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Some aspects of spinors
– classical and noncommutative –

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Contents

INTRODUCTION

The first clear record of the notion of a spin structure on a manifold is contained in Haefliger's paper [28] which also identifies the obstruction to the existence of such structure with the vanishing of the first Stiefel-Whitney class of the manifold. Similar results are also present in works by Frenkel and Dedecker ([22] and [17] treat the problem of the extension of a structure group in the broader setting of sheaf cohomology, in the spirit of [27]). In the physics literature this notion appeared quite late despite the pioneering works [20] and [57] aiming at defining the Dirac equation on curved Lorentzian manifolds. The reason is to be sought in the fact that the global character of spinors was not of primary concern when such early attempts were considered. See [50] and [52] for a nice exposition of the history of spinors and spin structures.

In sections 1.7 and 3.1 we shall present several definitions of a spin structure and illustrate differences and similarities among them. Before that, we start by giving a concise account of various bundle theory concepts needed to introduce spin structures. This account will serve the dual purpose of setting notation and proving some results which are needed for our discussion.

In Chapter 2 we introduce metrics on vector bundles and analyse the relationship among isometry classes, frame bundles and path-components of metrics. In particular, we prove Theorem 2.22 which is new to the best of our knowledge.

In Chapter 3 the issue of general covariance of spinor fields (for brevity: spinors) and related objects is reconsidered. This question has in fact at least two aspects regarding the transformation rules with respect to two different (though intrinsically related) operations: a change of coordinate system, and a diffeomorphism. In physics literature one can sometimes find statements like "spinors transform as 'spinors' with respect to the former and as scalars with respect to the latter". While these statements can in a certain sense be justified, they are meaningful only after introducing certain mathematical structures and determining their transformation properties, as we shall explain in the next sections.

Even though in principle one usually works with vector (bilinear) or tensor (multilinear) combinations of spinors, or even with invariants (scalars) like the Lagrangian, a transformation rule of spinor fields is really needed if one wants to treat them as independent variables (e.g. with respect to some variational principle).

However a subtlety with spinors, as compared to tensors, is that one needs to work with particular double covers of the groups we are accustomed to in the case of tensors. The global mathematical constructs needed for this task have been developed in the second half of the last century [28], [2], [39], via the notion of spin structure. The notion of a spin structure σ is topological in nature, but for our purposes it is here considered as an auxiliary tool to the definition of spinors and, as such, it requires a Riemannian metric g to be specified on a given (oriented) smooth manifold M. More precisely one needs a prolongation of the principal SO_n bundle $\mathrm{SO}_g(M)$ of oriented g-orthonormal frames to the group Spin_n. Then there is the associated space of smooth spinor fields $S_{\sigma,g}$ and the Hilbert space $\mathscr{H}_{\sigma,g} = L^2(S_{\sigma,g}, \mathrm{vol}_g(M))$. It should be stressed that the notion of spin structure is not only sufficient, but in fact necessary for the consistency of the definition of spinor fields in the following sense: given a principal Spin_n-bundle P on M, it corresponds to some spin structure on M (i.e. there is some equivariant bundle map $m\colon P \to \mathrm{SO}_g(M)$) if and only if TM is isomorphic (as a vector bundle) to $P \times_{\mathrm{Spin}_n} \mathbb{R}^n$. In other words, if we demand spinors to project down to tensor fields we end up introducing spin structures.

The question of the change of coordinates is then translated to the transformation rules under the change of a local orthonormal frame and corresponding change of the local spinor frame. We shall understand such a change as an automorphism of the tangent bundle, the related automorphism of the bundle of frames, and its lift to a spin structure. It should be mentioned that a large automorphism (i.e. not belonging to the connected component of the group of automorphisms) may require however a change of the spin structure σ. Concerning the question of diffeomorphisms, it is its derivative (tangent map) that plays the role of the automorphism in question.

In all these cases we shall be able to give a transformation rule of spinor fields, i.e. define a new spinor field. This new spinor field, unless the automorphism respects the metric (so the diffeomorphism is an isometry), will in general be a spinor field associated to a different metric, namely the pull back of the original metric. More precisely, we are able to give the components of the new spinor field with respect to the transformed frame (or more precisely transformed spinor frame). We should stress at this point that remaining solely in the aforementioned framework does not permit to describe the components of a given one and the same spinor field with respect to two linear frames which are orthonormal with respect to two different metrics[1]. This becomes possible however if the theory allows spinors with an infinite number of components (which carry a faithful representation of a double covering of the oriented general linear group). Such an extension is not usually appreciated (see however [45], [44]).

As far as the group $\mathrm{Diff}^+(M)$ of orientation preserving diffeomorphisms of M is concerned, it acts both on g (by a pull-back) and on $[\sigma]$ (by a suitably defined pull-back $f^*\sigma$). We shall show that any $f \in \mathrm{Diff}^+(M)$ lifts (in exactly two ways) to a unitary operator from $\mathscr{H}_{\sigma,g}$ to $\mathscr{H}_{f^*\sigma,f^*g}$. This provides a kind of a unitary implementation (in a sense that we shall specify) of the action of a certain double covering $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ of the subgroup $\mathrm{Diff}^+_\sigma(M)$ of $\mathrm{Diff}^+(M)$ preserving the spin structure σ, so in particular of the connected component

[1]Actually conformally related frames can still be treated.

of $\mathrm{Diff}^{+}(M)$. Moreover we prove that the canonically defined Dirac operator is shown to be equivariant with respect to these actions, so in particular its spectrum is invariant under the diffeomorphisms.

In this thesis we work with smooth (oriented) manifolds and use component-free notation, the usual spinor or vector indices can be easily inserted. We take the components of spinors as usual numbers, but our discussion applies in the anticommuting (Grassmann) case as well.

We now turn our attention to the noncommutative part of this thesis.

In noncommutative geometry spaces are traded for algebras. The main idea can be traced back to the Gelfand-Naimark duality established in the 1943 paper [23] between the category of compact Hausdorff spaces – and continuous functions between them – and the category of commutative unital C^*-algebras – and unital $*$-homomorphisms between them (compare also the sheaf theoretic approach to algebraic geometry started around the same period of time, or even the earlier concept of group algebra developed by Emmy Noether in the 1920s). Loosely speaking, relaxing the commutativity assumption on the algebraic side should then correspond to a generalization of the concept of a compact Hausdorff space (although strictly speaking this is a generalization of the *category* of such spaces). Noncommutative geometry focuses on spin Riemannian manifolds to achieve a generalization which is akin in spirit to the Gelfand-Naimark duality, even though to date it has been investigated mainly at the level of objects. As a candidate for the objects in this correspondence Connes has introduced the notion of a spectral triple. This is basically an algebra together with a Hilbert representation and a self-adjoint operator, together with certain axioms suggested by the commutative case (see Section 4.2 below for a precise definition). Moreover, other additional properties (dimension, regularity, reality, first order, orientation, smoothness, see [10], [26], [11]) again suggested by the commutative case can be used to characterise the commutative spectral triples of manifolds among all commutative spectral triples. These further properties have been formulated for noncommutative spectral triples [10]. They are satisfied e.g by the noncommutative torus, and part of them holds for various quantum groups.

The composition of not necessarily commutative spectral triples corresponding to the Cartesian product of manifolds is of relevance for construction of a would-be tensor category, but also bears interest for some applications in theoretical physics. Classically, spinors on product manifolds have been considered for instance in Kaluza-Klein type models, see e.g. [56] and subsequent generalizations to supergravity and string theories. As an example on the noncommutative side instead, the almost commutative spectral triple corresponding to the standard model of particle physics [7] is a tensor product of a canonical commutative spectral triple with a finite dimensional noncommutative one. Moreover the tensor product with a spectral triple of complex dimension is used in the Connes-Marcolli treatment [11] of dimensional renormalization of quantum fields.

In Chapter 4 we study in more detail the behaviour under tensor product of one of the additional properties, namely the reality axiom. This axiom is much more important in the noncommutative case as it is there employed in the formulation of few of the other axioms.

We carefully analyse all the possibilities and note that in even dimensions there are always two real structure operators J, that differ by multiplication by the grading operator. None of them should be preferred as they are perfectly on the same footing. This leads to a richer table of their possible tensor products, which we study systematically, completing the results of [54] (see also [46]) obtained for the even-even case and the even-odd (or odd-even) case. We construct also the tensor product of two odd real spectral triples (that requires a doubling of the tensor product of the Hilbert spaces). When dealing with odd spectral triples we are careful about the two inequivalent representations of gamma matrices (Clifford algebra).

Composing two even-dimensional Dirac operators we consider two choices, which differ by using the chirality operator (grading) either of the first or of the second space. The two operators thus obtained are unitarily equivalent if no other requirements are imposed, but this is no longer the case when boundaries are present ([6]). Moreover, the first expression is relevant for the composition of an even dimensional space with an odd dimensional one, while the second expression is relevant for the composition of an odd dimensional space with an even dimensional one.

For concreteness, we provide the explicit formulae for the eigenvalues and eigenvectors in terms of those of the individual components. Furthermore, we also analyse few of the additional conditions (axioms) like dimension, regularity, first order and orientation.

Note. The content of Chapters 3 and 4 corresponds to the following papers respectively:

- [14] *Dirac operator on spinors and diffeomorphisms*, L. Dabrowski and G. Dossena, to be published in Class. Quantum Grav. Preprint available at `http://arxiv.org/abs/1209.2021`
- [13] *Product of real spectral triples*, L. Dabrowski and G. Dossena, Vol. 8, No. 8 (2011) 1833–1848, Int. J. Geom. Methods Mod. Phys.

CLASSICAL PART

1 A concise account of bundle theory

What follows is a terse account of the part of bundle theory we need for our purposes. One can use [30], [47], [32], [29] as references. Most of the results in this chapter can be found there. In some cases we provide proofs for known results either for the sake of completeness or, sometimes, because they slightly differ from the accounts in the existing literature.

We shall use basically the theory of smooth manifolds and Lie groups as presented e.g. in [55], [4]. In particular, a smooth n-manifold M is a connected second countable Hausdorff topological space equipped with a smooth structure, that is a maximal C^∞ atlas of charts. Here a chart is an injective map $U \xrightarrow{\phi} \Omega_U \subset \mathbb{R}^n$ from an open subset $U \subset M$ onto some open subset $\Omega_U \subset \mathbb{R}^n$. As usual, two charts are compatible if their transition functions are C^∞ and an atlas is a collection of pairwise compatible charts whose domains cover M. Any open subset $U \subset M$ inherits a smooth structure by restriction of charts and it follows by the definition that each map $U \xrightarrow{\phi} \Omega_U$ is a diffeomorphism with respect to their corresponding smooth structures.

Remark 1.1. The familiar definition just sketched is somewhat redundant: we might as well take M as a set from the start and induce a topology on it by declaring a subset $S \subset M$ open if for each chart (U, ϕ) of a chosen atlas belonging to the maximal atlas the set $\phi(S \cap U)$ is open in $\mathbb{R}^n$. The global topological properties required in the familiar definition above (connectedness, second countability, Hausdorffness) are then to be viewed as conditions on any atlas defining the smooth structure. See [4] for a beautiful account of this approach to manifolds.

Remark 1.2. Many aspects of the theory of bundles are topological in nature and can be developed in the continuous case. However, since we shall eventually apply them to smooth manifolds and smooth bundles (where the Dirac equation belongs), the following treatment will present the theory mainly for the smooth case. Switching to the continuous case is mostly done by substituting all occurrences of the words "smooth" and "diffeomorphic" with the words "continuous" and "homeomorphic" respectively, and by substituting all smooth manifolds with topological manifolds or with sufficiently nice topological spaces (e.g. CW complexes). The use of more general spaces is also possible in the context of Dold's approach (see [18] and Appendix 1 in [30]).

1.1 FIBER BUNDLES

We now briefly review the notion of a fiber bundle. The theory of fiber bundles is ubiquitous in classical geometry and in most of the geometrical aspects of theoretical physics. It is hard to overestimate its importance.

Definition 1.3. A surjective smooth map $E \xrightarrow{\pi} B$ between manifolds is a (smooth) *fiber bundle* with *base space* B, *total space* E and *typical fiber* F if there are a smooth manifold F and an open cover $\{U_\alpha \mid \alpha \in A\}$ of B together with diffeomorphisms $U_\alpha \times F \xrightarrow{\psi_\alpha} \pi^{-1}(U_\alpha)$ such that $(\pi \circ \psi_\alpha)(x, y) = x$. The map π is called the *projection* of the bundle. The closed subspace $F_x := \pi^{-1}(x) \subset E$, which inherits from E a smooth structure making it diffeomorphic to F, is called the *fiber* over $x \in B$. The total space E is the disjoint union of all the fibers F_x when x runs over B.

For simplicity of notation, when no confusion arises we will indulge in the slight abuse of calling a fiber bundle by its total space.

Definition 1.4. Given two fiber bundles $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ with typical fibers F and F' respectively, a morphism from E to E' is a smooth map $m\colon E \to E'$ such that there is a smooth map $\mu\colon B \to B'$ satisfying $\mu \circ \pi = \pi' \circ m$ (i.e. fibers are mapped into fibers). It is clear that an isomorphism of fiber bundles is a diffeomorphism between the total spaces such that $\mu \circ \pi = \pi' \circ m$ where $\mu\colon B \to B'$ is a diffeomorphism between the base spaces. When $B = B'$ and $\mu = \mathrm{id}_B$, a morphism will be called a based-morphism (of fiber bundles over B).

A fiber bundle is then a generalization of the familiar Cartesian product of manifolds. Any fiber bundle based-isomorphic to $B \times F \to B$, where the projection is onto the first factor, is called a *trivial bundle*[1]. In our definition then a fiber bundle is locally trivial, and the diffeomorphisms ψ_α as in Definition 1.3 are called *local trivializations.* For our purposes we can safely ignore bundles which are not locally trivial. Moreover, the local triviality condition allows us to use the powerful techniques of homotopy theory as we shall see.

1.2 Vector bundles

Among the fundamental examples of fiber bundles stands the notion of a vector bundle.

Definition 1.5. A (real or complex) *vector bundle* is a fiber bundle such that each fiber has a structure of a (real or complex) vector space. Morphisms are just fiber bundle morphisms which preserve the vector space structure on each fiber. Analogously for the notion of a based-morphism. The dimension of the fiber as a vector space is called the rank of the vector bundle.

Since in this and the next few sections we shall be concerned with real vector bundles only, we drop the real/complex qualification for the time being.

We record without proof a well known useful result (see [41], Lemma 2.3).

[1]Some authors distinguish between a trivial bundle and a trivializable one: both are bundles isomorphic to the product bundle but the former has a choice of isomorphism singled out, the latter has not.

Lemma 1.6. Given vector bundles $E \to B$ and $E' \to B$ of same rank n, a based-morphism of fiber bundles $f\colon E \to E'$ which restricts to a vector space isomorphism on each fiber is a based-isomorphism of vector bundles.

We recall a few examples of vector bundles.

Example 1.7. The primary example is the tangent bundle $TM \xrightarrow{\pi} M$ of a smooth n-manifold M. It is a rank n vector bundle over M. Its total space is the set of all tangent vectors at $x \in M$ when x runs over M. The projection π assigns to each tangent vector its point of tangency on M. The topological and smooth structures are defined on TM by local trivializations $U \times \mathbb{R}^n \to \pi^{-1}(U)$ so as to make them diffeomorphisms, where $U \subset M$ is a domain of a chart of M. It is important to note that the tangent bundle is a functor from the category of smooth manifolds (and smooth maps between them) and the category of smooth vector bundles. This means that to each map $f\colon M \to M'$ there is associated a corresponding morphism $Tf\colon TM \to TM'$ of vector bundles, and this association is functorial. We shall use this fact later.

Example 1.8. Another related example is the normal bundle of an embedded submanifold $M \subset M'$ of a Riemannian manifold M'.

Example 1.9. All natural constructions of linear algebra (dual vector space, tensor product, direct sum, symmetric and anti-symmetric tensor product) give rise to corresponding constructions for a vector bundle. In particular when we apply these to the tangent bundle of a manifold we obtain fundamental objects used in geometrical accounts of classical mechanics and relativity theory (tensor fields, etcetera).

In Chapter 2 we shall need, along with the usual one given by based-isomorphisms, a stronger notion of equivalence between vector bundles[2]. We present it here.

[2]It is possible to define an analogous stronger notion of equivalence for fiber bundles as well, but we shall not need it.

Definition 1.10. Given two rank n vector bundles E_0 and E_1 over a paracompact space B, they are *homotopic* if there is a rank n vector bundle E over $B \times [0,1]$ such that $E \restriction B \times \{i\} = E_i$ for $i = 0,1$, where we identify $B \times \{i\} \simeq B$.

The following well known and fundamental lemma shows that being homotopic is not weaker than being based-isomorphic.

Lemma 1.11. Given a vector bundle E over $B \times [0,1]$, where B is paracompact (e.g. a smooth manifold), the restrictions of E to the subspaces $B \times \{0\}$ and $B \times \{1\}$ are based-isomorphic vector bundles (identifying $B \times \{0\} \simeq B \times \{1\} \simeq B$).

Proof. E.g. see Theorem 4.3 and Corollary 4.6 on pp.29–30 in [30]. □

In fact, in general being homotopic is strictly stronger than being based-isomorphic, as the following example shows.

Example 1.12. Given the trivial rank 2 vector bundle $S^1 \times \mathbb{R}^2 \to S^1$ and a natural number $k \in \mathbb{N}$, consider the line subbundle L_k defined by its corresponding Gauss map $\Phi_k \colon S^1 \to \mathbb{RP}^1$, $\theta \mapsto [\cos k\theta : \sin k\theta]$ where $[x : y]$ are homogeneous coordinates in $\mathbb{RP}^1$. In other words, it is a line in $\mathbb{R}^2$ rotating around the origin while we move around S^1. Notice that $k \neq k'$ implies that Φ_k and $\Phi_{k'}$ correspond to different elements in $\pi_1(\mathbb{RP}^1) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. Also notice that L_k is trivial if and only if k is even, a nowhere vanishing section being given by $\theta \mapsto (\cos k\theta, \sin k\theta)$. By the well known fact that on S^1 there are only two isomorphism classes of line bundles, one represented by the trivial bundle and the other one represented by the Möbius bundle, we deduce that $L_k \simeq L_{k'}$ if and only if $k = k' \mod 2$. However, for $k \neq k'$ the line bundles L_k and $L_{k'}$ are not homotopic. Indeed, given the projection $p \colon S^1 \times [0,1] \to S^1$, assume there is a line bundle L on $p^*(S^1 \times \mathbb{R}^2)$ such that $L \restriction S^1 \times \{0\} = L_k$ and $L \restriction S^1 \times \{0\} = L_{k'}$. Then the corresponding Gauss map $\Phi_L \colon S^1 \times [0,1] \to \mathbb{RP}^1$ determines a homotopy between Φ_k and $\Phi_{k'}$ as elements of $\pi_1(\mathbb{RP}^1)$, which implies $k = k'$ as we observed before. Another way to interpret this example is the following: the tautological line bundle $\gamma_1(\mathbb{R}^3) \to \mathbb{RP}^2$ is universal for 1-dimensional manifolds, and the line bundles L_{2k} and L_{2k+1} correspond to the two elements in

$\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$. On the other hand, the tautological line bundle $\gamma_1(\mathbb{R}^2) \to \mathbb{RP}^1$ is universal only for 0-dimensional manifolds.

1.3 PRINCIPAL BUNDLES

We now give the definition of a principal bundle and illustrate it by relevant examples.

Definition 1.13. Given a smooth manifold M and a real Lie group G, a principal G-bundle over M is a fiber bundle $P \xrightarrow{\pi} M$ with typical fiber $F = G$ together with a smooth right action $P \times G \to P$ such that the bundle P admits local trivializations $U_\alpha \times G \xrightarrow{\psi_\alpha} \pi^{-1}(U_\alpha)$ with the property $\psi_\alpha(x, ab) = \psi_\alpha(x, a)b$ for each $x \in U_\alpha$ and each $a, b \in G$.

Definition 1.14. Given two principal bundles $P \xrightarrow{\pi} B$ and $P' \xrightarrow{\pi'} B'$ of groups G and G' respectively and $\rho : G \to G'$ a morphism of Lie groups, a ρ-morphism from P to P' is a morphism $m : P \to P'$ of fiber bundles together with the condition $m(ug) = m(u)\rho(g)$ for each $u \in P$ and each $g \in G$. When $B = B'$ and $\mu = \mathrm{id}_B$, we call it a based-ρ-morphism. When $B = B'$, $\mu = \mathrm{id}_B$, $G = G'$, and $\rho = \mathrm{id}_G$, we call it a based-G-morphism (of principal G-bundles over B).

As in the case of a fiber bundle, there is the notion of a trivial principal bundle: this is any principal G-bundle based-G-isomorphic to the bundle given by the Cartesian product $B \times G \xrightarrow{\pi} B$ and projection π on the first factor, plus the global action $B \times G \times G \to B \times G$ simply given by $(x, g, h) \mapsto (x, gh)$. We can then rephrase part of Definition 1.13 by saying that a principal G-bundle is locally trivial.

In a few words, a principal bundle is a fiber bundle whose typical fiber is a Lie group G acting globally (and freely) on the total space and such that the local trivializations are equivariant with respect to this action. The action is automatically free, in our definition, thanks to the local triviality. Notice also that G acts freely and transitively on each fiber. In conclusion, a principal bundle should not be confused with simply a fiber bundle whose typical fiber is a Lie group: the global action is part of the data.

Another way to view a principal bundle is by considering a so called *principal homogeneous G-space* or *G-torsor*. These are just fancy names for a group G who forgot where its identity element is. In other words, it is a space which is acted upon freely and transitively by a group G. A principal bundle can then be viewed as a locally trivial bundle of G-torsors, i.e. a locally trivial bundle where each fiber has a structure of a G-torsor and the local trivializations preserve this structure. The global action is simply the action of G on each fiber. The local triviality condition ensures the continuity (or smoothness) of the global action.

If $B = B'$, $\mu \in \mathrm{Diff}(B)$, $G = G'$, and $\rho \in \mathrm{Aut}(G)$ then the global action severely restricts the notion of a ρ-morphism: indeed any such morphism is automatically an isomorphism. In the literature this result is usually shown in the special case of a based-G-morphism, i.e. $\rho = \mathrm{id}_G$ and $\mu = \mathrm{id}_B$, so it may be worthwhile to spell out a proof for this slightly more general case.

Proposition 1.15. Given two principal G-bundles $P \xrightarrow{\pi} B$ and $P' \xrightarrow{\pi'} B$ over the same base space B and given $\rho \in \mathrm{Aut}(G)$, consider a ρ-morphism $m\colon P \to P'$. If $\mu \in \mathrm{Diff}(B)$, then m is a ρ-isomorphism.

Proof. Let us consider the case of trivial bundles first. Given $m\colon B \times G \to B \times G$ as in the claim, there are a smooth map $f\colon B \to G$ and a diffeomorphism $\mu\colon B \to B$ such that $m(x,g) = (\mu(x), f(x)\rho(g))$. The inverse is then given by

$$m^{-1}(x,g) = (\mu^{-1}(x), \rho^{-1}(f(\mu^{-1}(x))^{-1}g))$$

where ρ^{-1} is the inverse automorphism (i.e. $\rho^{-1} \circ \rho = \rho \circ \rho^{-1} = \mathrm{id}_G$), $\mu^{-1}\colon B \to B$ is the inverse diffeomorphism and $f(x)^{-1}$ is the inverse of $f(x)$ as an element of G. For the general case simply apply the above argument to local trivializations. □

In the special case of $\rho = \mathrm{id}_G$ and $\mu = \mathrm{id}_B$ we obtain the following result.

Corollary 1.16. A based-G-morphism between principal G-bundles is an isomorphism.

Another important fact for principal bundles is the equivalence between the existence of a (global) section and triviality. We record it here.

Proposition 1.17. A principal G-bundle $P \xrightarrow{\pi} B$ admits a section if and only if it is trivial.

Proof. If it is trivial then it surely admits a section. If $\sigma\colon B \to P$ is a section, the map $B \times G \to P$ given by $(x, g) \mapsto \sigma(x)g$ is a based-G-morphism, hence a based-G-isomorphism by Corollary 1.16. □

We now present two examples of principal bundles that we shall need later on.

Example 1.18. Recall that a basis of a vector space of dimension n is an ordered set of n linearly independent vectors. Given a rank n vector bundle $E \xrightarrow{\pi} B$, for each $x \in B$ we can consider the set of all bases $e_x = (e_{x,1}, \dots e_{x,n})$ of E_x, also called frames in this context. The frame bundle $Fr(E) \xrightarrow{\pi'} B$ of the vector bundle $E \xrightarrow{\pi} B$ is the principal GL_n-bundle whose total space is the collection $Fr(E)$ of all frames of $E_x = \pi^{-1}(x)$, as x runs over B, equipped with the subspace topology of the Whitney sum $\oplus^n E$. Its projection is $\pi'(e_x) = x$ and its (right) action is $e_x g = (\sum_j e_{x,j} g_{j1}, \dots, \sum_j e_{x,j} g_{jn})$. In particular, this construction applies to the case when the vector bundle is the tangent bundle $TM \to M$ of some smooth n-manifold M. In this case the frame bundle inherits a smooth structure, becoming a smooth principal GL_n-bundle over M.

Example 1.19. Another example of a principal bundle is offered by the theory of covering spaces. A covering is a fiber bundle $E \xrightarrow{p} B$ whose typical fiber F is a discrete countable space and whose total space E is connected. An automorphism of a covering is a self-diffeomorphism $f\colon E \to E$ of the total space such that $p \circ f = p$, i.e. it does not move the fibers. A covering space such that the group of all automorphisms of the coverings acts transitively on the fibers is then a principal bundle (and called a regular covering, or Galois). A typical example is $U(1) \xrightarrow{p} U(1)$, $p(z) = z^n$ with $n \in \mathbb{N}_{>0}$. In this case the group of automorphisms is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and the action is given by

$e^{i\theta}\cdot[k]=e^{i(\theta+2\pi k/n)}$, where $[k]=k \mod n$. Another simple example is $\mathbb{R}\xrightarrow{p} U(1)$, $p(\theta)=e^{i\theta}$. In this case the group is isomorphic to $\mathbb{Z}$ and the action of $n\in\mathbb{Z}$ is given by $x\mapsto x+2\pi n$. The first homotopy groups $\pi_1(E)$ and $\pi_1(B)$ provide a very effective way to establish when a covering is Galois. Indeed, it is possible to prove that $E\xrightarrow{p} B$ is Galois if and only if $p_*(\pi_1(E))$ is normal as a subgroup of $\pi_1(B)$, where p_* is the map that sends a homotopy class $[\gamma]$ of $\pi_1(E)$ to the homotopy class $[p\circ\gamma]$ of $\pi_1(B)$. In particular, every covering of a manifold B with Abelian $\pi_1(B)$ is Galois.

1.4 Associated bundles and structure groups

The notion of an associated bundle will shed some light on the role principal bundles have in the theory of fiber bundles.

Definition 1.20. Given a principal G-bundle $P\xrightarrow{\pi} B$ and a left action $\lambda\colon G\times F\to F$ of a Lie group G on a manifold F, we define the fiber bundle $P\times_\lambda F\xrightarrow{\pi'} B$ whose total space is $(P\times F)/\sim$ where $(u,v)\sim(ug,\lambda(g^{-1},v))$ for any $g\in G$ and whose projection is $\pi'[(u,v)]=\pi(u)$. The typical fiber of $P\times_\lambda F$ is F. We agree to call *associated* (to the pair (P,λ)) any fiber bundle E based-isomorphic to $P\times_\lambda F$. In this case we say G is a *structure group* for E.

Remark 1.21. The use of a left action is just conventional: any right action $\rho\colon F\times G\to F$, $(y,g)\mapsto\rho(y,g)$ becomes a left action by defining $(g,y)\mapsto\rho(y,g^{-1})$.

If a fiber bundle $E\to B$ is associated to a pair (P,λ) for some principal bundle P and action $\lambda\colon G\times F\to F$, we can somehow decouple the study of E into the study of P and of λ separately. The following simple examples illustrate this idea.

Example 1.22. If $P\to B$ is a trivial principal bundle and $\lambda\colon G\times F\to F$ is *any* left action, then the associated bundle $P\times_\lambda F$ is trivial. In order to see it, let us choose a global section $s\colon B\to P$ whose existence is guaranteed by Proposition 1.17. Then a based-isomorphism of $P\times_\lambda F$ to the trivial bundle $B\times F$ is given by $[(s(x)g,v)]\mapsto(x,\lambda(g,v))$.

Example 1.23. Somewhat dual to the previous example: if λ is the trivial action, i.e. $\lambda(g,v)=v$ for each $g\in G$, and $P\xrightarrow{\pi}B$ is *any* principal G-bundle, then $P\times_\lambda F$ is trivial. The isomorphism is given by $[(u,v)]\mapsto(\pi(u),v)$.

If F carries more structure (e.g. a vector space structure) and the G-action preserves it, then $P\times_\lambda F$ can be given this structure fiberwise. A precise treatment of this point in full generality is beyond the scope of this work. We shall limit ourselves to when F is a vector space. The vector space structure on each fiber of $P\times_\lambda F$ is then defined by $r[(u,v)]+r'[(u',v')]=[(u,rv+r'\lambda(g,v'))]$ where $r,r'\in\mathbb{R}$ and $u'=ug$ for a unique $g\in G$. We agree to extend the attribute of associated bundle and structure group to this case: so a structure group of a rank n *vector* bundle E must come together with a *linear* representation on $\mathbb{R}^n$.

To appreciate the usefulness of "decoupling" a fiber bundle into some principal bundle P and some action λ, we need the following definition.

Definition 1.24. Given a principal G-bundle $P\to B$ and a morphism $\rho: G'\to G$, where G and G' are topological or Lie groups, a (G',ρ)*-structure* of P (or simply a G'-structure or a ρ-structure when the various ingredients are understood) is a principal G'-bundle $P'\to B$ together with a based-ρ-morphism $m: P'\to P$. When ρ is injective (resp. surjective) the pair (P',m) is also called a G'*-reduction* (resp. G'*-prolongation*) of P, the morphism ρ being understood.

Remark 1.25. A principal bundle P might not admit some (G',ρ)-structures. For instance, if $\rho: G'\to G$ is the trivial morphism, i.e. $\rho(g')=1_G$ for all $g'\in G'$, then the existence of a ρ-structure $m: P'\to P$ is equivalent to the triviality of P (one way is clear; for the other one, simply consider the global section defined by $x\mapsto m(u')$ for any $u'\in P'$).

We can now state the following proposition.

Proposition 1.26. Assume E is a fiber bundle which is associated to (P,λ) for some principal G-bundle and some left action $\lambda\colon G\times F\to F$. If (P',m) is a (G',ρ)-structure of P for some $\rho\colon G'\to G$, then E is associated to the pair $(P',\lambda\circ\rho)$ where – with a little abuse of notation – we indicate by $\lambda\circ\rho$ the G'-action on F given by $(g',v)\mapsto\lambda(\rho(g'),v)$.

Proof. The map $P'\times_{\lambda\circ\rho}F\to P\times_\lambda F$ given by $[(u',v)]\mapsto[(m(u'),v)]$ is well defined. Indeed, $[(u'g',\lambda(\rho(g'^{-1})v,))]\mapsto[(m(u')\rho(g'),\lambda(\rho(g'^{-1}),v))]=[(m(u'),v)]$. One can check that it is a based-isomorphism of fiber bundles. By composing with the based-isomorphism $P\times_\lambda F\to E$, which exists by assumption, we obtain the result. □

It is then clear that a bundle $E\to B$ can be associated to more than one pair (P,ρ). It can also have more than one structure group. For instance, a fiber bundle is trivial if and only if it has the trivial group among its structure groups.

The concepts of an associated bundle and of a structure group for a vector bundle will be essential ingredients of subsequent sections.

1.5 Vector bundles and principal GL_n-bundles

The frame bundle construction provides a bridge between vector bundles and principal GL_n-bundles. This bridge can be made into an equivalence of categories as follows. First of all, recall that a *groupoid* is a category whose morphisms are isomorphisms.

Definition 1.27. We write $\mathrm{Prin}_{GL_n}(B)$ for the category of principal GL_n-bundles over B and principal based-GL_n-morphisms between them. We write $\mathrm{Vect}_n(B)$ for the category of rank n vector bundles over B and based-morphisms between them.

Corollary 1.16 shows that $\mathrm{Prin}_{GL_n}(B)$ is a groupoid. On the other hand, there are based-morphisms between same rank vector bundles which are not based-isomorphisms of vector bundles (for instance, the trivial morphism which assigns the zero vector to each vector, fiberwise). However, given a category C, we can always consider its maximal subgroupoid $C^\times$ obtained by discarding all morphisms which are not isomorphisms. Let us then consider the maximal subgroupoid $\mathrm{Vect}_n(B)^\times$.

The frame bundle construction can be promoted to a functor $Fr\colon \mathrm{Vect}_n(B)^\times \to \mathrm{Prin}_{\mathrm{GL}_n}(B)$ by the following observation: to each based-isomorphism $E \xrightarrow{f} E'$ of vector bundles we can associate a based-GL_n-morphism of principal GL_n-bundles $Fr(E) \xrightarrow{Fr(f)} Fr(E')$ by

$$Fr(f)(e_1,\dots,e_n) = (f(e_1),\dots,f(e_n))\,.$$

Since f is a vector space isomorphism on each fiber, this definition is well posed. One can check that this makes Fr a functor.

The pseudo inverse functor, which we indicate by Fr^{-1} for lack of a better name, is given as follows. Let us consider the identity representation $\mathrm{id}\colon \mathrm{GL}_n \to \mathrm{GL}_n$ viewed as a left action of GL_n on $\mathbb{R}^n$ and for each object $P \xrightarrow{\pi} B$ of $\mathrm{Prin}_{\mathrm{GL}_n}(B)$ construct the associated bundle $Fr^{-1}(P) = P \times_{\mathrm{id}} \mathbb{R}^n$.

Any based-GL_n-morphism $P \xrightarrow{m} P'$ of principal GL_n-bundles over B gives a vector bundle based-isomorphism $Fr^{-1}(P) \xrightarrow{Fr^{-1}(m)} Fr^{-1}(P')$ defined by

$$Fr^{-1}(m)([(u,v)]) = [(m(u),v)]\,.$$

One can check that this is well defined and it is indeed a vector bundle based-isomorphism, with inverse given by $[(u',v)] \mapsto [(m^{-1}(u'),v)]$. Hence $Fr^{-1}\colon \mathrm{Prin}_{\mathrm{GL}_n}(B) \to \mathrm{Vect}_n(B)^\times$ is a functor.

Theorem 1.28. *$\mathrm{Vect}_n(B)^\times$ and $\mathrm{Prin}_{\mathrm{GL}_n}(B)$ are equivalent categories.*

Proof. The functors Fr and Fr^{-1} provide the equivalence. First, consider $Fr^{-1}\circ Fr$. There is an isomorphism $Fr(E)\times_{\mathrm{id}}\mathbb{R} \to E$ given by $[(e,v)] \mapsto e_i v_i$, the naturality being clear. Second, consider $Fr \circ Fr^{-1}$ and call l_i the canonical basis of $\mathbb{R}^n$. There is an isomorphism $Fr(P\times_{\mathrm{id}}\mathbb{R}^n) \to P$ given by $[u,l_i]_{i=1,\dots,n} \mapsto u$. Again the naturality is clear. □

By this equivalence we obtain at once the following corollaries.

Corollary 1.29. There is a bijective correspondence between based-isomorphism classes of rank n vector bundles and based-GL_n-isomorphism classes of principal GL_n-bundles. In particular, two frame bundles $Fr(E) \to B$ and $Fr(E') \to B$ are based-GL_n-isomorphic if and only if the vector bundles $E \to B$ and $E' \to B$ are based-isomorphic.

Corollary 1.30. For each vector bundle $E \to B$ there is a group isomorphism $\mathrm{Aut}(E) \simeq \mathrm{Aut}(Fr(E))$, the group structure being given by composition of automorphisms within their respective categories.

Notice that by Theorem 1.28 any vector bundle has GL_n as a structure group, and any vector bundle is the associated bundle of some principal GL_n-bundle (namely its frame bundle). Moreover, a vector bundle E is trivial if and only if we can reduce its structure group to the trivial group. Indeed, a reduction to the trivial group is equivalent to a global section of the frame bundle $Fr(E)$ and, by Proposition 1.17, the existence of a global section is equivalent to the triviality of $Fr(E)$, which in turn is equivalent to the triviality of E by Corollary 1.29.

1.6 Orientability of vector bundles

We recall that for a n-dimensional vector space V an orientation is a choice of an equivalence class of bases, where two bases $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ are called equivalent if their transition matrix has positive determinant. When a choice of orientation has been made, any basis belonging to the chosen class is called oriented.

Proposition 1.31. On any n-dimensional vector space there are precisely two orientations. Given a basis e, exchanging any two elements of e changes the orientation.

Proof. Suppose two bases e' and e'' do not lie in the equivalence class of a third basis e. Since any ordered pair of bases possesses a transition matrix (either with positive or negative determinant), we can write $e = e'g$ and $e = e''h$

with $\det g < 0$ and $\det h < 0$. Then $e'g = e''h$, that is $e' = e''hg^{-1}$. Since $\det(hg^{-1}) > 0$, it follows that e' and e'' lie in the same equivalence class. For the second part of the statement, just observe that the transition matrix between e and the basis e' obtained from e by exchanging two elements has determinant -1. □

For vector bundles the notion of orientability is as follows.

Definition 1.32. A rank n vector bundle $E \xrightarrow{\pi} B$ is called *orientable* if an orientation can be defined on each fiber in such a way that each $x \in B$ has a neighbourhood $U \subset B$ admitting n sections $s_i : U \to E$ with the property that $(s_1(y), \dots, s_n(y))$ is an oriented basis of E_y for each $y \in U$. A vector bundle with a choice of orientation is called *oriented*.

There is another way to look at the orientability of a vector bundle which makes use of its frame bundle.

Theorem 1.33. *An orientation of a rank n vector bundle E is equivalent to a (GL_n^+, ι)-reduction of its frame bundle, where $\iota : GL_n^+ \hookrightarrow GL_n$ is the obvious inclusion of $GL_n^+ = \{g \in GL_n \mid \det g > 0\}$ into GL_n.*

Proof. Assume E is oriented. The set of its oriented frames $Fr^+(E)$ is a principal GL_n^+-bundle with action on fibers simply obtained by restricting the usual action to GL_n^+. The inclusion map $Fr^+(E) \hookrightarrow Fr(E)$ is clearly an injective based-ι-morphism, hence a (GL_n^+, ι)-reduction of $Fr(E)$. Conversely, assume there is a principal GL_n^+-bundle $P \to B$ and a based-ι-morphism $m : P \to Fr(E)$. Then the image $m(P) \subset Fr(E)$ defines an orientation on E. Indeed, for each $x \in B$ consider a local trivialization of P on some neighbourhood $U \subset B$ of x and take any section $s_U : U \to P$. Then $m \circ s_U : U \to Fr(E)$ selects a basis of E_y for each $y \in U$ and its orientation is independent of the chosen section: any other section s'_U is related to s_U by $s'_U(y) = s_U(y)\,\phi(y)$ for some map $\phi : U \to \mathrm{GL}_n^+$, hence $m(s'_U(y)) = m(s_U(y))\,\iota(\phi(y)) = m(s_U(y))\,\phi(y)$. Having defined orientations on each fiber of E, any local section $m \circ s_U$ can be viewed as a set of linearly independent local sections s_i of E such that $(s_1(y), \dots, s_n(y))$ lies in the orientation class selected by $m(P)$. □

The analog of Proposition 1.31 for vector bundles is the following.

Proposition 1.34. On any orientable vector bundle $E \to B$ there are precisely $2|\pi_0(B)|$ orientations. When B is connected, any based-automorphism of E either preserves or reverses the orientation.

Proof. See Section 4.4 in [29]. □

1.7 Spin structures

In the preceding section we have seen that an orientation of a vector bundle $E \to B$ amounts to a reduction of its structure group from GL_n to its connected component GL_n^+. This can be viewed as a way to pass from a structure group with $\pi_0(\mathrm{GL}_n) = \mathbb{Z}/2\mathbb{Z}$ to a structure group with $\pi_0(\mathrm{GL}_n^+) = 0$ while keeping all higher homotopy groups unchanged: $\pi_k(\mathrm{GL}_n^+) \simeq \pi_k(\mathrm{GL}_n)$ for $k \geq 1$. In topological terms this counts as a simplification: for instance, any orientable line bundle is trivial (over a paracompact space; the converse is also true, independently of the base space).

Given a locally path-connected topological space X, we say it is n-connected if $\pi_k(X) = 0$ for $0 \leq k \leq n$ and $\pi_k(X) \neq 0$ for $k = n+1$. By Proposition 1.17 it is not hard to prove that, when G is a n-connected topological group, a principal G-bundle over a CW complex B is trivial over the $(n+1)$-skeleton of B. This in turn implies the triviality over the $(n+1)$-skeleton of any of its associated bundles. From this point of view, then, the existence of a (G', ρ)-structure with G' n-connected has remarkable consequences on the topology of a fiber bundle. In particular, when B has sufficiently low dimension the fact that a fiber bundle $E \to B$ admits a G'-structure with G' sufficiently more highly connected implies the triviality of E. In other words, if $\dim B = k$ then the existence of a G'-structure for $E \to B$, with G' some $(k-1)$-connected group, would imply at once the triviality of E.

In the case of a rank n vector bundle E, the next step would consist in finding a 1-connected group G such that $\pi_k(G) \simeq \pi_k(\mathrm{GL}_n^+)$ for $k \geq 2$, together with a morphism $\rho\colon G \to \mathrm{GL}_n^+$ (this morphism guarantees, via Proposition 1.26, that G is a structure group for E). Clearly, the universal covering group of GL_n^+ is such an object (for the general definition see Appendix B).

Remark 1.35. Surprising as it may seem, the homotopy groups of GL_n^+ are usually hard to compute and their complete understanding is still lacking. They are isomorphic to the homotopy groups of SO_n by the homotopy equivalence given by the Gram-Schmidt procedure, and the latter groups are related to the homotopy groups of spheres by viewing SO_{n+1} as the total space of a principal SO_n-bundle over S^n, where $SO_n \subset SO_{n+1}$ and the action is given by right multiplication (see Example 3.65 in [55] for details about this bundle). The cases $n = 1, 2$ are somewhat special: GL_1^+ is contractible hence $\pi_k(GL_1^+) = 0$ for $k \geq 1$; GL_2^+ is homotopy equivalent to $SO_2 \simeq S^1$ hence $\pi_1(GL_2^+) = \mathbb{Z}$ and $\pi_k(GL_2^+) = 0$ for $k \geq 2$.

By the general theory of covering spaces, the universal covering of a space X has $|\pi_1(X)|$ many sheets. Since $\pi_1(GL_1^+) = 0$, $\pi_1(GL_2^+) = \mathbb{Z}$ and $\pi_1(GL_n^+) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$, we see that for $n \geq 3$ the universal covering is 2-sheeted. This suggests the following definition.

Definition 1.36. Given a oriented rank n vector bundle $E \to B$, a spin structure on E is a $(\widetilde{GL_n^+}, \rho_n)$-structure of $Fr^+(E)$, where $\widetilde{GL_n^+} \xrightarrow{\rho_n} GL_n^+$ is the 2-sheeted covering group of GL_n^+ as defined in Appendix B. Two spin structures (P, m) and (P', m') on E are equivalent if there is a based-$\widetilde{GL_n^+}$-morphism $P \xrightarrow{f} P'$ such that $m = m' \circ f$.

We record here some fundamental results (see e.g. [36], [25], [12]).

Proposition 1.37. A oriented rank n vector bundle $E \to B$ admits a spin structure if and only if its second Stiefel-Whitney class $w_2(E)$ vanishes. There is a free and transitive action of $H^1(B, \mathbb{Z}/2\mathbb{Z})$ on the set of equivalence classes of spin structures on E. In particular, $H^1(B, \mathbb{Z}/2\mathbb{Z})$ enumerates the inequivalent spin structures on E.

By the above discussion, it is not surprising that for a rank $n \geq 3$ vector bundle $E \to B$ there is an equivalent definition in terms of trivializations of E over skeleta of B.

Proposition 1.38. Given a oriented rank n vector bundle $E \to B$ of rank $n \geq 3$, a spin structure on E is the same as a trivialization of E over the 2-skeleton of B.

Proof. See Section 5.6 in [25]. □

Notice that, since $\pi_2(\mathrm{GL}_n^+) = 0$ as every Lie group, we can automatically extend a trivialization over the 2-skeleton to a trivialization over the 3-skeleton. In particular, if $\dim(B) \leq 3$ a rank $n \geq 3$ vector bundle over B admits a spin structure if and only if it is trivial.

A slight inconvenience of Definition 1.36 is that it uses groups which are "larger" than necessary. In Chapter 3 we shall present other equivalent definitions based on the concept of a metric on $E \to B$. Metrics will be introduced in Chapter 2.

Remark 1.39. As a side note we remark that the idea of "unwrapping" the structure group of a vector bundle does not end at the level of spin structures. Since, as already observed, $\pi_2(\widetilde{\mathrm{GL}_n^+}) = 0$ we are led to consider a 3-connected group G with a morphism $G \to \widetilde{\mathrm{GL}_n^+}$ inducing isomorphisms on the higher homotopy groups. If G were a Lie group, the general theory would imply G to be contractible (see [36]) leading to trivial bundles directly. Instead, we can look at topological groups. Such a G has actually been considered (see [48] for the original paper and [31] for some information) and called String_n, and a manifold admitting such a structure is called a string manifold. String structures are related to the Hoehn-Stolz conjecture about the existence of positive Ricci curvature metrics.

2 METRICS, ISOMETRIES, AND FRAME BUNDLES

In this chapter we introduce metrics of general signature on vector bundles and explore the corresponding category. In particular, we are interested in studying the effect of changing a metric η on a fixed vector bundle E (we have in mind the problem of studying all metrics on the tangent bundle of a manifold). In this setting, we shall clarify the relationship among E_η, its frame bundle $Fr(E_\eta)$, and the path-component of η in the space of metrics. We shall see that all these notions can be translated in terms of a splitting of E. The analysis then shows that there is a natural bijection between the set of isomorphism classes of E_η and the set of isomorphism classes of its frame bundle $Fr(E_\eta)$, while the set of path-components of metrics on E is generally larger (the special case of positive definite metrics stands alone, in that there is only one isomorphism class of E_η and one path-component of metrics). A few examples and applications are given. Most of the material in this chapter is classical, even though a unified discussion of these matters seem to be missing in the literature. Theorem 2.16 (or its generalization 2.22) appears to be new, to the best of our knowledge.

2.1 METRICS ON VECTOR BUNDLES AD SPLITTINGS

The familiar notion of an inner product (i.e. a non-degenerate symmetric bilinear form) on a vector space can be extended as well to the vector bundle setting.

Definition 2.1. Given two non-negative integers $p, q \in \mathbb{N}$ summing to n, a (p,q)-*metric* η on a rank n vector bundle $E \xrightarrow{\pi} B$ is a smooth assignment of an inner product $\eta_x \colon E_x \times E_x \to \mathbb{R}$ of signature (p,q) for each $x \in B$, where p is the number of positive eigenvalues of any representative matrix for η_x, and q is the number of negative ones. We agree to call *Euclidean* a metric of signature $(n,0)$, *anti-Euclidean* a metric of signature $(0,n)$, and *Lorentzian* a metric of signature $(1,n-1)$.

It is well known that every vector bundle (whose base space is paracompact) admits a Euclidean metric. Now let us fix a rank n vector bundle $E \xrightarrow{\pi} B$ and consider the linear space of sections of the bundle $\otimes^2 E^* \to B$. The subset of Euclidean metrics $\mathscr{M}_{n,0}(E)$ inherits the structure of an open positive convex cone. In particular, given any two Euclidean metrics $\eta, \eta' \in \mathscr{M}_{n,0}(E)$, the assignment $[0,1] \ni t \mapsto (1-t)\eta + t\eta'$ is a path in $\mathscr{M}_{n,0}(E)$ from η to η'. We shall use this fact in the next proposition. Before that, we need to briefly recall a classical result and to introduce the notion of a splitting of a vector bundle.

Lemma 2.2. Two symmetric bilinear forms on a finite dimensional vector space can be simultaneously diagonalized when one of them is positive definite.

Sketch of proof. Consider the inner product space defined by the positive definite form and apply the spectral theorem to the other symmetric bilinear form. □

Definition 2.3. A *splitting* of a vector bundle $E \to B$ is an ordered pair of its vector subbundles $N \subset E \to B$ and $M \subset E \to B$ such that $E = N \oplus M$ as bundles. N and M are the *parts* of the splitting. Two such splittings $N \oplus M$

and $N' \oplus M'$ are called *equivalent* if there are two vector bundle based-isomorphisms $f: N \to N'$ and $g: M \to M'$. Equivalently, there is a vector bundle based-automorphism $m: E \to E$ which sends each part of the first splitting onto each part of the second splitting (respecting the order of the pair). Two equivalent splittings are called *homotopic* if the based-isomorphisms in the definition of equivalence just given can be strengthen to be homotopies. Equivalently there is a vector bundle $\tilde{E}$ over $B \times [0,1]$ with a splitting which, when restricted to $B \times \{0\}$ and $B \times \{1\}$, coincides with each of the two splittings of E. If $E = N \oplus M$ is a splitting of E whose parts have rank p and q respectively, then the splitting is also called a (p,q)-splitting. Moreover, if η is a (p,q)-metric on E such that its restrictions to each part of the splitting have definite signatures $(p,0)$ and $(0,q)$ respectively, and such that the two parts are η-orthogonal, then the splitting is said to be η-*adapted*. When dealing with η-adapted splittings, we agree to order the splitting by the sign of the metric on each part: if $N \oplus M$ is η-adapted, and η is positive definite on N and negative definite on M, we then choose (N, M) as ordering. Homotopies and equivalences of η-adapted splittings must then respect this ordering.

Proposition 2.4. Given a (p,q)-metric η on $E \to B$, there is a η-adapted (p,q)-splitting of E and any two η-adapted (p,q)-splittings of E are homotopic (hence also equivalent).

Proof. We shall break the proof into three parts.

1. We first prove that a choice of a Euclidean metric on $E \to B$ defines a η-adapted (p,q)-splitting of E. Choose a Euclidean metric β on $E \to B$. For each $x \in B$ consider the two symmetric bilinear forms β_x and η_x on E_x. Since β_x is positive-definite, by Lemma 2.2 we can simultaneously diagonalize β_x and η_x with respect to some basis $\{e_i\}$ of E_x. Now consider the linear isomorphism $m_x := (\widetilde{\eta_x})^{-1} \circ \widetilde{\beta_x}: E_x \to E_x$ where $\widetilde{\beta_x}: E_x \to E_x^*$ and $\widetilde{\eta_x}: E_x \to E_x^*$ are the dual maps given by $v \mapsto \beta_x(v, \cdot)$ and $v \mapsto \eta_x(v, \cdot)$ respectively (they are invertible because of the non-degeneracy of the metrics). In an arbitrary basis the operator m_x has the expression ${m^i}_j = \eta^{ik}\beta_{jk}$, so in the basis $\{e_i\}$ it becomes diagonal

with positive and negative eigenspaces coinciding with the corresponding eigenspaces for η_x. The splitting is then defined by the positive and negative eigenspaces of m_x as x runs over B.

2. We then prove that, even though the splitting constructed in point 1 depends on the choice of a Euclidean metric on E, its homotopy class (and hence also its equivalence class) is independent of it. Choose two Euclidean metrics β_i on E ($i=0,1$) and apply the construction of point 1 above to obtain splittings $E=E_+^0\oplus E_-^0=E_+^1\oplus E_-^1$. The path $t\mapsto\beta_t=(1-t)\beta_0+t\beta_1$ defines a Euclidean metric β on $\pi^*E\to B\times[0,1]$, the pullback bundle of $E\to B$ under the projection $\pi\colon B\times[0,1]\to B$. On π^*E we can also consider the pullback $\pi^*\eta$, which is again a (p,q)-metric, and we then repeat the construction of point 1 with respect to π^*E, $\pi^*\eta$ and β to obtain a splitting of π^*E which obviously coincides with $E_+^i\oplus E_-^i$ over $B\times\{i\}$ ($i=0,1$).

3. We finally prove that any η-adapted (p,q)-splitting of E can be constructed as we did in point 1. Given an η-adapted (p,q)-splitting $E=E_+\oplus E_-$, take any two Euclidean metrics β_+ and β_- on E_+ and E_- respectively. Then the splitting $E_+\oplus E_-$ arises via the construction of point 1 by using the Euclidean metric defined by $\beta(v,w)=\beta_+(v_+,w_+)+\beta_-(v_-,w_-)$ where $v=v_+\oplus v_-$, $w=w_+\oplus w_-$ with $v_+,w_+\in E_+$ and $v_-,w_-\in E_-$.

□

Remark 2.5. The proposition above is basically Theorem 8.11 in [32].

The following is a sort of converse of Proposition 2.4.

Proposition 2.6. Given a (p,q)-splitting $E=N\oplus M$, there is a (p,q)-metric η on E such that $N\oplus M$ is a η-adapted (p,q)-splitting.

Proof. Take any Euclidean metrics β_N and β_M on N and M respectively, and define $\eta(v,w):=\beta_N(v_N,w_N)-\beta_M(v_M,w_M)$ where $v=v_N+v_M$ and $w=w_N+w_M$ with $v_N,w_N\in N$ and $v_M,w_M\in M$. □

Let us now consider the problem of enumerating the path-connected components of the space $\mathscr{M}_{p,q}(E)$ of (p,q)-metrics on a given rank $p+q$ vector bundle $E \to B$. We have the following classical theorem (see §40 in [47] for a slightly different approach).

Theorem 2.7. *There is a bijection between the set of path-components of $\mathscr{M}_{p,q}(E)$ and the set of homotopy classes of rank p subbundles of E.*

Proof. In the cases $(p,q)=(n,0)$ and $(p,q)=(0,n)$ the claim holds true: since positive definite or negative definite metrics form a positive convex cone in the space of sections of $\otimes^2 E \to B$ (see Section 2.1), it follows that there is only one path-component; in both cases there is only one homotopy class of rank p subbundles of E: either E itself or the rank 0 bundle. Let then p and q be both nonzero and fix a Euclidean metric β on E. We define a bijection between the set $\mathscr{M}_{p,q}(E)$ and the set of rank p subbundles of E and then show it descends to a well defined bijection on equivalence classes, where the equivalence is by path-components and by homotopic subbundles respectively. The bijection is as follows: given a p-subbundle $N_0 \subset E \xrightarrow{\pi} B$ we construct a (p,q)-metric η_0 on E by considering the β-orthogonal decomposition $N_0 \oplus N_0^\perp$ and defining $\eta_0(v,w) = \beta(v',w') - \beta(v'',w'')$, where $v = v' \oplus v'' \in N_0 \oplus N_0^\perp$ and $w = w' \oplus w'' \in N_0 \oplus N_0^\perp$. The inverse operation is as in Proposition 2.4: take a (p,q)-metric η_0 and define $N_0 = E^+$ where $E^+ \oplus E^-$ is the (p,q)-splitting defined by the positive and negative eigenspaces of $(\widetilde{\eta_0})^{-1} \circ \widetilde{\beta}$. We now show that this bijection descends to equivalence classes. Assume N_0 and N_1 are homotopic, hence there is a p-subbundle $N \subset j^*E \xrightarrow{j^*\pi} B \times [0,1]$ such that $N \restriction B \times \{i\} = L_i$, $i = 0,1$ (here $j\colon B \times [0,1] \to B$ denotes the projection onto the first factor). There is a $j^*\beta$-orthogonal decomposition $j^*E = N \oplus N^\perp$. The metric defined by $\eta(v,w) = (j^*\beta)(v',w') - (j^*\beta)(v'',w'')$, where $v = v' \oplus v'' \in N \oplus N^\perp$ and $w = w' \oplus w'' \in N \oplus N^\perp$, is a (p,q)-metric on j^*E which restricts to η_i on $(j^*\pi)^{-1}(B \times \{i\})$, $i = 0,1$. Evaluating η on each p-subbundle $(j^*\pi)^{-1}(B \times \{t\}) \simeq E$ for each $t \in [0,1]$ gives a path in $\mathscr{M}_{p,q}(E)$ joining η_0 to η_1. Conversely, let $\eta\colon [0,1] \to \mathscr{M}_{p,q}(E)$ be a path of (p,q)-metrics such that $\eta(0) = \eta_0$ and $\eta(1) = \eta_1$. By viewing η as a (p,q)-metric on $j^*E \xrightarrow{j^*\pi} B \times [0,1]$ we decompose

j^*E into positive and negative eigenvalues of $(\widetilde{\eta})^{-1} \circ \widetilde{j^*\beta}$. Clearly the positive part restricts to N_i on $(j^*\pi)^{-1}(B \times \{i\})$, $i = 0, 1$ and the proof is complete. It is also clear that the induced bijection on equivalence classes is independent of a choice of β. □

Remark 2.8. It can be proven that the homotopy class of a (p,q)-splitting is completely determined by the homotopy class of one of its two parts. In other words, given splittings $N_0 \oplus M_0$ and $N_1 \oplus M_1$ of $E \to B$, if N_0 and N_1 are homotopic then M_0 and M_1 are homotopic. This implies that Theorem 2.7 is actually equivalent to the (only apparently) weaker statement: "There is a bijection between the set of path-components of $\mathscr{M}_{p,q}(E)$ and the set of homotopy classes of (p,q)-splittings of E". We might ask what corresponds to *equivalence* classes of (p,q)-splittings instead. The answer will be given in Section 2.2.

2.2 CLASSIFICATION OF FRAME BUNDLES, AND SOME APPLICATIONS

We are now in a good position to discuss vector bundles with (p,q)-metrics, their frame bundles and the relationship between the two. To ease notation we shall write $E_\eta \to B$ for a vector bundle $E \to B$ with a (p,q)-metric η on it.

Definition 2.9. Given two rank n vector bundles $E_\eta \to B$ and $E'_{\eta'} \to B$ where η and η' are (p,q)-metrics, an *isometry* from E_η to $E'_{\eta'}$ is a morphism $f\colon E \to E'$ of vector bundles such that $\eta'(fv, fw) = \eta(v,w)$ for each $v, w \in E$. Analogously we define a *based-isometry* as an isometry which is a based-morphism.

Remark 2.10. Notice that an isometry restricts to an isomorphism of vector spaces on each fiber. Moreover, by Lemma 1.6 a based-isometry is automatically a based-isomorphism of vector bundles.

In analogy with Section 1.5 we now make the following definition.

Definition 2.11. We write $\mathrm{Prin}_{\mathrm{O}_{p,q}}(B)$ for the category of principal $\mathrm{O}_{p,q}$-bundles over B and principal based-$\mathrm{O}_{p,q}$-morphisms between them. We write $\mathrm{Vect}_{p,q}(B)$ for the category of rank $p+q$ vector bundles over B with (p,q)-metrics and based-isometries between them.

Remark 2.12. By Remark 2.10 $\mathrm{Vect}_{p,q}(B)$ is a groupoid.

Just as we did in Example 1.18, we can define the frame bundle of a vector bundle with a metric as follows. Consider a vector bundle $E \to B$ with a (p,q)-metric η. In this case we can take the set of all η-orthonormal frames, defined as bases $(e_{x,1},\dots,e_{x,n})$ of E_x such that $\eta(e_{x,i},e_{x,j}) = \epsilon_{ij}$, where

$$\epsilon_{ij} = \mathrm{diag}(\underbrace{1,\dots,1}_{p \text{ times}},\underbrace{-1,\dots,-1}_{q \text{ times}}).$$

Notice that the *first* p vectors of a frame have squared length 1, while the *last* q vectors have squared length -1. The resulting bundle is a principal $\mathrm{O}_{p,q}$-bundle, where $\mathrm{O}_{p,q}$ is the group of rank n matrices g such that $g^t \epsilon g = \epsilon$, the diagonal matrix ϵ being defined as in the previous sentence.

We can therefore assign to each bundle $E_\eta \to B$ its principal $\mathrm{O}_{p,q}$-bundle of η-orthonormal frames, and the associated bundle construction (see Section 1.4) with action induced by the inclusion $\mathrm{O}_{p,q} \hookrightarrow \mathrm{GL}_n$ can be used to reconstruct E_η (up to natural isomorphism) from its frame bundle. In other words, repeating the argument of Section 1.5 we obtain at once the following theorem.

Theorem 2.13. *$\mathrm{Vect}_{p,q}(B)$ and $\mathrm{Prin}_{\mathrm{O}_{p,q}}(B)$ are equivalent categories.*

Again we have the following corollaries.

Corollary 2.14. There is a bijective correspondence between isomorphism classes of rank n vector bundles over B with (p,q)-metrics and isomorphism classes of principal $\mathrm{O}_{p,q}$-bundles over B. In particular, two frame bundles $Fr(E_\eta) \to B$ and $Fr(E'_{\eta'}) \to B$ are isomorphic if and only if $E_\eta \to B$ and $E'_{\eta'} \to B$ are isomorphic.

Corollary 2.15. For each vector bundle with metric $E_\eta \to B$ there is a group isomorphism $\mathrm{Aut}(E_\eta) \simeq \mathrm{Aut}(Fr(E_\eta))$, the group structure being given by composition of automorphisms within their respective categories.

Notice that introducing a (p,q)-metric on a vector bundle E amounts to restricting its structure group from GL_{p+q} to $\mathrm{O}_{p,q}$ via the inclusion morphism $\mathrm{O}_{p,q} \hookrightarrow \mathrm{GL}_{p+q}$. We shall not pursue this viewpoint here though.

The following theorem characterizes when two given (p,q)-metrics defined *on the same vector bundle* $E \to B$ have based-$\mathrm{O}_{p,q}$-isomorphic principal bundles of orthonormal frames. Equivalently, by Corollary 2.14 it classifies the isometry classes of a vector bundle with respect to a signature (p,q).

Theorem 2.16. *Given a rank $p+q$ vector bundle $E \to B$ and (p,q)-metrics η and η' on it, their respective principal $O_{p,q}$-bundles of orthonormal frames are based-$O_{p,q}$-isomorphic if and only if some (hence all) of the η-adapted (p,q)-splittings are equivalent to some (hence all) of the η'-adapted (p,q)-splittings.*

Before presenting its proof we need the following lemma (see Theorem 8.8 in [32], or Problem 2-E in [41]).

Lemma 2.17. Let $E \to B$ and $E' \to B$ be two based-isomorphic vector bundles of rank n. For any $(n,0)$-metrics η on E and η' on E' there is a based-isometry $E_\eta \to E'_{\eta'}$. The same holds true also for the case of $(0,n)$-metrics.

Proof. Let $f\colon E \to E'$ be a based-isomorphism of vector bundles and consider on E the pullback metric $\beta(v,w) := \eta'(f(v), f(w))$. This is a $(n,0)$-metric since $\beta(v,v) = \eta'(f(v), f(v)) > 0$ for each nonzero $v \in E$ while $0 = \beta(v,v) = \eta'(f(v), f(v))$ implies $f(v) = 0$, which in turn implies $v = 0$. As in the proof of Proposition 2.4 we introduce the based-isomorphism defined fiberwise by $m_x := (\widetilde{\beta_x})^{-1} \circ \widetilde{\eta_x}\colon E_x \to E_x$. It satisfies $\beta(m(v), w) = \eta(v,w)$ for each $v, w \in E$ and its expression in any basis $\{e_i\}$ of E_x is $m^i{}_j = \beta^{ik}\eta_{jk}$. By Lemma 2.2 we can simultaneously diagonalize the matrices β^{ij} and η_{ij} obtaining a diagonal matrix with strictly positive diagonal, hence m_x is positive and we can safely define $\sqrt{m_x}$, which satisfies $\beta(\sqrt{m}(v), \sqrt{m}(w)) = \eta(v,w)$. For a positive matrix M the operation $M \mapsto \sqrt{M}$ is smooth and by the local triviality of E it follows that $\sqrt{m}$ defines a based-automorphism of E into itself. Now define $r := f \circ \sqrt{m}$ and compute: $\eta'(r(v), r(w)) = \eta'((f \circ \sqrt{m})v, (f \circ \sqrt{m})w) = \beta(\sqrt{m}(v), \sqrt{m}(w)) = \beta(m(v), w) = \eta(v, w)$. For the case of $(0, n)$-metrics put $\alpha := -\eta$ and $\alpha' := -\eta'$ and apply the above to find $r \colon E \to E'$ such that $\alpha'(r(v), r(w)) = \alpha(v, w)$. Then $-\eta'(r(v), r(w)) = -\eta(v, w)$ which means $\eta'(r(v), r(w)) = \eta(v, w)$. □

Proof of Theorem 2.16. Assume $Fr(E_\eta)$ and $Fr(E_{\eta'})$ are based-$\mathrm{O}_{p,q}$-isomorphic as principal bundles. By Corollary 2.14 there is some based-isometry $f \colon E_\eta \to E_{\eta'}$. Now choose a η-adapted (p, q)-splitting of E (which exists by Proposition 2.4), call it $E_+^\eta \oplus E_-^\eta = E$, and define $E_+^{\eta'} := f(E_+^\eta)$, $E_-^{\eta'} := f(E_-^\eta)$. Then $E_+^{\eta'} \oplus E_-^{\eta'}$ is a η'-adapted (p, q)-splitting of E: indeed, their sum is E because f is a vector bundle isomorphism and, since $\eta'(f(v), f(w)) = \eta(v, w)$ for each $v, w \in E$, it follows that η' restricted to $E_+^{\eta'} \times E_+^{\eta'}$ is Euclidean, η' restricted to $E_-^{\eta'} \times E_-^{\eta'}$ is anti-Euclidean and finally $E_+^{\eta'}$ and $E_-^{\eta'}$ are η'-orthogonal. Clearly the restriction $f \restriction E_+^\eta$ is a based-isomorphism $E_+^\eta \to E_+^{\eta'}$ of vector bundles; analogously for $f \restriction E_-^\eta \colon E_-^\eta \to E_-^{\eta'}$. Hence the splittings $E_+^\eta \oplus E_-^\eta$ and $E_+^{\eta'} \oplus E_-^{\eta'}$ are equivalent. By Proposition 2.4 all η-adapted (p, q)-splittings are homotopic, hence equivalent by Lemma 1.11. In the same way, all η'-splittings are homotopic, hence equivalent. By transitivity it follows that each η-adapted (p, q)-splitting is equivalent to each η'-adapted (p, q)-splitting and this concludes the proof of the “only if” part of the claim. Conversely, let $E_+^\eta \oplus E_-^\eta$ and $E_+^{\eta'} \oplus E_-^{\eta'}$ be η-adapted and η'-adapted (resp.) (p, q)-splittings of E such that there are vector bundle based-isomorphisms $f_+ \colon E_+^\eta \to E_+^{\eta'}$ and $f_- \colon E_-^\eta \to E_-^{\eta'}$. The restrictions $\eta \restriction E_+^\eta \times E_+^\eta$ and $\eta' \restriction E_+^{\eta'} \times E_+^{\eta'}$ define two $(p, 0)$-metrics on E_+^η and $E_+^{\eta'}$ respectively. Analogously the restrictions $\eta \restriction E_-^\eta \times E_-^\eta$ and $\eta' \restriction E_-^{\eta'} \times E_-^{\eta'}$ define two $(0, q)$-metrics on E_-^η and $E_-^{\eta'}$ respectively. By applying the preceding Lemma we obtain based-isometries $r_+ \colon E_+^\eta \to E_+^{\eta'}$ and $r_- \colon E_-^\eta \to E_-^{\eta'}$. Finally the map $r \colon E_\eta \to E_{\eta'}$ defined by $r(v \oplus w) := r_+(v) + r_-(w)$ for each $v \in E_+^\eta$ and each $w \in E_-^\eta$ is a based-isometry. By Corollary 2.14 this means $Fr(E_\eta)$ and $Fr(E_{\eta'})$ are based-$\mathrm{O}_{p,q}$-isomorphic. □

Remark 2.18. Theorem 2.16 generalizes to any signature the known fact that any two Euclidean (or anti-Euclidean) metrics on the same vector bundle have isomorphic frame bundles. Indeed when $(p,q)=(n,0)$ (or $(p,q)=(0,n)$) any splitting is trivially E itself. We point out here that the Gram-Schmidt procedure, which can be applied to the Euclidean (or anti-Euclidean) case to produce a map from the bundle of η-orthonormal frames to the bundle of η'-orthonormal frames, is generally not an isomorphism of principal bundles since it does not commute with the respective actions. Instead, it is so if and only if η and η' are conformally equivalent, as shown in Appendix A.

Remark 2.19. In some cases Theorem 2.16 and Proposition 2.6 easily allow us to give an upper bound on the number of isomorphism classes of frame bundles for a given rank $p+q$ vector bundle $E \to B$ and a given signature (p,q). Indeed, if we agree to call $V_p(E)$ the set of isomorphism classes of rank p vector subbundles of E, then the cardinality of $V_p(E) \times V_q(E)$ is such an upper bound (hardly sharp, however). For instance, for the tangent bundle of the 2-torus T^2 this upper bound is 2^4 (here we are also using the fact that if the Euler characteristic $\chi(M)$ of a closed connected even dimensional manifold M vanishes, then every line bundle over M can be realized as a line subbundle of its tangent bundle, up to vector bundle isomorphism; see Theorem 2.1 in [33]); this upper bound is further reducible to 4 by making use of the theory of Stiefel-Whitney classes. As another example, on the 3-sphere S^3 every vector bundle is trivial, hence there is only one class of frame bundles for any metric on any vector bundle $E \to S^3$ and this is true, in particular, for $(1,2)$-metrics on the tangent bundle; this should be compared with the number of path-components of $(1,2)$-metrics on the tangent bundle of S^3, which by Theorem 2.7 is given by the number of homotopy classes of line fields. According to Example 1.7 in [34], there are countably many of them.

As another application of the theory so far, we consider time-orientable Lorentzian metrics. First we recall the definition.

Definition 2.20. A Lorentzian metric η on a vector bundle $E \to B$ is called *time-orientable* when there exists a timelike smooth section, i.e. $s\colon B \to E$

such that $\eta(s(x), s(x)) > 0$ for each $x \in B$.

Assume two Lorentzian metrics η and η' on E are time-orientable, with timelike sections s and s' respectively, and consider the line subbundles L and L' spanned by s and s' respectively. Given a Euclidean metric β on E, we find adapted $(1, n-1)$-splittings $L \oplus L^{\perp}$ and $L' \oplus L'^{\perp}$ which, by Proposition 2.4, specify the homotopy class of any η-adapted (resp. η'-adapted) splitting. Since L and L' are trivial, by Theorem 2.16 we deduce that the orthonormal frame bundles for η and η' are based-$\mathrm{O}_{1,n-1}$-isomorphic if and only if $L^{\perp}$ and $L'^{\perp}$, which are rank $n-1$ spacelike distributions on E, are based-isomorphic vector bundles. We have then proven the following result.

Proposition 2.21. Any two time-orientable Lorentzian metrics on a rank n vector bundle have based-$\mathrm{O}_{1,n-1}$-isomorphic frame bundles if and only if they admit rank $n-1$ space-like distributions which are based-isomorphic as vector bundles.

Let us now reconsider the question of when two different metrics on the same vector bundle give isomorphic frame bundles of orthonormal frames. Theorem 2.16 gives the solution when both metrics have same signature (p, q). When the metrics have different signatures some care is needed. The notion of isomorphism for principal bundles that we are interested in is that of based-G-isomorphism, hence $G = G'$ and $\rho = \mathrm{id}_G$ (see Definition 1.14). We are then led to the question of when two different pairs (p, q) and (p', q') such that $p + q = p' + q' = n$ give the *equality* $\mathrm{O}_{p,q} = \mathrm{O}_{p',q'}$. Without loss of generality we can assume $p > p'$. If $p' \neq 0$ take any nontrivial $R \in \mathrm{O}_2$ and consider the matrix M which has R as the submatrix M_{ij} for $(i, j) \in \{p', p'+1\} \times \{p', p'+1\}$ and $M_{ij} = \delta_{ij}$ otherwise. A short computation shows that $M \in \mathrm{O}_{p,q}$ and $M \notin \mathrm{O}_{p',q'}$. If $p' = 0$ a similar argument shows that for $q \neq 0$ the groups $\mathrm{O}_{p,q}$ and $\mathrm{O}_{p',q'}$ are different. The only case left is $(p, q) = (n, 0)$ and $(p', q') = (0, n)$, which clearly gives equal groups $\mathrm{O}_{n,0} = \mathrm{O}_{0,n} = \mathrm{O}_n$. Hence the only case when metrics of different signature can have isomorphic frame bundles is when $(p, q) = (n, 0)$

and $(p', q') = (0, n)$. Let then η and η' be metrics on E of signature $(n, 0)$ and $(0, n)$ respectively. It is easy to check that the identity map $\mathrm{id}_E \colon E \to E$ gives a based-O_n-isomorphism $Fr(E_{-\eta'}) \to Fr(E_{\eta'})$. By composing with the based-O_n-isomorphism $Fr(E_\eta) \to Fr(E_{-\eta'})$ of Theorem 2.16 we deduce that $Fr(E_\eta)$ and $Fr(E_{\eta'})$ are based-O_n-isomorphic. We collect these findings in the following theorem, which generalizes Theorem 2.16.

Theorem 2.22. *Given a rank n vector bundle $E \to B$ and two metrics η and η' of signatures (p, q) and (p', q') respectively, where $p + q = p' + q' = n$, their respective principal bundles of orthonormal frames are based-$\mathrm{O}_{p,q}$-isomorphic if and only if either both metrics have same signature (p, q) and equivalent adapted (p, q)-splittings, or the two metrics have signatures $(n, 0)$ and $(0, n)$.*

Also the treatment of Section 1.5 can be extended just as well to the case of different signatures. There is an obvious isomorphism of categories $\mathrm{Vect}_{p,q}(B) \simeq \mathrm{Vect}_{q,p}(B)$ given by the assignment $E_\eta \mapsto E_{-\eta}$ on objects and the identity on morphisms. We can compose this *isomorphism* together with the *equivalences* $\mathrm{Vect}_{p,q}(B) \sim \mathrm{Prin}_{\mathrm{O}_{p,q}}(B)$ and $\mathrm{Vect}_{q,p}(B) \sim \mathrm{Prin}_{\mathrm{O}_{q,p}}(B)$, thus obtaining the equivalence $\mathrm{Prin}_{\mathrm{O}_{p,q}}(B) \sim \mathrm{Prin}_{\mathrm{O}_{q,p}}(B)$ according to the following diagram.

$$\begin{array}{ccc}
\mathrm{Prin}_{\mathrm{O}_{p,q}}(B) & \xrightarrow{\sim} & \mathrm{Prin}_{\mathrm{O}_{q,p}}(B) \\
\Big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\Big\uparrow \\
\mathrm{Vect}_{p,q}(B) & \xrightarrow{\simeq} & \mathrm{Vect}_{q,p}(B)
\end{array}$$

3 Spinors and diffeomorphisms

3.1 Spin structures revisited

As already noted earlier in Section 1.7, a slight inconvenience of Definition 1.36 is that it uses groups which are "larger" than necessary. Since B is paracompact, we can pick up any Euclidean metric on $E \to B$ and reduce the structure group from GL_n^+ to SO_n, which is compact. The covering map $\widetilde{\mathrm{GL}_n^+} \xrightarrow{\rho_n} \mathrm{GL}_n^+$ induces a 2-sheeted covering[1] $\mathrm{Spin}_n \xrightarrow{\rho_n} \mathrm{SO}_n$, where $\mathrm{Spin}_n := \rho_n^{-1}(\mathrm{SO}_n)$ (obviously Spin_n is isomorphic, as a Lie group, to the group of the same name constructed through Clifford algebra).

Proposition 3.1. Given a oriented rank n vector bundle $E \to B$ with a Euclidean metric η on it, a spin structure on E is equivalent to a $(\mathrm{Spin}_n, \rho_n)$-structure on $Fr^+(E_\eta)$.

Sketch of proof. Given a $(\widetilde{\mathrm{GL}_n^+}, \rho_n)$-structure (P, m) on $Fr^+(E)$, consider the natural inclusion $Fr(E_\eta) \subset Fr(E)$. Then $(m^{-1}(Fr^+(E_\eta)), m)$ is a $(\mathrm{Spin}_n, \rho_n)$-structure (with principal action on $m^{-1}(Fr^+(E_\eta))$ given by restricting the action of $\widetilde{\mathrm{GL}_n^+}$). Conversely, given a $(\mathrm{Spin}_n, \rho_n)$-structure (P, m) on $Fr^+(E_\eta)$, the bundle $P \times_{\mathrm{Spin}_n} \widetilde{\mathrm{GL}_n^+}$ associated to P via the natural left action of Spin_n on $\widetilde{\mathrm{GL}_n^+}$, together with

[1] For notational simplicity we keep the name ρ_n for the map $\rho_n \restriction \mathrm{Spin}_n$.

the map $m'[u,g] = [m(u), g]$, is a $(\widetilde{\mathrm{GL}}_n^+, \rho_n)$-structure (where the $\widetilde{\mathrm{GL}}_n^+$-action is given by $[u,g]h = [u, gh]$). It can be proven that this construction defines a bijection between equivalence classes (see [15] for details about this geometric proof; for a homotopic proof, see [42]). □

As the names of the groups involved reveal, the definition in terms of Spin_n is more common than the one in terms of $\widetilde{\mathrm{GL}}_n^+$. Although we shall still need Spin_n to define spinors (see Section 3.2), we opted for Definition 1.36 in order to decouple the notion of a spin structure from that of a Euclidean metric in view of subsequent investigations carried out in this chapter.

For vector bundles of rank $n \geq 4$ there is a similar result by using Lorentzian metrics.

Theorem 3.2. *Given a rank $n \geq 4$ vector bundle $E \to B$ with a time-oriented and space-oriented Lorentzian metric η on it, a spin structure on E (relative to the orientation of E induced by the orientations for time and space) is equivalent to a $(Spin^0_{1,n-1}, \rho_n)$-structure of $Fr^+(E_\eta)$, where $Spin^0_{1,n-1} \xrightarrow{\rho_n} SO^0_{1,n-1}$ has been introduced in Appendix B.*

Proof. See §4 in [8] for a sketchy proof, and Chapter 8 in [42] for a rigorous proof; the latter is also a beautiful and thorough account of the matter. □

3.2 SPINORS

Hereafter we consider the above theory applied only to the case of the tangent bundle TM of a manifold M. Following tradition, we call g a Euclidean[2] *metric on TM instead of using greek letters. For notational simplicity, we shall put $Fr(TM) = GL^+(M)$ for the bundle of oriented frames and $Fr(TM_g) = SO_g(M)$ for the bundle of oriented g-orthonormal frames.*

We start by describing some algebraic structures behind spinors of a finite dimensional Euclidean space. They will be used to describe the structures on

[2] Called *Riemannian* in this context.

typical fibers of various bundles we shall encounter on manifolds (alternatively think of what happens at a point of a manifold).

Recall that a usual vector or tensor of $\mathbb{R}^n$ of type R, where R is some representation of GL_n in $\mathbb{R}^k$, can be viewed as a map from the space F of oriented linear frames in $\mathbb{R}^n$ to $\mathbb{R}^k$, which is GL_n-equivariant i.e. it intertwines the canonical action of GL_n on F with R. Equivalently, given any (positive or negative) definite bilinear form (metric) g on $\mathbb{R}^n$, one can work with the space F_g of oriented g-orthonormal frames in $\mathbb{R}^n$, that carries a natural action of SO_n. Then, we can regard a vector (or a tensor) as a map from F_g to $\mathbb{R}^k$, which is equivariant under (restriction of) the representation R to SO_n.

For spinors one usually uses the (nontrivial) double cover $\rho_n\colon \mathrm{Spin}_n \to \mathrm{SO}_n$, and a free orbit $\tilde{F}_g$ of Spin_n (called space of 'spinor frames' of $\mathbb{R}^n$) together with a 2:1 identification map $\eta\colon \tilde{F}_g \to F_g$, such that $\eta(\tilde{e}h) = \eta(\tilde{e})\rho_n(h)$, where $\tilde{e} \in \tilde{F}_g$ and $h \in \mathrm{Spin}_n$. Given a representation $R\colon \mathrm{Spin}_n \to \mathrm{GL}(k,\mathbb{C})$ of Spin_n we shall view a R-spinor of $\mathbb{R}^n$ as an R-equivariant map ψ from $\tilde{F}_g$ to $\mathbb{C}^k$. There is an obvious $\mathbb{C}$-linear structure on the space of R-spinors.

Obviously the interesting case here is when R is not a tensor representation, i.e. does not descend to a representation of SO_n. This is the case e.g. for $R=\mu$, where μ is the restriction to Spin_n of the fundamental (also called spin) representation of the Clifford algebra $\mathbb{C}\mathrm{l}_n$. The carrier complex space of μ has dimension $k=2^{n/2}$ for even n and $k=2^{(n-1)/2}$ for odd n.

Remark 3.3. Since Spin_n is compact, by averaging over it we can consider any of its representations (hence also μ) as being unitary with respect to a suitable hermitian inner product. In the case of the standard metric and the representation μ, this is just the standard inner product on $\mathbb{C}^k$.

Hereafter we fix the spin representation R to be $\mu\colon \mathrm{Spin}_n \to U(k)$, $k=2^{[n/2]}$, and consider μ-spinors $\psi : \tilde{F}_g \to \mathbb{C}^k$, i.e. $\psi(uh) = \mu(h^{-1})\psi(u), \forall u \in \tilde{F}_g, h \in \mathrm{Spin}_n$. The inner product is given in terms of the standard inner product in $\mathbb{C}^k$, as $(\psi,\phi) := (\psi(u) \mid \phi(u))$ (the right hand side is independent of u).

In order to liberate the setting from the dependence on the metric a natural temptation would be to use $\rho_n\colon \widetilde{\mathrm{GL}_n^+} \to \mathrm{GL}_n^+$, as we did in Section 1.7

(for notational simplicity we use the same letter ρ_n for the extension). This extends the double cover $\mathrm{Spin}_n \to \mathrm{SO}_n$ and is a central extension of GL_n^+ by $\mathbb{Z}/2\mathbb{Z}$. Unfortunately $\widetilde{\mathrm{GL}_n^+}$ is not usually used as a 'structure' group for spinors, for the reason that it is not a matrix group, i.e. it has only infinite-dimensional faithful representations, while geometric objects are usually assumed to have finite number of components. Instead, every finite-dimensional representation of $\widetilde{\mathrm{GL}_n^+}$ descends to a (tensor) representation of GL_n^+, at least for $n \geq 3$ (see Lemma 5.23 in [36]). Thus we have to stick to the subgroup Spin_n and so the space of spinors will be always labelled by a metric. Concretely, a spinor labelled by a metric g will be a μ-equivariant map from the orbit $\tilde{F}_g := \eta^{-1}(F_g) \subset \tilde{F}$ of Spin_n to $\mathbb{R}^k$. We shall however employ $\widetilde{\mathrm{GL}_n^+}$, as well as its free orbit space $\tilde{F}$ together with a 2:1 covering map $\eta : \tilde{F} \to F$, that intertwines the relative actions, in order to define the transformation of spinors under an oriented automorphism β of $\mathbb{R}^n$. More precisely we can and shall lift β to an automorphism $\tilde{\beta}$ of $\tilde{F}$ and define the transformed spinor as

$$\psi' = \psi \circ \tilde{\beta}_g,$$

where $\tilde{\beta}_g$ is the restriction of $\tilde{\beta}$ to $\widetilde{F_{\beta^* g}}$. The domain of ψ', understood as an equivariant map, is $\widetilde{F_{\beta^* g}}$. Clearly the new spinor ψ' is labelled by the pullback metric $g' = \beta^* g$. Note that the components of ψ' with respect to the spinor basis $\tilde{e}'$ are equal to the components of ψ with respect to the spinor basis $\tilde{e} = \tilde{\beta}(\tilde{e}')$, i.e. $\psi'(\tilde{e}') = (\psi \circ \tilde{\beta})(\tilde{e}') = \psi(\tilde{e})$. Moreover, since for any β there are precisely two lifts $\tilde{\beta}$ (which differ just by a sign) we get actually a double covering $\widetilde{\mathrm{Aut}^+(\mathbb{R}^n)} \simeq \widetilde{\mathrm{GL}_n^+}$ of the group $\mathrm{Aut}^+(\mathbb{R}^n) \simeq \mathrm{GL}_n^+$ that acts on spinors.

In the next section we shall globalize the structures described so far.

As already noted in Section 3.1, in the literature the notion of a spin structure is usually formulated for a Riemannian manifold (M, g) in terms of a principal Spin_n-bundle over M double covering the bundle $\mathrm{SO}_g(M)$ of oriented g-orthonormal frames of M. Since in the following we shall vary the metric g, we use the metric-independent definition as in Section 1.7 (we refer to [49] for a nice survey about the two different definitions and their equivalence at the topological level). An oriented manifold M is called spin if $w_2(M) = 0$ and sometimes we shall understand by this term a pair (M,σ) with a given spin structure σ on M.

Remark 3.4. A word of caution must be given: in some texts by a spin structure is understood an equivalence class of spin structures in our sense, i.e. $\widetilde{\mathrm{GL}_n^+}$- or Spin_n-prolongations (see e.g. [40] and [31] p.61). Moreover, sometimes it is not clearly stated if a prolongation is meant or rather an equivalence class of prolongations, though this may be grasped from context. Clearly this is crucial for the issue of proper parametrization, e.g. it is the set of equivalence classes of spin structures on M which is known to be in bijective correspondence with $H^1(M,\mathbb{Z}/2\mathbb{Z})$ (see e.g. Theorem 8.2 in [12]). Note also the difference with the case of reductions of the structure group – a reduction of a principal G-bundle P to some subgroup $G' \subset G$ is a principal G'-subbundle $P' \subset P$. Two different reductions encode different information with respect to the inclusion into P, even though they might be equivalent as reductions (the equivalence is defined analogously as for prolongations). For instance two different O_n-reductions of the principal GL_n-bundle of frames over a manifold M correspond to different Riemannian metrics on M, even though any two such reductions are equivalent as explained in Chapter 2.

3.3 Spinor fields

Let (M,g,σ) be a Riemannian spin manifold. Let $R\colon \mathrm{Spin}_n \to \mathrm{GL}(k,\mathbb{C})$ be a (fixed) representation of Spin_n. It is customary to call R-spinor field on M an R-equivariant map $\psi\colon \mathrm{Spin}_g(M) \to \mathbb{C}^k$, where $\mathrm{Spin}_g(M)$ is the total space of the spin structure σ (here by spin structure we temporarily mean a Spin_n-prolongation of $\mathrm{SO}_g(M)$, i.e. we need the metric g). By R-equivariance we mean that $\psi(ug) = R(g^{-1})\psi(u)$ for $u \in \mathrm{Spin}_g(M)$ and $g \in \mathrm{Spin}_n$. As in Section 3.2 we are interested in those R that are not tensor representations, e.g. in the unitary Dirac representation μ of Spin_n in the complex space of dimension $k = 2^{[n/2]}$. We denote by $S_{\sigma,g}$ the space of μ-spinor fields, often named Dirac spinor fields, for the spin structure σ and the metric g. There is an obvious

$\mathbb{C}$-linear structure on $S_{\sigma,g}$ induced by pointwise operations. Note that for different metrics we have a priori different spaces $S_{\sigma,g}$ (see [49] for a geometric description of a configuration space for both spinors and metrics).

An inner product on $S_{\sigma,g}$ can be defined as follows: take a cover $\{(U_\alpha, h_\alpha)\}_{\alpha\in A}$ of M which trivializes $\mathrm{Spin}_g(M)$. Given $\psi, \phi \in S_{\sigma,g}$ consider the global function $a_{\psi,\phi}\colon M \to \mathbb{C}$ defined locally by

$$a_{\psi,\phi}(x) := (\psi(u_x) \mid \phi(u_x))$$

where $u\colon U_\alpha \to \mathrm{Spin}_g(M)(U_\alpha)$ is any local section of $\mathrm{Spin}_g(M)$ (in writing u we omit the dependency on the index α to simplify notation). Consider the global (yet locally defined, α-dependency omitted) n-form $e^1 \wedge \cdots \wedge e^n$ where e^j is the g-dual of $e_j := \eta_g(u_j)$. Finally put:

$$\langle \psi \mid \phi \rangle_{\sigma,g} := \int_M a_{\psi,\phi}\, e^1 \wedge \cdots \wedge e^n \quad .$$

It is easy to see that the above definition does not depend on the trivialization. Note that $e^1 \wedge \cdots \wedge e^n = \mathrm{vol}_g(M)$, the g-volume form of M.

We then make the following definition.

Definition 3.5. The Hilbert space of spinors $\mathscr{H}_{\sigma,g}$ for a given spin structure σ and metric g is the L^2-completion of the inner product space $(S_{\sigma,g}, \langle\,|\,\rangle_{\sigma,g})$.

It is natural to investigate what happens to $\mathscr{H}_{\sigma,g}$ under a change of spin structure. For equivalent spin structures the answer is given by the next proposition.

Proposition 3.6. If we choose an equivalent $\widetilde{\mathrm{GL}}_n^+$-prolongation $\sigma' = (\widetilde{\mathrm{GL}}^+(M)', \eta')$, the principal $\widetilde{\mathrm{GL}}_n^+$-isomorphism $m\colon \widetilde{\mathrm{GL}}^+(M)' \to \widetilde{\mathrm{GL}}^+(M)$ induces a unitary operator $U\colon \mathscr{H}_{\sigma,g} \to \mathscr{H}_{\sigma',g}$ given by

$$U\psi = \psi \circ m_g$$

where $m_g = m \restriction \eta'^{-1}(\mathrm{SO}_g(M))$.

Proof. The operator U is clearly linear. It is invertible with inverse given by $U^{-1}\psi = \psi \circ (m_g)^{-1}$. To prove unitarity let us put $\psi' = \psi \circ m_g$. It is now enough to observe that $a_{\psi',\phi'} = a_{\psi,\phi}$. From this we obtain $\langle U\psi \mid U\phi\rangle = \int_M a_{\psi',\phi'} \mathrm{vol}_g(M) = \int_M a_{\psi,\phi} \mathrm{vol}_g(M) = \langle \psi \mid \phi \rangle$. □

Remark 3.7. Given two equivalent $\widetilde{\mathrm{GL}}_n^+$-prolongations of GL_M, there are exactly two distinct principal isomorphisms between the two prolongations (this is a consequence of the morphism ρ_n being a central extension of GL_n by $\mathbb{Z}/2\mathbb{Z}$). It follows that there is another unitary operator $U^-\colon \mathscr{H}_{\sigma,g} \to \mathscr{H}_{\sigma',g}$, given by $U\psi = \psi \circ m_g^-$ where $m_g^- u = (mu)(-1)$ with $\{\pm 1\} = \ker \rho_n \subset Z(\widetilde{\mathrm{GL}}_n^+)$. Clearly, once the existence of an isomorphism $\mathscr{H}_{\sigma,g} \to \mathscr{H}_{\sigma',g}$ has been established, any other isomorphism can be obtained by composing with a suitable automorphism of $\mathscr{H}_{\sigma,g}$. However, the operators U and U^- are the only two arising from principal morphisms as indicated above.

3.4 Diffeomorphisms and spin structures

We now study the interplay between orientation-preserving diffeomorphisms of M and spin structures on M. Given a spin manifold (M,σ) where $\sigma = (\widetilde{\mathrm{GL}}^+(M), \eta)$, let us choose an orientation-preserving diffeomorphism $f\colon M \to M$ and consider the natural lift of f to $\mathrm{GL}^+(M)$ given by applying the tangent map of f to each element of each frame $e \in \mathrm{GL}^+(M)$. We denote such a lift by the symbol Tf. The pullback bundle $Tf^*\widetilde{\mathrm{GL}}^+(M)$, defined explicitly by $Tf^*\widetilde{\mathrm{GL}}^+(M) = \{(e,u) \in \mathrm{GL}^+(M) \times \widetilde{\mathrm{GL}}^+(M) \mid Tf(e) = \eta(u)\}$, together with the canonical map $Tf^*\eta\colon Tf^*\widetilde{\mathrm{GL}}^+(M) \to \mathrm{GL}^+(M)$ given by $(Tf^*\eta)(e,u) = e$ is again a spin structure on M which we call $f^*\sigma$. By construction the map $Tf\colon \mathrm{GL}^+(M) \to \mathrm{GL}^+(M)$ admits exactly two distinct lifts, given by $\varphi^{\pm}(e,u) = u(\pm 1)$ where $\{\pm 1\} = \ker \rho_n \subset Z(\widetilde{\mathrm{GL}}_n^+)$. The following diagram illustrates the situation.

$$\begin{array}{ccc}
Tf^*\widetilde{\mathrm{GL}^+}(M) & \xrightarrow{\varphi^\pm} & \widetilde{\mathrm{GL}^+}(M) \\
\big\downarrow{\scriptstyle Tf^*\eta} & & \big\downarrow{\scriptstyle \eta} \\
\mathrm{GL}^+(M) & \xrightarrow{Tf} & \mathrm{GL}^+(M) \\
\big\downarrow & & \big\downarrow \\
M & \xrightarrow{f} & M
\end{array}$$

Recall that the set Σ_M of equivalence classes of spin structures on M is naturally an affine space over the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^1(M;\mathbb{Z}/2\mathbb{Z})$. The assignment $\mathrm{Diff}^+(M)\times\Sigma_M \to \Sigma_M$ given by $(f,[\sigma]) \mapsto [f^*\sigma]$ defines an affine representation ρ of $\mathrm{Diff}^+(M)$ on Σ_M (see [15] for a proof). Moreover, the normal subgroup $\mathrm{Diff}^+_0(M) \subset \mathrm{Diff}^+(M)$ of diffeomorphisms which are homotopy equivalent to the identity acts trivially on Σ_M, hence ρ descends to a representation of $\Omega(M) = \mathrm{Diff}^+(M)\,/\,\mathrm{Diff}^+_0(M)$ on Σ_M.

3.5 DIFFEOMORPHISMS AND SPINORS

This section explores the relation between diffeomorphisms of M and the system of spaces $\mathscr{H}_{\sigma,g}$. Let us start with a spin structure $\sigma = (\widetilde{\mathrm{GL}^+}(M),\eta)$ of M and a metric g on M. Given an orientation-preserving diffeomorphism $f \in \mathrm{Diff}^+(M)$ we can consider the pullback metric f^*g on M defined by $(f^*g)(v,w) = g(Tfv, Tfw)$. The map $Tf\colon \mathrm{GL}^+(M) \to \mathrm{GL}^+(M)$ restricts to a lift

$$Tf_g\colon \mathrm{SO}_{f^*g}(M) \to \mathrm{SO}_g(M)$$

by construction. The pullback spin structure $f^*\sigma$ restricts to a Spin_n-prolongation of $\mathrm{SO}_{f^*g}(M)$ by considering $(Tf^*\eta)^{-1}(\mathrm{SO}_{f^*g}(M))$ with Spin_n-action obtained by restricting the $\widetilde{\mathrm{GL}^+_n}$-action on $Tf^*\widetilde{\mathrm{GL}^+_M}$ to the subbundle

$$(Tf^*\eta)^{-1}(\mathrm{SO}_{f^*g}(M))\,.$$

There are exactly two lifts $\varphi_g^{\pm}\colon (Tf^*\eta)^{-1}(\mathrm{SO}_{f^*g}(M)) \to \eta^{-1}(\mathrm{SO}_g(M))$, given by restriction of $\varphi^{\pm}$. The following diagram illustrates the situation.

$$\begin{array}{ccccccc}
(Tf^*\eta)^{-1}(\mathrm{SO}_{f^*g}(M)) & \hookrightarrow & Tf^*\widetilde{\mathrm{GL}^+}(M) & \xrightarrow{\varphi^{\pm}} & \widetilde{\mathrm{GL}^+}(M) & \hookleftarrow & \eta^{-1}\mathrm{SO}_g(M) \\
 & & \downarrow{\scriptstyle Tf^*\eta} & & \downarrow{\scriptstyle \eta} & & \\
\mathrm{SO}_{f^*g}(M) & \hookrightarrow & \mathrm{GL}^+(M) & \xrightarrow{Tf} & \mathrm{GL}^+(M) & \hookleftarrow & \mathrm{SO}_g(M) \\
 & & \downarrow & & \downarrow & & \\
 & & M & \xrightarrow{f} & M & &
\end{array}$$

The next definition and proposition generalize the analysis in Section 3.3 to the case of changing the metric from g to f^*g.

Definition 3.8. For each of the two lifts $\varphi^{\pm}$ of Tf we define a linear operator $U_{\varphi^{\pm}}\colon \mathscr{H}_{\sigma,g} \to \mathscr{H}_{f^*\sigma,f^*g}$ by

$$U_{\varphi^{\pm}}\psi = \psi \circ \varphi^{\pm}_{f^*g} \tag{3.1}$$

where $\varphi^{\pm}_{f^*g} = \varphi^{\pm} \restriction (Tf^*\eta)^{-1}(\mathrm{SO}_{f^*g}(M))$.

Proposition 3.9. The operators $U_{\varphi^{\pm}}$ defined above are unitary, that is they are invertible and satisfy $\langle U_{\varphi^{\pm}}\psi \mid U_{\varphi^{\pm}}\phi\rangle_{f^*\sigma,f^*g} = \langle \psi \mid \phi\rangle_{\sigma,g}$ for each $\psi,\phi \in \mathscr{H}_{\sigma,g}$.

Proof. Linearity is clear. The inverse is given by $\psi \mapsto \psi \circ (\varphi^{\pm}_{f^*g})^{-1}$. For the second part: let us consider $\varphi^{+}_{f^*g}$, the case $\varphi^{-}_{f^*g}$ being analogous. Put $\psi' := \psi \circ \varphi^{+}_{f^*g}$, $\phi' := \phi \circ \varphi^{+}_{f^*g}$. An easy computation shows that $a_{\psi',\phi'} = a_{\psi,\phi} \circ f$. Now apply the formula for the invariance of integrals under pullback:

$$\begin{aligned}
\langle U_{\varphi^+}\psi \mid U_{\varphi^+}\phi\rangle_{f^*\sigma,f^*g} &= \int_M a_{\psi',\phi'}\, e'^1 \wedge \cdots \wedge e'^n \\
&= \int_M (a_{\psi,\phi} \circ f)\, e'^1 \wedge \cdots \wedge e'^n \\
&= \int_M f^*\left(a_{\psi,\phi}\, e^1 \wedge \cdots \wedge e^n\right) \\
&= \int_M a_{\psi,\phi}\, e^1 \wedge \cdots \wedge e^n \\
&= \langle \psi \mid \phi\rangle_{\sigma,g}
\end{aligned} \tag{3.2}$$

where we used local sections $e' \colon U_\alpha \to \mathrm{SO}_{f^*g}(M)(U_\alpha)$ and $e := Tfe'$. □

A remark similar to 3.7 holds here as well. In other words, the operators $U_{\varphi^\pm}$ are the only two unitary operators $\mathscr{H}_{\sigma,g} \to \mathscr{H}_{f^*\sigma, f^*g}$ which arise from some principal morphism as above.

The above results permit to introduce a certain covering of the group of diffeomorphisms. We restrict to the case of oriented diffeomorphisms preserving a given spin structure.

Definition 3.10. Let $\mathrm{Diff}^+_\sigma(M)$ be the subgroup of $\mathrm{Diff}^+(M)$ consisting of diffeomorphisms which preserve the spin structure $\sigma = (\widetilde{\mathrm{GL}^+}(M), \eta)$. Define the group $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ to consist of all principal $\widetilde{\mathrm{GL}}^+_n$-morphisms $\varphi \colon \widetilde{\mathrm{GL}^+}(M) \to \widetilde{\mathrm{GL}^+}(M)$ closing the diagram:

$$\begin{array}{ccc}
\widetilde{\mathrm{GL}^+}(M) & \xrightarrow{\ \varphi\ } & \widetilde{\mathrm{GL}^+}(M) \\
\downarrow \eta & & \downarrow \eta \\
\mathrm{GL}^+(M) & \xrightarrow{\ Tf\ } & \mathrm{GL}^+(M) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\ f\ } & M
\end{array}$$

where f runs over $\mathrm{Diff}^+_\sigma(M)$, together with the multiplication given by composition of maps.

It is clear that $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ is a double cover of $\mathrm{Diff}^+_\sigma(M)$ by the map $\pi_\sigma(\varphi) = f$. The corresponding operators $U_{\varphi^\pm}$ given by (3.1) implement – in a generalized sense – the action on spinor fields of the double cover $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ of oriented, spin structure preserving diffeomorphisms. This is however not an implementation in the strict sense, as we have not really an action on a fixed space of spinors but rather the target space of spinors changes according to the pull back action of f on the metric.

In order to get of a genuine action one should develop further our setting. A possible way could be to consider the disjoint union $\mathscr{C}_\sigma = \amalg_g \mathscr{H}_{\sigma,g}$ where g runs over all Riemannian metrics on M. The (right) action is then given by:

$$\mathscr{C}_\sigma \times \widetilde{\mathrm{Diff}}^+_\sigma(M) \to \mathscr{C}_\sigma,\ (\psi, \varphi) \mapsto \psi \cdot \varphi := \psi \circ \varphi_{f^*g} \tag{3.3}$$

where $\psi \in \mathscr{H}_{\sigma,g}$, $f = \pi_\sigma(\varphi)$ and $\varphi_{f^*g} = \varphi \restriction \eta^{-1}(\mathrm{SO}_{f^*g}(M))$.

Remark 3.11. By Proposition 3.9, this action is 'fiberwise unitary' in the sense that it is linear on each component $\mathscr{H}_{\sigma,g}$, it is invertible and $\langle \psi\cdot\varphi \mid \chi\cdot\varphi\rangle_{\sigma,f^*g} = \langle \psi \mid \chi \rangle_{\sigma,g}$ for each $\psi, \chi \in \mathscr{H}_{\sigma,g}$.

In order to speak of a unitary action of $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ one should put a Hilbert space structure on $\mathscr{C}_\sigma$. It would be natural to view $\mathscr{C}_\sigma$ as a direct integral of Hilbert spaces over the space Met(M) of Riemannian metrics on M. We hope it can be made rigorous using the following facts. First, Met(M) is a positive convex cone into the vector space of smooth symmetric $(0,2)$-tensors on M. The latter is naturally a Fréchet space by equipping it with the smooth topology. The space Met(M) is open in that space, hence it inherits the structure of a Fréchet manifold. The tangent space of Met(M) at some $g \in \mathrm{Met}(M)$ can be identified with the vector space of smooth symmetric $(0,2)$-tensors on M. A Riemannian metric μ can be put on Met(M) which is invariant under the action of diffeomorphisms of M by pullback, $g \mapsto f^*g$. Given $\varphi \in \widetilde{\mathrm{Diff}}^+_\sigma(M)$, we could then define for each $\psi \in \int^\oplus_{\mathrm{Met}(M)} \mathscr{H}_{\sigma,g} \mathrm{d}\mu$ the element $(\mathfrak{U}_\varphi \psi)(g) = U_\varphi \psi(\pi_\sigma(\varphi^{-1})^* g)$, where we denote by $\mathrm{d}\mu$ the induced invariant measure on Met(M). The assignment $\varphi \mapsto \mathfrak{U}_\varphi$ would then be a unitary action of $\widetilde{\mathrm{Diff}}^+_\sigma(M)$ on the space of spinors with spin structure σ.

3.6 Equivariance of the Dirac operator

In order to define the Dirac operator one uses the lift of the covariant derivative associated to the Levi-Civita (metric preserving and torsion free) connection. Its local components with respect to an orthonormal frame e are given explicitly by the Christoffel symbols

$$\Gamma^{(e(x))}_{jkl} = c_{jkl} + c_{jlk} + c_{lkj} \tag{3.4}$$

where c_{ijk} are the structure constants of the commutators (as vector fields)

$$[e_i, e_j] = c_{ijk} e_k.$$

Then for a given σ and g on M the Dirac operator D is defined by its local components, i.e. its action in the 'gauge' $\tilde{e}$ on the local components $\psi \circ \tilde{e}$ of $\psi \in S_{\sigma,g}$ as

$$(D\psi)(\tilde{e}(x)) := \sum_j \gamma_j \left(\mathscr{L}_{e_j(x)} + \frac{1}{4} \sum_{k\ell} \gamma_k \gamma_\ell \Gamma^{(e(x))}_{jkl} \right) \psi(\tilde{e}(x)),$$

where γ_j are the anticommuting gamma matrices and $e = \eta \circ \tilde{e}$. As it should, up to a unitary equivalence the Dirac operator is independent on the choice of a representation of the gamma matrices and of local orthonormal frames.

Proposition 3.12. The Dirac operator is equivariant, i.e.

$$D' U^{\pm}_f = U^{\pm}_f D\,, \tag{3.5}$$

where D' is the Dirac operator on $\mathscr{H}_{f^*\sigma, f^*g}$.

Denoting $\psi' := U^{\pm}_f \psi$ we can write (3.5) in the equivalent form as

$$D'\psi' = (D\psi)' \tag{3.6}$$

Proof. It is a matter of a straightforward check that (3.6) is satisfied. For that evaluate both sides on $\tilde{e}'(x)$ using the fact that $\mathscr{L}_{e'_j}(\psi \circ f)(x) = \mathscr{L}_{e_j}\psi(f(x))$, that the local Christoffel symbols in any orthonormal frame are given in terms of commutators of the vectors constituting the frame and the commutators of f-related frames are f-related, and the equality of local components $\psi'(\tilde{e}'(x)) = \psi(\tilde{e}(f(x)))$. □

From the formula (3.6), which is already present (modulo a typo) in [12] (p.101 at the bottom), follows that the eigenvalues (point spectrum) of the Dirac operator are invariant under diffeomorphisms. Now using also Proposition (3.9) we can state a stronger result:

Corollary 3.13. The spectrum of the Dirac operator is invariant under the diffeomorphisms.

3.7 Brief account of some other approaches

The relations between spinors, the Dirac equation and the metric has been investigated by other authors (for the case of a fixed metric we mention the beautiful and classical paper [37]). Now, recall that for any two Riemannian metrics g_1 and g_2 on M we have the geometrically constructed principal based-isomorphism $m_{g_1,g_2}\colon \mathrm{SO}_{g_1}(M) \to \mathrm{SO}_{g_2}(M)$ (see Chapter 2). A lift of this isomorphism to respective spin bundles makes possible to compare spinors and the Dirac equation for different metrics. However, the two spin structures are necessarily equivalent.

The paper [3] combines ideas in [1] and [15] to construct a "metric" Lie derivative of spinor fields (see also the fundamental paper [35] for an early study of a Lie derivative for spinors). As in [15], the metrics considered are g and f^*g. The isomorphism m_{g,f^*g} is used to project the tangent map of a diffeomorphism $f\colon M \to M$ onto the same principal bundle $\mathrm{SO}_g(M)$; an analogous isomorphism between the two principal Spin_n-bundles associated to g and f^*g is used to project the lift of f, thus realising an automorphism of the same Spin_n-bundle over f. This permits to define a Lie derivative of spinor fields, which however does not induce the canonical Lie derivative on tensor fields build from spinor fields. Its geometric nature has been clarified in [24]. The procedure above works however only for strictly Riemannian metrics and spin-structure-preserving diffeomorphisms.

It is worth to mention that the canonical Dirac operator on Dirac spinor fields provides a prominent example of a spectral triple, and of a noncommutative Riemannian spin manifold in the framework of noncommutative geometry of Connes [9]. The results of this chapter fit well into this scheme and can be interpreted as a unitary implementation of diffeomorphisms on spectral triples. Concerning the additional requirements (axioms) for noncommutative Riemannian spin manifolds [10], most of them are preserved under diffeomorphisms in a straightforward manner. Only the axiom of projectivity and absolute continuity requires a comment. Namely it is easy to check that the $C^\infty(M)$-modules of smooth spinor fields, equipped with the $C^\infty(M)$-

valued hermitian form, are intertwined by the action of diffeomorphisms, i.e

$$(a\psi)\circ \tilde{f}_{\pm} = (f^* a)(\psi \circ \tilde{f}_{\pm}).$$

3.8 FINAL REMARKS

In this chapter we have further developed the approach of [15] and [12] to give a consistent definition of the transformation rules for spinor fields under (the double cover of) diffeomorphisms and checked the covariance of the Dirac operator. This requires however, as mentioned in the introduction, the changing of the space of spinors according to the pull back action on metrics and on spin structures labelling the spaces of spinors. In particular we are able to give the components of the transformed spinor field with respect to the transformed (spinor) linear frame, orthonormal with respect to the different (pullback) metric. It should be stressed however that we cannot compare the components of a given one and the same spinor field with respect to two linear frames if they are not related by a orthonormal transformation (giving a proper scaling dimension we could treat however conformally related metrics).

Since we have not employed an isomorphism between Hilbert spaces associated to different Riemannian metrics, we can not discuss in general the behaviour of spinors under infinitesimal diffeomorphisms and the notion of the Lie derivative on spinor fields along vector fields (unless they are Killing vector fields).

Moreover, for simplicity we considered only the oriented diffeomorphisms, the orientation changing diffeomorphisms in general would require the coverings $\mathrm{Pin}^{\pm}$ of the full orthogonal group. Some parts of our results hold as well in the Lorenzian or pseudoriemannian case. The isomorphism φ should also play an important role for a rigorous discussion of the variational aspects of the theory (under a general variation of the metric), and thus for deriving the equation of motions.

NONCOMMUTATIVE PART

4 Product of real spectral triples

4.1 Commutative real spectral triple

In the following, the symbol $\mathbb{Z}_{\geq 1}$ will denote the set of strictly positive integers and the symbol $\mathbb{Z}_{\geq 0}$ will denote the set of non-negative integers.

Gamma matrices

For each $n \in \mathbb{Z}_{\geq 1}$ consider the irreducible (complex) representations of the (complex) Clifford algebra $\mathscr{C}(\mathbb{R}^n)$ of Euclidean space $\mathbb{R}^n$ with negative-definite metric. For even (resp. odd) n, let us denote by $\Gamma_{(n)}$ (resp. $\Gamma_{(n,+)}$ and $\Gamma_{(n,-)}$) a possible choice of sets of complex matrices generating the only irreducible representation (respectively the only two irreducible representations) of $\mathscr{C}(\mathbb{R}^n)$, given by:

$$\begin{aligned}
\Gamma_{(1,+)} &= \{\gamma^1_{(1,+)}\} \\
\Gamma_{(1,-)} &= \{\gamma^1_{(1,-)}\} \\
\Gamma_{(2m)} &= \{\gamma^1_{(2m)}, \ldots, \gamma^{2m}_{(2m)}\} \\
\Gamma_{(2m+1,+)} &= \{\gamma^1_{(2m+1,+)}, \ldots, \gamma^{2m}_{(2m+1,+)}, \gamma^{2m+1}_{(2m+1,+)}\} \\
\Gamma_{(2m+1,-)} &= \{\gamma^1_{(2m+1,-)}, \ldots, \gamma^{2m}_{(2m+1,-)}, \gamma^{2m+1}_{(2m+1,-)}\} ,
\end{aligned} \tag{4.1}$$

where $\gamma^1_{(1,\pm)} = \pm i$ and for $n = 2m$ ($m \in \mathbb{Z}_{\geq 1}$), $j = 1,\ldots,m$, each $\gamma^\mu_{(n)}$ ($\mu = 1,\ldots,n$) is a $2^m \times 2^m$ complex matrix given by

$$\begin{aligned}\gamma^j_{(n)} &= i\,\underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{m-j} \otimes \sigma_1 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1}\,,\\ \gamma^{m+j}_{(n)} &= i\,\underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{m-j} \otimes \sigma_2 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1}\,,\end{aligned} \tag{4.2}$$

while for $n = 2m+1$ ($m \in \mathbb{Z}_{\geq 1}$), $k = 1,\ldots,n-1$ we put

$$\begin{aligned}\gamma^k_{(n,\pm)} &= \gamma^k_{(n-1)}\,,\\ \gamma^n_{(n,\pm)} &= \pm i\,\underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{m}\,,\end{aligned} \tag{4.3}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4}$$

(σ_1, σ_2 and σ_3 are the so called Pauli matrices). Indeed, the gamma matrices defined above anti-commute with each other and square to -1. Note that they are anti-hermitean. Moreover we have chosen them so that for a given n the first m are imaginary, the next m are real and $\gamma^{2m+1}_{(2m+1,\pm)}$ is imaginary. The properties discussed in the sequel do not depend on this choice up to unitary equivalence of matrices.

Note also that

$$\gamma^{2m+1}_{(2m+1,\pm)} = \pm i (i)^{(m \bmod 2)} \gamma^1_{(2m)} \gamma^2_{(2m)} \cdots \gamma^{2m}_{(2m)} \tag{4.5}$$

For even n we define the grading operator $\chi_{(n)} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n/2}$. For odd n we define also $\chi_{(n,\pm)} = \pm\mathbf{1}$. (In the sequel we shall often omit the lower indices to simplify notation). Note that, for any $n \in \mathbb{Z}_{\geq 1}$,

$$\chi = \alpha_n \gamma^1 \cdots \gamma^n\,, \tag{4.6}$$

where $\alpha_n = 1, -i, i, 1$ if $n = 0,1,2,3 \bmod 4$ respectively.

Dirac operator

The (free) Dirac operator on $\mathbb{R}^n$ is given by the formula

$$D = \sum_{\mu=1}^{n} \gamma^\mu \frac{\partial}{\partial x^\mu}. \tag{4.7}$$

where the gammas are as above. Note that, for even n,

$$D\chi + \chi D = 0.$$

Note also that, for odd n, changing the representation $\Gamma_{(2m+1,+)}$ to $\Gamma_{(2m+1,-)}$ is equivalent to changing the orientation of the manifold.

The 'minimal coupling' interaction with gauge fields, notably the electromagnetic potential A_μ, amounts to the substitution of the usual derivatives by the covariant ones,

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} + ieA_\mu,$$

where e is the charge.

As well known, the Dirac operator can be defined for a flat metric of arbitrary signature, and generalized to (pseudo) Riemannian spin manifolds with the help of covariant derivative given by the Levi-Civita (spin) connection. The elliptic or Riemannian case is extremely important and well studied in mathematics. In theoretical physics the Lorentzian case describes the evolution of spinor fields (fermions), and is also useful in connection with general relativity, modern versions of Kaluza-Klein theories, and (super) string theory. Recently, A. Connes made the Dirac operator a fundamental ingredient of a 'spectral triple' and of the notion of noncommutative (spin) manifold.

Charge conjugation

In physics, the charge conjugation J of spinors exchanges the Dirac operators corresponding to charge e and $-e$, keeping invariant the other physical quantities. However, we allow a possibility that J either commutes or anticommutes with the 'neutral' D given by (4.7). We shall indicate by a subscript $\pm$ these two options, which amount to demanding that

$$J_{\pm}\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}}+ieA_{\mu}\right)=\pm\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}}-ieA_{\mu}\right)J_{\pm}\,. \tag{4.8}$$

The operators $J_{\pm}$ have to be $\mathbb{C}$-antilinear, given by a composition of the complex conjugation with a constant matrix $C_{\pm}$, satisfying $C_{\pm}\overline{\gamma}^{\mu}=\pm\gamma^{\mu}C_{\pm}$. Hence, C_{+} should anti-commute with γ^{μ} for $\mu\leq m$ which are imaginary and commute with γ^{μ} for $m<\mu\leq 2m$ which are real. By the uniqueness and anti-commutativity of gamma matrices, such C_{+} is proportional to $\gamma^{1}\gamma^{2}\cdots\gamma^{m}$ if m is even and to $\gamma^{m+1}\gamma^{m+2}\cdots\gamma^{2m}$ if m is odd. It is just the other way for C_{-}.

For $\Gamma_{(2m)}$, this fixes the two solutions $J_{\pm}$ (up to a scalar multiple). Moreover, J_{-} is obtained by multiplying J_{+} with $\gamma^{1}\cdots\gamma^{2m}$ (up to a scalar multiple).

For $\Gamma_{(2m+1,+)}$ and for $\Gamma_{(2m+1,-)}$ we have to consider in addition the matrix $\gamma^{2m+1}_{(2m+1,\pm)}$, which is imaginary. Then the above (anti)-commutativity requirement selects J_{+} as the only solution if $m=1,3$ mod 4 (i.e. $n=3,7$ mod 8) and J_{-} if $m=0,2$ mod 4 (i.e. $n=1,5$ mod 8).

It can be checked that, with respect to the standard Hermitean scalar product $<\,,\,>$ on $\mathbb{C}^{2^m}$, the charge conjugation is a $\mathbb{C}$-antilinear isometry, that is $JJ^{\dagger}=1=J^{\dagger}J$, where the adjoint of a $\mathbb{C}$-antilinear operator is defined by $<\phi,J^{\dagger}\psi>=<\psi,J\phi>$. This reduces the ambiguity of $J_{\pm}$ to be a scalar of modulus 1.

The commutation relation of $J_{\pm}$ with D is by construction $DJ_{\pm}=\varepsilon' J_{\pm}D$, with $\varepsilon'=+1$ for J_{+} and $\varepsilon'=-1$ for J_{-}. Next, the commutation relation with χ (if n is even) is governed by $\varepsilon''=i^{n}=(-1)^{n/2}$. A straightforward computation gives $(J_{\pm})^{2}=\varepsilon\mathbf{1}$, where ε (together with $\varepsilon',\varepsilon''$) is given by table 4.1 below.

Table 4.1: Connes' selection in [11] is marked by •

n	0	2	4	6	0	2	4	6	1	3	5	7
ϵ	+	−	−	+	+	+	−	−	+	−	−	+
ϵ'	+	+	+	+	−	−	−	−	−	+	−	+
ϵ''	+	−	+	−	+	−	+	−				
	•	•	•	•					•	•	•	•

Notice that altogether there are twelve different possibilities, which can be labeled by the so-called KO-dimension $n\in\mathbb{Z}_{8}$ with the additional index ϵ' if n

is even (so for example the case $(\epsilon,\epsilon',\epsilon'')=(+,-,-)$ is labelled by 2_-). We find it notationally convenient to place this additional index also in the case of odd n, though it is redundant there. (For pseudoeuclidean spaces the periodicity modulo 8 holds for the signature $p-q$ of the metric).

The geometrical significance of the charge conjugation $J_\pm$ is that it governs the reduction of a spin^c structure to a spin structure (the Lie algebra $spin(n)$ is generated by $\gamma_\mu\gamma_\nu$ with $\mu<\nu$, which commute with J and so are invariant under Ad_J, while $spin^c(n)$ is generated by $spin(n)$ and one more matrix $i\mathbf{1}$, which anti-commutes with J).

The operator in (4.7) is a first-order partial differential operator with matrix coefficients. It acts on $\mathscr{C}^\infty(\mathbb{R}^n,\mathbb{C}^{2^m})$. After completion to $L^2(\mathbb{R}^n,\mathbb{C}^{2^m})$, D becomes an unbounded self-adjoint operator. The $*$-algebra of smooth complex-valued functions on $\mathbb{R}^n$ (with pointwise operations) is represented on $L^2(\mathbb{R}^n,\mathbb{C}^{2^m})$ as multiplication operators.

DIRAC OPERATOR ON CARTESIAN PRODUCT

Let $n=n_1+n_2$ with $n_1,n_2\in\mathbb{Z}_{\geq 1}$. It is straightforward to see that the Dirac operator (4.7) on $\mathbb{R}^n$ decomposes (up to unitary equivalences of matrices and a suitable renaming of coordinates) into Dirac operators D_1 on $\mathbb{R}^{n_1}$ and D_2 on $\mathbb{R}^{n_2}$ as follows ($\mathbf{1}$ and χ denote the relevant identity and grading matrices):

- if $n_1=2m_1$ and $n_2=2m_2+1$ ($m_1\in\mathbb{Z}_{\geq 1}$, $m_2\in\mathbb{Z}_{\geq 0}$)

$$D=\sum_{\mu=1}^{n_1}\gamma^\mu_{(n_1)}\otimes\mathbf{1}\,\frac{\partial}{\partial x^\mu}+\sum_{\nu=1}^{n_2}\chi_{(n_1)}\otimes\gamma^\nu_{(n_2,\pm)}\,\frac{\partial}{\partial x^{n_1+\nu}}\,, \tag{4.9}$$

  (using (4.5) it is not difficult to see that the tensor product gamma matrices appearing in (4.9) belong to the representation $\Gamma_{(2m_1+2m_2+1,\pm)}$, with the index $\pm$ identical to the one of $\gamma^\nu_{(n_2,\pm)}$, belonging to $\Gamma_{(2m_2+1,\pm)}$; in other words, the $\pm$-type of the irreducible representation is preserved);

- if $n_1=2m_1+1$ and $n_2=2m_2$ ($m_1\in\mathbb{Z}_{\geq 0}$, $m_2\in\mathbb{Z}_{\geq 1}$)

$$D = \sum_{\mu=1}^{n_1} \gamma^{\mu}_{(n_1,\pm)} \otimes \chi_{(n_2)} \frac{\partial}{\partial x^{n_2+\mu}} + \sum_{\nu=1}^{n_2} \mathbf{1} \otimes \gamma^{\nu}_{(n_2)} \frac{\partial}{\partial x^{\nu}}, \tag{4.10}$$

(again the $\pm$-type of the irreducible representation is preserved);

- if both $n_1 = 2m_1$ and $n_2 = 2m_2$ ($m_1, m_2 \in \mathbb{Z}_{\geq 1}$) are even, both formulae 4.9 and 4.10 hold and are related by a unitary matrix;
- if both $n_1 = 2m_1 + 1$ and $n_2 = 2m_2 + 1$ ($m_1, m_2 \in \mathbb{Z}_{\geq 0}$) are odd then

$$D = \sum_{\mu=1}^{n_1} \gamma^{\mu}_{(n_1)} \otimes \mathbf{1} \otimes \sigma_1 \frac{\partial}{\partial x^{\mu}} + \sum_{\nu=1}^{n_2} \mathbf{1} \otimes \gamma^{\nu}_{(n_2)} \otimes \sigma_2 \frac{\partial}{\partial x^{n_1+\nu}}. \tag{4.11}$$

Moreover, we can take $\chi = 1 \otimes 1 \otimes \sigma_3$ as grading.

4.2 DEFINITION OF A REAL SPECTRAL TRIPLE

The classical setting presented in section 4.1 was generalized by Connes to the noncommutative case, which we now recall and supplement by keeping all the twelve possibilities for the reality structure. We recall from [11]

Definition 4.1. A spectral triple $(A, \mathcal{H}, D)$ is given by an involutive unital algebra A (over $\mathbb{R}$ or $\mathbb{C}$) faithfully represented as bounded operators on a complex separable Hilbert space $\mathcal{H}$ and by a self-adjoint operator D with compact resolvent such that for each $a \in A$ the commutator[1] $[D, a]$ has bounded extension.

A spectral triple is called even if the Hilbert space $\mathcal{H}$ is endowed with a nontrivial $\mathbb{Z}_2$-grading χ which[2] commutes with any $a \in A$ and anticommutes with D. Otherwise it is called odd.

The following definition is a modification of Definition 1.124 in [11] in order to cover all the twelve possibilities as discussed in the previous section.

[1]We assume $a\,\mathrm{dom}D \subset \mathrm{dom}D$ for each $a \in A$, so that $[D, a]$ is defined on $\mathrm{dom}D$.

[2]By definition χ is a self-adjoint unitary such that $\chi^2 = \mathrm{id}_{\mathcal{H}}$ and $\chi \neq \pm\mathrm{id}_{\mathcal{H}}$. The Hilbert space $\mathcal{H}$ can then be split into its eigenspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$; by requesting $[\chi, a] = 0$ for each $a \in A$ this splitting is invariant under the action of A on $\mathcal{H}$.

Definition 4.2. A real structure of KO-dimension $n \in \mathbb{Z}_8$ on a spectral triple $(A, \mathscr{H}, D)$ is an antilinear isometry $J : \mathscr{H} \to \mathscr{H}$, with the property that

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \text{ and, if } (A, \mathscr{H}, D) \text{ is even, } J\chi = \epsilon'' \chi J. \tag{4.12}$$

Given n, the possibilities for arrays of numbers $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ are given by the tables in section 4.1. Moreover, the action of A satisfies the commutation rule

$$[a, Jb^* J^{-1}] = 0, \quad \forall a, b \in A \tag{4.13}$$

and the operator D satisfies the order one condition

$$[[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in A. \tag{4.14}$$

A spectral triple $(A, \mathscr{H}, D)$ endowed with a real structure J is called a real spectral triple.

Remarks.

- usually we will omit to indicate the $*$-representation map $\rho : A \to \mathscr{B}(\mathscr{H})$ for simplicity;
- we recall that an antiunitary operator J is antilinear, bijective and $(Ju|Jv) = (v|u)$;
- the map $b \mapsto Jb^* J^{-1}$ is a representation of the opposite algebra A° on $\mathscr{B}(\mathscr{H})$;
- equation (4.13) which says that Ad_J sends A to its commutant is sometimes called the "zero order condition";
- note that putting:

  $$\vec{\epsilon}_\pm(n) = \left(\epsilon_\pm(n), \epsilon'_\pm(n), \epsilon''_\pm(n)\right), \tag{4.15}$$

  (where $n \in \mathbb{Z}_8$) we have the relation:

  $$\vec{\epsilon}_-(n) = -\vec{\epsilon}_+(n+2). \tag{4.16}$$

4.3 PRODUCT OF REAL SPECTRAL TRIPLES

Following what happens in the commutative case, we shall produce a real spectral triple of dimension n_1+n_2 out of two triples of dimensions n_1 and n_2 respectively. The new algebra is the tensor product algebra $A = A_1 \otimes A_2$, where $\otimes$ is the algebraic tensor product[3] and the involution is defined componentwise: $(a\otimes b)^* = a^* \otimes b^*$. It turns out that the other ingredients of the resulting spectral triple depend on the parity of the two given triples.

EVEN-EVEN CASE

As the Hilbert space carrying the $*$-representation of A we take the Hilbert tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ and as the $*$-representation we take the tensor product representation:

$$\rho_1 \otimes \rho_2 \colon A_1 \otimes A_2 \to \mathscr{B}(\mathscr{H}_1 \otimes \mathscr{H}_2). \tag{4.17}$$

The representation $\rho_1 \otimes \rho_2$ is faithful whenever ρ_1 and ρ_2 are. The grading operator is given by $\chi = \chi_1 \otimes \chi_2$ (it is easy to check that it is unitary, squares to $\mathrm{id}_{\mathscr{H}_1\otimes\mathscr{H}_2}$ and commutes with every element of the product algebra A). As for the Dirac operator, using χ_1 or χ_2 we take the following operators [4]:

$$\begin{aligned} \mathscr{D} &= D_1 \otimes \mathrm{id}_{\mathscr{H}_2} + \chi_1 \otimes D_2\,, \\ \widetilde{\mathscr{D}} &= D_1 \otimes \chi_2 + \mathrm{id}_{\mathscr{H}_1} \otimes D_2\,, \end{aligned} \tag{4.18}$$

both defined on the dense domain $\mathrm{dom} D_1 \otimes \mathrm{dom} D_2$. They are unitarily equivalent:

$$\widetilde{\mathscr{D}} = U \mathscr{D} U^\dagger\,, \tag{4.19}$$

where (see [54])

$$U = \frac{1}{2}(\mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2} + \chi_1 \otimes \mathrm{id}_{\mathscr{H}_2} + \mathrm{id}_{\mathscr{H}_1} \otimes \chi_2 - \chi_1 \otimes \chi_2). \tag{4.20}$$

We now show that $\mathscr{D}$ is essentially self-adjoint by exhibiting an orthonormal basis of eigenvectors. Self-adjointness is immediate if one Hilbert space is

[3] Over $\mathbb{R}$ if at least one of the two algebras is real, over $\mathbb{C}$ if both algebras are complex.

[4] The simplest choice $D = D_1 \otimes \mathrm{id}_{\mathscr{H}_2} + \mathrm{id}_{\mathscr{H}_1} \otimes D_2$ has non-compact resolvent in general (e.g. $\ker D$ is infinite dimensional if $D_2 = -D_1$ with $\mathscr{H}_1 = \mathscr{H}_2$ infinite dimensional).

finite dimensional, so we assume both Hilbert spaces to be infinite dimensional.

From the general theory of (linear unbounded) operators on a complex separable Hilbert space we know that each D_i has pure point spectrum consisting of countably many real eigenvalues, each with finite multiplicity, and the only limit point of their absolute values is $+\infty$. Let

$$\{v_{\lambda,m_\lambda},\ v_{j_\pm} \mid \lambda \in \sigma(D_1)/\{0\},\ m_\lambda = 1,\ldots,M_\lambda,\ j_\pm = 1,\ldots,K_\pm \in \mathbb{Z}_{\geq 0}\}$$

be an orthonormal basis of eigenvectors of D_1, where $\sigma(D_1)$ is the spectrum of D_1, M_λ is the multiplicity of $\lambda \in \sigma(D_1)/\{0\}$, and finally $\{v_{j_\pm} \mid j_\pm = 1,\ldots K_\pm\}$ is a basis of $\ker D_1$ consisting of eigenvectors of χ_1 such that $\chi_1 v_{j_\pm} = \pm v_{j_\pm}$. Note that since D_1 and χ_1 anticommute, and χ_1 is unitary, then $\sigma(D_1)$ is symmetric and $M_\lambda = M_{-\lambda}$. Let $\{w_{\mu,n_\mu} \mid \mu \in \sigma(D_2),\ n_\mu = 1,\ldots,N_\mu \in \mathbb{Z}_{\geq 0}\}$ be an orthonormal basis of eigenvectors of D_2. Finally, let us consider the following vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\begin{aligned} u^+_{\lambda,m_\lambda,\mu,n_\mu} &= \cos\theta_{\lambda\mu}(v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu}) + \sin\theta_{\lambda\mu}(\chi_1 v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu})\,, \\ u^-_{\lambda,m_\lambda,\mu,n_\mu} &= -\sin\theta_{\lambda\mu}(v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu}) + \cos\theta_{\lambda\mu}(\chi_1 v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu})\,, \\ u_{j_\pm,\mu,n_\mu} &= v_{j_\pm} \otimes w_{\mu,n_\mu}\,, \end{aligned} \tag{4.21}$$

where $\lambda \in \sigma(D_1) \cap \mathbb{R}_{>0}$, $\mu \in \sigma(D_2)$, $\theta_{\lambda\mu} = \frac{1}{2}\arctan\frac{\mu}{\lambda} \in (-\pi/4, \pi/4)$ and $j_\pm = 1,\ldots K_\pm$. Then the set

$$\{u^+_{\lambda,m_\lambda,\mu,n_\mu},\ u^-_{\lambda,m_\lambda,\mu,n_\mu},\ u_{j_\pm,\mu,n_\mu} \mid \lambda \in \sigma(D_1) \cap \mathbb{R}_{>0},\ \mu \in \sigma(D_2),\ j_\pm = 1,\ldots,K_\pm\} \tag{4.22}$$

is an orthonormal basis of eigenvectors of $\mathscr{D}$, with corresponding eigenvalues given by:

$$\begin{aligned} \mathscr{D}(u^\pm_{\lambda,m_\lambda,\mu,n_\mu}) &= \pm\sqrt{\lambda^2 + \mu^2}\ u^\pm_{\lambda,m_\lambda,\mu,n_\mu}\,, \\ \mathscr{D}(u_{j_\pm,\mu,n_\mu}) &= \pm\mu\ u_{j_\pm,\mu,n_\mu}\,. \end{aligned} \tag{4.23}$$

It can be easily seen that $\ker\mathscr{D} = \ker D_1 \otimes \ker D_2$. From the existence of a basis of eigenvectors for $\mathscr{D}$ we can promptly conclude that $D \equiv \overline{\mathscr{D}}$ (the closure of $\mathscr{D}$) is self-adjoint. From the analysis above it is clear that D is a self-adjoint operator with pure point spectrum consisting of countably many eigenvalues, each with finite multiplicity, and the only limit point of their absolute values is $+\infty$. By the general theory, we conclude that D has compact resolvent.

By unitary equivalence, the basis for $\mathscr{D}$ gives a basis of eigenvectors of $\widetilde{\mathscr{D}}$ with the same eigenvalues and multiplicities as those of $\mathscr{D}$, so the analysis above goes through and we conclude that $\widetilde{D} \equiv \overline{\widetilde{\mathscr{D}}}$ is also self-adjoint with compact resolvent.

It is easy to check that $[D, a]$ is defined on dom$D$ and extends to a bounded operator for each $a \in A_1 \otimes A_2$ (using the condition $[\chi, a] = 0$). Analogously for $\widetilde{D}$.

From J_1 and J_2 we can construct $J = J_1 \otimes J_2$, which is easily seen to be antiunitary on $\mathscr{H}_1 \otimes \mathscr{H}_2$. Moreover we have

$$\begin{aligned}[][a_1 \otimes a_2, J(b_1 \otimes b_2)^* J^{-1}] &= [a_1 \otimes a_2, (J_1 \otimes J_2)(b_1^* \otimes b_2^*)(J_1^{-1} \otimes J_2^{-1})] \\ &= [a_1, J_1 b_1^* J_1^{-1}] \otimes a_2 J_2 b_2^* J_2^{-1} + \\ &\quad + J_1 b_1^* J_1^{-1} a_1 \otimes [a_2, J_2 b_2^* J_2^{-1}] \\ &= 0 . \end{aligned} \tag{4.24}$$

Labeling the ϵ-triples with n_+ or n_- according to the KO-dimension and the J involved in the product, we get the following tables for the KO-dimension and reality structure of the resulting triple (we distinguish the two cases for the total Dirac operator, D or $\widetilde{D}$):

Remark 4.3. The two top blocks in table 4.2 correspond to the even-even cases covered by Vanhecke's paper [54].

EVEN-ODD CASE

The Hilbert space $\mathscr{H}$, the $*$-representation of A on $\mathscr{B}(\mathscr{H})$ and the reality structure J are the same as in the even-even case. Now we have only one nontrivial grading operator though: we then choose D or $\widetilde{D}$ from the previous construction, according to whether the even triple is the first one or the second one, respectively. The basis of eigenvectors is again given by (4.22) (or by the analogous construction using χ_2 instead of χ_1). The argument for proving self-

Table 4.2: D

1/2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+	2_+	4_+	6_+				
2_+					2_+	4_+	6_+	0_+
4_+	4_+	6_+	0_+	2_+				
6_+					6_+	0_+	2_+	4_+
0_-					0_-	2_-	4_-	6_-
2_-	2_-	4_-	6_-	0_-				
4_-					4_-	6_-	0_-	2_-
6_-	6_-	0_-	2_-	4_-				

Table 4.3: $\widetilde{D}$

1/2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+		4_+			2_-		6_-
2_+	2_+		6_+			4_-		0_-
4_+	4_+		0_+			6_-		2_-
6_+	6_+		2_+			0_-		4_-
0_-		2_+		6_+	0_-		4_-	
2_-		4_+		0_+	2_-		6_-	
4_-		6_+		2_+	4_-		0_-	
6_-		0_+		4_+	6_-		2_-	

adjointness and compactness of resolvent goes through exactly as before. For the KO-dimension and reality structure of the resulting triple we obtain the following tables:

Remark 4.4. Table 4.4 corresponds to the even-odd cases covered by Vanhecke's paper [54].

ODD-ODD CASE

In this case we have no nontrivial grading operator available. In order to overcome this, motivated by the commutative situation, we consider the following

Table 4.4: D

1/2	1_-	3_+	5_-	7_+
0_+		3_+		7_+
2_+	3_+		7_+	
4_+		7_+		3_+
6_+	7_+		3_+	
0_-	1_-		5_-	
2_-		5_-		1_-
4_-	5_-		1_-	
6_-		1_-		5_-

Table 4.5: $\widetilde{D}$

1/2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
1_-		3_+		7_+	1_-		5_-	
3_+	3_+		7_+			5_-		1_-
5_-		7_+		3_+	5_-		1_-	
7_+	7_+		3_+			1_-		5_-

construction:

$$\begin{aligned} A &= A_1 \otimes A_2, \\ \mathscr{H} &= (\mathscr{H}_1 \otimes \mathscr{H}_2) \otimes \mathbb{C}^2, \\ \mathscr{D} &= D_1 \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_1 + \mathrm{id}_{\mathscr{H}_1} \otimes D_2 \otimes \sigma_2, \\ J^{\pm} &= J_1 \otimes J_2 \otimes M^{\pm} K, \\ \chi &= \mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_3, \end{aligned} \tag{4.25}$$

where the σs are the Pauli matrices, $M^{\pm}$ are two complex matrices specified by the table below and K is the complex conjugation operator defined for the canonical basis of $\mathbb{C}^2$ (i.e., if (e_1, e_2) is the canonical basis, we have $K(\lambda e_i) = \overline{\lambda} e_i$ for every $\lambda \in \mathbb{C}$). The representation is understood to be trivial on the $\mathbb{C}^2$ factor, i.e. $\rho(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2) \otimes \mathrm{id}_{\mathbb{C}^2}$.

Remarks. The entries in table 4.6 stand for the pair M^+, M^-. For convenience, the identity matrix is called σ_0. Note that this construction still works under

Table 4.6: Odd-odd case

1/2	1_-	3_+	5_-	7_+
1_-	σ_2,σ_1	σ_3,σ_0	σ_2,σ_1	σ_3,σ_0
3_+	σ_0,σ_3	σ_1,σ_2	σ_0,σ_3	σ_1,σ_2
5_-	σ_2,σ_1	σ_3,σ_0	σ_2,σ_1	σ_3,σ_0
7_+	σ_0,σ_3	σ_1,σ_2	σ_0,σ_3	σ_1,σ_2

any permutation of the Pauli matrices (e.g., one can take $\mathscr{D} = D_1 \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_1 + \mathrm{id}_{\mathscr{H}_1} \otimes D_2 \otimes \sigma_3$ and $\chi = \mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_2$). The table obtained considering only the first element in each entry (i.e. M^+) corresponds to the odd-odd cases covered by Sitarz's notes [46].

Calling $n_1 = 2m_1 + 1$ and $n_2 = 2m_2 + 1$ the dimensions of the two triples involved, we have:

$$\begin{aligned} M^+(n_1,n_2) &= \sigma_j\,, \quad j = \tfrac{1}{2}\left(5+(-1)^{m_2+1}\right)+2m_1 \mod 4\,, \\ M^-(n_1,n_2) &= \sigma_k\,, \quad k = \tfrac{1}{2}(1+(-1)^{m_2})+2m_1 \mod 4\,. \end{aligned} \tag{4.26}$$

Self-adjointness of $D \equiv \overline{\mathscr{D}}$ and compactness of its resolvent can be proven by the same argument of section 4.3 with suitable changes. In particular the eigenvectors $u^\pm$ are given by:

$$\begin{aligned} u^+_{\lambda,m_\lambda,\mu,n_\mu} &= \frac{1}{\sqrt{2}}\cos\theta_{\lambda\mu} v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu} \otimes \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\sin\theta_{\lambda\mu} v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu} \otimes \begin{pmatrix}-i\\i\end{pmatrix}, \\ u^-_{\lambda,m_\lambda,\mu,n_\mu} &= -\frac{1}{\sqrt{2}}\sin\theta_{\lambda\mu} v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu} \otimes \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\cos\theta_{\lambda\mu} v_{\lambda,m_\lambda} \otimes w_{\mu,n_\mu} \otimes \begin{pmatrix}-i\\i\end{pmatrix}. \end{aligned} \tag{4.27}$$

4.4 Further properties and their preservation under products

We list here several additional axioms for a real spectral triple $(A, \mathscr{H}, D, J)$ introduced by Connes. See [26].

- **Dimension.** There is a positive integer n such that $|D|^{-n}$ is infinitesimal of order 1, where $|D|^{-1}$ is the resolvent of the absolute value of D and to be infinitesimal of order 1 means the sequence of eigenvalues decreasingly ordered is $\mathcal{O}(\frac{1}{n})$.

- **Regularity.** The space $A+[D,A]$ is contained in the smooth domain of the derivation $\delta(\ )=[|D|,\ ]$, where the smooth domain of δ is defined as $\mathrm{dom}^{\infty}(\delta) := \cap_{k=1}^{\infty}\mathrm{dom}(\delta^k)$ and the domain of δ^k is defined recursively as $\mathrm{dom}(\delta^k) := \{b \in \mathrm{dom}(\delta) \mid \delta(b) \in \mathrm{dom}(\delta^{k-1})\}$, where $\mathrm{dom}(\delta)$ is the set of linear operators T on $\mathscr{H}$ such that $T\mathrm{dom}(|D|) \subset \mathrm{dom}(|D|)$ and $[|D|,T]$ extends to a bounded operator on $\mathscr{H}$.

- **First order.** The representation of A° implemented by J commutes with $[D,A]$.

- **Orientation.** There is a Hochschild cycle $c_n \in Z_n(A, A\otimes A^{\circ})$ such that $\pi_D(c)=\chi$, where $C_n(A,E) := E\otimes A^{\otimes n}$ is the Hochschild chain complex, $b(e\otimes a_1\otimes\cdots\otimes a_n) = \sum_{j=0}^{n-1}(-1)^j e\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_n + (-1)^n a_n e\otimes a_1\otimes\cdots\otimes a_{n-1}$ is the Hochschild boundary map, $Z_n(E,A)$ is the A-bimodule of Hochschild cycles and $\pi_D(c) = \tau_J(a_0)[D,\rho(a_1)]\cdots[D,\rho(a_p)]$ where $c = a_0\otimes a_1\otimes\cdots\otimes a_p$ is an elementary n-chain of $C_n(A, A\otimes A^{\circ})$ and $\tau_J(a\otimes b) = \rho(a)J\rho(b^*)J^{-1}$.

We now check the preservation of stated properties under products.

- **Dimension.** Assume $(A_i, \mathscr{H}_i, D_i, J_i, (\chi_i))_{i=1,2}$ are two real spectral triples of dimensions n_1 and n_2 respectively. Independently of the parities of the triples, the eigenvalues of D^2 are given by the sum of the eigenvalues of D_1^2 and D_2^2, and this implies that the dimension of the product triple is n_1+n_2 (see p. 486 in [26] for details).

- **Regularity.** For even spectral triples the result that the product of two regular triples is regular is contained in [53], which uses the existence of an algebra of generalized differential operators; this works also when at least one of the two triples is even and for the odd-odd case the argument still can be carried over.

- **First order.** Assume $(A_i, \mathscr{H}_i, D_i, J_i, \chi_i)_{i=1,2}$ are two real spectral triples of even dimension satisfying this property. Then the product $(A, \mathscr{H}, D, J, \chi)$ satisfies it as well: indeed, taking $a, b \in A$, where $a = a_1 \otimes a_2$, $b = b_1 \otimes b_2$ we compute:

$$\begin{aligned}
[[D,a],b^\circ] &= \left[\left[D_1 \otimes \mathrm{id}_{\mathscr{H}_2} + \chi_1 \otimes D_2, a\right], b^\circ\right] \\
&= \left[\left[D_1 \otimes \mathrm{id}_{\mathscr{H}_2}, a\right] + \left[\chi_1 \otimes D_2, a\right], b^\circ\right] \\
&= \left[[D_1, a_1] \otimes a_2 + \chi_1 a_1 \otimes [D_2, a_2], b^\circ\right] \\
&= [[D_1, a_1] \otimes a_2, b^\circ] + \left[\chi_1 a_1 \otimes [D_2, a_2], b^\circ\right] \\
&= \left[[D_1, a_1], b_1^\circ\right] \otimes a_2 b_2^\circ + \chi_1 a_1 b_1^\circ \otimes \left[[D_2, a_2], b_2^\circ\right] \\
&= 0,
\end{aligned} \tag{4.28}$$

where $b^\circ = J b^* J^{-1}$, where we have used the facts that the representations of A_i and A_i° commute (this is part of the content of condition 5) and also that χ_i commutes with the representation of A_i. Analogously one can prove that the representations of A and A° commute, i.e. $[a, b^\circ] = 0$. The same computation also applies for $\widetilde{D}$, and this concludes the proof for the even-even case. For the even-odd case the computations are analogous. For the odd-odd case we compute:

$$\begin{aligned}
[[D,a],b^\circ] &= \left[\left[D_1 \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_1 + \mathrm{id}_{\mathscr{H}_1} \otimes D_2 \otimes \sigma_2, a\right], b^\circ\right] \\
&= \left[\left[D_1 \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_1, a_1 \otimes a_2 \otimes \mathrm{id}_{\mathbb{C}^2}\right], b_1^\circ \otimes b_2^\circ \otimes \mathrm{id}_{\mathbb{C}^2}\right] + \\
&\quad + \left[\left[\mathrm{id}_{\mathscr{H}_1} \otimes D_2 \otimes \sigma_2, a_1 \otimes a_2 \otimes \mathrm{id}_{\mathbb{C}^2}\right], b_1^\circ \otimes b_2^\circ \otimes \mathrm{id}_{\mathbb{C}^2}\right] \\
&= \left[[D_1, a_1] \otimes a_2 \otimes \sigma_1, b_1^\circ \otimes b_2^\circ \otimes \mathrm{id}_{\mathbb{C}^2}\right] + \\
&\quad + \left[a_1 \otimes [D_2, a_2] \otimes \sigma_2, b_1^\circ \otimes b_2^\circ \otimes \mathrm{id}_{\mathbb{C}^2}\right] \\
&= \left[[D_1, a_1], b_1^\circ\right] \otimes a_2 b_2^\circ \otimes \sigma_1 + \\
&\quad + a_1 b_1^\circ \otimes \left[[D_2, a_2], b_2^\circ\right] \otimes \sigma_2 \\
&= 0,
\end{aligned} \tag{4.29}$$

where we used the fact that the representations of A_i and A_i° commute. As before, checking that the representations of A and A° commute is entirely analogous.

- **Orientation.** Given two Hochschild cycles $a_i \in Z_{n_i}(A_i, A_i \otimes A_i^\circ)$, $i = 1, 2$, where n_i is the dimension of the algebra A_i according to "condition 1", one can construct the Hochschild cycle $a \in Z_n(A, A \otimes A^\circ)$ (where $n = n_1 + n_2$ and $A = A_1 \otimes A_2$) using the shuffle product (see [38], section 4.2). Let us provide some details for the construction, following [38]. First define[5] the shuffle product $\times : C_{n_1}(A_1) \otimes C_{n_2}(A_2) \to C_{n_1+n_2}(A_1 \otimes A_2)$ of two chains as follows:

$$
\begin{aligned}
&(a_0^1, a_1^1, \dots, a_p^1) \times (a_0^2, a_1^2, \dots, a_q^2) = \\
&\sum_\sigma (-1)^\sigma \sigma \cdot (a_0^1 \otimes a_0^2, a_1^1 \otimes 1, \dots, a_p^1 \otimes 1, 1 \otimes a_1^2, \dots, 1 \otimes a_q^2),
\end{aligned}
\tag{4.30}
$$

where

$$
\sigma \cdot (a_0, a_1 \dots, a_n) = (a_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}) \tag{4.31}
$$

and the sum is over all (p, q)-shuffles, i.e. permutations of $\{1, \dots, p+q\}$ preserving the order of $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$ separately. Then for the Hochschild boundary map ∂ the following formula holds:

$$
\partial(x \times y) = \partial(x) \times y + (-1)^{|x|} x \times \partial(y). \tag{4.32}
$$

From this formula it follows at once that the shuffle product of two Hochschild cycles is again a Hochschild cycle. The orientation condition for a triple now states that there is a Hochschild cycle c satisfying the following formula:

$$
\pi_D(c) = \chi, \tag{4.33}
$$

where the map π_D is defined as:

$$
\pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \tau_J(a_0)[D, \rho(a_1)] \cdots [D, \rho(a_p)], \tag{4.34}
$$

[5] For the sake of generality, here we can take the first coefficient a_0 to be in some module over A; we will be interested in the case where this module is $A \otimes A^\circ$.

where $a_0 \in A \otimes A^\circ$ and $\tau_J(a \otimes b) = \rho(a) J \rho(b^*) J^{-1}$. Given two real spectral triples of dimensions n_1 and n_2 respectively, we claim that if c_j $(j = 1, 2)$ are Hochschild cycles satisfying $\pi_{D_j}(c_j) = \chi_j$ then the analogous cycle on the product triple is given by

$$c = \tfrac{1}{r} c_1 \times c_2, \tag{4.35}$$

where

$$r = \begin{cases} \nu_{n_1+n_2}\,, & \text{when } n_1 n_2 \text{ is even} \\ i\nu_{n_1+n_2}\,, & \text{when } n_1 n_2 \text{ is odd}\,, \end{cases} \qquad \nu_n = \frac{1}{2}(n-1)n\,, \tag{4.36}$$

where $\nu(n) = \frac{1}{2}(n-1)n$. In order to check formula (4.33) on the product triple with c given by equation (4.35) we distinguish three cases depending on the parities involved:

Even-even. A simple computation shows that

$$\begin{aligned} [D, \rho_1(a) \otimes \mathrm{id}_{\mathscr{H}_2}] &= [D_1, \rho_1(a)] \otimes \mathrm{id}_{\mathscr{H}_2} \\ [D, \mathrm{id}_{\mathscr{H}_1} \otimes \rho_2(b)] &= \chi_1 \otimes [D_2, \rho_2(b)] \end{aligned} \tag{4.37}$$

from which it follows that

$$\begin{aligned} &\pi_D(\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \ldots, a_{n_1} \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_{n_2})) = \\ &= \Pi \sigma \cdot \big(\tau_{J_1}(a_0), [D_1, \rho_1(a_1)], \ldots, [D_1, \rho_1(a_{n_1})], \chi_1, \ldots, \chi_1\big) \otimes \\ &\quad \otimes \Pi(\tau_{J_2}(b_0), [D_2, \rho_2(b_1)], \ldots, [D_2, \rho_2(b_{n_2})]), \end{aligned} \tag{4.38}$$

where Π means algebra product of all the elements in the ordered list. Since χ_1 anti-commutes with D_1 and commutes with $\rho_1(a)$ for each $a \in A_1$, rearranging all the n_2 operators χ_1 side by side produces a $(-1)^\sigma$ sign which cancels the same sign from the shuffle product; moreover, since n_2 is even, we have ${\chi_1}^{n_2} = \mathrm{id}_{\mathscr{H}_1}$; therefore we are left with a sum of $\frac{1}{2}(n_1 + n_2 - 1)(n_1 + n_2)$ identical terms:

$$\begin{aligned} \pi_D(c_1 \times c_2) &= \nu_{n_1+n_2}\, \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \\ &= \nu_{n_1+n_2}\, \chi_1 \otimes \chi_2\,. \end{aligned} \tag{4.39}$$

The same reasoning applies to $\widetilde{D}$ with obvious modifications.

Even-odd. The previous argument carries over unaltered, but this time we have ${\chi_1}^{n_2} = \chi_1$ since n_2 is odd. Then we get

$$\begin{aligned} \pi_D(c_1 \times c_2) &= \nu_{n_1+n_2}\ \chi_1 \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \\ &= \nu_{n_1+n_2}\ {\chi_1}^2 \otimes \chi_2 \\ &= \nu_{n_1+n_2}\ \mathrm{id}_{\mathscr{H}_1} \otimes (\pm \mathrm{id}_{\mathscr{H}_2}) \\ &= \pm \nu_{n_1+n_2}\ \mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2}\ , \end{aligned} \tag{4.40}$$

as expected. The same reasoning applies to the case where n_1 is odd and n_2 is even, with obvious modifications.

Odd-odd. In this case a simple computation shows that

$$\begin{aligned} [D, \rho_1(a) \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \mathrm{id}_{\mathbb{C}^2}] &= [D_1, \rho_1(a)] \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_1 \\ [D, \mathrm{id}_{\mathscr{H}_1} \otimes \rho_2(b) \otimes \mathrm{id}_{\mathbb{C}^2}] &= \mathrm{id}_{\mathscr{H}_1} \otimes [D_2, \rho_2(b)] \otimes \sigma_2 \end{aligned} \tag{4.41}$$

from which it follows that

$$\begin{aligned} &\pi_D(\sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, \dots, a_{n_1} \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_{n_2})) = \\ =&\tau_{J_1}(a_0)[D_1, \rho_1(a_1)] \cdots [D_1, \rho_1(a_{n_1})] \otimes \\ &\otimes \tau_{J_2}(b_0)[D_2, \rho_2(b_1)] \cdots [D_2, \rho_2(b_{n_2})]) \otimes \\ &\otimes \Pi\sigma \cdot (1, \underbrace{\sigma_1, \dots, \sigma_1}_{n_1 \text{ times}}, \underbrace{\sigma_2, \dots, \sigma_2}_{n_2 \text{ times}}) \end{aligned} \tag{4.42}$$

Since $\sigma_1\sigma_2 = -\sigma_2\sigma_1$ we can rearrange the σ_is with all σ_1s on the left and all σ_2s on the right, producing a $(-1)^\sigma$ sign which cancels the same sign from the shuffle product; moreover since n_1 and n_2 are both odd we get $\sigma_1^{n_1}\sigma_2^{n_2} = \sigma_1\sigma_2 = i\sigma_3$, so we end up with

$$\begin{aligned} \pi_D(c_1 \times c_2) &= \nu_{n_1+n_2}\ \pi_{D_1}(c_1) \otimes \pi_{D_2}(c_2) \otimes i\sigma_3 \\ &= i\nu_{n_1+n_2}\ \mathrm{id}_{\mathscr{H}_1} \otimes \mathrm{id}_{\mathscr{H}_2} \otimes \sigma_3\,. \end{aligned} \tag{4.43}$$

Remark 4.5. The orientation axiom is consistent with the observation made in section 4.1 for the classical setting, namely that changing the representation $\Gamma_{(2m+1,+)}$ to $\Gamma_{(2m+1,-)}$ is equivalent to changing the orientation of the manifold. In the noncommutative setting, this translates into changing the sign of the Hochschild cycle c in (4.33).

4.5 FINAL COMMENTS

In this chapter we were concerned with unital spectral triples but the canonical Dirac operator on $\mathbb{R}^n$ (cf. section 4.1) is not of that type. The definition of a nonunital spectral triple is slightly different, as well as the additional axioms for it. In order to remain in the realm of unital spectral triples, as a commutative compact case study we should take rather the flat torus with the trivial spin structure. This however does not change the form (4.7) of the canonical Dirac operator, but just supplements it with periodic boundary conditions.

It is worth mentioning that in our setup the metric and KO dimensions need not be equal modulo 8. This is the case in some of the recent examples of spectral triples [11] [7] [16], see also [5].

A The Gram-Schmidt procedure and frame bundles

In this appendix we introduce the Gram-Schmidt procedure and show that it provides an isomorphism of frame bundles for different Euclidean metrics η and η' on the same vector bundle $E \to B$ if and only if the two metrics are conformally equivalent. This contrasts with a claim made in §5 of [19]. We recall to the reader that such frame bundles are indeed isomorphic, but the isomorphism is *not* given by the Gram-Schmidt procedure in general. Rather, it is given by the square root of the based-automorphism $(\widetilde{\eta'})^{-1} \circ \widetilde{\eta}\colon E \to E$ (apply Lemma 2.17 to the case $E' = E$).

Definition A.1. Given two (p,q)-metrics η and η' on a vector bundle $E \to B$, they are called *conformally equivalent* if there is a strictly positive smooth function $\lambda\colon B \to \mathbb{R}_{>0}$, $x \mapsto \lambda_x$, such that $\eta'(v_x, w_x) = \lambda_x \eta(v_x, w_x)$ for each $v_x, w_x \in E_x$ and each $x \in B$.

Definition A.2. Given a n-dimensional real vector space V equipped with a definite positive bilinear form $(\,|\,)\colon V \times V \to \mathbb{R}$, the Gram-Schmidt procedure assigns to each basis e of V an orthonormal basis e' defined recursively as follows. If the first j vectors of e are already orthonormal, the $(j+1)$-th vector e'_{j+1} is defined by $e'_{j+1} := l_{j+1}/\sqrt{(l_{j+1} \mid l_{j+1})}$ where $l_{j+1} := e_{j+1} - \sum_{k=1}^{j} (e_{j+1} \mid e_k) e_k$. In other words, to each vector we subtract the vector sum of its projections onto each previous vector and we normalize the final result.

The Gram-Schmidt procedure gives an upper triangular matrix $G_e^{(\,|\,)}$ such that $eG_e^{(\,|\,)}$ is orthonormal, where $eG_e^{(\,|\,)} = (e_i(G_e^{(\,|\,)})_{i1}, \ldots, e_i(G_e^{(\,|\,)})_{in})$. Now, given two Euclidean metrics η and η' on a vector bundle $E \to B$, we can apply this procedure to their frame bundles as follows. Each η-orthonormal frame $e_x \in E_x$ is a basis of the vector space E_x. Then $e_x G_{e_x}^{\eta'}$ is a η'-orthonormal frame over $x \in B$. Hence we have a well defined map $G_{\eta,\eta'}\colon Fr(E_\eta) \to Fr(E_{\eta'})$, and a brief inspection of the algorithm shows that it is indeed smooth. The following proposition clarifies when this smooth map is a based-O_n-morphism of principal bundles.

Proposition A.3. Given two Euclidean metrics η and η' on a vector bundle $E \to B$, the map $G_{\eta,\eta'}\colon Fr(E_\eta) \to Fr(E_{\eta'})$ is a based-O_n-morphism of principal bundles if and only if η and η' are conformally equivalent.

Proof. The smooth map $G_{\eta,\eta'}$ is a based-O_n-morphism of principal bundles if and only if $G_{\eta,\eta'}(eR) = G_{\eta,\eta'}(e)R$ for each $R \in \mathrm{O}_n$, that is:

$$eRG_{eR}^{\eta'} = eG_e^{\eta'}R \quad \forall e \in Fr(E_\eta)\,,\ \forall R \in \mathrm{O}_n \tag{A.1}$$

(round brackets suppressed because of associativity of the products involved).

Assume $\eta'(v_x, w_x) = \lambda_x \eta(v_x, w_x)$ for some smooth strictly positive λ. By looking at the Gram-Schmidt procedure it is straightforward to see that $G_{e_x}^{\eta'} = \lambda_x^{-1/2}\mathbf{1}_n$ for any $e_x \in Fr(E_\eta)$, which clearly satisfies (A.1).

Now assume that $G_{\eta,\eta'}$ satisfies (A.1) and consider the SO_n matrix

$$R_\theta = \begin{pmatrix} \widetilde{R_\theta} & 0 \\ 0 & \mathbf{1}_{n-2} \end{pmatrix}$$

where

$$\widetilde{R_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We fix an arbitrary $e \in Fr(E_\eta)$ (subscript x understood) and compute the first vector of equation (A.1) for $R = R_\theta$:

$$(G_{eR_\theta})_{11}[\cos\theta\, e_1 + \sin\theta\, e_2] = [(G_e)_{11}\cos\theta + (G_e)_{12}\sin\theta]\, e_1 + (G_e)_{22}\sin\theta\, e_2\,.$$

Since e_1 and e_2 are linearly independent we can equate coefficients of both sides. In particular for e_2 we obtain

$$(G_{eR_\theta})_{11} = (G_e)_{22} \quad \forall\theta \in (0,\pi)$$

hence $(G_{eR_\theta})_{11}$ does not depend on θ. An elementary computation shows that $(G_{eR_\theta})_{11}$ is given by

$$\begin{aligned}(G_{eR_\theta})_{11} &= [\eta'(\cos\theta\, e_1 + \sin\theta\, e_2, \cos\theta\, e_1 + \sin\theta\, e_2)]^{-1/2}\\ &= [\eta'(e_1,e_1)\cos^2\theta + \eta'(e_2,e_2)\sin^2\theta + 2\eta'(e_1,e_2)\sin\theta\cos\theta]^{-1/2}\,.\end{aligned}$$

A series expansion in $\theta \to 0$ gives

$$(G_{eR_\theta})_{11} = \frac{1}{\sqrt{\eta'(e_1,e_1)}}\left[1 - \frac{\eta'(e_1,e_2)}{\eta'(e_1,e_1)}\theta\right] + \mathcal{O}(\theta^2)$$

from which we get

$$\eta'(e_1,e_2) = 0 \quad \text{and} \quad \eta'(e_1,e_1) = \eta'(e_2,e_2)\,.$$

Since $e \in Fr(E_\eta)$ is arbitrary we can apply the above argument to any frame obtained from e by permuting its elements, therefore we get

$$\eta'(e_{\sigma(1)},e_{\sigma(2)}) = 0 \quad \text{and} \quad \eta'(e_{\sigma(1)},e_{\sigma(1)}) = \eta'(e_{\sigma(2)},e_{\sigma(2)}) \quad \forall\sigma \in S_n\,.$$

Since S_n acts 2-transitively on $\{1,2,\ldots,n\}$, we conclude that

$$\eta'(e_i,e_j) = 0 \quad \text{and} \quad \eta'(e_i,e_i) = \eta'(e_j,e_j) \quad \forall i,j \in \{1,2,\ldots,n\},\ i \neq j\,.$$

Finally, $\lambda_x := \eta'(e_{x,i},e_{x,i})/\eta(e_{x,i},e_{x,i})$ is a well-defined[1] function $\lambda\colon M \to \mathbb{R}$ such that $\eta'(v_x,w_x) = \lambda_x\eta(v_x,w_x)$. Being a product of two smooth strictly positive functions, λ itself must be so and the proof is complete. □

[1] λ does not depend on the choice of e: if $\mu_x := \eta'(\hat{e}_{x,i},\hat{e}_{x,i})/\eta(\hat{e}_{x,i},\hat{e}_{x,i})$ for some $\hat{e} \in Fr(E_\eta)$ then $\eta'(v_x,w_x) = \mu_x\eta(v_x,w_x) = \lambda_x\eta(v_x,w_x)$ therefore $\mu = \lambda$.

B Covering groups

Given a connected topological space B, a (connected) covering is a (connected) space Y together with a continuous surjection $p\colon Y \to B$ such that each $x \in B$ has a neighbourhood $U \subset B$ with the property that p is a homeomorphism on each connected component of $p^{-1}(U)$. Two coverings $Y \xrightarrow{p} B$ and $Y' \xrightarrow{p'} B$ are equivalent if there is a homeomorphism $f\colon Y \to Y'$ such that $p = p' \circ f$. For a sufficiently nice space B, its connected coverings are completely determined (up to equivalence) by the set of conjugacy classes of subgroups of $\pi_1(B)$, by the following theorem (see Theorem 79.2 in [43]).

Theorem B.1. *Given two coverings $Y \xrightarrow{p} B$, $Y' \xrightarrow{p'} B$ and points $y \in Y$, $y' \in Y'$ such that $p(y) = p'(y') = b$ where $b \in B$, the coverings are equivalent if and only if the subgroups $p_*(\pi_1(Y, y))$ and $p'_*(\pi_1(Y', y'))$ are conjugate in $\pi_1(B, b)$.*

Given B and a subgroup $H \subset \pi_1(B, b)$, we can construct a connected covering $Y \xrightarrow{p} B$ such that $p_*(\pi_1(Y, y)) = H$ as follows. Consider the set $P(B, b)$ of paths $\alpha\colon [0,1] \to B$ such that $\alpha(0) = b$ and declare two paths $\alpha, \beta \in P(B, b)$ equivalent if $[\alpha * \overline{\beta}] \in H$, where $*$ denotes composition of paths, $\overline{\beta}$ is the path β traced backwards, and square brackets denote the homotopy class. The equivalence class of α is denoted by $\alpha^\sharp$. The covering is then defined as $P(B, b)/\!\sim\, \to B$ with projection p given by $P(B, b)/\!\sim\, \ni \alpha^\sharp \mapsto \alpha(1) \in B$. The topology of $P(B,b)/\sim$ is defined by transferring the topology of B to $P(B,b)/\sim$ through the collection of bijections $p \restriction P(U,\alpha)$ where $P(U,\alpha) = \{(\alpha * \delta)^\sharp \mid \delta\colon [0,1] \to U$ such that $\delta(0) = \alpha(1)\}$ and $U \subset B$ is a path-connected neighbourhood of $\alpha(1)$. The number of sheets of the covering is given by the index of H in $\pi_1(B,b)$. When B is a topological group, this construction yields a topological group by defining the product $\alpha^\sharp\beta^\sharp = (\alpha\beta)^\sharp$, where $\alpha\beta\colon [0,1] \to B$ is defined by $(\alpha\beta)(t) = \alpha(t)\beta(t)$ and the last multiplication is by the group law of B. This definition is well posed since, for any paths $\alpha, \beta, \gamma, \delta$ with $\alpha(1) = \beta(1) = \gamma(0) = \delta(0)$ we have the equality $(\alpha\beta)*(\gamma\delta) = (\alpha*\gamma)(\beta*\delta)$, so if $\alpha^\sharp = \alpha'^\sharp$ we get $[(\alpha\beta)*(\overline{\alpha'\beta})] = [(\alpha*\overline{\alpha'})(\beta*\overline{\beta})] = [\alpha*\overline{\alpha'}] \in H$ which means $(\alpha\beta)^\sharp = (\alpha'\beta)^\sharp$. Since $p(\alpha^\sharp\beta^\sharp) = (\alpha\beta)(1) = \alpha(1)\beta(1) = p(\alpha^\sharp)p(\beta^\sharp)$, p becomes a homomorphism of topological groups and this definition of group structure is the unique one (up to a choice of the identity element in $p^{-1}(1)$) making p a homomorphism of topological groups (e.g. see [43]). When B is a Lie group, we can put a Lie group structure on $P(B,b)/\sim$ by means of the maps $p \restriction P(U,\alpha)$. In this case p becomes a homomorphism of Lie groups.

The case $H = 1$ clearly gives the universal covering group.

We now apply the general theory above to study the 2-sheeted coverings of $B = \mathrm{GL}_n^+$. The fundamental group is given by the following table.

	$n=1$	$n=2$	$n\geq 3$
$\pi_1(\mathrm{GL}_n^+)$	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

By the above discussion, we deduce that the only 2-sheeted covering of GL_1^+ is the trivial (hence non-connected) covering $\widetilde{\mathrm{GL}_1^+} \simeq \mathrm{GL}_1^+ \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\rho_1} \mathrm{GL}_1^+$, with obvious componentwise group structure and projection. Since $\mathbb{Z}$ has only one index 2 subgroup (the even integers), we deduce GL_2^+ has a unique nontrivial 2-sheeted covering $\widetilde{\mathrm{GL}_2^+} \xrightarrow{\rho_2} \mathrm{GL}_2^+$. Finally, for $n \geq 3$ GL_n^+ has a unique nontrivial 2-sheeted covering $\widetilde{\mathrm{GL}_n^+} \xrightarrow{\rho_n} \mathrm{GL}_n^+$ and it is universal. It should be noted that $\widetilde{\mathrm{GL}_n^+}$ ($n \geq 3$) is not a matrix Lie group (e.g. see Lemma 5.23 in [36]).

When we apply the general theory to $B = \mathrm{SO}_n$, we obtain the so called spin groups Spin_n. When we apply it to $B = \mathrm{O}_{p,q}^0 = \mathrm{SO}_{p,q}^0$, where 0 means "connected component of the identity element", we obtain $\mathrm{Spin}_{p,q}^0$ which is the connected component of $\rho^{-1}(\mathrm{SO}_{p,q}) \subset \widetilde{\mathrm{GL}_n^+}$ (for p and q both greater than 2 it is already connected). In particular, $\mathrm{Spin}^0_{1,n-1} \to \mathrm{SO}^0_{1,n-1}$ is isomorphic to a (unique for $n \geq 3$) nontrivial double covering of $\mathrm{SO}^0_{1,n-1}$ (see [51] and [21] vol.1 p.518).

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