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Nonlinear elliptic problems related to some integral inequalities

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Notation

We collect here a list of notations that we will frequently use.

$\mathbb{N}$	the set of the natural numbers, 0 included.
$\mathbb{R}$	the set of the real numbers.
$\mathbb{R}^j$	for any $j \geq 1$, the j-fold cartesian product of $\mathbb{R}$ with itself. We define $\mathbb{R}^0 := \{0\}$, then $\mathbb{R}^j \equiv \mathbb{R}^j \times \mathbb{R}^0$. If $j = 1$ we will simply write $\mathbb{R}$.
$\mathbb{R}^j_0$	$\mathbb{R}^j$ excluding 0. If $j = 1$ we will write $\mathbb{R}_0 = (-\infty, 0) \cup (0, +\infty)$.
$\mathbb{R}^j_+$	$(0, +\infty) \times \mathbb{R}^{j-1}$. If $j = 1$ we will write $\mathbb{R}_+ = (0, +\infty)$.
$B^j_r(z)$	the j-dimensional ball of radius r and centered in $z \in \mathbb{R}^j$. If the ball is centered in 0, we will write B^j_r instead of $B^j_r(0)$. We will omit j if it is clear that the ball is in $\mathbb{R}^j$. If the ball is closed we will write $\overline{B}^j_r(z)$.
ω_j	the surface measure of the unit sphere in $\mathbb{R}^j$.
Ω	a domain in $\mathbb{R}^N$.
Δ	the Laplace operator $\Delta \cdot = \sum_{i=1}^N \frac{\partial^2}{\partial \xi_i^2} \cdot$.
Δ_p	the p-Laplace operator $\Delta_p \cdot = \mathrm{div}(\lvert\nabla \cdot\rvert^{p-2} \nabla \cdot)$, for any $p \geq 1$.
C	a generic positive constant that can vary in a chain of inequalities.

$C_c^\infty(\Omega)$	the space of smooth functions on Ω with compact support.
p^*	the critical Sobolev exponent $\frac{Np}{N-p}$ for any $p \in [1, N)$. We set $p^* = \infty$ if $N \geq p$.
$\mathcal{D}^{1,p}(\Omega)$	for $p < N$, the closure of $C_c^\infty(\Omega)$ with respect to the norm $$\|u\|_{\mathcal{D}^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p \, d\xi\right)^{1/p}.$$ If Ω is bounded, by Poincaré inequality, $\mathcal{D}^{1,p}(\Omega) = H_0^{1,p}(\Omega)$.
$L^q(\Omega; \|x\|^\alpha d\xi)$	the space of measurable maps u such that $\int_\Omega \|x\|^\alpha \|u\|^q d\xi < \infty$. Here we denote points ξ in $\mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $1 \leq k \leq N$. If $k = N$ we will simply write $L^q(\Omega; \|x\|^\alpha)$. $L^q(\Omega; \|x\|^0 d\xi) \equiv L^q(\Omega)$ is the standard Lebesgue space.
$S(p)$	the Sobolev constant for $p \in [1, N)$, $$S(p) := \inf_{\substack{u \in D^{1,p}(\mathbb{R}^N) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, d\xi}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, d\xi\right)^{p/p^*}}.$$ For $p = 2$ we will write S instead of $S(2)$.

We will also use the Landau symbols. For example $O(\varepsilon)$ is a generic function such that $\lim_{\varepsilon \to 0}[O(\varepsilon)/\varepsilon] \leq C$ and $o(\varepsilon)$ is a function such that $\lim_{\varepsilon \to 0}[o(\varepsilon)/\varepsilon] = 0$.

Introduction

The aim of this Ph.D. thesis is to present some recent results concerning nonlinear elliptic equations involving spherical and cylindrical weights. In particular, we address our interest in problems related to some integral inequalities in weighted Sobolev spaces.

One of the most known is the *Caffarelli-Kohn-Nirenberg inequality* proved in 1984 in the celebrated paper [22] (see also Chapter 2). Here we state it in a particular case.

Theorem 0.0.1 (Caffarelli, Kohn, Nirenberg) *Assume $p \in (1, N)$, $p \le q \le p^* = \frac{Np}{N-p}$, $a > p - N$ and set*

$$b_{a,p,q} := N - q\frac{N-p+a}{p}\,. \tag{0.0.1}$$

Then there exists a constant $C = C(a, p, q, N) > 0$ such that

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}}|u|^q\,dx\right)^{p/q} \le \int_{\mathbb{R}^N} |x|^a|\nabla u|^p\,dx \qquad \forall\, u \in C_c^\infty(\mathbb{R}^N)\,.$$

In literature there is a large number of papers that deal with extremals of the previous inequality. We quote for example [1], [2], [3], [4], [5], [13], [17], [26], [27], [30], [35], [38], [45], [46], [52], [53], [56], [78], [79] and [84].

The counterpart of the Caffarelli-Kohn-Nirenberg inequality for cylindrical weights was proved by Maz'ya in 1980 ([67], Section 2.1.6; see also Chapter 2).
We denote points $\xi \in \mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $1 \le k \le N$.

Theorem 0.0.2 (Maz'ya) *Assume $1 \le k < N$. Let a, p, $q \in \mathbb{R}$ satisfy*

$$1 < p < N, \quad a > (p-N)\frac{k}{N}, \quad \max\left\{p, \frac{p\,(N-k)}{N-p+a}\right\} < q \le p^* = \frac{Np}{N-p}\ , \tag{0.0.2}$$

and $b_{a,p,q}$ as in (0.0.1). Then there exists a constant $C = C(a,p,q,k) > 0$ such that

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}} |u|^q \, d\xi\right)^{p/q} \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi \qquad \forall\, u \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{N-k})\,. \tag{0.0.3}$$

Notice that Theorem 0.0.2 for the spherical case $k = N$ coincides exactly with Theorem 0.0.1.

In Part I we collect some integral inequalities we are interested in. In particular, in Chapter 1 we recall the *Hardy inequality* in several forms, while in Chapter 2 we present the above theorems and other Hardy-Sobolev type inequalities that will be useful later on to study some degenerate and singular elliptic problems.

The original results of this thesis are essentially contained in Parts II and III and in particular we refer to papers [47], [48], [49] and [50].

In Part II we address our attention to investigate the existence of extremals for the best constant in (0.0.3) and their qualitative properties. Thanks to inequality (0.0.3), we can define the Banach space $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ by completing the space $C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{N-k})$ with respect to the norm $||u||^p = \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi$. Then we deal with the following minimization problem

$$S_{a,q}(p) := \inf_{\substack{u \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}} |u|^q \ d\xi\right)^{p/q}}\,. \tag{0.0.4}$$

The study of the Rayleigh quotient in (0.0.4) is strictly related to the following Euler-Lagrange equation

$$-\mathrm{div}(|x|^a |\nabla u|^{p-2}\, \nabla u) = |x|^{-b_{a,p,q}}\, |u|^{q-2}\, u \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k}\,. \tag{0.0.5}$$

In fact, extremals for (0.0.3), namely minima for (0.0.4), are the so called *ground state* solutions to (0.0.5). Our first main result is the following.

Theorem 0.0.3 *Assume that (0.0.2) and (0.0.1) are satisfied. Then $S_{a,q}(p)$ is achieved provided*

$$q < p^* \quad \text{or} \quad q = p^* \quad \text{and} \quad S_{a,p^*}(p) < S(p)\,.$$

The limiting case $q = p^*$ is more difficult. In general $S_{a,p^*}(p) \le S(p)$ for any $a > (p-N)\frac{k}{N}$ (see Proposition 3.2.1 at page 32) and we prove in the next result that the strict inequality holds true if a is negative.

Theorem 0.0.4 *Let $p \in (1, N)$. If $(p-N)\frac{k}{N} < a < 0$, then $S_{a,p^*}(p) < S(p)$ and hence $S_{a,p^*}(p)$ is achieved.*

One of the main features in (0.0.4) and (0.0.5) is their invariance with respect to transforms

$$u(x,y) \to (T(\tau,\eta)\,u)(x,y) := \tau^{\frac{N-p+a}{p}} u(\tau x, \tau y + \eta) \ ,$$

where $\tau \in (0,+\infty)$ and $\eta \in \mathbb{R}^{N-k}$. In fact, for any $u \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{N-k})$,

$$\int_{\mathbb{R}^N} |x|^a |\nabla (T(\tau,\eta)u)|^p \, d\xi = \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi \ ,$$

and

$$\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}} |T(\tau,\eta)u|^q \, d\xi = \int_{\mathbb{R}^N} |x|^{-b_{a,p,q}} |u|^q \, d\xi \ ,$$

with

$$b_{a,p,q} = N - q\frac{N-p+a}{p}$$

and whenever the weights $|x|^a$ and $|x|^{-b_{a,p,q}}$ are locally integrable on $\mathbb{R}^k$ (as in assumption (0.0.2)).

We notice that, for every minimizing sequence u_h to problem (0.0.4) and for arbitrary sequences $\tau_h \in (0,+\infty)$, $\eta_h \in \mathbb{R}^{N-k}$, it turns out that

$$\tilde{u}_h(x,y) := \tau_h^{\frac{N-p+a}{p}} u_h(\tau_h x, \tau_h y + \eta_h)$$

still approaches the infimum in (0.0.4). These considerations lead us to conclude that the action of the group of dilations in $\mathbb{R}^N$ and of translations in $\mathbb{R}^{N-k}$ produces a *lack of compactness* phenomenon. In the limiting case $q = p^*$, the group of translations in the x-variable and of dilations in $\mathbb{R}^N$ makes the lack of compactness worse, since minimizing sequences for (0.0.4) might blow-up an extremal for the Sobolev constant $S(p)$.

In Chapter 3 we overcome these difficulties with a strategy already followed in [69]. The idea consists in looking for a minimizing sequence that does not concentrate at $\{x = 0\}$ and does not vanish, namely it does not converge strongly to zero in $L^q_{\rm loc}(\mathbb{R}^N; |x|^{-b_{a,p,q}} d\xi)$. This is possible by means of a suitable rescaling argument, a

Rellich-type theorem and Ekeland variational principle ([36]). In this way, we can find a weakly convergent subsequence u_h whose L^q-norms are bounded away from 0 on a compact subset of $(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$. If $q < p^*$ we can use Rellich Theorem and we obtain that u_h converges weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ to some $u \neq 0$. Then u achieves $S_{a,q}(p)$ by standard arguments. However, concentration phenomena at points (x_0, y_0) with $x_0 \neq 0$ might arise if $q = p^*$. The assumption $S_{a,q}(p) < S(p)$ allows us to avoid this problem and we can conclude as in the subcritical case. In [47], we skip the blow-up analysis of all minimizing sequences: we do not require the Brezis-Lieb Lemma ([18]) and the Concentration-Compactness Lemmata by P. L. Lions ([60], [61], [62], [63]).

We state our existence results not only in $\mathbb{R}^N$, as in Theorem 0.0.3, but also in $\mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a *cone* in $\mathbb{R}^k$ (see Definition 2.1.1 at page 20 and Theorem 3.0.9 at page 29). We remark that the only domains in $\mathbb{R}^N$ that are invariant with respect to dilations and translations in the y-variable are of this type).

In Chapter 4 we deal with the particular case $p = 2$. We present further existence results that we summarize in Theorem 4.2.11 at page 53. Here we need an other integral inequality proved by Maz'ya in case $a = 0$ and that can be easily generalized to the case $a \neq 0$ (see Chapter 2 for details).

Theorem 0.0.5 (Maz'ya) *Let $1 \leq k < N$, $N \geq 3$. Assume*

$$a \in \mathbb{R}, \quad 2 < q \leq 2^* := \frac{2N}{N-2} \quad \text{and} \quad b_{a,q} := N - q\frac{N-2+a}{2}\,.$$

Then, for any $u \in C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$,

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}}|u|^q\,d\xi\right)^{2/q} \leq \int_{\mathbb{R}^N} |x|^a|\nabla u|^2\,d\xi - \lambda_1(a)\int_{\mathbb{R}^N} |x|^{a-2}|u|^2\,d\xi\,, \quad (0.0.6)$$

where $C = C(a,q,k) > 0$ is a constant and

$$\lambda_1(a) := \left(\frac{k-2+a}{2}\right)^2\,.$$

We remark that $\lambda_1(a)$ is the best constant in Hardy inequality (see Chapter 1 for a discussion on this subject).

Inequality (0.0.6) is the starting point for studying the following class of equations

$$-\mathrm{div}(|x|^a\nabla u) = \lambda|x|^{a-2}u + |x|^{-b_{a,q}}|u|^{q-2}\,u \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k}\ ,\ x \neq 0\,, \quad (0.0.7)$$

where $N \geq 3$, a is a real parameter, $q \in (2, 2^*]$ and $\lambda \leq \lambda_1(a)$.

In order to give some multiplicity results, in Chapter 4 (see also [49]) we consider, first of all, positive solutions to (0.0.7) for $\lambda = 0$, whose existence was proved in [69] for $a \neq 2 - k$ and in [85] for $a = 2 - k$. In [69], solutions u to

$$-\mathrm{div}(|x|^a \nabla u) = |x|^{-b_{a,q}} |u|^{q-2}\, u \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k} \ , \ x \neq 0 \tag{0.0.8}$$

are minimizers of

$$S^X_{a,q} := \inf_{\substack{u \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \ d\xi\right)^{2/q}} \, , \tag{0.0.9}$$

where

$$X^{1,2}(\mathbb{R}^N; |x|^a d\xi) := \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi) \, ,$$

and therefore are characterized by having

$$\int_{\mathbb{R}^N} |x|^{a-2}\, u^2 \, d\xi < \infty \, . \tag{0.0.10}$$

We analyze the qualitative properties and the behaviour of $S^X_{a,q}$ in (0.0.9) and of $S_{a,q}(2)$ in (0.0.4). We prove, under particular assumptions on the parameters a and q, that $S_{a,q}(2) < S^X_{a,q}$ and they are both achieved (see, for $q < 2^*$, Corollaries 4.2.6, 4.2.7 at page 51 and Theorem 4.2.8 at page 52; while we have summarized the case $q = 2^*$ in Theorem 4.2.13 at page 54). We can conclude, in these cases, that there exist at least two positive solutions to (0.0.8): the first satisfies condition (0.0.10) and the second one is characterized by having

$$\int_{\mathbb{R}^N} |x|^{a-2}\, u^2 \, d\xi = \infty \, . \tag{0.0.11}$$

Now, setting $v := |x|^{\frac{a}{2}}\, u$, we notice by direct computations that if u is a solution to problem (0.0.8), then v solves

$$-\Delta v = \lambda |x|^{-2} \ v + |x|^{-b_q} \ |v|^{q-2}\, v \quad \text{in } \mathbb{R}^k \times \mathbb{R}^{N-k} \ , \ x \neq 0 \, , \tag{0.0.12}$$

where $\lambda, q, b_q \in \mathbb{R}$ satisfy, for $N \geq 3$,

$$\lambda \leq \lambda_1(0) := \left(\frac{k-2}{2}\right)^2 \ , \quad q \in (2, 2^*] \ , \quad b_q = N - q\frac{N-2}{2} \ .$$

As a consequence we can state existence results for problem (0.0.12) and we find, under particular assumptions on the parameters $\lambda < \lambda_1(0)$ and q, solutions v to (0.0.12) such that $u = |x|^{-\frac{a}{2}}\, v$ satisfies condition (0.0.11) and hence

$$\int_{\mathbb{R}^N} |x|^{-2}\, v^2\, d\xi = \infty$$

(see Section 4.3 and Theorem 4.3.4 at page 58).

In Chapter 5 we deal with symmetry questions related to problem (0.0.12) and we present some results obtained in [48]. We prove existence also beyond the usual critical exponent 2^* (see Theorem 5.2.1 at page 62 and Theorem 5.2.3 at page 63) and a symmetry result for all *classical* solutions to (0.0.12) in case $0 \le \lambda \le \lambda_1(0)$ (see Theorem 5.2.5 at page 63). Moreover, we point out the symmetry breaking phenomenon of *ground states* in case $\lambda < 0$, provided $|\lambda|$ is large enough (see also Theorem 5.3.1 at page 69).

Theorem 0.0.6 *Let $2 \le k < N$ and $q \in (2, 2^*)$. Then ground states solutions to (0.0.12) are not radially symmetric in x if*

$$\lambda \le \left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2}\,.$$

Finally, we give analogous symmetry results for the degenerate problem (0.0.8) (see Section 5.4).

In Part III we consider a nonlinear elliptic equation involving spherical weights with positive powers. More precisely, we deal with the Hénon equation (see [55]):

$$-\Delta u = |x|^{\alpha} |u|^{q-2}\, u \quad \text{in } B_1\,, \tag{0.0.13}$$

where $B_1 := B_1^N(0)$ is the unit ball in $\mathbb{R}^N$ centered in the origin, $q > 2$ and $\alpha > 0$. Equation (0.0.13) raises several questions about existence, non-existence, multiplicity and symmetric properties of solutions to problems in which it appears. For these reasons it was largely studied in the past, in fact in literature we can find a lot of papers that deal with Hénon equation associated to homogeneous Dirichlet boundary conditions ($u = 0$ on ∂B_1). We quote for example [71], [80], [81], [28], [9], [20], [21], [76], [74], [25], [77], [24] and [23]. Nevertheless the following *Neumann* problem has been studied only very recently ([50], [14], [16]):

$$\begin{cases} -\Delta u + u = |x|^{\alpha} |u|^{q-2}\, u & \text{in } B_1 \\[2mm] \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1, \end{cases} \tag{0.0.14}$$

where again $q > 2$ and $\alpha > 0$. We have denoted by ν the outer normal to ∂B_1. Obviously, solutions to (0.0.14) arise from critical points of the functional $Q_\alpha : H^1(B_1) \setminus \{0\} \to \mathbb{R}$ defined by

$$Q_\alpha(u) := \frac{\int_{B_1} |\nabla u|^2 \, dx + \int_{B_1} u^2 \, dx}{\left(\int_{B_1} |x|^\alpha |u|^q \, dx\right)^{2/q}} \,. \tag{0.0.15}$$

We present the results obtained in [50], whose purpose is to investigate the existence of positive radial solutions to (0.0.14) also beyond the usual critical exponent 2^* (see Theorem 6.1.3 and Corollary 6.1.4 at page 81), as Ni proved in [71] for the Dirichlet problem. Moreover, we are especially interested in the analysis of symmetry properties of ground states that gives birth to unexpected phenomena, completely different with respect to the ones for (0.0.13) with Dirichlet boundary conditions (see [80]). The starting point is the fact that Gidas-Ni-Nirenberg type results ([54]) do not apply, and therefore nonradial solutions could be expected.

First of all, by means of the *Steklov* problem, a classical eigenvalue problem (see Chapter 6, Subsection 6.1.1 for the definition), we can describe the precise asymptotic behaviour of *radial minimizers* to (0.0.15) as $\alpha \to \infty$. They are minima of the functional Q_α on $H^1_{\mathrm{rad}}(B_1)$, that is the space of radial functions in $H^1(B_1)$. A further important point in all the symmetry questions is played by the number $2_* := \frac{2(N-1)}{N-2}$, the critical exponent for the embedding of $H^1(B_1)$ in $L^q(\partial B_1)$. In the Dirichlet case, Smets, Su and Willem proved in [80] that the symmetry of ground states breaks down for all $q \in (2, 2^*)$ as α is large enough. This phenomenon occurs because the second derivative of Q_α at a radial minimizer becomes indefinite on $H^1(B_1)$. For the Neumann problem we have a symmetry breaking result only for $q \in (2_*, 2^*)$ (see Theorem 6.2.1 at page 86) and the situation in this case is completely different because radial minimizers continue to be local minima also on the whole space $H^1(B_1)$ (see Theorem 6.3.8 at page 95). For α large, notice that a multiplicity result for (0.0.14) (a radial solution and the nonradial ground state) holds only if $q \in (2_*, 2^*)$ (see Remark 6.2.3 at page 87); while in the Dirichlet case there is multiplicity for any $q \in (2, 2^*)$.

In order to analyze what happens for $q \in (2, 2_*)$, we recall the *Sobolev trace inequality* (see for example [33]). For every $q \in [1, 2_*]$, there exists a constant $C > 0$ such that

$$C \left(\int_{\partial B_1} |u|^q d\sigma\right)^{2/q} \leq \int_{B_1} |\nabla u|^2 \, dx + \int_{B_1} u^2 \, dx \,.$$

Moreover, we introduce the functional $S_q : H^1(B_1) \setminus H^1_0(B_1) \to \mathbb{R}$ defined by

$$S_q(u) := \frac{\int_{B_1} |\nabla u|^2 \, dx + \int_{B_1} u^2 \, dx}{\left(\int_{\partial B_1} |u|^q d\sigma \right)^{2/q}} \, ,$$

whose infimum is the best constant in the previous inequality. We prove that the functional S_q plays the role of a limiting functional for Q_α when $\alpha \to \infty$ (see Lemma 6.1.6 at page 82). Therefore many properties of minimizers of Q_α for α large and of S_q coincide. Taking into account of these considerations, we can prove that for $q < 2_*$ close enough to 2 the ground state is radial and it is the unique positive solution (up to rotations) to problem (0.0.14). The result is the following (see also Theorem 6.6.1 at page 103).

Theorem 0.0.7 *There exists $\hat{q} \in (2, 2_*]$ such that if $q \in (2, \hat{q}\,)$, then, for every α large enough, the problem*

$$\min_{\substack{u \in H^1(B_1) \\ u \not\equiv 0}} \frac{\int_{B_1} |\nabla u|^2 dx + \int_{B_1} u^2 dx}{\left(\displaystyle\int_{B_1} |x|^\alpha |u|^q \, dx \right)^{2/q}}$$

has a unique positive solution, up to a multiplicative constant, and it is a radial function.

Part I

Integral inequalities

Chapter 1

Hardy inequalities

In this chapter and in the next, we will recall some essentially already proved inequalities that will be fundamental to study degenerate elliptic problems with singular potentials and, in particular, involving cylindrical weights (see Part II for details).

Here we present the Hardy inequality in several forms: from the classical one to some more general (see for example [70], [68], [32] and reference there-in). First of all we will discuss the spherical case and then we generalize to the cylindrical one.

1.1 Generalized Hardy inequality

In this section we will prove a Hardy-type inequality that we will call *generalized Hardy inequality.* From now on we will assume $k \in \mathbb{N}$, $k \geq 1$. We define $\mathbb{R}^k_0 := \mathbb{R}^k \backslash \{0\}$ for brevity.

Theorem 1.1.1 *Let $a \in \mathbb{R}$ and $u \in C^\infty_c(\mathbb{R}^k)$. If $a \leq 2-k$, assume that the support of u is contained in $\mathbb{R}^k_0$. Then*

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2}|u|^2\,dx \leq \int_{\mathbb{R}^k} |x|^a|\nabla u|^2\,dx\,. \tag{1.1.1}$$

Moreover, the constant

$$\lambda_1(a) := \left(\frac{k-2+a}{2}\right)^2$$

is sharp and it is not achieved.

Before proving Theorem 1.1.1, we start with an identity that will be useful also later on. For the sake of completeness, we recall the proof that is contained in [70].

Lemma 1.1.2 *Let $a \in \mathbb{R}$ and $u \in C_c^\infty(\mathbb{R}^k_0)$. We set $\lambda_1(a) = (\frac{k-2+a}{2})^2$ and $\lambda_1(0) = (\frac{k-2}{2})^2$, then*

$$\begin{aligned} \int_{\mathbb{R}^k} |x|^a |\nabla u|^2\, dx \;&-\; \lambda_1(a) \int_{\mathbb{R}^k} |x|^{a-2}|u|^2\, dx \\ &= \int_{\mathbb{R}^k} |\nabla(|x|^{\frac{a}{2}}\, u)|^2\, dx - \lambda_1(0) \int_{\mathbb{R}^k} |x|^{-2}||x|^{\frac{a}{2}}\, u|^2\, dx\,. \end{aligned} \tag{1.1.2}$$

Proof. For every $u \in C_c^\infty(\mathbb{R}^k_0)$ we compute

$$\nabla(|x|^{\frac{a}{2}}\, u) = \frac{a}{2}|x|^{\frac{a}{2}-2}\, x\, u + |x|^{\frac{a}{2}}\nabla u\,.$$

Then we get

$$\int_{\mathbb{R}^k} |\nabla(|x|^{\frac{a}{2}} u)|^2 dx = \left(\frac{a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2}|u|^2 dx + \int_{\mathbb{R}^k} |x|^a|\nabla u|^2 dx + a\int_{\mathbb{R}^k} |x|^{a-2}(x\cdot\nabla u)u\, dx. \tag{1.1.3}$$

Let $a \neq 0$ (if $a = 0$ the identity (1.1.2) is trivial). Notice that, integrating by parts, we get

$$\begin{aligned} \int_{\mathbb{R}^k} |x|^{a-2}(x\cdot\nabla u)\, u\, dx &= \frac{1}{2a}\int_{\mathbb{R}^k} \nabla(|x|^a)\cdot\nabla(|u|^2)\, dx = -\frac{1}{2a}\int_{\mathbb{R}^k} \Delta(|x|^a)\, |u|^2\, dx \\ &= -\left(\frac{k-2+a}{2}\right) \int_{\mathbb{R}^k} |x|^{a-2}|u|^2\, dx\,. \end{aligned}$$

Hence (1.1.3) becomes

$$\int_{\mathbb{R}^k} |\nabla(|x|^{\frac{a}{2}}\, u)|^2\, dx = \left[\left(\frac{a}{2}\right)^2 - \frac{a(k-2+a)}{2}\right] \int_{\mathbb{R}^k} |x|^{a-2}|u|^2\, dx + \int_{\mathbb{R}^k} |x|^a|\nabla u|^2\, dx\,.$$

The conclusion readily follows, since $|x|^{-2}||x|^{\frac{a}{2}}\, u|^2 = |x|^{a-2}|u|^2$ and

$$\left(\frac{a}{2}\right)^2 - \frac{a(k-2+a)}{2} = \lambda_1(0) - \lambda_1(a)\,.$$

$\square$

Remark 1.1.3 Note that if $a = 2-k$, by (1.1.2) we obtain that

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{-2} ||x|^{\frac{2-k}{2}} u|^2 \, dx \le \int_{\mathbb{R}^k} |\nabla(|x|^{\frac{2-k}{2}} u)|^2 \, dx \tag{1.1.4}$$

holds for every $u \in C_c^\infty(\mathbb{R}_0^k)$. Now, set $v := |x|^{\frac{2-k-a'}{2}} u$, with $a' \in \mathbb{R}$. Inequality (1.1.4) becomes

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{-2} ||x|^{\frac{a'}{2}} v|^2 \, dx \le \int_{\mathbb{R}^k} |\nabla(|x|^{\frac{a'}{2}} v)|^2 \, dx \,. \tag{1.1.5}$$

Applying identity (1.1.2) to (1.1.5) with respect to $a' \in \mathbb{R}$, we get that

$$\left(\frac{k-2+a'}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a'-2} |v|^2 \, dx \le \int_{\mathbb{R}^k} |x|^{a'} |\nabla v|^2 \, dx$$

for any $v \in C_c^\infty(\mathbb{R}_0^k)$.

Then, we need a density result that was proved by Musina in [70], in case $a = 0$. The proof is analogous also for $a \neq 0$, but we quote it for completeness.

Lemma 1.1.4 *Assume $a > 2-k$ and fix $u \in C_c^\infty(\mathbb{R}^k)$. Then there exists a sequence $u_h \in C_c^\infty(\mathbb{R}_0^k)$ such that $u_h \to u$ in $L^2(\mathbb{R}^k; |x|^{a-2})$ and*

$$\int_{\mathbb{R}^k} |x|^a |\nabla u_h - \nabla u|^2 \, dx \to 0 \,. \tag{1.1.6}$$

Proof. Consider, for every integer $h \ge 1$, $\varphi_h \in C^\infty(\mathbb{R})$ such that $0 \le \varphi_h \le 1$ and $|\varphi_h'| \le 2h$. In particular $\varphi_h(r) \equiv 0$ for $r \le h^{-1}$ and $\varphi_h(r) \equiv 1$ for $r \ge 2h^{-1}$. Set

$$u_h(x) := \varphi_h(|x|) u(x) \in C_c^\infty(\mathbb{R}_0^k) \,.$$

Since $a > 2-k$, $|x|^{a-2}$ is locally integrable in $\mathbb{R}^k$ and, applying Lebesgue's Theorem, we get that the sequence u_h converges to u in $L^2(\mathbb{R}^k; |x|^{a-2})$. Moreover,

$$\nabla u_h = \varphi_h(|x|) \nabla u + \varphi_h'(|x|) \frac{x}{|x|} u \,. \tag{1.1.7}$$

Let $A_h = \{x \in \text{supp}(u) \,|\, |x| \le 2h^{-1}\}$, where $\text{supp}(u)$ is the support of u. Notice that $|x|^{a-2}|u|^2 \in L^1(\mathbb{R}^k)$ and that the measure of A_h goes to 0 as $h \to +\infty$. Then, by assumption on φ_h' and the definition of A_h, we get

$$\int_{\mathbb{R}^k} |x|^a |\varphi_h'(|x|)|^2 |u|^2 \, dx \le 16 \int_{A_h} |x|^{a-2} |u|^2 \, dx = o(1) \,.$$

The conclusion follows from (1.1.7) and Lebesgue's Theorem that imply (1.1.6). $\square$

Now we present the proof of Theorem 1.1.1 (see [70] for $a = 0$, but notice that it is very similar in the general case $a \neq 0$).

Proof of Theorem 1.1.1. Step 1 First of all we want to prove inequality (1.1.1). By Remark 1.1.3, (1.1.1) holds for every $u \in C_c^\infty(\mathbb{R}_0^k)$ and then also for the sequence u_h defined in the previous lemma:

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2}|u_h|^2\,dx \leq \int_{\mathbb{R}^k} |x|^a|\nabla u_h|^2\,dx\,. \tag{1.1.8}$$

Thanks to Lemma 1.1.4, we can let $h \to +\infty$ in (1.1.8) and get the inequality (1.1.1) also for any $u \in C_c^\infty(\mathbb{R}^k)$ if $a > 2-k$.

Step 2 Now we have to prove that $\lambda_1(a)$ coincides with the best constant in the Hardy inequality, defined by

$$\lambda_H := \inf_{\substack{u \in C_c^\infty(\mathbb{R}_0^k)\\ u \neq 0}} \frac{\int_{\mathbb{R}^k} |x|^a|\nabla u|^2 dx}{\int_{\mathbb{R}^k} |x|^{a-2}|u|^2 dx}\,.$$

From (1.1.1) we deduce that

$$\lambda_1(a) \leq \frac{\int_{\mathbb{R}^k} |x|^a|\nabla u|^2\,dx}{\int_{\mathbb{R}^k} |x|^{a-2}|u|^2\,dx} \tag{1.1.9}$$

holds for every $u \in C_c^\infty(\mathbb{R}_0^k)$. If we pass to the inf in (1.1.9) we get that $\lambda_1(a) \leq \lambda_H$. We only have to prove that $\lambda_H \leq \lambda_1(a)$.

Set

$$\varphi_\varepsilon(|x|) := \begin{cases} 0 & \text{if } |x| \leq \varepsilon^2 \\ -\frac{\log|x|/\varepsilon^2}{\log\varepsilon} & \text{if } \varepsilon^2 < |x| < \varepsilon \\ 1 & \text{if } \varepsilon \leq |x| \leq \frac{1}{\varepsilon} \\ \frac{\log\varepsilon^2|x|}{\log\varepsilon} & \text{if } \frac{1}{\varepsilon} < |x| < \frac{1}{\varepsilon^2} \\ 0 & \text{if } |x| \geq \frac{1}{\varepsilon^2}\,. \end{cases} \tag{1.1.10}$$

Define $u_\varepsilon(x) := |x|^{-\frac{k-2+a}{2}}\varphi_\varepsilon(|x|)$ and notice that it can be approximated by smooth maps with compact support in $\mathbb{R}_0^k$. By direct computations we get

$$\begin{aligned} \int_{\mathbb{R}^k} |x|^a|\nabla u_\varepsilon|^2\,dx &= \int_{\mathbb{R}^k} |x|^{2-k}|\nabla\varphi_\varepsilon|^2\,dx + \left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{-k}|\varphi_\varepsilon|^2\,dx \\ &- (k-2+a)\int_{\mathbb{R}^k} |x|^{-k}\varphi_\varepsilon\,(x\cdot\nabla\varphi_\varepsilon)\,dx\,. \end{aligned} \tag{1.1.11}$$

Moreover, integrating by parts, if $k \neq 2$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^k} |x|^{-k}\varphi_\varepsilon\,(x\cdot\nabla\varphi_\varepsilon)\,dx &= \frac{1}{2(2-k)}\int_{\mathbb{R}^k}\nabla(|x|^{2-k})\cdot\nabla(|\varphi_\varepsilon|^2)\,dx\\ &= -\frac{1}{2(2-k)}\int_{\mathbb{R}^k}\Delta(|x|^{2-k})\,|\varphi_\varepsilon|^2\,dx = 0\,;\end{aligned}$$

if $k = 2$

$$\begin{aligned}\int_{\mathbb{R}^2} |x|^{-2}\varphi_\varepsilon\,(x\cdot\nabla\varphi_\varepsilon)dx &= \frac{1}{2}\int_{\mathbb{R}^2}\nabla(\log|x|)\cdot\nabla(|\varphi_\varepsilon|^2)dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^2}\Delta(\log|x|)|\varphi_\varepsilon|^2dx = 0\,.\end{aligned}$$

Then (1.1.11) becomes

$$\begin{aligned}\int_{\mathbb{R}^k}|x|^a|\nabla u_\varepsilon|^2\,dx &= \int_{\mathbb{R}^k}|x|^{2-k}|\nabla\varphi_\varepsilon|^2\,dx + \left(\frac{k-2+a}{2}\right)^2\int_{\mathbb{R}^k}|x|^{a-2}|u_\varepsilon|^2\,dx\\ &= O(|\log\varepsilon|^{-1}) + \left(\frac{k-2+a}{2}\right)^2\int_{\mathbb{R}^k}|x|^{a-2}|u_\varepsilon|^2\,dx\,. \qquad (1.1.12)\end{aligned}$$

About the other integral, we compute

$$\int_{\mathbb{R}^k}|x|^{a-2}|u_\varepsilon|^2\,dx = \int_{\mathbb{R}^k}|x|^{-k}|\varphi_\varepsilon|^2\,dx = O(|\log\varepsilon|)\,. \qquad (1.1.13)$$

Therefore, from (1.1.12) and (1.1.13), it follows

$$\lambda_H \le \frac{\int_{\mathbb{R}^k}|x|^a|\nabla u_\varepsilon|^2\,dx}{\int_{\mathbb{R}^k}|x|^{a-2}|u_\varepsilon|^2\,dx} \le \left(\frac{k-2+a}{2}\right)^2 + O(|\log\varepsilon|^{-2})\,.$$

Letting $\varepsilon \to 0$, we can conclude that $\lambda_H \le \lambda_1(a)$ and hence $\lambda_1(a)$ coincides with the best constant.

Step 3 In order to prove that the Hardy constant is not achieved, we argue by contradiction. Assume there exists a map $u \in L^2(\mathbb{R}^k;|x|^{a-2})$, with $\int_{\mathbb{R}^k}|x|^a|\nabla u|^2dx < \infty$, that satisfies

$$\lambda_H = \frac{\int_{\mathbb{R}^k}|x|^a|\nabla u|^2\,dx}{\int_{\mathbb{R}^k}|x|^{a-2}|u|^2\,dx}\,. \qquad (1.1.14)$$

Since the quotient in (1.1.14) is zero degree homogeneous with respect to u, we can rescale the map such that

$$\int_{\mathbb{R}^k}|x|^{a-2}|u|^2\,dx = 1\,, \qquad \int_{\mathbb{R}^k}|x|^a|\nabla u|^2\,dx = \lambda_H = \left(\frac{k-2+a}{2}\right)^2\,.$$

We can observe that u is the minimum of the functional $\mathcal{H}$: $C_c^\infty(\mathbb{R}^k_0)\backslash\{0\} \to \mathbb{R}_+$, defined as

$$\mathcal{H}(v) := \frac{\int_{\mathbb{R}^k} |x|^a |\nabla v|^2\, dx}{\int_{\mathbb{R}^k} |x|^{a-2}|v|^2\, dx}\,.$$

Then, for any $\psi \in C_c^\infty(\mathbb{R}^k_0)$, we get

$$0 = \mathcal{H}'(0)[\psi] = 2\left[\int_{\mathbb{R}^k} |x|^a \nabla u \cdot \nabla\psi\, dx - \left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2} u\,\psi\, dx\right],$$

namely, u is a weak solution to

$$-\mathrm{div}(|x|^a\, \nabla u) = \left(\frac{k-2+a}{2}\right)^2 |x|^{a-2} u \quad \text{in } \mathbb{R}^k_0. \tag{1.1.15}$$

Moreover, by standard elliptic regularity theory, u is smooth in $\mathbb{R}^k_0$.

Now, we consider, for every $h \in \mathbb{N}$, $h \geq 1$, cut-off functions $\varphi_h \in C_c^\infty(\mathbb{R}^k)$ such that $\varphi_h(x) \equiv 0$ if $|x| \leq 1/2h$ or if $|x| \geq 2h$, $\varphi_h(x) \equiv 1$ if $1/h \leq |x| \leq h$, $|\nabla\varphi_h| \leq 4h$ if $1/2h \leq |x| \leq 1/h$ and $|\nabla\varphi_h| \leq 2/h$ if $h \leq |x| \leq 2h$. Multiply (1.1.15) by $\varphi_h^2 u$ and integrate by parts to get

$$\int_{\mathbb{R}^k} |x|^a \nabla u \cdot \nabla(\varphi_h^2\, u)\, dx = \left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2} |\varphi_h u|^2\, dx\,. \tag{1.1.16}$$

By computations

$$\begin{aligned}\int_{\mathbb{R}^k} |x|^a\, \nabla u \cdot \nabla(\varphi_h^2\, u)\, dx &= \int_{\mathbb{R}^k} |x|^{a-2} |\varphi_h|^2 |\nabla u|^2\, dx + 2\int_{\mathbb{R}^k} |x|^a \varphi_h u\, (\nabla\varphi_h \cdot \nabla u)\, dx \\ &= \int_{\mathbb{R}^k} |x|^a |\nabla(\varphi_h u)|^2\, dx - \int_{\mathbb{R}^k} |x|^a u^2 |\nabla\varphi_h|^2\, dx\,. \end{aligned} \tag{1.1.17}$$

Therefore, from (1.1.16) and (1.1.17) it follows that

$$\int_{\mathbb{R}^k} |x|^a |\nabla(\varphi_h u)|^2\, dx = \left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2} |\varphi_h u|^2\, dx + \int_{A_h} |x|^a\, u^2 |\nabla\varphi_h|^2\, dx\,, \tag{1.1.18}$$

where $A_h \subset (B_{1/h} \setminus B_{1/2h}) \cup (B_{2h} \setminus B_h)$ is the support of $u\nabla\varphi_h$. Notice that $|x||\nabla\varphi_h| \leq 4$, then

$$\int_{A_h} |x|^a\, u^2 |\nabla\varphi_h|^2\, dx \leq 16 \int_{A_h} |x|^{a-2} u^2\, dx \to 0 \tag{1.1.19}$$

as $h \to +\infty$, since $|x|^{a-2}|u|^2 \in L^1(\mathbb{R}^k)$ and since the measure of A_h goes to 0. (1.1.18) and (1.1.19) imply

$$\int_{\mathbb{R}^k} |x|^a |\nabla(\varphi_h u)|^2\, dx - \left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2} |\varphi_h u|^2\, dx \to 0\,.$$

Since $\varphi_h u \in C_c^\infty(\mathbb{R}_0^k)$, we can apply Lemma 1.1.2 to obtain

$$\int_{\mathbb{R}^k} |\nabla(|x|^{\frac{a}{2}}\varphi_h u)|^2\,dx - \left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{-2}||x|^{\frac{a}{2}}\varphi_h u|^2\,dx \to 0\,.$$

If we apply again identity (1.1.2) to the map $|x|^{\frac{k-2+a}{2}}\varphi_h u$, we finally get

$$\int_{\mathbb{R}^k} |x|^{2-k}|\nabla(|x|^{\frac{k-2+a}{2}}\varphi_h u)|^2\,dx \to 0\,. \tag{1.1.20}$$

Now fix $R > 0$ in such a way that $\varphi_h \equiv 1$ on $B_R \setminus B_{1/R}$. This is possible if we take h large enough. Then, from (1.1.20),

$$\int_{B_R\setminus B_{1/R}} |x|^{2-k}|\nabla(|x|^{\frac{k-2+a}{2}} u)|^2\,dx \le \int_{\mathbb{R}^k} |x|^{2-k}|\nabla(|x|^{\frac{k-2+a}{2}}\varphi_h u)|^2\,dx = o(1)$$

as $h \to +\infty$. This implies $u \equiv 0$ and concludes the proof. □

Theorem 1.1.1 in case $a = 0$ includes the *classical Hardy inequality*, that it is trivial for $k = 2$:

Corollary 1.1.5 *Let $u \in C_c^\infty(\mathbb{R}_0^k)$. Then*

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{-2}|u|^2\,dx \le \int_{\mathbb{R}^k} |\nabla u|^2\,dx\,. \tag{1.1.21}$$

In addition, (1.1.21) holds for any $u \in C_c^\infty(\mathbb{R}^k)$ provided that $k \ge 3$. Moreover, the constant

$$\lambda_1(0) = \left(\frac{k-2}{2}\right)^2$$

is sharp and it is not achieved.

1.2 Hardy inequality for the p-Laplacian

In this section we present a Hardy-type inequality that involves the p-Laplace operator, defined for $p \ge 1$ as

$$-\Delta_p \cdot := -\mathrm{div}(|\nabla \cdot|^{p-2}\nabla \cdot)\,.$$

D'Ambrosio, in [32] (Theorems 2.5, 2.7 and 2.9), provides a powerful argument to obtain Hardy-type inequalities together with the explicit value of their best constants (see also [44]). Notice that the following result includes also Theorem 1.1.1.

Theorem 1.2.1 *Let $a \in \mathbb{R}$, $p \geq 1$ and $u \in C_c^\infty(\mathbb{R}^k)$. If $a \leq p-k$, assume that the support of u is contained in $\mathbb{R}^k_0$. Then*

$$\left|\frac{k-p+a}{p}\right|^p \int_{\mathbb{R}^k} |x|^{a-p}|u|^p\,dx \leq \int_{\mathbb{R}^k} |x|^a|\nabla u|^p\,dx\,. \tag{1.2.1}$$

Moreover, the constant

$$\lambda_1(a,p) := \left|\frac{k-p+a}{p}\right|^p$$

is sharp and for $p \geq 2$ it is not achieved.

Proof. **Step 1** First of all we want to prove inequality (1.2.1). If $a = p-k$ then $\lambda_1(a,p) = 0$ and (1.2.1) is trivial. Hence assume $a \neq p-k$, fix any $u \in C_c^\infty(\mathbb{R}^k)$ if $a > p-k$ and $u \in C_c^\infty(\mathbb{R}^k_0)$ if $a < p-k$ in order to have all locally integrable weights. Notice that $\operatorname{div}(|x|^{a-p}\,x) = (k-p+a)|x|^{a-p}$. Moreover, integrating by parts (up to approximate u^p with smooth functions) and using Hölder inequality, for $p > 1$, we get

$$\begin{aligned}
\int_{\mathbb{R}^k} |x|^{a-p}|u|^p\,dx &= \frac{1}{k-p+a}\int_{\mathbb{R}^k} \operatorname{div}(|x|^{a-p}\,x)|u|^p\,dx \\
&= -\frac{p}{k-p+a}\int_{\mathbb{R}^k} |x|^{a-p}|u|^{p-2}\,u\,(\nabla u\cdot x)\,dx \\
&\leq \left|\frac{p}{k-p+a}\right| \int_{\mathbb{R}^k} |x|^{a-p+1}|u|^{p-1}|\nabla u|\,dx \\
&\leq \left|\frac{p}{k-p+a}\right| \left(\int_{\mathbb{R}^k} |x|^{a-p}|u|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^k} |x|^a|\nabla u|^p\,dx\right)^{\frac{1}{p}},
\end{aligned}$$

that readily leads to (1.2.1) (for $p = 1$ computations are simpler).

Step 2 Now we have to prove that $\lambda_1(a,p)$ coincides with the best constant in the *Hardy inequality for the p-Laplacian*, defined by

$$\lambda_{H,p} := \inf_{\substack{u \in C_c^\infty(\mathbb{R}^k_0) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^k} |x|^a|\nabla u|^p dx}{\int_{\mathbb{R}^k} |x|^{a-p}|u|^p dx}\,.$$

From (1.2.1) we deduce that

$$\lambda_1(a,p) \leq \frac{\int_{\mathbb{R}^k} |x|^a|\nabla u|^p\,dx}{\int_{\mathbb{R}^k} |x|^{a-p}|u|^p\,dx} \tag{1.2.2}$$

holds for every $u \in C_c^\infty(\mathbb{R}^k_0)$. If we pass to the inf in (1.2.2) we get that $\lambda_1(a,p) \leq \lambda_{H,p}$. We only have to prove that $\lambda_{H,p} \leq \lambda_1(a,p)$.

First of all, set
$$c(\varepsilon) := \left|\frac{k-p+a}{p}\right| + \frac{\varepsilon}{p}$$
and
$$u_\varepsilon(x) := \begin{cases} |x|^{c(\varepsilon)} & \text{if } |x| \le 1 \\ |x|^{-c(\varepsilon)} & \text{if } |x| > 1 \,. \end{cases}$$
We can observe that, for every $\varepsilon > 0$, the weights $|x|^{c(\varepsilon)p+a-p}$ and $|x|^{-c(\varepsilon)p+a-p}$ are respectively integrable at 0 and at ∞. This implies that $\int_{\mathbb{R}^k} |x|^{a-p}\, |u_\varepsilon|^p \, dx$ is finite, thus we have
$$\begin{aligned} c(\varepsilon)^p \int_{\mathbb{R}^k} |x|^{a-p} |u_\varepsilon|^p \, dx &= c(\varepsilon)^p \int_{|x|\le 1} |x|^a \, |x|^{(c(\varepsilon)-1)p} \, dx \\ &+ c(\varepsilon)^p \int_{|x|>1} |x|^a \, |x|^{(-c(\varepsilon)-1)p} \, dx \\ &= \int_{\mathbb{R}^k} |x|^a |\nabla u_\varepsilon|^p \, dx \,. \end{aligned}$$
Moreover, notice that u_ε can be approximated by smooth functions with compact support in $\mathbb{R}^k_0$. Therefore, by definition of the best constant $\lambda_{H,p}$, we obtain
$$\frac{c(\varepsilon)^p}{\lambda_{H,p}} \int_{\mathbb{R}^k} |x|^a |\nabla u_\varepsilon|^p dx \ge c(\varepsilon)^p \int_{\mathbb{R}^k} |x|^{a-p} |u_\varepsilon|^p dx = \int_{\mathbb{R}^k} |x|^a |\nabla u_\varepsilon|^p \, dx \,.$$
It follows that $c(\varepsilon)^p \ge \lambda_{H,p}$ and, letting $\varepsilon \to 0$, we conclude that $\lambda_1(a,p) \ge \lambda_{H,p}$. Hence $\lambda_1(a,p)$ coincides with the best constant in the Hardy inequality for the p-Laplacian.

Step 3 In order to prove that the best constant is not achieved, we define, for any $u \in C^\infty_c(\mathbb{R}^k)$ if $a > p-k$ and for any $u \in C^\infty_c(\mathbb{R}^k_0)$ if $a < p-k$, the functional
$$J(u) := \int_{\mathbb{R}^k} |x|^a |\nabla u|^p \, dx - \left|\frac{k-p+a}{p}\right|^p \int_{\mathbb{R}^k} |x|^{a-p} |u|^p \, dx \,.$$
Notice that $J \ge 0$ and the best constant will be achieved if and only if there exists a map u such that $J(u) = 0$.

Define $v := |x|^\gamma u$ with $\gamma := \frac{k-p+a}{p}$. By computations we have
$$|\nabla u|^2 = |\gamma|^2 |v|^2 |x|^{-2\gamma-2} + |x|^{-2\gamma} |\nabla v|^2 - 2\gamma v |x|^{-2\gamma-1} (\nabla |x| \cdot \nabla v) \,. \tag{1.2.3}$$
We remind that the inequality
$$(\zeta - \eta)^t \ge \zeta^t - t\eta\zeta^{t-1} \tag{1.2.4}$$

holds for every $\zeta, \eta, t \in \mathbb{R}$ with $\zeta > 0$, $\zeta > \eta$ and $t \geq 1$ (see [51]). Applying (1.2.4) to (1.2.3) with $t = p/2$ ($p \geq 2$ by assumption), $\zeta = |\gamma|^2 |v|^2 |x|^{-2\gamma-2}$ and $\eta = -|x|^{-2\gamma}|\nabla v|^2 + 2\gamma v|x|^{-2\gamma-1}(\nabla|x| \cdot \nabla v)$, we get

$$\begin{aligned} |\nabla u|^p &\geq |\gamma|^p\,|v|^p\,|x|^{-k-a} \\ &- p\,|\gamma|^{p-2}\,\gamma\,|v|^{p-2}\,v\,|x|^{-k-a+1}(\nabla|x| \cdot \nabla v) \\ &+ \frac{p}{2}\,|\gamma|^{p-2}\,|v|^{p-2}\,|x|^{-k-a+2}|\nabla v|^2 \,. \end{aligned}$$

Multiplying by $|x|^a$, integrating on $\mathbb{R}^k$ and taking into account that $u := |x|^{-\gamma}v$, we have

$$J(u) \geq J_1(v) + J_2(v)\,,$$

where

$$J_1(v) := -p\,|\gamma|^{p-2}\,\gamma \int_{\mathbb{R}^k} |v|^{p-2}\,v\,|x|^{-k+1}(\nabla|x| \cdot \nabla v)\,dx\,,$$

$$J_2(v) := \frac{p}{2}\,|\gamma|^{p-2} \int_{\mathbb{R}^k} |v|^{p-2}\,|x|^{-k+2}|\nabla v|^2\,dx\,.$$

If we consider J_1 and we integrate by parts, we obtain for $k \neq 2$

$$\begin{aligned} J_1(v) &= -\frac{|\gamma|^{p-2}\,\gamma}{2-k} \int_{\mathbb{R}^k} \left(\nabla|v|^p \cdot \nabla|x|^{2-k}\right)\,dx \\ &= \frac{|\gamma|^{p-2}\,\gamma}{2-k} \int_{\mathbb{R}^k} |v|^p \Delta(|x|^{2-k})\,dx = 0\,. \end{aligned}$$

Notice that $J_1(v) = 0$ also for $k = 2$. On the other hand, we can rewrite J_2 as

$$J_2(v) = \frac{2}{p}\,|\gamma|^{p-2} \int_{\mathbb{R}^k} |x|^{-k+2}\,|\nabla|v|^{\frac{p}{2}}|^2\,dx\,.$$

Then, we can conclude that

$$J(u) = J_2(v) = \frac{2}{p}\,|\gamma|^{p-2} \int_{\mathbb{R}^k} |x|^{-k+2}\,|\nabla|v|^{\frac{p}{2}}|^2\,dx > 0\,,$$

because we are considering $u \not\equiv 0$ (and hence $v \not\equiv 0$). This implies that the infimum $\lambda_{H,p} = \lambda_1(a,p)$ is not achieved. □

1.3 The cylindrical case

Let k, N be positive integers with $1 \leq k \leq N$. We put $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ and we denote points ξ in $\mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. We collect here analogous inequalities with respect to the previous ones but that hold also in the cylindrical case $k < N$ and can be easily obtained from the spherical case.

We start with a lemma that includes Lemma 1.1.2.

Lemma 1.3.1 *Let $1 \le k \le N$, $a \in \mathbb{R}$ and $u \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$. We set $v := |x|^{\frac{a}{2}}\, u$, then*

$$\int_{\mathbb{R}^N} |x|^a |\nabla u|^2\, d\xi - \lambda_1(a) \int_{\mathbb{R}^N} |x|^{a-2}|u|^2\, d\xi = \int_{\mathbb{R}^N} |\nabla v|^2\, d\xi - \lambda_1(0) \int_{\mathbb{R}^N} |x|^{-2}|v|^2\, d\xi\,. \tag{1.3.1}$$

Proof. We apply identity (1.1.2) on $\mathbb{R}^k$ to the map $u(\,\cdot\,, y) \in C_c^\infty(\mathbb{R}_0^k)$ for any $y \in \mathbb{R}^{N-k}$.

$$\begin{aligned}\int_{\mathbb{R}^k} |x|^a |\nabla u\,(x,y)|^2\, dx \quad &- \quad \lambda_1(a) \int_{\mathbb{R}^k} |x|^{a-2}|u\,(x,y)|^2\, dx \\ &= \quad \int_{\mathbb{R}^k} |\nabla v\,(x,y)|^2\, dx - \lambda_1(0) \int_{\mathbb{R}^k} |x|^{-2}|v\,(x,y)|^2\, dx\,.\end{aligned}$$

By integrating the previous identity on $\mathbb{R}^{N-k}$, we get the conclusion. □

Lemma 1.1.4 becomes the following one (see [69] for the case $a = 0$).

Lemma 1.3.2 *Let $1 \le k \le N$, assume $a > 2 - k$ and fix $u \in C_c^\infty(\mathbb{R}^N)$. Then there exists a sequence $u_h \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$ such that $u_h \to u$ in $L^2(\mathbb{R}^N; |x|^{a-2}\, d\xi)$ and*

$$\int_{\mathbb{R}^N} |x|^a |\nabla u_h - \nabla u|^2\, d\xi \to 0\,.$$

Proof. Consider as in the proof of Lemma 1.1.4 the same function $\varphi_h \in C^\infty(\mathbb{R})$ and set

$$u_h(\xi) := \varphi_h(|x|)u(\xi) \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})\,.$$

Since $a > 2-k$, $|x|^{a-2}$ is locally integrable in $\mathbb{R}^k$ and, applying Lebesgue's Theorem, we get that the sequence u_h converges to u in $L^2(\mathbb{R}^N; |x|^{a-2}\, d\xi)$. Moreover,

$$\nabla_x u_h = \varphi_h(|x|)\nabla_x u + \varphi_h'(|x|)\frac{x}{|x|}\, u \ \text{ and } \ \nabla_y u_h = \varphi_h(|x|)\nabla_y u\,. \tag{1.3.2}$$

Then the proof can be carried out as in Lemma 1.1.4 with (1.3.2) instead of (1.1.7). □

The *generalized Hardy inequality with cylindrical weights* is the following.

Theorem 1.3.3 *Let $1 \leq k \leq N$, $a \in \mathbb{R}$ and $u \in C_c^\infty(\mathbb{R}^N)$. If $a \leq 2-k$, assume that the support of u is contained in $\mathbb{R}_0^k \times \mathbb{R}^{N-k}$. Then*

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^N} |x|^{a-2}|u|^2 \, d\xi \leq \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi \,. \tag{1.3.3}$$

Moreover, the constant

$$\lambda_1(a) = \left(\frac{k-2+a}{2}\right)^2$$

is sharp and it is not achieved.

The main difference with the proof of Theorem 1.1.1 is in Step 2 ([32], Theorem 3.5).

Proof. **Step 1** We apply inequality (1.1.1) on $\mathbb{R}^k$ to the map $u(\,\cdot\,, y) \in C_c^\infty(\mathbb{R}_0^k)$ for any $y \in \mathbb{R}^{N-k}$.

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2}|u(x,y)|^2 \, dx \leq \int_{\mathbb{R}^k} |x|^a |\nabla u(x,y)|^2 \, dx \,.$$

By integrating the previous inequality on $\mathbb{R}^{N-k}$, we get the conclusion.

Step 2 Now we have to prove that $\lambda_1(a)$ coincides with the best constant in the Hardy inequality, defined as

$$\lambda_{H,k} := \inf_{\substack{u \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k}) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \, d\xi} \,.$$

From (1.3.3) we deduce that

$$\lambda_1(a) \leq \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \, d\xi} \tag{1.3.4}$$

for every $u \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$. If we pass to the inf in (1.3.4) we get that $\lambda_1(a) \leq \lambda_{H,k}$. We only have to prove that $\lambda_{H,k} \leq \lambda_1(a)$.

We consider $z \in C_c^\infty(\mathbb{R}_0^k)$ and $v \in C_c^\infty(\mathbb{R}^{N-k})$ in order to use $u(\xi) := z(x)\, v(y)$ to estimate $\lambda_{H,k}$. By computations,

$$|\nabla u|^2 = |\nabla_x z|^2 |v|^2 + |\nabla_y v|^2 |z|^2$$

hence

$$\lambda_{H,k} \leq \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 d\xi}{\int_{\mathbb{R}^N} |x|^{a-2} |u|^2 d\xi} = \frac{\int_{\mathbb{R}^N} |x|^a |\nabla_x z|^2 |v|^2 d\xi}{\int_{\mathbb{R}^N} |x|^{a-2} |z|^2 |v|^2 d\xi} + \frac{\int_{\mathbb{R}^N} |x|^a |\nabla_y v|^2 |z|^2 d\xi}{\int_{\mathbb{R}^N} |x|^{a-2} |z|^2 |v|^2 d\xi}$$
$$= \frac{\int_{\mathbb{R}^k} |x|^a |\nabla_x z|^2 dx}{\int_{\mathbb{R}^k} |x|^{a-2} |z|^2 dx} + \frac{\int_{\mathbb{R}^{N-k}} |\nabla_y v|^2 dy}{\int_{\mathbb{R}^{N-k}} |v|^2 dy} \frac{\int_{\mathbb{R}^k} |x|^a |z|^2 dx}{\int_{\mathbb{R}^k} |x|^{a-2} |z|^2 dx}. \tag{1.3.5}$$

Thanks to Theorem 1.1.1, we get

$$\inf_{\substack{u \in C_c^\infty(\mathbb{R}^k_0) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^k} |x|^a |\nabla_x z|^2 \, dx}{\int_{\mathbb{R}^k} |x|^{a-2} |z|^2 \, dx} = \left(\frac{k-2+a}{2}\right)^2$$

and it is well known that

$$\inf_{\substack{u \in C_c^\infty(\mathbb{R}^{N-k}) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^{N-k}} |\nabla_y v|^2 \, dy}{\int_{\mathbb{R}^{N-k}} |v|^2 \, dy} = 0\,.$$

Therefore, from (1.3.5), we can conclude that $\lambda_{H,k} \leq \lambda_1(a)$ and then $\lambda_1(a)$ coincides with the best constant.

Step 3 By Theorem 1.1.1, for every $u(\,\cdot, y) \in C_c^\infty(\mathbb{R}^k_0)$ with $y \in \mathbb{R}^{N-k}$ fixed,

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^k} |x|^{a-2} |u(x,y)|^2 \, dx < \int_{\mathbb{R}^k} |x|^a |\nabla u(x,y)|^2 \, dx\,.$$

By integrating the previous inequality on $\mathbb{R}^{N-k}$, we get that the best constant is not achieved for every $u \in C_c^\infty(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$. □

Theorem 1.3.3 in case $a = 0$ includes the *classical Hardy inequality with cylindrical weights*, that it is trivial for $k = 2$:

Corollary 1.3.4 *Let $1 \leq k \leq N$ and $u \in C_c^\infty(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$. Then*

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^N} |x|^{-2} |u|^2 \, d\xi \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, d\xi\,. \tag{1.3.6}$$

In addition, (1.3.6) holds for any $u \in C_c^\infty(\mathbb{R}^N)$ provided that $k \geq 3$. Moreover, the constant

$$\lambda_1(0) = \left(\frac{k-2}{2}\right)^2$$

is sharp and it is not achieved.

Now we want to deal with the p-Laplace operator. Also in this case we refer to [32] for the proof of the *Hardy inequality for the p-Laplacian with cylindrical weights.* It is the most general Hardy inequality that we present in this chapter and it includes all the other inequalities proved until now.

Theorem 1.3.5 *Let $1 \leq k \leq N$, $a \in \mathbb{R}$, $p \geq 1$ and $u \in C_c^\infty(\mathbb{R}^N)$. If $a \leq p-k$, assume that the support of u is contained in $\mathbb{R}^k_0 \times \mathbb{R}^{N-k}$. Then*

$$\left|\frac{k-p+a}{p}\right|^p \int_{\mathbb{R}^N} |x|^{a-p}|u|^p \, d\xi \leq \int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi \,. \tag{1.3.7}$$

Moreover, for $p \geq 2$, the constant

$$\lambda_1(a,p) := \left|\frac{k-p+a}{p}\right|^p$$

is sharp and it is not achieved.

Proof. **Step 1** We apply inequality (1.2.1) on $\mathbb{R}^k$ to the map $u(\,\cdot\,,y) \in C_c^\infty(\mathbb{R}^k_0)$ for any $y \in \mathbb{R}^{N-k}$.

$$\left|\frac{k-p+a}{p}\right|^p \int_{\mathbb{R}^k} |x|^{a-p}|u(x,y)|^p \, dx \leq \int_{\mathbb{R}^k} |x|^a |\nabla u(x,y)|^p \, dx \,.$$

By integrating the previous inequality on $\mathbb{R}^{N-k}$, we get the conclusion.

Step 2 Now we have to prove that $\lambda_1(a,p)$ coincides with the best constant defined as

$$\lambda_{H,k,p} := \inf_{\substack{u \in C_c^\infty(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-p}|u|^p \, d\xi} \,.$$

From (1.3.7) we deduce that

$$\lambda_1(a,p) \leq \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, dx}{\int_{\mathbb{R}^N} |x|^{a-p}|u|^p \, dx} \tag{1.3.8}$$

holds for every $u \in C_c^\infty(\mathbb{R}^N_0)$. If we pass to the inf in (1.3.8) we get that $\lambda_1(a,p) \leq \lambda_{H,k,p}$. We only have to prove that $\lambda_{H,k,p} \leq \lambda_1(a,p)$.

Consider, as in the proof of Theorem 1.3.3, $z \in C_c^\infty(\mathbb{R}^k_0)$ and $v \in C_c^\infty(\mathbb{R}^{N-k})$ in order to use $u(\xi) := z(x)\, v(y)$ to estimate $\lambda_{H,k,p}$. Since $p \geq 2$, by convexity, for every $0 < \mu < 1$,

$$\begin{aligned} |\nabla u|^p &= (|\nabla_x z|^2 |v|^2 + |z|^2 |\nabla_y v|^2)^{p/2} \\ &= \left((1-\mu)\left(\frac{1}{1-\mu}|\nabla_x z|^2|v|^2\right) + \mu\left(\frac{1}{\mu}|z|^2|\nabla_y v|^2\right)\right)^{p/2} \\ &\leq (1-\mu)^{1-\frac{p}{2}} \, |\nabla_x z|^p \, |v|^p + \mu^{1-\frac{p}{2}} \, |z|^p \, |\nabla_y v|^p \,. \end{aligned}$$

Therefore

$$
\begin{aligned}
\lambda_{H,k,p} \le \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^p \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-p} |u|^p \, d\xi} &\le (1-\mu)^{1-\frac{p}{2}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla_x z|^p |v|^p \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-p} |z|^p |v|^p \, d\xi} \\
&+ \mu^{1-\frac{p}{2}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla_y v|^p |z|^p \, d\xi}{\int_{\mathbb{R}^N} |x|^{a-p} |z|^p |v|^p \, d\xi} \\
&= (1-\mu)^{1-\frac{p}{2}} \frac{\int_{\mathbb{R}^k} |x|^a |\nabla_x z|^p \, dx}{\int_{\mathbb{R}^k} |x|^{a-p} |z|^p \, dx} \\
&+ \mu^{1-\frac{p}{2}} \frac{\int_{\mathbb{R}^{N-k}} |\nabla_y v|^p \, dy}{\int_{\mathbb{R}^{N-k}} |v|^p \, dy} \frac{\int_{\mathbb{R}^k} |x|^a |z|^p \, dx}{\int_{\mathbb{R}^k} |x|^{a-p} |z|^p \, dx} \, .
\end{aligned}
\tag{1.3.9}
$$

Thanks to Theorem 1.2.1, we get

$$
\inf_{\substack{u \in C_c^\infty(\mathbb{R}^k_0) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^k} |x|^a |\nabla_x z|^p \, dx}{\int_{\mathbb{R}^k} |x|^{a-2} |z|^p \, dx} = \left| \frac{k-p+a}{p} \right|^p
$$

and it is well known that

$$
\inf_{\substack{u \in C_c^\infty(\mathbb{R}^{N-k}) \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^{N-k}} |\nabla_y v|^p \, dy}{\int_{\mathbb{R}^{N-k}} |v|^p \, dy} = 0 \, .
$$

Therefore (1.3.9) becomes

$$
\lambda_{H,k,p} \le (1-\mu)^{1-\frac{p}{2}} \left| \frac{k-p+a}{p} \right|^p
$$

and, letting $\mu \to 0$, we can conclude that $\lambda_{H,k,p} \le \lambda_1(a,p)$. Hence $\lambda_1(a,p)$ coincides with the best constant in the Hardy inequality for the p-Laplacian with cylindrical weights.

Step 3 By Theorem 1.2.1, for every $u(\,\cdot, y) \in C_c^\infty(\mathbb{R}^k_0)$ with $y \in \mathbb{R}^{N-k}$ fixed,

$$
\left| \frac{k-p+a}{p} \right|^p \int_{\mathbb{R}^k} |x|^{a-p} |u(x,y)|^p \, dx < \int_{\mathbb{R}^k} |x|^a |\nabla u(x,y)|^p \, dx \, .
$$

By integrating the previous inequality on $\mathbb{R}^{N-k}$, we get that the best constant is not achieved for every $u \in C_c^\infty(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$. $\square$

Chapter 2

Hardy-Sobolev and Maz'ya inequalities

In this chapter we introduce some *Hardy-Sobolev inequalities*, that include Hardy inequalities, proved in Chapter 1, and the well known Sobolev inequalities. We will state them on cones (see Definition 2.1.1).

In the spherical case $k = N$, they were proved in 1984 on $\mathbb{R}^N$ by Caffarelli, Kohn and Nirenberg (see [22]). The following result is a particular case of the one contained in [22]. We recall that, for every $p \in [1, N)$, the critical Sobolev exponent is $p^* := \frac{Np}{N-p}$.

Theorem 2.0.6 (Caffarelli, Kohn, Nirenberg) *Assume* $p \in (1, N)$, $q \in [p, p^*]$, $a > p - N$ *and set*

$$b_{a,p,q} := N - q\frac{N - p + a}{p}\,.$$

Then there exists a constant $C = C(a, p, q, N) > 0$ *such that*

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}}|u|^q\,dx\right)^{p/q} \le \int_{\mathbb{R}^N} |x|^a|\nabla u|^p\,dx \qquad \forall\, u \in C_c^\infty(\mathbb{R}^N)\,. \qquad (2.0.1)$$

Remark 2.0.7 In [22] the authors proved this result also under the assumptions $a > p - N$, $p \geq N$ and $q < +\infty$.

We present also *Sobolev-type inequalities*, peculiar to the cylindrical case $k < N$, that do not include Hardy. They were proved in 1980 by Maz'ya (see [67], Section 2.1.6, Corollary 2).

2.1 Hardy-Sobolev inequalities on cones

We start giving the following definition, accordingly with [27] (see also [13]).

Definition 2.1.1 A *cone* in $\mathbb{R}^k$, with $1 \leq k \leq N$, is a domain $\mathcal{C}^k \subset \mathbb{R}^k$ such that $\mu x \in \mathcal{C}^k$ for every $\mu > 0$ and for every $x \in \mathcal{C}^k$. A cone $\mathcal{C}^k$ is said to be *proper* if $0 \notin \mathcal{C}^k$.

Notice that $\mathbb{R}^k$ itself is a cone in $\mathbb{R}^k$, $\mathbb{R}^k_0 = \mathbb{R}^k \backslash \{0\}$ is a proper cone in $\mathbb{R}^k$ and that $(0, +\infty)$ is a proper cone in $\mathbb{R}$.

In the next results we will consider $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Therefore, in the spherical case $k = N$, Ω will coincide with $\mathcal{C}^N$, a proper cone in $\mathbb{R}^N$ (for example $\Omega = \mathbb{R}^N_0 = \mathbb{R}^N \backslash \{0\}$).

Remark 2.1.2 We can observe that all the inequalities contained in Chapter 1 hold also on Ω, for any $u \in C^\infty_c(\Omega)$.

Now we introduce a Hardy-Sobolev type inequality on proper cones. We will present a proof via Hardy inequality, contained in [47], Lemma 3.1 (see also [69] and [70]) and that extends the result in [22] on proper cones also for $a < p - N$.

Theorem 2.1.3 *Let $1 \leq k \leq N$ and $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Assume $p \in (1, N)$, $q \in [p, p^*]$, $a \neq p - k$ and set*

$$b_{a,p,q} := N - q\frac{N - p + a}{p} \ . \tag{2.1.1}$$

Then there exists a constant $C = C(a, p, q, k) > 0$ such that

$$C \left(\int_\Omega |x|^{-b_{a,p,q}} |u|^q \, d\xi \right)^{p/q} \leq \int_\Omega |x|^a |\nabla u|^p \, d\xi \qquad \forall\, u \in C^\infty_c(\Omega) \,. \tag{2.1.2}$$

Proof. Notice that, if $q = p$, (2.1.2) is equivalent to Hardy inequality (1.3.7) (see page 16) on Ω. If $q = p^*$ and $a = 0$, it is the standard Sobolev inequality.

First of all we are going to prove (2.1.2) for $q = p^*$. Fix any map $u \in C_c^\infty(\Omega)$. Since $a \neq p - k$, Remark 2.1.2 and inequality (1.3.7) at page 16 lead to

$$\int_\Omega |\nabla(|x|^{\frac{a}{p}} u)|^p \, d\xi \le C \left(\int_\Omega |x|^a \, |\nabla u|^p \, d\xi + \int_\Omega |x|^{a-p} |u|^p \, d\xi \right) \le C \int_\Omega |x|^a |\nabla u|^p \, d\xi$$

where the constant C does not depend on u. Thus, by standard Sobolev inequality, we get

$$\left(\int_\Omega |x|^{\frac{Na}{N-p}} |u|^{p^*} \, d\xi \right)^{p/p^*} = \left(\int_\Omega ||x|^{\frac{a}{p}} u|^{p^*} \, d\xi \right)^{p/p^*} \le C \int_\Omega |x|^a |\nabla u|^p \, d\xi \ .$$

Set $\tau = N - q\frac{N-p}{p}$ for $q \in (p, p^*)$. By interpolating the cases $q = p$ and $q = p^*$, via Hölder inequality we get

$$\int_\Omega |x|^{-b_{a,p,q}} |u|^q \, d\xi \le \left(\int_\Omega |x|^{a-p} |u|^p \, d\xi \right)^{\frac{\tau}{p}} \left(\int_\Omega |x|^{\frac{Na}{N-p}} |u|^{p^*} \, d\xi \right)^{\frac{p-\tau}{p}} .$$

By previous computations and Hardy inequality (1.3.7) at page 16, we readily obtain (2.1.2). $\square$

Remark 2.1.4 Let $a > p - k$ and $\Omega = \mathbb{R}_0^k \times \mathbb{R}^{N-k}$. Using Theorem 1.3.5 at page 16, we can show that inequality (2.1.2) holds also on $\mathbb{R}^N$ for every $u \in C_c^\infty(\mathbb{R}^N)$.

2.2 Maz'ya inequalities

Theorem 2.2.1 (Maz'ya) *Assume* $1 \le k < N$. *Let* $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, *with* $\mathcal{C}^k$ *a cone in* $\mathbb{R}^k$ *and* a, p, $q \in \mathbb{R}$ *satisfy*

$$1 < p < N, \quad a > (p - N)\frac{k}{N}, \quad \max\left\{ p, \frac{p\,(N-k)}{N-p+a} \right\} < q \le p^* = \frac{Np}{N-p} \ ,$$

and $b_{a,p,q}$ *as in (2.1.1). Then there exists a constant* $C = C(a, p, q, k) > 0$ *such that*

$$C \left(\int_\Omega |x|^{-b_{a,p,q}} |u|^q \, d\xi \right)^{p/q} \le \int_\Omega |x|^a |\nabla u|^p \, d\xi \qquad \forall \, u \in C_c^\infty(\Omega) \, . \tag{2.2.1}$$

Remark 2.2.2 The *Maz'ya inequality* for the spherical case $k = N$ and $\Omega = \mathbb{R}^N$ coincides with the *Caffarelli-Kohn-Nirenberg inequality.*

Remark 2.2.3 If $k < N$, we can take $a \in \big((p-N)\frac{k}{N}, p-k\big]$. For a in this range and $\Omega = \mathbb{R}^N$, Hardy inequality might not hold true for some $u \in C_c^\infty(\mathbb{R}^N)$ that does not vanish on the singular set $\{x = 0\}$. We postpone the discussion on this argument to Part II, Chapter 4.

Remark 2.2.4 Inequalities (2.0.1), (2.1.2) and (2.2.1) hold true also for $p = 1$, but we have stated them for $p > 1$ because we address our attention to investigate the existence of extremals (see Part II).

Now we present the *Hardy-Sobolev-Maz'ya inequality* that holds in the case $p = 2$ (see [67], Section 2.1.6, Corollary 3 for $a = 0$ and $N \geq 3$). For the proof, we refer to [69], Theorem A.2. We write it for completeness.

Theorem 2.2.5 *Let $1 \leq k < N$, $N \geq 3$ and $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Assume $a \in \mathbb{R}$ and $q \in (2, 2^*]$. Moreover, set*

$$b_{a,q} := N - q\frac{N-2+a}{2}\,.$$

Then there exists a constant $C = C(a, q, k) > 0$ such that

$$C\left(\int_\Omega |x|^{-b_{a,q}}|u|^q\,d\xi\right)^{2/q} \leq \int_\Omega |x|^a|\nabla u|^2\,d\xi - \lambda_1(a)\int_\Omega |x|^{a-2}|u|^2\,d\xi \quad \forall\, u \in C_c^\infty(\Omega)\,. \tag{2.2.2}$$

We recall that $\lambda_1(a) = \left(\frac{k-2+a}{2}\right)^2$ is the best constant in the *generalized Hardy inequality with cylindrical weights* (see Theorem 1.3.3 at page 14).

Proof. If $a = 2-k$ then $\lambda_1(a) = 0$ and (2.2.2) becomes (2.2.1) with $p = 2$ (notice that $2-k > (2-N)\frac{k}{N}$ because $k < N$ by assumption). Now let $a \neq 2-k$ and $u \in C_c^\infty(\Omega)$. By applying identity (1.3.1) at page 13 on Ω, with $v := |x|^{\frac{a}{2}}\,u$, we get

$$\int_\Omega |x|^a|\nabla u|^2\,d\xi - \lambda_1(a)\int_\Omega |x|^{a-2}|u|^2\,d\xi = \int_\Omega |\nabla v|^2\,d\xi - \lambda_1(0)\int_\Omega |x|^{-2}|v|^2\,d\xi\,. \tag{2.2.3}$$

If we apply again identity (1.3.1) to the map $|x|^{\frac{k-2}{2}}\,v$, we obtain

$$\int_\Omega |\nabla v|^2\,d\xi - \lambda_1(0)\int_\Omega |x|^{-2}|v|^2\,d\xi = \int_\Omega |x|^{2-k}|\nabla(|x|^{\frac{k-2+a}{2}}u)|^2 d\xi\,. \tag{2.2.4}$$

Since $2-k > (2-N)\frac{k}{N}$ for $k < N$, by Theorem 2.2.1 with $p = 2$ we get

$$\begin{aligned}\int_\Omega |x|^{2-k}|\nabla(|x|^{\frac{k-2+a}{2}}\,u)|^2\,d\xi \quad &\geq \quad C\left(\int_\Omega |x|^{-b_{2-k,q}}||x|^{\frac{k-2+a}{2}}\,u|^q\,d\xi\right)^{2/q} \\ &= \quad C\left(\int_\Omega |x|^{-b_{a,q}}|u|^q\,d\xi\right)^{2/q}.\end{aligned} \tag{2.2.5}$$

The conclusion follows from (2.2.3), (2.2.4) and (2.2.5). $\square$

Remark 2.2.6 In the proof of Theorem 2.2.5, we need $k < N$. Inequality (2.2.2) fails if $k = N$. Also in this case, we postpone the discussion on this subject to Chapter 4, Remark 4.1.1 at page 43 (see also [42]).

Remark 2.2.7 Let $a > 2 - k$ and $\Omega = \mathbb{R}_0^k \times \mathbb{R}^{N-k}$. Using Lemma 1.3.2 at page 13, we can show that inequality (2.2.2) holds also on $\mathbb{R}^N$ for every $u \in C_c^\infty(\mathbb{R}^N)$.

Remark 2.2.8 In [67] there are no inequalities analogous to (2.2.2) for $p \neq 2$. Nevertheless you can find some results in literature about this case (see for example [12], [41], [73] and references there-in).

Part II

Degenerate singular problems

Chapter 3

The weighted p-Laplace operator

In this chapter we deal with some degenerate elliptic equations related to the Maz'ya inequality (see Theorem 2.2.1 at page 21) and to a Hardy-Sobolev type inequality (see Theorem 2.1.3 at page 20). We will present here some results obtained in [47].

We denote points $\xi \in \mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $1 \leq k \leq N$. Set $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a *cone* in $\mathbb{R}^k$ (see Definition 2.1.1 at page 20). We consider the following class of problems:

$$\begin{cases} -\mathrm{div}(|x|^a |\nabla u|^{p-2} \nabla u) = |x|^{-b_{a,p,q}} |u|^{q-2} u \quad \text{in } \Omega \ , \\ u \geq 0 \ , \end{cases} \tag{3.0.1}$$

where

$$p \in (1, N) \ , \quad a > (p-N)\frac{k}{N} \ , \quad \max\left\{p, \frac{p\,(N-k)}{N-p+a}\right\} < q \leq p^* = \frac{Np}{N-p} \tag{3.0.2}$$

and

$$b_{a,p,q} := N - q\frac{N-p+a}{p} \ . \tag{3.0.3}$$

We recall an useful inequality that holds under assumptions (3.0.2) and (3.0.3). It was proved by Maz'ya in 1980 in case $k < N$ (see [67], Section 2.1.6, Corollary 2)

and by Caffarelli-Kohn-Nirenberg in 1984 in case $k = N$ (see [22]). We refer to Chapter 2 for a discussion on this subject.
There exists a constant $C > 0$, independent on u, such that

$$C \left(\int_\Omega |x|^{-b_{a,p,q}} |u|^q \; d\xi \right)^{p/q} \leq \int_\Omega |x|^a |\nabla u|^p \; d\xi \;, \quad \forall\, u \in C_c^\infty(\Omega)\,. \tag{3.0.4}$$

Thanks to (3.0.4), we can define the Banach space $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^p = \int_\Omega |x|^a |\nabla u|^p \, d\xi$. We are interested in extremals for the best constant

$$S_{a,q}(\Omega, p) := \inf_{\substack{u \in \mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi) \\ u \not\equiv 0}} \frac{\int_\Omega |x|^a |\nabla u|^p \; d\xi}{\left(\int_\Omega |x|^{-b_{a,p,q}} |u|^q \; d\xi \right)^{p/q}} \;. \tag{3.0.5}$$

When $\Omega = \mathbb{R}^N$ we will simply write $S_{a,q}(p)$ instead of $S_{a,q}(\mathbb{R}^N, p)$. Notice that, in case $a = 0$, the space $\mathcal{D}^{1,p}(\Omega; |x|^a d\xi)$ coincides with the standard space $\mathcal{D}^{1,p}(\Omega)$.

We recall that u is a *weak* solution to (3.0.1) on Ω if

$$\int_\Omega |x|^a |\nabla u|^{p-2} \nabla u \cdot \nabla \Phi \; d\xi = \int_\Omega |x|^{-b_{a,p,q}} |u|^{q-2} \, u \, \Phi \; d\xi \;, \qquad \forall \; \Phi \in C_c^\infty(\Omega) \;,$$

and it is *entire* if

$$\int_\Omega |x|^a |\nabla u|^p \; d\xi = \int_\Omega |x|^{-b_{a,p,q}} |u|^q \; d\xi < +\infty \;.$$

It is clear that if $S_{a,q}(\Omega, p)$ is achieved by $u \in \mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$, then u is, up to a Lagrange multiplier, a weak entire solution to (3.0.1) on Ω. Moreover, u is in particular a *ground state* solution to (3.0.1), namely a solution with minimal energy.

Several existence results are available in literature if $\Omega = \mathbb{R}^N$. For $a = 0$ and $q = p^*$ the infimum $S_{a,q}(p)$ coincides with the Sobolev constant $S(p)$. It is achieved on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by an explicitly known radially symmetric map (see [8] and [83]). This result was generalized by Egnell in the spherical case for $a = 0$ and $q \in (p, p^*]$ ([34]). Moreover, if $k = N$ and $p = 2$, Chou and Chu (see [31]) found the explicit minimizers for $a \leq 0$ and $q \in (2, 2^*]$ (see also [59] for $a = 0$ and $q \in (2, 2^*)$), while Catrina and Wang proved existence for $q \in (2, 2^*)$ and non-existence for $q = 2^*$ and $a > 0$ ([30]). We refer to [56] for some statements also in case $p \neq 2$. Finally, we cite the recent papers [75] and [43], where related problems are studied in $\mathbb{R}^N$.

As concerns the cylindrical case $k < N$ we quote [10], where $a = 0$, $k \geq 2$ and $q \in (p, p^*)$ are assumed, and [85], [69], that deal with $p = 2$, $a \geq 2-k$. In particular, in the last paper $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$ is considered.

There are some papers about general cones in the spherical case $k = N$. In [13] the authors proved existence for $a > p - N$ and $q \in (p, p^*)$. In case $p = 2$, we cite [26] and [35] for $a = 0$ and [27] for $a \in (0, 2)$ and $q = \frac{2N}{N-2+a}$.

We finally mention also [1], [2], [3], [4], [5], [17], [38], [45], [46], [53], [52], [78], [79], [84] for $k = N$, and [15], [29], [48], [64], [65], [66], [86] for $k < N$.

The approach we use for the minimization problem (3.0.5) works both in the cylindrical and in the spherical case. Our main theorem is the following.

Theorem 3.0.9 *Assume that (3.0.2) and (3.0.3) are satisfied. Let $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a cone in $\mathbb{R}^k$. Then $S_{a,q}(\Omega, p)$ is achieved provided that*

$$q < p^* \quad \textit{or} \quad q = p^* \textit{ and } S_{a,p^*}(\Omega, p) < S(p) .$$

We notice that problem (3.0.5) is invariant with respect to the groups of dilations in Ω and of translations in $\mathbb{R}^{N-k}$. Indeed, for any minimizing sequence u_h and for arbitrary sequences $t_h \in (0, +\infty)$, $y_h \in \mathbb{R}^{N-k}$, it turns out that $\tilde{u}_h(x, y) := u_h(t_h x, t_h y + y_h)$ still approaches the infimum $S_{a,q}(\Omega, p)$. These invariances produce the so called *lack of compactness* phenomena, that are also worse if $q = p^*$ for the group of translations in the x-variable and of dilations in Ω. In fact, in the limiting critical case, u_h might blow-up an extremal for the Sobolev constant $S(p)$.

In order to overcome these difficulties, we prove a Rellich-type theorem and we apply a suitable rescaling argument to sequences of approximated solutions to the Euler-Lagrange equation in (3.0.1). For $q = p^*$, the assumption $S_{a,p^*}(\Omega, p) < S(p)$ prevents concentration phenomena at points (x_0, y_0), with $x_0 \neq 0$.

3.1 Rellich-type theorem and approximated solutions

We start with two technical lemmata. The first is a compactness result on bounded domains in $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$.

Lemma 3.1.1 *Let $A \subset \Omega$ be a bounded domain. Then*

$$\mathcal{D}^{1,p}(\Omega; |x|^a d\xi) \hookrightarrow L^p(A, |x|^a d\xi)$$

with compact inclusion.

Proof. Fix a map $u \in C^\infty_c(\Omega)$. Hölder inequality and (3.0.4) give

$$\int_A |x|^a |u|^p \, d\xi \le |A|^{\frac{p}{N}} \left(\int_A |x|^{\frac{Na}{N-p}} |u|^{p^*} \, d\xi \right)^{p/p^*} \le C |A|^{\frac{p}{N}} \int_\Omega |x|^a |\nabla u|^p \, d\xi \ , \quad (3.1.1)$$

where C does not depend on u. This proves the continuity of the embedding. To prove compactness, take a sequence u_h in $\mathcal{D}^{1,p}(\Omega;|x|^a d\xi)$, with $u_h \to 0$ weakly in $\mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$. Fix $\varepsilon > 0$ and take a smooth function $\varphi_\varepsilon \in C^\infty(\mathcal{C}^k)$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon(x) = 0$ if $|x| \le \varepsilon^2$, and $\varphi_\varepsilon(x) = 1$ if $|x| \ge \varepsilon$. By Rellich Theorem, it turns out that

$$\int_A |x|^a |\varphi_\varepsilon u_h|^p \ d\xi = o(1)$$

as $h \to +\infty$, since $|x|$ is bounded away from 0 on the support of φ_ε. On the other hand, the sequence u_h is bounded in $\mathcal{D}^{1,p}(\Omega;|x|^a d\xi)$, and therefore from (3.1.1) one gets

$$\int_A |x|^a |(1-\varphi_\varepsilon)u_h|^p \ d\xi \le C|A_\varepsilon|^{\frac{p}{N}} \int_\Omega |x|^a |\nabla u_h|^p \ d\xi \ \le C|A_\varepsilon|^{\frac{p}{N}} \ ,$$

where $A_\varepsilon := \{(x,y) \in \Omega \mid |x| < \varepsilon\}$. Writing $u_h = \varphi_\varepsilon u_h + (1-\varphi_\varepsilon)u_h$ one infers that

$$\int_A |x|^a |u_h|^p \ d\xi \le C \int_A |x|^a (|\varphi_\varepsilon u_h|^p + |(1-\varphi_\varepsilon)u_h|^p) \ d\xi \le o(1) + C|A_\varepsilon|^{\frac{p}{N}}$$

for ε fixed, as $h \to +\infty$. The conclusion easily follows, since $|A_\varepsilon| \to 0$ as $\varepsilon \to 0$. $\square$

Lemma 3.1.2 *Assume $\Psi \in C_c^\infty(\Omega)$. Then $\Psi u \in \mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$ for every $u \in \mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$.*

Proof. We can approximate any fixed $u \in \mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$ with a sequence $u_h \in C_c^\infty(\Omega)$. By computations, using Lemma 3.1.1, it follows that $\Psi u_h \to \Psi u$ in $\mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$, hence $\Psi u \in \mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$. $\square$

Now we deal with the asymptotic behaviour of bounded sequences of approximated solutions to (3.0.1). This strategy has been introduced by Musina in [69]. The next proposition allows us to find weakly convergent minimizing sequences that are bounded away from 0 on a compact subset of $(\mathcal{C}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$. In this way we can exclude concentration in 0 and vanishing.

Proposition 3.1.3 *Assume $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a cone in $\mathbb{R}^k$. Let u_h be a bounded sequence in $\mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$, and let $f_h \to 0$ be a sequence in the dual of $\mathcal{D}^{1,p}(\Omega;|x|^a\, d\xi)$. Assume that, for $a, p, q, b_{a,p,q}$ as in (3.0.2), (3.0.3), it holds that*

$$-\mathrm{div}(|x|^a |\nabla u_h|^{p-2} \nabla u_h) = |x|^{-b_{a,p,q}} |u_h|^{q-2} u_h + f_h \ .$$

Then, up to a subsequence, either $u_h \to 0$ *strongly in* $L^q(\Omega; |x|^{-b_{a,p,q}}\, d\xi)$, *or there exist sequences* $\{t_h\}_h \subset (0, +\infty)$ *and* $\{\eta_h\}_h \subset \mathbb{R}^{N-k}$, *such that*

$$\lim_{h\to+\infty} \int_K |x|^{-b_{a,p,q}} |\tilde{u}_h|^q \, d\xi > 0 \ ,$$

where $\tilde{u}_h(x,y) := t_h^{\frac{N-p+a}{p}} u_h(t_h x, t_h y + \eta_h)$ *and*

$$K := \{(x,y) \in \Omega = \mathcal{C}^k \times \mathbb{R}^{N-k} \mid \frac{1}{2} < |x| < 1, |y| < 1\} .$$

Proof. We can assume that there exists $u \in \mathcal{D}^{1,p}(\Omega; |x|^a\, d\xi)$ such that $u_h \to u$ weakly in $\mathcal{D}^{1,p}(\Omega; |x|^a\, d\xi)$ and in $L^q(\Omega; |x|^{-b_{a,p,q}}\, d\xi)$. If $u \neq 0$ then we are done since, up to a rescaling, $\int_K |x|^{-b_{a,p,q}} |u|^q \, d\xi > 0$. Then the conclusion follows by the weak lower semicontinuity of the L^q-norm. Therefore, we assume $u = 0$ and $\lim_{h\to+\infty} \int_\Omega |x|^{-b_{a,p,q}} |u_h|^q \, d\xi > 0$.
Fix $\varepsilon_0 > 0$ in such a way that

$$\varepsilon_0^{\frac{q}{q-p}} < \lim_{h\to+\infty} \int_\Omega |x|^{-b_{a,p,q}} |u_h|^q \, d\xi \ , \qquad 2\varepsilon_0 < S_{a,q}(\Omega, p) \ .$$

Using in a standard way the concentration function

$$Q_h(t) := \sup_{\eta \in \mathbb{R}^{N-k}} \int_{(B_t^k(0)\cap \mathcal{C}^k)\times B_t^{N-k}(\eta)} |x|^{-b_{a,p,q}} \ |u_h|^q \, d\xi \ ,$$

it is possible to select $t_h > 0$ and $\eta_h \in \mathbb{R}^{N-k}$ such that the rescaled sequence

$$\tilde{u}_h(x,y) := t_h^{\frac{N-p+a}{p}} u_h(t_h x, t_h y + \eta_h)$$

satisfies $\int_\Omega |x|^a |\nabla \tilde{u}_h|^p \, d\xi = \int_\Omega |x|^a |\nabla u_h|^p \, d\xi = O(1)$, and

$$\int_{(B_1^k(0)\cap \mathcal{C}^k)\times B_1^{N-k}(y)} |x|^{-b_{a,p,q}} |\tilde{u}_h|^q \, d\xi \le (2\varepsilon_0)^{\frac{q}{q-p}} \quad \forall\, y \in \mathbb{R}^{N-k} \ , \tag{3.1.2}$$

$$\int_{(B_1^k(0)\cap \mathcal{C}^k)\times B_1^{N-k}(0)} |x|^{-b_{a,p,q}} |\tilde{u}_h|^q \, d\xi \ge \varepsilon_0^{\frac{q}{q-p}} > 0 \ , \tag{3.1.3}$$

$$-\mathrm{div}(|x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h) = |x|^{-b_a} |\tilde{u}_h|^{q-2} \tilde{u}_h + \tilde{f}_h \ , \tag{3.1.4}$$

with $\tilde{f}_h \to 0$ in $\mathcal{D}^{1,p}(\Omega; |x|^a\, d\xi)'$. As before, if (up to a subsequence) $\tilde{u}_h \to \tilde{u} \neq 0$ then we are done. If $\tilde{u}_h \to 0$, choose a finite number of points $y_1, ..., y_s \in \mathbb{R}^{N-k}$ such that

$$\overline{B}_1^{N-k}(0) \subset \bigcup_{j=1}^{s} B_{1/2}^{N-k}(y_j) \ . \tag{3.1.5}$$

Let $\psi_1, ..., \psi_s$ be cut-off functions, with $\psi_j = \psi_j(y) \in C_c^\infty(B_1^{N-k}(y_j)), \psi_j \equiv 1$ on $B_{1/2}^{N-k}(y_j)$ and $0 \le \psi_j \le 1$. Fix a map $\varphi = \varphi(x) \in C_c^\infty(B_1^k(0) \cap \mathcal{C}^k)$ satisfying $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $B_{1/2}^k(0) \cap \mathcal{C}^k$. Thanks to Lemma 3.1.2 we can use $\varphi^p \psi_j^p \tilde{u}_h$ as test function in (3.1.4) to find

$$\int_\Omega |x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h \cdot \nabla(\varphi^p \psi_j^p \tilde{u}_h)\, d\xi = \int_\Omega |x|^{-b_{a,p,q}} |\tilde{u}_h|^{q-p} |\varphi \psi_j \tilde{u}_h|^p\, d\xi + o(1)\,. \quad (3.1.6)$$

Direct computations and Lemma 3.1.1 give

$$\int_\Omega |x|^a |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h \cdot \nabla(\varphi^p \psi_j^p \tilde{u}_h)\, d\xi = \int_\Omega |x|^a |\nabla(\varphi \psi_j \tilde{u}_h)|^p\, d\xi \ + o(1).$$

Thus, we can use Hölder inequality, (3.1.2), (3.1.6) and the definition of $S_{a,q}(\Omega, p)$ to infer that

$$S_{a,q}(\Omega, p) \left(\int_\Omega |x|^{-b_{a,p,q}} |\varphi \psi_j \tilde{u}_h|^q\, d\xi \right)^{\frac{p}{q}} \le 2\varepsilon_0 \left(\int_\Omega |x|^{-b_{a,p,q}} |\varphi \psi_j \tilde{u}_h|^q\, d\xi \right)^{\frac{p}{q}} + o(1)\,.$$

Since $2\varepsilon_0 < S_{a,q}(\Omega, p)$ this implies that $\int_\Omega |x|^{-b_{a,p,q}} |\varphi \psi_j \tilde{u}_h|^q\, d\xi = o(1)$, and therefore, by (3.1.5),

$$\begin{aligned}
\int_{(B_{1/2}^k(0) \cap \mathcal{C}^k) \times B_1^{N-k}(0)} |x|^{-b_{a,p,q}} |\tilde{u}_h|^q d\xi &\le \sum_{j=1}^s \int_{(B_{1/2}^k(0) \cap \mathcal{C}^k) \times B_{1/2}^{N-k}(y_j)} |x|^{-b_{a,p,q}} |\tilde{u}_h|^q d\xi \\
&= o(1)\,.
\end{aligned}$$

Finally, from (3.1.3) we get

$$0 < \varepsilon_0^{\frac{q}{q-p}} < \int_{(B_1^k(0) \cap \mathcal{C}^k) \times B_1^{N-k}(0)} |x|^{-b_{a,p,q}} |\tilde{u}_h|^q\, d\xi = \int_K |x|^{-b_{a,p,q}} |\tilde{u}_h|^q\, d\xi + o(1)\,.$$

Proposition 3.1.3 is completely proved. □

3.2 Existence on cones

Before proving Theorem 3.0.9, we show that $S_{a,p^*}(\Omega, p) \le S(p)$ for any exponent a. This is a consequence of the action of translations in the x-variable in the limiting case $q = p^*$.

Proposition 3.2.1 *Let $1 \le k \le N$, $p \in (1, N)$ and $a > (p - N)\frac{k}{N}$. Then $S_{a,p^*}(\Omega, p) \le S(p)$.*

Proof. Fix $u \in C_c^\infty(B_1^N(0) \cap \Omega)$ and $\varepsilon > 0$. Consider a point $x_0 \in \mathcal{C}^k$ with $|x_0| = 1$ and set $\xi_0 = (x_0, 0)$. We define $u_\varepsilon(\xi) := u(\varepsilon^{-1}(\xi - \xi_0)) \in C_c^\infty(B_\varepsilon^N(\xi_0) \cap \Omega)$. We estimate

$$S_{a,p^*}(\Omega, p) \le \frac{\int_\Omega |x|^a |\nabla u_\varepsilon|^p \, d\xi}{\left(\int_\Omega |x|^{\frac{Na}{N-p}} |u_\varepsilon|^{p^*} \, d\xi\right)^{p/p^*}} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|a|} \frac{\int_\Omega |\nabla u|^p \, d\xi}{\left(\int_\Omega |u|^{p^*} \, d\xi\right)^{p/p^*}} ,$$

that is,

$$S_{a,p^*}(\Omega, p) \le \inf_{\substack{C_c^\infty(B_1^N(0) \cap \Omega) \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^p \, d\xi}{\left(\int_\Omega |u|^{p^*} \, d\xi\right)^{p/p^*}} = S(p) ,$$

by the invariance of the ratio $\left(\int_\Omega |\nabla u|^p\right) \left(\int_\Omega |u|^{p^*}\right)^{-p/p^*}$ with respect to dilations and by the independence of the Sobolev constant with respect to the domain (see for example [82]). □

Proof of Theorem 3.0.9. Take a minimizing sequence u_h satisfying

$$\int_\Omega |x|^{-b_{a,p,q}} |u_h|^q \, d\xi = (S_{a,q}(\Omega, p))^{\frac{q}{q-p}} , \quad \int_\Omega |x|^a |\nabla u_h|^p \, d\xi = (S_{a,q}(\Omega, p))^{\frac{q}{q-p}} + o(1) . \tag{3.2.1}$$

By Ekeland's variational principle ([36]), we can assume that

$$-\mathrm{div}(|x|^a |\nabla u_h|^{p-2} \nabla u_h) = |x|^{-b_a} |u_h|^{q-2} u_h + f_h , \tag{3.2.2}$$

where $f_h \to 0$ in $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)'$. Up to a subsequence, by (3.2.1), we can find $u \in \mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$ such that $u_h \to u$ weakly in $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$. Thanks to Proposition 3.1.3 we can assume that, up to a change of coordinates,

$$\lim_{h \to +\infty} \int_K |x|^{-b_{a,p,q}} |u_h|^q \, d\xi > 0 , \tag{3.2.3}$$

where $K = \{(x, y) \in \mathcal{C}^k \times \mathbb{R}^{N-k} \mid \frac{1}{2} < |x| < 1, \ |y| < 1 \}$. We claim that $u \ne 0$. This is immediate if $q < p^*$, since in this case $\int_K |x|^{-b_{a,p,q}} |u|^q \, d\xi = \lim_{h \to +\infty} \int_K |x|^{-b_{a,p,q}} |u_h|^q \, d\xi > 0$ by Rellich Theorem. Therefore we take $q = p^*$ and we assume by contradiction that $u = 0$. Choose smooth maps $\varphi \in C_c^\infty(\mathcal{C}^k)$ and $\psi \in C_c^\infty(\mathbb{R}^{N-k})$ in such a way that $\varphi(x) = 0$ for $|x| \le \frac{1}{4}$, $\varphi(x) = 1$ for $\frac{1}{2} \le |x| \le 1$ and $\psi(y) = 1$ for $|y| \le 1$. Notice that $\varphi\psi \equiv 1$ on K. Since $\langle f_h, \varphi^p \psi^p u_h \rangle = o(1)$, we can argue as in the proof of Proposition 3.1.3 to get

$$\int_\Omega |x|^a |\nabla(\varphi\psi u_h)|^p \, d\xi \le S_{a,p^*}(\Omega, p) \left(\int_\Omega |x|^{\frac{Na}{N-p}} |\varphi\psi u_h|^{p^*} \, d\xi\right)^{\frac{p}{p^*}} + o(1) . \tag{3.2.4}$$

Now, notice that $|x|^{\frac{a}{p}}\nabla(\varphi\psi u_h) = \nabla(|x|^{\frac{a}{p}}\varphi\psi u_h) - F_h$, where $F_h := \varphi\psi u_h\nabla(|x|^{\frac{a}{p}})$. Since $\varphi\psi$ has compact support and since it vanishes in a neighborhood of the singular set $\{x = 0\}$, then $F_h \to 0$ in $L^p(\Omega)^N$ by Rellich Theorem. Therefore

$$\begin{aligned}\int_\Omega |x|^a|\nabla(\varphi\psi u_h)|^p \, d\xi &= \int_\Omega |\nabla(|x|^{\frac{a}{p}}\varphi\psi u_h)|^p \, d\xi + o(1)\\ &\geq S(p)\left(\int_\Omega |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*} \, d\xi\right)^{\frac{p}{p^*}} + o(1)\end{aligned}$$

by Sobolev inequality. In this way from (3.2.4) we get

$$S(p)\left(\int_\Omega |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*} \, d\xi\right)^{\frac{p}{p^*}} \leq S_{a,p^*}(\Omega,p)\left(\int_\Omega |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*} \, d\xi\right)^{\frac{p}{p^*}} + o(1).$$

Since $S_{a,p^*}(\Omega, p) < S(p)$ by assumption, this implies that

$$\int_K |x|^{\frac{Na}{N-p}}|u_h|^{p^*} \, d\xi \leq \int_\Omega |x|^{\frac{Na}{N-p}}|\varphi\psi u_h|^{p^*} \, d\xi = o(1) \ ,$$

that contradicts (3.2.3). Thus, $u_h \to u \neq 0$ weakly in $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$.

Finally, standard arguments imply that $u_h \to u$ strongly in $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$, and therefore that u achieves $S_{a,q}(\Omega, p)$. For completeness we recall the argument here. From (3.2.2) it follows that u solves the equation in problem (3.0.1), and in particular

$$\int_\Omega |x|^a|\nabla u|^p \, d\xi = \int_\Omega |x|^{-b_{a,p,q}}|u|^q \, d\xi \leq (S_{a,q}(\Omega,p))^{-\frac{q}{p}}\left(\int_\Omega |x|^a|\nabla u|^p \, d\xi\right)^{\frac{q}{p}} \ ,$$

by definition of $S_{a,q}(\Omega, p)$. Since $u \neq 0$, this implies that

$$\int_\Omega |x|^a|\nabla u|^p \, d\xi \geq (S_{a,q}(\Omega,p))^{\frac{q}{q-p}} \ .$$

Thus (3.2.1) and the lower semicontinuity of the norm in $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$ imply

$$\int_\Omega |x|^a|\nabla u_h|^p \, d\xi = \int_\Omega |x|^a|\nabla u|^p \, d\xi + o(1) \ ,$$

that suffices to conclude that $u_h \to u$ strongly in the uniformly convex space $\mathcal{D}^{1,p}(\Omega; |x|^a \, d\xi)$. □

This result raises a natural question about the limiting critical case $q = p^*$. When is the condition $S_{a,p^*}(\Omega, p) < S(p)$ verified? A partial answer is given in the case $\Omega = \mathbb{R}^N$ by the following theorem (see also next chapter for the case $p = 2$ and $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$). From now on we set $S_{a,q}(p)$ instead of $S_{a,q}(\mathbb{R}^N, p)$.

Theorem 3.2.2 *Let $p \in (1, N)$. If $(p - N)\frac{k}{N} < a < 0$, then $S_{a,p^*}(p) < S(p)$ and hence $S_{a,p^*}(p)$ is achieved.*

Proof. In order to prove that $S_{a,p^*}(p) < S(p)$, we claim that the following estimate holds:

$$S_{a,p^*}(p) \le \ S(p)\ \frac{N-p}{N}\ \frac{N+a}{N-p-a(p-1)}\ . \tag{3.2.5}$$

Notice that the right hand side in (3.2.5) is strictly increasing in a and it is equal to $S(p)$ if $a = 0$, therefore (3.2.5) implies $S_{a,p^*}(p) < S(p)$ for $a < 0$. To prove (3.2.5) we estimate $S_{a,p^*}(p)$ with the map

$$U(\xi) = \left(1 + |\xi|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}} ,$$

that achieves the best constant $S(p)$ (see [8] and [83]). We compute

$$|\nabla U|^p = \left(\frac{N-p}{p-1}\right)^p |\xi|^{\frac{p}{p-1}}\Phi^{-N}\ , \quad |U|^{p^*} = \Phi^{-N}\ , \tag{3.2.6}$$

where we have set $\Phi(\xi) := 1 + |\xi|^{\frac{p}{p-1}}$. An application of the divergence theorem leads to

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a|\xi|^{\frac{p}{p-1}}\,\Phi^{-N}\,d\xi \ &= \ -\ \frac{p-1}{p}\ \frac{1}{N-1}\ \int_{\mathbb{R}^N} |x|^a\,\nabla(\Phi^{1-N})\cdot\xi\,d\xi \\ &= \ \frac{p-1}{p}\ \frac{N+a}{N-1}\int_{\mathbb{R}^N} |x|^a\,\Phi^{1-N}\,d\xi\ .\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^N} |x|^a|\xi|^{\frac{p}{p-1}}\Phi^{-N}\ d\xi = \int_{\mathbb{R}^N} |x|^a\left(\Phi^{1-N} - \Phi^{-N}\right)\ d\xi\ ,$$

and hence

$$\int_{\mathbb{R}^N} |x|^a|\xi|^{\frac{p}{p-1}}\Phi^{-N}\ d\xi = \frac{(p-1)(N+a)}{N-p-a(p-1)}\int_{\mathbb{R}^N} |x|^a\Phi^{-N}\ d\xi\ .$$

Thus, from (3.2.6) we infer

$$\int_{\mathbb{R}^N} |x|^a|\nabla U|^p\ d\xi = \left(\frac{N-p}{p-1}\right)^p\ \frac{(p-1)(N+a)}{N-p-a(p-1)}\ \int_{\mathbb{R}^N} |x|^a\Phi^{-N}\ d\xi\ . \tag{3.2.7}$$

We can compute $S(p)$ by setting $a = 0$ in (3.2.7):

$$S(p) = \left(\frac{N-p}{p-1}\right)^p \frac{(p-1)N}{N-p} \left(\int_{\mathbb{R}^N} \Phi^{-N}\, d\xi\right)^{\frac{p}{N}} ,$$

Therefore, (3.2.7) and Hölder inequality imply

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^a |\nabla U|^p &\le \left(\frac{N-p}{p-1}\right)^p \frac{(p-1)(N+a)}{N-p-a(p-1)} \left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}\, \Phi^{-N}\right)^{\frac{N-p}{N}} \cdot \\
&\quad \cdot \left(\int_{\mathbb{R}^N} \Phi^{-N}\right)^{\frac{p}{N}} \\
&= S(p)\, \frac{N-p}{N}\, \frac{N+a}{N-p-a(p-1)} \left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-p}}\, |U|^{p^*}\, d\xi\right)^{\frac{N-p}{N}} ,
\end{aligned}$$

and inequality (3.2.5) readily follows. By Theorem 3.0.9 we get the conclusion of the proof. □

Remark 3.2.3 Notice that Theorem 3.2.2 gives a positive answer to a question that has been raised by Tertikas and Tintarev in [85], Section 6, at point 4, at least when $p < k$. Moreover, the condition $p^2 < N$ suggested in [85] to get the existence of a minimizer for $S_{p-k,p^*}(p)$ is not necessary, even if up to now we are not able to prove its sufficiency (except when $p = 2$, compare with [85], where $N \ge 4$ is assumed).

Conjecture. Quite reasonably it happens that $S_{a,p^*}(p) = S(p)$ for a large enough. On the other hand, one might suspect that $S_{a,p^*}(p) < S(p)$ for a close to $p - k$ and $k < p \le \sqrt{N}$. This is the case when $p = 2$ (see [49] and next chapter).

By standard arguments it readily follows that Theorems 3.0.9 and 3.2.2 provide sufficient conditions for the existence of non trivial weak entire solutions to (3.0.1). Notice that if $k \ge 2$ then $u > 0$ on $\{x \ne 0\}$, by the maximum principle. This is no longer true in general if $k = 1$ (compare with Section 3.4).

3.3 Problems on proper cones

In this section we extend some results already proved in [27] and in the more recent papers [13], [69] and we consider $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a *proper cone* in $\mathbb{R}^k$ (see Definition 2.1.1 at page 20). Our arguments for Theorem 3.0.9 can be used with no modifications to study problems on proper cones.

We recall that inequality (3.0.4) at page 28 holds true also under the assumptions

$$p \in (1, N)\,, \quad q \in (p, p^*]\,, \quad a \neq p - k \tag{3.3.1}$$

and $b_{a,p,q}$ as in (3.0.3), for any $u \in C_c^\infty(\Omega)$, with $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$ and $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$, $1 \le k \le N$ (see Theorem 2.1.3 at page 20).

Thanks to this consideration, we can define the Banach space $\mathcal{D}^{1,p}(\Omega; |x|^a d\xi)$ by completing $C_c^\infty(\Omega)$ with respect to the norm $\int_\Omega |x|^a |\nabla u|^p \, d\xi$ also for any $a \neq p - k$. Notice that $\mathcal{D}^{1,p}(\Omega; |x|^a d\xi) \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ for $a > (p - N)\frac{k}{N}$. In particular, by a density argument, we can prove that $\mathcal{D}^{1,p}(\mathbb{R}_0^k \times \mathbb{R}^{N-k}; |x|^a d\xi) = \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ if and only if $a \ge p - k$. The lemma we refer to is the following.

Lemma 3.3.1 *Let $a \ge p - k$. Then $C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$ is dense in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$.*

Proof. Fix any map $v \in C_c^\infty(\mathbb{R}^N)$. For $\varepsilon > 0$ set

$$\varphi_\varepsilon(|x|) = \begin{cases} 0 & \text{if } |x| \le \varepsilon^2 \\ \frac{\log |x|/\varepsilon^2}{|\log \varepsilon|} & \text{if } \varepsilon^2 < |x| < \varepsilon \\ 1 & \text{if } |x| \ge \varepsilon\ . \end{cases}$$

It is clear that $\varphi_\varepsilon v \in \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ and that $\nabla(v - \varphi_\varepsilon v) = (1 - \varphi_\varepsilon)\nabla v - v\nabla\varphi_\varepsilon \to 0$ a.e. on $\mathbb{R}^N$, as $\varepsilon \to 0$. To prove that $\varphi_\varepsilon v \to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ it suffices to remark that

$$\int_{\mathbb{R}^N} |x|^a |v \nabla \varphi_\varepsilon|^p \; d\xi \le c_v \int_{\mathbb{R}^k} |x|^a |\varphi'_\varepsilon|^p \; dx \le c_v |\log \varepsilon|^{1-p}$$

since $a \ge p - k$, where the constants c_v do not depend on ε. The conclusion follows via Lebesgue's theorem, since $|(1 - \varphi_\varepsilon)\nabla v| \le |\nabla v|$ on $\mathbb{R}^N$, and since $|x|^a |\nabla v|^p \in L^1(\mathbb{R}^N)$. □

By Theorem 2.1.3 at page 20, the infimum $S_{a,q}(\Omega, p)$ is positive also under the assumptions (3.3.1). One can argue as for Theorem 3.0.9 to prove the next result. We omit the details.

Theorem 3.3.2 *Let $p \in (1, N)$, $q \in (p, p^*]$, $a \neq p - k$ and let $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$, with $\mathcal{C}^k$ a proper cone in $\mathbb{R}^k$. Then $S_{a,q}(\Omega, p)$ is achieved provided that*

$$q < p^* \quad \textit{or} \quad q = p^* \textit{ and } \ S_{a,p^*}(\Omega, p) < S(p).$$

We collect here some comments and remarks that will be reconsidered and discussed also in the next chapter. The first is about the condition $S_{a,p^*}(\Omega, p) < S(p)$.

Remark 3.3.3 In Theorem 3.2.2 we have given a sufficient condition for the strict inequality $S_{a,p^*}(p) < S(p)$. Since $S_{a,q}(\Omega, p) \geq S_{a,q}(p)$ under assumptions (3.0.2), up to now we are not able to give analogous general conditions for the strict inequality $S_{a,p^*}(\Omega, p) < S(p)$. There are some results on this subject in case $p = 2$ and $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$ (see [69] and [66]), but we postpone the discussion to the next chapter.

The next two remarks deal with a comparison between $S_{a,q}(\Omega, p)$ and $S_{a,q}(p)$.

Remark 3.3.4 Now we consider $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$. Since $\mathbb{R}^k_0$ is a proper cone in $\mathbb{R}^k$, we can apply Theorem 3.3.2 to get that $S_{a,q}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}, p)$ is achieved for any $a \neq p - k$, $q \in (p, p^*)$; in the limiting case $S_{a,p^*}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}, p)$ is achieved provided that $S_{a,p^*}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}, p) < S(p)$. It can be easily proved via Hardy inequality (see (1.3.7) at page 16) that for $a > (p - N)\frac{k}{N}$,

$$\mathcal{D}^{1,p}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}; |x|^a d\xi) = \mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi) \cap L^p(\mathbb{R}^N; |x|^{a-p} d\xi) \ .$$

Hence, $\mathcal{D}^{1,p}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}; |x|^a d\xi)$ is a proper subspace of $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$ if and only if $a < p - k$. In [49] the writing authors compare the infimum $S_{a,q}(\mathbb{R}^k_0 \times \mathbb{R}^{N-k}, 2)$ with $S_{a,q}(2)$ for $a \in ((2 - N)\frac{k}{N}, 2 - k)$ (see next chapter for details).

Remark 3.3.5 Let $k \geq 2$ and $\mathcal{C}^k$ be a cone, properly contained in $\mathbb{R}^k_0$. Assume that (3.0.2) are satisfied and that $q < p^*$ or $S_{a,p^*}(\Omega, p) < S(p)$, with $\Omega = \mathcal{C}^k \times \mathbb{R}^{N-k}$. Then both the infima $S_{a,q}(\Omega, p)$ and $S_{a,q}(p)$ are achieved. One can write down the Euler-Lagrange equations to infer that $S_{a,q}(p) < S_{a,q}(\Omega, p)$. This is no longer true if $k = 1$ and $a \geq p - 1$, compare with Section 3.4 below.

3.4 The case $k = 1$

When $k = 1$ the singular set $\{x = 0\}$ is an hyperplane that disconnects the domain. Let us point out an immediate corollary to Theorem 3.0.9.

Corollary 3.4.1 *Let $k = 1$, $p \in (1, N)$ and $\frac{p(N-1)}{N-p} < q < p^*$. Then problem*

$$\begin{cases} -\Delta_p u = |x|^{-b_{a,p,q}} \ |u|^{q-2} u \quad \text{in } \mathbb{R}^N \\ u \geq 0 \ , \end{cases}$$

has a weak entire ground state solution.

As observed in [48] (see also Theorem 5.2.4 at page 63), in case $p = 2$ the solution of Corollary 3.4.1 is even in the x-variable and decreasing for $x > 0$. In particular, u can never vanish on $\mathbb{R}^N$. This remark and the next lemma underline the contrast between the cases $a = 0 < p - 1$ and $a \geq p - 1$.

Lemma 3.4.2 *Let $k = 1$, $p \in (1, N)$, $q \in (p, p^*]$ and $a \geq p - 1$. Then every minimizer for $S_{a,q}(p)$ vanishes on a half-plane.*

Proof. Set $\mathbb{R}^N_- := (-\infty, 0) \times \mathbb{R}^{N-1}$ and $\mathbb{R}^N_+ := (0, +\infty) \times \mathbb{R}^{N-1}$. Assume that u is a minimizer for $S_{a,q}(p)$. By Lemma 3.3.1, there exist sequences $u_h^- \in C_c^\infty(\mathbb{R}^N_-)$ and $u_h^+ \in C_c^\infty(\mathbb{R}^N_+)$ such that $u_h^- + u_h^+ \to u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N; |x|^a d\xi)$. Therefore

$$\int_{\mathbb{R}^N_-} |x|^a |\nabla u_h^-|^p \, d\xi \to \int_{\mathbb{R}^N_-} |x|^a |\nabla u|^p \, d\xi \ , \quad \int_{\mathbb{R}^N_+} |x|^a |\nabla u_h^+|^p \, d\xi \to \int_{\mathbb{R}^N_+} |x|^a |\nabla u|^p \, d\xi \ ,$$

and similarly for the weighted L^q norms. Since u_h^- and u_h^+ have disjoint supports, then

$$\begin{aligned} S_{a,q}(p) &= \frac{\int_{\mathbb{R}^N} |x|^a |\nabla (u_h^- + u_h^+)|^p}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,p,q}} |u_h^- + u_h^+|^q\right)^{p/q}} + o(1) \\ &\geq S_{a,q}(p) \, \frac{\left(\int_{\mathbb{R}^N_-} |x|^{-b_{a,p,q}} |u_h^-|^q\right)^{p/q} + \left(\int_{\mathbb{R}^N_+} |x|^{-b_{a,p,q}} |u_h^+|^q\right)^{p/q}}{\left(\int_{\mathbb{R}^N_-} |x|^{-b_{a,p,q}} |u_h^-|^q + \int_{\mathbb{R}^N_+} |x|^{-b_{a,p,q}} |u_h^+|^q\right)^{p/q}} + o(1) \end{aligned}$$

by (3.0.5) at page 28. Letting $h \to +\infty$, we get

$$\frac{\left(\int_{\mathbb{R}^N_-} |x|^{-b_{a,p,q}} |u|^q\right)^{p/q} + \left(\int_{\mathbb{R}^N_+} |x|^{-b_{a,p,q}} |u|^q\right)^{p/q}}{\left(\int_{\mathbb{R}^N_-} |x|^{-b_{a,p,q}} |u|^q + \int_{\mathbb{R}^N_+} |x|^{-b_{a,p,q}} |u|^q\right)^{p/q}} \leq 1 \, .$$

The conclusion easily follows by a convexity argument, since $p < q$. □

By Lemma 3.4.2 it turns out that $S_{a,q}(\mathbb{R}^N_+, p) = S_{a,q}(p)$ for $a \geq p - 1$, even if both the infima are achieved. This means that the maximum principle fails in this case. We suspect that this is not longer true for a below $p - 1$, as the case $p = 2$, $a = 0$ suggests (see next chapter for a more exhaustive discussion on this subject).

Chapter 4

Multiplicity and singular solutions

In this chapter we deal with problem (3.0.1) at page 27 in the particular case $p = 2$, $1 \le k < N$, $N \ge 3$. We are interested in existence and multiplicity of solutions to

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{-b_{a,q}} |u|^{q-2}\, u & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ u \ge 0 \ , \end{cases} \tag{4.0.1}$$

where the exponent $b_{a,q}$ is given by

$$b_{a,q} := N - q \frac{N - 2 + a}{2} \ , \tag{4.0.2}$$

and the real parameters a, q satisfy for $N \ge 3$

$$(2 - N)\frac{k}{N} < a \le 2 - k \ , \quad \frac{2(N-k)}{N-2+a} < q \le 2^* = \frac{2N}{N-2} \ . \tag{4.0.3}$$

Under assumptions (4.0.3) and (4.0.2), Maz'ya inequality (3.0.4) at page 28 becomes on $\mathbb{R}^N$

$$C \left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \, d\xi \right)^{2/q} \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi \ , \quad \forall\, u \in C_c^\infty(\mathbb{R}^N) \, . \tag{4.0.4}$$

We recall that, thanks to (4.0.4), the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ is well defined by completing $C_c^\infty(\mathbb{R}^N)$ with respect to the scalar product $\langle u, v\rangle = \int_{\mathbb{R}^N} |x|^a \nabla u \cdot \nabla v \, d\xi$ (see page 28). Then, every minimizer for

$$S_{a,q}(2) = \inf_{\substack{u\in\mathcal{D}^{1,2}(\mathbb{R}^N;|x|^a\,d\xi)\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \; d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \; d\xi\right)^{2/q}} \tag{4.0.5}$$

(see (3.0.5) at page 28) is, up to a Lagrange multiplier, a solution to (4.0.1). The problem of the existence of minimizers for $S_{a,q}(2)$ is discussed in Theorem 3.0.9 at page 29 (see also Theorem 4.2.1) and Theorem 4.2.11.

In the present chapter we focus our attention on the case $a \le 2-k$. In particular for $a < 2-k$ we compare the solution u^D to the minimization problem (4.0.5) with the solution $u^X \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a \, d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$ to (4.0.1), whose existence was proved in [69] (see Theorems B.1 and B.2). Under suitable assumptions on the exponents a and q, we are able to prove that u^X do not solve (4.0.5). By Hardy inequality (1.3.3) at page 14, in case $a > 2-k$ it follows that the solution u^D coincides with the solution u^X.

In the last section we address our attention on classical solutions to

$$\begin{cases} -\Delta v = \lambda |x|^{-2}\, v + |x|^{-b_q}\, |v|^{q-2}\, v \quad \text{in } \mathbb{R}^N \;,\; x \neq 0 \\ v \ge 0 \;, \end{cases} \tag{4.0.6}$$

where $\lambda, q, b_q \in \mathbb{R}$ satisfy

$$\lambda \le \left(\frac{k-2}{2}\right)^2 \quad , \quad q \in (2, 2^*] \;, \quad b_q = N - q\frac{N-2}{2} \;.$$

First of all we use a functional change and the results obtained for (4.0.1) to prove existence in case $\lambda = \left(\frac{k-2}{2}\right)^2$ (see also [85]). Then we state some existence results of *singular* solutions to (4.0.6) under the assumption $\lambda < \left(\frac{k-2}{2}\right)^2$ (see Subsection 4.3.2).

We start recalling some essentially known results (see [69]) that will be useful later on to study problem (4.0.1).

4.1 The space $X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$

The *Hardy-Sobolev-Maz'ya inequality* was proved in [67], Section 2.1.6, Corollary 3 in case $1 \le k < N$, $N \ge 3$ (see also Theorem 2.2.5 at page 22 and Remark 2.2.7 at page

23). It states for every $q \in (2, 2^*]$ that there exists a constant $C = C(a, q, k) > 0$ such that

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}}|u|^q\ d\xi\right)^{2/q} \le \int_{\mathbb{R}^N} |x|^a|\nabla u|^2\ d\xi - \lambda_1(a)\int_{\mathbb{R}^N} |x|^{a-2}|u|^2\ d\xi \quad (4.1.1)$$

for any $u \in C_c^\infty(\mathbb{R}^N)$ if $a > 2-k$, and for any $u \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$ if $a \le 2-k$. We recall that

$$\lambda_1(a) = \left(\frac{k-2+a}{2}\right)^2$$

is the best constant in the Hardy inequality (see Theorem 1.3.3 at page 14).

Remark 4.1.1 Inequality (4.1.1) fails if $k = N$. By contradiction we assume that inequality (4.1.1) holds true. By identity (1.1.2) at page 4, with $v := |x|^{\frac{a}{2}}u$

$$\inf_{\substack{v\in C_c^\infty(\mathbb{R}_0^N)\\ v\not\equiv 0}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2\ dx - \lambda_1(0)\int_{\mathbb{R}^N} |x|^{-2}|v|^2\ dx}{\left(\int_{\mathbb{R}^N} |x|^{-b_q}|v|^q\ dx\right)^{2/q}} \ge C > 0\,,$$

where $\lambda_1(0) = \left(\frac{N-2}{2}\right)^2$ since $k = N$. We can argue as in [69] to get existence of a positive solution to problem (4.0.6), but this contradicts a non-existence result of Brezis, Dupaigne and Tesei (see [17], Theorem 2).

Inequality (4.1.1) provides the starting point to apply variational methods to the degenerate problem

$$\begin{cases} -\mathrm{div}(|x|^a\nabla u) = \lambda|x|^{a-2}u + |x|^{-b_{a,q}}|u|^{q-2}\,u & \text{in } \mathbb{R}^N\ ,\ x \ne 0 \\ u \ge 0\ , \end{cases} \quad (4.1.2)$$

where λ is a real parameter. For future convenience we recall here the approach used in [69] to study (4.1.2) in case $\lambda < \lambda_1(a)$. For any $a \in \mathbb{R}$ we define the Hilbert space

$$X^{1,2}(\mathbb{R}^N;|x|^a d\xi) := \mathcal{D}^{1,2}(\mathbb{R}^N;|x|^a\, d\xi) \cap L^2(\mathbb{R}^N;|x|^{a-2}d\xi)\ .$$

In case $a = 0$ we will simply write $X^{1,2}(\mathbb{R}^N;d\xi)$. Notice that $X^{1,2}(\mathbb{R}^N;|x|^a d\xi) = \mathcal{D}^{1,2}(\mathbb{R}^N;|x|^a\, d\xi)$ if $a > 2-k$, by Hardy inequality (1.3.3) at page 14. It turns out that $C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$ is dense in $X^{1,2}(\mathbb{R}^N;|x|^a d\xi)$ ([69], Appendix B) and that (4.1.1) holds for any $u \in X^{1,2}(\mathbb{R}^N;|x|^a d\xi)$, by Lemma 3.3.1 for $p = 2$, at page 37.

The paper [69] deals with the existence of extremals for

$$\inf_{\substack{u\in X^{1,2}(\mathbb{R}^N;|x|^a d\xi)\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a|\nabla u|^2\ d\xi - \lambda\int_{\mathbb{R}^N} |x|^{a-2}|u|^2\ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}}|u|^q\ d\xi\right)^{2/q}}$$

under the assumption $\lambda < \lambda_1(a)$. In particular, for $\lambda = 0$ and $a \neq 2-k$, every solution $u^X \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ to the minimization problem

$$S^X_{a,q} := \inf_{\substack{u \in X^{1,2}(\mathbb{R}^N;|x|^a d\xi) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \, d\xi\right)^{2/q}} \tag{4.1.3}$$

is, up to a Lagrange multiplier, an entire classical solution to (4.0.1). Notice that $S^X_{a,q} > 0$ if $a \neq 2-k$, by inequality (2.1.2) at page 20. Moreover u^X is a weak entire solution to (4.0.1) on $\mathbb{R}^k_0 \times \mathbb{R}^{N-k}$. If $a \leq 2-k$, in general, one can not conclude that u^X is indeed a weak solution on the whole $\mathbb{R}^N$. On the other hand, for $a > 2-k$ one can take advantage of the Hardy inequality (1.3.3) at page 14 to show that $S_{a,q}(2) = S^X_{a,q}$, and that u^X satisfies

$$\int_{\mathbb{R}^N} |x|^a \nabla u \cdot \nabla \Phi \, d\xi = \int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^{q-2} \, u \, \Phi \, d\xi \ , \qquad \forall \ \Phi \in C^\infty_c(\mathbb{R}^N) \ .$$

Finally we notice that in general $S_{a,q}(2) \leq S^X_{a,q}$ for $a > (2-N)\frac{k}{N}$, since the space $X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ is contained in $D^{1,2}(\mathbb{R}^N; |x|^a d\xi)$. In Section 4.2 we illustrate some examples in which the strict inequality $S_{a,q}(2) < S^X_{a,q}$ holds true.

Concerning the existence of minimizers for $S^X_{a,q}$ we can state the following result.

Theorem 4.1.2 *Assume $a \neq 2-k$ and let $N \geq 3$. Then the infimum $S^X_{a,q}$ is achieved by a map $u^X \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ if*

$$q \in (2, 2^*) \quad \text{or} \quad q = 2^* \text{ and } S^X_{a,2^*} < S \ .$$

Moreover, if $k = 1$ then the support of u^X is a half-plane.

Theorem 4.1.2 is a direct consequence of Theorems B.1 and B.2 in [69].
We conclude this section with a few remarks on the infimum $S^X_{a,q}$. First we state a useful lemma that was proved in [69], Appendix B (see also Lemma 1.3.1 at page 13).

Lemma 4.1.3 *For any $a \in \mathbb{R}$ the linear operator $L_a(u) := |x|^{\frac{a}{2}} u$ is a bi-continuous isomorphism between $X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ and $X^{1,2}(\mathbb{R}^N; d\xi)$. Moreover,*

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi \quad &- \quad \lambda_1(a) \int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \, d\xi \\ &= \int_{\mathbb{R}^N} |\nabla (L_a u)|^2 \, d\xi - \lambda_1(0) \int_{\mathbb{R}^N} |x|^{-2} |L_a u|^2 \, d\xi \end{aligned} \tag{4.1.4}$$

for any $u \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$.

In the next lemma we point out some remarks on the behaviour of the map $a \to S^X_{a,q}$.

Lemma 4.1.4 *Assume $N \ge 3$ and $2 < q < 2^*$. Then the map $a \to S^X_{a,q}$ is strictly increasing for $a \ge 2-k$, and it is strictly decreasing for $a \le 2-k$.*

Proof. For $a \in \mathbb{R}$ set $\bar{a} := 2(2-k) - a$ and notice that $\lambda_1(a) = \lambda_1(\bar{a})$. Then, by (4.1.4), it turns out that

$$\begin{aligned} S^X_{a,q} &= \inf_{\substack{v \in X^{1,2}(\mathbb{R}^N;d\xi) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - (\lambda_1(0) - \lambda_1(a)) \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi \right)^{2/q}} \\ &= \inf_{\substack{v \in X^{1,2}(\mathbb{R}^N;d\xi) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - (\lambda_1(0) - \lambda_1(\bar{a})) \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi \right)^{2/q}} = S^X_{\bar{a},q} \ . \end{aligned}$$

Thus $S^X_{a,q} = S^X_{\bar{a},q}$. The lemma is readily proved, since the map $a \to \lambda_1(a)$ is increasing for $a \ge 2-k$, for any k, and since $S^X_{a,q}$ is achieved for any $a \ne 2-k$. $\square$

The case $N \ge 3$, $q = 2^*$ is more difficult. We recall that $S^X_{a,2^*} \le S$ for any $a \in \mathbb{R}$ (see [69], Theorem B.5), and that the map $v_T := (1 + |x|^2 + |y|^2)^{-\frac{N-2}{2}}$ achieves the best Sobolev constant S on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (see [8] and [83]). Moreover $v_T \in L^2(\mathbb{R}^N;|x|^{-2}d\xi)$, namely $v_T \in X^{1,2}(\mathbb{R}^N;d\xi)$, if and only if $k \ge 3$. In this case, a direct computation shows that the map

$$u_T(x,y) := |x|^{k-2} v_T(x,y)$$

belongs to $X^{1,2}(\mathbb{R}^N;|x|^{2(2-k)}d\xi)$ and achieves $S^X_{2(2-k),2^*} = S$ (see proof of Lemma 4.1.4). On the contrary, if $k = 1,2$ then $v_T \notin L^2(\mathbb{R}^N;|x|^{-2}d\xi)$, and $S^X_{2(2-k),2^*} = S^X_{0,2^*} = S$ is not achieved.

In the next lemma we collect some remarks on the behaviour of the map $a \to S^X_{a,2^*}$.

Lemma 4.1.5 *Assume $N \ge 3$ and let $a \in \mathbb{R}$. Then the map $a \to S^X_{a,2^*}$ is increasing for $a \ge 2-k$, and it is decreasing for $a \le 2-k$. Moreover:*

1. *$S^X_{a,2^*} \le S$ for any $a \in \mathbb{R}$.*
2. *$S^X_{a,2^*} = S$ and $S^X_{a,2^*}$ is not achieved in the following cases:*

   $k = 1$ and $N = 3$, or $k = 1$, $N \ge 4$ and $a \notin (0,2)$;

$k = 2$

$k \geq 3$ *and* $a \notin [2(2-k), 0]$.

3. $S^X_{a,2^*}$ *is achieved in the following cases:*

$k = 1$, $N \geq 4$, $a \in (0,2)$, $a \neq 1$;

$k \geq 3$ *and* $a \in [2(2-k), 0]$, $a \neq 2-k$.

Proof. The monotonicity properties of the map $a \to S^X_{a,2^*}$ can be checked as in Lemma 4.1.4. For the proof of 1 we refer to [69], Theorem B.5.

We prove 2 by contradiction. Assume that $k = 1$, $N = 3$ and that $S^X_{a,6}$ is achieved by a map $u \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$, for some $a \in \mathbb{R}$. Then $a \in (0,2)$ by [69], Theorem B.5. By Proposition B.3 of [69], we can assume that the support of u is contained in the half-space $(0, +\infty) \times \mathbb{R}^2$. Then u solves

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{3a}\, |u|^4\, u & \text{in } (0,+\infty) \times \mathbb{R}^2 \\ u > 0 \\ u \in X^{1,2}(\mathbb{R}^3_+; |x|^a\, d\xi). \end{cases}$$

Set $v := |x|^{a/2}\, u$. By direct computations and Lemma 4.1.3, v solves

$$\begin{cases} -\Delta v = \frac{a(2-a)}{4} |x|^{-2} v + |v|^4\, v & \text{in } (0,+\infty) \times \mathbb{R}^2 \\ v > 0 \\ v \in X^{1,2}(\mathbb{R}^3_+; d\xi). \end{cases}$$

This contradicts the non-existence result in [66], Section 6. For the other statements of 2 we refer to [69], Theorem B.5.

Now we prove the first part of point 3 for completeness (see also [69], Appendix B). In case $k = 1$, $N \geq 4$, $a \in (0,2)$, with $a \neq 1$, we fix $r, R > 0$ and we take any bounded domain $\Gamma \subset (r, R) \times \mathbb{R}^{N-1}$. We consider any map $v \in C^\infty_c(\Gamma)$. Then the integration by parts implies that

$$\int_{\mathbb{R}^N} |x|^a |\nabla(|x|^{-a/2} v)|^2 = \int_\Gamma |\nabla v|^2 - \frac{a(2-a)}{4} \int_\Gamma |x|^{-2} v^2 \;\leq \int_\Gamma |\nabla v|^2 - \frac{a(2-a)}{4R^2} \int_\Gamma v^2\,.$$

Thus

$$S^X_{a,2^*} \leq \inf_{\substack{v \in C^\infty_c(\Gamma) \\ v \not\equiv 0}} \frac{\int_\Gamma |\nabla v|^2 - \frac{a(2-a)}{4R^2} \int_\Gamma v^2}{\left(\int_\Gamma |v|^{2^*}\right)^{\frac{2}{2^*}}} \; < S\,,$$

since $a \in (0,2)$ and $N \geq 4$, by a well known result by Brezis and Nirenberg [19], Lemma 1.1. Then $S^X_{a,2^*}$ is achieved in this case.

The second part of 3 follows by Theorem B.2 and Theorem B.5 in [69]. □

The following figures show the behaviour of $S^X_{a,2^*}$ in cases $k=1$, $N\geq 4$ and $k\geq 3$.

$k=1,\ N\geq 4$ $\qquad\qquad$ $k\geq 3$

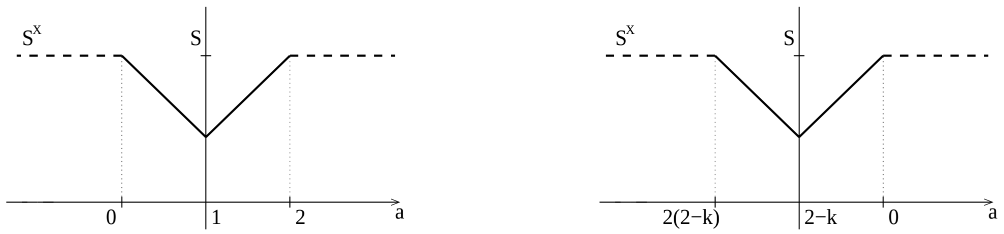

Fig.1 Graphics of $a\to S^X_{a,2^*}$.
$---=S^X_{a,2^*}$ not achieved; $\rule{1.5em}{0.4pt}=S^X_{a,2^*}$ achieved.

4.2 Existence and multiplicity results for (4.0.1)

The following existence result is a consequence of Theorem 3.0.9 at page 29 for $p=2$, so we omit the proof.

Theorem 4.2.1 *Assume $N\geq 3$ and that (4.0.2) and (4.0.3) are satisfied. Then the infimum $S_{a,q}(2)$ is achieved by a weak entire solution $u^D\in\mathcal{D}^{1,2}(\mathbb{R}^N;|x|^a\,d\xi)$ to (4.0.1) if*

$$q\in(2,2^*)\quad\text{ or }\quad q=2^*\text{ and }S_{a,2^*}(2)<S\,.$$

Remark 4.2.2 Assume $N\geq 3$, $q=2^*$ and $a\in((2-N)\frac{k}{N},2-k]$, $a\neq 0$. If $k=1$ assume in addition that $N\geq 4$ or $a<0$. In Subsection 4.2.2 we will show that, under these assumptions condition, $S_{a,2^*}(2)<S$ is always satisfied. The case $k=1$, $N=3$ and $a\in(0,1)$ is still open.

Remark 4.2.3 We recall that problem (4.0.1) is invariant with respect to the $(N-k+1)$-dimensional group $G_k=\{T(\tau,\eta)\mid\tau>0\ ,\ \eta\in\mathbb{R}^{N-k}\ \}$ of transforms given by

$$u(x,y)\to(T(\tau,\eta)\,u)(x,y):=\tau^{\frac{N-2+a}{2}}u(\tau x,\tau y+\eta)$$

(see however the remarks at page 51 for the case $k=1$). Moreover

$$\int_{\mathbb{R}^N}|x|^a|\nabla(T(\tau,\eta)u)|^2\,d\xi=\int_{\mathbb{R}^N}|x|^a|\nabla u|^2\,d\xi\ ,$$

$$\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |T(\tau,\eta)u|^q \, d\xi = \int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \, d\xi \ .$$

We shall identify solutions u which belong to the orbit of the same transform T in G_k.

Our strategy to prove multiplicity results for (4.0.1) is to compare the solution $u^D \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a \, d\xi)$ of Theorem 4.2.1 with the solution $u^X \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ of [69]. More precisely, we look for conditions on a, q that guarantee that

$$S_{a,q}(2) < S^X_{a,q} \ .$$

We start with a simple general lemma.

Lemma 4.2.4 *1. Let $q \in (2, 2^*]$, with $N \geq 3$. Then there exists $\varepsilon > 0$ such that $S_{a,q}(2) < S^X_{a,q}$ if*

$$0 < a - 2 + N - \frac{2(N-k)}{q} < \varepsilon \ .$$

2. Let $a \in \left((2-N)\frac{k}{N}, 2-k\right)$. Then there exists $\varepsilon > 0$ such that $S_{a,q}(2) < S^X_{a,q}$ if

$$0 < q - \frac{2(N-k)}{N-2+a} < \varepsilon \ .$$

Proof. Fix any map $w \in C^\infty_c(\mathbb{R}^N)$ such that $w \equiv 1$ on $\{(x,y) \mid |x| \leq 1 \ , \ \ |y| \leq 1 \ \}$. Then compute

$$S_{a,q}(2) \leq \frac{\int_{\mathbb{R}^N} |x|^a |\nabla w|^2 \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |w|^q \ d\xi\right)^{2/q}} \leq \frac{C}{\left(\int_{\{|x|<1\}} |x|^{-b_{a,q}} \ dx\right)^{2/q}} \ ,$$

and notice that the weight $|x|^{-b_{a,q}}$ looses its summability at the origin as $b_{a,q} \to k$. Therefore $S_{a,q}(2) \to 0$ as $(N-2+a)q \to 2(N-k)$. The conclusion readily follows from Lemmata 4.1.4 and 4.1.5. □

The uniqueness result in the recent paper [65] by Mancini and Sandeep allows us to compute exactly the value of the infimum $S_{2-k,q_k}(2)$, where

$$q_k = \frac{2(N-k+1)}{N-k} \tag{4.2.1}$$

(compare with (4.2.6) below). In the next lemma we use this information to estimate $S_{a,q_k}(2)$ from above when $a < 2-k$.

Lemma 4.2.5 *Assume* $2 \le k < N$, $N > 2(k-1)$ *and* $1-k+\frac{1}{N-k+1} < a < 2-k$. *Then*

$$S_{a,q_k}(2) < S^X_{a,q_k} \ .$$

Proof. Notice that $q_k < 2^*$. Thus, by Theorem 4.2.1, the infimum $S_{2-k,q_k}(2)$ is achieved by a map u that solves, up to a Lagrange multiplier, the degenerate problem

$$\begin{cases} -\mathrm{div}(|x|^{2-k}\nabla u) = |x|^{1-k}|u|^{q_k-2}\,u \quad \text{in } \mathbb{R}^N \ , \\ u \ge 0, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^{2-k}d\xi). \end{cases}$$

Set $v(x,y) := |x|^{\frac{2-k}{2}}u(x,y)$. Then by direct computation and by Lemma 4.1.3 it turns out that v solves the elliptic singular problem

$$\begin{cases} -\Delta v = \left(\frac{k-2}{2}\right)^2 |x|^{-2}v + |x|^{-b_{q_k}}|v|^{q_k-2}\,v \quad \text{in } \mathbb{R}^N \ , \\ v \ge 0, \\ \int_{\mathbb{R}^N}[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2]\,d\xi < \infty, \end{cases} \tag{4.2.2}$$

where $b_{q_k} = \frac{N-2k+2}{N-k}$. By Theorem 5.2.5 at page 63 (see also [48]), v is *cylindrically symmetric*, that roughly speaking means dependent only on $|x|$ and $|y|$ (see Definition 5.0.6 at page 60). Therefore, by the uniqueness result in [66], Section 6, it turns out that

$$v(x,y) = C(\lambda,N,k)\ |x|^{\frac{2-k}{2}}\left((1+|x|)^2+|y|^2\right)^{-\frac{N-k}{2}} \quad ,$$

for some constant $C(\lambda,N,k)$ that can be computed explicitly. As a consequence, we have that the map

$$u_M(x,y) := \left((1+|x|)^2+|y|^2\right)^{-\frac{N-k}{2}}$$

achieves the best constant $S_{2-k,q_k}(2)$. Now we set, for $a \le 2-k$,

$$R_a := \frac{\int_{\mathbb{R}^N}|x|^a|\nabla u_M|^2\ d\xi}{\left(\int_{\mathbb{R}^N}|x|^{-b_{a,q_k}}|u_M|^{q_k}\ d\xi\right)^{2/q_k}} \ .$$

We are going to prove by direct computation that $R_a < R_{2-k}$ for $a < 2-k$, hence

$$S_{a,q_k}(2) \le R_a < R_{2-k} = S_{2-k,q_k}(2).$$

Since $S_{2-k,q_k}(2) \le S^X_{2-k,q_k}$, and since the map $a \to S^X_{2-k,q_k}$ decreases for $a < 2-k$, this will lead to conclude the proof.

To compute R_a we set $r=|x|$ and $s=|y|$ and we notice that

$$\begin{aligned}\int_{\mathbb{R}^N}|x|^a|\nabla u_M|^2 &= C_{N,k}\,(N-k)^2\int_0^{+\infty}r^{a+k-1}dr\int_0^{+\infty}\frac{s^{N-k-1}}{((1+r)^2+s^2)^{N-k+1}}\,ds\\ &= C_{N,k}\,(N-k)^2\,\Gamma\int_0^{+\infty}\frac{r^{a+k-1}dr}{(1+r)^{N-k+2}}\,,\end{aligned}$$

where $C_{N,k}=\omega_k\omega_{N-k}$ and

$$\Gamma:=\int_0^{+\infty}\frac{t^{N-k-1}dt}{(1+t^2)^{N-k+1}}\,.$$

Therefore one gets, via integration by parts,

$$\int_{\mathbb{R}^N}|x|^a|\nabla u_M|^2\,d\xi=C_{N,k}\,\Gamma(N-k)^2\,\frac{a+k-1}{N-2k+2-a}\int_0^{+\infty}\frac{r^{a+k-2}dr}{(1+r)^{N-k+2}}\,. \tag{4.2.3}$$

Notice that for $q=q_k$ it turns out that $b_{a,q_k}=N-q_k\frac{N-2+a}{2}=-\frac{aq_k}{2}+\frac{N-2k+2}{N-k}$. Therefore we are lead to compute

$$\begin{aligned}\int_{\mathbb{R}^N}|x|^{-b_{a,q_k}}|u_M|^{q_k}\,d\xi &= C_{N,k}\int_0^{+\infty}r^{\frac{aq_k}{2}+(k-2)\frac{N-k+1}{N-k}}dr\,\cdot\\ &\quad\cdot\int_0^{+\infty}\frac{s^{N-k-1}}{((1+r)^2+s^2)^{N-k+1}}ds\\ &= C_{N,k}\,\Gamma\,\Phi_a\,,\end{aligned} \tag{4.2.4}$$

where

$$\Phi_a:=\int_0^{+\infty}\frac{r^{\frac{a+k-2}{2}q_k}dr}{(1+r)^{N-k+2}}\,.$$

Now we use Hölder inequality (with conjugate exponents $q_k/2$ and $N-k+1$) to estimate

$$\int_0^{+\infty}\frac{r^{a+k-2}dr}{(1+r)^{N-k+2}}\le\left(\int_0^{+\infty}\frac{dr}{(1+r)^{N-k+2}}\right)^{\frac{1}{N-k+1}}\left(\int_0^{+\infty}\frac{r^{\frac{a+k-2}{2}q_k}dr}{(1+r)^{N-k+2}}\right)^{\frac{2}{q_k}}\,.$$

Therefore

$$\int_{\mathbb{R}^N}|x|^a|\nabla u_M|^2\,d\xi\le C_{N,k}\,\Gamma(N-k)^2\,\frac{a+k-1}{N-2k+2-a}\left(\frac{1}{N-k+1}\right)^{\frac{1}{N-k+1}}\,(\Phi_a)^{\frac{2}{q_k}}\,. \tag{4.2.5}$$

From (4.2.3) and (4.2.4) we infer that

$$S_{2-k,q_k}(2)=R_{2-k}=(N-k)\left[\frac{C_{N,k}\,\Gamma}{N-k+1}\right]^{\frac{1}{N-k+1}}\,. \tag{4.2.6}$$

On the other hand, from (4.2.5) we get also

$$\begin{aligned} S_{a,q_k}(2) \le R_a &\le \left[\frac{C_{N,k}\Gamma}{N-k+1}\right]^{\frac{1}{N-k+1}} \frac{(N-k)^2(a+k-1)}{N-2k+2-a} \\ &= S_{2-k,q_k}(2)\frac{(N-k)(a+k-1)}{N-2k+2-a} \ . \end{aligned}$$

The conclusion follows from Lemma 4.1.4, since the map $a \to \frac{a+k-1}{N-2k+2-a}$ is strictly increasing. □

It is now convenient to distinguish the case $q < 2^*$ from the limiting case $q = 2^*$.

4.2.1 Multiplicity for $q < 2^*$

The next corollary is an immediate consequence of Lemma 4.2.4 and of Theorem 4.2.1.

Corollary 4.2.6 *Assume $1 \le k < N$, $N \ge 3$ and $q \in (2, 2^*)$. Then there exists $\varepsilon > 0$ such that if*

$$(N-k)\frac{2}{q} - (N-2) < a < (N-k)\frac{2}{q} - (N-2) + \varepsilon \ ,$$

then problem (4.0.1) has at least two distinct (modulo G_k) entire classical solutions.

Next we point out an immediate corollary to Lemma 4.2.5 where $q = q_k$ is given by (4.2.1).

Corollary 4.2.7 *Let $2 \le k < N$, $N > 2(k-1)$, $1 - k + \frac{1}{N-k+1} < a < 2 - k$, and let q_k be as in (4.2.1). Then problem (4.0.1) has at least two distinct (modulo G_2) entire solutions*

$$u^X \in X^{1,2}(\mathbb{R}^N; |x|^a d\xi) \ , \quad u^D \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a \, d\xi) \setminus X^{1,2}(\mathbb{R}^N; |x|^a d\xi) \ .$$

Finally, we focus our attention on the case $k = 1$, when the singular set $\{x = 0\}$ is an hyperplane that disconnects the domain. Notice that indeed a larger noncompact group G_1 of invariances acts on problem (4.0.1). More precisely, transforms in G_1 depend on $2N$ parameters, and are of the form

$$u(x,y) \longrightarrow \left(T(\tau_-, \tau_+, \eta_-, \eta_+)u\right)(x,y) := \begin{cases} \tau_-^{\frac{N-2+a}{2}} u(\tau_- x, \tau_- y + \eta_-) & \text{if } x < 0 \ , \\ \tau_+^{\frac{N+2+a}{2}} u(\tau_+ x, \tau_+ y + \eta_+) & \text{if } x > 0 \ , \end{cases}$$

for $\tau_-, \tau_+ \in (0, +\infty)$, and for $\eta_-, \eta_+ \in \mathbb{R}^{N-k}$. In other words, dilations in ξ and translations in y can be made independently for $x < 0$ and $x > 0$, so that the equation in (4.0.1) is still invariant. Essentially the same remark has been made by Catrina and Wang in [30] for an O.D.E. involving spherically symmetric weights.

By Theorem 5.2.4 at page 63 (see also [48]), we get that $u^D \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is cylindrically symmetric whenever u^D achieves the best constant $S_{0,q}(2)$. On the other hand, by Lemma 1.2 in [69], if $u^X \in X^{1,2}(\mathbb{R}^N; d\xi)$ achieves $S^X_{0,q}$, then the support of u^X is contained in a half-space, and hence it cannot achieve $S_{0,q}(2)$. This proved the next result.

Theorem 4.2.8 *Assume $k = 1$, $N \geq 3$, and $2_* := \frac{2(N-1)}{N-2} < q < 2^*$. Then problem*

$$\begin{cases} -\Delta u = |x|^{-b_q}\ |u|^{q-2}\, u & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u \geq 0\ , \end{cases}$$

with $b_q = N - q\frac{N-2}{2}$, has at least two distinct (modulo G_1) entire classical solutions:

$u^X \in \mathcal{D}^{1,2}((0, +\infty) \times \mathbb{R}^{N-1})$, with $u^X(x, y) \equiv 0$ for $x < 0$,

$u^D \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, with $u^D(x, y) = u^D(-x, y)$.

4.2.2 Existence and multiplicity for $q = 2^*$

In this section we deal with the limiting case $q = 2^*$. We study problem

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{\frac{Na}{N-2}} |u|^{2^*-2}\, u & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u \geq 0\ . \end{cases} \tag{4.2.7}$$

Theorem 4.2.1 provides the existence of a solution to (4.2.7) if

$$S_{a,2^*}(2) < S\ . \tag{4.2.8}$$

Notice that a first set of sufficient conditions for (4.2.8) can be easily obtained from Lemma 4.1.5. By the same argument and by the symmetry result in [48] (see also next chapter, Theorem 5.2.5 at page 63) one can prove the following result (see also [85] for existence).

Theorem 4.2.9 *Assume $N \geq 4$ and $k \neq 2$. Then the infimum $S_{2-k,2^*}(2)$ is achieved by an entire solution u to*

$$\begin{cases} -\mathrm{div}(|x|^{2-k} \nabla u) = |x|^{-N\frac{k-2}{N-2}}\ |u|^{2^*-2}\, u & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u \geq 0\ . \end{cases}$$

Moreover u is symmetric: $u(x, y) = u(|x|, |y|)$, and decreasing in the $|y|$-variable.

Conversely for $k = 1$, $N = 3$ and $a = 1$ we have the following non-existence result.

Theorem 4.2.10 *Assume $k = 1$ and $N = 3$. Then the infimum $S_{1,6}(2)$ is not achieved.*

Proof. By contradiction, we assume that the infimum $S_{1,6}(2)$ is achieved by a map $u \in \mathcal{D}^{1,2}(\mathbb{R}^3; |x|^1\, d\xi)$. By Lemma 3.4.2 at page 39, u is a positive entire solution to

$$\begin{cases} -\mathrm{div}(|x|^1 \nabla u) = |x|^3\; |u|^4\, u & \text{in } (0, +\infty) \times \mathbb{R}^2\,, \\ u = 0 & \text{on } \{0\} \times \mathbb{R}^2\,. \end{cases}$$

Set $v := |x|^{\frac{1}{2}}\, u$. Then Lemma 3.3.1 at page 37 and direct computations imply that $v \in L^6(\mathbb{R}^3_+)$ is a positive solution to

$$\begin{cases} -\Delta v = \frac{1}{4}|x|^{-2}\, v + |v|^4\, v & \text{in } (0, +\infty) \times \mathbb{R}^2\,, \\ v = 0 & \text{on } \{0\} \times \mathbb{R}^2\,. \end{cases}$$

This contradicts the non-existence result in [66], Section 6 (see also [15]). □

From now on we take $a < 2-k$. We recall that $S^X_{a,2^*}$ is achieved by Theorem 4.1.2 if $S^X_{a,2^*} < S$. Thus, besides (4.2.8), that gives existence, we are lead to investigate if it may happen that

$$S_{a,2^*}(2) < S^X_{a,2^*} < S\,. \tag{4.2.9}$$

Indeed, (4.2.9) would give multiplicity for (4.2.7). The aim of this section is to estimate from above $S_{a,2^*}(2)$ in order to find sufficient conditions for (4.2.8) or for (4.2.9).

Concerning (4.2.8), by Theorem 3.2.2 at page 35, Lemma 4.1.5 and Theorem 4.2.9, we obtain the following result.

Theorem 4.2.11 (existence) *Assume $(2-N)\frac{k}{N} < a \le 2-k$. If $k = 1$ and $N = 3$ assume in addition that $a \le 0$. Then problem (4.2.7) has at least an entire solutions u^D that achieves the best constant $S_{a,2^*}(2)$.*

We can get new sufficient conditions for (4.2.9) in the special case $N = 2(k-1)$. Indeed, in this case the exponent q_k defined in (4.2.1) coincides with the critical exponent 2^*. Therefore, the arguments and the computations of Lemma 4.2.5, together with the existence Theorem 4.2.9 and the uniqueness result in [66] lead to the following result.

Lemma 4.2.12 *Inequality (4.2.9) holds true if $3 \le k < N$, $N = 2(k-1)$ and $-\frac{N+2}{2^*} < a < -\frac{N}{2^*}$.*

We summarize here the multiplicity results known up to now in the limiting critical case.

Theorem 4.2.13 (multiplicity) *Assume* $(2-N)\frac{k}{N} < a < 2-k$. *Then problem (4.2.7) has two distinct entire solutions:*

$$u^D \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \setminus L^2(\mathbb{R}^N; |x|^{a-2} d\xi)\ ,\ u^X \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$$

if one of the following conditions is satisfied:

- $k = \frac{N+2}{2}$ *with* $k \geq 3$;
- $k = 3$, $N \geq 7$ *and* $a = -2$;
- $k = 3$, $N = 5, 6$, *or* $k \geq 4$ *and* a *is close enough to* $(2-N)k/N$.

The following figures show the behaviour of $S_{a,2^*}(2)$ (that is indicated by S^D) and of $S^X_{a,2^*}$ in some principal cases.

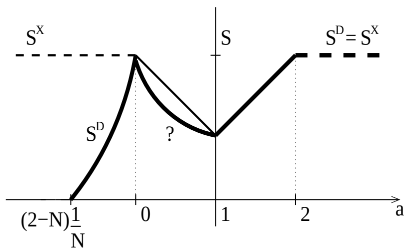

Fig.2 $k = 1$ and $N \geq 4$.

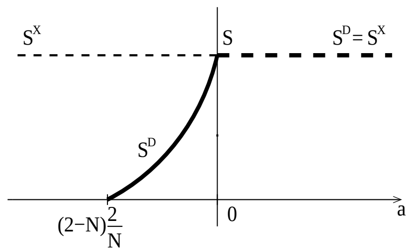

Fig.3 $k = 2$.

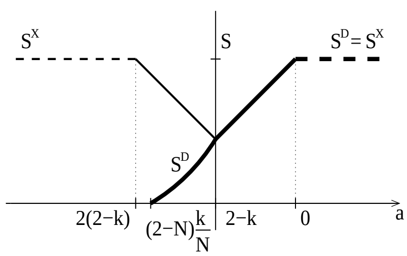

Fig.4 $k = \frac{N+2}{2}$ with $k \geq 3$.

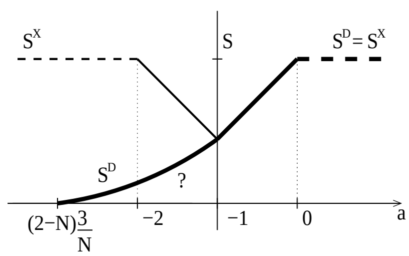

Fig.5 $k = 3$ and $N \geq 7$.

Remark 4.2.14 We do not know whether $S_{a,2^*}(2) < S^X_{a,2^*} = S$ holds true if $k = 1$, $N = 3$ and $a \in (0, 1)$. If $k = 1$, $N \geq 4$ and $a \in (0, 1)$ we know that $S_{a,2^*}(2), S^X_{a,2^*}$ are both achieved, but we do not know if $S_{a,2^*}(2) < S^X_{a,2^*}$. The same question is still open if $k \geq 3$ and $a \in (2(2-k), 2-k)$, unless $N = 2(k-1)$.

4.3 Existence results for (4.0.6)

We start recalling that for $a = 0$ inequality (4.1.1) takes the form

$$C\left(\int_{\mathbb{R}^N} |x|^{-b_q}|v|^q \, d\xi\right)^{2/q} \leq \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2\,\right] d\xi \tag{4.3.1}$$

for any $v \in C^\infty_c(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$ and $b_q = N - q\frac{N-2}{2}$.

Thanks to (4.3.1), for $\lambda \leq \lambda_1(0)$ we can define the Hilbert space

$$X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$$

as the closure of maps $v \in C^\infty_c(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$ with respect to the scalar product

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v - \lambda|x|^{-2}u\,v\right] d\xi\ .$$

In general, $X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$ contains the space $X^{1,2}(\mathbb{R}^N; d\xi)$. More precisely, for $\lambda < \lambda_1(0)$,

$$X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) = X^{1,2}(\mathbb{R}^N; d\xi) = \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2}d\xi)$$

by the *classical Hardy inequality* (1.3.6) at page 15.

For $\lambda = \lambda_1(0)$, $X^{1,2}(\mathbb{R}^N; d\xi) \subset X_{\lambda_1(0)}(\mathbb{R}^k \times \mathbb{R}^{N-k})$. In particular, if $k = 2$, $X_0(\mathbb{R}^2 \times \mathbb{R}^{N-2}) = \mathcal{D}^{1,2}(\mathbb{R}^N)$, while $\mathcal{D}^{1,2}(\mathbb{R}^N) = X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) \cap L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ if $k \geq 3$.

In Section 4.3.1 we will study the existence of extremals for the inequality (4.3.1). For future convenience we point out a lemma.

Lemma 4.3.1 *The linear operator* $L_{(k-2)}v := |x|^{\frac{k-2}{2}}v$ *is a bi-continuous isomorphism between* $X_{\lambda_1(0)}(\mathbb{R}^k \times \mathbb{R}^{N-k})$ *and* $\mathcal{D}^{1,2}(\mathbb{R}^N; |x|^{2-k}d\xi)$*. Moreover,*

$$\int_{\mathbb{R}^N} |x|^{2-k}|\nabla(L_{(k-2)}v)|^2 \, d\xi = \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2\right] \, d\xi \tag{4.3.2}$$

for any $v \in X_{\lambda_1(0)}(\mathbb{R}^k \times \mathbb{R}^{N-k})$.

Proof. An application of the divergence theorem shows that (4.3.2) holds for any $v \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$. We have to prove that $C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N; |x|^{2-k} d\xi)$. This follows from Lemma 3.3.1 at page 37 for $p = 2$ and $a = 2 - k$. □

In this section we deal with classical solutions to problem (4.0.6) under the assumption

$$\lambda \leq \lambda_1(0) = \left(\frac{k-2}{2}\right)^2 .$$

In particular, we are looking for solutions u that satisfy $\int_{\mathbb{R}^N} |x|^{-b_q} |u|^q \, d\xi < +\infty$, even if they might be singular, in the sense that one or both the integrals

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, d\xi \ , \quad \int_{\mathbb{R}^N} |x|^{-2} |u|^2 \, d\xi$$

might be unbounded. We distinguish the cases $\lambda = \lambda_1(0)$ and $\lambda < \lambda_1(0)$.

4.3.1 Existence for $\lambda = \lambda_1(0)$

We study problem

$$\begin{cases} -\Delta v = \left(\frac{k-2}{2}\right)^2 |x|^{-2} \, v + |x|^{-b_q} \, |v|^{q-2} \, v & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ v \geq 0, \end{cases} \tag{4.3.3}$$

where $N \geq 3$, $q \in (2, 2^*]$, and $b_q = N - q\frac{N-2}{2}$. We are going to give an alternative proof of a result by Tertikas and Tintarev [85].

Theorem 4.3.2 *Problem (4.3.3) has a cylindrically symmetric classical solution v such that*

$$\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi \ = \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0) |x|^{-2} |v|^2 \ \right] \, d\xi < +\infty \ ,$$

if

$$q \in (2, 2^*) \quad \text{ or } \quad q = 2^*, \, N \geq 4 \text{ and } k \neq 2 \, .$$

Proof. Our aim is to prove that the infimum

$$S_q^{\lambda_1(0)} := \inf_{\substack{v \in X_{\lambda_1(0)}(\mathbb{R}^k \times \mathbb{R}^{N-k}) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0) |x|^{-2} |v|^2 \ \right] d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi \right)^{2/q}} \tag{4.3.4}$$

is achieved by a map $v \in X_{\lambda_1(0)}(\mathbb{R}^k \times \mathbb{R}^{N-k})$ that solves problem (4.3.3). This can be done directly, arguing as in Chapter 3, Theorem 3.0.9 at page 29, or it can be obtained as a corollary to Theorem 4.2.1 at page 47. Indeed, by Lemma 4.3.1 it is clear that the minimization problems $S_q^{\lambda_1(0)}$ and $S_{2-k,q}(2)$ are completely equivalent, in the sense that $u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^{2-k}d\xi)$ achieves $S_{2-k,q}(2)$ if and only if $v = L_{(2-k)}u$ achieves $S_q^{\lambda_1(0)}$. Then Theorems 4.2.1 and 4.2.9 easily lead to existence. The symmetry follows from Theorem 5.2.5 at page 63 ([48]). □

Remark 4.3.3 The existence result in Theorem 4.3.2 does not hold for spherically symmetric weights (case $k = N$). Indeed, it has been proved in [17] that for $k = N$ and $\lambda = \left(\frac{N-2}{2}\right)^2$ problem (4.0.6) has no distributional solutions in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ (see also Remark 4.1.1).

4.3.2 Existence of singular solutions for $\lambda < \lambda_1(0)$

Here we study problem (4.0.6) in case $\lambda < \lambda_1(0) = \left(\frac{k-2}{2}\right)^2$. A first solution $v^X \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ can be find by studying the minimization problem for the infimum $S^X_{a,q}$ in (4.1.3). This was done in [69], Theorems 1 and 2. More precisely, $S^X_{a,q}$ is achieved, provided that $q < 2^*$; if $q = 2^*$ then existence is proved if in addition $\lambda > 0$, $N \geq 4$ and $k \neq 2$.

Our aim is to use here the results in Sections 4.2 to find new classical solutions $v \in L^q(\mathbb{R}^N; |x|^{-b_q}d\xi)$ to (4.0.6) that are *singular* in the sense that $\int_{\mathbb{R}^N} |x|^{-2}|v|^2 \, d\xi$ diverges. Notice that Theorem 4.2.8 already provides the existence of a solution $u \notin L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ when $k = 1$, $\lambda = 0$ and $q \in (2_*, 2^*)$.

To handle the other cases we use a simple trick: assume

$$\lambda_1(0) - \left(\frac{N-k}{N}\right)^2 < \lambda < \lambda_1(0) \ , \qquad \frac{2(N-k)}{N-k-2\sqrt{\lambda_1(0)-\lambda}} < q \leq 2^* \ , \tag{4.3.5}$$

and define

$$a = a_{k,\lambda} := 2 - k - 2\sqrt{\lambda_1(0) - \lambda} \ .$$

Notice that with this choice, assumptions (4.0.3) on a and q are satisfied by (4.3.5). Moreover, it turns out that $b_{a,q} = N - q\frac{N-2+a}{2} = b_q - \frac{qa}{2}$, accordingly with (4.0.2). Assume that u is a solution to (4.0.1) with respect to this choice of the parameters a, b_q and with respect to the same q. Then the map

$$v = L_a u := |x|^{\frac{a}{2}} u$$

is a $C^\infty(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$-solution to (4.0.6). In addition, if $u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi)$, then

$$\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi = \int_{\mathbb{R}^N} |x|^{-b_{a,q}} |u|^q \, d\xi < +\infty \ .$$

On the other hand, since $a < 2-k$, then it might happen that $u \notin L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$. If this is the case then $\int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi = +\infty$. By Hardy inequality (1.3.6) at page 15, in case $k > 2$ we have also that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi = +\infty \ .$$

In conclusion, trough the functional change L_a, we can construct a singular solution to (4.0.6) starting from any solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a \, d\xi) \setminus X^{1,2}(\mathbb{R}^N; |x|^a d\xi)$ of problem (4.0.1). Thus, the results in Subsection 4.2.1 lead to the following existence result.

Theorem 4.3.4 *Assume $1 \le k < N$, $N \ge 3$ and $q \in (2, 2^*]$. Then problem (4.0.6) has a solution $v_\infty \in L^q(\mathbb{R}^N; |x|^{-b_q} d\xi)$ that satisfies*

$$\int_{\mathbb{R}^N} |x|^{-2} v_\infty^2 \, d\xi = +\infty \ ,$$

provided that one of the following conditions is satisfied:

i) $q \le 2^*$ *and* $q\left(N - k - 2\sqrt{\lambda_1(0) - \lambda}\right)$ *is close enough to* $2(N-k)$*;*

ii) $q = q_k = \frac{2(N-k+1)}{N-k}$, $2 \le k \le \frac{N+2}{2}$ *and* $\lambda_1(0) - q_k^{-2} < \lambda < \lambda_1(0)$*;*

iii) $q = 2^*$, $k = 1, 2$ *and* $\lambda_1(0) - \left(\frac{N-k}{N}\right)^2 < \lambda < 0$*;*

iv) $q = 2^*$, $k = 3$, $N \ge 7$ *and* $\frac{1}{4} - \left(\frac{N-3}{N}\right)^2 < \lambda \le 0$*;*

We remark also the following immediate consequence to *iv)* of Theorem 4.3.4, and to Theorem 5.2.5 at page 63 ([48]).

Corollary 4.3.5 *Assume $N \ge 7$. Then the equation*

$$-\Delta v = |v|^{2^*-2} v \qquad \text{in } \mathbb{R}^3_0 \times \mathbb{R}^{N-3}$$

has a positive smooth cylindrically symmetric solution v_∞ such that

$$\int_{\mathbb{R}^N} |v_\infty|^{2^*} \, d\xi < S^{N/2} \ , \quad \int_{\mathbb{R}^N} |\nabla v_\infty|^2 \, d\xi = +\infty \ .$$

Chapter 5

Symmetry breaking of extremals

In this chapter we study the following problem on $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$ with $k < N$ and $N \geq 3$.

$$\begin{cases} -\Delta v = \lambda |x|^{-2}\, v + |x|^{-b_q}\, |v|^{q-2}\, v \quad \text{in } \mathbb{R}^k_0 \times \mathbb{R}^{N-k} \\ v \geq 0\ , \end{cases} \tag{5.0.1}$$

where $q > 2$, $\lambda \leq \lambda_1(0) = \left(\frac{k-2}{2}\right)^2$ and $b_q = N - q\frac{N-2}{2}$. We are mainly interested in symmetry properties of solutions to (5.0.1) and we present here some results obtained in [48].

A large number of bibliographical references for (5.0.1) is available in case $k = N$: we quote for example [3], [27], [30], [38], [84] and references there-in. In particular, in [30] and [38] one can find a careful analysis on symmetry breaking of ground state solutions.

Concerning existence in case $k < N$ we cite [10], [29], [66], [69] and [85]. Existence results can be found also in [47] (see also Chapters 3 and 4). Symmetry properties of weak entire solutions were proved in [64], under the assumptions $k \geq 2$, $\lambda = 0$ and $q \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$ is the standard critical Sobolev exponent in dimension N.

As noticed in [29] and in [66], solutions that are radially symmetric in the x-variable receive importance with regard to certain elliptic equations on the $n =$

$N-k+1$-dimensional hyperbolic space $\mathbb{H}^n$. More precisely, if $v(x,y) = v(|x|,y)$ solves (5.0.1), then the transform $u(r,y) := r^{\frac{N-2}{2}} v(r,y)$ gives a solution to

$$-\Delta_{\mathbb{H}^n} u = \mu u + |u|^{q-2}\, u \ . \tag{5.0.2}$$

Here, $\Delta_{\mathbb{H}^n}$ is the Laplace-Beltrami operator on $\mathbb{H}^n$ and the parameter μ is given by

$$\mu = \lambda + \frac{(N-k)^2 - (k-2)^2}{4} \ .$$

We refer to [29] and to [66] for a discussion on the relevances between equation (5.0.2) and some significant problems in hyperbolic geometry, Yamabe-type equations of Heisenberg type, Grushing-type equations.

Motivated by these considerations, in this chapter we address our attention towards *cylindrically symmetric* solutions u (see the following definition from [65] and [64]).

Definition 5.0.6 A smooth map v on $\mathbb{R}^k_0 \times \mathbb{R}^{N-k}$ is *cylindrically symmetric* if

i) for any choice of $y \in \mathbb{R}^{N-k}$, $v(\cdot, y)$ is symmetric decreasing in $\mathbb{R}^k$;

ii) there exists $y_0 \in \mathbb{R}^{N-k}$ such that, for any choice of $x \in \mathbb{R}^k_0$, $v(x,\cdot)$ is symmetric decreasing about y_0 in $\mathbb{R}^{N-k}$.

5.1 Setting

We start recalling the *Hardy-Sobolev-Maz'ya inequality*, that is peculiar to the cylindrical case $k < N$, for $a = 0$ and $\Omega = \mathbb{R}^k_0 \times \mathbb{R}^{N-k}$ (see Theorem 2.2.5 at page 22). Assume $q \in (2, 2^*]$, with $N \geq 3$, and b_q as in problem (5.0.1), then there exists a constant $C = C(q,k) > 0$ such that

$$C \left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi \right)^{2/q} \leq \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0) |x|^{-2} |v|^2 \right] \, d\xi \, , \tag{5.1.1}$$

for any $v \in C^\infty_c(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$. We recall that, thanks to (5.1.1), for $\lambda \leq \lambda_1(0)$ we have defined in Section 4.3 the Hilbert space $X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$ as the closure of maps $v \in C^\infty_c(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$ with respect to the scalar product

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v - \lambda |x|^{-2} u\, v \right] d\xi \ .$$

In general, $X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$ contains the space $X^{1,2}(\mathbb{R}^N; d\xi)$. More precisely,

$$X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) = X^{1,2}(\mathbb{R}^N; d\xi) = \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} d\xi)$$

for $\lambda < \lambda_1(0)$, by the *classical Hardy inequality* (1.3.6) at page 15. For $\lambda = \lambda_1(0)$ it turns out that $X_0(\mathbb{R}^2 \times \mathbb{R}^{N-2}) = \mathcal{D}^{1,2}(\mathbb{R}^N)$ if $k = 2$, while $\mathcal{D}^{1,2}(\mathbb{R}^N) = X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) \cap L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ if $k \geq 3$.
For future convenience we define also the Hilbert space

$$X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k})$$

as the closure in $X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$ of maps $v \in C_c^\infty(\mathbb{R}_0^k \times \mathbb{R}^{N-k})$, such that $v(x,y) = v(|x|,y)$.

Now we are in position to give the following definitions (compare with [66]).

Definition 5.1.1 A classical solution v to (5.0.1) is *entire* if $v \in X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$.

Definition 5.1.2 An entire solution v to (5.0.1) is a *ground state solution* if v achieves the best constant

$$S_q^\lambda = S_q^\lambda(k,N) := \inf_{\substack{v \in X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left[|\nabla v|^2 \ - \lambda |x|^{-2} |v|^2 \right] \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \ d\xi \right)^{2/q}} \ . \tag{5.1.2}$$

Notice that S_q^λ is positive by (5.1.1) and, for $\lambda \neq \lambda_1(0)$, it generalizes $S_q^{\lambda_1(0)}$ defined in (4.3.4) at page 56. The existence of ground state solutions were proved in [69], Theorem 1, in case $\lambda < \lambda_1(0)$, and in [85], in case λ coincides with the Hardy constant. If $q = 2^*$ and $\lambda \neq 0$, one needs the additional assumption

$$0 < \lambda < \lambda_1(0)\,, \quad S_{2^*}^\lambda < S\ .$$

Remark 5.1.3 If $k = 1$, the singular set $\{x = 0\}$ is an hyperplane that disconnects the domain. In this case for $\lambda \leq 1/4$ it is convenient to introduce, in a similar way, the space $X_\lambda(\mathbb{R}_+ \times \mathbb{R}^{N-1})$. It turns out that

$$S_q^\lambda(1,N) = \inf_{\substack{v \in X_\lambda(\mathbb{R}_+ \times \mathbb{R}^{N-1}) \\ v \not\equiv 0}} \frac{\int_0^{+\infty} \int_{\mathbb{R}^{N-1}} \left[|\nabla v|^2 \ - \lambda |x|^{-2} |v|^2 \right] \, d\xi}{\left(\int_0^{+\infty} \int_{\mathbb{R}^{N-1}} |x|^{-b_q} |v|^q \ d\xi \right)^{2/q}}\ ,$$

by Lemma 1.2 in [69] (see also Lemma 3.4.2 at page 39).

5.2 Cylindrical solutions

In our first result we consider q smaller than the critical Sobolev exponent in $\mathbb{R}^{N-k+1}$. We prove the existence of entire solutions.

Theorem 5.2.1 *Assume $2 \le k < N$, $\lambda \le \lambda_1(0)$ and $2 < q < 2^*_{N-k+1} := \frac{2(N-k+1)}{N-k-1}$. Then problem (5.0.1) has a cylindrically symmetric entire solution.*

By [66], Theorem 1.1 and Section 6, the bound on λ is a necessary condition for existence. Notice that we can allow q to be supercritical, since $2^* < 2^*_{N-k+1}$. Theorem 5.2.1 is an immediate consequence of the following Lemma.

Lemma 5.2.2 *Assume $2 \le k < N$, $\lambda \le \lambda_1(0)$ and $2 < q < 2^*_{N-k+1}$. Then the infimum*

$$S^\lambda_{q,\,\mathrm{cyl}} := \inf_{\substack{v \in X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k}) \\ v \not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left[|\nabla v|^2 \; - \lambda |x|^{-2} |v|^2 \right] \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \; d\xi \right)^{2/q}}$$

is achieved on $X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k})$.

Proof. For $\lambda \le \lambda_1(0)$ set

$$\mu(\lambda) := \frac{1}{4} - \left(\frac{k-2}{2} \right)^2 + \lambda \; .$$

For any smooth map $v = v(|x|, y)$ on $\mathbb{R}^k_0 \times \mathbb{R}^{N-k}$, we define $Lv \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^{N-k})$ by setting

$$Lv(s, y) = s^{\frac{k-1}{2}} v(s, y) \; .$$

Now we claim that L extends to a bijective isometry

$$L : X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k}) \longrightarrow X_{\mu(\lambda)}(\mathbb{R}_+ \times \mathbb{R}^{N-k}) \; .$$

This is readily proved, since for any $v \in C^\infty_c(\mathbb{R}^k_0 \times \mathbb{R}^{N-k})$ radially symmetric in the x-variable it turns out that

$$\int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda |x|^{-2} |v|^2 \right] \; d\xi = \omega_k \int_0^{+\infty} \int_{\mathbb{R}^{N-k}} \left[|\nabla Lv|^2 - \mu(\lambda) s^{-2} |Lv|^2 \right] \; ds dy \; .$$

In particular

$$S^\lambda_{q,\,\mathrm{cyl}} = \omega_k^{\frac{q-2}{q}} \; S^{\mu(\lambda)}_q (1, N)$$

(compare with Remark 5.1.3). Moreover we have that v achieves $S^{\lambda}_{q,\,\mathrm{cyl}}$ if and only if Lv achieves $S_q^{\mu(\lambda)}(1,N)$. On the other hand, when $q < 2^*_{N-k+1}$ the existence of minimizers for $S_q^{\mu(\lambda)}(1,N)$ was proved in [69] for $\mu(\lambda) < 1/4$ (hence, for $\lambda < \lambda_1(0)$), and in [85] for $\mu(\lambda) = 1/4$ (hence, for $\lambda = \lambda_1(0)$) . After proving the existence of an entire solution $v = v(|x|, y)$, the cylindrical symmetry of v follows as in [64] (see also proof of Theorem 5.2.5, Step 3). □

We complete Theorem 5.2.1 with a result in case q equals the critical Sobolev exponent in $\mathbb{R}^{N-k+1}$.

Theorem 5.2.3 *Assume $N \geq k+2$ and $q = 2^*_{N-k+1}$.*

If $N = k+2$ then (5.0.1) does not have any cylindrically symmetric entire solution v.

If $N \geq k+3$ then problem (5.0.1) has a cylindrically symmetric entire solution v if in addition

$$\lambda_1(0) - \frac{1}{4} < \lambda \leq \lambda_1(0) \ .$$

Proof. The non-existence result was proved in [66], Section 6. For existence in case $N \geq k+3$, see [69] (for $\lambda < \lambda_1(0)$) and [85] (for $\lambda = \lambda_1(0)$). □

Now we want to investigate the symmetry properties of solutions to (5.0.1). In case $k \geq 2$ and $q \in (2, 2^*)$, Mancini, Fabbri and Sandeep adopted in [64] the moving plane method to show that nonnegative entire solutions to

$$-\Delta v = |x|^{-b_q}\ v^{q-1} \quad \text{in } \mathbb{R}^N \tag{5.2.1}$$

are cylindrically symmetric. As a matter of fact, their arguments work as well in case $k = 1$. We omit the proof of the next result.

Theorem 5.2.4 *Assume $k = 1$, $N \geq 3$, and $2_* = \frac{2(N-1)}{N-2} < q < 2^*$. Then every weak nonnegative solution $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ to (5.2.1) is cylindrically symmetric.*

Existence is proved in [47], Theorem 0.1, under the same bounds on q as in Theorem 5.2.4 (see also Chapter 3, Corollary 3.4.1 for $p = 2$ at page 38).

In case $k \geq 2$ we are able to prove a stronger result.

Theorem 5.2.5 *Assume $2 \leq k < N$, $q \in (2, 2^*]$ and $0 \leq \lambda \leq \lambda_1(0)$. Let v be a classical solution to (5.0.1) such that $v \in L^q(\mathbb{R}^N; |x|^{-b_q} d\xi)$. If $\lambda = 0$ and $q = 2^*$ assume in addition that v has a nonremovable singularity on $\{x = 0\}$. Then v is cylindrically symmetric.*

Notice that we only require a summability assumption on v. No assumption on ∇v is needed, so that v might be not entire (compare with Theorem 3.1 in [84] for the case $k = N$). The existence of singular, non-entire solutions $v \in L^q(\mathbb{R}^N; |x|^{-b_q} d\xi)$ to (5.0.1) was proved in [49], Section 4.2, under suitable assumptions on the parameters involved (see also Subsection 4.3.2 at page 57). Even if Theorem 5.2.5 improves Theorem 2.1 of [64] to include singular solutions, one only needs to upgrade the arguments in [64] through a careful use of suitable cut-off functions. We write all the details for the sake of completeness.

Proof of Theorem 5.2.5. For $s > 0$ and $\xi = (x, y) \in \mathbb{R}^N$ we set $\xi^s := (x^s, y)$, where $x^s = (2s - x_1, x_2, ..., x_k)$. Thus, ξ^s is the reflection of ξ with respect to the hyperplane $\{(s, x_2, ..., x_k, y) \mid x_i \in \mathbb{R},\ y \in \mathbb{R}^{N-k}, i = 2, ..., k\}$. As in [64], for any

$$\xi \in \Omega_s := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid x_1 > s\ \}\ ,$$

we define

$$v_s(\xi) := v(\xi^s)\ , \qquad w_s := v_s - v \in C^\infty(\Omega_s \setminus \Sigma_{2s})\ ,$$

where

$$\Sigma_{2s} := \{(2s, 0, ..., 0, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid y \in \mathbb{R}^{N-k}\ \}\ .$$

Notice that $w_s \equiv 0$ on $\partial\Omega_s$.

Step 1 First of all we claim that $w_s \geq 0$ in Ω_s for s large enough. In order to prove this, we consider $s > 0$ in such a way that

$$\int_{\Omega_s} |x|^{-b_q} |v|^q\ d\xi \leq \varepsilon_0\ , \tag{5.2.2}$$

where $\varepsilon_0 > 0$ satisfies

$$2(q-1)\varepsilon_0^{\frac{q-2}{q}} \leq S_q^\lambda\ . \tag{5.2.3}$$

We recall that S_q^λ is the infimum in (5.1.2). Since $|\xi^s| < |\xi|$ in Ω_s, it turns out that

$$-\Delta w_s - \lambda |x|^{-2} w_s \geq |x|^{-b_q} A(\xi) w_s\ , \tag{5.2.4}$$

pointwise on $\Omega_s \setminus \Sigma_{2s}$, where

$$A(\xi) := \frac{v_s^{q-1} - v^{q-1}}{v_s - v}$$

and

$$0 \leq A(\xi) \leq (q-1)v^{q-2} \quad \text{ on } \{w_s \leq 0\}\ . \tag{5.2.5}$$

As in [64], the idea is to use $w_s^- := \min\{w_s, 0\} \le 0$ as test function for (5.2.4), but, differently from [64], the maps w_s and w_s^- are not smooth enough. Thus we have to use suitable cut-off functions. For $\varepsilon > 0$ small set

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| \le \varepsilon^2 \\ \frac{\log |x|/\varepsilon^2}{|\log \varepsilon|} & \text{if } \varepsilon^2 < |x| < \varepsilon \\ 1 & \text{if } |x| \ge \varepsilon \ . \end{cases} \quad , \quad \tilde\varphi_\varepsilon(x) = \varphi_\varepsilon(x^s) \ . \tag{5.2.6}$$

For any large integer h choose a cut off function $\psi_h \in C_c^\infty(\mathbb{R}^N)$, with

$$\begin{cases} \psi_h(\xi) = 1 & \text{if } |\xi| \le h \\ \psi_h(\xi) = 0 & \text{if } |\xi| \ge 2h \end{cases} \quad , \quad 0 \le \psi \le 1 \ , \quad ||\nabla \psi_h||_\infty \le \frac{4}{h} \ . \tag{5.2.7}$$

We are allowed to use $\tilde\varphi_\varepsilon^2 \psi_h^2 w_s^-$ as test function for (5.2.4) on Ω_s. Set

$$\omega_h^\varepsilon := \tilde\varphi_\varepsilon \psi_h w_s^- \ .$$

After integration by parts and simple computations one gets

$$\int_{\Omega_s} \left(|\nabla \omega_h^\varepsilon|^2 - \lambda |x|^{-2} |\omega_h^\varepsilon|^2\right) d\xi \le \int_{\Omega_s} |x|^{-b_q} A(\xi) |\omega_h^\varepsilon|^2 \ d\xi \ + \int_{\mathbb{R}^N} |\nabla(\tilde\varphi_\varepsilon \psi_h)|^2 |w_s^-|^2 \ d\xi \, . \tag{5.2.8}$$

By (5.1.2), the left hand side in (5.2.8) is bounded from below by

$$S_p^\lambda \left(\int_{\Omega_s} |x|^{-b_q} |\omega_h^\varepsilon|^q \ d\xi\right)^{2/q} .$$

To estimate the right hand side we notice that

$$\begin{aligned} \int_{\Omega_s} |x|^{-b_q} A(\xi) |\omega_h^\varepsilon|^2 \ d\xi \ &\le \ (q-1) \int_{\Omega_s} |x|^{-b_q} |v|^{q-2} |\omega_h^\varepsilon|^2 \ d\xi \\ &\le \ (q-1) \varepsilon_0^{\frac{q-2}{q}} \left(\int_{\Omega_s} |x|^{-b_q} |\omega_h^\varepsilon|^q \ d\xi\right)^{2/q} \end{aligned}$$

by (5.2.5), Hölder inequality and (5.2.2). Comparing with (5.2.8) and with (5.2.3) we infer

$$\frac{1}{2} S_q^\lambda \left(\int_{\Omega_s^{\varepsilon,h}} |x|^{-b_q} |w_s^-|^q \ d\xi\right)^{2/q} \le \int_{\mathbb{R}^N} |\nabla(\tilde\varphi_\varepsilon \psi_h)|^2 |w_s^-|^2 \ d\xi \ , \tag{5.2.9}$$

where

$$\Omega_s^{\varepsilon,h} = \{(x,y) \in \Omega_s \mid |x^s| > \varepsilon \ , \ |\xi| < h \ \} \ .$$

In order to handle the right hand side in (5.2.9), we compute

$$\int_{\mathbb{R}^N} |\nabla(\tilde\varphi_\varepsilon\psi_h)|^2 |w_s^-|^2 \, d\xi \ \le 2(I_1^{\varepsilon,h} + I_2^{\varepsilon,h}),$$

where

$$I_1^{\varepsilon,h} = \int_{\mathbb{R}^N} |\nabla\tilde\varphi_\varepsilon|^2 \psi_h^2 |w_s^-|^2 \, d\xi \ , \quad I_2^{\varepsilon,h} = \int_{\mathbb{R}^N} |\nabla\psi_h|^2 \tilde\varphi_\varepsilon^2 |w_s^-|^2 \, d\xi \ .$$

To estimate the first integral we notice that v is smooth on $\{x \neq 0\}$ and that $|w_s^-| \le v$ on Ω_s. Therefore, since $k \ge 2$,

$$I_1^{\varepsilon,h} \le C_h \int_{\mathbb{R}^k} |\nabla\tilde\varphi_\varepsilon|^2 \, dx \le \frac{C_h}{|\log\varepsilon|^2} \int_{\varepsilon^2}^{\varepsilon} r^{k-3} \, dr \le \frac{C_h}{|\log\varepsilon|} \ ,$$

with constants C_h that depend only on the measure of B^N_{2h} and on the L^∞-norm of v on $B^N_{2h} \cap \Omega_s$. Thus, $I_1^{\varepsilon,h} \to 0$ for h fixed, as $\varepsilon \to 0$. Concerning the second integral, we use Hölder inequality, $b_q \ge 0$ and $|w_s^-| \le v$ on Ω_s to get

$$\begin{aligned} I_2^{\varepsilon,h} \le \int_{\mathbb{R}^N} |\nabla\psi_h|^2 |v|^2 \, d\xi \ &\le \ \left(\int_{B^N_{2h}\setminus B^N_h} |x|^{-b_q} |v|^q \, d\xi\right)^{\frac{2}{q}} \cdot \\ &\cdot \ \left(\int_{\mathbb{R}^N} |\xi|^{\frac{2b_q}{q-2}} |\nabla\psi_h|^{\frac{2q}{q-2}} \, d\xi\right)^{\frac{q-2}{q}} \\ &\le \ C\left(\int_{B^N_{2h}\setminus B^N_h} |x|^{-b_q} |v|^q \, d\xi\right)^{\frac{2}{q}} \ . \end{aligned}$$

Thus $I_2^{\varepsilon,h} \to 0$ as $h \to +\infty$ uniformly in ε, since $v \in L^q(\mathbb{R}^N; |x|^{-b_q} d\xi)$. In conclusion, we have proved that

$$\frac{1}{2} S_q^\lambda \left(\int_{\Omega_s^{\varepsilon,h}} |x|^{-b_q} |w_s^-|^q \, d\xi\right)^{2/q} \le \frac{C_h}{|\log\varepsilon|} + C\left(\int_{B^N_{2h}\setminus B^N_h} |x|^{-b_q} |v|^q \, d\xi\right)^{\frac{2}{q}} \ .$$

Since $|w_s^-| \le v \in L^q(\mathbb{R}^N; |x|^{-b_q} d\xi)$, passing to the limit in (5.2.9), first as $\varepsilon \to 0$ and then as $h \to +\infty$, we get that $w_s \ge 0$ a.e. on Ω_s. More precisely, $w_s > 0$ on $\Omega_s \setminus \Sigma_{2s}$ by (5.2.4) and by the maximum principle.

Step 2 Now we want to prove that v is even with respect the first variable x_1. To this aim, we define $s_0 := \inf A$, where

$$A := \{s > 0 \,|\, v_{\overline{s}} \ge v \text{ in } \Omega_{\overline{s}} \ \text{ for all } \overline{s} > s\} \, .$$

Notice that A is not empty by the previous step. We claim that $s_0 = 0$. By contradiction assume that $s_0 > 0$ and consider $w_{s_0} = v_{s_0} - v$. Obviously it turns out that $w_{s_0} \geq 0$ and $-\Delta w_{s_0} - \lambda|x|^{-2} w_{s_0} \geq 0$ in $\Omega_{s_0} \setminus \Sigma_{2s_0}$. Moreover, $w_{s_0} > 0$ in $\Omega_{s_0} \setminus \Sigma_{2s_0}$ for the maximum principle. Let $\varepsilon > 0$ and choose R, $\delta_0 > 0$ such that

$$(q-1)\left(\int_{\{|\xi|>R\}} |x|^{-b_q}|v|^q \, d\xi\right)^{\frac{q-2}{q}} < 4^{\frac{1-q}{q}} S_q^\lambda \,, \tag{5.2.10}$$

and

$$(q-1)\left(\int_{\{|x_1-s_0|<\delta_0\}} |x|^{-b_q}|v|^q \, d\xi + \int_{\{|x_1-2s_0|<\delta_0\}} |x|^{-b_q}|v|^q \, d\xi\right)^{\frac{q-2}{q}} < 4^{\frac{1-q}{q}} S_q^\lambda \quad . \tag{5.2.11}$$

Let us consider the following compact set in which $w_{s_0} > 0$.

$$K = \{\xi = (x,y) \in \overline{B}_R^N \mid s_0 + \delta_0 \leq x_1 \leq 2s_0 - \delta_0 \;\text{ or }\; x_1 \geq 2s_0 + \delta_0\} \,.$$

Choose $\delta_1 \in (0, \delta_0)$ such that $w_{s_0-\delta} > 0$ in K for any $\delta \in (0, \delta_1)$. Now fix $\delta \in (0, \delta_1)$ and set $s_1 := s_0 - \delta$. We define $\overline{\varphi}_\varepsilon(x) = \varphi_\varepsilon(x^{s_1})$, with φ_ε as in (5.2.6), and $\overline{\omega}_h^\varepsilon := \overline{\varphi}_\varepsilon \, \psi_h \, w_{s_1}^-$, with ψ_h as in (5.2.7) and $w_{s_1}^- = \min\{w_{s_1}, 0\} \leq 0$. We carry on the proof as in the previous step, to get

$$\begin{aligned}
& S_q^\lambda \left(\int_{\Omega_{s_1}} |x|^{-b_q}|\overline{\omega}_h^\varepsilon|^q \, d\xi\right)^{2/q} \\
\leq \; & \left(\int_{\Omega_{s_1}\cap\{w_{s_1}<0\}} |x|^{-b_q} A(\xi)^{\frac{q}{q-2}} \, d\xi\right)^{\frac{q-2}{q}} \left(\int_{\Omega_{s_1}} |x|^{-b_q}|\overline{\omega}_h^\varepsilon|^q \, d\xi\right)^{\frac{2}{q}} \\
+ \; & \int_{\mathbb{R}^N} |\nabla(\overline{\varphi}_\varepsilon \psi_h)|^2 |w_{s_1}^-|^2 \, d\xi \,.
\end{aligned} \tag{5.2.12}$$

By (5.2.5), (5.2.10), (5.2.11) and considering that $w_{s_1} > 0$ in K, we obtain

$$\left(\int_{\Omega_{s_1}\cap\{w_{s_1}<0\}} |x|^{-b} A(\xi)^{\frac{q}{q-2}} \, d\xi\right)^{\frac{q-2}{q}} \leq \frac{S_q^\lambda}{2} \,,$$

then (5.2.12) becomes

$$\frac{S_q^\lambda}{2}\left(\int_{\Omega_{s_1}} |x|^{-b_q}|\overline{\omega}_h^\varepsilon|^q \, d\xi\right)^{2/q} \leq \int_{\mathbb{R}^N} |\nabla(\overline{\varphi}_\varepsilon \psi_h)|^2 |w_{s_1}^-|^2 \, d\xi \,.$$

Passing to the limit for $\varepsilon \to 0$ and then for $h \to +\infty$, we can conclude as in the first step that $w_{s_1}^- = 0$ and this is in contradiction with the definition of s_0. Hence $s_0 = 0$ and consequently $v(-x_1, ..., x_k, y) \geq v(x_1, ..., x_k, y)$ for every $x_1 > 0$. The same argument used until now, applied to the function $\tilde{v}(x_1, ..., x_k, y) := v(-x_1, ..., x_k, y)$, leads to $v(x_1, ..., x_k, y) \geq v(-x_1, ..., x_k, y)$ for every $x_1 > 0$. Thus v is even with respect to the x_1 variable.

Step 3 If we apply the moving plane method for any other direction x_i, with $i = 2, ..., k$, as we have done for x_1 in the previous steps, we get the symmetry in the x-variable. In order to prove the symmetry in the y-variable we redefine, for any $s \in \mathbb{R}$, $\Omega_s := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid y_1 > s\}$, $\xi^s := (x, y^s) = (x, 2s - y_1, y_2, ..., y_{N-k})$, $v_s(\xi) := v(\xi^s)$ and $w_s := v_s - v$. In the same way as in the first step, we get that for $s > 0$ large enough $w_s \geq 0$ in Ω^s. If we apply the same argument to the function $\overline{v}(x, y_1, ..., y_{N-k}) := v(x, -y_1, ..., y_{N-k})$ we prove that for any $s < 0$, provided $|s|$ large enough, $w_s \leq 0$ in Ω^s. Now, set

$$\tilde{A} := \{s \in \mathbb{R} \mid v_{\overline{s}} \geq v \text{ in } \Omega_{\overline{s}} \text{ for all } \overline{s} > s\}.$$

Notice that $\tilde{A}$ is not empty and bounded from below. We define $s' := \inf \tilde{A}$ and we want to prove that $w_{s'} = 0$ in $\Omega_{s'}$. If we carry on the proof as in the first step we get that $w_{s'} \geq 0$ in $\Omega_{s'}$, redefining $\omega_h^\varepsilon := \varphi_\varepsilon \psi_h \, w_{s'}^-$ with φ_ε and ψ_h as respectively in (5.2.6) and (5.2.7) and $w_{s'}^- = \min\{w_{s'}, 0\} \leq 0$. Moreover, $-\Delta w_{s'} - \lambda |x|^{-2} w_{s'} \geq 0$ and since $\Omega_{s'} \setminus \Sigma_0$ is connected, by strong maximum principle either $w_{s'} \equiv 0$ or $w_{s'} > 0$. If $w_{s'} > 0$, we can argue as in the second step to get that $s' = 0$ and then we have a contradiction because $w_0 \equiv 0$. Hence v is symmetric decreasing in the y_1 direction with respect to $y_1' = s'$.

In the same way, we can show that v is symmetric decreasing in the y_i variable with respect to some y_i' for any $i = 1, ..., N - k$. We can conclude the proof noting that $v(x, \cdot)$ is symmetric decreasing with respect $y' = (y_1', ...y_{N-k}')$ for every $x \neq 0$.

□

5.3 Symmetry breaking

Now we focus our attention on *ground state solutions*, namely, on solutions having minimal energy among all solutions (see Section 5.1 for the definition and for the existence results already available in literature). It is known that a ground state solution $\bar{v}$ exists for any $\lambda \le \lambda_1(0)$, provided that $q \in (2, 2^*)$ (see [69] and [85]). If in addition $\lambda \ge 0$ then $\bar{v}$ is cylindrically symmetric, by Theorem 5.2.5. A natural question is to ask whether $\bar{v}$ preserves the symmetry as λ decreases. The answer is negative, as it is shown by the next result.

Theorem 5.3.1 *Let $2 \le k < N$ and $q \in (2, 2^*)$. Then ground states solutions to (5.0.1) are not radially symmetric in x if*

$$\lambda \le \left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2} .$$

Notice that since $q < 2^*$ and $k < N$ it turns out that

$$\left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2} < \left(\frac{k-2}{2}\right)^2 - \frac{(k-1)(N-2)}{4} < -\frac{k-2}{4} \le 0 ,$$

coherently with Theorem 5.2.5. A similar phenomenon was already pointed out by Catrina and Wang [30] (see also [38]) for a related problem involving spherical weights.

The reason for the phenomenon described in Theorem 5.3.1 is that cylindrically symmetric solutions become highly unstable as $\lambda \to -\infty$, namely, their Morse index becomes too large (see Remark 5.3.3). This is a consequence of the next crucial theorem. Its proof was inspired by the papers by Kawohl [57] and by Smets, Su and Willem [80].

Theorem 5.3.2 *Assume $k \ge 2$ and let $v \not\equiv 0$ be a local minimum for*

$$R_\lambda(v) = \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^q \, d\xi\right)^{2/q}}$$

on $\mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^2; |x|^{-2} d\xi)$, such that $v(x,y) = v(|x|, y)$ for a.e. $y \in \mathbb{R}^{N-k}$. Then

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi \le \frac{k-1}{q-2} \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi .$$

Proof. Take any $h \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} \, d\xi)$, and set

$$\begin{aligned} z(t) &= \int_{\mathbb{R}^N} |\nabla(v+th)|^2 \, d\xi - \lambda \int_{\mathbb{R}^N} |x|^{-2} (v+th)^2 \, d\xi , \\ n(t) &= \left(\int_{\mathbb{R}^N} |x|^{-b_q} |v+th|^q \, d\xi\right)^{2/q} , \\ g(t) &= \frac{z(t)}{n(t)} = R_\lambda(v+th) . \end{aligned}$$

Since 0 is a local minima for g, then $g'(0) = 0$ and $g''(0) \geq 0$. To simplify notations we assume $n(0) = 1$. Thus we get $z(0) \leq \frac{z''(0)}{n''(0)}$, that is,

$$\int_{\mathbb{R}^N} |\nabla v|^2 - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 \ \leq \ \frac{\int_{\mathbb{R}^N} |\nabla h|^2 - \lambda \int_{\mathbb{R}^N} |x|^{-2} h^2}{(q-1)\int_{\mathbb{R}^N} |x|^{-b_q} |v|^{q-2} h^2 - (q-2)\left(\int_{\mathbb{R}^N} |x|^{-b_q} |v|^{q-2} v h\right)^2} . \tag{5.3.1}$$

Now, let $f_1 \in H^1(\mathbb{S}^{k-1})$ be an eigenfunction of the Laplace operator on $\mathbb{S}^{k-1}$ (the unit sphere in $\mathbb{R}^k$) with respect to the eigenvalue $k-1$. Thus, f_1 solves the minimization problem

$$\inf_{\substack{f \in H^1(\mathbb{S}^{k-1}) \\ \int_{\mathbb{S}^{k-1}} f \ = 0}} \frac{\int_{\mathbb{S}^{k-1}} |\nabla_\sigma f|^2 \ d\sigma}{\int_{\mathbb{S}^{k-1}} |f|^2 \ d\sigma} = k - 1.$$

To simplify computations it is convenient to take

$$\int_{\mathbb{S}^{k-1}} |f_1|^2 \ d\sigma = 1 \ , \int_{\mathbb{S}^{k-1}} |\nabla_\sigma f_1|^2 \ d\sigma = k - 1 \ .$$

Notice that we are allowed to use $h(x, y) = v(|x|, y) f_1(x/|x|)$ as test function in (5.3.1) ($h \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} \ d\xi)$ since $v \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} \ d\xi)$). It turns out that

$$\int_{\mathbb{R}^N} |\nabla h|^2 = \int_{\mathbb{R}^N} |\nabla v|^2 + (k-1) \int_{\mathbb{R}^N} |x|^{-2} v^2 \ ,$$

$$\int_{\mathbb{R}^N} |x|^{-2} h^2 = \int_{\mathbb{R}^N} |x|^{-2} v^2 \ , \quad \int_{\mathbb{R}^N} |x|^{-b_q} |v|^{q-2} h^2 = 1 \ , \quad \int_{\mathbb{R}^N} |x|^{-b_q} |v|^{q-2} v h = 0 \ .$$

Thus from (5.3.1) we infer that

$$\int_{\mathbb{R}^N} |\nabla v|^2 - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 \leq \frac{1}{q-1} \left(\int_{\mathbb{R}^N} |\nabla v|^2 \ - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 + (k-1) \int_{\mathbb{R}^N} |x|^{-2} v^2 \right) .$$

The conclusion easily follows. □

Proof of Theorem 5.3.1. Assume that $v \in X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k})$ achieves S_q^λ. Then, by Theorem 5.3.2 and by Hardy inequality one has

$$\begin{aligned} \left(\left(\frac{k-2}{2} \right)^2 - \lambda \right) \int_{\mathbb{R}^N} |x|^{-2} v^2 \ d\xi \ &\leq \ \int_{\mathbb{R}^N} |\nabla v|^2 \ d\xi - \lambda \int_{\mathbb{R}^N} |x|^{-2} v^2 \ d\xi \\ &\leq \ \frac{k-1}{q-2} \int_{\mathbb{R}^N} |x|^{-2} v^2 \ d\xi \ . \end{aligned}$$

Thus $\lambda \geq \left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2}$. Equality can not hold, since the Hardy constant is not achieved. □

Remark 5.3.3 Set $\Sigma = \{u \in X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k}) \mid \int_{\mathbb{R}^N} |x|^{-b_q}|v|^q \, d\xi = 1 \}$ and

$$E(v) = \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \lambda |x|^{-2} v^2\right) \, d\xi \ , \quad E : \Sigma \longrightarrow \mathbb{R} \ .$$

Then the Morse index of any cylindrically symmetric solution v to (5.0.1) diverges to $+\infty$ as $\lambda \to -\infty$. More precisely, for $j \geq 1$ let Λ_j be the eigenspace of $-\Delta_{\mathbb{S}^{k-1}}$ relative to the eigenvalue $\mu_j = j(k+j-2)$. Then $E''(v)$ is negative definite on Λ_i for any $i = 1, ..., j$, provided that $\lambda < \left(\frac{k-2}{2}\right)^2 - \frac{\mu_j}{q-2}$.

Remark 5.3.4 Assume $2 \leq k < N$, $\lambda \leq \lambda_1(0)$ and $q \in (2, 2^*]$. We compare here the best constants S_q^λ and $S_{q,\,\mathrm{cyl}}^\lambda$. Since $X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k}) \subset X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$, then

$$S_q^\lambda \leq S_{q,\,\mathrm{cyl}}^\lambda \ .$$

The infimum $S_{q,\,\mathrm{cyl}}^\lambda$ is always achieved on $X_{\lambda,\,\mathrm{cyl}}(\mathbb{R}^k \times \mathbb{R}^{N-k})$ by Lemma 5.2.2. By the results in [69], [85], [49] (see also Chapter 4), we have that S_q^λ is achieved on $X_\lambda(\mathbb{R}^k \times \mathbb{R}^{N-k})$ if $q < 2^*$, or if $q = 2^*$ and

$$0 \leq \lambda \leq \lambda_1(0) \ . \tag{5.3.2}$$

Notice that (5.3.2) is a necessary condition for existence in the limiting case $q = 2^*$, since for $\lambda < 0$ it happens that $S_{2^*}^\lambda = S$. In particular, if $k = 2$ then $S_{2^*}^\lambda$ is never achieved, unless $\lambda = 0$. Finally, $S_{2^*}^\lambda < S$ if and only if $k \geq 3$ and $0 < \lambda \leq \lambda_1(0)$.

Next, by the uniqueness result in [66] it turns out that, up to dilations and translations, problem (5.0.1) has at most one entire cylindrically symmetric solution. Taking into account also Theorem 5.2.5, we can state that

$$S_q^\lambda = S_{q,\mathrm{cyl}}^\lambda \quad \text{if} \quad 0 \leq \lambda \leq \left(\frac{k-2}{2}\right)^2 \ , \quad q \in (2, 2^*]$$

$$S_q^\lambda < S_{q,\mathrm{cyl}}^\lambda \quad \text{if} \quad \begin{cases} \lambda \leq \left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2} \ , & q \in (2, 2^*) \quad \text{or} \\ \\ \lambda < 0 \ , \quad q = 2^* \ . \end{cases}$$

The following figures show the behaviour of S_q^λ and of $S_{q,\mathrm{cyl}}^\lambda$ for $k \geq 3$.

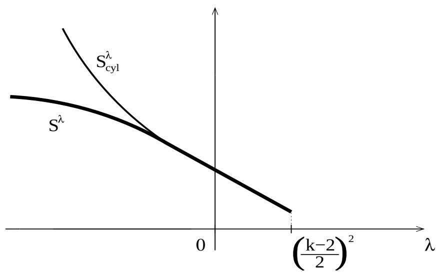

Fig.6 $k \geq 3$, $p < 2^*$.

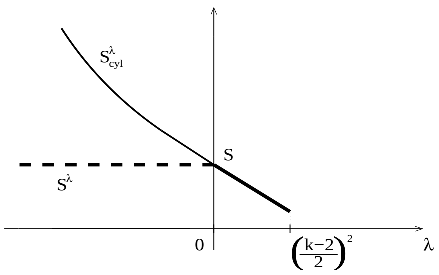

Fig.7 $k \geq 3$, $p = 2^*$.

We conclude by noticing the following multiplicity result.

Corollary 5.3.5 *Assume $2 \leq k < N$, $p \in (2, 2^*)$ and $\lambda \leq \left(\frac{k-2}{2}\right)^2 - \frac{k-1}{q-2}$. Then problem (5.0.1) has at least two distinct (modulo dilations and translations) entire solutions.*

5.4 On a degenerate problem

In the last Section we deal with classical solutions to

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{-b_{a,q}} \; |u|^{q-2}\, u \quad \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ u \geq 0 \ , \end{cases} \tag{5.4.1}$$

where $N \geq 3$ and

$$a > (2-N)\frac{k}{N}, \quad q > \max\left\{2, \frac{2\,(N-k)}{N-2+a}\right\}, \quad b_{a,q} := N - q\frac{N-2+a}{2}\,. \tag{5.4.2}$$

As corollaries of our theorems for (5.0.1) we prove an existence result of symmetric solutions, symmetry properties of solutions and a symmetry breaking phenomenon.

In the spherically symmetric case $k = N$ problem (5.4.1) is related to the Caffarelli-Kohn-Nirenberg inequalities [22]. Existence, non-existence and symmetry breaking of extremals functions were discussed in [30]. For $k < N$ the counterpart of the Caffarelli-Kohn-Nirenberg inequalities are the Maz'ya inequalities ([67], Section 2.1.6 and see also Chapter 4, inequality (4.0.4) at page 41). Existence results can be found in [85], where $a = 2 - k$, and in [69] for $a \neq 2 - k$.

We say that a classical solution u to (5.4.1) is *a-cylindrically symmetric* if *i*) for any choice of $y \in \mathbb{R}^{N-k}$, $u(\cdot, y)$ is radially symmetric in $\mathbb{R}^k$, and the map $|x| \to |x|^{\frac{a}{2}} u(|x|, y)$ is decreasing.

ii) there exists $y_0 \in \mathbb{R}^{N-k}$ such that, for any choice of $x \in \mathbb{R}^k \setminus \{0\}$, $u(x, \cdot)$ is symmetric decreasing about y_0 in $\mathbb{R}^{N-k}$.

As a corollary to Theorem 5.2.1 we easily get the following result.

Corollary 5.4.1 *Assume $2 \le k < N$ and (5.4.2). If $N \ge k+2$ assume in addition that $q < 2^*_{N-k+1}$. Then there exists an a-cylindrically symmetric solution*

$$u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi) \ .$$

Proof. Let us define

$$\lambda := \lambda_1(0) - \lambda_1(a) = \left(\frac{k-2}{2}\right)^2 - \left(\frac{k-2+a}{2}\right)^2 .$$

By direct computation and by results in [69], Appendix B, one can prove that a map $u \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$ is a classical solution to (5.4.1) if and only if $v := |x|^{\frac{a}{2}} u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} d\xi)$ is a solution to (5.0.1), with respect to the parameter λ. The conclusion easily follows from Theorem 5.2.1. □

With the same trick, from Theorems 5.2.5 and 5.3.1 one can prove the following results.

Corollary 5.4.2 *Assume $2 \le k < N$, (5.4.2), $q \le 2^*$, $2 - k \le a \le 0$ and let $u \in L^q(\mathbb{R}^N; |x|^{-b_{a,q}} d\xi)$ be a classical solution to (5.4.1). If $a = 0$ and $q = 2^*$, assume in addition that u has a nonremovable singularity on $\{x = 0\}$. Then u is a-cylindrically symmetric.*

Corollary 5.4.3 *Assume $2 \le k < N$, (5.4.2) and $q \le 2^*$, and let u be a solution to the minimum problem*

$$\inf_{\substack{w \in \mathcal{D}^{1,2}(\mathbb{R}^N; |x|^a d\xi) \\ w \not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla w|^2 \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_{a,q}} |w|^q \ d\xi\right)^{2/q}} \ .$$

Then u is not radially symmetric in x if

$$\left(\frac{k-2+a}{2}\right)^2 \ge \frac{k-1}{q-2} \ . \tag{5.4.3}$$

Remark 5.4.4 Corollary 5.4.3 holds also in the spherical case $k = N \geq 3$. However, for $k = N$ (5.4.3) Felli and Schneider (see also [30]) proved the stronger estimate:

$$q(N-2+a) \geq 2\sqrt{(N-2+a)^2 + 4(N-1)}\ .$$

Part III

The Hénon equation

Chapter 6

The Neumann problem and trace inequalities

The elliptic equation appearing in the Dirichlet problem

$$\begin{cases} -\Delta u = |x|^{\alpha}|u|^{q-2}\,u & \text{in } B_1 \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{6.0.1}$$

was introduced in the paper [55] by M. Hénon and now bears his name. Here $x \in B_1 = B_1^N(0)$, that is the unit ball of $\mathbb{R}^N$ with $N \geq 3$, $q > 2$ and $\alpha > 0$. In [55], problem (6.0.1) was proposed as a model for spherically symmetric stellar clusters and was investigated numerically for some definite values of q and α.

In the last few years, in spite of (or thanks to) its simple appearance, the Hénon equation raised a lot of questions concerning existence, multiplicity and, above all, symmetry properties of solutions. Research has been directed up to now only on the Dirichlet problem (6.0.1) with the intent of classifying the range of solvability (in q) and especially of analyzing the symmetry properties of the *ground state* solutions.

We start recalling some of the main achievements concerning problem (6.0.1). The first existence result is due to Ni, who in [71] proved that problem (6.0.1) admits at least one *radial* solution for any $q \in (2, 2^* + \frac{2\alpha}{N-2})$ and pointed out that it is the presence of the weight $|x|^\alpha$ that enlarges the existence range beyond the usual critical exponent.

The most important matter for our results is about the symmetry of solutions. The starting point is the fact that since the function $r \mapsto r^\alpha$ is increasing, Gidas-Ni-Nirenberg type results ([54]) do not apply, and therefore nonradial solutions could be expected. This is the content of the paper [80] by Smets, Su and Willem, who studied the ground state solutions associated to (6.0.1). They proved in particular the following symmetry breaking result.

Theorem 6.0.5 (Smets, Su, Willem) *For every $q \in (2, 2^*)$ no ground state for problem (6.0.1) is radial provided α is large enough.*

Further results on the Dirichlet problem can be found in [81], [28], [20], [21] for residual symmetry properties and asymptotic behaviour of ground states (for $q \to 2^*$ or $\alpha \to \infty$) and in [76], [9], [74] for existence and multiplicity of nonradial solutions for critical, supercritical and slightly subcritical growth; see also [25] and [77] for symmetry breaking results for Moser-Trudinger type nonlinearities. We quote finally [24] for multiple solutions in an annulus and [23] for elliptic systems.

We emphasize that all the above results have been obtained for the homogeneous Dirichlet problem, while it seems that so far the *Neumann* problem has never been studied. In this part we fill this gap and we point out a series of new and unexpected phenomena contained in [50], that arise passing from Dirichlet to Neumann boundary conditions. We quote also other two very recent papers: [16] that deals with a Hénon-type problem with a Moser-Trudinger term in case $N = 2$, and [14] (see Remark 6.1.5).

To describe our results, we let B_1 be the unit ball of $\mathbb{R}^N$, with $N \geq 3$, and we consider the Neumann problem analogue to (6.0.1), namely

$$\begin{cases} -\Delta u + u = |x|^\alpha |u|^{q-2}\, u & \text{in } B_1 \\ \\ u > 0 & \text{in } B_1 \\ \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1, \end{cases} \tag{6.0.2}$$

where again $q > 2$ and $\alpha > 0$. We have denoted by ν the outer normal to ∂B_1.

Remark 6.0.6 Comparing the equations in problems (6.0.1) and (6.0.2), we notice that in the left hand side of the second one there is the additional term u. Otherwise, without this term and with Neumann boundary conditions, integrating the equation in (6.0.1) on B_1 and using Gauss-Green formula, we obtain that there are no solutions.

Solutions to (6.0.2) arise from critical points of the functional $Q_\alpha : H^1(B_1)\setminus\{0\} \to \mathbb{R}$ defined by

$$Q_\alpha(u) := \frac{\int_{B_1} |\nabla u|^2\, dx + \int_{B_1} u^2\, dx}{\left(\int_{B_1} |x|^\alpha |u|^q\, dx\right)^{2/q}} = \frac{||u||^2}{\left(\int_{B_1} |x|^\alpha |u|^q\, dx\right)^{2/q}}\,,$$

where $||u|| = \left(\int_{B_1} |\nabla u|^2\, dx + \int_{B_1} u^2\, dx\right)^{1/2}$ is the norm in $H^1(B_1)$. If $q \in (2, 2^*]$, by Sobolev inequality this functional is well defined. Moreover, in the subcritical case $q < 2^*$, by standard arguments (see for example [7], [82] and [87]) the functional Q_α can be minimized in $H^1(B_1)$ and the infimum is attained.

We will call *ground states* the functions that minimize Q_α over $H^1(B_1)$, while we reserve the term *"radial minimizer"* to functions that minimize Q_α over $H^1_{rad}(B_1)$, that is the space of radial functions in $H^1(B_1)$.

Our purpose is to investigate problem (6.0.2), in the spirit of [71], in order to obtain existence of solutions beyond the usual critical threshold, and especially we are interested to carry out the analysis of the symmetry properties of the ground states of Q_α, as in [80].

First of all we analyze the existence and the properties of radial minimizers.

6.1 Radial minimizers and their asymptotic properties

We start establishing some properties that have been first proved by Ni in [71] in the context of the Dirichlet problem; we now give the H^1 versions.

Lemma 6.1.1 *There exists a positive constant C such that for all $u \in H^1_{rad}(B_1)$ there results*

$$|u(x)| \le C \frac{||u||}{|x|^{\frac{N-2}{2}}} \tag{6.1.1}$$

for all $x \in B_1 \setminus \{0\}$.

Proof. By the *radial Lemma* in [71], we have that for all u radial there exists $C > 0$ such that

$$|u(x)| \leq |u(1)| + C\frac{||\nabla u||_2}{|x|^{\frac{N-2}{2}}}\,, \tag{6.1.2}$$

where $||\cdot||_2$ is the norm in $L^2(B_1)$. Since $H^1(B_1)$ is embedded in $L^2(\partial B_1)$ by the *trace inequality* and u is radial, there exists a constant $C > 0$ such that

$$u^2(1)\,\omega_N = \int_{\partial B_1} u^2 d\sigma \leq C||u||^2\,. \tag{6.1.3}$$

The conclusion readily follows by (6.1.2) and (6.1.3). □

The previous lemma allows us to establish the following essential property.

Proposition 6.1.2 *The space $H^1_{rad}(B_1)$ embeds compactly into $L^q(B_1, |x|^\alpha)$ for every $q \in [1, 2^* + \frac{2\alpha}{N-2})$.*

Proof. By the growth estimate (6.1.1), we see that

$$\int_{B_1} |x|^\alpha |u|^q\, dx \leq C||u||^q \int_{B_1} |x|^{\alpha - q\frac{N-2}{2}}\, dx.$$

The last integral is finite for every $q \in [1, 2^* + \frac{2\alpha}{N-2})$, which shows that for all these q's the embedding is continuous. With a standard interpolation argument one obtains the compactness of the embedding in the same range. We write it for completeness.

We take a sequence u_n weakly convergent to zero in $H^1_{rad}(B_1)$ and for every $\varepsilon > 0$ we define $\tilde{2}_\varepsilon := 2^* + \frac{2\alpha}{N-2} - \varepsilon$. We consider $q \in [1, \tilde{2}_\varepsilon)$, then there exists $\theta \in (0, 1]$ such that $q = \theta + (1-\theta)\tilde{2}_\varepsilon$. Moreover, $\alpha = \alpha\,\theta + \alpha(1-\theta)$. By Hölder inequality and since $H^1_{rad}(B_1)$ embeds continuously into $L^{\tilde{2}_\varepsilon}(B_1, |x|^\alpha)$, we get

$$\begin{aligned}\int_{B_1} |x|^\alpha |u_n|^q\, dx &\leq \left(\int_{B_1} |x|^\alpha |u_n|\, dx\right)^\theta \left(\int_{B_1} |x|^\alpha |u_n|^{\tilde{2}_\varepsilon}\, dx\right)^{1-\theta} \\ &\leq C\left(\int_{B_1} |u_n|\, dx\right)^\theta ||u_n||^{(1-\theta)\tilde{2}_\varepsilon}. \end{aligned} \tag{6.1.4}$$

By the standard Rellich Theorem the right hand side of (6.1.4) goes to zero for $n \to +\infty$. The result follows because the previous computations hold for every $\varepsilon > 0$. □

We are now ready to give the main existence result. It matches completely the analogous one for the Dirichlet problem obtained in [71].

Theorem 6.1.3 *For every $\alpha > 0$ and every $q \in (2, 2^* + \frac{2\alpha}{N-2})$, there exists $u \in H^1_{rad}(B_1)$ such that*

$$Q_\alpha(u) = \inf_{\substack{v \in H^1_{rad}(B_1) \\ v \not\equiv 0}} Q_\alpha(v).$$

Proof. The proof is standard. Notice that, by Proposition 6.1.2, we have

$$\inf_{H^1_{rad}(B_1)} Q_\alpha > 0\,.$$

Let u_n be a minimizing sequence for Q_α, normalized by $\int_{B_1} |x|^\alpha |u_n|^q\, dx = 1$. Then there exists a subsequence u_n weakly convergent to a function u in $H^1_{rad}(B_1)$. By Proposition 6.1.2 the limit u cannot vanish identically, since in that case we would have $\int_{B_1} |x|^\alpha |u_n|^q\, dx \to 0$. Then, by lower semicontinuity,

$$Q_\alpha(u) = ||u||^2 \le \liminf ||u_n||^2 = \liminf Q_\alpha(u_n) = \inf_{H^1_{rad}(B_1)} Q_\alpha\,.$$

$\square$

Corollary 6.1.4 *For every α and q as in Theorem 6.1.3, (a suitable multiple of) the minimizer u is a classical solution to problem*

$$\begin{cases} -\Delta u + u = |x|^\alpha |u|^{q-2}\, u & \textit{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial B_1. \end{cases}$$

Moreover u is strictly positive in $\overline{B}_1$.

Proof. This follows from the Principle of symmetric criticality by Palais (see [72] and also [87]), standard elliptic regularity and the maximum principle. $\square$

Remark 6.1.5 Theorem 6.1.3 is analogous to Ni's result. Nevertheless, in the Dirichlet case, a Pohozaev-like identity shows that there are no solutions to problem (6.0.1) for any $q \ge 2^* + \frac{2\alpha}{N-2}$, whereas it does not give relevant informations in presence of Neumann boundary conditions. As a matter of fact, in [14] the authors recently proved that problem (6.0.2) has a radial solution for every $q > 2$ and $\alpha > 0$.

For every $\alpha > 0$ and for a given $q \in (2, 2^* + \frac{2\alpha}{N-2})$, let

$$m_{\alpha,r} := \min_{\substack{v \in H^1_{rad}(B_1) \\ v \not\equiv 0}} Q_\alpha(v).$$

Then any (positive) minimizer u_α of Q_α over $H^1_{rad}(B_1)$, when normalized by $||u_\alpha|| = 1$, satisfies

$$\begin{cases} -\Delta u_\alpha + u_\alpha = m_{\alpha,r}^{q/2} |x|^\alpha |u_\alpha|^{q-2}\, u_\alpha & \text{in } B_1 \\ \dfrac{\partial u_\alpha}{\partial \nu} = 0 & \text{on } \partial B_1. \end{cases}$$

We are interested in the behaviour of $m_{\alpha,r}$ and u_α when $\alpha \to \infty$. It turns out that it is possible to describe it in terms of a classical eigenvalue problem (see Subsection 6.1.1).

We begin with a fundamental result. We recall that $2_* := \frac{2(N-1)}{N-2}$ is the critical exponent for the embedding of $H^1(B_1)$ into $L^q(\partial B_1)$.

Lemma 6.1.6 *The asymptotic relation*

$$(\alpha + N) \int_{B_1} |x|^\alpha |u|^q \, dx = \int_{\partial B_1} |u|^q \, d\sigma + o(1) \quad \text{as} \quad \alpha \to \infty$$

holds

i) uniformly on bounded subsets of $H^1_{rad}(B_1)$, if $q \in (2, 2^*)$,

ii) uniformly on bounded subsets of $H^1(B_1)$, if $q \in (2, 2_*)$.

Proof. Notice that $(\alpha + N)|x|^\alpha = \operatorname{div}(|x|^\alpha x)$. For $u \in H^1_{rad}(B_1)$ or $u \in H^1(B_1)$, and q according to assumptions, we can write, applying the divergence Theorem,

$$\begin{aligned} (\alpha + N) \int_{B_1} |x|^\alpha |u|^q \, dx &= \int_{B_1} |u|^q \operatorname{div}(|x|^\alpha x) \, dx \\ &= \int_{\partial B_1} |u|^q |x|^\alpha x \cdot \nu \, d\sigma - q \int_{B_1} |u|^{q-2} u \nabla u \cdot x |x|^\alpha \, dx \\ &= \int_{\partial B_1} |u|^q \, d\sigma - q \int_{B_1} |u|^{q-2} u \nabla u \cdot x |x|^\alpha \, dx \end{aligned}$$

since on ∂B_1 we have $\nu = x$ and $|x| = 1$. We just have to show that the last integral is $o(1)$ as $\alpha \to \infty$ with the required uniformity.

To this aim, we first use the Hölder inequality to write

$$\begin{aligned}\left|\int_{B_1}|u|^{q-2}u\nabla u\cdot x|x|^\alpha\,dx\right| &\leq \left(\int_{B_1}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{B_1}|u|^{2q-2}|x|^\alpha\,dx\right)^{1/2}\\ &\leq ||u||\left(\int_{B_1}|u|^{2q-2}|x|^\alpha\,dx\right)^{1/2}.\end{aligned}$$

Assume now that $q\in(2,2^*)$ and that u is radial. By Lemma 6.1.1 we have

$$\int_{B_1}|u|^{2q-2}|x|^\alpha\,dx\leq C||u||^{2q-2}\int_{B_1}|x|^{\alpha-(2q-2)\frac{N-2}{2}}\,dx=||u||^{2q-2}o(1)$$

as $\alpha\to\infty$.

If, on the other hand, u is not radial, but q is strictly less than 2_*, we notice that $2q-2<2^*$, so that, by the Hölder and Sobolev inequalities,

$$\int_{B_1}|u|^{2q-2}|x|^\alpha\,dx\leq\left(\int_{B_1}|u|^{2^*}\,dx\right)^{\frac{2q-2}{2^*}}\left(\int_{B_1}|x|^{\alpha\frac{2^*}{2^*-2q+2}}\right)^{\frac{2^*-2q+2}{2^*}}\leq||u||^{2q-2}o(1)$$

as $\alpha\to\infty$.

Thus, in both cases,

$$\left|\int_{B_1}|u|^{q-2}u\nabla u\cdot x|x|^\alpha\,dx\right|\leq||u||^q o(1)\quad\text{as}\quad\alpha\to\infty,$$

which gives the required uniformity. $\square$

6.1.1 The Steklov problem

In order to state the main result of this section we need to introduce an auxiliary problem. This is one of the classical eigenvalue problems, and we refer to [11] and [58] for more details.

Definition 6.1.7 The eigenvalue problem

$$\begin{cases}-\Delta u+u=0 & \text{in } B_1\\ \dfrac{\partial u}{\partial\nu}=\lambda u & \text{on }\partial B_1\end{cases}\tag{6.1.5}$$

is called the *Steklov* problem.

The eigenvalues λ_k of this problem on B_1 are known to be

$$\lambda_k = 1 - \frac{N}{2} + \frac{I'_{k+N/2-2}(1)}{I_{k+N/2-2}(1)}, \qquad k = 1, 2, \ldots, \tag{6.1.6}$$

where I_ν is the modified Bessel function of the first kind of order ν. The associated eigenfunctions are also known (see [58]); the *first* eigenfunction, corresponding to λ_1, is radial and never vanishes in $\overline{B}_1$. The first eigenvalue is *simple* and it is characterized by

$$\lambda_1 = \min_{\substack{u \in H^1(B_1) \\ u \not\equiv 0}} \frac{||u||^2}{\displaystyle\int_{\partial B_1} u^2 \, d\sigma}.$$

With the aid of the Steklov problem we can now describe the asymptotic behaviour of the radial minimizers of Q_α. The asymptotics for the solutions of the Dirichlet problem for the Hénon equation has been obtained in [20] and [21]; in that case the situation is completely different and much more complex.

In the statement of the next result, λ_1 and φ_1, positive in $\overline{B}_1$ and normalized by $||\varphi_1|| = 1$, are respectively the first eigenvalue and eigenfunction of the Steklov problem (6.1.5).

Theorem 6.1.8 *Let $q \in (2, 2^*)$ and let u_α, with $||u_\alpha|| = 1$, be a minimizer of Q_α over $H^1_{rad}(B_1)$, so that $m_{\alpha,r} = Q_\alpha(u_\alpha)$. Then, as $\alpha \to \infty$,*

$$m_{\alpha,r} \sim (\alpha + N)^{2/q} |\partial B_1|^{1-2/q} \lambda_1 \tag{6.1.7}$$

$$u_\alpha \to \varphi_1 \quad in \quad H^1(B_1). \tag{6.1.8}$$

Proof. Let u be any (nonnegative) function in $H^1_{rad}(B_1)$, with $||u|| = 1$. By Lemma 6.1.6, as $\alpha \to \infty$,

$$\begin{aligned} \frac{Q_\alpha(u)}{(\alpha+N)^{2/q}} = \frac{1}{\Big((\alpha+N)\int_{B_1} |x|^\alpha |u|^q\Big)^{2/q}} &= \frac{1}{\Big(\int_{\partial B_1} |u|^q \, d\sigma + o(1)\Big)^{2/q}} \\ &= \frac{1}{\Big(\int_{\partial B_1} |u|^q \, d\sigma\Big)^{2/q}} + o(1), \end{aligned}$$

where $o(1)$ does not depend on u.

Since u is radial,

$$\left(\int_{\partial B_1} |u|^q \, d\sigma\right)^{2/q} = |\partial B_1|^{2/q} u^2(1) = |\partial B_1|^{2/q-1} \int_{\partial B_1} u^2 \, d\sigma,$$

so that for $u = u_\alpha$,

$$\begin{aligned}\frac{m_{\alpha,r}}{(\alpha+N)^{2/q}} &= \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/q}} = |\partial B_1|^{1-2/q}\frac{1}{\int_{\partial B_1} u_\alpha^2\,d\sigma} + o(1)\\ &\geq |\partial B_1|^{1-2/q} \min_{\substack{v\in H^1_{rad}(B_1)\\ ||v||=1}} \frac{1}{\int_{\partial B_1} v^2\,d\sigma} + o(1)\\ &= |\partial B_1|^{1-2/q}\lambda_1 + o(1) \qquad (6.1.9)\end{aligned}$$

because λ_1 is attained by a radial function.

On the other hand, for every $u \in H^1_{rad}(B_1)$ with $||u|| = 1$,

$$\frac{m_{\alpha,r}}{(\alpha+N)^{2/q}} = \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/q}} \leq \frac{Q_\alpha(u)}{(\alpha+N)^{2/q}} = |\partial B_1|^{1-2/q}\frac{1}{\int_{\partial B_1} u^2\,d\sigma} + o(1).$$

Choosing $u = \varphi_1$ we obtain

$$\frac{m_{\alpha,r}}{(\alpha+N)^{2/q}} \leq |\partial B_1|^{1-2/q}\lambda_1 + o(1)\,, \qquad (6.1.10)$$

and, by (6.1.9) and (6.1.10), (6.1.7) easily follows.

To prove (6.1.8) notice that, since $||u_\alpha|| = 1$, there is a subsequence, still denoted u_α, that converges to some u weakly in $H^1(B_1)$ and strongly in $L^q(\partial B_1)$ for $q < 2_*$. By the above arguments,

$$|\partial B_1|^{1-2/q}\frac{1}{\int_{\partial B_1} u_\alpha^2\,d\sigma} + o(1) = \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/q}} \leq |\partial B_1|^{1-2/q}\lambda_1 + o(1),$$

from which we see that u cannot be identically zero. Then, by previous computations and the properties of u_α,

$$\begin{aligned}\lambda_1 \leq \frac{||u||^2}{\int_{\partial B_1} u^2\,d\sigma} &\leq \frac{1}{\int_{\partial B_1} u^2\,d\sigma} = \lim_{\alpha\to\infty}\frac{1}{\int_{\partial B_1} u_\alpha^2\,d\sigma}\\ &= |\partial B_1|^{2/q-1}\lim_{\alpha\to\infty}\left(\frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/q}} + o(1)\right) = \lambda_1.\end{aligned}$$

This shows that $||u||^2 = 1$ and that

$$\frac{1}{\int_{\partial B_1} u^2\,d\sigma} = \lambda_1\,;$$

since λ_1 is simple, it must be $u = \varphi_1$. Convergence of the norm implies that $u_\alpha \to \varphi_1$ strongly in $H^1(B_1)$. $\square$

6.2 Symmetry breaking of ground states

The precise asymptotic behaviour of $m_{\alpha,r}$ will be used later. However the fact that $m_{\alpha,r}$ grows like $\alpha^{2/q}$ is enough to prove some symmetry properties of the ground states. As observed previously, when $q \in (2, 2^*)$, the functional Q_α can be minimized directly in $H^1(B_1)$, without the symmetry constraint. The natural question that arises is to ascertain whether this minimizer, the ground state, is still a radial function. This is the question addressed in [80] for the Dirichlet problem and answered in the negative for all q, provided α is large enough.

In the Neumann problem the situation is much more complex, and we point out from the beginning that we cannot give a complete solution in the whole interval $(2, 2^*)$. We will see also later that an important role is played by the number $2_* = \frac{2(N-1)}{(N-2)}$, the critical exponent for the embedding of $H^1(B_1)$ in $L^q(\partial B_1)$. We start with the following result.

Theorem 6.2.1 *Assume that $q \in (2_*, 2^*)$. Then for every α large enough (depending on q) we have*

$$\min_{\substack{u\in H^1(B_1)\\ u\not\equiv 0}} \frac{||u||^2}{\left(\int_{B_1} |x|^\alpha |u|^q\, dx\right)^{2/q}} < \min_{\substack{u\in H^1_{rad}(B_1)\\ u\not\equiv 0}} \frac{||u||^2}{\left(\int_{B_1} |x|^\alpha |u|^q\, dx\right)^{2/q}}. \tag{6.2.1}$$

Proof. We estimate the growth of the left hand side of (6.2.1) as in [80]. We take a nonnegative function $v \in C^1_c(B_1)$ and we extend it to zero outside B_1. Let $x_\alpha = (1 - 1/\alpha, 0, \dots, 0)$ and set $v_\alpha(x) = v(\alpha(x - x_\alpha))$. Then the support of v_α is contained in $B_{1/\alpha}(x_\alpha)$. By standard changes of variable,

$$\int_{B_1} |\nabla v_\alpha|^2\, dx = \alpha^{2-N} \int_{B_1} |\nabla v|^2\, dx$$

and

$$\int_{B_1} |v_\alpha|^2\, dx = \alpha^{-N} \int_{B_1} |v|^2\, dx\,.$$

Moreover, notice that if $x \in B_{1/\alpha}$ then $|x| \ge |x_\alpha| - \frac{1}{\alpha} = 1 - \frac{2}{\alpha}$. Therefore,

$$\begin{aligned}\int_{B_1} |x|^\alpha |v_\alpha|^q\, dx = \int_{B_{1/\alpha}(x_\alpha)} |x|^\alpha |v_\alpha|^q\, dx \;&\ge\; (1 - 2/\alpha)^\alpha \int_{B_{1/\alpha}(x_\alpha)} |v_\alpha|^q\, dx \\ &=\; \alpha^{-N}(1 - 2/\alpha)^\alpha \int_{B_1} |v|^q\, dx\end{aligned}$$

and, since $(1-\frac{2}{\alpha})^\alpha \to e^{-2}$ for $\alpha \to +\infty$,

$$Q_\alpha(v_\alpha) \leq \frac{\alpha^{2-N}\int_{B_1}|\nabla v|^2\,dx + \alpha^{-N}\int_{B_1} v^2\,dx}{\alpha^{-2N/q}(1-2/\alpha)^{2\alpha/q}\left(\int_{B_1}|v|^q\,dx\right)^{2/q}} \leq C\alpha^{2-N+2N/q}.$$

By Theorem 6.1.8 the right hand side of (6.2.1) is $m_{\alpha,r} \sim \alpha^{2/q}$. We can see that (6.2.1) holds for all α large because $2-N+2N/q < 2/q$ for all $q \in (2_*, 2^*)$. □

Remark 6.2.2 The level of radial minimizers for the quotient associated to the Dirichlet problem grows like $\alpha^{1+2/q}$, as is shown in [80]; this gives a symmetry breaking result for *all* $q \in (2, 2^*)$. We will see in the next sections that it is the loss of one power in the case of the Neumann problem that causes a more subtle behaviour from the point of view of symmetry of the ground states.

Remark 6.2.3 Theorem 6.2.1 gives a multiplicity result: for every $q \in (2_*, 2^*)$ and every α large enough, problem (6.0.2) admits at least two solutions. One is radial and the other is the (nonradial) ground state.

The new limitation $q > 2_*$ does not come from a weakness of the arguments, but is a structural fact, peculiar of the Neumann problem (see Section 6.6).

6.3 Variational properties of radial minimizers

We have seen that for $q \in (2_*, 2^*)$ the minimizers of Q_α over $H^1_{rad}(B_1)$ are not global minimizers over $H^1(B_1)$, at least for α large. In the interval $(2, 2_*]$ the situation is less clear, and will be analyzed in the next sections.

Now, since for $\alpha = 0$ global minimizers are radial, it is quite natural to think that the symmetry breaking phenomenon described above takes place because when α becomes very large the radial minimizer u_α ceases to be a minimizer over $H^1(B_1)$, due to the appearance of "negative directions". In other words one expects that when α grows the second derivative $Q''_\alpha(u_\alpha)$ over $H^1(B_1)$ becomes indefinite; this is exactly the phenomenon described in [80] for the Dirichlet problem.

In this section we show that this is not the case for the Neumann problem: although for $q \in (2_*, 2^*)$ the functions u_α are not global minimizers over $H^1(B_1)$ for α large, they are still *local* minimizers.

We now study the sign of $Q''_\alpha(u_\alpha)$ for α large. From now on we assume that u_α is normalized by $||u_\alpha|| = 1$. We denote by $\mathcal{S}$ the unit sphere in $H^1(B_1)$, and by $T_{u_\alpha}\mathcal{S}$ the tangent space to $\mathcal{S}$ at u_α, namely

$$T_{u_\alpha}\mathcal{S} = \{v \in H^1(B_1) \ : \ \langle v, u_\alpha\rangle = 0\},$$

where $\langle \cdot, \cdot\rangle$ denotes the scalar product in $H^1(B_1)$.

Notice that since u_α solves

$$\begin{cases} -\Delta u_\alpha + u_\alpha = m_{\alpha,r}^{q/2}|x|^\alpha |u_\alpha|^{q-2}\, u_\alpha & \text{in } B_1 \\ \dfrac{\partial u_\alpha}{\partial \nu} = 0 & \text{on } \partial B_1, \end{cases} \tag{6.3.1}$$

the condition $v \in T_{u_\alpha}\mathcal{S}$ is equivalent to $\int_{B_1} |x|^\alpha |u_\alpha|^{q-2} u_\alpha v\, dx = 0$.

Lemma 6.3.1 *Let $q \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{rad}(B_1)$, normalized by $u_\alpha \in \mathcal{S}$. Then for every $v \in T_{u_\alpha}\mathcal{S}$,*

$$Q''_\alpha(u_\alpha)\cdot v^2 = 2m_{\alpha,r}\left(||v||^2 - (q-1)m_{\alpha,r}^{q/2}\int_{B_1}|x|^\alpha |u_\alpha|^{q-2} v^2\, dx\right). \tag{6.3.2}$$

Proof. Set $N(u) = ||u||^2$ and $D(u) = \left(\int_{B_1}|x|^\alpha |u|^q\, dx\right)^{2/q}$, so that

$$Q_\alpha(u) = N(u)/D(u)\,.$$

For every critical point $u \in H^1(B_1)$ of Q_α and every $v \in H^1(B_1)$, we have

$$Q''_\alpha(u)\cdot v^2 = \frac{D(u)N''(u)\cdot v^2 - N(u)D''(u)\cdot v^2}{D(u)^2}.$$

Now $N''(u)\cdot v^2 = 2||v||^2$ and

$$\begin{aligned} D''(u)\cdot v^2 = {} & 2(2-q)\left(\int_{B_1}|x|^\alpha|u|^q\, dx\right)^{2/q-2}\left(\int_{B_1}|x|^\alpha|u|^{q-2}u\, v\, dx\right)^2 \\ & + 2(q-1)\left(\int_{B_1}|x|^\alpha|u|^q\, dx\right)^{2/q-1}\int_{B_1}|x|^\alpha|u|^{q-2}v^2\, dx, \end{aligned}$$

so that, for every critical point u of Q_α and every $v \in H^1(B_1)$,

$$Q''_\alpha(u)\cdot v^2 = 2\frac{||v||^2 - ||u||^2\left((2-q)\left(\frac{\int_{B_1}|x|^\alpha|u|^{q-2}u\,v\,dx}{\int_{B_1}|x|^\alpha|u|^q\,dx}\right)^2 + (q-1)\frac{\int_{B_1}|x|^\alpha|u|^{q-2}v^2\,dx}{\int_{B_1}|x|^\alpha|u|^q\,dx}\right)}{\left(\int_{B_1}|x|^\alpha|u|^q\,dx\right)^{2/q}}. \tag{6.3.3}$$

If $u = u_\alpha \in \mathcal{S}$, we have

$$\frac{1}{\left(\int_{B_1} |x|^\alpha |u|^q \, dx\right)^{2/q}} = m_{\alpha,r}$$

and if $v \in T_{u_\alpha}\mathcal{S}$, then $\int_{B_1} |x|^\alpha |u|^{q-2} u \, v \, dx = 0$. Therefore in this case (6.3.3) reduces to

$$Q''_\alpha(u_\alpha) \cdot v^2 = 2m_{\alpha,r} \left(||v||^2 - (q-1) m_{\alpha,r}^{q/2} \int_{B_1} |x|^\alpha |u_\alpha|^{q-2} v^2 \, dx \right).$$

□

In order to study the sign of $Q''_\alpha(u_\alpha)$, we need some more precise estimates on u_α. In what follows u_α is a minimizer of Q_α over $H^1_{rad}(B_1)$, normalized by $||u_\alpha|| = 1$.

Lemma 6.3.2 *The functions u_α are uniformly bounded in $C^1(\overline{B}_1)$ as $\alpha \to \infty$.*

Proof. We first prove a uniform bound in $L^\infty(B_1)$. Since $||u_\alpha|| = 1$ and u_α is radial, by Lemma 6.1.1 there is $C > 0$ such that

$$||u_\alpha||_{L^\infty(B_1 \setminus B_{1/2})} \leq C ||u_\alpha|| = C. \tag{6.3.4}$$

Moreover, there is a positive constant C such that

$$|u_\alpha(x)| \leq C \frac{||u_\alpha||}{|x|^{\frac{N-2}{2}}} = \frac{C}{|x|^{\frac{N-2}{2}}}$$

for all $x \in B_1 \setminus \{0\}$ and all α.

Set $f_\alpha(x) = m_{\alpha,r}^{q/2} |x|^\alpha |u_\alpha(x)|^{q-2} u_\alpha(x)$; then, recalling that $m_{\alpha,r} \leq C\alpha^{2/q}$ by (6.1.7) in Theorem 6.1.8,

$$||f_\alpha||_{L^\infty(B_{1/2})} \leq C\alpha |x|^\alpha \frac{1}{|x|^{(q-1)\frac{N-2}{2}}} \leq \frac{C\alpha}{2^{\alpha - (q-1)\frac{2-N}{2}}} = o(1) \tag{6.3.5}$$

as $\alpha \to \infty$.

Therefore, by (6.3.4) and (6.3.5) we see that u_α solves

$$\begin{cases} -\Delta u_\alpha + u_\alpha = f_\alpha & \text{in } B_{1/2} \\ u_\alpha \leq C & \text{on } \partial B_{1/2} \end{cases}$$

with $||f_\alpha||_{L^\infty(B_{1/2})} \to 0$ as $\alpha \to \infty$. By standard elliptic estimates, we obtain that u_α is uniformly bounded in $C^{1,\beta}(\overline{B}_{1/2})$, for all $\beta \in (0,1)$. In view of (6.3.4) we obtain that u_α is uniformly bounded in $L^\infty(B_1)$ as $\alpha \to \infty$.

To complete the proof we only have to show that there is a C^1 bound also on $\overline{B}_1 \setminus B_{1/2}$. Since u_α is radial we see that it solves

$$-u_\alpha'' - \frac{N-1}{\rho} u_\alpha' + u_\alpha = m_{\alpha,r}^{q/2} \rho^\alpha |u_\alpha|^{q-2} u_\alpha$$

and $u_\alpha'(1) = 0$. Integrating this equation over $[t, 1]$, with $t \geq \frac{1}{2}$ we obtain

$$u_\alpha'(t) = (N-1) \int_t^1 \frac{1}{\rho} u_\alpha' \, d\rho - \int_t^1 u_\alpha + m_{\alpha,r}^{q/2} \int_t^1 |u_\alpha|^{q-2} u_\alpha \rho^\alpha \, d\rho.$$

Therefore, using the fact that u_α is bounded in $L^\infty(B_1)$ and the growth of $m_{\alpha,r}$,

$$|u_\alpha'(t)| \leq (N-1) \Big(\frac{u_\alpha(\rho)}{\rho} \Big|_t^1 + \int_t^1 \frac{u_\alpha(\rho)}{\rho^2} \, d\rho \Big) + C + C m_{\alpha,r}^{q/2} \frac{\rho^{\alpha+1}}{\alpha+1} \Big|_t^1 \leq C + C \frac{\alpha}{\alpha+1} \leq C$$

for all $t \in [\frac{1}{2}, 1]$. This, together with the estimate in $C^{1,\beta}(B_{1/2})$, gives the required bound in $C^1(\overline{B}_1)$. □

Remark 6.3.3 Notice that one cannot hope to obtain uniform $C^2(\overline{B}_1)$ estimates, even though each u_α lies in $C^2(\overline{B}_1)$. This is due to the fact that the right hand side of the equation behaves like $\alpha |x|^\alpha$. Now, while this term goes to zero locally uniformly in B_1, on the boundary it blows up like α. Therefore Δu_α cannot be bounded in C^0 up to the boundary of B_1.

At first sight, a rather confusing consequence of the lack of C^2 bounds is that if one tries to pass naïvely to the limit in (6.3.1) as $\alpha \to \infty$, then one can do it in the equation (in the weak form, for instance), but *not in the boundary conditions*. Thus it may (and does) happen that limits of solutions of homogeneous Neumann problems do not satisfy a homogeneous Neumann condition. We have already observed this fact when we have shown that $u_\alpha \to \varphi_1$, a solution of the *Steklov* problem.

Remark 6.3.4 In view of the preceding lemma, we can assure that the convergence of u_α to φ_1 takes place also in $C^0(\overline{B}_1)$. Since φ_1 is strictly positive in $\overline{B}_1$, then for some $C > 0$ and all α large,

$$\min_{x \in \overline{B}_1} u_\alpha(x) \geq C.$$

We can now continue the study of the second derivative of Q_α.

Lemma 6.3.5 *Let $q \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{rad}(B_1)$, normalized by $u_\alpha \in \mathcal{S}$. Then, as $\alpha \to \infty$,*

$$(\alpha + N)\int_{B_1} |x|^\alpha |u_\alpha|^{q-2} v^2\, dx = \int_{\partial B_1} |u_\alpha|^{q-2} v^2\, d\sigma + o(1), \tag{6.3.6}$$

uniformly for v in bounded subsets of $H^1(B_1)$.

Proof. We apply the divergence Theorem exactly like in Lemma 6.1.6. We obtain

$$\begin{aligned}(\alpha + N)\int_{B_1} |x|^\alpha |u_\alpha|^{q-2} v^2\, dx &= \int_{\partial B_1} |u_\alpha|^{q-2} v^2\, d\sigma \\ &- (q-2)\int_{B_1} v^2 |u_\alpha|^{q-3}\nabla|u_\alpha| \cdot x|x|^\alpha\, dx \\ &- 2\int_{B_1} |u_\alpha|^{q-2} v\nabla v \cdot x|x|^\alpha\, dx\end{aligned}$$

and we just have to show that the two last integral vanish as $\alpha \to \infty$.

Now by Lemma 6.3.2 and Hölder inequality, we have

$$\begin{aligned}\left|\int_{B_1} v^2 |u_\alpha|^{q-3}\nabla|u_\alpha| \cdot x|x|^\alpha\, dx\right| &\leq C\int_{B_1} |x|^\alpha v^2\, dx \leq C||v||^2_{2^*}\left(\int_{B_1} |x|^{\alpha\frac{2^*}{2^*-2}}\, dx\right)^{\frac{2^*-2}{2^*}} \\ &\leq C||v||^2 o(1)\end{aligned}$$

as $\alpha \to \infty$, while

$$\begin{aligned}\left|\int_{B_1} |u_\alpha|^{q-2} v\nabla v \cdot x|x|^\alpha\, dx\right| &\leq C\int_{B_1} |x|^\alpha |v||\nabla v|\, dx \leq C||v||\left(\int_{B_1} |x|^\alpha v^2\, dx\right)^{1/2} \\ &\leq C||v||^2 o(1),\end{aligned}$$

as in the previous computation. Thus (6.3.6) is proved. □

We can now prove the main result on the sign of $Q''_\alpha(u_\alpha)$.

Proposition 6.3.6 *Let $q \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{rad}(B_1)$, normalized by $u_\alpha \in \mathcal{S}$. Then the inequality*

$$\min_{\substack{v \in T_{u_\alpha}\mathcal{S} \\ ||v||=1}} Q''_\alpha(u_\alpha) \cdot v^2 > 0 \tag{6.3.7}$$

holds

i) for all $q \in (2, 2^*)$ if $N \geq 4$,

ii) for all $q \in (2, \overline{q})$, for some $\overline{q} \in (2_*, 2^*)$ if $N = 3$

provided α is large enough (depending on q).

Proof. Step 1 It is standard to see that the minimum in (6.3.7) is attained; we supply some details for completeness. Set

$$F(v) = 2m_{\alpha,r}\left(||v||^2 - (q-1)m_{\alpha,r}^{q/2}\int_{B_1}|x|^\alpha|u_\alpha|^{q-2}v^2\,dx\right)$$

and

$$\mu = \inf_{\substack{v\in T_{u_\alpha}\mathcal{S}\\ ||v||=1}} F(v).$$

By Lemma 6.3.1, we have to show that μ is attained. It is obvious that μ is finite and that $\mu < 2m_{\alpha,r}$. Let $v_n \in T_{u_\alpha}\mathcal{S}$, $||v_n|| = 1$, be a minimizing sequence for F. Up to subsequences, $v_n \to v$ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$. Notice that $v \not\equiv 0$, since otherwise

$$\mu + o(1) = F(v_n) = 2m_{\alpha,r}\left(1 - (q-1)m_{\alpha,r}^{q/2}\int_{B_1}|x|^\alpha|u_\alpha|^{q-2}v_n^2\,dx\right) = 2m_{\alpha,r} + o(1),$$

namely $\mu = 2m_{\alpha,r}$, which is false. Moreover, $v \in T_{u_\alpha}\mathcal{S}$. Write now $v_n = v + w_n$, with $w_n \to 0$ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$. A simple computation shows that

$$\begin{aligned}
\mu + o(1) &= F(v_n)\\
&= 2m_{\alpha,r}\left(||v||^2 + ||w_n||^2 - (q-1)m_{\alpha,r}^{q/2}\int_{B_1}|x|^\alpha|u_\alpha|^{q-2}v^2\,dx + o(1)\right)\\
&= ||v||^2 F\Big(\frac{v}{||v||}\Big) + 2m_{\alpha,r}||w_n||^2 + o(1)\\
&\geq \mu||v||^2 + 2m_{\alpha,r}||w_n||^2 + o(1)\\
&= \mu(1 - ||w_n||^2 + o(1)) + 2m_{\alpha,r}||w_n||^2 + o(1)\,,
\end{aligned}$$

so that $(2m_{\alpha,r} - \mu)||w_n||^2 \leq o(1)$. Since $\mu < 2m_{\alpha,r}$, this shows that $v_n \to v$ strongly in $H^1(B_1)$; thus $||v|| = 1$ and $F(v) = \mu$.

Step 2 We now turn to the main part of the proof. We assume that (6.3.7) is false for an unbounded sequence of α's (which we denote by A), so that

$$\min_{\substack{v\in T_{u_\alpha}\mathcal{S}\\ ||v||=1}} Q''_\alpha(u_\alpha)\cdot v^2 \leq 0 \qquad \text{for all } \alpha \in A.$$

This means that for all $\alpha \in A$ there exists $v_\alpha \in T_{u_\alpha}\mathcal{S}$, with $||v_\alpha|| = 1$ such that

$$Q''_\alpha(u_\alpha)\cdot v_\alpha^2 \leq 0\,,$$

namely, by (6.3.2),

$$1-(q-1)m_{\alpha,r}^{q/2}\int_{B_1}|x|^\alpha|u_\alpha|^{q-2}v_\alpha^2\,dx\le 0\,.$$

Recalling from Theorem 6.1.8 the asymptotic behaviour of $m_{\alpha,r}$, we can write the preceding inequality for $\alpha\to\infty$ in A as

$$\begin{aligned}1 &\le (q-1)\left(|\partial B_1|^{q/2-1}\lambda_1^{q/2}+o(1)\right)(\alpha+N)\int_{B_1}|x|^\alpha|u_\alpha|^{q-2}v_\alpha^2\,dx\\ &= (q-1)\left(|\partial B_1|^{q/2-1}\lambda_1^{q/2}+o(1)\right)\left(\int_{\partial B_1}|u_\alpha|^{q-2}v_\alpha^2\,d\sigma+o(1)\right),\end{aligned}$$

where for the last equality we have used Lemma 6.3.5.

Now, by Remark 6.3.4, $u_\alpha\to\varphi_1$ in $C^0(\overline{B}_1)$ and v_α, being bounded in $H^1(B_1)$, admits a subsequence (still denoted v_α) converging to some v weakly in $H^1(B_1)$ and strongly in $L^2(\partial B_1)$. Passing to the limit as $\alpha\to\infty$ in A in the preceding inequality we find

$$1\le(q-1)|\partial B_1|^{q/2-1}\lambda_1^{q/2}\int_{\partial B_1}\varphi_1^{q-2}v^2\,d\sigma. \tag{6.3.8}$$

If v is identically zero we have reached a contradiction and the proof is complete. Assume therefore that $v\not\equiv 0$.

Since φ_1 is radial and normalized by $||\varphi_1||=1$, we have

$$\lambda_1=\frac{1}{\int_{\partial B_1}\varphi_1^2\,d\sigma}=\frac{1}{\varphi_1(1)^2|\partial B_1|},$$

so that $\varphi_1(1)^{q-2}=|\partial B_1|^{1-q/2}\lambda_1^{1-q/2}$. Inserting this in (6.3.8) we see that

$$1\le(q-1)|\partial B_1|^{q/2-1}\lambda_1^{q/2}|\partial B_1|^{1-q/2}\lambda_1^{1-q/2}\int_{\partial B_1}v^2\,d\sigma=(q-1)\lambda_1\int_{\partial B_1}v^2\,d\sigma,$$

that is,

$$\frac{1}{\int_{\partial B_1}v^2\,d\sigma}\le(q-1)\lambda_1.$$

Notice now that $\langle v,\varphi_1\rangle=\lim_{\alpha\to+\infty}\langle v_\alpha,u_\alpha\rangle=0$ by strong convergence of u_α and weak convergence of v_α. Thus v is orthogonal to φ_1 in $H^1(B_1)$; this, together with the fact that λ_1 is simple yields

$$\lambda_2=\min_{\substack{w\in H^1(B_1)\\ \langle w,\varphi_1\rangle=0}}\frac{||w||^2}{\int_{\partial B_1}w^2\,d\sigma}\le\frac{||v||^2}{\int_{\partial B_1}v^2\,d\sigma}\le\frac{1}{\int_{\partial B_1}v^2\,d\sigma}\le(q-1)\lambda_1.$$

Therefore, assuming that (6.3.7) is false for an unbounded sequence of α's, it implies the inequality

$$\frac{\lambda_2}{\lambda_1} \leq q - 1 \tag{6.3.9}$$

on the eigenvalues of the Steklov problem (6.1.5).

Step 3 We now complete the proof by showing that inequality (6.3.9) cannot hold. We recall from (6.1.6) that

$$\lambda_k = 1 - \frac{N}{2} + \frac{I'_{k+N/2-2}(1)}{I_{k+N/2-2}(1)}, \qquad k = 1, 2, \ldots,$$

where I_ν is the modified Bessel function of the first kind of order ν.

Since (see [6]) $I'_\nu(x) = I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x)$ holds for all x and all ν, we see that

$$\lambda_k = k - 1 + \frac{I_{k+N/2-1}(1)}{I_{k+N/2-2}(1)}. \tag{6.3.10}$$

We also recall from [6] that $I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x)$, so that $I_{\nu-1}(1)/I_\nu(1) \geq 2\nu$. Therefore

$$\frac{\lambda_2}{\lambda_1} = \frac{1 + I_{N/2+1}(1)/I_{N/2}(1)}{I_{N/2}(1)/I_{N/2-1}(1)} > \frac{1}{I_{N/2}(1)/I_{N/2-1}(1)} = \frac{I_{N/2-1}(1)}{I_{N/2}(1)} \geq 2\frac{N}{2} = N.$$

Thus, if (6.3.9) holds, then by our choice of q,

$$N < \frac{\lambda_2}{\lambda_1} \leq q - 1 < \frac{N+2}{N-2},$$

which is false for every $N \geq 4$. If $N = 3$, the inequality is false not in the whole interval $(2, 2^*) = (2, 6)$, but only in a subinterval $(2, \overline{q})$, with $\overline{q} \in (2_*, 2^*) = (4, 6)$ (an approximate value of λ_2/λ_1 is 3.8, which would locate $\overline{q}$ around 4.8). In both cases this is the required contradiction and the proof is complete. □

Remark 6.3.7 In the previous proposition, one cannot hope to get, when $N = 3$, the whole interval $(2, 2^*)$ as for $N = 4$. Indeed testing $Q''_\alpha(u_\alpha)$ with φ_2, the second eigenfunction of the Steklov problem, one sees easily that $Q''_\alpha(u_\alpha) \cdot \varphi_2^2 < 0$ for q close to 2^* and α large. The fact that for α large $Q''_\alpha(u_\alpha)$ becomes indefinite for some $q \in (2_*, 2^*)$ is thus a peculiarity of dimension three.

Proposition 6.3.6, together with the fact that Q_α is homogeneous of degree zero constitutes the proof of the following result.

Theorem 6.3.8 *For all $q \in (2, 2^*)$ if $N \geq 4$ (resp. $q \in (2, \overline{q})$ if $N = 3$) and all α large enough, the minimizers u_α of Q_α over $H^1_{rad}(B_1)$ are local minima of Q_α over the whole space $H^1(B_1)$. The limitation for $N = 3$ is not removable.*

Comparing with Theorem 6.2.1, this means for $q \in (2_*, 2^*)$ that the formation of nonsymmetric ground states does not manifest locally around radial minimizers, but can be justified only by *global* properties of the functional.

6.4 Uniqueness of radial minimizers

As an application of the discussion carried out in the previous section, we now give a uniqueness result for radial minimizers of Q_α.

Theorem 6.4.1 *For every $q \in (2, 2^*)$ if $N \geq 4$, or every $q \in (2, \overline{q})$ if $N = 3$, there exists $\alpha(q)$ such that for all $\alpha \geq \alpha(q)$, the problem*

$$\min_{\substack{u \in H^1_{rad}(B_1) \\ u \not\equiv 0}} \frac{||u||^2}{\left(\int_{B_1} |x|^\alpha |u|^q \, dx \right)^{2/q}}$$

has a unique positive solution (normalized by $||u|| = 1$).

Proof. Fix q in the appropriate range, according to the value of N, and assume by contradiction that for an unbounded sequence of α's, denoted by A, there exist two (positive) minimizers u_α and v_α of Q_α over $H^1_{rad}(B_1)$, normalized by $||u_\alpha|| = ||v_\alpha|| = 1$.

By Theorem 6.1.8 and Lemma 6.3.2 we have that, as $\alpha \to \infty$ in A,

$$u_\alpha \to \varphi_1 \qquad \text{and} \qquad v_\alpha \to \varphi_1 \qquad \text{in } H^1(B_1) \text{ and in } C^0(\overline{B}_1).$$

Moreover, both u_α and v_α solve

$$\begin{cases} -\Delta u + u = m_{\alpha,r}^{q/2} |x|^\alpha |u|^{q-2} u & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \end{cases} \tag{6.4.1}$$

with $m_{\alpha,r} \sim (\alpha + N)^{2/q} |\partial B_1|^{1-2/q} \lambda_1$.

Subtracting (6.4.1) for v_α from (6.4.1) for u_α and setting $w_\alpha = u_\alpha - v_\alpha$, we see that w_α solves

$$\begin{cases} -\Delta w_\alpha + w_\alpha = (q-1)m_{\alpha,r}^{q/2}|x|^\alpha c_\alpha w_\alpha & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1, \end{cases} \tag{6.4.2}$$

where

$$c_\alpha = \int_0^1 (v_\alpha + t(u_\alpha - v_\alpha))^{q-2}\, dt.$$

By assumption $u_\alpha \not\equiv v_\alpha$ for all $\alpha \in A$, so that we can divide the equations in (6.4.2) by $||w_\alpha||$ and set $\psi_\alpha = w_\alpha/||w_\alpha||$. Then we obtain that ψ_α satisfies

$$\begin{cases} -\Delta \psi_\alpha + \psi_\alpha = (q-1)m_{\alpha,r}^{q/2}|x|^\alpha c_\alpha \psi_\alpha & \text{in } B_1 \\ \dfrac{\partial \psi_\alpha}{\partial \nu} = 0 & \text{on } \partial B_1 \\ ||\psi_\alpha|| = 1 \end{cases} \tag{6.4.3}$$

for all $\alpha \in A$.

Since $||\psi_\alpha|| = 1$, we can assume that (up to a subsequence), $\psi_\alpha \to \psi$ weakly in $H^1(B_1)$ and strongly in $L^q(\partial B_1)$ for all $q < 2_*$.

We now show that it cannot be $\psi \equiv 0$. Indeed, noticing that by Lemma 6.3.2 we have $||c_\alpha||_\infty \le C$ uniformly in α, and multiplying (6.4.3) by ψ_α, we obtain by integration (using $(\alpha + N)|x|^\alpha = \operatorname{div}(|x|^\alpha x)$)

$$\begin{aligned} 1 = ||\psi_\alpha||^2 &= (q-1)m_{\alpha,r}^{q/2}\int_{B_1} c_\alpha \psi_\alpha^2 |x|^\alpha\, dx \\ &\le C m_{\alpha,r}^{q/2}\int_{B_1} \psi_\alpha^2|x|^\alpha\, dx \le C(\alpha+N)\int_{B_1}\psi_\alpha^2|x|^\alpha\, dx \\ &= C\left(\int_{\partial B_1}\psi_\alpha^2\, d\sigma - 2\int_{B_1}\psi_\alpha \nabla\psi_\alpha\cdot x|x|^\alpha\, dx\right) \\ &\le C\left(\int_{\partial B_1}\psi_\alpha^2\, d\sigma + 2||\psi_\alpha||\left(\int_{B_1}\psi_\alpha^2\, dx\right)^{1/2}\right). \end{aligned}$$

If ψ were identically zero, we would have $\psi_\alpha \to 0$ strongly in $L^2(B_1)$ and in $L^2(\partial B_1)$, so that the preceding inequality would yield $1 = ||\psi_\alpha||^2 \le o(1)$, as $\alpha \to \infty$ in A, a contradiction. Therefore $\psi \not\equiv 0$.

To proceed we notice that, still by Lemma 6.3.2, we have

$$c_\alpha \to \varphi_1^{q-2} \qquad \text{in } C^0(\overline{B}_1). \tag{6.4.4}$$

Multiplying (6.4.3) by $\phi \in H^1(B_1)$ and integrating we obtain

$$\begin{aligned} \langle \psi_\alpha, \phi \rangle &= (q-1) m_{\alpha,r}^{q/2} \int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha \, dx \\ &= (q-1)(|\partial B_1|^{q/2-1} \lambda_1^{q/2} + o(1))(\alpha + N) \int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha \, dx. \end{aligned} \tag{6.4.5}$$

Now

$$\begin{aligned} & \left| (\alpha+N) \int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha \, dx - (\alpha+N) \int_{B_1} \varphi_1^{q-2} \psi_\alpha \phi |x|^\alpha \, dx \right| \\ \leq \;& (\alpha+N) \int_{B_1} |c_\alpha - \varphi_1^{q-2}| |\psi_\alpha \phi| |x|^\alpha \, dx \\ \leq \;& ||c_\alpha - \varphi_1^{q-2}||_\infty (\alpha+N) \int_{B_1} |\psi_\alpha| |\phi| |x|^\alpha \, dx \end{aligned}$$

and

$$\begin{aligned} (\alpha+N) \int_{B_1} |\psi_\alpha| |\phi| |x|^\alpha \, dx &\leq \left((\alpha+N) \int_{B_1} \psi_\alpha^2 |x|^\alpha \, dx \right)^{1/2} \cdot \\ &\cdot \left((\alpha+N) \int_{B_1} \phi^2 |x|^\alpha \, dx \right)^{1/2} \\ &\leq \left(\int_{\partial B_1} \psi_\alpha^2 \, d\sigma - 2 \int_{B_1} \psi_\alpha \nabla \psi_\alpha \cdot x |x|^\alpha \, dx \right)^{1/2} \cdot \\ &\cdot \left(\int_{\partial B_1} \phi^2 \, d\sigma - 2 \int_{B_1} \phi \nabla \phi \cdot x |x|^\alpha \, dx \right)^{1/2} \leq C \end{aligned}$$

as $\alpha \to \infty$, since all the integrals are uniformly bounded.

This and the preceding inequality, joint to (6.4.4), show that

$$(\alpha+N) \int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha \, dx = (\alpha+N) \int_{B_1} \varphi_1^{q-2} \psi_\alpha \phi |x|^\alpha \, dx + o(1)$$

as $\alpha \to \infty$ in A. Finally we notice that

$$\begin{aligned} (\alpha+N) \int_{B_1} \varphi_1^{q-2} \psi_\alpha \phi |x|^\alpha \, dx &= \int_{\partial B_1} \varphi_1^{q-2} \psi_\alpha \phi \, d\sigma - \int_{B_1} \nabla(\varphi_1^{q-2} \psi_\alpha \phi) \cdot x |x|^\alpha \, dx \\ &= \int_{\partial B_1} \varphi_1^{q-2} \psi_\alpha \phi \, d\sigma + o(1) \, , \end{aligned}$$

due to by now familiar computations. Inserting this in the left hand side of (6.4.5), it yields, as $\alpha \to \infty$,

$$\langle \psi_\alpha, \phi \rangle = (q-1)\left(|\partial B_1|^{q/2-1}\lambda_1^{q/2} + o(1)\right)\left(\int_{\partial B_1} \varphi_1^{q-2}\psi_\alpha \phi \, d\sigma + o(1)\right).$$

Letting $\alpha \to \infty$ in A (and recalling that $\varphi_1^{q-2} \equiv |\partial B_1|^{1-q/2}\lambda_1^{1-q/2}$ on ∂B_1), we obtain

$$\langle \psi, \phi \rangle = (q-1)\lambda_1 \int_{\partial B_1} \psi \phi \, d\sigma,$$

for all $\phi \in H^1(B_1)$. In other words, ψ is a (nontrivial) solution of the problem

$$\begin{cases} -\Delta \psi + \psi = 0 & \text{in } B_1 \\ \dfrac{\partial \psi}{\partial \nu} = (q-1)\lambda_1 \psi & \text{on } \partial B_1. \end{cases} \tag{6.4.6}$$

Thus the number $(q-1)\lambda_1$ must be one of the eigenvalues λ_k of the Steklov problem. However $(q-1)\lambda_1 > \lambda_1$ because $q > 2$ and, as we have already proved in Proposition 6.3.6, $(q-1)\lambda_1 < \lambda_2$ for all $q \in (2, 2^*)$ if $N \geq 4$ and all $q \in (2, \overline{q})$ if $N = 3$. This is a contradiction, and the proof is complete. □

6.5 A detour on the trace inequalities

In this section we analyze a little more closely the relations between the minimization of Q_α and some Sobolev trace inequalities. Although we have already used some more or less evident link between the two, we have not yet formalized the question.

In our context the trace inequalities state that the embedding of $H^1(B_1)$ into $L^q(\partial B_1)$ is continuous for $q \in [1, 2_*]$; that is, for all $q \in [1, 2_*]$ there exists $C > 0$ such that

$$\left(\int_{\partial B_1} |u|^q \, d\sigma\right)^{2/q} \leq C||u||^2$$

for every $u \in H^1(B_1)$. We set

$$S_q := \inf_{\substack{u \in H^1(B_1) \\ u \not\equiv 0}} \frac{||u||^2}{\left(\int_{\partial B_1} |u|^q \, d\sigma\right)^{2/q}} \tag{6.5.1}$$

and we recall that S_q is attained for $q \in [1, 2_*)$ because the corresponding embedding is compact. If $q = 2_*$ the embedding is no longer compact and the situation is more complex (see [39] and references there-in) and only partial results are known. However, combining the condition of Theorem 1 of [39] with the results of [37], one can say that *for the unit ball* B_1 the constant S_{2_*} is attained.

The question of the symmetry of minimizer of S_q has been treated in [58], [40], and [33] (see also the references in these papers).

Roughly speaking it turns out that radial symmetry of minimizers depends on the size of the domain. Confining ourselves to the context where B_1 is the unit ball, the main results about symmetry (deduced from [33], [40] and [58]) take the following form: denoting by μB_1 the ball of radius μ centered at zero, then the functions that attain S_q in (6.5.1) are radial for all μ small enough, and nonradial for all μ large enough.

The same kind of phenomenon takes place for a fixed domain, for example B_1, but when q varies: it has been proved in [58] (for more general problems) that minimizers of (6.5.1) are radial for all q close enough to 2, and nonradial for q large.

For further reference we quote a part of Theorem 2 of [58], specialized to our context. In its statement λ_1 denotes, as usual, the first eigenvalue of the Steklov problem (6.1.5).

Theorem 6.5.1 (Lami Dozo, Torné) *If*

$$q - 1 > \frac{1}{\lambda_1^2}(1 - (N-1)\lambda_1), \tag{6.5.2}$$

then no minimizer of (6.5.1) is radial.

Below we will give an interpretation of the number appearing in the right hand side of (6.5.2).

Notice that minimizers of S_q, normalized by $||u|| = 1$, are solutions of

$$\begin{cases} -\Delta u + u = 0 & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = S_q^{q/2} u^{q-1} & \text{on } \partial B_1. \end{cases} \tag{6.5.3}$$

Let us now return to the Hénon problem. In the rest of this section we always assume that $q \in (2, 2_*)$. The link with the trace inequalities is given by ii) of Lemma 6.1.6; indeed, denoting by $S_q : H^1(B_1) \setminus H_0^1(B_1) \to \mathbb{R}$ the functional

$$S_q(u) := \frac{||u||^2}{\left(\displaystyle\int_{\partial B_1} |u|^q \, d\sigma\right)^{2/q}},$$

Lemma 6.1.6 shows that as $\alpha \to \infty$,

$$\frac{Q_\alpha(u)}{(\alpha+N)^{2/q}} = S_q(u) + o(1)$$

uniformly on bounded subsets of $H^1(B_1)$. Therefore, the functional S_q plays the role of a limiting functional for Q_α when $\alpha \to \infty$. It is clear that many properties of minimizers of Q_α for α large and of S_q should coincide. Indeed this fact, that we have already used, has further consequences that we now examine, especially in connection with the results on $Q''_\alpha(u_\alpha)$ of Section 6.3.

Theorem 6.5.2 *For all $q \in (2, 2_*)$ the minimizers of S_q over $H^1_{rad}(B_1)$ are local minima of S_q over the whole space $H^1(B_1)$.*

Proof. It is a simplified version of the proof of Proposition 6.3.6. Indeed we will show that if u is a minimizer of S_q over $H^1_{rad}(B_1)$, then

$$\min_{\substack{v \in T_u\mathcal{S} \\ ||v||=1}} S''_q(u)\cdot v^2 > 0,$$

where $T_u\mathcal{S} = \{v \in H^1(B_1) \ : \ \langle u, v\rangle = 0\}$. Notice that, by (6.5.3), $\langle u, v\rangle = 0$ is equivalent to $\int_{\partial B_1} u^{q-1} v \, d\sigma = 0$.

Since u minimizes S_q among radial functions, we have that $u = \varphi_1$ and $S_q(u) = S_q(\varphi_1) = |\partial B_1|^{1-2/q}\lambda_1$; therefore, with the same arguments as in Lemma 6.3.1 we see that

$$\begin{aligned} S''_q(u)\cdot v^2 &= S''_q(\varphi_1)\cdot v^2 \\ &= 2|\partial B_1|^{1-2/q}\lambda_1 \left(||v||^2 - (q-1)|\partial B_1|^{q/2-1}\lambda_1^{q/2} \int_{\partial B_1} \varphi_1^{q-2} v^2 \, d\sigma \right) \end{aligned}$$

for all $v \in T_u\mathcal{S}$. Recalling that $\varphi_1^{q-2}(1) = |\partial B_1|^{1-q/2}\lambda_1^{1-q/2}$, and taking $||v|| = 1$, we get

$$S''_q(u)\cdot v^2 = 2|\partial B_1|^{1-2/q}\lambda_1 \left(1 - (q-1)\lambda_1 \int_{\partial B_1} v^2 \, d\sigma \right).$$

Since $v \in T_u\mathcal{S}$, we have $\int_{\partial B_1} v^2 \, d\sigma \le 1/\lambda_2$, so that

$$S''_q(u)\cdot v^2 \ge 2|\partial B_1|^{1-2/q}\lambda_1 \left(1 - (q-1)\frac{\lambda_1}{\lambda_2} \right)$$

for all $v \in T_u\mathcal{S}$, with $||v|| = 1$. By the results at the end of the proof of Proposition 6.3.6 the number in the right hand side of the preceding inequality is uniformly positive, for all $q \in (2, 2_*)$, which shows that $S''_q(u)$ is positive definite on $T_u\mathcal{S}$. Hence $u = \varphi_1$ is a local minimum for S_q over $H^1(B_1)$. $\square$

It is not known whether the best constant S_q is attained by a radial function for all $q \in (2, 2_*)$. If the ground states are not radial for some q, it is quite natural to expect that they bifurcate from the branch of radial minimizers. Theorem 6.5.2 shows however that this is definitely not the case: nondegeneracy of radial minimizers over the whole space $H^1(B_1)$ rules out any bifurcation phenomenon. Nonradial ground states, if any exist, are rather "separated objects", located far away from the radial minimizers and whose existence begins only after a certain value of q.

It is also interesting to compare our results with Theorem 6.5.1, by Lami Dozo and Torné (the actual result from [58] is much more general and applies to a wider class of problems). Theorem 6.5.1 states that if

$$q - 1 > \frac{1}{\lambda_1^2}(1 - (N-1)\lambda_1), \tag{6.5.4}$$

then no minimizer of (6.5.1) is radial; the argument consists in showing that a suitable (small nonradial) variation of a radial minimizer makes the functional $S_q(u)$ decrease. Therefore we are in the presence of a local phenomenon, around radial minimizers.

On the other hand, Theorem 6.5.2 shows that radial minimizers are local minima over the whole space $H^1(B_1)$, for all $q \in (2, 2_*)$; the key argument is the fact already proved that for all these q,

$$q - 1 < \frac{\lambda_2}{\lambda_1}. \tag{6.5.5}$$

A natural question is to compare the conditions (6.5.4) and (6.5.5); our intent is to give a natural interpretation of (6.5.4). The next result shows that the two conditions are in some sense dual.

Proposition 6.5.3 *There results*

$$\frac{1}{\lambda_1^2}(1 - (N-1)\lambda_1) = \frac{\lambda_2}{\lambda_1}. \tag{6.5.6}$$

Proof. We recall from (6.3.10) that

$$\lambda_k = k - 1 + \frac{I_{k+N/2-1}(1)}{I_{k+N/2-2}(1)}$$

for all $k = 1, 2, \ldots$. In particular,

$$\lambda_1 = \frac{I_{N/2}(1)}{I_{N/2-1}(1)} \qquad \text{and} \qquad \lambda_2 = \frac{I_{N/2+1}(1)}{I_{N/2}(1)} + 1.$$

To prove (6.5.6) we have to show that $\frac{1}{\lambda_1} - N + 1 = \lambda_2$.

The already used recursive relation $I_{\nu-1}(x) = I_{\nu+1}(x) + \frac{2\nu}{x} I_\nu(x)$, for $\nu = N/2$ and $x = 1$ reads

$$I_{N/2-1}(1) = I_{N/2+1}(1) + N I_{N/2}(1).$$

Therefore

$$\frac{1}{\lambda_1} - N + 1 = \frac{I_{N/2-1}(1)}{I_{N/2}(1)} - N + 1 = \frac{I_{N/2+1}(1) + N I_{N/2}(1)}{I_{N/2}(1)} - N + 1 = \frac{I_{N/2+1}(1)}{I_{N/2}(1)} + 1 = \lambda_2.$$

□

Although (6.5.4) or (6.5.5) are only sufficient conditions for the existence of nonradial minimizers, the fact that $\lambda_2/\lambda_1 > N/(N-2) = 2_* - 1$ for all $N \geq 3$ (proved in Section 6.3) and the variational properties of the radial minimizers described in this section seem to provide some evidence towards the validity of the following

Conjecture. For all $N \geq 3$ and for all $q \in (2, 2_*)$, the best constant S_q for the trace inequality on the *unit* ball of $\mathbb{R}^N$ is attained by a *radial* function.

6.6 Symmetry of ground states for slow growth

In this final section we return to the Neumann problem for the Hénon equation. We have seen that for every $q \in (2_*, 2^*)$ the minimizers of Q_α are not radial provided α is sufficiently large. In the interval $(2, 2_*)$ the situation is less clear, since it depends on the symmetry properties of the minimizers of the trace inequality, which are not precisely known for the unit ball. We point out that even if one knows that the minimizers of S_q are radial, it is not clear a priori that also the minimizers of Q_α should be radial.

In this section we investigate the symmetry of minimizers when q is close to 2. It is interesting to keep in mind the behaviour of minimizers for the Dirichlet problem described in [80]: in that case the authors showed that for q close to 2 minimizers are nonradial only if α is very large (the threshold α^* between radial and nonradial minimizers tends to infinity as $q \to 2$). However the symmetry breaking phenomenon persists, as for a fixed q close to 2 one has nonradial solutions for very large α.

We show in Theorem 6.6.1 below that this is not the case for minimization in $H^1(B_1)$: for q close to 2 minimizers are radial for *all* α large enough.

Of course we take advantage of some result for the "limit" problem given by the minimization of S_q. The precise result we need is contained in Theorem 4 of [58]. There it is proved that there exists $\hat{q} \in (2, 2_*]$ such that for every $q \in (2, \hat{q}\,]$ the problem $\min_{u\in H^1(B_1)} S_q(u)$ has a unique solution, which is radial. Of course this solution (with norm equal to one) is φ_1, and $S_q(\varphi_1) = |\partial B_1|^{1-2/q}\lambda_1$.

Theorem 6.6.1 *Let $q \in (2, \hat{q}\,)$. For every α large enough the problem*

$$\min_{\substack{u\in H^1(B_1)\\ u\not\equiv 0}} \frac{||u||^2}{\left(\displaystyle\int_{B_1} |x|^\alpha |u|^q\,dx\right)^{2/q}} \tag{6.6.1}$$

has a unique positive solution (normalized by $||u|| = 1$), and it is a radial function.

Proof. The proof of this theorem follows very closely that of Theorem 6.4.1; the main difference comes from the fact that in the present case we are not dealing with radial functions, which tends to complicate things. On the other hand we will profit of the fact that we are now working with $q < 2_*$.

By Lemma 6.1.6 we have that as $\alpha \to \infty$,

$$\frac{Q_\alpha(u)}{(\alpha+N)^{2/q}} = S_q(u) + o(1)$$

uniformly on bounded subsets of $H^1(B_1)$; thus, setting $m_\alpha = \min_{H^1(B_1)\cap\mathcal{S}} Q_\alpha$ we see that

$$m_\alpha \sim (\alpha+N)^{2/q}|\partial B_1|^{1-2/q}\lambda_1.$$

Let $u_\alpha \in H^1(B_1)\cap\mathcal{S}$ be (positive and) such that $Q_\alpha(u_\alpha) = m_\alpha$. Then, up to subsequences, $u_\alpha \to u$ weakly in $H^1(B_1)$, and strongly in $L^q(B_1)$ and in $L^q(\partial B_1)$, since $q < 2_*$. Notice that $u \not\equiv 0$, because otherwise

$$|\partial B_1|^{1-2/q}\lambda_1 + o(1) = \frac{m_\alpha}{(\alpha+N)^{2/q}} = \frac{1}{\left(\int_{\partial B_1}|u_\alpha|^q\,d\sigma\right)^{2/q}} + o(1) \to \infty,$$

which is absurd.

Furthermore, since $q < \hat{q}$,

$$|\partial B_1|^{1-2/q}\lambda_1 = \min_{\substack{v\in H^1(B_1)\\ v\not\equiv 0}} \frac{||v||^2}{\left(\int_{\partial B_1}|v|^q\,d\sigma\right)^{2/q}} \le \frac{||u||^2}{\left(\int_{\partial B_1}|u|^q\,d\sigma\right)^{2/q}} \le \frac{1}{\left(\int_{\partial B_1}|u|^q\,d\sigma\right)^{2/q}}$$

$$= \lim_{\alpha\to+\infty} \frac{1}{\left(\int_{\partial B_1}|u_\alpha|^q\,d\sigma\right)^{2/q}} = \lim_{\alpha\to+\infty}\left(\frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/q}} + o(1)\right) = |\partial B_1|^{1-2/q}\lambda_1.$$

Therefore we have that $||u|| = 1$ and that u is a minimizer for S_q. By the above quoted results and the assumption $q < \hat{q}$, it must be $u = \varphi_1$. We conclude that every sequence of minimizers for Q_α converges to φ_1 strongly in $H^1(B_1)$.

We now show that Q_α has a unique minimizer for α large. Suppose this is not true; then for every α in an unbounded set A, there exist two distinct (positive) minimizers u_α and v_α. As in the proof of Theorem 6.4.1, if we set $\psi_\alpha = (u_\alpha - v_\alpha)/||u_\alpha - v_\alpha||$, we see that it solves

$$\begin{cases} -\Delta\psi_\alpha + \psi_\alpha = (q-1)m_\alpha^{q/2}|x|^\alpha c_\alpha \psi_\alpha & \text{in } B_1 \\ \dfrac{\partial \psi_\alpha}{\partial \nu} = 0 & \text{on } \partial B_1, \end{cases} \tag{6.6.2}$$

with

$$c_\alpha = \int_0^1 (v_\alpha + t(u_\alpha - v_\alpha))^{q-2}\, dt.$$

Notice that $|c_\alpha| \le (u_\alpha + v_\alpha)^{q-2}$.

Since $\psi_\alpha \in \mathcal{S}$, up to subsequences we can assume that $\psi_\alpha \to \psi$ weakly in $H^1(B_1)$ and strongly in $L^q(B_1)$ and in $L^q(\partial B_1)$. We claim that $\psi \not\equiv 0$. To see this we multiply the equation in (6.6.2) by ψ_α and we use Lemma 6.1.6 to obtain

$$\begin{aligned} 1 = ||\psi_\alpha||^2 &= (q-1)m_\alpha^{2/q}\int_{B_1} c_\alpha \psi_\alpha^2 |x|^\alpha\, dx \\ &= (q-1)(|\partial B_1|^{2/q-1}\lambda_1^{q/2} + o(1))(\alpha+N)\int_{B_1} c_\alpha \psi_\alpha^2 |x|^\alpha\, dx \\ &\le C\left((\alpha+N)\int_{B_1} c_\alpha^{\frac{q}{q-2}}|x|^\alpha\, dx\right)^{1-2/q}\left((\alpha+N)\int_{B_1}\psi_\alpha^q |x|^\alpha\, dx\right)^{2/q} \\ &\le C\left((\alpha+N)\int_{B_1}(u_\alpha+v_\alpha)^q|x|^\alpha\, dx\right)^{1-2/q}\left(\int_{\partial B_1}\psi_\alpha^q\, d\sigma + o(1)\right)^{2/q} \\ &\le C\left(\int_{\partial B_1}(u_\alpha+v_\alpha)^q\, d\sigma + o(1)\right)^{1-2/q}\left(\int_{\partial B_1}\psi_\alpha^q\, d\sigma + o(1)\right)^{2/q} \\ &\le C\left(\int_{\partial B_1}\psi_\alpha^q\, d\sigma + o(1)\right)^{2/q}. \end{aligned}$$

If ψ is zero, then the strong convergence of ψ_α in $L^q(\partial B_1)$ gives a contradiction.

We now pass to the limit in the weak form of (6.6.2), which is

$$\langle \psi_\alpha, \phi\rangle = (q-1)m_\alpha^{2/q}\int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha\, dx$$

for all $\phi \in H^1(B_1)$. We write

$$(\alpha+N)\int_{B_1} c_\alpha \psi_\alpha \phi |x|^\alpha = (\alpha+N)\left[\int_{B_1} \varphi_1^{q-2}\psi_\alpha\phi|x|^\alpha + \int_{B_1}(c_\alpha - \varphi_1^{q-2})\psi_\alpha\phi|x|^\alpha\right]$$

$$= (\alpha+N)\left[\int_{B_1} \varphi_1^{q-2}\psi\phi|x|^\alpha + \int_{B_1} \varphi_1^{q-2}(\psi_\alpha-\psi)\phi|x|^\alpha + \int_{B_1}(c_\alpha - \varphi_1^{q-2})\psi_\alpha\phi|x|^\alpha\right]$$

and we evaluate the three terms in the right hand side separately. For the first one we have

$$(\alpha+N)\int_{B_1} \varphi_1^{q-2}\psi\phi|x|^\alpha\,dx = \int_{\partial B_1}\varphi_1^{q-2}\psi\phi\,d\sigma - \int_{B_1}\nabla(\varphi_1^{q-2}\psi\phi)\cdot|x|^\alpha x\,dx,$$

and, by Hölder inequality,

$$\left|\int_{B_1}\nabla(\varphi_1^{q-2}\psi\phi)\cdot|x|^\alpha x\,dx\right| \le C\int_{B_1}|\psi||\phi||x|^\alpha\,dx +$$

$$+C\int_{B_1}(|\phi||\nabla\psi| + |\psi||\nabla\phi|)|x|^\alpha\,dx \le C||\psi||_2||\phi||_{2^*}\left(\int_{B_1}|x|^{\alpha N}\,dx\right)^{1/N} +$$

$$+C(||\nabla\psi||_2||\phi||_{2^*} + ||\nabla\phi||_2||\psi||_{2^*})\left(\int_{B_1}|x|^{\alpha N}\,dx\right)^{1/N} = o(1)$$

as $\alpha\to\infty$. Therefore

$$(\alpha+N)\int_{B_1}\varphi_1^{q-2}\psi\phi|x|^\alpha\,dx = \int_{\partial B_1}\varphi_1^{q-2}\psi\phi\,d\sigma + o(1).$$

For the second term we apply Hölder inequality to obtain, by Lemma 6.1.6,

$$(\alpha+N)\int_{B_1}\varphi_1^{q-2}(\psi_\alpha-\psi)\phi|x|^\alpha\,dx \le \left((\alpha+N)\int_{B_1}\varphi_1^q|x|^\alpha\,dx\right)^{1-2/q}\cdot$$

$$\cdot\left((\alpha+N)\int_{B_1}|\psi_\alpha-\psi|^q|x|^\alpha\,dx\right)^{1/q}\cdot\left((\alpha+N)\int_{B_1}|\phi|^q|x|^\alpha\,dx\right)^{1/q}$$

$$= \left(\int_{\partial B_1}\varphi_1^q\,d\sigma + o(1)\right)^{1-2/q}\cdot\left(\int_{\partial B_1}|\psi_\alpha-\psi|^q\,d\sigma + o(1)\right)^{1/q}\cdot$$

$$\cdot\left(\int_{\partial B_1}|\phi|^q\,d\sigma + o(1)\right)^{1/q} = o(1)$$

because $\psi_\alpha\to\psi$ strongly in $L^q(\partial B_1)$.

Finally, for the third term we write

$$(\alpha+N)\int_{B_1}(c_\alpha-\varphi_1^{q-2})\psi_\alpha\phi|x|^\alpha\,dx \le \left((\alpha+N)\int_{B_1}|c_\alpha-\varphi_1^{q-2}|^{\frac{q}{q-2}}|x|^\alpha\,dx\right)^{\frac{q-2}{q}}\cdot$$

$$\cdot\left((\alpha+N)\int_{B_1}|\psi_\alpha|^q|x|^\alpha\,dx\right)^{1/q}\left((\alpha+N)\int_{B_1}|\phi|^q|x|^\alpha\,dx\right)^{1/q}$$

and we readily recognize, as above, that the last two integrals are uniformly bounded as $\alpha \to \infty$. Recalling the definition of c_α and (6.4.4) at page 97, it is easy to see that the first integral goes to zero as $\alpha \to \infty$.

Putting together the above estimates we can say that

$$\langle \psi_\alpha, \phi \rangle = (q-1)(|\partial B_1|^{q/2-1} \lambda_1^{q/2} + o(1)) \left(\int_{\partial B_1} \varphi_1^{q-2} \psi \phi \, d\sigma + o(1) \right),$$

so that, when $\alpha \to \infty$,

$$\langle \psi, \phi \rangle = (q-1)\lambda_1 \int_{\partial B_1} \psi \phi \, d\sigma, \tag{6.6.3}$$

for all $\phi \in H^1(B_1)$ (we have used the fact that $\varphi_1^{q-2} \equiv |\partial B_1|^{1-q/2} \lambda_1^{1-q/2}$ on ∂B_1). Equation (6.6.3) says that ψ is a (nontrivial) weak solution of

$$\begin{cases} -\Delta \psi + \psi = 0 & \text{in } B_1 \\ \dfrac{\partial \psi}{\partial \nu} = (q-1)\lambda_1 \psi & \text{on } \partial B_1. \end{cases}$$

Since we know that $(q-1)\lambda_1$ is not an eigenvalue of the Steklov problem (6.1.5), we conclude that $\psi \equiv 0$, a contradiction.

Therefore $u_\alpha \equiv v_\alpha$ for all α large. In other words, problem (6.6.1) has a unique solution for $q \in (2, \hat{q}\,)$ and α large. Since (6.6.1) is invariant under rotations, this solution must be radial. □

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