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REDUCED BASIS METHOD FOR PARAMETRIZED ELLIPTIC OPTIMAL CONTROL PROBLEMS*

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Abstract. We propose a suitable model reduction paradigm—the certified reduced basis method (RB)—for the rapid and reliable solution of parametrized optimal control problems governed by partial differential equations. In particular, we develop the methodology for parametrized quadratic optimization problems with elliptic equations as a constraint and infinite-dimensional control variable. First, we recast the optimal control problem in the framework of saddle-point problems in order to take advantage of the already developed RB theory for Stokes-type problems. Then, the usual ingredients of the RB methodology are called into play: a Galerkin projection onto a low-dimensional space of basis functions properly selected by an adaptive procedure; an affine parametric dependence enabling one to perform competitive offline-online splitting in the computational procedure; and an efficient and rigorous a posteriori error estimate on the state, control, and adjoint variables as well as on the cost functional. Finally, we address some numerical tests that confirm our theoretical results and show the efficiency of the proposed technique.

Key words. reduced basis methods, parametrized optimal control problems, saddle-point problems, model order reduction, PDE-constrained optimization, a posteriori error estimate

AMS subject classifications. 49J20, 65K10, 65M15, 65M60, 65N12, 93C20

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1. Introduction.

Usually, the numerical solution of PDE-constrained optimization problems is computationally demanding, since it requires the solution of a system of PDEs arising from the optimality conditions—the state problem, the adjoint problem, and a further set of equations ensuring the optimality of the solution. This task becomes even more challenging if the state system (or the cost functional to be minimized) depend on a set of parameters—which can specify physical or geometrical properties of interest—and the optimal control problem has to be solved for many different scenarios (*many-query* context) corresponding to different sets of parameter values. Substantial computational savings become possible using the reduced basis (RB) method [30, 25]: the parametrized PDE problem is solved *online* for any new value of the parameters once a set of (full-order) solutions have been computed *offline* for selected values of the parameter set and stored.

After denoting with $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$ a p-vector of parameters representing either physical or geometrical quantities of interest, y the state variable, u the control variable, $\mathcal{J}$ the objective (cost) functional, and $\mathcal{E}(\cdot, \cdot; \boldsymbol{\mu})$ the residual of the state equation,

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the general form of a parametrized optimal control problem reads as follows:

$$(OCP_{\boldsymbol{\mu}}) \qquad \min_{y,u} \mathcal{J}(y,u;\boldsymbol{\mu}) \qquad \text{subject to (s.t.)} \qquad \mathcal{E}(y,u;\boldsymbol{\mu}) = 0.$$

In this work we focus our analysis on the most typical linear/quadratic case, i.e., to optimal control problems featuring quadratric cost functionals and linear (scalar coercive) elliptic PDEs as a constraint.

From an abstract point of view, the mapping $\boldsymbol{\mu} \mapsto (y(\boldsymbol{\mu}), u(\boldsymbol{\mu}))$ defines a *smooth* and rather *low-dimensional* parametrically induced manifold $\mathcal{M} = \{(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in X : \boldsymbol{\mu} \in \mathcal{D}\}$, where $y(\boldsymbol{\mu})$ and $u(\boldsymbol{\mu})$ are the state and control solutions of $(OCP_{\boldsymbol{\mu}})$ and X is a suitable functional space. In a classical discretization approach, after introducing an approximation space $X^{\mathcal{N}}$ of (typically very large) dimension $\mathcal{N}$—e.g., a finite element (FE) space—for every value of the parameters $\boldsymbol{\mu}$, we are supposed to solve the whole optimal control problem in order to compute the solution $(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu}))$, ignoring the possibly *smooth* relation between parameters and solutions. A reduced (basis) approach is premised, e.g., upon a classical FE method, and consists in a low-order approximation of the *truth* manifold $\mathcal{M}^{\mathcal{N}}$, based on (i) computation of some snapshots of the *truth* manifold $\mathcal{M}^{\mathcal{N}}$ and (ii) a Galerkin projection onto the space spanned by the precomputed snapshots.

The main features of the RB methods [25, 30] are the following: (i) a rapidly convergent global approximation onto a space spanned by solution of the original problem at some selected parameters value; (ii) a rigorous a posteriori error estimation which provides inexpensive yet sharp bounds for the error between the RB and the *truth* solution; (iii) an offline/online computational procedure, yielding an efficient splitting between a time-consuming and parameter independent offline stage and an inexpensive online calculation for each new input/output evaluation.

Computational reduction strategies such as RB methods or proper orthogonal decomposition (POD) have already been employed to speed up the solution of optimal control problems, as well as other PDE-constrained optimization problems. First examples of optimal control problems solved by exploiting computational reduction techniques have been addressed by Ito and Ravindran, in the context of either (a preliminary version of) the RB method [18] or the POD method [27]. Recent works dealing with optimal control problems through POD techniques have been addressed, for instance, by Kunisch and Volkwein [20] (and references therein). More recent contributions dealing with RB methods have been presented in the elliptic case by Quarteroni, Rozza and Quaini [26], Tonn, Urban, and Volkwein [35], and Grepl and Kärcher [9]. However, in all these works the control variable is a low-dimensional variable, e.g., a set of real numbers, and can therefore be treated itself as a parameter (or a set of parameters). In [5, 6] the case of parabolic state equations and time dependent control variables has been treated, however, since a reduction of the control space is not operated, in case of high-dimensional control variables the computational gain in the online stage is rather limited.

In this work, we aim at developing a certified reduced framework that enables one to handle infinite-dimensional (either distributed or boundary) control functions. In this context, designing a strategy for the reduction of the complexity of the optimal control problem (that is treated as a whole, with respect to all its variables simultaneously) becomes mandatory. The main novelty addressed by this work deals with the possibility of recasting a general parametrized optimal control problem $(OCP_{\boldsymbol{\mu}})$ in a saddle-point framework, which allows us to analyze the well-posedness and stability of the RB approximation in a general way and to derive suitable a posteriori error estimates.

Indeed, we provide an efficient and rigorous a posteriori error estimation—necessary both for constructing the reduced-order model and for assessing its accuracy—in the case of linear-quadratic optimal control problems characterized by control variables of infinite dimension. In fact, this question is still partially unresolved; for instance, the a posteriori estimators for the error in the cost functional and in the control variable proposed in some previous works [5, 6] are efficient in practice but unfortunately lack rigorousness, whereas the estimator proposed in [35] is proved to be rigorous but not efficient. Only recently an efficient and rigorous estimator has been proposed in the case of finite-dimensional control functions in [9], an assumption that is unfortunately too restrictive for our purpose. In this work we propose both efficient and rigorous a posteriori error bounds in order to provide a simultaneous estimate of the errors on the optimal control, the state variable, and the cost functional.

With reference to the basic RB features previously outlined, we point out the following:

(i) In our approach the reduced scheme is built directly over the optimality conditions system rather than on the original optimization problem, following an *optimize-then-discretize-then-reduce* approach. Indeed, we first derive the optimality system (*optimize* step), then we introduce its *truth* FE approximation (*discretize* step), and finally we provide the RB approximation for the whole optimality system (*reduce* step).

(ii) The reduced basis is made of optimal solutions of the original problem, and hence the computation of each basis function requires the resolution of the FE truth approximation; moreover, the reduced spaces are built for both the state, control, and adjoint variables.

(iii) To ensure the well-posedness of the RB approximation, and in order to provide an a posteriori error estimate for the optimal control problem, we take advantage of the RB theory developed for Stokes-type problems [22, 29, 32] by recasting the optimal control problem in the framework of saddle-point problems.

(iv) We rely on the affine parameter dependence assumption, which provides the possibility of extracting the parameter dependent components from our operators and thus exploit an offline/online computational procedure.

Although we are focusing on the linear-quadratic case, our approach can be extended to more general situations. In particular, the case of noncoercive and nonlinear state equations can be treated in a common unified framework. This is a subject of our ongoing research.

In this paper we present three numerical test cases to confirm the predicted theoretical results and assess the computational efficiency of the proposed approach. We consider both distributed and boundary control problems for a scalar convection-diffusion PDEs, where parameters can be related to either the computational geometry, the physical coefficients, or the observation function. In particular, we provide a detailed numerical verification of the stability and convergence property of the RB approximation. The proposed error estimator shows to be sharp enough to select a reasonably small number of basis functions in the greedy algorithm, thus keeping the offline computational costs under control and resulting in very small system to solve in the online stage. As a result, by applying our RB framework, we manage to reduce the computational cost entailed by the solution of an optimal control problem of at least two orders of magnitude, still ensuring a high accuracy. Indeed, by using just a small number of basis functions (ranging from 10 to 30, depending on the problem), we can guarantee the relative error with respect to the FE solution to be less than 10^{-3} for the whole range of parameters.

The paper is structured as follows. In section 2 we introduce the formulation of parametrized linear/quadratic optimal control problems governed by elliptic coercive PDEs with affine parameter dependence; after having recast the problem in the framework of saddle-point problems, we briefly discuss its FE truth approximation, recalling the necessary assumptions to ensure well-posedness. In section 3 we discuss the RB approximation and the main features of the method, focusing on the corresponding stability condition for the RB approximation. Then in section 4 we deal with the a posteriori error estimation for the RB solution and functional based on the *Babuška stability theory* [2]. Finally, in section 5 some numerical examples are presented.

2. Parametrized optimal control problems. In this section we introduce the parametrized optimal control problems that we focus on and, once recast in the framework of saddle-point problems, we prove a well-posedeness result. Finally we introduce the truth FE approximation.

2.1. Problem definition. Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be an open and bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, and $\mathcal{D} \subset \mathbb{R}^p$ be a prescribed p-dimensional compact set of parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$ with $p \geq 1$. Let Y, U be two Hilbert spaces[1] for the state and control variables y and u, respectively, while the Hilbert space $\mathcal{Z} \supset Y$ shall denote the observation space. Given another Hilbert space Q, we define the linear *constraint equation* in the form

$$a(y, q; \boldsymbol{\mu}) = c(u, q; \boldsymbol{\mu}) + \langle G(\boldsymbol{\mu}), q\rangle \qquad \forall q \in Q, \tag{2.1}$$

where the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu}) : Y \times Q \to \mathbb{R}$ represents a linear elliptic operator, the bilinear form $c(\cdot, \cdot; \boldsymbol{\mu}) : U \times Q \to \mathbb{R}$ expresses the action of the control, and $G(\boldsymbol{\mu}) \in Q'$ is a linear continuous functional acting as a forcing term. The quadratic *cost functional* to be minimized is given by

$$J(y, u; \boldsymbol{\mu}) = \frac{1}{2} m(y - y_d(\boldsymbol{\mu}), y - y_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(u, u; \boldsymbol{\mu}), \tag{2.2}$$

where $\alpha > 0$ is a given constant, $y_d(\mathbf{x}, \boldsymbol{\mu}) \in \mathcal{Z}$ is a given parameter-dependent observation function, and the bilinear form $m(\cdot, \cdot; \boldsymbol{\mu}) : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ defines the objective of the minimization while the bilinear form $n(\cdot, \cdot; \boldsymbol{\mu}) : U \times U \to \mathbb{R}$ acts as a penalization term for the control variable. The parametrized optimal control problem reads as follows: for any given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{y,u} J(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad (y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in Y \times U \text{ solves (2.1).} \tag{2.3}$$

Let us specify the assumptions on the linear and bilinear forms introduced above. We first remark that, since we are interested in considering second-order coercive

[1] Typically the state space Y is a closed subspace of $H^1(\Omega)$ such that $H^1_0(\Omega) \subset Y \subset H^1(\Omega)$, while the control space can be given, for example, by $U = L^2(\omega)$, with ω being a portion of the domain or of the boundary. We do not treat here the case of control-constrained problems, i.e., problems where the control space is a closed and convex set in a Hilbert space rather than a Hilbert space itself. Indeed, control-constrained problems feature extra challenges since they require dealing with variational inequalities. Though by employing primal-dual active set algorithms we can reduce the problem to the solution of a sequence of saddle-point problems, the usual local (rather than variational) nature of the control constraints makes the problem harder to tackle with the reduced basis method.

elliptic equation as a constraint, we can assume without loss of generality that $Q \equiv Y$.[2] Then, we assume that the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$ is bounded and coercive over Y for any $\boldsymbol{\mu} \in \mathcal{D}$, i.e., there exists a constant $\tilde{\alpha}_0 > 0$ such that

$$\tilde{\alpha}(\boldsymbol{\mu}) = \inf_{z \in Y} \frac{a(z,z;\boldsymbol{\mu})}{\|z\|_Y^2} \geq \tilde{\alpha}_0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{2.4}$$

We assume that the bilinear form $c(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric and bounded and that the bilinear form $n(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric, bounded, and coercive. Moreover, we assume the bilinear form $m(\cdot,\cdot;\boldsymbol{\mu})$ to be symmetric, continuous, and positive in the norm induced by the space $\mathcal{Z}$. Holding these assumptions, the existence of a unique solution $(y,u) \in Y \times U$ of the optimal control problem (2.3) can be easily proved by applying either Lions theory [21] or Lagrange multiplier theory [17, 13]. Here, however, in view of the application of the RB method, we are interested in recasting the problem in the framework of saddle-point problems.

Before addressing this issue, let us make an additional assumption, crucial to offline-online procedures, by assuming the bilinear and linear forms, as well as the observation function, to be affine[3] in the parameter $\boldsymbol{\mu}$, i.e., for some finite $\tilde{Q}_*$, $* \in \{a,c,n,m,g,d\}$, they can be expressed as

$$\begin{aligned}
a(z,q;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_a} \tilde{\Theta}_a^q(\boldsymbol{\mu})\, a^q(z,q), & c(v,q;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_c} \tilde{\Theta}_c^q(\boldsymbol{\mu})\, c^q(v,q),\\
m(y,z;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_m} \tilde{\Theta}_m^q(\boldsymbol{\mu})\, m^q(y,z), & n(u,v;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_n} \tilde{\Theta}_n^q(\boldsymbol{\mu})\, n^q(u,v),\\
\langle G(\boldsymbol{\mu}), q\rangle &= \sum_{q=1}^{\tilde{Q}_g} \tilde{\Theta}_g^q(\boldsymbol{\mu})\, \langle G^q, q\rangle, & y_d(\mathbf{x},\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_d} \tilde{\Theta}_d^q(\boldsymbol{\mu})\, y_d^q(\mathbf{x})
\end{aligned} \tag{2.5}$$

for given smooth $\boldsymbol{\mu}$-dependent function $\tilde{\Theta}_*^q(\boldsymbol{\mu})$ and continuous $\boldsymbol{\mu}$-independent bilinear and linear forms $a^q(\cdot,\cdot)$, $c^q(\cdot,\cdot)$, $m^q(\cdot,\cdot)$, $n^q(\cdot,\cdot)$, $n^q(\cdot,\cdot)$, G^q and functions $y_d^q \in \mathcal{Z}$.

2.2. Saddle-point formulation. In order to formulate the optimal control problem (2.3) as a saddle-point problem, let us denote with $X = Y \times U$ the product space between the state space Y and the control space U, equipped with the inner product $(\underline{x},\underline{w})_X = (y,z)_Y + (u,v)_U$ and norm $\|\cdot\|_X = \sqrt{(\cdot,\cdot)_X}$, being $\underline{x} = (y,u) \in X$, $\underline{w} = (z,v) \in X$. Furthermore, we define the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu}) : X \times X \to \mathbb{R}$ as

$$\mathcal{A}(\underline{x},\underline{w};\boldsymbol{\mu}) = m(y,z;\boldsymbol{\mu}) + \alpha n(u,v;\boldsymbol{\mu}) \qquad \forall \underline{x},\underline{w} \in X$$

and the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu}) : X \times Q \to \mathbb{R}$ as

$$\mathcal{B}(\underline{w},q;\boldsymbol{\mu}) = a(z,q;\boldsymbol{\mu}) - c(v,q;\boldsymbol{\mu}) \qquad \forall \underline{w} \in X, q \in Q.$$

By defining the linear functional $\underline{F}(\boldsymbol{\mu}) \in X'$ as

$$\langle \underline{F}(\boldsymbol{\mu}), \underline{w}\rangle = m(y_d(\boldsymbol{\mu}), z; \boldsymbol{\mu}) \qquad \forall \underline{w} = (z,v) \in X,$$

[2] We therefore limit ourselves to considering Galerkin variational problems as state equations rather than Petrov–Galerkin problems. We remark that while at the continuous level it seems useless to keep a different notation for the spaces Y and Q, it will be crucial in order to correctly construct the RB approximation (as well as to generalize the method to the case $Y \neq Q$).

[3] If this assumption does not hold, it could be recovered through the so-called empirical interpolation method; see [31] for an application to optimal control problems.

we can express the cost functional as $J(y,u;\boldsymbol{\mu}) = \mathcal{J}(\underline{x};\boldsymbol{\mu}) + t(\boldsymbol{\mu})$, where $t(\boldsymbol{\mu}) = \frac{1}{2}m(y_d(\boldsymbol{\mu}),y_d(\boldsymbol{\mu});\boldsymbol{\mu})$ and

$$\mathcal{J}(\underline{x};\boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(\underline{x},\underline{x};\boldsymbol{\mu}) - \langle \underline{F}(\boldsymbol{\mu}),\underline{x}\rangle. \tag{2.6}$$

Since for any fixed $\boldsymbol{\mu}\in\mathcal{D}$ the constant term $t(\boldsymbol{\mu})$ does not affect the minimizer of $J(\cdot,\cdot;\boldsymbol{\mu})$, we can reformulate the problem (2.3) as follows: given $\boldsymbol{\mu}\in\mathcal{D}$,

$$\min_{\underline{x}\in X} \mathcal{J}(\underline{x};\boldsymbol{\mu}) \qquad \text{s.t.} \qquad \mathcal{B}(\underline{x},q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}),q\rangle \quad \forall q\in Q. \tag{2.7}$$

Let us define the Lagrangian functional $\mathcal{L}(\cdot;\boldsymbol{\mu}) : X\times Q\to\mathbb{R}$ as

$$\mathcal{L}(\underline{x},p;\boldsymbol{\mu}) = \mathcal{J}(\underline{x},\boldsymbol{\mu}) + \mathcal{B}(\underline{x},p;\boldsymbol{\mu}) - \langle G(\boldsymbol{\mu}),p\rangle. \tag{2.8}$$

It is well known (see, for instance, [4, 11, 33]) that the constrained optimization problem (2.7) is equivalent to the unconstrained optimization problem of finding saddle points $(x(\boldsymbol{\mu}),p(\boldsymbol{\mu}))\in X\times Q$ of the Lagrangian functional. The existence and uniqueness of a solution is well-established by Brezzi theorem under the following conditions:

(i) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is continuous over $X\times X$:

$$\gamma_a(\boldsymbol{\mu}) = \sup_{\underline{x}\in X}\sup_{\underline{w}\in X}\frac{\mathcal{A}(\underline{x},\underline{w};\boldsymbol{\mu})}{\|\underline{w}\|_X\|\underline{x}\|_X} < +\infty \qquad \forall\boldsymbol{\mu}\in\mathcal{D};$$

(ii) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is coercive over $X_0 = \{\underline{w}\in X\colon \mathcal{B}(\underline{w},q;\boldsymbol{\mu}) = 0 \quad \forall q\in Q\}\subset X$, i.e., there exists a constant $\alpha_0>0$ such that

$$\alpha(\boldsymbol{\mu}) = \inf_{\underline{x}\in X_0}\frac{\mathcal{A}(\underline{x},\underline{x};\boldsymbol{\mu})}{\|\underline{x}\|_X^2} \geq \alpha_0 \qquad \forall\boldsymbol{\mu}\in\mathcal{D};$$

(iii) the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ is continuous over $X\times Q$

$$\gamma_b(\boldsymbol{\mu}) = \sup_{\underline{w}\in X}\sup_{q\in Q}\frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X\|q\|_Q} < +\infty \qquad \forall\boldsymbol{\mu}\in\mathcal{D};$$

(iv) the bilinear form $\mathcal{B}(\cdot,\cdot)$ satisfies the inf-sup condition over $X\times Q$, i.e., there exists a constant $\beta_0>0$ such that

$$\beta(\boldsymbol{\mu}) = \inf_{q\in Q}\sup_{\underline{w}\in X}\frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X\|q\|_Q} \geq \beta_0 \qquad \forall\boldsymbol{\mu}\in\mathcal{D}; \tag{2.9}$$

(v) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric and nonnegative over X.

Holding these assumptions, the optimal control problem has a unique solution $\underline{x}(\boldsymbol{\mu})\in X$ for any $\boldsymbol{\mu}\in\mathcal{D}$, and the solution can be determined by solving the following saddle-point problem (i.e., the optimality system): given $\boldsymbol{\mu}\in\mathcal{D}$, find $(\underline{x}(\boldsymbol{\mu}),p(\boldsymbol{\mu}))\in X\times Q$ such that

$$\begin{cases} \mathcal{A}(\underline{x}(\boldsymbol{\mu}),\underline{w};\boldsymbol{\mu}) + \mathcal{B}(\underline{w},p(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}),\underline{w}\rangle & \forall\underline{w}\in X,\\ \mathcal{B}(\underline{x}(\boldsymbol{\mu}),q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}),q\rangle & \forall q\in Q, \end{cases} \tag{2.10}$$

where $p(\boldsymbol{\mu})$ is the Lagrange multiplier (i.e., the adjoint variable) associated to the constraint. In fact, the equations in (2.10) are nothing but the first-order necessary

(and sufficient[4]) optimality conditions for the unconstrained optimization problem of finding saddle points $(\underline{x}, p) \in X \times Q$ of the Lagrangian, i.e., (2.10) is equivalent to

$$\nabla \mathcal{L}(\underline{x}(\boldsymbol{\mu}), p(\boldsymbol{\mu}); \boldsymbol{\mu})[\underline{w}, q] = 0 \qquad \forall (\underline{w}, q) \in X \times Q. \tag{2.11}$$

Furthermore, we remark that the optimality system (2.10) is in fact the usual optimality system given by the state, adjoint, and optimality equations.

Let us now verify the fulfillment of the hypotheses (i)–(v).

LEMMA 2.1. *The bilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ satisfy the Brezzi assumptions* (i)–(v).

Proof. It is sufficient to exploit the assumptions made on the bilinear forms $a(\cdot,\cdot;\boldsymbol{\mu})$, $c(\cdot,\cdot;\boldsymbol{\mu})$, $m(\cdot,\cdot;\boldsymbol{\mu})$, and $n(\cdot,\cdot;\boldsymbol{\mu})$; see, for instance, [11]. In view of the design of a suitable RB scheme it is useful to show here the proof of the fulfillment of the inf-sup condition for the bilinear form $\mathcal{B}(\cdot,\cdot)$. We exploit the fact that $Y \equiv Q$ and the coercivity property of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$

$$\begin{aligned}\sup_{0 \neq \underline{w} \in X} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X} &= \sup_{0 \neq (z,v) \in Y \times U} \frac{a(z, q; \boldsymbol{\mu}) - c(v, q; \boldsymbol{\mu})}{(\|z\|_Y^2 + \|v\|_U^2)^{1/2}} \\ &\geq_{(z,v)=(q,0)} \frac{a(q, q; \boldsymbol{\mu})}{\|q\|_Y} \geq \tilde{\alpha}(\boldsymbol{\mu}) \|q\|_Y = \tilde{\alpha}(\boldsymbol{\mu}) \|q\|_Q.\end{aligned}$$

Note that the inequality $\beta(\boldsymbol{\mu}) \geq \tilde{\alpha}(\boldsymbol{\mu})$ plays a crucial role in the following. ∎

Then, for any $\boldsymbol{\mu} \in \mathcal{D}$, the optimal control problem (2.3) is equivalent to the saddle-point problem (2.10) and the latter admits a unique solution $(\underline{x}(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$. Moreover, the solution satisfies the stability estimate

$$\|\underline{x}(\boldsymbol{\mu})\|_X + \|p(\boldsymbol{\mu})\|_Q \leq C(\|\underline{F}(\boldsymbol{\mu})\|_{X'} + \|G(\boldsymbol{\mu})\|_{Q'}) \qquad \forall \boldsymbol{\mu} \in \mathcal{D},$$

where C is a positive constant (possibly $\boldsymbol{\mu}$-dependent).

Let us finally observe that, thanks to the affine parameter dependence assumption (2.5), an affine decomposition holds also for the bilinear and linear forms in (2.10), i.e., for some finite Q_a, Q_b, Q_f, Q_g, they can be expressed as

$$\mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\, \mathcal{A}^q(\underline{x}, \underline{w}), \qquad \mathcal{B}(\underline{w}, p; \boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\, \mathcal{B}^q(\underline{w}, p), \tag{2.12}$$

$$\langle G(\boldsymbol{\mu}), q \rangle = \sum_{q=1}^{Q_g} \Theta_g^q(\boldsymbol{\mu})\, \langle G^q, q \rangle, \qquad \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu})\, \langle \underline{F}^q, \underline{w} \rangle, \tag{2.13}$$

where the coefficients $\Theta^q(\boldsymbol{\mu})$ and the $\boldsymbol{\mu}$-independent linear and bilinear forms are related to those appearing in (2.5). For example, $Q_a = \tilde{Q}_m + \tilde{Q}_n$, $\Theta_a^q(\boldsymbol{\mu}) = \tilde{\Theta}_m^q(\boldsymbol{\mu})$, and $\mathcal{A}^q(\underline{x}, \underline{w}) = m^q(y, z)$ for $1 \leq q \leq \tilde{Q}_m$, while $\Theta_a^{q+\tilde{Q}_m}(\boldsymbol{\mu}) = \tilde{\Theta}_n^q(\boldsymbol{\mu})$ and $\mathcal{A}^{q+\tilde{Q}_m}(\underline{x}, \underline{w}) = n^q(u, v)$ for $1 \leq q \leq \tilde{Q}_n$.

2.3. Truth approximation. Let $\mathcal{T}_\mathcal{N}$ be a triangulation of the domain Ω, and we denote $V_\mathcal{N}^r$ the space of globally continuous functions that are polynomials of degree r on the single elements of the triangulation. Then we define $Y^\mathcal{N} = Y \cap V_\mathcal{N}^r$, $Q^\mathcal{N} \equiv Y^\mathcal{N}$,

[4]We recall that in the linear/quadratic case the usual second-order sufficient optimality condition—requiring the second derivative of the Lagrangian functional to be coercive on the null space of the linearized state equation [17, 13]—reduces to the assumption (ii) stated above.

and $U_{\mathcal{N}} = U \cap V_{\mathcal{N}}^r$ in such a way that $X^{\mathcal{N}} = Y^{\mathcal{N}} \times U^{\mathcal{N}} \subset X$, $Q^{\mathcal{N}} \subset Q$ are sequences of FE approximation spaces. Moreover we indicate with $\mathcal{N}$ the global dimension—typically very "large"—of the product space $X^{\mathcal{N}} \times Q^{\mathcal{N}}$, i.e., $\mathcal{N} = \mathcal{N}_X + \mathcal{N}_Q$, where $\mathcal{N}_X = \mathcal{N}_Y + \mathcal{N}_U$ and $\mathcal{N}_Y = \mathcal{N}_Q$.

Following an optimize-then-discretize approach—rather than a discretize-then-optimize approach; see, e.g., [10]—we introduce the truth Galerkin-FE approximation of the optimality system (2.10): given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu})) \in X^{\mathcal{N}} \times Q^{\mathcal{N}}$ such that

$$\begin{cases} \mathcal{A}(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle & \forall \underline{w} \in X^{\mathcal{N}}, \\ \mathcal{B}(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q \rangle & \forall q \in Q^{\mathcal{N}}. \end{cases} \tag{2.14}$$

Provided $Y^{\mathcal{N}} \equiv Q^{\mathcal{N}}$, the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ remains continuous over $X^{\mathcal{N}} \times X^{\mathcal{N}}$ and coercive over $X_0^{\mathcal{N}} = \{\underline{w} \in X^{\mathcal{N}} : \mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = 0 \quad \forall q \in Q^{\mathcal{N}}\}$, and the bilinear form $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ remains continuous and inf-sup stable over $X^{\mathcal{N}} \times Q^{\mathcal{N}}$, i.e., there exists a constant $\beta_0 > 0$ such that

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{q \in Q^{\mathcal{N}}} \sup_{\underline{w} \in X^{\mathcal{N}}} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} \geq \beta_0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{2.15}$$

In particular, mimicking the proof of Lemma 2.1, we can easily show that $\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$, with $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ being the FE coercivity constant of the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu})$. Therefore, thanks to Brezzi theory, also the FE approximation (2.14) is well-posed.

Let us now investigate the structure of the algebraic system associated to the Galerkin approximation (2.14). We denote with $\{\underline{\varphi}_j \in X^{\mathcal{N}}\}_{j=1}^{\mathcal{N}_X}$, $\{\phi_k \in Q^{\mathcal{N}}\}_{k=1}^{\mathcal{N}_Q}$, the basis functions of the spaces $X^{\mathcal{N}}$, $Q^{\mathcal{N}}$, respectively. Then, (2.14) is equivalent to the linear system

$$\underbrace{\begin{pmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{\mathcal{K}(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \mathbf{p}^{\mathcal{N}}(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}(\boldsymbol{\mu}) \\ \mathbf{G}(\boldsymbol{\mu}) \end{pmatrix}, \tag{2.16}$$

where $\mathbf{x}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\mathbf{p}^{\mathcal{N}}(\boldsymbol{\mu})$ denote the vectors of the coefficients in the expansion of $\underline{x}(\boldsymbol{\mu})$ and $p(\boldsymbol{\mu})$, while, for example, the elements of the matrix A are given by $A_{ij}(\boldsymbol{\mu}) = \mathcal{A}(\underline{\varphi}_j, \underline{\varphi}_i; \boldsymbol{\mu})$ for $1 \leq i, j \leq \mathcal{N}_X$. Let us notice that also the matrices appearing in (2.16) inherit the same affine decompositions (2.12), so that

$$A(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\, A^q, \qquad B(\boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\, B^q,$$

where the $\boldsymbol{\mu}$-independent matrices A^q, B^q represent the discrete counterparts of the corresponding bilinear. It is analogous for the vectors $\mathbf{F}(\boldsymbol{\mu})$ and $\mathbf{G}(\boldsymbol{\mu})$.

For the resolution of the linear system (2.16) several strategies can be employed (see, for instance, [17, 1]): a popular alternative is based on the so-called reduced Hessian methods, in which block elimination on the state and adjoint variables yields a reduced[5] system for the control variable whose matrix is the Schur complement of the optimality system. A radically alternative strategy consists of using *full space* (also

[5]Here *reduced* must not be understood in the sense of *reduced-order model*.

called *all-at-once*) methods, where the optimality system is solved simultaneously for the state, adjoint, and control variables. Both approaches present advantages and disadvantages and require problem-tailored design of suitable preconditioners and iterative linear solvers. Yet, beside the choice of the favorite solution algorithm, it is well known that the numerical solution of an optimal control problem entails large computational costs and may be very time-consuming already in the nonparametric case. Therefore, when performing the optimization process for many different parameter values or else when, for a new given configuration, the solution has to be computed in a rapid way, reducing the computational complexity is mandatory. This is why we advocate using suitable model-order reduction techniques.

3. The reduced basis approximation. The idea of the RB method is to efficiently compute an approximation of $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}))$ by using approximation spaces made up of well-chosen solutions of (2.14), i.e., corresponding to specific choices of the parameter values. As already mentioned in the introduction, the main assumption is that the solution of (2.14) depends smoothly on the parameters, thus implying the parametric manifold $\mathcal{M}^{\mathcal{N}}$ to be smooth and approximable by selecting some snapshot FE solutions.

3.1. Construction of RB approximation spaces and stability properties. Let us suppose that we are given a set of hierarchical RB approximation subspaces $X_N \subset X^{\mathcal{N}}$ and $Q_N \subset Q^{\mathcal{N}}$, $N \in [1, N_{\max}]$, made up of properly selected FE solutions. By using Galerkin projection onto the low-dimensional subspace $X_N \times Q_N$, we obtain the following reduced basis approximation: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{A}(\underline{x}_N(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle & \forall \underline{w} \in X_N, \\ \mathcal{B}(\underline{x}_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q \rangle & \forall q \in Q_N. \end{cases} \tag{3.1}$$

The existence, uniqueness, and stability of the solution to problem (3.1) depend on the properties of the RB spaces X_N and Q_N, which are analyzed in the following.

Let us take, for given $N \in [1, N_{\max}]$, a finite set of parameter values $S_N = \{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N\}$ and consider the corresponding FE solutions $\{(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^n), p^{\mathcal{N}}(\boldsymbol{\mu}^n))\}_{n=1}^N$, the so-called snapshots of the corresponding optimal control problem. We (naively) define the RB spaces for the state, control, and adjoint variables, respectively, as

$$\begin{aligned} Y_N &= \operatorname{span}\{\zeta_n := y^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \\ U_N &= \operatorname{span}\{\lambda_n := u^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \\ Q_N &= \operatorname{span}\{\xi_n := p^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \end{aligned} \tag{3.2}$$

and denote $X_N = Y_N \times U_N$. Let us discuss the well-posedness of the RB approximation (3.1). While the continuity properties of the bilinear forms over the RB spaces are automatically inherited from the parents spaces (i.e., the FE spaces), the coercivity property of the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ over

$$X_0^N = \{\underline{w} \in X_N : \mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = 0 \quad \forall q \in Q_N\}$$

and the fulfillment of the inf-sup condition of $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ are not granted and have to be proved. In particular, the problem (3.1) has to satisfy the following RB inf-sup condition: there exists $\beta_0 > 0$ such that

$$\beta_N(\boldsymbol{\mu}) = \inf_{q \in Q_N} \sup_{\underline{w} \in X_N} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} \geq \beta_0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{3.3}$$

The first idea in order to prove the fulfillment of (3.3) is to mimic the proof already used for the continuous problem and its FE approximation; see Lemma 2.1. Unfortunately, while in the continuous case (respectively, for the FE approximation) the state and adjoint spaces Y and Q (respectively, $Y^{\mathcal{N}}$ and $Q^{\mathcal{N}}$) are equivalent, with the choice (3.2), we lose this property on the corresponding RB spaces, i.e., $Y_N \neq Q_N$.

In order to recover the stability of the RB approximation, we therefore need to enrich in some way at least one of the RB spaces involved. This is not surprising when dealing with the RB approximation of a saddle-point problem, since the structure of this class of problems—in particular the requirement to fulfill the inf-sup condition—implies that building the RB approximation spaces solely from snapshots is not always sufficient. In fact, there are at least two other examples where a similar treatment shows to be necessary: the application of the RB method to parametrized Stokes equations [28, 32, 29, 8] and to parametrized variational inequalities [12]. Two possible strategies to achieve the stability of the approximation are the use of a suitable supremizer operator or the use of the same (properly defined) space for the state and adjoint variables. While the first option can be seen as a trial to mimic what has been done in the case of the Stokes problem, the second option follows naturally from the discussion above and already has been considered in some previous works [5, 19] (even if not specifically for this reason). We chose to pursue the second one, being aware that these issues deserve further investigation in order to also explore other strategies, which might be more convenient from the computational point of view.

We thus define the following *aggregated* space for the state and adjoint variables

$$Z_N = \text{span}\{\zeta_n := y^{\mathcal{N}}(\boldsymbol{\mu}^n),\ \xi_n := p^{\mathcal{N}}(\boldsymbol{\mu}), \quad n = 1, \dots, N\}, \tag{3.4}$$

and we let

$$Y_N = Z_N, \qquad X_N = Y_N \times U_N, \qquad Q_N = Z_N. \tag{3.5}$$

LEMMA 3.1. *If the reduced spaces X_N and Q_N are chosen as in* (3.4)–(3.5)*, then the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ satisfies the inf-sup condition* (3.3)*. Moreover, we have the estimate*

$$\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) \qquad \forall \boldsymbol{\mu} \in \mathcal{D},$$

where $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ is the coercivity constant associated to the FE approximation of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$.

Proof. It is sufficient to follow the proof of Lemma 2.1. In fact,

$$\begin{aligned}\beta_N(\boldsymbol{\mu}) &= \inf_{q\in Q_N} \sup_{\underline{w}\in X_N} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} = \inf_{q\in Z_N} \sup_{(z,v)\in Z_N\times U_N} \frac{a(z,q;\boldsymbol{\mu}) - c(v,q;\boldsymbol{\mu})}{\|(z,v)\|_X \|q\|_Q} \\ &\geq \inf_{(z,v)=(q,0)} \inf_{q\in Z_N} \frac{a(q,q;\boldsymbol{\mu})}{\|q\|_Q} = \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) > 0.\end{aligned}$$

Note that the choice $z = q$ is allowed because both z and q belong to the space Z_N. ∎

The well-posedness of the RB approximation is ensured by the following.

PROPOSITION 3.2. *If the reduced spaces X_N and Q_N are chosen as in* (3.4)–(3.5)*, then, for any $\boldsymbol{\mu} \in \mathcal{D}$, the RB approximation* (3.1) *has a unique solution $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ depending continuously on the data.*

Proof. It suffices to check that the assumptions of the Brezzi theorem hold. As already mentioned, the continuity properties of the bilinear and linear forms over the

RB space are automatically inherited from the parents spaces (i.e., the FE spaces). The fulfillment of the inf-sup condition of the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ has been proved in Lemma 3.1, while the fulfillment of the coercivity condition of the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ can be proved using the same arguments as in Lemma 2.1. ∎

3.2. Algebraic formulation and offline-online computational procedure. Let us now investigate the algebraic formulation associated to the enriched spaces introduced in the previous section. Let $\{\tau_j\}_{j=1}^{2N} = \{\zeta_j\}_{j=1}^{N} \cup \{\xi_j\}_{j=1}^{N}$ such that $Z_N = \text{span}\{\tau_j,\, j = 1,\dots,2N\}$, and we can express the RB state, adjoint, and control solutions as

$$\underline{x}_N(\boldsymbol{\mu}) = \sum_{j=1}^{3N} x_{Nj}(\boldsymbol{\mu})\underline{\sigma}_j, \qquad p_N(\boldsymbol{\mu}) = \sum_{j=1}^{2N} p_{Nj}(\boldsymbol{\mu})\tau_j,$$

where $\underline{\sigma}_j = (\tau_j, 0)$ for $j = 1,\dots,2N$, while $\underline{\sigma}_j = (0,\lambda_j)$ for $j = 2N+1,\dots,3N$, in such a way that $X_N = \text{span}\{\underline{\sigma}_j,\, j = 1,\dots,3N\}$. Hence, given a parameter $\boldsymbol{\mu}$, the RB solution of the problem (3.1) can be written as a combination of basis functions with weights given by the following reduced basis linear system:

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{\mathcal{K}_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}, \tag{3.6}$$

where $A_N(\boldsymbol{\mu}) = \sum \Theta_a^q(\boldsymbol{\mu}) A_N^q$, $B_N(\boldsymbol{\mu}) = \sum \Theta_b^q(\boldsymbol{\mu}) B_N^q$, and the submatrices A_N^q and B_N^q are given by $(A_N)_{ij}^q = \mathcal{A}^q(\underline{\sigma}_j, \underline{\sigma}_i)$, $(B_N)_{li}^q = \mathcal{B}^q(\underline{\sigma}_i, \tau_l)$ for $1 \le i,j \le 3N$, $1 \le l \le 2N$.

In order to state the connection between the RB linear system (3.6) and the FE discretization (2.16), let us define the *basis matrices* $Z_z = (\boldsymbol{\tau}_1 \,|\, \cdots \,|\, \boldsymbol{\tau}_N) \in \mathbb{R}^{\mathcal{N}\times 2N}$, $Z_u = (\boldsymbol{\lambda}_1 \,|\, \cdots \,|\, \boldsymbol{\lambda}_N) \in \mathbb{R}^{\mathcal{N}\times N}$, and

$$Z_x = \begin{pmatrix} Z_z & 0 \\ 0 & Z_u \end{pmatrix} \in \mathbb{R}^{2\mathcal{N}\times 3N}, \qquad Z = \begin{pmatrix} Z_z & 0 & 0 \\ 0 & Z_u & 0 \\ 0 & 0 & Z_z \end{pmatrix} \in \mathbb{R}^{3\mathcal{N}\times 5N}.$$

Then, the matrix $\mathcal{K}_N = Z^T \mathcal{K} Z$ is given by

$$\mathcal{K}_N = \begin{pmatrix} A_N & B_N^T \\ B_N & 0 \end{pmatrix} = \begin{pmatrix} Z_x^T A Z_x & Z_x^T B^T Z_z \\ Z_z^T B Z_x & 0 \end{pmatrix}. \tag{3.7}$$

Thus the matrix $\mathcal{K}_N$ is still symmetric, with saddle-point structure, and has dimension $5N \times 5N$. Although being dense (rather than sparse as in the FE case), the system matrix is very small, with a size independent of the FE space dimension $\mathcal{N}$; for this reason the RB linear system can be easily solved using direct solvers. Furthermore, to keep under control the condition number of the matrix $\mathcal{K}_N$, we have adopted the Gram–Schmidt orthonormalization procedure [30]. In particular we apply the Gram–Schmidt procedure separately on the basis functions of the space Z_N and on the basis functions of the space U_N.

Thanks to the assumption of affine parameter dependence, we can decouple the formation of the matrix $\mathcal{K}_N(\boldsymbol{\mu})$ in two stages, the offline and online stages, that enable the efficient resolution of the system (3.6) for each new parameter $\boldsymbol{\mu}$.

In particular, in the offline stage, performed only once, we first compute and store the basis function $\{\tau_i\}_{i=1}^{2N}$ and $\{\lambda_j\}_{j=1}^{N}$ and form the $\boldsymbol{\mu}$-independent matrices

A_N^q, $1 \le q \le Q_a$, B_N^q, $1 \le q \le Q_b$ and the vectors F_N^q, $1 \le q \le Q_f$, G_N^q, $1 \le q \le Q_g$. The operation count depends on N, Q_a, Q_b, Q_f, Q_g, and $\mathcal{N}$.

In the online stage, performed for each new value $\boldsymbol{\mu}$, we use the precomputed matrices A_N^q, B_N^q and vectors F_N^q, G_N^q to assemble the (full) matrix $\mathcal{K}_N$ and the vectors $\mathbf{F}_N$, $\mathbf{G}_N$ appearing in (3.6); we then solve the resulting system to obtain $(\mathbf{x}_N, \mathbf{p}_N)$. The online operation count depends on N, Q_a, Q_b, Q_f, Q_g but is independent of $\mathcal{N}$. In particular we need $O((Q_a + Q_b)N^2)$ and $O((Q_f + Q_g)N)$ operations to assemble matrices and vectors, and $O((5N)^3)$ operations to solve the RB linear system (3.6).

3.3. Sampling strategy. For the construction of the hierarchical Lagrange RB approximation spaces—and thus the optimal choice of the sample points $\boldsymbol{\mu}^n$, $1 \le n \le N$—we rely on the sampling strategy based on the standard greedy algorithm [30, 29]. Let $\Xi_{\text{train}} \subset \mathcal{D}$ be a finite-dimensional sample set, called the set of *train* samples. The cardinality of Ξ_{train} will be denoted with n_{train}, which we assume to be sufficiently large such that Ξ_{train} is a good approximation of the set $\mathcal{D}$ (a finite-dimensional surrogate for $\mathcal{D}$). The idea of the greedy procedure is that, starting with a train sample Ξ_{train}, we adaptively select (in the sense of minimizing a suitable error indicator) N parameters $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N$ and form the hierarchical sequence of reduced basis spaces X_N, Q_N as in (3.4)–(3.5). At each iteration N, the greedy algorithm appends to the previously *retained* snapshots that particular candidate—over all candidate snapshots $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}))$, $\boldsymbol{\mu} \in \Xi_{\text{train}}$—which is least well approximated by the "old" RB space $X_{N-1} \times Q_{N-1}$. The key ingredient of this adaptive procedure is a rigorous, sharp, and inexpensive estimator $\Delta_N(\boldsymbol{\mu})$ for the RB error such that

$$\left(\|\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \underline{x}_N(\boldsymbol{\mu})\|_X^2 + \|p^{\mathcal{N}}(\boldsymbol{\mu}) - p_N(\boldsymbol{\mu})\|_Q^2\right)^{1/2} \le \Delta_N(\boldsymbol{\mu}), \tag{3.8}$$

where $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))$ is the RB approximated solution associated with the generic RB space $X_N \times Q_N$. The construction of the a posteriori error estimator Δ_N will be described in detail in section 4.

Given such an estimator, we can state precisely the steps required by the greedy algorithm. By denoting ε_{tol} a chosen tolerance for the stopping criterium, the greedy sampling strategy can be implemented as reported in Algorithm 1.

ALGORITHM 1: GREEDY ALGORITHM FOR PARAMETRIZED OPTIMAL CONTROL PROBLEMS.

$S_1 = \{\boldsymbol{\mu}^1\}$, compute $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^1), p^{\mathcal{N}}(\boldsymbol{\mu}^1))$ by solving the truth approximation (2.14)
$U_1 = \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^1)\}, \quad Z_1 = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^1), p^{\mathcal{N}}(\boldsymbol{\mu}^1)\}$
$X_1 = Z_1 \times U_1, \quad Q_1 = Z_1$
for $N = 2 : N_{\max}$ **do**
  $\boldsymbol{\mu}^N = \arg\max_{\boldsymbol{\mu} \in \Xi_{\text{train}}} \Delta_{N-1}(\boldsymbol{\mu})$
  $\varepsilon_{N-1} = \Delta_{N-1}(\boldsymbol{\mu})$
  if $\varepsilon_{N-1} \le \varepsilon_{\text{tol}}$
    $N_{\max} = N - 1$
  end if
  compute $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^N), p^{\mathcal{N}}(\boldsymbol{\mu}^N))$ by solving the truth approximation (2.14)
  $S_N = S_{N-1} \cup \{\boldsymbol{\mu}^N\}$
  $U_N = U_{N-1} \cup \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^N)\}, \quad Z_N = Z_{N-1} \cup \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^N), p^{\mathcal{N}}(\boldsymbol{\mu}^N)\}$
  $X_N = Z_N \times U_N, \quad Q_N = Z_N$
end for

We underline again that the key point in the algorithm is to exploit an a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ efficiently computable, since at each iteration the algorithm requires to evaluate $\Delta_N(\boldsymbol{\mu})$ $\forall \boldsymbol{\mu} \in \Xi_{\text{train}}$.

4. Rigorous a posteriori error estimates. In the RB framework, a posteriori error estimates play a crucial role in order to guarantee the efficiency and reliability of the method. As regards efficiency, the error bound is essential in the sampling procedure, by allowing an exhaustive exploration of the parameters domain and a proper selection of the basis functions. As regards reliability, at the online stage for each new value of parameter $\boldsymbol{\mu} \in \mathcal{D}$, the a posteriori estimator permits one to bound the error of the RB approximation with respect to the underlying truth approximation.

Different strategies can be pursued in order to provide a posteriori error estimation for parametrized optimal control problems. In [5] an efficient yet not rigorous estimator has been proposed dealing with time-dependent optimal control problems, while recently in [19] similar techniques combined with some previous results proposed in [36] have been applied to the same problem considered here, providing an efficient and rigorous estimator. In this work, we propose a new a posteriori error estimate that can be easily obtained exploiting the structure of the optimality system. In particular, once the saddle-point structure of the optimality system has been highlighted, one can apply three different approaches, already proposed in the RB context: (i) to exploit Brezzi stability theory [4]; (ii) to use the Nečas–Babuška stability theory [2, 23]; or (iii) to adopt a penalty approach [11]. While the approaches (i) and (iii) have been only recently applied in the RB context, respectively, in [8] and [7], the second approach is quite standard in the RB context [30]. We thus choose to pursue the latter, exploiting the analogies with the RB scheme proposed for affinely parametrized Stokes equations in [32, 29].

In section 4.1 we construct a rigorous and inexpensive (i.e., $\mathcal{N}$-independent) a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ such that

$$\left(\|\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \underline{x}_N(\boldsymbol{\mu})\|_X^2 + \|p^{\mathcal{N}}(\boldsymbol{\mu}) - p_N(\boldsymbol{\mu})\|_Q^2\right)^{1/2} \leq \Delta_N(\boldsymbol{\mu}). \tag{4.1}$$

Then in section 4.2, using the same ingredients, we construct a rigorous and inexpensive a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ for the error on the cost functional, i.e.,

$$|J(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) - J(y_N(\boldsymbol{\mu}), u_N(\boldsymbol{\mu}); \boldsymbol{\mu})| \leq \Delta_N^J(\boldsymbol{\mu}). \tag{4.2}$$

4.1. Bound for the solution. Since saddle-point problems can be regarded as a particular case of *weakly coercive* (also called *noncoercive*) problems, the construction of the error estimator $\Delta_N(\boldsymbol{\mu})$ can be carried out by using the Nečas–Babuška stability theory [2, 23].

Upon defining the space $\mathcal{X} = X \times Q$, the bilinear form $\mathsf{B}(\cdot, \cdot; \boldsymbol{\mu})\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$,

$$\mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu}) := \mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p; \boldsymbol{\mu}) + \mathcal{B}(\underline{x}, q; \boldsymbol{\mu}), \tag{4.3}$$

and the linear continuous functional $\mathsf{F}(\cdot; \boldsymbol{\mu})\colon \mathcal{X} \to \mathbb{R}$,

$$\mathsf{F}(\mathsf{w}; \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle + \langle G(\boldsymbol{\mu}), q \rangle, \tag{4.4}$$

where $\mathsf{x} = (\underline{x}, p) \in \mathcal{X}$ and $\mathsf{w} = (\underline{w}, q) \in \mathcal{X}$, problem (2.10) can equivalently be reformulated as follows: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\text{find } \mathsf{x} \in \mathcal{X} \text{ s.t.} \qquad \mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu}) = \mathsf{F}(\mathsf{w}; \boldsymbol{\mu}) \qquad \forall \mathsf{w} \in \mathcal{X}. \tag{4.5}$$

According to the Nečas theorem, the problem (4.5) is well posed if for any $\boldsymbol{\mu} \in \mathcal{D}$, the bilinear form $\mathsf{B}(\cdot,\cdot;\boldsymbol{\mu})$ is continuous and weakly coercive, i.e., there exists a constant $\hat{\beta}_0 > 0$ such that[6]

$$\hat{\beta}(\boldsymbol{\mu}) = \inf_{\mathsf{w}\in\mathcal{X}} \sup_{\mathsf{x}\in\mathcal{X}} \frac{\mathsf{B}(\mathsf{x},\mathsf{w};\boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}}\|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0. \tag{4.6}$$

Moreover, holding these assumptions, for any $\boldsymbol{\mu} \in \mathcal{D}$ the unique solution satisfies the following stability estimate:

$$\|\mathsf{x}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{1}{\hat{\beta}(\boldsymbol{\mu})} \|\mathsf{F}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}. \tag{4.7}$$

Actually, since the bilinear forms $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ satisfy the hypotheses of Brezzi theorem, it can be shown (see, e.g., [37, 11]) that the compound form $\mathsf{B}(\cdot,\cdot;\boldsymbol{\mu})$ is bounded and weakly coercive. Similarly, the FE and RB approximations satisfy the same inf-sup condition,

$$\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) := \inf_{\mathsf{w}\in\mathcal{X}^{\mathcal{N}}} \sup_{\mathsf{x}\in\mathcal{X}^{\mathcal{N}}} \frac{\mathsf{B}(\mathsf{x},\mathsf{w};\boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}}\|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0^{\mathcal{N}} > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{4.8}$$

$$\hat{\beta}_N(\boldsymbol{\mu}) := \inf_{\mathsf{w}\in\mathcal{X}_N} \sup_{\mathsf{x}\in\mathcal{X}_N} \frac{\mathsf{B}(\mathsf{x},\mathsf{w};\boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}}\|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0^{N} > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{4.9}$$

where $\mathcal{X}^{\mathcal{N}} = X^{\mathcal{N}} \times Q^{\mathcal{N}}$ and $\mathcal{X}_N = X_N \times Q_N$. Moreover the stability estimate (4.7) holds also for the FE and RB approximations, in particular

$$\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{1}{\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})} \|\mathsf{F}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'} \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{4.10}$$

The construction of the a posteriori error estimation is based on two main ingredients (as usual in RB context): an effective calculation of a lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ and the calculation of the dual norm of the residual [24]. As regards the first one, we suppose to have at our disposal a $\boldsymbol{\mu}$-dependent lower bound $\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) : \mathcal{D} \to \mathbb{R}$ such that

$$\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) \geq \hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) \geq \hat{\beta}_0 > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D} \tag{4.11}$$

and the online computational time to evaluate $\boldsymbol{\mu} \to \hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})$ is independent of $\mathcal{N}$. The calculation of $\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})$ can be carried out using the natural norm successive constraint method, an improvement of the successive constraint method (SCM) algorithm specifically tailored for noncoercive problems; see, e.g., [16, 29] for a detailed explanation of this procedure as well as for many numerical tests.

As regards the second ingredient, the residual $\mathsf{r}(\cdot;\boldsymbol{\mu}) \in (\mathcal{X}^{\mathcal{N}})'$ is defined as

$$\mathsf{r}(\mathsf{w};\boldsymbol{\mu}) := \mathsf{F}(\mathsf{w};\boldsymbol{\mu}) - \mathsf{B}(\mathsf{x}_N,\mathsf{w};\boldsymbol{\mu}) \qquad \forall \mathsf{w} \in \mathcal{X}^{\mathcal{N}}.$$

Finally, let us define the error between the truth FE approximation and the RB approximation, $\mathsf{e}(\boldsymbol{\mu}) := \mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})$. We can now formulate an a posteriori estimator for the error $\mathsf{e}(\boldsymbol{\mu})$.

[6] In the following we will refer to the inf-sup constant $\hat{\beta}(\boldsymbol{\mu})$ (4.6) as the Babuška inf-sup constant, in contrast to the Brezzi inf-sup constant $\beta(\boldsymbol{\mu})$ (2.9); similar notation will be used for their FE and RB approximations.

PROPOSITION 4.1. *For any given* $\boldsymbol{\mu} \in \mathcal{D}$, $N \in [1, N_{\max}]$, *and* $\hat{\beta}_{LB}(\boldsymbol{\mu})$ *satisfying* (4.11), *we define*

$$\Delta_N(\boldsymbol{\mu}) = \frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}}{\hat{\beta}_{LB}(\boldsymbol{\mu})}. \tag{4.12}$$

Then, $\Delta_N(\boldsymbol{\mu})$ *is an upper bound for the error* $\mathsf{e}(\boldsymbol{\mu})$,

$$\|\mathsf{e}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \Delta_N(\boldsymbol{\mu}) \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \quad \forall N \in [1, N_{\max}]. \tag{4.13}$$

Proof. The problem statement for the FE solution $\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu})$ and for the RB solution $\mathsf{x}_N(\boldsymbol{\mu})$ and the bilinearity of $\mathsf{B}(\cdot,\cdot;\boldsymbol{\mu})$ imply that the error $\mathsf{e}(\boldsymbol{\mu})$ statisfy the following equation: $\mathsf{B}(\mathsf{e}(\boldsymbol{\mu}),\mathsf{w};\boldsymbol{\mu}) = \mathsf{r}(\mathsf{w};\boldsymbol{\mu})$ $\forall \mathsf{w} \in \mathcal{X}^{\mathcal{N}}$. Then it suffices to apply the stability estimate (4.7) and exploit the lower bound (4.11) for the Babuška inf-sup constant. ∎

As usual (see, for instance, [30, 29]), the computation of the dual norm of the residual can be decomposed in two stages: an expensive, $\boldsymbol{\mu}$-independent offline stage and an inexpensive online stage. As a result, given $\boldsymbol{\mu} \in \mathcal{D}$, the evaluation of $\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}$ requires $O(25N^2Q_B^2 + 5NQ_BQ_f + Q_F^2)$ operations, independent of $\mathcal{N}$.

4.2. A posteriori error bound for the cost functional. To develop an a posteriori error bound on the cost functional $J(y,u;\boldsymbol{\mu})$, we first observe that this is equivalent to providing an estimator for the error on $\mathcal{J}(\underline{x};\boldsymbol{\mu})$, since $\mathcal{J}(\cdot;\boldsymbol{\mu})$ and $J(\cdot,\cdot;\boldsymbol{\mu})$ differ only in a constant term once $\boldsymbol{\mu} \in \mathcal{D}$ is fixed. Although the cost functional $\mathcal{J}(\cdot;\boldsymbol{\mu})$ is a *quadratic* functional, thanks to the structure of the optimal control problem we can avoid to use the techniques of error estimation for quadratic outputs already proposed in the RB context; see, for instance, [34, 14, 22]. Rather, following the work in [5] we may use a goal-oriented analysis, a standard tool for the development of a posteriori error estimates for optimal control problems.

The error on the cost functional evaluated with respect to the FE and RB approximations will be denoted with

$$\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu}) = J(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu});\boldsymbol{\mu}) - J(y_N(\boldsymbol{\mu}), u_N(\boldsymbol{\mu});\boldsymbol{\mu}).$$

Recalling the definition of the Lagrangian functional (2.8), we observe that we can use a different formalism to express the gradient of the Lagrangian as

$$\nabla\mathcal{L}(\mathsf{x};\boldsymbol{\mu})[\mathsf{w}] = \mathsf{B}(\mathsf{x},\mathsf{w};\boldsymbol{\mu}) - \mathsf{F}(\mathsf{w};\boldsymbol{\mu}) \qquad \forall \mathsf{w} \in \mathcal{X}. \tag{4.14}$$

Then, we can show the following result.

PROPOSITION 4.2. *For any given* $\boldsymbol{\mu} \in \mathcal{D}$, $N \in [1, N_{\max}]$, *and* $\hat{\beta}_{LB}(\boldsymbol{\mu})$ *satisfying* (4.11), *we define*

$$\Delta_N^J(\boldsymbol{\mu}) = \frac{1}{2}\frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}^2}{\hat{\beta}_{LB}(\boldsymbol{\mu})}. \tag{4.15}$$

Then, $\Delta_N^J(\boldsymbol{\mu})$ *is an upper bound for the error on the cost functional,*

$$|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})| \leq \Delta_N^J(\boldsymbol{\mu}) \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \quad \forall N \in [1, N_{\max}]. \tag{4.16}$$

Proof. The RB error on the cost functional can be rewritten as (see, e.g., [3, 5])

$$\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu}) = \frac{1}{2}\nabla\mathcal{L}(\mathsf{x}_N(\boldsymbol{\mu});\boldsymbol{\mu})[\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})].$$

Thanks to (4.14) we have that

$$\nabla\mathcal{L}(\mathsf{x}_N;\boldsymbol{\mu})[\mathsf{x}^{\mathcal{N}}-\mathsf{x}_N]=\mathsf{B}(\mathsf{x}_N,\mathsf{x}^{\mathcal{N}}-\mathsf{x}_N;\boldsymbol{\mu})-\mathsf{F}(\mathsf{x}^{\mathcal{N}}-\mathsf{x}_N;\boldsymbol{\mu})=\mathsf{r}(\mathsf{x}^{\mathcal{N}}-\mathsf{x}_N;\boldsymbol{\mu}).$$

By exploiting the continuity of the residual $\mathsf{r}(\cdot;\boldsymbol{\mu})$ and the estimate (4.13), we obtain the required bound (4.16). ∎

Note that the error estimator $\Delta_N^J(\boldsymbol{\mu})$ does not need any further ingredients besides those already available: the efficient computation of the dual norm of the residual and the calculation of a lower bound for the Babuška inf-sup constant.

5. Numerical examples. In this section we discuss three numerical examples in order to verify the properties—and to test the performances—of the proposed RB scheme. In the cases in which we consider a parametrized geometry we first define an "original" problem (subscript o) posed over a parameters dependent domain, and then we trace back the problem to a reference domain through suitable affine geometrical mappings (see [30, 29, 22] for the details) in order to recover the formulation (2.10). The implementation of the method has been carried out in the MATLAB environment using an enhanced version of the `rbMIT` library [15]. Since the problems we deal with are of moderate size, in the offline stage we use a multifrontal sparse direct solver to solve the saddle-point problem (2.16) $N_{\max}$ time (one for every selected value of the parameters), thus employing an *all-at-once* approach. We remark that, as the dimension of the problem increases, this strategy is no more viable and we have to rely on suitably preconditioned iterative solvers (see section 2.3). On the other hand, the resolution of the RB system (3.6) can be always performed through a dense direct solver due to its very small dimension.

All the computations are performed on a personal computer with an Intel Core i5-2400S CPU and 16 GB of RAM.

5.1. Test 1: Distributed optimal control for the Laplace equation with geometrical parametrization. We consider an "original" domain $\Omega_o(\boldsymbol{\mu})=\Omega_o^1\cup\Omega_o^2(\boldsymbol{\mu})$ given by a rectangle separated in two subdomains, with the first one parameter independent, as shown in Figure 5.1. We consider two parameters $\boldsymbol{\mu}=(\mu_1,\mu_2)$, with μ_1 being related to the geometry of Ω_o^2 while μ_2 is such that $y_d(\boldsymbol{\mu})=1$ in Ω_o^1 and $y_d(\boldsymbol{\mu})=\mu_2$ in $\Omega_o^2(\boldsymbol{\mu})$, i.e., the observation function is parameter dependent (constant on each subdomain). The set spanned by the parameters is given by $\mathcal{D}=[1,3.5]\times[0.5,2.5]$.

We consider the following optimal control problem:

$$\begin{aligned}&\min_{y_o.u_o} J(y_o(\boldsymbol{\mu}),u_o(\boldsymbol{\mu});\boldsymbol{\mu})=\frac{1}{2}\|y_o(\boldsymbol{\mu})-y_d(\boldsymbol{\mu})\|^2_{L^2(\Omega_o)}+\frac{\alpha}{2}\|u_o(\boldsymbol{\mu})\|^2_{U_o},\\ &\text{s.t.}\begin{cases}-\Delta y_o(\boldsymbol{\mu})=u_o(\boldsymbol{\mu}) & \text{in } \Omega_o(\boldsymbol{\mu}),\\ y_o(\boldsymbol{\mu})=g_D & \text{on } \Gamma_D^o(\boldsymbol{\mu})=\partial\Omega_o(\boldsymbol{\mu}),\end{cases}\end{aligned}\tag{5.1}$$

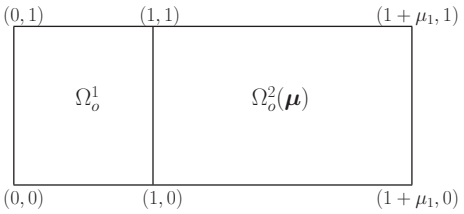

FIG. 5.1. *Test* 1: *original domain* $\Omega_o(\boldsymbol{\mu})$.

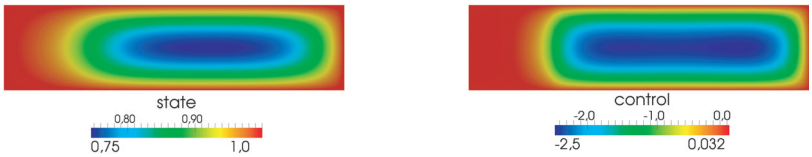

FIG. 5.2. *Test* 1: *representative solution for* $\boldsymbol{\mu} = (3, 0.6)$*; on the left the state variable* y_N*, on the right the optimal control* u_N*.*

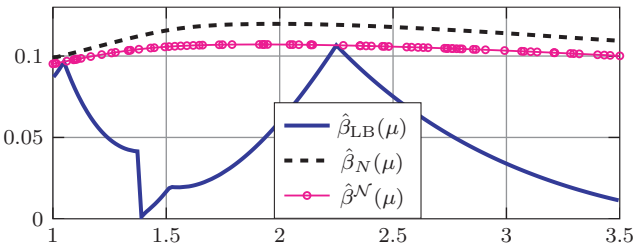

FIG. 5.3. *Test* 1: *lower bound for the Babuška inf-sup constant* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of the geometrical parameter* μ_1 *(on the x-axis).*

where y_o and u_o are the state and control functions defined on the original domain, while the Dirichlet boundary condition is given by $g_D = 1$. After introducing a suitable lifting of the nonhomogeneous Dirichlet condition, we denote with Y_o and U_o the spaces $H_0^1(\Omega_o(\boldsymbol{\mu}))$ and $L^2(\Omega_o(\boldsymbol{\mu}))$, respectively, and moreover, $Q_o \equiv Y_o$. By tracing the problem back to a reference domain $\Omega = \Omega_o(\boldsymbol{\mu}_{\mathrm{ref}})$ (with the arbitrary choice $\boldsymbol{\mu}_{\mathrm{ref}} = (1, 1)$) we obtain the parametrized formulation (2.10), where the affine decompositions (2.12), (2.13) hold with $Q_a = 2$, $Q_b = 3$, $Q_f = 2$, $Q_g = 3$.

Computations are based upon a finite element approximation on $\mathbb{P}^1$ spaces for the state, control, and adjoint variables; the total number of degrees of freedom, i.e., the dimension of the space $\mathcal{X}^{\mathcal{N}} = Y^{\mathcal{N}} \times U^{\mathcal{N}} \times Q^{\mathcal{N}}$, is $\mathcal{N} = 5\,982$, obtained using a mesh of 4 136 triangular elements. The regularization parameter is kept fixed and equal to $\alpha = 0.01$. In Figure 5.2 a representative solution for a fixed value of the parameters is given.

With a fixed tolerance $\varepsilon_{tol} = 5 \cdot 10^{-4}$, $N_{\max} = 12$ basis functions have been selected by the greedy algorithm, thus resulting in a RB linear system of dimension 60×60. In Figure 5.3 we show the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ (defined in (4.8)) obtained using the natural norm SCM algorithm, which requires in this case the solution of $10 + 2Q_B$ eigenproblems of dimension $\mathcal{N}$. (See [16, 29] for further details.) In Figure 5.3 the RB Babuška inf-sup constant $\hat{\beta}_N(\boldsymbol{\mu})$ defined in (4.9) is also reported; in particular we can observe that $\hat{\beta}_N(\boldsymbol{\mu}) \geq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, thus indicating the good stability property of the RB approximation.

Furthermore, as regards the stability properties, in Figure 5.4 we give some numerical results on the discrete Brezzi inf-sup constants $\beta^{\mathcal{N}}(\boldsymbol{\mu})$ and $\beta_N(\boldsymbol{\mu})$, also compared with the coercivity constant $\tilde{\alpha}(\boldsymbol{\mu})$ of the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu})$ in the state equation. In Figure 5.4(a) we report some results obtained in a preliminary numerical investigation without any enrichment option, i.e., using different RB spaces Y_N and Q_N (see section 3.1). We compare the discrete Brezzi inf-sup constant and coercivity constant for the FE and RB approximation. We can confirm that, as claimed in section 2.3

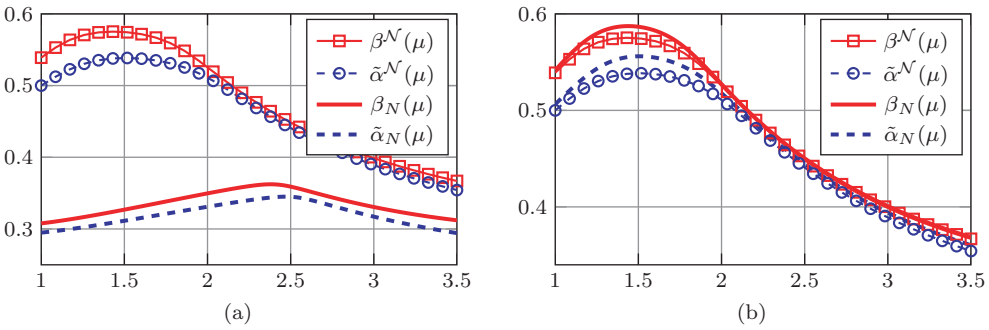

FIG. 5.4. *Test* 1: *comparison of the FE and RB discrete Brezzi inf-sup constant* $\beta(\boldsymbol{\mu})$ *and coercivity constant of the state equation* $\tilde{\alpha}(\boldsymbol{\mu})$. *The two quantities are given as function only of* μ_1, *since* μ_2 *does not appear in the affine expansion of* $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$. (a) *No enrichment:* $Y_N \neq Q_N$. (b) *Aggregated space:* $Y_N = Q_N = Z_N$ *with* Z_N *defined as in* (3.4).

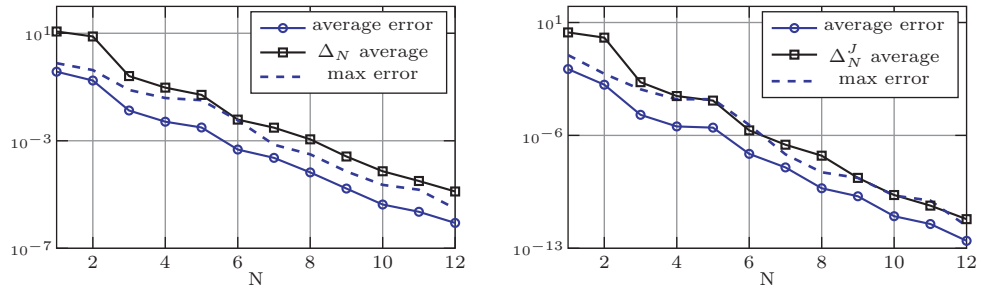

FIG. 5.5. *Test* 1. *Average and max computed errors and estimate between the truth FE solution and the RB approximation for* $N = 1, \ldots, N_{\max}$ *(left). Average computed errors and estimate* $\Delta_N^J(\mu)$ *between* $\mathcal{J}^{\mathcal{N}}(\mu)$ *and* $\mathcal{J}_N(\mu)$ *for* $N = 1, \ldots, N_{\max}$ *(right). Here* Ξ_{train} *is a sample of size* $n_{train} = 1000$ *and* $N_{\max} = 12$.

(see also Lemma 2.1), $\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$. Moreover, we observe that

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) \geq \beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}),$$

and hence (as expected) we cannot bound from below the RB inf-sup constant $\beta_N(\boldsymbol{\mu})$ with similar quantities related to the FE approximations. We note also that in this case the RB coercivity constant $\tilde{\alpha}_N(\boldsymbol{\mu})$ is in fact an inf-sup constant, since we are approximating the state equation with a Petrov–Galerkin scheme, i.e.,

$$\tilde{\alpha}_N(\boldsymbol{\mu}) = \inf_{q \in Q_N} \sup_{y \in Y_N} \frac{a(y, q; \boldsymbol{\mu})}{\|q\|_Q \|y\|_Y} \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

In Figure 5.4(b) we compare the RB stability factors obtained using the aggregated space Z_N for the state and adjoint variables. In this case we have a numerical evidence of the result proven in Lemma 3.1, that is,

$$\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

Finally, in Figure 5.5 we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$ and the a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ with the true error on the cost functional $|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})|$.

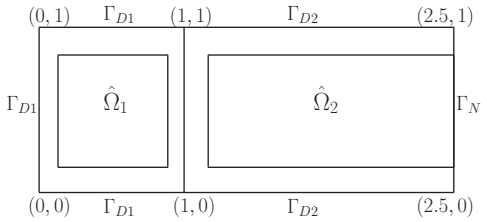

FIG. 5.6. *Test* 2: *domain* Ω *(the observations subdomains are denoted with* $\hat{\Omega}_1$ *and* $\hat{\Omega}_2$*).*

As regards the computational performances, the offline computational time is equal to $t_{RB}^{offline} = 139$ s, as the (average) online evaluation time is $t_{RB}^{online} = 8.5$ ms comprehensive of the evaluation of the a posteriori error estimation; we remark that most of the offline time is spent performing the SCM and greedy algorithms, with the former requiring around 88 seconds and the latter requiring around 46 seconds. The evaluation time for the FE approximation is equal to about $t_{FE}^{online} = 1$ s taking into account the time needed for assembling the FE matrices and vectors.

5.2. Test 2: Distributed optimal control for a Graetz convection-diffusion problem with physical parametrization. As a second example we consider a distributed optimal control problem for the Graetz conduction-convection equation. With respect to the previous test we consider here a simple physical parametrization instead of a geometrical one; in particular, μ_1 will be the Péclet number, while μ_2 and μ_3, similarly to the previous example, are such that $y_d(\boldsymbol{\mu}) = \mu_2$ in $\hat{\Omega}_1$ and $y_d(\boldsymbol{\mu}) = \mu_3$ in $\hat{\Omega}_2$, where the spatial domain (shown in Figure 5.6) is the rectangle $\Omega = [0, 2.5] \times [0, 1]$. The parameter domain is $\mathcal{D} = [3, 20] \times [0.5, 1.5] \times [1.5, 2.5]$. We consider the following optimal control problem:

$$
\begin{aligned}
&\min_{y,u} J(y, u; \boldsymbol{\mu}) = \frac{1}{2}\|y(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega})} + \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{L^2(\Omega)},\\
&\text{s.t.} \quad \begin{cases} -\dfrac{1}{\mu_1}\Delta y(\boldsymbol{\mu}) + x_2(1-x_2)\dfrac{\partial y(\boldsymbol{\mu})}{\partial x_1} = u(\boldsymbol{\mu}) & \text{in } \Omega,\\ \dfrac{1}{\mu_1}\nabla y(\mu)\cdot \boldsymbol{n} = 0 & \text{on } \Gamma_N,\\ y(\boldsymbol{\mu}) = 1 \quad \text{on } \Gamma_{D1}, \qquad y(\boldsymbol{\mu}) = 2 & \text{on } \Gamma_{D2}, \end{cases}
\end{aligned} \tag{5.2}
$$

where $y(\boldsymbol{\mu})$ is the temperature field, the control $u(\boldsymbol{\mu})$ acts as a heat source, and $\hat{\Omega} = \hat{\Omega}_1 \cup \hat{\Omega}_2$ is the observation domain. The problem admits an affine decomposition with $Q_a = 1$, $Q_b = 2$, $Q_f = 2$, $Q_g = 2$ components. For the computation we fixed $\alpha = 0.01$ and used piecewise linear finite elements for the FE approximation, the dimension of the global FE space $\mathcal{X}^{\mathcal{N}}$ used is $\mathcal{N} = 10\,494$.

With a fixed tolerance $\varepsilon_{tol}^{rel} = 10^{-4}$, $N_{\max} = 19$ basis functions have been selected, thus resulting in a RB linear system of dimension 95×95. In Figure 5.7(a) we show the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ obtained using the natural norm SCM algorithm; SCM requires in this case the solution of $28 + 2Q_B$ eigenproblems. Once again we can observe that $\hat{\beta}_N(\boldsymbol{\mu}) \geq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, thus indicating the good stability property of the RB approximation.

In Figure 5.7(b) we compare the Brezzi inf-sup constants $\beta^{\mathcal{N}}(\boldsymbol{\mu})$ and $\beta_N(\boldsymbol{\mu})$ and the coercivity constants $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\tilde{\alpha}_N(\boldsymbol{\mu})$ of the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu})$. As in the previous example we have confirmed numerically that $\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$.

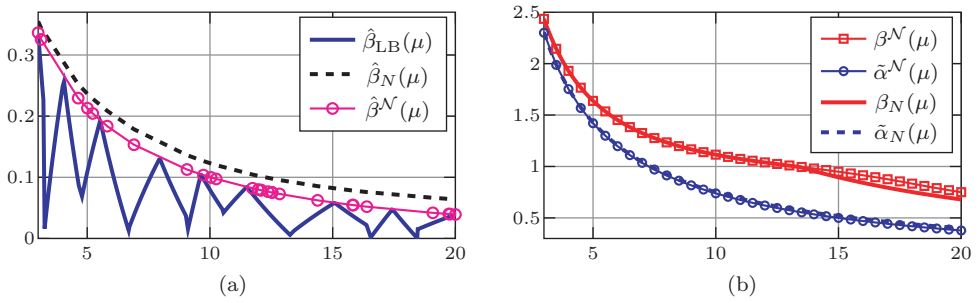

FIG. 5.7. *Test* 2: *stability factors as functions of the physical parameter* μ_1. (a) *Lower bound for the discrete Babuška inf-sup constant* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. (b) *Comparison of discrete Brezzi inf-sup constant* $\beta(\boldsymbol{\mu})$ *and coercivity constant* $\tilde{\alpha}(\boldsymbol{\mu})$ *for the FE and RB approximations.*

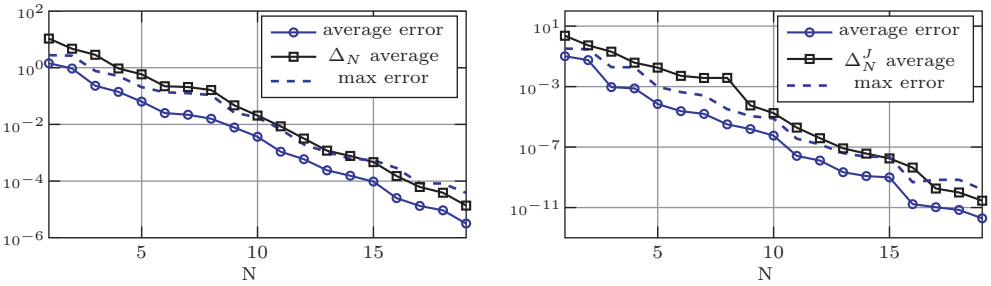

FIG. 5.8. *Test* 2. *Average and max computed errors and bound between the truth FE solution and the RB approximation (left). Average true error and bound* $\Delta_N^J(\mu)$ *between* $\mathcal{J}^{\mathcal{N}}(\mu)$ *and* $\mathcal{J}_N(\mu)$ *(right).*

TABLE 5.1
Numerical details for Test 2. *The RB spaces have been built by means of the greedy procedure and* $N = 19$ *basis functions have been selected.*

Approximation data		Computational performances	
Number of FE dof $\mathcal{N}$	10 494	Linear system size reduction	110:1
Number of parameters P	3	Offline total time	417 s
Error tolerance greedy ε_{tol}	10^{-4}	Offline SCM time	315 s
Affine operator components Q_B	3	Offline greedy time	90 s

Finally, in Figure 5.8 we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$ and the a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ with the true error on the cost functional $|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})|$.

As regards the computational performances, (see Table 5.1) while the average online time needed to compute and certify the RB solution is approximately equal to the one reported in the previous test, the offline computational time required to build all the ingredients is now equal to $t_{RB}^{offline} = 417$ s. Notice that here performing the SCM algorithm requires around 75% of the overall offline time, a percentage that can further increase rapidly when the number of parameters P, the number of terms Q_B in the affine decomposition, or the number of FE degrees of freedom $\mathcal{N}$ increase. In the next example we will discuss an alternative strategy for the construction of the lower bound $\hat{\beta}_{LB}(\boldsymbol{\mu})$, in order to avoid this computational bottleneck in the offline stage.

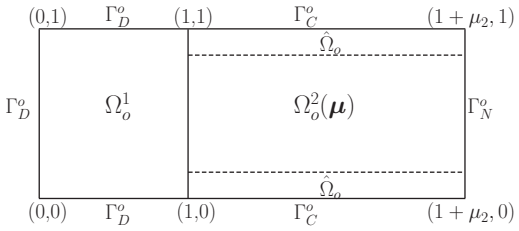

FIG. 5.9. *Test* 3: *original domain* $\Omega_o(\boldsymbol{\mu})$.

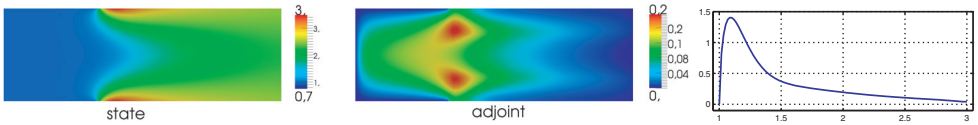

FIG. 5.10. *Test* 3: *representative solution for* $\boldsymbol{\mu} = (12, 2, 2.5)$. *We report the state variable* y_N *(left), the adjoint variable* p_N *(middle), and the optimal control* u_N *on* Γ_C^o *(right); thanks to the symmetry of the problem, the control variable has the same values on the boundaries* $\Gamma_C \cap \{x_2 = 0\}$ *and* $\Gamma_C \cap \{x_2 = 1\}$.

5.3. Test 3: Boundary optimal control for a Graetz flow with both physical and geometrical parametrization. This third example deals again with a control problem for a Graetz flow, but this time we consider a boundary control instead of a distributed one and we consider both a geometrical and physical parametrization. The original domain is shown in Figure 5.9; we consider three parameters: μ_1 is the Péclet number, μ_2 is the geometrical parameter (the length of second portion of the channel), and μ_3 is such that $y_d(\boldsymbol{\mu}) = \mu_3 \chi_{\hat{\Omega}_o}$, with $\hat{\Omega}_o(\boldsymbol{\mu})$ being the observation domain $\hat{\Omega}_o(\boldsymbol{\mu}) \subset \Omega_o^2(\boldsymbol{\mu})$. The parameter domain is $\mathcal{D} = [6, 20] \times [1, 3] \times [0.5, 3]$.

We consider the following optimal control problem:

$$\min_{y_o, u_o} J(y_o(\boldsymbol{\mu}), u_o(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\|y_o(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega}_o)} + \frac{\alpha}{2}\|u_o(\boldsymbol{\mu})\|^2_{U_o},$$

$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_1}\Delta y_o(\boldsymbol{\mu}) + x_{o2}(1 - x_{o2})\dfrac{\partial y_o(\boldsymbol{\mu})}{\partial x_{o1}} = 0 & \text{in } \Omega_o(\boldsymbol{\mu}), \\ y_o(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_D^o, \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = u_o(\boldsymbol{\mu}) & \text{on } \Gamma_C^o(\boldsymbol{\mu}), \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_N^o(\boldsymbol{\mu}), \end{cases} \tag{5.3}$$

where we impose constant Dirichlet conditions on the inlet boundary of the channel, homogeneous Neumann condition on the outlet boundary, and finally a Neumann condition equal to the control function u_o on Γ_C^o. We denote with Y_o and U_o the spaces $H_0^1(\Omega_o)$ and $L^2(\Gamma_C^o)$, respectively, and moreover $Q_o \equiv Y_o$. By tracing the problem back to a reference domain we obtain the parametrized formulation (2.10), where the affine decompositions (2.12), (2.13) hold with $Q_a = 1, Q_b = 5, Q_f = 1, Q_g = 4$. Figure 5.10 reports a representative solution of the optimal control problem (state, adjoint, control) for a given parameters configuration.

As mentioned in section 5.2, in order to avoid the time-consuming SCM algorithm, we seek an alternative strategy to compute a lower bound of the inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. As recently proposed in [22], we consider—rather than a rigorous lower

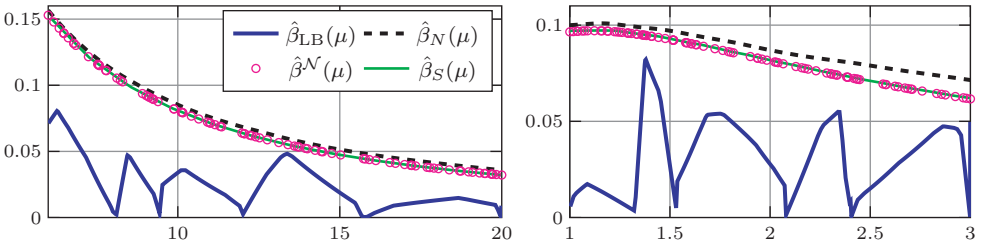

FIG. 5.11. *Test* 3: *comparison between lower bound and interpolant surrogate for the discrete Babuška inf-sup constant* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. *On the left:* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of* μ_1, $(\mu_2, \mu_3) = (1.5, 3)$ *fixed; on the right:* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of* μ_2, $(\mu_1, \mu_3) = (9, 1)$ *fixed.*

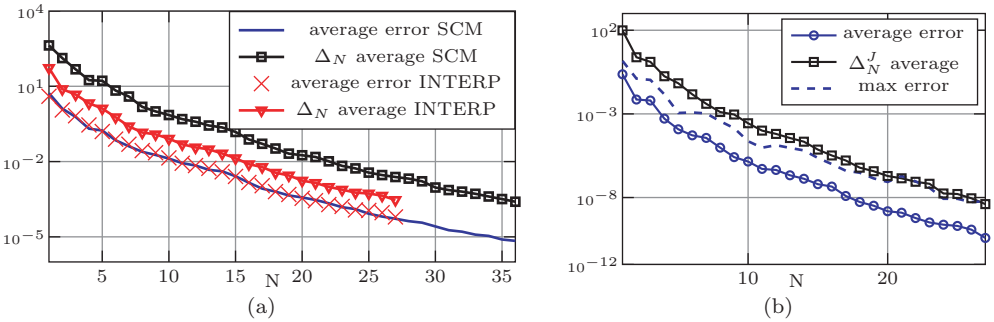

FIG. 5.12. *Test* 3 ($\alpha = 0.07$). (a) *True error and error estimate between the FE solution and the RB approximation: the quantities in red are obtained using the interpolant surrogate* $\hat{\beta}_S(\boldsymbol{\mu})$ *instead of* $\hat{\beta}_{LB}(\boldsymbol{\mu})$. (b) *Average true error and bound* $\Delta_N^J(\boldsymbol{\mu})$ *between* $\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu})$ *and* $\mathcal{J}_N(\boldsymbol{\mu})$ *using* $\hat{\beta}_S(\boldsymbol{\mu})$ *in the estimate.*

bound—a *surrogate* of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ given by an interpolation procedure. We (arbitrarily and a priori) select a (possibly *small*) set of interpolation points $\Xi_\beta \subset \mathcal{D}$ and compute the inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ by solving the related eigenproblem for each $\boldsymbol{\mu} \in \Xi_\beta$. Then we compute a suitable *interpolant surrogate* $\hat{\beta}_S(\boldsymbol{\mu})$ such that

$$\hat{\beta}_S(\boldsymbol{\mu}) = \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) \qquad \forall \boldsymbol{\mu} \in \Xi_\beta.$$

Depending on the number of parameters and their range of variation, different interpolation methods can be employed. Here we use a simple linear interpolant and an equally spaced grid of interpolation points in the parameter space. Actually, since the parameter μ_3 does not affect the value of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, we perform just a two-dimensional interpolation with respect to the parameters μ_1 and μ_2.

We present here a first test comparing the performances of this alternative strategy with respect to the SCM algorithm. We fixed $\alpha = 0.07$ and used piecewise linear finite elements for the FE approximation; the dimension of the global FE space $\mathcal{X}^{\mathcal{N}}$ is $\mathcal{N} = 7\,156$. In Figure 5.11 we show a comparison between the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ obtained using the SCM algorithm and the interpolant surrogate $\hat{\beta}_S(\boldsymbol{\mu})$; SCM takes around 1 hour to be performed, while the computation of the interpolant surrogate needs only 24 seconds using 120 sampling points in the parameter space. Furthermore, the interpolant surrogate is a much sharper approximation of the true FE inf-sup constant—despite not being a rigorous lower bound—thus resulting also in a sharper a posteriori error estimate (see Figure 5.12). For this reason, with a fixed tolerance $\varepsilon_{tol}^{rel} = 5 \cdot 10^{-4}$, the greedy algorithm

TABLE 5.2
Numerical details for Test 3 ($\alpha = 0.07$). Comparison between the use of the SCM algorithm and the interpolation procedure.

	SCM algorithm	Interpolation surrogate
Number of eigenvalue problems	239	120
"Lower bound" computation time	3523 s	24 s
Greedy algorithm comput. time	349 s	175 s
Number of RB functions N	36	27
Linear system size reduction	39:1	53:1

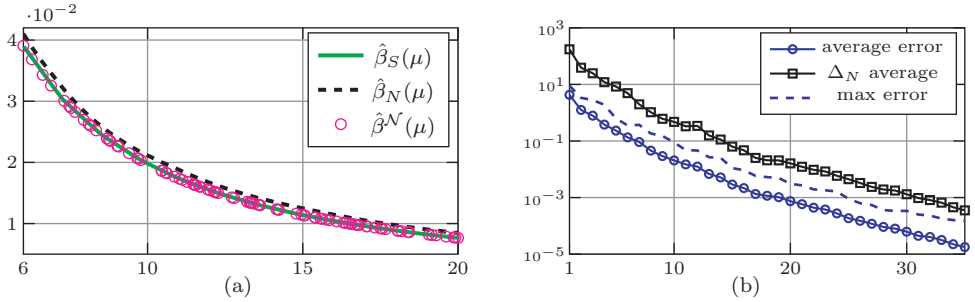

FIG. 5.13. *Test 3 ($\alpha = 8 \cdot 10^{-3}$). (a) Interpolant surrogate for the discrete Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ as a function of μ_1, $(\mu_2, \mu_3) = (1.5, 3)$ fixed. (b) Average and max true errors and estimate on the solution variables.*

TABLE 5.3
Numerical details for Test 3 ($\alpha = 8 \cdot 10^{-3}$).

Approximation data		Computational performances	
Number of FE dof $\mathcal{N}$	22 792	Linear system size reduction	130:1
Number of parameters P	3	RB solution	2.5 ms
Affine operator components Q_B	6	Offline interpolation time	102 s
Number of RB functions N	35	Offline greedy time	860 s

selects $N_{\max} = 36$ basis functions when using the lower bound given by the SCM, while only $N_{\max} = 27$ basis functions are selected when employing the interpolant surrogate. A detailed comparison of the computational costs is given in Table 5.2.

Finally, we have performed a further test using a smaller regularization constant $\alpha = 8 \cdot 10^{-3}$ and a finer triangulation of the spatial domain, resulting in a global FE space $\mathcal{X}^{\mathcal{N}}$ of dimension $\mathcal{N} = 22\,792$. We use $\hat{\beta}_S(\boldsymbol{\mu})$ as surrogate for the lower bound of the FE inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$: with 120 equally distributed interpolation points we obtain a sharp approximation of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ (see Figure 5.13(a)), yet require less than two minutes to be computed in the offline stage. (All the numerical details are given in Table 5.3.) The greedy algorithm selects $N_{\max} = 35$ basis functions in order to guarantee the relative error of the RB solution (with respect to the FE approximation) to be under the desired tolerance $\varepsilon_{tol}^{rel} = 5 \cdot 10^{-4}$. In Figure 5.13(b) we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$.

6. Conclusions. In this work we have developed a reduced basis framework for the efficient solution of parametrized linear-quadratic optimal control problems governed by elliptic coercive PDEs. A rigorous well-posedness analysis has been

carried out by exploiting a suitable saddle-point formulation. On the other hand, the certified error bounds on the solution variables as well as on the cost functional have been obtained by recasting the problem in the form of weakly coercive problems and then applying standard arguments based on Nečas–Babuška stability theory. Finally, we have also provided a full offline-online decomposition strategy ensuring the online efficiency of the method. Our numerical tests showed the possibility of obtaining large computational savings (a speedup of at least two orders of magnitude) in the online stage with respect to classical high-fidelity discretization methods. In particular, the proposed error estimators show to be sharp enough to enable an efficient exploration of the parameter space through the Greedy algorithm, thus resulting in the selection of a reasonably small number of basis functions.

A possible drawback resides in the offline stage, which demands large computational resources. To alleviate this problem, we have provided a detailed (empirical) analysis of the computational costs required by the main operations to be performed, i.e., the computation of a lower bound for the inf-sup constant (via the SCM algorithm) and the construction of the RB spaces through the Greedy algorithm. Since the main computational effort is required by the former, we have proposed the use of a suitable interpolant surrogate instead of a rigorous lower bound. This alternative strategy is signicantly more efficient, resulting in both a substantial computational savings in the offline stage and a sharper approximation of the true stability factor.

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