. [1], for the Bessel functions

$$I_0(z) = 1 + \frac{z^2}{4} + \mathcal{O}(z^4), \quad z \rightarrow 0,$$

and

$$K_0(z) = -\left(\log \frac{z}{2} + \gamma\right) I_0(z) + \frac{z^2}{4} + \mathcal{O}(z^4), \quad z \rightarrow 0,$$

where γ is Euler’s constant, it is straightforward to verify that the function (172)

$$\Psi(\zeta) = e^{-\frac{i\pi}{4}\sigma_3} \pi^{\frac{1}{2}\sigma_3} \begin{pmatrix} 1 + \frac{\zeta}{4} + \mathcal{O}(\zeta^2) & \frac{1}{2\pi i} \log \zeta + \mathcal{O}(1) \\ \frac{i\pi}{2}\zeta + \mathcal{O}(\zeta^2) & 1 + \mathcal{O}(\zeta \log \zeta) \end{pmatrix}, \quad \zeta \rightarrow 0. \tag{221}$$

Using this expression, the definition

$$E_b(z; s) = P^\infty(z; s) e^{-s^{3/2}g(b)\sigma_3} \sigma_3 N_0^{-1} \sigma_3 f_b(z)^{\frac{1}{4}\sigma_3} s^{\frac{3}{4}\sigma_3},$$

and the fact that $P^\infty(z; s)$ is bounded, as $s \rightarrow \infty$, we estimate the error term in $\lim_{z \rightarrow b} X^{-1}(z)X'(z)$, arising from that in (219), as follows

$$\lim_{z \rightarrow b} \left(e^{s^{3/2}g(z)\sigma_3} P_b^{-1} \mathcal{O}\left(\frac{1}{s^3}\right) P_b e^{-s^{3/2}g(z)\sigma_3} \right)_{21} = \mathcal{O}\left(\frac{b-c}{s^{3/2}}\right). \tag{222}$$

Thus, in the region $W_b := U_b \cap \{z : \text{Im } z > 0\} \cap \zeta^{-1}(\Omega_1)$,

$$-\frac{1}{2\pi i} \lim_{z \rightarrow b} \left(X^{-1}(z) \frac{d}{dz} X(z) \right)_{21} = \lim_{z \rightarrow b} (\tau_1(z) + \tau_2(z) + \tau_3(z))_{21} + \mathcal{O}\left(\frac{b-c}{s^{3/2}}\right), \tag{223}$$

where

$$\tau_1(z) = -\frac{1}{2\pi i} \sigma_3 \Psi^{-1}(s^3 f_b(z)) \frac{d\Psi}{dz}(s^3 f_b(z)) \sigma_3, \tag{224}$$

$$\tau_2(z) = -\frac{1}{2\pi i} \sigma_3 \Psi^{-1}(s^3 f_b(z)) \sigma_3 E_b^{-1}(z) \frac{dE_b}{dz}(z) \sigma_3 \Psi(s^3 f_b(z)) \sigma_3, \tag{225}$$

$$\tau_3(z) = -\frac{1}{2\pi i} e^{s^{3/2}g(z)\sigma_3} P_b^{-1}(z) \frac{dR^{(1)}}{dz}(z) P_b(z) e^{-s^{3/2}g(z)\sigma_3}. \tag{226}$$

With these we have the formula

$$\frac{d}{db} \log P_s^{\text{Ai}}(a, b) = \lim_{z \rightarrow b} (\tau_1(z) + \tau_2(z) + \tau_3(z))_{21} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \tag{227}$$

for fixed $c < b < a < 0$, where the limit is taken along a path in W_b .

To extend (227) to the regime $b \rightarrow c$, $a \rightarrow 0$ of Lemma 3, we rely on the results of the previous section. Considering $R^{-1}R'$ by using (216), we obtain

$$R^{-1}(z)R'(z) = \frac{dR^{(1)}}{dz}(z) + \mathcal{O}\left(\frac{1}{s^3 \min\{|a|^2, (b-c)^2\}}\right), \quad s \rightarrow \infty, \tag{228}$$

and hence (cf. (223))

$$\frac{d}{db} \log P_s^{\text{Ai}}(a, b) = \lim_{z \rightarrow b} (\tau_1(z) + \tau_2(z) + \tau_3(z))_{21} + \mathcal{O}\left(\frac{1}{s^{3/2} \min\{|a|^2, (b-c)\}}\right), \tag{229}$$

uniformly in the range from fixed $c < b < a < 0$ to the regime (214), where the limit is taken along a path in W_b .

In the next 3 sections, we prove the following lemmata (with the notation from the introduction).

Lemma 20.

$$\lim_{z \rightarrow b} \tau_1(z)_{21} = -s^3 \frac{d}{db} \alpha_2. \tag{230}$$

Lemma 21.

$$\lim_{z \rightarrow b} \tau_2(z)_{21} = \frac{d}{db} \log \theta_3(s^{3/2}\Omega; \tau) - \frac{d\tau}{db} \frac{\partial}{\partial \tau} \log \theta_3(s^{3/2}\Omega; \tau). \tag{231}$$

Lemma 22. *With $\omega = s^{3/2}\Omega$,*

$$\int_0^1 \lim_{z \rightarrow b} \tau_3(z)_{21} d\omega = -\frac{1}{2} \frac{d}{db} \log |J_0| - \frac{1}{8} \frac{d}{db} \log |q(a)q(b)q(c)| + \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega; \tau) d\omega. \tag{232}$$

Using these results, we may now write the differential identity (229) in the following explicit form.

Proposition 23. *(Asymptotic form of the differential identity for b) Let $\varepsilon > 0$ be fixed. Then uniformly for $c < b_0 < b < b_1 < a < 0$, where $b_0, b_1 \in [c + \frac{2t_0}{s^{3/2}}, a - \varepsilon]$, $a \leq -\frac{t_1}{s}$, as $s \rightarrow \infty$, with $t_0 = t_1 = (\log s)^{1/8}$,*

$$\frac{d}{db} \log P_s^{\text{Ai}}(a, b) = \frac{d}{db} D(a, b) + \Theta_b + \tilde{\Theta}_b + \mathcal{O}\left(\frac{1}{s^{3/2}|a|^2}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)}\right), \tag{233}$$

where

$$D(a, b) = -\alpha_2 s^3 + \log \theta_3(s^{3/2}\Omega; \tau) - \frac{1}{2} \log |J_0| - \frac{1}{8} \log |q(a)q(b)q(c)|, \tag{234}$$

and

$$\Theta_b = \lim_{z \rightarrow b} \tau_3(z)_{21} - \int_0^1 \lim_{z \rightarrow b} \tau_3(z)_{21} d\omega, \tag{235}$$

$$\tilde{\Theta}_b = \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega; \tau) d\omega - \frac{d\tau}{db} \frac{\partial}{\partial \tau} \log \theta_3(s^{3/2}\Omega; \tau). \tag{236}$$

Moreover,

$$\int_{b_0}^{b_1} \Theta_b db, \quad \int_{b_0}^{b_1} \tilde{\Theta}_b db = \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b_0-c)}\right). \tag{237}$$

Proof. By (229) and 3 lemmata above, we only need to prove (237). First we show that

$$\int_{b_0}^{b_1} \Theta_b db = \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b_0-c)}\right), \tag{238}$$

i.e. replacing $\lim_{z \rightarrow b} \tau_3(z)_{21}$ by its average w.r.t. $\omega = s^{3/2}\Omega$ under the sign of the integral introduces only a small error. Let

$$f(\omega; a, b, c) = \lim_{z \rightarrow b} \tau_3(z)_{21}, \quad \omega = s^{3/2}\Omega.$$

First, let $c < b < a < 0$ be fixed. Then f is an analytic function of ω . Let f_n denote its Fourier coefficients

$$f(\omega; a, b, c) = \sum_{n \in \mathbb{Z}} f_n(a, b, c) e^{2\pi i n \omega}. \tag{239}$$

For $n \neq 0$, it follows from integration by parts that

$$\left| \int_{b_0}^{b_1} f_n(a, b, c) e^{2\pi i n s^{3/2}\Omega} db \right| = \tag{240}$$

$$= \frac{1}{2\pi|n|s^{3/2}} \left| \left[\frac{f_n(a, b, c)e^{2\pi i n s^{3/2}\Omega}}{\frac{d\Omega}{db}} \right]_{b_0}^{b_1} - \int_{b_0}^{b_1} e^{2\pi i n s^{3/2}\Omega} \frac{\partial}{\partial b} \left(\frac{f_n(a, b, c)}{\frac{d\Omega}{db}} \right) db \right|. \tag{241}$$

From (120) of Lemma 14, we see that $\frac{d\Omega}{db}(a, b, c)$ is a strictly positive differentiable function of the parameters $a, b,$ and $c,$ bounded away from zero if $c < b < a < 0$ are bounded away from each other. Furthermore, the functions $f_n, \partial f_n/\partial b$ decrease with n exponentially. Therefore

$$\int_{b_0}^{b_1} f(\omega; a, b, c)db = \sum_{n \in \mathbb{Z}} \int_{b_0}^{b_1} f_n(a, b, c)e^{2\pi i n \omega} db = \int_{b_0}^{b_1} f_0(a, b, c)db + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \quad s \rightarrow \infty. \tag{242}$$

Now let $c < b < a < 0$ be in the range of the Proposition. We will show that

$$\int_{b_0}^{b_1} f(\omega; a, b, c)db = \int_{b_0}^{b_1} f_0(a, b, c)db + \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b_0 - c)}\right), \quad s \rightarrow \infty, \tag{243}$$

which implies (238).

By (120), (199), (203),

$$\frac{d\Omega}{db} = \frac{a - 2c}{\pi\sqrt{a - c}} + \mathcal{O}(b - c), \quad \frac{d^2\Omega}{db^2} = \mathcal{O}(1), \tag{244}$$

as $b \rightarrow c$ uniformly for $a - b > \varepsilon > 0.$ We note, in particular, that $\frac{d\Omega}{db}$ remains bounded away from zero in the regime of the Proposition. To apply the above Fourier series arguments and deduce (243), it suffices to show that

$$f_n(a, b, c) = \mathcal{O}\left(\frac{1}{n(b - c)}\right) + \mathcal{O}\left(\frac{1}{n|a|}\right), \quad \frac{\partial}{\partial b} f_n(a, b, c) = \mathcal{O}\left(\frac{1}{n(b - c)^2}\right) + \mathcal{O}\left(\frac{1}{n|a|}\right) \tag{245}$$

in the regime of the Proposition. Since

$$|f_n(a, b, c)| = \left| \int_0^1 f(\omega; a, b, c)e^{-2\pi i n \omega} d\omega \right| = \left| \frac{1}{2\pi n} \int_0^1 \frac{\partial}{\partial \omega} f(\omega; a, b, c)e^{-2\pi i n \omega} d\omega \right|, \quad n \neq 0, \tag{246}$$

and similarly for $\frac{\partial}{\partial b} f_n(a, b, c),$ we only need to show that

$$\begin{aligned} \frac{\partial}{\partial \omega} f(\omega; a, b, c) &= \mathcal{O}\left(\frac{1}{b-c}\right) + \mathcal{O}\left(\frac{1}{|a|}\right), \\ \frac{\partial}{\partial \omega} \frac{\partial}{\partial b} f(\omega; a, b, c) &= \mathcal{O}\left(\frac{1}{(b-c)^2}\right) + \mathcal{O}\left(\frac{1}{|a|}\right). \end{aligned} \tag{247}$$

One can use directly the definition of $\lim_{z \rightarrow b} \tau_3(z)_{21}$, but it is easier to consider its simplified form in Section 5.3 below. By (208), we have uniformly in ω , the estimates for the θ -functions

$$\theta_j(\omega) = 1 + \mathcal{O}(b-c), \quad j = 1, 2, 3, 4, \tag{248}$$

and their derivatives $\theta'_j(\omega) = \mathcal{O}(b-c)$. The analysis of the previous section shows that the matrix elements (98), $\theta_{jk}(u(z))$ are bounded on ∂U_b , and so are their derivatives w.r.t. ω . Using (211), (208), we also conclude that

$$\frac{\partial u}{\partial b}, \quad \frac{\partial \tau}{\partial b}, \quad \frac{\partial \beta}{\partial b}, \quad \frac{\partial \beta^{-1}}{\partial b}$$

are all $\mathcal{O}(1/(b-c))$ uniformly on ∂U_b . Finally, note that we can use the heat equation to differentiate θ -functions w.r.t. τ . The estimates (247) follow then from (293), (301), and (320) with (312) below. Thus, we obtained (243).

In view of the above Lemmata 20, 21, 22, and the formula (229), it remains to show the second estimate in (237), i.e. that

$$\tilde{\Theta}_b = \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega; \tau) d\omega - \frac{d\tau}{db} \frac{\partial}{\partial \tau} \log \theta_3(s^{3/2}\Omega; \tau)$$

becomes $\mathcal{O}\left(\frac{1}{s^{3/2} \min\{|a|, b-c\}}\right)$ after integration. But this follows from the above considerations similarly to (and simpler than) (243). \square

Using the analogous results for the differential identity at a , we follow the above methods and arrive in Section 5.4, similarly, at

Proposition 24. *(Asymptotic form of the differential identity for a) Let $\varepsilon > 0$ be fixed. Then uniformly for $c < b < a_0 < a < a_1 < 0$, where $a_0, a_1 \in [b + \varepsilon, -\frac{t_1}{s}]$, $b \geq c + \frac{2t_0}{s^{3/2}}$ as $s \rightarrow \infty$, with $t_0 = t_1 = (\log s)^{1/8}$,*

$$\frac{d}{da} \log P_s^{\text{Ai}}(a, b) = \frac{d}{da} D(a, b) + \Theta_a + \tilde{\Theta}_a + \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)^2}\right), \tag{249}$$

where $D(a, b)$ is given by (234), and

$$\Theta_a = \lim_{z \rightarrow a} \tau_3(z)_{21} - \int_0^1 \lim_{z \rightarrow a} \tau_3(z)_{21} d\omega, \tag{250}$$

$$\tilde{\Theta}_a = \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega; \tau) d\omega - \frac{d\tau}{da} \frac{\partial}{\partial \tau} \log \theta_3(s^{3/2}\Omega; \tau). \tag{251}$$

Moreover,

$$\int_{a_0}^{a_1} \Theta_a da, \quad \int_{a_0}^{a_1} \tilde{\Theta}_a da = \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)}\right). \tag{252}$$

Because of the form of the error terms, we will need to move the edges a and b first simultaneously. To do this we will use the following Proposition which easily follows from the 2 ones above and some similar additional arguments. For fixed α, β , we denote $a = \alpha - x, b = \beta + x$ and remark that

$$\frac{d}{dx} = \frac{\partial}{\partial b} - \frac{\partial}{\partial a}.$$

Proposition 25. (Asymptotic form of the differential identity for $a = \alpha - x, b = \beta + x$)
 Let $\varepsilon > 0$ be fixed. Then uniformly for $0 \leq x_0 < x < x_1 \leq \frac{\alpha-\beta}{2} - \varepsilon$, where $a = \alpha - x, b = \beta + x$, and $\beta = c + \frac{2t_0}{s^{3/2}}, \alpha = -\frac{t_1}{s}$ as $s \rightarrow \infty$ with $t_0 = t_1 = (\log s)^{1/8}$,

$$\begin{aligned} \frac{d}{dx} \log P_s^{\text{Ai}}(a, b) &= \frac{d}{dx} D(a, b) - \Theta_a - \tilde{\Theta}_a + \Theta_b + \tilde{\Theta}_b \\ &+ \mathcal{O}\left(\frac{1}{s^{3/2}|a|^2}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)^2}\right), \end{aligned} \tag{253}$$

where $D(a, b)$ is given by (234), and $\Theta_a, \tilde{\Theta}_a, \Theta_b, \tilde{\Theta}_b$ as in Propositions 24, 23 above. Moreover,

$$\int_{x_0}^{x_1} \Theta_a dx, \quad \int_{x_0}^{x_1} \tilde{\Theta}_a dx, \quad \int_{x_0}^{x_1} \Theta_b dx, \quad \int_{x_0}^{x_1} \tilde{\Theta}_b dx = \mathcal{O}\left(\frac{1}{s^{3/2}|a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(b-c)}\right). \tag{254}$$

With these propositions, we may prove Theorem 1. We will integrate to reach the desired values of $a = a_0, b = b_0$ of the Theorem. Set $\alpha = -\frac{t_1}{s}, \beta = c + \frac{2t_0}{s^{3/2}}, t_0 = t_1 = (\log s)^{1/8}$. First, by Proposition 25, we integrate over x from $x = 0$ to $x = \alpha - a_0$, which fixes the desired value of a :

$$\begin{aligned} \log P_s^{\text{Ai}}(a_0, \beta + \alpha - a_0) - \log P_s^{\text{Ai}}(\alpha, \beta) &= \int_0^{\alpha - a_0} \frac{d}{dx} \log P_s^{\text{Ai}}(a, b) dx = \\ D(a_0, \beta + \alpha - a_0) - D(\alpha, \beta) &+ \mathcal{O}\left(\frac{1}{s^{3/2}|\alpha|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2}(\beta - c)}\right). \end{aligned} \tag{255}$$

Next, using Proposition 23, we integrate over b from the present value $b = \beta + \alpha - a_0$ to $b = b_0$:

$$\begin{aligned} \log P_s^{\text{Ai}}(a_0, b_0) - \log P_s^{\text{Ai}}(a_0, \beta + \alpha - a_0) &= \int_{\beta + \alpha - a_0}^{b_0} \frac{d}{db} \log P_s^{\text{Ai}}(a, b) db = \\ D(a_0, b_0) - D(a_0, \beta + \alpha - a_0) &+ \mathcal{O}\left(\frac{1}{s^{3/2}}\right). \end{aligned} \tag{256}$$

The sum of these expressions gives

$$\log P_s^{\text{Ai}}(a_0, b_0) = D(a_0, b_0) - D(\alpha, \beta) + \log P_s^{\text{Ai}}(\alpha, \beta) + \mathcal{O}\left(\frac{1}{(\log s)^{1/8}}\right). \tag{257}$$

Note that $\log P_s^{\text{Ai}}(\alpha, \beta)$ in (257) may be replaced with the expression in Lemma 3. Thus it remains only to expand the expression for $D(\alpha, \beta)$ when $s \rightarrow \infty$. To this end we prove the following

Proposition 26. For $\alpha = -\frac{t_1}{s}$ and $\beta = c + \frac{2t_0}{s^{3/2}}$, $t_0 = t_1 = (\log s)^{1/8}$,

$$D(\alpha, \beta) = -\frac{|c|t_0^2}{2} - \frac{1}{4} \log(\sqrt{|c|}t_0) - \frac{t_1^3}{12} - \frac{1}{8} \log t_1 + \frac{1}{2} \log s + \frac{1}{8} \log 2 - \frac{1}{2} \log \pi + o(1), \quad s \rightarrow \infty. \tag{258}$$

Proof. The formula is obtained by expanding all terms in $D(a, b)$ with $a = -\frac{t_1}{s}$ and $b = c + \frac{2t_0}{s^{3/2}}$, $s \rightarrow \infty$. First, by definition of α_2 in (15) and the expansion of q_0 (201),

$$\alpha_2 = -\frac{a^3}{12} + \frac{(a - 2c)^2}{32(a - c)}(b - c)^2 + \mathcal{O}((b - c)^3) = \frac{t_1^3}{12s^3} + \frac{|c|t_0^2}{2s^3} (1 + o(1)). \tag{259}$$

Next, by (248), $\theta_3(s^{3/2}\Omega; \tau) = 1 + \mathcal{O}(b - c)$. The expansion (199) implies that

$$-\frac{1}{2} \log |J_0| = -\frac{1}{2} \log \pi + \frac{1}{4} \log |c| + \mathcal{O}(b - c). \tag{260}$$

Finally, the expansions (202), (203), and (204) imply that

$$\begin{aligned}
 -\frac{1}{8} \log|q(a)q(b)q(c)| &= -\frac{1}{8} \log \left| \frac{a(a-c)}{2} \left(\frac{a-2c}{4}(b-c) \right)^2 (1 + \mathcal{O}(b-c)) \right| \\
 &= -\frac{1}{4} \log(\sqrt{|c|}t_0) - \frac{1}{8} \log t_1 + \frac{1}{2} \log s + \frac{1}{8} \log 2 - \frac{1}{4} \log|c| + o(1).
 \end{aligned}
 \tag{261}$$

Combining (259), (260), and (261), we obtain (258). \square

On substituting the expressions (27) of Lemma 3 and (258) of Proposition 26, into (257), Theorem 1 follows with $a = a_0$ and $b = b_0$.

In the following sections we prove the aforementioned Lemmata 20, 21, and 22, and their analogues for the differential identity in a .

5.1. The main term. Proof of Lemma 20

From (221) we easily obtain, recalling that $\zeta = \zeta(z) = s^3 f_b(z) = s^3(g(z) - g(b))^2$ and using (177),

$$\begin{aligned}
 \lim_{z \rightarrow b} \tau_1(z)_{21} &= -\frac{1}{2\pi i} \lim_{z \rightarrow b} \left(\sigma_3 \Psi^{-1}(s^3 f_b(z)) \frac{d\Psi}{dz}(s^3 f_b(z)) \sigma_3 \right)_{21} \\
 &= -\frac{1}{2\pi i} \lim_{\zeta \rightarrow 0} \left(\sigma_3 \Psi^{-1}(\zeta) \frac{d\Psi}{d\zeta}(\zeta) \sigma_3 \right)_{21} s^3 f'_b(b) = -s^3 \frac{q(b)^2}{(a-b)(b-c)}.
 \end{aligned}
 \tag{262}$$

On the other hand, using (13), (143), and (142), we differentiate α_2 given by (82) in Lemma 11, and obtain

$$\frac{d}{db} \alpha_2 = \frac{q(b)^2}{(a-b)(b-c)}.
 \tag{263}$$

Comparing this with (262), we obtain Lemma 20.

Remark 27. Identity (263) is suitable to prove that $\alpha_2 > 0$. Note that this property is important since it determines exponential *decay* of the determinant itself. It is clear that the right hand side of (263) is positive for all values of a , b , and c . Since (cf Section 4.6) $\alpha_2 \rightarrow -\frac{a^3}{12} > 0$ as $b \rightarrow c$, we have that

$$\alpha_2(b) = \alpha_2(b=c) + \int_c^b \frac{q(b)^2}{(a-b)(b-c)} db > 0.
 \tag{264}$$

5.2. *The oscillatory term. Proof of Lemma 21*

We now study the term

$$\tau_2(z) = -\frac{1}{2\pi i} \sigma_3 \Psi^{-1}(\zeta) \sigma_3 E_b^{-1}(z) E_b'(z) \sigma_3 \Psi(\zeta) \sigma_3. \tag{265}$$

From the definition

$$E_b(z) = P^\infty(z) e^{-s^{3/2} g_+(b)\sigma_3} \sigma_3 N_0^{-1} \sigma_3 f_b(z)^{\frac{1}{4}\sigma_3} s^{\frac{3}{4}\sigma_3},$$

we have that

$$\begin{aligned} E_b^{-1}(z) E_b'(z) &= \frac{\sigma_3 f_b'(z)}{4 f_b(z)} \\ &+ s^{-\frac{3}{4}\sigma_3} f_b^{-\frac{1}{4}\sigma_3}(z) \sigma_3 N_0 \sigma_3 e^{s^{3/2} g_+(b)\sigma_3} (P^\infty(z))^{-1} (P^\infty(z))' e^{-s^{3/2} g_+(b)\sigma_3} \\ &\times \sigma_3 N_0^{-1} \sigma_3 f_b^{\frac{1}{4}\sigma_3}(z) s^{\frac{3}{4}\sigma_3}. \end{aligned} \tag{266}$$

It follows from the expansions of $\Psi(\zeta)$ in (221) and of $f_b(z)$ in (177), that the contribution to τ_2 from the first term in the above expression is

$$-\frac{1}{2\pi i} \frac{1}{4} \frac{f_b'(z)}{f_b(z)} (\sigma_3 \Psi^{-1}(\zeta) \sigma_3 \Psi(\zeta) \sigma_3)_{21} \rightarrow s^3 \frac{q(b)^2}{2(a-b)(b-c)}, \quad z \rightarrow b. \tag{267}$$

Next, with the notation for P^∞ in (97), we obtain, using the fact that $\det P^\infty(z) = 1$, $\det(P^\infty)'(z) = 0$ (see Lemma 14 (iv)),

$$e^{s^{3/2} g_+(b)\sigma_3} (P^\infty(z))^{-1} (P^\infty(z))' e^{-s^{3/2} g_+(b)\sigma_3} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{268}$$

with

$$A = \theta_{22} \frac{d\theta_{11}}{dz} N_{11}^2 + \theta_{11} \theta_{22} N_{11} \frac{dN_{11}}{dz} + \theta_{12} \frac{d\theta_{21}}{dz} N_{12}^2 + \theta_{12} \theta_{21} N_{12} \frac{dN_{12}}{dz}, \tag{269}$$

$$B = \left[\left(\theta_{22} \frac{d\theta_{12}}{dz} - \theta_{12} \frac{d\theta_{22}}{dz} \right) N_{11} N_{12} + \theta_{12} \theta_{22} \left(N_{11} \frac{dN_{12}}{dz} - N_{12} \frac{dN_{11}}{dz} \right) \right] e^{2g_+(b)s^{3/2}}, \tag{270}$$

and

$$C = \left[\left(\theta_{21} \frac{d\theta_{11}}{dz} - \theta_{11} \frac{d\theta_{21}}{dz} \right) N_{11} N_{12} - \theta_{21} \theta_{11} \left(N_{11} \frac{dN_{12}}{dz} - N_{12} \frac{dN_{11}}{dz} \right) \right] e^{-2g_+(b)s^{3/2}}, \tag{271}$$

where we omit the arguments for brevity.

In this notation, we determine using the expansion of $\Psi(\zeta)$ in (221) that

$$\lim_{z \rightarrow b} \tau_2(z)_{21} = \lim_{z \rightarrow b} \left(\frac{1}{4} (2A + i(B + C)) s^{3/2} f_b^{1/2}(z) - \frac{i}{4} (B - C) s^3 f_b(z) \right) + s^3 \frac{q(b)^2}{2(a - b)(b - c)}, \tag{272}$$

where we substituted (267). We first evaluate the term with $B - C$ in this expression. Writing

$$\frac{d}{dz} \theta_{jk} = \theta'_{jk} \frac{du}{dz}, \quad \theta'_{jk}(u) = \frac{d}{du} \theta_{jk}(u(z)), \tag{273}$$

we have by (104)

$$\frac{du}{dz} = \frac{u_{0,b}}{2} (z - b)^{-1/2} (1 + \mathcal{O}(z - b)), \quad z \rightarrow b.$$

Replacing $\frac{d}{dz} \theta_{jk}$ by $\theta'_{jk} \frac{du}{dz}$ in the above expressions for B and C , we obtain by the expansion of $\beta(z)$ in (108) and the identities for the values of θ_{jk} at $u_+(b)$ in (110),

$$\begin{aligned} B - C &= -\frac{1}{2i} \frac{1}{z - b} \left(\theta_{11} \theta_{22} + \frac{u_{0,b} \beta_{0,b}^2}{2} (\theta_{11} \theta'_{22} + \theta_{22} \theta'_{11}) \right) \Big|_{u_+(b)} (1 + \mathcal{O}(z - b)) \\ &= -\frac{1}{2i} \frac{1}{z - b} (1 + \mathcal{O}(z - b)), \end{aligned} \tag{274}$$

where we used the identity (133) in the last equation. Then, by the expansion of $f_b(z)$ in (177),

$$-\frac{i}{4} (B - C) s^3 f_b(z) \rightarrow -s^3 \frac{q(b)^2}{2(a - b)(b - c)}, \quad z \rightarrow b, \tag{275}$$

which cancels the last term in (272).

We now evaluate $2A + i(B + C)$. We obtain as before, but now using also (112), that

$$B + C = -\frac{u_{0,b}}{2i} \frac{1}{(z - b)^{1/2}} \left(\theta_{11} \theta'_{22} - \theta_{22} \theta'_{11} + \frac{u_{0,b} \beta_{0,b}^2}{2} (\theta_{11} \theta''_{22} - \theta_{22} \theta''_{11}) \right) \Big|_{u_+(b)} + \mathcal{O}(1), \tag{276}$$

and similarly,

$$A = -\frac{u_{0,b}}{4} \frac{1}{(z - b)^{1/2}} \left(\theta_{11} \theta'_{22} - \theta_{22} \theta'_{11} + \frac{u_{0,b} \beta_{0,b}^2}{2} (\theta_{11} \theta''_{22} - \theta_{22} \theta''_{11}) \right) \Big|_{u_+(b)} + \mathcal{O}(1). \tag{277}$$

Recalling also (134), we obtain

$$2A + i(B + C) = -\frac{u_{0,b}}{(z - b)^{1/2}} \left(\theta_{11}\theta'_{22} - \theta_{22}\theta'_{11} + \frac{1}{2J_0}(\theta_{11}\theta''_{22} - \theta_{22}\theta''_{11}) \right) \Big|_{u_+(b)} + \mathcal{O}(1). \tag{278}$$

Substituting this expression into (272) and using again (177) and the value of $u_{0,b}$ in (106), we have

$$\lim_{z \rightarrow b} \tau_2(z)_{21} = s^{3/2} \frac{q(b)}{(a - b)(b - c)} \frac{1}{4J_0^2} \left[\theta_{11}\theta_{22} \left(2J_0 \left(\frac{\theta'_{11}}{\theta_{11}} - \frac{\theta'_{22}}{\theta_{22}} \right) + \frac{\theta''_{11}}{\theta_{11}} - \frac{\theta''_{22}}{\theta_{22}} \right) \right]_{u_+(b)}. \tag{279}$$

Using parts (v) and (vii) of Lemma 14, and also expanding θ_{jk} by their definition (98), we obtain

$$\lim_{z \rightarrow b} \tau_2(z)_{21} = \frac{s^{3/2}}{2} \frac{d\Omega}{db} T_1(s^{3/2}\Omega), \tag{280}$$

where

$$\begin{aligned} T_1(\omega) &= \frac{1}{2J_0} \left[\theta_{11}\theta_{22} \left(2J_0 \left(\frac{\theta'_{11}}{\theta_{11}} - \frac{\theta'_{22}}{\theta_{22}} \right) + \frac{\theta''_{11}}{\theta_{11}} - \frac{\theta''_{22}}{\theta_{22}} \right) \right]_{u_+(b)} \\ &= -\frac{\theta_3^2\theta_3(\omega + \widehat{d})\theta_3(\omega - \widehat{d})}{J_0\theta_3^2(\widehat{d})\theta_3^2(\omega)} \left[\frac{\theta'_1(\widehat{d})}{\theta_1} \left(\frac{\theta'_3}{\theta_3}(\omega + \widehat{d}) + \frac{\theta'_3}{\theta_3}(\omega - \widehat{d}) \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{\theta''_3}{\theta_3}(\omega + \widehat{d}) - \frac{\theta''_3}{\theta_3}(\omega - \widehat{d}) \right) \right], \end{aligned} \tag{281}$$

with $\widehat{d} = u_+(b) + d$.

The function $T_1(\omega)$ has the same structure as (219) in [19]. As in the proof of part (c) of Proposition 23 in [19], we verify that (note: $J_0 < 0$)

$$T_1(\omega) = 2 \frac{\theta'_3}{\theta_3}(\omega). \tag{282}$$

Substituting this into (280) and writing out the total derivative, we obtain Lemma 21.

5.3. The constant term. Proof of Lemma 22

We are left to consider the term

$$\tau_3(z) = -\frac{1}{2\pi i} e^{s^{3/2}g(z)\sigma_3} P_b^{-1}(z) \frac{dR^{(1)}}{dz}(z) P_b(z) e^{-s^{3/2}g(z)\sigma_3}. \tag{283}$$

Note that

$$\begin{aligned} \frac{d}{dz} R^{(1)}(z) &= \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{\Delta_1(\xi)}{(\xi - z)^2} d\xi \\ &= \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{\Delta_1(\xi)}{(\xi - b)^2} d\xi (1 + \mathcal{O}(z - b)), \quad z \rightarrow b. \end{aligned} \tag{284}$$

Furthermore, with notation as in (97) and using the identities (110) for θ_{jk} at $u_+(b)$, and the expansions for $\Psi(\xi)$ in (221), $f_b(z)$ in (177), $\beta(z)$ in (108), we have for the first column of $P_b(z)e^{-s^{3/2}g(z)\sigma_3}$,

$$\lim_{z \rightarrow b} \begin{pmatrix} (P_b(z)e^{-s^{3/2}g(z)\sigma_3})_{11} \\ (P_b(z)e^{-s^{3/2}g(z)\sigma_3})_{21} \end{pmatrix} = \frac{\beta_{0,b}}{\sqrt{2}} (s^3\pi^2 f'_b(b))^{1/4} N_0 e^{-s^{3/2}g_+(b)\sigma_3} \begin{pmatrix} \theta_{11}(u_+(b)) \\ i\theta_{22}(u_+(b)) \end{pmatrix}. \tag{285}$$

Set

$$F(\xi) = \begin{pmatrix} \theta_{11}(u(\xi))N_{11}(\xi) & \theta_{12}(u(\xi))N_{12}(\xi) \\ \theta_{21}(u(\xi))N_{21}(\xi) & \theta_{22}(u(\xi))N_{22}(\xi) \end{pmatrix}. \tag{286}$$

The above observations and the fact that $\det P^\infty = \det F \equiv 1$ imply

$$\lim_{z \rightarrow b} \tau_3(z)_{21} = \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{L(\xi)}{(\xi - b)^2} d\xi, \tag{287}$$

where

$$\begin{aligned} L(\xi) &= \frac{\beta_{0,b}^2}{-4\pi i} (s^3\pi^2 f'_b(b))^{1/2} (-i\theta_{22}(u_+(b)) \quad \theta_{11}(u_+(b))) e^{s^{3/2}g_+(b)\sigma_3} \widehat{\Delta_1(\xi)} \\ &\quad \times e^{-s^{3/2}g_+(b)\sigma_3} \begin{pmatrix} \theta_{11}(u_+(b)) \\ i\theta_{22}(u_+(b)) \end{pmatrix}, \end{aligned} \tag{288}$$

with, by recalling Lemma 19,

$$\widehat{\Delta_1(\xi)} = \frac{1}{8s^{3/2}f_a(\xi)^{1/2}} F(\xi) \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} F(\xi)^{-1}, \quad \text{near } \xi = a, \tag{289}$$

$$= \frac{1}{8s^{3/2}f_b(\xi)^{1/2}} F(\xi) e^{-s^{3/2}g_\pm(b)\sigma_3} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} e^{s^{3/2}g_\pm(b)\sigma_3} F(\xi)^{-1}, \quad \text{near } \xi = b, \tag{290}$$

$$= \frac{1}{8s^{3/2}f_c(\xi)^{1/2}} F(\xi) e^{-s^{3/2}g_\pm(c)\sigma_3} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} e^{s^{3/2}g_\pm(c)\sigma_3} F(\xi)^{-1}, \quad \text{near } \xi = c, \tag{291}$$

where in \pm , $+$ is taken if $\text{Im } \xi > 0$, and $-$ if $\text{Im } \xi < 0$. (In fact, $g_{\pm}(b) = g_{\pm}(c)$.) Recall that $\Delta_1(\xi)$ (and hence $\widehat{\Delta_1(\xi)}$) is a meromorphic function near each point a, b, c . In fact, it has a first order pole at these points. Note that since there is an additional pole at b introduced by the denominator in (284), we must expand $\widehat{\Delta_1(\xi)}$ up to the third term at b , but only up to the first term at a and c to compute the residues. Therefore, it is convenient to define

$$T_2(\omega) = \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_c} \frac{L(\xi)}{(\xi - b)^2} d\xi, \quad T_3(\omega) = \frac{1}{2\pi i} \int_{\partial U_b} \frac{L(\xi)}{(\xi - b)^2} d\xi, \tag{292}$$

where we denote $\omega = s^{3/2}\Omega$. Then we have

$$\lim_{z \rightarrow b} \tau_3(z)_{21} = T_2(\omega) + T_3(\omega). \tag{293}$$

5.3.1. Evaluation of $T_2(\omega)$

Since $u(a) = 0$, we have expanding $F(\xi)$ (using (108))

$$F(\xi) = \begin{pmatrix} \theta_{11}(0) & -i\theta_{12}(0) \\ -i\theta_{21}(0) & \theta_{22}(0) \end{pmatrix} \frac{1}{2\beta_{0,a}} (\xi - a)^{-1/4} (1 + o(1)), \quad \xi \rightarrow a.$$

Substituting this into (289) and expanding $f_a(\xi)$ by (177), we obtain

$$\widehat{\Delta_1(\xi)} = \frac{1}{\xi - a} \frac{\beta_{0,a}^{-2}}{16s^{3/2} f'_a(a)^{1/2}} \begin{pmatrix} \theta_{11}(0)\theta_{22}(0) & -i\theta_{11}(0)^2 \\ -i\theta_{22}(0)^2 & -\theta_{11}(a)\theta_{22}(0) \end{pmatrix} + \mathcal{O}(1), \quad \xi \rightarrow a. \tag{294}$$

Similarly, using the identities

$$\theta_{j1}(u_+(c)) = \theta_{j2}(u_+(c)) e^{2s^{3/2}g_+(b)}, \quad j = 1, 2, \tag{295}$$

which are derived as (110), we obtain

$$\begin{aligned} \widehat{\Delta_1(\xi)} &= \frac{1}{\xi - c} \frac{\beta_{0,c}^{-2}}{16s^{3/2} f'_c(c)^{1/2}} e^{-s^{3/2}g_+(b)\sigma_3} \\ &\times \begin{pmatrix} \theta_{11}(u_+(c))\theta_{22}(u_+(c)) & -i\theta_{11}(u_+(c))^2 \\ -i\theta_{22}(u_+(c))^2 & -\theta_{11}(u_+(c))\theta_{22}(u_+(c)) \end{pmatrix} e^{s^{3/2}g_+(b)\sigma_3} \\ &+ \mathcal{O}(1), \quad \xi \rightarrow c. \end{aligned} \tag{296}$$

Therefore by (288), writing out θ_{jk} by (98), using relations in (114) and the following summation formulae,

$$\theta_2(x+y)\theta_3(x-y) + \theta_2(x-y)\theta_3(x+y) = \frac{2}{\theta_2\theta_3} \theta_2(x)\theta_2(y)\theta_3(x)\theta_3(y), \tag{297}$$

$$\theta_1(x+y)\theta_2(x-y) - \theta_1(x-y)\theta_2(x+y) = \frac{2}{\theta_3\theta_4} \theta_1(y)\theta_2(y)\theta_3(x)\theta_4(x), \tag{298}$$

(where we recall the convention that the θ -functions without argument stand for their values at zero), we obtain

$$L(\xi) = \frac{1}{\xi - a} \frac{1}{16} \frac{\beta_{0,b}^2 f'_b(b)^{1/2} \theta_3^2 \theta_2^2(s^{3/2}\Omega)}{\beta_{0,a}^2 f'_a(a)^{1/2} \theta_2^2 \theta_3^2(s^{3/2}\Omega)}, \quad \xi \rightarrow a, \tag{299}$$

and

$$L(\xi) = \frac{1}{\xi - c} \frac{1}{16} \frac{\beta_{0,b}^2 f'_b(b)^{1/2} \theta_3^2 \theta_4^2(s^{3/2}\Omega)}{\beta_{0,c}^2 f'_c(c)^{1/2} \theta_4^2 \theta_3^2(s^{3/2}\Omega)}, \quad \xi \rightarrow c. \tag{300}$$

Now note that an analysis of branches in (177) gives that $f'_a(a), f'_c(c) > 0$, while $f'_b(b) = -i |f'_b(b)|$. Using their values and those of $\beta_{0,p}$ in (109), we write the residues and determine T_2 (note the negative direction of integration in (292)):

$$T_2(\omega) = -\frac{1}{16} \left(\frac{1}{a-b} \left| \frac{q(b)}{q(a)} \right| \frac{\theta_3^2 \theta_2^2(\omega)}{\theta_2^2 \theta_3^2(\omega)} + \frac{1}{b-c} \left| \frac{q(b)}{q(c)} \right| \frac{\theta_3^2 \theta_4^2(\omega)}{\theta_4^2 \theta_3^2(\omega)} \right). \tag{301}$$

We now average $T_2(\omega)$ over ω . We use the integrals (238) obtained in [19]. Namely,

$$\int_0^1 \frac{\theta_2^2(\omega)}{\theta_3^2(\omega)} d\omega = \frac{\theta_4^2}{(\theta_1^2)^2} \frac{\theta_3''}{\theta_3} + \frac{\theta_2^2}{\theta_3^2}, \quad \int_0^1 \frac{\theta_4^2(\omega)}{\theta_3^2(\omega)} d\omega = -\frac{\theta_2^2}{(\theta_1^2)^2} \frac{\theta_3''}{\theta_3} + \frac{\theta_4^2}{\theta_3^2}. \tag{302}$$

By the identity $\theta_1' = \pi\theta_2\theta_3\theta_4$ [26], we can write the first integral as follows

$$\frac{\theta_3^2}{\theta_2^2} \int_0^1 \frac{\theta_2^2(\omega)}{\theta_3^2(\omega)} d\omega = \frac{1}{\pi^2 \theta_2^4} \frac{\theta_3''}{\theta_3} + 1. \tag{303}$$

To simplify this further, we use the heat equation for θ -functions,

$$\frac{\theta_3''}{\theta_3} = 4\pi i \frac{\partial}{\partial \tau} \log \theta_3 = \pi i \frac{1}{\frac{\partial \tau}{\partial b}} \frac{\partial}{\partial b} \log \theta_3^4 = -2J_0(a-b)(b-c) \frac{d}{db} J_0 = J_0(J_1 - bJ_0), \tag{304}$$

where we made use of parts (i) and (viii) of Lemma 14 to express θ_3^4 and $\frac{d\tau}{db}$ in terms of elliptic integrals, and used (142) to evaluate $\frac{dJ_0}{db}$. Substituting this expression as well as the one for θ_2^4 from Lemma 14 into the r.h.s. of (303), we obtain

$$\frac{\theta_3^2}{\theta_2^2} \int_0^1 \frac{\theta_2^2(\omega)}{\theta_3^2(\omega)} d\omega = \frac{J_1 - cJ_0}{J_0(b-c)}. \tag{305}$$

A similar analysis of the second integral in (302) gives

$$\frac{\theta_3^2}{\theta_4^2} \int_0^1 \frac{\theta_4^2(\omega)}{\theta_3^2(\omega)} d\omega = -\frac{J_1 - aJ_0}{J_0(a - b)}. \tag{306}$$

Therefore, integrating (301) we obtain

$$\int_0^1 T_2(\omega) d\omega = -\frac{1}{16(a - b)(b - c)} \left(\left| \frac{q(b)}{q(a)} \right| \left(\frac{J_1}{J_0} - c \right) - \left| \frac{q(b)}{q(c)} \right| \left(\frac{J_1}{J_0} - a \right) \right). \tag{307}$$

Next, by means of direct differentiation of $q(a)$ and $q(c)$, recalling the derivative $\frac{dq_a}{db}$ in (143), we have that

$$\frac{\frac{J_1}{J_0} - c}{2(a - b)(b - c)} = \frac{\frac{d}{db}q(a)}{q(b)} + \frac{a + b - c}{4q(b)} + \frac{1}{2(a - b)}, \tag{308}$$

and

$$\frac{\frac{J_1}{J_0} - a}{2(a - b)(b - c)} = \frac{\frac{d}{db}q(c)}{q(b)} + \frac{b + c - a}{4q(b)} - \frac{1}{2(b - c)}. \tag{309}$$

Substituting these expressions into (307), noting the signs of $q(p)$ obtained earlier in (154), and using the simple identities

$$q(b) = q(a) - \frac{1}{2}(a - b)(a + b - c), \quad q(b) = q(c) + \frac{1}{2}(b - c)(b + c - a), \tag{310}$$

we find

$$\int_0^1 T_2(\omega) d\omega = -\frac{1}{8} \frac{d}{db} \log \frac{|q(a)q(c)|}{\sqrt{(a - b)(b - c)}}. \tag{311}$$

5.3.2. Evaluation of $T_3(\omega)$

In view of the definition (292), we will now simplify the expression for $L(\xi)$ in (288), with $\widehat{\Delta_1}(\xi)$ given by (290). We use the following argument from [19] to decompose matrix products. In a neighbourhood of b , define functions $A_j(z)$ and $B_j(z)$, $j = 1, 2$, by

$$\begin{aligned} A_1(z) &= \frac{1}{2} \left(\theta_{11}(u)(\beta(z) + \beta^{-1}(z))e^{\mp s^{3/2}g_+(b)} + \theta_{12}(u)(\beta(z) - \beta^{-1}(z))e^{\pm s^{3/2}g_+(b)} \right), \\ A_2(z) &= \frac{1}{2} \left(\theta_{22}(u)(\beta(z) + \beta^{-1}(z))e^{\pm s^{3/2}g_+(b)} + \theta_{21}(u)(\beta(z) - \beta^{-1}(z))e^{\mp s^{3/2}g_+(b)} \right), \\ B_1(z) &= \frac{1}{2} \left(\theta_{11}(u)(\beta(z) + \beta^{-1}(z))e^{\mp s^{3/2}g_+(b)} - \theta_{12}(u)(\beta(z) - \beta^{-1}(z))e^{\pm s^{3/2}g_+(b)} \right), \\ B_2(z) &= \frac{1}{2} \left(\theta_{22}(u)(\beta(z) + \beta^{-1}(z))e^{\pm s^{3/2}g_+(b)} - \theta_{21}(u)(\beta(z) - \beta^{-1}(z))e^{\mp s^{3/2}g_+(b)} \right), \end{aligned} \tag{312}$$

where the upper sign in \pm, \mp is taken for $\text{Im}(z - b) > 0$, and the lower, for $\text{Im}(z - b) < 0$. It follows from the jump relations (cf. proof of Proposition 13), that

$$A_j(z)(z - b)^{1/4}, \quad B_j(z)(z - b)^{-1/4}, \quad j = 1, 2, \tag{313}$$

are analytic functions in U_b . In this notation, we have

$$\widehat{\Delta_1}(z) = \frac{1}{8s^{3/2}f_b(z)^{1/2}}F(z)e^{-s^{3/2}g_{\pm}(b)\sigma_3} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} e^{s^{3/2}g_{\pm}(b)\sigma_3}F(z)^{-1},$$

with

$$Fe^{-s^{3/2}g_{\pm}(b)\sigma_3} = \frac{1}{2} \begin{pmatrix} A_1 + B_1 & -i(A_1 - B_1) \\ i(A_2 - B_2) & A_2 + B_2 \end{pmatrix}. \tag{314}$$

This is a convenient way of expressing $\widehat{\Delta_1}(z)$ near b for the following reason. Since $\widehat{\Delta_1}(z)$ is meromorphic near b , its expansion at $z = b$ contains only integer powers. On account of the $1/f_b(z)^{1/2}$ term, containing only half-powers, it follows from the analyticity of (313) that all ‘cross-terms’, containing products A_jB_k , must be zero. Thus we have the following decomposition:

$$\begin{aligned} \widehat{\Delta_1}(z) &= \frac{1}{4 \cdot 8s^{3/2}f_b(z)^{1/2}} \begin{pmatrix} A_1 & -iA_1 \\ iA_2 & A_2 \end{pmatrix} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} \begin{pmatrix} A_2 & iA_1 \\ -iA_2 & A_1 \end{pmatrix} + \\ &+ \frac{1}{4 \cdot 8s^{3/2}f_b(z)^{1/2}} \begin{pmatrix} B_1 & iB_1 \\ -iB_2 & B_2 \end{pmatrix} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix} \begin{pmatrix} B_2 & -iB_1 \\ iB_2 & B_1 \end{pmatrix}. \end{aligned} \tag{315}$$

It is now a straightforward calculation to see that we may write (288) near b in the form

$$\begin{aligned} L(z) &= -\frac{\beta_{0,b}^2 f'_b(b)^{1/2}}{64f_b(z)^{1/2}} \left([A_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} - A_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2 \right. \\ &\left. + 3[B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2 \right). \end{aligned} \tag{316}$$

By means of (104), (108), and the identities for θ_{jk} in (110), (111), we expand $\beta^{-1}(z)A_j(z)$, $z \rightarrow b$, for $j = 1, 2$, and determine that the analytic (by (313)) in U_b function

$$\begin{aligned} &\beta^{-1}(z)[A_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} - A_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] \\ &= \left\{ \beta_{0,b}^{-2}u_{0,b}(\theta'_{11}\theta_{22} - \theta_{11}\theta'_{22}) + \frac{u_{0,b}^2}{2}(\theta''_{11}\theta_{22} - \theta_{11}\theta''_{22}) \right\} (z - b) + \mathcal{O}((z - b)^2) \\ &= \frac{u_{0,b}^2}{2} \{2J_0(\theta'_{11}\theta_{22} - \theta_{11}\theta'_{22}) + \theta''_{11}\theta_{22} - \theta_{11}\theta''_{22}\} (z - b) + \mathcal{O}((z - b)^2), \quad z \rightarrow b, \end{aligned} \tag{317}$$

where all θ_{jk} and their derivatives with omitted argument are evaluated at $u_+(b)$, and we used the identity (134) in the last equation. Notice that the $z - b$ term is, up to a prefactor, is what we found earlier in (279). By the equations (281) and (282),

$$\begin{aligned} &\beta^{-1}(z)[A_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} - A_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] \\ &= 2u_{0,b}^2 J_0 \frac{\theta'_3}{\theta_3}(\omega)(z - b) + \mathcal{O}((z - b)^2), \quad z \rightarrow b, \quad \omega = s^{3/2}\Omega. \end{aligned} \tag{318}$$

Substituting the above expression into the first term in (316) and that in turn into (292), we find that

$$\begin{aligned} T_3(\omega) &= \frac{1}{2\pi i} \int_{\partial U_b} \frac{L(z)}{(z - b)^2} dz \\ &= -\frac{\beta_{0,b}^2 f'_b(b)^{1/2}}{64} \left(\int_{\partial U_b} \frac{dz}{2\pi i} \frac{\beta^2(z)}{f_b(z)^{1/2}} \left\{ \left(2u_{0,b}^2 J_0 \frac{\theta'_3}{\theta_3}(\omega) \right)^2 + \mathcal{O}(z - b) \right\} \right. \\ &\quad + \int_{\partial U_b} \frac{dz}{2\pi i} \frac{3}{(z - b)^2 f_b(z)^{1/2}} [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} \\ &\quad \left. + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2 \right). \end{aligned} \tag{319}$$

Since the meromorphic in U_b function

$$\frac{\beta^2(z)}{f_b(z)^{1/2}} = \frac{1}{z - b} \frac{\beta_{0,b}^2}{f'_b(b)^{1/2}} (1 + \mathcal{O}(z - b)), \quad z \rightarrow b,$$

we immediately compute the first integral by taking the residue (recall the negative direction of the integration), and recalling the expressions (109), (106) for $\beta_{0,b}$, $u_{0,b}$, we obtain

$$\begin{aligned} T_3(\omega) &= -\frac{1}{16J_0^2(a - b)(b - c)} \left(\frac{\theta'_3}{\theta_3}(\omega) \right)^2 \\ &\quad - \frac{\beta_{0,b}^2 f'_b(b)^{1/2}}{64} \int_{\partial U_b} \frac{dz}{2\pi i} \frac{3}{(z - b)^2 f_b(z)^{1/2}} [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} \\ &\quad + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2. \end{aligned} \tag{320}$$

Recall that we are interested in the average of $T_3(\omega)$ over ω . As was noticed in a similar situation in [19], it is easier to do the averaging of the second term in (320) before computing its residue. Furthermore, to average the first term, we can use the integral (A.19) in Lemma 26 from [19]:

$$\int_0^1 \left(\frac{\theta'_3(\omega)}{\theta_3(\omega)}\right)^2 d\omega = \frac{\pi^2}{3} + \frac{\theta'''_1}{3\theta'_1}. \tag{321}$$

By the heat equation, the identity $\theta'_1 = \pi\theta_2\theta_3\theta_4$, and the identities of Lemma 14, we have that

$$\frac{\theta'''_1}{\theta'_1} = 4\pi i \frac{\partial}{\partial \tau} \log \theta'_1 = \frac{\pi i}{\frac{d\tau}{db}} \frac{d}{db} \log(\theta_2^4\theta_3^4\theta_4^4) = -J_0^2(a-b)(b-c) \frac{d}{db} \log [(J_0^6(a-b)(b-c))]. \tag{322}$$

Thus, we obtain

$$\frac{1}{J_0^2(a-b)(b-c)} \int_0^1 \left(\frac{\theta'_3(\omega)}{\theta_3(\omega)}\right)^2 d\omega = \frac{\pi^2/3}{J_0^2(a-b)(b-c)} - \frac{1}{3} \frac{d}{db} \log [(J_0^6(a-b)(b-c))]. \tag{323}$$

On the other hand, since

$$\frac{\partial}{\partial \tau} \log \theta_3(\omega) = \frac{1}{4\pi i} \frac{\theta''_3(\omega)}{\theta_3(\omega)} \tag{324}$$

and

$$0 = \int_0^1 \left(\frac{\theta'_3(\omega)}{\theta_3(\omega)}\right)' d\omega = \int_0^1 \left[\frac{\theta''_3(\omega)}{\theta_3(\omega)} - \left(\frac{\theta'_3(\omega)}{\theta_3(\omega)}\right)^2 \right] d\omega,$$

we obtain, using the expression for $d\tau/db$ from Lemma 14,

$$\begin{aligned} \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega &= -\frac{\pi i}{J_0^2(a-b)(b-c)} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= -\frac{1}{4J_0^2(a-b)(b-c)} \int_0^1 \left(\frac{\theta'_3(\omega)}{\theta_3(\omega)}\right)^2 d\omega. \end{aligned} \tag{325}$$

Thus, integrating (320) and using (325) and (323), we obtain

$$\int_0^1 T_3(\omega) d\omega - \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega = \frac{\pi^2/16}{J_0^2(a-b)(b-c)} - \frac{1}{16} \frac{d}{db} \log [(J_0^6(a-b)(b-c))]$$

$$\begin{aligned}
 & - \frac{\beta_{0,b}^2 f'_b(b)^{1/2}}{64} \int_{\partial U_b} \frac{dz}{2\pi i} \frac{3}{(z-b)^2 f_b(z)^{1/2}} \int_0^1 d\omega [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} \\
 & + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2. \tag{326}
 \end{aligned}$$

We now turn to computation of the average of the term with B 's in this expression. First note that the analytic in U_b function

$$\frac{1}{\beta^2(z)f_b(z)^{1/2}} = \frac{1}{\beta_{0,b}^2 f'_b(b)^{1/2}} \left(1 - \left(\frac{f_{1,b}}{2} + 2\beta_{1,b} \right) (z-b) + \mathcal{O}((z-b)^2) \right), \quad z \rightarrow b, \tag{327}$$

where we used (108) and wrote $f_b(z) = f'_b(b)(z-b)(1 + f_{1,b}(z-b) + \mathcal{O}((z-b)^2))$ recalling (177). In the last expansion

$$f_{1,b} = \frac{2}{3} \left(\frac{q_1 + 2b}{q(b)} - \frac{1}{2} \left(\frac{1}{b-c} - \frac{1}{a-b} \right) \right). \tag{328}$$

Expanding $B_j(z)$ similarly to $A_j(z)$ above, and using the identity (135), we obtain for the analytic (by (313)) in U_b function

$$\beta(z)[B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] - 2 = \mathcal{O}(z-b), \quad z \rightarrow b. \tag{329}$$

Taking the square of this expression, we may write

$$\begin{aligned}
 & \beta(z)^2 [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2 \\
 & = -4 + 4\beta(z)[B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] \\
 & \quad + \mathcal{O}((z-b)^2), \quad z \rightarrow b. \tag{330}
 \end{aligned}$$

The advantage of this representation is that we see (taking into account (327)) that only the first 2 terms on the r.h.s. may give a nonzero contribution to the residue in (326), and the second term is not squared.

We now evaluate the average

$$\begin{aligned}
 & \int_0^1 [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] d\omega \\
 & = \int_0^1 [\widehat{q(\omega)} + \widehat{q(-\omega)}] d\omega, \quad \omega = s^{3/2}\Omega, \tag{331}
 \end{aligned}$$

where, by expanding the terms and using quasiperiodicity of θ -functions,

$$\begin{aligned}
 \widehat{q(\omega)} = & \frac{\theta_3^2}{2\theta_3^2(\omega)} \frac{\theta_3(-\omega + \widehat{d})}{\theta_3(\widehat{d})} \left(\frac{\theta_3(\widehat{u(z)} + \omega + \widehat{d})}{\theta_3(\widehat{u(z)} + \widehat{d})} (\beta(z) + \beta^{-1}(z)) \right. \\
 & \left. - \frac{\theta_3(\widehat{u(z)} - \omega - \widehat{d})}{\theta_3(\widehat{u(z)} - \widehat{d})} (\beta(z) - \beta^{-1}(z)) \right), \tag{332}
 \end{aligned}$$

with

$$\widehat{u(z)} = u(z) - u_+(b), \quad \widehat{d} = d + u_+(b), \quad u_+(b) = -\frac{\tau}{2}. \tag{333}$$

Since $\widehat{q(-\omega)} = q(\widehat{1-\omega})$, we have that $\int_0^1 [\widehat{q(\omega)} + \widehat{q(-\omega)}] d\omega = 2 \int_0^1 \widehat{q(\omega)} d\omega$. We notice that our expression for $\widehat{q(\omega)}$ has the form of $q(\omega)$, given by (264) in Section 9.2 of [19], with $u(z) \rightarrow \widehat{u(z)}$ and $d \rightarrow \widehat{d}$. Therefore, our analysis now is very similar to that of [19].

Applying Lemma 26 of [19] to evaluate $\int_0^1 \widehat{q(\omega)} d\omega$, we obtain:

$$\begin{aligned}
 \beta(z) \int_0^1 [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] d\omega = & \frac{\pi\theta_3^2\widehat{g}(\widehat{d})}{(\theta'_1)^2 \sin(\pi\widehat{u})} \\
 \times \left\{ (\beta(z)^2 + 1) \widehat{g}(\widehat{d} + \widehat{u}) [\widehat{f}(\widehat{d}) - \widehat{f}(\widehat{d} + \widehat{u})] + (\beta(z)^2 - 1) \widehat{g}(\widehat{d} - \widehat{u}) [\widehat{f}(\widehat{d}) - \widehat{f}(\widehat{d} - \widehat{u})] \right\}, \tag{334}
 \end{aligned}$$

where

$$\widehat{g}(x) = \frac{\theta_1(x)}{\theta_3(x)}, \quad \widehat{f}(x) = \frac{\theta'_1(x)}{\theta_1(x)}. \tag{335}$$

Furthermore, expanding $\widehat{u(z)}$ and using (270), (271) in [19] gives, as $z \rightarrow b$,

$$\begin{aligned}
 \beta(z) \int_0^1 [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}] d\omega \\
 = \widehat{g}(\widehat{d}) \left(1 + \frac{\pi^2}{6} u_{0,b}^2 (z - b) + \mathcal{O}((z - b)^2) \right) \\
 \times \left[H_0 - u_{0,b}\beta_{0,b}^2 (1 + (z - b)(u_{1,b} + 2\beta_{1,b})) H_1 + (z - b)(u_{0,b}^2 H_2 + u_{0,b}^3 \beta_{0,b}^2 H_3) \right. \\
 \left. + \mathcal{O}((z - b)^{3/2}) \right], \tag{336}
 \end{aligned}$$

where

$$\begin{aligned}
 H_0 &= 2\widehat{g}(\widehat{d}) \left(\frac{1}{\widehat{g}(\widehat{d})^2} - \frac{\theta_3''}{\theta_3} \left(\frac{\theta_3}{\theta_1'} \right)^2 \right), & H_1 &= 2\widehat{g}'(\widehat{d}) \frac{\theta_3''}{\theta_3} \left(\frac{\theta_3}{\theta_1'} \right)^2, \\
 H_2 &= \widehat{g}''(\widehat{d}) \left(\frac{1}{3\widehat{g}(\widehat{d})^2} - \frac{\theta_3''}{\theta_3} \left(\frac{\theta_3}{\theta_1'} \right)^2 \right), & & \\
 H_3 &= \frac{\widehat{g}'''(\widehat{d})}{6} \left(\frac{1}{\widehat{g}(\widehat{d})^2} - 2\frac{\theta_3''}{\theta_3} \left(\frac{\theta_3}{\theta_1'} \right)^2 \right) - \frac{1}{6} \frac{\widehat{g}''(\widehat{d})\widehat{g}'(\widehat{d})}{\widehat{g}(\widehat{d})^3}.
 \end{aligned}
 \tag{337}$$

Recalling (118) of Lemma 14 and (134), we note that

$$\widehat{g}'(\widehat{d}) = -J_0\widehat{g}(\widehat{d}) = -\frac{1}{u_{0,b}\beta_{0,b}^2}\widehat{g}(\widehat{d}).
 \tag{338}$$

By further applying (118) and (119) of Lemma 14, we simplify the combinations of the H_j as follows:

$$\begin{aligned}
 H_0 - u_{0,b}\beta_{0,b}^2H_1 &= \frac{2}{\widehat{g}(\widehat{d})}, \\
 u_{0,b}^2H_2 + u_{0,b}^3\beta_{0,b}^2H_3 &= \frac{2\beta_{1,b} + u_{1,b}}{\widehat{g}(\widehat{d})} \left(1 - 2\widehat{g}(\widehat{d})^2 \frac{\theta_3''}{\theta_3} \left(\frac{\theta_3}{\theta_1'} \right)^2 \right),
 \end{aligned}
 \tag{339}$$

which allows us to write (336) in the form

$$\begin{aligned}
 \beta(z) \int_0^1 [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]d\omega &= \\
 2 + \left(\frac{\pi^2}{3}u_{0,b}^2 + u_{1,b} + 2\beta_{1,b} \right) (z - b) + \mathcal{O}((z - b)^2), & \quad z \rightarrow b.
 \end{aligned}
 \tag{340}$$

Substituting this into (326) and calculating the residue by (330) and (327) we obtain

$$\begin{aligned}
 & -\frac{\beta_{0,b}^2f_b'(b)^{1/2}}{64} \int_{\partial U_b} \frac{dz}{2\pi i} \frac{3}{(z - b)^2 f_b(z)^{1/2}} \\
 & \times \int_0^1 d\omega [B_1(z)\theta_{22}(u_+(b))e^{s^{3/2}g_+(b)} + B_2(z)\theta_{11}(u_+(b))e^{-s^{3/2}g_+(b)}]^2 \\
 & = \frac{1}{16} (\pi^2u_{0,b}^2 + 3u_{1,b} - 3f_{1,b}/2) = -\frac{1}{16} \left(\frac{\pi^2}{(a - b)(b - c)J_0^2} + \frac{q_1 + 2b}{q(b)} \right).
 \end{aligned}
 \tag{341}$$

For the last equation here we used (106), (328). On the other hand, differentiating $q(b)$ w.r.t. b and using (143), we obtain

$$\frac{q_1 + 2b}{2q(b)} = \frac{d}{db} \log|q(b)| + \frac{d}{db} \log|J_0|. \tag{342}$$

With this we may write the r.h.s. of (341) as

$$-\frac{\pi^2}{16(a-b)(b-c)J_0^2} - \frac{1}{8} \frac{d}{db} \log|q(b)J_0|. \tag{343}$$

Substituting this expression into (326) we obtain

$$\begin{aligned} & \int_0^1 T_3(\omega) d\omega - \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= -\frac{1}{2} \frac{d}{db} \log|J_0| - \frac{1}{8} \frac{d}{db} \log|q(b)| - \frac{1}{16} \frac{d}{db} \log((a-b)(b-c)). \end{aligned} \tag{344}$$

Together with (311) this equation gives

$$\begin{aligned} & \lim_{z \rightarrow b} \int_0^1 \tau_3(z)_{21} d\omega - \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= \int_0^1 T_2(\omega) d\omega + \int_0^1 T_3(\omega) d\omega - \frac{d\tau}{db} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= -\frac{1}{2} \frac{d}{db} \log|J_0| - \frac{1}{8} \frac{d}{db} \log|q(a)q(b)q(c)|, \end{aligned}$$

which proves Lemma 22.

5.4. Asymptotic form of the differential identity at a. Proof of Proposition 24

In this section we prove that the differential identity is symmetric in a and b as shown in Proposition 24. We start from the differential identity at a , namely

$$\frac{d}{da} \log P^{Ai}(sJ) = \frac{1}{2\pi i} \lim_{z \rightarrow a} (X^{-1}(z)X'(z))_{21}, \tag{345}$$

where the limit is taken from outside the lens and with $\text{Im } z > 0$. By the definitions of $S(z)$ and $R(z; s)$, we have that

$$\begin{aligned} X^{-1}X' &= e^{s^{3/2}g(z)\sigma_3} P_a^{-1} R^{-1} R' P_a e^{-s^{3/2}g(z)\sigma_3} \\ &+ e^{s^{3/2}g(z)\sigma_3} P_a^{-1} P_a' e^{-s^{3/2}g(z)\sigma_3} - s^{3/2}g'(z)\sigma_3. \end{aligned} \tag{346}$$

Then, by definition of $P_a(z; s)$, noting that $g(a) = 0$, we have

$$P_a = E_a \Psi(s^3 g(z)^2) e^{s^{3/2} g(z) \sigma_3},$$

$$P_a^{-1} P'_a = e^{-s^{3/2} g(z) \sigma_3} \Psi^{-1} E_a^{-1} E'_a \Psi e^{s^{3/2} g(z) \sigma_3} + e^{-s^{3/2} g(z) \sigma_3} \Psi^{-1} \Psi' e^{s^{3/2} g(z) \sigma_3} + s^{3/2} g'(z) \sigma_3.$$

Moreover, we have

$$R^{-1}(z) R'(z) = \frac{dR^{(1)}}{dz}(z) + \mathcal{O}\left(\frac{1}{s^3 \min\{|a|^2, (b-c)^2\}}\right), \quad s \rightarrow \infty, \tag{347}$$

where

$$R^{(1)'}(z; s) = \frac{1}{2\pi i} \int_{\partial U_a \cup \partial U_b \cup \partial U_c} \frac{\Delta_1(\zeta; s)}{(\zeta - z)^2} d\zeta. \tag{348}$$

It follows, similarly to the argument leading to (229), that in the regime of Proposition 24

$$\frac{d}{da} \log P^{Ai}(sJ) = \lim_{z \rightarrow a} (\tilde{\tau}_1(z) + \tilde{\tau}_2(z) + \tilde{\tau}_3(z))_{21} + \mathcal{O}\left(\frac{1}{s^3 |a|}\right) + \mathcal{O}\left(\frac{1}{s^{3/2} (b-c)^2}\right) \tag{349}$$

with

$$\tilde{\tau}_1(z) = \frac{1}{2\pi i} \Psi^{-1}(s^3 f_a(z)) \Psi'(s^3 f_a(z)), \tag{350}$$

$$\tilde{\tau}_2(z) = \frac{1}{2\pi i} \Psi^{-1}(s^3 f_a(z)) E_a^{-1}(z) E'_a(z) \Psi(s^3 f_a(z)), \tag{351}$$

$$\tilde{\tau}_3(z) = \frac{1}{2\pi i} e^{s^{3/2} g(z) \sigma_3} P_a^{-1}(z) \frac{R^{(1)}}{dz}(z) P_a(z) e^{-s^{3/2} g(z) \sigma_3}. \tag{352}$$

We now compute the limit in (349). As before, we summarise the contributions to the differential identity, from each term, in the following lemmata.

Lemma 28.

$$\lim_{z \rightarrow a} \tilde{\tau}_1(z)_{21} = -s^3 \frac{d}{da} \alpha_2. \tag{353}$$

Proof. The proof of Lemma 28 runs almost exactly the same as that of Lemma 20: From (221) we easily obtain, recalling that $\zeta = \zeta(z) = s^3 f_a(z) = s^3 g(z)^2$ and using (177),

$$\begin{aligned} \lim_{z \rightarrow a} \tilde{\tau}_1(z)_{21} &= \frac{1}{2\pi i} \lim_{z \rightarrow a} \left(\Psi^{-1}(s^3 f_a(z)) \frac{d\Psi}{dz}(s^3 f_a(z)) \right)_{21} \\ &= \frac{1}{2\pi i} \lim_{\zeta \rightarrow 0} \left(\Psi^{-1}(\zeta) \frac{d\Psi}{d\zeta}(\zeta) \right)_{21} s^3 f'_a(a) = s^3 \frac{q(a)^2}{(a-b)(a-c)}, \end{aligned} \tag{354}$$

where we have noted that, in this case,

$$f'_a(a) = \frac{4q(a)^2}{(a-b)(a-c)}.$$

At this stage we must prove, similarly to (263), the identity

$$\frac{d}{da} \alpha_2 = -\frac{q(a)^2}{(a-b)(a-c)}. \tag{355}$$

For this we, as before, differentiate (82). Then, using (147) and (148), we obtain (355). Comparing (355) with (354), we obtain Lemma 28. \square

Lemma 29.

$$\lim_{z \rightarrow a} \tilde{\tau}_2(z)_{21} = \frac{d}{da} \log \theta_3(s^{3/2}\Omega; \tau) - \frac{d\tau}{da} \frac{\partial}{\partial \tau} \log \theta_3(s^{3/2}\Omega; \tau). \tag{356}$$

Proof. To prove Lemma 29 we proceed as in the proof of Lemma 21. We first have that

$$\frac{1}{2\pi i} \frac{1}{4} \frac{f'_a(z)}{f_a(z)} (\Psi^{-1}(\zeta) \sigma_3 \Psi(\zeta))_{21} \rightarrow -s^3 \frac{q(a)^2}{2(a-b)(a-c)}, \quad z \rightarrow a. \tag{357}$$

Then, similarly to (272) we obtain

$$\begin{aligned} \lim_{z \rightarrow a} \tilde{\tau}_2(z)_{21} &= \lim_{z \rightarrow a} \left(\frac{1}{4} (2A - i(B + C)) s^{3/2} f_a^{1/2}(z) + \frac{i}{4} (B - C) s^3 f_a(z) \right) \\ &\quad - s^3 \frac{q(a)^2}{2(a-b)(a-c)}, \end{aligned} \tag{358}$$

where A, B, C are given by (269)–(271) but without the $e^{\pm 2g_+(b)s^{3/2}}$ factors and with θ_{jk} evaluated at $u(a) = 0$ rather than at $u_+(b)$. Using the identities (113) for θ_{jk} and its derivatives, we see that

$$B - C = \frac{1}{2i} \frac{1}{z - a} \left(\theta_{11} \theta_{22} + \frac{u_{0,a} \beta_{0,a}^{-2}}{2} (\theta_{11} \theta'_{22} + \theta_{22} \theta'_{11}) \right) \Big|_{u(a)=0} (1 + \mathcal{O}(z - a)). \tag{359}$$

Using the fact that

$$u_{0,a} \beta_{0,a}^{-2} = \frac{1}{J_0(a - c)} \tag{360}$$

and the identity (136), we obtain

$$B - C = \frac{1}{2i} \frac{1}{z - a} (1 + \mathcal{O}(z - a)), \tag{361}$$

so that by the expansion of $f_a(z)$ in (177),

$$\frac{i}{4}(B - C)s^3 f_a(z) \rightarrow s^3 \frac{q(a)^2}{2(a - b)(a - c)}, \quad z \rightarrow a, \tag{362}$$

which cancels the last term in (358), as before.

Furthermore, we obtain similarly

$$2A - i(B + C) = -\frac{u_{0,a}}{(z - a)^{1/2}} \left(\theta_{11} \theta'_{22} - \theta_{22} \theta'_{11} + \frac{1}{2J_0(a - c)} (\theta_{11} \theta''_{22} - \theta_{22} \theta''_{11}) \right) \Big|_0 + \mathcal{O}(1). \tag{363}$$

By identities (121) and (122) of Lemma 14, we then obtain

$$\lim_{z \rightarrow a} \tilde{\tau}_2(z)_{21} = s^{3/2} \frac{d\Omega}{da} \tilde{T}_1(s^{3/2}\Omega), \tag{364}$$

where

$$\begin{aligned} \tilde{T}_1(\omega) &= \frac{1}{2J_0(a - c)} \left[\theta_{11} \theta_{22} \left(2J_0(a - c) \left(\frac{\theta'_{11}}{\theta_{11}} - \frac{\theta'_{22}}{\theta_{22}} \right) + \frac{\theta''_{11}}{\theta_{11}} - \frac{\theta''_{22}}{\theta_{22}} \right) \right]_0 \\ &= -\frac{\theta_3^2 \theta_3(\omega + d) \theta_3(\omega - d)}{J_0(a - c) \theta_3^2(d) \theta_3^2(\omega)} \left[\frac{\theta'_1}{\theta_1}(d) \left(\frac{\theta'_3}{\theta_3}(\omega + d) + \frac{\theta'_3}{\theta_3}(\omega - d) \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{\theta''_3}{\theta_3}(\omega + d) - \frac{\theta''_3}{\theta_3}(\omega - d) \right) \right]. \end{aligned} \tag{365}$$

The same argument as for (281) above gives

$$\tilde{T}_1(\omega) = 2 \frac{\theta'_3}{\theta_3}(\omega). \tag{366}$$

Substituting this into (364) and writing out the total derivative, we obtain the Lemma. \square

Lemma 30. *With $\omega = s^{3/2}\Omega$,*

$$\int_0^1 \lim_{z \rightarrow a} \tilde{\tau}_3(z)_{21} d\omega = -\frac{1}{2} \frac{d}{da} \log |J_0| - \frac{1}{8} \frac{d}{da} \log |q(a)q(b)q(c)| + \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega; \tau) d\omega. \tag{367}$$

Proof. We have already computed most of the necessary ingredients in the proof of Lemma 22. For $\omega = s^{3/2}\Omega$ let us denote the residue contribution to $\lim_{z \rightarrow a} \tau_3(z)_{21}$ from the points b and c by $T_{2,a}(\omega)$, and from the point a by $T_{3,a}(\omega)$. Thus, similarly to (293) we have

$$\lim_{z \rightarrow a} \tilde{\tau}_3(z)_{21} = T_{2,a} + T_{3,a} \tag{368}$$

where

$$T_{2,a} = \frac{1}{2\pi i} \int_{\partial U_b \cup \partial U_c} \frac{L_a(\xi)}{(\xi - a)^2} d\xi, \quad T_{3,a} = \frac{1}{2\pi i} \int_{\partial U_a} \frac{L_a(\xi)}{(\xi - a)^2} d\xi, \tag{369}$$

with (note that $u(a) = 0$ is the argument of θ_{jk})

$$L_a(\xi) = \frac{\beta_{0,a}^{-2}}{4\pi i} (s^3 \pi^2 f'_a(a))^{1/2} (i\theta_{22}(0) \quad \theta_{11}(0)) \widehat{\Delta}_1(\xi) \begin{pmatrix} \theta_{11}(0) \\ -i\theta_{22}(0) \end{pmatrix}, \tag{370}$$

and $\widehat{\Delta}_1(\xi)$ is given by (289)–(291).

For $T_{2,a}$ we may proceed as in the evaluation of T_2 in the proof of Lemma 22. It is straightforward to derive, analogously to (307), that

$$\int_0^1 T_{2,a}(\omega) d\omega = \frac{1}{16(a-b)(a-c)} \left[\frac{q(a)}{q(c)} \left(\frac{J_1}{J_0} - b \right) + \frac{q(a)}{q(b)} \left(\frac{J_1}{J_0} - c \right) \right]. \tag{371}$$

Using (147) and (148), we differentiate $q(b)$ and $q(c)$ directly and find that

$$-\frac{\frac{J_1}{J_0} - b}{2(a-b)(a-c)} = \frac{\frac{d}{da} q(c)}{q(a)} + \frac{a + c - b}{4q(a)} - \frac{1}{2(a-c)},$$

and

$$-\frac{\frac{J_1}{J_0} - c}{2(a-b)(a-c)} = \frac{\frac{d}{da} q(b)}{q(a)} + \frac{a + b - c}{4q(a)} - \frac{1}{2(a-b)}.$$

Substituting these into (371) and noting the simple relations

$$q(a) = q(b) + \frac{1}{2}(a-b)(a+b-c), \quad q(a) = q(c) + \frac{1}{2}(a-c)(a+c-b),$$

we arrive at the expression

$$\int_0^1 T_{2,a}(\omega) d\omega = -\frac{1}{8} \frac{d}{da} \log \frac{|q(b)q(c)|}{\sqrt{(a-b)(a-c)}}. \tag{372}$$

Concerning $T_{3,a}$ we, again, follow the proof of Lemma 22. We first redefine

$$\begin{aligned}
 A_1(z) &= \frac{1}{2} (\theta_{11}(u)(\beta(z) + \beta^{-1}(z)) + \theta_{12}(u)(\beta(z) - \beta^{-1}(z))), \\
 A_2(z) &= \frac{1}{2} (\theta_{22}(u)(\beta(z) + \beta^{-1}(z)) + \theta_{21}(u)(\beta(z) - \beta^{-1}(z))), \\
 B_1(z) &= \frac{1}{2} (\theta_{11}(u)(\beta(z) + \beta^{-1}(z)) - \theta_{12}(u)(\beta(z) - \beta^{-1}(z))), \\
 B_2(z) &= \frac{1}{2} (\theta_{22}(u)(\beta(z) + \beta^{-1}(z)) - \theta_{21}(u)(\beta(z) - \beta^{-1}(z))),
 \end{aligned}
 \tag{373}$$

and then derive similarly to (316) that for z close to a

$$L_a(z) = -\frac{\beta_{0,a}^{-2} f'_a(a)^{1/2}}{64 f_a(z)^{1/2}} ([B_1(z)\theta_{22}(0) - B_2(z)\theta_{11}(0)]^2 + 3[A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2).
 \tag{374}$$

Note that $u(a) = 0$, $g_+(a) = 0$, and the roles of A and B are interchanged.

We then obtain similarly to (318) that, in terms of $\tilde{T}_1(\omega)$ given by (365), (366),

$$\begin{aligned}
 &\beta(z)[B_1(z)\theta_{22}(0) - B_2(z)\theta_{11}(0)] \\
 &= \left\{ \beta_{0,a}^2 u_{0,a} (\theta'_{11}\theta_{22} - \theta_{11}\theta'_{22}) + \frac{u_{0,a}^2}{2} (\theta''_{11}\theta_{22} - \theta_{11}\theta''_{22}) \right\} (z - a) + \mathcal{O}((z - a)^2) \\
 &= \beta_{0,a}^2 u_{0,a} \tilde{T}_1(\omega)(z - a) + \mathcal{O}((z - a)^2) \\
 &= 2u_{0,a}^2 J_0(a - c) \frac{\theta'_3}{\theta_3}(\omega)(z - a) + \mathcal{O}((z - a)^2), \quad z \rightarrow a, \quad \omega = s^{3/2}\Omega.
 \end{aligned}
 \tag{375}$$

It follows that

$$\begin{aligned}
 T_{3,a}(\omega) &= \frac{1}{2\pi i} \int_{\partial U_a} \frac{L_a(z)}{(z - a)^2} dz \\
 &= -\frac{\beta_{0,a}^{-2} f'_a(a)^{1/2}}{64} \left(\int_{\partial U_a} \frac{dz}{2\pi i} \frac{\beta^{-2}(z)}{f_a(z)^{1/2}} \left\{ \left(2u_{0,a}^2 J_0(a - c) \frac{\theta'_3}{\theta_3}(\omega) \right)^2 + \mathcal{O}(z - a) \right\} \right. \\
 &\quad \left. + \int_{\partial U_a} \frac{dz}{2\pi i} \frac{3}{(z - a)^2 f_a(z)^{1/2}} [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2 \right).
 \end{aligned}
 \tag{376}$$

Since the meromorphic in U_a function

$$\frac{\beta^{-2}(z)}{f_a(z)^{1/2}} = \frac{1}{z - a} \frac{\beta_{0,a}^{-2}}{f'_a(a)^{1/2}} (1 + \mathcal{O}(z - a)), \quad z \rightarrow a,$$

we immediately compute the first integral by taking the residue (recall the negative direction of the integration), and recalling the expressions (109), (105) for $\beta_{0,a}$, $u_{0,a}$, we obtain

$$T_{3,a}(\omega) = \frac{1}{16J_0^2(a-b)(a-c)} \left(\frac{\theta'_3}{\theta_3}(\omega)\right)^2 - \frac{\beta_{0,a}^{-2}f'_a(a)^{1/2}}{64} \int_{\partial U_a} \frac{dz}{2\pi i} \frac{3}{(z-a)^2 f_a(z)^{1/2}} [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2. \tag{377}$$

To average the first term, we again use (321). By the identities of Lemma 14, we have that

$$\frac{\theta'''_1}{\theta'_1} = 4\pi i \frac{\partial}{\partial \tau} \log \theta'_1 = \frac{\pi i}{\frac{d\tau}{da}} \frac{d}{da} \log(\theta_2^4 \theta_3^4 \theta_4^4) = J_0^2(a-b)(a-c) \frac{d}{da} \log [(J_0^6(a-b)(a-c))]. \tag{378}$$

Thus, we obtain

$$\frac{1}{J_0^2(a-b)(a-c)} \int_0^1 \left(\frac{\theta'_3}{\theta_3}(\omega)\right)^2 d\omega = \frac{\pi^2/3}{J_0^2(a-b)(a-c)} + \frac{1}{3} \frac{d}{da} \log [(J_0^6(a-b)(a-c))]. \tag{379}$$

Similarly to the derivation of (325), we find

$$\begin{aligned} \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega &= \frac{\pi i}{J_0^2(a-b)(a-c)} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= \frac{1}{4J_0^2(a-b)(a-c)} \int_0^1 \left(\frac{\theta'_3}{\theta_3}(\omega)\right)^2 d\omega. \end{aligned} \tag{380}$$

Thus, integrating (377) and using (380) and (379), we obtain

$$\begin{aligned} &\int_0^1 T_{3,a}(\omega) d\omega - \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial \tau} \log \theta_3(\omega) d\omega \\ &= -\frac{\pi^2/16}{J_0^2(a-b)(a-c)} - \frac{1}{16} \frac{d}{da} \log [(J_0^6(a-b)(a-c))] \\ &\quad - \frac{\beta_{0,a}^{-2}f'_a(a)^{1/2}}{64} \int_{\partial U_a} \frac{dz}{2\pi i} \frac{3}{(z-a)^2 f_a(z)^{1/2}} \int_0^1 d\omega [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2. \end{aligned} \tag{381}$$

We now turn to computation of the average of the term with A 's in this expression. First note that the analytic in U_a function

$$\frac{1}{\beta^{-2}(z)f_a(z)^{1/2}} = \frac{1}{\beta_{0,a}^{-2}f'_a(a)^{1/2}} \left(1 - \left(\frac{f_{1,a}}{2} - 2\beta_{1,a} \right) (z - a) + \mathcal{O}((z - a)^2) \right), \quad z \rightarrow a, \tag{382}$$

where we used (108) and wrote $f_a(z) = f'_a(a)(z - a)(1 + f_{1,a}(z - a) + \mathcal{O}((z - a)^2))$ recalling (177). In the last expansion

$$f_{1,a} = \frac{2}{3} \left(\frac{q_1 + 2a}{q(a)} - \frac{1}{2} \left(\frac{1}{a - b} + \frac{1}{a - c} \right) \right). \tag{383}$$

Expanding $B_j(z)$ and using the identity (136), we obtain for the analytic in U_a function

$$\beta(z)^{-1}[A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)] - 2 = \mathcal{O}(z - a), \quad z \rightarrow a. \tag{384}$$

Taking the square of this expression, we may write

$$\begin{aligned} &\beta(z)^{-2}[A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2 \\ &= -4 + 4\beta(z)^{-1}[A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)] + \mathcal{O}((z - a)^2), \quad z \rightarrow a. \end{aligned} \tag{385}$$

We now evaluate the average

$$\int_0^1 [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)] d\omega = \int_0^1 [q(\widehat{\omega})_a + q(\widehat{-\omega})_a] d\omega, \quad \omega = s^{3/2}\Omega, \tag{386}$$

where, by writing out the terms and using periodicity properties of θ -functions,

$$\begin{aligned} \widehat{q(\omega)}_a &= \frac{\theta_3^2}{2\theta_3^2(\omega)} \frac{\theta_3(-\omega + d)}{\theta_3(d)} \left(\frac{\theta_3(u(z) + \omega + d)}{\theta_3(u(z) + d)} (\beta(z) + \beta^{-1}(z)) \right. \\ &\quad \left. + \frac{\theta_3(u(z) - \omega - d)}{\theta_3(u(z) - d)} (\beta(z) - \beta^{-1}(z)) \right). \end{aligned} \tag{387}$$

Since $\widehat{q(-\omega)}_a = q(\widehat{1 - \omega})_a$, we have that $\int_0^1 [q(\widehat{\omega})_a + q(\widehat{-\omega})_a] d\omega = 2 \int_0^1 q(\widehat{\omega})_a d\omega$. We notice that the expression for $q(\widehat{\omega})_a$ has the form of $q(\widehat{\omega})$, given in Section 9.2 of [19]. In particular, we may proceed in the same way (ans similarly to the derivation of (340) above). Expanding $u(z)$ and $\beta^{-2}(z)$ as $z \rightarrow a$ by (104), (108), and now using the identities (121) and (122) of Lemma 14, we obtain

$$\begin{aligned} &\beta(z)^{-1} \int_0^1 [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]d\omega = \\ &= 2 + \left(\frac{\pi^2}{3} u_{0,a}^2 + u_{1,a} - 2\beta_{1,a} \right) (z - a) + \mathcal{O}((z - a)^2), \quad z \rightarrow a. \end{aligned} \tag{388}$$

Substituting (385) into (381) and calculating the residue by (388) and (382), we obtain

$$\begin{aligned} &-\frac{\beta_{0,a}^2 f'_a(a)^{1/2}}{64} \int_{\partial U_a} \frac{dz}{2\pi i} \frac{3}{(z - a)^2 f_a(z)^{1/2}} \int_0^1 d\omega [A_1(z)\theta_{22}(0) + A_2(z)\theta_{11}(0)]^2 \\ &= \frac{1}{16} (\pi^2 u_{0,a}^2 + 3u_{1,a} - 3f_{1,a}/2) = \frac{1}{16} \left(\frac{\pi^2}{J_0^2(a - b)(a - c)} - \frac{q_1 + 2a}{q(a)} \right). \end{aligned} \tag{389}$$

For the last equation here we used (105) and (383). Next, we differentiate $q(a)$ w.r.t. a and use (148) which gives the equation

$$\frac{q_1 + 2a}{2q(a)} = \frac{d}{da} \log|q(a)| + \frac{d}{da} \log|J_0|. \tag{390}$$

Substituting this into (389) and that, in turn, into (381) we conclude that

$$\begin{aligned} &\int_0^1 T_{3,a}(\omega)d\omega - \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial\tau} \log \theta_3(\omega)d\omega = \\ &-\frac{1}{2} \frac{d}{da} \log|J_0| - \frac{1}{8} \frac{d}{da} \log|q(a)| - \frac{1}{16} \frac{d}{da} \log((a - b)(a - c)). \end{aligned} \tag{391}$$

Substituting (372) and (391) into the average of (368), we finally obtain

$$\begin{aligned} &\int_0^1 \lim_{z \rightarrow a} \tilde{\tau}_3(z)_{21}d\omega - \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial\tau} \log \theta_3(\omega)d\omega \\ &= \int_0^1 T_{2,a}(\omega)d\omega + \int_0^1 T_{3,a}(\omega)d\omega - \frac{d\tau}{da} \int_0^1 \frac{\partial}{\partial\tau} \log \theta_3(\omega)d\omega \\ &= -\frac{1}{2} \frac{d}{da} \log|J_0| - \frac{1}{8} \frac{d}{da} \log|q(a)q(b)q(c)|, \end{aligned} \tag{392}$$

which proves Lemma 30. \square

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