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Periodic solutions of nonlinear wave equations with non-monotone forcing terms

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Equazioni a derivate parziali. — *Periodic solutions of nonlinear wave equations with non-monotone forcing terms.* Nota (*) di Massimiliano Berti e Luca Biasco, presentata dal Socio A. Ambrosetti.

Abstract. — Existence and regularity of periodic solutions of nonlinear, completely resonant, *forced* wave equations is proved for a large class of non-monotone forcing terms. Our approach is based on a variational Lyapunov-Schmidt reduction. The corresponding infinite dimensional bifurcation equation exhibits an intrinsic lack of compactness. This difficulty is overcome finding a-priori estimates for the constrained minimizers of the reduced action functional, through techniques inspired by regularity theory as in [10].

Key words: Wave equation; Periodic solutions; Variational methods; A-priori estimates; Lyapunov-Schmidt reduction.

Riassunto. — *Soluzioni periodiche dell'equazione delle onde non lineari con termini forzanti non monotoni.* Presentiamo risultati di esistenza ed unicità di soluzioni periodiche per equazioni delle onde nonlineari, completamente risonanti e periodicamente forzate nel tempo, per un'ampia classe di termini forzanti non monotoni. Il nostro approccio si basa su una riduzione variazionale di tipo Lyapunov-Schmidt. La corrispondente equazione di biforcazione manca radicalmente di proprietà di compattezza. Questa difficoltà viene superata trovando opportune stime a-priori per i minimi vincolati del funzionale di azione ridotto, mediante tecniche ispirate alla teoria della regolarità di [10].

1. Introduction and results

We outline in this *Note* the recent results obtained in [2] concerning existence and regularity of nontrivial time-periodic solutions for completely resonant, nonlinear, *forced* wave equations like

$$\square u = \varepsilon f(t, x, u; \varepsilon) \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{1.2}$$

where $\square := \partial_{tt} - \partial_{xx}$ is the D'Alembertian operator, ε is a small parameter and the nonlinear forcing term $f(t, x, u; \varepsilon)$ is T-periodic in time. We consider the case when T is a rational multiple of 2π and, for simplicity of exposition, we assume

$$T = 2\pi .$$

We look for 2π-periodic in time solutions of (1.1)-(1.2), namely u satisfying

$$u(t + 2\pi, x) = u(t, x) . \tag{1.3}$$

For $\varepsilon = 0$, (1.1)-(1.2) reduces to the linear homogeneous wave equation

$$\begin{cases} \square u = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \tag{1.4}$$

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which possesses an *infinite* dimensional space of solutions which are 2π-periodic in time and of the form $v(t,x) = \hat{v}(t+x) - \hat{v}(t-x)$ for any 2π-periodic function $\hat{v}(\,\cdot\,)$. For this reason equation (1.1)-(1.2) is called completely resonant.

The main difficulty for proving existence of solutions of (1.1)-(1.2)-(1.3) for $\varepsilon \neq 0$ is to find from which periodic orbits of the linear equation (1.4) the solutions of the nonlinear equation (1.1) branch off. This requires to solve an infinite dimensional bifurcation equation with an intrinsic *lack of compactness*.

The first breakthrough regarding problem (1.1)-(1.2)-(1.3) was achieved by Rabinowitz in [10] where existence and regularity of solutions was proved for nonlinearities satisfying the strongly monotone assumption $(\partial_u f)(t,x,u) \geq \beta > 0$. Using methods inspired by the theory of elliptic regularity, [10] proved the existence of a unique curve of smooth solutions for ε small. Other existence results have been obtained, still for strongly monotone f's, in [5, 7].

Subsequently, Rabinowitz [11] was able to prove existence of weak solutions of (1.1)-(1.2)-(1.3) for weakly monotone nonlinearities like $f(t,x,u) = u^{2k+1} + G(t,x,u)$ where $G(t,x,u_2) \geq G(t,x,u_1)$ if $u_2 \geq u_1$. Actually, in [11] the bifurcation of a global continuum branch of weak solutions is proved.

In all the quoted papers the monotonicity assumption (strong or weak) is the key property for overcoming the lack of compactness in the infinite dimensional bifurcation equation.

We underline that, in general, the weak solutions obtained in [11] are only continuous functions. Concerning regularity, Brézis and Nirenberg [5] proved – but only for strongly monotone nonlinearities – that any L^∞-solution of (1.1)-(1.2)-(1.3) is smooth, even in the nonperturbative case $\varepsilon = 1$, whenever f is smooth.

On the other hand, very little is known about existence and regularity of solutions if we drop the monotonicity assumption on the forcing term f. Willem [12], Hofer [8] and Coron [6] have considered the class of equations (1.1)-(1.2) where $f(t,x,u) = g(u) + h(t,x)$, $\varepsilon = 1$, and $g(u)$ satisfies suitable linear growth conditions. Existence of weak solutions is proved, in [8, 12], for a set of h dense in L^2, although explicit criteria that characterize such h are not provided. The infinite dimensional bifurcation problem is overcome by assuming non-resonance hypothesis between the asymptotic behaviour of $g(u)$ and the spectrum of $\Box$. On the other side, Coron [6] finds weak solutions assuming the additional symmetry $h(t,x) = h(t+\pi, \pi - x)$ and restricting to the space of functions satisfying $u(t,x) = u(t+\pi, \pi - x)$, where the Kernel of the d'Alembertian operator $\Box$ reduces to 0.

Let us now present the results obtained in [2] on existence and regularity of solutions of (1.1)-(1.2)-(1.3) for a large class of *nonmonotone* forcing terms $f(t,x,u)$.

We look for solutions $u : \Omega \to \mathbb{R}$ of (1.1)-(1.2)-(1.3) in the Banach space

$$E := H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega}), \qquad \Omega := \mathbb{T} \times (0,\pi)$$

where $H^1(\Omega)$ is the usual Sobolev space and $C_0^{1/2}(\overline{\Omega})$ is the space of all the 1/2-Hölder continuous functions $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying (1.2), endowed with norm

$$|u|_E := |u|_{H^1(\Omega)} + |u|_{C^{1/2}(\overline{\Omega})}$$

where $|u|^2_{H^1(\Omega)} := |u|^2_{L^2(\Omega)} + |u_x|^2_{L^2(\Omega)} + |u_t|^2_{L^2(\Omega)}$ and

$$|u|_{C^{1/2}(\overline{\Omega})} := |u|_{C^0(\Omega)} + \sup_{(t,x)\neq(t_1,x_1)} \frac{|u(t,x)-u(t_1,x_1)|}{(|t-t_1|+|x-x_1|)^{1/2}} .$$

Critical points of the Lagrangian action functional $\Psi \in C^1(E,\mathbb{R})$

$$\Psi(u) := \int_\Omega \left[\frac{u_t^2}{2} - \frac{u_x^2}{2} + \varepsilon F(t,x,u;\varepsilon)\right] dtdx \,, \tag{1.5}$$

where $F(t,x,u;\varepsilon) := \int_0^u f(t,x,\xi;\varepsilon)d\xi$, are weak solutions of (1.1)-(1.2)-(1.3).

For $\varepsilon = 0$, the critical points of Ψ in E reduce to the solutions of the linear equation (1.4) and form the subspace $V := N \cap H^1(\Omega)$ where

$$N := \left\{ v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) \;\Big|\; \hat{v} \in L^2(\mathbb{T}) \text{ and } \int_0^{2\pi} \hat{v}(s)\, ds = 0 \right\}. \tag{1.6}$$

Note that $V = \{v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) \in N \mid \hat{v} \in H^1(\mathbb{T})\} \subset E$ since any function $\hat{v} \in H^1(\mathbb{T})$ is 1/2-Hölder continuous.

Let $N^\perp := \{h \in L^2(\Omega) \mid \int_\Omega hv = 0, \ \forall v \in N\}$ denote the L^2-orthogonal of N which coincides with the range of $\Box$ in $L^2(\Omega)$.

In [2] we prove the following Theorem:

THEOREM 1. *Let $f(t,x,u) = \beta u^{2k} + h(t,x)$ and $h \in N^\perp$ satisfies $h(t,x) > 0$ (or $h(t,x) < 0$) a.e. in Ω. Then, for ε small enough, there exists at least one weak solution $u \in E$ of* (1.1)-(1.2)-(1.3) *with $|u|_E \le C|\varepsilon|$. If, moreover, $h \in H^j(\Omega) \cap C^{j-1}(\overline{\Omega})$, $j \ge 1$, then $u \in H^{j+1}(\Omega) \cap C^j_0(\overline{\Omega})$ with $|u|_{H^{j+1}(\Omega)} + |u|_{C^j(\overline{\Omega})} \le C|\varepsilon|$ and therefore, for $j \ge 2$, u is a classical solution.*

Theorem 1 is a Corollary of the following more general result which enables to deal with non-monotone nonlinearities like, for example, $f(t,x,u) = (\sin x)\, u^{2k} + h(t,x)$, $f(t,x,u) = u^{2k} + u^{2k+1} + h(t,x)$.

THEOREM 2. *Let $f(t,x,u) = g(t,x,u) + h(t,x)$, $h(t,x) \in N^\perp$ and*

$$g(t,x,u) := \beta(x)u^{2k} + \mathcal{R}(t,x,u)$$

where $\mathcal{R}$, $\partial_t\mathcal{R}$, $\partial_u\mathcal{R} \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ satisfy (1)

$$|\mathcal{R}(\cdot,u)|_{C(\overline{\Omega})} = o(u^{2k})\,, \quad |\partial_t\mathcal{R}(\cdot,u)|_{C(\overline{\Omega})} = O(u^{2k})\,, \quad |\partial_u\mathcal{R}(\cdot,u)|_{C(\overline{\Omega})} = o(u^{2k-1})\,, \tag{1.7}$$

and $\beta \in C([0,\pi],\mathbb{R})$ verifies, for $x \in (0,\pi)$, $\beta(x) > 0$ (or $\beta(x) < 0$) and $\beta(\pi - x) = \beta(x)$.

(i) (Existence). *Assume there exists a weak solution $H \in E$ of $\Box H = h$ such that*

(1) The notation $f(z) = o(z^p)$, $p \in \mathbb{N}$, means that $f(z)|z|^p \to 0$ as $z \to 0$. $f(z) = O(z^p)$ means that there exists a constant $C > 0$ such that $|f(z)| \le C|z|^p$ for all z in a neighborhood of 0.

$$(1.8) \qquad H(t,x) > 0 \qquad (\text{or } H(t,x) < 0) \qquad \forall (t,x) \in \Omega \,.$$

Then, for ε small enough, there exists at least one weak solution $u \in E$ of (1.1)-(1.2)-(1.3) *satisfying $|u|_E \le C|\varepsilon|$.*

(ii) (Regularity). *If, moreover, $h \in H^j(\Omega) \cap C^{j-1}(\overline{\Omega})$, $\beta \in H^j\big((0,\pi)\big)$, $\mathcal{R}$, $\partial_t \mathcal{R}$, $\partial_u \mathcal{R}$ $\in C^j(\overline{\Omega} \times \mathbb{R})$, $j \ge 1$, then $u \in H^{j+1}(\Omega) \cap C^j_0(\overline{\Omega})$ and, for $j \ge 2$, u is a classical solution.*

Note that Theorem 2 does not require any growth condition on g at infinity. In particular it applies for any analytic function $g(u)$ satisfying $g(0) = g'(0) = \ldots = g^{2k-1}(0) = 0$ and $g^{2k}(0) \neq 0$.

We now collect some comments on the previous results.

REMARK 1.1. The assumption $h \in N^{\perp}$ is not of technical nature both in Theorem 1 and in Theorem 2 (at least if $g = g(x,u) = g(x,-u) = g(\pi - x, u)$). Actually one can prove that, if $h \notin N^{\perp}$, periodic solutions of problem (1.1)-(1.2)-(1.3) do not exist in any fixed ball $\{|u|_{L^\infty} \le R\}$, $R > 0$, for ε small.

REMARK 1.2. In Theorem 2 hypothesis (1.8) and $\beta > 0$ (or $\beta < 0$) are assumed to prove the existence of a minimum of the «reduced action functional» Φ, see (1.17). A sufficient condition implying (1.18) is $h > 0$ a.e. in Ω. This follows by the «maximum principle» proposition

$$(1.9) \qquad h \in N^{\perp}, \ h > 0 \text{ a.e. in } \Omega \quad \Longrightarrow \quad \exists H \in E \ \text{ solving } \ \Box H = h \ \text{ with } \ H > 0 \,.$$

This is the key step to derive Theorem 1 from Theorem 2.

REMARK 1.3 (Regularity). It is not at all obvious that the weak solution u of Theorems 1, 2 is actually smooth. Indeed, while regularity always holds true for strictly monotone nonlinearities (see [10, 5]), yet for weakly monotone f it is not proved in general, unless the weak solution u verifies $|\Pi_N u|_{L^2} \ge C > 0$ (see [11]). Note, on the contrary, that the weak solution u of Theorem 2 satisfies $|\Pi_N u|_{L^2} = O(\varepsilon)$. Moreover, assuming

$$(1.10) \qquad |\partial_t^l \, \partial_x^m \, \partial_u^n \, \mathcal{R}|_{C(\overline{\Omega})} = O(u^{2k-n}) \,, \qquad \forall\, 0 \le l, n \le j+1 \,, 0 \le m \le j \,, l+m+n \le j+1$$

we can also prove the estimate

$$(1.11) \qquad |u|_{H^{j+1}(\Omega)} + |u|_{C^j(\overline{\Omega})} \le C|\varepsilon| \,.$$

REMARK 1.4 (Multiplicity). For nonmonotone nonlinearities f one can NOT in general expect unicity of the solutions. Actually, for $f(t,x,u) = g(x,u) + h(t,x)$ with $g(x,u) = g(x,-u)$, $g(\pi - x, u) = g(x,u)$, there exist infinitely many $h \in N^{\perp}$ for which problem (1.1)-(1.2)-(1.3) has (at least) 3 solutions.

REMARK 1.5 (Mimimal period). If $h(t,x)$ has minimal period 2π w.r.t time, then also the solution $u(t,x)$ has minimal period 2π.

Finally, we extend the result of [10] proving existence of periodic solutions for nonmonotone nonlinearities $f(t,x,u)$ obtained adding to a nonlinearity $\widetilde{f}(t,x,u)$ as in

[10] (*i.e.* $\partial_u \widetilde{f} \geq \beta > 0$) any nonmonotone term $a(x, u)$ satisfying

$$a(x, -u) = a(x, u)\,, \qquad a(\pi - x, u) = a(x, u) \tag{1.12}$$

or

$$a(x, -u) = -a(x, u)\,, \qquad a(\pi - x, u) = -a(x, u)\,. \tag{1.13}$$

A prototype nonlinearity is $f(t, x, u) = u^{2k} + \widetilde{f}(t, x, u)$ with $\partial_u \widetilde{f} \geq \beta > 0$.

THEOREM 3. *Let* $f(t, x, u) = \widetilde{f}(t, x, u) + a(x, u)$ *where* f, $\partial_t f$, $\partial_u f$ *are continuous,* $\partial_u \widetilde{f} \geq \beta > 0$ *and* $a(x, u)$ *satisfy* (1.12) *or* (1.13). *Then, for* ε *small enough,* (1.1)-(1.2)-(1.3) *has at least one weak solution* $u \in E$. *If moreover* f, $\partial_t f$, $\partial_u f \in C^j(\overline{\Omega} \times \mathbb{R})$, $j \geq 1$, *then* $u \in H^{j+1}(\Omega) \cap C_0^j(\overline{\Omega})$.

In the next subsection we describe the method followed in [2] to prove Theorems 1, 2, 3. We anticipate that such approach is not merely a sharpening of the ideas of [10, 11] which, to deal with non monotone nonlinearities, require a significant change of prospective.

1.1. *Sketch of the Proof.*

We look for critical points of the Lagrangian action functional $\Psi : E \to \mathbb{R}$ defined in (1.5) performing a variational Lyapunov-Schmidt reduction. We decompose the space E as

$$E = V \oplus W$$

where

$$V := N \cap H^1(\Omega) \qquad \text{and} \qquad W := N^\perp \cap H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega})\;.$$

Setting $u = v + w$ with $v \in V$, $w \in W$ and denoting by Π_N and $\Pi_{N^\perp}$ the projectors from $L^2(\Omega)$ onto N and $N^\perp$ respectively, Problem (1.1)-(1.2)-(1.3) is equivalent to solve the *bifurcation equation*

$$\Pi_N f(v + w, \varepsilon) = 0 \tag{1.14}$$

and the *range equation*

$$w = \varepsilon\, \Box^{-1}\, \Pi_{N^\perp} f(v + w, \varepsilon) \tag{1.15}$$

where $\Box^{-1} : N^\perp \to N^\perp$ is the inverse of $\Box$ and $f(u, \varepsilon)$ denotes the Nemitski operator associated to f, namely

$$[f(u, \varepsilon)](t, x) := f(t, x, u, \varepsilon).$$

The usual approach of [10, 7, 11] is to find, first, by the monotonicity of f, the unique solution $v = v(w)$ of the bifurcation equation (1.14) and, next, to solve the range equation (1.15). On the other hand, for non-monotone forcing terms, one can *not* in general solve uniquely the equation (1.14) – recall by Remark 1.4 that in general unicity of solutions does not hold. Therefore, to deal with non monotone nonlinearities, we must solve first the range equation and thereafter the bifurcation equation. For other applications of this approach to perturbation problems in critical point theory, see *e.g.* the forthcoming monograph by Ambrosetti and Malchiodi [1].

We find a solution $w := w(v, \varepsilon) \in W$ of the range equation (1.15) satisfying $|w(v, \varepsilon)|_E = = O(\varepsilon)$ by means of a quantitative version of the Implicit Function Theorem. Here no serious difficulties arise since $\Box^{-1}$ acting on W is a compact operator, due to the assumption $T = 2\pi$ (actually $|\Box^{-1} f|_E \le C|f|_{L^2}$, $\forall f \in L^2$, see [5]).

Remark 1.6. More in general, $\Box^{-1}$ is compact on the orthogonal complement of $\ker(\Box)$ whenever T is a rational multiple of 2π. On the contrary, if T is an irrational multiple of 2π, then $\Box^{-1}$ is, in general, unbounded (a «small divisor» problem appears), but the kernel of $\Box$ reduces to 0 (there is no bifurcation equation). For existence of periodic solutions in the case $T/2\pi$ is irrational see *e.g.* [9].

Once the range equation (1.15) has been solved by $w(v, \varepsilon) \in W$ it remains the infinite dimensional bifurcation equation

$$\Pi_N f(v + w(v, \varepsilon), \varepsilon) = 0 \tag{1.16}$$

which, by the Lyapunov-Schmidt reduction procedure, turns out to be the Euler-Lagrange equation of the *reduced Lagrangian action functional*

$$\Phi : V \longrightarrow \mathbb{R} \qquad \Phi(v) := \Psi(v + w(v, \varepsilon)) \ . \tag{1.17}$$

Since Φ lacks compactness properties, we can not rely on critical point theory, unlike the autonomous case considered in [3, 4] where, thanks to the «viscous term» $|v|^2_{H^1}$, existence and regularity of solutions is proved through the Mountain Pass Theorem and standard elliptic regularity theory.

We attempt to minimize Φ. We do not try to apply the direct methods of the calculus of variations because Φ, even though it could possess some coercivity property, will not be convex (being f non monotone). Moreover, without assuming any growth condition on the nonlinearity f, the functional Φ could neither be well defined on any L^p-space.

We minimize Φ constrained in $\overline{B_R} := \{v \in V, |v|_{H^1} \le R\}$, $\forall R > 0$. By standard compactness arguments Φ attains minimum at, say, $\bar{v} \in \overline{B_R}$. Since $\bar{v}$ could belong to the boundary $\partial \overline{B_R}$ we can only conclude the variational inequality

$$D_v \Phi(\bar{v})[\varphi] = \int_\Omega f(\bar{v} + w(\bar{v}, \varepsilon), \varepsilon)\varphi \le 0 \tag{1.18}$$

for any *admissible variation* $\varphi \in V$, *i.e.* if $\bar{v} + \theta\varphi \in \overline{B_R}$, $\forall \theta < 0$ sufficiently small.

The heart of the existence proof of the weak solution u of Theorem 1, Theorem 2 and Theorem 3 is to obtain, choosing suitable admissible variations the *a-priori estimate* $|\bar{v}|_{H^1} < R$ for some $R > 0$, *i.e.* to show that $\bar{v}$ is an *{inner} minimum point of Φ in B_R.*

The strong monotonicity assumption $(\partial_u f)(t, x, u) \ge \beta > 0$ would allow here to get such a-priori estimates by arguments similar to [10]. On the contrary, the main difficulty for proving Theorems 1, 2 and 3 which deal with non-monotone nonlinearities is to obtain such a priori-estimates for $\bar{v}$.

The most difficult cases are the proof of Theorems 1 and 2. To understand the problem, let consider the particular nonlinearity $f(t, x, u) = u^{2k} + h(t, x)$ of Theorem 1.

The even term u^{2k} does not give any contribution into the variational inequality (1.18) at the 0^{th}-order in ε, since the right hand side of (1.18) reduces, for $\varepsilon = 0$, to

$$\int_{\Omega} \left(\bar{v}^{2k} + h(t,x)\right)\varphi = 0, \qquad \forall \varphi \in V$$

since $h \in N^{\perp}$ and $\int_{\Omega} \bar{v}^{2k}\varphi \equiv 0$ by the specific form $\bar{v} = \hat{\bar{v}}(t+x) - \hat{\bar{v}}(t-x)$, $\varphi = \hat{\varphi}(t+x) - \hat{\varphi}(t-x)$ of the functions of V.

Therefore, for deriving, if ever possible, the required a-priori estimates, we have to develop the variational inequality (1.18) at higher orders in ε. We obtain

$$(1.19) \quad 0 \geq \int_{\Omega} 2k\bar{v}^{2k-1}\varphi\, w(\bar{v},\varepsilon) + O(w^2(\bar{v},\varepsilon)) = \int_{\Omega} \varepsilon\, 2k\bar{v}^{2k-1}\varphi\Box^{-1}(h+\bar{v}^{2k}) + O(\varepsilon^2)$$

because $w(\bar{v},\varepsilon) = \varepsilon\Box^{-1}(\bar{v}^{2k} + h) + o(\varepsilon)$ (recall that $\bar{v}^{2k}, h \in N^{\perp}$).

We now sketch how the ε-order term in the variational inequality (1.19) allows to prove an L^{2k}-estimate for $\bar{v}$. Inserting the admissible variation $\varphi := \bar{v}$ in (1.19) we get

$$(1.20) \qquad \int_{\Omega} H\bar{v}^{2k} + \bar{v}^{2k}\Box^{-1}\bar{v}^{2k} \leq O(\varepsilon)$$

where H is a weak solution of $\Box H = h$ which verifies $H(t,x) > 0$ in Ω (H exists by the «maximum principle» proposition (1.9)).

The crucial fact is that the first term in (1.20) satisfies the coercivity inequality

$$(1.21) \qquad \int_{\Omega} Hv^{2k} \geq c(H)\int_{\Omega} v^{2k}, \qquad \forall v \in V$$

for some constant $c(H) > 0$. We remark that (1.21) is not trivial because H vanishes at the boundary ($H(t,0) = H(t,\pi) = 0$) and, indeed, its proof relies on the specific form $v(t,x) = \hat{v}(t+x) - \hat{v}(t-x)$ of the functions of V. The second term $\int_{\Omega} \bar{v}^{2k}\Box^{-1}\bar{v}^{2k}$ will be negligible, ε-close to the origin, with respect to $\int_{\Omega} Hv^{2k}$ and (1.20)-(1.21) will provide the L^{2k}-estimate for $\bar{v}$.

Next, we can obtain an L^{∞}-estimate and the required H^1-estimate for $\bar{v}$ inserting further admissible variations φ (inspired by [10]) into (1.19) and using inequalities similar to (1.21). In this way we prove the existence of a weak solution u in the interior of some B_R.

The regularity of the solution u – fact not at all obvious for non-monotone nonlinearities – is proved using similar techniques inspired to regularity theory. We insert suitable variations φ in $D_v\Phi(\bar{v})[\varphi] = 0$ and, using inequalities like (1.21), we get estimates for the L^{∞} and H^1-norm of the higher order derivatives of $\bar{v}$.

Theorem 2 is proved developing such ideas and a careful analysis of the further term $\mathcal{R}$.

Finally, the proof of Theorem 3 is easier than for Theorems 1 and 2. Indeed the additional term $a(x,u)$ does not contribute into the variational inequality (1.18) at the 0^{th}-order in ε, because $\int_{\Omega} a(x,\bar{v})\varphi \equiv 0$, $\forall \varphi \in V$. Therefore the dominant term in the variational inequality (1.18) is provided by the monotone forcing term $\widetilde{f}$ and the required a-priori estimates are obtained with arguments similar to [10].

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