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Physics Area - Ph.D. course in Astroparticle Physics

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Symmetries, charges and thermodynamics at the corners of spacetime

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[1] Stefano, if you ever need a recommendation letter, just hit me up.

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[2] I will never meet anyone as ingenious.
[3] I will never meet anyone as selfless.
[4] I vowed to remember which Francesco is which without writing their last name.
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[6] Stefano, se mai avrai bisogno di una lettera di raccomandazione, contattami pure.

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[7] Non incontrerò mai nessuno così geniale.
[8] Non incontrerò mai nessuno così altruista.
[9] Ho fatto voto di ricordarmi quale Francesco fosse quale senza scrivere il cognome.
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Abstract

In the presence of boundaries, the physics of systems with gauge symmetries becomes significantly richer: new degrees of freedom emerge, and would-be redundancies are promoted to true symmetries. In recent years, the study of boundaries has rapidly grown, especially in the context of gravity, where it is expected to shed light on foundational problems such as the definition of observables and the origin of gravitational entropy. This thesis covers some aspects of this wide area.

After a brief introduction to the necessary technical tools in classical mechanics, we present the covariant phase space formalism, which enables us to apply these tools to study the symplectic structure of gravity in a covariant way. In this setting, we can formulate a refined definition of symmetry that allows us to distinguish between physical symmetries and gauge redundancies, and identify the associated new degrees of freedom.

Next, we discuss the relevance of boundaries in the formulation of a variational principle for field theories and how it is related to ambiguity resolution. At timelike boundaries, imposing suitable conditions provides a way to unambiguously define charges associated with symmetries and to ensure their conservation, in accordance with the spirit of Noether's work. We investigate the compatibility of this construction with renormalization, showing that the running of couplings, induced by quantum effects, can jeopardize the variational principle.

Afterward, we apply covariant phase space techniques to the thermodynamics of black holes in Lanczos–Lovelock theories. These are the simplest extensions of General Relativity, as they introduce new interactions but no additional degrees of freedom. Our results demonstrate that the covariant phase space formalism is suited to deriving both the first law of black hole thermodynamics and the Smarr formula, of which we present a natural generalization. We also find a background-independent prescription for the thermodynamic potentials associated with the theory's couplings.

Finally, we turn to the problem of identifying the degrees of freedom that exhibit the newly uncovered symmetries. We present the universal form of the symmetry group and study its representation theory, paying attention to the complications that may appear in a quantum context. As a first step of a larger program, we construct the unitary irreducible representations of the quantum corner symmetry group in two-dimensional gravity using the orbit method. We find that a dressed version of the generators of boosts (in the plane normal to the corner) plays a key role in this analysis.

This research adheres to the paradigm that much can be learned about gravity by exploiting its classical symmetry structure. In the spirit of Wigner's program, identifying the correct symmetry group — which is possible only after a careful construction of the charges — is a crucial step in understanding gravity at the quantum level.

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Forewords

This thesis collects the result of my four years as a PhD student in SISSA, during which I started practicing independent research, going *through the trials and trying* of such a life. At first, it was not evident that the projects I have been working on could be related into a consistent narrative. This thesis is my effort to weave together the following papers

- Giulio Neri, Stefano Liberati. *On the resilience of the gravitational variational principle under renormalization.* JHEP 10 (2023) 054., [arXiv:2306.17505]
- Giulio Neri, Stefano Liberati. *Covariant phase space analysis of Lanczos–Lovelock gravity with boundaries.* JHEP 06 (2024) 136, [arXiv:2404.16981]
- Giulio Neri, Ludovic Varrin. *Orbit method for Quantum Corner Symmetries.* [arXiv:2507.10683]

Other projects that I have worked on in these years did not make it here, either for their incomplete status or because of their topic

- Florian Girelli, Simon Langenscheidt, Giulio Neri, Christopher Pollack, Céline Zwikel. *Logarithmic supertranslations at Spatial Infinity.*
- Walter Arata, Stefano Liberati, Giulio Neri. *A Covariant Phase Space approach to Einstein-Æther gravity.*
- Julio Arrechea, Giulio Neri, Stefano Liberati. *Inner horizon instability via the trace anomaly effective action.* Phys.Rev.D 111 (2025) 8, 084036

Notation and conventions

Our convention is to work with the mostly plus signature $(-,+,+,\dots)$. We adopt the abstract index notation to distinguish spacetime tensors: we use Greek letters for bulk tensors, lowercase Latin letters for boundary tensors, and uppercase Latin letters for corner[11] tensors. Lowercase Latin letters are also used as generic indices for elements of a set; in case of a conflict with tensor indices, set indices are enclosed in round brackets. We make a large use of Einstein's summation convention.

Bold uppercase letters denote elements in some Lie algebra, while bold lowercase letters denote vectors in the dual space to the algebra. Especially in Chapter 7, coordinates and vector components — which can be identified in linear spaces — are written with the same letter as the element they refer to, but without boldface.

We are going to assume that all the functions we work with are sufficiently smooth to perform all the operations, and label them as infinitely differentiable even if not necessary.

Kinematical structure

The spacetime $\mathcal{M}$ we will consider is a globally hyperbolic pseudo-Riemannian manifold with boundary. Hence, topologically, $\mathcal{M}$ is a cylinder $\mathbb{R}\times\Sigma$. We will usually restrict to a certain *time* interval $[t_{\text{in}}, t_{\text{fin}}]\subset\mathbb{R}$. The boundary of this portion of $\mathcal{M}$, denoted by $\mathcal{M}$ as well with a slight abuse of notation, comprises three parts: a *lateral* boundary $\mathcal{I}$, with the topology of $[t_{\text{in}}, t_{\text{fin}}]\times\partial\Sigma$, and two "lids" Σ_{in} and Σ_{fin}.

$$\partial\mathcal{M} = \Sigma_{\text{fin}}\cup\mathcal{I}\cup\Sigma_{\text{in}}. \tag{1}$$

We will also refer to Σ_{in} and Σ_{fin} as the initial and final slice, respectively, or as the past and future boundary of $\mathcal{M}$. We follow the orientation conventions of [1], which we summarize here. Given an orientation of $\mathcal{M}$, we induce the orientation on the boundary $\partial\mathcal{M}$ in such a way that Stokes' theorem reads (for each $(n-1)$-form α)

$$\int_{\mathcal{M}} \mathrm{d}\alpha = \int_{\partial\mathcal{M}} \alpha. \tag{2}$$

[11] By corner, we mean a codimension-2 surface.

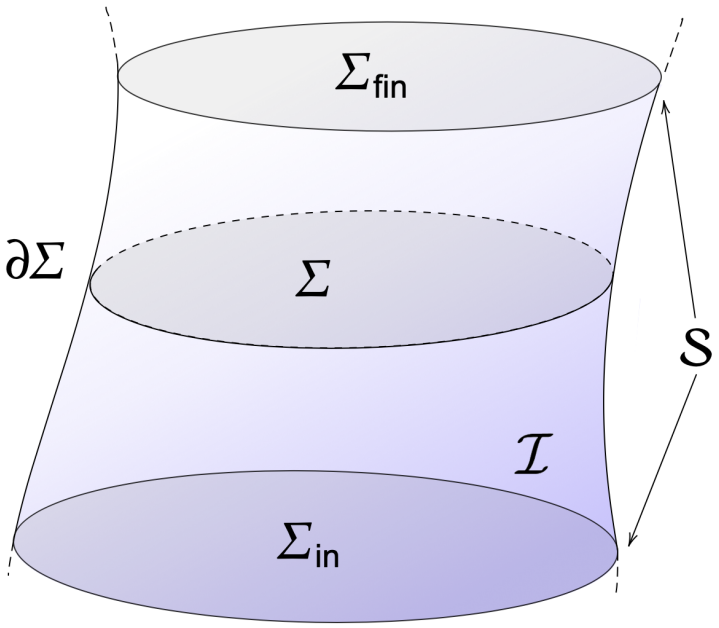


Figure 1: Picture of the spacetime region $\mathcal{M}$, its boundaries $\mathcal{I}$, and corners $\mathcal{S}$. Since the orientation on each $\partial\Sigma$ is induced by its bulk part Σ and not by its past in $\mathcal{I}$, the orientation of $\partial\Sigma_{\text{in}} \cup \partial\Sigma_{\text{fin}}$ is opposite to $\partial\mathcal{I}$.

We also assign to each codimension-1 spacelike surface the orientation induced by viewing it as the future boundary of its past. Therefore, given the decomposition 1, Σ_{in} enters the previous integral with a minus sign.

We emphasize that the region $\mathcal{M}$ has **corners**, given by $\mathcal{S} \equiv \partial\Sigma_{\text{in}} \cup \partial\Sigma_{\text{fin}}$. As a set, $\partial\Sigma_{\text{in}} \cup \partial\Sigma_{\text{fin}}$ coincides with the boundary of $\mathcal{I}$. However, under our conventions, the two have opposite orientations, as illustrated in Figure 1.

Acronyms

QFT quantum field theory

ADM Arnowitt–Deser–Misner

CPS covariant phase space

GR General Relativity

GHY Gibbons–Hawking–York

GB Gauss–Bonnet

HDG higher-derivative gravity

AdS Anti-de Sitter

LL Lanczos–Lovelock

CS corner symmetry

UCS universal corner symmetry

ECS extended corner symmetry

ACS asymptotic corner symmetry

QCS quantum corner symmetry

KKS Kirrilov–Kostant–Souriau

At the round earth's imagin'd corners, blow
Your trumpets, angels, and arise, ...
— John Donne, *Holy Sonnet*

Chapter 1

Introduction

What are the symmetries of gravity? This question echoes in my mind as the incipit of famous books: "In the beginning was the Word", "Happy families are all alike", "Someone must have slandered Josef K."... Then I pause and repeat a sentence I have heard many times: gravity is a gauge theory for diffeomorphisms.

At the classical level, gravity can be thought of as a gauge theory where the gauge group is the diffeomorphisms of spacetime. At the quantum level, what we can say is that physical states of the gravitational field should not transform under diffeomorphisms. However, we do not know much more about quantum gravity with the same certainty that we have in other areas, despite the amount of effort that has been devoted by the community to the formulation of a consistent theory.

The lack of decisive experimental results, the absence of evident, unanimously recognized contradictions within our theoretical understanding, and the plethora of competing ideas, make the search for a quantum gravity theory one of the most challenging of modern day theoretical physics. Broadly speaking, we can distinguish between *top-down* and *bottom-up* approaches. Top-down approaches start from basic, trustworthy assumptions that are taken as axioms and work their way down to predictions. In general, these axioms are properties imposed on the full quantum theory, where we might have expectations, but no decisive result so far — let alone experimental inputs. Also, the procedure of selecting a particular class of states, for which it makes sense to talk about a (semi-)classical behavior, is extremely lengthy and complicated, often involving approximations and simplifying assumptions.

In bottom-up approaches, one starts from classical data and quantize them. Unfortunately, this program did not lead us very far: perturbative covariant quantization met the issue of non-renormalizability, canonical quantization led to the Wheeler–DeWitt equation, which is notoriously hard to deal with except in some very simple cases. So how do we move on?

Leveraging symmetries has been a powerful tool to make progress in physics for a long time. Even if symmetries might be broken or modified at the quantum level, emerge or disappear depending on the energy scale, some of them are strong enough to carry over to the quantum regime. The relevance of symmetries comes from the fact that they provide non-perturbative information about the quantum theory, even if we deduce them from low-energy observations. For example, if we have a classical description of a physical system in terms of its phase space, we can quantize a certain subset of phase space functions to obtain a collection of quantum operators that are enough to generate the entire algebra of operators (modulo ordering ambiguities) under the simplifying assumption that any two states of a physical system are connected by a symmetry transformation. This is the essence of Isham's quantization [2, 3].

What are the symmetries of gravity, then? Being gauge, diffeomorphisms cannot connect two states because their action must be trivial. However, in the presence of boundaries, something interesting happens. Starting from condensed matter [4–7], it has been increasingly appreciated that gauge theories reveal new structures and phenomena when confined in a region with boundaries. Specifically, what were expected to be gauge transformations, i.e. mere redundancies of the description, become physical symmetries.

According to Noether's legacy, these symmetries are associated with charges [1] which, as we will see in great detail, live on the *corners* of spacetime. The name corner comes from the way these surfaces are represented in a conformal Penrose diagram, but they are just surfaces of codimension 2. Understanding the nature of these charges is the first step of a new approach to quantum gravity, which uses symmetries as a guiding principle and the full power of Noether's theorems as a tool.

1.1 Theoretical framework

In this thesis, we mostly analyze the symplectic structure of classical gravity, exploring its symmetries and how they are represented. As we mentioned, these data are important for quantization. There are several options to do this analysis [8–11], some of which require minimal setup, but the author of this thesis, as a human being, is biased towards the covariant phase space framework. Therefore, the first chapters will be devoted to the introduction to this formalism, and the last chapters to its applications in the study of corner symmetries.

At all moments, we keep in mind that classical gravity is just a particular limit of quantum gravity, valid in a likely narrow sector. On the other hand, we will not delve into quantum considerations except in two instances: to motivate the relevance of boundaries and to show, as a simple example, what the study of corner symmetries entails for $2d$ gravity.

[1] We will be rather liberal in what we call a *charge*. Every function associated with, or generating, a transformation will be called a charge, regardless of whether it is actually conserved in the dynamics.

Since symmetries will play a key role throughout this thesis, we take a moment to state clearly what we mean by it, and to introduce some terminology.

A symmetry is a transformation of the state of a physical system that leaves its essential properties unchanged.

When dealing with field theories, it is customary to distinguish

Global symmetries If the symmetry acts in the same way at any point in spacetime. In this class there are also discrete symmetries, which we will not deal with;

Local symmetries If the symmetry acts differently at different spacetime points.

Another distinction is between

Physical symmetries These transformations change the physical state of the system, in the sense that there are (inessential) quantities which are different before and after the transformation;

Gauge symmetries These transformations do not change the physical state of the system, but only our description of it. Since such transformations do not fit our previous definition, saying "gauge symmetry" is, for us, an oxymoron. We will rather call these transformations *redundancies*, *gauge* or *pure gauge*.

The common lore is to identify global symmetries with physical symmetries and local symmetries with gauge redundancies. While this identification is usually correct, it is too naïve and it fails in the presence of boundaries, as we previously mentioned. A refined criterion is based on the charge associated with the transformation.

$$\text{The charge is non-zero} \implies \text{Physical symmetry}$$

$$\text{The charge is zero} \implies \text{Gauge redundancy}$$

The problem of computing these charges can be elegantly formulated in the covariant phase space formalism, as we will see, but it is also an active area of research (see [12–21] for instance) due to ambiguities intrinsic to the formalism and technical difficulties intrinsic to certain systems. One issue with this classification is that we often need to use some transformation to even compute the charges, which implies a form of circularity because, strictly speaking, the only transformations that we are allowed without altering the system state are gauge transformations. Much of the current research in this field aims to dispense with this circularity by computing the charges in more and more generality.

1.2 Motivation

The motivations of the works presented in this thesis are multiple, but we can conveniently organize them in two macro-areas, with names that are deliberately vague.

Gravity & Thermodynamics The connection between gravity and thermodynamics started to be unveiled in the early 1970s [22, 23]. At the classical level, this appears paradoxical: why should a black hole have a non-zero temperature if it is a perfect absorber, why should we associate a finite entropy to it if every microscopic configuration (up to few constraints given by the total mass, electric charge and angular momentum) is equally possible? It was only through the inclusion of quantum effects that these seemingly counterintuitive results started to make sense [24, 25].

In every other area of physics, thermodynamics is understood not as a fundamental description, but as an emergent one coming from the application of statistical tools to the underlying microscopic system. Put shortly, thermodynamics comes from statistical physics. The fact that black holes behave as thermodynamic objects raises the question of what are the microscopic constituents of the spacetime [26–28]. The study of gravitational charges can provide insights about the nature and behavior of these constituents [29, 30]. Moreover, since the ultimate nature of every other system seems to be quantum, it is reasonable to think that the same is true for the gravitational fields. Hence, black hole thermodynamics, if not a convincing proof, is a strong clue that the gravitational field is quantum. In this light, a better understanding of this topic goes in the direction of a better understanding of quantum gravity.

Gravity & Entanglement The idea that entanglement is associated with the emergence of spacetime can be traced back to Penrose's spin networks [31] and now permeates many *corners* of quantum gravity [32–40]. In approaches where spacetime is emergent, the fundamental entities are (pre-)geometric *data* that do not have any gravitational interpretation *per se*. This ontology is shared by loop quantum gravity [41], spin foam models [42], and group field theory [43–45].

These fundamental degrees of freedom are then glued together, as building blocks, into many-particle states: the entanglement pattern dictates the topology of this combined state. Then, for a certain class of these states, and only in a suitable semiclassical regime, the expectation values of geometric operators on these states approximate a smooth spatial geometry and, as they evolve, a smooth geometry for the entire spacetime.

Gravity is not a fundamental field in this picture, but it emerges as the collective behavior of a large number of the microscopic degrees of freedom. In this sense, it is entanglement that creates spacetime *connectivity*. As we will see in the last chapter, *breaking* these connections gives us clues on what is the nature of these fundamental entities.

Part I

The basic tools

Chapter 2

Classical mechanics

Since the early days of classical mechanics in the 17th century, the theoretical description of physical systems has evolved in two distinct yet interconnected formulations: the Lagrangian and the Hamiltonian formalisms. Beyond deriving results within each framework, considerable effort has been devoted to comparing the two, identifying the conditions under which they are equivalent and the properties of systems for which they are not.

The two formalisms have complementary strength. The **Lagrangian description** manifestly preserves the symmetries of the system, it is easier to apply, and it is the natural starting point for the path integral quantization. On the other hand, the **Hamiltonian description** is based on quantities with more transparent physical meaning (such as energy), it leads to first-order equations of motion, and it is the starting point for canonical quantization. Finally, the Hamiltonian description has a nice geometric structure, that will be the main topic of exploration in this thesis.

2.1 Lagrangian mechanics

The easiest path to introduce our discussion starts from the description of a simple mechanical system, that we can think of as a collection of point particles subjected to some (conservative) forces. Let $\{q^i\}_{i=1,\dots,N}$ be some generalized positions that describe the system at hand. These are the dynamical variables that describe the allowed configuration of the system. As each of these variables might change in time in principle, we introduce the corresponding velocities $\{\dot{q}^i\}$. We define the **configuration space** $\mathcal{Q}$ as the set of all dynamical variables. Its tangent bundle $T\mathcal{Q}$ describes, at any given moment, the set of these variables and their velocities.

In the Lagrangian picture, one assumes there is a function $L : T\mathcal{Q} \to \mathbb{R}$, called *Lagrangian*, that governs the dynamics, in the sense that the equations of motion of mechanics

are given by the **Euler–Lagrange equations**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{2.1}$$

This is a system of N second-order coupled differential equations for the functions $t \to q^i(t)$, i.e. the *trajectories*. The system admits solutions for all q's provided that the Hessian matrix is non-singular

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0. \tag{2.2}$$

A Lagrangian with this property is said to be *regular*. Violating this assumption opened the door to the study of constrained systems, among which *gauge theories* feature as perhaps the most interesting example for modern theoretical physics.

Remark The Lagrangian of an unconstrained classical mechanical system can be written in the form $L = T - U$, with $T = \frac{1}{2}\delta^{ij}\dot{q}^i\dot{q}^j$ the kinetic energy of the system and $U = U(q^i)$ the potential energy. For these systems, we recover the second law of dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \implies \ddot{q}^i = -\frac{\partial U}{\partial q^i} \equiv F^i. \tag{2.3}$$

2.1.1 The action principle

A simple way to understand (and derive) the Euler–Lagrange equations is to see them as variational calculus equations coming from the extremization of some functional. Let $\underline{q} : [t_{\mathrm{in}}, t_{\mathrm{fin}}] \subset \mathbb{R} \to \mathcal{Q}$ be a curve in configuration space describing a possible trajectory of the system. We introduce the action functional on the set of all possible trajectories $\Gamma([t_{\mathrm{in}}, t_{\mathrm{fin}}], \mathcal{Q})$ as

$$S[\underline{q}] := \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t\, L(\underline{q}(t), \dot{\underline{q}}(t)). \tag{2.4}$$

Let us prove that the solutions of the Euler–Lagrange equations correspond to local extremal points of the action functional, subject to the conditions that $\delta\underline{q}^i(t_{\mathrm{in}}) = 0$ and $\delta\underline{q}^i(t_{\mathrm{fin}}) = 0$.

Proof. Consider a small deformation $\delta\underline{q}$ of the trajectory $\underline{q}$ that solves the Euler–Lagrange equations (2.1). The effect of this deformation changes the action output by

$$\begin{aligned}
\delta S[\underline{q}] = S[\underline{q} + \delta\underline{q}] - S[\underline{q}] &= \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t \left(\frac{\partial L}{\partial q^i}\delta\underline{q}^i(t) + \frac{\partial L}{\partial \dot{q}^i}\delta\dot{\underline{q}}^i(t)\right) = \\
&= \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t \frac{\partial L}{\partial q^i}\delta\underline{q}^i(t) + \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\delta\underline{q}^i(t)\right) - \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\delta\underline{q}^i(t) = \\
&= \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \mathrm{d}t \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right)\delta\underline{q}^i(t) + \frac{\partial L}{\partial \dot{q}^i}\delta\underline{q}^i(t)\Big|_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} = 0,
\end{aligned}$$

where, in the last line, we recognized the Euler–Lagrange equations, and we used that $\underline{q}^i$ are fixed at the extrema of integration. This proves that such a trajectory (locally) extremizes the action. □

In what follows, we will not use boldface for the argument of the action and will just conflate the curve describing the trajectory with its image.

2.1.2 Noether's theorem

This is a good point to introduce the concept **Noether's theorem** for the first time. The core idea of the theorem is that continuous symmetries imply conservation laws. So, whenever there is a transformation that leaves the description of the system [1] invariant, then we expect that there is a quantity that does not change in time.

An elementary version of the theorem follows by this simple observation. If the Lagrangian does not depend on some variable $q^{\underline{i}}$ — so the action is invariant under $q^{\underline{i}} \to q^{\underline{i}} + c$ — then the Euler–Lagrange equations directly imply the conservation of the associated Lagrangian momentum $p_{\underline{i}} \equiv \frac{\partial L}{\partial \dot{q}^{\underline{i}}}$

$$\frac{\mathrm{d}p_{\underline{i}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^{\underline{i}}} = \frac{\partial L}{\partial q^{\underline{i}}} = 0. \tag{2.5}$$

Let us now refine the theorem by considering a possibly more complicated transformation. We let $\delta_F q^i$ be a generic, possibly time-dependent, variation of the configuration variables

$$q^i \to q^i + \delta_F q^i(t). \tag{2.6}$$

If this variation preserves the Lagrangian up to a total time derivative $\mathrm{d}F/\mathrm{d}t$, i.e.

$$\delta_F L \equiv \frac{\partial L}{\partial q^i}\delta_F q^i + \frac{\partial L}{\partial \dot{q}^i}\delta_F \dot{q}^i = \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{2.7}$$

where $\delta_F \dot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}\delta_F q^i$, then Noether proved that the following function

$$J_F := \frac{\partial L}{\partial \dot{q}^i}\delta_F q^i - F \tag{2.8}$$

is constant on the solutions to the Euler–Lagrange equations (2.1).

[1] Strictly speaking, the system is described by the set of solutions of the equations of motion, so that a symmetry should be a transformation that preserves this set. In practice, it is much more convenient to study the symmetries of the action.

2.2 Hamiltonian mechanics

The framework in which the Hamiltonian description is formulated is based on the notion of **phase space** $\mathscr{P}$. The *state* of a system is represented by a point of $\mathscr{P}$, while the evolution in time can be described as a vector field $v_H \in \mathfrak{X}(\mathscr{P})$ ($\mathfrak{X}(\cdot)$ denotes the set of vector fields):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = v_H(f). \tag{2.9}$$

where $f \in C^\infty(\mathscr{P})$ is any function.

What makes a phase space special is that it is a *symplectic manifold*, i.e. a manifold equipped with a two-form ω that is both closed and non-degenerate. Once a special function $H : \mathscr{P} \to \mathbb{R}$, called *Hamiltonian*, is chosen within $C^\infty(\mathscr{P})$, the *symplectic form* ω dictates the evolution in time according to

$$\mathrm{d}H = -\iota_{v_H}\omega, \tag{2.10}$$

where ι denotes the interior product/contraction of a vector field with a form. Equation (2.10) is a set of $2N$ first-order differential equations (where $2N$ is the dimension of phase space) known as **Hamilton equations**. They are usually written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}, \tag{2.11}$$

where the **Poisson brackets** are a bilinear map $\{\cdot,\cdot\} : C^\infty(\mathscr{P}) \times C^\infty(\mathscr{P}) \to C^\infty(\mathscr{P})$ defined by

$$\{f, g\} := \omega^{-1}(\mathrm{d}f, \mathrm{d}g). \tag{2.12}$$

An important result in the theory of symplectic manifolds is that all the symplectic manifolds locally look the same at fixed dimensionality. This is the essence of Darboux's theorem, according to which, around any point, there is a local patch of coordinates (q^i, p_i) $(i = 1, \ldots, N)$ where $\omega = \mathrm{d}\theta$ with

$$\theta = p_i \mathrm{d}q^i, \tag{2.13}$$

so that

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i. \tag{2.14}$$

In these coordinates, the Poisson brackets read

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{2.15}$$

and the Hamilton equations can be expressed in terms of coordinate functions in their arguably most famous version

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p^i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}. \tag{2.16}$$

2.2.1 Symplectomorphisms

Symmetries of a symplectic manifolds are simply described by *symplectomorphisms*, that is vector fields whose flow preserve the symplectic form [2]

$$\mathcal{L}_v \omega = 0, \tag{2.17}$$

where $\mathcal{L}$ denotes the Lie derivative along a vector field. It is straightforward to verify that these vector fields form a subalgebra of $\mathfrak{X}(\mathscr{P})$.

Given that the symplectic form is closed, this equation implies, via Cartan's homotopy formula, that $\iota_v \omega$ is a closed one-form

$$\mathcal{L}_v \omega = \mathrm{d}\iota_v \omega + \iota_v \mathrm{d}\omega = \mathrm{d}\iota_v \omega = 0. \tag{2.18}$$

It is then natural to ask whether this form is exact as well. If this is the case, then there exists a function $H_v \in C^\infty(\mathscr{P})$ such that

$$\iota_v \omega = -\mathrm{d}H_v. \tag{2.19}$$

By analogy with Hamilton's equations (2.10), we call H_v the **Hamiltonian function** of v: if time was the parameter along the integral curves of v, the Hamiltonian of the system would be H_v.[3] A symplectomorphism that admits a Hamiltonian function is called *Hamiltonian*, but we rather call it *integrable*.

Not all symplectomorphisms are integrable, but integrable symplectomorphisms form a subalgebra (actually, an ideal, but we leave this proof to the reader).

Proof. Given two integrable vector fields $v, w \in \mathfrak{X}(\mathscr{P})$, with Hamiltonian functions H_v, H_w, we can consider the contraction of the symplectic form with their commutator

$$\iota_{[v,w]_{\mathrm{Lie}}} \omega = \mathcal{L}_v \iota_w \omega - \iota_w \mathcal{L}_v \omega = -\mathcal{L}_v(\mathrm{d}H_w) = -\,\mathrm{d}(\mathcal{L}_v H_w)\,. \tag{2.20}$$

This shows that the $\mathcal{L}_v H_w$ is a Hamiltonian function for the commutator $[v, w]_{\mathrm{Lie}}$, which is thus an integrable symplectomorphism. □

In the examples that we will see, it will turn out very hard to compute Poisson brackets. Luckily, if we restrict to Hamiltonian functions, then we can directly use

$$\{H_v, H_w\} = \omega(v, w) = \mathcal{L}_v H_w. \tag{2.21}$$

[2] A vector field generates a one-parameter subfamily of symplectomorphisms by exponentiation. So the vector field is just an "infinitesimal version" of a symplectomorphism. In this thesis, we will often mix the two notions.

[3] Somehow redundantly, we have $H_{v_H} = H$, which means that the physical Hamiltonian is the Hamiltonian function of the time evolution vector field.

It turns out that the Poisson algebra of Hamiltonian functions is a *projective representation* of the integrable vector fields subalgebra

$$\{H_v, H_w\} = H_{[v,w]_{\text{Lie}}} + K_{v,w} \tag{2.22}$$

where $K_{v,w}$ is a closed 2-cocyle

$$K_{v,[w,z]_{\text{Lie}}} + K_{w,[z,v]_{\text{Lie}}} + K_{z,[v,w]_{\text{Lie}}} = 0. \tag{2.23}$$

Proof. Taking an exterior derivative of (2.21), we get

$$\mathrm{d}\{H_v, H_w\} = \mathrm{d}(\omega(v,w)) = \mathcal{L}_w \iota_v \omega - \iota_w \mathcal{L}_v \omega = -\iota_{[v,w]_{\text{Lie}}}\omega = \mathrm{d}H_{[v,w]_{\text{Lie}}}, \tag{2.24}$$

from which it follows that the difference $\{H_v, H_w\} - H_{[v,w]_{\text{Lie}}} = K_{v,w}$ is closed. To show the cocycle condition, take the sum of these three

$$\{H_v, \{H_w, H_z\}\} = \left\{H_v, H_{[w,z]_{\text{Lie}}}\right\} = H_{[v,[w,z]_{\text{Lie}}]_{\text{Lie}}} + K_{v,[w,z]_{\text{Lie}}}, \tag{2.25a}$$

$$\{H_w, \{H_z, H_v\}\} = \left\{H_w, H_{[z,v]_{\text{Lie}}}\right\} = H_{[w,[z,v]_{\text{Lie}}]_{\text{Lie}}} + K_{w,[z,v]_{\text{Lie}}}, \tag{2.25b}$$

$$\{H_z, \{H_v, H_w\}\} = \left\{H_z, H_{[v,w]_{\text{Lie}}}\right\} = H_{[z,[v,w]_{\text{Lie}}]_{\text{Lie}}} + K_{z,[v,w]_{\text{Lie}}}. \tag{2.25c}$$

The sum of the l.h.s. vanishes by the Jacobi identity for Poisson brackets, while the sum of the first term in the r.h.s. vanishes by linearity and the Jacobi identity for Lie brackets. □

The term $K_{v,w}$ is said to be *central* because it has vanishing Poisson brackets with every other function.

2.2.2 Hamiltonian vector fields

We have seen that the flow of a vector field may not always be associated with a function. For this to be true, the vector field must be a symplectomorphism, but this is generally only a necessary condition. However, the opposite question: whether, given a function $f \in C^\infty(\mathscr{P})$, we can find a vector field v_f such that

$$\mathrm{d}f = -\iota_{v_f}\omega, \tag{2.26}$$

is always answered in the positive, because we can simply consider the so-called *Hamiltonian vector field* of f

$$v_f = -\{f, \cdot\}. \tag{2.27}$$

In local coordinates, this reads

$$v_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{2.28}$$

from which we see that f implements a *canonical transformation*.

Remark Hamiltonian vector fields form an involutive distribution because

$$[v_f, v_g] = -v_{\{f,g\}}, \tag{2.29}$$

as it easily follows from the Jacobi identity. Frobenius' theorem guarantees that such a distribution is also integrable, meaning that there is a *foliation* of the manifold $\mathscr{P}$ such that, on each *leaf*, the vector fields of the distribution are sections of the tangent bundle of that leaf. Such a foliation is called *symplectic* because, even if the ambient space $\mathscr{P}$ is only Poisson, the leaves are nevertheless symplectic, with the non-degenerate form that maps $(v_f, v_g) \to \{f, g\}$.

The Hamiltonian description is also referred to as the **canonical formalism**, because it is based on the use of canonical transformations to identify special coordinates for which the Poisson brackets take a very simple, *canonical*, form. We will see in the next chapter that it is possible to have a description in terms of phase spaces and Hamiltonians even without introducing these canonical Poisson brackets.

2.2.3 Momentum Maps

It is usually the case that the phase space $\mathscr{P}$ is acted upon by a Lie group G

$$G \times \mathscr{P} \to \mathscr{P}, \quad (g, m) \mapsto g \triangleright m, \tag{2.30}$$

where $\triangleright$ denotes a generic action from the left, which we assume to be smooth.

Given a point $m \in \mathscr{P}$, we can define its orbit as the set of points reached via the action of a group element

$$\mathcal{O}_m^G := \{g \triangleright m,\, g \in G\}, \tag{2.31}$$

and the stabilizer subgroup as the set of group elements that leave that point fixed

$$G_m := \{g \in G |\, g \triangleright m = m\}. \tag{2.32}$$

It is easy to see that coadjoint orbits do not intersect, hence they foliate $\mathscr{P}$.

Given the structure of orbits and stabilizers, we characterize the action as

- Faithful/effective: if $g \triangleright m = m \ \forall m \in \mathscr{P}$ implies $g = \mathrm{id}$ (the identity element of G);
- Free action: if the property $\exists\, m \in \mathscr{P}$ such that $g \triangleright m = m$ implies $g = \mathrm{id}$;
- Transitive action: if every two points are connected by a group element, which means there is a single orbit $O_m^G = \mathscr{P}\ \forall m \in \mathscr{P}$.

Given a point $m \in \mathscr{P}$, we can consider the map $G \to \mathscr{P}$, $g \to g \triangleright m$ that just acts with g on m. The differential of this map at the identity provides a map from the Lie algebra $\mathfrak{g}$

to the tangent space at $T_m\mathscr{P}$. Repeating this construction for all $m \in \mathscr{P}$ defines a vector field associated with each Lie algebra element $\mathbf{X} \in \mathfrak{g}$

$$(v_{\mathbf{X}})_m[f] := \frac{\mathrm{d}}{\mathrm{d}s} f\left(e^{s\mathbf{X}} \triangleright m\right)\Big|_{s=0} \qquad \forall f \in C^\infty(\mathscr{P}). \tag{2.33}$$

We call these vector fields *generators* of the Lie group action, because the integral curves of these vector fields span the orbits of G. Notice that these vector fields satisfy

$$[v_{\mathbf{X}}, v_{\mathbf{Y}}]_{\mathrm{Lie}} = -v_{[\mathbf{X},\mathbf{Y}]}, \tag{2.34}$$

where the brackets in the r.h.s. are the Lie brackets of $\mathfrak{g}$, so they form an involutive distribution.

The G-action is a symmetry if its generators are symplectomorphism. Moreover, it is said to be *Hamiltonian* if all its generators are integrable. If this is true, there exists a map $\tilde{\mathcal{J}} : \mathfrak{g} \to C^\infty(\mathscr{P})$, $\mathbf{X} \mapsto \mathcal{J}_{\mathbf{X}}$ such that

$$\mathrm{d}\mathcal{J}_{\mathbf{X}} = -\iota_{v_{\mathbf{X}}}\omega. \tag{2.35}$$

which means $\mathcal{J}_{\mathbf{X}} = H_{v_{\mathbf{X}}}$. We can also introduce the dual map $\mathcal{J} : \mathscr{P} \to \mathfrak{g}^*$, defined by

$$m \mapsto \mathcal{J}(m), \quad \langle \mathcal{J}(m), \mathbf{X}\rangle = \mathcal{J}_{\mathbf{X}}(m), \tag{2.36}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between an element of $\mathfrak{g}^*$ and an element of $\mathfrak{g}$. This map is called *momentum map*, and we will see shortly one of its applications. For now, we notice that combining (2.35) with (2.22) and (2.34), we deduce that

$$\{\mathcal{J}_{\mathbf{X}}, \mathcal{J}_{\mathbf{Y}}\} = \mathcal{J}_{[\mathbf{X},\mathbf{Y}]} + K_{\mathbf{X},\mathbf{Y}}, \tag{2.37}$$

where K is a closed 2-(group) cocycle. Hence, generically, $\mathcal{J}$ is a homomorphism of an extension of the Lie algebra $\mathfrak{g}$ into the Poisson algebra of phase space functions.

Equivariance A condition that is usually assumed is that the momentum maps are G-equivariant, namely

$$\mathcal{J}(g \triangleright m) = \mathrm{Ad}^*_g \mathcal{J}(m), \quad \forall m \in \mathscr{P}, \forall g \in G \tag{2.38}$$

This condition implies that the central term $K_{\mathbf{X},\mathbf{Y}}$ is absent from (2.37).

2.2.4 Legendre transformation

In classical mechanics, it is usually the case that the phase space is nothing but the cotangent bundle over the configuration space $\mathscr{P} = T^*\mathcal{Q}$.

On a cotangent bundle there exists a natural one-form θ, called *tautological one-form*, defined by the following property: for each $\alpha \in \Omega^1(\mathcal{Q})$

$$\alpha^*\theta = \alpha \tag{2.39}$$

where α on the l.h.s. is the pull-back under the α seen as a section of the cotangent bundle $\alpha : \mathcal{Q} \to T^*\mathcal{Q}$. So, the tautological one-form is the only one that "cancels" pull-backs. In a local trivialization (q_i, p^i), the tautological form coincides with the Darboux form (2.13). So, on a cotangent bundle, the symplectic form is always exact

$$\omega = \mathrm{d}\theta. \tag{2.40}$$

Given that the Lagrangian description lives on $T\mathcal{Q}$ and the Hamiltonian description lives on $T^*\mathcal{Q}$, one may ask whether there exists a natural map between the tangent and the cotangent bundles. Such a map always exists — it is the *fiber derivative* [46] — but it provides a one-to-one map only if the Lagrangian is regular. In local coordinates, we can write this map as a **Legendre transformation**

$$\mathbb{F}_L : T\mathcal{Q} \to T^*\mathcal{Q}, \quad (q^i, \dot{q}^i) \mapsto \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i}\right). \tag{2.41}$$

The implicit function theorem tells us that we can locally invert this map only if the Jacobian has non-vanishing determinant, which amounts to the regularity condition on the Lagrangian

$$0 \neq \frac{\partial p_i}{\partial \dot{q}^j} = \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}. \tag{2.42}$$

Once one has inverted all the functions $\dot{q}^i = \dot{q}^i(q, p)$, the Hamiltonian can be written as

$$H(q, p) = p_i \dot{q}^i(q, p) - L(q, \dot{q}(q, p)). \tag{2.43}$$

Under the assumed regularity, it is straightforward to show the equivalence between (2.1) and (2.16).

2.3 Constrained systems

The case in which the equivalence between Lagrangian and Hamiltonian description fails is among the most interesting physical systems that we know: gauge theories are a very famous example. For now, we restrict to mechanical systems that have a singular Lagrangian.

These are called **constrained systems**, and were originally treated by Dirac and Bergmann [47, 48]. Given that the Hessian of the Lagrangian of constrained system is singular, one cannot invert all the relations (2.42) in terms of velocities. The Dirac–Bergmann algorithm is based on the identification of primary constraints, arising from the

definition of momenta, and a sequence of constraints (called secondary, tertiary and so on) arising from the requirement of constraints being preserved in time. This separation is useful when we try to obtain the Hamiltonian picture from the Lagrangian one, but it is not intrinsic to the former. A more physically relevant classification of constraints is based on their Poisson commutation relations.

Suppose we find the set of all constraints $\{c_j \in C^\infty(\mathscr{P})\}_{j=1,\dots,J}$. Since there cannot be more, this set must be preserved by the dynamics

$$\frac{\mathrm{d}c_i}{\mathrm{d}t} = \{c_i, H\} = 0 \quad \text{if} \quad c_j = 0, \forall j. \tag{2.44}$$

The evolution of the system is confined to the intersection between the zero level set of all of the constrain functions. It is customary to introduce the *weak equality* $\approx$ as an equality that holds at least in this intersection.

A function $f \in C^\infty(\mathscr{P})$ is said to be *first class* if its Poisson brackets with the constraints vanishes weakly

$$\{f, c_i\} \approx 0. \tag{2.45}$$

Otherwise, the function is said to be *second class*. We use this conditions to separate the constraints $\{c_j\}$ between first class $\{\phi_a\}_{a=1,\dots,K}$, and second class $\{\chi_\alpha\}_{\alpha=1,\dots,2L}$ (with $K + 2L = J$).

2.3.1 Second class constraints

Let's consider first the matrix given by all Poisson brackets between two second class constraints

$$\{\chi_\alpha, \chi_\beta\} = \Upsilon_{\alpha\beta}. \tag{2.46}$$

If we assume $\det \Upsilon \approx 0$, there would be at least one eigenvector with weakly vanishing eigenvalue; since such a vector would be a combination of second class constraints that weakly commute with the rest of them (and with first class constraint, by their definition), it is first class. This contradicts the assumption of the χs being second-class, hence $\det \Upsilon \not\approx 0$. A non-singular, antisymmetric, finite-dimensional matrix must be even-dimensional, hence our choice to denote the number of second class constraints as $2L$. Second class constraints always come in pairs.

In the absence of first class constraints, the surface defined by the vanishing of all second class constraints $\mathscr{P}_\chi := \{\chi_\alpha(m) = 0\} \subset \mathscr{P}$ is a symplectic manifold on its own, with dimension $2(N - L)$. Therefore, the interpretation of each second class pair is that of (configurational) degrees of freedom which are superfluous and can thus be removed from the description. To implement this reduction, Dirac proposed a modification of the Poisson brackets that goes under the name of Dirac brackets [49, 50]

$$\{f, g\}_D := \{f, g\} - \{f, \chi_\alpha\}\Upsilon^{\alpha\beta}\{\chi_\beta, g\} \tag{2.47}$$

where $\Upsilon^{\alpha\beta}$ is the inverse of $\Upsilon_{\alpha\beta}$. One can show that Dirac brackets are the inverse of the symplectic form ω pulled back to the constraint submanifold $\mathscr{P}_\chi$ [51].

If there are both first and second class constraints, we can still remove the latter with the introduction of Dirac brackets or, equivalently, pulling back ω to $\mathscr{P}_\chi$. In this case, the manifold $\mathscr{P}_\chi$ will not be symplectic, but it only has first class constraints.

2.3.2 First class constraints

Since we showed how to get rid of second class constraints, let us suppose here that the only constraints in the phase space $\mathscr{P}$ are first class.

By definition, the Poisson brackets between two first class constraints is weakly zero, which means that it must be strongly equal to a linear combination of other first class constraints

$$\{\phi_a, \phi_b\} = C^c_{ab}\phi_c. \tag{2.48}$$

Therefore, first class constraints form an algebra. The structure coefficients of this algebra are not necessarily phase space constant. Indeed, as Nature is unkind [52], they are not constant in the case we are most interested in: General Relativity [53, 54].

The evolution of the system happens within the constraint surface $\mathscr{C}_\phi := \{\phi_a(m) = 0\}$, so we can modify the Hamiltonian with an arbitrary combination of the first class constraints $H \to H + \lambda^a\phi_a$. The Hamilton's equations (2.16) are then modified to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \lambda^a\{f, \phi_a\} + \{f, \lambda^a\}\phi_a \approx \{f, H\} + \lambda^a\{f, \phi_a\}. \tag{2.49}$$

Therefore, unless the function f is first-class, its time evolution will be inherently ambiguous because the functions λ^a can depend arbitrarily on time. As canonical variables — the coordinates on $\mathscr{P}$ — are not generally first-class, this implies that points of $\mathscr{C}_\phi$ do not correspond one-to-one with physical states.

Let us notice that the ambiguity in the evolution is exactly the effect of a canonical transformation induced by the constraint $g_\lambda \equiv \lambda^a\phi_a$

$$\delta_{g_\lambda} f \equiv v_{g_\lambda}(f) = \{f, g_\lambda\} \approx \lambda^a\{f, \phi_a\}. \tag{2.50}$$

Unlike second class constraints, the constraint surface is not a symplectic submanifold. The pull-back of the ambient symplectic form ω will still have degenerate directions, corresponding to the Hamiltonian vector fields of the constraints. Indeed, the vector field v_{ϕ_a} is parallel to the constraint surface

$$v_{\phi_a}(\phi_b) = \{\phi_b, \phi_a\} \approx 0, \quad \forall\phi_b, \tag{2.51}$$

and it is symplectically orthogonal to any other parallel vector w

$$\omega(v_{\phi_a}, w) = -\mathrm{d}\phi_a(w) \approx 0. \tag{2.52}$$

Therefore, the easiest way to understand of first class constraints is to think of them as generators of gauge transformations. In fact, these constraints do double duty: they restrict the allowed states by imposing relations among the original degrees of freedom, and they also connect the solutions to these relations through canonical transformations. While it is still a conjecture that all first class constraints are generators of gauge transformations, it is true in every non pathological example that have been studied so far. For this reason, the integral curves of v_{ϕ_a} are called *gauge orbits*. It follows from (2.29) and (2.48) that the distribution of vector fields v_{ϕ_a} is involutive, which implies, by Frobenius' theorem, that their integral curves form submanifolds, called *gauge leaves*. The constraint surface $\mathscr{C}_\phi$ is thus foliated by gauge leaves, as depicted in Figure 2.1.

Symplectic reduction

Consider the set of canonical transformations generated by the first class constraints g_λ with different λ^a (we call this set G_0). We can think of the action on functions as an action on the manifold itself [4]

$$G_0 \times \mathscr{P} \to \mathscr{P}, \quad g_\lambda \triangleright m = e^{v_{g_\lambda}} m, \tag{2.53}$$

so that each function is pulled-back as

$$f(g_\lambda \triangleright m) = f(m) + \{f, g_\lambda\}(m) + \frac{1}{2}\{\{f, g_\lambda\}, g_\lambda\}(m) + \dots \tag{2.54}$$

Since first class constraints Poisson-commute into each others, this set is a group. In particular, it is a Lie group. Its Lie algebra is the algebra of Hamiltonian vector fields of the constraints v_{ϕ_a}, which means the Lie group is K-dimensional.

The action (2.53) is clearly Hamiltonian

$$\tilde{\mathcal{J}}(v_{\phi_a}) = \phi_a, \tag{2.55}$$

and the momentum maps are weakly equivariant [5]

$$\left\{\mathcal{J}_{v_{\phi_a}}, \mathcal{J}_{v_{\phi_b}}\right\} = \{\phi_a, \phi_b\} = C^c_{ab}\phi_c \approx \mathcal{J}_{C^c_{ab} v_{\phi_c}} = \mathcal{J}_{[v_{\phi_a}, v_{\phi_b}]}. \tag{2.56}$$

Moreover, the action (2.53) is free and proper in the constraint surface $\mathscr{C}_\phi$. These conditions are enough to invoke a theorem by Marsden, Weinstein, and Meyer [55–57] according to which the set of gauge leaves $\mathscr{P}_{\text{red}} := \mathscr{C}_\phi / \sim$, defined by the condition that two points are equivalent if they are connected by a gauge transformation

$$m \sim m' \iff \exists\, \lambda^a \,|\, m' = g_\lambda \triangleright m, \tag{2.57}$$

[4] The notation $e^{sv} m$ denotes the point $\gamma(s)$, where γ is a curve with $\gamma(0) = m$ and $\gamma'(0) = v|_m$.

[5] The reason for the weak equality is that the structure coefficients are not necessarily constant, so

$$v_{C^c_{ab}\phi_c}[f] = -C^c_{ab}\{\phi_c, f\} - \{C^c_{ab}, f\}\phi_c \approx -C^c_{ab}\{\phi_c, f\} = C^c_{ab} v_{\phi_c}[f].$$

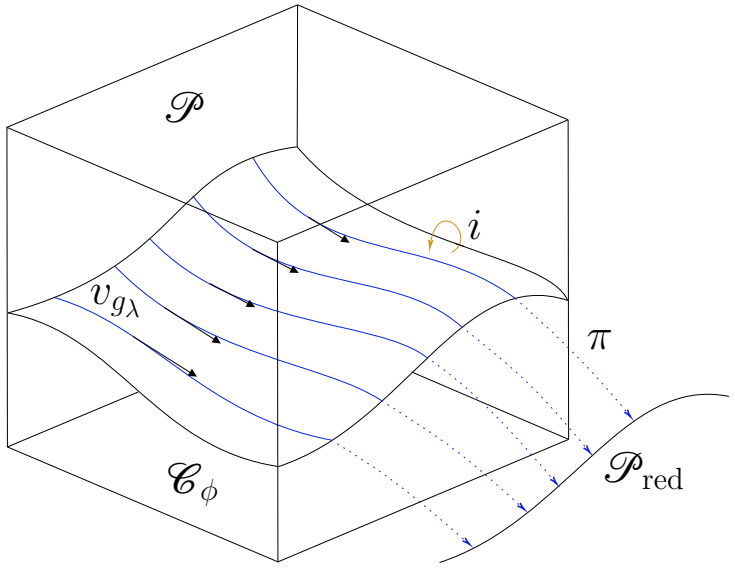


Figure 2.1: Sketch of the symplectic reduction procedure. The map $i : \mathscr{C}_\phi \hookrightarrow \mathscr{P}$ (yellow) embeds the constraint surface inside $\mathscr{P}$. On this surface, the Hamiltonian vector field of the first-class constraint $g_\lambda = \lambda^a \phi_a$ generate gauge orbits (blue). The physical, reduced, phase space $\mathscr{P}_{\text{red}}$ is obtained by projecting every point to the corresponding leaf — which typically do not correspond to the orbit of a sigle vector field as in figure — with the (dotted blue) map $\pi : \mathscr{C}_\phi \to \mathscr{P}_{\text{red}}$.

is a symplectic manifold. We call this manifold the **reduced phase space**. The symplectic form on $\mathscr{P}_{\text{red}}$ is the unique form ω_{red} such that $\pi^* \omega_{\text{red}} = i^* \omega$, where $\pi : \mathscr{C}_\phi \to \mathscr{P}_{\text{red}}$ is the projection to the space of equivalence classes and $i : \mathscr{C}_\phi \hookrightarrow \mathscr{P}$ is just the natural embedding.

Notice that, starting from an even-dimensional space $\mathscr{P}$, we removed K (phase space) degrees of freedom by imposing the constraints, and other K degrees of freedom, conjugate to the former, by quotienting with respect to the canonical transformations g_u, ending up with a $2(N - K)$ dimensional phase space.[6] In light of this double action of the gauge group, one usually denotes $\mathscr{C}_\phi / \sim$ also with a "double quotient" notation $\mathscr{P} // G_0$.

Dirac observables Since the physical phase space is $\mathscr{P}_{\text{red}}$, the only functions of the original phase space that can classify as observable are those that are invariant under the gauge orbits, because they can be mapped to functions of the equivalence class. First class functions trivially satisfy these conditions, so they are called *Dirac observables*.

[6] Someone says "gauge kills twice".

Remark In the context of gauge theories, whereas first class constraints are related to redundancies in the equations of motion and to gauge transformations, gauge fixing conditions need to be constraints of second class.

Chapter 3

Classical Field Theories

The canonical formalism of mechanics, which historically provided the path to quantization, began to reveal its limitation in the early 20th century, with the rise of *field theories* as an essential component in our description of nature.

From a conceptual point of view, transitioning from a system with a large number of particles to a (classical) field theory does not seem drastic. We can imagine that, at any moment of time, the field is just a collection of particles, each labeled by their position in space x, interacting among themselves. The configuration space $\mathcal{Q}$ of a field ϕ would then be the set of all possible spatial configurations ϕ_x of these particles. Once a configuration is specified on a Cauchy slice of a globally hyperbolic spacetime, we can evolve it uniquely,[1] at least for a short time interval. This picture is particularly suggestive in lattice realizations of field theories [58, 59].

From the practical perspective, the canonical formulation of field theories is rather cumbersome, and it explicitly breaks one of the fundamental symmetries of the class of field theories we are interested in, **covariant field theories**. In fact, whereas it is true that some field theories predate the theory of relativity, modern field theories are built to comply with it: their content relies on Lorentz invariance, and their formulation is explicitly spacetime covariant. The canonical formulation, requiring a separation of *space* and *time* to separate variables into *positions* and *momenta*, clashes with the covariance of field theories, as recognized since the early days of quantum field theory (QFT) [50, 60]. In the case of gravity, the application of Dirac–Bergmann algorithm led to the celebrated Arnowitt–Deser–Misner (ADM) phase space [54, 61].

Another point of complication is that most physically relevant field theories are **gauge field theories** – just think of the Standard Model. In these cases, the Hamiltonian description leads to a constrained system. As we saw in Section 2.3, (first class) constraints indicate the presence of redundant information in our description of the system. Eliminating this redundancy and identifying the physical degrees of freedom generally complicates

[1] For gauge theories, the evolution is unique only after a complete gauge fixing.

the description for most gauge theories [62].

In the previous chapter, we used classical mechanics as a reference example and as an excuse to introduce concepts and notation. Hereafter, we are going to focus on field theories. In this chapter, we will first show how to give a Hamiltonian description for field theories without losing covariance. Then, we will use the Lagrangian description to highlight the relevance of boundaries in the study of field theories.

3.1 Locality in field theories

A field theory is usually described in terms of a fiber bundle over spacetime $\mathbb{E} \xrightarrow{\pi} \mathcal{M}$. Fields are sections of this bundle,[2] so we identify the space of all *spacetime* field configurations with the space of sections

$$\mathscr{F} = \Gamma(\mathbb{E}). \tag{3.1}$$

This set is also referred to as configuration space in some sources, even if, according to the analogy with classical mechanics, the actual configuration space is $\mathcal{Q}$. Unless we specify the content of the theory, we let $\phi \in \mathscr{F}$ denote the collection of all fields.

A *local field theory* is then described by a function $S : \mathscr{F} \to \mathbb{R}$ that can be written as a spacetime integral of a local density

$$S[\phi] = \int_{\mathcal{M}} L, \tag{3.2}$$

where L is a function of the field value and its derivatives up to some finite order m. Notice that here, unlike in the previous chapter, we use L to denote a spacetime top-form because the volume form $\epsilon_{\mathcal{M}}$, usually induced by the spacetime metric, might be dynamical as well.

The analogy with classical mechanics would require that we look at S as a functional of $\mathcal{Q}$, i.e. a function that assigns a real number to each trajectory $\underline{\phi}$. However, in simple cases, trajectories in $\mathcal{Q}$ can be identified with points in $\mathscr{F}$: $\underline{\phi}(t) = \phi_{\underline{x}}(t) = \phi(t, x)$. With this in mind, varying the action to find its extrema is quite similar to compute the derivative of a function to find where it vanishes.

3.2 Solution space and natural symplectic form

We require the equations of motion for the field theory, the *field equations*, to follow from the action principle. Therefore, let us vary the argument in the action (3.2). For small variations, the expression we need will be first-order, so we can imagine writing the variation

[2] A section ϕ of a fiber bundle is a map $\phi : \mathcal{M} \to \mathbb{E}$ such that $\pi \circ \phi = \mathrm{id}_{\mathcal{M}}$, where π is the natural fiber bundle projection.

of the Lagrangian density as

$$\delta L = \sum_{|\mu|=0}^{m} L_{\{\mu\}}(\phi, \partial\phi, \dots)\delta\partial^{\{\mu\}}\phi, \tag{3.3}$$

where $\{\mu\}$ is here a multi-index with absolute value $|\mu|$ [3] and we left implicit a possible sum over the fields. Locality ensures that the sum has finite terms. Since field variations are local, we can commute them with spacetime derivatives and then, performing $|\mu|$ integration by parts, we can recast the sum in the form

$$\sum_{|\mu|=0}^{m} (-1)^{|\mu|}\partial^{\{\mu\}} L_{\{\mu\}}(\phi, \partial\phi, \dots)\delta\phi + \mathrm{d}(\dots)\,. \tag{3.4}$$

This procedure defines for us the *Euler–Lagrange operator* $\mathfrak{E}_{\delta\phi}(\delta L)$, which, for a generic functional f that depends on a finite number of derivatives of a spacetime function α, satisfies the defining relation

$$f(\alpha, \partial\alpha, \dots) = \mathfrak{E}_{\alpha}(f)\alpha + \mathrm{d}(\dots)\,. \tag{3.5}$$

In this case, integrating (3.4) over spacetime yields

$$\delta S[\phi] = \int_{\mathcal{M}} \mathfrak{E}_{\delta\phi}(\delta L)\delta\phi + (\text{boundary terms}), \tag{3.6}$$

from which, we deduce that the Euler–Lagrange operator applied to the Lagrangian variation yields nothing but the Euler–Lagrange equations, which are thus the equations of motion we were looking for

$$\mathfrak{E}_{\delta\phi}(\delta L) = \frac{\delta S}{\delta\phi} =: E. \tag{3.7}$$

We define the **solution space** as the subset

$$\mathcal{S} := \{\phi \in \mathscr{F} \,|\, E = 0\}, \tag{3.8}$$

and refer to fields configurations in $\mathcal{S}$ as *on-shell*. Likewise, equalities holding when pulled-back to $\mathcal{S}$ are said to hold on-shell, and will be denoted with the symbol $\doteq$. Incidentally, we notice that fields which satisfy the field equations (on-shell) are not necessarily stationary points of the action

$$E(\phi_0) = 0 \implies \delta S[\phi_0] = 0. \tag{3.9}$$

because of the boundary terms in (3.6). We will soon come back to this point.

[3] Namely, $\partial^{\{\mu\}}\phi = \partial^{\mu_1}\dots\partial^{\mu_n}\phi$ with $|\mu| = \mu_1 + \dots + \mu_n$

Natural symplectic form on $\mathcal{S}$ [4] Given the set of spatial field configurations $\mathcal{Q}$, which we can imagine to live on a Cauchy surface Σ, the Hamiltonian description is naturally formulated on $T^*\mathcal{Q}$. This space has a canonical symplectic structure Ω_{can}.[5] It is standard to assume that *allowed* field configurations can be prescribed in terms of Cauchy data (fields and momenta) on Σ. Let $\mathrm{CDS}(\Sigma)$ be the set of these data. If we also assume the existence of a *polarization* map that allows us to construct Cauchy data out of solutions of the field equations, then we have the following sequence of inclusions

$$\mathcal{S} \hookrightarrow \mathrm{CDS}(\Sigma) \hookrightarrow T^*\mathcal{Q}, \tag{3.10}$$

and a natural symplectic form Ω on $\mathcal{S}$, given by pulling-back Ω_{can} twice via the inclusion maps. Determining the symplectic form with this procedure is extremely challenging, hence it is preferable to use other methods to derive a different symplectic form Ω' on $\mathcal{S}$ and then prove the equivalence between Ω' and Ω.

Identifying the set of solutions to the equations of motion as the correct classical phase space was perhaps the most important step towards a *covariant*, Hamiltonian description of field theories [64]. We are going to show in the next sections how to construct Ω' without losing covariance.

3.3 Variational bicomplex

The framework which formalizes the manipulations needed to derive a covariant symplectic structure for field theories is the *variational bicomplex*, whose modern formulation is due to Anderson [65]. According to Anderson himself, the framework originated from a problem first addressed by Helmholtz [66]: determining the properties of those systems which can be described by a variational principle. Whereas one can construct the variational bicomplex for a generic fibered manifold, we will consider only the construction that is relevant for field theory applications.

3.3.1 Field space differential geometry

Given the fiber bundle $\mathbb{E} \xrightarrow{\pi} \mathcal{M}$, the first step is to mimic in the space of sections the operations that we can do on the base manifold: spacetime, in this case. In doing this, we are going to sweep a lot of technical issues under the rug as some of the tools of differential geometry that are available for finite-dimensional manifolds do not simply extend — or simply do not extend — to infinite-dimensional ones.

The easiest objects to generalize are functions. Functions of field space are just maps $\mathscr{F} \to \mathbb{R}$. An important example of such functions are the so-called *evaluation maps* $\phi \mapsto$

[4] This remark is mostly drawn from Ref. [63]

[5] In field theory, we will reserve ω for the quantity whose spatial integral is the symplectic form Ω.

$\phi^i(x)$, which, given a field configuration, output the value of the i-th field at the spacetime point x.

Consider now a curve in field space $I \to \mathscr{F}$, $s \mapsto \phi_s$ that we can think as the transformation of the field under a continuous deformation, parametrized by s. From a curve, we can construct vectors as directional derivatives in the usual way: for each function $f \in C^\infty(\mathscr{F})$

$$\mathbb{X}_{\phi_0}[f] = \frac{\mathrm{d}}{\mathrm{d}s} f(\phi_s)\bigg|_{s=0}. \tag{3.11}$$

Applying the chain rule, we get

$$\mathbb{X}_{\phi_0}[f] = \int_{\mathcal{M}} \mathrm{d}^n x \frac{\mathrm{d}\phi^i_s(x)}{\mathrm{d}s}\bigg|_{s=0} \frac{\delta f}{\delta \phi^i(x)}\bigg|_{\phi_0} \tag{3.12}$$

If we strip the function f, we see that the tangent space $T_{\phi_0}\mathscr{F}$ at ϕ_0 is given by the span of directional derivatives along the field components, which we are going to denote as $\mathbb{d}/\mathbb{d}\phi^i(x)$. In this basis, the components of the vector are given by the field infinitesimal variations along the deformation

$$\mathbb{X}^i_{\phi_0} = \frac{\mathrm{d}\phi^i_s(x)}{\mathrm{d}s}\bigg|_{s=0}. \tag{3.13}$$

The logical next step is to define a basis of 1-forms for $T^*_{\phi_0}\mathscr{F}$ using the linear maps $\mathbb{X} \to \mathbb{X}^i(x)$. These are maps that, given a vector (i.e. a direction in field space), tell us how much the field is going to change along it. By analogy with the forms $\mathrm{d}x^i$ in the finite dimensional case, we can denote these forms as $\mathbb{d}\phi^i(x)$. Removing the dependence on x and ϕ_0 for readability, we thus have

$$\mathbb{d}\phi^i(\mathbb{X}) = \mathbb{I}_{\mathbb{X}}\mathbb{d}\phi^i = \mathbb{X}^i, \tag{3.14}$$

where we have taken this occasion to introduce $\mathbb{I}$, the field space analog of the interior product ι. Notice that $\mathbb{d}\phi^i(x)$ is nothing but the exterior derivative of the evaluation map. So, for a generic function $f \in C^\infty(\mathscr{F})$

$$\mathbb{d}f = \int_{\mathcal{M}} \mathrm{d}^n x \, \frac{\delta f}{\delta \phi^i(x)} \mathbb{d}\phi^i(x). \tag{3.15}$$

Taking the exterior product of the forms $\mathbb{d}\phi^i$, we can then span forms of higher degree.

With suitable assumptions about regularity, we can extend this construction from vectors to vector fields, from forms to form fields, and from general tensors to tensor fields. In particular, we can construct the exterior algebra

$$\Omega(\mathscr{F}) = \bigoplus_{k=0} \Omega^k(\mathscr{F}), \tag{3.16}$$

where the exterior product maps $\Omega^i(\mathscr{F}) \times \Omega^j(\mathscr{F}) \to \Omega^{i+j}(\mathscr{F})$. [6] Unlike the exterior algebra over spacetime $\Omega(\mathcal{M})$, this algebra does not have a maximal degree, meaning that the de Rham complex associated with the *exterior derivative* $\mathbb{d}$ does not end

$$0 \hookrightarrow \Omega^0(\mathscr{F}) \xrightarrow{\mathbb{d}} \Omega^1(\mathscr{F}) \xrightarrow{\mathbb{d}} \dots \xrightarrow{\mathbb{d}} \Omega^i(\mathscr{F}) \xrightarrow{\mathbb{d}} \Omega^{i+1}(\mathscr{F}) \xrightarrow{\mathbb{d}} \dots \tag{3.17}$$

Cartan calculus In the graded algebra $\Omega(\mathscr{F})$ we find the usual operations of Cartan calculus. From $\mathbb{d}$ and $\mathbb{I}$, we define the field space Lie derivative according to Cartan's homotopy formula

$$\mathbb{L}_{\mathbb{X}} = \mathbb{I}_{\mathbb{X}}\mathbb{d} + \mathbb{d}\mathbb{I}_{\mathbb{X}}, \tag{3.18}$$

where $\mathbb{X} \in \mathfrak{X}(\mathscr{F})$. We then have the usual properties, given $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(\mathscr{F})$

$$\mathbb{d}^2 = 0, \qquad \mathbb{I}_{\mathbb{X}}^2 = 0 \tag{3.19a}$$

$$\mathbb{d}\mathbb{L}_{\mathbb{X}} = \mathbb{L}_{\mathbb{X}}\mathbb{d}, \qquad \mathbb{I}_{\mathbb{X}}\mathbb{I}_{\mathbb{Y}} + \mathbb{I}_{\mathbb{Y}}\mathbb{I}_{\mathbb{X}} = 0, \tag{3.19b}$$

$$\mathbb{L}_{\mathbb{X}}\mathbb{L}_{\mathbb{Y}} - \mathbb{L}_{\mathbb{Y}}\mathbb{L}_{\mathbb{X}} = \mathbb{L}_{[\mathbb{X},\mathbb{Y}]_{\text{Lie}}}, \qquad \mathbb{L}_{\mathbb{X}}\mathbb{I}_{\mathbb{Y}} + \mathbb{I}_{\mathbb{Y}}\mathbb{L}_{\mathbb{X}} = \mathbb{I}_{[\mathbb{X},\mathbb{Y}]_{\text{Lie}}}, \tag{3.19c}$$

Given the exterior derivative, we can define closed forms and exact forms in the common way: $\alpha \in \Omega(\mathscr{F})$ is *closed* if $\mathbb{d}\alpha = 0$ and *exact* if $\alpha = \mathbb{d}\beta$ for another form $\beta \in \Omega(\mathscr{F})$.

3.3.2 Total space differential geometry

We now want to consider the total space $\mathbb{F} \equiv \mathcal{M} \times \mathscr{F}$ and extend differential geometry to it as well. The reason for this is clear: if we want to apply something like (3.15) to the Lagrangian density, which is both a function (a 0-form) of field space and a n-form on spacetime, we need a formalism that treats both kinds of objects on equal footing.

Usually, the differential structure of a product manifold is induced from the differential structures of the factors. The complications here is that the second factor ($\mathscr{F}$) depends on the first ($\mathcal{M}$): a generic function of $\mathbb{F}$ depends on spacetime points both explicitly, as a function of $\mathcal{M}$, and implicitly, as a function of $\mathscr{F}$. However, if we restrict to functions (and forms) that are *local*, in the sense that their value at (x, ϕ) only depends on $(x, \phi(x))$ — and on the value of vectors of $T_\phi(\mathscr{F})$ at x, plus finitely many derivatives thereof at x — we can disentangle the two factors. In the following, we assume that all the tensors under consideration are local.

We can combine a p-form of $\Omega(\mathcal{M})$ with a q-form of $\Omega(\mathscr{F})$ to get a (p, q)-form in $\Omega^{(p,q)}(\mathbb{F}) = \Omega^p(\mathcal{M}) \times \Omega^q(\mathscr{F})$ and define the variational bicomplex as the system of (p, q)-

[6] As customary in the literature, the exterior product of two forms in field space is just denoted by their juxtaposition, without a special symbol, e.g $\alpha\beta = (-1)^{rs}\beta\alpha$ if $\alpha \in \Omega^r(\mathscr{F})$, $\beta \in \Omega^s(\mathscr{F})$.

forms, together with the two operations of exterior derivative

$$\begin{array}{ccccc} \dots & \xrightarrow{\mathrm{d}} & \Omega^{(p,q+1)}(\mathbb{F}) & \xrightarrow{\mathrm{d}} & \dots \\ \uparrow{\scriptstyle \mathbb{d}} & & \uparrow{\scriptstyle \mathbb{d}} & & \uparrow{\scriptstyle \mathbb{d}} \\ \Omega^{(p-1,q)}(\mathbb{F}) & \xrightarrow{\mathrm{d}} & \Omega^{(p,q)}(\mathbb{F}) & \xrightarrow{\mathrm{d}} & \Omega^{(p+1,q)}(\mathbb{F}) \\ \uparrow{\scriptstyle \mathbb{d}} & & \uparrow{\scriptstyle \mathbb{d}} & & \uparrow{\scriptstyle \mathbb{d}} \\ \dots & \xrightarrow{\mathrm{d}} & \Omega^{(p,q-1)}(\mathbb{F}) & \xrightarrow{\mathrm{d}} & \dots \end{array} \tag{3.20}$$

In field theory applications of the variational bicomplex, one also refers to the exterior derivative d — and tensors in spacetime — as *horizontal* and to the exterior derivative $\mathbb{d}$ — and (local) tensors in field space — as *vertical*.

Remark The Cartan calculus extends naturally to the bicomplex (3.20), with all horizontal operations that commute with all vertical ones.[7]

3.4 The covariant phase space

The covariant phase space (CPS) was developed through the effort of many authors [67–76] to overcome the fundamental clash between the canonical formulation and the covariance of field theories. A pivotal point in the formulation of the CPS was Crnković & Witten's paper in 1987 [77], where an explicit symplectic structure was constructed for scalar fields, Yang–Mills, and General Relativity in a completely covariant fashion (cfr. end of Section 3.2). The key insight was the identification of the usual variation δ with the exterior derivative in field space $\mathbb{d}$, which allowed to import the machinery of the to-be variational bicomplex in the space of solutions. A nice review of the topic can be found in [78].

3.4.1 Appetizer

Let us briefly show here why it makes sense to identify $\delta = \mathbb{d}$. Starting from (3.2), we can see the action as a 0-form in field space which is obtained by the integral of a $(n, 0)$-form, the Lagrangian density. Applying a field-space/vertical derivative, we will get a $(n, 1)$-form that we can easily rewrite as

$$\mathbb{d}L = E\,\mathbb{d}\phi + \mathrm{d}\theta, \tag{3.21}$$

because, given the commutativity between $\mathbb{d}$ and d, we can repeat the same considerations that led us to (3.4) and it is easy to see that

$$\mathfrak{E}_{\mathbb{d}\phi}(\mathbb{d}L) = \mathfrak{E}_{\delta\phi}(\delta L) = E. \tag{3.22}$$

[7] In most of the mathematical literature, they are taken to anti-commute, but it is just a convention.

In eq. (3.21), we again left implicit a possible sum over the fields $E_i \mathbb{d}\phi^i$, and we introduced a symbol for the total derivative term. The $(n-1,1)$-form θ is called (pre-)*symplectic potential* (density) for reasons that will soon be clear. One could write an expression for $\mathrm{d}\theta$ in terms of the *Anderson homotopy operator* [65] starting from L but, in most cases, it is easier to just evaluate $\mathbb{d}L$, put in the form (3.21), and identify the two terms.

Therefore, taking a vertical derivative of the action gives us the same information about the dynamics as considering its variation. On the other hand, $\mathbb{d}$ has stronger algebraic properties, from which we deduce that

$$0 = \mathbb{d}^2 L = \mathbb{d}E\mathbb{d}\phi + \mathrm{d}\mathbb{d}\theta, \tag{3.23}$$

which means that $\mathbb{d}\theta$ is closed as a spacetime form on-shell (i.e. when pulled back to $\mathscr{S}$) because $\mathbb{d}E \doteq 0$. Let us see why this is useful.

Analogy with classical mechanics From (3.21), we see that the action vertical derivative is on-shell a total boundary term

$$\mathbb{d}S = \int_{\mathcal{M}} \mathbb{d}L = \int_{\mathcal{M}} E\mathbb{d}\phi + \int_{\partial\mathcal{M}} \theta \doteq \int_{\partial\mathcal{M}} \theta. \tag{3.24}$$

If we ignore the lateral boundary $\mathcal{I}$ for the moment, the only terms that survive are on the initial and final slices (the lids)

$$\mathbb{d}S \doteq \int_{\Sigma_{\mathrm{fin}}} \theta - \int_{\Sigma_{\mathrm{in}}} \theta. \tag{3.25}$$

This expression can be compared to the on-shell action variation in classical mechanics

$$\delta S \doteq p\delta q|_{t_{\mathrm{fin}}} - p\delta q|_{t_{\mathrm{in}}}, \tag{3.26}$$

where p are the Lagrangian momenta. This naturally leads to the analogy

$$\int_{\Sigma} \theta \longleftrightarrow p\delta q, \tag{3.27}$$

which, looking at eq. (2.40), clarifies why θ is called a (pre-)symplectic potential density.[8] One of the main results of classical mechanics is Liouville's theorem, according to which the expression $\delta p \delta q$, giving the infinitesimal volume in phase space, does not change under time evolution. The analogous result in field theory would be that the integral of $\mathbb{d}\theta$ does not depend on the choice of time slice Σ

$$\int_{\Sigma_1} \mathbb{d}\theta \overset{?}{=} \int_{\Sigma_2} \mathbb{d}\theta, \tag{3.28}$$

which is indeed the case on-shell, if Σ_1 and Σ_2 are taken to have the same boundary, because $\mathbb{d}\theta$ is horizontally closed (3.23).

[8] Also here, we reserve θ for the quantity whose spatial integral is the actual symplectic potential.

3.4.2 Field theory symmetries

With the introduction of the variational bicomplex, we are also able to give a more precise, operative meaning to the concept of symmetry. In this framework, a (infinitesimal) transformation is implemented by a vector field in field space $u \in \mathfrak{X}(\mathscr{F})$.

- To be a symmetry, this transformation should preserve the set of solutions, so we require
$$\mathbb{L}_{\mathbb{X}} E \doteq 0 \tag{3.29}$$
In other words, when restricted to $\mathscr{S}$, $\mathbb{X}$ must be tangent to it. We can call these *symmetries of the equations of motion*;

- The set of symmetries of the equations of motion is hard to find, so we define the smaller set of *symmetries of the action* as those vector fields such that
$$\mathbb{L}_{\mathbb{X}} S = 0, \tag{3.30}$$
up to integrals localized on the lids. Symmetries of the action are symmetries, but not all symmetries preserve the action;

- If we want to consider *symmetries of the Lagrangian*, we have to take into account that L naturally lives in $\mathbb{F}$ and not in $\mathscr{F}$. We define these symmetries by the condition [9]
$$\mathbb{L}_{\mathbb{X}} L = \mathrm{d}\Lambda_{\mathbb{X}}, \tag{3.31}$$
for some $\Lambda_{\mathbb{X}} \in \Omega^{(n-1,0)}(\mathbb{F})$. Whereas it is true that, for any $\phi \in \mathscr{F}$, $\mathbb{L}_{\mathbb{X}} S(\phi) = 0$ implies $\mathbb{L}_{\mathbb{X}} L(\phi) = \mathrm{d}(\Lambda_{\mathbb{X},\phi})$ with $\Lambda_{\mathbb{X},\phi} \in \Omega^{n-1}(\mathcal{M})$, such a $\Lambda_{\mathbb{X},\phi}$ might not exist globally in $\mathbb{F}$. Hence, whereas a symmetry of the Lagrangian is a symmetry of the action, the converse is not true is general.

3.4.3 The symplectic structure

Given the presymplectic potential θ, we can take a further variation and construct a $(n-1,2)$-form
$$\omega = \mathbb{d}\theta. \tag{3.32}$$
Such a form is trivially closed in the vertical direction, and it is closed also on-shell because the pull-back to $\mathscr{S}$ commutes with the exterior derivative. Moreover, as we deduced from (3.21), it is closed in the horizontal direction as well
$$\mathrm{d}\omega = \mathbb{d}E \wedge \mathbb{d}\phi \doteq 0. \tag{3.33}$$

[9] These are called d-symmetries in [63].

The natural thing to do with a closed $(n-1)$ spacetime form is to integrate it over a hypersurface of codimension 1. Therefore, we pick a Cauchy hypersurface $\Sigma \subset \mathcal{M}$ and define

$$\Omega_\Sigma = \int_\Sigma \omega. \tag{3.34}$$

Thanks to the properties of ω, Ω_Σ is closed in field space. As long as we are off-shell, Ω_Σ is necessarily degenerate because it only depends on field variation at $x \in \Sigma$, so we pull it back to $\mathcal{S}$: the resulting form is independent of Σ thanks to the on-shell closure of ω, so we just denote it by Ω. At this stage, we cannot yet exclude the possibility that there are still degenerate directions in Ω, hence we call it *presymplectic*. We will show in Section 5.2 how to take care of them.

After Ω is made non-degenerate, it provides the symplectic structure which makes the space of non-degenerate directions within $\mathcal{S}$ into a symplectic manifold: this is the celebrated **covariant phase space**!

Remark In undergrad field theory, it is usually assumed that fields decay sufficiently fast in space so that nothing interesting happens at $\mathcal{I}$. The situation becomes more interesting in the presence of long-range forces. The asymptotic behavior of fields become non-trivial and we must consider the possibility that the system is not closed. If Σ_1 and Σ_2 are two Cauchy surfaces whose boundaries do not coincide, then the difference between the corresponding form (3.34) is

$$\Omega_{\Sigma_2} - \Omega_{\Sigma_1} = \int_{\mathcal{M}_{12}} \underbrace{\mathrm{d}\omega}_{\doteq 0} - \int_{\mathcal{I}_{12}} \omega, \tag{3.35}$$

where $\mathcal{I}_{12}$ is the portion of $\mathcal{I}$ enclosed by $\partial\Sigma_1 \cup \partial\Sigma_2$ and $\mathcal{M}_{12}$ is the portion of $\mathcal{M}$ enclosed by $\Sigma_1 \cup \mathcal{I}_{12} \cup \Sigma_2$. This mean that the presymplectic form Ω changes from one surface to the other based on the *presymplectic flux* across the lateral boundary. This term will be very important when dealing with boundaries, at finite or infinite distance.

Peierls bracket An important point is what are the Poisson brackets associated with the inverse of Ω. A natural, covariant Poisson structure in the space of solutions was single-handedly found by Peierls [79] (and later, independently, by Segal [80, 81]). The so-called *Peierls brackets* and are associated with how phase space functions respond to a modification of the action [1]. The equivalence between the Poisson structure given by the Peierls brackets and the symplectic structure given by Ω was only proven much later [82].

3.4.4 Phase space symmetries

Once we have a symplectic structure on the space of solutions $\mathcal{S}$, we can import all the concepts from Section (2.2) and adapt them to field space differential geometry. The exterior calculus of $\mathscr{F}$ naturally descend to $\mathcal{S}$ via pull-back. It is then natural to ask the

following questions: which vector fields $\mathfrak{X}(\mathcal{S})$ are symplectomorphisms? Which symplectomorphisms are integrable? What is the algebra of integrable symplectomorphisms? An important point in the development of the CPS is to study how the symmetries of a field theory, defined by (3.29), (3.30) or (3.31), connect to the symmetries of the symplectic structure.

Before tackling this problem, let us quickly summarize what the usual Hamiltonian description looks like in the CPS.

- A vector field $\mathbb{X} \in \mathfrak{X}(\mathcal{S})$ is a symplectomorphism if

$$\mathbb{L}_{\mathbb{X}}\Omega = 0. \tag{3.36}$$

  Symplectomorphisms form a subalgebra.

- A symplectomorphism is integrable if there exists a function $H_{\mathbb{X}} \in C^\infty(\mathcal{S})$ such that

$$\mathbb{I}_{\mathbb{X}}\Omega = -\mathbb{d}H_{\mathbb{X}}. \tag{3.37}$$

  Integrable symplectomorphisms form a subalgebra (an ideal).

- Given two integrable symplectomorphisms $\mathbb{X}, \mathbb{Y}$, we can write what we call the *representation theorem* (cfr. eq. (2.22))

$$\{H_{\mathbb{X}}, H_{\mathbb{Y}}\} = H_{[\![\mathbb{X},\mathbb{Y}]\!]_{\text{Lie}}} + K_{\mathbb{X},\mathbb{Y}} \tag{3.38}$$

  where $K_{\mathbb{X},\mathbb{Y}}$ is a field-independent two-cocyle.

3.4.5 Noether's theorems in the covariant phase space

In this section, we are going to assume that nothing interesting happens at the boundary $\mathcal{I}$, and that the form Ω (which we remind to be $i_{\mathcal{S}}^*\Omega_\Sigma$, with $i : \mathcal{S} \hookrightarrow \mathscr{F}$) is non-degenerate.

Given a vector field $u \in \mathfrak{X}(\mathcal{S})$ that is a symmetry of the Lagrangian, we can introduce the following quantity

$$J_{\mathbb{X}} := \mathbb{I}_{\mathbb{X}}\theta - \Lambda_{\mathbb{X}}, \tag{3.39}$$

where $\Lambda_{\mathbb{X}}$ is defined by (3.31). In the CPS formalism, **Noether's first theorem** is the statement that this quantity is closed on-shell, so that the associated charge

$$Q_{\mathbb{X},\Sigma} := \int_\Sigma J_{\mathbb{X}} \tag{3.40}$$

does not depend on the choice of Σ when pulled-back to $\mathcal{S}$: $i_{\mathcal{S}}^* Q_{\mathbb{X},\Sigma} = Q_{\mathbb{X}}$.

Proof. We can prove the statement in one line using (3.21)

$$\mathrm{d}J_{\mathbb{X}} = \mathbb{I}_{\mathbb{X}}\mathrm{d}\theta - \mathrm{d}\Lambda_{\mathbb{X}} = \mathbb{I}_{\mathbb{X}}(\mathbb{d}L - E\mathbb{d}\phi) - \mathbb{L}_{\mathbb{X}}L = -E\mathbb{L}_{\mathbb{X}}\phi \doteq 0. \tag{3.41}$$

□

Since $Q_{\mathbb{X}}$ is the conserved quantity associated with the symmetry $\mathbb{X}$, we call it the *Noether charge*, while we call $J_{\mathbb{X}}$ the *Noether current*. A very important property of a Noether charge is that it coincides with the phase space generator of the transformation, i.e. it is a Hamiltonian function for the transformation $\mathbb{X}$, which is therefore an integrable symplectomorphism.

Proof. First of all, we show how the symplectic potential transform under the Lagrangian symmetry, applying a field space Lie derivative to eq. (3.21) and using (3.31)

$$\mathbb{L}_{\mathbb{X}}(E\mathbb{d}\phi) + \mathbb{L}_{\mathbb{X}}\mathrm{d}\theta = \mathbb{L}_{\mathbb{X}}\mathbb{d}L = \mathbb{d}\mathbb{L}_{\mathbb{X}}L = \mathbb{d}(\mathrm{d}\Lambda_{\mathbb{X}}). \tag{3.42}$$

Considering that a Lagrangian symmetry is a symmetry of the equations of motion (3.29), we get that

$$\mathbb{L}_{\mathbb{X}}\theta \doteq \mathbb{d}\Lambda_{\mathbb{X}} + \mathbb{z}_{\mathbb{X}}, \tag{3.43}$$

where $\mathbb{z}_{\mathbb{X}}$ is a horizontally closed $(n-1,1)$-form. We can invoke a theorem (Theorem II.13 in [63]) to conclude that $\mathbb{z}_{\mathbb{X}}$ is actually horizontally exact $\mathbb{z}_{\mathbb{X}} = \mathrm{d}\mathbb{s}_{\mathbb{X}}$, and explicitly compute

$$\mathbb{d}J_{\mathbb{X}} = \mathbb{d}\mathbb{I}_{\mathbb{X}}\theta - \mathbb{d}\Lambda_{\mathbb{X}} = -\mathbb{I}_{\mathbb{X}}\omega + \mathbb{L}_{\mathbb{X}}\theta - \mathbb{d}\Lambda_{\mathbb{X}} \doteq -\mathbb{I}_{\mathbb{X}}\omega + \mathrm{d}\mathbb{s}_{\mathbb{X}}. \tag{3.44}$$

Integrating over a Cauchy slice (no matter which) and assuming that $\mathbb{s}_{\mathbb{X}}$ does not contribute at the boundary, we get

$$\boxed{\mathbb{d}Q_{\mathbb{X}} \doteq -\mathbb{I}_{\mathbb{X}}\Omega.} \tag{3.45}$$

□

By previous considerations, we deduce that the Poisson algebra of charges is a projective representation of the algebra of symplectomorphisms. If the latter are also the generators of a Lie group action (2.33), then the charge algebra is a central extension of the associated Lie algebra.

Let us now assume that the symmetry is local. This means that there exists a collection of spacetime function(al)s α which parametrize the field variations $\delta_\alpha\phi$, so we can write the field-space vector field as

$$\mathbb{X}_\alpha = \int_{\mathcal{M}} \delta_\alpha\phi(x)\frac{\mathbb{d}}{\mathbb{d}\phi(x)}. \tag{3.46}$$

In this case, we can use integration by parts to rewrite the last term of (3.41) as

$$\mathbb{I}_{\mathbb{X}_\alpha}(E\mathbb{d}\phi) = E\delta_\alpha\phi = \mathfrak{E}_\alpha(E\delta_\alpha\phi)\alpha - \mathrm{d}c_\alpha, \tag{3.47}$$

where the $(n-1,0)$-form c_α contains the spacetime derivatives of the local parameter. Since the Lagrangian is invariant (up to exact pieces) under the flow of $\mathbb{X}_\alpha$, we can show that $\mathfrak{E}_\alpha(E\delta_\alpha\phi)\alpha = 0$ off-shell [83], so that

$$E\delta_\alpha\phi = -\mathrm{d}c_\alpha. \tag{3.48}$$

Given that c_α vanishes identically on the equations of motion, it is called **constrain current**. For instance, in the case of Maxwell theory, the time component of the constraint current is the well-known *Gauss constraint* $\nabla \cdot E = e^2\rho$ [84]. In this case, the steps that led to Noether's first theorem lead now to the *off-shell* equality

$$\mathrm{d}(J_{\chi_\alpha} - c_\alpha) = 0 \tag{3.49}$$

which implies [10]

$$J_\alpha \equiv J_{\chi_\alpha} = c_\alpha + \mathrm{d}q_\alpha. \tag{3.50}$$

where $q_\alpha \in \Omega^{(n-2,0)}(\mathbb{F})$ is the so-called **charge aspect**. In this case, the integrated Noether charge is a total derivative on-shell, and thus vanishes in the absence of boundaries. The vanishing of the charge makes it impossible to use it as a label to distinguish states, hence the standard lore of associating the existence of a local symmetry with a redundancy of the description. The true, physical state must be obtained via the identification of field configurations that are mapped into each other under the gauge transformation, as we saw near the end of Section 2.3.2.

Obviously, in the presence of boundaries, local symmetries can produce a non-trivial charge. The usual identification of local symmetries with gauge redundancies fail and new physics appears. We will see this in Chapter 5.

3.4.6 Covariant phase space ambiguities

In the description of the CPS algorithm we just laid out, we glossed over the fact that the identity (3.21) does not provide us with a unique presymplectic potential θ. As a matter of fact, the form of the identity remains the same if we make the following changes

$$\begin{cases} L \to L' = L + \mathrm{d}Z, \\ \theta \to \theta' = \theta + \mathbb{d}Z + \mathrm{d}Y, \end{cases} \tag{3.51}$$

with $Z \in \Omega^{(n-1,0)}(\mathbb{F})$ and $Y \in \Omega^{(n-2,1)}(\mathbb{F})$. These are the well-known CPS ambiguities [85].

It is easy to see that the Z shift that does not affect the symplectic form because $\mathbb{d}^2 = 0$. Consequently, everything we said about symmetries and charges remain the same. On the other hand, the Y shift does change the symplectic form.

$$\Omega \to \Omega' = \Omega + \int_{\partial\Sigma} \mathbb{d}Y, \tag{3.52}$$

and it is thus a true ambiguity of the formalism. It is important to point out that this ambiguity is only in the form of a *corner* contribution to the symplectic form. While the charges of the theory might change in the presence of boundaries, bulk objects such as the symmetries we described so far and the equations of motion stay the same.

[10] We assume spacetime to be contractible as to satisfy the assumptions of Poincaré's lemma.

3.5 Background fields

One important lesson that we learn from field theories is that not all fields are born equal. Sometimes, in order to write an action, we need to introduce *background fields*. Unlike for *dynamical fields*, the time evolution of background fields is taken as a given, which means that we do not require the action to provide an equations of motion for them. However, this does not imply that it is meaningless to consider their variation: varying background fields allows us to identify quantities, generically called *currents*, that give us information about the theory and about how it can be coupled to other theories. These are defined by

$$\mathscr{J} := \frac{\delta S[\phi; b]}{\delta b}, \tag{3.53}$$

where b denotes the collection of all background fields. It is important to notice that S is still a function of $\mathscr{F}$ only, and we consider $b \notin \mathscr{F}$: the dependence on b is only parametrical.

It is usually the case that background fields have symmetries, in the sense that the description of the theory does not change if we use b or $b + \delta_\alpha b$ as background. If $\delta_\alpha b$ depends on a set of spacetime dependent parameters α, we can use the Euler–Lagrange operator at the level of the action variation to write

$$\delta_\alpha S = \int_{\mathcal{M}} \mathscr{J} \delta_\alpha b + (\text{boundary terms}) = \int_{\mathcal{M}} \mathfrak{E}_\alpha(\mathscr{J} \delta_\alpha b)\alpha + (\text{boundary terms}). \tag{3.54}$$

Since the action cannot change by bulk terms under a symmetry, it must be that

$$\mathfrak{E}_\alpha(\mathscr{J} \delta_\alpha b) \doteq 0. \tag{3.55}$$

If $\delta_\alpha b$ depends on first derivatives of α at least, this is a differential equation for the current.

3.5.1 Covariance

We are finally able to precisely state what we mean by *covariance* when talking about covariant phase space, covariant field theories, etc.

The idea we need to formalize is that the description should not change if we choose different coordinates on the base manifold. Geometrically, these transformations are realized by diffeomorphism. For our purposes, we consider only infinitesimal diffeomorphisms, which are generated by smooth vector fields $\xi \in \mathfrak{X}(\mathcal{M})$. In the framework of the variational bicomplex, there exists a natural lift of spacetime vector fields to field space vector fields, provided by the Lie derivative: if $\xi \in \mathfrak{X}(\mathcal{M})$, its lift is

$$\mathbb{X}_\xi = \int_{\mathcal{M}} \mathrm{d}^n x \, \mathcal{L}_\xi \phi(x) \frac{\mathbb{d}}{\mathbb{d}\phi(x)} \in \mathfrak{X}(\mathscr{F}). \tag{3.56}$$

Consider now a tensor field T on $\mathbb{F}$, i.e. an object which transforms as

$$\delta_\xi T = \mathbb{L}_{\mathbb{X}_\xi} T. \tag{3.57}$$

Since T is a local combination of fields, dynamical or background, it is also a tensor in spacetime, which means that it changes under the diffeomorphism generated by ξ as

$$\delta_\xi T = \mathcal{L}_\xi T. \tag{3.58}$$

We say that a tensor T is *covariant* under ξ if these variations are the same:

$$\mathbb{L}_{\mathbb{X}_\xi} T = \mathcal{L}_\xi T \tag{3.59}$$

Namely, a covariant tensor T transforms in the same way under diffeomorphisms, regardless of whether we view it as a spacetime object or as a field space object.

Remark The dynamical fields ϕ are covariant under all $\xi \in \mathfrak{X}(\mathcal{M})$ by definition of $\mathbb{X}_\xi$

$$\delta_\xi \phi = \mathbb{L}_{\mathbb{X}_\xi} \phi = \mathbb{X}_\xi \phi = \mathcal{L}_\xi \phi. \tag{3.60}$$

It follows that tensors built only out of dynamical fields are necessarily covariant under all diffeomorphisms (**general covariance**). On the contrary, if a tensor depends on background objects, then

$$\mathcal{L}_\xi T = \frac{\delta T}{\delta \phi}\mathcal{L}_\xi \phi - \frac{\delta T}{\delta b}\mathcal{L}_\xi b = \frac{\delta T}{\delta \phi}\mathbb{L}_{\mathbb{X}_\xi}\phi - \frac{\delta T}{\delta b}\mathcal{L}_\xi b = \mathbb{L}_{\mathbb{X}_\xi} T - \frac{\delta T}{\delta b}\mathcal{L}_\xi b, \tag{3.61}$$

so that it can be covariant only under symmetries of the background

$$\mathcal{L}_\xi b = 0, \tag{3.62}$$

or, if there are multiple non-invariant background fields, under transformations that leave their combination invariant.

The key point is that true symmetries are those realized by field space vector fields, because symmetries are only allowed to act on dynamical fields. If this were not true, the notion of covariance would be useless because any theory can be *covariantized* by adding suitable background objects.

A theory is covariant under $\xi \in \mathfrak{X}(\mathcal{M})$ if its space of solution is. Namely

$$\mathbb{L}_{\mathbb{X}_\xi} E = \mathcal{L}_\xi E \doteq 0. \tag{3.63}$$

Importantly, since

$$\delta_\xi S = \int_{\mathcal{M}} \mathscr{J} \delta_\xi b + (\text{boundary terms}), \tag{3.64}$$

another way for the theory to be covariant is if all background objects live at the boundary. A theory that is generally covariant has only dynamical fields in the bulk, so it cannot

depend on any fixed background spacetime metric. This is the notion of **background independence**.

We notice that a covariant Lagrangian is enough for $\mathbb{X}_\xi$ to be a Lagrangian symmetry, as we can easily see using Cartan identity and the fact that L is a top form:

$$\mathbb{L}_{\mathbb{X}_\xi} L = \mathcal{L}_\xi L = \mathrm{d}\iota_\xi L + \iota_\xi \mathrm{d}L = \mathrm{d}(\iota_\xi L)\,. \tag{3.65}$$

The associated Noether current for a covariant Lagrangian is

$$J_\xi \equiv J_{\mathbb{X}_\xi} = \mathbb{I}_{\mathbb{X}_\xi}\theta - \iota_\xi L. \tag{3.66}$$

On the other hand, covariance of the Lagrangian is not a necessary condition. The derivation of Noether's theorems and the construction of charge brackets can be extended to the case where $\Lambda_{\mathbb{X}_\xi} \neq \iota_\xi L$. In this case, we say that the Lagrangian is *anomalous* [86–88]. The CPS formalism can account for several sources of (covariance) anomaly, as we will briefly show at the end of Section 5.4.

Part II

The role of boundaries

Chapter 4

The variational principle

The action is the main ingredient of the path integral approach to quantization. In this setup, classical physics can be seen to emerge in the limit of large actions (in units of $\hbar$). To see how this happens, let us consider a generic amplitude for the transition between two field configurations $\phi_{\text{in}}, \phi_{fin} \in \mathcal{Q}$

$$\mathcal{A} = \int_{\phi_{\text{in}}}^{\phi_{fin}} \mathfrak{D}[\phi] e^{iS[\phi]}.$$

We can try to approximate this path integral by picking a certain path ϕ_0 and expanding the exponent around it. In most cases, this would result in nearby paths contributing with an essentially random phase. However, if ϕ_0 is a stationary point of the action, these contributions start at second order at least, resulting in a sizable contribution to the amplitude. At leading order, the path ϕ_0 has probability 1, which is what happens in the classical realm. Even if we do not have a consistent path integral formulation of gravity — despite many attempts (a non-exhaustive list of them [89–99]) — we expect this classical limit to exist and emerge in the same way.

In light of this consideration, we demand the classical solutions to be determined by what we call a **well-posed variational principle**:

> *The solutions to the equations of motion must be such that the action variation vanishes, up to terms that are localized on the initial and final slices.*

The difference in treatment between conditions imposed at the lateral boundary and conditions imposed at the future and past boundary is due to how we describe states and evolution. We cannot say this better than Harlow & Wu: "the former are part of the definition of the theory, while the latter specify a state within that theory" [1] (see also eqs. (3.25) and (3.26)). If we want to allow variations that change the state — and we do! — we cannot restrict the field variations on the lids. So, to have a well-posed variational principle, we need to impose **boundary conditions** that make the boundary terms

in (3.6) vanish on $\mathcal{I}$. Whether a well-posed variational principle corresponds to a well-posed *initial-boundary value problem* [100–105] is beyond the scope of this thesis, but we will at times implicitly assume that it does.

In the presence of boundaries, the action of a field theory can be generally written as

$$S[\phi] = \int_{\mathcal{M}} L + \int_{\mathcal{I}} \ell, \tag{4.1}$$

where ℓ is a $(n-1,0)$-form called **boundary Lagrangian**. Consistently, L is referred to as the **bulk Lagrangian**. In the next chapter, we will see that the boundary Lagrangian, despite not affecting the equations of motion, will play a fundamental role in the symplectic structure of the solution space. Now, let us consider three examples.

In this chapter, we adopt the Lagrangian perspective and represent field variations with δ. The discussion is primarily based on [106] and the references therein.

4.1 Case study: Scalar field theory

We consider the field theory of a massless scalar field ϕ (not a collection of fields) in Minkowski spacetime

$$S[\phi;\eta] = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{-\eta}\partial_\mu\phi\partial^\mu\phi + \int_{\mathcal{I}} \ell. \tag{4.2}$$

Varying the action with respect to ϕ yields

$$\delta S = \int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{-\eta}\partial^2\phi\delta\phi - \int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{-\eta}\partial_\mu(\delta\phi\partial^\mu\phi) + \int_{\mathcal{I}} \delta\ell, \tag{4.3}$$

where we separated the field equation from the boundary contribution. Our goal is to determine which are the boundary conditions that make the variational principle well-posed. For simplicity, we assume that ℓ is quadratic in ϕ: $\ell = -\beta\phi^2\epsilon_{\mathcal{I}}$, where $\beta \in \mathbb{R}$ and $\epsilon_{\mathcal{I}}$ is the volume form induced on the boundary. The total contribution on $\mathcal{I}$ is then

$$-\int_{\mathcal{I}} \sqrt{-h}(\mathcal{L}_n\phi + 2\beta\phi)\delta\phi, \tag{4.4}$$

where we used Stokes' theorem to rewrite the middle term in (4.3) and replaced $n_\mu\nabla^\mu$ with the Lie derivative along the vector dual to n, which is legit when acting on a scalar. This contribution vanishes in two cases:

- $\delta\phi\big|_{\mathcal{I}} = 0$. These are called **Dirichlet boundary conditions** because we are fixing the value of the field at the boundary;
- $\mathcal{L}_n\phi + 2\beta\phi\big|_{\mathcal{I}} = 0$. These are called **Neumann boundary conditions** because we are fixing the value of the field normal derivative at the boundary.[1]

[1] These are *generalized* Neumann boundary conditions; the standard form would correspond to $\beta = 0$.

The scalar field action features a background field, the Minkowski metric η. If we assume for a moment that is not a fixed metric and vary with respect to it, we easily find

$$\delta^\eta S = -\frac{1}{2}\int_{\mathcal{M}} \delta\eta^{\mu\nu} T^\phi_{\mu\nu} + (\text{boundary terms}), \tag{4.5}$$

where

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi \tag{4.6}$$

is the stress-energy tensor of a massless scalar field in flat spacetime. From the general considerations we made in eq. (3.55), applied to the variation (4.5), we see that the isometries of Minkowski metric imply the conservation laws of the stress-energy tensor and the angular momentum, defined by $M^\phi_{\mu\nu\lambda} = x_\mu T^\phi_{\nu\lambda} - x_\nu T^\phi_{\mu\lambda}$.

Robin boundary conditions This example shows the deep link between boundary conditions and boundary Lagrangians. Tuning the coefficient β, we can explore a family of different boundary conditions, which in turn determine the field in the bulk once initial and final data are provided. In the general case, the boundary term over $\mathcal{I}$ looks like

$$\int_{\mathcal{I}} \Pi\delta\Phi \tag{4.7}$$

where Φ and Π are combinations of the field values and derivatives at the boundary and any other field that we used to construct ℓ. By analogy, we distinguish

$$\delta\Phi = 0 \quad \text{(Dirichlet)}, \tag{4.8}$$

$$\Pi = 0 \quad \text{(Neumann)}. \tag{4.9}$$

It is easy to see that we can flip the role of Π and Φ by adding (another) boundary Lagrangian of the form $\ell = -\Pi\Phi$. Doing this, we can impose Neumann boundary conditions on Φ or Dirichlet boundary conditions on Π. We refer to this procedure as changing the *polarization* of the boundary term. More generally, as shown in [107], we can fix any *constant* linear combination of variables and momenta to obtain the so-called **Robin boundary conditions**, which continuously interpolate between the original Dirichlet and Neumann boundary conditions. For instance, if we want to change the polarization to

$$\int_{\mathcal{I}} \Pi^\nu\delta\Phi^\nu, \qquad \text{with} \quad \begin{cases} \Phi^\nu = \Phi\cos\nu + \Pi\sin\nu, \\ \Pi^\nu = \Pi\cos\nu - \Phi\sin\nu, \end{cases} \tag{4.10}$$

in order to impose the Neumann-like conditions $\Pi^\nu = 0$, we can use the boundary Lagrangian

$$\ell = \frac{1}{2}\cos\nu\sin\nu(\Pi^2 - \Phi^2) - \sin^2\nu\Pi\Phi. \tag{4.11}$$

4.2 Case study: General Relativity

We now consider the field theory that is most relevant for us, General Relativity (GR), and see what the discussion about the variational principle entails for it. The bulk action of GR is the celebrated Einstein–Hilbert term

$$S_{EH}[g] = \frac{1}{2\kappa^2}\int_{\mathcal{M}} R\,\epsilon_{\mathcal{M}}, \tag{4.12}$$

where R is the Ricci scalar of the metric g and $\epsilon_{\mathcal{M}} = \sqrt{-g}\mathrm{d}^n x$ the metric volume form, and $\kappa^2 \equiv 8\pi G_N$. Varying this action with respect to $g^{\mu\nu}$ gives

$$\delta S_{EH} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} G_{\mu\nu}\delta g^{\mu\nu}\epsilon_{\mathcal{M}} + \frac{1}{2\kappa^2}\int_{\mathcal{I}} n_\mu \mathbb{v}^\mu \epsilon_{\mathcal{I}}, \tag{4.13}$$

where we used that $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu \mathbb{v}^\mu$ and then we wrote the total derivative as a boundary term with Stokes' theorem. Explicitly, the vector $\mathbb{v}$ is

$$\mathbb{v}^\mu = g^{\lambda\nu}\delta\Gamma^\mu_{\lambda\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\nu\lambda}, \tag{4.14}$$

and, with some manipulations (which can be found in standard textbooks such as [108]), we can write the last integral in (4.13) as

$$-\frac{1}{2\kappa^2}\int_{\mathcal{I}}(2\delta K + K^{ab}\delta h_{ab})\epsilon_{\mathcal{I}}, \tag{4.15}$$

up to total boundary derivative terms. The usual way in which this boundary contribution is set to zero is by imposing Dirichlet boundary conditions for the pulled-back metric $h := i^*_{\mathcal{I}} g$, and adding the Gibbons–Hawking–York (GHY) boundary term [89, 100]

$$S_{GHY} = \frac{1}{\kappa^2}\int_{\mathcal{I}} K\epsilon_{\mathcal{I}}. \tag{4.16}$$

We thus define $S_{GR} := S_{EH} + S_{GHY}$.

GR is perhaps the most famous example of a generally covariant theory. General covariance follows from its background independence, as the only background object we introduced is the boundary $\mathcal{I}$ itself. Varying with respect to the metric induced on $\mathcal{I}$ allows us to define the associated current, which is the well-known **Brown–York tensor** [109]

$$T^{ab}_{BY} := 2\,\frac{\delta S_{GR}}{\delta h_{ab}} = \frac{1}{\kappa^2}(Kh^{ab} - K^{ab}). \tag{4.17}$$

Alternative boundary conditions have been considered in the literature [110–113], with particular emphasis on the well-posedness on the initial-boundary value problem in both the Lorentzian and Euclidean settings, but we will stick with Dirichlet boundary conditions.

4.3 Case study: Higher derivative gravity in 4d

In four dimensions, a theory like GR, with operators of *mass dimension* ≤ 2 only,[2] is not renormalizable at one-loop, unless there is no matter and no cosmological constant either [114]. This is a good enough reason to study gravitational theories with higher-derivative operators too. In particular, the first logical step is to add dimension 4 operators. In a theory without torsion and non-metricity, there are only three such operators — we don't count $\nabla^2 R$ as a bulk operator as it can be integrated — but one can choose to expand them in different basis. The most commonly used are (hdg stands for *higher-derivative gravity*)

- $L_{\rm hdg} = (c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\epsilon_{\mathcal{M}}$, known as Riemann basis;
- $L_{\rm hdg} = (\alpha R^2 + \beta C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \gamma E_4)\epsilon_{\mathcal{M}}$, known as Weyl basis;
- $L_{\rm hdg} = (o_1 S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} + o_2 E_{\mu\nu\rho\sigma}E^{\mu\nu\rho\sigma} + o_3 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})\epsilon_{\mathcal{M}}$, that we might call Ricci basis.[3]

When we deal with operators that are quadratic in the bulk curvature, we generically need to add operators of dimension 3 on the boundary. We can classify them as such:

(a) Operators that are cubic in the extrinsic curvature. These are obtained combining Melmed's polynomials [4] with appropriate powers of the extrinsic curvature trace:

$$K^3, \qquad \mathrm{m}_2 K = K K^a_b K^b_a - \frac{1}{3}K^3, \qquad \mathrm{m}_3 = K^a_b K^b_c K^c_a - K K^a_b K^b_a + \frac{2}{9}K^3. \tag{4.19}$$

(b) Operators built out of the boundary intrinsic and extrinsic curvatures:

$$\mathcal{R}K, \qquad B_3 \equiv 8\det K + 8\left(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}h_{ab}\right)K^{ab} \tag{4.20}$$

where B_3 is a Chern–Simons 3-form.

[2] In this context, the mass dimension of an operator coincides with the number of derivatives of the metric tensor that appear in it.

[3] The tensors in this decomposition have the same symmetries of the Riemann tensor

$$\begin{aligned} S_{\mu\nu\rho\sigma} &= \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \\ E_{\mu\nu\rho\sigma} &= \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma}) - 3S_{\mu\nu\rho\sigma}, \\ C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - E_{\mu\nu\rho\sigma} - S_{\mu\nu\rho\sigma}. \end{aligned}$$

and are orthogonal to each other.

[4] Melmed's polynomials are given by $\mathrm{m}_k := \mathrm{tr}(\check{K}^k)$, where $\check{K}_{ab}$ denotes the traceless part of K_{ab} [115]. Under a Weyl scaling $g_{\mu\nu} \to W^2 g_{\mu\nu}$, these polynomials transform covariantly $\mathrm{m}_k \to W^{-k}\mathrm{m}_k$, hence the integral of m_k on a k-dimensional submanifold is a conformal invariant.

(*c*) Operators coming from the bulk curvature:

$$RK, \qquad n_\mu \nabla^\mu R, \qquad U \equiv C_{\mu\nu\rho\sigma} n^\mu n^\rho K^{\nu\sigma}. \tag{4.21}$$

(*d*) Operators that are total boundary derivative or can be written in terms of others using the bulk Bianchi identity contracted with n:

$$n^\lambda \left(n_\mu n_\nu \nabla_\lambda R^{\mu\nu} - \frac{1}{2} \nabla_\lambda R \right) = -n_\mu \Pi_{\nu\lambda} \nabla^\lambda R^{\mu\nu} = R^{\mu\nu}(K_{\mu\nu} - K n_\mu n_\nu) + \mathcal{D}_a(\dots)^a.$$

Therefore, neglecting terms of class (*d*) which do not contribute on $\mathcal{I}$, we can express the boundary Lagrangian as the following linear combination

$$\ell = \big(a_1 \mathrm{m}_3 + a_2 \mathrm{m}_2 K + a_3 K^3 + b_1 B_3 + b_2 \mathcal{R} K + c_1 U + c_2 R K + c_3 \nabla_n R \big) \epsilon_{\mathcal{I}}. \tag{4.22}$$

The precise coefficients will be fixed according to the desired boundary conditions.

As soon as we add terms that contain two or more powers of the curvature to the usual GR action, the resulting equations of motion for the metric will be fourth-order differential equations, unless in the exceptional case of Gauss–Bonnet (GB) gravity [116]

$$c_1 = c_3 = -\frac{c_2}{4} \iff \alpha = \beta = 0 \iff o_1 = -o_2 = o_3. \tag{4.23}$$

It is known that these kind of equations describe the propagation of more degrees of freedom than just the two metric ones [27, 117–119]. For example, the addition of quadratic curvature operators with coefficients different from (4.23) produces up to six extra degrees of freedom in the expansions around flat spacetime, associated with a massive traceless tensor and a massive scalar [120, 121]. See [122–124] for extensive reviews and comments about the problem of ghosts generically associated with this class of theories.

These additional degrees of freedom will be subjected to some boundary conditions that select physically interesting solutions according to the variational principle. While nothing prevents us from solving the fourth-order field equations once we these conditions are given, we prefer to have an understanding of these solutions in terms of phase space variables. Moreover, boundary conditions are supposed to describe the physical processes that occur when the bulk degrees of freedom interact with the boundary — think of an electric field interacting with the electrons on a sphere of conducting material — so it is more natural to first identify the degrees of freedom and then assign boundary conditions to them.

For these reasons, we present an alternative formulation of the theory based on the introduction of auxiliary fields which will help us identify the degrees of freedom and their boundary conditions.

4.3.1 Boundary action by auxiliary fields method

Following the approach adopted in [121, 125], we introduce two auxiliary fields with the same symmetries as the Riemann tensor, ϱ and Y (a Lagrange multiplier [5]), and write the bulk action of higher-derivative gravity as

$$S_{\rm hdg}[g,\varrho,Y]=\frac{1}{2}\int_{\mathcal{M}}(\mathscr{L}_{\rm hdg}(\varrho_{\mu\nu\rho\sigma})+Y^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma}-\varrho_{\mu\nu\rho\sigma}))\epsilon_{\mathcal{M}}, \tag{4.24}$$

where $\frac{1}{2}\mathscr{L}_{\rm hdg}(R_{\mu\nu\rho\sigma})$ is the Lagrangian *scalar* of higher-derivative gravity (HDG) written as a symmetric form in the space of Riemann-symmetric tensors. Thanks to the introduction of auxiliary fields which are independent of the metric, now the action contains second derivatives (of the metric) only at the linear level, in the Riemann tensor. Its variation is then much easier

$$\begin{aligned}\delta S_{\rm hdg}[g,\varrho,Y]=&\frac{1}{2}\int_{\mathcal{M}}\mathscr{G}_{\mu\nu}\delta g^{\mu\nu}\epsilon_{\mathcal{M}}+\int_{\mathcal{I}}n_\sigma(Y^{\mu\nu\rho\sigma}\nabla_\nu\delta g_{\mu\rho}-\nabla_\nu Y^{\mu\nu\rho\sigma}\delta g_{\mu\rho})\epsilon_{\mathcal{I}}+\\&+\frac{1}{2}\int_{\mathcal{M}}\left[\left(\frac{\delta\mathscr{L}_{\rm hdg}}{\delta\varrho_{\mu\nu\rho\sigma}}-Y^{\mu\nu\rho\sigma}\right)\delta\varrho_{\mu\nu\rho\sigma}+(R_{\mu\nu\rho\sigma}-\varrho_{\mu\nu\rho\sigma})\delta Y^{\mu\nu\rho\sigma}\right]\epsilon_{\mathcal{M}},\end{aligned} \tag{4.25}$$

where $\mathscr{G}_{\mu\nu}$ are the field equations for the metric and the extra degrees of freedom which contains only second derivatives of Y. On the other hand, the equations of motion for the auxiliary fields are algebraic, so we could easily solve for them

$$\varrho_{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma},\qquad Y^{\mu\nu\rho\sigma}=\left.\frac{\delta\mathscr{L}_{\rm hdg}}{\delta\varrho_{\mu\nu\rho\sigma}}\right|_{\varrho=R}, \tag{4.26}$$

and replace them in $\mathscr{G}_{\mu\nu}$ to get back the original fourth-order equation for the metric. At the level of the action, setting ϱ to its on-shell value gives back the original Lagrangian. What is less obvious is that we can replace the Lagrange multiplier Y with its on-shell value and obtain an equivalent action that is a functional of g an ϱ only [126]. This can be done as long as the Hessian matrix $\delta^2\mathscr{L}_{\rm hdg}/\left(\delta\varrho_{\alpha\beta\gamma\delta}\delta\varrho_{\mu\nu\rho\sigma}\right)$ is non-singular.[6] In the following, we assume that this is the case and use Y either as an independent field or as a shorthand for the derivative of $\mathscr{L}$ in (4.26) interchangeably.

Let us now focus on the boundary piece that can be read from eq. (4.25). Using the results in Appendix A, we can use the projections of Y and the Riemann tensors to define

[5] Not related to the CPS ambiguity in (3.51).

[6] It is easy to formulate this condition in the Ricci basis because it is orthogonal. In this basis, the Hessian matrix is diagonal and its determinant is just a product of the coefficients o_i (each raised to a positive power), so it is non-zero as long as none of the o_i vanish.

the following tensors on $\mathcal{I}$ [7]

$$Y^{\alpha\beta\gamma\delta}_{\parallel\parallel\parallel\parallel} \to \mathscr{Y}^{abcd}, \qquad R^{\parallel\parallel\parallel\parallel}_{\alpha\beta\gamma\delta} \to \rho_{abcd}, \tag{4.27}$$

$$Y^{\alpha\beta\gamma}_{\parallel\parallel\parallel\perp} \to \phi^{abc} \qquad R^{\parallel\parallel\parallel\perp}_{\alpha\beta\gamma} \to \tau_{abc}, \tag{4.28}$$

$$Y^{\alpha\beta}_{\parallel\perp\parallel\perp} \to \psi^{ab}, \qquad R^{\parallel\perp\parallel\perp}_{\alpha\beta} \to \sigma_{ab}. \tag{4.29}$$

It is then possible to rewrite the boundary term as

$$\int_{\mathcal{I}} \Big[\Big(K_{cd}\mathscr{Y}^{acbd} - 2\mathcal{D}_c\phi^{acb} - \nabla_n\psi^{ab} - K\psi^{ab}\Big)\,\delta h_{ab} + 2\psi^{ab}\delta K_{ab}\Big]\epsilon_{\mathcal{I}}, \tag{4.30}$$

where $\nabla_n\psi$ is another boundary field, independent from ψ, that is obtained from $n_\mu\nabla^\mu Y$ just as ψ is obtained from Y. This form of the boundary term suggests some natural boundary conditions to be impose on the auxiliary fields.

4.3.2 Dirichlet boundary conditions: a problem

The simplest possibility we can envision, as put forward in [127], is to impose Dirichlet boundary conditions on both the metric and the double normal projection of Y, that is

$$\delta h_{ab} = \delta\psi^{ab} = 0 \qquad \text{on } \mathcal{I} \tag{4.31}$$

This could be achieved by introducing a generalized GHY boundary Lagrangian to flip the polarization of the last term in (4.30)

$$\ell_{\text{hdg}} \stackrel{?}{=} -2\psi^{ab}K_{ab}\epsilon_{\mathcal{I}}. \tag{4.32}$$

However, it turns out that this guess is too naïve, as we can see in the simple case of GB gravity (4.23). In this case, the on-shell value of ψ reduces to

$$\psi^{ab} = \cancel{2\alpha(\rho + 2\sigma)h^{ab}} + \cancel{2\beta(\check{\sigma}^{ab} - \check{\rho}^{ab})} + 2\gamma(\rho h^{ab} - 2\rho^{ab}) \tag{4.33}$$

which, according to the Gauss–Codazzi equations, is completely determined in terms of the boundary (intrinsic and extrinsic) geometry. Therefore, given the Dirichlet condition $\delta h_{ab} = 0$, imposing Dirichlet boundary conditions on ψ^{ab} would be equivalent to fixing the extrinsic curvature. However, this is a statement — a Neumann boundary condition — about the metric tensor.

Proof. The most convenient basis to evaluate $Y^{\mu\nu\rho\sigma}$ on-shell is the Ricci basis, because of its orthogonality. We find

$$Y^{\mu\nu\rho\sigma} \doteq 4o_1 S^{\mu\nu\rho\sigma} + 4o_2 E^{\mu\nu\rho\sigma} + 4o_3 C^{\mu\nu\rho\sigma}. \tag{4.34}$$

[7] These tensors should not be confused with the analogous of e.g [121], where the projection is on a (Cauchy) spacelike hypersurface.

We compute the double normal projections of each tensor

$$S^{\mu\rho}_{\parallel\perp\parallel\perp} = \frac{R}{12}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})\Pi^\mu_\alpha n_\beta \Pi^\rho_\gamma n_\delta = \frac{R}{12}\Pi^{\mu\rho}, \tag{4.35a}$$

$$E^{\mu\rho}_{\parallel\perp\parallel\perp} = \frac{1}{2}(g^{\alpha\gamma}R^{\beta\delta} - g^{\alpha\delta}R^{\beta\gamma} + g^{\beta\delta}R^{\alpha\gamma} - g^{\beta\gamma}R^{\alpha\delta})\Pi^\mu_\alpha n_\beta \Pi^\rho_\gamma n_\delta - 3S^{\mu\rho}_{\parallel\perp\parallel\perp} = \tag{4.35b}$$

$$= \frac{1}{2}R^{\mu\rho}_{\parallel\parallel} - \frac{R - R_{\perp\perp}}{4}\Pi^{\mu\rho}, \tag{4.35c}$$

$$C^{\mu\rho}_{\parallel\perp\parallel\perp} = R^{\alpha\beta\gamma\delta}\Pi^\mu_\alpha n_\beta \Pi^\rho_\gamma n_\delta - E^{\mu\rho}_{\parallel\perp\parallel\perp} - S^{\mu\rho}_{\parallel\perp\parallel\perp} = R^{\mu\rho}_{\parallel\perp\parallel\perp} - \frac{1}{2}R^{\mu\rho}_{\parallel\parallel} + \frac{R - 3R_{\perp\perp}}{6}\Pi^{\mu\rho}. \tag{4.35d}$$

We now pull-back these tensors to $\mathcal{I}$ and use the results of Appendix A to find

$$(i^*_{\mathcal{I}} S_{\parallel\perp\parallel\perp})^{ab} = \frac{\rho + 2\sigma}{12}h^{ab}, \tag{4.36}$$

$$(i^*_{\mathcal{I}} E_{\parallel\perp\parallel\perp})^{ab} = \frac{1}{2}\rho^{ab} + \frac{1}{2}\sigma^{ab} - \frac{1}{4}\rho h^{ab}, \tag{4.37}$$

$$(i^*_{\mathcal{I}} C_{\parallel\perp\parallel\perp})^{ab} = \frac{1}{2}\check{\sigma}^{ab} - \frac{1}{2}\check{\rho}^{ab}, \tag{4.38}$$

where we used the notation $\check{T}$ to denote the traceless part of a symmetric rank-two tensor T. Plugging these results in the pull-back of $Y_{\parallel\perp\parallel\perp}$ (that is ψ) and using the relationships $o_1 = 6\alpha + \gamma$, $o_2 = -\gamma$, and $o_3 = \beta + \gamma$, we finally arrive at

$$\begin{aligned}\psi^{ab} &\doteq o_1\frac{\rho + 2\sigma}{3}h^{ab} + 2o_2\left(\rho^{ab} + \sigma^{ab} - \frac{1}{2}\rho h^{ab}\right) + 2o_3\left(\check{\sigma}^{ab} - \check{\rho}^{ab}\right) = \\ &= 2\alpha(\rho + 2\sigma)h^{ab} + 2\beta(\check{\sigma}^{ab} - \check{\rho}^{ab}) + 2\gamma(\rho h^{ab} - \rho^{ab}).\end{aligned} \tag{4.39}$$

□

Remark By construction, the action of GB theory yields second-order equations of motion for the metric, as in GR, hence they propagate two degrees of freedom only. The introduction of auxiliary fields is superfluous and no additional boundary condition are required to make the variational principle well-defined. In the language we just introduced, we can say that the dynamical content of Y is included in the field which projects to Σ at the boundary. As we saw in (4.33), this contribution drops in GB gravity.

4.3.3 Dirichlet boundary conditions: a solution

A better way to set boundary conditions is to separate the contribution of the E_4 term from the start: $\mathscr{L}_{\rm hdg} = \tilde{\mathscr{L}}_{\rm hdg} + \bar{\mathscr{L}}_{\rm hdg}$, with $\tilde{\mathscr{L}}_{\rm hdg} = \alpha R^2 + \beta C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $\bar{\mathscr{L}}_{\rm hdg} = \gamma E_4$. Making this split changes (4.30) to

$$\int_{\mathcal{I}}\sqrt{-h}\Big((\dots)\delta h_{ab} + 2\tilde{\psi}^{ab}\delta K_{ab} + 2\bar{\psi}^{ab}\delta K_{ab}\Big), \tag{4.40}$$

where the tilded and barred fields are obtained from the tilded and barred Lagrangian respectively. Consistently with our previous observations, we notice that the last term is already a total variation, up to terms that depend solely on δh_{ab} and thus vanish by virtue of Dirichlet boundary conditions on the metric,

$$2\bar{\psi}^{ab}\delta K_{ab} = \gamma\delta B_3 + \mathcal{O}(\delta h_{ab}). \tag{4.41}$$

It thus makes sense to impose boundary conditions only on $\tilde{\psi}$, the *dynamical* part of Y

$$\delta h_{ab} = \delta\tilde{\psi}^{ab} = 0 \qquad \text{on } \mathcal{I}. \tag{4.42}$$

Therefore, to obtain a well-posed variational principle, we add the following, improved (w.r.t. (4.32)), boundary Lagrangian

$$\ell_{\text{hdg}} = -\Big(2\tilde{\psi}^{ab}K_{ab} + \gamma B_3\Big)\epsilon_{\mathcal{I}}, \tag{4.43}$$

whose γ-dependent piece in the boundary combines with the analogous term in the bulk to give the Euler characteristic of the manifold, according to the Gauss–Bonnet–Chern theorem [128]

$$\chi[\mathcal{M}] := \frac{1}{32\pi^2}\left(\int_{\mathcal{M}}\sqrt{-g}E_4 - \int_{\mathcal{I}}\sqrt{-h}B_3\right). \tag{4.44}$$

Going on-shell of the auxiliary field leads to the following boundary term in metric variables, which is a simple subcase of (4.22)

$$\ell_{\text{hdg}} = -(4\alpha RK + 8\beta U + \gamma B_3)\epsilon_{\mathcal{I}}. \tag{4.45}$$

4.4 A pinch of quantum

In this section, we make a first contact with the quantum world. Despite the rest of the thesis will be devoted to the construction of a phase space for certain (sectors of) field theories, which is the path that historically led towards quantization, this will be the only place where we explicitly touch upon (a small aspect of) *quantum field theories*. In particular, we are going to consider the possible consequences of **renormalization** in what we have been discussing so far.

Since quantum gravity remains elusive for the most part, the first steps we made to incorporate quantum effects in gravity are within the approximation of **semiclassical gravity**. The idea is to take into account the quantum nature of matter fields, while treating the gravitational field as a classical field. In a fully quantum picture, this approximation requires the existence of states for the gravitational field which have highly-peaked distributions for the observables that we care about. Despite the mental gymnastic required to fit these assumptions into a rigorous framework, semiclassical gravity still provides interesting results, among whom the prediction that black holes radiate at a finite temperature [24].

Assuming there are such states, one is led to solve the *semiclassical Einstein equations*

$$G_{\mu\nu} = \kappa^2 \langle T_{\mu\nu}[g] \rangle \,, \tag{4.46}$$

where $\langle T_{\mu\nu}[g] \rangle$ is the renormalized stress-energy tensor of matter in the background metric g, which is a complicated, non-local functional of g. Strictly speaking, these equations should be solved self-consistently, in such a way that the l.h.s. is the renormalized stress-energy tensor in the background that it itself generates.

In realistic computations, one is almost forced to use a perturbative approach, where the quantum fluctuations over a fixed background $\bar{g}$ are used as a source to determine the (hopefully small) deviations from it. In addition to be extremely challenging and time-consuming, this procedure is based on assumption that one can always work with *renormalized* quantities. As we will show, this is a non-trivial assumption.

4.4.1 Matter effective action

Let us start from what we expect to be the partition function of the full theory

$$Z = \int \mathfrak{D}[g]\mathfrak{D}[\phi] e^{i(S_0[g]+S_{\rm mat}[\phi;g])}. \tag{4.47}$$

and see what is the effect of integrating out matter fields.[8] If we formally perform the integration, we get by definition

$$Z = \int \mathfrak{D}[g] e^{i(S_0[g]+W_{\rm mat}[g])} \tag{4.48}$$

where $W_{\rm mat}[g]$ is the so-called **matter effective action**.

Such a quantity is generically divergent. However, as is typical in QFT, these divergences are local. So, upon regularization, they can be isolated ($W_{\rm mat}[g] = W^{\rm div}_{\rm mat}[g] + W^{\rm finite}_{\rm mat}[g]$) and reabsorbed into the coupling constants of the *bare* action $S_0[g]$, which are assumed to be divergent as well. What is left behind is a finite renormalized action $S_{\rm ren}[g] = S_0[g] + W^{\rm div}_{\rm mat}[g]$, that we can just think as the functional S_0 where each bare coupling has been replaced by its renormalized value. In addition to this, the exponent of eq. (4.48) also contain the finite piece $W^{\rm finite}_{\rm mat}[g]$, which is non-local and contain all possible interactions of the metric field that are mediated by matter loops

$$Z = \int \mathfrak{D}[g] e^{i(S_{\rm ren}[g]+W^{\rm finite}_{\rm mat}[g])}. \tag{4.49}$$

In the limit of large actions, the semiclassical Einstein equations arise from a saddle-point evaluation of the path integral (4.49). In fact, assuming that $S_{\rm ren}[g] = S_{GR}[g]$, then the stationarity condition translates into

$$\frac{\delta S_{\rm ren}[g]}{\delta g^{\mu\nu}} + \frac{\delta W^{\rm finite}_{\rm mat}[g]}{\delta g^{\mu\nu}} = 0 \implies \frac{1}{2\kappa^2} G_{\mu\nu} - \frac{1}{2} \langle T_{\mu\nu}[g] \rangle = 0. \tag{4.50}$$

[8] Gravitational perturbations can be treated as well as a matter field, with zero mass and spin 2.

In actual computations, one assumes that $g = g_{\text{ren}}$ is a solution of the renormalized equations of motion and then evaluates the renormalized stress-energy tensor as

$$\langle T_{\mu\nu}[g_{\text{ren}}]\rangle = -2\frac{\delta W_{\text{mat}}^{\text{finite}}[g]}{\delta g^{\mu\nu}}\bigg|_{g_{\text{ren}}} \tag{4.51}$$

using *quantum field theory in curved spacetime* techniques. The resulting tensor is then a $\mathcal{O}(\hbar)$ correction to the l.h.s. of (4.46). On the other hand, even neglecting these quantum corrections, the assumption that solutions to the renormalized equations of motion are also stationary points of the renormalized action functional, which is necessary for the approximation (4.50), is worth questioning. Indeed, throughout the chapter, we have argued that this is true only in the presence of suitable boundary conditions. Let us notice that even in a spacetime without boundaries, this might be an issue because sometimes we need to introduce boundaries by hand in order to apply the variational principle to subregions.

As explained in more detail in Appendix B, the divergent part of the matter effective action can be expressed, in four dimensions, as

$$W_{\text{mat}}^{\text{div}}[g] = -\frac{1}{16\pi^2}\sum_{k=0}^{3}\frac{\mathfrak{a}_k}{(4-k)\epsilon^{4-k}} - \frac{\mathfrak{a}_4}{16\pi^2}\ln\frac{\epsilon}{\mu}. \tag{4.52}$$

where the $\mathfrak{a}_k$ are known as **heat kernel coefficients**. Under reasonable conditions, these coefficients can be written as bulk and boundary integrals of local geometrical invariants: for even k, there is both a bulk and boundary contribution; for odd k, there is only a boundary contribution. What is relevant is that the invariants appear with relative weights that, as far as we know, are unrelated to the variational principle. For example, the weights of boundary terms depend on the boundary conditions of matter fields, whereas the weights of bulk terms are insensitive to them.

Therefore, it is not obvious that whatever relation holds between the operators in S_0 to make the variational principle well-posed will persist in S_{ren}. If not, it seems that the validity of the perturbative expansion around a saddle-point would be compromised. The best we can do is to answer the question case by case.

Remark One could take the point of view that only renormalized quantities are relevant, and that sensible questions can only refer to them. After all, physical observables are oblivious to the renormalization procedure that we use to predict them. Therefore, one might say that only the renormalized classical action, and not the bare one, needs to have a well-posed variational principle. However, we agree with the authors of [129] that renormalization is not merely the formal step of absorbing artificial divergences, but it also entails observable effects such as the flow of couplings with the energy scale [130]. The *beta functions* of this running can be extracted from the divergences, even though the physical meaning of these functions, e.g. for scattering processes, is vague at best [131–133].

4.4.2 Case study: Non-minimally coupled scalar field

As mentioned in Section 3.5, we can use the stress-energy tensor (4.6) to couple a scalar field to gravity. For consistency, we first need to generalize the stress-energy tensor to a generic background, as to obtain an object that is covariantly conserved. This procedure leads to the so-called *minimal coupling* between the fields. However, there are many examples in physics where it is useful to consider a *non-minimally coupled scalar field* [134–137]. For this reason, we now study the field theory

$$S[\phi; g] = S_0[g] + S_{\rm mat}[\phi; g], \tag{4.53}$$

where $S_0[g]$ is an action for gravity, and

$$S_{\rm mat}[\phi; g] = -\frac{1}{2}\int_{\mathcal{M}} \big(\nabla_\mu\phi\nabla^\mu\phi + \xi R\phi^2\big)\epsilon_{\mathcal{M}} + \int_{\mathcal{I}} \ell_{\rm mat}. \tag{4.54}$$

Repeating the arguments of Section 4.1 yields the modified equations of motion

$$2\frac{\delta S_0}{\delta g^{\mu\nu}} = T_{\mu\nu}, \qquad (\nabla^2 - \xi R)\phi = 0 \tag{4.55}$$

and modified boundary terms. In contrast to the minimally coupled case, the stress-energy tensor has correction terms that cannot be predicted by a flat spacetime expansion

$$T^\phi_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\phi\nabla^\sigma\phi + \xi G_{\mu\nu}\phi^2 + \xi(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2)\phi^2. \tag{4.56}$$

Let us assume that the metric field is required to satisfy Dirichlet boundary conditions and that these are implemented by S_0 alone. From $S_{\rm mat}$, we get

$$-\bigg(\delta\phi\mathcal{L}_n\phi + \frac{1}{2}\xi\phi^2 n_\mu \mathfrak{v}^\mu + \frac{1}{2}\xi(n^\mu g^{\lambda\nu} - n^\nu g^{\mu\lambda})\delta g_{\nu\lambda}\nabla_\mu\phi^2\bigg)\epsilon_{\mathcal{I}} + \delta\ell_{\rm mat}, \tag{4.57}$$

with $\mathfrak{v}$ given in (4.14). This expression can be written in terms of variations of the induced metric and the extrinsic curvature of $\mathcal{I}$, plus a total boundary derivative that we drop

$$\Big(p\delta\phi + \Pi^{ab}\delta h_{ab}\Big) + \delta(\xi K\phi^2\epsilon_{\mathcal{I}}) + \delta\ell_{\rm mat} \tag{4.58}$$

where p and Π^{ab} are the momenta conjugated to the boundary variables $i^*_{\mathcal{I}}\phi$ and $i^*_{\mathcal{I}}g$

$$p = -(\mathcal{L}_n + 2\xi K)\phi\,\epsilon_{\mathcal{I}}, \qquad \Pi^{ab} = -\frac{1}{2}\xi\Big(h^{ab}\mathcal{L}_n - K^{ab} + Kh^{ab}\Big)\phi^2\epsilon_{\mathcal{I}}. \tag{4.59}$$

Equation (4.58) suggests us to choose $\ell_{\rm mat}$ as a ϕ-dependent GHY term

$$\ell_{\rm mat} = -\xi K\phi^2\epsilon_{\mathcal{I}} \tag{4.60}$$

to put the boundary term in the form $\Pi\delta\Phi$. In this way, we can implement Dirichlet boundary conditions on the boundary variables or Neumann boundary conditions on the conjugated momenta.

Conformal couplings At the special value $\xi = 1/6 =: \xi_c$, called *conformal coupling*,[9] the bulk action $S_{\rm mat}$ becomes invariant under local Weyl scaling $g_{\mu\nu} \to W^2 g_{\mu\nu}$, $\phi \to W^{-1}\phi$. The boundary contribution vanishes as well if we choose $\ell_{\rm mat} = -\xi K \phi^2 \epsilon_{\mathcal{I}}$ as above.

Proof. We prove this by explicit computation for a small infinitesimal transformation $W \simeq 1 + w$ applied to (4.58). Using that $\delta_w h_{ab} = 2w h_{ab}$ and $\delta_w \phi = -w\phi$, we get

$$-(\mathcal{L}_n + 2\xi K)\phi\delta_w\phi - \frac{1}{2}\xi\Big(h^{ab}\mathcal{L}_n - K^{ab} + K h^{ab}\Big)\phi^2\delta_w h_{ab} = \frac{w}{2}(1-6\xi)\mathcal{L}_n\phi^2, \tag{4.61}$$

which clearly vanishes for $\xi = \xi_c = 1/6$. □

If we generically write the scalar field momentum as $p = -(\mathcal{L}_n + \zeta K)\phi\epsilon_{\mathcal{I}}$, with a new parameter ζ, we can check that the Neumann boundary condition $p = 0$ is Weyl invariant if and only if $\zeta = 2\xi_c = 1/3$ [139]. For this reason, we also refer to $\zeta_c = 1/3$ as *conformal coupling*.[10] Notice that the Dirichlet boundary condition $\phi = 0$ is trivially Weyl invariant.

Proof. We only need to know that the boundary normal and the extrinsic curvature of $\mathcal{I}$ transform under a Weyl scaling as (Ref. [140] contains a convenient list of identities)

$$\delta_w n_\mu = w n_\mu, \tag{4.63}$$
$$\delta_w K_{ab} = w K_{ab} + \mathcal{L}_n w \, h_{ab}. \tag{4.64}$$

From here, it is straightforward to compute

$$\begin{aligned}\delta_w p &= 3wp - (\delta_w(\mathcal{L}_n\phi) + \zeta\delta_w K\phi + \zeta K\delta_w\phi)\epsilon_{\mathcal{I}} = \\ &= 3wp + (1-3\zeta)\phi\mathcal{L}_n w\,\epsilon_{\mathcal{I}} + 2w(\mathcal{L}_n\phi + \zeta K\phi)\epsilon_{\mathcal{I}} = wp + (1-3)\phi\mathcal{L}_n w\,\epsilon_{\mathcal{I}}\end{aligned} \tag{4.65}$$

which is proportional to p itself if and only if $\zeta = 1/3$. □

4.4.3 Bulk renormalization / Boundary renormalization

We now study the effects of integrating out the non-minimally coupled scalar field ϕ, with the goal of determining whether, and under which conditions, the variational principle is preserved. In particular, we first review the renormalization of operators of dimensions ≤ 2 as performed in [129], and then move to the analysis of dimension 4 operators. As already pointed out by Solodukhin to the authors of [129] (see footnote 11 there), the latter

[9] It has been shown in [138] that any choice other than $\xi = \xi_c$ leads to a violation of Huygens's principle due to the timelike tails of the scalar field two-point function. This has been argued to conflict with the equivalence principle.

[10] For a generic dimension n, the conformal couplings are

$$\xi_c = \frac{n-2}{4(n-1)}, \qquad \zeta_c = \frac{n-2}{2(n-1)} = 2\xi_c. \tag{4.62}$$

analysis is necessary because these terms are necessarily generated by radiative corrections in four dimensions.

Our analysis will be carried out in the Euclidean sector of theory, as required to ensure a good behavior for the path integral.

Renormalization of dimension ≤ 2 operators

Despite what we said about the necessity of higher-derivative operators, let us assume for now that the bare couplings in front of them are tuned to cancel exactly the corresponding divergences from the matter effective action, so that we can focus on the renormalization of the cosmological constant and Newton's constant only.

Let us consider the Euclidean action

$$S_0[g] = -\frac{1}{2\kappa^2}\int_{\mathcal{M}}(R-2\Lambda)\epsilon_{\mathcal{M}} - \frac{1}{\varkappa^2}\int_{\mathcal{I}}(K-2\lambda)\epsilon_{\mathcal{I}}, \tag{4.66}$$

where we introduced two cosmological constant terms, one for the bulk and one for the boundary, and we also differentiated the overall bulk and boundary couplings. As discussed in (4.2), the crucial point for this action to be stationary on-shell of the equations of motion when the metric satisfies Dirichlet boundary conditions on $\mathcal{I}$ is that the coefficient in front of the bulk and the boundary terms are the same

$$\varkappa = \kappa. \tag{4.67}$$

In [106], we called this (an example of) *bulk-boundary matching*.

Let us now see what happens when the scalar field is integrated out. Following the discussion in Section 4.4.1, and in particular eq. (4.52), we see that both cosmological constants are needed: the one in the bulk absorbs the divergence multiplying the identity operator in the heat kernel coefficient $\mathfrak{a}_0$; the one in the boundary absorbs the coefficient of the identity in $\mathfrak{a}_1$. On the other hand, the *Newton constants* κ^2 and $\varkappa^2$ are renormalized by the same heat kernel coefficient $\mathfrak{a}_2$ in the following way [141, 142]

$$\frac{1}{2\kappa^2_{\text{ren}}} = \frac{1}{2\kappa^2} - \frac{1}{32\pi^2\epsilon^2}\left(\xi-\frac{1}{6}\right), \qquad \frac{1}{\varkappa^2_{\text{ren}}} = \frac{1}{\varkappa^2} - \frac{1}{16\pi^2\epsilon^2}\left(\zeta-\frac{1}{6}\right) \tag{4.68}$$

where we set $\zeta = 0$ if the scalar field satisfies Dirichlet boundary conditions.[11]

What does this mean?

[11] The renormalization of Newton's constant is quite commonly accepted as the resolution of the so-called *species problem*: the black hole entropy is independent on the types of quantum fields because there is a precise cancellation between the entanglement entropy divergences and the divergences in gravitational couplings. [143–150]

- If the scalar field is minimally coupled ($\xi = 0$), we see that the renormalization of Newton constants does not spoil the matching (4.67) as long as the scalar field satisfies Dirichlet or pure Neumann boundary conditions ($\zeta = 0$ for both)

$$\frac{1}{\kappa^2_{\rm ren}} = \frac{1}{\kappa^2} + \frac{1}{96\pi^2\epsilon^2} \overset{\kappa=\varkappa}{=} \frac{1}{\varkappa^2} + \frac{1}{96\pi^2\epsilon^2} = \frac{1}{\varkappa^2_{\rm ren}}; \tag{4.69}$$

- If the scalar field is coupled non-minimally to the Ricci curvature, the matching is generally not preserved,[12] [13] unless we impose Robin boundary conditions with $\zeta = \xi$. In this way, the quantum corrections to κ^2 and $\varkappa^2$ are precisely the same. This was proposed in [129] (where ζ is written as $-\xi/\alpha$) as a way to obtain a consistent matching.

A natural question is whether we can modify the natural Neumann condition for the scalar field $p = 0$, with p given in (4.59), into a Robin boundary condition with generic coupling ζ

$$\mathcal{L}_n\phi + 2\xi K\phi = 0 \overset{?}{\to} \mathcal{L}_n\phi + \zeta K\phi = 0. \tag{4.70}$$

by adding a suitable boundary Lagrangian. By doing so, we could allow $\zeta = \xi$ as suggested in [129], and thereby achieve a consistent bulk-boundary matching. However, if we want to rotate p into $p+(2\xi-\zeta)K\phi\epsilon_{\mathcal{I}}$, we need a mixing angle ν that depends on K, and K is not a phase space constant *a priori*. If we did that anyway, the required Lagrangian (4.11) would produce an additional, unmatched, term in δK, when we consider its variation with respect to the metric, thus spoiling the full variational principle. For the sake of completeness, we cannot exclude at this stage that there exist viable solutions if one changes the metric boundary conditions as well: for example, the appearance of a δK term suggests looking into *York boundary conditions* [112].

Renormalization of dimension 4 operators

We now turn to higher-derivative operators. As discussed in Section 4.4.1, the coefficients of dimension 4 operators obtain logarithmic corrections. The precise structure of these divergences is dictated by the heat kernel coefficient $\mathfrak{a}_4$, which in turn depends on whether

[12] In [142], it was claimed that there is no problem in having independent Newton's constants in the bulk and in the boundary because they enter different observables. This conclusion was shown to be unreliable and amended in [151].

[13] One could envision some sort of ξ-dependent boundary conditions for the metric that would imply a matching of the form $\varkappa^2 = (1-6\xi)\kappa^2$, that is preserved by Dirichlet boundary conditions for ϕ, but it is not clear to us what guiding principle would lead to this scenario.

the field satisfies Dirichlet (D) or Robin (R) boundary conditions

$$\begin{aligned}\mathfrak{a}_4^D &= \int_{\mathcal{M}} \left(\frac{1}{120} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \epsilon_{\mathcal{M}} + \frac{1}{2}(\xi - \xi_c)^2 R^2\right) - \frac{4\pi^2}{45}\chi[\mathcal{M}] + \\ &+ \int_{\mathcal{I}} \left(\frac{2}{35}\mathrm{m}_3 - \frac{1}{15}U - (\xi - \xi_c)\left(\frac{1}{3}RK + \frac{1}{2}\nabla_n R\right)\right)\epsilon_{\mathcal{I}}, \end{aligned} \tag{4.71a}$$

$$\begin{aligned}\mathfrak{a}_4^R &= \int_{\mathcal{M}} \left(\frac{1}{120} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{2}(\xi - \xi_c)^2 R^2\right)\epsilon_{\mathcal{M}} - \frac{4\pi^2}{45}\chi[\mathcal{M}] + \\ &+ \int_{\mathcal{I}} \left(\frac{2}{45}\mathrm{m}_3 - \frac{1}{15}U + (\xi - \xi_c)\left((2\zeta - \zeta_c)RK + \frac{1}{2}\nabla_n R\right)\right)\epsilon_{\mathcal{I}} \\ &- (\zeta - \zeta_c)\int_{\mathcal{I}} \left(\frac{2}{15}K\mathrm{m}_2 + \frac{1}{3}(2\zeta - \zeta_c)^2 K^3\right)\epsilon_{\mathcal{I}}. \end{aligned} \tag{4.71b}$$

In Section 4.3.1, we learned that imposing Dirichlet boundary conditions for the pull-back of both g and $\tilde{Y}_{\perp\perp}$ (we remind the reader that $\tilde{Y}$ excludes the topological contribution from Y, as discussed in 4.3.3) corresponds to the following bulk-boundary matching condition

$$a_1 = a_2 = a_3 = 0, \qquad b_1 = -\gamma,\ b_2 = 0, \qquad c_1 = -8\beta,\ c_2 = -4\alpha,\ c_3 = 0. \tag{4.72}$$

Considering the heat kernel coefficients, we find that bulk and boundary couplings are renormalized as reported in Tables 4.1 and 4.2.

$$\begin{aligned}\alpha_{\text{ren}} &= \alpha - \frac{(\xi - \xi_c)^2}{2}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}, \\ \beta_{\text{ren}} &= \beta - \frac{1}{120}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}, \\ \gamma_{\text{ren}} &= \gamma + \frac{1}{360}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}. \end{aligned}$$

Table 4.1: Bulk coupling renormalization

What we can see is that most coefficients do not renormalize if the scalar field is conformally coupled to the curvature ($\xi = \xi_c$) and its boundary condition is Weyl invariant (either Dirichlet or Robin with $\zeta = \zeta_c$). This is because the heat kernel coefficient $\mathfrak{a}_4$ — or $\mathfrak{a}_n$ in n dimensions — of a conformally invariant operator is conformally invariant itself [152]. Therefore, only the coefficients of conformally invariant operators can get corrections: namely β, γ, a_1, b_1, c_1. Under these conditions, the matching involving $\alpha, a_2, a_3, c_2, c_3$ are trivially preserved, while it can be explicitly checked that the matchings of b_1 with γ and of c_1 with β are preserved as well. The fly in the ointment is the coefficient

Dirichlet boundary condition	Robin boundary condition
$a_{1,\text{ren}} = a_1 - \frac{2}{35}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$	$a_{1,\text{ren}} = a_1 - \frac{2}{45}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$a_{2,\text{ren}} = a_2$	$a_{2,\text{ren}} = a_2 + \frac{2(\eta-\eta_c)}{15}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$a_{3,\text{ren}} = a_3$	$a_{3,\text{ren}} = a_3 + \frac{4(\eta-\eta_c)^3}{3}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$b_{1,\text{ren}} = b_1 - \frac{1}{360}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$	$b_{1,\text{ren}} = b_1 - \frac{1}{360}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$b_{2,\text{ren}} = b_2$	$b_{2,\text{ren}} = b_2 - \frac{2(\eta-\eta_c)}{45}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$c_{1,\text{ren}} = c_1 + \frac{1}{15}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$	$c_{1,\text{ren}} = c_1 + \frac{1}{15}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$c_{2,\text{ren}} = c_2 - \frac{\xi-\xi_c}{3}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$	$c_{2,\text{ren}} = c_2 - (\xi-\xi_c)(2\zeta-\zeta_c)\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$
$c_{3,\text{ren}} = c_3 + \frac{\xi-\xi_c}{2}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$	$c_{3,\text{ren}} = c_3 - \frac{\xi-\xi_c}{2}\frac{1}{16\pi^2}\ln\frac{\epsilon}{\mu}$

Table 4.2: Boundary coupling renormalization

a_1 multiplying m_3 in the boundary action (4.22), which necessarily acquires a non-zero contribution, breaking condition (4.72).

Even though we cannot exclude that one could achieve a matching by imposing different Robin boundary conditions, it is clear that the minimal and $\zeta = \xi$ cases are still unfeasible as they spoil even more matching conditions than the conformal one. On a similar note, one could think that the addition of fields which are not conformally invariant [14] and/or satisfy alternative boundary conditions (with respect to Dirichlet or Neumann) might help

[14] The $\mathfrak{a}_4$ heat kernel coefficient for a generic set of conformally invariant matter fields (conformally coupled massless scalars, Weyl fermions, Maxwell fields) can be written as [153]

$$\mathfrak{a}_4 = \frac{4\pi^2\, t}{45}\chi[\mathcal{M}] + \frac{q}{120}\int_{\mathcal{M}} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\epsilon_{\mathcal{M}} - \int_{\mathcal{I}}\left(\frac{q_1}{15}U - \frac{2q_2}{35}\mathrm{m}_3\right)\epsilon_{\mathcal{I}},$$

with specific values of the charges t, q, q_1, q_2. This form preserves the matching between $b_1 = -\gamma$ and, for $q = q_1$ — which is the case for all relevant matter fields — also $c_1 = -8\beta$. However, since $q_2 > 0$ is positive for all fields ($q_2 = 1$ for Dirichlet scalar fields, $q_2 = 7/9$ for conformal Robin scalar fields, $q_2 = 5$ for Weyl fermions, $q_2 = 8$ for Maxwell fields) means that there is no way to combine them to give a net vanishing loop contribution. This seems to imply that the m_3 term is universal and, as such, its eventual removal should be based on very general grounds.

finding a consistent matching for a_1 without spoiling the rest. This possibility appears to us quite remote, and the task to check it is made more challenging by the lack of results on the $\mathfrak{a}_4$ coefficient for fields with boundary conditions which are neither Dirichlet nor Neumann [154].

The m_3 term It is clear that the reason for our shortcoming is that the GHY term in (4.45) does not contain the m_3 operator (see (4.19)). Even if the meaning of this object in the variational principle and the physical reasons why it should be expected to appear in the boundary action are not clear yet, we can offer a reason why this term might not be so problematic.

Obviously, m_3 trivially vanishes if the extrinsic curvature tensor is pure trace, i.e. $K_{ab} = \frac{1}{3}Kh_{ab}$. Under this assumption, the only possible contribution of the extrinsic curvature to the W_{mat} reduces to K^3, which, being conformally variant, cannot be present if we restrict to conformally invariant matter fields. The condition $K_{ab} = \frac{1}{3}Kh_{ab}$ has been frequently used in the renormalization of pure gravity, either perturbative or within the exact functional renormalization group [129, 151]. It also plays a role in the formulation of the boundary value problem for gravitons: it has been first identified in [155] as a way to get simple BRST-invariant local boundary conditions (see also [156, 157]), and it allows to preserve the strong ellipticity of the boundary value problem without introducing tangential derivatives in the boundary conditions [110, 158]. Since gravitons can be effectively treated as matter propagating on a fixed background, it might be reasonable to constrain the extrinsic curvature of the boundary in such a way.

Renormalization of Newton's constant The idea of setting $\xi = \xi_c$ and $\zeta = \zeta_c$, which make most of the matching condition (4.72) survive renormalization, contrasts with the condition $\zeta = \xi$ that is needed to ensure a consistent matching of Newton's constants[129]. Whereas the entire question is far from solved, we believe there are at least two reasons to not be too concerned about that. First, the renormalization of Newton's constant is affected by regularization and gauge dependencies, making its physical meaning difficult to establish [132, 159]. Second, unlike logarithmic divergences, power-law divergences are not "robust": had we used a regularization procedure that does not introduce a mass scale, Newton's constant would have not been renormalized in the case of Weyl invariant matter fields.

4.5 Summary

In this chapter, we defined the notion of well-posed variational principle and we showed how boundary conditions are used to enforce that. We then showed, through some examples, that these boundary conditions are closely related to the choice of a boundary Lagrangian, giving rise to a form of *bulk-boundary matching* between the coefficients in

the bulk Lagrangian and those in the boundary Lagrangian. In particular, we derived the form of this matching for higher-derivative gravity — already implicit in the Hamiltonian description of [121] — in a fully covariant way.

Later, we analyzed whether the well-posed of the variational principle for gravity is stable under renormalization. Our analysis complements that of [129]. Our findings reveal that a consistent bulk-boundary matching of Newton's constant is more subtle than expected. Furthermore, they show that, in order to have the same consistency at the level of quartic-derivatives operators, it might be necessary to restrict to specific boundary embeddings for which the extrinsic curvature is pure trace.

Chapter 5

Covariant phase space with boundaries

In the previous chapter, we saw that boundaries play a relevant role in field theories. What happens at the boundary determines the boundary conditions for the fields, which in turn determine the allowed field configurations. Here we will see that the presence of boundaries in field theory is also deeply connected with the notion of symmetries.

In this chapter, we introduce the extension of the CPS formalism to spacetimes with boundaries, which will provide the perfect framework to elaborate both of these observations. A rigorous treatment of the CPS in such spacetimes would require the introduction of the *relative* variational bicomplex [63]. Nonetheless, we refrain from such a technical discussion here and rather focus on physical applications, highlighting the interesting results that can be derived from the formalism without developing it in full detail.

5.1 Harlow–Wu formalism

Let us start by considering a local field theory described by the action

$$S = \int_{\mathcal{M}} L + \int_{\mathcal{I}} \ell, \tag{5.1}$$

where $L \in \Omega^{(n,0)}(\mathbb{F})$ and $\ell \in \Omega^{(n-1,0)}(\mathbb{F})$. We can also see ℓ as a horizontal top form in $\mathcal{I} \times \mathscr{F}$. Since what ultimately matters is the action, which is the sum of the integrals of these forms, the formalism is inherently redundant under

$$L \to L + \mathrm{d}w, \qquad \ell \to \ell - w, \tag{5.2}$$

where w is a $(n-1,0)$-form. One might think that there is no need to introduce a boundary Lagrangian at all, as this could be simply eliminated by choosing $i_{\mathcal{I}}^* w = \ell$. It is usually

the case though that ℓ does not admit a covariant extension in the bulk, so that we cannot write $\mathrm{d}w$ in the bulk without breaking covariance. For this reason, we do not allow these redefinitions and take the pair (L, ℓ) as given.

If we take a vertical derivative of the action and use the CPS identity (3.21) in the bulk, we get the following expression

$$\mathbb{d}S = \int_{\mathcal{M}} (E\mathbb{d}\phi + \mathrm{d}\theta) + \int_{\mathcal{I}} \mathbb{d}\ell. \tag{5.3}$$

We notice again that ℓ does not contribute to the equations of motion, but it is fundamental to enforce the well-posedness of the variational principle (cfr. the beginning of Chapter 4), which we repeat here: on-shell, $\mathbb{d}S$ must reduce to terms that are integrated only on the lids $\Sigma_{\text{in}} \cup \Sigma_{\text{fin}}$. Accordingly, we choose the boundary Lagrangian in such a way that

$$\mathbb{d}\ell + i_{\mathcal{I}}^{*}\theta = B + \mathrm{d}\vartheta, \tag{5.4}$$

where ϑ is called *boundary (pre-)symplectic potential*, and the $(n-1,1)$-form B is required to vanish when fields satisfy the boundary conditions. The reason for a total derivative $\mathrm{d}\vartheta$ on the r.h.s. is that such a term, even if present, would contribute only at the corner $\partial\mathcal{I}$, which is the same set as $\partial\Sigma_{\text{in}} \cup \partial\Sigma_{\text{fin}}$. As part of the lids, it is thereby allowed by the variation principle. This term was already introduced in [160–162], but it was Harlow & Wu who first studied its covariance properties and relevance in the construction of charges.

Despite vanishing in the space we are interested in, the B term contains a lot of information about the theory. If we interpret the boundary conditions as background structure, then varying the boundary conditions provides a current-source pair as in (3.53). For this reason, we are going to generically write $B = \not{J}_{\phi}\, \mathbb{d}\bar{\phi}$, with the implied meaning that $\delta\bar{\phi} = 0$ corresponds to Dirichlet boundary conditions and $\not{J}_{\phi} = 0$ to Neumann boundary conditions. In the following, we restrict to Dirichlet boundary conditions for simplicity, with the awareness that we can always change the polarization of B with a suitable boundary Lagrangian.

We thus restrict the space of field configurations to those satisfying Dirichlet boundary conditions on $\mathcal{I}$

$$\mathscr{F} \to \bar{\mathscr{F}} := \{\phi \in \mathscr{F} \,|\, \delta\bar{\phi} = 0\}. \tag{5.5}$$

Consequently, the solution space is restricted to

$$\bar{\mathcal{S}} = \{\phi \in \bar{\mathscr{F}} | E = 0\} = \mathcal{S} \cap \bar{\mathscr{F}}. \tag{5.6}$$

We now refer to equalities holding in $\bar{\mathcal{S}}$ as being on-shell and use for them the symbol $\hat{=}$, reserving $\doteq$ for equalities that only require the fields to satisfy the equations of motion (without imposing boundary conditions).

5.1.1 Total symplectic form

Using (5.4) and considering that the boundary of $\mathcal{M}$ comprises three parts, we get

$$\mathbb{d}S = \int_{\mathcal{M}} E\mathbb{d}\phi + \int_{\mathcal{I}} \mathscr{J}_\phi \mathbb{d}\bar{\phi} + \int_{\Sigma_{\text{fin}}} (\theta - \mathrm{d}\vartheta) - \int_{\Sigma_{\text{in}}} (\theta - \mathrm{d}\vartheta) \tag{5.7}$$

which, on-shell, reduces to

$$\mathbb{d}S \mathrel{\hat{=}} \int_{\Sigma_{\text{fin}}} (\theta - \mathrm{d}\vartheta) - \int_{\Sigma_{\text{in}}} (\theta - \mathrm{d}\vartheta). \tag{5.8}$$

The change in sign in $\mathrm{d}\vartheta$ with respect to (5.4) to account for the opposite orientations of the corners within the two boundary pieces, as in Figure 5.1.[1]

This construction singles out a new 1-form in field space that acts as symplectic potential, which in turn gives us the (pre-)symplectic form current

$$\omega = \mathbb{d}\theta - \mathbb{d}\mathrm{d}\vartheta. \tag{5.9}$$

This form is again always $\mathbb{d}$-closed and d-closed on-shell. Another property is that $i_{\mathcal{I}}^*\omega = 0$ when the fields satisfy the boundary conditions (also off-shell), which ensures that the system is *closed*. Finally, the (pre-)symplectic form on $\bar{\mathcal{S}}$ is obtained by the pull-back of

$$\Omega_\Sigma = \int_\Sigma \omega. \tag{5.10}$$

Thanks to the properties of ω, this form does not depend on-shell on the choice of Σ even if the boundary $\partial\Sigma$ changes (while staying on $\mathcal{I}$), because the symplectic flux (3.35) vanishes.

Remark Notice that the condition (5.4) allows us to improve on the CPS ambiguities as we can compensate a shift $\theta \to \theta + \mathrm{d}Y$ by a simultaneous shift $\vartheta \to \vartheta + Y$ in such a way that the new total symplectic potential $\theta - \mathrm{d}\vartheta$ is invariant.

5.1.2 Symmetries and Noether's first theorem

When considering symmetries of a system with spacetime boundaries, we have to take into account the presence of the boundary itself. Since the configuration space of the field theory is defined in terms of $\bar{\mathscr{F}}$, we require that a true symmetry not only preserves the solution space, but also the boundary conditions: if $\mathbb{X} \in \mathfrak{X}(\mathbb{F})$ is a symmetry, it must be

$$\mathbb{L}_{\mathbb{X}} B \mathrel{\hat{=}} 0, \tag{5.11}$$

which, for Dirichlet boundary conditions, this condition is equivalent to $\delta_{\mathbb{X}}\bar{\phi} = 0$.

We now revise the results of Section 3.4.5 for Lagrangian symmetries in the new CPS framework for spacetimes with boundaries. It is useful to distinguish two classes of symmetry transformations.

[1] In writing $\mathrm{d}\vartheta$, we assumed that we have extended the boundary symplectic potential into the bulk. The actual extension is irrelevant since only the value of ϑ on $\partial\Sigma \subset \mathcal{I}$ actually contributes.

Internal symmetries

These transformations, which can be global or local, do not act on spacetime, but on some internal degrees of freedom of the fields. In this case, the situation is not so different from the case without boundaries 3.4.5. If $\mathbb{X} \in \mathfrak{X}(\mathbb{F})$ is such that

$$\mathbb{L}_{\mathbb{X}} L = \mathrm{d}\Lambda_{\mathbb{X}}, \tag{5.12}$$

for some $\Lambda_{\mathbb{X}} \in \Omega^{(n-1,0)}(\mathbb{F})$, then the Noether current $J_{\mathbb{X}} = \mathbb{I}_{\mathbb{X}}\theta - \Lambda_{\mathbb{X}}$ we introduced in (3.39) is closed on-shell. This statement is not affected by boundary conditions. It follows that the associated charge does not vary under arbitrary variations in the bulk of the slice we choose to integrate it over. However, thanks to the vanishing symplectic flux, we are now free to deform the slice at the boundary as well, and we want to construct a charge that reflects this freedom. Therefore, we also require

$$\mathbb{L}_{\mathbb{X}} \ell = \mathrm{d}\lambda_{\mathbb{X}} - \Lambda_{\mathbb{X}}, \tag{5.13}$$

for some $\lambda_{\mathbb{X}} \in \Omega^{(n-2,0)}(\mathbb{F})$. This allows us to introduce the *boundary Noether current*

$$j_{\mathbb{X}} := \mathbb{I}_{\mathbb{X}}\vartheta - \lambda_{\mathbb{X}}, \tag{5.14}$$

which mirrors the form of its bulk counterpart. This term compensates the variation in the bulk Noether charge, in the sense that the *total Noether charge*

$$Q_{\mathbb{X},\Sigma} := \int_{\Sigma} J_{\mathbb{X}} - \int_{\partial\Sigma} j_{\mathbb{X}} \tag{5.15}$$

is on-shell independent on the choice of Σ, as long as it ends on the lateral boundary $\mathcal{I}$. Note that we need both $E = 0$ and $B = 0$ for the conservation of this charge, demonstrating the relevance of boundary conditions and their connection to symmetries.

Proof. Let us first evaluate the spacetime/horizontal derivative of $j_{\mathbb{X}}$, using (5.4) and (5.13)

$$\mathrm{d}j_{\mathbb{X}} = \mathbb{I}_{\mathbb{X}}(\mathbb{d}\ell + i_{\mathcal{I}}^*\theta - B) - \mathrm{d}\lambda_{\mathbb{X}} = i_{\mathcal{I}}^* J_{\mathbb{X}} - \mathbb{I}_{\mathbb{X}} B. \tag{5.16}$$

To prove that $Q_{\mathbb{X},\Sigma_2} = Q_{\mathbb{X},\Sigma_1}$ for any choice of Σ_1 and Σ_2 with $\partial\Sigma_{1,2} \subset \mathcal{I}$, we consider their difference. Using the inverse Stokes' theorem twice and accounting for the opposite orientation of the corners, we get

$$\begin{aligned} Q_{\mathbb{X},\Sigma_2} - Q_{\mathbb{X},\Sigma_1} &= \int_{\Sigma_2} J_{\mathbb{X}} - \int_{\partial\Sigma_2} j_{\mathbb{X}} - \int_{\Sigma_1} J_{\mathbb{X}} + \int_{\partial\Sigma_1} j_{\mathbb{X}} = \\ &= \int_{\mathcal{M}_{12}} \mathrm{d}J_{\mathbb{X}} - \int_{\mathcal{I}_{12}} (i_{\mathcal{I}}^* J_{\mathbb{X}} - \mathrm{d}j_{\mathbb{X}}) = \\ &= -\int_{\mathcal{M}_{12}} \mathbb{I}_{\mathbb{X}}(E\mathbb{d}\phi) + \int_{\mathcal{I}_{12}} \mathbb{I}_{\mathbb{X}} B, \end{aligned} \tag{5.17}$$

where we wrote again $\mathcal{M}_{12}$ and $\mathcal{I}_{12}$ for the portions of $\mathcal{M}$ and $\mathcal{I}$ bounded by the two slices. This expression clearly vanishes on-shell. □

To show that the Noether charge is also the Hamiltonian function of the transformation $\mathbb{X}$ for the total symplectic form, we first need to use that, on-shell,

$$\mathbb{L}_{\mathbb{X}}\theta \mathrel{\hat{=}} \mathbb{d}\Lambda_{\mathbb{X}} + \mathrm{d}\mathbb{S}_{\mathbb{X}}, \tag{5.18a}$$

$$\mathbb{L}_{\mathbb{X}}\vartheta \mathrel{\hat{=}} \mathbb{d}\lambda_{\mathbb{X}} + i^*_{\mathcal{I}}\mathbb{S}_{\mathbb{X}} + \mathrm{d}\mathbb{s}_{\mathbb{X}} \tag{5.18b}$$

where $\mathbb{S}_{\mathbb{X}} \in \Omega^{(n-2,1)}(\mathbb{F})$ and $\mathbb{s}_{\mathbb{X}} \in \Omega^{(n-3,1)}(\mathbb{F})$ are two undetermined forms. We proved the first of these relations to arrive at eq. (3.45) in Section 3.4, and the second can be proven analogously. From these, we can deduce the relations

$$\mathbb{d}J_{\mathbb{X}} \mathrel{\hat{=}} -\mathbb{I}_{\mathbb{X}}\mathbb{d}\theta + \mathrm{d}\mathbb{S}_{\mathbb{X}}, \tag{5.19a}$$

$$\mathbb{d}j_{\mathbb{X}} \mathrel{\hat{=}} -\mathbb{I}_{\mathbb{X}}\mathbb{d}\vartheta + i^*_{\mathcal{I}}\mathbb{S}_{\mathbb{X}} + \mathrm{d}\mathbb{s}_{\mathbb{X}}, \tag{5.19b}$$

which combine to give

$$\mathbb{d}(J_{\mathbb{X}} - \mathrm{d}j_{\mathbb{X}}) \mathrel{\hat{=}} -\mathbb{I}_{\mathbb{X}}\omega. \tag{5.20}$$

Integrating this over a Cauchy slice proves our statement.

Spacetime symmetries

Spacetime transformations are those that act on spacetime as well as on the fields. The presence of boundaries complicates their analysis because some of them may not even preserve the location of the boundary. Let us then consider a generic diffeomorphism $\xi \in \mathfrak{X}(\mathcal{M})$ and the associated vector field $\mathbb{X}_\xi$

$$\mathbb{X}_\xi = \int_{\mathcal{M}} \mathcal{L}_\xi \phi(x) \frac{\mathbb{d}}{\mathbb{d}\phi(x)}. \tag{5.21}$$

These vector fields realize a Lie algebra anti-homomorphism from $\mathfrak{X}(\mathcal{M})$

$$[\![\mathbb{X}_\xi, \mathbb{X}_{\xi'}]\!]_{\mathrm{Lie}} = -\mathbb{X}_{[\xi,\xi']_{\mathrm{Lie}}}. \tag{5.22}$$

As we saw in (3.65), covariance of the bulk Lagrangian is enough for the action variation to be a boundary term. For consistency with the new CPS formulation, we need the stronger condition that such variation is localized on the lids. In other words, it must be a total boundary derivative

$$\iota_\xi L + \mathbb{L}_{\mathbb{X}_\xi}\ell = \mathrm{d}(\ldots). \tag{5.23}$$

This is the case if the vector field becomes tangent to the boundary as we approach it, and if the boundary Lagrangian is covariant under this tangential vector field. We thus require

$$\xi|_{\mathcal{I}} = \bar{\xi} \in \mathfrak{X}(\mathcal{I}), \qquad \mathbb{L}_{\mathbb{X}_\xi}\ell = \mathcal{L}_{\bar{\xi}}\ell. \tag{5.24}$$

We learned that background objects may appear at the boundary in a generally covariant theory, but not all background objects can appear in ℓ without breaking *boundary covariance* (the second condition in (5.24)). A nice description of what terms are allowed follows

from the *covariant hypersurface formalism* [108, 163]. If we see the boundary as the zero level set of a background scalar function $\mathscr{R}$, namely $\mathcal{I} \equiv \{x \in \mathcal{M} | \mathscr{R}(x) = 0\}$, then the allowed diffeomorphisms are those that locally preserve the foliation by level sets of $\mathscr{R}$. From this function, we can define quantities that are covariant under *boundary-preserving diffeomorphisms*.

Under the assumptions that ξ is boundary-preserving and the boundary Lagrangian is covariant with respect to it (5.24), the condition that the action variation is an integral over the lids (5.23) is equivalent to ξ being a symmetry of the boundary Lagrangian (5.13), with

$$\Lambda_{\mathbb{X}_\xi} = \iota_\xi L, \qquad \lambda_{\mathbb{X}_\xi} = \iota_{\bar{\xi}} \ell, \tag{5.25}$$

so we can construct the Noether currents (we shortcut $J_\xi \equiv J_{\mathbb{X}_\xi}$, $j_\xi \equiv j_{\mathbb{X}_\xi}$)

$$J_\xi = \mathbb{I}_{\mathbb{X}_\xi} \theta - \iota_\xi L, \tag{5.26a}$$

$$j_\xi = \mathbb{I}_{\mathbb{X}_\xi} \vartheta - \iota_{\bar{\xi}} \ell, \tag{5.26b}$$

and the total Noether charge ($Q_{\xi,\Sigma} \equiv Q_{\mathbb{X}_\xi,\Sigma}$)

$$Q_{\xi,\Sigma} = \int_\Sigma J_\xi - \int_{\partial\Sigma} j_\xi. \tag{5.27}$$

We can check that conditions (5.18) are satisfied if both θ and ϑ are covariant under the boundary-preserving diffeomorphisms, with $S_{\mathbb{X}_\xi} = \iota_\xi \theta$ and $s_{\mathbb{X}_\xi} = \iota_{\bar{\xi}} \vartheta$. It thus follows that

$$\mathbb{d}(J_\xi - \mathrm{d} j_\xi) \hat{=} -\mathbb{I}_{\mathbb{X}_\xi} \omega, \tag{5.28}$$

and

$$\boxed{\mathbb{d} Q_\xi \hat{=} -\mathbb{I}_{\mathbb{X}_\xi} \Omega.} \tag{5.29}$$

From here, we deduce the representation theorem, which states

$$\{Q_\xi, Q_{\xi'}\} \hat{=} -Q_{[\xi,\xi']} + K(\xi, \xi'), \tag{5.30}$$

where $K(\xi, \xi')$ is a field-independent ($\mathbb{d} K = 0$) 2-cocycle.

This result, even if very important, will prove to be too restrictive for our needs. In fact, eq. (5.28) holds as long as we stay within $\bar{\mathcal{S}}$, which means that we can only consider perturbations that are tangent to the zero level set of both the equations of motion ($E = 0$) and the boundary conditions ($B = 0$). It is straightforward to generalize this to variations which are tangent to $E = 0$ but not to $B = 0$. In this case, the B term in j_ξ contributes a boundary term to (5.28)

$$\mathbb{d}(J_\xi - \mathrm{d} j_\xi) \doteq -\mathbb{I}_{\mathbb{X}_\xi} \omega + \mathrm{d}\big(\iota_{\bar{\xi}} B\big)\,. \tag{5.31}$$

Such a perturbation does not exist within a single configuration space, but, in the case of Dirichlet boundary conditions, it moves from a certain $\bar{\mathscr{F}}$, defined by a boundary value for ϕ, to a different $\bar{\mathscr{F}}$, defined by a slightly different value.

5.2 Gauge theories, gravity and corner charges

In our description so far, we have strongly relied on two pillars of field theory: locality and Lorentz invariance. However, it is well-known that these two necessarily lead to gauge redundancies when we want to describe massless fields with spin $s \geq 1$. Simply put, massless fields have two physical degrees of freedom (corresponding to the two helicity states), but there is no Lorentz tensor that can encode those and only those, given that the smallest representation of the Lorentz group — not of its universal cover — is four-dimensional. Conversely, the price to pay when working with fully gauge invariant objects is that some of them — or all of them, as in the case of gravity! [164] — will be non-local (at least with respect to the *kinematical* notion of locality, see e.g. [165] for an alternative, *dynamical*, notion).

A **gauge field theory** with *charge group* G is a field theory which is invariant under the action of the *gauge group* $\mathcal{G}$, where $\mathcal{G} \equiv \Gamma(\mathcal{M}, G)$ is the set of G-valued functions of spacetime [62]. The modern formulation of gauge theories is in terms of a principal bundle, where G plays the role of the *structure group* [166]. A field theory of **gravity** is a field theory which is diffeomorphism-invariant. Although gravity bears many resemblances with Yang–Mills theories, its formulation in terms of principal bundles (natural candidates for the structure group are the Lorentz group $\mathrm{O}(1, n-1)$, its cover= $\mathrm{Spin}(1, n-1)$, or the Poincaré group [167]) does not naturally capture invariance under diffeomorphisms, because these do not act fiber-wise, but move points in the base manifold as well [168]. Recent geometric realizations of gravitational theories involve the so-called Atiyah-Lie algebroids [169]. Nevertheless, we can say that gravity is invariant under the gauge group $\mathcal{G} = \mathrm{Diff}(\mathcal{M})$.

In both cases, the gauge group $\mathcal{G}$ is an infinite-dimensional Lie group, whose Lie algebra is the set of $\mathfrak{g}$-valued maps $\mathrm{Lie}(\mathcal{G}) = \Gamma(\mathcal{M}, \mathfrak{g})$. The action of $\mathcal{G}$ on field space, usually called a *gauge transformation* (in this thesis, we reserve this terminology until the verdict of Noether's theorems has been established), define the following vector field

$$\mathbb{X}_\alpha = \int_{\mathcal{M}} \mathrm{d}^n x \, \delta_\alpha \phi(\alpha) \frac{\mathbb{d}}{\mathbb{d}\phi(\alpha)}, \tag{5.32}$$

where $\phi \to \phi + \delta_\alpha \phi$, $\alpha \in \mathrm{Lie}(\mathcal{G})$ is the infinitesimal version of the gauge transformation. These vector fields form a representation of the Lie algebra. The statement of invariance under $\mathcal{G}$ means that each $\mathbb{X}_\alpha$ corresponds to a Lagrangian symmetry. Given that the transformations are clearly local, the combination of Noether's theorems implies that the corresponding bulk Noether current is $J_\alpha = c_\alpha + \mathrm{d}q_\alpha$, with $\mathrm{d}c_\alpha = -\mathbb{I}_{\mathbb{X}_\alpha}(E\mathbb{d}\phi)$. As we learned in the previous sections, the physics of local symmetries becomes remarkably more interesting in the presence of boundaries: transformations which would naïvely be regarded as gauge can provide non-trivial charges, thus becoming true symmetries. In this case, the on-shell Noether charge is localized at the boundary of Σ and is thereby referred to as a

corner charge

$$Q_\alpha \hat{=} \int_{\partial\Sigma} (q_\alpha - j_\alpha). \tag{5.33}$$

We can now state the fundamental criterion for determining which local transformations are physical: if the corner charge associated with it vanishes, the transformation is a *true* gauge transformations; if it does not, it is a physical symmetry.

Moreover, Noether's theorem also tells us that each vector field generating a symmetry (as defined in Section 5.1.2) is an integrable symplectomorphism, whose Hamiltonian function is the charge itself $H_{\mathbb{X}_\alpha} = Q_\alpha$. Following the previous criterion, we see that the generators of gauge transformations correspond to degenerate directions of the presymplectic form. Given that the latter form an ideal inside $\mathfrak{X}(\bar{\mathcal{S}})$, we can identify the corresponding Lie subalgebra as the Lie algebra of a subgroup $\mathcal{G}_0$, which is the actual group of gauge (as opposed to physical) transformations

$$q_\alpha - j_\alpha = \mathrm{d}(\dots) \iff \alpha \in \mathrm{Lie}(\mathcal{G}_0). \tag{5.34}$$

The orbits of the gauge group $\mathcal{G}_0$ form leaves with foliate the presymplectic space $\bar{\mathcal{S}}$. Let us denote by $\bar{\mathcal{S}}/\mathcal{G}_0$ the set of these leaves, and by $\pi : \bar{\mathcal{S}} \to \bar{\mathcal{S}}/\mathcal{G}_0$ the map that sends each point to the leaf it is on. Normally, we could not push-forward a form through π. However, because the form is both orthogonal to the $\mathbb{X}_\alpha$ directions ($\mathbb{I}_{\mathbb{X}_\alpha}\Omega = 0$) and invariant along them ($\mathbb{L}_{\mathbb{X}_\alpha}\Omega = 0$), there exists a unique form $\Omega_{\mathrm{red}} \in \Omega^2(\bar{\mathcal{S}}/\mathcal{G}_0)$ such that $\pi^*\Omega_{\mathrm{red}} = \Omega$. The form Ω_{red} is closed and, assuming the vector fields $\mathbb{X}_\alpha$ span the entire kernel of Ω, non-degenerate, thereby making $\bar{\mathcal{S}}/\mathcal{G}_0$ a symplectic space. The symmetry algebra acting on the reduced phase space is then isomorphic to the quotient

$$\frac{\mathcal{G}}{\mathcal{G}_0} = \frac{(\text{physical symmetry})}{(\text{gauge transformations})},$$

which is possible because the generators of gauge transformations form an ideal inside the algebra of local symmetries.

Proof. Let $\mathbb{X}_\alpha$ and $\mathbb{X}_\beta$ be local symmetries and assume Q_α vanishes. We already proven that their commutator is an integrable vector field (2.20), so we compute the associated generator using that $Q_\alpha \hat{=} 0$ implies both $\mathbb{I}_{\mathbb{X}_\alpha}\Omega \hat{=} 0$ and $\mathbb{L}_{\mathbb{X}_\alpha}\Omega \hat{=} 0$

$$\mathbb{I}_{[\mathbb{X}_\alpha,\mathbb{X}_\beta]}\Omega = \mathbb{L}_{\mathbb{X}_\alpha}\mathbb{I}_{\mathbb{X}_\beta}\Omega - \mathbb{I}_{\mathbb{X}_\beta}\mathbb{L}_{\mathbb{X}_\alpha}\Omega \hat{=} \mathbb{L}_{\mathbb{X}_\alpha}\mathbb{d}Q_{\mathbb{X}_\beta} = \mathbb{d}\mathbb{I}_{\mathbb{X}_\alpha}\mathbb{d}Q_{\mathbb{X}_\beta} = \mathbb{d}(\mathbb{I}_{\mathbb{X}_\beta}\mathbb{I}_{\mathbb{X}_\alpha}\Omega) = 0. \tag{5.35}$$

Therefore $[\mathbb{X}_\alpha, \mathbb{X}_\beta]$ is a local symmetry with vanishing charge: a gauge transformation. □

To highlight the differences with the symplectic reduction in the canonical approach, let us consider first the covariant phase space construction of [11], in which degenerate directions of Ω_Σ are removed before imposing the equations of motion. These directions comprise any variation of degrees of freedom away from Σ, so that the remaining degrees of

freedom parametrize the kinematical data on that surface, which we called CDS(Σ). The reduced form $\Omega^{\text{kin}}_\Sigma$ is non-degenerate. Drawing a parallelism to the symplectic reduction we reviewed in 2.3.2, this form is the analogue of ω in the ambient space $\mathcal{P}$, which is symplectic outside the constraint subspace. However, once the equations of motion are imposed, additional degenerate directions of $\Omega^{\text{kin}}_\Sigma$ might appear, thereby requiring a second round of reduction: this is when the gauge constraints are set to zero. It has been shown in [11] that the integrated constraint currents $C_{\alpha,\Sigma} \equiv \int_\Sigma c_\alpha$ can be projected to CDS(Σ) and form a system of first class constraints. The set of gauge orbits generated by these constraints is the physical space. Notice that the algebra of these projected constraints is not always straightforwardly related to the algebra of $\mathcal{G}_0$; in fact, it may not even be a Lie algebra, as in the case of canonical GR [54], where one obtains the *hypersurface deformation algebroid* instead of $\mathfrak{diff}_0(\mathcal{M})$ [53, 170] (we recall that $\mathfrak{diff}(\mathcal{M}) \cong \mathfrak{X}(\mathcal{M})$, and same for their subspaces).

In the covariant analysis, the form Ω is directly evaluated on-shell of the equations of motion, which naturally include all the constraints. In this way, we bypass the need to separate constraints from dynamics, and we can perform the two rounds of reduction in a single step.

5.3 Case study: Maxwell theory

Let's consider Maxwell theory as a simple example where the reduction is non-trivial

$$L_M = -\frac{1}{2} F \wedge *F. \tag{5.36}$$

From its variation, we extract the equations of motion and the presymplectic potential

$$E_M = \mathrm{d} * F, \qquad \theta_M = -\mathbb{d}A \wedge *F. \tag{5.37}$$

Since we are going to impose Dirichlet boundary conditions on the pull-back to the boundary

$$\delta(i^*_{\mathcal{I}} A) = 0, \tag{5.38}$$

there is no need for a boundary Lagrangian, nor for a boundary symplectic potential.

The gauge group of electromagnetism is $\Gamma(\mathcal{M}, \mathrm{U}(1))$, which induces the gauge transformation $A \to A + \mathrm{d}\lambda$, $\lambda \in \Gamma(\mathcal{M}, \mathfrak{u}(1))$, where $\mathfrak{u}(1) = \mathrm{Lie}(\mathrm{U}(1))$. We immediately notice that $\mathbb{L}_{\mathbb{X}_\lambda} F = \mathrm{d}^2\lambda = 0$, so that the Lagrangian is trivially invariant, whereas the boundary conditions is preserved if and only if λ is constant on the boundary

$$\delta_{\mathbb{X}_\lambda}(i^*_{\mathcal{I}} A) = \mathbb{I}_{\mathbb{X}_\lambda} \mathbb{d}(i^*_{\mathcal{I}} A) = \mathrm{d}(i^*_{\mathcal{I}} \lambda)\,. \tag{5.39}$$

Therefore, we restrict the group of allowed transformations to those for which λ approaches a constant on $\mathcal{I}$.

Let us consider the associated Noether current, which, given our boundary conditions, comprises only the bulk piece

$$J_{\mathbb{X}_\lambda} = \mathbb{I}_{\mathbb{X}_\lambda}\theta = -\mathrm{d}\lambda \wedge *F = \underbrace{\lambda\,\mathrm{d} * F}_{c_\lambda} + \mathrm{d}\underbrace{(-\lambda * F)}_{q_{\mathbb{X}_\lambda}}. \tag{5.40}$$

In this simple case, we can show that $\mathbb{d}Q_{\mathbb{X}_{\lambda,\Sigma}} = -\mathbb{I}_{\mathbb{X}_\lambda}\Omega_\Sigma$ even off-shell. Going on-shell, we have

$$\mathbb{I}_{\mathbb{X}_\lambda}\Omega \doteq \int_{\partial\Sigma} \lambda * \mathbb{d}F, \tag{5.41}$$

whence it follows that $\mathbb{X}_\lambda$ is a degenerate direction of Ω for $\lambda|_{\mathcal{I}} = 0$. Conversely, for $\lambda|_{\mathcal{I}} \neq 0$, the Hamiltonian function is

$$-\int_{\partial\Sigma} \lambda * F \doteq Q_\lambda, \tag{5.42}$$

which, for constant λ, coincides with the total charge enclosed by $\partial\Sigma$ according to Gauss' law for electromagnetism. The true CPS is thus the quotient of the space of solutions by the gauge transformation for which $\lambda|_{\mathcal{I}} = 0$.

5.3.1 Field-dependent transformations

When dealing with gauge field theories, it is often necessary to relax the assumption that the parameters of the transformations are constant in field space (notice that $\mathbb{d}\lambda = 0$ in going from (5.41) to (5.42)). Indeed, when working with a gauge-fixed version of the theory or when passing from one gauge to another, the transformations of interest become field-dependent. Although this issue is more relevant in the context of non-abelian Yang–Mills theory, we once again use Maxwell theory as a testing ground to explore potential complications due to $\mathbb{d}\lambda \neq 0$.

Even when the transformation depends on the field, the Lagrangian of Maxwell theory (5.36) is still invariant. The symplectic form, on the other hand, is not. The relation (5.41) is modified by the presence of a flux term $\mathcal{F}_\lambda$

$$-\mathbb{I}_{\mathbb{X}_\lambda}\Omega = \left(\mathbb{d}\int_\Sigma c_\lambda - \int_\Sigma c_{\mathbb{d}\lambda}\right) + \mathbb{d}Q_\lambda - \mathcal{F}_\lambda, \tag{5.43}$$

with

$$\mathcal{F}_\lambda = -\int_\Sigma \mathrm{d}(\mathbb{d}\lambda) \wedge *F \doteq Q_{\mathbb{d}\lambda}. \tag{5.44}$$

Clearly, since a field-dependent transformation is not a symplectomorphism is its flux is non-trivial $\mathbb{L}_{\mathbb{X}_\lambda}\Omega \doteq \mathbb{d}\mathcal{F}_\lambda \neq 0$, it does not make sense to wonder about integrability. Nevertheless, we can still compute the Poisson brackets between Noether charges. Unlike the case of integrable transformations (cfr. eq. (2.21)), the result cannot be $\mathbb{L}_{\mathbb{X}_\lambda}Q_{\lambda'}$, since the flux spoils the antisymmetry of the Lie derivative. The explicit computation gives

$$\{Q_\lambda, Q_{\lambda'}\} \equiv \Omega^{-1}(\mathbb{d}Q_\lambda, \mathbb{d}Q_{\lambda'}) = \{Q_\lambda, Q_{\lambda'}\}_{BT} + \Omega^{-1}(\mathcal{F}_\lambda, \mathcal{F}_{\lambda'}), \tag{5.45}$$

where

$$\{Q_\lambda, Q_{\lambda'}\}_{BT} \equiv \mathbb{L}_{\mathbb{X}_\lambda} Q_{\lambda'} - \mathbb{I}_{\mathbb{X}_{\lambda'}} \mathcal{F}_\lambda = 0 \tag{5.46}$$

are the celebrated Barnich–Troessaert brackets [171, 172], which consistently reproduce the abelian character of the $\mathfrak{u}(1)$ algebra.

The presence of fluxes is a more general feature than described here. We will expand more on this topic in Sec. 5.4.3, after we discuss about corner symmetries of GR.

5.4 Case study: General Relativity

As a second example, let us consider again GR

$$L_{EH} = \frac{1}{2\kappa^2}(R - 2\Lambda)\epsilon_{\mathcal{M}}, \tag{5.47}$$

where $\epsilon_{\mathcal{M}}$ is the metric-compatible volume form of $\mathcal{M}$.

5.4.1 Komar charges

It is instructive to first study the charges associated with diffeomorphisms in GR ignoring boundary contributions.

Since the Lagrangian (5.47) is written in a covariant fashion, it follows that all vector fields $\mathbb{X}_\xi$ will be Lagrangian symmetries. According to Noether's theorem, we find the following bulk current

$$J_\xi \equiv J_{\mathbb{X}_\xi} := \mathbb{I}_{\mathbb{X}_\xi}\theta - \iota_\xi L_{EH} = c_\xi + \mathrm{d}q^K_\xi \tag{5.48}$$

where q^K_ξ is the *Komar charge aspect* [30, 83], whose Hodge dual has the following components

$$(\star q^K_\xi)_{\mu\nu} := \frac{1}{2\kappa^2}(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu). \tag{5.49}$$

Proof. Using the expression (4.14) and considering that $\mathbb{L}_{\mathbb{X}_\xi} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$, we compute

$$\mathbb{I}_{\mathbb{X}_\xi} \mathbb{v}^\mu = g^{\mu\sigma} g^{\nu\lambda}(\nabla_\nu \mathcal{L}_\xi g_{\lambda\sigma} - \nabla_\sigma \mathcal{L}_\xi g_{\nu\lambda}) = \nabla_\lambda \nabla^{[\lambda}\xi^{\mu]} + 2[\nabla^\lambda, \nabla^\mu]\xi_\lambda \equiv 2\kappa^2 U^\mu_\xi, \tag{5.50}$$

where U_ξ is defined by the last equality. Then, considering that $\iota_\xi L_{EH} = \mathscr{L}_{EH}\iota_\xi \epsilon_{\mathcal{M}}$, we can rewrite the current as the contraction of the spacetime volume form by the following vector

$$U^\mu_\xi - \mathscr{L}_{EH}\xi^\mu = \frac{1}{\kappa^2}\left(\nabla_\lambda \nabla^{[\lambda}\xi^{\mu]} + R^{\mu\nu}\xi_\nu - \frac{1}{2}\xi^\mu(R - 2\Lambda)\right) \tag{5.51}$$

where we replaced the commutator with the Riemann tensor. Let us consider this expression. The last three terms in (5.51) combine to give the metric field equations

$$\frac{1}{\kappa^2}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} + \Lambda g^{\mu\nu}\right)\xi_\nu = E^{\mu\nu}\xi_\nu. \tag{5.52}$$

When contracted with $\epsilon_{\mathcal{M}}$, this term produces a bulk $(n-1,0)$-form, which we recognize as the constraint current c_ξ. The first term in (5.51) is the divergence of an antisymmetric tensor, so its contraction with the volume form can be written as the exterior derivative of a $(n-2,0)$-form. We can then check that its components agree with those in (5.49). □

In this case, the on-shell charge is localized at the corner, as expected for a local symmetry. The resulting expression is known as the **Komar charge** associated with the diffeomorphism $\xi \in \mathfrak{X}(\mathcal{M})$

$$Q^K_\xi := \int_{\partial\Sigma} q^K_\xi \tag{5.53}$$

Since q^K_ξ depends on derivatives of ξ in the directions normal to the corner, we distinguish the following set of diffeomorphisms [38]

- **Corner boosts** When $\xi = 0$ on $\partial\Sigma$, but normal derivatives are not zero, the transformation locally generates a (Lorentz) boost that keeps the corner fixed;
- **Corner translations** When $\xi_\perp \neq 0$, the transformation moves the corner;
- **Corner diffeomorphisms** When $\xi_\perp = 0$ but $\xi_\parallel \neq 0$, the vector field generates a diffeomorphism of $\partial\Sigma$ itself.

Since we are not including any boundary Lagrangian at the moment, the system is open and not all symmetries (in the field theory sense) are necessarily symplectomorphism. Taking $\omega = \mathbb{d}\theta$ as the presymplectic form current, we can easily derive the so-called **Iyer–Wald identity** [30]

$$\mathbb{I}_{\mathbb{X}_\xi}\omega \doteq -\,\mathrm{d}\big(\mathbb{d}q^K_\xi - \iota_\xi\theta\big) \tag{5.54}$$

Proof. Using Cartan homotopy identity and the covariance of θ, we compute in one line

$$\mathbb{I}_{\mathbb{X}_\xi}\omega = \mathbb{L}_{\mathbb{X}_\xi}\theta - \mathbb{d}\mathbb{I}_{\mathbb{X}_\xi}\theta = \mathcal{L}_\xi\theta - \mathbb{d}(J_\xi + \iota_\xi L) = \mathrm{d}(\iota_\xi\theta) - \cancel{\iota_\xi \mathbb{d}\theta} - \mathbb{d}(c_\xi + \mathrm{d}q^K_\xi) - \iota_\xi(E\mathbb{d}g + \cancel{\mathrm{d}\theta}), \tag{5.55}$$

which, evaluated on the solutions of the equations of motion, it is just a reordering away from what we want. □

From the Iyer–Wald identity (5.54), we immediately see that the generator of the transformation, if it exists, is a pure corner term. Therefore, bulk diffeomorphisms with support far from $\partial\Sigma$ are necessarily gauge. We also notice that not all diffeomorphisms are integrable

$$\mathbb{I}_{\mathbb{X}_\xi}\Omega \doteq -\mathbb{d}Q^K_\xi + \int_{\partial\Sigma}\iota_\xi\theta \overset{?}{=} \mathbb{d}H_\xi \tag{5.56}$$

The obstruction comes from the integrated $\iota_\xi\theta$ term, thereby giving three possibilities:

1. $\iota_\xi\theta|_{\partial\Sigma} = 0$. This happens if ξ is parallel to the corner, as for corner diffeomorphisms, or $\xi = 0$, as for corner boosts. Therefore, these transformations can be integrated and their charge is the Komar charge;

2. $\iota_\xi\theta|_{\partial\Sigma} = \mathbb{d}b_\xi$. This case is not general, but it may happen in some cases (e.g. at large distance in asymptotically flat spacetimes [29, 30]). If this happens, the Komar charge gets a corrections $Q^K_\xi \to Q^K_\xi - b_\xi$;[2]

3. $\iota_\xi\theta|_{\partial\Sigma} \neq \mathbb{d}b_\xi$. This is the case for corner translations ($\xi_\perp \neq 0$), which are typically not integrable unless we are in one of the exceptional cases mentioned above.

5.4.2 Brown–York charges

As vastly discussed in the previous chapter, one way to impose Dirichlet boundary conditions on the metric pull-back $h \equiv i^*_{\mathcal{I}} g$ is to supplement the Lagrangian (5.47) with the GHY boundary term

$$\ell_{GHY} = \frac{1}{\kappa^2} K \epsilon_{\mathcal{I}}. \tag{5.57}$$

Proof. We can evaluate explicitly the sum in the l.h.s. of (5.4). First, we compute

$$\theta = \frac{1}{2\kappa^2}\iota_{\mathbb{v}}\epsilon_{\mathcal{M}} \implies i^*_{\mathcal{I}}\theta = -\frac{1}{2\kappa^2}\Big(2\mathbb{d}K + K^{ab}\mathbb{d}h_{ab} + \mathcal{D}^a \mathbb{c}_a\Big)\epsilon_{\mathcal{I}} \tag{5.58}$$

where $\mathbb{c}_a$ is the pull-back of the tangential vector $\mathbb{c}_\alpha \equiv -\mathbb{d}g_{\mu\nu}\Pi^\mu_\alpha n^\nu$, and its boundary covariant derivative yields a corner term [15]. Summing the variation of the boundary Lagrangian, we cancel the $\mathbb{d}K$ term and remain with

$$\mathbb{d}\ell_{GHY} + i^*_{\mathcal{I}}\theta = \underbrace{\frac{1}{2\kappa^2}(Kh^{ab} - K^{ab})\mathbb{d}h_{ab}\epsilon_{\mathcal{I}}}_{B = \not{\mathcal{L}}_g \mathbb{d}h_{ab}} + \underbrace{\frac{1}{2\kappa^2}\mathcal{D}_a \mathbb{c}^a \epsilon_{\mathcal{I}}}_{\mathrm{d}\vartheta}. \tag{5.59}$$

Indeed $B = 0$ with Dirichlet boundary conditions. □

From the expression of the current associated with the boundary condition, we recognize the Brown–York tensor (4.17)

$$B = \frac{1}{2} T^{ab}_{BY} \mathbb{d}h_{ab}\, \epsilon_{\mathcal{I}}. \tag{5.60}$$

The gauge group we consider will be a subgroup of $\mathrm{Diff}(\mathcal{M})$. From the previous considerations, we learned that a diffeomorphism must preserve both the field equations *and* the boundary conditions to qualify as a symmetry in the presence of boundaries. So we require that

$$\xi|_{\mathcal{I}} = \bar{\xi} \in \mathfrak{X}(\mathcal{I}), \qquad \delta_{\mathbb{X}_\xi} h_{ab} = \mathbb{l}_{\mathbb{X}_\xi}(\mathbb{d}h_{ab}) = \mathcal{L}_{\bar{\xi}} h_{ab}. \tag{5.61}$$

[2] This is what accounts for the "factor of 2" discrepancy between the Komar mass and the ADM mass[173].

In other words, ξ must approach a *Killing symmetry* of the boundary. Since the boundary Lagrangian is also constructed covariantly, the vector fields $\mathbb{X}_\xi$ are Lagrangian symmetries for each ξ in the allowed subgroup. Using (5.48), we compute the total charge aspect as

$$q_\xi^{BY} := q_\xi^K - j_\xi, \tag{5.62}$$

from which it follows that the total diffeomorphism charge coincide with the celebrated Brown–York charge [1, 109]

$$Q_\xi^{BY} = -\int_{\partial\Sigma} T_{BY}^{ab}\bar{\xi}_a u_b \epsilon_{\mathcal{S}}, \quad \text{where} \quad T_{BY}^{ab} = \frac{1}{\kappa^2}(Kh^{ab} - K^{ab}). \tag{5.63}$$

We thus recovered, in a completely covariant fashion, the statement that Brown–York charges are the generators of boundary isometries. We think that leaving this proof to the reader is a bit too much.

Proof. To facilitate the computation of the total charge aspect (5.62), we introduce u, the unit timelike normal form to the corner $\partial\Sigma$ in $\mathcal{I}$, so that the corner volume element $\epsilon_{\mathcal{S}}$ satisfies $\epsilon_{\mathcal{M}} = u \wedge n \wedge \epsilon_{\mathcal{S}}$, where n is the unit spacelike normal to $\mathcal{I}$, as in Figure 5.1. We can now evaluate the relevant forms in (5.62) in terms of this decomposition. We get

$$q_\xi^K = -\frac{1}{2\kappa^2}(\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu)u^\mu n^\nu \epsilon_{\mathcal{S}}, \tag{5.64}$$

$$\mathbb{I}_{\mathbb{X}_\xi}\vartheta = \frac{1}{2\kappa^2}\iota_{(\mathbb{I}_{\mathbb{X}_\xi}\mathbb{c})}\epsilon_{\mathcal{I}} = \frac{1}{2\kappa^2}(u\cdot\mathbb{I}_{\mathbb{X}_\xi}\mathbb{c})\epsilon_{\mathcal{S}} = \frac{1}{2\kappa^2}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)u^\mu n^\nu\epsilon_{\mathcal{S}}, \tag{5.65}$$

$$\iota_{\bar{\xi}}\ell_{GHY} = \frac{1}{\kappa^2}K\iota_{\bar{\xi}}\epsilon_{\mathcal{I}} = -\frac{1}{\kappa^2}(u\cdot\bar{\xi})K\epsilon_{\mathcal{S}}, \tag{5.66}$$

where, in the second expression, $\mathbb{c}$ is the same vector that appears in (5.58). We then combine these results to get

$$\begin{aligned} q_\xi^{BY} &= -\frac{1}{\kappa^2}\left[\frac{1}{2}(\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu)u^\mu n^\nu + \frac{1}{2}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)u^\mu n^\nu + (u\cdot\bar{\xi})K\right]\epsilon_{\mathcal{S}} = \\ &= \frac{1}{\kappa^2}[\nabla_\mu\xi_\nu u^\mu n^\nu + \xi_\mu u^\mu K]\epsilon_{\mathcal{S}} = \frac{1}{\kappa^2}[-\xi_\nu u^\mu\nabla_\mu n^\nu + \xi_\mu u^\mu K]\epsilon_{\mathcal{S}} = \\ &= \frac{1}{\kappa^2}(-K_\mu^\nu + K\Pi_\mu^\nu)\xi_\nu u^\mu\epsilon_{\mathcal{S}} = \frac{1}{\kappa^2}(-K_b^a + Kh_b^a)\bar{\xi}_a u^b\epsilon_{\mathcal{S}}, \end{aligned} \tag{5.67}$$

where, in the second line, we used that $n\cdot\xi = 0$ to flip the covariant derivative. □

Remark Unlike Komar charges, Brown–York charges only depend on the boundary value of the vector field. If $\xi|_{\partial\Sigma} = 0$, then $Q_\xi^{BY} = 0$.

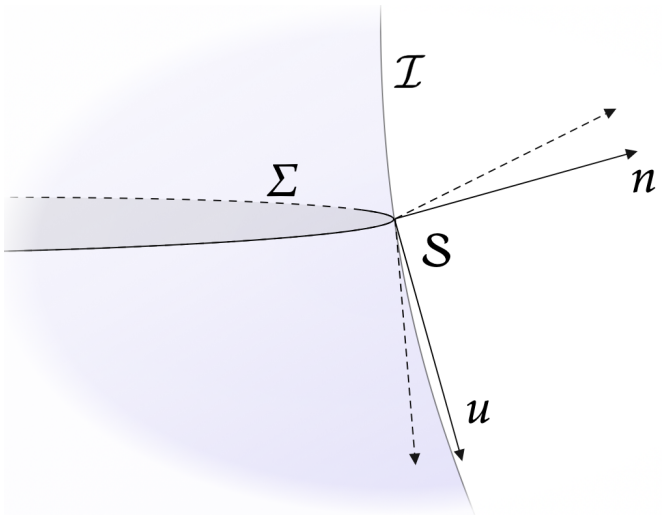


Figure 5.1: Illustration of the frame at the corner $\mathcal{S} \subset \partial\Sigma$. The vector n is the normal to the boundary such that $\epsilon_{\mathcal{I}} = \iota_n \epsilon_{\mathcal{M}}$, while u is the normal of $\mathcal{S}$ inside $\mathcal{I}$, defined by $\epsilon_{\mathcal{S}} = \iota_u \epsilon_{\mathcal{I}}$. Notice that, given our conventions, the vector u is past-pointing. The dotted frame represent the frame that would be induced on $\mathcal{S}$ by Σ.

5.4.3 Field-dependent transformations and anomalies

As anticipated, the inclusion of field-dependent transformation becomes much more relevant in the context of non-abelian theories such as gravity. Vector fields are usually taken to be field-dependent in order to respect gauge choices or boundary conditions, as in the study of asymptotic symmetries. Even though we are not going to discuss asymptotic symmetries, we refer the interested reader to the following works and the references therein [171, 174–183]. When vector fields are allowed to depend on field configurations, the anti-homomorphism property (5.22) between the field space and the spacetime vector fields can be restored by a suitable modification of the $\mathfrak{X}(\mathcal{M})$ Lie brackets [18, 184]

$$[\![\mathbb{X}_\xi, \mathbb{X}_{\xi'}]\!]_{\text{Lie}} = -\mathbb{X}_{[\xi,\xi]^*_{\text{Lie}}}, \tag{5.68}$$

with

$$[\xi, \xi']^*_{\text{Lie}} := [\xi, \xi']_{\text{Lie}} - \mathbb{I}_{\mathbb{X}_\xi} \mathrm{d}\xi' + \mathbb{I}_{\mathbb{X}_{\xi'}} \mathrm{d}\xi. \tag{5.69}$$

The Lagrangian of GR (5.47) is covariant even under field-dependent diffeomorphisms, but there is a difference between the field space and the spacetime Lie derivative of the symplectic form acquires an extra term, namely

$$\mathbb{L}_{\mathbb{X}_\xi} \theta - \mathcal{L}_\xi \theta = \mathbb{I}_{\mathrm{d}\xi} \theta. \tag{5.70}$$

Assuming that the same holds for ϑ for vector fields that are boundary symmetry (5.61), we can deduce how the Iyer–Wald identity is modified to

$$-\mathbb{I}_{\mathbb{X}_\xi}\Omega = \left(\mathbb{d}\int_\Sigma c_\xi - \int_\Sigma c_{\mathbb{d}\xi}\right) + \mathbb{d}Q_\xi - \mathcal{F}_\xi, \tag{5.71}$$

where the flux term is

$$\mathcal{F}_\xi = Q_{\mathbb{d}\xi} - \left(\int_\Sigma \iota_\xi E\mathbb{d}g - \int_{\partial\Sigma}\iota_\xi B\right) \doteq Q_{\mathbb{d}\xi} + \int_{\partial\Sigma}\iota_\xi B. \tag{5.72}$$

Repeating the same argument used in the Maxwell case, we find that the Poisson brackets between Noether charges are given — up to a flux commutation term — by the Barnich–Troessaert brackets. These brackets, originally derived in the context of gravity [12, 171, 177, 185], reproduce the vector field (modified) Lie algebra

$$\{Q_\xi, Q_{\xi'}\}_{BT} = -Q_{[\![\xi,\xi']\!]_{\text{Lie}}} + \mathbb{K}(\xi,\xi'), \tag{5.73}$$

up to a generically field-dependent 2-cocycle [19, 87, 186, 187]

$$\mathbb{K}(\xi,\xi') = -\int_{\partial\Sigma}(\iota_\xi\Delta_{\xi'}\ell - \iota_{\xi'}\Delta_\xi\ell) - \int_{\partial\Sigma}\iota_\xi\iota_{\xi'}(L+\mathrm{d}\ell). \tag{5.74}$$

We present the proof of this statement in a more general setting, after the following remark.

Remark The appearance of $\mathbb{I}_{\mathbb{d}\xi}\theta$ in (5.70) is actually expected if θ is covariant. Indeed, for any covariant form in $\Omega^{(p,q)}(\mathbb{F})$ with $q \geq 1$,[3] we do not expect the two Lie derivatives to agree, as one can readily check using $\mathbb{d}\phi$

$$\mathbb{L}_{\mathbb{X}_\xi}\mathbb{d}\phi - \mathcal{L}_\xi\mathbb{d}\phi = \mathbb{d}\mathbb{L}_{\mathbb{X}_\xi}\phi - \mathbb{d}\mathcal{L}_\xi\phi + \mathcal{L}_{\mathbb{d}\xi}\phi = \mathbb{I}_{\mathbb{X}_{\mathbb{d}\xi}}\mathbb{d}\phi. \tag{5.75}$$

For this reason, it is preferable to define the *anomaly operator* (cfr. the end of Section 3.5.1) as

$$\Delta_\xi := \mathbb{L}_{\mathbb{X}_\xi} - \mathcal{L}_\xi - \mathbb{I}_{\mathbb{X}_{\mathbb{d}\xi}}, \tag{5.76}$$

and claim that forms $\alpha \in \Omega^{(p,q)}(\mathbb{F})$ are covariant when $\Delta_\xi\alpha = 0$. The presence of anomalies produces additional terms in the flux (5.72) [18, 88]

$$\mathcal{F}_\xi \doteq Q_{\mathbb{d}\xi} + \int_{\partial\Sigma}\iota_\xi B + \left(\int_\Sigma \Delta_\xi\theta - \int_{\partial\Sigma}\Delta_\xi\vartheta\right). \tag{5.77}$$

Proof. By definition, the Barnich–Troessaert brackets are

$$\{Q_\xi, Q_{\xi'}\}_{BT} \equiv \mathbb{L}_{\mathbb{X}_\xi}Q_{\xi'} - \mathbb{I}_{\mathbb{X}_{\xi'}}\mathcal{F}_\xi = \int_\Sigma m(\xi,\xi'), \tag{5.78}$$

[3] Obviously, for a 0-form in field space ($q = 0$), the contraction does not give any contribution.

and we now evaluate the integrated quantity $m(\xi,\xi')$ allowing for non-zero anomalies in the symplectic potential. We remind the reader that our choice of Lagrangian pair (L,ℓ) was motivated by the requirement of the bulk Lagrangian general covariance ($\Delta_\xi L = 0$). The charge aspect we use in this proof is that improved by the boundary term (e.g. the Brown–York charge aspects when the boundary Lagrangian is the GHY one). With some algebra, we get

$$\begin{aligned} m(\xi,\xi') &= \mathrm{d}\mathbb{L}_{\mathbb{X}_\xi} q_{\xi'} - \mathbb{I}_{\mathbb{X}_{\xi'}}(\mathrm{d}q_{\mathrm{d}\xi} + \mathrm{d}\iota_\xi B + \Delta_\xi\theta - \mathrm{d}\Delta_\xi\vartheta) = \\ &= \mathrm{d}\mathbb{L}_{\mathbb{X}_\xi} q_{\xi'} - \mathrm{d}q_{\mathbb{X}_{\xi'}[\xi]} - \mathrm{d}\mathbb{I}_{\mathbb{X}_{\xi'}}\iota_\xi B - \mathbb{I}_{\mathbb{X}_{\xi'}}\Delta_\xi(\theta - \mathrm{d}\vartheta) = \\ &= \mathrm{d}\mathcal{L}_\xi q_{\xi'} - \mathrm{d}q_{\mathbb{X}_{\xi'}[\xi]} - \mathrm{d}\mathbb{I}_{\mathbb{X}_{\xi'}}\iota_\xi B - \mathrm{d}\Big(q_{[\xi,\xi']} - q_{\mathbb{X}_\xi[\xi']}\Big) - \iota_{\xi'}\Delta_\xi(L - \mathrm{d}\ell) = \\ &= \mathrm{d}\mathcal{L}_\xi q_{\xi'} - \mathrm{d}\mathbb{I}_{\mathbb{X}_{\xi'}}\iota_\xi B - \mathrm{d}q_{[\xi,\xi']^*} - \iota_{\xi'}\Delta_\xi(L - \mathrm{d}\ell) = \\ &= -\mathrm{d}q_{[\xi,\xi']^*} - \mathrm{d}\iota_\xi\iota_{\xi'}(L + \mathrm{d}\ell) - \mathrm{d}\big(\iota_\xi\Delta_{\xi'}\ell - \iota_{\xi'}\Delta_\xi\ell\big) - \iota_{\xi'}\Delta_\xi L, \end{aligned} \tag{5.79}$$

where we used the identities $[\Delta_\xi, \mathbb{I}_{\mathbb{X}_{\xi'}}] = -\mathbb{I}_{\mathbb{X}_{[\xi,\xi']}} + \mathbb{I}_{\mathbb{X}_{\mathbb{X}_\xi[\xi']}}$ and $[\Delta_\xi, \iota_{\xi'}] = -\iota_{[\xi,\xi']} + \iota_{\mathbb{X}_\xi[\xi']}$ in the second line to replace

$$\begin{aligned} \mathbb{I}_{\mathbb{X}_{\xi'}}\Delta_\xi(\theta - \mathrm{d}\vartheta) &= ([\mathbb{I}_{\mathbb{X}_{\xi'}}, \Delta_\xi] + \Delta_\xi\mathbb{I}_{\mathbb{X}_{\xi'}})(\theta - \mathrm{d}\vartheta) = \\ &= (\mathbb{I}_{\mathbb{X}_{[\xi,\xi']}} - \mathbb{I}_{\mathbb{X}_{\mathbb{X}_\xi[\xi']}} + \Delta_\xi\mathbb{I}_{\mathbb{X}_{\xi'}})(\theta - \mathrm{d}\vartheta) = \\ &= \mathrm{d}q_{[\xi,\xi']} - \mathrm{d}q_{\mathbb{X}_\xi[\xi']} + \Delta_\xi\mathrm{d}q_{\xi'} + \\ &\quad + \iota_{[\xi,\xi']}(L - \mathrm{d}\ell) - \iota_{\mathbb{X}_\xi[\xi']}(L - \mathrm{d}\ell) + \Delta_\xi\iota_{\xi'}(L - \mathrm{d}\ell) = \\ &= \mathrm{d}\Big(q_{[\xi,\xi']} - q_{\mathbb{X}_\xi[\xi']} + \Delta_\xi q_{\xi'}\Big) + \iota_{\xi'}\Delta_\xi(L - \mathrm{d}\ell), \end{aligned} \tag{5.80}$$

and, to get to the last line, we used that

$$\mathcal{L}_\xi q_{\xi'} = -\iota_\xi\iota_{\xi'}(L + \mathrm{d}\ell) + \mathbb{I}_{\mathbb{X}_{\xi'}}\iota_\xi B - \iota_\xi\Delta_{\xi'}\ell + \mathrm{d}(\ldots). \tag{5.81}$$

Finally, integrating $m(\xi,\xi')$ over Σ, we obtain the brackets

$$\big\{Q_\xi, Q_{\xi'}\big\}_{BT} = -Q_{[\xi,\xi']^*} - \int_{\partial\Sigma}(\iota_\xi\Delta_{\xi'}\ell - \iota_{\xi'}\Delta_\xi\ell) - \int_{\partial\Sigma}\iota_\xi\iota_{\xi'}(L + \mathrm{d}\ell). \tag{5.82}$$

□

Chapter 6

Black hole thermodynamics

If you can heat it, it has microscopic degrees of freedom.
— T. Padmanabhan, rephrasing L. Boltzmann

The CPS formalism is particularly valuable for studying generally covariant theories, among which GR stands as the most famous example. Following Einstein's lesson, most alternative theories of gravity are built to respect the principle of general covariance. [188–191]. Over the last century, many theories of gravity have been developed to extend GR, and most of them — if not all — contain black holes. Moreover, despite their phenomenological differences, these objects appear to obey universal laws, analogous to the laws of *thermodynamics*, as in the case of GR [191–194].

The apparent universality of **black hole thermodynamics** in generally covariant theories suggests a fundamental connection with covariance, as if one cannot have the latter without the former. Indeed, bridging the seemingly irreconcilable divide between the covariance of gravitational theories and thermodynamics — which is naturally built upon the Hamiltonian formulation — was one of the first success of the CPS formalism [11, 29, 30, 195–197]. Starting from these applications in black hole thermodynamics [11, 29, 30, 195–197], the formalism has then found widespread application in many fields, ranging from Anti-de Sitter (AdS)/CFT correspondence [160, 198–201] to near-horizon symmetries [202–204], from asymptotic symmetries [12, 176, 180] to the study of non-stationary black holes [205–208].

In what follows, we are going to derive the first law of black hole thermodynamics within the CPS formalism, and then generalize it to Lanczos–Lovelock (LL) theories, the most general set of theories with second-order field equations for the metric tensor.

The content of this chapter is drawn from Ref. [209].

6.1 Conserved quantities

First of all, let us generalize the construction of Brown–York charges (cfr. Section 5.4.2 and, in particular, eq. (5.63)) to generally covariant theories of the metric tensor field that are not necessarily GR. To do that, we consider the expression for the charge of a boundary-preserving diffeomorphism, as it follows from Noether's theorem

$$Q_{\xi,\Sigma} = \int_{\Sigma} J_\xi - \int_{\partial\Sigma} j_\xi. \tag{6.1}$$

Then, we consider the difference between two such charges evaluated on different slices Σ_1 and Σ_2. With the same algebraic manipulations that we used to prove the charge conservation (5.17), we can arrive at the result

$$Q_{\xi,\Sigma_2} - Q_{\xi,\Sigma_1} = -\int_{\mathcal{M}_{12}} \mathbb{I}_{\mathbb{X}_\xi}(E^{\mu\nu}\mathbb{d}g_{\mu\nu}) + \int_{\mathcal{I}_{12}} \mathbb{I}_{\mathbb{X}_\xi} B. \tag{6.2}$$

If we evaluate this expression on the surface $E = 0$, the first term drops; while, assuming Dirichlet boundary conditions for the metric tensor, we can recast the second as

$$\mathbb{I}_{\mathbb{X}_\xi} B = (\not{j}_h)^{ab}\mathbb{L}_{\mathbb{X}_\xi} h_{ab} = (\not{j}_h)^{ab}\mathcal{L}_\xi h_{ab} = (\not{j}_h)^{ab}(\mathcal{D}_a\bar{\xi}_b + \mathcal{D}_b\bar{\xi}_a), \tag{6.3}$$

where $\not{j}_h$ is a tensor density (a $(n-1,0)$ form) that we can write as $(\not{j}_h)^{ab} \equiv \frac{1}{2}\mathcal{T}^{ab}\epsilon_{\mathcal{I}}$, introducing the *generalized Brown–York tensor*

$$\mathcal{T}^{ab} := 2\frac{\delta B}{\delta h_{ab}}. \tag{6.4}$$

From the invariance of the action under diffeomorphisms, it follows that $\mathcal{T}^{ab}$ is conserved in the absence of matter fluxes. For this reason, the term $\mathbb{I}_{\mathbb{X}_\xi} B$ can be written as a total boundary derivative, and the its integral is thereby localized on the lids. Indeed, we remind the reader that $\partial\mathcal{I}_{12} = \partial\Sigma_2 \cap \partial\Sigma_1$ as a set, so any total derivative on $\mathcal{I}$ can be recast as a total derivative on the two Σs. Explicitly, the difference

$$Q_{\xi,\Sigma_2} - Q_{\xi,\Sigma_1} \doteq \frac{1}{2}\int_{\mathcal{I}_{12}} \mathcal{T}^{ab}(\mathcal{D}_a\bar{\xi}_b + \mathcal{D}_b\bar{\xi}_a)\epsilon_{\mathcal{I}} = \int_{\partial\mathcal{I}_{12}} \mathcal{T}^{ab}\bar{\xi}_a u_b \epsilon_{\mathcal{S}} \tag{6.5}$$

can be written as

$$Q_{\xi,\Sigma_2} - Q_{\xi,\Sigma_1} \doteq -\int_{\partial\Sigma_2} \mathcal{T}^{ab}\bar{\xi}_a u_b \epsilon_{\mathcal{S}} + \int_{\partial\Sigma_1} \mathcal{T}^{ab}\bar{\xi}_a u_b \epsilon_{\mathcal{S}}, \tag{6.6}$$

taking into account the opposite orientation of the corners. This implies that, up to a constant, diffeomorphism charges are always equal to Brown–York charges for boundary isometries

$$Q_\xi \hat{=} -\int_{\partial\Sigma} \mathcal{T}^{ab}\bar{\xi}_a u_b \epsilon_{\mathcal{S}}. \tag{6.7}$$

Let us now define some properties associated with possible isometries of the boundary.

- *Stationarity*: there exists a timelike Killing vector field $\bar{t} \in \mathfrak{X}(\mathcal{I})$;
- *Staticity*: a stationary spacetime is said to be static if it is also invariant under time reversal, where "time" is the parameter along the flow of $\bar{t}$;
- *Rotational symmetry*: there exists a spacelike Killing vector field with closed orbits $\bar{\varphi} \in \mathfrak{X}(\mathcal{I})$;
- *Spherical symmetry*: there are multiple of the previous Killing vector field $\bar{\varphi}_{(i)} \in \mathfrak{X}(\mathcal{I})$ in involution whose orbits are topologically $(n-2)$-spheres.

Given a boundary isometry, we define the associated *canonical charge* as (the boundary part of) the Noether charge of a diffeomorphism which approaches that isometry at the boundary. Accordingly, we introduce the **energy** and the **angular momenta** as the canonical charges associated with time translation and rotational invariance

$$E \equiv Q_{\bar{t}} = -\int_{\mathcal{S}} \mathcal{T}^{ab} \bar{t}_a u_b \epsilon_{\mathcal{S}}, \tag{6.8}$$

$$J_i \equiv -Q_{\bar{\varphi}_{(i)}} = \int_{\mathcal{S}} \mathcal{T}^{ab} \bar{\varphi}_{(i)a} u_b \epsilon_{\mathcal{S}}. \tag{6.9}$$

Let us now take a moment to evaluate the form $\iota_{\bar{\xi}} B$, which appears in the l.h.s. of (5.31). Using the definition of generalized Brown–York tensor that we just introduced (6.4), we can write

$$\iota_{\bar{\xi}} B = \frac{1}{2} \mathcal{T}^{ab} \mathbb{d} h_{ab} \iota_{\bar{\xi}} \epsilon_{\mathcal{I}} = -\frac{1}{2} \mathcal{T}_{ab} \mathbb{d} h^{ab} \iota_{\bar{\xi}} \epsilon_{\mathcal{I}}. \tag{6.10}$$

If we assume that the vector field $\bar{\xi}$ can be decomposed as $\bar{\xi} = \bar{t} + \Omega^i_H \bar{\varphi}_{(i)}$, with vectors $\bar{\varphi}_{(i)}$ that are tangential to the corner, then the only contribution in the contraction comes from the time translation vector field, namely $\iota_{\bar{\xi}} \epsilon_{\mathcal{I}} = \iota_{\bar{t}} \epsilon_{\mathcal{I}}$. We then let u^a be the dual vector to the normal 1-form to the corner, and accordingly decompose $h^{ab} = -u^a u^b + \gamma^{ab}$, where γ is completely tangential to the corner $\gamma^{ab} u_b = 0$. Expanding the variation of h^{ab} we thus find two terms, whose effects we consider separately.

- The first term yields the contribution

$$-\mathcal{T}_{ab} u^a \mathbb{d} u^b (u \cdot \bar{t}) \epsilon_{\mathcal{S}}. \tag{6.11}$$

  Under the assumption that the normal vector to the corner is proportional to $\bar{t}$, it follows from $\mathbb{d}\bar{t} = 0$, which we are going to assume later, that $\mathbb{d} u^b \propto u^b$. If we denote the proportionality constant by $\nu := u \cdot \bar{t}$,[1] we can rewrite our expression as

$$-\mathcal{T}_{ab} u^a \mathbb{d} u^b (u \cdot \bar{t}) \epsilon_{\mathcal{S}} = -\mathcal{T}_{ab} u^a \mathbb{d} u^b \nu \epsilon_{\mathcal{S}} = \mathcal{T}_{ab} u^a u^b \mathbb{d}\nu \epsilon_{\mathcal{S}} = -\frac{\mathbb{d}\nu}{\nu} \mathcal{T}_{ab} u^a \bar{t}^b \epsilon_{\mathcal{S}}, \tag{6.12}$$

  which, up to the $\mathbb{d} \ln \nu$ factor, coincides with the Brown–York energy density (6.8).

[1] We are assuming $\bar{t} = -\nu u$ because, given our convention, the normal vector u is past-pointing.

- The second term is not as simple, but we can think of it as a work term related to a deformation of the boundary surface. If we think that the Brown–York contains a *surface tension* term $\mathcal{T}_{ab} = -\sigma\gamma_{ab}$ (where $\gamma_{ab}\gamma^{bc} = \gamma^c_a$), then one can write

$$\frac{1}{2}\mathcal{T}_{ab}\mathbb{d}\gamma^{ab}(u\cdot\bar{t})\epsilon_{\mathcal{S}} = -\frac{1}{2}\sigma\gamma_{ab}\mathbb{d}\gamma^{ab}\nu\epsilon_{\mathcal{S}} = \nu\sigma\mathbb{d}\epsilon_{\mathcal{S}}, \tag{6.13}$$

  where we used that $\mathbb{d}\epsilon_{\mathcal{S}} = -\frac{1}{2}\gamma_{ab}\mathbb{d}\gamma^{ab}\epsilon_{\mathcal{S}}$.

We then arrive at the following relation

$$\iota_{\bar{\xi}}B = q^{BY}_{\bar{t}}\,\mathbb{d}\ln\nu + \nu\sigma\mathbb{d}\epsilon_{\mathcal{S}}. \tag{6.14}$$

6.2 First law of thermodynamics

Let us consider a stationary, non-extremal black hole with at least one bifurcate Killing horizon. We assume that the Killing vector field ξ is timelike outside the horizon: if there are multiple horizons, we just consider the outermost. Moreover, when there is no theorem that guarantees that, we also assume that, as we approach the boundary, $\xi \to \bar{t} + \Omega^i_H\bar{\varphi}_{(i)}$ (with an implicit sum over i) for a set of constant coefficients Ω^i_H that we interpret as the *angular velocity* of the black hole.[2] We let κ_H be the surface gravity of ξ (see [210] for a comparison between the various definitions of surface gravity).

We start from the local relation $\mathbb{d}(J_\xi - \mathrm{d}j_\xi) \mathrel{\hat{=}} -\mathbb{I}_{\chi_\xi}\omega$ (derived in (5.28)) and notice that the r.h.s. trivially vanishes if ξ is an isometry, because ω is linear in its arguments and $\mathbb{I}_{\chi_\xi}\mathbb{d}g = \mathcal{L}_\xi g = 0$ by covariance. The quantity $J_\xi - \mathrm{d}j_\xi$ is thus stationary on-shell. Next, instead of integrating this quantity over a Cauchy slice as we did to derive the Noether charge (5.29), we integrate it over a slice Σ' that extends from the horizon bifurcation surface $\mathcal{B}$, that we assumed to exist, to an arbitrary section of the boundary $\mathcal{S} \subset \mathcal{I}$. Since $J_\xi \mathrel{\hat{=}} \mathrm{d}q^K_\xi$, this integral will be localized at $\partial\Sigma' = \mathcal{S}\cup(-\mathcal{B})$: the integral at $\mathcal{S}$ receives contribution from the boundary current, reconstructing the (generalized) Brown–York charge (cfr. (5.62)), while the integral at $\mathcal{B}$ is just the Komar charge

$$\int_{\Sigma'}(J_\xi - \mathrm{d}j_\xi) \mathrel{\hat{=}} \int_{\mathcal{S}}(q^K_\xi - j_\xi) - \int_{\mathcal{B}} q^K_\xi = \int_{\mathcal{S}} q^{BY}_\xi - \int_{\mathcal{B}} q^K_\xi. \tag{6.15}$$

Given the assumption about the asymptotic behavior of ξ, and the previous definitions of energy and angular momentum, we can rewrite the term at $\mathcal{S}$ as $E - \Omega^i_H J_i$, so that its vertical derivative is $\mathbb{d}E - \Omega^i_H\mathbb{d}J_i$. The analogy with thermodynamics demands that this combination is equal to $T\mathbb{d}S$, for some temperature T and some entropy S. Indeed, this very consideration led Wald to define the following quantity [29], known as **Wald entropy**,

$$S := 2\pi\int_{\mathcal{B}}\tilde{q}_\xi, \tag{6.16}$$

[2] Despite the notation, these coefficients have nothing to do with the symplectic form.

after proving that $\tilde{q}_\xi \equiv q_{\xi/\kappa_H}$, restricted to the bifurcation surface, is a form that can be constructed from local geometric quantities only. To see why, notice that every expression containing covariant derivatives of ξ/κ_H can be expressed, using the Killing equation, in terms of linear combinations of ξ/κ_H and its first covariant derivative. In turn, when evaluated on the bifurcation surface, these two quantities reduce to

$$\left.\frac{\xi_a}{\kappa_H}\right|_{\mathcal{B}} = 0, \qquad \left.\nabla_b\left(\frac{\xi_a}{\kappa_H}\right)\right|_{\mathcal{B}} = -\varepsilon_{ab}, \tag{6.17}$$

where ε_{ab} is the binormal to the bifurcation surface. Replacing these expression inside $\tilde{q}_\xi|_{\mathcal{B}}$ makes it independent of ξ. Obviously, by linearity, $q_\xi = \kappa_H \tilde{q}_\xi$.

Our derivation of the first law of thermodynamics will be limited to *stationary perturbations*, meaning we consider only those perturbations that preserve the Killing horizon. This implies that we hold fixed ξ/κ_H rather than ξ itself. Indeed, if we fixed ξ with the normalization given by $\xi \to \bar{t} + \Omega_H^i \bar{\varphi}_{(i)}$, we could not preserve the Killing horizon under perturbations that change the surface gravity. Hence, in the language of field space differential geometry, stationarity is expressed as the condition $\mathbb{d}(\xi/\kappa_H) = 0$. This seemingly harmless request leads us to the domain of field-dependent vector fields $\mathbb{d}\xi \neq 0$, which we briefly explored in Section 5.4.3. As we saw in the modified Iyer–Wald equation (5.71), the contraction of the symplectic form with a field-dependent diffeomorphism does not yield $\mathbb{d}Q_\xi$ as in (5.29), but $\mathbb{d}Q_\xi - Q_{\mathbb{d}\xi}$, thereby changing the term at the bifurcation surface to [3]

$$\mathbb{d}\int_{\mathcal{B}} q_\xi - \int_{\mathcal{B}} q_{\mathbb{d}\xi} = \int_{\mathcal{B}} (\mathbb{d}\kappa_H \tilde{q}_\xi + \kappa_H \mathbb{d}\tilde{q}_\xi - q_{\mathbb{d}\xi}) = \int_{\mathcal{B}} (\mathbb{d}\kappa_H \tilde{q}_\xi + \kappa_H \mathbb{d}\tilde{q}_\xi - q_{\frac{\mathbb{d}\kappa_H}{\kappa_H}\xi}) = \frac{\kappa_H}{2\pi}\mathbb{d}S, \tag{6.18}$$

where we used the property, known as the **zeroth law of black hole thermodynamics** [22], that κ_H is constant on the horizon. Combining this expression with the terms at $\mathcal{S}$ that we evaluated before gives

$$\boxed{\mathbb{d}E \doteq \Omega_H^i \mathbb{d}J_i + \frac{\kappa_H}{2\pi}\mathbb{d}S}, \tag{6.19}$$

which is the celebrated **first law of black hole thermodynamics**. The analogy with thermodynamics suggests identifying $\kappa_H/2\pi$ with a temperature, a view that is consistent with Hawking's prediction that black hole radiate exactly at that temperature [24]

$$T_H = \frac{\kappa_H}{2\pi}. \tag{6.20}$$

Moreover, for GR, the Wald entropy is proportional to the surface area of the horizon, as expected from Bekenstein's considerations [25]. The numerical factor $1/4G_N$ is fixed by once the temperature is fixed to T_H.

$$S = \frac{A_H}{4G_N}. \tag{6.21}$$

[3] This modification does not affect the term at infinity because Brown–York charges only depend on the value of ξ at $\mathcal{I}$ and we demand that $\mathbb{d}\bar{\xi} = 0$.

As we saw, we have been able to define canonical charges that satisfy the first law without the need to postulate any *integrability condition*. In the early approaches to the CPS, the role of boundary quantities was not properly recognized. The Noether current and the symplectic potential were respectively taken to be J_ξ and θ only. Accordingly, it was required that a vector field contraction with $\mathbb{d}\theta$ was integrable by itself. This poses no problem for diffeomorphisms that asymptote to $\bar{\varphi}_{(i)}$ to the boundary, but the situation is problematic for those that asymptote to $\bar{t}$ (let us call them $\bar{t}$ as well with a slight abuse of notation). In fact, the latter is a corner translation and, as we saw in Section 5.4.1, the integrability has to been either proven or postulated as the existence of a $(n-2,0)$-form $b_{\bar{t}}$ such that $\mathbb{d}b_{\bar{t}} = \mathbb{I}_{\chi_{\bar{t}}}\theta$. A careful application of the CPS with boundaries shows that the contraction with the true symplectic current $\mathbb{d}(\theta - \mathrm{d}\vartheta)$ is always integrable thanks to the contribution of the boundary Noether current. In fact, it is $j_{\bar{t}}$ the $(n-2,0)$-form that unambiguously plays the role of $b_{\bar{t}}$.

6.3 Review of Lanczos–Lovelock theories

In this section, we present a short introduction to LL theories with the goal to apply the CPS formalism to their solution space. Particular emphasis will be put on black hole solutions. These theories were born as a generalization of GR, when Lovelock set out to find the most general set of Lagrangians that produce second order field equations for the metric tensor [211, 212]. The resulting structure is that of a sum of terms, that we will call k-terms, with arbitrary coefficients c_k. Interest in these models resparked when they were found to describe low energy limit of supersymmetric string theory [134]. For an extensive review of the subject, the reader can consult [116].

The bulk Lagrangian *scalar* of a generic LL theory is given by

$$\mathscr{L} = \sum_k c_k \mathscr{L}^{(k)}, \quad \text{with} \quad \mathscr{L}^{(k)} = \frac{1}{2^k}\delta^{\mu_1\nu_1\dots\mu_k\nu_k}_{\rho_1\sigma_1\dots\rho_k\sigma_k}\prod_{i=1}^{k} R^{\rho_i\sigma_i}_{\mu_i\nu_i}. \tag{6.22}$$

Here $\delta^{\mu_1\nu_1\dots\mu_k\nu_k}_{\rho_1\sigma_1\dots\rho_k\sigma_k}$ is the generalized Kronecker delta, antisymmetric in all upper and lower indices. We also wrote $R^{\mu\nu}_{\rho\sigma}$ both to highlight the symmetry of the (completely covariant) Riemann tensor under the swap of its first and second pair of indices

$$R^{\mu\nu}{}_{\rho\sigma} = R_{\alpha\beta\rho\sigma}g^{\alpha\mu}g^{\beta\nu} = R_{\rho\sigma\alpha\beta}g^{\alpha\mu}g^{\beta\nu} = R_{\rho\sigma}{}^{\mu\nu}, \tag{6.23}$$

and also because, for LL theories, it is more natural to consider the contraction $R_{\rho\sigma}{}^{\mu}{}_{\beta}g^{\beta\nu}$ as the fundamental object, rather than $R_{\rho\sigma}{}^{\mu}{}_{\beta}$ itself. The Lagrangian (6.22) is proven to be the most general that can be constructed out of the Riemann tensor (but not its derivatives) such that the corresponding equations of motion for the metric are second order. The index k can be taken to go from 0 to $\lfloor n/2 \rfloor$. Beyond that value, the antisymmetry of the generalized Kronecker delta over more than n indices would make the contribution vanish

anyway. Once a maximum value of k is fixed (say m), the *critical dimension* is defined as $n = 2m$. In this dimension, the Lagrangian density $\mathscr{L}^{(m)}$ is a total derivative and hence does not contribute to the equations of motion of the theory[116]. This is a generalization of the observation we made about the GB Lagrangian in Section 4.3.

As exemplified in (6.22), some expressions are valid only when the contribution of each k-term is included. Other results, such as the expression of $\mathscr{L}^{(k)}$, are true term by term. We will henceforth include an explicit sum over k only for the former class of results.

In the study of LL theories, it turns out very useful to introduce the so-called Lovelock tensors

$$P^{(k)\mu\nu}_{\quad\rho\sigma} \equiv \frac{\partial \mathscr{L}^{(k)}}{\partial R^{\rho\sigma}_{\mu\nu}}\bigg|_{g_{\alpha\beta}} = \frac{k}{2^k}\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\rho\sigma\rho_2\sigma_2\ldots\rho_k\sigma_k}\prod_{i=2}^{k} R^{\rho_i\sigma_i}_{\mu_i\nu_i}, \tag{6.24}$$

which have the same symmetries as the Riemann tensor and are covariantly conserved $\nabla_\mu P^{(k)\mu\nu}_{\quad\rho\sigma} = 0$. This property follows from the antisymmetry of the generalized delta antisymmetry and the differential Bianchi identities $\nabla_{[\lambda} R^{\rho\sigma}_{\mu\nu]} = 0$. Since the Lagrangians $\mathscr{L}^{(k)}$ are homogenous polynomials in the Riemann tensor, we can conveniently express them as

$$\mathscr{L}^{(0)} = 1, \qquad \mathscr{L}^{(k>0)} = \frac{1}{k} P^{(k)\mu\nu}_{\quad\rho\sigma} R^{\rho\sigma}_{\mu\nu}. \tag{6.25}$$

The equations of motion of the theory can be written in terms of the generalized Einstein tensor [4]

$$\mathscr{G}_{\alpha\beta} = \sum_k c_k \mathscr{G}^{(k)}_{\alpha\beta}, \quad \text{with} \quad \mathscr{G}^{(k)\alpha}_{\quad\beta} \equiv P^{(k)\mu\nu}_{\quad\beta\sigma} R^{\alpha\sigma}_{\mu\nu} - \frac{1}{2}\delta^\alpha_\beta \mathscr{L}^{(k)}. \tag{6.27}$$

Just like their usual counterpart, each Einstein tensor $\mathscr{G}^{(k)}_{\alpha\beta}$ is symmetric and divergenceless. When matter is present, the field equations are $\mathscr{G}_{\alpha\beta} \propto T_{\alpha\beta}$, but we will be only concerned with vacuum solutions, i.e. $\mathscr{G}_{\alpha\beta} = 0$. As previously mentioned, despite the Lagrangian can contain high-curvature terms, these field equations are by construction second-order in the metric tensor. This property stems from the divergenceless of the Lovelock tensors.

6.3.1 Pure and generic theories

Taking the trace of the vacuum field equations and the expression (6.27), we can easily obtain the relation

$$\sum_k c_k (n - 2k)\mathscr{L}^{(k)} \mathrel{\hat{=}} 0. \tag{6.28}$$

[4] Notice that each generalized Einstein tensor admits the compact expression

$$\mathscr{G}^{(k)\alpha}_{\quad\beta} = -\frac{1}{2^{k+1}}\mathbb{d}^{\alpha\mu_1\nu_1\ldots\mu_k\nu_k}_{\beta\rho_1\sigma_1\ldots\rho_k\sigma_k}\prod_{i=1}^{k} R^{\rho_i\Sigma_{\text{in}}}_{\mu_i\nu_i}. \tag{6.26}$$

Unlike in GR, we cannot conclude from this that the bulk Lagrangian itself vanishes. A sufficient condition for this to be true is if the sum contained a single term. This motivates the distinction between *pure* and *generic* LL theories: in the former, there is only one k such that $c_k \neq 0$.

Pure theories have a lot more in common with GR, especially in their geometrical description. It has been proven that vacuum solutions of pure LL theories in $(2k+1)$ dimensions are generalized flat [213], in the sense that, for each k, there exists a generalized Riemann tensor which satisfies differential Bianchi identities [214, 215] and vanishes on these solutions. Furthermore, their vacuum state (the maximally symmetric vacuum solution) is just flat spacetime. On the other hand, generic LL admit cosmological vacuum solutions, even in the absence of the cosmological constant Lagrangian term $c_0\mathscr{L}^{(0)}$. To see this, we can rewrite the total generalized Einstein tensor as

$$\mathscr{G}^{\alpha}_{\beta} = -\sum_{k=0}^{m} \frac{c_k}{2^{k+1}} \delta^{\alpha\mu_1\nu_1\ldots\mu_k\nu_k}_{\beta\rho_1\sigma_1\ldots\rho_k\sigma_k} \prod_{i=1}^{k} R^{\rho_i\sigma_i}_{\mu_i\nu_i} = -\frac{c_m}{2^{m+1}} \delta^{\alpha\mu_1\nu_1\ldots\mu_m\nu_m}_{\beta\rho_1\sigma_1\ldots\rho_m\sigma_m} \prod_{i=1}^{m} (R^{\rho_i\sigma_i}_{\mu_i\nu_i} - \alpha_i \delta^{\rho_i\sigma_i}_{\mu_i\nu_i}), \quad (6.29)$$

for appropriate $m \leq \lfloor n/2 \rfloor$ and coefficients $\alpha_1, \ldots, \alpha_m$ which are combinations of the original coefficients $c_0, \ldots, c_m$. Whereas these combinations are usually complex, some of them might be real. If this happens, the theory admits up to m constant curvature vacua

$$R^{\rho\sigma}_{\mu\nu} = \alpha_{\underline{i}} \delta^{\rho\sigma}_{\mu\nu}, \quad \forall \alpha_{\underline{i}} \in \mathbb{R}. \quad (6.30)$$

6.3.2 Boundary Lagrangian for Lanczos–Lovelock theories

Since the field equations are second order, we expect to find a one-to-one correspondence between Cauchy data on a slice Σ_{in} and the solutions to the field equations [216, 217]. A well-posed initial value formalism ensures that the Hamiltonian functions constructed in Section 6.1 are sufficiently smooth to generate a good flow in phase space.

As extensively discussed, we can obtain the solutions from a variational principle only if we remove boundary contributions by boundary conditions. Here, we are interested in the solution space defined by Dirichlet boundary conditions at $\mathcal{I}$

$$\delta h_{ab} = 0, \quad (6.31)$$

where we repeat that $h \equiv i^*_{\mathcal{I}} g$. Spurious terms in $\mathbb{d}L$ will depend on normal (to $\mathcal{I}$) derivatives of the metric. Explicitly, the (pre-)symplectic potential reAdS

$$\theta = \sum_k c_k \theta^{(k)}, \quad \text{with} \quad \theta^{(k)} = P^{(k)\,\mu\nu}_{\rho\sigma} \Big(g^{\rho\lambda}\nabla^{\sigma} - g^{\sigma\lambda}\nabla^{\rho} \Big) \mathbb{d} g_{\lambda\nu}\, \iota_{\partial_\mu} \epsilon_{\mathcal{M}}. \quad (6.32)$$

As in standard GR with Dirichlet boundary conditions, we can get rid of these terms with a boundary covariant Lagrangian constructed from the boundary extrinsic curvature [108,

163]. In terms of the boundary Lagrangian *scalar* ($\ell = \ell\epsilon_{\mathcal{I}}$), we have [116, 218]

$$\ell = \sum_k c_k \ell^{(k)}, \quad \text{with } \ell^{(k)} = \frac{2k}{2^{k-1}} \int_s \delta^{ia_2b_2\dots a_kb_k}_{jc_2d_2\dots c_kd_k} K_i^j \prod_{i=2}^{k} \left(\mathcal{R}^{c_id_i}_{a_ib_i} - 2s^2 K^{c_i}_{a_i} K^{d_i}_{b_i} \right), \tag{6.33}$$

where the symbol $\int_s$ is just a shortcut for the integral of s over $[0,1]$. The term within round brackets is the curvature of the so-called *interpolating connection*, which, as the name suggests, interpolates between the curvature of the boundary, at $s = 0$, and the curvature of the bulk, projected on the boundary, at $s = 1$ (cfr. Section 6.1 of Ref. [219]).

As we show in Appendix C, one can prove that

$$\mathbb{d}\ell = \frac{1}{2}\mathcal{T}^{ab}\mathbb{d}h_{ab}\epsilon_{\mathcal{I}} - i^*_{\mathcal{I}}\theta + \mathrm{d}\vartheta \tag{6.34}$$

where $\mathcal{T}_{ab}$ generalizes the Brown–York tensor of GR (4.17)

$$\mathcal{T}_{ab} = \sum_k c_k \mathcal{T}^{(k)}_{ab}, \quad \text{with} \quad \mathcal{T}^{(k)}_{ab} = -4\pi^{(k)}_{c(a} K_{b)c} + h_{ab}\ell^{(k)} - 4(k-1)\int_s p^{(k)\,idc}_{\ \ je(a} R_{b)}{}^{e}{}_{cd} K^j_i;, \tag{6.35}$$

and ϑ is the corner symplectic potential, whose expression is [5]

$$\vartheta = \sum_k c_k \vartheta^{(k)}, \quad \text{with} \quad \vartheta^{(k)} = 2\pi^{(k)ab}\mathbb{c}_b \iota_{\partial_a}\epsilon_{\mathcal{I}} + 4(k-1)\int_s p^{(k)\,iab}_{\ \ jcd} K^j_i \mathcal{D}^c \mathbb{d}h^d_b \iota_{\partial_a}\epsilon_{\mathcal{I}}. \tag{6.36}$$

To simplify the notation of the previous formulas, we introduced the tensors

$$p^{(k)\,iab}_{\ \ jcd}(s) := \begin{cases} 0 & \text{for} \quad k = 0, 1, \\ \delta^{iab}_{jcd} & \text{for} \quad k = 2, \\ \frac{k}{2^{k-1}} \delta^{iaba_3b_3\dots a_kb_k}_{jcdc_3d_3\dots c_kd_k} \prod_{i=3}^{k} \left(\mathcal{R}^{c_id_i}_{a_ib_i} - 2s^2 K^{c_i}_{a_i} K^{d_i}_{b_i} \right) & \text{for} \quad k > 2, \end{cases} \tag{6.37}$$

and we denoted by $\pi^{(k)a}_{\ \ b}$ and $\mathbb{c}_a$ the pull-back to $\mathcal{I}$ of

$$P^{(k)\alpha\mu}_{\ \ \beta\nu}\, n_\mu n^\nu \qquad \text{and} \qquad \mathbb{c}_\alpha \equiv -\mathbb{d}g_{\mu\nu} h^\mu_\alpha n^\nu. \tag{6.38}$$

Obviously, if all $c_k = 0$ for $k > 1$, we recover the familiar expressions of GR

$$\mathcal{T}^{(1)}_{ab} = 2(Kh_{ab} - K_{ab}), \qquad \vartheta^{(1)} = \mathbb{c}^a \iota_{\partial_a}\epsilon_{\mathcal{I}} = \iota_{\mathbb{c}}\epsilon_{\mathcal{I}}, \tag{6.39}$$

which we can compare to eq. (5.58).

Given that $B = \frac{1}{2}\mathcal{T}_{ab}\mathbb{d}h^{ab}\epsilon_{\mathcal{I}}$ obviously vanishes if we impose Dirichlet boundary conditions on the metric, what remains in the action variation is just a corner term (i.e. ϑ), which is consistent with our general construction to achieve a well-posed variational principle.

[5] This symplectic potential has been obtained also in Refs. [220, 221]. Our expression agrees with that of Ref. [221] by direct inspection, whereas the comparison with [220] is more involved.

6.3.3 Black holes in Lanczos–Lovelock theories

Since LL theories are generally-covariant, we expect a large degree of redundancy associated with diffeomorphisms that are pure gauge. Indeed, as predicted by Noether's second theorem, not all the equations of motion are independent because the generalized Einstein tensor satisfies the generalized Bianchi identities for the $\nabla_\mu \mathcal{G}^{\mu\nu} = 0$. Considering the Noether current horizontal derivative, we get

$$\mathrm{d}J_\xi = \mathbb{I}_{\mathbb{X}_\xi}(\mathcal{G}^{\mu\nu}\mathbb{d}g_{\mu\nu}\epsilon_{\mathcal{M}}) = \mathcal{G}^{\mu\nu}\mathbb{L}_{\mathbb{X}_\xi} g_{\mu\nu}\epsilon_{\mathcal{M}} = 2\mathcal{G}^{\mu\nu}\nabla_\mu\xi_\nu\epsilon_{\mathcal{M}} = 2\nabla_\mu(\mathcal{G}^{\mu\nu}\xi_\nu)\epsilon_{\mathcal{M}}, \tag{6.40}$$

whence it follows that J_ξ is exact (assuming we can use Poincaré's lemma) up to the following constraint current

$$c_\xi = -2\iota_{\mathcal{G}\cdot\xi}\epsilon_{\mathcal{M}}, \tag{6.41}$$

where $(\mathcal{G}\cdot\xi)^\mu = \mathcal{G}^\mu{}_\nu\xi^\nu$. The resulting (generalized) Komar charge aspect is then

$$q_\xi = \sum_k c_k q_\xi^{(k)}, \quad \text{with} \quad (\star q_\xi^{(k)})_{\mu\nu} = P^{(k)\rho\sigma}{}_{\mu\nu}(\nabla_\rho\xi_\sigma - \nabla_\sigma\xi_\rho). \tag{6.42}$$

An immediate and relevant consequence of this form of q_ξ is that the Wald entropy of LL theories does not coincide with the Bekenstein–Hawking one, as it is not simply proportional to the surface area of the horizon, but also includes a sum of (integrated) intrinsic curvature invariants [222]. As a matter of fact, using the relation $\nabla_{[\mu}\xi_{\nu]} = \kappa_H\varepsilon_{\mu\nu}$, we can show that the Wald entropy of a LL theory is

$$S = 4\pi\int_{\mathcal{B}}\sum_k k c_k \mathscr{L}^{(k-1)}\epsilon_{\mathcal{S}}, \tag{6.43}$$

which coincides with the action of a "reduced" LL theory: this theory lives on the bifurcation surface and it is defined by a set of LL coefficients that can be obtained from the original ones.

Finding black hole solutions in Lanczos–Lovelock proved to be a hard task. The first exact solution was found in [134], motivated by low energy supersymmetric string theory considerations. After that, many other solutions have been found (see the nice reviews in Refs [223] and [224]). Moreover, results have been obtained in the study of Birkhoff's theorem in LL theories, at least for the particular class known as *Einstein–Gauss–Bonnet* [225–227], but important milestones in our understanding of black holes do not survive in more than four dimensions, where LL theories become non-trivial. For example, it is known that Hawking topology theorem [23] does not hold in higher dimensions [228]; the rigidity theorem [229] has been generalized to prove the existence of one rotational isometry for stationary black holes [230], but it is unclear whether the result can be made stronger. See [198, 231] for a review on the subject.

For simplicity, we are going to study *Schwarzschild-like* solutions, i.e. static and spherically symmetric spacetimes with the following line element [6]

$$\mathrm{d}s^2 = -\Xi(r) f(r) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_{n-2}^2. \tag{6.44}$$

With this ansatz, the generalized Einstein tensor $\mathcal{G}_\nu^\mu$ is diagonal, and its angular components are all equal. The combination $\mathcal{G}_t^t - \mathcal{G}_r^r$ of the field equations implies that Ξ is a constant, so we fix $\Xi = 1$. The remaining field equations are first order and second order differential equations for f, but the latter depends on the former because of the Bianchi identities. The first order equation can be integrated exactly, yielding f as the solution of an algebraic equation

$$\sum_k \hat{c}_k \left(\frac{1-f}{r^2} \right)^k = \frac{M}{\Omega_{n-2}\, r^{n-1}}, \tag{6.45}$$

where we introduced the rescaled couplings

$$\hat{c}_k = \frac{(n-2)!}{(n-2k-1)!} c_k, \tag{6.46}$$

and fixed the integration constant in terms of the ADM mass. We will soon justify this identification.

To preserve spherical symmetry and time translation invariance, we consider $\mathcal{I}$ to be a straight cylinder $[t_{\text{in}}, t_{\text{fin}}] \times \mathcal{S}$ and take $\mathcal{S}$ to be a sphere at $r = R$. We fix the metric pull-back on $\mathcal{I}$ to the prescribed value

$$\bar{h}_{ab} \mathrm{d}x^a \mathrm{d}x^b = -\mathrm{d}\tau^2 + R^2 \mathrm{d}\Omega_{n-2}^2, \tag{6.47}$$

where τ is the proper time of an observer hovering at $r = R$. These boundary conditions fix the value of $f(R)$ in terms of the relative redshift ν_R between the times τ and t

$$f(R) = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t} \right)^2 \equiv \nu_R^2. \tag{6.48}$$

The condition of asymptotic flatness imposes that $\nu_R \to 1$ as $R \to \infty$.

Given the solution, we can compute the Noether charges associated with the Killing vector field $\xi = \partial_t$. The energy measured by the Brown–York charge is dual to the Schwarzschild coordinate time t. On the boundary, this is related to the proper time of the static observer by $\mathrm{d}t = \nu_R \mathrm{d}\tau$, which means that the proper energy measured by the

[6] The $r = $ const. surfaces are $(n-2)$ spheres, obtained by a scaling of the unit sphere with metric $\mathrm{d}\Omega_{n-2}^2$, total area $\Omega_{n-2} = 2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)$ and curvature $(n-2)(n-3)$.

observer is related to the Brown–York energy by a simple redshift factor $\mathcal{E}_R = E_R/\nu_R$. Taking into account the boundary condition (6.48), the explicit computation yields

$$\mathcal{E}_R = -\sum_k k\hat{c}_k \int_0^1 \mathrm{d}s \left(\frac{1-s^2 f(R)}{R^2}\right)^{k-1} \frac{2\sqrt{f(R)}}{R} \Omega_{n-2} R^{n-2}. \tag{6.49}$$

In order to extract the properties of the black hole, we need to get rid of finite-size effects by sending the boundary to infinity. In this limit, quantities such as (6.49) generically diverge. To get around this problem, we consider the quantity that the same observer would measure in a reference spacetime with the same boundary conditions, but defined in such a way that $M = 0$ in (6.45). Subtracting this quantity from the original one, we obtain something which is finite in the limit $R \to \infty$

$$\mathcal{E}_R^{\mathrm{reg}} := \mathcal{E}_R - \mathcal{E}_R^{bg}. \tag{6.50}$$

This regularization procedure is called **background subtraction**. We then define a regularized Brown–York energy E_R^{reg} by the relation

$$E_R^{\mathrm{reg}} = \nu_R \mathcal{E}_R^{\mathrm{reg}}. \tag{6.51}$$

Other quantities measured by the boundary observer can be constructed in a similar way. For this class of solutions though, we obviously have $J_i = 0$.[7] We can now check explicitly that, in the $R \to \infty$ limit,

$$E_\infty^{\mathrm{reg}} = \lim_{R\to\infty} E_R^{\mathrm{reg}} = \lim_{R\to\infty} \nu_R(\mathcal{E}_R - \mathcal{E}_R^{bg}) = M, \tag{6.52}$$

where we used the fact that, at large distances, the effect of a massive source decays as $f - f_{bg} \propto 1/R^{n-3}$. This result shows that the energy of the source is indeed the ADM mass, and that is given by the M parameter in (6.45).

Area law As we mentioned, the Wald entropy (6.43) of a LL black hole does not follow Bekenstein's area law. Indeed, singling out the first coefficient, we can express it as

$$S = \frac{A_H}{4G_N} + 2\pi \sum_{k=1}^{m} \frac{2k\hat{c}_k}{n-2k} \left(\frac{1}{r_H^2}\right)^{k-1} \Omega_{n-2} r_H^{n-2}. \tag{6.53}$$

Just as in other modified theories of gravity, it is not obvious that S is positive definite, as it should be if we want to interpret it as an entropy (see Refs. [235, 236] for results on the validity of the second law of thermodynamics in LL theories [235, 236] and Refs. [205, 206]

[7] Only a few exact analytic solutions for rotating black holes in LL theories have been found so far [232–234].

for generic generally covariant theories). For this reason, it is tempting to look for possible modifications of this result.

First of all, we notice that, even in the critical dimension $n = 2m$, the ratio $\hat{c}_m$ to $(n-2m)$ is still finite. This leads to the paradoxical conclusion that a term in the Lagrangian can affect the entropy despite not contributing to the equations of motion ($\mathscr{L}^{(m)}$ is a total derivative). The resolution is that this contribution is harmless because it is just a constant. As a matter of fact, just as $\mathscr{L}^{(m)}$ is topological in $n = 2m$, $\mathscr{L}^{(m-1)}$ is topological in the horizon cross-section dimension $n - 2 = 2(m - 1)$: the result is proportional to the Euler characteristic of the manifold.

Following this line of reasoning, one could envision a way to remove higher-order corrections from (6.53) and get a manifestly positive entropy for LL theories. The idea is based on the fact that, whereas $L^{(k)}$ can be written as the exterior derivative of a potential $w^{(k)}$ only in $n = 2k$, one could always shift it by such a potential in any dimension $\mathscr{L}^{(k)} \to \mathscr{L}^{(k)} - \mathrm{d}w^{(k)}$ without affecting the equations of motion [237]. However, the precise treatment of boundary terms that is required by the well-posedness of the variational principle forbids such a shift. If we were to do so, the black hole whose entropy we are trying to compute would not even be a solution anymore.

Let us make a further comment. In order to preserver the first law, entropy cannot be modified unless the Hawking temperature is modified as well. While it is true that in alternative theories (such as Horndeski's [189]), the Hawking temperature needs to be modified when gravitons do not move at the speed of light, it has been checked for LL theories that this is not true, at least for radially propagating gravitons: despite the intrinsic non-linearity of LL gravity, they move exactly at the speed of light [238, 239]. Since this scenario is ruled out, there seems to be no other reason to alter the Hawking temperature.

6.4 Smarr formula

Despite the phenomenological differences between black holes in alternative theories of gravity, the aforementioned rarity of exact solutions, and the difficulty in studying their properties, there is a formula that seems to hold for all of them, just like the first law of thermodynamics. The relevance of this formula is that it provides a relation between the various parameters of the solution that can be used also when the analytic form of the latter is unknown. This is the **Smarr formula**. In four-dimensional GR, the formula relates the black hole mass to its other macroscopic properties, like the horizon area and the angular momentum, and it looks very simple [240]

$$M = 2\,\Omega_H J + \frac{\kappa_H A_H}{4\pi G_N}. \tag{6.54}$$

However, this simple expression was found to be incorrect as soon as GR was supplemented with a non-zero cosmological constant [241].

Historically, there are two ways to derive the Smarr formula in GR. One is by means of the famous *Komar integral relation* [68], which follow from the existence of a non-trivial $(n-2)$-spacetime form that is closed on-shell; integrating this form on a submanifold extending between two surfaces yields an equality between quantities evaluated on those surfaces. As we will see and generalize in Section 6.4.3, this quantity is deeply related to the Noether current J_ξ. The other method is by applying a *scaling argument* to the first law of thermodynamics written in terms of variation. Considering the variation induced on each quantity by an overall change in the length scale, we obtain an integral relation. Our main focus will be on this second method, as tying the Smarr formula to the first law of thermodynamics will offer an explanation for its apparent ubiquity.

6.4.1 Scale invariance and the Smarr formula in General Relativity

As a warm-up, we review the derivation of the Smarr formula in GR, in any spacetime dimension, following the scaling argument. Then, we reformulate the argument using the field space formalism.

In pure GR without cosmological constant, we can harmlessly set $\kappa^2 \equiv 8\pi G_N = 1$ from the beginning and get a theory with no dimensional couplings. In such a theory, the parameters of a black hole solution can only depend on dynamical scales. Under an overall change of length scale, each of them transforms homogeneously with a degree that is fixed by the *length dimension*,[8]. Therefore, expressing one of the parameters in term of the others yield a homogeneous function as well. Applying Euler's theorem on homogeneous functions to $E = E(J_i, S)$ and using dimensional analysis, we get

$$(n-3)E = (n-2)\frac{\partial E}{\partial J_i}\bigg|_S J_i + (n-2)\frac{\partial E}{\partial S}\bigg|_{J_i} S. \tag{6.55}$$

The derivative terms, known as *thermodynamic potentials*, can then be obtained from the first law

$$\frac{\partial E}{\partial J_i}\bigg|_S = \Omega^i_H, \qquad \frac{\partial E}{\partial S}\bigg|_{J_i} = T_H, \tag{6.56}$$

and replaced in the first expression, eventually yielding the familiar Smarr formula

$$(n-3)E = (n-2)\Omega^i_H J_i + (n-2)T_H S. \tag{6.57}$$

The reason why length dimensions appear to be off by $(n-2)$ — indeed one would expect $[E] = -1$, $[J] = 0$, and $[S] = 0$ — is because we are actually considering $G_N E$, $G_N J_i$ and $G_N S$, so we need to account for the length dimension of Newton's constant $[G_N] = n-2$ [242].

[8] The opposite of the mass dimension.

In order to make contact with the CPS — and to open the path to a generalization to LL theories — we reinterpret these considerations in terms of the solution space. We realize the overall change in length scale as a transformation of the metric field

$$g_{\mu\nu} \to W^2 g_{\mu\nu} \tag{6.58}$$

with W a spacetime constant. We will refer to this transformation (6.58) as *global* Weyl scaling or as an overall dilatation. Under an infinitesimal transformation $W \simeq 1 + w$, we have $g_{\mu\nu} \to g_{\mu\nu} + 2w g_{\mu\nu}$, so that we can define the following vector field

$$\mathbb{X}_W = \int_{\mathcal{M}} \delta_W g_{\mu\nu}(x) \frac{\mathbb{d}}{\mathbb{d} g_{\mu\nu}(x)}, \tag{6.59}$$

which implements the dilatation at the level of field space $\mathbb{I}_{\mathbb{X}_W} \mathbb{d} g_{\mu\nu} = \delta_W g_{\mu\nu}$. Since the transformation parameter w is a constant, we can just remove it and define $\delta_W g_{\mu\nu} \equiv 2 g_{\mu\nu}$. The vector field $\mathbb{X}_W$ preserve the field equations of pure GR because the Einstein tensor does not transform under a global Weyl scaling [9]

$$G_{\mu\nu}[W^2 g] = G_{\mu\nu}[g]. \tag{6.60}$$

This implies that $\mathbb{X}_W$ is tangent to the space of solutions $\mathcal{S}$ and can be pulled-back to it. Since the first law of thermodynamics (6.19) is nothing but a 1-form field in solution space, it would be tempting to interpret the (pulled-back) vector field $\mathbb{X}_W$ as the generator of a flow which describes an overall dilatation, and integrate the first law form field along it. The only problem with this view is that the first law "lives" in $\bar{\mathcal{S}}$, the space defined by the solutions of the field equations which *also* satisfy the boundary conditions. Conversely, it is easy to see that $\mathbb{X}_W$ cannot be pulled-back to $\bar{\mathcal{S}}$ as it does not preserve Dirichlet boundary conditions

$$\mathbb{I}_{\mathbb{X}_W} \mathbb{d} h_{ab} = 2 h_{ab} \neq 0. \tag{6.61}$$

A quick fix to this issue is to repeat the construction of the first law of thermodynamics that we did in Section 6.2, but starting from

$$\mathbb{d}(J_\xi - \mathrm{d} j_\xi) \doteq - \mathbb{I}_{\mathbb{X}_\xi} \omega + \mathrm{d}\big(\iota_{\bar{\xi}} B\big)\,, \tag{6.62}$$

which generalizes (5.28) to variations in $\mathcal{S}$. Integrating this relation on the surface Σ' that extends from $\mathcal{B}$ to $\mathcal{S}$ produces a 1-form field in $\Omega^1(\mathcal{S})$, which can be sensibly contracted with $\mathbb{X}_W$

$$\mathbb{L}_{\mathbb{X}_W} \int_{\Sigma'} (J_\xi - \mathrm{d} j_\xi) \doteq - \int_{\Sigma'} \mathbb{I}_{\mathbb{X}_W} \mathbb{I}_{\mathbb{X}_\xi} \omega + \int_{\mathcal{S}} \mathbb{I}_{\mathbb{X}_W} \iota_{\bar{\xi}} B \tag{6.63}$$

Let us evaluate the two sides separately.

[9] Obviously, the Einstein equations are not invariant under a *local* Weyl scaling.

- The integral of the l.h.s. produces the first law as we have seen it already, so we can easily write the contraction with $\mathbb{X}_W$ as

$$\mathbb{L}_{\mathbb{X}_W} E - \Omega_H^i \mathbb{L}_{\mathbb{X}_W} J_i - T_H \mathbb{L}_{\mathbb{X}_W} S. \tag{6.64}$$

  To evaluate these Lie derivatives, we notice that the absence of dimensionful couplings implies that the charges E, J_i and S are homogenously constructed out of the metric field only. We then explicitly compute

$$\mathbb{L}_{\mathbb{X}_W} E = \int_{\mathcal{M}} \mathrm{d}^n x \frac{\delta E}{\delta g_{\mu\nu}(x)} \mathbb{L}_{\mathbb{X}_W} g_{\mu\nu}(x) = 2 \int_{\mathcal{M}} \mathrm{d}^n x \frac{\delta E}{\delta g_{\mu\nu}(x)} g_{\mu\nu}(x) = (n-2)E, \tag{6.65}$$

  where, in the last equalities, we used Euler's theorem and considered that the net scaling comes from the corner volume form, for which $\mathbb{L}_{\mathbb{X}_W} \epsilon_{\mathcal{S}} = (n-2)\epsilon_{\mathcal{S}}$. The same is true for J_i and S, so that the l.h.s. becomes

$$(n-2)E - (n-2)\Omega_H^i J_i - (n-2)T_H S; \tag{6.66}$$

- The only contribution to the r.h.s. comes from the $\iota_{\bar{\xi}} B$ term in (5.31) because the contraction $\mathbb{I}_{\mathbb{X}_\xi} \omega$ identically vanishes for an isometry. Using the expression of B (5.60), we can show that this term yields the trace of the Brown–York tensor

$$\mathbb{I}_{\mathbb{X}_W} \iota_{\bar{\xi}} B = 2 \iota_{\bar{\xi}} \mathcal{J}_h^{ab} h_{ab} = (u \cdot \bar{\xi}) T_{BY} \epsilon_{\mathcal{S}}. \tag{6.67}$$

  However, it will be more convenient for us to work under the assumptions we made in Section 6.1, which allowed us to write $\iota_{\bar{\xi}} B = q_{\bar{t}}^{BY} \mathbb{d} \ln \nu + \nu\sigma \mathrm{d}\epsilon_{\mathcal{S}}$. Evaluating the contraction we get

$$\mathbb{I}_{\mathbb{X}_W} \iota_{\bar{\xi}} B = \mathbb{I}_{\mathbb{X}_W} (q_{\bar{t}}^{BY} \mathbb{d} \ln \nu + \nu\sigma \mathbb{d}\epsilon_{\mathcal{S}}) = q_{\bar{t}}^{BY} + (n-2)\nu\sigma\epsilon_{\mathcal{S}} \tag{6.68}$$

  where we used $\mathbb{I}_{\mathbb{X}_W} \mathbb{d} \ln \nu = 1$ and $\mathbb{I}_{\mathbb{X}_W} \mathbb{d}\epsilon_{\mathcal{S}} = (n-2)\epsilon_{\mathcal{S}}$. If we make the further assumption that σ becomes negligible at infinity, we can approximate the integral of the r.h.s. with E.

Putting the two sides together gives

$$(n-2)E - (n-2)\Omega_H^i J_i - (n-2)T_H S = E, \tag{6.69}$$

which gives back the correct Smarr formula (6.57).

Remark These considerations hold for any pure LL theory (defined in Section 6.3.1) and lead to analogous results with the sole replacement $T_{BY}^{ab} \to \mathcal{T}^{ab}$. We decided to focus on GR because it was the first example in which the Smarr formula appeared and because it is by far the most extensively studied pure theory.

6.4.2 Scale invariance breaking

The seemingly irrelevant introduction of the cosmological constant in GR drastically changes the Smarr formula (6.57). More worryingly, while the derivation via Komar integrals is still viable [73], it appears that there is no simple way to recover a simple relation from the first law via the scaling argument. This problem was recognized early on and was soon related to the breaking of scale invariance by the cosmological constant [243] (a similar problem appeared in [244]). The same issue appears more generally in the context of LL theories (in $n > 4$) [241, 245]: if the theory is not pure, there are at least two non-vanishing couplings and their ratio provides a dimensionful constant that cannot be set to 1. The presence of this constant breaks the scale invariance. In this case, recovering the Smarr formula appeared to be problematic as in the case of GR with a cosmological constant. This is not a surprise: GR with a cosmological constant *is* a non-pure LL theory!

Taking the latter as the simplest example, it is easy to see where the presence of a dimensionful coupling such as Λ can spoil the scaling argument. In writing the mass/energy as a function of the other parameters, we can construct all sorts of combinations of Λ and S with the same length dimension of $(n-3)$:

$$S^{\frac{n-3}{n-2}}, \quad S\Lambda^{1/2}, \quad \Lambda S^{\frac{n-1}{n-2}}, \quad (S/\Lambda)^{\frac{n-3}{n}}, \quad S^{1/2}\Lambda^{1-n/4}\dots \tag{6.70}$$

In this case, it is impossible to fix the dependence of E on S just by dimensional analysis [242, 245–247].

6.4.3 Komar integrals

Before attempting to recover the scaling argument, let us show here how to obtain the Smarr formula from the Komar integrals. The goal is to find a non-trivial $(n-2)$-form that is closed on-shell. If we call this form $\not{k}$, then

$$\int_{\partial\Sigma} \not{k} = \int_{\Sigma} \mathrm{d}\not{k} \doteq 0 \tag{6.71}$$

By choosing a hypersurface Σ that stretches from the horizon to the boundary, the previous equation establishes a relation between entropy and canonical charges.

First, let us go back to eq. (6.40) and notice that, for a Killing vector field ξ, the next-to-last term vanishes for symmetry reasons $\mathcal{G}^{\mu\nu}\nabla_\mu\xi_\nu = \mathcal{G}^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} = 0$. So the Noether current is closed also off-shell. Assuming the validity of Poincaré's lemma, it then follows that not only $J_\xi - c_\xi$ admits a potential, but J_ξ and c_ξ separately do. We thus define the **Killing–Lovelock potential** p_ξ by

$$\mathrm{d}p_\xi = c_\xi. \tag{6.72}$$

The existence of this potential is related to the possibility of integrating the equations of motion and get (6.45): indeed, the integral of p_ξ is proportional to the ADM mass.

Obviously, p_ξ is a codimension 2 form that is closed on-shell, but it cannot be the $\not{k}$ we are looking for as the resulting Komar integrals will be trivially equal on-shell ($0 = 0$).

In order to find something meaningful, we first look at the subcase of pure GR (with no cosmological constant) where the Komar integral is just the integrated Komar charge, whence the name

$$\int_{\partial\Sigma} q_\xi \hat{=} \int_\Sigma J_\xi = -\int_\Sigma \iota_\xi L \hat{=} 0. \tag{6.73}$$

As we see, this result is deeply tied to the fact that the bulk Lagrangian of pure LL theories vanishes on-shell. For a generic theory, the best we can say about the on-shell bulk Lagrangian is (6.28), so one might think that there is no Komar relation. However, whereas the Killing–Lovelock potential p_ξ is irrelevant to the Komar argument in pure theories (because its integrals will be trivially equal), the existence of multiple potentials — one for each k-term [10] — allows to obtain non-trivial results. Indeed, it is only the combination $\sum_k c_k p_\xi^{(k)}$ which drops out of the Komar integral. Combining them with different weights allows us to find the sought-after form

$$\not{k} = \sum_k \frac{n-2k}{n-2} c_k (q_\xi^{(k)} + p_\xi^{(k)}). \tag{6.74}$$

The last step would be to relate the quantities the Komar integrals to canonical charges. To do that, we are first going to derive the Smarr formula with the scaling argument, and then prove its equivalence with the formula above.

Proof. We first try to express the Komar form as a linear combination of $q_\xi^{(k)}$ and $c_\xi^{(k)}$

$$\not{k} \stackrel{?}{=} \sum_k c_k (\alpha_k q_\xi^{(k)} + \beta_k p_\xi^{(k)}), \tag{6.75}$$

whose overall normalization is arbitrary. Taking an exterior derivative, we get

$$\mathrm{d}\not{k} = \sum_k c_k (\alpha_k \mathrm{d}q_\xi^{(k)} + \beta_k \mathrm{d}p_\xi^{(k)}) = \sum_k c_k \left[-\alpha_k \iota_\xi L^{(k)} + (\beta_k - \alpha_k) c_\xi^{(k)} \right]. \tag{6.76}$$

Comparing with eq. (6.28) multiplied by $\iota_\xi \epsilon_{\mathcal{M}}$, we see that is possible to achieve the on-shell closure of $\not{k}$ by taking $\alpha_k = \beta_k \propto (n-2k)$. The proportionality factor is fixed in such a way that the result reproduces that of GR when $c_{k\neq 1} = 0$. □

[10] The off-shell nature of the construction allows for the argument to be repeated individually for each term. Only the equations of motion involve a mixing between the k-terms.

6.5 Extended thermodynamics

Originally, the Smarr formula for LL was derived as a natural generalization of the case of GR with a cosmological constant [245] using the Komar relation, as we did in the previous section. However, the physical insight that comes with the scaling argument is much more profound and we would like to retain it. In this section, we combine the lesson of the CPS formalism and the ideas of *extended thermodynamics* [242, 246] to obtain the Smarr formula from the first law of black hole thermodynamics.

Let us consider the effect of a global Weyl scaling on the k-th generalized Einstein tensor

$$\mathcal{G}^{(k)}_{\mu\nu}[W^2 g] = W^{2-2k}\mathcal{G}^{(k)}_{\mu\nu}[g]. \tag{6.77}$$

Since the scaling is homogenous, the vector field $\mathbb{X}_W$ is still tangent to the space of solution of pure theories when evaluated on it, i.e. $\mathbb{L}_{\mathbb{X}_W} E \doteq 0$. However, since the weights depend on k, the total generalized Einstein tensor does not scale homogeneously, and thus $\mathbb{X}_W$ cannot be a symmetry of a non pure LL theory.

The solution to this problem is to consider an *extended framework* in which vector fields are allowed to act on couplings as well. In this framework, we interpret the original field space $\mathscr{F}$ as a subspace of an *extended field space* $\tilde{\mathscr{F}}$, which is essentially $\mathscr{F} \times \mathbb{R}^m$, where m is the number of couplings. The idea that couplings can be treated on par with other fields has many other application in physics [248–251]. The action can be easily extended to a function of $\tilde{\mathscr{F}}$. However, since the Lagrangian does not involve derivatives of the couplings, the presymplectic potential (and form) will not contain $\tilde{\mathbb{d}}c_k$, making the additional directions degenerate even off-shell. We extend the vector field $\mathbb{X}_W$ to a vector field $\tilde{\mathbb{X}}_W \in \mathfrak{X}(\tilde{\mathscr{F}})$ in such a way that $\mathbb{L}_{\tilde{\mathbb{X}}_W} c_k = (2k-n)c_k$, where $(2k-n)$ is the length dimension of c_k. In this way, the product $c_k \mathcal{G}^{(k)}_{\mu\nu}$ does transform homogeneously and independently of k under $\tilde{\mathbb{X}}_W$, thereby leaving the space of solutions invariant under the extended transformation.

6.5.1 First law of extended black hole thermodynamics

Let us first see how the black hole thermodynamics changes in this extended framework. Since we do not need to violate boundary conditions to derive the first law, we can start from the on-shell relation $\mathbb{d}(J_\xi - \mathrm{d}j_\xi) \hat{=} 0$, from which we factor out the c_1 coefficient for convenience. The advantage of our derivation is that it can be done in two steps. First, we consider that, if the function f is a sum of k-terms, its non-extended (vertical) derivative can be written as [11]

$$\mathbb{d}\left(\frac{f}{c_1}\right) = \sum_k \frac{c_k}{c_1}\mathbb{d}f^{(k)} = \tilde{\mathbb{d}}\left(\frac{f}{c_1}\right) - \sum_k \tilde{\mathbb{d}}\left(\frac{c_k}{c_1}\right) f^{(k)}, \tag{6.78}$$

[11] Notice that, for functions $f^{(k)}$ that are independent of the LL coefficients, $\mathbb{d}f^{(k)} = \tilde{\mathbb{d}}f^{(k)}$

using the Leibniz rule. If we apply this to the total Noether charge density associated with the isometry, we obtain

$$\mathbb{d}(c_1^{-1}(J_\xi - \mathrm{d}j_\xi)) = \tilde{\mathbb{d}}(c_1^{-1}(J_\xi - \mathrm{d}j_\xi)) - \sum_k (J_\xi^{(k)} - \mathrm{d}j_\xi^{(k)})\tilde{\mathbb{d}}(c_1^{-1}c_k) \,\hat{=}\, 0. \tag{6.79}$$

Then, as we did in the non-extended case, we integrate this relation on a slice Σ' that connects the bifurcation surface $\mathcal{B}$ to the boundary corner $\mathcal{S}$ and get

$$\tilde{\mathbb{d}}(c_1^{-1}E) - \Omega_H^i \tilde{\mathbb{d}}(c_1^{-1}J_i) - T_H \tilde{\mathbb{d}}(c_1^{-1}S) - \sum_k \Psi_k\, \tilde{\mathbb{d}}(c_1^{-1}c_k) \,\hat{=}\, 0, \tag{6.80}$$

where we denoted by Ψ_k the k-th component of the total Noether charge

$$\Psi_k \equiv \int_\Sigma J_\xi^{(k)} - \int_{\mathcal{S}} j_\xi^{(k)}. \tag{6.81}$$

Equation (6.80) is the **first law of extended black hole thermodynamics**, that is usually presented in the cleaner form

$$\boxed{\delta E = \Omega_H^i \delta J_i + T_H \delta S + \textstyle\sum_k \Psi_k\, \delta c_k.} \tag{6.82}$$

With this quick derivation, we were able to obtain a prescription for the *thermodynamic potentials* conjugated to the (relative) Lovelock couplings that is simple and unambiguous. It easily follows from the extended first law that these quantities satisfy the relations (we set $c_1^{-1} \propto G_N = 1$)

$$\Psi_k = \left.\frac{\partial E}{\partial c_k}\right|_{J_i,S}, \tag{6.83}$$

rather than being defined by it. These potentials quantify the variation of the total mass/energy of the solution with respect to a change in the (relative) couplings of the theory, as first suggested in [241] and confirmed in [252].

Comparison with Kastor–Ray–Traschen

We can compare our expression for the thermodynamic potentials (6.81) with the result found by the authors of Ref. [242] (ψ_k in eq. (37) there), that we report here as:

$$\Psi_{KRT}^{(k)} = B^{(k)} + 2\kappa_H A^{(k)} + \Theta^{(k)}. \tag{6.84}$$

The authors reconstruct the thermodynamic potential conjugated to the ratio c_k/c_1 (there called b_k) in terms of three quantities:

1. $B^{(k)}$, defined as the boundary integral of the $(n-2)$-form that measures the difference of the on-shell (k-th) Hamiltonian density between two nearby solutions, one with a black hole and one without. Notice that the authors assume an asymptotically AdS spacetime, hence the solution wiht no black hole is just pure AdS;

2. $A^{(k)}$, which is just the integral of $k\mathscr{L}^{k-1}\epsilon_{\mathcal{S}}$ over a horizon cross-section (cfr. eq. (6.43));

3. $\Theta^{(k)}$, defined as the difference between the k-th Killing–Lovelock potential integrated on the sphere at infinity $\mathcal{S}_\infty$ and that integrated on a horizon cross-section. Since the integral at $\mathcal{S}_\infty$ make $\Theta^{(k)}$ diverge, they regulate it by *background subtraction*, which means that they subtract the same quantity evaluated on the AdS background.

For the sake of comparison, we also consider a black hole in AdS spacetime. We take Σ to go from $\mathcal{S}$ to $\mathcal{B}$ in the black hole spacetime,[12] while we extend Σ down to the origin in the reference AdS background. The potentials we obtain are respectively

$$\Psi_k = \int_{\mathcal{S}} (q_\xi^{(k)} - j_\xi^{(k)}) - \int_{\mathcal{B}} q_\xi^{(k)} + \int_\Sigma c_\xi^{(k)}, \tag{6.85}$$

$$\Psi_k^{AdS} = \int_{\mathcal{S}^{AdS}} (q_\xi^{(k)} - j_\xi^{(k)}) + 0 + \int_{\Sigma^{AdS}} c_\xi^{(k)}, \tag{6.86}$$

and we should compare $\Psi_{KRT}^{(k)}$ with the difference $\Psi_k - \Psi_k^{AdS}$. We now show that the three pieces, subtracted one by one, exactly correspond to the three pieces in $\Psi_{KRT}^{(k)}$.

1. We first observe that the Hamiltonian density is identified with the constraint current in [242], so that its variation generates a flow with respect to the bulk the symplectic current only:

$$\mathbb{d} c_\xi^{(k)} = -\mathbb{I}_{\chi_\xi}\omega^{(k)} + \mathrm{d}\left(\mathbb{d} q_\xi^{(k)} - \iota_\xi \theta^{(k)}\right). \tag{6.87}$$

   The aforementioned $(n-2)$ form that produces $B^{(k)}$ is thus $\mathbb{d} q_\xi^{(k)} - \iota_\xi\theta^{(k)}$. Whereas the first term is exact, the second is generally not, and the only way to integrate it is to require the existence of $b_\xi^{(k)} \in \Omega^{(n-2,0)}$ such that $\mathbb{d} b_\xi^{(k)} = \iota_\xi \theta^{(k)}$ (crf. Section 5.4.1). As we discussed at the end of Section 6.2, these terms are nothing but the boundary current components $j_\xi^{(k)}$. With this identification, the $B^{(k)}$ term is reconstructed — and generalized to finite-distance boundaries — by the first term in $\Psi_k - \Psi_k^{AdS}$;

2. The term at the horizon clearly coincides with $2\kappa_H A^{(k)}$, as one can check using (6.43);

3. As we discussed in Section 6.4.3, we can replace the bulk integral of the k-th constraint current by the integral of its potential at the boundaries. The boundaries are two ($\mathcal{S}$ and $\mathcal{B}$) when there is a black hole; when there is not, the boundary is just one ($\mathcal{S}^{AdS}$). Thus, the last term in $\Psi_k - \Psi_k^{AdS}$ generalizes $\Theta^{(k)}$ to finite-distance boundaries, and reduces to it when the boundary is sent to infinity $\mathcal{S} \to \mathcal{S}_\infty$.

[12] In our treatment, we chose the bifurcation surface $\mathcal{B}$ as a preferred horizon cross-section, but, due to time translation invariance, any other cross-section would give the same result.

Therefore, the prescription for the thermodynamic potentials found in our work [209] generalizes the previous result to arbitrary backgrounds (indeed, there is no need to refer to any background as long as we work at finite volume). Another advantage is that the definition (6.81) came out naturally from the derivation of the (extended) first law, so that it is compatible with that law even if it is independent *a priori*.

6.5.2 Smarr formula in Lanczos–Lovelock theories

As we learned in Section 6.4.1, we can derive the Smarr formula from the first law using field space differential geometry, but only if we move from $\bar{\mathcal{S}}$ to $\mathcal{S}$. Indeed, $\mathbb{X}_W$ does not preserve Dirichlet boundary conditions. This remains true in the extended field space, because $\tilde{\mathbb{X}}_W$ acts on the metric just as $\mathbb{X}_W$ does. Thus, if we repeat the derivation of the extended first law, but starting from (5.31) rather than (5.28), we find the same l.h.s. as in the first law, but a non-zero r.h.s. due to the $\iota_\xi B$ term, exactly as in the case of GR. Using eq. (6.14) to expand the boundary term, we can write

$$\tilde{\mathbb{d}}(c_1^{-1}(J_\xi - \mathrm{d}j_\xi)) - \sum_k (J_\xi^{(k)} - \mathrm{d}j_\xi^{(k)})\tilde{\mathbb{d}}(c_1^{-1}c_k) \doteq c_1^{-1}\,\mathrm{d}\Big(q_{\bar{t}}^{BY}\tilde{\mathbb{d}}\ln\nu + \nu\sigma\tilde{\mathbb{d}}\epsilon_{\mathcal{S}}\Big)\,, \tag{6.88}$$

where we noted that ν and $\epsilon_{\mathcal{S}}$ only depend on the metric field. Since this relation is valid within the extended solution space, where the boundary conditions are not imposed, we can contract it with the vector field $\tilde{\mathbb{X}}_W$ to get

$$(n-2)\sum_k c_k(J_\xi^{(k)} - \mathrm{d}j_\xi^{(k)}) - 2\sum_k (k-1)c_k(J_\xi^{(k)} - \mathrm{d}j_\xi^{(k)}) \doteq \mathrm{d}q_{\bar{t}}^{BY} + (n-2)\,\mathrm{d}(\nu\sigma\epsilon_{\mathcal{S}})\,, \tag{6.89}$$

where we replaced $\mathbb{l}_{\mathbb{X}_W}\mathbb{d}(c_k/c_1)$ with its length dimension $2(k-1)$, and multiplied both sides by c_1. Integrating the last relation over the usual slice Σ' yields a Smarr-like formula

$$(n-3)E - (n-2)\Omega_H^i J_i - (n-2)T_H S - 2\sum_k (k-1)c_k\Psi_k = (n-2)W_\sigma, \tag{6.90}$$

where we used the previously introduced definition of the thermodynamic potentials associated with the LL couplings (6.81), and we defined the surface tension work term.[13]

$$W_\sigma := \int_{\mathcal{S}} \nu\sigma\epsilon_{\mathcal{S}} \tag{6.92}$$

Equation (6.90) is perhaps the most important contribution of [209]. However, as it is, it is not the Smarr formula yet because it involves quantities which are not intrinsic to the

[13] A more rigorous definition is given in terms of the Brown–York tensor

$$W_\sigma := \frac{1}{n-2}\int_{\mathcal{S}} \mathcal{T}_{ab}\gamma^{ab}\iota_{\bar{t}}\epsilon_{\mathcal{I}}. \tag{6.91}$$

black hole solution, but depend on the position and configuration of the boundary. Before *sending the boundary to infinity* (what this means in practice depends on the geometry under consideration), we need to regularize the quantities that appear in (6.90), as they might diverge because of the infinite volume that we are considering. Repeating what we did in Section 6.3.3, we introduce the regularized quantities (we omit the subscript R)

$$\begin{aligned} E^{\rm reg} &= E - \frac{\nu_R}{\nu_R^{bg}} E^{bg}, \qquad & J_i^{\rm reg} &= J_i - \frac{\nu_R}{\nu_R^{bg}} J_i^{bg}, \\ \Psi_k^{\rm reg} &= \Psi_k - \frac{\nu_R}{\nu_R^{bg}} \Psi_k^{bg}, \qquad & W_\sigma^{\rm reg} &= W_\sigma - \frac{\nu_R}{\nu_R^{bg}} W_\sigma^{bg}, \end{aligned} \tag{6.93}$$

and, taking their $R \to \infty$ limit, the finite quantities

$$\begin{aligned} M &= \lim_{R\to\infty} E^{reg}, \qquad & \mathcal{J}_i &= \lim_{R\to\infty} J_i^{reg}, \\ \psi_k &= \lim_{R\to\infty} \Psi_k^{reg}, \qquad & \mathcal{W}_\sigma &= \lim_{R\to\infty} W_\sigma^{reg}. \end{aligned} \tag{6.94}$$

Taking a suitable combination of the Smarr-like formula for the physical spacetime and that for the background, we eventually arrive at the Smarr formula

$$\boxed{(n-3)M = (n-2)\Omega_H^i \mathcal{J}_i + (n-2) T_H S + 2\textstyle\sum_k (k-1) c_k \psi_k + (n-2)\mathcal{W}_\sigma.} \tag{6.95}$$

We emphasize that this result holds for black holes with spherical horizon topology in vacuum. The generalization to different topologies and charged black holes [253, 254] should not be too difficult as these results have already been obtained with the standard derivation (see e.g. [193]).

Equivalence with Komar relations As promised, we now show that the Smarr formula we just derived is equivalent to the Komar relation (6.74) we found in Section 6.4.3. Considering (6.78), it is easy to derive the analogous relation involving Lie derivatives. If f is a function that can be expressed as a sum of k-terms, we can write

$$\mathbb{L}_{\mathbb{X}_W}\left(\frac{f}{c_1}\right) = \sum_k \frac{c_k}{c_1} \mathbb{L}_{\mathbb{X}_W} f^{(k)} = \mathbb{L}_{\tilde{\mathbb{X}}_W}\left(\frac{f}{c_1}\right) - 2\sum_k (k-1)\frac{c_k}{c_1} f^{(k)}. \tag{6.96}$$

If we take $f = J_\xi - \mathrm{d} j_\xi$, we notice that the r.h.s. of this equation is just the l.h.s. of (6.89). On the other hand, the l.h.s. can be evaluated by contracting eq. (5.31) with $\mathbb{X}_W$

$$\mathbb{L}_{\mathbb{X}_W}(J_\xi - \mathrm{d} j_\xi) = \mathbb{I}_{\mathbb{X}_W} \mathbb{d}(J_\xi - \mathrm{d} j_\xi) \doteq \mathrm{d}\big(\mathbb{I}_{\mathbb{X}_W} \iota_{\bar\xi} B\big)\,. \tag{6.97}$$

Using the relation

$$\mathbb{L}_{\mathbb{X}_W} j_\xi = -\mathbb{I}_{\mathbb{X}_W}(\iota_{\bar\xi} B), \tag{6.98}$$

we obtain that $\mathbb{L}_{\mathbb{X}_W} J_\xi \doteq 0$, which means that the bulk Noether current is invariant under an overall dilatation on solutions to the field equations. Given the existence of the Killing-Lovelock potential, we can recast the bulk integral of the latter as the following boundary integral

$$0 \doteq \mathbb{L}_{\mathbb{X}_W} \int_\Sigma J_\xi = \mathbb{L}_{\mathbb{X}_W} \int_{\partial\Sigma} (q_\xi + p_\xi). \tag{6.99}$$

To compute the Lie derivative, we expand both q_ξ and p_ξ into k-terms, and then notice that each of them have the same eigenvalue $(n-2k)$ under an global Weyl scaling. Therefore, including a factor $1/(n-2)$ for convenience, we find

$$\frac{1}{n-2}\mathbb{L}_{\mathbb{X}_W} \int_{\partial\Sigma} (q_\xi + p_\xi) = \int_{\partial\Sigma} \sum_k c_k \frac{n-2k}{n-2} (q_\xi^{(k)} + p_\xi^{(k)}) = \int_{\partial\Sigma} \mathscr{k}, \tag{6.100}$$

where we recognized that the sum in the middle term coincides with the Komar form (6.74). The on-shell vanishing of this quantity (i.e. the Komar relation) is thus equivalent to the starting point (6.89) of the Smarr formula derivation.

Proof. Let us prove (6.98) by explicitly evaluating the field space Lie derivative of the boundary current. Since ξ is an isometry, we find that $\mathbb{I}_{\mathbb{X}_\xi}\vartheta = 0$, while the $\iota_{\bar\xi}\ell$ term gives

$$\mathbb{L}_{\mathbb{X}_W} j_\xi = -\mathbb{L}_{\mathbb{X}_W}(\iota_{\bar\xi}\ell) = -\iota_{\bar\xi}\mathbb{I}_{\mathbb{X}_W}\mathbb{d}\ell = -\iota_{\bar\xi}\mathbb{I}_{\mathbb{X}_W}(B + \mathrm{d}\vartheta - i_{\mathcal{I}}^*\theta). \tag{6.101}$$

The first term was computed in (6.68) (up to the replacement $T^{ab}_{BY} \to \mathcal{T}^{ab}$), the second vanishes because $\mathbb{I}_{\mathbb{X}_W} s_a = 2g_{\mu\nu}\Pi^\mu_a n^\nu = 0$ and $I_{\mathbb{X}_W}\mathcal{D}^c\mathbb{d}h^d_b = 2\mathcal{D}^c h^d_b = 0$, while the third vanishes because $\mathbb{I}_{\mathbb{X}_W}\mathbb{v}^\mu = 2\nabla_\lambda g^{\mu\lambda} = 0$. Since horizontal and vertical contractions commute, we have our result. □

6.6 Canonical ensemble derivation

In this section, we present a derivation of the equations of black hole thermodynamics using the Euclidean path integral approach [255]. The idea is to interpret the path integral as the partition function of some thermodynamic ensemble (specified by the boundary conditions) and its saddle points as the dominant contributions of the partition function, i.e. the favored states. Replacing $t \to it_E$ in (6.44), we readily obtain the ansatz for static and spherically symmetric saddle points. In the following, we are going to restrict to this class only.

Notice that the boundary $\mathcal{I}$ has now the topology of $S^1 \times S^{n-2}$. Imposing Dirichlet boundary conditions on it amounts to fix the system size, via the total area of the boundary sphere, and its inverse temperature β, via the length of the Euclidean circle:

$$A_{\mathcal{S}} = R^{n-2}\Omega_{n-2}, \qquad \beta = \nu_R \Delta t_E. \tag{6.102}$$

These considerations prove that the path integral with Dirichlet boundary conditions is the partition function of a *canonical ensemble*. Accordingly, in a (semi-)classical limit, we identify the on-shell Euclidean action with βF, where F is the Helmholtz free energy.

Once the value of β is fixed, the location — and the very presence — of black hole horizons is restricted. Indeed, the near horizon topology is generically that of a cone (times the sphere)

$$\mathrm{d}s^2 = f(r)\mathrm{d}t_E^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_{n-2}^2 \simeq \frac{f'(r_H)^2}{4}\rho^2\mathrm{d}t_E^2 + \mathrm{d}\rho^2 + r^2\mathrm{d}\Omega_{n-2}^2, \tag{6.103}$$

where we introduced the coordinate ρ as the proper length from the horizon. To avoid the conical singularity (or to *flatten the cone*), we must impose that the angular coordinate has period 2π

$$\frac{f'(r_H)}{2}\Delta t_E = 2\pi \implies \beta = \frac{4\pi\nu_R}{f'(r_H)}. \tag{6.104}$$

The resulting regular topology is that of a cigar $D^2 \times S^{n-2}$, whose boundary is indeed $S^1 \times S^{n-2}$. The corresponding temperature at the boundary is thus

$$T = \frac{1}{\beta} = \frac{f'(r_H)}{4\pi\nu_R} = \frac{\kappa_H}{2\pi\nu_R} = \frac{T_H}{\left|\bar{\xi}\right|} \tag{6.105}$$

consistently with the Tolman relation and with the fact that, in asymptotically flat spacetime, (for which $\left|\bar{\xi}\right| = \nu_R \to 1$ as $R \to \infty$), the temperature measured at infinity is the one that Hawking predicted.

The on-shell action, expressed as a function of β and R gives the free energy

$$\begin{aligned} F(\beta, R) = &- \frac{2\pi}{\beta}\sum_k \frac{2k\hat{c}_k}{n-2k}\left(\frac{1}{r_H^2}\right)^{k-1}\Omega_{n-2}r_H^{n-2} + \\ &- \sum_k k\hat{c}_k \int_0^1 \mathrm{d}s \left(\frac{1-s^2\nu_R^2}{R^2}\right)^{k-1}\frac{2\nu_R}{R}\Omega_{n-2}R^{n-2}, \end{aligned} \tag{6.106}$$

where ν_R and r_H are intended here as functions of β and R. The expression comprises two pieces, one evaluated at the horizon and one at the boundary sphere. Notice that, in the former, the total time interval combined precisely with the $f'(r_H)$ factor to give 2π. To obtain the functions ν_R and r_H, one can follow these steps [256]:

1. Solve the field equations relating the value of the integrated constraint at different radii, this yield $\nu_R = \nu_R(r_H, R)$;

2. Replace $\nu_R = \nu_R(r_H, R)$ in F and then compute the total derivative with respect to r_H. Solving $\mathrm{d}F/\mathrm{d}r_H = 0$ gives an equation for $r_H = r_H(\beta, R)$;

3. Insert $r_H = r_H(\beta, R)$ inside $\nu_R = \nu_R(r_H, R)$ to get $\nu_R = \nu_R(\beta, R)$. This procedure does not yield a unique solution, compatibly with the existence of multiple possible vacua in a generic LL theory.

Once we have the free energy, thermodynamic quantities such as the energy, entropy, and surface tension, follow by the usual relations [14]

$$\mathcal{E}_{th}(\beta, R) := F + \beta \frac{\partial F}{\partial \beta} = -\sum_k k\hat{c}_k \int_0^1 \mathrm{d}s \left(\frac{1 - s^2 f(R)}{R^2}\right)^{k-1} \frac{2\sqrt{f(R)}}{R} \Omega_{n-2} R^{n-2}, \quad (6.107)$$

$$S_{th}(\beta, R) := \beta^2 \frac{\partial F}{\partial \beta} = 2\pi \sum_k \frac{2k\hat{c}_k}{n - 2k} \left(\frac{1}{r_H^2}\right)^{k-1} \Omega_{n-2} r_H^{n-2}, \quad (6.108)$$

$$\sigma_{th}(\beta, R) = -\frac{\partial F}{\partial A_{\mathcal{S}}}. \quad (6.109)$$

The first two quantities coincide respectively with the energy $\mathcal{E}_R$ computed in (6.49) and the entropy that we identified in (6.43). The last quantity, at least for the examples that we considered in [209], is compatible with $\nu_R^{-1} W_\sigma$ when σ (and ν) is taken to be constant over $\mathcal{S}$, as expected from spherical symmetry. These observations back up the interpretation of these quantities as proper thermodynamical quantities measured by a observer at the boundary. With these definitions, we can justify the thermodynamic interpretation:

$$F = \mathcal{E} - \frac{1}{\beta} S = \mathcal{E}_{th} - \frac{1}{\beta} S_{th} \equiv F_{th}. \quad (6.110)$$

As an example, let us specialize the analysis in the well-known case of four dimensional GR with cosmological constant, which we assume to be non-positive for definiteness. In the context of LL theories, the cosmological constant is just the ratio of the first two Lovelock coefficients $c_0/c_1 = -2\Lambda$. The only non-trivial thermodynamic potential of the theory is the one conjugated to the cosmological constant. It is easy to see that this is just the flat volume of the hypersurface Σ

$$\Psi_0 = \int_\Sigma J_\xi^{(0)} = -\frac{4\pi}{3}(R^3 - r_H^3). \quad (6.111)$$

The reason why we get the same result as in flat spacetime is because the effect of the curvature $1/\sqrt{f}$ is compensated by the norm of $\xi = \partial_t$. Even for finite-size boundaries, we can check that this potential is what satisfies eq. (6.90)

$$E = 2\, T_H S - 2 P_\Lambda \Psi_0 + 2 W_\sigma, \quad (6.112)$$

where we introduced the so-called *cosmological pressure*

$$P_\Lambda \equiv -\frac{\Lambda}{8\pi G_N} = c_0. \quad (6.113)$$

[14] Notice that the implicit dependence on β from r_H cancels because of the saddle-point condition.

Taking the $R \to \infty$ limit — which would be problematic for $\Lambda > 0$, as the $R = \text{const}$ hypersurfaces turn spacelike — of the regularized quantities, we find [15]

$$E^{\text{reg}} \sim M, \quad \Psi_0^{\text{reg}} \sim \frac{4}{3}\pi r_H^3 \equiv V_{eff}, \quad W_\sigma^{\text{reg}} \sim -\frac{M}{2R}, \tag{6.114}$$

where we defined the *effective volume* of the black hole, which is just the volume of a flat ball with the same radius as the horizon. Therefore, we easily recover the usual Smarr formula [245] for asymptotically flat/AdS black holes

$$M = 2\,T_H S - 2P_\Lambda V_{eff}. \tag{6.115}$$

Let us now consider the canonical ensemble derivation. Following the steps we laid out there, we find the expression of ν_R

$$\nu_R^2 = 1 - \frac{r_H}{R} - \frac{\Lambda}{3}\frac{R^3 - r_H^3}{R}, \tag{6.116}$$

and then the equation that relates r_H to β and R:

$$\beta = 4\pi r_H \left(1 - \Lambda r_H^2\right)^{-1} \left(1 - \frac{r_H}{R} - \frac{\Lambda}{3}\frac{R^3 - r_H^3}{R}\right)^{1/2}. \tag{6.117}$$

We can compute the on-shell value of the Euclidean action to get the free energy

$$F(\beta, R) = -\frac{\pi r_H^2}{\beta G_N} - \frac{R}{G_N}\sqrt{1 - \frac{r_H}{R} - \frac{\Lambda}{3}\frac{R^3 - r_H^3}{R}}. \tag{6.118}$$

Perhaps unsurprisingly for the reader, the energy, entropy and surface tension that we obtain are

$$\mathcal{E}_{th} = -\frac{R}{G_N}\sqrt{1 - \frac{2G_N M}{R} - \frac{\Lambda}{3}R^2}, \tag{6.119}$$

$$S_{th} = \frac{A_H}{4G_N}, \tag{6.120}$$

$$\sigma_{th} = -\frac{1}{8\pi R}\frac{1}{G_N \mathcal{E}_{th}/R}\left(\frac{M}{R} + \frac{G_N \mathcal{E}_{th}^2}{R^2} - \frac{\Lambda R^2}{3G_N}\right), \tag{6.121}$$

where we expressed $r_H(\beta, R)$ in terms of the more familiar ADM mass using

$$3r_H - \Lambda r_H^3 = 6\,G_N M. \tag{6.122}$$

These quantities satisfy the *physical process* version of the first law

$$\delta\mathcal{E}_{th} = T\delta S_{th} - \sigma_{th}\delta A_{\mathcal{S}}, \tag{6.123}$$

[15] The fact that $W_\sigma^{\text{reg}} \to 0$ as $R \to \infty$ justifies the approximation we made after eq. (6.68).

where the δ represent here a transformation between two nearby black hole stationary solutions. The correct Hawking temperature can be obtained from the limit

$$T_H = \lim_{R\to\infty} \nu_R T = \frac{1-\Lambda r_H^2}{4\pi r_H}. \tag{6.124}$$

In [209], we considered the case of five-dimensional Einstein–Gauss–Bonnet gravity as well. However, due to the difficulty in giving a physical interpretation to the thermodynamic potential Ψ_2, conjugated to the Gauss–Bonnet coupling constant, we do not report it here.

6.7 Summary

In this chapter, we employed the previously reviewed CPS formalism to rigorously derive the first law of black hole thermodynamics. This derivation has been extended in two directions. The first extension accounts for field variations that do not preserve the boundary conditions. As we showed in [209], this extension is necessary to consistently derive the Smarr formula from the first law, even in pure GR. The second extension, arguably more widely known, is the foundation of black hole extended thermodynamics [246, 257, 258], a framework which allows to restore scale invariance in theories with multiple dimensionful couplings, such as LL theories. By combining these two extensions, we derived the exact relation (6.90) for LL black hole solutions, generalizing the Smarr formula to arbitrary spacetime regions in any background.

Part III

Symmetries at every corner

Chapter 7

Corner Symmetries

I pray to everything.
You see the spider crawling on the wall,
I look at it and thank it for crawling.
— F. Dostoevsky, *The possessed*

In this last chapter, we make contact with more speculative applications of the concepts we have introduced and developed so far. Special emphasis was given to the construction of corner charges in gauge theories and gravity in a framework, the CPS formalism, which embraces the power of symmetries through Noether's theorems. The reason is that these charges play a key role in several fields at the frontier of current research, ranging from the investigation of the entropy of black holes [202, 259–261], to the study of the entire spacetime in celestial holography [180, 262], from the problem of entanglement entropy [38, 263–268] to the construction of reference frames [269–272].

Most of the content of this chapter is drawn from [273].

7.1 Entanglement in gauge theories

What is the easiest way to understand how an ancient orrery works? The first impulse might likely be to break it apart and take a look inside. Seeing which gears are close together and where threads are connected gives us an idea of how the motion of one part would cause the rest to move. It turns out that this is a good way to understand gauge theories as well.

States in the quantum theory can be defined as functionals that live on a certain spacetime slice Σ [274]. To break apart such a state, we can imagine to split the slice in two (or more) disjoint regions

$$\Sigma = \mathcal{A} \cup \mathcal{B}, \tag{7.1}$$

and see what are the states, defined in the *subregions* $\mathcal{A}$ and $\mathcal{B}$, whose combination reproduces the original state. However, this question makes sense as long as

$$\mathcal{H}_\Sigma \subseteq \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}, \tag{7.2}$$

where we denote with $\mathcal{H}_\mathcal{R}$ the Hilbert space of states defined on any subregion $\mathcal{R}$ (such as $\mathcal{A}$ or $\mathcal{B}$). If this is not the case, we must have broken some connection while opening our metaphorical orrery. This is what happens in gauge theories, where the tensor product **factorization** (7.2) is generally not expected. This property can be seen as the quantum counterpart of the obstruction to arbitrary specifications of Cauchy data. As discussed in Section 3.4.5 and 5.2, gauge theories are characterized by a certain non-local nature. In the canonical formulation, this non-locality is reflected in the presence of differential constraints, so that, even restricting to compatible data, specifying them independently on regions $\mathcal{A}$ and $\mathcal{B}$ does not generally constitute a valid specification of data on Σ. The study of how (classical or quantum) the data of a region separate into subregion data, and how the former can be reconstructed from the latter is sometimes referred to as the *cutting* and *gluing* problem [275].

As an example, let us consider a pure Yang–Mills theory. In the Hilbert space $\mathcal{H}_\Sigma$, we expect to find a state that represents a *Wilson loop* straddling the common boundary $\mathcal{S} \equiv \partial\mathcal{A} \cap \partial\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$. Since Wilson loops are gauge-invariant, this state is perfectly allowed, but there is no way to construct such a state from a tensor product of states in $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. States in $\mathcal{H}_\mathcal{A}$ are required to be invariant under gauge transformations supported $\mathcal{A}$, including those that are non-zero at $\mathcal{S}$, whereas the *Wilson line* lying in $\mathcal{A}$, obtained by splitting the Wilson loop, is not invariant under the latter. This example shows that a naïve tensor product Hilbert space misses the contribution of non-invariant data on $\mathcal{S}$.

The lack of factorization is a serious problem in *entanglement entropy* computations. Without that, no single prescription exists, and several alternatives have been proposed (see [263, 276] and references therein for a more comprehensive discussion). Strictly speaking, the lack of factorization is common to all quantum field theories because of ultraviolet divergences [277].[1] Regularizing the theory or providing an ultraviolet completion is expected to solve this part of the problem. In the case of gravity, giving a notion of entanglement entropy becomes even more challenging, as the specification of the subregions $\mathcal{A}$ and $\mathcal{B}$ themselves cannot be made in a diffeomorphism-invariant way [2]. Because of the relevance that entanglement entropy is expected to have in quantum gravity, especially in our understanding of black hole entropy [143, 281–283] and holography [32, 33, 94, 284–287], it is desirable to have a robust definition for it.

[1] In algebraic approaches to QFT, this is reflected in the algebra of observables being type III [278].

[2] One possible approach is to use the algebra of local observables defined in the subregion. This is a non-trivial problem as well, given the absence of diffeomorphism-invariant observables supported on finite regions in gravity [279, 280].

7.1.1 Extended Hilbert space

A possible way to overcome the difficulties in computing entanglement entropies in gauge theories is to extend the Hilbert space by the introduction of the so-called *edge modes*, dynamical fields living on the common boundary $\mathcal{S}$ [34, 263, 288, 289]. These modes transform non-trivially under gauge transformations supported on $\mathcal{S}$, hence the *extended Hilbert space* is only invariant under those gauge transformations which trivialize at the boundary. According to our terminology, these are the *true* gauge transformations, the ones that yield zero Noether charge, that we collectively denoted as $\mathcal{G}_0$ in Section 5.2. Heuristically speaking, the new degrees of freedom, the edge modes, will therefore provide a representation of what we define as the corner symmetry (CS) group

$$\mathrm{CS} := \mathcal{G}/\mathcal{G}_0. \tag{7.3}$$

Making this statement more precise requires a detailed analysis of each case, already at the classical level. As a matter of fact, we have intentionally kept our discussion vague, avoiding any mention of which phase space $\mathcal{G}$ acts on, possible integrability issues of its action on this space, and potential central extensions.

Once the Hilbert space is extended in this way for all regions $\mathcal{H}_\mathcal{R} \to \bar{\mathcal{H}}_\mathcal{R}$, one recovers the factorization property

$$\bar{\mathcal{H}}_\Sigma \subset \bar{\mathcal{H}}_\mathcal{A} \otimes \bar{\mathcal{H}}_\mathcal{B}. \tag{7.4}$$

Even more, the extension provides a prescription to identify *which* subspace of the tensor product coincides with $\bar{\mathcal{H}}_\Sigma$: since the boundary $\partial\Sigma$ does not contain $\mathcal{S}$, states in $\bar{\mathcal{H}}_\Sigma$ are required to be invariant even under transformations that are non-trivial at $\mathcal{S}$, so they can only carry the trivial representation of the CS group. The procedure of extracting the invariant subspace is captured by the *entangling product* construction

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_\mathcal{A} \otimes_{\mathrm{CS}} \bar{\mathcal{H}}_\mathcal{B}, \tag{7.5}$$

where the symbol $\otimes_{\mathrm{CS}}$ denotes a tensor product of modules [290]: both $\bar{\mathcal{H}}_\mathcal{A}$ and $\bar{\mathcal{H}}_\mathcal{B}$ are modules over CS as they are acted upon by it (we use the symbol $\triangleright$ to generically denote this action), and one identifies states that are connected by a diagonal action of the group, that is $(g \triangleright \psi_\mathcal{A}, g \triangleright \psi_\mathcal{B}) \sim (\psi_\mathcal{A}, \psi_\mathcal{B})$, with $g \in \mathrm{CS}$, $\psi_\mathcal{A} \in \bar{\mathcal{H}}_\mathcal{A}$, $\psi_\mathcal{B} \in \bar{\mathcal{H}}_\mathcal{B}$. Donnelly & Freidel introduced this product and gave some examples of its construction in [38].

7.1.2 Embedding fields

In the case of gravity, the introduction of edge modes turns out to be crucial for the construction of canonical charges and the invariant definition of localized subsystems. The pivotal work [38] identified the edge modes of gravity with a certain set of *embedding fields*, which we are going to define soon, and sparked renewed interest in gravitational corner symmetries, leading to an extensive study of the topic [15, 88, 264, 270, 291–298].

Let $\mathcal{X} : \mathrm{M} \to \mathcal{M}$ be a map that embeds a reference manifold M into $\mathcal{M}$. In [38], M is interpreted as a fixed, non-dynamical set of parameters — which means that M is essentially $\mathbb{R}^n$ — and $\mathcal{X}$ as a parametrization of spacetime. Assuming that $\mathcal{X}$ is locally invertible, we can think of (the components of) its inverse as coordinate functions $\mathcal{X}^{-1} : \mathcal{X}(U) \to U$, $x \mapsto \mathcal{X}^\mu$, where U is an open subset of M. Under a diffeomorphism $\Phi \in \mathrm{Diff}(\mathcal{M})$, the embedding field $\mathcal{X}$ transform as

$$\Phi^* \mathcal{X} = \Phi^{-1} \circ \mathcal{X}. \tag{7.6}$$

Since diffeomorphisms act on spacetime fields ϕ by pull-back $\Phi^*(\phi)$, it is easy to see that the combination $\mathcal{X}^*\phi$ is diffeomorphism-invariant

$$\mathcal{X}^*\phi \to (\Phi^*\mathcal{X})^*(\Phi^*\phi) = (\Phi^{-1} \circ \mathcal{X})^*(\phi \circ \Phi) = (\phi \circ \Phi) \circ (\Phi^{-1} \circ \mathcal{X}) = \phi \circ \mathcal{X} = \mathcal{X}^*\phi. \tag{7.7}$$

In fact, the pull-back field $\mathcal{X}^*\phi$ lives in the reference manifold, where spacetime diffeomorphisms do not act. Therefore, a prescription to write a theory that is fully diffeomorphism-invariant is to express its Lagrangian density, which is a function of ϕ and its derivatives, entirely in terms of the pulled back fields $\mathcal{X}^*\phi$. In this way, we are essentially writing the theory in the reference manifold. From this point of view, the introduction of embedding fields $\mathcal{X}$ is essentially the Stueckelberg trick for diffeomorphisms [299].

As we mentioned, one of the advantages of this construction is the possibility to localize subregions in a diffeomorphism-invariant way. Somehow unsurprisingly, the idea is to just identify the subregions $\mathcal{A}, \mathcal{B} \subset \Sigma$ (cfr. eq. (7.1)) by their coordinate locations $\mathrm{A}, \mathrm{B} \subset \mathrm{M}$, defined by $\mathcal{X}(\mathrm{A}) = \mathcal{A}$ and $\mathcal{X}(\mathrm{B}) = \mathcal{B}$. When integrating over spacetime, we can always pull-back to the reference manifold, so that, for example

$$\int_\Sigma \theta = \int_{\mathcal{A}} \theta + \int_{\mathcal{B}} \theta = \int_{\mathrm{A}} \mathcal{X}^*\theta + \int_{\mathrm{B}} \mathcal{X}^*\theta. \tag{7.8}$$

The second advantage is that, in the extended phase space $(g, \mathcal{X})$, every diffeomorphism is integrable, meaning that the associated vector field $\not{X}_\xi$ admits a Hamiltonian function. Proving this would take us too far from the main discussion, so we refer the interested reader to the proofs available in [38, 293]. The important point is that the extended phase space provides an example of a phase space on which the action of diffeomorphisms is Hamiltonian, which means that we can construct momentum maps.

It is important to point out that, while they agree on integrability, the last two references differ in the value that they assign to the Hamiltonian function of diffeomorphisms (see Ref. [20] for a clear and complete analysis): for the authors of [38], the charge is zero, hence diffeomorphisms are pure gauge, even when they act non-trivially on $\mathcal{S}$; on the other hand, for the authors of [293], the charge coincides with the Komar charge, which, being non-zero, means that some diffeomorphisms become physical at the boundary. This discrepancy highlights that there are still questions that need to be answered in this area of research, in particular about the role of boundaries in gravity, the meaning of diffeomorphism charges, and the nature of embedding fields.

7.2 Universal Corner Symmetries

To identify the corner symmetries of gravity, let us consider the coordinate location S of $\mathcal{S}$ via the embedding, i.e. $\mathcal{X}(\mathrm{S}) = \mathcal{S}$. Explicitly, we introduce local coordinates (u^a, x^A) (with $\alpha = 1, 2$ and $A = 1, \dots, n-2$) in a neighborhood of $\mathcal{M}$ close to $\mathcal{S}$, isolating a set of codimension 2. Equipping S with local coordinates $\sigma = (\sigma^1, \dots, \sigma^{n-2})$, we can realize the embedding as

$$\mathcal{X}(\sigma) = (u^a(\sigma), x^A(\sigma)). \tag{7.9}$$

Among all possible embeddings, the trivial embedding $\mathcal{X}_0(\sigma) = (0, \sigma^A)$ plays a special role, as it corresponds to bulk coordinates adapted to $\mathcal{S}$. In the following, we are sometimes going to conflate the reference location S with its trivially embedding $\mathcal{S}$ with a slight abuse of notation.

Given the embedding of a corner, we can separate the tangent bundle into a tangential bundle and a normal bundle, defined by the splitting $\xi = \xi^a \partial_a + \xi^A \partial_A$ in the neighborhood where the coordinates (u^a, x^A) are valid [292]. This split reveals the presence of a rich symmetry structure inside $\mathfrak{diff}(\mathcal{M})$. In fact, we can see that there exists a maximal subalgebra with a "finite" [3] number of generators associated with the corner, given by

$$\mathfrak{diff}(\mathrm{S}) \oplus (\mathfrak{gl}(2, \mathbb{R}) \oplus \mathbb{R}^2)^{\mathrm{S}}. \tag{7.10}$$

This algebra contains the diffeomorphisms of the corner (which are one-to-one with the diffeomorphisms of S), and the set of smooth maps $\mathrm{S} \to \mathfrak{gl}(2, \mathbb{R}) \oplus \mathbb{R}^2$, that we can think as a copy of the target algebra for each point on the corner.

Proof. Before finding the form of the subalgebra, let us check whether there exists one. Assuming the trivial embedding, we can expand both components of a vector field ξ around points of $\mathcal{S} = \{(0, x)\}$ as

$$\xi^a(u, x) = \xi^a_{(0)}(x) + \xi^a_{(1)b_1}(x) u^{b_1} + \frac{1}{2} \xi^a_{(2)b_1 b_2}(x) u^{b_1} u^{b_2} + \dots, \tag{7.11}$$

$$\xi^A(u, x) = \xi^A_{(0)}(x) + \xi^A_{(1)b_1}(x) u^{b_1} + \frac{1}{2} \xi^A_{(2)b_1 b_2}(x) u^{b_1} u^{b_2} + \dots. \tag{7.12}$$

Each component generates a certain transformation whose composition with the others follow from the Lie derivative of the corresponding vector field. Let us then introduce a second vector field ξ', with the same split and expansion, and compute their commutator up to cubic order. With some patience, we can organize the result in a similar power

[3] What is finite is the number of functions of S that are required to write a generator of the algebra, in contrast to the infinite number that would be required to write a generator of $\mathfrak{diff}(\mathcal{M})$.

expansion

$$
\begin{aligned}
[\xi,\xi']_{\text{Lie}} =&(\xi^b_{(0)}\xi'^a_{(1)b}+\xi^A_{(0)}\partial_A\xi'^a_{(0)})\partial_a+(\xi^a_{(0)}\xi'^A_{(1)a}+\xi^B_{(0)}\partial_B\xi'^A_{(0)})\partial_A+\\
&+(\xi^b_{(0)}\xi'^a_{(2)bc}+\xi^b_{(1)c}\xi'^a_{(2)b}+\xi^A_{(0)}\partial_A\xi'^a_{(1)c}+\xi^A_{(1)c}\partial_A\xi'^a_{(0)})u^c\partial_a+\\
&+(\xi^a_{(0)}\xi'^A_{(2)ab}+\xi^a_{(1)b}\xi'^A_{(1)a}+\xi^B_{(0)}\partial_B\xi'^A_{(1)b}+\xi^B_{(1)b}\partial_B\xi'^A_{(0)})u^b\partial_A+\\
&+\frac{1}{2}(2\xi^d_{(1)b}\xi'^a_{(2)dc}+\xi^d_{(2)bc}\xi'^a_{(1)d}+\xi^A_{(0)}\partial_A\xi'^a_{(2)bc}+2\xi^A_{(1)b}\partial_A\xi'^a_{(1)c}+\xi^A_{(2)bc}\partial_A\xi'^a_{(0)})u^bu^c\partial_a+\\
&+\frac{1}{2}(2\xi^c_{(1)a}\xi'^A_{(2)cb}+\xi^c_{(2)ab}\xi'^A_{(1)c}+\xi^B_{(0)}\partial_B\xi'^A_{(2)ab}+2\xi^B_{(1)a}\partial_B\xi'^A_{(1)b}+\xi^B_{(2)ab}\partial_B\xi'^A_{(0)})u^au^b\partial_A+\\
&+\frac{1}{2}\xi^b_{(0)}\xi'^a_{(3)bcd}u^cu^d\partial_a+\frac{1}{2}\xi^a_{(0)}\xi'^A_{(3)abc}u^bu^c\partial_A-(\xi\leftrightarrow\xi')+\mathcal{O}(u)^3,
\end{aligned}
$$

In order for the subalgebra to be finitely-generated — in the same sense as before, i.e. to contain only a finite number of functions of S — we need to find a consistent truncation of the expansion in u^a, for which the commutator of two truncated vector fields is a truncated vector field itself. From the previous computation, we learn that we cannot allow first-order (or higher) terms in ξ^A, as that would indefinitely raise the order of the expansion. More explicitly, the action of $\xi^A_{(1)}$ on either $\xi'^a_{(k)}$ or $\xi'^B_{(k)}$ (we omit writing tensor indices in the subscript for readability) produces a term of order $(k+1)$.[4] On the other hand, combining $\xi^a_{(k)}$ and $\xi^b_{(l)}$, we get a term of order $k+l-1$, which is greater than $\max(k,l)$ if one of the two is greater than 1. These two considerations tell us that a consistent truncation, if it exists, has to stop at first order for ξ^a and at order zero for ξ^A. The following computation provides an explicit proof of the existence.

To find the structure of the algebra, we denote the truncated vector field as

$$
\underline{\xi}:=\xi^A_{(0)}(x)\partial_A+(\xi^a_{(0)}(x)+\xi^a_{(1)b}(x)u^b)\partial_a. \tag{7.13}
$$

and organize the commutator of two such fields as

$$
\begin{aligned}
[\underline{\xi},\underline{\xi'}]_{\text{Lie}} =&\underbrace{[\xi_{(0)},\xi'_{(0)}]_{\text{Lie}}}_{\mathfrak{diff}(\mathrm{S})}+\underbrace{\Big(\xi_{(0)}(\xi'^a_{(0)})-\xi'_{(0)}(\xi^a_{(0)})\Big)}_{\mathfrak{diff}(\mathrm{S})\text{ acts on }\mathbb{R}^2}\partial_a+\underbrace{\Big(\xi_{(0)}(\xi'^a_{(0)b})-\xi'_{(0)}(\xi^a_{(0)b})\Big)}_{\mathfrak{diff}(\mathrm{S})\text{ acts on }\mathfrak{gl}(2,\mathbb{R})}u^b\partial_a\\
&-\underbrace{[\xi_{(1)},\xi'_{(1)}]^b{}_a}_{\mathfrak{gl}(2,\mathbb{R})}u^b\partial_a-\underbrace{\Big(\xi^a_{(1)b}\xi'^b_{(0)}-\xi'^a_{(1)b}\xi^b_{(0)}\Big)}_{\mathfrak{gl}(2,\mathbb{R})\text{ acts on }\mathbb{R}^2}\partial_a,
\end{aligned} \tag{7.14}
$$

where we used $[\cdot,\cdot]$ to distinguish the Lie brackets of $\mathfrak{gl}(2,\mathbb{R})$ (the usual matrix commutator) from the Lie brackets of vector fields, and we denoted the tangential vector field by $\xi_{(0)}=\xi^A_{(0)}\partial_A\in\mathfrak{X}(\mathcal{S})$. From eq. (7.14), we see that there are three subalgebras: $\mathfrak{diff}(\mathrm{S})$, $\mathfrak{gl}(2,\mathbb{R})$ and $\mathbb{R}^2$ ($\xi^a_{(0)}$ and $\xi'^a_{(0)}$ commute as they do not appear together on the l.h.s.). We also see that the first acts on the latter two, while the second acts on only the last. □

[4] The subalgebra obtained by $\xi^A=\xi^A_{(0)}$ and an untruncated ξ^a is a relevant one, as it reproduces the algebra of *Carrollian diffeomorphisms* [300–303].

Since we did not make any reference to any field content or geometry, the algebra (7.10) we identified is kinematical and universal in the following sense: any theory with diffeomorphism invariance is also invariant under the subalgebra, but, in the presence of a corner, while most diffeomorphisms are pure gauge transformations, some of the diffeomorphisms within $\mathfrak{diff}(\mathrm{S}) \oplus (\mathfrak{gl}(2,\mathbb{R}) \oplus \mathbb{R}^2)^{\mathrm{S}}$ become physical. In fact, thanks to the representation theorem (5.30), we know that

$$\{Q_\xi, Q_{\xi'}\} = -Q_{[\xi,\xi']}, \tag{7.15}$$

up to a central extension, and since most diffeomorphisms have zero Noether charge, the maximal subalgebra that these charges can projectively represent is the universal corner symmetry (UCS). By virtue of this universality, we call $\mathfrak{ucs} := \mathfrak{diff}(\mathrm{S}) \oplus (\mathfrak{gl}(2,\mathbb{R}) \oplus \mathbb{R}^2)^{\mathrm{S}}$ the UCS algebra.[5] The construction we just performed gives its so-called *defining representation*. The geometric interpretation of the UCS algebra components is clear

- $\mathfrak{diff}(\mathrm{S})$ contains the diffeomorphisms of the corner;
- $\mathfrak{gl}(2,\mathbb{R})$ corresponds to boosts and dilatations in the normal plane to the corner;
- $\mathbb{R}^2$ corresponds to translations in the two normal directions.

The fact that these transformations fall into the classification of diffeomorphisms we did in Section 5.4.1 is not a coincidence. Indeed, except for corner dilatations, these are the only diffeomorphisms with non-vanishing Komar charge in GR [293].

So far, there are no known examples of theories whose charges at a specific corner realize the full $\mathfrak{ucs}$ algebra. Nevertheless, we can find two important subalgebras. The first is the extended corner symmetry (ECS) algebra

$$\mathfrak{ecs} := \mathfrak{diff}(\mathrm{S}) \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2)^{\mathrm{S}} \tag{7.16}$$

which is realized at finite-distance corners.[6] The second, called asymptotic corner symme-

[5] It was previously referred to as the *maximal embedding algebra*.

[6] Given an *isolated* corner (in contrast to a corner that is part of a codimension-2 foliation of spacetime), we can locally write the metric in the Randers-Papapetrou form [304, 305]

$$\mathrm{d}s^2 = h_{ab}(u,x)(\mathrm{d}u^a - \mathrm{a}^a_A(u,x)\mathrm{d}x^A)(\mathrm{d}u^b - \mathrm{a}^b_B(u,x)\mathrm{d}x^B) + \gamma_{AB}(u,x)\mathrm{d}x^A\mathrm{d}x^B,$$

and expand each metric function in $\{u^1,u^2\}$ in a similar fashion to (7.11). The pull-back to the corner of the diffeomorphism charge aspect of GR gives the following generators of the corner symmetry

$$Q_\xi = \xi^a_{(1)b}N^b{}_a + \xi^a_{(0)}p_a + \xi^A_{(0)}b_A,$$

where

$$N^b{}_a = \sqrt{-h^{(0)}}h^{bc}_{(0)}\epsilon_{ca}, \tag{7.17}$$

$$p_a = \frac{1}{2}\sqrt{-h^{(0)}}\epsilon^{bc}(h^{(1)}_{ab,c} - h^{(1)}_{ac,b}), \tag{7.18}$$

$$b_A = \sqrt{-h^{(0)}}h^{bc}_{(0)}\epsilon_{ca}\mathrm{a}^{a\,(1)}_A{}_b. \tag{7.19}$$

try (ACS) algebra, is

$$\mathfrak{acs} := \mathfrak{diff}(\mathrm{S}) \oplus (\mathfrak{w} \oplus \mathbb{R}^2)^{\mathrm{S}}, \tag{7.20}$$

where $\mathfrak{w} \cong \mathbb{R}$ is the subalgebra associated with the determinant in $\mathrm{GL}(2,\mathbb{R})$. This latter algebra contains the BMSW (Weyl–Bondi–Metzner–Sachs) algebra, which was shown to be the most general that is realized in asymptotically flat geometries [13]. All these algebras share the structure $\mathfrak{diff}(\mathrm{S}) \ltimes \mathfrak{h}^{\mathrm{S}}$ (we repeat that $\mathfrak{h}^{\mathrm{S}}$ denotes the set of smooth maps $\mathrm{S} \to \mathfrak{h}$), where the local algebra $\mathfrak{h}$ is $\mathfrak{gl}(2,\mathbb{R}) \oplus \mathbb{R}^2$ for the universal algebra, $\mathrm{SL}(2,\mathbb{R}) \oplus \mathbb{R}^2$ for the extended, and $\mathfrak{w} \oplus \mathbb{R}^2$ for the asymptotic one. As recognized in [296], the latter two are ideals inside the former.

7.2.1 Corner proposal

The UCS algebra (and group) is at the core of the **corner proposal** for quantum gravity: a bottom-up approach that puts emphasis on symmetries and their consequences on observables rather than achieving a complete description valid at all scales. The idea of this proposal is that, regardless of which are the fundamental degrees of freedom of gravity in a subregion, there must exist a *mesoscopic* regime in which these organize into representations of the corner symmetry group [7], because this is the symmetry group that we observe at the classical level. In fact, according to Dirac's canonical quantization, phase space functions turn into quantum operators in such a way that quantum commutators reproduce, to leading order in $\hbar$, the Poisson brackets of the original functions: if these functions are equivariant momentum maps of a certain symmetry group, it follows that quantum operators represent its algebra as well, according to the following diagram

$$\begin{array}{c} \xi \longmapsto Q_\xi \longmapsto \hat{Q}_\xi \\ [\xi,\xi']_{\mathrm{Lie}} \mapsto \{Q_\xi, Q_{\xi'}\} \mapsto [\hat{Q}_\xi, \hat{Q}_{\xi'}]. \end{array} \tag{7.21}$$

The extent to which the mesoscopic regime extends into the quantum realm is not known, but it is expected to be large enough to yield interesting predictions.

As shown by Isham [2, 3], one way to realize Dirac's program as in the previous diagram (7.21) is to identify a group of symplectomorphisms whose action on phase space is transitive and Hamiltonian. Quantizing the transformations of this group *à la* Dirac yields a special set of operators from which the full algebra can, in principle, be generated. Recalling the discussion of the previous section, the extended phase space offers an example of phase space where diffeomorphisms act via integrable symplectomorphisms. However, since their action is not transitive, we do not expect the associated quantum operators

Since $Q_\xi = Q_{\underline{\xi}}$, we see that only the vector fields in the UCS subalgebra have non-trivial charges. Moreover, since $N^a{}_b$ is traceless, the determinant subgroup of $\mathrm{GL}(2,\mathbb{R})$ is inactive and the algebra that is realized is just the ECS.

[7] The corner we refer to here is the one associated to the subregion, i.e. its boundary.

to exhaust all possible quantum gravity observables. Nevertheless, they are believed to provide a physically interesting sector [306–308].

Alternatively, we can proceed in the opposite way: rather than finding which symmetry group has transitive and Hamiltonian action on a given phase space, we may look for a phase space that is acted upon in the desired way by a certain symmetry group. More so, we could directly ask which Hilbert space features those symmetries. In this case, the answer has been known for a long time thanks to representation theory: given the symmetry group, we can classify its irreducible unitary representation and, by combining them, construct Hilbert spaces with the desired symmetry properties. In the classical case, the analogous question remained unsolved for some time. The answer came with the work of Kirillov, Kostant, and Souriau on *coadjoint orbits* [309–314], as we are going to review in Section 7.3.

From this perspective, identifying the correct symmetry group of gravity is analogous to the identification of the Poincaré group as the underlying symmetry group of relativistic QFT. The existence of the UCS is a strong indication towards the feasibility of this program. Its universality makes it a good candidate for being the gravitational analogue of the Poincaré group. Once the correct corner symmetry group is identified, be it the UCS or some other group derived from it, following Wigner's famous program [315], one should then set out to classify all of its irreducible unitary representations. Developing the representation theory of the UCS group is thus a central task in the corner proposal. Unfortunately, the infinite-dimensionality and the semidirect product with diffeomorphisms make this a complicated problem, so that a full classification of UCS representations is not known yet.

7.2.2 Quantum extension

When we consider a quantum theory, we are interested in the construction of a Hilbert space $\mathcal{H}$ and the computation of transition amplitudes between any two (normalized) vectors. However, since we only have access to transition probabilities $|\mathcal{A}|^2$, we cannot determine individual phases of vectors, but only relative phases. Quantum states are thus properly defined as equivalence classes of normalized vectors $|\psi\rangle \sim e^{i\theta}|\psi\rangle$, i.e. as elements of the projective Hilber space $(\mathcal{H}\backslash\{0\})/\mathbb{C}^\times$. Consequently, the operators U_g that represent a symmetry group G will generically form a *projective* representation of it, in the sense that they compose as

$$U_{g_1}U_{g_2} = \omega(g_1, g_2)U_{g_1 g_2}, \qquad g_1, g_2 \in G \tag{7.22}$$

where $\omega(g_1, g_2)$ is a 2-cocycle of the symmetry group. An important result in representation theory is that the projective representations of a path-connected group are equivalent to the ordinary representations of its maximal central extension [316, 317]. In light of these considerations, we expect that the symmetry group which is going to be represented in the Hilbert space of quantum gravity is not the corner symmetry group, but rather its maximal central extension. If the representation theory of the groups like the UCS and the ECS

is complicated, we can foresee that finding all irreducible unitary representations of their central extension will be a challenging endeavor.

As a starting point, we tackle a simplified version of the problem by focusing on the two-dimensional case

$$\mathrm{UCS}_2 = \mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2. \tag{7.23}$$

We can see this restriction either as the study of plain two-dimensional gravity (where the Diff(S) part trivializes), or a per-point analysis of the corner in the full-dimensional case, which is especially useful when restricting to the spherically symmetric sector of the phase space. In addition, since the determinant part W of the general linear group $\mathrm{GL}(2,\mathbb{R}) = W \ltimes \mathrm{SL}(2,\mathbb{R})$ is not realized at finite-distance corners (as indicated by all examples that have been considered so far), we further restrict to the extended group

$$\mathrm{ECS}_2 = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2. \tag{7.24}$$

The study of representations of this group and their applications to the description of local subsystems was initiated in [298] and continued in [318]. In the those works, it was found that the group ECS_2 admits the following maximal extension, dubbed quantum corner symmetry (QCS) group,

$$\mathrm{QCS}_2 = \widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes \mathbb{H}_3, \tag{7.25}$$

where the tilde denotes the universal cover and $\mathbb{H}_3$ denotes is three-dimensional Heisenberg group. The extension $\mathbb{R}^2 \to \mathbb{H}_3$ is due to a non-trivial 2-(group) cocycle in the translation sector of ECS_2 (the same that exists in one-dimensional quantum mechanics), whereas the universal cover extension is due to the fact that $\mathrm{SL}(2,\mathbb{R})$ is not simply connected. Whereas the latter extension is present at the classical level, the former is a purely quantum effect, analogous to the appearance of a non-trivial commutator between positions and momenta in quantum mechanics. The complete classification of unitary irreducible representations of the QCS has been obtained first in [318] and later confirmed by us in [273].

Higher-dimensions and diffeomorphisms Let us take a moment to pause and consider what happens in $n > 2$. If we were to look back at the entire corner, given the common structure of the corner symmetry algebras, one could naively expect the full quantum group to be something like $\mathrm{Ext}(\mathrm{Diff}(\mathrm{S})) \ltimes H_Q$, where H_Q is (some modification of) QCS_2 and $\mathrm{Ext}(\mathrm{Diff}(\mathrm{S}))$ is the maximal central extension of Diff(S). However — in addition to the complication that the maximal central extension of the diffeomorphisms group is highly dependent on the dimension and the topology of S — the central extension of a (semidirect) product is not the (semidirect) product of the extensions, which means that finding the correct QCS group requires analyzing the possible extensions of the full ECS group. Already at the classical level, we know that reintroducing diffeomorphisms has dramatic consequences. Whereas $\mathfrak{ecs}_2$ is an ideal of $\mathfrak{ucs}_2$, $\mathfrak{diff}(\mathrm{S}) \oplus \mathfrak{ecs}_2$ is not an ideal inside $\mathfrak{diff}(\mathrm{S}) \oplus \mathfrak{ucs}_2$. Moreover, diffeomorphisms can change both the form and the number of Casimirs, altering the quantum description as well. For example, in the simpler case when we drop the

translation sector from the ECS group (what remains are only *corner-preserving* transformations), we know that $\mathfrak{sl}(2,\mathbb{R})$ has a single independent Casimir, while $\mathfrak{diff}(\mathrm{S}) \oplus \mathfrak{sl}(2,\mathbb{R})^{\mathrm{S}}$ has an infinite number of invariants [291]. In the case at hand, the appearance of a central element in the translation sector might be an artifact of the two-dimensional setting. For this reason, before attributing physical significance to it (particularly tempting is to read that as an indication of spacetime fundamental *fuzzyness*), we should study its fate when the suppressed dimensions are restored and diffeomorphism are reintroduced.

7.3 The orbit method

As previously mentioned, the **orbit method** originated as an attempt to provide a classical counterpart to representation theory. Following Kirillov's idea, group coadjoint orbits, which we will introduce shortly, behave as *prototypical* classical phase spaces, in the sense that any phase space that is symmetric under a group action can be seen as a coadjoint orbit (or a collection thereof) of that group. Successfully quantizing these orbits with suitable techniques is expected to yield Hilbert spaces that are representation spaces of the symmetry group. Because coadjoint orbits are elementary objects, these Hilbert spaces will not have non-trivial invariant subspaces. For this reason, Kirillov postulated a correspondence between the coadjoint orbits of a group and its unitary irreducible representations [314]. Although this correspondence has been proven only for unipotent Lie groups [310], the orbit method has been an exceptional tool for studying the representation theory of Lie groups and Lie algebras [314, 319–322]. In particular, its application to Virasoro and Kac-Moody algebras has proven very useful in the context of lower-dimensional quantum gravity [323–328].

After reviewing the basics of the method, we will show how it can be applied to the corner symmetry groups ECS_2 and QCS_2.[8]

7.3.1 Lie group coadjoint orbits

Let G be a Lie group and $\mathfrak{g} \cong T_eG$ its Lie algebra. G has a natural action on itself by conjugation $h \mapsto ghg^{-1}$, which can be pushed-forward to construct a G-action on the algebra

$$G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, \mathbf{X}) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\left(ge^{t\mathbf{X}}g^{-1}\right)\Big|_{t=0} =: \mathrm{Ad}_g\mathbf{X}, \tag{7.26}$$

commonly known as the *adjoint action*.

Since the Lie algebra $\mathfrak{g}$ is a vector space, we can introduce the dual space $\mathfrak{g}^*$ as the space of linear functionals $\mathbf{m} : \mathfrak{g} \to \mathbb{R}$, $\mathbf{X} \mapsto \langle \mathbf{m}, \mathbf{X}\rangle \in \mathbb{R}$. In this space, we can define the

[8] The analysis of the ECS coadjoint orbits was carried out in [296] in terms of Lie algebra actions. Here we extend it to the full group action.

coadjoint action by requiring consistency with the pairing $\langle\cdot,\cdot\rangle$, i.e.

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, \mathbf{m}) \to \mathrm{Ad}^*_g \mathbf{m}, \tag{7.27}$$

where

$$\langle \mathrm{Ad}^*_g \mathbf{m}, \mathbf{X}\rangle = \langle \mathbf{m}, \mathrm{Ad}_{g^{-1}} \mathbf{X}\rangle \quad \forall \mathbf{X} \in \mathfrak{g}. \tag{7.28}$$

Taking the infinitesimal version of the group action on $\mathfrak{g}$ and $\mathfrak{g}^*$, we can define the algebra adjoint and coadjoint actions

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (\mathbf{X}, \mathbf{Y}) \to \mathrm{ad}_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}], \tag{7.29}$$

$$\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (\mathbf{X}, \mathbf{m}) \to \mathrm{ad}^*_{\mathbf{X}} \mathbf{m}, \text{ where } \langle \mathrm{ad}^*_{\mathbf{X}} \mathbf{m}, \mathbf{Y}\rangle = -\langle \mathbf{m}, \mathrm{ad}_{\mathbf{X}} \mathbf{Y}\rangle. \tag{7.30}$$

Given the group coadjoint action, we can introduce orbits and stabilizers (cfr. eqs. (2.31) and (2.32)) as

$$\mathcal{O}^G_{\mathbf{m}} = \big\{ \mathrm{Ad}^*_g \mathbf{m} \,|\, g \in G \big\}, \tag{7.31}$$

$$G_{\mathbf{m}} = \big\{ g \in G \,|\, \mathrm{Ad}^*_g \mathbf{m} = \mathbf{m} \big\}. \tag{7.32}$$

Coadjoint orbits foliate $\mathfrak{g}^*$. However, since the action of G is generically not free, the leaves of this foliation do not have the same dimensions. These orbits are generated by the following vector fields (cfr. eq. (2.33))

$$(\xi_{\mathbf{X}})_{\mathbf{m}}[f] := \frac{\mathrm{d}}{\mathrm{d}s} f\big(\mathrm{Ad}^*_{e^{s\mathbf{X}}} \mathbf{m}\big)\Big|_{s=0}, \qquad f \in C^\infty(\mathcal{P}). \tag{7.33}$$

An important result in the theory of Lie groups is that orbits can be written as *homogeneous spaces*

$$\mathcal{O}_{\mathbf{m}} \simeq G/G_{\mathbf{m}}. \tag{7.34}$$

The relevance of the coadjoint action is that the corresponding orbits are symplectic manifolds! Before proving this, we first notice that the dual space $\mathfrak{g}^*$ is a Poisson vector space, with a Poisson algebra structure induced by the Lie algebra structure of $\mathfrak{g}$

$$\{f, g\}(\mathbf{m}) = \langle \mathbf{m}, [f_{*\mathbf{m}}, g_{*\mathbf{m}}]\rangle, \quad \forall f, g \in C^\infty(\mathfrak{g}^*) \tag{7.35}$$

where $f_{*\mathbf{m}}$ is the differential of f at $\mathbf{m}$, that is the linear map $T_{\mathbf{m}}\mathfrak{g}^* \to T_{f(\mathbf{m})}\mathbb{R}$ such that $f_{*\mathbf{m}}(v) = v(f)$, $v \in T_{\mathbf{m}}\mathfrak{g}^*$.[9] These brackets are referred to as *Kirillov–Kostant* Poisson brackets.

[9] Given that both $\mathfrak{g}^*$ and $\mathbb{R}$ are linear spaces, they are identical to their tangent spaces, so that the differential is also a map $\mathfrak{g}^* \to \mathbb{R}$, hence an element of $(\mathfrak{g}^*)^*$, which we can identify (at least for finite dimensional vector spaces) with $\mathfrak{g}$, explaining why we can take a Lie bracket between two differentials at a point in eq. (7.35)

Proof. Given the Poisson structure, we can associate to every function $f \in C^\infty(\mathfrak{g}^*)$ its Hamiltonian vector field $\xi_f \in \mathfrak{X}(\mathfrak{g}^*)$. We already showed that the distribution of these vector fields is involutive

$$[\xi_f, \xi_g]_{\text{Lie}} = -\xi_{\{f,g\}}, \tag{7.36}$$

and it is thus integrable by Frobenius' theorem. We have also seen that the leaves of this distribution are symplectic even if the ambient space is not, as in this case.

We now show that coadjoint orbits are equivalent to the symplectic leaves. To show this, we prove that the distributions of coadjoint action generators and Hamiltonian vector fields coincide. First, we introduce a basis $\{\mathbf{e}_i\}$ in $\mathfrak{g}$ and the corresponding dual basis $\{\mathbf{e}^{*i}\}$ in $\mathfrak{g}^*$. If we let $\{m_i\}$ be the coordinates of $\mathfrak{g}^*$ associated with the latter basis, we can realize the isomorphism between $T_{\mathbf{m}}\mathfrak{g}^*$ and $\mathfrak{g}^*$ as

$$T_{\mathbf{m}}\mathfrak{g}^* \ni \frac{\partial}{\partial m_i}\bigg|_{\mathbf{m}} \longleftrightarrow \mathbf{e}^{*i} \in \mathfrak{g}^*. \tag{7.37}$$

Then, we take a Hamiltonian vector field ξ_f and evaluate its action on a generic function g

$$(\xi_f)_{\mathbf{m}}(g) = -\{f,g\}(\mathbf{m}) = -\langle \mathbf{m}, [f_{*\mathbf{m}}, g_{*\mathbf{m}}]\rangle = -\langle \mathbf{m}, \mathrm{ad}_{f_{*\mathbf{m}}} g_{*\mathbf{m}}]\rangle = \langle \mathrm{ad}^*_{f_{*\mathbf{m}}}\mathbf{m}, g_{*\mathbf{m}}\rangle. \tag{7.38}$$

Expanding the last term on the chosen basis, we see that $(\xi_f)_{\mathbf{m}}$ can be written as

$$(\xi_f)_{\mathbf{m}} = \langle \mathrm{ad}^*_{f_{*m}}\mathbf{m}, \mathbf{e}_i\rangle \frac{\partial}{\partial m_i}\bigg|_{\mathbf{m}}, \tag{7.39}$$

which coincides with the generator associated with $f_{*\mathbf{m}} \in \mathfrak{g}$ (7.33). This proves that every Hamiltonian vector field is a generator. To prove the converse, we just need to construct a function f such that $f_{*\mathbf{m}} = \mathbf{X}$ for any $\mathbf{X}$, but this is always possible solving

$$\frac{\partial f}{\partial m_i}\bigg|_{\mathbf{m}} = X^i. \tag{7.40}$$

□

The correspondence between algebra elements and functions on the dual space can be extended to the universal enveloping algebra of $\mathfrak{g}$ with the following procedure. Given a basis $\{\mathbf{e}_i\}$ in $\mathfrak{g}$, we can expand an element of the universal enveloping algebra as $\mathbf{K} = \sum_k c^{i_1,\dots i_k}\mathbf{e}_{i_1} \otimes \dots \mathbf{e}_{i_k}$, with $c^{i_1,\dots i_k} \in \mathbb{R}$. Using these coefficients, we construct the function $\mathcal{K} : \mathfrak{g}^* \to \mathbb{R}$, $\mathbf{m} \mapsto \mathcal{K}(\mathbf{m}) = \sum_k c^{i_1,\dots i_k} m_{i_1} \dots m_{i_k}$, where $m_i = \langle \mathbf{m}, \mathbf{e}_i\rangle$. The variation of this function along the orbit can be computed with the Leibniz rule, considering that $\mathbb{L}_{\mathbf{X}} m_i = \langle \mathrm{ad}^*_{\mathbf{X}}\mathbf{m}, \mathbf{e}_i\rangle = -\langle \mathbf{m}, \mathrm{ad}_{\mathbf{X}}\mathbf{e}_i\rangle$

$$\begin{aligned}\mathbb{L}_{\mathbf{X}}\mathcal{K}(\mathbf{m}) &= -\sum_k c^{i_1\dots i_k}(\langle \mathbf{m}, \mathrm{ad}_{\mathbf{X}}\mathbf{e}_{i_1}\rangle m_{i_2}\dots m_{i_k} + \dots + m_{i_1}m_{i_2}\dots\langle \mathbf{m}, \mathrm{ad}_{\mathbf{X}}\mathbf{e}_{i_k}\rangle) = \\ &= -\sum_k c^{i_1\dots i_k}\langle \mathbf{m}\otimes\mathbf{m}\otimes\dots\otimes\mathbf{m}, \mathrm{ad}_{\mathbf{X}}(\mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\dots\otimes\mathbf{e}_{i_k})\rangle = -\langle \mathbf{m}, \mathrm{ad}_{\mathbf{X}}\mathbf{K}\rangle.\end{aligned} \tag{7.41}$$

Therefore, if $\mathbf{K}$ is a Casimir, $\mathcal{K}$ is invariant along the orbit. We call this a *Casimir function*.

7.3.2 Kirillov–Kostant–Souriau symplectic structure

Since coadjoint orbits coincide with the symplectic leaves, they are symplectic. This structure is given by the celebrated Kirrilov–Kostant–Souriau (KKS) 2-form $\Omega^G \in \Omega^2(\mathfrak{g}^*)$, defined, for each $\mathbf{q} \in \mathcal{O}_\mathbf{m}$, by

$$\Omega^G_\mathbf{q}(\xi_\mathbf{X}, \xi_\mathbf{Y}) = -\langle \mathbf{q}, [\mathbf{X}, \mathbf{Y}] \rangle. \tag{7.42}$$

An important property of coadjoint orbits is that the G-action on them is transitive (by definition) and Hamiltonian. In fact, we can check that

$$-\mathbb{I}_{\xi_\mathbf{X}} \Omega^G = \mathbb{d} \mathcal{J}_\mathbf{X}, \tag{7.43}$$

where $\mathcal{J}$ is the inclusion map

$$\mathcal{J} : \mathcal{O}_\mathbf{m} \hookrightarrow \mathfrak{g}^*, \quad \mathbf{q} \to \mathbf{q} \tag{7.44}$$

and $\mathcal{J}_\mathbf{X} \equiv \langle \mathcal{J}(\cdot), \mathbf{X} \rangle$ is just its $\mathbf{X}$-component.

Proof. Let us evaluate both sides of (7.43) for any $\mathbf{q} \in \mathcal{O}_\mathbf{m}$ and any tangent vector field that we can write as $(\xi_Y)_\mathbf{q}$. On the l.h.s. we have

$$(-\mathbb{I}_{\xi_\mathbf{X}} \Omega^G)_\mathbf{q}[\xi_Y] = -\langle \mathbf{q}, [\mathbf{X}, \mathbf{Y}] \rangle = \langle \mathbf{q}, [\mathbf{Y}, \mathbf{X}] \rangle = -\langle \mathrm{ad}^*_\mathbf{Y} \mathbf{q}, \mathbf{X} \rangle, \tag{7.45}$$

while, on the r.h.s.

$$(\mathbb{d} \mathcal{J}_\mathbf{X})_\mathbf{q}[\xi_\mathbf{Y}] = (\xi_\mathbf{Y}[\mathcal{J}_\mathbf{X}])_\mathbf{q} = \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{J}_\mathbf{X}(\mathrm{Ad}^*_{e^{s\mathbf{Y}}} \mathbf{q}) \Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} \langle \mathrm{Ad}^*_{e^{s\mathbf{Y}}} \mathbf{q}, \mathbf{X} \rangle \Big|_{s=0} = \langle \mathrm{ad}^*_\mathbf{Y} \mathbf{q}, \mathbf{X} \rangle. \tag{7.46}$$

□

Thus, the coadjoint action on coadjoint orbits admit a very simple momentum map. As a corollary, we have that the G-action on the orbit is always symplectic.

The momentum map provides an explicit realization of a function whose differential at $\mathbf{q} \in \mathcal{O}_\mathbf{m}$ equals $\mathbf{X}$ (cfr. eq. (7.40)). From here, we can deduce that the coadjoint orbit momentum maps are G-equivariant, meaning that they furnish a true (non-projective) representation of the Lie algebra

$$\{\mathcal{J}_\mathbf{X}, \mathcal{J}_\mathbf{Y}\} = \mathcal{J}_{[\mathbf{X}, \mathbf{Y}]}. \tag{7.47}$$

Proof. In one line

$$\{\mathcal{J}_\mathbf{X}, \mathcal{J}_\mathbf{Y}\}(\mathbf{q}) = \langle \mathbf{q}, [(\mathcal{J}_\mathbf{X})_{*\mathbf{q}}, (\mathcal{J}_\mathbf{Y})_{*\mathbf{q}}] \rangle = \langle \mathbf{q}, [X, Y] \rangle = \mathcal{J}_{[\mathbf{X}, \mathbf{Y}]}(\mathbf{q}). \tag{7.48}$$

□

We now review the coadjoint orbit analysis for the Heisenberg group and for the special linear group, which are the fundamental building blocks of QCS_2. Later, we will perform the analysis of QCS_2 and its classical counterpart ECS_2.

Coadjoint orbits of the Heisenberg group

The Heisenberg group $\mathbb{H}_3$ can be realized as the set of 3×3 upper triangular matrices of the form

$$n(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.49}$$

with standard matrix multiplication as the group product. The Heisenberg algebra $\mathfrak{h}_3$ is then isomorphic to the algebra of strictly upper triangular matrices with the standard matrix commutator. A convenient basis is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.50}$$

where the matrices satisfy

$$[\mathbf{A}, \mathbf{B}] = \mathbf{E}. \tag{7.51}$$

We now want to describe the elements of $\mathfrak{h}_3^*$ using the standard bilinear form on matrices

$$\langle \mathbf{p}, \mathbf{X} \rangle = \mathrm{Tr}[\mathbf{p}\mathbf{X}], \quad \mathbf{X} \in \mathfrak{h},\ \mathbf{p} \in \mathfrak{h}^*. \tag{7.52}$$

Given this pairing, we observe that $\mathfrak{h}_3^\perp$, the space orthogonal to the algebra, is isomorphic to the upper triangular matrices, hence the dual algebra is isomorphic to strictly lower triangular matrices

$$\mathfrak{h}_3^* \cong \mathrm{M}_{3\times 3}(\mathbb{R})/\mathfrak{h}_3^\perp, \tag{7.53}$$

where the r.h.s. defines equivalence classes of matrices with the same strictly lower triangular part: any matrix $\mathbf{p} \in \mathrm{M}_{3\times 3}(\mathbb{R})$ can be identified with an element of $\mathfrak{h}_3^*$ by considering its equivalence class $[\mathbf{p}]$; conversely, we can write matrices in $\mathfrak{h}_3^*$ as normal matrices, with the implicit assumption that only their strictly lower triangular part matters. So we write the dual basis of $\mathfrak{h}_3^*$ as follows

$$\mathbf{A}^* = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{E}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.54}$$

The coadjoint action of a group element is simply given by the matrix product

$$\mathrm{Ad}^*_n[\mathbf{p}] = \left[n\mathbf{p}n^{-1}\right]. \tag{7.55}$$

A simple calculation in the chosen basis gives

$$\mathrm{Ad}^*_{n(a,b,c)} \left[\begin{pmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ E & B & 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 0 & 0 & 0 \\ A + bE & 0 & 0 \\ E & B - aE & 0 \end{pmatrix} \right]. \tag{7.56}$$

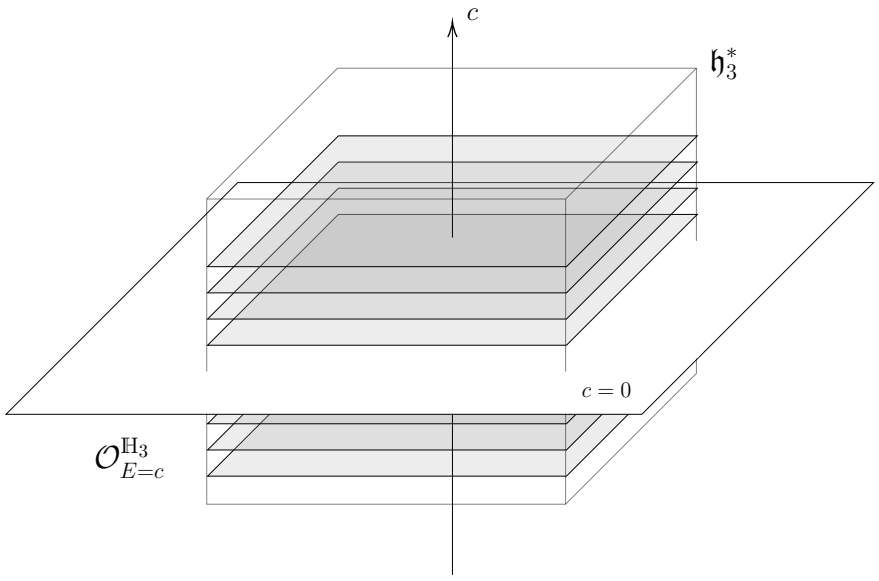


Figure 7.1: Foliation of the dual space $\mathfrak{h}_3^* \cong \mathbb{R}^3$ to the Heisenberg algebra by coadjoint orbits $\mathcal{O}^{\mathbb{H}_3}_E$ labeled by $E = c \neq 0$.

The fact that E is constant under the coadjoint action comes from the fact that $\mathbf{E}$ is a Casimir of the Heisenberg algebra. Since orbits are characterized by a fixed value of E, points $\mathbf{p} = (0, 0, E) \in \mathfrak{h}_3^*$ with different E belong to separate orbits, and for every one of them we can take one such point as a representative.

Equation (7.56) tells us that, given $E \neq 0$,[10] the orbits are just two-dimensional planes

$$\mathcal{O}^{\mathbb{H}_3}_E \equiv \mathcal{O}^{\mathbb{H}_3}_{(0,0,E)} = \{(A, B, E) \mid A, B \in \mathbb{R}\}. \tag{7.57}$$

As expected, these planes provide a foliation of the dual algebra, which is isomorphic to $\mathbb{R}^3$, as depicted in Figure 7.1

$$\mathfrak{h}_3^* \cong \mathbb{R}^3 = \bigcup_{E \in \mathbb{R}} \mathcal{O}^{\mathbb{H}_3}_E. \tag{7.58}$$

Using the coadjoint action, we can compute the generators $\xi_{\mathbf{X}}$ in the basis of $T\mathcal{O}^{\mathbb{H}_3}_E$ associated with the (A, B) coordinates on the orbit:

$$\xi_{\mathbf{A}} = -E\partial_B, \quad \xi_{\mathbf{B}} = E\partial_A, \quad \xi_{\mathbf{E}} = 0. \tag{7.59}$$

The only non-vanishing component of the KKS symplectic form at a point $\mathbf{q} = (A, B, E)$ is given by

$$\Omega^{\mathbb{H}_3}_{\mathbf{q}}(\xi_{\mathbf{A}}, \xi_{\mathbf{B}}) = -\langle \mathbf{q}, [\mathbf{A}, \mathbf{B}] \rangle = -\langle (A, B, E), \mathbf{E} \rangle = -E, \tag{7.60}$$

[10] The $E = 0$ orbits are simply points.

which implies the following KKS symplectic form

$$\Omega^{\mathbb{H}_3} = -\frac{1}{E}\mathbb{d}A \wedge \mathbb{d}B. \tag{7.61}$$

Since the orbits are contractible — they are isomorphic to $\mathbb{R}^2$ — the KKS form is not only closed but exact. Indeed we can write $\Omega^{\mathbb{H}_3} = \mathbb{d}\theta^{\mathbb{H}_3}$ with $\theta^{\mathbb{H}_3} := \frac{1}{E}B\mathbb{d}A$.

Coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$

The group $\mathrm{SL}(2,\mathbb{R})$ is the set of 2×2 matrices of unit determinant with matrix multiplication as the group product. At the algebra level, the condition of unit determinant reflects in the matrices of the $\mathfrak{sl}(2,\mathbb{R})$ defining representation being traceless. We choose the basis

$$\mathbf{L}_0 = \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \mathbf{L}_+ = \begin{pmatrix}0 & 0\\ i & 0\end{pmatrix}, \quad \mathbf{L}_- = \begin{pmatrix}0 & -i\\ 0 & 0\end{pmatrix}, \tag{7.62}$$

which indeed satisfy

$$[\mathbf{L}_-, \mathbf{L}_+] = -2i\mathbf{L}_0, \quad [\mathbf{L}_0, \mathbf{L}_\pm] = \mp i\mathbf{L}_\pm. \tag{7.63}$$

This algebra has the well-known quadratic Casimir element [11]

$$\mathbf{C}_2^{\mathfrak{sl}(2,\mathbb{R})} = \mathbf{L}^2 \equiv \eta^{ij}\mathbf{L}_i\mathbf{L}_j = -\mathbf{L}_0^2 + \frac{1}{2}(\mathbf{L}_+\mathbf{L}_- + \mathbf{L}_-\mathbf{L}_+), \tag{7.64}$$

where we introduced indices $i, j = -, 0, +$ and the invariant Killing metric η^{ij} with signature $(1,2)$. As in the previous example, we want the pairing to be given by the bilinear form on $\mathrm{M}_{2\times 2}(\mathbb{R})$. This implies that the orthogonal space $\mathfrak{sl}(2,\mathbb{R})^\perp$ comprises only multiples of the identity, so that $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathrm{M}_{2\times 2}(\mathbb{R})/\mathfrak{sl}(2,\mathbb{R})^\perp$ is isomorphic to the space of traceless matrices. Taking the coordinate basis of $\mathfrak{sl}(2,\mathbb{R})^*$, we can write the coadjoint action generators as

$$\xi_{\mathbf{L}_0} = iL_+\partial_{L_+} - iL_-\partial_{L_-}, \quad \xi_{\mathbf{L}_+} = -iL_+\partial_{L_0} - 2iL_0\partial_{L_-}, \quad \xi_{\mathbf{L}_-} = iL_-\partial_{L_0} + 2iL_0\partial_{L_+}. \tag{7.65}$$

Another way to prove the isomorphism $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathfrak{sl}(2,\mathbb{R})$ is to use the previously introduced Killing form η^{ij}, which is non-degenerate, to explicitly construct the dual basis $\theta^i = 2\eta^{ij}\mathbf{L}_j$. In this basis, the coadjoint action transforms the components as

$$\mathrm{Ad}^*_h(L_i\theta^i) = 2L_j\eta^{jk}\,\mathrm{Tr}\Big[h\mathbf{L}^k h^{-1}\mathbf{L}_i\Big]\theta^i \equiv L_j\Lambda(h)^j{}_i\theta^i \tag{7.66}$$

where $\Lambda(h)^j{}_k$ is a $\mathfrak{so}(2,1)$ Lorentz transformation, consistently with the adjoint action (cfr. footnote 11). We just proved that the $\mathrm{SL}(2,\mathbb{R})$-coadjoint orbits are in one-to-one

[11] As a vector space, $\mathrm{SL}(2,\mathbb{R}) = \mathbb{R}^{(1,2)}$ and the adjoint action of the algebra on itself corresponds to the set of $2d$ Lorentz transformations: one rotation and two boosts. In this representation, the quadratic Casimir corresponds to the squared momentum.

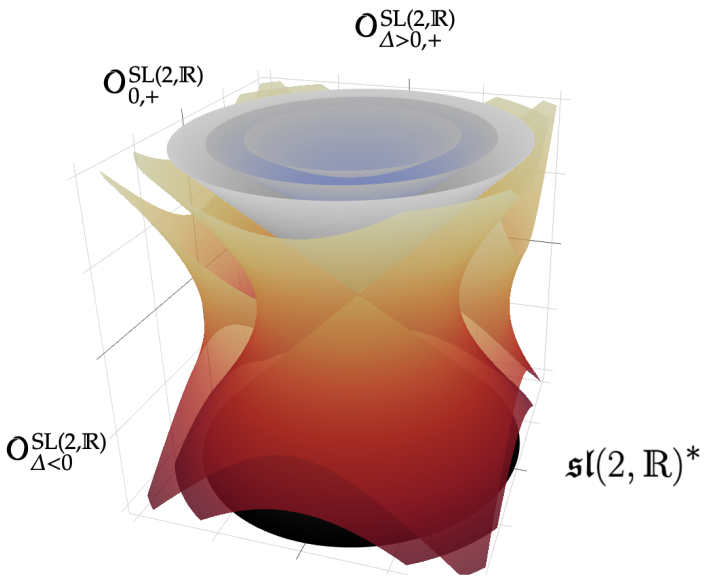


Figure 7.2: Foliation of the dual space $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathbb{R}^{(1,2)}$ to the $\mathfrak{sl}(2,\mathbb{R})$ algebra by coadjoint orbits $\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,r}$. Orbits with $\Delta < 0$ are one-sheeted hyperboloids (yellow-red gradient); those with $\Delta > 0$ form one of the two sheets, distinguished by $r = \pm$, of a two-sheeted hyperboloids (light blue). When $\Delta = 0$, there are only two orbits, corresponding to the upper (grey) and lower (black) cone, distinguished again by $r = \pm$.

correspondence with the orbits of a vector in $\mathbb{R}^{(1,2)}$ under Lorentz transformations, from which we borrow the nomenclature. We thus have *massive*, *tachyonic*, and *massless* orbits, depending on the value of $L^2 \equiv -\Delta$, which is the Casimir function associated with $\mathbf{L}^2$.

- Massive orbits: When $\Delta > 0$, the orbits are the two sheets of a two-sheeted hyperboloid, distinguished by the sign of L_0. The corresponding tangent spaces are spanned by

$$\partial_{L_+} = \mp \frac{i}{2\sqrt{\Delta + L_+ L_-}} \xi_{\mathbf{L}_-}, \quad \partial_{L_-} = \pm \frac{i}{2\sqrt{\Delta + L_+ L_-}} \xi_{\mathbf{L}_+}, \tag{7.67}$$

  where the upper (lower) sign corresponds to the upper (lower) sheet. On these orbits, the KKS form is

$$\Omega^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,\pm} = \pm \frac{1}{2i} \frac{\mathbb{d}L_- \wedge \mathbb{d}L_+}{\sqrt{\Delta + L_+ L_-}}; \tag{7.68}$$

- Tachyonic orbits: When $\Delta < 0$, the orbits are one-sheeted hyperboloids. A convenient parametrization is in terms of the angle φ, defined by $L_\pm = (L_0 \pm i\hbar s)\, e^{\mp i\varphi}$ with $\hbar^2 s^2 = -\Delta$. This allows us to span the tangent space to these orbits with

$$\partial_{L_0} = \frac{i}{2}\left(\frac{L_0 + i\hbar s}{L_0} \frac{\xi_{\mathbf{L}_+}}{L_+} - \frac{L_0 - i\hbar s}{L_0} \frac{\xi_{\mathbf{L}_-}}{L_-} \right), \quad \partial_\varphi = -\xi_{\mathbf{L}_0}. \tag{7.69}$$

  On these orbits, the KKS form is simply

$$\Omega^{\mathrm{SL}(2,\mathbb{R})}_{\Delta} = -\mathbb{d}L_0 \wedge \mathbb{d}\varphi; \tag{7.70}$$

- Trivial orbits: When $\Delta = 0$ because $L_i = 0$, the orbit $\Omega^{\mathrm{SL}(2,\mathbb{R})}_{0,0}$ is a single point;

- Null orbits: When $\Delta = 0$ but $L_i \neq 0$, there are only two orbits $\Omega^{\mathrm{SL}(2,\mathbb{R})}_{0,\pm}$, given by the null cones in the $L_0 > 0$ and $L_0 < 0$ half-spaces. We can see the null orbits either as the $\Delta \to 0^+$ limit of the punctured massive orbits (the orbits minus the point $L_+ = L_- = 0$) or as the $\Delta \to 0^-$ limit of the sliced tachyonic orbits (the orbitd minus the circle $L_0 = 0$) and extend to them the previous parametrization and results.

To collectively denote all these orbits, we used the notation $\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,r}$ where Δ denotes (minus) the value of the Casimir function on the orbit and, for $\Delta > 0$, the additional label $r \in \{-,+\}$ distinguishes the two sheets, while, for $\Delta = 0$, $r \in \{-,0,+\}$ distinguishes the two cones and the origin. These orbits are represented in Figure 7.2. The interested reader can explore in greater depth the $\mathrm{SL}(2,\mathbb{R})$ representation theory by consulting the extensive literature on the subject (see [329–332] for a selection of relevant references).

7.3.3 Geometric quantization

Because they are symplectic, coadjoint orbits are good candidates to represent physically interesting phase spaces. Moreover, as they carry a transitive, Hamiltonian action of a group (the original group G), their quantization is expected to provide irreducible representations of that group. This is the core of Kirillov's program to *geometrize* representation theory [313]. Whereas finding coadjoint orbits is a conceptually simple task — although it can be challenging in practice — the major obstacle to the program is finding a way to quantize them. Indeed, not all orbits are amenable to the same quantization technique.

We review here the basics of **geometric quantization** applied to a generic symplectic manifold space, with the purpose of applying it in the following section to the coadjoint orbits we have seen so far.

The starting point to geometrically quantize a phase space $(\mathcal{P}, \omega)$ is to introduce a complex line bundle L over it: sections of this bundle will serve as wavefunctions representing quantum states. Then, we introduce the two following ingredients:

- A *connection* ∇ on L. This is essentially a rule to compare sections at different fibers in order to be able to take derivatives. Specifically, given any (complex) vector field $\xi \in \mathfrak{X}^{\mathbb{C}}(\mathcal{P})$, the connection produces a linear map

$$\nabla_\xi : \Gamma(\mathcal{P}, \mathrm{L}) \to \Gamma(\mathcal{P}, \mathrm{L}), \quad \psi \mapsto \nabla_\xi \psi, \tag{7.71}$$

  which satisfies the Leibniz rule

$$\nabla_\xi(f\psi) = \xi(f)\psi + f\nabla_\xi\psi; \tag{7.72}$$

- A *Hermitian structure* $(\cdot|\cdot)$ on L. This is a smooth map

$$\Gamma(\mathcal{P}, \mathrm{L}) \times \Gamma(\mathcal{P}, \mathrm{L}) \to C^\infty(\mathcal{P}, \mathbb{C}) \tag{7.73}$$

  which is linear in the second and antilinear in the first

$$(\psi|\alpha\psi_1 + \beta\psi_2) = \alpha(\psi_1|\psi) + \beta(\psi_2|\psi), \tag{7.74}$$
$$(\alpha\psi_1 + \beta\psi_2|\psi) = \alpha^*(\psi_1|\psi) + \beta^*(\psi_2|\psi). \tag{7.75}$$

The orbit is said to be *prequantizable* if we can choose these two ingredients to be compatible in the following sense

$$\xi[(\psi_1|\psi_2)] = (\nabla_\xi\psi_1|\psi_2) + (\psi_1|\nabla_\xi\psi_2). \tag{7.76}$$

If ξ is a vector field that makes the l.h.s. vanish, this condition is essentially saying that $i\nabla_\xi$ is *self-adjoint* with respect to the Hermitian structure.

Let us now find a quantization prescription to turn the space of sections into a Hilbert space. The basic request we make is that said Hilbert space must support an operator algebra that can be associated with the Poisson algebra of phase space functions

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}} + \mathcal{O}(\hbar)^2. \tag{7.77}$$

The prescription of geometric quantization is

$$f \to \hat{f} = f - i\hbar\nabla_{\xi_f}, \tag{7.78}$$

where ξ_f is the Hamiltonian vector field of f. This prescription satisfies eq. (7.77) — with no $\hbar^2$ correction — as long as the curvature of the connection is given by the symplectic form

$$R_\nabla = \frac{\omega}{i\hbar}. \tag{7.79}$$

Proof. A direct evaluation shows that, given a section ψ

$$\begin{aligned}
[\hat{f},\hat{g}]\psi &= [f - i\hbar\nabla_{\xi_f}, g - i\hbar\nabla_{\xi_g}]\psi = -i\hbar[\nabla_{\xi_f}, g]\psi - i\hbar[f,\nabla_{\xi_g}]\psi - \hbar^2[\nabla_{\xi_f},\nabla_{\xi_g}]\psi = \\
&= -i\hbar\xi_f[g]\psi + i\hbar\xi_g[f]\psi - \hbar^2\nabla_{[\xi_f,\xi_g]_{\text{Lie}}}\psi - \hbar^2 R_\nabla(\xi_f,\xi_g)\psi = \\
&= 2i\hbar\{f,g\}\psi + \hbar^2\nabla_{\xi_{\{f,g\}}}\psi - \hbar^2 R_\nabla(\xi_f,\xi_g)\psi = \\
&= i\hbar(\{f,g\} - i\hbar\nabla_{\xi_{\{f,g\}}})\psi + i\hbar(\{f,g\} + i\hbar R_\nabla(\xi_f,\xi_g))\psi = \\
&= i\hbar\widehat{\{f,g\}}\psi + (\{f,g\} + i\hbar R_\nabla(\xi_f,\xi_g))\psi.
\end{aligned} \tag{7.80}$$

The last term in round brackets vanishes if and only if

$$R_\nabla(\xi_f,\xi_g) = \frac{i}{\hbar}\{f,g\} = \frac{1}{i\hbar}\omega(\xi_f,\xi_g) \tag{7.81}$$

for any $f, g \in C^\infty(\mathcal{P},\mathbb{C})$, thus proving the claim that the forms must be the same (Hamiltonian vector fields are dense in $\mathfrak{X}^{\mathbb{C}}(\mathcal{P})$). □

In order to be promoted to the curvature form of a complex line bundle, the symplectic form must satisfy the following integrality condition in terms if its (de Rham) cohomology class

$$\left[\frac{\omega}{2\pi\hbar}\right] \in H^2(\mathcal{P},\mathbb{Z}). \tag{7.82}$$

This **quantization condition** implies that the integral of ω over any closed two-dimensional surface is an integer multiple of $2\pi\hbar$. Among the prequantized orbits, only those that satisfy this condition are actually quantizable.

The last step of geometric quantization is the choice of a *polarization*. This is necessary because the space of sections is too big. Since coadjoint orbits are symplectic, a section over it will generically depend on both "positions" and "momenta", in contrast with the uncertainty principle. To solve this issue, we must split the variables in two sets and require that wavefunctions depend only on one of them. Since such a split is usually impossible to make globally, we introduce a polarization: an involutive distribution $P \subset T^{\mathbb{C}}(\mathcal{P})$ that is Lagrangian (which means $\Omega(\xi_1,\xi_2) = 0\ \forall\xi_1,\xi_2 \in P$) and of maximal dimension. Then we declare that the allowed wavefunctions are such that

$$\nabla_\xi\psi = 0, \quad \forall\xi \in P. \tag{7.83}$$

The choice of polarization is deeply related to the choice of connection because, after we construct the operators with the quantization map (7.78), we must retain only those operators that preserve the polarization. Since the connection is related to the choice of a symplectic potential for the symplectic form, we need the latter potential to be compatible with the polarization, as we will see in Section 7.4.

Quantization of the Heisenberg orbits

Let us now apply geometric quantization to the Heisenberg group coadjoint orbits we described in 7.3.2, which are essentially planes. Picking one orbit $\mathcal{O}_E^{\mathbb{H}_3}$, we consider L to be the trivial line bundle over it. Sections of this bundle are simply complex-valued functions of the plane

$$\psi : \mathcal{O}_E^{\mathbb{H}_3} \cong \mathbb{R}^2 \longrightarrow \mathbb{C}. \tag{7.84}$$

For readability and to match with the usual Schrödinger representation, we relabel the coordinates (A, B, E) as (x, p, c). Since the bundle is trivial, we can introduce a trivial point-wise hermitian structure

$$(\psi_1|\psi_2)(x,p) := \overline{\psi_1(x,p)}\psi_2(x,p). \tag{7.85}$$

Our choice of connection is

$$\nabla_\xi \psi := \xi[\psi] - \frac{i}{\hbar}\mathbb{l}_\xi \theta^{\mathbb{H}_3}\psi \tag{7.86}$$

where $\theta^{\mathbb{H}_3} = \frac{1}{E}B\mathbb{d}A = \frac{1}{c}p\mathbb{d}x$ is a potential for the KKS form that was introduced right after eq. (7.61). It is easy to see that these two structures are compatible.

Let us then apply the geometric quantization prescription

$$f \mapsto \hat{f} = f - i\hbar\nabla_{\xi_f} = f - i\hbar\xi_f - \mathbb{l}_{\xi_f}\theta^{\mathbb{H}_3}, \tag{7.87}$$

to the momentum maps of the coadjoint action, since we know that they will furnish an equivariant representation. From the associated Hamiltonian vector fields

$$\xi_{\mathbf{A}} = -c\partial_p, \qquad \xi_{\mathbf{B}} = c\partial_x, \qquad \xi_{\mathbf{E}} = 0, \tag{7.88}$$

we can easily obtain the quantum operators

$$\hat{\mathcal{J}}_{\mathbf{A}} = i\hbar c\,\partial_p + x, \qquad \hat{\mathcal{J}}_{\mathbf{B}} = -i\hbar c\,\partial_x, \qquad \hat{\mathcal{J}}_{\mathbf{E}} = c. \tag{7.89}$$

At this stage, it is clear that the polarization $P = \mathrm{Span}(\partial_p)$ is the one compatible with the chosen observables

$$\partial_p\psi = 0 \implies \partial_p(\hat{\mathcal{J}}_{\mathbf{A}}\psi) = \partial_p(\hat{\mathcal{J}}_{\mathbf{B}}\psi) = \partial_p(\hat{\mathcal{J}}_{\mathbf{E}}\psi) = 0, \tag{7.90}$$

consistently with the fact that $\theta^{\mathbb{H}_3}(\partial_p) = 0$. Polarized wavefunctions are therefore complex valued function of $x \in \mathbb{R}$, on which the observables act as

$$\hat{\mathcal{J}}_{\mathbf{A}}\psi(x) = x\psi(x), \tag{7.91}$$

$$\hat{\mathcal{J}}_{\mathbf{B}}\psi(x) = -i\hbar c\partial_x\psi(x), \tag{7.92}$$

$$\hat{\mathcal{J}}_{\mathbf{E}}\psi(x) = c\psi(x), \tag{7.93}$$

which is the well known action of the Heisenberg group in quantum mechanics.

Half-form construction To construct the Hilbert space, we need to introduce a scalar product. Since we chose a *real polarization*, we need to use the so-called half-form construction, which we will shortly summarize here. We refer the reader to the following publications [330, 333–335] for a more comprehensive exposition. On a given orbit, we denote by $Q \subset \mathcal{O}^G$ a submanifold that is isomorphic to the space of leaves corresponding to the chosen polarization $P = \mathrm{Span}(\partial_p)$ (an involutive distribution induces a foliation) and construct its complexified cotangent bundle $T^*Q^{\mathbb{C}}$. Sections of this bundle are complexified forms on $\mathcal{O}_E^{\mathbb{H}_3}$ that are by construction orthogonal to the polarization, i.e. $\mathbb{I}_\xi\varpi = 0$, $\forall \xi \in P$. Last, we introduce the *square-root bundle* $\delta_P \to T^*Q$, whose sections are **half-forms** in the sense that $\delta_P^{\otimes 2} \cong T^*Q^{\mathbb{C}}$. Wavefunctions are then defined as elements of the product bundle $\mathrm{L} \otimes \delta_P$.

More concretely, in the case under consideration, the choice of $P = \mathrm{Span}(\partial_p)$ leads to $T^*Q^{\mathbb{C}} = \{\lambda\, \mathbb{d}x \mid \lambda \in C^\infty(Q, \mathbb{C})\}$. Since Q is one-dimensional, we can write a polarized [12] wavefunction as $\tilde{\psi} = \psi_{1/2}\sqrt{\mathrm{d}x} \in \Gamma(\mathrm{L}\otimes\delta_P)$, with $\psi_{1/2}$ that transforms as a density of weight $1/2$:

$$\psi_{1/2}(x) \to \left(\frac{\mathrm{d}x'}{\mathrm{d}x}\right)^{1/2} \psi_{1/2}(x'). \tag{7.94}$$

The hermitian structure (7.85) extends to polarized wavefunctions as

$$(\tilde{\psi}_1|\tilde{\psi}_2)(x) = \overline{\psi_1(x)}\psi_2(x)\bar{\nu}_1\nu_2. \tag{7.95}$$

Since the latter is a volume form on the submanifold Q, it can naturally be integrated to obtain the scalar product: modulo possible irrelevant functions that can be reabsorbed in the definition of the wavefunctions, we thus obtain the standard $L^2(\mathbb{R})$ scalar product

$$\left\langle \tilde{\psi}_1 \middle| \tilde{\psi}_2 \right\rangle = \int_Q (\tilde{\psi}_1|\tilde{\psi}_2) = \int_{\mathbb{R}} \mathrm{d}x\, \overline{\psi_1(x)}\psi_2(x), \tag{7.96}$$

[12] Given the covariant derivative on $T^*Q^{\mathbb{C}}$, $\nabla_\xi\varpi = \mathbb{I}_\xi \mathbb{d}\varpi$, we define the covariant derivative on δ_P such that

$$2\nu \otimes \nabla_\xi\nu = \nabla_\xi(\nu\otimes\nu) = \mathbb{I}_\xi \mathbb{d}(\nu\otimes\nu).$$

for which it is easy to see that the quantized momentum map are Hermitian operators. This completes the construction of the Fock Hilbert space $\mathfrak{F}_e$ associated with the central charge c.

$$\mathcal{O}_c^{\mathrm{H}_3} \xrightarrow{\text{quantization}} \mathscr{F}_c. \tag{7.97}$$

To make the structure more familiar, we consider the following Lie algebra elements

$$\mathbf{T}_\pm = \frac{\mathbf{A} \pm i\mathbf{B}}{\sqrt{2}}, \tag{7.98}$$

and construct the associated operators

$$\hat{\mathcal{J}}_{\mathbf{T}_\pm} = \frac{1}{\sqrt{2}}\Big(\hat{\mathcal{J}}_{\mathbf{A}} \pm i\hat{\mathcal{J}}_{\mathbf{B}}\Big), \tag{7.99}$$

which satisfy $[\hat{\mathcal{J}}_{\mathbf{T}_-}, \hat{\mathcal{J}}_{\mathbf{T}_+}] = \hbar\hat{\mathcal{J}}_{\mathbf{E}}$. Since $\hat{\mathcal{J}}_{\mathbf{E}}$ is represented by a multiple of the identity in $\mathscr{F}_e$ (by Schur's lemma), we can use $\hat{\mathcal{J}}_{\mathbf{T}_-}$ and $\hat{\mathcal{J}}_{\mathbf{T}_+}$ to build creation and annihilation operators. Assuming $c > 0$, we set [13]

$$\hat{\mathcal{J}}_{\mathbf{T}_-} =: \sqrt{c\hbar}a, \qquad \hat{\mathcal{J}}_{\mathbf{T}_+} =: \sqrt{c\hbar}a^\dagger, \tag{7.100}$$

and it is easy to check that

$$[a, a^\dagger] = \frac{1}{c\hbar}\hat{\mathcal{J}}_{\mathbf{E}} = 1. \tag{7.101}$$

Letting $\{|x\rangle\}_{x\in\mathbb{R}}$ be the set of (position) eigenstates of the operator $\hat{\mathcal{J}}_{\mathbf{A}}$, we can construct the usual **Fock basis** $\{|k\rangle\}_{k\in\mathbb{N}}$ using Hermite polynomials [14]

$$|k\rangle = \frac{1}{\pi^{1/4}\sqrt{2^k k!}} \int_{-\infty}^{\infty} \mathrm{d}x\, H_k(x/\sqrt{2c\hbar})\, e^{-\frac{x^2}{2c\hbar}}\, |x\rangle\,. \tag{7.102}$$

On these states, we find the expected action of a and $a^\dagger$

$$a\,|k\rangle = \sqrt{k}\,|k-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{k+1}\,|k+1\rangle\,. \tag{7.103}$$

Quantization of $\mathrm{SL}(2,\mathbb{R})$ orbits

In this section, we present in detail the quantization of $\mathrm{SL}(2,\mathbb{R})$ massive and tachyonic orbits, and make some comments about the other types of orbits.

[13] For $c < 0$, we just switch the roles of $\hat{\mathcal{J}}_{\mathbf{T}_-}$ and $\hat{\mathcal{J}}_{\mathbf{T}_+}$.

[14] The k-th Hermite polynomial is

$$H_k(x) = (-1)^k e^{x^2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2}.$$

Massive orbits Consider the massive orbits $\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,\pm}$ described in 7.3.2. As mentioned there, these are two-sheeted hyperboloids in $\mathfrak{sl}(2,\mathbb{R})^*$ given by the relation $L^2 = -\Delta$, with $\Delta > 0$. For convenience, we write $\Delta = \hbar^2 q^2$ with $q > 0$. We will here quantize only the upper sheet, as the quantization of the lower sheet will be similar. This sheet of the massive orbit is diffeomorphic to the upper half complex plane $\mathbb{H}^+ := \{\tau \in \mathbb{C} | \, \Im(\tau) > 0\}$ ($\Im$ denotes the imaginary part). An explicit diffeomorphism is given by

$$\Phi : \mathbb{H}^+ \to \mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,+}, \quad (\tau,\bar\tau) \mapsto (L_+, L_-) = \hbar q \left(\frac{(\tau - i)(\bar\tau - i)}{2\,\Im(\tau)}, \frac{(\tau + i)(\bar\tau + i)}{2\,\Im(\tau)} \right), \tag{7.104}$$

where we take the positive branch of the square root for L_0

$$L_0 = \sqrt{\hbar^2 q^2 + L_+ L_-} = \hbar q \frac{1 + |\tau|^2}{2\,\Im(\tau)} > 0. \tag{7.105}$$

The advantage of this construction is that the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^+$ is naturally realized in terms of *real* Möbius transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{R}. \tag{7.106}$$

As these transformations do not mix τ and $\bar\tau$, the *holomorphic polarization* $P = \mathrm{Span}(\partial_{\bar\tau})$ in $T^{\mathbb{C}}\mathbb{H}^+$ is clearly compatible with them. The anti-holomorphic polarization $\mathrm{Span}(\partial_\tau)$ would be preserved as well, but we choose the former for definiteness. Notice that the holomorphic polarization does not have a counterpart on the hyperboloid tangent bundle, but only on its complexification. While this choice might appear strange — even unmotivated in the framework of geometric quantization — it is necessary because there is no real polarization on these orbits that is preserved by the $\mathrm{SL}(2,\mathbb{R})$ action [330].

The pull-back of the KKS symplectic form to $\mathbb{H}^+$ gives the unique (up to real constants) $\mathrm{SL}(2,\mathbb{R})$ invariant symplectic potential in $\Omega^2(\mathbb{H}^+, \mathbb{R})$

$$\omega^+ = \Phi^* \Omega^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,+} = i\hbar q \frac{\mathbb{d}\tau \wedge \mathbb{d}\bar\tau}{2\,\Im(\tau)^2}, \tag{7.107}$$

which can be written as the double derivative of a *Kähler potential* $\mathfrak{K}$: [15]

$$\omega^+ = -i \frac{\partial^2 \mathfrak{K}}{\partial\bar\tau \partial\tau} \mathbb{d}\tau \wedge \mathbb{d}\bar\tau, \quad \mathfrak{K} := 2\hbar q \ln(\Im(\tau)). \tag{7.108}$$

As we work in the holomorphic polarization, we choose the following symplectic potential ($\omega^+ = \mathbb{d}\theta^+$) to build the connection

$$\theta^+ = \frac{\hbar q\, \mathbb{d}\tau}{\Im(\tau)}, \tag{7.109}$$

[15] The existence of this a Kähler structure naturally fits with the decomposition $T^{\mathbb{C}}\mathbb{H}^+ = P \oplus \bar P$.

which trivially satisfies $\theta^+(P) = 0$.

We now quantize the momentum maps corresponding to the $\mathbf{L}_i$ generators with the prescription (7.78). Pushing-forward the $\xi_{\mathbf{L}_i}$ vector fields through the inverse map

$$\Phi_*^{-1}\xi_{\mathbf{L}_0} = \frac{1}{2}\big((1+\tau^2)\partial_\tau + (1+\bar\tau^2)\partial_{\bar\tau}\big), \tag{7.110a}$$

$$\Phi_*^{-1}\xi_{\mathbf{L}_-} = \frac{1}{2}\big((\tau+i)^2\partial_\tau + (\bar\tau+i)^2\partial_{\bar\tau}\big), \tag{7.110b}$$

$$\Phi_*^{-1}\xi_{\mathbf{L}_+} = \frac{1}{2}\big((\tau-i)^2\partial_\tau + (\bar\tau-i)^2\partial_{\bar\tau}\big), \tag{7.110c}$$

we construct the quantum operators acting on holomorphic sections of the line bundle $\psi \in \Gamma_P(\mathrm{L})$:

$$\hat{\mathcal{J}}_{\mathbf{L}_0}\psi(\tau) = -i\hbar\frac{1+\tau^2}{2}\partial_\tau\psi(\tau) - i\hbar q\tau\psi(\tau), \tag{7.111a}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_+}\psi(\tau) = -i\hbar\frac{(\tau-i)^2}{2}\partial_\tau\psi(\tau) - i\hbar q(\tau-i)\psi(\tau), \tag{7.111b}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_-}\psi(\tau) = -i\hbar\frac{(\tau+i)^2}{2}\partial_\tau\psi(\tau) - i\hbar q(\tau+i)\psi(\tau). \tag{7.111c}$$

To turn the space of holomorphic sections into a Hilbert space, we need to introduce a scalar product. On $\mathbb{H}^+$, there exists a family of $\mathrm{SL}(2,\mathbb{R})$ covariant measures given by [336]

$$\mathrm{d}\mu_m = \mathrm{d}\mu\, e^h = \frac{\mathrm{d}\tau\mathrm{d}\bar\tau}{\Im(\tau)^2}\Im(\tau)^m, \quad m > 1, \tag{7.112}$$

where $h = m\ln(\Im(\tau))$ is a metric on the line bundle. If we want θ^+ to be compatible with this metric, we must have $h = \Re/\hbar$, which implies $m = 2q$. The Hilbert space that we associate with an orbit with Casimir value $-\hbar^2q^2$ is thus $L^2(\mathbb{H}^+, \mathrm{d}\mu_{2q}) \subset \Gamma_P(\mathrm{L})$, where the inner product is given by

$$\langle\psi_1|\psi_2\rangle = \int_{\mathbb{H}^+}\mathrm{d}\mu_{2q}(\tau)\overline{\psi_1(\tau)}\psi_2(\tau). \tag{7.113}$$

The action of $\mathrm{SL}(2,\mathbb{R})$ preserves this product provided that the wavefunctions transform as the component of differential forms of degree q, i.e.

$$\psi(\tau) \to \left(\frac{1}{c\tau+d}\right)^{2q}\psi\left(\frac{a\tau+b}{c\tau+d}\right). \tag{7.114}$$

With respect to this inner product, the operators in (7.111) satisfy $\hat{\mathcal{J}}_{\mathbf{L}_0}^\dagger = \hat{\mathcal{J}}_{\mathbf{L}_0}$ and $\hat{\mathcal{J}}_{\mathbf{L}_-}^\dagger = \hat{\mathcal{J}}_{\mathbf{L}_+}$, which means that they provide a unitary representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. If the transformation (7.114) is single-valued (which happens for $q \in \mathbb{Z}/2$), the Hilbert space carries a representation of the non-covered $\mathrm{SL}(2,\mathbb{R})$. Notice that this construction is valid for $q > 1/2$, but there exists an alternative unitary structure for $0 < q \leq 1/2$ [323].

Proof. Let us first prove that the inner product is preserved by the $\mathrm{SL}(2,\mathbb{R})$ action

$$\langle \psi_1' | \psi_2' \rangle = \langle \psi_1 | \psi_2 \rangle \tag{7.115}$$

where we denoted by ψ_1', ψ_2' the transformed wavefunctions in (7.114).

$$\langle \psi_1' | \psi_2' \rangle = \int_{\mathbb{H}^+} \mathrm{d}\mu_{2q}(\tau) \left(\frac{1}{c\bar{\tau}+d}\right)^{2q} \overline{\psi_1(\tau')} \left(\frac{1}{c\tau+d}\right)^{2q} \psi_2(\tau') \tag{7.116}$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$. Applying the inverse Möbius transformation on the coordinates $\tau \to \tau'' = -\frac{d\tau-b}{c\tau-a}$, we can map τ' back to τ and get

$$\langle \psi_1' | \psi_2' \rangle = \int_{\mathbb{H}^+} \mathrm{d}\mu_{2q}(\tau'') \left(\frac{1}{c\bar{\tau}''+d}\right)^{2q} \overline{\psi_1(\tau)} \left(\frac{1}{c\tau''+d}\right)^{2q} \psi_2(\tau). \tag{7.117}$$

We now determine the change in the measure under a real Möbius transformation. For simplicity, we write the general result and later use it for the inverse Möbius transformation we were considering. Noticing that

$$\Im\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{2i}\frac{(ad-bc)(\tau-\bar{\tau})}{|c\tau+d|^2} = \frac{\Im(\tau)}{|c\tau+d|^2}, \tag{7.118}$$

we compute

$$\mathrm{d}\mu_m\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\Im(\tau)^{m-2}}{|c\tau+d|^{2(m-2)}} \frac{\mathrm{d}\tau \mathrm{d}\bar{\tau}}{(c\tau+d)^2(c\bar{\tau}+d)^2} = \frac{1}{|c\tau+d|^{2m}} \mathrm{d}\mu_m(\tau). \tag{7.119}$$

If we apply this result for τ'', we get

$$\mathrm{d}\mu_m(\tau'') = |c\tau''+d|^{2m} \mathrm{d}\mu_m(\tau). \tag{7.120}$$

which means that the measure change exactly compensates, for $m = 2q$, the prefactors due to the wavefunction transformation

$$\langle \psi_1' | \psi_2' \rangle = \int_{\mathbb{H}^+} \frac{\mathrm{d}\mu_{2q}(\tau'')}{|c\tau''+d|^{4q}} \overline{\psi_1(\tau)} \psi_2(\tau) = \int_{\mathbb{H}^+} \mathrm{d}\mu_{2q}(\tau) \overline{\psi_1(\tau)} \psi_2(\tau) = \langle \psi_1 | \psi_2 \rangle . \tag{7.121}$$

□

If we denote by $|n\rangle \equiv |n, q\rangle$ the state such that

$$\langle \tau | n \rangle = c_n \left(\frac{\tau - i}{\tau + i}\right)^n (1+\tau^2)^{-q}, \tag{7.122}$$

where c_n is a suitable normalization constant (i.e. $|c_n| < \infty$) [329], we can show that

$$\hat{\mathcal{J}}_{\mathbf{L}_0} |n\rangle = \hbar\, n\, |n\rangle\,, \tag{7.123a}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_-} |n\rangle = \hbar\sqrt{n(n-1) + q(1-q)}\, |n-1\rangle\,, \tag{7.123b}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_+} |n\rangle = \hbar\sqrt{n(n+1) + q(1-q)}\, |n+1\rangle\,. \tag{7.123c}$$

This prove that $L^2(\mathbb{H}^+, \mathrm{d}\mu_{2q})$ with $q > 1/2$ coincides with the Hilbert space $\mathcal{H}_{q,\mu=q}$ that carries the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ positive discrete series (for convenience, the representation theory of $\mathrm{SL}(2,\mathbb{R})$ is shortly summarized in Appendix D)

$$\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{q^2,+} \xrightarrow{\text{quantization}} \mathcal{H}_{q^2,+} \equiv \mathcal{H}_{q,\mu=q}. \tag{7.124}$$

Notice that, in contrast to the familiar case of $\mathfrak{su}(2)$, the discreteness of these representations does not come from the integrality condition (7.82), but from the single-valuedness of (7.114). Finally, we can check that the Casimir $\mathbf{L}^2$ is represented by a multiple of the identity

$$\left(-\hat{\mathcal{J}}^2_{\mathbf{L}_0} + \frac{1}{2}(\hat{\mathcal{J}}_{\mathbf{L}_+}\hat{\mathcal{J}}_{\mathbf{L}_-} + \hat{\mathcal{J}}_{\mathbf{L}_-}\hat{\mathcal{J}}_{\mathbf{L}_+})\right) |n\rangle = \hbar^2 q(1-q)\, |n\rangle\,, \tag{7.125}$$

finding that the Casimir value $-q^2$ is corrected to $q(1-q)$ at the quantum level.

Tachyonic orbits Since we have already presented in two examples the procedure of geometric quantization, we will be less detailed in this instance. Tachyonic orbits are obtained when $\Delta \equiv -\hbar^2 s^2 < 0$. Topologically, they are cylinders and can thus be seen as cotangent bundles of a circle $\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})} = T^*S^1$ [337].

Cotangent bundles are easy to quantize: the Hilbert space is just the space of complex functions of the base manifold, the circle, in this case. As in the case of the Heisenberg orbits, we are using a real polarization $P = \mathrm{Span}(\partial_{L_0})$, so we need to introduce half-forms here as well. Our wavefunctions can be written as products $\psi\nu$, with $\psi \in L^2(S^1, \mathrm{d}\varphi)$ and $\nu \in \Gamma(\delta_P)$. Choosing the potential $\theta = (\hbar\mu - L_0)\mathrm{d}\varphi$, we quantize the functions $\mathcal{J}_{\mathbf{L}_i}$ and get

$$\hat{\mathcal{J}}_{\mathbf{L}_0}\psi(\varphi) = i\hbar\partial_\varphi\psi(\varphi) + \hbar\mu\psi(\varphi), \tag{7.126a}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_\pm}\psi(\varphi) = i\hbar e^{\mp i\varphi}\left(i\partial_\varphi\psi(\varphi) + \mu\psi(\varphi) \pm i\left(s - \frac{i}{2}\right)\psi(\varphi)\right). \tag{7.126b}$$

These operators satisfy

$$\left(-\hat{\mathcal{J}}^2_{\mathbf{L}_0} + \frac{1}{2}(\hat{\mathcal{J}}_{\mathbf{L}_+}\hat{\mathcal{J}}_{\mathbf{L}_-} + \hat{\mathcal{J}}_{\mathbf{L}_-}\hat{\mathcal{J}}_{\mathbf{L}_+})\right)\psi = \hbar^2\left(s^2 + \frac{1}{4}\right)\psi, \tag{7.127}$$

which shows that the Casimir is shifted from s^2 to $s^2 + 1/4$ at the quantum level. This value of the Casimir corresponds to a representation in the continuous principal series. We can show this explicitly introducing the states $|n\rangle \equiv \left|n, \frac{1}{2} + is\right\rangle$ by

$$\langle \phi | n \rangle = \frac{1}{\sqrt{2\pi}} e^{i(n-\mu)\phi}, \qquad \text{with} \quad n \in \mu + \mathbb{Z}. \tag{7.128}$$

These are such that

$$\hat{\mathcal{J}}_{\mathbf{L}_0} |n\rangle = \hbar\, n\, |n\rangle\,, \tag{7.129a}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_-} |n\rangle = \hbar \sqrt{n(n-1) + s^2 + 1/4}\, |n-1\rangle\,, \tag{7.129b}$$

$$\hat{\mathcal{J}}_{\mathbf{L}_+} |n\rangle = \hbar \sqrt{n(n+1) + s^2 + 1/4}\, |n+1\rangle\,, \tag{7.129c}$$

proving that their span is the Hilbert space $\mathcal{H}_{1/2+is,\mu}$.

$$\mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{-s^2} \xrightarrow{\text{quantization}} \mathcal{H}_{-s^2} \equiv \mathcal{H}_{1/2+is,\mu}. \tag{7.130}$$

Light-like orbits Unlike the others, the quantization of these orbits is elusive [323]. They seem to lack a Kähler or cotangent bundle structure that can guide the quantization. Moreover, they do not even have a scale that might serve as a coupling constant, hindering perturbative quantization. We refer the reader to [338] for a discussion on the quantization of degenerate orbits such as these.

Trivial orbit The orbit corresponds to the trivial representation, the only possible finite-dimensional unitary representation of a non-compact Lie group.

7.4 Corner symmetry representation theory

Developing the full representation of the corner symmetry group is a central task in our understanding of gravity, given the numerous applications that corner charges have in different areas. In fact, although we cannot say much with certainty, we expect the Hilbert space of quantum gravity to host a representation of the CS algebra, provided the symmetry group survives at the quantum level.

Assuming we succeed in the classification of all (projective, cfr. our remark at the end of Section 7.2.2) representations of the CS group, we are left with the task of determining the weight of each representation in the Hilbert space. Unitary irreducible representations are like building blocks that we can compose to create the total Hilbert space, but — just as the unitary irreducible representations only tell us what type of relativistic fields are allowed to exist, and not which ones and how many of them *do* exist — they do not tell us how to do that. This is the *spectral measure* problem.

Assuming we find the spectral measure as well, we can then imagine to combine the Hilbert space of different subregions, each containing representations of the corresponding corner symmetry group. When the subregions share a corner (as $\mathcal{A}$ and $\mathcal{B}$ in (7.1)), the two Hilbert spaces possess degrees of freedom that transform under the same group. These can be used to entangle states across the corner (accordingly also referred to as *entangling surface*) with the procedure described by Donnelly & Freidel in [38]. See [298, 339] for an implementation in the two-dimensional case. Gluing different subregions together, we may expect to reconstruct a state — a highly entangled one — for the entire region.

Given the complexity of this endeavor, we keep working in the simplified scenario of $2d$ gravity, hoping that the lesson we learn here will provide some guidance in the investigations of the full-dimensional case. In the following sections, we apply the previously review orbit method to the QCS_2 group, with the goal of constructing Hilbert spaces that carry unitary irreducible representations of it. The representation theory of the QCS_2 group has been derived in [318] with algebraic methods. Providing an alternative derivation not only serves to corroborate those results, but it also introduces a complementary, geometric perspective that can provide valuable insights about a particle-like interpretation, analogously to how relativistic particles arise from the coadjoint orbits of the Poincaré group. Moreover, understanding the structure of coadjoint orbits is particularly significant in the context of determining the *Plancherel measure* for the QCS group, which is believed to play a fundamental role when corners are interpreted as *entangling surfaces* of local subregions [38], with important implications for the entanglement entropy [339].

7.4.1 Corner symmetries of two-dimensional gravity

In this section, we provide a description of the classical algebra $\mathfrak{ecs}_2 = \mathfrak{sl}(2,\mathbb{R}) \dotplus \mathbb{R}^2$ and of its quantum extension $\mathfrak{qcs}_2 = \mathfrak{sl}(2,\mathbb{R}) \dotplus \mathfrak{h}_3$ ($\mathfrak{h}_3$ is the Lie algebra of $\mathbb{H}_3$). Starting from the $\mathfrak{sl}(2,\mathbb{R})$ algebra, we build $\mathfrak{ecs}_2$ including the generators $\mathbf{T}_\pm \in \mathbb{R}^2$, with $[\mathbf{T}_-, \mathbf{T}_+] = 0$,[16] and extend the Lie brackets to them in the following way

$$[\mathbf{L}_0, \mathbf{T}_\pm] = \mp\frac{i}{2}\mathbf{T}_\pm, \quad [\mathbf{L}_\pm, \mathbf{T}_\pm] = 0, \quad [\mathbf{L}_\pm, \mathbf{T}_\mp] = \pm i\mathbf{T}_\pm. \tag{7.131}$$

In this larger algebra, the quadratic $\mathfrak{sl}(2,\mathbb{R})$ element $\mathbf{C}_2^{\mathfrak{sl}(2,\mathbb{R})}$ is not a Casimir anymore. Nevertheless, the $\mathfrak{ecs}_2$ algebra still admits a Casimir, which is cubic in the generators [296]

$$\mathbf{C}_3^{\mathfrak{ecs}_2} = -\mathbf{L}_+\mathbf{T}_-^2 - \mathbf{L}_-\mathbf{T}_+^2 + \left(\mathbf{L}_0 + \frac{3}{8}i\right)\mathbf{T}_+\mathbf{T}_- + \left(\mathbf{L}_0 - \frac{3}{8}i\right)\mathbf{T}_-\mathbf{T}_+. \tag{7.132}$$

At this stage, one could simply drop the $3i/8$ terms as $\mathbf{T}_-$ and $\mathbf{T}_+$ commute, but we prefer to keep them around for later convenience. Notice that this definition of $\mathbf{C}_3^{\mathfrak{ecs}_2}$ differs from the one used in the cited references.

[16] We use the same notation for the generators of $\mathbb{R}^2$ and those of $\mathbb{H}_3$ (cfr. eq. (7.98)): the intended meaning will be clear from the context.

Let us now construct the Lie algebra $\mathfrak{qcs}_2$ as a central algebra extension of $\mathfrak{ecs}_2$. To do this, we introduce a central element in the translation sector

$$[\mathbf{T}_-, \mathbf{T}_+] = -i\mathbf{E}, \tag{7.133}$$

turning $\mathbb{R}^2$ into $\mathbb{H}_3$. Because of this deformation, the cubic Casimir is deformed as well

$$\mathbf{C}_3^{\mathfrak{qcs}_2} = \mathbf{C}_3^{\mathfrak{ecs}_2} + 2\mathbf{E}\,\mathbf{C}_2^{\mathfrak{sl}(2,\mathbb{R})}. \tag{7.134}$$

Clearly, in the limit where the deformation is removed ($\mathbf{E} \to 0$), we recover the undeformed $\mathfrak{ecs}_2$ Casimir. What is more surprisingly is that, in the opposite limit ($\mathbf{E} \to \infty$), $\mathbf{C}_3^{\mathfrak{qcs}_2}$ becomes proportional to $\mathbf{C}_2^{\mathfrak{sl}(2,\mathbb{R})}$. This apparent coincidence hints at a quite remarkable property of this algebra, which we are now going to exhibit. If we consider the universal enveloping algebra of $\mathfrak{qcs}_2$, we can define the following elements [17]

$$\tilde{\mathbf{L}}_0 = \mathbf{L}_0 - \frac{1}{4\mathbf{E}}(\mathbf{T}_+\mathbf{T}_- + \mathbf{T}_-\mathbf{T}_+), \tag{7.135a}$$

$$\tilde{\mathbf{L}}_+ = \mathbf{L}_+ - \frac{1}{2\mathbf{E}}\mathbf{T}_+^2, \qquad \tilde{\mathbf{L}}_- = \mathbf{L}_- - \frac{1}{2\mathbf{E}}\mathbf{T}_-^2, \tag{7.135b}$$

These *dressed* generators satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[\tilde{\mathbf{L}}_-, \tilde{\mathbf{L}}_+] = -2i\tilde{\mathbf{L}}_0, \quad [\tilde{\mathbf{L}}_0, \tilde{\mathbf{L}}_\pm] = \mp i\tilde{\mathbf{L}}_\pm, \tag{7.136}$$

and commute with translations

$$[\tilde{\mathbf{L}}_0, \mathbf{T}_\pm] = 0, \quad [\tilde{\mathbf{L}}_\pm, \mathbf{T}_\pm] = 0, \quad [\tilde{\mathbf{L}}_\pm, \mathbf{T}_\mp] = 0. \tag{7.137}$$

This construction implies that we can embed the algebra of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{H}_3$ (notice the direct product) in the universal enveloping algebra of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{H}_3$. In terms of these modified generators, the cubic Casimir of $\mathfrak{qcs}_2$ can be written as

$$\mathbf{C}_3^{\mathfrak{qcs}_2} = 2\mathbf{E}\left(-\tilde{\mathbf{L}}_0^2 + \frac{1}{2}(\tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- + \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+)\right), \tag{7.138}$$

reproducing the $\mathrm{SL}(2,\mathbb{R})$ invariant product $\tilde{\mathbf{L}}^2$. In the limit of large $\mathbf{E}$, the quadratic $\mathbf{T}$ contributions in (7.135) becomes negligible and the new $\mathfrak{sl}(2,\mathbb{R})$ generators approach the old one $\tilde{\mathbf{L}} \to \mathbf{L}$. Accordingly, $\mathbf{C}_3^{\mathfrak{qcs}_2}$ tends to $2\mathbf{E}\mathbf{L}^2$.

Let us now give an interpretation of the dressed generators (7.135). Since both $\mathbf{L}$ and $\tilde{\mathbf{L}}$ generate a $\mathfrak{sl}(2,\mathbb{R})$ algebra, it follows that

$$\frac{1}{4\mathbf{E}}(\mathbf{T}_+\mathbf{T}_- + \mathbf{T}_-\mathbf{T}_+), \qquad \frac{1}{2\mathbf{E}}\mathbf{T}_+^2, \qquad \frac{1}{2\mathbf{E}}\mathbf{T}_-^2, \tag{7.139}$$

[17] Since the universal enveloping algebra does not contain the reciprocal of $\mathbf{E}$, the $\tilde{\mathbf{L}}$ generators do not belong to it strictly speaking. To find them, we have to consider the universal enveloping algebra of $\mathbb{R} \times \mathfrak{qcs}_2$, where the basis element $\mathbf{E}^{-1} \in \mathbb{R}$ commutes with everything, and quotient it by the ideal generated by $\mathbf{E}^{-1}\mathbf{E} - 1$ (or, equivalently, $\mathbf{E}\mathbf{E}^{-1} - 1$).

are $\mathfrak{sl}(2,\mathbb{R})$ generators as well. These elements are well-known in the mathematical literature as the generators of the *Weyl representation*, which often arises in the context of induced representations. Since $\mathrm{SL}(2,\mathbb{R})$ is the automorphism group of $\mathbb{H}_3$, representing $\mathbf{T}_+$ and $\mathbf{T}_-$ in some space automatically induces an $\mathrm{Mp}(2,\mathbb{R})$ representation in the same space, where $\mathrm{Mp}(2,\mathbb{R})$ is the *metaplectic group*, the double cover of $\mathrm{SL}(2,\mathbb{R})$. In this sense, the $\tilde{\mathbf{L}}$ generators parametrize the "distance" of a certain $\mathrm{SL}(2,\mathbb{R})$ representation from the metaplectic one.

7.4.2 Orbit analysis for corner symmetry groups

Let us now analyze the coadjoint orbits of the corner symmetry groups of $2d$ gravity. As mentioned earlier, these are relevant not only for the genuinely two-dimensional case, but also as a first step toward understanding the full-dimensional theory.

We start with the group $\mathrm{ECS}_2 = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, whose coadjoint orbit analysis was initiated in [296]. For compatibility with this reference and with the rest of the literature on the subject, we switch from the "vectorial" basis $\{\mathbf{L}_0, \mathbf{L}_\pm\}$ we introduced in the previous section to a "matricial" basis, defined by

$$\mathbf{J}^a{}_b := i\eta^{ij}(\tau_j)^a{}_b\mathbf{L}_i, \qquad a,b=1,2, \quad i,j=0,+,- \tag{7.140}$$

where the τ matrices can be expressed in terms of the usual Pauli matrices as $\tau_0 = -\sigma_3$ and $\tau_\pm = \mp(\sigma_1 \mp i\sigma_2)$. In this basis, the $\mathfrak{sl}(2,\mathbb{R})$ Casimir is $\frac{1}{2}\mathbf{J}^2$. Consistently, we also introduce

$$\mathbf{P}_a = \frac{1+i}{\sqrt{2}}(\mathbf{T}_-, \mathbf{T}_+), \qquad a,b=1,2. \tag{7.141}$$

In this new basis, the $\mathfrak{h}_S$ commutation relations read

$$[\mathbf{J}^a{}_b, \mathbf{J}^c{}_d] = \delta^c_b\mathbf{J}^a{}_d - \delta^a_d\mathbf{J}^c{}_b, \quad [\mathbf{J}^a{}_b, \mathbf{P}_c] = -\delta^a_c\mathbf{P}_b + \frac{1}{2}\delta^a_b\mathbf{P}_c, \quad [\mathbf{P}_a, \mathbf{P}_b] = 0, \tag{7.142}$$

while the cubic Casimir can be written as

$$\mathbf{C}_3^{\mathfrak{ecs}_2} = \frac{1}{4}\epsilon^{bc}(2\mathbf{J}^a{}_b\mathbf{P}_c\mathbf{P}_a + \mathbf{P}_a\mathbf{J}^a{}_b\mathbf{P}_c + \mathbf{P}_c\mathbf{P}_a\mathbf{J}^a{}_b), \tag{7.143}$$

where

$$\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.144}$$

are $\mathrm{SL}(2,\mathbb{R})$ invariant tensors.

Coadjoint orbits of the extended corner symmetry group

Being 5-dimensional, the dual algebra $\mathfrak{ecs}_2^*$ cannot be symplectic. Moreover, given that coadjoint orbits must be even-dimensional, the maximal dimension of a ECS_2 orbits is 4.

We will see that this dimensionality reduction is due to the existence of the cubic Casimir in the algebra. We write a generic element of ECS_2 as (h, n), with $h \in \mathrm{SL}(2,\mathbb{R})$ and $n \in \mathbb{R}^2$. When these are taken in their defining representation, the action $h \triangleright n$ implicit in the semidirect product is just the natural action of matrices on vectors $(h \triangleright n)^a = h^a{}_b n^b$.

Given the matricial basis (7.142) in $\mathfrak{ecs}_2$, we write a generic vector as $\mathbf{X} = \theta^a{}_b \mathbf{J}^b{}_a + \zeta^a \mathbf{P_a}$, where $\theta^a{}_b$ is a traceless 2×2 matrix and ζ^a a 2-vector. The adjoint action of $(h, n) \in \mathrm{ECS}_2$ on it is given by

$$\mathrm{Ad}_{(h,n)}\mathbf{X} = (h\theta h^{-1})^a{}_b \mathbf{J}^b{}_a + (h^a{}_b \zeta^b - (h\theta h^{-1})^a{}_b n^b)\mathbf{P}_a. \tag{7.145}$$

Dualizing this result, we obtain the coadjoint action on a $\mathfrak{ecs}_2^*$ element $\mathbf{m} = J^a{}_b \theta^b{}_a + P_a \zeta^a$

$$\mathrm{Ad}^*_{(h,n)}\mathbf{m} = -(hJh^{-1})^a{}_b \theta^b{}_a - P_a (h^{-1})^a{}_b (\theta^b{}_c n^c - \zeta^b), \tag{7.146}$$

which, for an infinitesimal transformation generated by $\xi_{\mathbf{X}}$, reduces to the result of [296]

$$\begin{cases} \mathbb{L}_{\mathbf{X}} J^a{}_b = [J, \theta]^a{}_b - P_b \zeta^a + \frac{1}{2} P_c \zeta^c \delta^a_b, \\ \mathbb{L}_{\mathbf{X}} P_a = P_b\, \theta^b{}_a. \end{cases} \tag{7.147}$$

Since we fixed a basis, we can write explicitly the Casimir function associated with $\mathbf{C}_3^{\mathfrak{ecs}_2}$ (notice that here, unlike in (7.143), $J^a{}_b$ and P_c are components in a given basis, hence they commute)

$$\mathcal{C}_3^{\mathfrak{ecs}_2} : \mathfrak{g}^* \to \mathbb{R}, \quad \mathbf{m} \mapsto \mathcal{C}_3^{\mathfrak{ecs}_2}(\mathbf{m}) = \epsilon^{bc} J^a{}_b P_c P_a, \tag{7.148}$$

and check explicitly that it is constant under the coadjoint action. Orbits must therefore lie within the level sets of $\mathcal{C}_3^{H_S}$.

Proof. Using the algebra ajoint action (7.147), we can explicitly compute

$$\begin{aligned} \mathbb{L}_{\mathbf{X}} \mathcal{C}_3^{\mathfrak{ecs}_2}(\mathbf{m}) &= \epsilon^{bc}(\mathbb{L}_{\mathbf{X}} J^a{}_b P_c P_a + J^a{}_b \mathbb{L}_{\mathbf{X}} P_c P_a + J^a{}_b P_c \mathbb{L}_{\mathbf{X}} P_a) = \\ &= \epsilon^{bc}\left(\left([J,\theta]^a{}_b - P_b\zeta^a + \frac{1}{2} P_d \zeta^d \delta^a{}_b\right) P_c P_a + J^a{}_b P_d \theta^d{}_c P_a + J^a{}_b P_c P_d \theta^a_b\right) = \\ &= \epsilon^{bc}\left(J^a{}_d \theta^d{}_b P_c P_a - \cancel{\theta^a{}_d J^d{}_b P_c P_a} + J^a{}_b P_d \theta^d{}_c P_a + \cancel{J^a{}_b P_c P_d \theta^d{}_a}\right) = \\ &= P_a J^a{}_b (\theta^b{}_c \epsilon^{cd} + \epsilon^{bc} \theta^d{}_c) P_d = 0, \end{aligned} \tag{7.149}$$

where the last equality follows because $\theta^d{}_c \epsilon^{cb}$ is symmetric, so $\epsilon^{bc}\theta^d{}_c = \theta^b{}_c \epsilon^{dc}$. □

The KKS form on these orbits reads

$$\Omega^{\mathrm{ECS}}(\xi_{\mathbf{X}}, \xi_{\mathbf{X}'}) = J^a{}_b [\theta, \theta']^b{}_a + P_a(\theta^a{}_b \zeta'^b - \theta'^a{}_b \zeta^b). \tag{7.150}$$

Ideally, we would like to invert the system (7.147) and express the Lie algebra vector components in terms of the variations induced in the dual space element. However, since $\mathcal{C}_3^{H_S}$ is constant along the orbit, the system has rank 4 and it is not invertible, but we can use (7.149) to eliminate one variation and solve for the rest.

Let us consider again the coadjoint action and define $\mathbf{j} \equiv J^a{}_b\theta^b{}_a \in \mathfrak{sl}(2,\mathbb{R})^*$, and $\mathbf{p} \equiv P_a\zeta^a \in \mathbb{R}^{2*}$. We see from (7.146) that the result only depends on $\mathbf{p}$ through the combination $P_a(h^{-1})^a{}_b$. This means that the ECS orbits form a fiber bundle over the $\mathrm{SL}(2,\mathbb{R})$ orbits in $\mathbb{R}^{2*}$

$$\mathcal{W}_{\mathbf{p}}^{\mathrm{SL}(2,\mathbb{R})} := \{P_b h^b{}_a \zeta^a | h \in \mathrm{SL}(2,\mathbb{R})\}. \tag{7.151}$$

Each fiber is a direct product of the cotangent space to $\mathcal{W}_{\mathbf{p}}^{\mathrm{SL}(2,\mathbb{R})}$ with a coadjoint orbit of the stabilizer $\mathrm{SL}(2,\mathbb{R})_{\mathbf{p}}$ — specifically, the one acting on the restriction of $\mathbf{j}$ to the stabilizer itself. This structure of the coadjoint orbits is not peculiar to the ECS, but is exhibited by any semidirect product with a vector normal subgroup [340]. In the case under consideration though, the orbits $\mathcal{W}_{\mathbf{p}}^{\mathrm{SL}(2,\mathbb{R})}$ are very simple because either $\mathbf{p} = 0$ and the orbit is trivial, or $\mathbf{p} \neq 0$ and the orbit is the entire plane $\mathbb{R}^{2*}$.

- When $\mathbf{p} = 0$, the ECS_2 coadjoint action reduces to $\mathrm{Ad}^*_{(h,n)}\mathbf{m} = (h^{-1}Jh)^a{}_b\theta^b{}_a = \mathrm{Ad}^*_h\mathbf{j}$. The stabilizer is obviously the entire $\mathrm{SL}(2,\mathbb{R})$. The coadjoint orbits thus reduce to the $\mathrm{SL}(2,\mathbb{R})$ coadjoint orbits $\mathcal{O}_{\mathbf{j}}^{\mathrm{SL}(2,\mathbb{R})}$, which are either 2 or 0 dimensional, as showed in Section 7.3.2;

- When $\mathbf{p} \neq 0$, we can use a special linear element to set $\mathbf{p} = (1,0)$. The stabilizer group is one dimensional, generated by the element $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})_{(1,0)}$. The restriction of $\mathbf{j}$ to the stabilizer is given by the equivalence class of $\mathfrak{sl}(2,\mathbb{R})^*$ matrices with the same upper right entry $\left[\left(\begin{smallmatrix} 0 & \delta \\ 0 & 0 \end{smallmatrix}\right)\right]$. One can easily check that this class is preserved by the stabilizer coadjoint action. The total dimension of the orbits is the product of the dimensions of the base manifold and of its cotangent space $2 \times 2 = 4$.

Coadjoint orbits of the quantum corner symmetry group

We now turn our attention to the group $\mathrm{QCS}_2 = \widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes \mathbb{H}_3$, whose most notable difference from ECS_2 is that the normal subgroup is not abelian.[18] This generally makes the analysis more complicated, but in this case it actually introduces a surprising simplification, as we will show. We write again a generic group element as (h,n), where $h \in \mathrm{SL}(2,\mathbb{R})$, but now $n \in \mathbb{H}_3$. When these are taken in their defining representation (cfr. eq. (7.49)),

[18] Another difference is the universal covering of $\mathrm{SL}(2,\mathbb{R})$. This distinction has no effect on the orbits analysis, so we will forget about it in this section, but it will become relevant later, when discussing their quantization. The group $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{H}_3$ is also known as the $(n=1)$ *Jacobi group* [341].

the action in the semidirect product can be written as

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} \triangleright \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ua+vb & c+\frac{1}{2}(ua+vb)(wa+zb)-\frac{1}{2}ab \\ 0 & 1 & wa+zb \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.152}$$

For convenience, we report here the fundamental $\mathfrak{qcs}_2$ Lie brackets in the matricial basis

$$[\mathbf{J}^a{}_b, \mathbf{J}^c{}_d] = \delta^c_b \mathbf{J}^a{}_d - \delta^a_d \mathbf{J}^c{}_b, \quad [\mathbf{J}^a{}_b, \mathbf{P}_c] = -\delta^a_c \mathbf{P}_b + \frac{1}{2}\delta^a_b \mathbf{P}_c, \quad [\mathbf{P}_a, \mathbf{P}_b] = \epsilon_{ab}\mathbf{E}. \tag{7.153}$$

We define the dual basis $\{\theta^a{}_b, \zeta^a, \sigma\}$ of $\mathfrak{qcs}_2^*$ by the non-trivial pairings

$$\langle \theta^a{}_b, \mathbf{j}^c{}_d \rangle = \delta^a_d \delta^c_b - \frac{1}{2}\delta^a_b \delta^c_d, \quad \langle \zeta^a, \mathbf{p}_b \rangle = \delta^a_b, \quad \langle \sigma, \mathbf{E} \rangle = 1. \tag{7.154}$$

We then write a generic element of $\mathfrak{qcs}_2$ as $\mathbf{X} = \theta^a{}_b \mathbf{J}^b{}_a + \zeta^a \mathbf{P}_a + \sigma \mathbf{E}$ and compute the adjoint action of the group on it as

$$\begin{aligned} \mathrm{Ad}_{(h,n)}\mathbf{X} =& (h\theta h^{-1})^a{}_b \mathbf{J}^b{}_a + (h^a{}_b \zeta^b - (h\theta h^{-1})^a{}_b n^b)\mathbf{P}_a + \\ &+ \left(\sigma + n^a \epsilon_{ab} h^b{}_c \zeta^c - \frac{1}{2} n^a \epsilon_{ab} (h\theta h^{-1})^b{}_c n^c\right)\mathbf{E}, \end{aligned} \tag{7.155}$$

where $n^a = (n^1, n^2)$ are the components in the $\mathbb{R}^2$ sector of $\mathbb{H}^3$, while the last component n^3 does not contribute to the adjoint action. Similarly, we write an element of $\mathfrak{qcs}_2^*$ as $\mathbf{m} = J^a{}_b \theta^b{}_a + P_a \zeta^a + E\sigma$ and compute the group coadjoint action

$$\begin{aligned} \mathrm{Ad}^*_{(h,n)}\mathbf{m} =& -(hJh^{-1})^a{}_b \theta^b{}_a - P_c (h^{-1})^c{}_b n^a \theta^b{}_a - \frac{1}{2} E n^c \epsilon_{bc} n^a \theta^b{}_a + \\ &+ (P_b (h^{-1})^b{}_a - E n^b \epsilon_{ba})\zeta^a + E\sigma. \end{aligned} \tag{7.156}$$

The transformations of coordinates analogous to (7.147) are thus

$$\begin{cases} \mathbb{L}_{\mathbf{X}} J^a{}_b = [J, \theta]^a{}_b - P_b \zeta^a + \frac{1}{2}\delta^a_b P_c \zeta^c, \\ \mathbb{L}_{\mathbf{X}} P_a = P_b \theta^b{}_a - E\zeta^b \epsilon_{ba}, \\ \mathbb{L}_{\mathbf{X}} E = 0. \end{cases} \tag{7.157}$$

The function E is trivially invariant. On the other hand, the function we construct from the cubic Casimir $\mathbf{C}_3^{\mathfrak{qcs}_2}$, i.e.

$$\mathcal{C}_3^{\mathfrak{qcs}_2}(\mathbf{m}) = \mathcal{C}_3^{\mathfrak{ecs}_2}(\mathbf{m}) + 2E\mathcal{C}_2^{\mathfrak{sl}(2,\mathbb{R})}(\mathbf{m}) = \epsilon^{bc} J^a{}_b P_c P_a + E J^a{}_b J^b{}_a, \tag{7.158}$$

is non-trivially invariant.

Proof. Given that $\mathcal{C}_3^{\mathfrak{ccs}_2}$ is invariant when translations are commuting, we know that its variation must come from the term proportional to E in $\mathbb{L}_{\mathbf{X}} P_a$. Neglecting the remaining terms, we get

$$\mathbb{L}_{\mathbf{X}} \mathcal{C}_3^{\mathfrak{ccs}_2}(\mathbf{m}) = -E\epsilon^{bc}\Big(J^a{}_b\zeta^d\epsilon_{dc}P_a + J^a{}_bP_c\zeta^d\epsilon_{da}\Big) = 2EP_aJ^a{}_b\zeta^b, \tag{7.159}$$

where we used $\epsilon^{bc}\epsilon_{da} = \delta^b_a\delta^c_d - \delta^b_d\delta^c_a$. The contribution from $\mathcal{C}_2^{\mathfrak{sl}(2,\mathbb{R})}$ is already proportional to E, but we now that the θ term contribution will drop as it corresponds to the pure $\mathrm{SL}(2,\mathbb{R})$ action

$$\mathbb{L}_{\mathbf{X}} \mathcal{C}_2^{\mathfrak{sl}(2,\mathbb{R})}(\mathbf{m}) = \mathbb{L}_{\mathbf{X}} J^a{}_bJ^b{}_a + J^a{}_b\mathbb{L}_{\mathbf{X}} J^b{}_a = -P_b\zeta^aJ^b{}_a - J^a{}_b\zeta^bP_a = 2P_aJ^a{}_b\zeta^b, \tag{7.160}$$

Therefore, the sum in (7.158) is manifestly invariant. □

The presence of two Casimir functions implies that the QCS_2 orbits are, at most, 4 dimensional. We now show how to classify them.

One approach would be to take the KKS form on the QCS coadjoint orbits

$$\Omega^{\mathrm{QCS}}(\xi_{\mathbf{X}}, \xi_{\mathbf{X}'}) = J^a{}_b[\theta,\theta']^b{}_a + P_a(\theta^a{}_b\zeta'^b + \theta'^a{}_b\zeta^b) + E\zeta^a\epsilon_{ab}\zeta'^b, \tag{7.161}$$

and express it in terms of coordinate differentials by inverting the transformations in (7.157), as we did in the previous section. Another strategy is to take inspiration from the dressed generators (7.135) and introduce the corresponding coordinate functions, defined for $E \neq 0$.[19] In terms of the matricial basis, these read

$$\tilde{J}^a{}_b = J^a{}_b + \frac{1}{2E}\epsilon^{ac}P_cP_b. \tag{7.162}$$

In any open set with $E \neq 0$, we can thus express both the adjoint action and the Casimir function in terms of these coordinates:

$$\mathbb{L}_{\mathbf{X}} \tilde{J}^a{}_b = [\tilde{J},\theta]^a{}_b, \tag{7.163}$$

$$\mathcal{C}_3^{\mathfrak{qcs}_2}(\mathbf{m}) = E\tilde{J}^a{}_b(\mathbf{m})\tilde{J}^b{}_a(\mathbf{m}) \equiv E\tilde{J}^2(\mathbf{m}). \tag{7.164}$$

We can summarize these results as saying that QCS_2 acts on the $\tilde{J}^a{}_b$ functions only with its $\mathrm{SL}(2,\mathbb{R})$ part, preserving the associated Casimir function $\tilde{J}^2$. Therefore, whenever $E \neq 0$, the QCS_2 coadjoint orbits on $\tilde{J}^a{}_b$ are diffeomorphic to the $\mathrm{SL}(2,\mathbb{R})$ coadjoint orbits.

[19] While we are presenting the construction in this order, the actual development followed the reverse order. First, we noticed that these coordinates allowed a simpler description, then we connected them to dressed generators in the algebra.

We can back up these observations by expressing the KKS form in the $\{\tilde{J}^a{}_b, P_a, E\}$ coordinates. What we see is that the symplectic form splits into two KKS forms, one for $\mathrm{SL}(2,\mathbb{R})$ and one for $\mathbb{H}_3$

$$\Omega^{\mathrm{QCS}}_{\mathbf{m}}(\xi_{\mathbf{X}}, \xi_{\mathbf{X}'}) = \underbrace{\tilde{J}^a{}_b[\theta, \theta']^b{}_a}_{\mathrm{SL}(2,\mathbb{R})} - \underbrace{\frac{1}{E}\mathbb{L}_{\mathbf{X}} P_a \epsilon^{ab} \mathbb{L}_{\mathbf{X}'} P_b}_{\mathbb{H}_3}. \tag{7.165}$$

This implies that the each QCS orbit factorize as well into a $\mathrm{SL}(2,\mathbb{R})$ coadjoint orbit and a $\mathfrak{h}_3$ coadjoint orbit. Therefore, we can label a generic QCS orbits with $E \neq 0$ as

$$\mathcal{O}^{\mathrm{QCS}}_{\Delta,r,E} = \mathcal{O}^{\mathrm{SL}(2,\mathbb{R})}_{\Delta,r} \times \mathcal{O}^{\mathbb{H}_3}_{E}, \tag{7.166}$$

where Δ is given in terms of the dressed coordinates $-\frac{1}{2}\tilde{J}^2$. Because of this decoupling, the QCS_2 orbits for $E \neq 0$ are never 0-dimensional, compatibly with the fact that setting $P_1 = P_2 = 0$ would be inconsistent with their commutation relation, but they are 2- or 4-dimensional depending on whether $\tilde{J}^a{}_b$ vanishes or not.

Remark The existence of functions $\tilde{J}^a{}_b$ that factorize the symplectic form can be seen as an application of Darboux's theorem. What is non-trivial is that these functions are well-defined in a very large region, which is exactly where the QCS group differs from the "classical" regime, described by the ECS.

7.4.3 Quantization of the quantum corner symmetry group orbits

The result (7.166) we obtained at the end of the last section — that coadjoint orbits of QCS_2 factorize into independent orbits of the special linear group and the Heisenberg group when expressed in suitable coordinates — greatly simplifies the quantization procedure for the QCS orbits, reducing it to the following three steps:

1. We pick a quantum orbit of the Heisenberg group, associated with the coordinates P_a, corresponding to a single particle Fock space $\mathfrak{F}_e$. In the following, we keep assuming $c > 0$ — the case $c < 0$ is not very different;

2. We pick a quantum orbit of $\mathrm{SL}(2,\mathbb{R})$, associated with the dressed coordinates $\tilde{J}^a{}_b$, corresponding to a Hilbert space $\mathcal{H}_{\Delta,r}$ carrying a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ representation;

3. We construct the Hilbert space associated with the QCS orbit as the tensor product of the two Hilbert spaces obtained in the preceding steps

$$\mathcal{H}_{\Delta,r,c} = \mathcal{H}_{\Delta,r} \otimes \mathfrak{F}_e. \tag{7.167}$$

Indeed, geometric quantization can be performed independently on the two orbit factors and, after the prequantum vector space is constructed, the correct scalar product is just the one induced by the tensor product.

The operators that represent the generators of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{h}_3$ on the two factors of the product (7.167) — which we defined in the previous sections (see eq.s (7.91), (7.123) and (7.129)) — admit a natural extension to the total Hilbert space

$$\hat{Q}_{\tilde{\mathbf{L}}_i} = \hat{\mathcal{J}}_{\mathbf{L}_i} \otimes \mathbb{1}, \quad \hat{Q}_{\mathbf{T}_\pm} = \mathbb{1} \otimes \hat{\mathcal{J}}_{\mathbf{T}_\pm}, \quad \hat{Q}_{\mathbf{E}} = \mathbb{1} \otimes \hat{\mathcal{J}}_{\mathbf{E}}, \tag{7.168}$$

where it is important to notice that we associate the first set of operators to the $\tilde{\mathbf{L}}_i$ elements in the universal enveloping algebra (7.135) rather than the standard generators of $\mathfrak{qcs}_2$. In fact, the possibility to write the operators in this tensor product form follows from the commutativity of the dressed generators $\tilde{\mathbf{L}}_i$ with translations $\mathbf{T}_\pm$ (see eq. (7.137)). The operators associated with the original $\mathrm{SL}(2,\mathbb{R})$ coordinates (those that do not decouple from the Heisenberg sector) can be obtained by inverting eq.(7.135)

$$\hat{Q}_{\mathbf{L}_0} = \hat{Q}_{\tilde{\mathbf{L}}_0} + \frac{1}{4c}\Big(\hat{Q}_{\mathbf{T}_-}\hat{Q}_{\mathbf{T}_+} + \hat{Q}_{\mathbf{T}_+}\hat{Q}_{\mathbf{T}_-}\Big) = \hat{\mathcal{J}}_{\mathbf{L}_0} \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\hbar}{2}\left(a^\dagger a + \frac{1}{2}\right), \tag{7.169a}$$

$$\hat{Q}_{\mathbf{L}_-} = \hat{Q}_{\tilde{\mathbf{L}}_-} + \frac{1}{2c}\hat{Q}_{\mathbf{T}_-}\hat{Q}_{\mathbf{T}_-} = \hat{\mathcal{J}}_{\mathbf{L}_-} \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\hbar}{2} aa, \tag{7.169b}$$

$$\hat{Q}_{\mathbf{L}_+} = \hat{Q}_{\tilde{\mathbf{L}}_+} + \frac{1}{2c}\hat{Q}_{\mathbf{T}_+}\hat{Q}_{\mathbf{T}_+} = \hat{\mathcal{J}}_{\mathbf{L}_+} \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\hbar}{2} a^\dagger a^\dagger, \tag{7.169c}$$

where we have used the identification between $\hat{\mathcal{J}}_{\mathbf{T}_\pm}$ with creation and annihilation operators. The action of the QCS operators $\hat{Q}$ is conveniently expressed in the basis $|n,k\rangle \equiv |n\rangle \otimes |k\rangle$, where $|n\rangle$ is a basis for $\mathcal{H}_{\Delta,r}$ (introduced in (7.122) for the massive case, and in (7.128) for the tachyonic case), and $|k\rangle$ is the basis of $\mathfrak{F}_e$ (introduced in (7.102)). We find [20]

$$\hat{Q}_{\mathbf{T}_-}|n,k\rangle = \sqrt{c\hbar}\sqrt{k}\,|n,k-1\rangle\,, \tag{7.170a}$$

$$\hat{Q}_{\mathbf{T}_+}|n,k\rangle = \sqrt{c\hbar}\sqrt{k+1}\,|n,k+1\rangle\,, \tag{7.170b}$$

$$\hat{Q}_{\mathbf{L}_0}|n,k\rangle = \hbar\left(n + \frac{2k+1}{4}\right)|n,k\rangle\,, \tag{7.170c}$$

$$\hat{Q}_{\mathbf{L}_-}|n,k\rangle = -\hbar\sqrt{n(n-1)+q(1-q)}\,|n-1,k\rangle + \frac{\hbar}{2}\sqrt{k(k-1)}\,|n,k-2\rangle\,, \tag{7.170d}$$

$$\hat{Q}_{\mathbf{L}_+}|n,k\rangle = -\hbar\sqrt{n(n+1)+q(1-q)}\,|n+1,k\rangle + \frac{\hbar}{2}\sqrt{(k+1)(k+2)}\,|n,k+2\rangle\,, \tag{7.170e}$$

which correspond, up to an automorphism in the $\hat{Q}_{\tilde{\mathbf{L}}_i}$ algebra, to the representations derived in [318]. In that work, representations have been derived using Mackey's theory of induced

[20] To get the analogous action on the basis corresponding to tachyonic orbits, just replace $q \to 1/2 + is$.

representations [317, 342]. In this approach, the fact that the **little group** — the subgroup that preserves the representations of the normal subgroup, cfr. Appendix D — of $\mathbb{H}_3$ is the full $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, implies that the induced representations of the full group can be realized on a tensor product space, as in (7.167). In light of this, the orbit factorization (7.166) can be seen as the geometric counterpart of this algebraic property.

We can check that these operators indeed satisfy the $\mathfrak{qcs}_2$ commutation relations. Moreover, if we define the Casimir operator in the natural way (combining (7.64), (7.132), and (7.134))

$$\begin{aligned}\hat{Q}_{\mathbf{C}_3^{\mathfrak{qcs}_2}} := &-\hat{Q}_{\mathbf{L}_+}\hat{Q}^2_{\mathbf{T}_-} - \hat{Q}_{\mathbf{L}_-}\hat{Q}^2_{\mathbf{T}_+} + \left(\hat{Q}_{\mathbf{L}_0} + \frac{3}{8}i\right)\hat{Q}_{\mathbf{T}_+}\hat{Q}_{\mathbf{T}_-} + \\ &+ \left(\hat{Q}_{\mathbf{L}_0} - \frac{3}{8}i\right)\hat{Q}_{\mathbf{T}_-}\hat{Q}_{\mathbf{T}_+} + 2c\left(-\hat{Q}^2_{\mathbf{L}_0} + \frac{1}{2}\hat{Q}_{\mathbf{L}_+}\hat{Q}_{\mathbf{L}_-} + \frac{1}{2}\hat{Q}_{\mathbf{L}_-}\hat{Q}_{\mathbf{L}_+}\right).\end{aligned} \tag{7.171}$$

we can confirm that it is represented by a multiple of the identity

$$\hat{Q}_{\mathbf{C}_3^{\mathfrak{qcs}_2}}\,|n,k\rangle = 2c\hbar^2 q(1-q)\,|n,k\rangle\,, \tag{7.172}$$

and that it gives the same result as

$$2c\left(-\hat{Q}^2_{\tilde{\mathbf{L}}_0} + \frac{1}{2}\left(\hat{Q}_{\tilde{\mathbf{L}}_+}\hat{Q}_{\tilde{\mathbf{L}}_-} + \hat{Q}_{\tilde{\mathbf{L}}_-}\hat{Q}_{\tilde{\mathbf{L}}_+}\right)\right)|n,k\rangle\,, \tag{7.173}$$

thus proving the relation (7.138) at the operatorial level.

Remark The orbit method does not seem capable of reproducing the complementary series of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (cfr. Appendix D). This can be linked to the fact that these representations do not contribute to the Plancherel measure of $L^2(\mathrm{SL}(2,\mathbb{R}))$ [321, 343–345]. Analogously, the corresponding QCS_2 representations are missing from the coadjoint orbit side and the result (7.166) suggests that the reason is the same, despite the Plancherel decomposition of the group $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{H}_3$ has not been completely achieved yet [341].

7.5 Summary

We introduced in this chapter a promising application of the study of corner charges and symmetries, that is the tentative construction of a Hilbert space for quantum gravity which exhibit, at the quantum level, the same symmetries that underpin our classical description of gravity. We motivated the inclusion of edge modes in the classical phase space of gravity as a possible solution to the Hilbert space factorizability problem shared by gauge theories and gravity.

Given that edge modes are expected to carry a representation of the corner symmetry group, we were motivated to develop its representation theory. A task that, so far, we

have been able to complete only in the two-dimensional case. Despite the simplicity of the classical theory (2d gravity is topological), the quantum theory — at least to the extent presented here — appears to have interesting features, such as the appearance of a non-zero commutator between translations in the normal direction to a corner.

We presented in this chapter the application of the orbit method to obtain the unitary irreducible representations of several relevant groups, culminating in the analysis of the orbits of the quantum corner symmetry group $\mathrm{QCS}_2 = \widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes \mathbb{H}_3$. Our results, drawn from [273], reproduces and confirms the results obtained with Mackey's theory of induced representations [318] (cfr. Appendix D). In particular, we learned that the tensor product factorization of its representations is due to a factorization of the orbit themselves, which, in a particular set of *dressed* coordinates, can be written as products of a $\mathrm{SL}(2,\mathbb{R})$ orbit and a $\mathbb{H}_3$ orbit.

Part IV

Closing remarks

E come quei che con lena affannata,
uscito fuor dal pelago alla riva,
si volge a l'acqua perigliosa e guata,
così l'animo mio, ch'ancor fuggiva,
si volse a retro a rimirar lo passo
che non lasciò già mai persona viva.
— Dante, *Inferno*

Arrived at the end of this thesis, it is worth looking back and considering where we started and what we learned. Our starting point was a consideration about the symmetries of gravity, diffeomorphisms, and how they seem to clash with the usual quantization procedure that is more familiar to us from undergraduate studies: the quantization based on the canonical phase space. Reiterating the importance of having a description of gravity consistent with the quantum principles of the universe we live in may seem superfluous at this stage. What we learned in this journey is that it is possible to provide a phase space description that respects diffeomorphism-invariance. Armed with such a formalism, we have analyzed from many angles the implications of those symmetries, and we discovered that boundaries play a crucial role in this analysis.

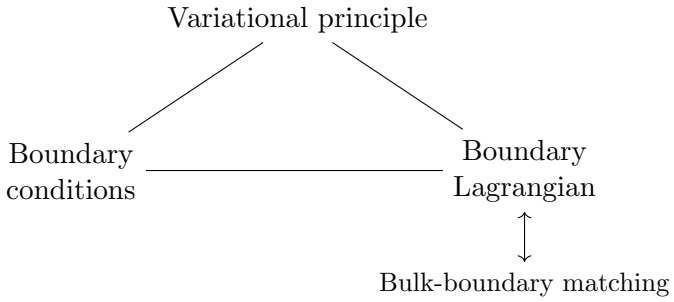
In Chapter 2, we reviewed the basic concepts of a phase space description in the familiar context of classical mechanics. In particular, we touched upon the notion of constrained

systems and symplectic reduction, which form the foundation of our modern understanding of gauge theories, and gravity in particular. In Chapter 3, we introduced the proper formalism that is needed to apply these concepts to (covariant) field theories, hence the name covariant phase space formalism. We illustrated its advantages, which stem from a rich geometrical architecture, the variational bicomplex, and its limitations, manifesting in the form of ambiguities. Fixing these ambiguities is an active area of investigation. Among the other results, we emphasized the importance of Noether's theorems: only harnessing their full power we could formulate a better definition of symmetry, which allows us to uncover new degrees of freedom and new physics. In a rather informal way, this definition says that

A symmetry transformation is a true symmetry, and not a pure gauge redundancy, if it has a non-zero charge.

Here, the charge we are referring to is the Noether charge, interpreted as the Hamiltonian function which implements the transformation in the phase space. Therefore, the unambiguous computation of these charges becomes a crucial task.

The first time we encountered how spacetime boundaries can play a role in fixing these ambiguities is in Chapter 4, even if we used a Lagrangian approach there. The idea is that we require the dynamics to come from the action according to a well-posed variational principle: when spacetime has boundaries, this requirement typically makes it necessary to impose boundary conditions on the system. Doing so, one effectively *closes* the system. What we learned through the analysis of some case studies is that there is a precise correspondence, determined by the request of a well-posed variational principle, between the boundary conditions imposed on the fields and the choice of a boundary Lagrangian.



In the case of gravity, this correspondence is famously exemplified by the Gibbons-Hawking-York boundary term, which exactly cancels the variations in the metric derivative that are normal to the boundary, allowing the variational principle to produce the Einstein equations with Dirichlet boundary conditions. In order for this to happen, a precise *bulk-boundary matching* must exist between the couplings appearing the bulk and boundary Lagrangian. We then showed how the construction can be extended to gravitational

Lagrangians with higher-derivative operators, determining a new set of bulk-boundary matching conditions.

It has been questioned how this picture is modified at the quantum level. A reasonable expectation is that the same considerations can be straightforwardly applied, but to the quantum effective action rather than the classical action. However, as first brought up in [129], the quantum effective action *run*, and so does its boundary term (which is supposed to sit at the right corner of this "quantum triangle"). It is therefore non-trivial to determine whether a well-posed variational principle can be achieved at more than one scale. What would be the physical implications of such a mismatch is not clear at this stage. However, if we consider that, even assuming conformally coupled matter fields, a consistent matching cannot be achieved for all the quadratic operators, except for a particular class of boundary embeddings — and, even in that case, one has to give up on a consistent matching for Newton's constant — we think it is worth investigating whether such a mismatch has observable consequences or it is another shortcoming of Dirichlet boundary conditions.

Having recognized the relevance of spacetime boundaries, in Chapter 5 we revised the covariant phase space formalism to incorporate them. This extension, pioneered by Harlow & Wu [1], gave us the instruments to confirm our intuition that boundary conditions and boundary Lagrangian are strongly connected. The boundary Lagrangian, together with the associated boundary contributions to the symplectic form and to the Noether current, provided a way to fix the covariant phase space ambiguities. More so, it allows to define charges associated will all diffeomorphisms that are tangent to the boundary, including those that act non-trivially there — this improves the Iyer–Wald result, in which integrability is guaranteed only for diffeomorphisms that are tangent to the *corner*. The conservation of these charges, on the other hand, is guaranteed by the boundary conditions.

In Chapter 6, we applied this formalism to derive the first law of black hole thermodynamics. Inspired by the treatment of [1], we extended the derivation to a larger class of generally covariant theories known as Lanczos-Lovelock theories. Their relevance for us is that, as they yield second-order field equations, we can reasonably expect to have a correspondence between the canonical and the covariant formulation of their phase space. On our way to the final result, we found a novel, background-independent prescription for determining the thermodynamic potentials associated with Lanczos-Lovelock couplings (for these theories, one has to resort to an *extended* thermodynamic framework). The fact that the first law could be derived from the covariant phase space formalism was perhaps unsurprising. As a matter of fact, the link between general covariance and thermodynamics is so strong that one can derive the Einstein equations (the only generally covariant theory for the sole metric tensor in four dimensions) from thermodynamical considerations [346]. What was less obvious is that the formalism is so powerful that one can also use it to derive the Smarr formula, interpreting the first law as a 1-form field and integrating it along a curve in solution space. Despite its wide applicability, our formula can be extended to more interesting scenarios, such as non-stationary black holes and non-stationary perturbations, and to the wilder class of objects that can be found in higher dimensions. Looking even

further, we imagine the covariant phase space could play a key role in the formulation of the second law of thermodynamics as well, resolving open issues about its validity in Lanczos-Lovelock theories [347, 348].

Finally, in Chapter 7, we started making some considerations about what the whole analysis of gravitational phase space, gravitational symmetries and charges might be worth for. The construction of the *entangling product* as a way to restore factorization in gauge theories and gravities has been a source of inspiration in this context. In few words, studying classical symmetries of gravity gives us a handle on what its fundamental degrees of freedom might be (assuming the symmetry survives at the quantum level), just like Wigner's classification of Poincaré irreducible representations allowed us to start building relativistic quantum field theory models. Once we know what are the fundamental degrees of freedom, we can start combining them, according to the structure of the (quantum version of the) symmetry generators, in order to generate more complex states. We then learned the relevant fact that gravity at spacetime corners admits a maximal closed subalgebra within the full algebra of diffeomorphisms. More so, the associated group was shown to be universal across all diffeomorphism-invariant theories, independent of their field content. The existence of a universal corner symmetry group is a solid starting point for the program we just summarized, as it gives a good ansatz for what might be the gravitational analogue of the Poincaré group. Since deriving the representation theory of the full group is a hard task, in this chapter we started with a simple case: developing the representation theory of the corner symmetry group of two-dimensional (quantum) gravity.

The projective nature of representations that are relevant for a quantum theory then led us to consider the group $\mathrm{QCS}_2 := \widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes \mathbb{H}_3$, where $\mathbb{H}_3$ is the Heisenberg group. An interesting feature of this group is that it exhibits a central element in the commutation relation of spacetime translations (in the two normal directions to the corner). Albeit it is unclear whether this property would survive in a realistic, four-dimensional model, it nonetheless suggests that surprising new physics may emerge in the quantum regime. In our analysis, a pivotal role was played by a new set of $\mathrm{SL}(2,\mathbb{R})$ generators — which physically correspond to boosts in the normal plane to the corner — dressed by $\mathbb{R}^2$ generators — representing normal translations. Clarifying the physical meaning of these operators is the natural next step. The successful application of the orbit method to derive the unitary irreducible representations of the two-dimensional quantum corner symmetry group paves the way for its generalization to higher dimensions, where algebraic techniques might become significantly more intricate to apply due to the presence of corner diffeomorphisms.

Embracing a most ambitious dream, once we identify the correct quantum extension of the full corner symmetry group, we develop its complete representation theory, and we determine the corresponding spectral measure, we will finally be able to construct genuine quantum gravity states. By combining these states with the aforementioned entangling product, we will be able to build more composite states, where some of them will admit a (semi-)classical interpretation as actual spacetime regions. This would provide a concrete

realization of the "gravity as entanglement" paradigm outlined in the introduction. Looking forward, numerous questions entice the curious mind: What is the interpretation of these quantum constituents of spacetime? How do these quanta interact and scatter? Are they responsible for black hole entropy? [21] Can they induce changes in topology? Do they actually entail spacetime *fuzzyness*? What are the first observable effects we can look after?

[21] A first step in this direction has already been made in [339]

Part V

Supplemental material

Appendix A

Geometry of the bulk and the boundary

In this appendix, we illustrate the geometry of our spacetime and the conventions we use. We also describe how the presence of an embedded boundary allows us to define a whole set of new objects.

For us, spacetime is an n-dimensional globally hyperbolic (pseudo-)Riemannian manifold $(\mathcal{M}, g)$. From the metric g we can construct a volume form $\epsilon_{\mathcal{M}} := \sqrt{-g}\mathrm{d}^n x$ and a compatible covariant derivative ∇_μ. The curvature is then given by the Riemann tensor

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \tag{A.1}$$

and its contractions are the Ricci tensor $R_{\mu\nu} := R^\lambda{}_{\mu\lambda\nu}$ and the Ricci scalar $R := R^\lambda_\lambda$. With this convention, we have $[\nabla_\rho, \nabla_\sigma]V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu$.

We consider the boundary of $\mathcal{M}$ to be non-empty. If we restrict to a certain spacetime region inside the *time* interval $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$, we can write its boundary as

$$\Sigma_{\mathrm{fin}} \cup \mathcal{I} \cup \Sigma_{\mathrm{in}}, \tag{A.2}$$

where Σ_{in} and Σ_{fin} are spacelike Cauchy surfaces and $\mathcal{I}$ is a timelike hypersurface. Thanks to the embedding map $i_{\mathcal{I}} : \mathcal{I} \hookrightarrow \mathcal{M}$, we can induce a metric $h := i^*_{\mathcal{I}} g$ on $\mathcal{I}$, which we assume to coincide with the *intrinsic* metric (isometric embedding). We denote intrinsic quantities with a calligraphic notation, so that the covariant derivative compatible with the metric h is $\mathcal{D}_a$, and the intrinsic curvature is $\mathcal{R}^a{}_{bcd}$. We also let $\epsilon_{\mathcal{I}}$ be the intrinsic volume form on $\mathcal{I}$.

The embedding of the boundary $\mathcal{I}$ allows us to introduce even more structure. For example, the kernel of the pull-back $i^*_{\mathcal{I}}$ allows us to single out a form $\Omega^1(\mathcal{M})$ that is normal to $\mathcal{I}$, which we denote by n_μ and orient in such a way that the corresponding vector n^μ points outward. Since we take $\mathcal{I}$ to be timelike, we can normalize n to 1. Given this unit

normal, we can define the following *projector*

$$\Pi^\mu_\nu := \delta^\mu_\nu - n^\mu n_\nu \tag{A.3}$$

which indeed satisfies $\Pi^\mu_\nu \Pi^\nu_\lambda = \Pi^\mu_\lambda$. This projector allows us to separate tangent vectors at each point on the boundary into a parallel and a normal component

$$V^\mu = V_\perp n^\mu + V^\mu_\parallel, \qquad V_\perp := (n \cdot V), \quad V^\mu_\parallel := \Pi^\mu_\nu V^\nu. \tag{A.4}$$

The same can be done with forms. Parallel vectors can be unambiguously pulled-back to the boundary (i.e. there is a unique $\bar{V} \in \mathfrak{X}(\mathcal{I})$ whose push-forward is $V_\parallel \in \mathfrak{X}(\mathcal{M})$). A well-known property of the embedding is that, even when restricting to parallel vectors, the covariant derivative along the boundary does not remain tangent to it, but acquires a normal component according to

$$e^\nu_b \nabla_\nu V^\alpha = \mathcal{D}_b \bar{V}^a e^\alpha_a - V^a K_{ab} n^\alpha, \tag{A.5}$$

where $K := i^*_{\mathcal{I}}(\nabla n)$ is the *extrinsic curvature tensor*, which is thus a measure of the difference between the tangential projection of the bulk covariant derivative and the boundary covariant derivative.

Since the split (A.4) applies to both vectors and forms, it naturally extends to arbitrary tensors. Projecting the indices of a rank k tensor either tangentially or perpendicularly, we can construct 2^k new tensors which, being parallel, can be unambiguously pulled-back to $\mathcal{I}$. These tensors are independent *a priori* but symmetries of the original tensor can reduce their number. As an example, we consider the Riemann tensor. Because of its symmetries, it is easy to see that this construction only produces three tensors

$$R_{\mu\nu\rho\sigma} = R^{\parallel\parallel\parallel\parallel}_{\mu\nu\rho\sigma} + R^{\parallel\parallel\parallel\perp}_{\mu\nu\rho} n_\sigma + R^{\parallel\perp\parallel\perp}_{\mu\rho} n_\nu n_\sigma + \text{symm}, \tag{A.6}$$

because further contractions with n would vanish due to antisymmetry. In his equation, we wrote "symm" to collect those terms that can be obtained from the others using the Riemann tensor symmetries. The projections we introduced are

$$R^{\parallel\parallel\parallel\parallel}_{\mu\nu\rho\sigma} := \Pi^\alpha_\mu \Pi^\beta_\nu \Pi^\gamma_\rho \Pi^\delta_\sigma R_{\alpha\beta\gamma\delta}, \tag{A.7}$$

$$R^{\parallel\parallel\parallel\perp}_{\mu\nu\rho} := \Pi^\alpha_\mu \Pi^\beta_\nu \Pi^\gamma_\rho n^\delta R_{\alpha\beta\gamma\delta}, \tag{A.8}$$

$$R^{\parallel\perp\parallel\perp}_{\mu\rho} := \Pi^\alpha_\mu n^\beta \Pi^\gamma_\rho n^\delta R_{\alpha\beta\gamma\delta}, \tag{A.9}$$

and we can check that they are completely tangential. Pulling-back to the boundary, we define the following tensors

$$\rho := i^*_{\mathcal{I}} R^{\parallel\parallel\parallel\parallel}, \qquad \tau := i^*_{\mathcal{I}} R^{\parallel\parallel\parallel\perp}, \qquad \sigma := i^*_{\mathcal{I}} R^{\parallel\perp\parallel\perp}. \tag{A.10}$$

If we let e^α_a be the components of the pull-back map and e^a_α an "inverse" of it (in the sense that $e^\alpha_b e^b_\beta = \Pi^\alpha_\beta$ and $n^\alpha e^a_\alpha = 0$), we can write eq. (A.6) as

$$R_{\alpha\beta\gamma\delta} = \rho_{abcd} e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta + \tau_{abc} e^a_\alpha e^b_\beta e^c_\gamma n_\delta + \sigma_{ac} e^a_\alpha n_\beta e^c_\gamma n_\delta + \text{symm} \tag{A.11}$$

Likewise, we have

$$R_{\alpha\beta} = \rho_{ab} e^a_\alpha e^b_\beta - 2\tau_a e^a_{(\alpha} n_{\beta)} + \sigma_{ab} e^a_\alpha e^b_\beta + \sigma n_\alpha n_\beta, \tag{A.12}$$

and

$$R = \rho + 2\sigma, \tag{A.13}$$

where we defined $\rho_{ab} \equiv h^{cd}\rho_{acbd}$, $\rho = h^{ab}\rho_{ab}$, $\tau_a \equiv h^{bc}\tau_{abc}$ and $\sigma \equiv \sigma_{ab}h^{ab}$.

The geometrical meaning of the Riemann tensor projections is related to the boundary intrinsic and extrinsic curvature by the famous **Gauss–Codazzi equations**

$$\rho_{abcd} = \mathcal{R}_{abcd} - K_{ac}K_{bd} + K_{ad}K_{bc}, \tag{A.14}$$

$$\tau_{abc} = \mathcal{D}_b K_{ac} - \mathcal{D}_a K_{bc}. \tag{A.15}$$

On the other hand, it is known that the σ_{ab} tensor cannot be written solely in terms of the surface intrinsic and extrinsic geometry [349].

Appendix B

Heat kernel regularization

We briefly review here the regularization scheme for the quantum effective action based on the heat kernel method. For definiteness, we apply this to the theory that is considered in the bulk, namely a massless scalar field ϕ generically-coupled to the metric tensor g.

$$S_0[g] + S_{\rm mat}[\phi; g] = S_0[g] - \frac{1}{2}\int_{\mathcal{M}} (\nabla_\mu \phi \nabla^\mu \phi + \xi R \phi^2)\epsilon_{\mathcal{M}}, \tag{B.1}$$

where S_0 is a gravitational action.

We start from the partition function of the Euclidean theory, which makes the path integral better behaved compared to its Lorentzian counterpart

$$Z = \int \mathfrak{D}[g]\mathfrak{D}[\phi] e^{-(S_0[g]+S_{\rm mat}[\phi;g])}. \tag{B.2}$$

The integration is over field configurations that are periodic in Euclidean time and subject to suitable boundary conditions at the lateral spacetime boundary $\mathcal{I} \subset \partial\mathcal{M}$. Integrating out matter produces an effective contribution to the gravitational field, that we call *matter effective action* $W_{\rm mat}$

$$Z = \int \mathfrak{D}[g]\, e^{-S_0[g]-W_{\rm mat}[g]}. \tag{B.3}$$

If the action $S_{\rm mat}$ is quadratic in the field ϕ — as is the case in the theory at hand or, for a generic theory, when restricting to small perturbations — we can, at least formally, express $W_{\rm mat}$ as

$$W_{\rm mat}[g] = \frac{1}{2}\ln\det\Delta, \tag{B.4}$$

where $\Delta = \frac{\delta^2 S_{\rm mat}}{\delta\phi^2}$ is a differential operator. For $S_{\rm mat}$ in (B.1), we find $\Delta = -\nabla^2 + \xi R$. In order to compute the logarithm of this operator, we need to specify the boundary conditions satisfied by the scalar field at $\mathcal{I}$. For our purposes, we assume these to be one

of the following:[1]

$$\phi\Big|_{\mathcal{I}} = 0, \quad \text{Dirichlet boundary condition} \tag{B.5a}$$

$$(\mathcal{L}_n + \zeta K)\phi\Big|_{\mathcal{I}} = 0, \quad \text{Robin boundary condition} \tag{B.5b}$$

The explicit computation of W_{mat} requires that we first regularize it. A common procedure is to use the *heat kernel regularization*. If we assume that all the eigenvalues of Δ are positive, we can write

$$\ln\det\Delta = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{d}t}{t^{1-s}}\sum_\lambda e^{-t\lambda}\right]_{s=0} \tag{B.6}$$

where the sum is over all the eigenvalues of Δ. To prove this relation, one uses the Laplace integral representation of λ^{-s}. When $s \to 0$, the integral diverges for small values of t (this is an ultraviolet divergence; for conciseness, we are neglecting possible divergences coming from the integral extending to infinity). To make the integral finite, we introduce a cutoff ϵ^2 in the lower limit of integration. Switching the derivative and the integral, we get the much simpler expression

$$\ln\det\Delta = -\int_{\epsilon^2}^\infty \frac{\mathrm{d}t}{t}\sum_\lambda e^{-t\lambda} \equiv -\int_{\epsilon^2}^\infty \frac{\mathrm{d}t}{t}\operatorname{Tr} K_\Delta(t). \tag{B.7}$$

In the previous expression, we recognized the sum to be the trace of the heat kernel of the differential operator Δ. This kernel is defined as the solution of the Cauchy problem

$$\begin{cases} (\partial_t + \Delta_x)K_\Delta(x,y;t) = 0, \\ K_\Delta(x,y;0) = \delta(x,y), \end{cases} \tag{B.8}$$

supplemented by proper boundary conditions on $\mathcal{I}$, while $\operatorname{Tr} K_\Delta(t) = \int_{\mathcal{M}} K_\Delta(x,x;t)$ (plus a trace over internal spaces, when appropriate).

It has been shown that, for small values of t

$$\operatorname{Tr} K_\Delta(t) \simeq \frac{1}{(4\pi t)^{n/2}}\sum_{k=0}^\infty \mathfrak{a}_k t^{k/2}, \tag{B.9}$$

up to exponentially suppressed terms [154]. Integrating this expression produces a similar expansion for the divergent part of the matter effective action

$$W_{\text{mat}}[g] \simeq -\frac{1}{(4\pi)^{n/2}}\sum_{k=0}^\infty \frac{\mathfrak{a}_k}{(n-k)\epsilon^{n-k}}. \tag{B.10}$$

[1] The *generic* Dirichlet condition $\delta\phi\big|_{\mathcal{I}} = 0$ only implies that the value of ϕ is fixed at the boundary. Requiring that $\phi = 0$ is an admissible solution, this fixed value must be 0.

The latter clearly shows that there are $n+1$ divergent terms, corresponding to the heat kernel coefficients with $k \leq n$. For $k = n$, the divergence is logarithmic.

After regularization, one claims that the bare couplings in $S_0[g]$ that correspond to the operators appearing in $\{\mathfrak{a}_k\}_{k\leq n}$ are also divergent as $\epsilon \to 0$, in such a way that the sum $S_0 + W_{\text{mat}}$ is finite.

Heat kernel coefficients The coefficients in (B.9) can be written in terms of integrals of local geometrical invariants: for even k, these integrals are over the entire spacetime, whereas, for odd k, there is only a boundary contribution. The explicit form depends on the boundary conditions we impose. Adapting from [154], we report here the first few coefficients for a generically coupled scalar field in $n = 4$ with Dirichlet (D) boundary condition

$$\mathfrak{a}_0^D = \int_{\mathcal{M}} \epsilon_{\mathcal{M}}, \tag{B.11a}$$

$$\mathfrak{a}_1^D = -\frac{\sqrt{\pi}}{2} \int_{\mathcal{I}} \epsilon_{\mathcal{I}}, \tag{B.11b}$$

$$\mathfrak{a}_2^D = -\int_{\mathcal{M}} (\xi - \xi_c) R\, \epsilon_{\mathcal{M}} + \frac{1}{3} \int_{\mathcal{I}} K \epsilon_{\mathcal{I}}, \tag{B.11c}$$

$$\mathfrak{a}_3^D = \frac{\sqrt{\pi}}{2} \int_{\mathcal{I}} \left((\xi - \xi_c) R - \frac{1}{12} R_{\mu\nu} n^\mu n^\nu + \frac{5}{48} \mathrm{m}_2 - \frac{11}{288} K^2 \right) \epsilon_{\mathcal{I}}, \tag{B.11d}$$

$$\begin{aligned} \mathfrak{a}_4^D = & \int_{\mathcal{M}} \left(\frac{1}{120} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{2} (\xi - \xi_c)^2 R^2 \right) \epsilon_{\mathcal{M}} - \frac{4\pi^2}{45} \chi[\mathcal{M}] + \\ & + \int_{\mathcal{I}} \left(\frac{2}{35} \mathrm{m}_3 - \frac{1}{15} U - (\xi - \xi_c) \left(\frac{1}{3} R K + \frac{1}{2} \nabla_n R \right) \right) \epsilon_{\mathcal{I}}, \end{aligned} \tag{B.11e}$$

and with Robin (R) boundary coefficients.

$$\mathfrak{a}_0^R = \int_{\mathcal{M}} \epsilon_{\mathcal{M}}, \tag{B.12a}$$

$$\mathfrak{a}_1^R = \frac{\sqrt{\pi}}{2} \int_{\mathcal{I}} \epsilon_{\mathcal{I}}, \tag{B.12b}$$

$$\mathfrak{a}_2^R = -\int_{\mathcal{M}} (\xi - \xi_c) R\, \epsilon_{\mathcal{M}} - \int_{\mathcal{I}} (2\zeta - \zeta_c) K \epsilon_{\mathcal{I}}, \tag{B.12c}$$

$$\begin{aligned}\mathfrak{a}_3^R = &-\frac{\sqrt{\pi}}{2} \int_{\mathcal{I}} \bigg((\xi - \xi_c) R - \frac{1}{12} R_{\mu\nu} n^\mu n^\nu - \frac{1}{48} \mathrm{m}_2 - \frac{1}{32} K^2 + \\ &- \left(\zeta - \frac{1}{3}\right)(2\zeta - \zeta_c) K^2 \bigg) \epsilon_{\mathcal{I}},\end{aligned} \tag{B.12d}$$

$$\begin{aligned}\mathfrak{a}_4^R = &\int_{\mathcal{M}} \left(\frac{1}{120} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{2} (\xi - \xi_c)^2 R^2 \right) \epsilon_{\mathcal{M}} - \frac{4\pi^2}{45} \chi[\mathcal{M}] + \\ &+ \int_{\mathcal{I}} \left(\frac{2}{45} \mathrm{m}_3 - \frac{1}{15} U + (\xi - \xi_c) \left((2\zeta - \zeta_c) R K + \frac{1}{2} \nabla_n R \right) \right) \epsilon_{\mathcal{I}} + \\ &- \left(\zeta - \frac{1}{3}\right) \int_{\mathcal{I}} \left(\frac{2}{15} K \mathrm{m}_2 + \frac{1}{3} (2\zeta - \zeta_c)^2 K^3 \right) \epsilon_{\mathcal{I}},\end{aligned} \tag{B.12e}$$

where we wrote the geometrical invariants in terms of the operators (4.19), (4.20), and (4.21) introduced in Section 4.3, as they have better transformation properties under Weyl scaling $g_{\mu\nu} \to W^2 g_{\mu\nu}$.

Appendix C

Lanczos-Lovelock boundary terms

In this appendix, we show that the boundary Lagrangian ℓ (6.33) is the correct one to ensure the well-posedness of the variational principle for LL theories under Dirichlet boundary conditions for the metric. In other words, we will show the validity of eq. (6.34), that we report here

$$\mathbb{d}\ell + i^*_{\mathcal{I}}\theta = \frac{1}{2}\mathcal{T}^{ab}\mathbb{d}h_{ab}\epsilon_{\mathcal{I}} + \mathrm{d}\vartheta. \tag{C.1}$$

In doing so, we also derive the expressions for the generalized Brown–York tensor $\mathcal{T}^{ab}$ and the boundary presymplectic potential ϑ. Throughout this appendix, we omit writing the (k) superscript to avoid cluttering the notation; the results we present here hold for each k term independently.

We find it convenient to introduce the following tensor in the boundary $\mathcal{I}$, analogous to the Lovelock tensor $P^{\mu\nu}_{\rho\sigma}$ in the bulk

$$p^{iab}_{jcd}(s) := \begin{cases} 0 & \text{for} \quad k = 0, 1, \\ \delta^{iab}_{jcd} & \text{for} \quad k = 2, \\ \frac{k}{2^{k-1}}\delta^{iaba_3b_3\ldots a_kb_k}_{jcdc_3d_3\ldots c_kd_k}\prod_{i=3}^{k}\left(\mathcal{R}^{c_id_i}_{a_ib_i} - 2s^2K^{c_i}_{a_i}K^{d_i}_{b_i}\right) & \text{for} \quad k > 2, \end{cases} \tag{C.2}$$

where the term in round brackets is referred to as the *interpolating curvature*. Using this tensor, we can recast the k-th boundary Lagrangian *scalar* (with $k > 1$) as

$$\mathscr{\ell} = 2\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\left(\mathcal{R}^{cd}_{ab} - 2s^2K^c_aK^d_b\right). \tag{C.3}$$

For $k = 0$ and $k = 1$, the boundary Lagrangians are respectively $\mathscr{\ell} = 0$ and $\mathscr{\ell} = 2K$. The

variation of this boundary Lagrangian yields

$$\begin{aligned}\mathbb{d}\mathscr{\ell} =& 2p^{iab}_{jcd}(1)\mathbb{d}K^j_i(\mathcal{R}^{cd}_{ab} - 2K^c_aK^d_b)+\\ &+4(k-1)\left[\mathcal{D}^c(p^{iab}_{jcd}(1)\mathcal{D}_aK^j_i)\mathbb{d}h^d_b + \frac{1}{2}\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\mathbb{d}h^{ce}\mathcal{R}_e{}^d{}_{ab}\right]+\\ &-4(k-1)\left[\mathcal{D}^c(p^{iab}_{jcd}(1)\mathcal{D}_aK^j_i\mathbb{d}h^d_b) - \int_0^1 \mathrm{d}s\mathcal{D}_a(p^{iab}_{jcd}(s)K^j_i\mathcal{D}^c\mathbb{d}h^d_b)\right],\end{aligned} \tag{C.4}$$

where the first line depends on $\mathbb{d}K$, the terms in the second line will contribute to the generalized Brown–York tensor, and those in the third line will contribute to the boundary symplectic potential.

Proof. Upon variation, we find terms in $\mathbb{d}\mathscr{\ell}$ that are proportional to either $\mathbb{d}\mathcal{R}^{ab}_{cd}$ or $\mathbb{d}K^a_b$. The remarkable property of LL theories is that the latter multiplies a term that can be integrated exactly in s. To show this we notice that, among such terms (those proportional to $\mathbb{d}K^a_b$), there are $(2k-1)$ — of these, $2(k-1)$ come from the variation of the $k-1$ interpolating curvature factors, and one comes from the K^j_i term outside — which multiply a product of two Ks, and there is a single one which multiplies an intrinsic curvature $\mathcal{R}$. All these terms effectively commute because of the antisymmetry of the generalized Kronecker delta (inside p), so that we can sum them and write

$$\mathbb{d}\mathscr{\ell} \supset \frac{2^{k-1}}{k}\delta K^j_i\int_0^1 \mathrm{d}s(\mathcal{R}^{cd}_{ab} - 2s^2(2k-1)K^c_aK^d_b)p^{iab}_{jcd}(s). \tag{C.5}$$

It is now a matter of elementary algebra [1] to show that the integrand is a total derivative, which we can easily integrate to get

$$\mathbb{d}\mathscr{\ell} = 2p^{iab}_{jcd}(1)\mathbb{d}K^j_i(\mathcal{R}^{cd}_{ab} - 2K^c_aK^d_b) + 2(k-1)\underbrace{\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\mathbb{d}\mathcal{R}^{cd}_{ab}}, \tag{C.7}$$

where we reinstated the term proportional to the intrinsic curvature variations. Let us rearrange this last term using $\mathbb{d}\mathcal{R}^{cd}_{ab} = \mathbb{d}h^{ce}\mathcal{R}_e{}^d{}_{ab} + h^{ce}\mathbb{d}\mathcal{R}_e{}^d{}_{ab}$ and the expression for the Riemann tensor variation. Using the antisymmetry of p, we can write

$$\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\mathbb{d}\mathcal{R}^{cd}_{ab} = \int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\mathbb{d}h^{ce}\mathcal{R}_e{}^d{}_{ab} + 2\underbrace{\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K^j_i\mathcal{D}_a\mathcal{D}^c\mathbb{d}h^d_b}. \tag{C.8}$$

Integrating by part once the second term yields

$$2\int_0^1 \mathrm{d}s\mathcal{D}_a(p^{iab}_{jcd}(s)K^j_i\mathcal{D}^c\mathbb{d}h^d_b) - 2\underbrace{\int_0^1 \mathrm{d}s\mathcal{D}_a(p^{iab}_{jcd}(s)K^j_i)\mathcal{D}^c\mathbb{d}h^d_b}, \tag{C.9}$$

[1] Just use

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[s(a+s^2b)^{k-1}\right] = (a+(2k-1)s^2b)(a+s^2b)^{k-2}. \tag{C.6}$$

where the second term is another exact s integral (cfr. footnote 1), that we integrate by part once more

$$\begin{aligned} 2\int_0^1 \mathrm{d}s\mathcal{D}_a(p^{iab}_{jcd}(s)K^j_i)\mathcal{D}^c\mathbb{d}h^d_b &= 2p^{iab}_{jcd}(1)\mathcal{D}_aK^j_i\mathcal{D}^c\mathbb{d}h^d_b \\ &= 2\mathcal{D}^c(p^{iab}_{jcd}(1)\mathcal{D}_aK^j_i\mathbb{d}h^d_b) - 2\mathcal{D}^c(p^{iab}_{jcd}(1)\mathcal{D}_aK^j_i)\mathbb{d}h^d_b. \end{aligned} \tag{C.10}$$

Adding this last expression to all the terms we have left behind (i.e. those outside the horizontal square brackets), we get the reported result. □

When Dirichlet boundary conditions are imposed on the metric, the second two lines in (C.4) vanish as they only depend on the induced metric and its derivatives along the surface. On the other hand, the extrinsic curvature depends on how the surface $\mathcal{I}$ is embedded in spacetime, hence its variation contains normal derivatives of the metric. Indeed we have

$$\mathbb{d}K_{\mu\nu} = n\cdot\mathbb{d}n\,K_{\mu\nu} - \mathbb{c}_\mu n_\nu - \mathbb{c}_\nu n_\mu - \frac{1}{2}(\Pi^\lambda_\mu\Pi^\beta_\nu n^\alpha + \Pi^\alpha_\mu\Pi^\lambda_\nu n^\beta - \Pi^\beta_\mu\Pi^\alpha_\nu n^\lambda)\nabla_\lambda\mathbb{d}g_{\alpha\beta} \tag{C.11}$$

where $\mathbb{c}_\mu := \mathbb{d}g_{\alpha\beta}\Pi^\alpha_\mu n^\beta$ and $\Pi^\mu_\nu = \delta^\mu_\nu - n^\mu n_\nu$ is the tangential projector to $\mathcal{I}$ (A.3).

The other term that we need is the pull-back to $\mathcal{I}$ of the symplectic potential θ. Given the embedding of the boundary, the indices of the bulk P tensor can be projected either tangentially or perpendicularly to the boundary, as illustrated in Appendix A. Because of its antisymmetry, we expect this procedure to produce only four independent boundary tensors, namely $P^{\parallel\parallel}_{\parallel\parallel}$, $P^{\perp\parallel}_{\parallel\parallel}$, $P^{\parallel\parallel}_{\perp\parallel}$, and $P^{\perp\parallel}_{\perp\parallel}$: further contractions with the normal n vanish by antisymmetry. We can then expand

$$i^*_{\mathcal{I}}\theta = [(P^{\perp\parallel}_{\parallel\parallel})^\nu_{\rho\sigma} + (P^{\perp\parallel}_{\perp\parallel})^\nu_\sigma n_\rho - (P^{\perp\parallel}_{\perp\parallel})^\nu_\rho n_\sigma](g^{\rho\lambda}\nabla^\sigma - g^{\sigma\lambda}\nabla^\rho)\mathbb{d}g_{\lambda\nu}\epsilon_{\mathcal{I}}. \tag{C.12}$$

If we introduce the boundary tensors $\upsilon := \iota^*_{\mathcal{I}}P^{\perp\parallel}_{\parallel\parallel}$ and $\pi := \iota^*_{\mathcal{I}}P^{\perp\parallel}_{\perp\parallel}$, of rank $(1,2)$ and $(1,1)$ respectively, we can write the pull-back of the symplectic potential to the boundary as

$$i^*_{\mathcal{I}}\theta = -2\Big[2\pi^i_j\mathbb{d}K^j_i + (\pi^{ac}K^b_c + \mathcal{D}^d\upsilon^b_{cd}h^{ac})\mathbb{d}h_{ab} - \mathcal{D}^a\Big(\pi^{ac}\mathbb{c}_c + \upsilon^c_{ba}\mathbb{d}h^b_c\Big)\Big]\epsilon_{\mathcal{I}}, \tag{C.13}$$

where we separated the term proportional to $\mathbb{d}K$, the contribution to the generalized Brown–York tensor and the contribution to the boundary symplectic potential.

Proof. Let us rewrite eq. (C.12) as

$$i^*_{\mathcal{I}}\theta = 2(P^{\perp\parallel}_{\parallel\parallel})^\nu_{\rho\sigma}g^{\rho\lambda}\nabla^\sigma\mathbb{d}g_{\lambda\nu}\epsilon_{\mathcal{I}} + 2\Big((P^{\perp\parallel}_{\perp\parallel})^\nu_\sigma n_\rho - (P^{\perp\parallel}_{\perp\parallel})^\nu_\rho n_\sigma\Big)g^{\rho\lambda}\nabla^\sigma\mathbb{d}g_{\lambda\nu}\epsilon_{\mathcal{I}}, \tag{C.14}$$

and call the three pieces (a), (b) and (c). We analyze them one-by-one:

(a) The first term can be integrated by parts because the derivative is along the boundary

$$\begin{aligned}(P^{\perp\parallel})^{\nu}_{\rho\sigma}g^{\rho\lambda}\nabla^{\sigma}\mathbb{d}g_{\lambda\nu} &= \nabla^{\sigma}\Big[(P^{\perp\parallel})^{\nu}_{\rho\sigma}g^{\rho\lambda}\mathbb{d}g_{\lambda\nu}\Big] - \nabla^{\sigma}(P^{\perp\parallel})^{\nu}_{\rho\sigma}g^{\rho\lambda}\mathbb{d}g_{\lambda\nu} = \\ &= \mathcal{D}^{a}\Big(\upsilon^{c}_{ba}h^{bd}\mathbb{d}h_{dc}\Big) - (\mathcal{D}^{a}\upsilon^{c}_{ba}e^{\nu}_{c}e^{b}_{\rho} - \upsilon^{c}_{ba}K^{a}_{c}n^{\nu}e^{b}_{\rho} + \cancel{\upsilon^{c}_{ba}K^{ba}n_{\rho}e^{\nu}_{c}})g^{\rho\lambda}\mathbb{d}g_{\lambda\nu} = \\ &= \mathcal{D}^{a}\Big(\upsilon^{c}_{ba}h^{bd}\mathbb{d}h_{dc}\Big) - \mathcal{D}^{a}\upsilon^{c}_{ba}h^{bd}\mathbb{d}h_{dc} - \upsilon^{c}_{ba}K^{a}_{c}\mathbb{c}^{b},\end{aligned} \tag{C.15}$$

where, in the second-line, we used the antisymmetry of υ on the lower indices to cancel its contraction with the extrinsic curvature.

(b) In terms of previously introduced tensor π, we can write the second term

$$(P^{\perp\parallel})^{\nu}_{\sigma}n_{\rho}g^{\rho\lambda}\nabla^{\sigma}\mathbb{d}g_{\lambda\nu} = \pi^{ac}e^{\nu}_{a}e^{\sigma}_{c}n^{\lambda}\nabla_{\sigma}\mathbb{d}g_{\lambda\nu}. \tag{C.16}$$

(c) Using (C.11), the third term yields

$$(P^{\perp\parallel})^{\nu}_{\rho}n^{\sigma}g^{\rho\lambda}\nabla_{\sigma}\mathbb{d}g_{\lambda\nu} = 2\pi^{ac}\mathbb{d}K_{ac} - 2n\cdot\mathbb{d}n\,\pi^{ac}K_{ac} + \pi^{ac}(e^{\beta}_{a}e^{\sigma}_{c} + e^{\sigma}_{a}e^{\beta}_{c})n^{\alpha}\nabla_{\sigma}\mathbb{d}g_{\alpha\beta}. \tag{C.17}$$

Notice that the first term in round brackets coincides and cancels the (b) terms; whereas the second can be recast as

$$\begin{aligned}\pi^{ac}e^{\sigma}_{a}e^{\beta}_{c}n^{\alpha}\nabla_{\sigma}\mathbb{d}g_{\alpha\beta} &= e^{\sigma}_{a}\nabla_{\sigma}(\pi^{ac}e^{\beta}_{c}n^{\alpha}\mathbb{d}g_{\alpha\beta}) - e^{\sigma}_{a}\nabla_{\sigma}(\pi^{ac}e^{\beta}_{c}n^{\alpha})\mathbb{d}g_{\alpha\beta} \\ &= -\mathcal{D}_{a}(\pi^{ac}\mathbb{c}_{c}) + \mathcal{D}_{a}\pi^{ab}\mathbb{c}_{b} + \pi^{ac}K_{ac}n^{\alpha}n^{\beta}\mathbb{d}g_{\alpha\beta} - \pi^{ac}K^{b}_{c}\mathbb{d}h_{ab}.\end{aligned} \tag{C.18}$$

When putting these pieces back together, we notice that there are some nice cancellations due to $\mathbb{d}n_{\mu} = \frac{1}{2}n_{\mu}n^{\alpha}n^{\beta}\mathbb{d}g_{\alpha\beta}$, and to the divergenceless of the Lovelock tensor, which implies the relation $\mathcal{D}_{a}\pi^{a}_{b} + \upsilon^{c}_{ba}K^{a}_{c} = 0$ between its boundary projections

$$\begin{aligned}2\Big[\mathcal{D}^{a}\Big(\upsilon^{c}_{ba}h^{bd}\mathbb{d}h_{dc}\Big) - \mathcal{D}^{a}\upsilon^{c}_{ba}h^{bd}\mathbb{d}h_{dc} - \cancel{\upsilon^{c}_{ba}K^{a}_{c}\mathbb{c}^{b}} - 2\pi^{ac}\mathbb{d}K_{ac} + \cancel{2n\cdot\mathbb{d}n\,\pi^{ac}K_{ac}} + \\ + \mathcal{D}_{a}(\pi^{ac}\mathbb{c}_{c}) - \cancel{\mathcal{D}_{a}\pi^{ab}\mathbb{c}_{b}} - \cancel{\pi^{ac}K_{ac}n^{\alpha}n^{\beta}\mathbb{d}g_{\alpha\beta}} + \pi^{ac}K^{b}_{c}\mathbb{d}h_{ab})\Big]\epsilon_{\mathcal{I}}.\end{aligned} \tag{C.19}$$

Rewriting $\pi^{ac}\mathbb{d}K_{ac}$ as $\pi^{i}_{j}\mathbb{d}K^{j}_{i} + \pi^{ac}K^{d}_{c}\mathbb{d}h_{ad}$, gives back the reported result. □

Let us notice that only the fully-projected Riemann tensor (ρ in (A.10)) can enter in π^{a}_{b} because, otherwise, there will be at least a second contraction with the normal. Therefore, using the Gauss equation (A.14), we can explicitly compute

$$\pi^{i}_{j} = \frac{k}{2^{k}}\delta^{ia_2b_2a_3b_3\dots a_kb_k}_{jc_2d_2c_3d_3\dots c_kd_k}\prod_{i=2}^{k}\rho^{c_id_i}_{a_ib_i} = \frac{1}{2}p^{iab}_{jcd}(1)\Big(\mathcal{R}^{cd}_{ab} - 2K^{c}_{a}K^{d}_{b}\Big). \tag{C.20}$$

This relation ensures the cancellation between the terms proportional to $\mathbb{d}K_i^j$ coming from the Lagrangian variation (C.4) from the symplectic potential pull-back (C.13). The other terms are either proportional to $\mathbb{d}h_{ab}$ or inside a total boundary derivative. The former will add up to the generalized Brown–York tensor [2]

$$\mathcal{T}_{ab} = -4\pi^c_{(a}K_{b)c} + h_{ab}\ell - 4(k-1)\int_0^1 \mathrm{d}s\, p^{idc}_{je(a}(s)R_{b)}{}^e{}_{cd}K_i^j; \tag{C.21}$$

while the latter form the LL boundary symplectic potential ϑ.

$$\vartheta = 2\pi^{ab}\mathbb{c}_b\iota_{\partial_a}\epsilon_{\mathcal{I}} + 4(k-1)\int_0^1 \mathrm{d}s\, p^{iab}_{jcd}(s)K_i^j\mathcal{D}^c\mathbb{d}h^d_b\iota_{\partial_a}\epsilon_{\mathcal{I}}. \tag{C.22}$$

Notice that all the terms involving υ dropped out of the final result because

$$\upsilon^a_{cd} = 2(k-1)p^{iab}_{jcd}(1)\mathcal{D}_bK_i^j, \tag{C.23}$$

as one can prove analogously to (C.20), using the Codazzi equation (A.14).

[2] Since we have been using the Lagrangian scalar, we must include the term $\frac{1}{2}h^{ab}\ell\,\mathbb{d}h_{ab}$ to account for the variation of the boundary volume element.

Appendix D

Induced representations of the quantum corner symmetry group

We report here the complete set of unitary irreducible representations of the two-dimensional quantum corner symmetry group $\mathrm{QCS}_2 = \widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes \mathbb{H}_3$ that was found in [318], together with a brief presentation of the method of *induced representations* that was employed to find it.

Let us consider a semidirect product group $H \ltimes N$. This structure indicates that elements of H are associated with automorphisms of N (which we denote with the generic notation $n \mapsto h \triangleright n$), such that the composition law of two group elements has a mixing between the two sectors in the second

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 \cdot h_2, n_1 \cdot (h_1 \triangleright n_2)). \tag{D.1}$$

Let π be an irreducible representation of N on the target space V_π, that is $\pi(n) \in \mathrm{End}(V_\pi)$ for each $n \in N$ (we use $\mathrm{End}(V) \equiv \mathrm{End}(V, V)$, with $\mathrm{End}(V, W)$ the space of linear maps $V \to W$). Using the $h \in H$ action on N, we can construct a new representation π_h in the same space, defined by

$$(\pi_h)(n) = \pi(h^{-1} \triangleright n). \tag{D.2}$$

Considering the equivalence class of both sides (we recall that two representations are equivalent if there is an isomoprhism $I : V_{\pi_1} \to V_{\pi_2}$ such that $\pi_2 \circ I = I \circ \pi_1$), we extend the action of $h \in H$ to the set of inequivalent irreducible representations of N, that is $h : [\pi] \mapsto [\pi]_h := [\pi_h]$. This set is thus foliated by H-orbits. Some equivalence classes have non-trivial stabilizers, which means that there exist some elements of H whose action leave the class invariant. These elements form a subgroup, called the **little group** of that class

$$H_{[\pi]} := \{h \in H | [\pi]_h = [\pi]\}. \tag{D.3}$$

This definition means that, for each $\pi \in [\pi]$, there exists an automorphism $I_h \in \mathrm{Aut}(V_\pi)$ such that $\pi_h = I_h \circ \pi \circ I_h^{-1}$ (remember that π_h acts on the same space as π).

According to Mackey's theory of induced representations [317, 342], we can construct the irreducible (unitary) representations of the group $H \ltimes N$ by first constructing the irreducible (unitary) representations of $H_{[\pi]} \ltimes N$, and then inducing a representation of the full group [1]. The last step is obviously not necessary if the little group $H_{[\pi]}$ coincides with the full group. In these cases, we find representations in the form of a tensor product: if $\sigma : H_{[\pi]} \to \mathrm{End}(\mathcal{H}_\sigma)$ is a (unitary) irreducible representation of $H_{[\pi]}$ on some vector space $\mathcal{H}_\sigma$, then

$$\rho_{\sigma,[\pi]} : H_{[\pi]} \ltimes N \to \mathrm{End}(\mathcal{H}_\sigma \otimes V_\pi), \tag{D.4}$$

$$(h, n) \mapsto \rho_{\sigma,[\pi]}(h, n) = \sigma(h) \otimes \pi(n)\mathcal{I}_h, \tag{D.5}$$

is a (unitary) irreducible representation. When N is non-abelian, there is a complication due to the maps $h \mapsto \mathcal{I}_h$ generally forming a projective (unitary) representation of $H_{[\pi]}$. In this case, σ cannot be generic, but it has to "balance" the projective phase factor of $\mathcal{I}$ in such a way $h \to \sigma(h) \otimes \mathcal{I}_h$ is an ordinary, non projective, representation.

The group under consideration, QCS_2, has the structure of a semidirect product with a non-abelian normal subgroup, $\mathbb{H}_3$. According to Stone–von Neumann's theorem [350], the Heisenberg group has essentially one, up to equivalence, (strongly continuous) unitary irreducible representation π_c for each choice of non-zero $\pi_c^*(\mathbf{E}) \in \mathbb{R}$, where π_c^* is the associated Lie algebra representation. For $c = 0$, there are infinite irreducible representations, but they are not interesting for us. One possible representation space for π_c is the Fock space $\mathfrak{F}_c$ of a free particle with ladder operators $a, a^\dagger$, which is unitary.

Since the little group of π_c is the entire $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the representation theory of QCS_2 falls in the simple case where the representations are a tensor product.

To complete the construction, we need to choose a representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. These depend on two numbers, associated to the eigenvalues of the Casimir $\mathbf{L}^2$ and the central element $e^{2\pi\mathbf{L_0}}$ [351]. We write the first as $q(1-q)$, with $q \in \mathbb{R}$ or $q - \frac{1}{2} \in i\mathbb{R}$, and the second as $e^{2\pi i\mu}$, with $\mu \in \mathbb{R}$. When the group that is represented is the non-covered $\mathrm{SL}(2,\mathbb{R})$, the central element must be in $\mathbb{Z}_2$, which means that $\mu \in \mathbb{Z}/2$. The $\sigma_{q,\mu}$ representation, which we take to be realized in the space $\mathcal{H}_{q,\mu}$, is unitary only in the following cases:

Trivial representation: When $q = 0$ and the spectrum of $\mathbf{L}_0$ contains zero, then the representation is one-dimensional — it is the only finite-dimensional unitary representation — because all generators annihilate the state $|0\rangle$.

Discrete series representation For $q > \frac{1}{2}$ and $\mu = \pm q$, then $\mathbf{L}_\mp |\pm q\rangle = 0$ and the spectrum of $\mathbf{L}_0$ is either $q + \mathbb{Z}$ or $-q - \mathbb{Z}$.

[1] Given a group G and a representation $D : H \to \mathrm{End}(V)$ of a subgroup of $H \subset G$, we can construct a representation of the G over the space of V-valued functions of the coset G/H by $g \mapsto U_g \in \mathrm{End}(\Gamma(G/H, V))$ with

$$(U_g\psi)[p] = D(h)\,\psi[g^{-1} \triangleright p], \quad \forall p \in G/H$$

where $\triangleright$ denotes here the natural action of G on the coset G/H.

Continuous series representations: If $q = \frac{1}{2} + i\mathbb{R}$, $|\mu| \leq \frac{1}{2}$ or $|\mu| < q < \frac{1}{2}$ — these two cases are known as the *principal* and the *complementary* series — the ladder created by $\mathbf{L}_\pm$ does not stop, and the spectrum of $\mathbf{L}_0$ is $\mu + \mathbb{Z}$. The representations for $\mu > 0$ and $\mu < 0$ are equivalent.

Thanks to Mackey's theory, we can obtain the entire set of QCS_2 group representations by combining representations of its factors

$$\rho_{q,\mu,c} \equiv \rho_{\sigma_{q,\mu},[\pi_c]} : \mathrm{QCS}_2 \to \mathrm{End}(\mathcal{H}_{q,\mu} \otimes \mathfrak{F}_c). \tag{D.6}$$

Because of the tensor product structure, this representation is unitary if and only if both $\sigma_{q,\mu}$ and π_c are. Choosing the basis $\{\mathbf{L}_0, \mathbf{L}_-, \mathbf{L}_+, \mathbf{T}_-, \mathbf{T}_+, \mathbf{E}\}$ we introduced in Section 7.4.1 for the Lie algebra of QCS_2, we find that these elements are represented as [2]

$$\rho^*_{q,\mu,c}(\mathbf{E})\,|n,k\rangle = c\,|n,k\rangle\,, \tag{D.7a}$$

$$\rho^*_{q,\mu,c}(\mathbf{T}_-)\,|n,k\rangle = \sqrt{c\hbar}\sqrt{k}\,|n,k-1\rangle\,, \tag{D.7b}$$

$$\rho^*_{q,\mu,c}(\mathbf{T}_+)\,|n,k\rangle = \sqrt{c\hbar}\sqrt{k+1}\,|n,k+1\rangle\,, \tag{D.7c}$$

$$\rho^*_{q,\mu,c}(\mathbf{L}_0)\,|n,k\rangle = \hbar\left(\frac{2k+1}{4} + n\right)|n,k\rangle\,, \tag{D.7d}$$

$$\rho^*_{q,\mu,c}(\mathbf{L}_-)\,|n,k\rangle = -\hbar\sqrt{(n-q)(n-1+q)}\,|n-1,k\rangle + \frac{\hbar}{2}\sqrt{k(k-1)}\,|n,k-2\rangle\,, \tag{D.7e}$$

$$\rho^*_{q,\mu,c}(\mathbf{L}_+)\,|n,k\rangle = -\hbar\sqrt{(n+q)(n+1-q)}\,|n+1,k\rangle + \frac{\hbar}{2}\sqrt{(k+1)(k+2)}\,|n,k+2\rangle\,, \tag{D.7f}$$

where we write $|n,k\rangle$ for $|n\rangle \otimes |k\rangle \in \mathcal{H}_{q,\mu} \otimes \mathfrak{F}_c$. The quantum number k is the eigenvalue of the Fock space number operator, while n is defined by (D.7d). We can check that the cubic Casimir of $\mathfrak{qcs}_2$ is indeed represented as multiple of the identity

$$\rho^*_{q,\mu,c}(\mathbf{C}_3^{\mathfrak{qcs}_2})\,|n,k\rangle = 2c\hbar^2 q(1-q)\,|n,k\rangle\,. \tag{D.8}$$

[2] This for $c > 0$. The inequivalent representation for $c < 0$ can be easily constructed.

Bibliography

[1] Daniel Harlow and Jie-Qiang Wu. "Covariant phase space with boundaries". In: *JHEP* 10 (2020). _eprint: 1906.08616, p. 146. DOI: 10.1007/JHEP10(2020)146.

[2] C. J. Isham. "Topological and global aspects of quantum theory". In: *Les Houches Summer School on Theoretical Physics: Relativity, Groups and Topology*. July 1983, pp. 1059–1290.

[3] C.J. Isham. "Canonical groups and the quantization of general relativity". In: *Nuclear Physics B - Proceedings Supplements* 6 (1989), pp. 349–356. ISSN: 0920-5632. DOI: https://doi.org/10.1016/0920-5632(89)90471-4. URL: https://www.sciencedirect.com/science/article/pii/0920563289904714.

[4] S. C. Zhang, T. H. Hansson, and S. Kivelson. "Effective-Field-Theory Model for the Fractional Quantum Hall Effect". In: *Phys. Rev. Lett.* 62 (1 Jan. 1989), pp. 82–85. DOI: 10.1103/PhysRevLett.62.82. URL: https://link.aps.org/doi/10.1103/PhysRevLett.62.82.

[5] Xiao-Gang Wen. "Theory of the edge states in fractional quantum Hall effects". In: *Int. J. Mod. Phys. B* 6 (1992), pp. 1711–1762. DOI: 10.1142/S0217979292000840.

[6] A.P. Balachandran et al. "Conformal Edge Currents in Chern-Simons Theories". en. In: *International Journal of Modern Physics A* 07.19 (July 1992), pp. 4655–4670. ISSN: 0217-751X, 1793-656X. DOI: 10.1142/S0217751X92002106. URL: https://www.worldscientific.com/doi/abs/10.1142/S0217751X92002106 (visited on 05/30/2025).

[7] A.P. Balachandran, L. Chandar, and E. Ercolessi. "Edge Stateas in Gauge Theories: Theory, Interpretations and Predictions". en. In: *International Journal of Modern Physics A* 10.13 (May 1995), pp. 1969–1993. ISSN: 0217-751X, 1793-656X. DOI: 10.1142/S0217751X95000966. URL: https://www.worldscientific.com/doi/abs/10.1142/S0217751X95000966 (visited on 05/30/2025).

[8] R. Arnowitt, S. Deser, and C. W. Misner. "Dynamical structure and definition of energy in general relativity". In: *Phys. Rev.* 116.5 (1959), pp. 1322–1330. ISSN: 0031899X. DOI: 10.1103/PhysRev.116.1322.

[9] Tullio Regge and Claudio Teitelboim. "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity". In: *Annals Phys.* 88 (1974), p. 286. DOI: 10.1016/0003-4916(74)90404-7.

[10] Abhay Ashtekar. "New Variables for Classical and Quantum Gravity". In: *Phys. Rev. Lett.* 57 (18 1986), pp. 2244–2247. DOI: 10.1103/PhysRevLett.57.2244. URL: https://link.aps.org/doi/10.1103/PhysRevLett.57.2244.

[11] J. Lee and Robert M. Wald. "Local symmetries and constraints". In: *J. Math. Phys.* 31 (1990), pp. 725–743. DOI: 10.1063/1.528801.

[12] Glenn Barnich and Cedric Troessaert. "BMS charge algebra". In: *JHEP* 12 (2011). _eprint: 1106.0213, p. 105. DOI: 10.1007/JHEP12(2011)105.

[13] Laurent Freidel et al. "The Weyl BMS group and Einstein's equations". en. In: *Journal of High Energy Physics* 2021.7 (July 2021), p. 170. ISSN: 1029-8479. DOI: 10.1007/JHEP07(2021)170. URL: https://doi.org/10.1007/JHEP07(2021)170 (visited on 03/29/2025).

[14] Laurent Freidel. "A canonical bracket for open gravitational system". In: (Nov. 2021). arXiv: 2111.14747 [hep-th].

[15] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. "Edge modes of gravity. Part I. Corner potentials and charges". en. In: *Journal of High Energy Physics* 2020.11 (Nov. 2020), p. 26. ISSN: 1029-8479. DOI: 10.1007/JHEP11(2020)026. URL: http://link.springer.com/10.1007/JHEP11(2020)026 (visited on 05/30/2025).

[16] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. "Edge modes of gravity. Part II. Corner metric and Lorentz charges". en. In: *Journal of High Energy Physics* 2020.11 (Nov. 2020), p. 27. ISSN: 1029-8479. DOI: 10.1007/JHEP11(2020)027. URL: http://link.springer.com/10.1007/JHEP11(2020)027 (visited on 05/30/2025).

[17] Laurent Freidel, Marc Geiller, and Daniele Pranzetti. "Edge modes of gravity. Part III. Corner simplicity constraints". en. In: *Journal of High Energy Physics* 2021.1 (Jan. 2021), p. 100. ISSN: 1029-8479. DOI: 10.1007/JHEP01(2021)100. URL: https://link.springer.com/10.1007/JHEP01(2021)100 (visited on 05/30/2025).

[18] Venkatesa Chandrasekaran et al. "A general framework for gravitational charges and holographic renormalization". In: *Int. J. Mod. Phys. A* 37.17 (2022), p. 2250105. DOI: 10.1142/S0217751X22501056. arXiv: 2111.11974 [gr-qc].

[19] Wolfgang Wieland. "Barnich–Troessaert bracket as a Dirac bracket on the covariant phase space". In: *Class. Quant. Grav.* 39.2 (2022), p. 025016. DOI: 10.1088/1361-6382/ac3e52. arXiv: 2104.08377 [hep-th].

[20] Antony J. Speranza. "Ambiguity resolution for integrable gravitational charges". In: *JHEP* 07 (2022), p. 029. DOI: 10.1007/JHEP07(2022)029. arXiv: 2202.00133 [hep-th].

[21] Luca Ciambelli. "Asymptotic Limit of Null Hypersurfaces". In: (Jan. 2025). _eprint: 2501.17357.

[22] James M. Bardeen, B. Carter, and S. W. Hawking. "The Four laws of black hole mechanics". In: *Commun. Math. Phys.* 31 (1973), pp. 161–170. DOI: 10.1007/BF01645742.

[23] S. W. Hawking. "Black holes in general relativity". In: *Commun. Math. Phys.* 25 (1972), pp. 152–166. DOI: 10.1007/BF01877517.

[24] S. W. Hawking. "Particle Creation by Black Holes". In: *Commun. Math. Phys.* 43 (1975). Ed. by G. W. Gibbons and S. W. Hawking, pp. 199–220. DOI: 10.1007/BF02345020.

[25] Jacob D. Bekenstein. "Black holes and entropy". In: *Phys. Rev. D* 7 (1973), pp. 2333–2346. DOI: 10.1103/PhysRevD.7.2333.

[26] Ted Jacobson. "Thermodynamics of space-time: The Einstein equation of state". In: *Phys. Rev. Lett.* 75 (1995), pp. 1260–1263. DOI: 10.1103/PhysRevLett.75.1260. arXiv: gr-qc/9504004.

[27] G. Chirco and S. Liberati. "Non-equilibrium Thermodynamics of Spacetime: The Role of Gravitational Dissipation". In: *Phys. Rev. D* 81 (2010), p. 024016. DOI: 10.1103/PhysRevD.81.024016. arXiv: 0909.4194 [gr-qc].

[28] T Padmanabhan. "Thermodynamical aspects of gravity: new insights". In: *Reports Prog. Phys.* 73.4 (Apr. 2010), p. 046901. ISSN: 0034-4885. DOI: 10.1088/0034-4885/73/4/046901. URL: http://stacks.iop.org/0034-4885/73/i=4/a=046901?key=crossref.3bdc2f234fac28aaaa57930c10866e36.

[29] Robert M. Wald. "Black hole entropy is the Noether charge". In: *Phys. Rev. D* 48.8 (1993). _eprint: gr-qc/9307038, R3427–R3431. DOI: 10.1103/PhysRevD.48.R3427.

[30] Vivek Iyer and Robert M. Wald. "Some properties of Noether charge and a proposal for dynamical black hole entropy". In: *Phys. Rev. D* 50 (1994). _eprint: gr-qc/9403028, pp. 846–864. DOI: 10.1103/PhysRevD.50.846.

[31] R Penrose. "Angular momentum: an approach to combinatorial space-time". In: *Quantum Theory and Beyond* (1971), pp. 151–180.

[32] Shinsei Ryu and Tadashi Takayanagi. "Holographic derivation of entanglement entropy from AdS/CFT". In: *Phys. Rev. Lett.* 96 (2006), p. 181602. DOI: 10.1103/PhysRevLett.96.181602.

[33] Shinsei Ryu and Tadashi Takayanagi. "Aspects of Holographic Entanglement Entropy". In: *JHEP* 08 (2006), p. 045. DOI: 10.1088/1126-6708/2006/08/045.

[34] William Donnelly. "Entanglement entropy in loop quantum gravity". In: *Physical Review D* 77.10 (May 2008). Publisher: American Physical Society, p. 104006. DOI: 10.1103/PhysRevD.77.104006. URL: https://link.aps.org/doi/10.1103/PhysRevD.77.104006 (visited on 06/14/2025).

[35] Mark Raamsdonk. "Building up spacetime with quantum entanglement". In: *General Relativity and Gravitation* 42 (May 2010). DOI: 10.1142/S0218271810018529.

[36] Thomas Faulkner et al. "Gravitation from entanglement in holographic CFTs". In: *Journal of High Energy Physics* 2014.3 (Mar. 2014). Publisher: Springer Science and Business Media LLC. ISSN: 1029-8479. DOI: 10.1007/jhep03(2014)051. URL: http://dx.doi.org/10.1007/JHEP03(2014)051.

[37] Veronika E. Hubeny. "The AdS/CFT Correspondence". In: *Class. Quant. Grav.* 32.12 (2015), p. 124010. DOI: 10.1088/0264-9381/32/12/124010. arXiv: 1501.00007 [gr-qc].

[38] William Donnelly and Laurent Freidel. "Local subsystems in gauge theory and gravity". In: *JHEP* 09 (2016). _eprint: 1601.04744, p. 102. DOI: 10.1007/JHEP09(2016)102.

[39] Goffredo Chirco, Daniele Oriti, and Mingyi Zhang. "Group field theory and tensor networks: towards a Ryu–Takayanagi formula in full quantum gravity". In: *Class. Quant. Grav.* 35.11 (2018), p. 115011. DOI: 10.1088/1361-6382/aabf55. arXiv: 1701.01383 [gr-qc].

[40] Eugenia Colafranceschi and Daniele Oriti. "Quantum gravity states, entanglement graphs and second-quantized tensor networks". In: *JHEP* 07 (2021), p. 052. DOI: 10.1007/JHEP07(2021)052. arXiv: 2012.12622 [hep-th].

[41] Abhay Ashtekar and Eugenio Bianchi. "A short review of loop quantum gravity". In: *Rept. Prog. Phys.* 84.4 (2021). _eprint: 2104.04394, p. 042001. DOI: 10.1088/1361-6633/abed91.

[42] Alejandro Perez. "The Spin Foam Approach to Quantum Gravity". In: *Living Rev. Rel.* 16 (2013). _eprint: 1205.2019, p. 3. DOI: 10.12942/lrr-2013-3.

[43] Laurent Freidel. "Group field theory: An Overview". In: *Int. J. Theor. Phys.* 44 (2005). _eprint: hep-th/0505016, pp. 1769–1783. DOI: 10.1007/s10773-005-8894-1.

[44] Daniele Oriti. "The Group field theory approach to quantum gravity". In: (2006). _eprint: gr-qc/0607032, pp. 310–331.

[45] Thomas Krajewski. "Group field theories". In: *PoS* QGQGS2011 (2011). Ed. by John Barrett et al., p. 005. DOI: 10.22323/1.140.0005.

[46] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*. Texts in Applied Mathematics. Springer New York, 2013. ISBN: 9780387217925. URL: https://books.google.it/books?id=k-7kBwAAQBAJ.

[47] Peter G. Bergmann. "Non-Linear Field Theories". In: *Phys. Rev.* 75 (1949), pp. 680–685. DOI: 10.1103/PhysRev.75.680.

[48] Paul A. M. Dirac. "Generalized Hamiltonian dynamics". In: *Can. J. Math.* 2 (1950), pp. 129–148. DOI: 10.4153/CJM-1950-012-1.

[49] Peter G. Bergmann and Irwin Goldberg. "Dirac bracket transformations in phase space". In: *Phys. Rev.* 98 (1955), pp. 531–538. DOI: 10.1103/PhysRev.98.531.

[50] P.A.M. Dirac. *Lectures on Quantum Mechanics*. Belfer Graduate School of Science: Monograph Series. Belfer Graduate School of Science, Yeshiva University, 1967. URL: https://books.google.it/books?id=IRBBAQAAIAAJ.

[51] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, Aug. 1994. ISBN: 978-0-691-03769-1, 978-0-691-21386-6.

[52] Giacomo Leopardi. *Essays, Dialogues, and Thoughts (Operette Morali)*. Translated by James Thomson. George Routledge, 1905.

[53] Paul A. M. Dirac. "The Theory of gravitation in Hamiltonian form". In: *Proc. Roy. Soc. Lond. A* 246 (1958), pp. 333–343. DOI: 10.1098/rspa.1958.0142.

[54] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. "The Dynamics of general relativity". In: *Gen. Rel. Grav.* 40 (2008), pp. 1997–2027. DOI: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109.

[55] Kenneth R. Meyer. "Symmetries and Integrals in Mechanics††This research was supported by ONR contract number N00014-67-A-0113-0019." In: *Dynamical Systems*. Ed. by M.M. Peixoto. Academic Press, 1973, pp. 259–272. ISBN: 978-0-12-550350-1. DOI: https://doi.org/10.1016/B978-0-12-550350-1.50025-4. URL: https://www.sciencedirect.com/science/article/pii/B9780125503501500254.

[56] Jerrold Marsden and Alan Weinstein. "Reduction of symplectic manifolds with symmetry". In: *Rept. Math. Phys.* 5.1 (1974), pp. 121–130. DOI: 10.1016/0034-4877(74)90021-4.

[57] Ana Cannas da Silva. "The Marsden-Weinstein-Meyer Theorem". In: *Lectures on Symplectic Geometry*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 167–171. ISBN: 978-3-540-45330-7. DOI: 10.1007/978-3-540-45330-7_23. URL: https://doi.org/10.1007/978-3-540-45330-7_23.

[58] Istvan Montvay and Gernot Münster. *Quantum Fields on a Lattice*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1994.

[59] Michael Creutz. *Quarks, Gluons and Lattices*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2023.

[60] L. Rosenfeld. "Über die Gravitationswirkungen des Lichts". In: *Zeitschrift für Physik* 65 (1930), pp. 589–599.

[61] R. Arnowitt, S. Deser, and C. W. Misner. "Canonical variables for general relativity". In: *Phys. Rev.* 117.6 (1960), pp. 1595–1602. ISSN: 0031899X. DOI: 10.1103/PhysRev.117.1595.

[62] Henrique Gomes and Aldo Riello. "The quasilocal degrees of freedom of Yang-Mills theory". In: *SciPost Phys.* 10.6 (2021), p. 130. DOI: 10.21468/SciPostPhys.10.6.130. arXiv: 1910.04222 [hep-th].

[63] Juan Margalef-Bentabol and Eduardo J. S. Villaseñor. "Geometric formulation of the Covariant Phase Space methods with boundaries". In: *Phy20s. Rev. D* 103.2 (2021). _eprint: 2008.01842, p. 025011. DOI: 10.1103/PhysRevD.103.025011.

[64] J. M. Souriau. *Structure des systèmes dynamiques*. Dunod, Paris, 1970.

[65] I. M. Anderson. "The variational bicomplex". In: *Utah State Technical Report* (1989).

[66] H. von Helmholtz. "Ueber die physikalische Bedeutung des Prinicips der kleinsten Wirkung." ger. In: *Journal für die reine und angewandte Mathematik* 100 (1887), pp. 137–166. URL: http://eudml.org/doc/148654.

[67] Peter G. Bergmann and Ralph Schiller. "Classical and Quantum Field Theories in the Lagrangian Formalism". In: *Phys. Rev.* 89 (1953), pp. 4–16. DOI: 10.1103/PhysRev.89.4.

[68] Arthur Komar. "Covariant Conservation Laws in General Relativity". In: *Phys. Rev.* 113.3 (Feb. 1959). Publisher: American Physical Society, pp. 934–936. DOI: 10.1103/PhysRev.113.934. URL: https://link.aps.org/doi/10.1103/PhysRev.113.934.

[69] A. Trautman. "Invariance of Lagrangian systems". In: *General Relativity, Papers in honour of J. L. Synge* (1972).

[70] A. Ashtekar, Anne Magnon, and Roger Penrose. "Quantum fields in curved space-times". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 346.1646 (1975), pp. 375–394. DOI: 10.1098/rspa.1975.0181.

[71] John L Friedman. "Generic instability of rotating relativistic stars". In: *Communications in Mathematical Physics* 62.3 (1978), pp. 247–278.

[72] J. L. Friedman and B. F. Schutz. "Lagrangian perturbation theory of nonrelativistic fluids." In: *Astrophysical Journal* 221 (1978), pp. 937–957. DOI: 10.1086/156098.

[73] A. Magnon. "On Komar integrals in asymptotically anti-de Sitter space-times". In: *J. Math. Phys.* 26 (1985), pp. 3112–3117. DOI: 10.1063/1.526690.

[74] M. Henneaux. "Hamiltonian Form of the Path Integral for Theories with a Gauge Freedom". In: *Phys. Rept.* 126 (1985), pp. 1–66. DOI: 10.1016/0370-1573(85)90103-6.

[75] Edward Witten. "Interacting Field Theory of Open Superstrings". In: *Nucl. Phys. B* 276 (1986), pp. 291–324. DOI: 10.1016/0550-3213(86)90298-1.

[76] Gregg J. Zuckerman. "ACTION PRINCIPLES AND GLOBAL GEOMETRY". In: *Conf. Proc. C* 8607214 (1986), pp. 259–284.

[77] C. Crnkovic and E. Witten. "Covariant description of canonical formalism in geometrical theories." In: *Three Hundred Years of Gravitation*. 1987, pp. 676–684.

[78] Igor Khavkine. "Covariant phase space, constraints, gauge and the Peierls formula". In: *Int. J. Mod. Phys. A* 29.5 (2014). _eprint: 1402.1282, p. 1430009. DOI: 10.1142/S0217751X14300099.

[79] Rudolf E. Peierls. "The Commutation laws of relativistic field theory". In: *Proc. Roy. Soc. Lond. A* 214 (1952), pp. 143–157. DOI: 10.1098/rspa.1952.0158.

[80] I. E. Segal. "Quantization of Nonlinear Systems". In: *Journal of Mathematical Physics* 1.6 (Nov. 1960), pp. 468–488. DOI: 10.1063/1.1703683.

[81] I. E. Segal, P. J. M. Bongaarts, and T. Niemeijer. "Representations of the canonical commutations relations". In: *Cargèse Lectures in Theoretical Physics : Summer School on Theoretical Physics*. Vol. 6. 1967, pp. 107–170.

[82] Glenn Barnich, Marc Henneaux, and Christiane Schomblond. "Covariant description of the canonical formalism". In: *Phys. Rev. D* 44 (4 Aug. 1991), R939–R941. DOI: 10.1103/PhysRevD.44.R939. URL: https://link.aps.org/doi/10.1103/PhysRevD.44.R939.

[83] Robert M. Wald. "On identically closed forms locally constructed from a field". In: *J. Math. Phys.* 31.10 (1990), p. 2378. DOI: 10.1063/1.528839.

[84] Francois Gieres. "Covariant canonical formulations of classical field theories". In: *SciPost Phys. Lect. Notes* 77 (2023), p. 1. DOI: 10.21468/SciPostPhysLectNotes.77. arXiv: 2109.07330 [hep-th].

[85] Ted Jacobson, Gungwon Kang, and Robert C. Myers. "On black hole entropy". In: *Phys. Rev. D* 49 (1994). _eprint: gr-qc/9312023, pp. 6587–6598. DOI: 10.1103/PhysRevD.49.6587.

[86] Florian Hopfmüller and Laurent Freidel. "Null Conservation Laws for Gravity". In: *Phys. Rev. D* 97.12 (2018), p. 124029. DOI: 10.1103/PhysRevD.97.124029. arXiv: 1802.06135 [gr-qc].

[87] Venkatesa Chandrasekaran and Antony J. Speranza. "Anomalies in gravitational charge algebras of null boundaries and black hole entropy". In: *JHEP* 01 (2021), p. 137. DOI: 10.1007/JHEP01(2021)137. arXiv: 2009.10739 [hep-th].

[88] Laurent Freidel et al. "Extended corner symmetry, charge bracket and Einstein's equations". en. In: *Journal of High Energy Physics* 2021.9 (Sept. 2021), p. 83. ISSN: 1029-8479. DOI: 10.1007/JHEP09(2021)083. URL: https://link.springer.com/10.1007/JHEP09(2021)083 (visited on 05/30/2025).

[89] G. W. Gibbons and S. W. Hawking. "Action Integrals and Partition Functions in Quantum Gravity". In: *Phys. Rev. D* 15 (1977), pp. 2752–2756. DOI: 10.1103/PhysRevD.15.2752.

[90] G. W. Gibbons, S. W. Hawking, and M. J. Perry. "Path Integrals and the Indefiniteness of the Gravitational Action". In: *Nucl. Phys. B* 138 (1978), pp. 141–150. DOI: 10.1016/0550-3213(78)90161-X.

[91] Michael P Reisenberger and Carlo Rovelli. "'Sum over surfaces' form of loop quantum gravity". In: *Phys. Rev. D* 56 (1997), pp. 3490–3508. DOI: 10.1103/PhysRevD.56.3490. arXiv: gr-qc/9612035.

[92] John C. Baez. "An Introduction to Spin Foam Models of BF Theory and Quantum Gravity". In: *Lect.Notes Phys.* Vol. 543. _eprint: 9905087. 2000, pp. 25–93. DOI: 10.1007/3-540-46552-9_2.

[93] Alexander Maloney and Edward Witten. "Quantum Gravity Partition Functions in Three Dimensions". In: *JHEP* 02 (2010), p. 029. DOI: 10.1007/JHEP02(2010)029. arXiv: 0712.0155 [hep-th].

[94] Aitor Lewkowycz and Juan Maldacena. "Generalized gravitational entropy". In: *JHEP* 08 (2013), p. 090. DOI: 10.1007/JHEP08(2013)090. arXiv: 1304.4926 [hep-th].

[95] Phil Saad, Stephen H. Shenker, and Douglas Stanford. "JT gravity as a matrix integral". In: (Mar. 2019). arXiv: 1903.11115 [hep-th].

[96] Jordan Cotler and Kristan Jensen. "AdS_3 gravity and random CFT". In: *JHEP* 04 (2021), p. 033. DOI: 10.1007/JHEP04(2021)033. arXiv: 2006.08648 [hep-th].

[97] Donald Marolf and Henry Maxfield. "Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information". In: *JHEP* 08 (2020), p. 044. DOI: 10.1007/JHEP08(2020)044. arXiv: 2002.08950 [hep-th].

[98] Lorenz Eberhardt. "Summing over Geometries in String Theory". In: *JHEP* 05 (2021), p. 233. DOI: 10.1007/JHEP05(2021)233. arXiv: 2102.12355 [hep-th].

[99] Eugenia Colafranceschi et al. "Algebras and Hilbert spaces from gravitational path integrals. Understanding Ryu-Takayanagi/HRT as entropy without AdS/CFT". In: *JHEP* 10 (2024), p. 063. DOI: 10.1007/JHEP10(2024)063. arXiv: 2310.02189 [hep-th].

[100] James York. "Boundary terms in the action principles of general relativity". In: *Found. Phys.* 16 (1986), pp. 249–257. DOI: 10.1007/BF01889475.

[101] Helmut Friedrich and Gabriel Nagy. "The Initial boundary value problem for Einstein's vacuum field equations". In: *Commun. Math. Phys.* 201 (1999), pp. 619–655. DOI: 10.1007/s002200050571.

[102] Michael T. Anderson. "On quasi-local Hamiltonians in General Relativity". In: *Phys. Rev. D* 82 (2010), p. 084044. DOI: 10.1103/PhysRevD.82.084044. arXiv: 1008.4309 [gr-qc].

[103] Xiaoyi Liu et al. "Ill-posedness of the Cauchy problem for linearized gravity in a cavity with conformal boundary conditions". In: (May 2025). arXiv: 2505.20410 [gr-qc].

[104] Zhongshan An and Michael T. Anderson. "Well-posed geometric boundary data in General Relativity, I: Conformal-mean curvature boundary data". In: (Mar. 2025). arXiv: 2503.12599 [math.AP].

[105] Zhongshan An and Michael T. Anderson. "Well-posed geometric boundary data in General Relativity, II: Dirichlet boundary data". In: (May 2025). arXiv: 2505.07128 [math.AP].

[106] Giulio Neri and Stefano Liberati. "On the resilience of the gravitational variational principle under renormalization". In: *JHEP* 10 (2023). _eprint: 2306.17505, p. 054. DOI: 10.1007/JHEP10(2023)054.

[107] Karl E. Gustafson. "Domain Decomposition, Operator Trigonometry, Robin Condition". In: 1998.

[108] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.

[109] J. David Brown and James W. York Jr. "Quasilocal energy and conserved charges derived from the gravitational action". In: *Phys. Rev. D* 47 (1993). _eprint: gr-qc/9209012, pp. 1407–1419. DOI: 10.1103/PhysRevD.47.1407.

[110] Ivan G. Avramidi and Giampiero Esposito. "Lack of strong ellipticity in Euclidean quantum gravity". In: *Class. Quant. Grav.* 15 (1998), pp. 1141–1152. DOI: 10.1088/0264-9381/15/5/006. arXiv: hep-th/9708163.

[111] Michael T Anderson. "On boundary value problems for Einstein metrics". In: *Geometry & Topology* 12.4 (July 2008), 2009–2045. ISSN: 1465-3060. DOI: 10.2140/gt.2008.12.2009. URL: http://dx.doi.org/10.2140/gt.2008.12.2009.

[112] Edward Witten. "A note on boundary conditions in Euclidean gravity". In: *Rev. Math. Phys.* 33.10 (2021), p. 2140004. DOI: 10.1142/S0129055X21400043. arXiv: 1805.11559 [hep-th].

[113] Roberto Oliveri and Simone Speziale. "Boundary effects in General Relativity with tetrad variables". In: *Gen. Rel. Grav.* 52.8 (2020), p. 83. DOI: 10.1007/s10714-020-02733-8. arXiv: 1912.01016 [gr-qc].

[114] Gerard 't Hooft and M. J. G. Veltman. "One loop divergencies in the theory of gravitation". In: *Ann. Inst. H. Poincare Phys. Theor. A* 20 (1974), pp. 69–94.

[115] J Melmed. "Conformal invariance and the regularised one-loop effective action". In: 21.23 (Dec. 1988), p. L1131. DOI: 10.1088/0305-4470/21/23/005. URL: https://dx.doi.org/10.1088/0305-4470/21/23/005.

[116] T. Padmanabhan and D. Kothawala. "Lanczos-Lovelock models of gravity". In: *Phys. Rept.* 531 (2013), pp. 115–171. DOI: 10.1016/j.physrep.2013.05.007. arXiv: 1302.2151 [gr-qc].

[117] Thomas P. Sotiriou and Valerio Faraoni. "f(R) Theories Of Gravity". In: *Rev. Mod. Phys.* 82 (2010), pp. 451–497. DOI: 10.1103/RevModPhys.82.451. arXiv: 0805.1726 [gr-qc].

[118] Antonio De Felice and Shinji Tsujikawa. "f(R) theories". In: *Living Rev. Rel.* 13 (2010), p. 3. DOI: 10.12942/lrr-2010-3. arXiv: 1002.4928 [gr-qc].

[119] Christopher Eling, Raf Guedens, and Ted Jacobson. "Non-equilibrium thermodynamics of spacetime". In: *Phys. Rev. Lett.* 96 (2006), p. 121301. DOI: 10.1103/PhysRevLett.96.121301. arXiv: gr-qc/0602001.

[120] K. S. Stelle. "Classical Gravity with Higher Derivatives". In: *Gen. Rel. Grav.* 9 (1978), pp. 353–371. DOI: 10.1007/BF00760427.

[121] Nathalie Deruelle et al. "Hamiltonian formulation of f(Riemann) theories of gravity". In: *Prog. Theor. Phys.* 123 (2010), pp. 169–185. DOI: 10.1143/PTP.123.169. arXiv: 0908.0679 [hep-th].

[122] Alessio Belenchia et al. "Higher-order theories of gravity: diagnosis, extraction and reformulation via non-metric extra degrees of freedom—a review". In: *Rept. Prog. Phys.* 81.3 (2018), p. 036001. DOI: 10.1088/1361-6633/aaa4ab. arXiv: 1612.07749 [gr-qc].

[123] Alberto Salvio. "Quadratic Gravity". In: *Front. in Phys.* 6 (2018), p. 77. DOI: 10.3389/fphy.2018.00077. arXiv: 1804.09944 [hep-th].

[124] John F. Donoghue and Gabriel Menezes. "On quadratic gravity". In: *Nuovo Cim. C* 45.2 (2022), p. 26. DOI: 10.1393/ncc/i2022-22026-7. arXiv: 2112.01974 [hep-th].

[125] Ali Teimouri et al. "Generalised Boundary Terms for Higher Derivative Theories of Gravity". In: *JHEP* 08 (2016), p. 144. DOI: 10.1007/JHEP08(2016)144. arXiv: 1606.01911 [gr-qc].

[126] L. D. Faddeev and R. Jackiw. "Hamiltonian Reduction of Unconstrained and Constrained Systems". In: *Phys. Rev. Lett.* 60 (1988), pp. 1692–1694. DOI: 10.1103/PhysRevLett.60.1692.

[127] Yuuiti Sendouda et al. "Higher curvature theories of gravity in the ADM canonical formalism". In: *Int. J. Mod. Phys. Conf. Ser.* 01 (2011). Ed. by Sang Pyo Kim, pp. 297–302. DOI: 10.1142/S2010194511000432.

[128] L. J. Alty. "The Generalized Gauss-Bonnet-Chern theorem". In: *J. Math. Phys.* 36 (1995), pp. 3094–3105. DOI: 10.1063/1.531015.

[129] Ted Jacobson and Alejandro Satz. "On the renormalization of the Gibbons-Hawking boundary term". In: *Phys. Rev. D* 89.6 (2014), p. 064034. DOI: 10.1103/PhysRevD.89.064034. arXiv: 1308.2746 [gr-qc].

[130] Steven Weinberg. *The Quantum Theory of Fields*. Cambridge University Press, 1995.

[131] Mohamed M. Anber and John F. Donoghue. "On the running of the gravitational constant". In: *Phys. Rev. D* 85 (2012), p. 104016. DOI: 10.1103/PhysRevD.85.104016. arXiv: 1111.2875 [hep-th].

[132] John F. Donoghue. "Do Λ_{CC} and G run?" In: *64th Cracow School of Theoretical Physics From the UltraViolet to the InfraRed: A panorama of modern gravitational physics*. Dec. 2024. arXiv: 2412.08773 [hep-th].

[133] Diego Buccio et al. "Physical Running of Couplings in Quadratic Gravity". In: *Phys. Rev. Lett.* 133.2 (2024), p. 021604. DOI: 10.1103/PhysRevLett.133.021604. arXiv: 2403.02397 [hep-th].

[134] David G. Boulware and Stanley Deser. "String Generated Gravity Models". In: *Phys. Rev. Lett.* 55 (1985), p. 2656. DOI: 10.1103/PhysRevLett.55.2656.

[135] Steven B. Giddings and Andrew Strominger. "Quantum theories of dilaton gravity". In: *Physical Review D* 47.6 (Mar. 1993), 2454–2460. ISSN: 0556-2821. DOI: 10.1103/physrevd.47.2454. URL: http://dx.doi.org/10.1103/PhysRevD.47.2454.

[136] Valerio Faraoni. *Cosmology in scalar tensor gravity*. Springer Dordrecht, 2004. ISBN: 978-1-4020-1988-3. DOI: 10.1007/978-1-4020-1989-0.

[137] Yasunori Fujii and Kei-ichi Maeda. *The Scalar-Tensor Theory of Gravitation*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003.

[138] S. Sonego and V. Faraoni. "Coupling to the curvature for a scalar field from the equivalence principle". In: *Class. Quant. Grav.* 10 (1993), pp. 1185–1187. DOI: 10.1088/0264-9381/10/6/015.

[139] Gerard Kennedy, Raymond Critchley, and J. S. Dowker. "Finite Temperature Field Theory with Boundaries: Stress Tensor and Surface Action Renormalization". In: *Annals Phys.* 125 (1980), p. 346. DOI: 10.1016/0003-4916(80)90138-4.

[140] Robert A McNees. *Conventions, Definitions, Identities, and Formulas*. http://jacobi.luc.edu/Useful.html#Variations.

[141] A. D. Barvinsky and S. N. Solodukhin. "Nonminimal coupling, boundary terms and renormalization of the Einstein-Hilbert action and black hole entropy". In: *Nucl. Phys. B* 479 (1996), pp. 305–318. DOI: 10.1016/0550-3213(96)00438-5. arXiv: gr-qc/9512047.

[142] Daniel Becker and Martin Reuter. "Running boundary actions, Asymptotic Safety, and black hole thermodynamics". In: *JHEP* 07 (2012), p. 172. DOI: 10.1007/JHEP07(2012)172. arXiv: 1205.3583 [hep-th].

[143] Leonard Susskind and John Uglum. "Black hole entropy in canonical quantum gravity and superstring theory". In: *Phys. Rev. D* 50 (1994), pp. 2700–2711. DOI: 10.1103/PhysRevD.50.2700. arXiv: hep-th/9401070.

[144] Dmitri V. Fursaev and Sergei N. Solodukhin. "On one loop renormalization of black hole entropy". In: *Phys. Lett. B* 365 (1996), pp. 51–55. DOI: 10.1016/0370-2693(95)01290-7. arXiv: hep-th/9412020.

[145] Valeri P. Frolov, D. V. Fursaev, and A. I. Zelnikov. "Black hole entropy: Off-shell versus on-shell". In: *Phys. Rev. D* 54 (1996), pp. 2711–2731. DOI: 10.1103/PhysRevD.54.2711. arXiv: hep-th/9512184.

[146] Valeri P. Frolov. "Black hole entropy and physics at Planckian scales". In: *NATO Sci. Ser. C* 476 (1996). Ed. by N. Sanchez and A. Zichichi, pp. 187–207. arXiv: hep-th/9510156.

[147] Finn Larsen and Frank Wilczek. "Renormalization of black hole entropy and of the gravitational coupling constant". In: *Nucl. Phys. B* 458 (1996), pp. 249–266. DOI: 10.1016/0550-3213(95)00548-X. arXiv: hep-th/9506066.

[148] Valeri P. Frolov, D. V. Fursaev, and A. I. Zelnikov. "Statistical origin of black hole entropy in induced gravity". In: *Nucl. Phys. B* 486 (1997), pp. 339–352. DOI: 10.1016/S0550-3213(96)00678-5. arXiv: hep-th/9607104.

[149] Valeri P. Frolov and D. V. Fursaev. "Thermal fields, entropy, and black holes". In: *Class. Quant. Grav.* 15 (1998), pp. 2041–2074. DOI: 10.1088/0264-9381/15/8/001. arXiv: hep-th/9802010.

[150] Joshua H. Cooperman and Markus A. Luty. "Renormalization of Entanglement Entropy and the Gravitational Effective Action". In: *JHEP* 12 (2014), p. 045. DOI: 10.1007/JHEP12(2014)045. arXiv: 1302.1878 [hep-th].

[151] Kevin Falls. "Physical renormalization schemes and asymptotic safety in quantum gravity". In: *Phys. Rev. D* 96.12 (2017), p. 126016. DOI: 10.1103/PhysRevD.96.126016. arXiv: 1702.03577 [hep-th].

[152] J. S. Dowker and Gerard Kennedy. "Finite Temperature and Boundary Effects in Static Space-Times". In: *J. Phys. A* 11 (1978), p. 895. DOI: 10.1088/0305-4470/11/5/020.

[153] Dmitri Fursaev. "Conformal anomalies of CFT's with boundaries". In: *JHEP* 12 (2015), p. 112. DOI: 10.1007/JHEP12(2015)112. arXiv: 1510.01427 [hep-th].

[154] D. V. Vassilevich. "Heat kernel expansion: User's manual". In: *Phys. Rept.* 388 (2003), pp. 279–360. DOI: 10.1016/j.physrep.2003.09.002. arXiv: hep-th/0306138.

[155] Ian G. Moss and Pedro J. Silva. "BRST invariant boundary conditions for gauge theories". In: *Phys. Rev. D* 55 (1997), pp. 1072–1078. DOI: 10.1103/PhysRevD.55.1072. arXiv: gr-qc/9610023.

[156] Hugh Luckock. "Mixed boundary conditions in quantum field theory". In: *J. Math. Phys.* 32 (1991), pp. 1755–1766. DOI: 10.1063/1.529238.

[157] Giampiero Esposito, Alexander Yu. Kamenshchik, and Giuseppe Pollifrone. *Euclidean quantum gravity on manifolds with boundary*. Vol. 85. Germany: Springer, Dordrecht, 1997. ISBN: 978-94-010-6452-1, 978-94-011-5806-0. DOI: 10.1007/978-94-011-5806-0.

[158] Ian G. Moss. "BRST-invariant boundary conditions and strong ellipticity". In: *Phys. Rev. D* 88.10 (2013), p. 104039. DOI: 10.1103/PhysRevD.88.104039. arXiv: 1301.0717 [hep-th].

[159] Alfio Bonanno et al. "Critical reflections on asymptotically safe gravity". In: (Apr. 2020). _eprint: 2004.06810. URL: http://arxiv.org/abs/2004.06810.

[160] Geoffrey Compere and Donald Marolf. "Setting the boundary free in AdS/CFT". In: *Class. Quant. Grav.* 25 (2008). _eprint: 0805.1902, p. 195014. DOI: 10.1088/0264-9381/25/19/195014.

[161] Geoffrey Compere and Fran Dehouck. "Relaxing the Parity Conditions of Asymptotically Flat Gravity". In: *Class. Quant. Grav.* 28 (2011). _eprint: 1106.4045, p. 245016. DOI: 10.1088/0264-9381/28/24/245016.

[162] Tomas Andrade et al. "On the stability of gravity with Dirichlet walls". In: *Class. Quant. Grav.* 32.23 (2015). _eprint: 1504.07580, p. 235006. DOI: 10.1088/0264-9381/32/23/235006.

[163] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, July 2019. ISBN: 978-0-8053-8732-2 978-1-108-48839-6 978-1-108-77555-7. DOI: 10.1017/9781108770385.

[164] Steven B. Giddings. "Gravitational dressing, soft charges, and perturbative gravitational splitting". In: *Phys. Rev. D* 100.12 (2019), p. 126001. DOI: 10.1103/PhysRevD.100.126001. arXiv: 1903.06160 [hep-th].

[165] Christophe Goeller, Philipp A. Hoehn, and Josh Kirklin. "Diffeomorphism-invariant observables and dynamical frames in gravity: reconciling bulk locality with general covariance". In: (June 2022). arXiv: 2206.01193 [hep-th].

[166] M Nakahara. *Geometry, topology and physics*. Publication Title: Boca Raton, USA Taylor Fr. 573 p. 2003.

[167] F. W. Hehl et al. "General Relativity with Spin and Torsion: Foundations and Prospects". In: *Rev. Mod. Phys.* 48 (1976), pp. 393–416. DOI: 10.1103/RevModPhys.48.393.

[168] A. Trautman. "Fiber bundles associated with space-time". In: *Rept. Math. Phys.* 1 (1970), pp. 29–62. DOI: 10.1016/0034-4877(70)90003-0.

[169] Luca Ciambelli and Robert G. Leigh. "Lie Algebroids and the Geometry of Off-shell BRST". en. In: *Nuclear Physics B* 972 (Nov. 2021). arXiv:2101.03974 [hep-th], p. 115553. ISSN: 05503213. DOI: 10.1016/j.nuclphysb.2021.115553. URL: http://arxiv.org/abs/2101.03974 (visited on 06/19/2025).

[170] S. A. Hojman, K. Kuchar, and C. Teitelboim. "Geometrodynamics Regained". In: *Annals Phys.* 96 (1976), pp. 88–135. DOI: 10.1016/0003-4916(76)90112-3.

[171] Glenn Barnich and Cedric Troessaert. "Aspects of the BMS/CFT correspondence". In: *JHEP* 05 (2010), p. 062. DOI: 10.1007/JHEP05(2010)062. arXiv: 1001.1541 [hep-th].

[172] Glenn Barnich and Cedric Troessaert. "BMS charge algebra". In: *JHEP* 12 (2011), p. 105. DOI: 10.1007/JHEP12(2011)105. arXiv: 1106.0213 [hep-th].

[173] J Katz. "A note on Komar's anomalous factor". In: *Classical and Quantum Gravity* 2.3 (May 1985), p. 423. DOI: 10.1088/0264-9381/2/3/018. URL: https://dx.doi.org/10.1088/0264-9381/2/3/018.

[174] R. Sachs. "Asymptotic Symmetries in Gravitational Theory". en. In: *Physical Review* 128.6 (Dec. 1962), pp. 2851–2864. ISSN: 0031-899X. DOI: 10.1103/PhysRev.128.2851. URL: https://link.aps.org/doi/10.1103/PhysRev.128.2851 (visited on 05/30/2025).

[175] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems". In: *Proc. Roy. Soc. Lond.* A269 (1962), p. 21. DOI: 10.1098/rspa.1962.0161.

[176] Glenn Barnich and Friedemann Brandt. "Covariant theory of asymptotic symmetries, conservation laws and central charges". In: *Nucl. Phys. B* 633 (2002). _eprint: hep-th/0111246, pp. 3–82. DOI: 10.1016/S0550-3213(02)00251-1.

[177] Glenn Barnich and Cedric Troessaert. "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited". In: *Phys. Rev. Lett.* 105 (2010), p. 111103. DOI: 10.1103/PhysRevLett.105.111103. arXiv: 0909.2617 [gr-qc].

[178] Glenn Barnich and Cedric Troessaert. "Supertranslations call for superrotations". In: *PoS* CNCFG (2010). [Ann. U. Craiova Phys.21,S11(2011)], p. 010. arXiv: 1102.4632 [gr-qc].

[179] Miguel Campiglia and Alok Laddha. "New symmetries for the Gravitational S-matrix". In: *JHEP* 04 (2015), p. 076. DOI: 10.1007/JHEP04(2015)076. arXiv: 1502.02318 [hep-th].

[180] Andrew Strominger. *Lectures on the Infrared Structure of Gravity and Gauge Theory*. _eprint: 1703.05448. Mar. 2017. ISBN: 978-0-691-17973-5.

[181] Marc Henneaux and Cédric Troessaert. "BMS Group at Spatial Infinity: the Hamiltonian (ADM) approach". In: *Journal of High Energy Physics* 2018.3 (Mar. 2018). arXiv:1801.03718 [gr-qc], p. 147. ISSN: 1029-8479. DOI: 10.1007/JHEP03(2018)147. URL: http://arxiv.org/abs/1801.03718 (visited on 02/18/2025).

[182] Miguel Campiglia and Javier Peraza. "Generalized BMS charge algebra". In: *Phys. Rev. D* 101.10 (2020), p. 104039. DOI: 10.1103/PhysRevD.101.104039. arXiv: 2002.06691 [gr-qc].

[183] Laura Donnay and Romain Ruzziconi. "BMS flux algebra in celestial holography". en. In: *Journal of High Energy Physics* 2021.11 (Nov. 2021), p. 40. ISSN: 1029-8479. DOI: 10.1007/JHEP11(2021)040. URL: https://link.springer.com/10.1007/JHEP11(2021)040 (visited on 05/30/2025).

[184] Henrique Gomes, Florian Hopfmüller, and Aldo Riello. "A unified geometric framework for boundary charges and dressings: non-Abelian theory and matter". In: *Nucl. Phys. B* 941 (2019), pp. 249–315. DOI: 10.1016/j.nuclphysb.2019.02.020. arXiv: 1808.02074 [hep-th].

[185] Glenn Barnich and Geoffrey Compere. "Surface charge algebra in gauge theories and thermodynamic integrability". In: *J. Math. Phys.* 49 (2008). _eprint: 0708.2378, p. 042901. DOI: 10.1063/1.2889721.

[186] Glenn Barnich. "Centrally extended BMS4 Lie algebroid". In: *JHEP* 06 (2017), p. 007. DOI: 10.1007/JHEP06(2017)007. arXiv: 1703.08704 [hep-th].

[187] Antoine Rignon-Bret and Simone Speziale. "General covariance and boundary symmetry algebras". In: *EPL* 149.6 (2025), p. 69002. DOI: 10.1209/0295-5075/adc2cc. arXiv: 2403.00730 [hep-th].

[188] Valerio Faraoni. "Black Hole Entropy in Scalar-Tensor and F(R) Gravity: An Overview". In: *Entropy* 12.5 (2010), pp. 1246–1263. ISSN: 1099-4300. DOI: 10.3390/e12051246. URL: https://www.mdpi.com/1099-4300/12/5/1246.

[189] K. Hajian et al. "On Black Hole Temperature in Horndeski Gravity". In: *Phys. Lett. B* 812 (2021). _eprint: 2005.12985, p. 136002. DOI: 10.1016/j.physletb.2020.136002.

[190] Lavinia Heisenberg, Simon Kuhn, and Laurens Walleghem. "Wald's entropy in Coincident General Relativity". In: *Class. Quant. Grav.* 39.23 (2022). _eprint: 2203.13914, p. 235002. DOI: 10.1088/1361-6382/ac987d.

[191] Masato Minamitsuji and Kei-ichi Maeda. "Black hole thermodynamics in Horndeski theories". In: *Phys. Rev. D* 108.8 (2023). _eprint: 2308.01082, p. 084061. DOI: 10.1103/PhysRevD.108.084061.

[192] Costantino Pacilio and Stefano Liberati. "First law of black holes with a universal horizon". In: *Phys. Rev. D* 96.10 (2017). _eprint: 1709.05802, p. 104060. DOI: 10.1103/PhysRevD.96.104060.

[193] Rong-Gen Cai. "A Note on thermodynamics of black holes in Lovelock gravity". In: *Phys. Lett. B* 582 (2004). _eprint: hep-th/0311240, pp. 237–242. DOI: 10.1016/j.physletb.2004.01.015.

[194] David Kubiznak, Robert B. Mann, and Mae Teo. "Black hole chemistry: thermodynamics with Lambda". In: *Class. Quant. Grav.* 34.6 (2017). _eprint: 1608.06147, p. 063001. DOI: 10.1088/1361-6382/aa5c69.

[195] Vivek Iyer and Robert M. Wald. "A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes". In: *Phys. Rev. D* 52 (1995). _eprint: gr-qc/9503052, pp. 4430–4439. DOI: 10.1103/PhysRevD.52.4430.

[196] Robert M Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.

[197] Robert M. Wald and Andreas Zoupas. "A General definition of 'conserved quantities' in general relativity and other theories of gravity". In: *Phys. Rev. D* 61 (2000). _eprint: gr-qc/9911095, p. 084027. DOI: 10.1103/PhysRevD.61.084027.

[198] Stefan Hollands, Akihiro Ishibashi, and Donald Marolf. "Comparison between various notions of conserved charges in asymptotically AdS-spacetimes". In: *Class. Quant. Grav.* 22 (2005). _eprint: hep-th/0503045, pp. 2881–2920. DOI: 10.1088/0264-9381/22/14/004.

[199] Tomas Andrade and Donald Marolf. "Asymptotic Symmetries from finite boxes". In: *Class. Quant. Grav.* 33.1 (2016). _eprint: 1508.02515, p. 015013. DOI: 10.1088/0264-9381/33/1/015013.

[200] Alexandre Belin, Aitor Lewkowycz, and Gábor Sárosi. "Complexity and the bulk volume, a new York time story". In: *JHEP* 03 (2019). _eprint: 1811.03097, p. 044. DOI: 10.1007/JHEP03(2019)044.

[201] Alexandre Belin, Aitor Lewkowycz, and Gábor Sárosi. "The boundary dual of the bulk symplectic form". In: *Phys. Lett. B* 789 (2019). _eprint: 1806.10144, pp. 71–75. DOI: 10.1016/j.physletb.2018.10.071.

[202] Steven Carlip. "Entropy from conformal field theory at Killing horizons". In: *Class. Quant. Grav.* 16 (1999). _eprint: gr-qc/9906126, pp. 3327–3348. DOI: 10.1088/0264-9381/16/10/322.

[203] Sasha Haco et al. "Black Hole Entropy and Soft Hair". In: *JHEP* 12 (2018). _eprint: 1810.01847, p. 098. DOI: 10.1007/JHEP12(2018)098.

[204] Sasha Haco, Malcolm J. Perry, and Andrew Strominger. "Kerr-Newman Black Hole Entropy and Soft Hair". In: (Feb. 2019). _eprint: 1902.02247.

[205] Aron C. Wall. "A Second Law for Higher Curvature Gravity". In: *Int. J. Mod. Phys. D* 24.12 (2015). _eprint: 1504.08040, p. 1544014. DOI: 10.1142/S0218271815440149.

[206] Aron C. Wall and Zihan Yan. "Linearised Second Law for Higher Curvature Gravity and Non-Minimally Coupled Vector Fields". In: (Feb. 2024). _eprint: 2402.05411.

[207] Manus R. Visser and Zihan Yan. "Properties of Dynamical Black Hole Entropy". In: (Mar. 2024). _eprint: 2403.07140.

[208] Stefan Hollands, Robert M. Wald, and Victor G. Zhang. "The Entropy of Dynamical Black Holes". In: (Feb. 2024). _eprint: 2402.00818.

[209] Giulio Neri and Stefano Liberati. "Covariant phase space analysis of Lanczos-Lovelock gravity with boundaries". In: *JHEP* 06 (2024). _eprint: 2404.16981, p. 136. DOI: 10.1007/JHEP06(2024)136.

[210] Bethan Cropp, Stefano Liberati, and Matt Visser. "Surface gravities for non-Killing horizons". In: *Class. Quant. Grav.* 30 (2013), p. 125001. DOI: 10.1088/0264-9381/30/12/125001. arXiv: 1302.2383 [gr-qc].

[211] Cornelius Lanczos. "A Remarkable property of the Riemann-Christoffel tensor in four dimensions". In: *Annals Math.* 39 (1938), pp. 842–850. DOI: 10.2307/1968467.

[212] D. Lovelock. "The Einstein tensor and its generalizations". In: *J. Math. Phys.* 12 (1971), pp. 498–501. DOI: 10.1063/1.1665613.

[213] Naresh Dadhich, Sushant G. Ghosh, and Sanjay Jhingan. "The Lovelock gravity in the critical spacetime dimension". In: *Phys. Lett. B* 711 (2012). _eprint: 1202.4575, pp. 196–198. DOI: 10.1016/j.physletb.2012.03.084.

[214] Naresh Dadhich. "Characterization of the Lovelock gravity by Bianchi derivative". In: *Pramana* 74 (2010). _eprint: 0802.3034, pp. 875–882. DOI: 10.1007/s12043-010-0080-1.

[215] David Kastor. "The Riemann-Lovelock Curvature Tensor". In: *Class. Quant. Grav.* 29 (2012). _eprint: 1202.5287, p. 155007. DOI: 10.1088/0264-9381/29/15/155007.

[216] Y. Choquet-Bruhat. "The Cauchy Problem for Stringy Gravity". In: *J. Math. Phys.* 29 (1988), pp. 1891–1895. DOI: 10.1063/1.527841.

[217] Nathalie Deruelle and John Madore. "On the quasilinearity of the Einstein-'Gauss-Bonnet' gravity field equations". In: _eprint: gr-qc/0305004. May 2003.

[218] Olivera Miskovic and Rodrigo Olea. "Counterterms in Dimensionally Continued AdS Gravity". In: *JHEP* 10 (2007). _eprint: 0706.4460, p. 028. DOI: 10.1088/1126-6708/2007/10/028.

[219] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. "Gravitation, gauge theories and differential geometry". In: *Physics Reports* 66.6 (1980), pp. 213–393. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(80)90130-1. URL: https://www.sciencedirect.com/science/article/pii/0370157380901301.

[220] Sumanta Chakraborty, Krishnamohan Parattu, and T. Padmanabhan. "A Novel Derivation of the Boundary Term for the Action in Lanczos-Lovelock Gravity". In: *Gen. Rel. Grav.* 49.9 (2017). _eprint: 1703.00624, p. 121. DOI: 10.1007/s10714-017-2289-5.

[221] Pablo A. Cano. "Lovelock action with nonsmooth boundaries". In: *Phys. Rev. D* 97.10 (2018). _eprint: 1803.00172, p. 104048. DOI: 10.1103/PhysRevD.97.104048.

[222] Ted Jacobson and Robert C. Myers. "Black hole entropy and higher curvature interactions". In: *Phys. Rev. Lett.* 70 (1993). _eprint: hep-th/9305016, pp. 3684–3687. DOI: 10.1103/PhysRevLett.70.3684.

[223] Robert C. Myers. "Black holes in higher curvature gravity". In: *Black Holes, Gravitational Radiation and the Universe: Essays in Honor of C.V. Vishveshwara*. Ed. by Bala R. Iyer and B. Bhawal. _eprint: gr-qc/9811042. Nov. 1998, pp. 121–136. DOI: 10.1007/978-94-017-0934-7_8.

[224] Cecilia Garraffo and Gaston Giribet. "The Lovelock Black Holes". In: *Mod. Phys. Lett. A* 23 (2008). _eprint: 0805.3575, pp. 1801–1818. DOI: 10.1142/S0217732308027497.

[225] Stanley Deser and Bayram Tekin. "Shortcuts to high symmetry solutions in gravitational theories". In: *Class. Quant. Grav.* 20 (2003). _eprint: gr-qc/0306114, pp. 4877–4884. DOI: 10.1088/0264-9381/20/22/011.

[226] Stanley Deser and J. Franklin. "Birkhoff for Lovelock redux". In: *Class. Quant. Grav.* 22 (2005). _eprint: gr-qc/0506014, pp. L103–L106. DOI: 10.1088/0264-9381/22/16/L03.

[227] Robin Zegers. "Birkhoff's theorem in Lovelock gravity". In: *J. Math. Phys.* 46 (2005). _eprint: gr-qc/0505016, p. 072502. DOI: 10.1063/1.1960798.

[228] Gregory J. Galloway and Richard Schoen. "A Generalization of Hawking's black hole topology theorem to higher dimensions". In: *Commun. Math. Phys.* 266 (2006). _eprint: gr-qc/0509107, pp. 571–576. DOI: 10.1007/s00220-006-0019-z.

[229] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

[230] Stefan Hollands, Akihiro Ishibashi, and Robert M. Wald. "A Higher dimensional stationary rotating black hole must be axisymmetric". In: *Commun. Math. Phys.* 271 (2007). _eprint: gr-qc/0605106, pp. 699–722. DOI: 10.1007/s00220-007-0216-4.

[231] Roberto Emparan and Harvey S. Reall. "Black Holes in Higher Dimensions". In: *Living Rev. Rel.* 11 (2008). _eprint: 0801.3471, p. 6. DOI: 10.12942/lrr-2008-6.

[232] Hyeong-Chan Kim and Rong-Gen Cai. "Slowly rotating charged Gauss-Bonnet black holes in AdS spaces". In: *Phys. Rev. D* 77.2 (Jan. 2008). Publisher: American Physical Society, p. 024045. DOI: 10.1103/PhysRevD.77.024045. URL: https://link.aps.org/doi/10.1103/PhysRevD.77.024045.

[233] Yves Brihaye and Eugen Radu. "Five-dimensional rotating black holes in Einstein-Gauss-Bonnet theory". In: *Phys. Lett. B* 661 (2008). _eprint: 0801.1021, pp. 167–174. DOI: 10.1016/j.physletb.2008.02.005.

[234] Ryotaku Suzuki and Shinya Tomizawa. "Rotating black holes at large D in Einstein-Gauss-Bonnet theory". In: *Phys. Rev. D* 106.2 (July 2022). Publisher: American Physical Society, p. 024018. DOI: 10.1103/PhysRevD.106.024018. URL: https://link.aps.org/doi/10.1103/PhysRevD.106.024018.

[235] Sanved Kolekar, T. Padmanabhan, and Sudipta Sarkar. "Entropy Increase during Physical Processes for Black Holes in Lanczos-Lovelock Gravity". In: *Phys. Rev. D* 86 (2012). _eprint: 1201.2947, p. 021501. DOI: 10.1103/PhysRevD.86.021501.

[236] Sudipta Sarkar and Aron C. Wall. "Generalized second law at linear order for actions that are functions of Lovelock densities". In: *Phys. Rev. D* 88 (2013). _eprint: 1306.1623, p. 044017. DOI: 10.1103/PhysRevD.88.044017.

[237] David Kubizňák and Marek Liška. "Can Bekenstein's area law prevail in modified theories of gravity?" In: *Phys. Rev. D* 108.12 (2023), p. L121501. DOI: 10.1103/PhysRevD.108.L121501. arXiv: 2307.16201 [hep-th].

[238] Harvey Reall, Norihiro Tanahashi, and Benson Way. "Causality and Hyperbolicity of Lovelock Theories". In: *Class. Quant. Grav.* 31 (2014). _eprint: 1406.3379, p. 205005. DOI: 10.1088/0264-9381/31/20/205005.

[239] Ram Brustein and Yotam Sherf. "Causality violations in Lovelock theories". In: *Phys. Rev. D* 97.8 (Apr. 2018). Publisher: American Physical Society, p. 084019. DOI: 10.1103/PhysRevD.97.084019.

[240] Larry Smarr. "Mass formula for Kerr black holes". In: *Phys. Rev. Lett.* 30 (1973), pp. 71–73. DOI: 10.1103/PhysRevLett.30.71.

[241] G. W. Gibbons, M. J. Perry, and C. N. Pope. "The First law of thermodynamics for Kerr-anti-de Sitter black holes". In: *Class. Quant. Grav.* 22 (2005). _eprint: hep-th/0408217, pp. 1503–1526. DOI: 10.1088/0264-9381/22/9/002.

[242] David Kastor, Sourya Ray, and Jennie Traschen. "Smarr Formula and an Extended First Law for Lovelock Gravity". In: *Class. Quant. Grav.* 27 (2010). _eprint: 1005.5053, p. 235014. DOI: 10.1088/0264-9381/27/23/235014.

[243] Marco M. Caldarelli, Guido Cognola, and Dietmar Klemm. "Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories". In: *Class. Quant. Grav.* 17 (2000). _eprint: hep-th/9908022, pp. 399–420. DOI: 10.1088/0264-9381/17/2/310.

[244] D. A. Rasheed. "Nonlinear electrodynamics: Zeroth and first laws of black hole mechanics". In: (Feb. 1997). _eprint: hep-th/9702087.

[245] David Kastor. "Komar Integrals in Higher (and Lower) Derivative Gravity". In: *Class. Quant. Grav.* 25 (2008). _eprint: 0804.1832, p. 175007. DOI: 10.1088/0264-9381/25/17/175007.

[246] David Kastor, Sourya Ray, and Jennie Traschen. "Enthalpy and the Mechanics of AdS Black Holes". In: *Class. Quant. Grav.* 26 (2009). _eprint: 0904.2765, p. 195011. DOI: 10.1088/0264-9381/26/19/195011.

[247] Stefano Liberati and Costantino Pacilio. "Smarr Formula for Lovelock Black Holes: a Lagrangian approach". In: *Phys. Rev. D* 93.8 (2016). _eprint: 1511.05446, p. 084044. DOI: 10.1103/PhysRevD.93.084044.

[248] C. Brans and R. H. Dicke. "Mach's Principle and a Relativistic Theory of Gravitation". In: *Phys. Rev.* 124 (3 Nov. 1961), pp. 925–935. DOI: 10.1103/PhysRev.124.925. URL: https://link.aps.org/doi/10.1103/PhysRev.124.925.

[249] C. Teitelboim. "The Cosmological Constant as a Thermodynamic Black Hole Parameter". In: *Phys. Lett. B* 158 (1985), pp. 293–297. DOI: 10.1016/0370-2693(85)91186-4.

[250] J. M. Overduin and P. S. Wesson. "Kaluza-Klein gravity". In: *Phys. Rept.* 283 (1997), pp. 303–380. DOI: 10.1016/S0370-1573(96)00046-4. arXiv: gr-qc/9805018.

[251] Irene Bredberg et al. "Wilsonian Approach to Fluid/Gravity Duality". In: *JHEP* 03 (2011), p. 141. DOI: 10.1007/JHEP03(2011)141. arXiv: 1006.1902 [hep-th].

[252] Shuang Wang. "Thermodynamics of Schwarzschild de Sitter spacetimes: Variable cosmological constant". In: (June 2006). _eprint: gr-qc/0606109.

[253] James T. Wheeler. "Symmetric Solutions to the Gauss-Bonnet Extended Einstein Equations". In: *Nucl. Phys. B* 268 (1986), pp. 737–746. DOI: 10.1016/0550-3213(86)90268-3.

[254] Brayden R. Hull and Robert B. Mann. "Thermodynamics of exotic black holes in Lovelock gravity". In: *Phys. Rev. D* 104.8 (2021). _eprint: 2102.05282, p. 084032. DOI: 10.1103/PhysRevD.104.084032.

[255] G.W. Gibbons and S.W. Hawking. *Euclidean Quantum Gravity*. G - Reference,Information and Interdisciplinary Subjects Series. World Scientific, 1993. URL: https://books.google.it/books?id=iVQ6zcLyxy4C.

[256] Harry W. Braden et al. "Charged black hole in a grand canonical ensemble". In: *Phys. Rev. D* 42.10 (Nov. 1990). Publisher: American Physical Society, pp. 3376–3385. DOI: 10.1103/PhysRevD.42.3376. URL: https://link.aps.org/doi/10.1103/PhysRevD.42.3376.

[257] Brian P. Dolan. "Pressure and volume in the first law of black hole thermodynamics". In: *Class. Quant. Grav.* 28 (2011), p. 235017. DOI: 10.1088/0264-9381/28/23/235017. arXiv: 1106.6260 [gr-qc].

[258] David Kubiznak and Robert B. Mann. "P-V criticality of charged AdS black holes". In: *JHEP* 07 (2012), p. 033. DOI: 10.1007/JHEP07(2012)033. arXiv: 1205.0559 [hep-th].

[259] S Carlip. "Black Hole Entropy from BMS Symmetry at the Horizon". In: *Phys. Rev. Lett.* 120.10 (2018). _eprint: 1702.04439, p. 101301. DOI: 10.1103/PhysRevLett.120.101301.

[260] Andrew Strominger. "Black hole entropy from near horizon microstates". In: *JHEP* 02 (1998), p. 009. DOI: 10.1088/1126-6708/1998/02/009. arXiv: hep-th/9712251.

[261] Manvir Grewal, Y. T. Albert Law, and Klaas Parmentier. "Black hole horizon edge partition functions". In: *JHEP* 06 (2023), p. 025. DOI: 10.1007/JHEP06(2023)025. arXiv: 2211.16644 [hep-th].

[262] Ana-Maria Raclariu. "Lectures on Celestial Holography". In: (July 2021). arXiv: 2107.02075 [hep-th].

[263] William Donnelly. "Entanglement entropy and nonabelian gauge symmetry". In: *Classical and Quantum Gravity* 31.21 (Nov. 2014). arXiv:1406.7304 [hep-th], p. 214003. ISSN: 0264-9381, 1361-6382. DOI: 10.1088/0264-9381/31/21/214003. URL: http://arxiv.org/abs/1406.7304 (visited on 06/12/2025).

[264] Antony J. Speranza. "Local phase space and edge modes for diffeomorphism-invariant theories". en. In: *Journal of High Energy Physics* 2018.2 (Feb. 2018), p. 21. ISSN: 1029-8479. DOI: 10.1007/JHEP02(2018)021. URL: http://link.springer.com/10.1007/JHEP02(2018)021 (visited on 05/30/2025).

[265] Valentin Benedetti and Horacio Casini. "Entanglement entropy of linearized gravitons in a sphere". In: *Phys. Rev. D* 101.4 (2020), p. 045004. DOI: 10.1103/PhysRevD.101.045004. arXiv: 1908.01800 [hep-th].

[266] Justin R. David and Jyotirmoy Mukherjee. "Entanglement entropy of gravitational edge modes". In: *JHEP* 08 (2022), p. 065. DOI: 10.1007/JHEP08(2022)065. arXiv: 2201.06043 [hep-th].

[267] Philipp A. Hoehn, Isha Kotecha, and Fabio M. Mele. "Quantum Frame Relativity of Subsystems, Correlations and Thermodynamics". In: (Aug. 2023). arXiv: 2308.09131 [quant-ph].

[268] Goncalo Araujo-Regado, Philipp A. Hoehn, and Francesco Sartini. "Relational entanglement entropies and quantum reference frames in gauge theories". In: (June 2025). arXiv: 2506.23459 [hep-th].

[269] Sylvain Carrozza and Philipp A. Hoehn. "Edge modes as reference frames and boundary actions from post-selection". In: *JHEP* 02 (2022), p. 172. DOI: 10.1007/JHEP02(2022)172. arXiv: 2109.06184 [hep-th].

[270] Sylvain Carrozza, Stefan Eccles, and Philipp A. Hoehn. "Edge modes as dynamical frames: charges from post-selection in generally covariant theories". In: *SciPost Phys.* 17.2 (2024), p. 048. DOI: 10.21468/SciPostPhys.17.2.048. arXiv: 2205.00913 [hep-th].

[271] Goncalo Araujo-Regado et al. "Soft edges: the many links between soft and edge modes". In: *JHEP* 07 (2025), p. 180. DOI: 10.1007/JHEP07(2025)180. arXiv: 2412.14548 [hep-th].

[272] Kristina Giesel, Viktoria Kabel, and Wolfgang Wieland. "Linking Edge Modes and Geometrical Clocks in Linearized Gravity". In: (Oct. 2024). arXiv: 2410.17339 [gr-qc].

[273] Giulio Neri and Ludovic Varrin. "Orbit method for Quantum Corner Symmetries". In: (July 2025). arXiv: 2507.10683 [hep-th].

[274] Robert M. Wald. *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago Lectures in Physics. Chicago, IL: University of Chicago Press, 1995. ISBN: 978-0-226-87027-4.

[275] Aldo Riello. "Edge modes without edge modes". In: (Apr. 2021). arXiv: 2104.10182 [hep-th].

[276] Jennifer Lin and Djordje Radičević. "Comments on defining entanglement entropy". In: *Nucl. Phys. B* 958 (2020), p. 115118. DOI: 10.1016/j.nuclphysb.2020.115118. arXiv: 1808.05939 [hep-th].

[277] R. Haag. *Local quantum physics: Fields, particles, algebras*. Springer, 1992.

[278] Jakob Yngvason. "The Role of type III factors in quantum field theory". In: *Rept. Math. Phys.* 55 (2005), pp. 135–147. DOI: 10.1016/S0034-4877(05)80009-6. arXiv: math-ph/0411058.

[279] Steven B. Giddings, Donald Marolf, and James B. Hartle. "Observables in effective gravity". In: *Phys. Rev. D* 74 (2006), p. 064018. DOI: 10.1103/PhysRevD.74.064018. arXiv: hep-th/0512200.

[280] Steven B. Giddings and Sean Weinberg. "Gauge-invariant observables in gravity and electromagnetism: Black hole backgrounds and null dressings". en. In: *Physical Review D* 102.2 (July 2020), p. 026010. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.102.026010. URL: https://link.aps.org/doi/10.1103/PhysRevD.102.026010 (visited on 05/30/2025).

[281] Rafael D. Sorkin. "1983 paper on entanglement entropy: "On the Entropy of the Vacuum outside a Horizon"". In: *10th International Conference on General Relativity and Gravitation*. Vol. 2. 1984, pp. 734–736. arXiv: 1402.3589 [gr-qc].

[282] Mark Srednicki. "Entropy and area". In: *Phys. Rev. Lett.* 71 (1993), pp. 666–669. DOI: 10.1103/PhysRevLett.71.666. arXiv: hep-th/9303048.

[283] Sergei N. Solodukhin. "The Conical singularity and quantum corrections to entropy of black hole". In: *Phys. Rev. D* 51 (1995), pp. 609–617. DOI: 10.1103/PhysRevD.51.609. arXiv: hep-th/9407001.

[284] Netta Engelhardt and Aron C. Wall. "Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime". In: *JHEP* 01 (2015), p. 073. DOI: 10.1007/JHEP01(2015)073. arXiv: 1408.3203 [hep-th].

[285] Raphael Bousso and Geoff Penington. "Entanglement wedges for gravitating regions". In: *Phys. Rev. D* 107 (8 Apr. 2023), p. 086002. DOI: 10.1103/PhysRevD.107.086002. URL: https://link.aps.org/doi/10.1103/PhysRevD.107.086002.

[286] Raphael Bousso and Geoff Penington. "Holograms in our world". In: *Phys. Rev. D* 108 (4 Aug. 2023), p. 046007. DOI: 10.1103/PhysRevD.108.046007. URL: https://link.aps.org/doi/10.1103/PhysRevD.108.046007.

[287] Vijay Balasubramanian and Charlie Cummings. *The entropy of finite gravitating regions*. arXiv:2312.08434 [hep-th]. Jan. 2024. DOI: 10.48550/arXiv.2312.08434. URL: http://arxiv.org/abs/2312.08434 (visited on 01/12/2025).

[288] P. V. Buividovich and M. I. Polikarpov. "Entanglement entropy in gauge theories and the holographic principle for electric strings". en. In: *Physics Letters B* 670.2 (Dec. 2008), pp. 141–145. ISSN: 03702693. DOI: 10.1016/j.physletb.2008.10.032. URL: https://linkinghub.elsevier.com/retrieve/pii/S0370269308012987 (visited on 05/30/2025).

[289] P. V. Buividovich and M. I. Polikarpov. "Entanglement entropy in lattice gauge theories". In: *Journal of Physics A: Mathematical and Theoretical* 42.30 (July 2009), p. 304005. ISSN: 1751-8113, 1751-8121. DOI: 10.1088/1751-8113/42/30/304005. URL: https://iopscience.iop.org/article/10.1088/1751-8113/42/30/304005 (visited on 05/30/2025).

[290] N. Jacobson. *Basic Algebra II: Second Edition*. Basic Algebra. Dover Publications, 2009. ISBN: 9780486471877. URL: https://books.google.com.bz/books?id=4WU_AwAAQBAJ.

[291] William Donnelly et al. "Gravitational edge modes, coadjoint orbits, and hydrodynamics". en. In: *Journal of High Energy Physics* 2021.9 (Sept. 2021), p. 8. ISSN: 1029-8479. DOI: 10.1007/JHEP09(2021)008. URL: https://link.springer.com/10.1007/JHEP09(2021)008 (visited on 05/30/2025).

[292] Luca Ciambelli and Robert G. Leigh. "Isolated surfaces and symmetries of gravity". en. In: *Physical Review D* 104.4 (Aug. 2021), p. 046005. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.104.046005. URL: https://link.aps.org/doi/10.1103/PhysRevD.104.046005 (visited on 05/30/2025).

[293] Luca Ciambelli, Robert G. Leigh, and Pin-Chun Pai. "Embeddings and Integrable Charges for Extended Corner Symmetry". In: *Physical Review Letters* 128.17 (Apr. 2022). arXiv:2111.13181 [hep-th], p. 171302. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.128.171302. URL: http://arxiv.org/abs/2111.13181 (visited on 03/29/2025).

[294] Marc Geiller and Puttarak Jai-akson. "Extended actions, dynamics of edge modes, and entanglement entropy". en. In: *Journal of High Energy Physics* 2020.9 (Sept. 2020), p. 134. ISSN: 1029-8479. DOI: 10.1007/JHEP09(2020)134. URL: https://link.springer.com/10.1007/JHEP09(2020)134 (visited on 05/30/2025).

[295] Laurent Freidel, Marc Geiller, and Wolfgang Wieland. "Corner Symmetry and Quantum Geometry". In: *Handbook of Quantum Gravity*. Ed. by Cosimo Bambi, Leonardo Modesto, and Ilya Shapiro. 2024, pp. 1–36. arXiv: 2302.12799 [hep-th].

[296] Luca Ciambelli and Robert G. Leigh. "Universal corner symmetry and the orbit method for gravity". In: *Nuclear Physics B* 986 (Jan. 2023), p. 116053. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2022.116053. URL: https://www.sciencedirect.com/science/article/pii/S0550321322004047 (visited on 12/03/2024).

[297] Simon Langenscheidt and Daniele Oriti. "New edge modes and corner charges for first-order symmetries of 4D gravity". In: *Class. Quant. Grav.* 42.7 (2025), p. 075010. DOI: 10.1088/1361-6382/adbfee. arXiv: 2408.01809 [gr-qc].

[298] Luca Ciambelli, Jerzy Kowalski-Glikman, and Ludovic Varrin. "Quantum corner symmetry: Representations and gluing". In: *Phys. Lett. B* 866 (2025), p. 139544. DOI: 10.1016/j.physletb.2025.139544. arXiv: 2406.07101 [hep-th].

[299] Antony J. Speranza. "Geometrical tools for embedding fields, submanifolds, and foliations". In: (Apr. 2019). arXiv: 1904.08012 [gr-qc].

[300] Luca Ciambelli et al. "Carroll Structures, Null Geometry and Conformal Isometries". In: *Phys. Rev. D* 100.4 (2019), p. 046010. DOI: 10.1103/PhysRevD.100.046010. arXiv: 1905.02221 [hep-th].

[301] Wen-Bin Liu and Jiang Long. "Symmetry group at future null infinity: Scalar theory". In: *Phys. Rev. D* 107.12 (2023), p. 126002. DOI: 10.1103/PhysRevD.107.126002. arXiv: 2210.00516 [hep-th].

[302] Wen-Bin Liu and Jiang Long. "Symmetry group at future null infinity II: Vector theory". In: *JHEP* 07 (2023), p. 152. DOI: 10.1007/JHEP07(2023)152. arXiv: 2304.08347 [hep-th].

[303] Wen-Bin Liu and Jiang Long. "Symmetry group at future null infinity III: Gravitational theory". In: *JHEP* 10 (2023), p. 117. DOI: 10.1007/JHEP10(2023)117. arXiv: 2307.01068 [hep-th].

[304] Gunnar Randers. "On an Asymmetrical Metric in the Four-Space of General Relativity". In: *Physical Review* 59.2 (Jan. 1941). Publisher: American Physical Society, pp. 195–199. DOI: 10.1103/PhysRev.59.195. URL: https://link.aps.org/doi/10.1103/PhysRev.59.195 (visited on 06/13/2025).

[305] A. Papapetrou. "Champs gravitationnels stationnaires à symétrie axiale". fr. In: *Annales de l'institut Henri Poincaré. Section A, Physique Théorique* 4.2 (1966), pp. 83–105. ISSN: 2400-4863. URL: https://www.numdam.org/item/AIHPA_1966__4_2_83_0/ (visited on 06/13/2025).

[306] Luca Ciambelli. "From Asymptotic Symmetries to the Corner Proposal". In: *PoS* Modave2022 (2023), p. 002. DOI: 10.22323/1.435.0002. arXiv: 2212.13644 [hep-th].

[307] Luca Ciambelli, Jerzy Kowalski-Glikman, and Ludovic Varrin. *Quantum Corner Symmetry: Representations and Gluing*. arXiv:2406.07101 [hep-th]. June 2024. DOI: 10.48550/arXiv.2406.07101. URL: http://arxiv.org/abs/2406.07101 (visited on 01/30/2025).

[308] Luca Ciambelli, Jerzy Kowalski-Glikman, and Ludovic Varrin. "Entanglement Entropy of Quantum Corners". In: (July 2025). arXiv: 2507.16800 [hep-th].

[309] Jean-Marie Souriau. "Matière parfaite en relativité générale". fr. In: *Séminaire Janet. Mécanique analytique et mécanique céleste* 3 (1959-1960). talk:7, pp. 1–11. URL: https://www.numdam.org/item/SJ_1959-1960__3__A7_0/.

[310] A. A. Kirillov. "Unitary Representations of Nilpotent Lie Groups". en. In: *Russian Mathematical Surveys* 17.4 (Aug. 1962). Publisher: IOP Publishing, p. 53. ISSN: 0036-0279. DOI: 10.1070/RM1962v017n04ABEH004118. URL: https://iopscience.iop.org/article/10.1070/RM1962v017n04ABEH004118/meta (visited on 06/20/2025).

[311] J.M. Souriau. *Structure des systèmes dynamiques: maîtrises de mathématiques*. Collection Dunod Université. Dunod, 1970. ISBN: 9782040107529. URL: https://books.google.ch/books?id=cE7vAAAAMAAJ.

[312] Bertram Kostant. "Quantization and unitary representations". In: *Lectures in Modern Analysis and Applications III*. Ed. by C. T. Taam. Berlin, Heidelberg: Springer Berlin Heidelberg, 1970, pp. 87–208. ISBN: 978-3-540-36417-7.

[313] A. A. Kirillov. "Geometric Quantization". en. In: *Dynamical Systems IV: Symplectic Geometry and its Applications*. Ed. by V. I. Arnol'd and S. P. Novikov. Berlin, Heidelberg: Springer, 1990, pp. 137–172. ISBN: 978-3-662-06793-2. DOI: 10.1007/978-3-662-06793-2_2. URL: https://doi.org/10.1007/978-3-662-06793-2_2 (visited on 12/17/2024).

[314] A. A. Kirillov. *Lectures on the Orbit Method*. Vol. 64. Graduate Studies in Mathematics. Softcover: 978-1-4704-7999-2; eBook: 978-1-4704-1799-4. Providence, RI: American Mathematical Society, 2004, p. 408. ISBN: 978-0-8218-3530-2.

[315] Eugene P. Wigner. "On Unitary Representations of the Inhomogeneous Lorentz Group". In: *Annals Math.* 40 (1939). Ed. by Y. S. Kim and W. W. Zachary, pp. 149–204. DOI: 10.2307/1968551.

[316] V. Bargmann. "On Unitary Ray Representations of Continuous Groups". In: *Annals of Mathematics* 59.1 (1954), pp. 1–46. ISSN: 0003-486X. DOI: 10.2307/1969831. URL: https://www.jstor.org/stable/1969831 (visited on 03/29/2025).

[317] George W. Mackey. "Unitary representations of group extensions. I". In: *Acta Mathematica* 99.none (Jan. 1958). Publisher: Institut Mittag-Leffler, pp. 265–311. ISSN: 0001-5962, 1871-2509. DOI: 10.1007/BF02392428. URL: https://projecteuclid.org/journals/acta-mathematica/volume-99/issue-none/Unitary-representations-of-group-extensions-I/10.1007/BF02392428.full (visited on 03/29/2025).

[318] Ludovic Varrin. "Physical representations of corner symmetries". In: *Phys. Rev. D* 111.8 (2025), p. 086003. DOI: 10.1103/PhysRevD.111.086003. arXiv: 2409.10624 [hep-th].

[319] L. Auslander and B. Kostant. *Polarization and Unitary Representations of Solvable Lie Groups*. URL: https://link.springer.com/article/10.1007/bf01389744 (visited on 06/20/2025).

[320] Michel Duflo. "Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple". In: *Annals of Mathematics* 105.1 (1977). Publisher: [Annals of Mathematics, Trustees of Princeton University on Behalf of the Annals of Mathematics, Mathematics Department, Princeton University], pp. 107–120. ISSN: 0003-486X. DOI: 10.2307/1971027. URL: https://www.jstor.org/stable/1971027 (visited on 06/20/2025).

[321] A.W. Knapp. *Representation Theory of Semisimple Groups: An Overview Based on Examples*. Princeton landmarks in mathematics and physics. Princeton University Press, 2001. ISBN: 9780691090894. URL: https://books.google.it/books?id=QCcW1h835pwC.

[322] Ismaël Ahlouche Lahlali and Josh A. O'Connor. *Coset symmetries and coadjoint orbits*. arXiv:2411.05918 [hep-th]. Nov. 2024. DOI: 10.48550/arXiv.2411.05918. URL: http://arxiv.org/abs/2411.05918 (visited on 01/30/2025).

[323] Edward Witten. "Coadjoint Orbits of the Virasoro Group". In: *Commun. Math. Phys.* 114 (1988), p. 1. DOI: 10.1007/BF01218287.

[324] A. Alekseev, L. Faddeev, and S. Shatashvili. "Quantization of symplectic orbits of compact Lie groups by means of the functional integral". In: *Journal of Geometry and Physics* 5.3 (Jan. 1988), pp. 391–406. ISSN: 0393-0440. DOI: 10.1016/0393-0440(88)90031-9. URL: https://www.sciencedirect.com/science/article/pii/0393044088900319 (visited on 01/30/2025).

[325] Balram Rai and V. G. J. Rodgers. "From Coadjoint Orbits to Scale Invariant WZNW Type Actions and 2-D Quantum Gravity Action". In: *Nucl. Phys. B* 341 (1990), pp. 119–133. DOI: 10.1016/0550-3213(90)90264-E.

[326] Samuel Brensinger and Vincent G. J. Rodgers. "Dynamical Projective Curvature in Gravitation". In: *Int. J. Mod. Phys. A* 33.36 (2019), p. 1850223. DOI: 10.1142/S0217751X18502238. arXiv: 1712.05394 [hep-th].

[327] Pawel Caputa and Javier M. Magan. "Quantum Computation as Gravity". In: *Phys. Rev. Lett.* 122.23 (2019), p. 231302. DOI: 10.1103/PhysRevLett.122.231302. arXiv: 1807.04422 [hep-th].

[328] Johanna Erdmenger, Marius Gerbershagen, and Anna-Lena Weigel. "Complexity measures from geometric actions on Virasoro and Kac-Moody orbits". In: *JHEP* 11 (2020), p. 003. DOI: 10.1007/JHEP11(2020)003. arXiv: 2004.03619 [hep-th].

[329] Serge Lang. "Discrete Series". en. In: *SL2(R)*. Ed. by Serge Lang. New York, NY: Springer, 1985, pp. 179–190. ISBN: 978-1-4612-5142-2. DOI: 10.1007/978-1-4612-5142-2_9. URL: https://doi.org/10.1007/978-1-4612-5142-2_9 (visited on 06/06/2025).

[330] N. M. J. Woodhouse. *Geometric quantization*. 1992.

[331] Francisco Rubilar and Leonardo Schultz. *Adjoint orbits of sl(2,R) and their geometry*. arXiv:2004.12180 [math]. Apr. 2020. DOI: 10.48550/arXiv.2004.12180. URL: http://arxiv.org/abs/2004.12180 (visited on 06/06/2025).

[332] Bill Casselman. "Representations of SL2(R)". en. In: ().

[333] Sean Bates and Alan Weinstein. *Lectures on the Geometry of Quantization*. Vol. 8. Berkeley Mathematics Lecture Notes. A free PDF version is available from Alan Weinstein's homepage at https://math.berkeley.edu/~alanw/. American Mathematical Society, Center for Pure, and Applied Mathematics, University of California, Berkeley, 1997, p. 137. ISBN: 978-0-8218-0798-9.

[334] M. Blau. *Symplectic geometry and geometric quantization*. http://www.blau.itp.unibe.ch/Lecturenotes.html.

[335] Andrea Carosso. *Geometric Quantization*. 2018. arXiv: 1801.02307 [math-ph]. URL: https://arxiv.org/abs/1801.02307.

[336] Serge Lang. *SL2(R)*. Vol. 105. Graduate Texts in Mathematics. New York, NY: Springer, 1985. ISBN: 978-1-4612-9581-5 978-1-4612-5142-2. DOI: 10.1007/978-1-4612-5142-2. URL: http://link.springer.com/10.1007/978-1-4612-5142-2 (visited on 06/06/2025).

[337] M. S. Plyushchay. "Quantization of the classical SL(2,R) system and representations of SL(2,R) group". In: *J. Math. Phys.* 34 (1993), pp. 3954–3963. DOI: 10.1063/1.530016.

[338] David A. Vogan. *Unitary Representations of Reductive Lie Groups. (AM-118)*. Princeton University Press, 1987. ISBN: 9780691084824. URL: http://www.jstor.org/stable/j.ctt1bh4b81 (visited on 08/11/2025).

[339] Luca Ciambelli, Jerzy Kowalski-Glikman, and Ludovic Varrin. *Entanglement Entropy of Quantum Corners*. 2025, *To appear*.

[340] Blagoje Oblak. *BMS Particles in Three Dimensions*. arXiv:1610.08526 [hep-th]. 2017. DOI: 10.1007/978-3-319-61878-4. URL: http://arxiv.org/abs/1610.08526 (visited on 01/30/2025).

[341] Rolf Berndt and Ralf Schmidt. *Elements of the Representation Theory of the Jacobi Group*. Ed. by H. Bass, J. Oesterlé, and A. Weinstein. Vol. 163. Progress in Mathematics. Basel: Birkhäuser, 1998. ISBN: 978-3-7643-5922-5 978-3-0348-8772-4. DOI: 10.1007/978-3-0348-8772-4. URL: http://link.springer.com/10.1007/978-3-0348-8772-4 (visited on 06/19/2025).

[342] George Whitelaw Mackey. *The Theory of Unitary Group Representations*. en. University of Chicago Press, 1976. ISBN: 978-0-226-50052-2.

[343] Harish-Chandra. *Harmonic Analysis on Real Reductive Groups*. Vol. 576. Lecture Notes in Mathematics. Berlin, New York: Springer-Verlag, 1977.

[344] A. A. Kirillov. "Merits and demerits of the orbit method". In: *Bulletin of the American Mathematical Society* 36.4 (1999), 433–488.

[345] Edgar Assing and Radu Toma. *The orbit method in number theory through the sup-norm problem for* GL(2). 2025. arXiv: 2503.06224 [math.NT]. URL: https://arxiv.org/abs/2503.06224.

[346] Ted Jacobson. "Thermodynamics of space-time: The Einstein equation of state". In: *Phys. Rev. Lett.* 75 (1995). _eprint: gr-qc/9504004, pp. 1260–1263. DOI: 10.1103/PhysRevLett.75.1260.

[347] Sudipta Sarkar and Aron C. Wall. "Second Law Violations in Lovelock Gravity for Black Hole Mergers". In: *Physical Review D* 83.12 (June 2011). arXiv:1011.4988 [gr-qc], p. 124048. ISSN: 1550-7998, 1550-2368. DOI: 10.1103/PhysRevD.83.124048. URL: http://arxiv.org/abs/1011.4988 (visited on 01/27/2025).

[348] Maxime Gadioux and Harvey S. Reall. "Creases, corners, and caustics: Properties of nonsmooth structures on black hole horizons". In: *Phys. Rev. D* 108.8 (2023), p. 084021. DOI: 10.1103/PhysRevD.108.084021. arXiv: 2303.15512 [gr-qc].

[349] Eric Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2004.

[350] J. v. Neumann. "Über Einen Satz Von Herrn M. H. Stone". In: *Annals of Mathematics* 33.3 (1932), pp. 567–573. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1968535 (visited on 06/20/2025).

[351] Alexei Kitaev. *Notes on* $\widetilde{SL(2,\mathbb{R})}$ *representations*. arXiv:1711.08169 [hep-th]. Aug. 2018. DOI: 10.48550/arXiv.1711.08169. URL: http://arxiv.org/abs/1711.08169 (visited on 01/30/2025).