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# Arithmetic genus of locally Cohen-Macaulay space curves

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Thesis submitted for the degree of "Magister Philosophiæ"
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## Introduction

The problem of determining the maximal arithmetic genus $G(d,t)$ over all integral curves in $\mathbf{P}^3$ having degree $d$ and not contained in a surface of degree $t-1 \geq 0$ was first considered by Halphen [11]. He solved it in the case $t=2$, and Castelnuovo [2] proved the same result for any integral curve $C \subseteq \mathbf{P}^n$ with $n \geq 3$. A complete answer in the general case was given by Gruson and Peskine [9] when $d \geq t^2 - 2t + 2$. They introduced the numerical character, which is a suitable sequence of integers associated with the general plane section $\Gamma$ of a curve $C$, to lower the Hilbert function of $\Gamma$, and hence to bound the genus of $C$. Another tool used in their proof is the Generalized Trisecant Lemma [16, 10]: any integral curve $C$ of degree $d > t^2+1$ and such that $s(C) \geq t$ has the property that $\sigma(C) \geq t$, where $s(C)$ denotes the minimum of the degrees of all surfaces containing $C$, and $\sigma(C)$ is the minimum of the degrees of all plane curves passing through $\Gamma$. Afterwards several authors treated the case $d < t^2 - 2t + 2$ (see, among others, [12, 10, 14, 5, 8]), but a complete answer is still missing.

In this paper we consider the same problem in the more general setting of locally Cohen-Macaulay curves in $\mathbf{P}^3$, i.e., equidimensional curves without embedded points. More precisely, our aim is to compute the maximal arithmetic genus $P_a(d,t)$ over all loc.CM curves having degree $d$ and such that $s(C) \geq t$. It is known (see, for instance, [13]) that the arithmetic genus $p_a(C)$ of an arbitrary curve $C$ of degree $d$ is bounded from above by $\frac{1}{2}(d-1)(d-2)$, and the equality holds if and only if $C$ is a plane curve. The formula for $P_a(d,2)$ was found by Hartshorne [13], who proved that all curves of maximal genus lie on a quadric surface. Here we compute $P_a(d,3)$ and $P_a(d,4)$. We point out that $P_a(d,t)$ is defined for any $d \geq t$, since a loc.CM curve of degree $d$ is always contained in a surface of degree $d$ and there exist curves $C$ such that $s(C) = d$ (Lemmas 2.1 and 2.2). This differs from the integral case, where $G(d,t)$ is defined for $d \geq (t^2+4t+6)/6$ [12]. We note also that the Generalized Trisecant Lemma does not necessarily hold for loc.CM curves $C$ with $s(C) = 3, 4$ (Lemmas 4.1 and 4.3). As a consequence, it can not be used to distinguish any ranges for the degree $d$ in terms of $\sigma(C)$ like in the integral case (ranges A, B and C). Moreover, for the curves we consider, the use of the numerical character does not give a sharp bound on the genus. However, it allows to deduce that the curves of sufficientely large degree are the schematic union of two subcurves modulo a finite number of isolated points. The idea is then to bound the arithmetic genus of the two subcurves. The answer to the problem of determining $P_a(d,t)$ for $t \leq 4$ can be summarized as follows:

**Theorem** *The maximal arithmetic genus of loc.CM curves $C \subseteq \mathbf{P}^3$ of degree $d$ not con-*

*tained in a surface of degree $t$ with $1 \le t \le 4$ is given by the following formulas*

$$P_a(d,1) = \frac{1}{2}(d-1)(d-2);$$
$$P_a(d,2) = \begin{cases} -1 & \text{if } d = 2, \\ \frac{1}{2}(d-2)(d-3) & \text{if } d \ge 3; \end{cases}$$
$$P_a(d,3) = \begin{cases} 2d-9 & \text{if } 3 \le d \le 5, \\ \frac{1}{2}(d-3)(d-4) & \text{if } d \ge 6; \end{cases}$$
$$P_a(d,4) = \begin{cases} 3d-19 & \text{if } 4 \le d \le 8, \\ \frac{1}{2}(d-4)(d-5)-1 & \text{if } d \ge 9. \end{cases}$$

It turns out that the genus $P_a(d,3)$ and $P_a(d,4)$ are attained by curves $C$ having $s(C) = 3,4$ and $\sigma(C) = 2,3$ for any $d$, and this situation is different from the integral maximal genus curves in range C, which have $\sigma(C) = s(C) = t$. The curves found suggest a method for constructing curves of high genus which allows to determine the following lower bound for $P_a(d,t)$ with $t \ge 5$ and $d \ge 2t-1$:

$$P_a(d,t) \ge \frac{(d-t)(d-t-1)}{2} - \frac{(t-1)(t-2)(t-3)}{2}.$$

The outline of the paper is the following. In Section 1 we recall some results on the numerical character (Propositions 1.1 and 1.5), and we study the geometry of the curves $C$ having $\sigma(C) < s(C)$ (Corollary 1.3 and Lemma 1.7). In Section 2 we prove the existence of curves of arbitrary degree $d \ge s$ on a surface of degree $s$ (Lemma 2.2), and we characterize all curves having $d = s$ (Lemma 2.3). In Section 3 we describe three methods for bounding the genus of a curve (Propositions 3.1, 3.3 and 3.4). Section 4 is devoted to the characterization of all curves with $s(C) > \sigma(C) = 2,3$ and $d$ sufficientely large (Lemmas 4.1 and 4.3). In Section 5 we compute $P_a(d,3)$ and $P_a(d,4)$ (Propositions 5.3 and 5.4), and we prove the existence of curves $C$ of degree $d$ and genus $p$ for any $d \ge t = 3,4$ and any $p \le P_a(d,t)$, such that $\sigma(C) = t-1$ and $s(C) = t$.

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## 1. Notations and preliminary results

Let K be an algebraically closed field of characteristic zero. We denote by R the polynomial ring $K[x_0, x_1, x_2]$. By a curve $C \subseteq \mathbf{P}^3$ we mean a locally Cohen-Macaulay (loc.CM for short) equidimensional subscheme of dimension one of the projective 3-space $\mathbf{P}^3$ over K. We indicate by $p_a(C)$ the arithmetic genus of a curve $C$, by $\Gamma$ the general plane section of $C$, and by $\mathrm{I}(\Gamma) = \bigoplus_{k\geq 0} H^0(\mathcal{I}_\Gamma(k))$ the saturated homogeneous ideal of $\Gamma$. Finally we set

$$s(C) = \min\{k \in \mathbf{Z} : h^0(\mathcal{I}_C(k)) \neq 0\}, \quad \sigma(C) = \min\{k \in \mathbf{Z} : h^0(\mathcal{I}_\Gamma(k)) \neq 0\}.$$

Sometimes, in the sequel, we shall write $s$ and $\sigma$ instead of $s(C)$ and $\sigma(C)$, respectively. We associate with $\Gamma$ the sequence $\chi(\Gamma) = (n_0, \ldots, n_{\sigma-1})$, where $n_0, \ldots, n_{\sigma-1} \in \mathbf{N}$, $n_0 \geq n_1 \geq \ldots \geq n_{\sigma-1} \geq \sigma$, which is called the numerical character of $\Gamma$ [9, Def. 2.4]. If $\sigma \geq 2$, a numerical character $(n_0, \ldots, n_{\sigma-1})$ is said connected if $n_i \leq n_{i+1} + 1$ for every $i = 0, \ldots, \sigma - 2$. The genus $g(\chi(\Gamma))$ of $\chi(\Gamma)$ is defined as $g(\chi(\Gamma)) = \sum_{n\geq 1} h^1(\mathcal{I}_\Gamma(n))$.
In this Section we will recall some known results on $\chi(\Gamma)$ and we will use them to relate the numerical character to the geometry of a curve.

**1.1 Proposition** *Let $C \subseteq \mathbf{P}^3$ be a curve, let $\Gamma$ be its general plane section and let $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$. Then*

(i) *the degree $d$ of $C$ is given by*

$$d = \sum_{i=0}^{\sigma-1} (n_i - i);$$

(ii) *the Hilbert function $h_\Gamma$ of $\Gamma$ is determined by the formula*

$$h_\Gamma(n) = \sum_{i=0}^{\sigma-1} [(n - i + 1)_+ - (n - n_i + 1)_+], \quad \text{for} \quad n \in \mathbf{N},$$

*where, for $k \in \mathbf{Z}$, we set $k_+ = max\{0, k\}$;*

(iii) *the following equality holds:*

$$h^1(\mathcal{I}_\Gamma(n)) = \sum_{i=0}^{\sigma-1} [(n_i - n - 1)_+ - (i - n - 1)_+], \quad \text{for} \quad n \in \mathbf{Z};$$

(iv) *if $T$ is a plane curve of degree $\sigma$ containing $\Gamma$, then the minimum of the degrees of all curves passing through $\Gamma$ and not containing $T$ is equal to $n_{\sigma-1}$.*

*Proof.* Assertions $(i)$, $(ii)$ and $(iii)$ are proved in [9]. Assertion $(iv)$ follows from the definition of $\chi(\Gamma)$. □

The following is a powerful algebraic result by Strano, which will be used to study curves with $\sigma < s$.

**1.2 Theorem** *Let $C \subseteq \mathbf{P}^3$ be a curve and let $m \in \mathbf{N}$. If*

$$\mathrm{Tor}_1^{\mathrm{R}}(\mathrm{I}(\Gamma), \mathrm{K})_h = 0 \quad \textit{for} \quad \textit{every} \quad 0 \leq h \leq m+2,$$

*then the restriction map $\rho_m : H^0(\mathcal{I}_C(m)) \to H^0(\mathcal{I}_\Gamma(m))$ is surjective.*

*Proof.* [17, Teorema 4]. □

**1.3 Corollary** *Let $C \subseteq \mathbf{P}^3$ be a curve with $s > \sigma$. Then $n_{\sigma-1} \in \{\sigma, \sigma+1\}$.*

*Proof.* The assumption $s > \sigma$ implies that the restriction map $\rho_\sigma$ is not surjective. By Theorem 1.2 there exists a syzygy in degree $h \leq \sigma + 2$ between the generators of $\mathrm{I}(\Gamma)$. Since the syzygies always occur in degree $m \geq \sigma + 1$, if $F$ is a degree $\sigma$ generator, there exists at least one generator $G$ of degree $\sigma$ or $\sigma + 1$ which is not a multiple of $F$, and, by $(iv)$ of Proposition 1.1, we have $n_{\sigma-1} \leq \sigma + 1$. □

Gruson and Peskine [9] showed that the numerical character of an integral curve is connected. However, this result is not necessarily true in the more general case of loc.CM curves. Indeed, a non connected numerical character $\chi$ is the character of a reducible curve, as we shall see in Lemma 1.6. We will use a result by Ellia and Peskine (see Proposition 1.5) to prove that such a $\chi$ is the character associated with two groups of points in the plane, and we will show that they can be lifted to two curves using a result by Strano [18]. We first recall a definition.

**1.4 Definition** [15] Let $X$ be a subscheme of $\mathbf{P}^n$ and let $F$ be a hypersurface in the same space, which is defined by the equation $\tilde{F}$ of degree $f$. The residual scheme $Z = Res_F X$ to $X$ with respect to $F$ is the subscheme defined by the ideal sheaf

$$\mathcal{I}_Z = \tilde{F}^{-1} ker[\mathcal{I}_{X,\mathbf{P}^n} \to \mathcal{I}_{X \cap F, F}],$$

and we have the exact sequence

$$0 \to \mathcal{O}_Z(-f) \to \mathcal{O}_X \to \mathcal{O}_{X \cap F} \to 0.$$

**1.5 Proposition** *Let $\Gamma \subseteq \mathbf{P}^2$ be a group of points with $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$.*

(1) *Assume that* $n_{t-1} > n_t$ *for some* $1 \leq t \leq \sigma - 1$ *and that all the curves of degree* $n_{t-1} - 1$ *containing* $\Gamma$ *have a greatest common divisor* $T$ *of degree* $t$. *Then* $\Gamma' = T \cap \Gamma$ *is a subgroup of points of* $\Gamma$ *such that* $\chi(\Gamma') = (n_0, ..., n_{t-1})$. *Moreover, if* $\Gamma'' = Res_T \Gamma$, *one has* $\chi(\Gamma'') = (n_t - t, ..., n_{\sigma-1} - t)$.

(2) *If* $n_{t-1} > n_t + 1$ *for some* $1 \leq t \leq \sigma - 1$, *then* $\Gamma$ *verifies the assumptions in* (1).

*Proof.* [4, 7]. □

**1.6 Lemma** *Let* $C \subseteq \mathbf{P}^3$ *be a curve such that* $\Gamma$ *verifies the assumptions of* (1) *in Proposition 1.5. Then* $C$ *contains a loc.CM curve* $C'$ *which lies on a surface* $S$ *of degree* $t$ *having the following properties:*

(a) $S \cap C = C'$ *modulo a finite number of zero- dimensional components;*

(b) *let* $\Gamma'$ *be the general plane section of* $C'$*; then* $\chi(\Gamma') = (n_0, ..., n_{t-1})$*;*

(c) *if* $C'' = Res_S C$, *the numerical character of the general plane section of* $C''$ *is given by* $(n_t - t, ..., n_{\sigma-1} - t)$.

*Proof.* By Proposition 1.5, $\Gamma$ contains two subgroups $\Gamma'$ and $\Gamma''$ such that $\chi(\Gamma') = (n_0, ..., n_{t-1})$ and $\chi(\Gamma'') = (n_t - t, ..., n_{\sigma-1} - t)$. By [18, Lemma 2], $\Gamma'$ is the general plane section of a loc.CM curve $C' \subseteq C$ and $\sigma(C') = t$. We observe that a curve $T$ of degree $t$ containing $\Gamma'$ can be lifted to a surface $S$ of the same degree containing $C'$. Indeed, $n_{t-1} \geq n_t + 1 \geq \sigma + 1 \geq t + 2$ and, by (*iv*) of 1.1, $T$ is the only curve of degree less or equal to $t+1$ containing $\Gamma'$. As a consequence there is no syzygy in degree $h \leq t + 2$ between the generators of $\mathrm{I}(\Gamma')$ and the claim follows applying Theorem 1.2. These arguments prove (a) and (b).

To prove (c) it is enough to observe that the general plane section of $C''$ is $\Gamma''$. □

**1.7 Lemma** *Let* $C \subseteq \mathbf{P}^3$ *be a curve of degree* $d$ *with* $\sigma \geq 2$ *and* $s > \sigma$. *Assume that* $n_{\sigma-1} = \sigma + 1$ *and* $n_{\sigma-2} \geq \sigma + 2$. *Then* $C$ *verifies the assumptions (1) of Lemma 1.5 with* $t = \sigma - 1$, *and hence there exists a curve* $C' \subset C$ *with* $deg(C') = d - 2$ *and* $s(C') = \sigma - 1$, *and the curve* $C'' = Res_S C$, *where* $S \supseteq C'$ *is a surface of degree* $\sigma - 1$, *is a non planar curve of degree two, i.e. it consists of two skew lines or it is a double line with* $p_a(C'') \leq -1$. *In particular,* $s = \sigma + 1$.

*Proof.* Let us prove that all curves of degree $\sigma + 1$ containing $\Gamma$ have a greatest common divisor od degree $\sigma - 1$. The hypothesis $n_{\sigma-1} = \sigma + 1$ implies $h^0(\mathcal{I}_\Gamma(\sigma)) = 1$ by (*iv*) of Proposition 1.1. Moreover, if we compute $h_\Gamma(\sigma + 1)$ both using (*ii*) of 1.1 and writing $h_\Gamma(\sigma + 1) = \dim \mathrm{R}_{\sigma+1} - h^0(\mathcal{I}_\Gamma(\sigma + 1))$, we get $h^0(\mathcal{I}_\Gamma(\sigma + 1)) = 4$. Since $s > \sigma$, by Theorem 1.2 there exists a syzygy of the form $G_2 F_\sigma + L F_{\sigma+1} = 0$ where $F_\sigma \in H^0(\mathcal{I}_\Gamma(\sigma))$,

$F_{\sigma+1} \in H^0(\mathcal{I}_\Gamma(\sigma+1))$ and $F_{\sigma+1}$ is not a multiple of $F_\sigma$, $G_2$ is a homogeneous quadratic polynomial and $L$ is a homogeneous linear polynomial. This implies that the two generators have a common component $P$ of degree $\sigma - 1$, and we can apply Lemma 1.6. We note that the subgroup $\Gamma' = P \cap \Gamma$ consists of $d-2$ points of $\Gamma$ by the unicity of $F_\sigma$. Therefore, using the notations of Lemma 1.6, we have $deg(C') = d-2$ and $deg(C'') = 2$. We claim that $s(C'') = 2$. Indeed, suppose $C''$ is planar and let $S \supseteq C'$ be a degree $\sigma - 1$ surface. Then $C$ is generically contained in the union $\tilde{S}$ of $S$ with the plane of $C''$. Since $C$ is loc.CM, we have $C \subseteq \tilde{S}$ and this is a contradiction as $deg(\tilde{S}) = \sigma$. It follows $s(C'') = 2$ since a degree two curve is always contained in a quadric surface (see Lemma 2.1 in next Section). □

**1.8 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve with $\sigma \geq 2$. Assume that $n_{\sigma-1} = \sigma$ and $n_{\sigma-2} \geq \sigma + 2$. Then $s = \sigma$.*

*Proof.* Since $\chi(\Gamma)$ is not connected, we can apply Lemma 1.6 with $t = \sigma - 1$. We obtain that $C$ contains a curve $C'$ on a surface $S$ with $\deg(S) = \sigma - 1$ and $Res_S C$ is a line $L$. Hence $C$ is generically contained in the union of $S$ with a plane $H$ containing $L$. Since $C$ is loc.CM, it is globally contained in the same union, and we have $s = \sigma$. □

## 2. Relations between $d$ and $s$

It is well known that an integral curve $C$ of degree $d$ is contained in a cone over $C$ with vertex not on $C$, so that $s \leq d$. A similar construction (see for example [3]) can be done for loc.CM curves, and hence the inequality $s \leq d$ still holds. In this Section we will prove that there exist loc.CM curves with $s = d$, and we will characterize them.

**2.1 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d$. Then $C$ is contained in a surface of degree $d$.*

*Proof.* [3, Lemma 2.6]. □

**2.2 Lemma** *Let $s \geq 1$ be an integer. For every $d \geq s$ there exists a degree $d$ curve $C$ with $s(C) = s$.*

*Proof.* For $s = 1$ the assertion of the Lemma is obvious. Hence we shall assume $s \geq 2$. Let $L$ be a line and $S$ be a general surface of degree $s$ containing $L$. Consider the divisor $C = dL$ on $S$ and let $H$ be a general plane. Let us prove that $h^0(\mathcal{I}_{C,S}(s-1)) = 0$, which is equivalent to showing that the divisor $-dL + (s-1)H$ is not effective. Assume by contradiction it is effective. We note that the linear system $|H-L|$ contains a smooth irreducible curve and that $(H-L)^2=0$. So we can apply [1, Rem. III.5], and say that $(-dL+(s-1)H)\cdot(H-L)\geq 0$. But the direct computation gives $(-dL+(s-1)H)\cdot(H-L)=(s-1)(s-d-1)$ which is strictly negative, since $s\geq 2$ and $d\geq s$, and this is a contradiction. The exactness of the sequence

$$0\to\mathcal{I}_S\to\mathcal{I}_C\to\mathcal{I}_{C,S}\to 0$$

and $h^0(\mathcal{I}_{C,S}(s-1))=0$ imply that $h^0(\mathcal{I}_C(s-1))=0$ and therefore $s(C)=s$. To conclude we note that $deg(C)=dL\cdot H=d$. □

**2.3 Lemma** *Let $C$ be a degree $d$ curve with $s=d$. Then $C_{red}$ consists of disjoint lines.*

*Proof.* Assume first that $C$ is irreducible. If $deg(C_{red})=n\geq 2$, the cone over $C_{red}$ with vertex at a closed point of $C_{red}$ is a degree $m\leq n-1$ surface containing $C_{red}$. If $e\geq 1$ is the multiplicity of $C$ at a general closed point, then $en=d$ and the surface $eS$ contains $C$ [3, Lemma 2.6]. We have $deg(eS)=em\leq e(n-1)=d-e<d$ which contradicts the assumption $s=d$. It follows that $deg(C_{red})=1$.
If $C$ is reducible, it is sufficient to repeat the above arguments for the irreducible components of $C$ and to observe that their supports are disjoint because of the assumption $s=d$.
□

## 3. Bounds on the genus

In this Section we will describe three methods for bounding the arithmetic genus of a curve. The first method consists in computing the genus of $\chi(\Gamma)$, since we always have $p_a(C)\leq g(\chi(\Gamma))$ (Lemma 3.1). We will show that any curve with genus equal to $g(\chi(\Gamma))$ is such that $s=\sigma$ (Corollary 3.2). The second method is based on the classical Castelnuovo's technique (Proposition 3.3) of estimating the Hilbert function of $\Gamma$ to lower $h^0(\mathcal{O}_C(n))$ for $n$ sufficientely large. The third method (Lemma 3.4) applies to curves with $\chi(\Gamma)$ such that all the integers $n_i$ are small enough with respect to $s$.

**3.1 Lemma** *Let $C\subseteq\mathbf{P}^3$ be a curve. Then $p_a(C)\leq g(\chi(\Gamma))$ and equality holds if and only if $h^0(\mathcal{O}_C)=1$ and $h^1(\mathcal{I}_C(n))=0$ for every $n\geq 0$.*

*Proof.* The proof is similar to [9, Lemma 3.5].
For any $n\in\mathbf{Z}$, let $\Delta_n$ be the kernel of the surjective map

$$H^1(\mathcal{O}_C(n-1))\to H^1(\mathcal{O}_C(n))$$

and let $d_n$ be its dimension. It is immediate to verify that $h^1(\mathcal{O}_C) = \sum_{n\geq 1} d_n$. The commutative diagram

$$\begin{array}{ccccccc}
H^0(\mathcal{O}_{\mathbf{P}^3}(n)) & \rightarrow & H^0(\mathcal{O}_{\mathbf{P}^2}(n)) & \rightarrow & 0 & & \\
\downarrow & & \downarrow & & & & \\
H^0(\mathcal{O}_C(n)) & \rightarrow & H^0(\mathcal{O}_\Gamma(n)) & \rightarrow & \Delta_n & \rightarrow & 0 \\
 & & \downarrow & & \| & & \\
 & & H^1(\mathcal{I}_\Gamma(n)) & \rightarrow & \Delta_n & \rightarrow & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

implies that $h^1(\mathcal{I}_\Gamma(n)) \geq d_n$. Therefore we have

$$g(\chi(\Gamma)) \geq \sum_{n\geq 1} d_n = h^1(\mathcal{O}_C) \geq p_a(C). \tag{3.1}$$

Assume now $g(\chi(\Gamma)) = p_a(C)$. Since $p_a(C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C)$, it follows that $h^0(\mathcal{O}_C) = 1$ and that $h^1(\mathcal{I}_\Gamma(n)) = d_n$ for every $n \geq 1$. This implies $h^1(\mathcal{I}_C(n)) \leq h^1(\mathcal{I}_C(n-1))$ for $n \geq 1$. On the other hand, $h^0(\mathcal{O}_C) = 1$ implies $h^1(\mathcal{I}_C) = 0$ and therefore $h^1(\mathcal{I}_C(n)) = 0$ for any $n \geq 0$.
For the converse, we note that the assumption $h^0(\mathcal{O}_C) = 1$ implies $p_a(C) = h^1(\mathcal{O}_C)$. Furthermore, since $h^1(\mathcal{I}_C(n)) = 0$ for any $n \geq 0$, we also have $g(\chi(\Gamma)) = \sum_{n\geq 1} d_n = h^1(\mathcal{O}_C)$, and this concludes the proof. □

**3.2 Corollary** *If $p_a(C) = g(\chi(\Gamma))$, then $s = \sigma$.*

*Proof.* By Lemma 3.1, $h^1(\mathcal{I}_C(\sigma - 1)) = 0$, and thus the restriction map $\rho_\sigma : H^0(\mathcal{I}_C(\sigma)) \rightarrow H^0(\mathcal{I}_\Gamma(\sigma))$ is surjective. □

**3.3 Proposition** *Let $C$ be a curve and assume that $\sigma > \tau$ for some integer $\tau \geq 1$. Then*

$$p_a(C) \leq \frac{1}{2}\left(d - \frac{\tau(\tau+1)}{2} - 1\right)\left(d - \frac{\tau(\tau+1)}{2} - 2\right) + \frac{(\tau-1)\tau(\tau+1)}{3}. \tag{3.2}$$

*Moreover, if equality holds in (3.2), then $\sigma = s = \tau + 1$ and*

$$\chi(\Gamma) = (d - \frac{\sigma(\sigma-1)}{2}, \sigma, ..., \sigma). \tag{3.3}$$

*Proof.* For any $k \in \mathbf{Z}$ we have the exact sequence $0 \rightarrow \mathcal{O}_C(k-1) \rightarrow \mathcal{O}_C(k) \rightarrow \mathcal{O}_\Gamma(k) \rightarrow 0$

and the commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & H^0(\mathcal{I}_\Gamma(k)) & & \\ & & & & & & \downarrow & & \\ 0 & \to & H^0(\mathcal{O}_{P^3}(k-1)) & \to & H^0(\mathcal{O}_{P^3}(k)) & \to & H^0(\mathcal{O}_{P^2}(k)) & \to & 0 \\ & & & & \downarrow & & \downarrow \alpha_k & & \\ 0 & \to & H^0(\mathcal{O}_C(k-1)) & \to & H^0(\mathcal{O}_C(k)) & \stackrel{\rho_k}{\to} & H^0(\mathcal{O}_\Gamma(k)) & & \end{array}$$

from which we obtain $h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) = \dim \mathrm{Im}\rho_k$, and $\mathrm{Im}\alpha_k \subseteq \mathrm{Im}\rho_k$. By the definition of Hilbert function we have $\dim \mathrm{Im}\alpha_k = h_\Gamma(k)$, hence

$$h^0(\mathcal{O}_C(k)) - h^0(\mathcal{O}_C(k-1)) \geq h_\Gamma(k). \tag{3.4}$$

Let us estimate $h_\Gamma(k)$. Recall that $h_\Gamma$ is strictly increasing till it reaches the value $d = \deg(C)$, and then it is constant and equal to $d$. Since $\sigma > \tau$ by assumption, letting $a = \frac{1}{2}(\tau^2 + \tau + 2)$, we have

$$\begin{cases} h_\Gamma(k) = \binom{k+2}{2}, & \text{if } 0 \leq k \leq \tau; \\ h_\Gamma(k) \geq \min\{k+a, d\}, & \text{if } k \geq \tau + 1, \end{cases}$$

since it is $k + a = h^0(\mathcal{O}_{\mathrm{P}^2}(\tau)) + (k - \tau)$. Summing (3.4) over $k$ we get $h^0(\mathcal{O}_C(n)) \geq \sum_{k=0}^n h_\Gamma(k)$ for any $n \geq 0$. On the other hand

$$\sum_{k=0}^{n} h_\Gamma(k) \geq \sum_{k=0}^{\tau} \binom{k+2}{2} + \sum_{l=\tau+1}^{n} \min\{l+a, d\}$$

for any $n \geq \tau + 1$. Therefore, for $n \geq d - a$, we get

$$\begin{aligned} h^0(\mathcal{O}_C(n)) &\geq \frac{1}{2}\sum_{k=0}^{\tau}(k+1)(k+2) + \sum_{l=\tau+1}^{d-a-1}(l+a) + \sum_{j=d-a}^{n} d \\ &= \frac{1}{2}\left(\frac{\tau^3}{3} + \frac{\tau^2}{2} + \frac{\tau}{6}\right) + \frac{3}{4}\tau(\tau+1) + (\tau+1) + \frac{1}{2}d(d-1) \\ &\quad - \frac{1}{2}(\tau+a+1)(\tau+a+2) + (n-d+a+1)d \\ &= 1 + nd - \frac{1}{2}(d-a)(d-a-1) - \frac{1}{3}(\tau^3 - \tau). \end{aligned} \tag{3.5}$$

Finally, for $n$ sufficentely large, $h^0(\mathcal{O}_C(n))$ is given by the Hilbert polynomial of $C$ and thus we have

$$h^0(\mathcal{O}_C(n)) = nd + 1 - p_a(C). \tag{3.6}$$

From (3.5) and (3.6) we immediately deduce (3.2).
To prove (3.3), assume that the equality holds in (3.2). In this case all the inequalities above become equalities, and we have $h_\Gamma(\tau+1) = \min\{\frac{1}{2}(\tau^2+3\tau+4), d\} \leq \frac{1}{2}(\tau^2+3\tau+4)$. We also have

$$h^0(\mathcal{O}_C(\tau+1)) = \sum_{k=0}^{\tau} \binom{k+2}{2} + h_\Gamma(\tau+1),$$

which gives $h^0(\mathcal{O}_C(\tau+1)) \leq \frac{1}{6}(\tau^3+9\tau^2+20\tau+18)$. The defining exact sequence yields

$$h^0(\mathcal{I}_C(\tau+1)) \geq h^0(\mathcal{O}_{P^3}(\tau+1)) - h^0(\mathcal{O}_C(\tau+1)) \geq \tau+1,$$

and by the restriction exact sequence it is also $h^0(\mathcal{I}_\Gamma(\tau+1)) \geq \tau+1$. Hence $\sigma = \tau+1$. It remains to compute $\chi(\Gamma)$. Note that since $\sigma > \tau$, we have $d \geq \frac{1}{2}(\tau^2+3\tau+2)$. Assume $d \geq \frac{1}{2}(\tau^2+3\tau+4)$. Then $h_\Gamma(\tau+1) = \frac{1}{2}(\tau^2+3\tau+4)$ and, taking into account (*ii*) of Proposition 1.1, we get $\sum_{i=0}^{\sigma-1}(\sigma+1-n_i)_+ = \sigma-1$. Since $n_i \geq \sigma$ for every $i = 0, ..., \sigma-1$, we have $n_1 = ... = n_{\sigma-1} = \sigma$ and $n_0$ can be expressed in terms of $d$ and $\sigma$ using (*i*) of Proposition 1.1. In the case $d = \frac{1}{2}(\tau^2+3\tau+2)$ we get $\sum_{i=0}^{\sigma-1}(\sigma+1-n_i)_+ = \sigma$ which implies $n_i = \sigma$ for every $i = 0, ..., \sigma-1$. □

3.3.1 *Remarks:*

1. The numerical character $\Theta_{d,\sigma} := (d - \frac{\sigma(\sigma-1)}{2}, \sigma, ..., \sigma)$ is the maximal character for the lexicographic order over all characters of degree $d$ and length $\sigma$.
2. Substituting $\tau$ with $\sigma-1$ in the formula (3.2) we get

$$g(\Theta_{d,\sigma}) = \frac{1}{2}\left(d - \frac{\sigma(\sigma-1)}{2} - 1\right)\left(d - \frac{\sigma(\sigma-1)}{2} - 2\right) + \frac{\sigma(\sigma-1)(\sigma-2)}{3}$$

   and one can show that this is the maximal genus of the characters of degree $d$ and length $\sigma$.
3. The character $\Theta_{d,t}$ is attained by a curve $C$ with $h^0(\mathcal{I}_C(t-1)) = 0$. Indeed, if $\Theta_{d,t}$ is connected, this follows from the results in [9]. If $\Theta_{d,t}$ is not connected, and this means that $d > t(t+1)/2$, we can consider the union of a plane curve $P \subseteq H$ of degree $d - t(t-1)/2$ with a projectively normal curve $C'$ with $\chi(\Gamma') = (t-1, \ldots, t-1)$ such that $C' \cap H \subseteq P$. Then one can check that $p_a(C) = g(\Theta_{d,t})$. As a consequence we get a lower bound for the maximal arithmetic genus $P_a(d,t)$ of curves of degree $d$ and not contained in a surface of degree $t-1$

$$P_a(d,t) \geq g(\Theta_{d,t}). \tag{3.7}$$

**3.4 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve and let $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$. Assume that, for some $b \geq 1$, we have $s \geq b$, $\sigma \leq b+2$ and $n_i \leq b+1$ for every $0 \leq i \leq \sigma - 1$. Then*

$$p_a(C) \leq (b-1)d + 1 - \binom{b+2}{3}.$$

*Proof.* Since $n_i \leq b+1$ for every $i = 0, ..., \sigma - 1$ and $\sigma \leq b+2$, using $(iii)$ of Proposition 1.1 we have $h^1(\mathcal{I}_\Gamma(n))) = 0$ for any $n \geq b$. Moreover, from the restriction exact sequence we get $h^2(\mathcal{I}_C(n)) = 0$ for any $n \geq b-1$. Therefore, as we have the exact sequence

$$0 \to \mathcal{I}_C(b-1) \to \mathcal{O}_{P^3}(b-1) \to \mathcal{O}_C(b-1) \to 0,$$

it follows that $h^1(\mathcal{O}_C(b-1)) = h^2(\mathcal{I}_C(b-1)) = 0$ and hence $\chi(\mathcal{O}_C(b-1)) = h^0(\mathcal{O}_C(b-1))$. As $s > b-1$, $C$ is not contained in a surface of degree $b-1$ and

$$h^0(\mathcal{O}_C(b-1) \geq h^0(\mathcal{O}_{P^3}(b-1)) = \binom{b+2}{3}.$$

We conclude recalling that $\chi(\mathcal{O}_C(b-1))$ is equal to the Hilbert polynomial of $C$ and thus $\chi(\mathcal{O}_C(b-1)) = d(b-1) - p_a(C) + 1$. The assertion of the Lemma immediately follows. □

## 4. Curves with low $\sigma$ and $s > \sigma$

It is well known that if a loc.CM curve has $\sigma = 1$ and degree $d \geq 3$, then $s = 1$ (see for. ex. [6]), while if $\sigma = 2, 3$ there exist curves of any degree having $s = 3, 4$. In this Section we will characterize curves of sufficientely large degree with $\sigma = 2, 3$ and $s > \sigma$.

**4.1 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 6$ with $\sigma = 2$. Then $s \geq 3$ if and only if $C$ contains a loc.CM plane curve $C'$ of degree $d-2$ and the curve $Res_H C$ where $H$ is the plane of $C'$ is a non planar curve of degree two. In particular, $s = 3$.*

*Proof.* Assume first $h^0(\mathcal{I}_\Gamma(2)) = 1$. Since $s > \sigma$, we have $n_1 = 3$ by $(iv)$ of Proposition 1.1 and Corollary 1.3. Moreover, property $(i)$ of Proposition 1.1 and the assumption on $d$ imply $n_0 \geq 4$. Hence we can apply Lemma 1.7 and we get the assertion.
We conclude the proof noting that the case $h^0(\mathcal{I}_\Gamma(2)) \geq 2$ can not occur, since it implies $s = 2$ by Lemma 4.2 below. □

**4.2 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 5$ with $\sigma = 2$. Assume that $h^0(\mathcal{I}_\Gamma(2)) \geq 2$. Then $C$ is contained in a reducible quadric surface, and hence $s = 2$.*

*Proof.* The assumption $h^0(\mathcal{I}_\Gamma(2)) \geq 2$ implies $n_1 = 2$ by $(iv)$ of Proposition 1.1. By $(i)$ of 1.1 and by the hypothesis on $d$ we have $n_0 \geq 4$. Applying Lemma 1.8 we get the assertion. □

**4.3 Lemma** *Let $C \subseteq \mathbf{P}^3$ be a curve of degree $d \geq 11$ with $\sigma = 3$. Then $s \geq 4$ if and only if one of the following properties holds:*

(1) *there exists a loc.CM curve $C' \subseteq C$ such that $deg(C') = d-2$, $s(C') = 2$, and $deg(Res_Q C) = 2$, $s(Res_Q C) = 2$, where $Q$ is a quadric surface such that $C' \subseteq Q$;*

(2) *there exists a loc.CM curve $C' \subseteq C$ such that $deg(C') = d-q$ with $3 \leq q \leq 5$, $s(C') = 1$, and $deg(Res_H C) = q$, $s(Res_Q C) \geq 3$, where $H$ is a plane such that $C' \subseteq H$.*

*Proof.* Taking into account Corollary 1.3 and Lemma 1.8, we have the following possibilities for $\chi(\Gamma) = (n_0, n_1, n_2)$: a) $(d-3, 3, 3)$, b) $(d-4, 4, 3)$, c) $(d-5, 4, 4)$, d) $(d-n_1-1, n_1, 4)$ with $n_1 \geq 5$. Since $d \geq 11$ by assumption, the characters a), b), c) are all non connected and we can apply Lemma 1.6 to get assertion (2). Assertion (1) can be obtained appling Lemma 1.7 to the character d).
For the converse, note that the curves described in (1) and (2) have $s \geq 4$ by construction.
□

## 5. Computation of $P_a(d,3)$ and $P_a(d,4)$

**5.1 Definition** We set $P_a(d,t)$ to be the maximal arithmetic genus of locally Cohen-Macaulay curves of degree $d$ in $\mathbf{P}^3$ not contained in a surface of degree $t-1$.

This Section is devoted to the computation of $P_a(d,t)$ for $t = 2,3,4$. Note that $P_a(d,t)$ is defined for any $d \geq t$ by Lemma 2.2.

The following result has been proven by Hartshorne in [13], and it holds over a field of any characteristic. For completeness we give a proof in characteristic zero.

**5.2 Proposition** *The maximal genus of curves of degree $d$ with $s \geq 2$ is*

$$P_a(d,2) = \begin{cases} -1, & \text{if } d = 2; \\ \frac{1}{2}(d-2)(d-3), & \text{if } d \geq 3 \end{cases}$$

*and for any $(d,p) \in \mathbf{N} \times \mathbf{Z}$ with $d \geq 2$ and $p \leq P_a(d,2)$, there exists a curve of degree $d$ and genus $p$ such that $s = 2$.*

*Proof.* If $\sigma = 1$, then $C$ is either a plane curve or a degree two curve of arithmetic genus $p_a(C) \leq -1$ [6]. Using the Ferrand construction one can show that there exist double structures on a line of any genus $p \leq -1$. Hence, if $d \geq 3$, as it is $s \geq 2$, we have $\sigma \geq 2$. Proposition 3.3 with $\tau = 1$ yields the bound of the thesis. It remains to prove that there exist curves of genus $p$ and degree $d$ for any $p = P_a(d,2) - r$ with $r \geq 0$ and any $d \geq 3$, and such that $s = 2$. We can take the union of a plane curve $C' \subseteq H$ of degree $d-2$ with a double line $\tilde{L}$ of arithmetic genus $-r$ such that $supp(\tilde{L}) \subseteq H$ but $\tilde{L} \not\subseteq H$, and $\tilde{L}$ intersects $C'$ in $d-2$ points transversally. □

**5.3 Proposition** *The maximal genus of curves of degree $d$ with $s \geq 3$ is*

$$P_a(d,3) = \begin{cases} 2d-9, & \text{if } 3 \leq d \leq 5; \\ \frac{1}{2}(d-3)(d-4), & \text{if } d \geq 6 \end{cases}$$

*and for any $(d,p) \in \mathbf{N} \times \mathbf{Z}$ with $d \geq 3$ and $p \leq P_a(d,3)$, there exists a curve of degree $d$ and genus $p$ such that $\sigma = 2$ and $s = 3$.*

*Proof.* For $3 \leq d \leq 5$, we can apply Lemma 3.4 as $\chi(\Gamma)$ verifies the required conditions with $b = 3$, and we get the bound of the statement of the Proposition. If $d \geq 6$, we have to distinguish the cases $\sigma = 2$ and $\sigma \geq 3$. If $\sigma = 2$, then

$$p_a(C) \leq \frac{1}{2}(d-3)(d-4). \tag{5.1}$$

Indeed, for a curve with $d \geq 6$, $\sigma = 2$ and $s \geq 3$, we have $\chi(\Gamma) = (d-2, 3)$ by Lemma 4.1, and $g(\chi(\Gamma)) = \frac{1}{2}(d-3)(d-4) + 1$. Since $s > \sigma$, $p_a(C) \leq g(\chi(\Gamma)) - 1$ by Lemma 3.1 and Corollary 3.2, and this gives (5.1). If $\sigma \geq 3$, then Proposition 3.3 with $\tau = 2$ yields

$$p_a(C) \leq \frac{(d-4)(d-5)}{2} + 2. \tag{5.2}$$

The highest bound is (5.1). Let us show that the bounds found are sharp. Consider the case $3 \leq d \leq 5$ and fix an integer $p = 2d - 9 - r$ with $r \geq 0$. For $d = 3$, $p$ is the genus of the disjoint union of a double structure $Z$ on a line with $p_a(Z) = -2 - r$, with a line $L$ not contained in any of the quadrics $Q \supseteq Z$, which form a two-dimensional linear system, consisting of couples of planes containing $Supp(Z)$. For $d = 4$, we can take the union of a degree two plane curve $C' \subset H$ with $Z$ as above, $Z$ not supported in $H$ and intersecting $C'$ in a point with multiplicity two. For $d = 5$, we consider the union of a smooth elliptic cubic with a double line of arithmetic genus $-1 - r$ which intersect in a point with multiplicity two. Finally, if $d \geq 6$, we set $p' = \frac{1}{2}(d-3)(d-4) - r$ with $r \geq 0$. Let $C' \subset H$ be a plane curve of degree $d - 2$ and $C''$ a double line with $p_a(C'') = -1 - r$, which is not supported in $H$ and such that $C''$ intersects $C'$ in a point with multiplicity two. Then the schematic union of $C'$ and $C''$ has the required invariants. □

5.3.1 *Remark* Let $C$ be a curve of degree $d \geq 6$ with $s \geq 3$, and let $p \in \mathbf{Z}$ be such that $\frac{1}{2}(d-4)(d-5) + 3 \leq p \leq \frac{1}{2}(d-3)(d-4)$. Then $C$ has genus $p$ if and only if $\sigma = 2$, i.e.

$p_a(C) = p$ if and only if there exists a plane $H$ and a curve $C' \subseteq C \cap H$ with $deg(C') = d-2$, and $Res_H C$ is a non planar curve of degree two, by Lemma 4.1.

**5.4 Proposition** *The maximal genus of curves of degree $d$ with $s \geq 4$ is*

$$P_a(d,4) = \begin{cases} 3d-19, & \text{if } 4 \leq d \leq 8; \\ \frac{1}{2}(d-4)(d-5)-1, & \text{if } d \geq 9 \end{cases}$$

*and for any $(d,p) \in \mathbf{N} \times \mathbf{Z}$ with $d \geq 4$ and $p \leq P_a(d,4)$, there exists a curve of degree $d$ and genus $p$ such that $\sigma = 3$ and $s = 4$.*

*Proof.* If $d = 4, 5$, then $\sigma = 2$, and if $d \geq 6$, we have $\sigma \geq 3$ by Proposition 4.1. One can check that for $4 \leq d \leq 8$, $\chi(\Gamma)$ verifies the assumptions of Lemma 3.4 with $b = 4$, which gives the bound

$$p_a(C) \leq 3d - 19. \tag{5.3}$$

If $d \geq 11$, then we have to distinguish the cases $\sigma = 3$ and $\sigma \geq 4$. If $\sigma = 3$, then the statement of Lemma 4.3 holds and one of the following sequences is exact:

$$0 \to \mathcal{O}_{C''}(-2) \to \mathcal{O}_C \to \mathcal{O}_{C \cap Q} \to 0, \tag{5.4}$$

where $Q$ is a quadric surface, $C \cap Q$ contains a curve $C'$ of degree $d-2$ and $C'' = Res_Q C$ has $deg(C'') = 2$, $s(C'') = 2$ and $p_a(C'') \leq -1$, or

$$0 \to \mathcal{O}_{\tilde{C}''}(-1) \to \mathcal{O}_C \to \mathcal{O}_{C \cap H} \to 0, \tag{5.5}$$

where $H$ is a plane, $C \cap H$ contains a curve $\tilde{C}'$ of degree $d - q$ with $3 \leq q \leq 5$ and $\tilde{C}'' = Res_H C$ has $deg(\tilde{C}'') = q$, $s(\tilde{C}'') \geq 3$ and, by Proposition 5.3, $p_a(\tilde{C}'') \leq 2q - 9$. The sequence (5.4) yields $p_a(C) = p_a(C \cap Q) - \chi(\mathcal{O}_{C''}(-2))$. We observe that $p_a(C \cap Q) \leq p_a(C')$, since zero-dimensional components, which may be embedded, do not effect $h^1(\mathcal{O}_{C'})$, so we shall bound $p_a(C')$. If $\chi(\Gamma) = (n_0, n_1, n_2)$ and $\Gamma'$ is the general plane section of $C'$, then $\chi(\Gamma') = (n_0, n_1)$ with $n_1 \geq 5$ by Lemmas 1.6 and 1.7. Using $(ii)$ of Proposition 1.1 we can compute $h_{\Gamma'}(k)$ for $k \leq 4$, and for $k \geq 5$ we can estimate the value of $h_{\Gamma'}$ in a similar way as in Proposition 3.3. We find $p_a(C') \leq \frac{1}{2}(d-7)(d-8)+6$. Moreover $\chi(\mathcal{O}_{C''}(-2))$ is equal to the value of the Hilbert polynomial of $C''$ in $-2$, and therefore $\chi(\mathcal{O}_{C''}(-2)) \geq -2$. Hence

$$p_a(C) \leq \frac{1}{2}(d-7)(d-8) + 8. \tag{5.6}$$

The sequence (5.5) yields $p_a(C) = p_a(C \cap H) - \chi(\mathcal{O}_{\tilde{C}''}(-1))$. Again, we can lower $\chi(\mathcal{O}_{\tilde{C}''}(-1))$ computing the Hilbert polynomial of $\tilde{C}''$, and we get the following inequalities

$$p_a(C) \leq \frac{1}{2}(d-6)(d-7) + 5, \quad \text{if} \quad q = 5, \tag{5.7}$$

$$p_a(C) \le \frac{1}{2}(d-5)(d-6)+2, \quad \text{if} \quad q=4, \tag{5.8}$$

$$p_a(C) \le \frac{1}{2}(d-4)(d-5)-1, \quad \text{if} \quad q=3. \tag{5.9}$$

Since $d \ge 11$, the highest bound is (5.9).
Consider now the range $9 \le d \le 10$ and assume $\sigma = 3$. For $d = 9$, we have the following possibilities for $\chi(\Gamma)$: $(4,4,4)$, $(5,4,3)$ and $(6,3,3)$. In the first two cases we can apply Lemma 3.4 to bound the genus. In the last case the sequence (5.5) is exact with $q = 3$. If $d = 10$, the possibilities for $\chi(\Gamma)$ are: $(5,4,4)$, $(6,4,3)$ and $(7,3,3)$. In the first case we apply Lemma 3.4, and in the other two cases the sequence (5.5) is exact with $q = 3$ and $q = 4$ respectively; hence (5.8) and (5.9), respectively, hold. One can check that (5.9) is the highest bound in both cases of $d = 9$ and $d = 10$.
It remains to consider the case $\sigma \ge 4$, which implies $d \ge 10$. Applying Proposition 3.3 with $\tau = 3$, we get

$$p_a(C') \le \frac{1}{2}(d-7)(d-8)+8, \tag{5.10}$$

which is lower than (5.9).
Let us show that the bounds (5.3) and (5.9) are sharp. Fix $r \ge 0$. For the case $d = 4$, consider the disjoint union of two double lines $L'$, $L''$ with $p_a(L') = -3$ and $p_a(L'') = -3 - r$. The curve obtained is not contained in a cubic surface and its genus is equal to $-7 - r = P_a(4,4) - r$. To prove the claim for $d = 5$, take the disjoint union of a line $L$ with a double line $Z$ of genus $-2 - r$; then fix a plane $H$ such that $H \not\supseteq L$ and $H \not\supseteq supp(Z)$, and fix a conic curve $P$ on $H$ which does not intersect $L$ nor $Z$. Then the curve $C = L \cup Z \cup P$ has $deg(C) = 5$, $s(C) = 4$ and $p_a(C) = -4 - r = P_a(5,4) - r$. As to the case $d = 6$, consider a line $L$ and a double line $Y$ with $p_a(Y) = -3 - r$ not intersecting each other. Fix a plane $H'$ which is transversal to $L \cup Y$; then, by counting dimensions, one checks that it is possible to find an elliptic cubic curve $X$ on $H'$ such that $X$ intersects both $L$ and $Y$ with maximal multiplicity and such that $X$ is not contained in any of the cubic surfaces containing $L \cup Y$. The curve $C = L \cup Y \cup X$ has $\deg(C) = 6$, $s(C) = 4$ and $p_a(C) = -1 - r = P_a(6,4) - r$. Finally, we consider the case $d \ge 7$. Let $C'$ be a plane curve of degree $d - 3$ and let $C''$ be a degree three curve such that $s(C'') = 3$, $p_a(C'') = -3 - r$ and assume that $C''$ intersects $C'$ with multiplicity three. Then the union of $C'$ with $C''$ has the required invariants. This concludes the proof. □

5.4.1 *Remark* Let $C$ be a curve of degree $d \ge 11$ with $s \ge 4$, and let $p \in \mathbb{Z}$ be such that $\frac{1}{2}(d-5)(d-6) + 3 \le p \le \frac{1}{2}(d-4)(d-5) - 1$. Then $C$ has genus $p$ if and only if $\sigma = 3$,

$s = 4$ and $\chi(\Gamma) = (d-3,3,3)$, i.e. $p_a(C) = p$ if and only if there exists a plane $H$ and a curve $C' \subseteq C \cap H$ with $deg(C') = d-3$ and $Res_H C$ is a degree three curve not on a quadric surface, by Lemma 4.3.

5.4.2 *Remark* We note that the genus $P_a(5,3)$, $P_a(7,4)$ and $P_a(8,4)$ are attained by smooth connected curves.

5.4.3 *Remark* The examples of maximal genus curves seen in Propositions 5.2, 5.3, 5.4 suggest a method for constructing curves of high genus, in general. The idea is to consider a plane curve $P \subseteq H$ of high degree and a suitable curve $C'$ not on a surface of degree $t-2$, which intersects $P$ in $deg(C')$ points, and such that the union of $P$ with $C'$ is not contained in a surface of degree $t-1$. For instance, let $L$ be a line and $S \supseteq L$ a general surface; then the divisor $C' = (t-1)L$ is such that $h^0(\mathcal{I}_{C'}(t-2)) = 0$, by Lemma 2.2. Fix an integer $d > 2t-2$, and consider the degree $t-1$ plane curve $P' = H \cap S$ and a curve $P''$ on $H$ of degree $d-2t+2$ which is not contained in $P'$. The schematic union of $P'$, $P''$ and $C'$ is a curve $C$ of degree $d$ with $h^0(\mathcal{I}_C(t-1)) = 0$. By Mayer-Vietoris sequence we have

$$p_a(C) = \frac{(d-t)(d-t-1)}{2} + p_a(C') + (t-1) - 1.$$

The genus $p_a(C')$ can be computed using adjunction formula which gives

$$p_a(C') = -(t-1)(t^2-2t+2).$$

Therefore, if $d \geq 2t-1$, we have

$$P_a(d,t) \geq \frac{(d-t)(d-t-1)}{2} - \frac{(t-1)(t-2)(t-3)}{2}. \tag{5.11}$$

For $d \gg 0$ this bound improves (3.7). Finally, we point out that the equality holds in (5.11) for $t \leq 3$ by Propositions 5.2 and 5.3, while we have a strict inequality for $t = 4$ by Proposition 5.4.

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