Selected applications of functional RG

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Contents

Plan of the work.

In this thesis we will address the study of quantum field theories using the exact renormalization group technique. In particular, we will calculate the flow of a Yukawa system coupled to gravity and that of a higher derivative nonlinear sigma model. The study of the Yukawa system in presence of gravity, as well as the study of any matter theory coupled to gravity, is important for two reason. First, it is interesting to see what gravitational dressing one should expect to the beta functions of any matter theory. Second, it is important to test the possibility that gravity is an asymptotically safe theory [1, 2] against the addition of matter degrees of freedom.

We also calculate the 1-loop flow of a general higher derivative nonlinear sigma model, using exact renormalization group techniques. We think that the nonlinear sigma model is an important arena to test the exact renormalization. The reason is that the nonlinear sigma model shares many of the features of gravity, like perturbative nonrenormalizability, but does not have the additional complication of a local gauge invariance. Furthermore, it is an interesting question whether a nonlinear sigma model admits a ultraviolet limit or it has to be regarded as an effective field theory only.

The plan of the work is as follows. In Chapter 1 we give a very brief introduction to the technique of functional exact renormalization group. In Chapter 2 we introduce the notion of "Asymptotic Safety" [1] and discuss some of the approximation schemes generally involved in calculations. In Chapter 3 we use a simple Yukawa model as a toy model for many of the techniques we will need later. We also discuss the background field method in the context of a theory with local gauge invariance, which will turn out to be useful in Chapter 4. In Chapter 4 we couple the simple Yukawa model with gravity and calculate its renormalization group flow. In Chapter 5 we study numerically the flow calculated in Chapter 4 and point out the possibility that the model admits a nontrivial ultraviolet limit. Chapter 6 is the final chapter and contains the study of the flow of the higher derivative nonlinear sigma model; it is a self contained chapter. In fact, Chapter 5 and 6 contain separate discussions for the results of the Yukawa and sigma model, respectively. We dedicate the appendices to arguments that would have implied very long digressions in the main text.

Chapter 1

Introduction to the functional RG method.

1.1 QFT functionals.

In the following we are going to give a very simple introduction to a quantum field theory, through the method of functional generators. Most of the subsequent concepts we are going to introduce, can of course be refined depending on the subject under study. Our aim is to develop some necessary tool needed for the further developments we want to build. In this introduction we want to develop only the minimal structures needed for the understanding of the next chapters, avoiding most of the complications.

As basic object we take a field ϕ that takes value on a spacetime $\mathcal{M}$. We will assume from now on that the spacetime $\mathcal{M}$ is euclidean and in general riemaniann manifold, so a notion of distance among points is provided. What we have in mind will thus be a euclidean field theory. It is common knowledge that the QFTs that are useful for describing physical world are minkowskian, rather than euclidean. For this reason we will also always assume that the things we are going to compute will admit a translation, or even a direct interpretation, in terms of some associated minkowskian field theory. This is generally done in terms of Wick rotations to imaginary time.

The physical content of a field theory is expressed in terms of correlation functions of the field ϕ. We therefore define the n-point correlation function

$$G^{(n)}_{A_1,\dots,A_n} = \left\langle \phi^{A_1} \dots \phi^{A_n} \right\rangle \tag{1.1}$$

Here the labels A_i of the field ϕ are written in a very condensed form (deWitt condensed notation). They specify the properties of every single copy ϕ^{A_i} of the field inside the n-point function. In particular every label is of the form

$$A_i = (a_i, x_i) \tag{1.2}$$

where x_i are coordinates on $\mathcal{M}$ of the i-th insertion of ϕ. Instead, each index a_i contains information about the geometric nature of the field. Any time an index A will appear twice in a formula, an Einstein summation convention will be assumed (otherwise stated). For example, given two fields ϕ and ψ having the same type of indices, their inner product is

$$\phi^A \psi_A = \int dx \sum_a \phi^a(x)\, \psi_a(x) \tag{1.3}$$

For example the field can belong to a vector space, and in that case a_i will be indices in some vector basis v_{a_i} such that $\phi = \phi^{a_i}(x)\, v_{a_i}$, or it can be a section of a general fiber bundle. However through this work we will also concentrate on cases in which ϕ is a map $\phi : \mathcal{M} \to \mathcal{N}$ with $\mathcal{N}$ a riemaniann manifold, thus, in this case, a_i will be indices in some coordinate basis of it.

We mentioned that the physical content of our theory is encoded in the n-point functions. Some further specification is in order. Here, in fact, we implicitly assume that there is a way to relate the functions $G^{(n)}$ to observables quantities. In general we say that a complete knowledge of the correlation functions imply a complete knowledge of the theory and its physics.

So far for the construction, it is time to build the formalism in such a way that things may be computed. We introduce therefore a measure

$$\mathcal{D}\phi \tag{1.4}$$

on the space of all possible fields and a probability density

$$\mathcal{P}[\phi] \tag{1.5}$$

for ϕ. We will use $\mathcal{D}\phi$ and $\mathcal{P}[\phi]$ to weigh field configurations. In particular we ask for the expectation value of a general field configuration $\mathcal{O}[\phi]$ to be computed as

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi\, \mathcal{P}[\phi]\; \mathcal{O}[\phi] \tag{1.6}$$

Here Z is a normalization factor for our probability, we will come back to it later. As it is well known measure and probability always come together when calculating correlations in probability theory. In fact one could imagine to define a new measure $\tilde{\mathcal{D}}\phi = \mathcal{P}[\phi]\, \mathcal{D}\phi$ with associated probability equal to one, obtaining the same result

$$\langle \mathcal{O}[\phi] \rangle = \int \tilde{\mathcal{D}}\phi\, \mathcal{O}[\phi] \tag{1.7}$$

In particular one may think at $\tilde{\mathcal{D}}\phi$ as the true measure of our field theory. Nonetheless we expect that there is some natural measure on the space $\mathcal{N}$, and therefore on the space of its maps ϕ, dictated by its geometry. That measure, $\mathcal{D}\phi$, will thus be taken as reference and $\mathcal{P}[\phi]$ will contain the deviations of it from $\tilde{\mathcal{D}}\phi$. As we are going to see in a moment some straight physical concept is associated to $\mathcal{P}[\phi]$.

With the aid of $\mathcal{D}\phi$ and $\mathcal{P}[\phi]$ we now give a constructive way to calculate correlations functions. First notice that the correlation (1.1) is obtained as

$$\left\langle \phi^{A_1} \dots \phi^{A_n} \right\rangle = \frac{1}{Z} \int \mathcal{D}\phi\, \mathcal{P}[\phi]\; \phi^{A_1} \dots \phi^{A_n} \tag{1.8}$$

From the point of view of this definition it is clear that the set of field theories is in one to one correspondence with that of true functional measures $\tilde{\mathcal{D}}\phi$. Here we may also give a distinction between classical and quantum field theory. In a classical field theory the configuration of ϕ is assumed to be known once enough boundary conditions are specified and the equations of motion are solved. We call that configuration $\phi^A_{\rm cl}$. In doing so it is obvious that the probability density must be a delta functional

$$\mathcal{P}_{\rm cl}\left[\phi\right] = \delta\left[\phi - \phi_{\rm cl}\right] \tag{1.9}$$

with respect to the measure $\mathcal{D}\phi$. In the general, quantum, case no peculiar configuration is picked up by the measure. In our euclidean formalism the probability is parametrized as

$$\mathcal{P}_{\rm qu}\left[\phi\right] = e^{-S[\phi]} \tag{1.10}$$

Here we introduced the action functional

$$S\left[\phi\right] \tag{1.11}$$

of our theory.

From now on we will address the quantum case, dropping the distinctive label. A systematic way to calculate the correlation functions is obtained introducing a source current J and coupling it to our field ϕ. We start by computing the normalization factor

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi]} \tag{1.12}$$

which is sometimes called partition function of our theory. Then, we modify Z adding a source coupling term to the action

$$Z\left[J\right] = \int \mathcal{D}\phi\, e^{-S[\phi]+J_A\phi^A} = Z\left\langle e^{J_A\phi^A}\right\rangle \tag{1.13}$$

therefore making it a functional of current J. A straightforward Taylor expansion around $J = 0$ gives

$$Z\left[J\right]/Z = 1 + J_{A_1}\left\langle\phi^{A_1}\right\rangle + \frac{1}{2}J_{A_1}J_{A_2}\left\langle\phi^{A_1}\phi^{A_2}\right\rangle + \dots \tag{1.14}$$

It is clear that $Z\left[J\right]$ is a generating functional for the correlations, in fact

$$\left\langle\phi^{A_1}\dots\phi^{A_n}\right\rangle = \frac{1}{Z}\frac{\delta^n}{\delta J_{A_1}\dots\delta J_{A_n}}Z\left[J\right]\bigg|_{J\to 0} \tag{1.15}$$

In the future we will call the source dependent functional $Z\left[J\right]$ partition function and Z will be referred just as a normalization factor. Further we define J-dependent correlation functions simply by avoiding the limit $J \to 0$

$$\left\langle\phi^{A_1}\dots\phi^{A_n}\right\rangle_J = \frac{1}{Z}\frac{\delta^n}{\delta J_{A_1}\dots\delta J_{A_n}}Z\left[J\right] \tag{1.16}$$

As is customary, at this point in the theory of generating functionals, to define another functional

$$W\left[J\right] = \mathrm{Log}\, Z\left[J\right] \tag{1.17}$$

$W[J]$ is often called generator of connected n-point functions. These connected functions are again generated by taking J functional derivatives

$$G^{(n)}_{\text{conn.},A_1,\ldots,A_n} = \frac{\delta^n}{\delta J_{A_1}\ldots\delta J_{A_n}} W[J]\bigg|_{J=0} \tag{1.18}$$

We did not build any diagrammatic expansion so the reason of these names is not evident. Before going on further it may be the case to give an example for the sake of interpretation.

First we specify the form of a quite general action as

$$S[\phi] = \frac{1}{2}\Delta_{AB}\phi^A\phi^B + V[\phi] \tag{1.19}$$

Here one could think about these terms as a kinetic one plus an interaction. In particular we may also call

$$S_0[\phi] = \frac{1}{2}\Delta_{AB}\phi^A\phi^B \tag{1.20}$$

the "free action". To fix ideas, imagine we were working on flat spacetime and ϕ would belong to linear space. The kernel Δ represent the propagation of the states of our field. If Δ is further specified to be a local differential operator, say for example $-\partial_\mu\partial_\mu$ or the Dirac operator $\gamma^\mu\partial_\mu$, it is well known that the theory contains only free particle states. A general theory with action $S_0[\phi]$ is easily solved if Δ is invertible. To this end just calculate the functional

$$Z_0[J] = \int_\phi \mathcal{D}\phi\, e^{-S_0[\phi]+J_A\phi^A} = \mathcal{C}\, e^{\frac{1}{2}J_AJ_B\left(\Delta^{-1}\right)^{AB}} \tag{1.21}$$

$\mathcal{C}$ is a normalization factor that we can neglect because J independent. Sometimes, this is taken as formal definition of the gaussian integration over the space of fields.

Using $Z_0[J]$ it is easy to see that any even correlation function is a complicated sum of products of Δ^{-1}. One is tempted to diagrammatically interpret each term in this expansion. To do that one has to draw a point for each label A_i of the field. Having done this, it is sufficient to join a couple of points (A,B) with a segment, any time $\left(\Delta^{-1}\right)^{AB}$ appears. These n-point correlation functions are said to be disconnected, when n is larger than two, because the segments do not join at any point, so they are all separated. Actually this is rather trivial having introduced no interactions. The only connected function being

$$\left\langle\phi^A\phi^B\right\rangle_{0,J=0} = \left(\Delta^{-1}\right)^{AB} \tag{1.22}$$

that is precisely the building block for all the others correlators. If one looks at the associated functional

$$W_0[J] = \operatorname{Log} Z_0[J] = \frac{1}{2}J_AJ_B\left(\Delta^{-1}\right)^{AB} \tag{1.23}$$

easily realizes that it is the generator of the (only) connected function of the system.

Let us go beyond the very simple free action, considering also the interaction term $V[\phi]$. A model like this is not generally explicitly solvable, but still we can manipulate it a little bit. Notice that

$$Z[J] = e^{-V\left[\frac{\delta}{\delta J}\right]} Z_0[J] \tag{1.24}$$

If the prefactor e^{-V} is expanded, any term is going to look like a combination of correlations functions of the free theory. The only modifications being vertices, whose number of external lines are determined by the characteristics of $V[\phi]$. This procedure sets the ground of what is called perturbative expansion. It is possible to show that the n-point correlation functions generated by $W[J]$ are those of $Z[J]$ provided one removes all the diagrams that are disconnected. This finally explains the reason of the name generator of connected n-point functions.

Before concluding this section we need to introduce one final, and perhaps more important, functional. We start considering the J-dependent 1-point function of any theory

$$\bar{\phi}^A = \left\langle \phi^A \right\rangle_J \tag{1.25}$$

that defines the field $\bar{\phi}^A$. We state that it is, under some still unspecified meaning, a quantum "cousin" of ϕ^A. Obviously $\bar{\phi}^A$ is a functional of the sources J_A. We define the Legendre transform

$$\Gamma\left[\bar{\phi}\right] = \sup_J \bar{\phi}^A J_A - W[J] \tag{1.26}$$

While taking the extremum in the ensamble of source configurations we are lead to the relation

$$J_A = J_A\left[\bar{\phi}\right] \tag{1.27}$$

so that $\bar{\phi}$ is the field configuration that realizes the maximum. The transform $\Gamma\left[\bar{\phi}\right]$ is called effective action (EA). It is also a generating functional and its correlators are defined as

$$\Gamma^{(n)}_{A_1,\dots,A_n} = \frac{\delta^n}{\delta\bar{\phi}^{A_1}\dots\delta\bar{\phi}^{A_n}}\Gamma\left[\bar{\phi}\right] \tag{1.28}$$

By a systematic construction one may see that the diagrammatic of $W[J]$ and $\Gamma\left[\bar{\phi}\right]$ are related, in particular the expansion of $\Gamma\left[\bar{\phi}\right]$ contains the proper vertices of the quantum theory. Essentially a proper vertex is a connected function, that is one of those generated by $W[J]$, once external lines are removed. These are also usually called "one particle irreducible functions" (1PI). Beyond the diagrammatic interpretation, one may think about the effective action as that (classical) action capable to reproduce the (quantum) correlations.

The computation of the effective action is often a really complicated task, especially if one does not have theories with many symmetries and spacetime is not two dimensional, or both. The purpose of the next sections is to highlight the technical problems shadowed in the discussion up to now. We will also further extend our definitions, giving up however to some generality as a price of clarity.

1.2 Wilson approach.

One lesson of the technique of perturbative renormalization is that the quantum nature of a theory manifests itself through scale dependent couplings. Somehow this is a derived feature of a QFT, coming generally to the need of renormalizing it. Our aim now is to address the issue of scale dependence from the very beginning, introducing scale dependent functionals in such a way this characteristic is built-in in the formalism. This is rather different from perturbation theory, where the scale emerges in the development.

We start our discussion of scale dependent functionals by introducing them in rather historical order. It will be clear step by step that any innovation will provide some new tool or structure. Each of these will prove useful for the interpretation of scale dependent physics. Some of these will also be striking from the point of view of calculations. We are going to try to underline for any improvement what we are getting and from what point of view.

A systematic attempt to introduce scale dependence in the functionals we constructed before is due to Wilson [3]. For simplicity we begin by considering spacetime to be a flat d-dimensional euclidean manifold $\mathbb{R}^d$. We shall also take the field ϕ to be a scalar. We have now a natural basis, the momentum basis, to expand the field

$$\phi\left(x\right)=\int_q \phi_q e^{-iqx}. \tag{1.29}$$

We have therefore also a natural functional measure for our field integration that is defined by

$$\mathcal{D}\phi=\prod_{q\in\mathbb{R}^d} d\phi_q \tag{1.30}$$

Following the procedure of the former sections the partition function would be

$$Z=\int\prod_{q\in\mathbb{R}^d} d\phi_q e^{-S[\phi]} \tag{1.31}$$

This is of course ill defined because in explicit calculations it is generally divergent, due to the infinitely many modes the field can have.

These divergences come from the unbounded integrations in momentum space. We therefore introduce a cutoff Λ and a certain action $S_\Lambda\left[\phi\right]$ such that

$$Z=\int\prod_{|q|\leq\Lambda} d\phi_q e^{-S_\Lambda[\phi]} \tag{1.32}$$

The introduction of a cutoff is a regularization for the divergent correlations, which we need in order to work with finite expressions. What we did was, at the same time, to modify the action and the measure. We further asked that both modifications together match and reproduce the same partition function. From the point of view of the interpretation it makes sense to think at $S_\Lambda\left[\phi\right]$ as a certain UV action that contains the information of our theory if large energies are addressed. Large energies means small scales by the dual relation. If Λ is a very big scale beyond which we do not know the behavior of the theory, it makes sense to associate it to the UV action of a theory with a hard cutoff.

In general we could pick any scale k and perform the same trick to write Z. and this is precisely what we are going to do now. We define a new action $S_k[\phi]$ as that particular action for which

$$Z = \int \prod_{|q| \leq k \leq \Lambda} d\phi_q e^{-S_k[\phi]} \tag{1.33}$$

One notices at this point that we are introducing a semigroup structure and that it is possible to relate any $S_k[\phi]$ to $S_\Lambda[\phi]$, in fact

$$Z = \int \prod_{|q| \leq k} \prod_{k \leq |q| \leq \Lambda} d\phi_q e^{-S_\Lambda[\phi]} \tag{1.34}$$

determines the relation

$$e^{-S_k} = \int \prod_{k \leq |q| \leq \Lambda} d\phi_q e^{-S_\Lambda[\phi]} \tag{1.35}$$

We thus arrived at a complicated integral equation. We interpret $S_\Lambda[\phi]$ as the UV action of a theory possessing a hard cutoff and $S_k[\phi]$ as the result that comes integrating all the modes $k \leq |q| \leq \Lambda$ towards IR. The complete integration of allowed modes will therefore move us in the direction of a theory in which all scales contributed to our effective understanding of the theory. We call $S_k[\phi]$ Wilson effective actions.

We have constructed a one parameter family of actions labelled by a scale k. These actions are supposed to contain a good description of the physics at the associated scale. A good reason for believe it is that, by definition, only the modes $|q| \simeq k$ are active at the given scale and therefore we are, under some approximation, describing their physics. To understand this more precisely it is the case to refine the technique using a blocking procedure. As a byproduct, it will prove useful to have some insight on the scheme dependence of Wilson technique. Finally it will provide us an actual way to calculate the Wilson effective action, something that is still quite unclear from the formal manipulations we made in this section. We will implement it in the next section.

1.3 Wilson approach through blocking.

We call a "blocked" field the result of averaging the field using a smearing function $\rho_k(x)$ [4]. The blocked version of our scalar field $\phi(x)$ is defined to be

$$\phi_k(x) = \int_y \rho_k(x-y)\,\phi(y) \tag{1.36}$$

The integration is extended over the whole spacetime, but we want the smearing function to provide an averaging of our field over a region of typical size k^{-d} where d is our spacetime dimension. It is clear that $\phi_k(x)$ is rather insensitive to effects involving wave modes of energies greater than k, although it depends on our coarse-graining scheme through the function $\rho_k(x)$. Therefore if we manage to build a theory for $\phi_k(x)$ from the coarse-graining of the theory for $\phi(x)$, we also manage to construct an effective theory that naturally describes effects of energies of order k or less. One can say that we are going to build a theory for slow-modes. We proceed constructing the coarse-grained functional

$$e^{-S_k[\Phi]} = \int \mathcal{D}\phi\delta\left[\Phi - \phi_k\right] e^{-S[\phi]} \tag{1.37}$$

The coarse-grained action $S_k\left[\Phi\right]$ precisely represents what we were looking for, as we shall see in a moment. We denoted Φ the argument of the coarse-grained functional and its phase space is spanned by all the averaged fields ϕ_k [1] .

Before going further we mention again that, as always happens in coarse-graining procedures, there is a hidden scheme dependence in the method. In particular, we did not specify any particular form for the smearing function $\rho_k\left(x\right)$. This is precisely what we were stressing at the end of the previous section. A very natural choice for the smearing is the sharp cutoff form in which

$$\rho_k\left(x\right) = \int_{q\leq k} e^{iqx} = \int_q \theta\left(k-q\right) e^{iqx} \tag{1.38}$$

This kind of smearing provides a clear cut between slow

$$\Phi\left(x\right) = \phi_<\left(x\right) = \int_y \rho_k\left(x-y\right)\phi\left(y\right) \tag{1.39}$$

and fast

$$\phi_>\left(x\right) = \phi\left(x\right) - \phi_<\left(x\right) \tag{1.40}$$

modes. Also, having specified the form of the smearing we can now try to evaluate $S_k\left[\Phi\right]$, in fact

$$e^{-S_k[\Phi]} \quad = \quad \int \mathcal{D}\phi_<\mathcal{D}\phi_>\delta\left[\Phi - \phi_<\right] e^{-S[\phi_< + \phi_>]} \tag{1.41}$$

It is sufficient to expand quadratically the action $S\left[\phi\right]$ in $\phi_>$ and perform the gaussian integration to have an approximate result

$$S_k\left[\Phi\right] = S\left[\Phi\right] + \frac{1}{2}\mathrm{Tr}_{k\leq q}\, S^{(2)}\left[\phi\right]\Big|_{\phi=\Phi} + \dots \tag{1.42}$$

where $S^{(2)}\left[\phi\right]$ is the second functional derivative of $S\left[\phi\right]$.

There are some issues at this point. As it is indicated the trace, that in this case in essentially Fourier modes integration, has a lower bound k. This essentially means that we are integrating the fast modes down to the scale we are interested in. Unfortunately there is no upper bound and, as always happens in QFT these unbounded integrals tend to be divergent. In a real computation either we should have a lattice, that is a natural UV cutoff Λ proportional to the inverse of lattice size $\Lambda \sim 1/a$, or we should cut-off the theory in some other way. In particular we may as well introduce a cutoff Λ and define

$$S_{k,\Lambda}\left[\Phi\right] = S\left[\Phi\right] + \frac{1}{2}\mathrm{Tr}_{k\leq q\leq\Lambda}\, S^{(2)}\left[\phi\right]\Big|_{\phi=\Phi} + \dots \tag{1.43}$$

[1] In general, the phase space of the original variable ϕ and that of Φ differ, however for theories in continuum they coincide.

This procedure is rather standard and allows to reabsorb the divergences through redefinitions of the couplings. If this is possible one can safely take the limit $\Lambda \to \infty$ and renormalize the theory. There is, however, a way out that avoids the UV regularization procedure. It is possible, in fact, to derive an evolution equation for $S_k[\Phi]$. It is sufficient to perform an infinitesimal step in the integration, to get a result of the form

$$k\frac{\partial}{\partial k}S_k[\Phi] = \frac{1}{2}\mathrm{Tr}\,\mathrm{Log}\left.\frac{\delta^2 S_k[\phi]}{\delta\phi\delta\phi}\right|_{\Phi} \tag{1.44}$$

This is called Wegner-Houghton equation[2]. Once the equation is derived it is necessary to specify $S_\Lambda[\phi] = S[\phi]$ as initial condition of the flow and the integration towards lower values of k will automatically give regular results for $S_k[\phi]$. However the problem of taking the UV limit $\Lambda \to \infty$ is not really solved, but rather hidden in the choice of some initial condition of the flow. We will extensively address this problem in the next chapter.

1.4 Functional renormalization.

In the previous section we just uncovered a small piece of what generally goes under the name of functional renormalization. We start by further modifying the partition function associated to a theory by adding an infrared cutoff that depends on some cutoff scale k. In this context k is generally called sliding cutoff scale. We will also try to keep the discussion as general as possible, so we restore the abstract index notation of the field ϕ^A. We define

$$Z_k[J] = \int \mathcal{D}\phi\, e^{-S[\phi]+J_A\phi^A-\Delta S_k[\phi]} \tag{1.45}$$

where we introduced by hand a new term, called infrared cutoff term [5]. The new term is required to satisfy

$$\Delta S_{k=0}[\phi] = 0 \tag{1.46}$$

in such a way that $Z_{k=0}[J] = Z[J]$. This means that, taking the limit $k \to 0$, we get back the partition function we defined previously in (1.13) [5].

We also want $\Delta S_k[\phi]$ to implement a coarse-graining similar to that produced by the function ρ_k of the previous section. To this end we first ask it to be at most quadratic in the field ϕ^A:

$$\Delta S_k[\phi] = \frac{1}{2}\phi^A R_{k,AB}\phi^B \tag{1.47}$$

(we recall that repeated indices imply summation). Now it is clear that the cutoff affects directly the propagation of ϕ^A modes. The coarse-graining of these may be achieved as follows. We start by looking at the quadratic kernel of the action and define

$$\Delta_{AB} = \left. S^{(2)}_{AB}\right| \tag{1.48}$$

[2]Note that, differently from the other exact functional equations we are going to derive in the following, the Wegner-Houghton equation that we present here still needs a subtraction point to make sense.

where the vertical bar means that it is evaluated at some particular reference field configuration. As an example, if we are doing perturbative computations for a field that belongs to a vector space, a good choice could be $\phi^A = 0$. In general Δ_{AB} is some reference operator and with respect to it we are coarse-graining. It is not really necessary that it is the second order expansion our action $S[\phi]$, what is fundamental instead is that it will provide us a distinction of rapid and slow modes as we will briefly outline. We will use in the following a spectral representation that has been introduced in [6].

We first assume the existence of some metric $g_{AB}\delta\phi^A\delta\phi^B$ in field configuration space so to obtain an endomorphism

$$\Delta^A{}_B = g^{AC}\Delta_{CB} \tag{1.49}$$

Then there exists a basis ψ_i^A of functions that diagonalize $\Delta^A{}_B$ with eigenvalues λ_i

$$\Delta^A{}_B\psi_i^B = \lambda_i\psi^A \tag{1.50}$$

Modes ψ_i^A are now referred to "rapid" if $\lambda_i \gtrsim k$ and "slow" if $\lambda_i \lesssim k$. We ask the kernel $R_k{}^A{}_B$ of $\Delta S_k[\phi]$ to be a function of $\Delta^A{}_B$. The quadratic kernel of the theory together with the cutoff term is

$$\Delta^A{}_B + R_k{}^A{}_B[\Delta] \tag{1.51}$$

and we require it to kill mainly the propagation of the slow modes ψ_i^A, so those such that $\lambda_i \leq k^2$. This is seen easily by moving to the eigenstates basis where

$$\Delta^A{}_B + R_k{}^A{}_B[\Delta] \to \lambda_i + R_k[\lambda_i] \tag{1.52}$$

We want $R_k[\lambda]$ to satisfy

- $R_{k=0}[\lambda] = 0$ that ensures the limit $Z_{k=0}[J] = Z[J]$.
- $R_{k\to\infty}[\lambda] = \infty$ at fixed λ. This ensures that in the converse limit no modes are propagating.
- $R_k[\lambda] \simeq 0$ for $\lambda \geq k$, so the rapid modes are unaffected by the coarse-graining. Conversely slow modes will tend to have a mass that forces their decoupling from spectrum.

We will later introduce some precise choice for the shape of $R_k[\lambda]$ function. In general there is some freedom in its choice, that reflects in some scheme dependence of our averaging technique. This is exactly analog to the Wilson case, where the blocking smearing function ρ_k was undetermined in (1.36). There, we were able to point out a natural form for ρ_k and some similar consideration may be done also for $R_k[\lambda]$.

We can use $Z_k[J]$ in exactly the same way we used $Z[J]$ previously, although it has a further k dependence. We first define a modified generator of connected green functions

$$W_k[J] = \mathrm{Log}\, Z_k[J] \tag{1.53}$$

to be compared with (1.17). Both $Z_k[J]$ and $W_k[J]$ generate n-points correlations, which will differ from those generated by $Z[J]$ and $W[J]$ only for a further k dependence. In particular we have the single point correlation (at non-zero source)

$$\bar{\phi}^A = \langle\phi\rangle_{k,J} = \frac{\delta W_k[J]}{\delta J_A} \tag{1.54}$$

For the "average field" $\bar{\phi}$ we used the same notation that was introduced when there was no k dependence (1.25). It is interesting to note that this the relation is generally k dependent. This means that when later k derivatives will be performed, we will have to specify the behavior of J and $\bar{\phi}$ under it. In particular, we cannot have them simultaneously constant under k, but we can choose that either the source of the average field are.

Like in (1.26) we can use $\bar{\phi}$ to define a Laplace transform

$$\hat{\Gamma}_k[\bar{\phi}] = \bar{\phi}^A J_A[\bar{\phi}] - W_k[J_A[\bar{\phi}]] \tag{1.55}$$

where again J has to be inverted as a function of $\bar{\phi}$. We slightly changed notation to $\hat{\Gamma}_k$ because we want to reserve Γ_k for what we call the "effective average action"

$$\Gamma_k[\bar{\phi}] = \hat{\Gamma}_k[\bar{\phi}] - \Delta S_k[\bar{\phi}] = \bar{\phi}^A J_A[\bar{\phi}] - W_k[J_A[\bar{\phi}]] - \Delta S_k[\bar{\phi}] \tag{1.56}$$

In particular this last functional will be our main object of study. Notice that both (1.55) and (1.56) tend to (1.26) in the limit $k \to 0$ [5].

1.5 Functional equations.

The functionals we introduced in the previous section posses some interesting property, namely, their behavior with respect to the sliding scale k is governed by functional equations we are going to derive now [5, 7, 8]. Our first step is to calculate the k-derivative of $Z_k[J] = \mathrm{Exp}(W_k[J])$ (see (1.53)) at fixed source J

$$\begin{aligned}
\partial_k e^{W_k[J]} &= -\frac{1}{2}\int \mathcal{D}\phi\, \phi^A \partial_k R_{k,AB}\phi^B e^{-S[\phi]+J_A\phi^A-\Delta S_k[\phi]} \\
&= -\frac{1}{2}\frac{\delta}{\delta J_A}\partial_k R_{k,AB}\frac{\delta}{\delta J_B}e^{W_k[J]}
\end{aligned} \tag{1.57}$$

We remind that repeated indices are summed. It is important to notice that J-derivatives are acting only on the far right of the second line because k-derivative is taken at fixed J. Easily, one sees that this equation is a functional equation for $Z_k[J]$. After some easy manipulation we arrive at Polchinski equation [9]

$$\partial_k W_k[J] = -\frac{1}{2}\partial_k R_{k,AB}\left(\frac{\delta^2 W_k[J]}{\delta J_A \delta J_B} + \frac{\delta W_k[J]}{\delta J_A}\frac{\delta W_k[J]}{\delta J_B}\right) \tag{1.58}$$

that is a functional differential equation for the evolution of $W_k[J]$. At this point it is worth to notice that we developed in a rather different way than Polchinski originally did. There is an obvious notation difference that amounts at replacing $W_k[J] \to -W_k[J]$. Also the original Polchinski equation is a functional of the field, rather than the current. Therefore it is more precise to say that our equation is in form analog to Polchinski's one. Apart from this, our derivation is useful to obtain the evolution of $\Gamma_k[\bar{\phi}]$ as we shall see below.

The first step is to show some formal manipulation involving the two functionals that are related by a Legendre transform. As the field $\bar{\phi}^A$ is defined by

$$\bar{\phi}^A = \frac{\delta W_k[J]}{\delta J_A} \tag{1.59}$$

it is easy to show that the current J_A comes from the $\bar{\phi}^A$ derivative of $\hat{\Gamma}_k\left[\bar{\phi}\right]$ of (1.55). In fact we can use its definition (1.55) as follows

$$\begin{aligned}
\frac{\delta\hat{\Gamma}_k\left[\bar{\phi}\right]}{\delta\bar{\phi}^A} &= J_A + \frac{\delta J_B}{\delta\bar{\phi}^A}\bar{\phi}^B - \frac{\delta W_k[J]}{\delta\bar{\phi}^A} \\
&= J_A + \frac{\delta J_B}{\delta\bar{\phi}^A}\bar{\phi}^B - \frac{\delta W_k[J]}{\delta J_B}\frac{\delta J_B}{\delta\bar{\phi}^A} \\
&= J_A
\end{aligned} \tag{1.60}$$

For the average effective action (1.56) we have therefore

$$\frac{\delta\hat{\Gamma}_k\left[\bar{\phi}\right]}{\delta\bar{\phi}^A} = J_A = \frac{\delta\Gamma_k\left[\bar{\phi}\right]}{\delta\bar{\phi}^A} + R_{k,AB}\bar{\phi}^B \tag{1.61}$$

We now also need to distinguish if the derivatives with respect to k are taken at constant J or $\bar{\phi}$. We therefore temporarily use $\bar{\partial}_k$ for the k derivative at constant $\bar{\phi}$. We can relate these derivatives by [8]

$$\partial_k = \bar{\partial}_k + \partial_k\bar{\phi}^A\frac{\delta}{\delta\bar{\phi}^A} \tag{1.62}$$

First it is useful to notice that we can reshuffle $\Gamma_k\left[\bar{\phi}\right]$ definition as

$$\Gamma_k\left[\bar{\phi}\right] + W_k[J] = J_A\bar{\phi}^A - \Delta S_k\left[\bar{\phi}\right] \tag{1.63}$$

and act with ∂_k on both sides. The left hand side gives

$$\begin{aligned}
\partial_k\left(\Gamma_k\left[\bar{\phi}\right] + W_k[J]\right) &= \bar{\partial}_k\Gamma_k\left[\bar{\phi}\right] + \partial_k\bar{\phi}^A J_A - \partial_k\bar{\phi}^A R_{k,AB}\bar{\phi}^B \\
&\quad -\frac{1}{2}\partial_k R_{k,AB}\frac{\delta^2 W_k[J]}{\delta J_A\delta J_B} - \frac{1}{2}\partial_k R_{k,AB}\bar{\phi}^A\bar{\phi}^B
\end{aligned} \tag{1.64}$$

while the right hand side gives

$$\begin{aligned}
\partial_k\left(J_A\bar{\phi}^A - \Delta S_k\left[\bar{\phi}\right]\right) &= \partial_k\bar{\phi}^A J_A - \partial_k\bar{\phi}^A R_{k,AB}\bar{\phi}^B \\
&\quad -\frac{1}{2}\partial_k R_{k,AB}\bar{\phi}^A\bar{\phi}^B
\end{aligned} \tag{1.65}$$

Equating (1.64) and (1.65) gives a result that is very close to what we were looking for, because it is an equation for the average effective action

$$\bar{\partial}_k\Gamma_k\left[\bar{\phi}\right] = \frac{1}{2}\partial_k R_{k,AB}\frac{\delta^2 W_k[J]}{\delta J_A\delta J_B} \tag{1.66}$$

but some formal manipulation is still needed because we want it to be expressed in terms of $\Gamma_k\left[\bar{\phi}\right]$ only.

What we need is a simple property that combines the matrices of second derivatives of two functionals that are related by a Legendre transformation. It is sufficient to calculate the second derivative of $\hat{\Gamma}_k\left[\bar{\phi}\right]$ and after some algebraic manipulation one gets, in matrix notation

$$\hat{\Gamma}_k^{(2)}\left[\bar{\phi}\right]^{-1} = \left[\frac{\delta^2\hat{\Gamma}_k\left[\bar{\phi}\right]}{\delta\bar{\phi}\delta\bar{\phi}}\right]^{-1} = \frac{\delta W_k\left[J\right]}{\delta J\delta J} = W_k^{(2)}\left[J\right] \tag{1.67}$$

Inserting this relation in the flow for $\Gamma_k\left[\bar{\phi}\right]$ (1.66) together with the definition

$$G_k\left[\bar{\phi}\right] = \hat{\Gamma}_k^{(2)}\left[\bar{\phi}\right]^{-1} = \left(\Gamma_k^{(2)}\left[\bar{\phi}\right] + R_k\right)^{-1} \tag{1.68}$$

we obtain the exact renormalization group equation (ERGE) [5]

$$\partial_k\Gamma_k\left[\bar{\phi}\right] = \frac{1}{2}\mathrm{Tr}\, G_k\left[\bar{\phi}\right]\partial_k R_k \tag{1.69}$$

Here we dropped the bar notation in ∂_k of (1.62) because from now on derivatives will be performed at fixed $\bar{\phi}$. The trace is extended to every index, so it includes integrations over continuous indices. For future reasons we define the scale parameter $t = \mathrm{Log}\, k/k_0$ that uses an arbitrary reference scale k_0. We also indicate t derivatives by dots, the ERGE is therefore written

$$\dot{\Gamma}_k\left[\bar{\phi}\right] = \frac{1}{2}\mathrm{Tr}\, G_k\left[\bar{\phi}\right]\dot{R}_k \tag{1.70}$$

It is obvious from the construction (1.68) that $G_k\left[\bar{\phi}\right]$ has a role of modified propagator in which slow modes are suppressed. As we shall see later one can interpret the ERGE as a 1-loop equation where the modified propagator performs a loop with a single insertion of the derivative of the cutoff term. Indeed, this "1-loop like" structure is very useful from the computational point of view and makes many calculations accessible. This structure is also best visualized in terms of diagrams, as we can see in Fig. 1.1. However this equation is not approximate, as we always have to remember, so the results are exact. As a final remark we have to say that the ERGE is a functional differential equation, therefore contains an enormous amount of information. This can be seen by the fact that it leads to an infinite tower of functional equations for the correlations

$$\Gamma_k^{(n)}\left[\bar{\phi}\right] \tag{1.71}$$

of $\Gamma_k\left[\bar{\phi}\right]$. To show this it is sufficient to take any number of derivatives of $\dot{\Gamma}_k\left[\bar{\phi}\right]$ using the fact that

$$\frac{\delta}{\delta\bar{\phi}^A}G_{k,BC}\left[\bar{\phi}\right] = -G_{k,BD}\left[\bar{\phi}\right]\Gamma^{(3)}_{k,ADE}\left[\bar{\phi}\right]G_{k,EC}\left[\bar{\phi}\right] \tag{1.72}$$

A diagrammatic representation of this identity is given in Fig. 1.2. It is easy to see that the flow of the n-point correlation $\dot{\Gamma}_k^{(n)}\left[\bar{\phi}\right]$ depends at most on $\Gamma_k^{(n+2)}\left[\bar{\phi}\right]$

$$\dot{\Gamma}_k^{(n)}\left[\bar{\phi}\right] = \mathcal{F}_n\left[\Gamma_k^{(n+2)},\ldots,\Gamma_k^{(2)};\bar{\phi}\right] \tag{1.73}$$

for some functional $\mathcal{F}_n$. We come back later to this point. We will also drop the bar notation in $\bar{\phi}$ from now on, because no more reference will be made to the other functionals.

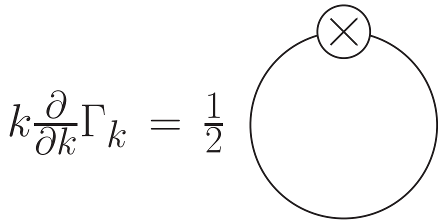

Figure 1.1: Diagrammatic representation of $\frac{1}{2}\mathrm{Tr}G^{(2)}\dot{R}_k$. The straight line represents the modified propagator $G^{(2)}$, while the crossed vertex is the cutoff insertion $\dot{R}_k$ in the trace. The line is closed because all the degrees of freedom are traced.

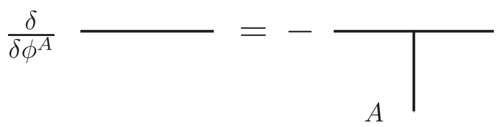

Figure 1.2: Diagrammatic representation of formula (1.72), $\frac{\delta}{\delta\phi^A}G_k = -G_k\Gamma^{(3)}_{k,A}G_k$.

1.6 Alternate form of the ERGE.

We want now to briefly give another derivation of exact renormalization group equation. In particular we will obtain the flow of the average effective action, without using that of the functional $W_k[J]$ and implicitly that of $Z_k[J]$. We first recall definition (1.53)

$$e^{W_k[J]} = \int \mathcal{D}\phi\, e^{-S[\phi]-\Delta S_k[\phi]+J\cdot\phi} \tag{1.74}$$

where we omitted some index for brevity. It is easy to substitute the definition of the functional $\hat{\Gamma}_k\left[\bar{\phi}\right]$ (1.55), being the transform of $W_k[J]$. In order to get rid of the current, we remember that $J = \delta\hat{\Gamma}/\delta\bar{\phi}$ is a general property of the transform. We obtain

$$e^{-\hat{\Gamma}_k[\bar{\phi}]} = \int \mathcal{D}\phi\, e^{-S[\phi]-\Delta S_k[\phi]+\frac{\delta\hat{\Gamma}_k}{\delta\bar{\phi}}\left(\phi-\bar{\phi}\right)} \tag{1.75}$$

In the last step we moved a current-field term on the right hand side. This is an integro-differential equation for $\hat{\Gamma}_k\left[\bar{\phi}\right]$.

We now want to write it completely in terms of $\Gamma_k\left[\bar{\phi}\right] = \hat{\Gamma}_k\left[\bar{\phi}\right] - \Delta S_k\left[\bar{\phi}\right]$ of (1.56). Before performing the substitution it is crucial to remember that the infrared cutoff term (1.47) is quadratic in the fields. We first introduce some notation defining the fluctuation field $\chi = \phi - \bar{\phi}$. In terms of this one easily sees that the difference of the cutoff evaluated in ϕ and $\bar{\phi}$ has finite terms

$$\Delta S_k[\phi] - \Delta S_k\left[\bar{\phi}\right] = \left.\frac{\delta \Delta S_k}{\delta\phi}\right|_{\bar{\phi}} \chi + \Delta S_k[\chi] \tag{1.76}$$

We can now substitute the definition of the average action (1.56) and manipulate it so that only its exponential remains on the left. We also use the invariance of the integration measure under translation to integrate over χ. We get

$$e^{-\Gamma_k[\bar{\phi}]} = \int \mathcal{D}\chi \, e^{-S[\bar{\phi}+\chi]-\Delta S_k[\bar{\phi}+\chi]+\frac{\delta \Gamma_k}{\delta \phi}\chi+\frac{\delta \Delta S_k}{\delta \phi}\big|_{\bar{\phi}}\chi} \tag{1.77}$$

which we can further simplify using the properties of the quadratic cutoff we outlined and obtain

$$e^{-\Gamma_k[\bar{\phi}]} = \int \mathcal{D}\chi \, e^{-S[\bar{\phi}+\chi]-\Delta S_k[\chi]+\frac{\delta \Gamma_k}{\delta \phi}\chi} \tag{1.78}$$

It is important to remember that the expectation value of a single χ field is always zero by construction

$$\langle \chi \rangle = \langle \phi - \bar{\phi} \rangle = 0 \tag{1.79}$$

so equations (1.78) and (1.79) are actually a coupled system. Again we have an integal equation for the effective average action.

In order to recast this integro-differential equation in a differential form, it is sufficient to perform a $k\partial/\partial k$ derivative on both sides. A potentially dangerous term is that coming from the derivative of $\Gamma_k\left[\bar{\phi}\right]$ appearing on the right. However, as we expect, it must be zero. Indeed it disappears after the χ integration is performed, because it is proportional to $\langle \chi \rangle = 0$ In the process of obtaining the flow, it is also necessary to recall how correlators are written in terms of the functionals and translate everything using the definition of the average action. The final result is again the exact renormalization group equation (1.69), (1.70).

1.7 1-loop approximation and ERGE.

In this section we are going to review an established approximation scheme of ERGE that goes under the name of 1-loop approximation. Loop counting is generally performed by powers of $\hbar$, so in this section we are going to restore it although it was originally omitted and set to one. Actually, we are not going to start from ERGE, but rather from the full effective action that $\Gamma_k[\phi]$ captures only in the limit $k \to 0$ by definition.

It is well known in QFT that the saddle point approximation of the path integral leads to an approximate form of the effective action

$$\Gamma^{1,L}[\phi] = S[\phi] + \frac{\hbar}{2}\mathrm{Tr}\,\mathrm{Log}\, S^{(2)} \tag{1.80}$$

while in general

$$\Gamma[\phi] = \Gamma^{1,L}[\phi] + \mathcal{O}\left(\hbar^2\right) \tag{1.81}$$

If we evaluate in the same way the action Γ_k defined in (1.56) we obtain

$$\Gamma_k^{1,L}[\phi] = S[\phi] + \frac{\hbar}{2}\mathrm{Tr}\,\mathrm{Log}\left(S^{(2)}[\phi] + R_k\right) \tag{1.82}$$

Notice that the term ΔS_k in the bare action cancels against that of the definition (1.56).

It obeys

$$\dot{\Gamma}_k^{1,L}[\phi] = \frac{\hbar}{2}\mathrm{Tr}\left(S^{(2)}[\phi] + R_k\right)^{-1}\dot{R}_k \tag{1.83}$$

that is, an approximated ERGE (1.69) in which $\Gamma_k^{(2)}[\phi]$ is approximated by $S^{(2)}[\phi]$ (and, of course $\hbar$ is restored).

We may also derive it entirely in the formalism of exact renormalization group. First, we expand $\Gamma_k[\phi]$ in powers of $\hbar$

$$\Gamma_k[\phi] = \sum_{n\geq 0}\hbar^n\Gamma_k^{n,L} \tag{1.84}$$

and n apex counts the number of loops through the powers of $\hbar$. If we insert this expansion inside the ERGE (1.69) we get that the order zero in $\hbar$ imply

$$\Gamma_k^{0,L} = S[\phi] \tag{1.85}$$

because it does not flow as expected and is given by the initial conditions only. The other terms in the expansion, instead, flow according to

$$\hbar\dot{\Gamma}_k^{1,L} + \hbar^2\dot{\Gamma}_k^{2,L} + \dots \quad = \quad \frac{\hbar}{2}\mathrm{Tr}\left(S^{(2)} + \hbar\Gamma_k^{1,L,(2)}[\phi] + \dots + R_k\right)^{-1}\dot{R}_k \tag{1.86}$$

Equating order by order in $\hbar$ this equation, we obtain a set functional differential equations for each $\Gamma_k^{n,L}[\phi]$ in the form of an infinite tower of equations. This set is ordered by powers of $\hbar$. We notice that the flow of $\Gamma_k^{n,L}[\phi]$ depends at most on $\Gamma_k^{n+1,L}[\phi]$, as is customary in perturbation theory. In particular (1.82) coincides with the order $\hbar$ of (1.86). This means that the result of perturbation theory calculated with ERGE coincide with those calculated with standard methods, modulo the usual scheme dependence due to cutoff procedures. What we just said is true also in the sense that the validity of the 1-loop approximation of ERGE is limited to weak coupling regimes, as it is well known for standard QFT procedures. We refer to [10] for a systematic study of the problem of reconstructing perturbation theory from exact renormalization group.

There is an apparent difference between the standard perturbative renormalization of a QFT and the functional approach we outlined. It concerns where the UV problem is addressed. In perturbation theory and with cutoff Λ regularization, the quantities are renormalized so that the limit $\Lambda \to \infty$ can be safely taken. A renormalization scale is a byproduct of this realization and the beta function tells us how the coupling behave according to the scale. As we outlined, in the functional method we introduced a sliding scale, thus constructing a one parameter family of actions and a flow connecting them. The UV limit $\Lambda \to \infty$ is addressed at the stage of finding the initial condition of the flow.

Chapter 2

Asymptotic safety.

2.1 Beta functions.

In the previous chapter we introduced a functional, the average effective action $\Gamma_k[\phi]$, that has a built-in dependence with respect to a scale k. By abuse of nomenclature we will often refer to this functional as simply effective action EA in the future, while, if necessary we will call the effective action in which all modes have been integrated out true of full effective action. We also obtained an equation, the ERGE, that describes the evolution of $\Gamma_k[\phi]$ with respect to k or equivalently the scale parameter $t = \text{Log}k/k_0$. We know that $\Gamma_k[\phi]$, by construction, interpolates between some initial condition $S_\Lambda[\phi]$ and the full effective action $\Gamma[\phi]$. The interpretation we give to this feature of the EA is that $\Gamma_k[\phi]$ represents a good description of phenomena with characteristic scale k. By good description we mean that by the EA we obtain classical equations of motion, which contain also quantum effects for the scale of interest. The characteristic scale has to be determined case by case and strongly depends on the precise effects we have in mind. A particle physics example could be the scattering amplitude of a certain process, in this case k could be set by the center of mass energy of the process and $\Gamma_k[\phi]$ would at tree level resum the quantum correction to calculate the cross-section. Rather than being a drawback, the "external" k has to be seen as a useful device with which we can play to improve our application.

At this point it is necessary to address quantitatively the features of the flow and to this end it is necessary to make some definitions. As obvious the flow of $\Gamma_k[\phi]$ is rather uncontrolled in nature. It is clear by its flow equation that it will contain very nontrivial effects. The first idea we may have is to parametrize the effective action with a basis of operators

$$\mathcal{O}_{i,k}[\phi] \tag{2.1}$$

that are compatible with the symmetries of the system. In some sense $\mathcal{O}_{i,k}[\phi]$ are coordinatizing the space of field theories. Of course this space is in principle infinite dimensional. We also assumed that the basis could flow with k for completeness. One is free to keep the basis fixed or not, much like in quantum mechanics we have the Schroedinger and Heisenberg representations, provided some care is paid. Dual to operators space is the space of couplings. Each component of this space is simply called coupling $g_{i,k}$ and gives a weight to its corresponding operator $\mathcal{O}_{i,k}[\phi]$ is in the effective action in the form

$$\Gamma_k[\phi] = \sum_i g_{i,k}\mathcal{O}_{i,k}[\phi] \tag{2.2}$$

Let us choose for simplicity a basis that does not flow $\mathcal{O}_i[\phi]$. In this parametrization the flow of the effective action is encoded in the couplings only

$$\Gamma_k[\phi] = \sum_i g_{i,k}\mathcal{O}_i[\phi] \tag{2.3}$$

The derivative with respect to the scale parameter t of the action defines a tangent vector $\beta_i = \partial_t g_{i,k}$ to coupling space in the form

$$\dot{\Gamma}_k[\phi] = \sum_i \beta_i\mathcal{O}_i[\phi] \tag{2.4}$$

Each β_i is called beta function of the coupling g_i and by definition is simply its t derivative. By the choice of parametrization it is clear that β_i are functions of k much like the couplings are. However we know that $\dot{\Gamma}_k[\phi]$ is expressed, through a functional RG equation, in terms of $\Gamma_k[\phi]$ and its derivatives. This imply that the beta functions have a natural parametrization

$$\beta_i = \beta_i(g, k) \tag{2.5}$$

where the g dependence comes from $\Gamma_k[\phi]$ appearing on the right hand side of ERGE and k is a genuine dependence on the scale.

The operators $\mathcal{O}_i[\phi]$ have some canonical mass dimension D_i that tells that also their relative couplings are, in general, dimensionful. In particular $g_{i,k}$ has dimension $d_i = -D_i$ and their naive scaling with k is thus $g_{i,k} \sim k^{d_i}$. To each coupling we can associate a corresponding dimensionless partner

$$\tilde{g}_{i,k} = g_{i,k}k^{-d_i} \tag{2.6}$$

which will flow accordingly

$$\tilde{\beta}_i = \partial_t\tilde{g}_{i,k} = -d_i\tilde{g}_{i,k} + k^{-d_i}\beta_i \tag{2.7}$$

We just defined the beta functions of dimensionless couplings, that are dimensionless exactly like their couplings. For this reason a simple scaling argument tells that, according to the dependence of β_i, their natural dependence is

$$\tilde{\beta}_i = \tilde{\beta}_i(\tilde{g}) \tag{2.8}$$

In the following we will always compute beta functions of dimensionless couplings, because they are, in some sense, more fundamental. There are two arguments to justify this last statement, one more physical and the other more mathematical. From the point of view of experimental physics it is clear that we always measure quantities compared to some reference scale. For example we measure height using a meter. The result of a length measurement is not really a length, but rather a real number that tells us how many times the unit meter measures our length. This is true for every kind of measurement.

The mathematical argument that justify the use of dimensionless couplings has instead to do with the phase space of our quantum field theory and may be better visualized in Wilson approach. Imagine we are coarse-graining our QFT over regions of flat space of typical size k^{-d}. Suppose also that initially the theory has some density of degrees of freedom over spacetime. It is clear that the number of degrees of freedom will change after coarse-graining, in particular the density will decrease with k^{-d}. This is quite a problem if someone is willing to compare the theories before and after the averaging, if these not even live in the same phase space. The general solution to this problem involves using dimensionless couplings, that are a byproduct of the procedure of manually scaling the number of degrees of freedom in order to keep the phase space fixed. One may see the argument also in terms of the entropy of a system in a lattice, therefore in more statistical physics settings. The general expectation of an averaging procedure is that the entropy increases with the size of averaging, but it is easily seen in lattice models that it actually decrease because the number of sites is decreasing. The correct quantity to look at is the entropy density that has a further k^{-d} dependence that balances the coarse-graining. Following this arguments we may interpret dimensionless couplings as densitized couplings along the flow.

We dedicate the final part of this section to define some further concept. Suppose that for some reason we do not want to parametrize our theory directly with coupling, but we rather want to hide them inside some appropriate operators in the form

$$\Gamma_k [\phi] = \sum_i \mathcal{O}_{i,k} [\phi] \tag{2.9}$$

In this case the flow of $\Gamma_k [\phi]$ gives functional beta functions $\beta_{\mathcal{O}_{i,k}}$ for the operators $\mathcal{O}_{i,k} [\phi]$ by

$$\dot{\Gamma}_k [\phi] = \sum_i \beta\mathcal{O}_{i,k} [\phi] \tag{2.10}$$

This may be particularly useful if a dependence of the functionals beta like

$$\beta\mathcal{O}_{i,k} = \beta\mathcal{O}_{i,k} [\mathcal{O}] \tag{2.11}$$

is found. In these terms, even the exact renormalization group equation itself is a functional beta function for the effective action. In future chapters we will extensively calculate functional beta functions and see some limitations of their use.

2.2 Asymptotic safety.

The concepts we are going to introduce in this chapter are better understood in terms of couplings and corresponding beta functions. Also, the basis of operators is chosen to be fixed with the scale. Having chosen the settings we may say that a complete knowledge of the beta functions is equivalent to a complete knowledge of the RG flow, provided the basis of operators spans completely the theory space that is explored by the flow. As one may easily imagine and as we already mentioned, this theory space is infinite dimensional. Our expectation is that, at any fixed scale k, the effective action develops infinite terms through its operators. We may imagine we are testing our theory at that scale k with some experiment to the end of measuring the action and therefore each single g_i.

If we are particularly skilled experimentally, we may also imagine that there exists one experiment to measure g_1, another to measure g_2, then another for g_3 and so on and on. There is no end to this story because there is no end to g_i couplings, in principle. It means we have to do infinite experiments to measure every coupling of the theory. This may not worry, but we actually use theories to predict experiments, not only to measure them. There is no chance that we can predict anything with this theory, because there will always be a infinitely large parameter space of unmeasured couplings that are unconstrained, after a finite number of experiments. In perturbatively renormalizable theories the number of allowed couplings is constrained by the requirement of renormalizability, however we want to go beyond perturbation theory. We want to explore the meaning of renormalizability in the most general settings possible. We therefore have a big problem. If we trace back the issue, it is clear that it is related to the initial condition $S_\Lambda[\phi]$ of the flow. If we were able to consistently constrain $S_\Lambda[\phi]$ to some finite dimensional subset of coupling space, then it would only be an issue of integrating it to any scale k but the finite number of parameters to be measured would be unaltered by flow. The idea is to solve at the same time the issue of taking $\Lambda \to \infty$ and the requirement of predictivity of the theory.

To this end we define a fixed point (FP) of the beta functions as the set of dimensionless parameters $\tilde{g}_i^\star$ such that

$$\tilde{\beta}_i\left(\tilde{g}^\star\right) = 0 \tag{2.12}$$

By definition a fixed point is a point where all beta functions vanish identically. It is obviously untouched by the flow, in the sense that if we take it as initial condition of the flow our theory will remain at the FP at any scale. It is clear that a FP defines a conformal field theory (CFT), that is by definition a theory that remains the same through change in scale. In such a theory the result of experiments is completely determined by the fixed point. In general a FP is a point where flow lines start or end. Going deeper in the study of FPs amounts of determining the number of attractive and repulsive directions it has. This is easily done, at least formally, defining the stability matrix

$$\mathcal{M}_{ij}\big|_{\tilde{g}^\star} = \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j} \tag{2.13}$$

This matrix can be diagonalized to obtain a set of eigenvalues

$$\{c_1, c_2, \dots\} \tag{2.14}$$

and corresponding eigenvectors

$$\{v_1, v_2, \dots\} \tag{2.15}$$

The eigenvalues are not necessarily reals. A complex eigenvalue encodes in its imaginary part the fact that the flow spirals around a fixed point. In the following, when saying that an eigenvalue is positive or negative, we actually refer to the property of its real part. A positive eigenvalue means that the FP is repulsive in the corresponding direction, the converse is true for a negative eigenvalue. In particular, if we are following the flow close to the FP along the direction v_i and we slightly perturb it, it is easy to see that

$$g_i \sim g_i^\star + b v_i k^{c_i} \tag{2.16}$$

The critical exponents b_i of a theory at a fixed point are defined through the scaling relation $g_i \sim g_i^\star \sim k^{-b_i}$ (along the direction v_i) and this relation shows that they correspond to minus the eigenvalues c_i. A fixed point with some attractive direction is a possible ending of the limit $\Lambda \to \infty$, here interpreted as a prolongation of k integration from Λ to ∞. The other possible endings are the theory being pushed to the far boundary of coupling space, so having some dimensionless quantity going infinity. It is hopefully clear from this discussion that an attractive FP has possibly an important physical meaning, because it may represent our UV limit. We may want to find a FP in first instance to take $\Lambda \to \infty$.

This is not the end of the story. With the fixed point we ensure the finiteness (in terms of dimensionless quantities) of the theory, but still we do not know if the theory is predictive. We notice that a subset of coupling space is attracted in the $k \to \infty$ limit. This subset of coupling space is called "critical surface" (the flow in coupling space is assumed to be smooth). This subset is finite dimensional if the number of attractive directions is finite dimensional. In such a case only a finite number of experiments is needed to identify the trajectory. So we do not only need an attractive fixed point, but it is important that it has a finite number of attractive directions. At this point we have all the ingredients to define a theory, or rather a set of theories, that is UV safe and predictive.

We call "asymptotically safe" [1] a theory (here theory means actually a one parameter set of theories) which satisfy these conditions:

- There exists a FP in its flow.
- The fixed point has a finite number of attractive directions.

or equivalently

- There exists a FP in its flow.
- The critical surface of the FP is finite dimensional.

We already stressed that such a theory is finite and predictive. An asymptotically safe theory is a theory for which the UV limit can be consistently performed. We call this limit also "continuum limit" borrowing some terminology from lattice theories. In that case, in fact, Λ would correspond to the inverse of lattice size $\Lambda \sim 1/a$ and the UV limit would send $a \to 0$. In that case continuum limit means that there exists a theory with smooth degrees of freedom that governs the UV behavior of the lattice ones. Namely, the same is happening in cutoff-regulated theories and theories with a sliding scale. If we want our theory's action to be a low energy manifestation of a more fundamental action with the *same* degrees of freedom, it is *necessary* that a FP exists with the mentioned properties. Otherwise no meaningful UV-limit is possible. An example of flux of in coupling space is given in Fig. (2.1).

We refer to [11] for a review of the application of the asymptotic safety scenario to gravity.

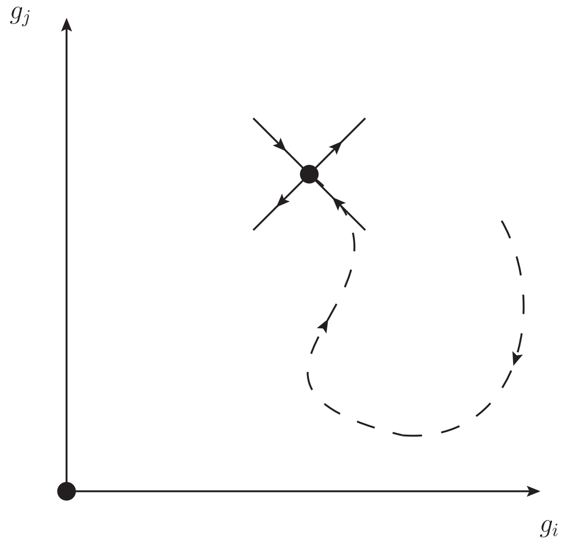

Figure 2.1: A possible flow of a theory, seen from the point of view of the 2-dimensional subspace spanned by the couplings g_i and g_j. In the diagrams both the gaussian FP and the non-trivial FP are shown. The non-trivial FP is attractive in one direction of the subspace and repulsive in the other. A safe trajectory, that flows to the FP for $k \to \infty$, is shown and represented with the dashed line.

2.3 Asymptotic freedom.

The purpose of this section is to show that the familiar concept of asymptotic freedom [12] can easy be embedded inside that of asymptotic safety. We need a further definition. We will call gaussian fixed point (GFP) a fixed point in which all the couplings are zero. The gaussian naively corresponds to the free theory configuration of a certain QFT. It is a general feature of any QFT that the gaussian fixed point exists. GFPs are very important under many circumstances, for example in the former chapter we said that the validity of the loop expansion holds only if coupled to a small coupling expansion. Therefore the loop expansion is a good approximation of the RG flow around the gaussian fixed point and in particular any 1-loop equation approximates well the flow for small couplings.

We have always to keep in mind in this chapter that we are interested in the conditions by which we can take the smooth limit $\Lambda \to \infty$. So we may argue what happens if we specialize the definition of asymptotic safety requiring the fixed point to be gaussian. First of all let us have a look to the dimensionless beta functions

$$\tilde{\beta}_i = -d_i \tilde{g}_i + k^{-d_i} \beta_i \tag{2.17}$$

where d_i are the canonical dimensions of the dimensionful coupling g_i and β_i their beta functions. It is easy to figure out that $\beta_i = 0$ for $g_i = 0$ due to the structure of functional RG flow. This means that around the GFP the scaling of the couplings is essentially determined by the canonical dimensions. After diagonalization, the stability matrix of the gaussian fixed point looks like

$$\mathcal{M}_{ij}|_{\tilde{g}^\star=0} = \mathrm{diag}\,(-d_1, -d_2, \dots) \tag{2.18}$$

We could classify our operators by increasing canonical dimension, so that also the sequence $-d_1 \leq -d_2 \leq \dots$ is increasing. Maintaining our abstract notation, we assume there exists a certain $d_{\hat{\imath}}$ so that $d_{\hat{\imath}+1} > 0$ while still $d_{\hat{\imath}} \leq 0$. $d_{\hat{\imath}}$ represents a threshold, all the couplings before it in the sequence have a negative eigenvalue and are therefore attracted to the GFP. Contrarily, all the couplings after it are repulsed by the gaussian fixed point because of the positive eigenvalue. Clearly we are trivializing the situation a little bit, because there may be a lot of couplings with the same canonical dimension, but what we said is easily generalized. Some special care is needed to analyze the behavior of dimension zero couplings. Their attractive or repulsive behavior is actually dictated by a further expansion of their beta functions and not determined by the linearized flow $\mathcal{M}_{ij}$.

We finally ended with a finite sequence $\{\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_{\hat{\imath}}\}$ of couplings in the direction of which we are attracted to the GFP. The total intersection of the hyperplanes $\tilde{g}_i = 0$ represents the (linearized) critical surface of couplings we used in the definition of asymptotic safety. We know that, close to the GFP, if our theory stays in the critical surface we safely can take the UV limit. From the linearized point of view this incidentally means that all the negative dimension couplings are set to zero, while the others flow to the GFP according to their scaling. Now we can translate what we just said in perturbation theory terminology. A negative dimensional coupling is generally called non-renormalizable, the set of these is precisely the set of couplings we excluded by putting ourselves on the critical surface. All the positive dimensional couplings, which are super-renormalizable, have been instead included and are free to run thanks to their controlled behavior. The final case is the case of dimension zero couplings, where, exactly like in perturbation theory some special care is needed. The fate of all active (non-zero) couplings around the GFP is to flow to the GFP itself, thanks to the critical surface, they are therefore asymptotically free because they go to zero in the UV limit, as dictated by the usual perturbative definition of asymptotic freedom. The prototype of asymptotically free theory is QCD and in general any Yang-Mills theory.

The fact that we can embed the definition of asymptotic freedom inside that of asymptotic safety is promising. One can easily figure out that it is hardly possible to enlarge even more the definition of renormalizability, at least from the point of view of functional RG flow and in such general settings.

2.4 Nonperturbative approximation scheme.

It is well known that, if a theory is asymptotically free, it is renormalizable. This is easily seen because we have a method to produce its expansion, the loop expansion, that goes parallel to the coupling expansion. In the UV limit we know the couplings are going to zero, so the loop expansion receives corrections that are smaller loop by loop at high energies. Unfortunately we are interested in nonperturbative scenarios. Now we definitely move back to the asymptotic safety definition, where the fixed point is generally non-gaussian. It still may happen that the couplings of the non-gaussian fixed point are small and some kind of perturbative analysis may be used. The kind of nonperturbative analysis we desire need some new approximation scheme that is nonperturbative in nature. Some of these schemes have been developed with varying degree of success. We are not going to review all of them, instead we will just try to explain the origin of the approximation scheme we are going to apply in the following.

There has been developed three main approximation schemes in the context of functional RG flow and that we summarize here. The first is the derivative expansion, that consists in including higher and higher derivative terms in the effective action. This scheme is mainly effective when used to explore critical phenomena. Conversely, a theory such as QCD, is best explored through vertex expansion. The latter consists in introducing more and more vertices in the effective action. Finally, one can implement any mixture of derivative and vertex expansions. The main lesson here is that one must always keep in mind the nature of the system under consideration and use the approximation that best captures its physics.

We want to give some example. Let us start by looking at how chiral perturbation theory (χPT) is constructed. We want to look at this example because χPT is a prototypical perturbatively nonrenormalizable theory, that still has a lot to say from the experimental point of view. In χPT the loop expansion is coupled with a derivative expansion of the terms. So to say, at 1-loop the running of 2-derivative term is calculated and 4-derivative terms are generated. At 2-loops, the 2-derivative term runs properly with 2-loop equation, the 4-derivative terms are evolved with 1-loop equations and the 6-derivative terms are generated. This goes on and on. The validity of the approximation stays in the fact that the energy scales Λ' we are interested in, are much smaller than the scale Λ that is the hard cutoff of the theory (beyond which we assume new degrees of freedom arise). A similar discussion can be addressed in the case of gravity where derivative expansion (and locality) can be achieved by an expansion in curvature invariants and their derivatives.

The truncation is what we are interested in now. A truncation of the effective action is by definition a nonperturbative way to approximate it. This may look bad at first sight and indeed it is if one wishes to prove general statements of renormalizability by use of truncations. However it is very effective when we have some idea of the dominant terms in the flow (see again χPT for an example) and therefore we start our study by their inclusion. Often, there are not many a priori requirements to prefer a truncation rather than another and accordingly we generally have some freedom in its choice. Consequently, not much nomenclature of precise truncation has been developed. The general rule is: after a calculation is done one should test the validity of a truncation by looking at what else is generated, but could never be completely sure fundamental dominant terms have been omitted [13]. There is however a way out. What we said holds for the general effective action $\Gamma_k[\phi]$ and particularly for the full effective action at $k = 0$. The UV limit, instead, is a typical action $S_\Lambda[\phi]$ that by simple arguments is expected to be local and related to some derivative expansion, mainly due to the presence of a very high (eventually infinite) scale Λ. Exactly like in χPT.

We now want to give some example of truncations of systems that are often studied in the effective field theory framework. Let us fix the ideas by working with four dimensional spacetime and canonical fields. A typical scalar field action in the local potential approximation is of the form [14, 15]

$$\Gamma_k [\phi] = \int dx \left(\frac{Z_\phi}{2} (\partial\phi)^2 + V[\phi] \right) \tag{2.19}$$

where we included the possible field renormalization. This does not include all possible terms with two derivatives, but includes all those with zero derivatives. In fact, in the potential $V[\phi]$ there are all the possible power-like interactions of ϕ. If we further include a spinor and analyze a truncation of the form [16, 17, 18]

$$\Gamma_k \left[\phi, \bar\psi, \psi\right] = \int dx \left(\frac{Z_\phi}{2} (\partial\phi)^2 + Z_\psi \bar\psi i \not{D} \psi + H[\phi] \bar\psi\psi + V[\phi] \right) \tag{2.20}$$

we end up with a truncation that contains also any interaction involving a power of ϕ with the couple $\bar\psi\psi$. The interaction term $H[\phi]\bar\psi\psi$ is a generalized Yukawa interaction and accomodates, among the others, a mass term and a true Yukawa interaction. This contains all dimension zero (local, non-derivative) interactions because spacetime is four dimensional. The dimension zero operator are ϕ^4 and $\phi\bar\psi\psi$. This model, describing the interaction of a scalar and a spinor fields, will be the main subject of Chapter 3 and the toy model for some of the techniques used in this Thesis.

Now we include gravity in the game through an Einstein-Hilbert action [19, 21]

$$\Gamma_k \left[\phi, \bar\psi, \psi, g_{\mu\nu}\right] = \int dx \sqrt{g} \Big(Z_g R + \frac{Z_\phi}{2} (\partial\phi)^2 + Z_\psi \bar\psi i \not{D} \psi + H[\phi] \bar\psi\psi + V[\phi] \Big) \tag{2.21}$$

This time not all dimension zero local interactions are included, because the operator $\phi^2 R[g_{\mu\nu}]$ is missing [22]. We will study the inclusion of gravity in Chapter 4 and its implications for the asymptotic safety scenario in Chapter 5.

A prototypical example of truncation for gravity with infinitely many terms is the simple $f(R)$-gravity. Here f is a function of the curvature scalar only and no other Ricci- and Riemann-tensor interaction are allowed. It would be [23, 24]

$$\Gamma_k [g_{\mu\nu}] = \int dx \sqrt{g} f(R) \tag{2.22}$$

Instead a truncation in local curvature terms, at order zero, one and two in Riemann tensor, would be of the form [26]

$$\Gamma_k [g_{\mu\nu}] = \int dx \sqrt{g} \left(g_0 + g_1 R + g_{2,1} R^2 + g_{2,2} \mathrm{Ric}^2 \right) \tag{2.23}$$

where boundary terms have been eliminated. A nonlocal equivalent could be [27]

$$\Gamma_k [g_{\mu\nu}] = \int dx \sqrt{g} \left(g_0 + g_1 R + R f_1(\Box) R + \mathrm{Ric} f_2(\Box) \mathrm{Ric} \right) \tag{2.24}$$

As it is well known in two dimensional examples, an action with non-local terms is expected to be important when exploring the IR regimes with RG flow.

Going back to the scalar field case and relaxing the requirement of having a canonical dimension one scalar, there is an important model for which ϕ is of canonical dimension zero. It is the non-linear sigma model (NLSM) and ϕ is dimensionless because interpreted as a coordinate in target space manifold [28, 29, 30]. Its action is

$$\Gamma_k[\phi] = \frac{1}{2}\int dx h_{\alpha\beta}\partial\phi^{\alpha}\partial\phi^{\beta} \tag{2.25}$$

It contains all possible two-derivatives interactions, which are all hidden inside $h_{\alpha\beta} = h(\phi)_{\alpha\beta}$ that will be interpreted as a metric of ϕ space. A NLSM action is also that of chiral perturbation theory (χPT), where the field U is group-valued. In this case, up to four derivatives we could have, schematically

$$\Gamma_k[U] = \mathrm{tr}\int dx\left(\left(U^{\dagger}\partial U\right)^2 + \left(U^{\dagger}\partial U\right)^4 + \left(U^{\dagger}\Box U\right)^2\right) \tag{2.26}$$

Chapter 6 will be dedicated to a general study of higher derivative sigma model. We will also analyze in detail the chiral model.

Many of the truncation we showed here have been studied in the literature in the context of exact renormalization group flow, some are currently under study, some other will be the main topics of this work. The calculations of their beta functions and fixed points will be our main results.

Chapter 3

Methods for the functional exact RG.

In this chapter we review some methods that are usually applied in the context of functional renormalization group. Both the methods and the toy models we will work with will turn out to be useful later.

3.1 A scalar model in the LPA.

We now want to specifically address a very simple toy model of functional renormalization group, namely a simple real scalar field [15]. We truncate its action in a local potential form, but we generally keep the possibility that it has a nontrivial anomalous dimension by adding a wavefunction renormalization. We also fix the spacetime dimensions to be four, although it is very easy to generalize the results we will obtain to general d. The action is therefore

$$\Gamma_k\left[\phi\right] = \int d^4x \left(\frac{Z_\phi}{2}\partial_\mu\phi\partial_\mu\phi + V\left[\phi\right]\right) \tag{3.1}$$

No assumption is made on the form of the potential at this point, so in principle we are including all possible terms. The reason we choose such a model is twofold. It turns out that we can apply and therefore explain through it all the techniques we will need in the following chapters. Additionally, we will also be able to outline a strategy to treat the potential in symmetry-breaking phase. It is particularly useful to have a functional flow for the potential, because we will be able to resum it when dealing with the flow of the vacuum expectation value and a nonperturbative beta function for it will be given.

Before going on to the specific techniques, some general step can be performed by analyzing how ϕ and its potential get renormalized. The canonical step in renormalization procedures is to take the renormalized field

$$\phi_R = \sqrt{Z_\phi}\phi \tag{3.2}$$

that is the actual field whose correlations we measure. This is easily seen because ϕ_R has a canonical kinetic term and we always assume the asymptotic states to be canonically normalized. In the asymptotic safety scenario we know we want to study the flow of dimensionless couplings. It turns out that it is useful to define the dimensionless renormalized field

$$\bar{\phi}_R = k^{-1}\phi_R = k^{-1}\sqrt{Z_\phi}\phi \tag{3.3}$$

Now, k- and t-derivatives are always performed at fixed ϕ (when using the effective average action), so we can apply them to $\bar{\phi}_R$ and get a non-zero result. We obtain for the t-derivative

$$\partial_t\bar{\phi}_R = -\left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R \tag{3.4}$$

where we defined the anomalous dimension η_ϕ as minus the logarithmic derivative of Z_ϕ,

$$\eta_\phi = -\frac{\dot{Z}_\phi}{Z_\phi} \tag{3.5}$$

The reason of the name anomalous dimension is that it changes the scaling one naively expects of the field ϕ. In fact, the t-derivative of the dimensionless renormalized field is telling us that the expected scaling of ϕ is

$$\phi \sim k^{1+\frac{\eta_\phi}{2}} \tag{3.6}$$

It looks like ϕ is not really living in four dimensions (the canonical dimension is in a general spacetime $(d-2)/2$, that equals 1 for $d=4$). This is just a little bite of the full meaning of the anomalous dimension, but actually all we need at this point.

The dimensionless renormalized field is the natural argument of the dimensionless renormalized potential. We define it as

$$\bar{v}_R\left[\bar{\phi}_R\right] \quad = \quad k^{-4}V\left[kZ_\phi^{-\frac{1}{2}}\bar{\phi}_R\right] \tag{3.7}$$

and its coefficients in a powerlaw expansion are called dimensionless renormalized couplings. These are precisely the dimensionless couplings we need to test the possibility of asymptotic safety. To see how the flow of $\bar{v}_R\left[\bar{\phi}_R\right]$ is related to that of $V[\phi]$ it is sufficient to derive with respect to k both sides of (3.7). It is important to remember that the dimensionless renormalized potential has a built-in dependence on k through its expansion coefficients and a dependence due to the argument $\bar{\phi}_R$. What we want to calculate is only the intrinsic one. We obtain

$$\dot{\bar{v}}_R\left[\bar{\phi}_R\right] = -4\bar{v}_R\left[\bar{\phi}_R\right] + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{v}'_R\left[\bar{\phi}_R\right] + k^{-4}\dot{V}\left[\phi\right] \tag{3.8}$$

We can identify the three terms that appeared in order. The first is the effect of the canonical potential scaling, the second represent how the (anomalous) scaling of ϕ affects that of the potential. Finally, the third term has to be determined, for example using ERGE. Again a scaling argument tells that $k^{-4}\dot{V}[\phi]$ is going to be a function of $\bar{\phi}_R$ with dependence on k only through its argument or its powerlaw coefficients.

The main tasks of the next sections will be to determine $k^{-4}\dot{V}[\phi]$ through exact renormalization group, but also η_ϕ because, in principle, it leads to important modifications of the flow.

3.2 Effective potential at constant field.

We know the exact renormalization group equation is a functional equation of the form

$$\dot{\Gamma}_k[\phi] = \mathcal{F}[\Gamma_k] \tag{3.9}$$

Both sides of it are functionals of ϕ through the dependences $\dot{\Gamma}_k[\phi]$ and $\Gamma_k[\phi]$. There is some freedom in choosing the particular field configuration to evaluate this equation. For example one may want to evaluate it for $\phi =$ const. ending up with the flow

$$\int d^4x \dot{V}[\phi] = \frac{1}{2}\mathrm{Tr}\left(Z_\phi\left(-\partial^2\right) + V''[\phi] + \mathcal{R}_k\right)^{-1}\dot{\mathcal{R}}_k \tag{3.10}$$

where the fact that both sides are evaluated at ϕ constant is understood.

A precise definition of the cutoff kernel is needed. The IR cutoff term is

$$\Delta S_k = \frac{1}{2}\int d^4x \phi \mathcal{R}_k \phi \tag{3.11}$$

and has to kill the modes of a kinetic term like

$$\int d^4x \frac{Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi = -\frac{1}{2}\int d^4x Z_\phi \phi\left(-\partial^2\right)\phi \tag{3.12}$$

Therefore it is natural to let the cutoff kernel have the same global scaling by setting $\mathcal{R}_k = Z_\phi R_k$. We also move to momentum space and use the operator $-\partial^2 \to q^2$ as reference for the coarse-graining. Altogether amounts to setting

$$\Delta S_k = \frac{Z_\phi}{2}\int_q \phi_{-q} R_k\left(q^2\right)\phi_q \tag{3.13}$$

and the RG equation for the potential becomes

$$\dot{V}[\phi] = \frac{1}{2}\int_q\left(q^2 + \frac{V''[\phi]}{Z_\phi} + R_k\left(q^2\right)\right)^{-1}\left(\dot{R}_k\left(q^2\right) - \eta_\phi R_k\left(q^2\right)\right) \tag{3.14}$$

and a graphical representation of this integral is given in Fig. (3.1).

We further specify the cutoff kernel to be the "optimized" cutoff $R_k\left(q^2\right) = \left(k^2 - q^2\right)\theta\left(k^2 - q^2\right)$ [31]. Some care is needed to approach the composition of any function with a theta function. In particular we want to write consistently $\left(q^2 + \frac{V''[\phi]}{Z_\phi} + R_k\left(q^2\right)\right)^{-1}$. We first note that our cutoff kernel has a support, $q^2 \leq k^2$. We perform the composition supportwise and we get the inverse modified propagator in the form

$$\frac{1}{q^2 + \frac{V''[\phi]}{Z_\phi} + R_k\left(q^2\right)} = \frac{1}{q^2 + \frac{V''[\phi]}{Z_\phi}}\theta\left(q^2 - k^2\right) + \frac{1}{k^2 + \frac{V''[\phi]}{Z_\phi}}\theta\left(k^2 - q^2\right) \tag{3.15}$$

We want to show that the term proportional to $\theta\left(q^2 - k^2\right)$ does not contribute to the running, because both $R_k\left(q^2\right)$ and $\dot{R}_k\left(q^2\right)$ are proportional to $\theta\left(k^2 - q^2\right)$. It is sufficient to calculate

$$\dot{R}_k\left(q^2\right) = 2k^2\theta\left(k^2 - q^2\right) + \left(k^2 - q^2\right)\delta\left(k^2 - q^2\right) \simeq 2k^2\theta\left(k^2 - q^2\right) \tag{3.16}$$

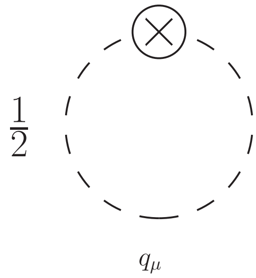

Figure 3.1: Diagram representing the flow of $V[\phi]$ in 3.14. The dashed line is the inverse modified scalar propagator $\left(q^2 + V''/Z_\phi + R_k\right)^{-1}$, while the vertex is the cutoff insertion. q_μ is the momentum running in the loop.

where an assumption of regularity of all integrated terms is done, namely no poles in $k^2 = q^2$ should be present. The result for the cutoff sector of the ERGE is thus

$$\dot{R}_k\left(q^2\right) - \eta_\phi R_k\left(q^2\right) \quad = \quad \left(\left(2-\eta_\phi\right)k^2 + \eta_\phi q^2\right)\theta\left(k^2 - q^2\right) \tag{3.17}$$

which obviously cannot have the poles mentioned.

The result of cutoff specification is therefore an easily calculable integral. Before writing it we also want to factor the angular part of the momentum integration

$$\int_q = \frac{1}{(2\pi)^4}\int d^4q = \frac{1}{(2\pi)^4}\int d\Omega_3 \int q^3 dq \tag{3.18}$$

and change the q integration to a $z = q^2$ one by

$$\int q^3 dq = \frac{1}{2}\int z dz \tag{3.19}$$

These choices imply for the running of the potential in very compact form

$$\dot{V}[\phi] \quad = \quad \frac{\mathrm{Vol}\left(S^3\right)}{4\,(2\pi)^4}\int_0^{k^2} dz \frac{\left(2-\eta_\phi\right)k^2 z + \eta_\phi z^2}{k^2 + \frac{V''[\phi]}{Z_\phi}} \tag{3.20}$$

$$\quad = \quad \frac{1}{32\pi^2}\frac{k^6\left(1 - \frac{\eta_\phi}{6}\right)}{k^2 + \frac{V''[\phi]}{Z_\phi}} \tag{3.21}$$

As expected this is a homogeneous function in the renormalized quantities apart for an overall scaling k^4, in fact $V''[\phi]/Z_\phi = k^2 \bar{v}''_R\left[\bar{\phi}_R\right]$ and

$$k^{-4}\dot{V}[\phi] \quad = \quad \frac{1}{32\pi^2}\frac{1 - \frac{\eta_\phi}{6}}{1 + \bar{v}''_R\left[\bar{\phi}_R\right]} \tag{3.22}$$

The final result for the beta functional of the potential is very well known and the typically studied object when making comparison among the functional coarse-graining methods. To have the final form we desired, we thus include the terms representing the canonical scaling and write down the flow for the dimensionless renormalized potential

$$\dot{\bar{v}}_R\left[\bar{\phi}_R\right] = -4\bar{v}_R\left[\bar{\phi}_R\right] + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{v}'_R\left[\bar{\phi}_R\right] + \frac{1}{32\pi^2}\frac{1-\frac{\eta_\phi}{6}}{1+\bar{v}''_R\left[\bar{\phi}_R\right]} \tag{3.23}$$

For completeness we also give the same result in general d-dimensional spacetime that is

$$\dot{\bar{v}}_R\left[\bar{\phi}_R\right] = -d\bar{v}_R\left[\bar{\phi}_R\right] + \frac{1}{2}\left(d-2+\eta_\phi\right)\bar{\phi}_R\bar{v}'_R\left[\bar{\phi}_R\right] + \frac{2^{1-d}\pi^{-d/2}\left(1-\frac{\eta_\phi}{d+2}\right)}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{v}''_R\left[\bar{\phi}_R\right]\right)} \tag{3.24}$$

where $\Gamma\left(\frac{d}{2}\right)$ is the Euler gamma function.

Yet another ingredient is missing, the anomalous dimension, and we shall calculate it in the next section.

3.3 Scalar anomalous dimension.

It should be clear from the calculation of the previous section that it is not possible to evaluate the anomalous dimension setting $\phi = \text{const.}$ in the flow of the effective action. The simple reason is that a constant ϕ hides from the functional flow the operator associated to the field renormalization, namely the kinetic term of our LPA. One easily sees that for the computation of $\dot{Z}_\phi$ there are two major alternatives, that actually give the same result.

The first possibility is maybe the most straightforward, being a simple generalization of the calculation of the previous section. The idea is to take $\phi = \phi_1 + \varphi$ as configuration in which evaluate the flow $\dot{\Gamma}_k[\phi]$, where ϕ_1 is a reference constant configuration and φ is a nonconstant "perturbation". If this is done obviously the kinetic term for ϕ will convert to a kinetic term for φ, that in the flow will become

$$\frac{\dot{Z}_\phi}{2}\int d^4x\partial_\mu\varphi\partial_\mu\varphi \tag{3.25}$$

Then, for the actual computation, it is sufficient to calculate the functional flow

$$\dot{\Gamma}_k\left[\phi_1+\varphi\right] \tag{3.26}$$

and obtain all its terms containing two powers of φ and two derivatives acting properly on them. The coefficient of these terms is proportional to $\dot{Z}_\phi$. We shall not follow this line, but still some consideration is possible before starting the computation.

The kind of calculation we just proposed has a simple, but perhaps unexpected byproduct. It is easy to see that if one expands $\dot{\Gamma}_k\left[\phi_1+\varphi\right]$ as suggested, the coefficient of the terms of interests (those like the kinetic one, eventually modulo an integration by parts) is generally a function of ϕ_1. It looks like the functional flow is suggesting us that the actual truncation we should employ is beyond the LPA and has a kinetic term of the form

$$\int d^4x\frac{Z\left[\phi\right]}{2}\partial_\mu\varphi\partial_\mu\varphi \tag{3.27}$$

This is possibly a problem, because if for consistency we generalize the truncation including it, we shall encounter new terms that are going to contribute and we should repeat the calculation from the beginning. This apparent issue is solved by simply noticing that, when computing the anomalous dimension, some physical meaning has to be attached to the constant configuration ϕ_1. If this is done then we just need to assume what we call "Z-rule" and choose

$$Z_\phi = Z[\phi_1] \tag{3.28}$$

so that we fall back in the original LPA truncation. The meaning of the rule is that the result of the computation of the anomalous dimension depends on the field configuration one decides to evaluate the ERGE from. This is another kind of scheme dependence of the functional RG. It is generated by the fact that the flow does not "know" about our original truncation and produces further terms that are possible in a scalar action. We have to consistently fix it. A natural configuration around which we could look at Z_ϕ is a minimum of the potential

$$V'[\phi_1] = 0 \tag{3.29}$$

and so its ground state $\phi_1 = \langle\phi\rangle$. If the potential is symmetric the minimum $\phi_1 = \langle\phi\rangle = 0$ will give back the same results of perturbation theory around $\phi = 0$. Some novelty is expected if the potential has a symmetry-broken phase. We will see this in more detail later. We will also see that similar considerations will apply to all other coupling (like Yukawa's) and eventually to the flow of the effective potential itself.

Still we have to explain an alternative way to calculate the flow of Z_ϕ. We want to perform it keeping the simple configuration $\phi =$ const. as point in which evaluate the flow (from now on we drop again the label in ϕ_1 coming back to the notation of the previous section). The key ingredient is that we have to look at the flow of the 2-point function, rather than at the flow of the effective action itself. This is possible because we know that the ERGE provides an infinite tower of differential equations coupled together. In particular the 2-point function of the scalar in momentum space and with incoming momentum p_μ is

$$Z_\phi p^2 + V''[\phi] \tag{3.30}$$

It tells us that it is sufficient to calculate, in momentum space, the coefficient of the p^2 term of the flow of the 2-point correlator to obtain $\dot{Z}_\phi$. As a byproduct we will also have a consistency check by comparing $\dot{V}''[\phi]$ (from the p^0 term) with the second derivative of $\dot{V}[\phi]$. This check will turn out to be important in future gravitational applications. A final remark, we avoided to place an explicit dependence on ϕ in $\dot{Z}_\phi$ in order to distinguish it from more "genuine" dependences, like that of the potential.

Now we need to calculate the flow of the propagator. To this end we introduce some new compact notation. We write the general n-point correlator of the effective action as

$$\frac{\delta^n \Gamma_k[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)} = \Gamma^{(n)}_{k;x_1,\ldots,x_n} \tag{3.31}$$

which will be later assumed to be evaluated at a certain constant ϕ configuration. We also define its momentum space transform as

$$\tilde{\Gamma}^{(n)}_{k;p_1,\ldots,p_n} \tag{3.32}$$

Clearly we have that momenta are conserved at each vertex and therefore each correlator is proportional to a momentum space delta function

$$\tilde{\Gamma}^{(n)}_{k;p_1,\ldots,p_n} = \Gamma^{(n)}_{k;p_1,\ldots,p_n} \delta_{p_1+\cdots+p_n} \tag{3.33}$$

When writing down functional equations we will always drop the momentum conservations at each vertex, therefore we will always use $\Gamma^{(n)}_{k;p_1,\ldots,p_n}$ that are actually functions of $n-1$ momenta. This means, it will be always assumed that the Dirac delta has been used when integrating over all undetermined momenta. In particular, the two point function $\Gamma^{(2)}_{k;x_1,x_2}$ is a function of $(x_1 - x_2)_\mu$ only, so the information pertaining its Fourier transforms are all in the conserved $\Gamma^{(2)}_{k;p} = \Gamma^{(2)}_{k;p,-p}$, where p_μ is the conjugate momentum to $(x_1 - x_2)_\mu$ and the conservation has been dropped. Using the two point function we can construct the modified inverse propagator, for which we define

$$G_{k;p} = \left(\Gamma^{(2)}_{k;p} + \mathcal{R}_{k;p}\right)^{-1} \tag{3.34}$$

where $\mathcal{R}_{k;p} = \mathcal{R}_k\left(p^2\right)$ is the IR cutoff kernel we saw before. In this notation the exact RG equation for the simple scalar field model becomes very compact

$$\dot{\Gamma}_k[\phi] = \frac{1}{2}\int_q G_{k;q}\dot{\mathcal{R}}_{k;q} \tag{3.35}$$

Once one knows the simple rule to take functional derivatives of the propagator, the derivation of the flow of any correlator is straightforward. It is sufficient to show that, in coordinate space

$$\frac{\delta}{\delta\phi(x_1)} G_{k;x_2,x_3} = -\int d^4x_4 d^4x_5 G_{k;x_2,x_4} \Gamma^{(3)}_{k;x_1,x_4,x_5} G_{k;x_5,x_3} \tag{3.36}$$

In momentum space the conservation rules come into rescue and we can drop the integrations thanks to them

$$\frac{\delta}{\delta\phi_p} G_{k;q} = -G_{k;q}\Gamma^{(3)}_{k;p,q,-(q+p)} G_{k;q+p} \tag{3.37}$$

We have now all the ingredients to derive the flow of any n-point function. We give the first ones in coordinate space

$$\dot{\Gamma}^{(1)}_{k;x} = -\frac{1}{2}\int \prod_{i=1,\ldots,4} d^4x_i G_{k;x_1,x_2}\Gamma^{(3)}_{k;x,x_2,x_3} G_{k;x_3,x_4}\dot{\mathcal{R}}_{k;x_4,x_1} \tag{3.38}$$

$$\begin{aligned}\dot{\Gamma}^{(2)}_{k;x,y} &= \int \prod_{i=1,\ldots,6} d^4x_i G_{k;x_1,x_2}\Gamma^{(3)}_{k;x,x_2,x_3} G_{k;x_3,x_4}\Gamma^{(3)}_{k;y,x_4,x_5} G_{k;x_5,x_6}\dot{\mathcal{R}}_{k;x_6,x_1} \\ &\quad -\frac{1}{2}\int \prod_{i=1,\ldots,4} d^4x_i G_{k;x_1,x_2}\Gamma^{(4)}_{k;x,y,x_2,x_3} G_{k;x_3,x_4}\dot{\mathcal{R}}_{k;x_4,x_1}\end{aligned} \tag{3.39}$$

$$\Gamma^{(3)}_{k;x,y,z} = \ldots$$

We can look at this set of equations as representative of a vertex expansion, as it is seen from the increasing number and order of the vertices.

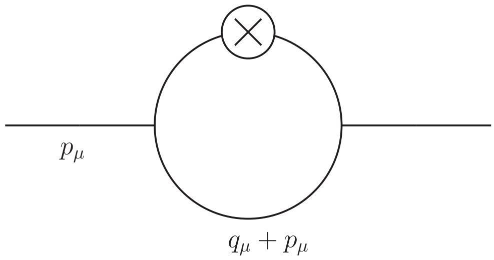

Figure 3.2: Representation of $\int_q G_{k;q}\Gamma^{(3)}_{k;p,q,-(q+p)}G_{k;q+p}\Gamma^{(3)}_{k;-p,q+p,-q}G_{k;q}\dot{\mathcal{R}}_{k;q}$ contribution to the flow of $\dot{\Gamma}^{(2)}_k$. Full lines indicate a general field of any type.

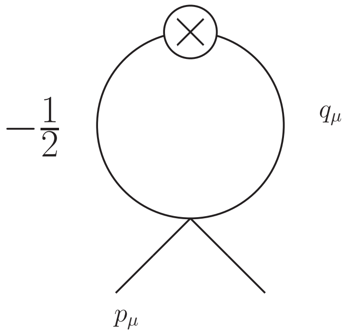

Figure 3.3: Representation of $-\frac{1}{2}\int_q G_{k;q}\Gamma^{(4)}_{k;p,-p,q,-q}G_{k;q}\dot{\mathcal{R}}_{k;q}$ contribution to the flow of $\dot{\Gamma}^{(2)}_k$.

It should be clear that we implicitly assumed that the theory we are working with does not posses grassmanian degrees of freedom, or any other complicated tensor structure. For example the order in which derivatives are applied to any correlator is actually unimportant. All this is true for the scalar model we have in mind, although we will see when adding a spinor field things will remain pretty much the same. The flow of $\dot{\Gamma}^{(1)}_{k;x}$ somehow represents the flow of the equations of motions. Diagrammatically it is a tadpole and it goes obviously to zero when evaluating things on-shell. We will however try to work with off-shell variables as much as possible, to obtain general results. The flow of $\Gamma^{(2)}_{k;x,y}$ is what we are actually interested in. In momentum space there are less integrations thanks to momentum conservation

$$\begin{aligned}\dot{\Gamma}^{(2)}_{k;p} &= \int_q G_{k;q}\Gamma^{(3)}_{k;p,q,-(q+p)}G_{k;q+p}\Gamma^{(3)}_{k;-p,q+p,-q}G_{k;q}\dot{\mathcal{R}}_{k;q} \\ &\quad -\frac{1}{2}\int_q G_{k;q}\Gamma^{(4)}_{k;p,-p,q,-q}G_{k;q}\dot{\mathcal{R}}_{k;q}\end{aligned} \tag{3.40}$$

The diagrammatic expression of this flow equation with general momentum dependence is given in Fig. 3.2 and Fig. 3.3.

We can now specialize completely to our scalar model truncation in the LPA. First we notice that any correlation beyond the 2-point function is determined

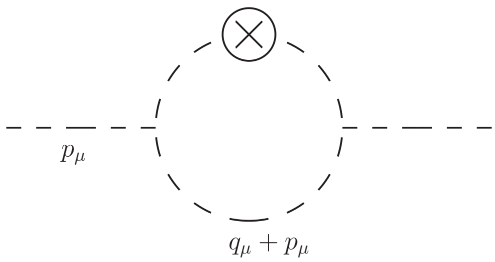

Figure 3.4: Representation of $V^{(3)}[\phi]^2 \int_q G^2_{k;q} G_{k;q+p} \dot{\mathcal{R}}_{k;q}$ contribution to the flow of $\dot{\Gamma}^{(2)}_k$. Scalar lines are dashed.

by the potential, that is a nonderivative functional. Therefore, regardless of the incoming momenta

$$\Gamma^{(n)}_{k;q_1,\ldots,q_n} = V^{(n)}[\phi] \tag{3.41}$$

that greatly simplify the flow because the p_μ dependence now appears only through a propagation $G_{k;q+p}$ in the first term, while the second term drops it completely. We have

$$\dot{Z}_\phi p^2 + \dot{V}''[\phi] \quad = \quad V^{(3)}[\phi]^2 \int_q G_{k;q+p} G^2_{k;q} \dot{\mathcal{R}}_{k;q} - \frac{V^{(4)}[\phi]}{2} \int_q G^2_{k;q} \dot{\mathcal{R}}_{k;q} \tag{3.42}$$

These two contributions to the flow of the 2-point function are represented in Fig. 3.4 and Fig. 3.5. The right hand side produces terms of order p^4 or greater, that go beyond the chosen truncation. The graph that produces p^2 and higher order contributions is in particular Fig. 3.4, while Fig. 3.5 is p_μ independent. p^4 contributions are simply telling us that we should include four derivative terms, as expected, to further improve our truncation. If we set $p_\mu = 0$ in this equation we read the flow of the second derivative of the potential

$$\dot{V}''[\phi] = V^{(3)}[\phi]^2 \int_q G^3_{k;q} \dot{\mathcal{R}}_{k;q} - \frac{V^{(4)}[\phi]}{2} \int_q G^2_{k;q} \dot{\mathcal{R}}_{k;q} \tag{3.43}$$

and it is easy to see that it coincides with the second derivative, at constant ϕ, of the previously determined flow of the potential. We can subtract it to obtain the flow of the terms with non-zero p_μ

$$\dot{Z}_\phi p^2 = V^{(3)}[\phi]^2 \int_q \left(G_{k;q+p} - G_{k;q}\right) G^2_{k;q} \dot{\mathcal{R}}_{k;q} \tag{3.44}$$

and obviously the tadpole dropped completely due to the lack of derivative interactions. The last formula we wrote is actually a formula for the flow of the entire self-energy $\Sigma\left(p^2\right)$, but we are not going to solve it although this can be done numerically. In that case one would be able to obtain the anomalous dimension from the derivative of $\Sigma\left(p^2\right)$ with respect to p^2.

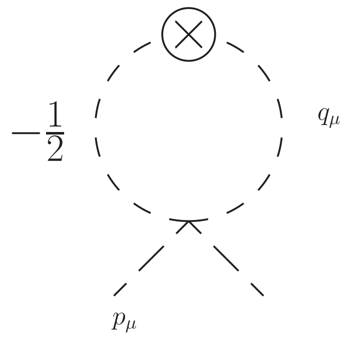

Figure 3.5: Representation of $-\frac{V^{(4)}}{2}\int_q G_{k;q}^2 \dot{\mathcal{R}}_{k;q}$ contribution to the flow of $\dot{\Gamma}_k^{(2)}$.

3.3.1 A closed formula for the anomalous dimension.

The general computation of the q_μ integral is a very complicated task. This is easily seen by introducing the optimized cutoff. The function that is integrated in that case contains products like

$$\theta\left(k^2 - q^2\right)\theta\left(k^2 - (q+p)^2\right) \tag{3.45}$$

that are hard to manage, although still giving a support to the integration. For example, polar q_μ coordinates cannot be used. The way out is to expand $G_{k;q+p}$ as a function of p_μ and just look at the p^2 term. The expansion is

$$G_{k;q+p} = G_{k;q} + p_\mu \frac{\partial}{\partial q_\mu} G_{k;q} + \frac{1}{2} p_\mu p_\nu \frac{\partial^2}{\partial q_\mu \partial q_\nu} G_{k;q} + \ldots \tag{3.46}$$

In many of the applications of this thesis we will be interested only in p^2 terms, so only terms that do not have a tensor structure like $p_\mu p_\nu$, but rather p^2 only. Also we will usually drop any p^4 term. Therefore we can often use the modified expansion in which any product

$$p_\mu p_\nu \to \frac{1}{4}\delta_{\mu\nu} p^2 \tag{3.47}$$

or its generalization to d dimensions. This simplifies the expansion of the propagator

$$G_{k;q+p} \to G_{k;q} + p_\mu \frac{\partial}{\partial q_\mu} G_{k;q} + \frac{1}{8} p^2 \frac{\partial^2}{\partial q_\mu \partial q_\mu} G_{k;q} + \ldots \tag{3.48}$$

A this point one could immediately start the computation of the derivatives of the modified propagator, as we shall do generally in gravitational applications. In the simple LPA scalar case we can actually go a little beyond and give a closed formula. First one notices that $G_{k;q}$ is naturally a function of q^2 and this is something that will not generalize to spinors. If we indicate by primes the derivatives with respect to q^2 it is easy to show that, in four dimensions,

$$\begin{aligned}\frac{1}{8}\frac{\partial^2}{\partial q_\mu \partial q^\mu} G_{k;q} &= \frac{1}{8}\frac{\partial q^2}{\partial q_\mu}\frac{\partial}{\partial q^2}\left(\frac{\partial q^2}{\partial q^\mu}\frac{\partial G_{k;q}}{\partial q^2}\right) \\ &= \frac{1}{4}\frac{\partial q_\mu}{\partial q^\mu} G'_{k;q} + \frac{4 q_\mu q^\mu}{8} G''_{k;q} \\ &= G'_{k;q} + \frac{1}{2} q^2 G''_{k;q} \end{aligned} \tag{3.49}$$

and a general d-dimensional formula is easily derived. If we plug this result in the flow of Z_ϕ we easily derive

$$\dot{Z}_\phi = V^{(3)}[\phi] \int_q \left(G'_{k;q} + \frac{1}{2} q^2 G''_{k;q}\right) G^2_{k;q} \dot{\mathcal{R}}_{k;q} \tag{3.50}$$

An alternative way to obtain this result is to take a p^2 derivative of the flow of $\dot{Z}_\phi p^2$ and then set $p_\mu = 0$ to drop higher order terms. This derivative cannot commute with the q_μ integration, so we need to replace it by

$$\frac{\partial}{\partial p^2} \to \frac{1}{8}\frac{\partial^2}{\partial p_\mu \partial p^\mu} \tag{3.51}$$

that has the same behavior on p^2, but not on functions of p^2. Again this is easily generalized to any dimensionality. We obtain

$$\begin{aligned}\dot{Z}_\phi &= \left.\frac{\partial}{\partial p^2}\right|_{p=0} \dot{Z}_\phi p^2 \\ &= \left.\frac{1}{8}\frac{\partial^2}{\partial p_\mu \partial p^\mu}\right|_{p=0} V^{(3)}[\phi]^2 \int_q \left(G_{k;q+p} - G_{k;q}\right) G^2_{k;q} \dot{\mathcal{R}}_{k;q} \\ &= V^{(3)}[\phi]^2 \int_q \left.\frac{1}{8}\frac{\partial^2}{\partial p_\mu \partial p^\mu}\right|_{p=0} G_{k;q+p} G^2_{k;q} \dot{\mathcal{R}}_{k;q} \\ &= V^{(3)}[\phi]^2 \int_q \left(\frac{1}{8}\frac{\partial^2}{\partial q_\mu \partial q^\mu} G_{k;q}\right) G^2_{k;q} \dot{\mathcal{R}}_{k;q} \\ &= V^{(3)}[\phi]^2 \int_q \left(G'_{k;q} + \frac{1}{2} q^2 G''_{k;q}\right) G^2_{k;q} \dot{\mathcal{R}}_{k;q} \end{aligned} \tag{3.52}$$

where we used the ERGE equation of the flow of the two point function. The result is shown to be the same of (3.50). This formula is particularly useful when computing η_ϕ for a smooth cutoff. In the next section we are going to derive the result for the optimized cutoff and in that case it is more transparent to work with the expansion. The reason is that derivatives of theta functions can be handled, but with special care as we shall see.

3.3.2 Optimized cutoff result.

The final step in the computation of the anomalous dimension is to specify the actual form of the cutoff function and perform the trace. Again we use the optimized cutoff shape and normalize it with Z_ϕ

$$\mathcal{R}_{k;q} = Z_\phi R_{k;q} = Z_\phi \left(k^2 - q^2\right) \theta\left(k^2 - q^2\right) \tag{3.53}$$

The momentum space modified propagator therefore is as before a function of a theta function and must be worked out supportwise. It looks like

$$G_{k;q} = \frac{1}{Z_\phi q^2 + V''[\phi]} \theta\left(q^2 - k^2\right) + \frac{1}{Z_\phi k^2 + V''[\phi]} \theta\left(k^2 - q^2\right) \tag{3.54}$$

We mentioned before that in the following we are going to calculate the coefficients of p^2 by expanding the p_μ dependent structures on the right hand side of the ERGE, like $G_{k;q+p}$. Therefore, rather than using the closed formula we simply give the expansion of $G_{k;q+p}$. It is easy to calculate the first term

$$p_\mu \frac{\partial}{\partial q_\mu} G_{k;q} = -\frac{2 Z_\phi \left(q \cdot p\right)}{\left(Z_\phi q^2 + V''[\phi]\right)^2} \theta\left(q^2 - k^2\right) \tag{3.55}$$

Some terms of the form $x\delta(x)$ have been neglected in the assumption of regularity of the rest of the integrand. Before further expanding, it is useful to remember that the term of order p^2 will be integrated together with $\dot{\mathcal{R}}_{k;q}$ that only takes values in the support $q^2 \leq k^2$. Therefore any term proportional to $\theta\left(q^2 - k^2\right)$ in the second order expansion can be neglected, unless we want to expand it further, that is not the case. So, taking another derivative, what matters is when we act on the theta function and all other terms are in the undesired support

$$\frac{1}{2} p_\mu p_\nu \frac{\partial^2}{\partial q_\mu \partial q_\nu} G_{k;q+p} = -\frac{2 Z_\phi \left(q \cdot p\right)^2}{\left(Z_\phi k^2 + V''[\phi]\right)^2} \delta\left(q^2 - k^2\right) + \ldots \tag{3.56}$$

and this is always integrated with functions of q^2. We also used the fact that $f(x)\delta(x) = f(0)\delta(x)$. Inside the integration we therefore can substitute by rotational invariance $q_\mu q_\nu \to q^2/4\delta_{\mu\nu}$ and this is precisely realizing the simplified expansion in p^2

$$\frac{1}{2} p_\mu p_\nu \frac{\partial^2}{\partial q_\mu \partial q_\nu} G_{k;q+p} \to -\frac{Z_\phi k^2 p^2}{2\left(Z_\phi k^2 + V''[\phi]\right)^2} \delta\left(q^2 - k^2\right) \tag{3.57}$$

Now it seems that we have to integrate in the flow a certain power of the support function with a Dirac delta

$$\delta\left(q^2 - k^2\right) \theta\left(k^2 - q^2\right)^m \tag{3.58}$$

and in particular $m = 3$ for the case under study. This would be actually a mistake and we would underestimate the result. In the ERGE for the effective action only two theta functions (or their derivatives) are present, the one from the modified propagator and the one from the cutoff. When expanding it to a flow equation for any correlator, one should always remember that no new theta functions or delta functions are created. In other words, we should write down all the modified propagators together

$$G_{k;q}^2 G_{k;q+p} \tag{3.59}$$

and give their support representation. The set of all propagators has actually only one theta function at time. This tells us that the integral of interest is actually only

$$\int dw \delta\left(w - w_0\right) \theta\left(w - w_0\right) = \frac{1}{2} \tag{3.60}$$

This relation is shown once one realizes that the Dirac delta function that appears in the calculation comes from a derivative of the cutoff theta function, therefore it is sufficient to write it as $delta\left(w-w_0\right)=\partial_w\theta\left(w-w_0\right)$. Further, in presence of a Dirac delta the derivative of the cutoff term simplifies. It is

$$\dot{\mathcal{R}}_{k;q}=Z_\phi\left(2k^2-\eta_\phi\left(k^2-q^2\right)\right)\theta\left(k^2-q^2\right) \tag{3.61}$$

and reduces to $Z_\phi k^2$ when $k^2=q^2$ is forced.

Finally, we can substitute the quadratic expansion and effectively calculate the result.

$$\begin{aligned}
\dot{Z}_\phi &= -\frac{V^{(3)}\left[\phi\right]^2}{\left(2\pi\right)^4}\int d^4q\frac{Z_\phi^2k^4}{\left(Z_\phi k^2+V''\left[\phi\right]\right)^4}\delta\left(q^2-k^2\right)\theta\left(k^2-q^2\right)\\
&= -\frac{\mathrm{Vol}\left(S^3\right)V^{(3)}\left[\phi\right]^2}{\left(2\pi\right)^4}\int dqq^3\frac{Z_\phi^2k^4}{\left(Z_\phi k^2+V''\left[\phi\right]\right)^4}\delta\left(q^2-k^2\right)\theta\left(k^2-q^2\right)\\
&= -\frac{\mathrm{Vol}\left(S^3\right)V^{(3)}\left[\phi\right]^2}{\left(2\pi\right)^4}\int dzz\frac{Z_\phi^2k^4}{2\left(Z_\phi k^2+V''\left[\phi\right]\right)^4}\delta\left(z-k^2\right)\theta\left(z-q^2\right)\\
&= -\frac{\mathrm{Vol}\left(S^3\right)V^{(3)}\left[\phi\right]^2}{\left(2\pi\right)^4}\frac{Z_\phi^2k^6}{4\left(Z_\phi k^2+V''\left[\phi\right]\right)^4}\\
&= -\frac{V^{(3)}\left[\phi\right]^2}{32\pi^2}\frac{Z_\phi^2k^6}{\left(Z_\phi k^2+V''\left[\phi\right]\right)^4}
\end{aligned} \tag{3.62}$$

We ended up with a final result that, as expected by the scaling arguments, can be homogenized by switching to dimensionless renormalized variables. Using $\bar{\phi}_R$ and the corresponding potential we can write

$$\eta_\phi=\frac{\bar{v}_R^{(3)}\left[\bar{\phi}_R\right]^2}{32\pi^2\left(1+\bar{v}_R''\left[\bar{\phi}_R\right]\right)^4} \tag{3.63}$$

that is the main result of this section. As we already said the ϕ dependence of the right hand side is not a genuine dependence but rather a scheme dependence. Due to the definition of anomalous dimension we adopted, it is always necessary to specify the field configuration one uses to calculate it. In perturbative settings and with a symmetric potential one usually works with the ground state $\phi=0$ and has $\bar{v}_R^{(3)}\left[0\right]=0$. In this case $\eta_\phi=0$. From the Feynman-diagrammatic point of view this is easily seen because in a ϕ^4 theory there are no 1-loop graphs contributing to the 2-point correlator.

As we did for the flow of the potential, we end the section giving the general d-dimensional result for this calculation of the anomalous dimension. It is obtained slightly generalizing the steps we performed and it should be evident form the text where the generalizations happen. We have for general R^d spacetime

$$\eta_\phi=\frac{2^{1-d}\pi^{-d/2}\bar{v}_R^{(3)}\left[\bar{\phi}_R\right]^2}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{v}_R''\left[\bar{\phi}_R\right]\right)^4} \tag{3.64}$$

3.4 The inclusion of spinors.

Working with the simple LPA approximation for the scalar field we saw that there is no much difference between working directly with the ERGE and using the Feynman rules in momentum space to draw diagrams. In particular we were able to write close formulas for the anomalous dimension with arbitrary cutoff. In a general situation this is possibly not true. If many fields with different tensor structures are propagating, but only few limited kinds of interactions are allowed by the truncations, it is often convenient to work with the diagrammatic technique in momentum space. It is a matter of fact that this procedures eliminates from the beginning a lot of terms that will not contribute to the final result.

In this section we want to add some Dirac spinor ψ degree of freedom. These degrees are going to couple to the scalar field with Yukawa-like interactions like the ones introduced at the end of the second chapter. Like for the simple scalar case we have to take a reference configuration, to perform the calculations. This configuration will also play the role of the background. Given our truncation ansatz for the spinor action is quadratic in ψ and $\bar{\psi}$, only limited vertices are possible if we take as reference $\phi = \text{const.}$ and $\psi = 0$. In particular the choice $\psi = 0$ will prohibit any interaction possessing an odd number of spinor lines by simple Lorentz invariance. Further, in the action we will have operators with at most two copies of the spinor field, so actually only diagrams whose vertices can be written with interaction with no more and no less than two spinors are possible. These reasoning enormously simplify the very big number of diagrams one has to calculate.

We shall start by introducing the coupled scalar and spinor action, that still has no derivative interactions. Again we work in the specialized case of four dimensions and eventually we will give the general d-dimensional result. The action is

$$\Gamma_k\left[\phi, \bar{\psi}, \psi\right] = \int d^4x \left(\frac{Z_\phi}{2}\partial_\mu\phi\partial_\mu\phi + Z_\psi\bar{\psi}i\not{D}\psi + H\left[\phi\right]\bar{\psi}\psi + V\left[\phi\right]\right) \tag{3.65}$$

We are as always implicitly working in euclidean space. In the case of Dirac spinor it is possible to show that $H\left[\phi\right]$ is a purely imaginary function. Therefore, if we decide to expand it

$$H\left[\phi\right] = im + iy\phi + \dots \tag{3.66}$$

the first two coefficients are the mass and the usual Yukawa coupling times the imaginary unit. We take this into account when renormalizing. We define the renormalized dimensionless partner of the function $H\left[\phi\right]$ as

$$i\bar{h}_R\left[\bar{\phi}_R\right] = k^{-1}Z_\psi^{-1}H\left[kZ_\phi^{-\frac{1}{2}}\bar{\phi}_R\right] \tag{3.67}$$

This can be see easily by introducing the dimensionless renormalized partner of ψ and $\bar{\psi}$ fields, which we avoid in order not to generate confusion with the bar notation. Obviously $\bar{h}_R\left[\bar{\phi}_R\right]$ is a real quantity.

The canonical scaling terms of $\bar{h}_R\left[\bar{\phi}_R\right]$ will depend both on the anomalous dimension of the scalar η_ϕ, but also on the anomalous dimension of the spinor field, that we define similarly

$$\eta_\psi = -\frac{\dot{Z}_\psi}{Z_\psi} \tag{3.68}$$

Exactly with the same technique we worked out for the renormalized dimensionless potential, we can easily calculate

$$\dot{\bar{h}}_R\left[\bar{\phi}_R\right] = (-1+\eta_\psi)\bar{h}_R\left[\bar{\phi}_R\right] + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{h}'_R\left[\bar{\phi}_R\right] + k^{-1}Z_\psi^{-1}\dot{H}\left[\phi\right] \quad (3.69)$$

If one desires now to know the flow of the coupled system of $\bar{h}_R\left[\bar{\phi}_R\right]$ and $\bar{v}_R\left[\bar{\phi}_R\right]$, four things must be calculated. We need to obtain the undetermined term $k^{-1}Z_\psi^{-1}\dot{H}\left[\phi\right]$ and η_ψ from ERGE, but we also need to include the new spinor effects in the previously calculated $k^{-4}V\left[\phi\right]$ and η_ϕ.

3.4.1 Spinor contribution to the effective potential.

As we mentioned before, we are going to take as reference field configuration $\phi = \text{const.}$ and $\psi = \bar{\psi} = 0$. This limit is technically very useful because the propagation of scalar and spinor degrees of freedom gets decoupled in $\Gamma_k^{(2)}\left[\phi, 0, 0\right]$. It is also quite natural to take such kind of reference field, for the simple reason that only scalar fields are expected to get a nonzero vacuum expectation value. Though, we may expect that some condensation happens for the spinor degrees of freedom. In that case it is sufficient to think about ψ as the field "out of the condensate" and correctly add to the scalar potential a contribution from the spinor condensation in the form $V\left[\phi\right] \to V\left[\phi\right] + H\left[\phi\right]\left\langle\bar{\psi}\psi\right\rangle$. After this discussion it should be evident that, in the chosen configuration, we can freely accomodate any realistic physical situation we may have in mind.

For the purpose of this subsection it is now sufficient to choose the cutoff for the spinor degrees of freedom, in the chosen limit. Let us have a first look at the kernel of the spinor kinetic term in momentum space $\not{q}$. It satisfy the property $\not{q}^2 = q^2\mathbf{1}$, where q^2 is the momentum space transform of the kernel of the scalar kinetic term. We want to maintain a similar property when these kernels are corrected by the infrared cutoff. First we define the IR cutoff term for the spinor degrees to be in momentum space

$$\Delta S_k^\psi = \int d^4x\bar{\psi}_{-q}\mathcal{R}_k^\psi\left(q\right)\psi_q = \int d^4x Z_\psi\bar{\psi}_{-q}R_k^\psi\left(q\right)\psi_q \quad (3.70)$$

The kernel $R_k^\psi\left(q\right)$ has to be a function of q_μ rather than q^2 to correctly cut off modes. It also possesses indices in spinor space, so it is generally an element of the Clifford algebra. It is useful to factor out its Clifford algebra nature by parametrizing

$$R_k^\psi\left(q\right) = \not{q}r_k^\psi\left(q^2\right) = \not{q}r_{k;q}^\psi \quad (3.71)$$

which also introduces a new function that is q^2 dependent. The modified inverse propagator therefore looks like

$$\not{q}\left(1 + r_{k;q}^\psi\right) \quad (3.72)$$

and in analogy to $\not{q}^2 = q^2\mathbf{1}$ we require

$$\left(\not{q}\left(1 + r_{k;q}^\psi\right)\right)^2 = q^2 + R_{k;q} \quad (3.73)$$

This relation has obviously to be true supportwise. It is easy to show that the correct choice for the factored function of the spinor cutoff is, for the optimized cutoff choice,

$$r^{\psi}_{k;q} = \left(\sqrt{\frac{k^2}{q^2}} - 1\right)\theta\left(k^2 - q^2\right) \tag{3.74}$$

that is only slightly more complicated than the scalar one.

We now define the modified propagator for the spinors as the second variation with respect to spinor fields in the chosen configuration

$$G^{\psi}_{k,q} = \left(Z_\psi \not{q} + H\left[\phi\right] + Z_\psi R^{\psi}_{k}\left(q\right)\right)^{-1} \tag{3.75}$$

Here we face two problems simultaneously. We have to take the inverse of a function that contains theta functions, but also lives in the Clifford algebra. It is convenient to first face the problem of inverting a function of $\not{q}$ and then work out the result in terms of the theta functions

$$\begin{aligned} G^{\psi}_{k,q} &= \frac{Z_\psi \not{q}\left(1 + r^{\psi}_{k,q}\right) - H\left[\phi\right]}{Z^2_\psi\left(q^2 + R_{k;q}\right) - H\left[\phi\right]^2} \\ &= \frac{Z_\psi \not{q} - H\left[\phi\right]}{Z^2_\psi q^2 - H\left[\phi\right]^2}\theta\left(q^2 - k^2\right) + \frac{Z_\psi \not{q}\sqrt{\frac{k^2}{q^2}} - H\left[\phi\right]}{Z^2_\psi k^2 - H\left[\phi\right]^2}\theta\left(k^2 - q^2\right) \end{aligned} \tag{3.76}$$

One may be worried by the possible pole, but we have to remember that $H\left[\phi\right]$ is a purely imaginary function. Before starting the first ERGE computation with spinors we still need as a last ingredient the t-derivative of the cutoff. It is easily calculated

$$\dot{\mathcal{R}}^{\psi}_{k;q} = Z_\psi \not{q}\left(\sqrt{\frac{k^2}{q^2}} - \eta_\psi\left(\sqrt{\frac{k^2}{q^2}} - 1\right)\right)\theta\left(k^2 - q^2\right) \tag{3.77}$$

Our final task is to calculate the fermion loop contribution to the running of the scalar effective potential. We therefore have to calculate

$$-\mathrm{tr}\int_q G^{\psi}_{k;q}\dot{\mathcal{R}}_{k;q} \tag{3.78}$$

The minus sign is due to the anticommuting nature of the spinor field and there is no factor 1/2 because there are actually two diagrams contributing with opposite charge flow. It is represented in Fig. 3.6. It is sufficient to plug in the explicit forms of modified propagator and derivative of the cutoff to get the result. We write the new contribution to the running of the potential directly in terms of renormalized quantities

$$\Delta\dot{\bar{v}}_R\left[\bar{\phi}_R\right] = -\frac{\left(1 - \frac{\eta_\psi}{5}\right)}{8\pi^2\left(1 + \bar{h}_R\left[\bar{\phi}_R\right]^2\right)} \tag{3.79}$$

As usual we also give its generalization in general d spacetime dimensions. We actually assume the spacetime to be even dimensional (so that the Dirac spinor

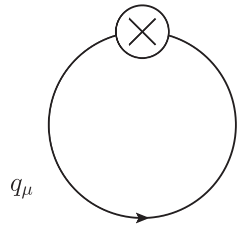

Figure 3.6: Spinor loop $\mathrm{tr}\int_q G^{\psi}_{k;q}\dot{\mathcal{R}}_{k;q}$ contribution to the running of $V[\phi]$. The arrow denotes a charge flux.

bundle is $2^{d/2}$-dimensional) and that there is a number N_f of spinors coupled to the scalar in a $SU(N_f)$ symmetric way. We have

$$\Delta\dot{\bar{v}}_R\left[\bar{\phi}_R\right] = -\frac{2^{1-\frac{d}{2}}\pi^{-\frac{d}{2}}N_f(1-\frac{\eta_\psi}{d+1})}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{h}_R\left[\bar{\phi}_R\right]^2\right)} \tag{3.80}$$

If we collect this contribution together with the running of the potential for the self-interacting scalar field, we obtain in $d=4$ the running of the potential in the coupled system

$$\begin{aligned}\dot{\bar{v}}_R\left[\bar{\phi}_R\right] &= -4\bar{v}_R\left[\bar{\phi}_R\right] + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{v}'_R\left[\bar{\phi}_R\right] \\ &\quad + \frac{1}{32\pi^2}\frac{1-\frac{\eta_\phi}{6}}{1+\bar{v}''_R\left[\bar{\phi}_R\right]} - \frac{\left(1-\frac{\eta_\psi}{5}\right)}{8\pi^2\left(1+\bar{h}_R\left[\bar{\phi}_R\right]^2\right)}\end{aligned} \tag{3.81}$$

3.4.2 Spinor contribution to η_ϕ.

The next task is to compute the spinor loop contribution to the scalar anomalous dimension. As we did for the simple scalar model we have to compute the p^2 term in the flow of the 2-point function of the scalar with incoming momentum p_μ. In the limit of constant ϕ and zero ψ there are four graphs contributing to the flow of the correlator of interest. Two of them involve the scalar loop, one has the 3-vertices and we already computed its contribution, while the other has the 4-vertex and gives no contribution because the vertex is non-derivative. The other two vertices are those new, due to the presence of the spinor. These are essentially of the same form of the previous two, with the difference that the field in the loop is the spinor. There cannot be mixed scalar-spinor propagation in the loop and the cutoff is necessarily on spinor propagation. This means that the two graphs have a global sign minus in the trace. Further, the tadpole-like graph gives no contribution to the anomalous dimension because its vertex is non-derivative. We end up with only a graph to be calculated

$$-2\mathrm{tr}\int_q G^{\psi}_{k;q+p}H'\left[\phi\right]G^{\psi}_{k;q}\dot{\mathcal{R}}_{k;q}G^{\psi}_{k;q}H'\left[\phi\right] \tag{3.82}$$

where the factor 2 takes into account different charge propagation. This contribution is represented in Fig. 3.7.

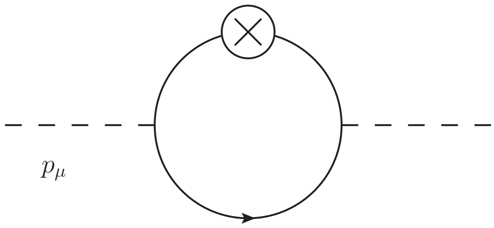

Figure 3.7: Graph representation of $\mathrm{tr} \int_q G^{\psi}_{k;q+p} H' [\phi] \, G^{\psi}_{k;q} \dot{\mathcal{R}}_{k;q} G^{\psi}_{k;q} H' [\phi]$

To proceed further we need the second order expansion in p_μ of $G^{\psi}_{k;q+p}$. As we mentioned before it is not necessary to retain the entire tensor structure of the expansion, that generally is $p_\mu p_\nu$, but rather we can already perform the substitution $p_\mu p_\nu \to p^2/4\delta_{\mu\nu}$. Exactly like what we saw for the simple scalar model this simplification is always realized by rotational symmetry when integrating in q_μ. We shall force it before because the expansion of $G^{\psi}_{k;q+p}$ is strongly simplified. A straightforward calculation yields

$$G^{\psi}_{k;q+p} \quad \to \quad G^{\psi}_{k;q} + p_\mu \frac{\partial}{\partial q_\mu} G^{\psi}_{k;q} + \frac{p^2}{8} \frac{\partial^2}{\partial q_\mu \partial q_\mu} G^{\psi}_{k;q+p} \tag{3.83}$$

The terms of interest are, when restricted to the support $\theta\left(k^2 - q^2\right)$,

$$p_\mu \frac{\partial}{\partial q_\mu} G^{\psi}_{k;q} = \left(\not{p} - \not{q} \frac{p \cdot q}{q^2} \right) \frac{Z_\psi \sqrt{\frac{k^2}{q^2}}}{Z_\psi^2 k^2 - H[\phi]^2} \theta\left(k^2 - q^2\right) + \dots \tag{3.84}$$

and

$$\begin{aligned}
\frac{p^2}{8} \frac{\partial^2}{\partial q_\mu \partial q_\mu} G^{\psi}_{k;q+p} \quad = \quad & \left(\not{q} - \frac{2 Z_\psi k^2 \left(Z_\psi \not{q} - H[\phi] \right)}{Z_\psi^2 k^2 - H[\phi]^2} \right) \times \\
& \times \frac{Z_\psi}{Z_\psi^2 k^2 - H[\phi]^2} \frac{p^2}{4} \delta\left(k^2 - q^2\right) \\
& - \frac{3}{8} \frac{\not{q}}{q^2} \frac{Z_\psi \sqrt{\frac{k^2}{q^2}}}{Z_\psi^2 k^2 - H[\phi]^2} p^2 \theta\left(k^2 - q^2\right) + \dots
\end{aligned} \tag{3.85}$$

Actually, only the second order expansion is needed for the computation. We see it is a quite complicated function, especially if compared to the second order expansion of its scalar counterpart. The reason is that the spinor propagator requires a structural $\not{q}$ in front because it belongs to the Clifford algebra. Consequently one is forced to introduce the structure $\sqrt{k^2/q^2}$ that practically regularize $\not{q}$. This procedure could not be achieved directly with a function of k^2 only, because k is just an energy scale and has no vector nature like q_μ. Cutting modes in a spinor propagation requires a vector. The net effect is that the first order expansion is much more complicated than the scalar case and in particular there is a dependence on q_μ also outside the support functions. The main property of this second order expansion is that both the support functions and the Dirac delta distribution are appearing. We therefore expect, when tracing in q_μ, terms with the integration restricted to the support and terms "concentrated" in $k^2 = q^2$ due to the Dirac delta. We had to manage both in previous examples, so we shall not spend more time in explaining them. We will instead give the result for the integration of the second order p^2 directly in terms of renormalized quantities. The new contribution to η_ϕ is

$$\Delta\eta_\phi = \frac{\left(1 - \frac{\eta_\psi}{4} - \frac{3}{2}\bar{h}_R\left[\bar{\phi}_R\right]^2 - \frac{1}{2}\left(1 - \frac{\eta_\psi}{2}\right)\bar{h}_R\left[\bar{\phi}_R\right]^4\right)\bar{h}'_R\left[\bar{\phi}_R\right]^2}{4\pi^2\left(1 + \bar{h}_R\left[\bar{\phi}_R\right]^2\right)^4} \tag{3.86}$$

We also give the general even dimensional spacetime result for N_f fermions

$$\Delta\eta_\phi = -\frac{N_f\left(4 - 3d + 2\eta_\psi + 6\left(d-2\right)\bar{h}_R\left[\bar{\phi}_R\right]^2 + \left(d - 2\eta_\psi\right)\bar{h}_R\left[\bar{\phi}_R\right]^4\right)\bar{h}'_R\left[\bar{\phi}_R\right]^2}{(2\pi)^{d/2}d(d-2)\,\Gamma\left(\frac{d}{2}\right)\left(1 + \bar{h}_R\left[\bar{\phi}_R\right]^2\right)^4} \tag{3.87}$$

As for the potential we collect the complete $d = 4$ result for the anomalous dimension η_ϕ in the scalar-spinor coupled system. We obtain

$$\begin{aligned}\eta_\phi &= \frac{\bar{v}_R^{(3)}\left[\bar{\phi}_R\right]^2}{32\pi^2\left(1 + \bar{v}''_R\left[\bar{\phi}_R\right]\right)^4} \\ &\quad + \frac{\left(1 - \frac{\eta_\psi}{4} - \frac{3}{2}\bar{h}_R\left[\bar{\phi}_R\right]^2 - \frac{1}{2}\left(1 - \frac{\eta_\psi}{2}\right)\bar{h}_R\left[\bar{\phi}_R\right]^4\right)\bar{h}'_R\left[\bar{\phi}_R\right]^2}{4\pi^2\left(1 + \bar{h}_R\left[\bar{\phi}_R\right]^2\right)^4}\end{aligned} \tag{3.88}$$

Some final remark is in order. Again η_ϕ does not depend on ϕ, because it actually depends on the configuration. Therefore any ϕ appearing on the right hand side of $\Delta\eta_\phi$ is set by some physical requirement like being the ground state of the potential. One may also notice that the total η_ϕ depends on η_ψ but not on η_ϕ itself. We will later see that η_ψ will depend on η_ϕ only, that means that the actual values for the anomalous dimensions will rather appear diagonalizing their associated subsystem. This will indeed introduce some further nonlinearity in their actual functional value. This simple pattern will ultimately be broken in presence of gravity.

3.4.3 Flow for the function h.

In the previous two subsections we computed the spinor loop contributions to the anomalous scalar dimension and the flow of the potential. We still need to calculate the spinor anomalous dimension and the flow of the function $H\left[\phi\right]$. These two quantities arise naturally from the spinor 2-point function. In the limit of the field configuration $\phi = \text{const.}$ and $\psi = \bar{\psi} = 0$, its flow is

$$\left.\frac{\delta^2\dot{\Gamma}_k\left[\phi,\psi,\bar{\psi}\right]}{\delta\psi_p\delta\bar{\psi}_{-p}}\right|_{\psi=\bar{\psi}=0,\phi} = \dot{Z}_\psi \not{p} + \dot{H}\left[\phi\right] \tag{3.89}$$

We are interested now in the $p_\mu = 0$ limit of this 2-point function. The external legs are spinorial, no four fermion interactions are allowed by the truncation and the reference configuration is $\psi = \bar{\psi} = 0$. These three ingredient

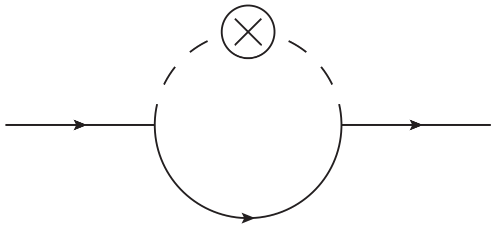

Figure 3.8: Contribution corresponding to $\langle 1\rangle$ in formula (3.90).

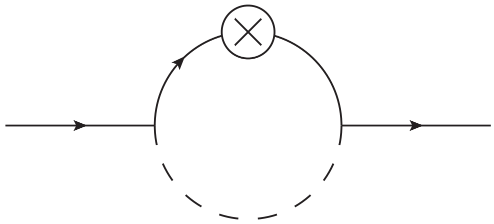

Figure 3.9: Contribution corresponding to $\langle 2\rangle$ in formula (3.91).

imply that in the loop of the tadpole-like graph there must be a scalar running, while in the graphs with 3-vertices there must be one and only one fermion line. Concerning this last kind of graph, we can have that the cutoff is on the scalar propagating or on the spinor line. Therefore there are three possible graphs contributing to $p_\mu = 0$ and are

$$\langle 1\rangle = \int_q G^{\psi}_{k;q+p} H'[\phi]\, G_{k;q} \dot{\mathcal{R}}_{k;q} G_{k;q} H'[\phi] \tag{3.90}$$

$$\langle 2\rangle = -\int_q G_{k;q+p} H'[\phi]\, G^{\psi}_{k;q} \dot{\mathcal{R}}^{\psi}_{k;q} G^{\psi}_{k;q} H'[\phi] \tag{3.91}$$

$$\langle 3\rangle = -\frac{1}{2}\int_q H''[\phi]\, G_{k;q} \dot{\mathcal{R}}_{k;q} G_{k;q} \tag{3.92}$$

We left the p_μ dependence although we need them at $p_\mu = 0$ now, because in the next subsection we will use their expansions.

At order $p_\mu = 0$ their contributions to the running of $H[\phi]$ written in terms of dimensionless renormalized quantities are

$$\Delta_1 \dot{\bar{h}}_R[\bar{\phi}_R] = \frac{\left(1-\frac{\eta_\phi}{6}\right) \bar{h}_R[\bar{\phi}_R]\, \bar{h}'_R[\bar{\phi}_R]^2}{16\pi^2 \left(1+\bar{h}_R[\bar{\phi}_R]^2\right)\left(1+\bar{v}''_R[\bar{\phi}_R]\right)^2} \tag{3.93}$$

$$\Delta_2 \dot{\bar{h}}_R[\bar{\phi}_R] = \frac{\left(1-\frac{\eta_\psi}{5}\right) \bar{h}_R[\bar{\phi}_R]\, \bar{h}'_R[\bar{\phi}_R]^2}{16\pi^2 \left(1+\bar{h}_R[\bar{\phi}_R]^2\right)^2\left(1+\bar{v}''_R[\bar{\phi}_R]\right)} \tag{3.94}$$

$$\Delta_3 \dot{\bar{h}}_R[\bar{\phi}_R] = -\frac{\left(1-\frac{\eta_\phi}{6}\right) \bar{h}''_R[\bar{\phi}_R]}{32\pi^2 \left(1+\bar{v}''_R[\bar{\phi}_R]\right)^2} \tag{3.95}$$

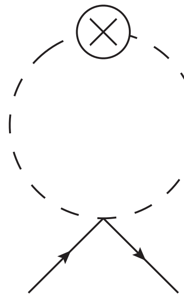

Figure 3.10: Contribution corresponding to $\langle 3\rangle$ in formula (3.92).

while their general even dimensional d counterparts are

$$\Delta_{1,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right] = \frac{2^{2-d}\pi^{-d/2}\left(1-\frac{\eta_\phi}{d+2}\right)\bar{h}_R\left[\bar{\phi}_R\right]\bar{h}'_R\left[\bar{\phi}_R\right]^2}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{h}_R\left[\bar{\phi}_R\right]\right)\left(1+\bar{v}''_R\left[\bar{\phi}_R\right]\right)^2} \tag{3.96}$$

$$\Delta_{2,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right] = \frac{2^{2-d}\pi^{-d/2}\left(1-\frac{\eta_\psi}{d+1}\right)\bar{h}_R\left[\bar{\phi}_R\right]\bar{h}'_R\left[\bar{\phi}_R\right]^2}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{h}_R\left[\bar{\phi}_R\right]^2\right)^2\left(1+\bar{v}''_R\left[\bar{\phi}_R\right]\right)} \tag{3.97}$$

$$\Delta_{3,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right] = -\frac{2^{1-d}\pi^{-d/2}\left(1-\frac{\eta_\phi}{d+2}\right)\bar{h}''_R\left[\bar{\phi}_R\right]}{d\,\Gamma\left(\frac{d}{2}\right)\left(1+\bar{v}''_R\left[\bar{\phi}_R\right]\right)^2} \tag{3.98}$$

Finally, the total contribution corresponds to the running of the function $\bar{h}_R\left[\bar{\phi}_R\right]$ itself. We write it adding the canonical scaling terms in four dimensions

$$\begin{aligned}\dot{\bar{h}}_R\left[\bar{\phi}_R\right] &= (-1+\eta_\psi)\bar{h}_R\left[\bar{\phi}_R\right]+\left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{h}'_R\left[\bar{\phi}_R\right]\\ &\quad+\Delta_1\dot{\bar{h}}_R\left[\bar{\phi}_R\right]+\Delta_2\dot{\bar{h}}_R\left[\bar{\phi}_R\right]+\Delta_3\dot{\bar{h}}_R\left[\bar{\phi}_R\right]\end{aligned} \tag{3.99}$$

and in general dimensionality

$$\begin{aligned}\dot{\bar{h}}_R\left[\bar{\phi}_R\right] &= (-1+\eta_\psi)\,\bar{h}_R\left[\bar{\phi}_R\right]+\frac{1}{2}\left(d-2+\eta_\phi\right)\bar{\phi}_R\bar{h}'_R\left[\bar{\phi}_R\right]\\ &\quad+\Delta_{1,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right]+\Delta_{2,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right]+\Delta_{3,d}\dot{\bar{h}}_R\left[\bar{\phi}_R\right]\end{aligned} \tag{3.100}$$

3.4.4 Spinor anomalous dimension.

In this subsection we shall refer to the contributions $\langle 1\rangle$, $\langle 2\rangle$ and $\langle 3\rangle$, as defined in the previous one. The graphs corresponding to them are in Fig. (3.8), (3.9) and (3.10) respectively. We want to calculate the spinor anomalous dimension, so we want to isolate the terms that are proportional to $\not{p}$ out of them. To achieve this task it is sufficient to use the first order expansion in p_μ of $G_{k;q+p}$ and $G^\psi_{k;q+p}$. First we notice that $\langle 3\rangle$ (Fig. 3.10) is not going to contribute for the simple reason that it is p_μ independent because the vertex is nonderivative. Second, the first order expansion of $G_{k;q+p}$ is nonzero, but out of the support $\theta\left(k^2-q^2\right)$ forced by $\dot{\mathcal{R}}^\psi_{k;q}$ and thus $\langle 2\rangle$ gives no contribution to $\dot{Z}_\psi$ (Fig. 3.9).

This means that there is actually just one graph contributing when using the optimized cutoff and it is $\langle 1 \rangle$ of Fig. 3.8. The result is, in four dimensions,

$$\eta_\psi = \frac{\left(1 - \frac{\eta_\phi}{5}\right) \bar{h}'_R \left[\bar{\phi}_R\right]^2}{16\pi^2 \left(1 + \bar{h}_R \left[\bar{\phi}_R\right]^2\right) \left(1 + \bar{v}''_R \left[\bar{\phi}_R\right]\right)^2} \tag{3.101}$$

and in general d dimensions

$$\eta_\psi = \frac{2^{2-d}\pi^{-d/2} \left(1 - \frac{\eta_\phi}{d+1}\right) \bar{h}'_R \left[\bar{\phi}_R\right]^2}{d\,\Gamma\left(\frac{d}{2}\right) \left(1 + \bar{h}_R \left[\bar{\phi}_R\right]^2\right) \left(1 + \bar{v}''_R \left[\bar{\phi}_R\right]\right)^2} \tag{3.102}$$

The pattern of dependence of anomalous dimensions we anticipated is now evident, because η_ψ is a function of η_ϕ and not of itself. The subsystem of anomalous dimensions can be algebrically solved easily, although it is not our purpose now. These form a system like

$$\begin{aligned} \eta_\phi &= \eta_\phi\left(\eta_\psi\right) \\ \eta_\psi &= \eta_\psi\left(\eta_\phi\right) \end{aligned} \tag{3.103}$$

In the LPA approximation the anomalous dimensions are small and this is telling us that our approximation is reliable. This also means that we could approximately solve the system by setting

$$\begin{aligned} \eta_\phi &= \eta_\phi\left(0\right) \\ \eta_\psi &= \eta_\psi\left(0\right) \end{aligned} \tag{3.104}$$

We will later go into deeper details about the truncation, but for illustrative purposes we can now set it to comprehend only a true Yukawa interaction $H\left[\phi\right] = iy\phi$ and a ϕ^4 one in $V\left[\phi\right] = \lambda_4\phi^4$. The fields are chosen to be massless so that the thresholds (denominators) disappear. In this case we have in four dimensions

$$\begin{aligned} \eta_\phi &= \frac{y^2}{4\pi^2} \\ \eta_\psi &= \frac{y^2}{16\pi^2} \end{aligned} \tag{3.105}$$

which reproduces the usual 1-loop result that is obtained in the context of perturbation theory. The LPA approximation is evidently good as long as the y coupling is small, although not necessarily too small because the factors $1/4\pi^2$ and $1/16\pi^2$ is relaxing the condition a little bit. The Yukawa coupling is known to show a Landau pole towards the UV in usual perturbative computations, unless gauge couplings are present and sufficiently strong to “save” its flow. This tells us that the Yukawa coupling, in such simple perturbative truncations, is not going to be small forever if integrated towards UV, and the same holds for η_ψ. At the same time, the appearance of a big η_ψ tells us that is probably necessary to go beyond and analyze truncations that do not belong to LPA approximation. The typical model one has in mind in such a scenario contains higher derivatives. In the next sections we shall not analyze these cases. We instead decide to try to violate one generally underestimated assumption that we made in the previous argument, namely that the framework is perturbative.

3.5 Collecting the results in flat space.

In this section, for the sake of clarity, we want to collect all the results involving the flow of the renormalized dimensionless functions $\bar{v}_R$ and $\bar{h}_R$ in flat space. We will also complete this result giving the explicit complete form of the anomalous dimensions η_ϕ and η_ψ. The results given here will be useful when later we will add the gravitational degrees of freedom.

The complete flow of the potential $\bar{v}_R$ comes from formulas (3.23) and (3.79). The flow of $\bar{h}_R$ comes from formulas (3.99), including the canonical scaling and contributions (3.93), (3.94) and (3.95). In four dimensions we obtain

$$\dot{\bar{v}}_R = -4\bar{v}_R + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{v}'_R + \frac{1}{32\pi^2}\frac{1-\frac{\eta_\phi}{6}}{1+\bar{v}''_R} - \frac{\left(1-\frac{\eta_\psi}{5}\right)}{8\pi^2\left(1+\bar{h}_R^2\right)} \tag{3.106}$$

$$\begin{aligned}\dot{\bar{h}}_R &= (-1+\eta_\psi)\bar{h}_R + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{h}'_R + \frac{\left(1-\frac{\eta_\phi}{6}\right)\bar{h}_R\bar{h}'^2_R}{16\pi^2\left(1+\bar{h}_R^2\right)\left(1+\bar{v}''_R\right)^2} \\ &\quad + \frac{\left(1-\frac{\eta_\psi}{5}\right)\bar{h}_R\bar{h}'^2_R}{16\pi^2\left(1+\bar{h}_R^2\right)^2\left(1+\bar{v}''_R\right)} - \frac{\left(1-\frac{\eta_\phi}{6}\right)\bar{h}''_R}{32\pi^2\left(1+\bar{v}''_R\right)^2}\end{aligned} \tag{3.107}$$

Similarly, the anomalous dimension η_ϕ in its complete form has contributions from (3.63) and (3.86). Instead η_ψ has only one contribution (3.101). We have

$$\eta_\phi = \frac{\bar{v}_R^{(3)2}}{32\pi^2\left(1+\bar{v}''_R\right)^4} + \frac{\left(1-\frac{\eta_\psi}{4}-\frac{3}{2}\bar{h}_R^2-\frac{1}{2}\left(1-\frac{\eta_\psi}{2}\right)\bar{h}_R^4\right)\bar{h}'^2_R}{4\pi^2\left(1+\bar{h}_R^2\right)^4} \tag{3.108}$$

$$\eta_\psi = \frac{\left(1-\frac{\eta_\phi}{5}\right)\bar{h}'^2_R}{16\pi^2\left(1+\bar{h}_R^2\right)\left(1+\bar{v}''_R\right)^2} \tag{3.109}$$

We think it is useful to give these flat space results in the simplified truncation in which the generalized Yukawa interaction reduces to a proper Yukawa coupling. We therefore set $\bar{h}_R = \bar{y}_R\bar{\phi}_R$, where $\bar{y}_R$ is the renormalized Yukawa coupling. For simplicity, we also decided to expand in the neighbor of $\bar{\phi}_R = 0$, so that we implicitly assume a symmetric phase of the potential. In the system (3.106) and (3.107) we obtain

$$\dot{\bar{v}}_R = -4v + \left(1+\frac{\eta_\phi}{2}\right)\bar{\phi}_R\bar{v}'_R + \frac{1}{32\pi^2}\frac{1-\frac{\eta_\phi}{6}}{1+\bar{v}''_R} - \frac{\left(1-\frac{\eta_\psi}{5}\right)}{8\pi^2\left(1+\bar{\phi}_R^2\bar{y}_R^2\right)} \tag{3.110}$$

$$\dot{\bar{y}}_R = \left(\frac{\eta_\phi}{2}+\eta_\psi\right)\bar{y}_R + \frac{\bar{y}_R^3(1-\frac{\eta_\phi}{6})}{16\pi^2\left(1+\bar{v}''_R(0)\right)^2} + \frac{\bar{y}_R^3(1-\frac{\eta_\psi}{5})}{16\pi^2\left(1+\bar{v}''_R(0)\right)} \tag{3.111}$$

The anomalous dimensions (3.108) and (3.109) are more complex than the system (3.105), even if we expanded around $\bar{\phi}_R = 0$ (the issue of the expansion is discussed in the next section). They also have an additional N_f dependence.

$$\begin{aligned}\eta_\phi &= \frac{\bar{v}_R^{(3)}[0]^2}{32\pi^2\left(1+\bar{v}''_R[0]\right)^4} + \frac{N_f\bar{y}_R^2(1-\frac{\eta_\psi}{4})}{4\pi^2} \\ &= \frac{N_f\bar{y}_R^2(1-\frac{\eta_\psi}{4})}{4\pi^2}\end{aligned} \tag{3.112}$$

$$\eta_\psi = \frac{\bar{y}_R^2(1-\frac{\eta_\phi}{5})}{16\pi^2\left(1+\bar{v}''_R[0]\right)^2} \tag{3.113}$$

In the second line we used the fact that the potential is symmetric and expanded around $\bar{\phi}_R = 0$.

In order to understand better the meaning of the expression (3.110), (3.111), (3.112) and (3.113) it is useful to compare them with perturbation theory, like we did with (3.105). In particular we can check if our flow for the Yukawa coupling $\bar{y}_R$ reproduces the result of perturbation theory. To this end it is convenient to consider a scale k, which is beyond any possible threshold scale K, $k \gg K$. In particular, the thresholds in our truncation are set by the quantity $V''[0] = K$ having the role of mass, through the denominators of the beta functions. In this limit we can neglect $\bar{v}''_R[0] = V''[0]/k^2 Z_\phi^2 \ll 1$. If we plug (3.112) and (3.113) inside (3.111) we get

$$\dot{\bar{y}}_R = \frac{\bar{y}_R^3}{16\pi^2}\left(2N_f + 1 + 2\right) + \mathcal{O}\left(\bar{y}_R^5\right) \tag{3.114}$$

This formula has a direct perturbative interpretation. In the parenthesis we isolated the contributions coming from the field renormalization of the scalar field $2N_f\bar{y}_R^3/16\pi^2$, from the field renormalization of the spinor field $N_f\bar{y}_R^3/16\pi^2$ and form the vertex renormalization $2\bar{y}_R^3/16\pi^2$. In the case in which there is just one flavor we see that

$$\dot{y} = \frac{5y^3}{16\pi^2} + \mathcal{O}\left(y^5\right) \tag{3.115}$$

we obtain the 1-loop 5/16 positive coefficient of the Yukawa beta function.

3.6 Perturbative vs non-perturbative expansions.

We just mentioned in a simple example, but we actually saw it at many steps. There is a relation between functional renormalization group results and standard perturbative renormalization schemes. Namely, 1-loop ERGE results coincide with 1-loop perturbative ones. We proved it in chapter one, but now we can further specify that we really get standard perturbative results when the configuration field taken as reference is $\phi = 0$. More generally when all fields are taken to zero. This may be simply interpreted. When using a reference configuration for the expansion it is necessary to provide it a physical meaning, because otherwise it is a new external ingredient. It is natural to decide that the constant ϕ corresponds to the ground state of our potential. This means that setting $\phi = 0$ and requiring the symmetry $V[\phi] = V[-\phi]$ precisely indicates that the potential is in a symmetric phase.

In this case it is natural to expand the potential as a power series of ϕ^2 in the form

$$V[\phi] = \sum_{j\geq 0} \lambda_{2j}\phi^{2j} \tag{3.116}$$

The convergence of such an expansion has been studied and shown in [32]. Let us call the renormalized dimensionless partners of the couplings $\bar{\lambda}_{2j,R}$, these are related to the bare dimensionful λ_{2j} in four dimensions by the relation

$$\bar{\lambda}_{2j,R} = k^{2(j-2)} Z_\phi^{-j} \lambda_{2j} \tag{3.117}$$

that is also correctly taking into account the field renormalization. It is easy to see that $\bar{\lambda}_{2j,R}$ are the coefficients of the renormalized dimensionless potential

$$\bar{v}_R\left[\bar{\phi}_R\right] = \sum_{j\geq 0} \bar{\lambda}_{2j,R}\bar{\phi}_R^{2j} \tag{3.118}$$

The same considerations work for all other couplings, for example the coefficients in $\bar{h}_R\left[\bar{\phi}_R\right]$ are the dimensionless partners of the couplings one expects to measure. Having expanded around $\phi = 0$ we naturally expand $\bar{h}_R\left[\bar{\phi}_R\right]$ similarly

$$\bar{h}_R\left[\bar{\phi}_R\right] = \bar{m}_R + \bar{y}_R\bar{\phi}R + \dots \tag{3.119}$$

The beta functions of these couplings are easily obtained using the functional flows $\dot{\bar{v}}_R\left[\bar{\phi}_R\right]$ and $\dot{\bar{h}}_R\left[\bar{\phi}_R\right]$. It is sufficient to expand them, for example taking the appropriate number of derivatives

$$\begin{aligned}
\dot{\bar{\lambda}}_{2j,R} &= \frac{1}{(2j)!}\left.\frac{\partial^{2j}}{\partial\bar{\phi}_R^{2j}}\right|_{\bar{\phi}_R=0} \dot{\bar{v}}_R\left[\bar{\phi}_R\right] && (3.120)\\
\dot{\bar{m}}_R &= \dot{\bar{h}}_R\left[0\right] && (3.121)\\
\dot{\bar{y}}_R &= \left.\frac{\partial}{\partial\bar{\phi}_R}\right|_{\bar{\phi}_R=0} \dot{\bar{h}}_R\left[\bar{\phi}_R\right] && (3.122)\\
& \dots && (3.123)
\end{aligned}$$

while the anomalous dimensions are consistently fixed to the values of the chosen configuration

$$\begin{aligned}
\eta_\phi &= \left.\eta_\phi\right|_{\bar{\phi}_R=0} && (3.124)\\
\eta_\psi &= \left.\eta_\psi\right|_{\bar{\phi}_R=0} && (3.125)
\end{aligned}$$

This last equation is a manifestation of the "$Z = Z\left[\phi\right]$ rule", we introduced when computing the anomalous dimension of the scalar. We stress again that neither the anomalous dimensions, nor the field renormalization, depend on ϕ, but they do depend on the reference $\langle\phi\rangle$ through the rule. Here we are simply setting the configuration interpreted as ground state.

This is a rather standard procedure when dealing with functional renormalization group equation. In principle one could imagine to solve the entire differential equation for the functions without exploiting any particular expansion, with appropriate boundary conditions. This is actually a hard task that can be achieved only numerically, due to the high nonlinearity of the flow. By setting a power expansion and calculating the beta functions we can reduce the complexity of the system simply choosing a finite number of coefficients in the expansion of the two functions. If this is done we may wonder now if we reduced the complexity too much, in the sense that we may be missing some information contained in the flow. In this respect we can have a look at the existing literature. The study of the flow of the potential is a quite well known subject in two, three and four dimensions. It has been studied both in polynomial truncations and numerically, despite not coupled usually to the function $H\left[\phi\right]$.

There is a general failure of polynomial truncation we could worry about, that is, if the flow for the potential has a fixed point which is a function that cannot be expanded. This turns out to be a rather pathological situation, although not impossible. We shall assume this is not the case. In two dimensions it was shown by Morris that the flow of the dimensionless renormalized potential leads to a series of fixed points, that are the known multicritical $2d$ CFT. To be precise, he increased the truncation adding a field dependent wavefunction renormalization and treated the anomalous dimension in a slightly different way than what we do. Apart from that, he was able to calculate with good precision the nature of the properties of the first few CFTs (the critical exponents) and found indications that further theories can be obtained by increasing more the truncation. It is useful to remember that these theories are known analytically, so he was able to compare his results with the exact ones. The example of two dimensions is representative because the truncation Morris chooses goes beyond LPA and such a truncation is necessary to capture the details of the multicritical theories. This might be seen as an indication that we should also improve in that direction, without much hope to find some nontrivial structures. All in all, still two dimensional critical theories are polynomial-like, so at least we may still trust our expansion.

In three dimensions the situation is slightly different. It is well known that there is an IR fixed point in the flow which is called Wilson-Fisher (WF) FP. It has been studied mainly numerically, but also analytically. The best determination of its properties are those obtained from Monte Carlo techniques, while functional RG techniques are known to give these numbers with quite good accuracy, at least in some cases. The existence of WF FP is very important because it overcomes the problem of triviality of a scalar theory in three dimensions. To see this it is sufficient to take the usual λ_4 running, that is

$$\dot{\lambda}_4 \sim \lambda_4^2 \tag{3.126}$$

The theory is known to encounter a singularity in finite RG time, so for a finite $\bar{k}$, called Landau pole. The question is whether we can still take the smooth limit. In three dimensions this is possible. In fact we can push the scale $\bar{k}$ to infinity, by simply sending all other scales to zero. There is actually just another scale in the problem, k, and when k is going to zero we may tune our RG trajectory to hit WF fixed point. Therefore we can take the continuum limit of the theory.

The procedure we just outlined is not possible in four dimensions, because there are no nontrivial analogues of the Wilson-Fisher FP. This forces the theory to flow to the gaussian FP. For this reason the four dimensional scalar theory is said to be trivial. We can rephrase what we just said in terms of asymptotic safety definition. The system $\dot{\lambda}_4 \sim \lambda_4^2$ has just the gaussian (repulsive) solution. Given this FP is repulsive, its critical surface happens to coincide with the fixed point itself. Therefore, given that in the context of functional RG we have to place the initial condition of the flow on the critical surface, we see that asymptotic safety imply triviality of the scalar model (it will always sit in the gaussian fixed point). This means that we can define the UV limit of the four dimensional theory, although it is of no interest because the theory is completely trivial (free).

The examples we gave all have in common that there is no spinor sector and therefore no Yukawa interaction. However the Yukawa coupling y is known to have similar UV behavior to λ_4. Among the problems of standard model the running of y is as dangerous as that of λ_4. We do not really expect that the addition of a Yukawa sector may really solve any issue. In fact the possible existence of nontrivial FP in the symmetric potential expansion has been studied in quite detail by [18, 34, 35]. They were able to show that the LPA symmetric regime does not posses any nontrivial FP. The question is, do we have any hope to find some nontrivial structure in four dimensions without giving up with the LPA?

3.6.1 Expansion around a VEV.

In order to produce some nontrivial result before leaving the LPA, we can still try to relax the condition of symmetry of the potential. Actually, the potential is always asked to be symmetric, but its minimum (ground state) may not be. We parametrize the potential in the form

$$V\left[\phi\right] = \sum_{j\neq 1} \theta_{2j} \left(\phi^2 - \kappa\right)^j \tag{3.127}$$

and correspondingly

$$\bar{v}_R\left[\bar{\phi}_R\right] = \sum_{j\neq 1} \bar{\theta}_{2j,R} \left(\bar{\phi}_R^2 - \bar{\kappa}_R\right)^{2j} \tag{3.128}$$

At first sight this parametrization is not going to give any improvement. It is evident that an expansion up to a certain order in θ_{2j} can always be rewritten in terms of one in λ_{2i}. So we have a linear relation between couplings λ and θ. There is no way a nontrivial FP is generated for the couplings θ if their beta functions are obtained from those of λ system.

We have to remember that we introduced the VEV in the potential because we want to avoid the $\phi = 0$ expansion. Therefore, rather than using the λ system, it is convenient to genuinely calculate $\dot{\bar{\theta}}_{2j,R}$ from the flow of the potential. We have quite naturally

$$\dot{\bar{\theta}}_{2j,R} = \frac{1}{j!}\left.\frac{\partial}{\left(\partial\bar{\phi}_R^2\right)^j}\right|_{\bar{\phi}_R^2=\bar{\kappa}_R} \dot{\bar{v}}_R\left[\bar{\phi}_R\right] \tag{3.129}$$

and the results will be different. The reason is simple. The flow equation does not know about the symmetric or symmetry-breaking phase, so it is sensitive of this different parametrization as long as the way we calculate the beta functions is different.

To physically improve further our calculation, it is sufficient to notice that the limit $\bar{\phi}_R^2 = \bar{\kappa}_R$ has the same role that the limit $\phi = 0$ had in the symmetric expansion. It is therefore natural to regard the configuration $\bar{\phi}_R = \sqrt{\bar{\kappa}_R}$ (or alternatively $\bar{\phi}_R = -\sqrt{\bar{\kappa}_R}$) as the reference configuration of our calculation. This means that the anomalous dimensions are for this system

$$\eta_\phi = \eta_\phi|_{\bar{\phi}_R^2=\bar{\kappa}_R} \tag{3.130}$$

$$\eta_\psi = \eta_\psi|_{\bar{\phi}_R^2=\bar{\kappa}_R} \tag{3.131}$$

and are easily seen to be much more complex than those in $\phi = 0$.

The symmetry breaking potential is known to provide a mass for the spinor field, provided there is a Yukawa coupling. We may truncate the interaction the spinor have with the scalar to the simple Yukawa one

$$\bar{h}_R\left[\bar{\phi}_R\right] = \bar{y}_R\bar{\phi}_R \tag{3.132}$$

and coherently calculate its running always making reference to the nontrivial VEV in the form

$$\dot{\bar{y}} = \left.\frac{\partial}{\partial\bar{\phi}_R}\right|_{\bar{\phi}_R=\sqrt{\bar{\kappa}_R}} \dot{\bar{h}}_R\left[\bar{\phi}_R\right] \tag{3.133}$$

This way to evaluate the running of the Yukawa coupling is a crucial difference compared to [16, 17]. In fact, in [16, 17] an expansion around a nonzero VEV was used for the potential, but not done for the Yukawa coupling. We agree with [18] and believe that for consistency one should expand all the quantities around the VEV.

There is a final ingredient we need. We introduced a VEV for the renormalized dimensionless potential, but we still do not have a prescription on how to calculate its running. We clearly do not want it to be a new external scale. The issue is solved as follows. The VEV is a minimum of the potential that now we regard as a function of $\bar{\phi}_R^2 = \rho$, therefore

$$\partial_\rho \bar{v}_R\left[\bar{\kappa}_R\right] = 0 \tag{3.134}$$

We can take a t derivative on both sides of this equation [18]. This procedure forces the running of $\bar{\kappa}_R$ to stay on a minimum of the flow. We obtain

$$\partial_\rho \dot{\bar{v}}_R\left[\bar{\kappa}_R\right] + \partial_\rho^2 \bar{v}_R\left[\bar{\kappa}_R\right]\dot{\bar{\kappa}}_R = 0 \tag{3.135}$$

This relation is easily solved in terms of the beta function for $\bar{\kappa}_R$. The result is

$$\dot{\bar{\kappa}}_R = -\left.\frac{\partial_\rho \dot{\bar{v}}_R\left[\rho\right]}{\partial_\rho^2 \bar{v}_R\left[\rho\right]}\right|_{\rho=\bar{\kappa}_R} \tag{3.136}$$

This relation was first derived by Wetterich [33].

As an example, we give the running of the VEV coming from (3.23), so for a truncation in which there is just the scalar self-interaction field. We first restore the usual ϕ dependence in (3.136) and obtain a formula which is written in terms of derivatives with respect to ϕ

$$\dot{\bar{\kappa}}_R = -2\sqrt{\bar{\kappa}_R}\,\frac{\dot{\bar{v}}_R'\left[\sqrt{\bar{\kappa}_R}\right]}{\bar{v}_R''\left[\sqrt{\bar{\kappa}_R}\right]} \tag{3.137}$$

Then we use (3.137) and (3.23) to obtain

$$\dot{\bar{\kappa}}_R = -\left(2+\eta_\phi\right)\bar{\kappa}_R + \frac{\left(1-\frac{\eta_\phi}{6}\right)\sqrt{\bar{\kappa}_R}\,\bar{v}_R^{(3)}\left[\sqrt{\bar{\kappa}_R}\right]}{32\pi^2\bar{v}_R''\left[\sqrt{\bar{\kappa}_R}\right]\left(1+\bar{v}_R''\left[\sqrt{\bar{\kappa}_R}\right]\right)^2} \tag{3.138}$$

the running of a non-trivial VEV in the self-interacting scalar model.

In [18] it was shown that the symmetry breaking parametrization, with all the prescription we introduced for the calculation of the anomalous dimensions and the Yukawa coupling running, could indeed lead to a nontrivial UV fixed point for the simple system under study. The only ingredient they had to assume was a general number of N_f spinors coupled in a $SU(N_f)$ symmetric way, they 'extended the results to general real N_f and showed that the nontrivial FP exists provided $N_f \lesssim 0.4 - 0.5$. This value is far from being completely unnatural. It is simply telling us that we should consider different models in which the scalar sector dominates in particle numbers, with respect to the spinor sector. A situation that is likely to happen in any GUT model, in which a very big scalar Higgs multiplet is needed to correctly break the symmetry down to the standard model one. From the point of view of the running of the VEV we may see the situation in this way. The general running of the VEV is of the form

$$\dot{\bar{\kappa}}_R = -2\bar{\kappa}_R + N_s(\dots) - N_f(\dots) \tag{3.139}$$

Between parentheses we are hiding how actually the N_s scalars and N_f spinors are contributing.

In order to give an example of such a relation, we can calculate using (3.137) the running of the non-trivial VEV coming from (3.106)

$$\dot{\bar{\kappa}}_R = -(2+\eta_\phi)\bar{\kappa}_R + \frac{\sqrt{\bar{\kappa}_R}\bar{v}_R'''\left(1-\frac{\eta_\phi}{6}\right)}{16\pi^2\bar{v}_R''\left(1+\bar{v}_R''\right)^2} - \left.\frac{hN_f\sqrt{\bar{\kappa}_R}\bar{h}_R'\left(1-\frac{\eta_\psi}{5}\right)}{2\left(1+\bar{h}_R^2\right)^2\pi^2\bar{v}_R''}\right|_{\bar{\phi}_R=\sqrt{\bar{\kappa}_R}} \tag{3.140}$$

Compared to (3.138), we see that (3.140) has the predicted additional contribution coming from the N_f spinors. The new contribution has, as expected, a negative sign. In the model we used as an example the scalar number of flavors is obviously one.

In general, the situation we have in mind is that of a conformal running for $\bar{\kappa}_R$, in which the two contributions (see always (3.140) for an explicit example) mainly cancel or that of the scalars dominates. It is clear that decreasing N_f should give approximately an idea of what happens when instead N_s is increased. A flow in the SSB regime is typically characterized by a freeze-out of all couplings. This is because all particles coupling to the vacuum expectation value acquire a mass and decouple from the flow. If the conformal running condition is realized $\kappa \sim k^2$ and bosonic degrees of freedom slightly dominate, a $\bar{\kappa}_R^\star \neq 0$ FP for κ is possible. Also, this FP is generally shown to be attractive with critical exponent 2. This signals that the mass of the spinor, rather than being a parameter, has to be determined from the constraint on the flow coming from asymptotic safety. This particular situation, if realized in nature, would allow to further extend the validity of the standard model, provided a compatible top-mass is the outcome of the model. To this end, the model was further extended in more realistic settings and other situations [34].

In [19] it was also extended to include gravity. The reason that makes the inclusion of gravity particularly important is twofold. First, one wishes to apply the nontrivial FP scenario to a Higgs sector of the standard model and therefore hopes that the standard model is asymptotically safe. If it is so, it could be explored to arbitrarily large energies, including those at which gravity is unavoidably strong. Second, the inclusion of gravity means the inclusion of a set of new spin particles. Among the components of the graviton, there are two spin zero fields which can relax the condition $N_f \lesssim 0.5$, or even avoid it completely. The task of the next chapter will be to introduce gravity in the game, calculate the gravitational corrections to the running of the potential, the Yukawa function and anomalous dimensions. Finally we will investigate the possible presence of nontrivial FPs as functions of N_f. For a complete account of the nontrivial features of the Higgs-like sector in flat space we refer [18, 34, 35]. For the investigations of the flow of matter effective actions coupled to gravity we refer to [20].

3.7 The background field method.

Among all the techniques that are applied in conjunction with functional renormalization group, we will extensively use the background field method [36, 37]. The main aim of its application is to let us manage the gauge dependence of the scale dependent effective action. In particular it will allow us to precisely define a gauge invariant functional with built-in scale dependence. As we shall see the application of the background field technique requires some care when working with the ERGE, especially when the action we are considering has a general k-dependence, rather than being the full effective action at $k = 0$.

Before going into the details of the method, it is worth saying that alternatives exist. The issue of obtaining a gauge invariant effective action has been addressed using standard variables and modified Ward-Takahashi identities in [38]. Another alternative is to use gauge invariant (or covariant) variables, that has been explored in [39].

To fix ideas it is useful to work with a specific theory and we decide to analyze gravity. The other possible choice would be a general Yang-Mills theory, but as it will be evident all the discussion of this section can be applied to any gauge theory too. Our basic quantum field is therefore the metric $\gamma_{\mu\nu}$. We adopted the symbol $\gamma_{\mu\nu}$ because we will reserve $g_{\mu\nu}$ for the classical metric, that is the argument of the effective action. A diffeomorphism invariant theory is invariant under the transformation

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + \mathcal{L}_v \gamma_{\mu\nu} = \gamma_{\mu\nu} + \nabla_\mu v_\nu + \nabla_\nu v_\mu \tag{3.141}$$

where v_μ is any vector field.

The essence of the background field method involves a splitting of the metric of the form

$$\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{3.142}$$

The metric $\bar{g}_{\mu\nu}$ is called the background metric, while $h_{\mu\nu}$ is the fluctuation. Our aim is to quantize with the functional technique $h_{\mu\nu}$ leaving the background as a reference metric. It is important to understand that the perturbations are not considered small in any sense and all orders in $h_{\mu\nu}$ have to be taken into account. The reason of the background splitting is twofold. The metric $\bar{g}_{\mu\nu}$ provides us a notion of distance that we will use for the coarse-graining. Also, it will allow us to carry gauge-invariance to quantum level.

The issue of gauge invariance is therefore primary. Having split the metric we have some freedom in translating the gauge transformation of the full metric, in terms of split components. There are two main, opposite in respect, alternatives. The first one is to ask that the background transforms in the same way the metric does and $h_{\mu\nu}$ behaves accordingly

$$\begin{aligned}\delta\bar{g}_{\mu\nu} &= \mathcal{L}_v\bar{g}_{\mu\nu} &(3.143)\\ \delta h_{\mu\nu} &= \mathcal{L}_v h_{\mu\nu} &(3.144)\end{aligned}$$

that is called background gauge transformation. Alternatively one may ask that the background is fixed and the fluctuation retains all the transformation

$$\begin{aligned}\delta\bar{g}_{\mu\nu} &= 0 &(3.145)\\ \delta h_{\mu\nu} &= \mathcal{L}_v\left(\bar{g}_{\mu\nu}+h_{\mu\nu}\right) &(3.146)\end{aligned}$$

The last one is called the true gauge transformation. It should be clear that it is possible to gauge fix the true gauge transformation in such a way that the background gauge transformation is left intact. This is important because we will be able to carry the background symmetry up to the end of the process of functional quantization.

We shall start the construction of the functionals introducing a certain classical action

$$S\left[\gamma_{\mu\nu}\right] = S\left[\bar{g}_{\mu\nu},h_{\mu\nu}\right] \tag{3.147}$$

where we made explicit the double dependence, although at this point it is only a fake dependence because the split symmetry is assumed

$$\begin{aligned}\bar{g}_{\mu\nu} &\to \bar{g}_{\mu\nu}+s_{\mu\nu} &(3.148)\\ h_{\mu\nu} &\to h_{\mu\nu}-s_{\mu\nu} &(3.149)\end{aligned}$$

The idea, at this point, is to construct the scale dependent effective action following closely the steps we outlined in the first chapter, but applying them to $h_{\mu\nu}$ only. We therefore introduce the IR cutoff term in the form

$$\Delta S_k\left[\bar{g}_{\mu\nu},h_{\mu\nu}\right] = \frac{1}{2}\int\sqrt{\bar{g}}h^{\alpha\beta}\mathcal{R}_k\left[\bar{g}\right]^{\mu\nu}_{\alpha\beta}h_{\mu\nu} \tag{3.150}$$

This makes explicit what we meant by saying that the coarse-graining was made with respect to the background. The kernel of the quadratic cutoff depends on $\bar{g}_{\mu\nu}$ and modes of $h_{\mu\nu}$ are suppressed according to an operator that is obtained through the background structure. In other words, we need an operator constructed from the background metric $\bar{g}_{\mu\nu}$ to be able to define in a covariant way high and low frequency modes. An example of such an operator is the background laplacian defined as $-\bar{g}_{\mu\nu}\bar{\nabla}^\mu\bar{\nabla}^\nu$. Using the terminology introduced in the first chapter, the background metric allows us to define a notion of fast- and slow-modes for the fields $h_{\mu\nu}$ or $g_{\mu\nu}$. It is clear that $\Delta S_k\left[\bar{g}_{\mu\nu},h_{\mu\nu}\right]$ has a true double dependence because it violates the split symmetry explicitly.

Now we can gauge-fix $h_{\mu\nu}$ modes introducing the gauge fixing term and the related ghost term in the path integral for the partition function. As we already said the gauge fixing is performed on the true gauge transformation. The property that this gauge fixing leaves the background gauge intact, is also particularly useful because it means it can be written in an explicitly covariant form. Covariance is implied once $\bar{g}_{\mu\nu}$ is regarded as the metric and $h_{\mu\nu}$ as a generic symmetric 2-tensor. If we neglect the ghost dependences, that we will see in more detail in a specific application below, the result of functional integration and Legendre transformation is a functional

$$\mathit{\Gamma}_k\left[\bar{g}_{\mu\nu}, \bar{h}_{\mu\nu}\right] \tag{3.151}$$

which has the genuine dependence on the background and on the expectation value

$$\bar{h}_{\mu\nu} = \langle h_{\mu\nu} \rangle \tag{3.152}$$

We adopted the symbol $\mathit{\Gamma}_k$ instead of Γ_k, because we want to reserve it for a slightly different definition of this functional as we shall see in a moment. It is natural to introduce a new metric out of the expectation value of $h_{\mu\nu}$ by defining

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu} \tag{3.153}$$

and it easy to see that this metric is the full classical metric because $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle$. For this reason the actual dependence we may want to adopt uses both the metrics

$$\mathit{\Gamma}_k\left[\bar{g}_{\mu\nu}, \bar{h}_{\mu\nu}\right] = \mathit{\Gamma}_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu} - \bar{g}_{\mu\nu}\right] = \Gamma_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] \tag{3.154}$$

In this sense, this is often referred as a bimetric functional.

Again, the presence of the cutoff term influences the fact that the functional does not respects the split symmetry, this time in $\bar{g}_{\mu\nu}$ and $\bar{h}_{\mu\nu}$. Therefore it does genuinely depends on two metrics, so it is a bimetric functional. The exact renormalization group equation looks like

$$\dot{\Gamma}_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(0,2)}\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] + \mathcal{R}_k\left[\bar{g}\right]\right)^{-1}\dot{\mathcal{R}}_k\left[\bar{g}\right] \tag{3.155}$$

The notation $\Gamma_k^{(0,2)}\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right]$ means that derivatives are performed only on the $g_{\mu\nu}$ dependence. Again, ghosts are neglected in our discussion, but can easily be restored. We temporarily abbreviate the flow equation as

$$\partial_t \Gamma_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] = \mathcal{F}\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] \tag{3.156}$$

The background was useful in our construction because it allowed to perform the coarse graining in such a way that $\Gamma_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right]$ is always background gauge invariant along the flow. However, at the very end, we are looking for a functional that is invariant under the full gauge transformation on $g_{\mu\nu}$. This issue is partly resolved by requiring that the true effective action is only $g_{\mu\nu}$ dependent and gauge invariant

$$\Gamma_{k=0}\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] \simeq \Gamma\left[g_{\mu\nu}\right] \tag{3.157}$$

where the equality is not exact for the presence of gauge fixing terms. We want to briefly see under what circumstances this is achieved.

We now parametrize the action in order to keep under control the terms that are background dependent when moving to $k = 0$

$$\Gamma_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] = \bar{\Gamma}_k\left[g_{\mu\nu}\right] + \hat{\Gamma}_k\left[\bar{g}_{\mu\nu}, g_{\mu\nu}\right] + \Gamma_{\mathrm{g.f.}} \tag{3.158}$$

The first term is defined from the relation $\bar{\Gamma}_k\,[g_{\mu\nu}] = \Gamma_k\,[g_{\mu\nu}, g_{\mu\nu}]$, while $\hat{\Gamma}_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}]$ contains all the deviations $\Gamma_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}]$ has from being a single-metric functional of $g_{\mu\nu}$. By construction $\bar{\Gamma}_k\,[g_{\mu\nu}]$ is precisely the functional that flows to the full effective action

$$\bar{\Gamma}_{k=0}\,[g_{\mu\nu}] = \Gamma\,[g_{\mu\nu}] \tag{3.159}$$

while $\hat{\Gamma}_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}]$ possesses only the general property

$$\hat{\Gamma}_k\,[g_{\mu\nu}, g_{\mu\nu}] = 0 \tag{3.160}$$

so it vanishes if evaluated at equal metrics. If we plug these two functionals inside the ERGE we get two evolution equations

$$\begin{aligned}
\partial_t \bar{\Gamma}_k\,[g_{\mu\nu}] &= \partial_t \Gamma_k\,[g_{\mu\nu}, g_{\mu\nu}] \\
&= \mathcal{F}\,[g_{\mu\nu}, g_{\mu\nu}] &\quad (3.161)\\
\partial_t \hat{\Gamma}_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}] &= \partial_t \Gamma_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}] - \partial_t \Gamma_k\,[g_{\mu\nu}, g_{\mu\nu}] \\
&= \mathcal{F}\,[\bar{g}_{\mu\nu}, g_{\mu\nu}] - \mathcal{F}\,[g_{\mu\nu}, g_{\mu\nu}] &\quad (3.162)
\end{aligned}$$

using just their definitions [40].

We want to simplify the situation, by looking at some ad hoc approximation that allows to study a single metric functional. Notice that the flow equations of $\bar{\Gamma}_k\,[g_{\mu\nu}]$, even if it looks decoupled from that of $\hat{\Gamma}_k\,[\bar{g}_{\mu\nu}, g_{\mu\nu}]$, actually is not. In particular $\mathcal{F}\,[g_{\mu\nu}, g_{\mu\nu}]$ contains $\hat{\Gamma}_k^{(0,2)}\,[g_{\mu\nu}, g_{\mu\nu}]$. If we decide to approximate $\hat{\Gamma}_k^{(0,2)}\,[g_{\mu\nu}, g_{\mu\nu}] = 0$, then we end up with an equation

$$\partial_t \bar{\Gamma}_k\,[g_{\mu\nu}] = \mathcal{F}\,[g_{\mu\nu}, g_{\mu\nu}]|_{\hat{\Gamma}^{(0,2)}=0} \tag{3.163}$$

Now the flow of $\bar{\Gamma}_k\,[g_{\mu\nu}]$ is decoupled and it maps only over single metric functionals. We rename the single metric functional

$$\Gamma_k\,[g_{\mu\nu}] = \Gamma_k\,[g_{\mu\nu}, g_{\mu\nu}] = \mathit{\Gamma}_k\,[g_{\mu\nu}, 0] \tag{3.164}$$

and this will be the EA we are going to flow in future chapters for gauge theories. No notational ambiguity should arise because this time we have a functional of a single metric.

The flow of $\Gamma_k\,[g_{\mu\nu}]$ is not, in general, a function of $\Gamma_k\,[g_{\mu\nu}]$ only, for the reasons we stressed before. Actually, it is instructive to calculate it and we get

$$\dot{\Gamma}_k\,[g_{\mu\nu}] = \frac{1}{2}\mathrm{Tr}\left(\mathit{\Gamma}_k^{(0,2)}\,[g_{\mu\nu}, 0] + \mathcal{R}_k\,[g]\right)^{-1} \dot{\mathcal{R}}_k\,[g] \tag{3.165}$$

The evolution of Γ_k depends generally on $\mathit{\Gamma}_k$ (and in particular on $\hat{\Gamma}^{(0,2)}$ as already seen). This flow is not “closed” in a partial differential equation sense. We are uncomfortable with this situation, because in chapter one and two, we always used flows that are closed. However, under the approximation $\hat{\Gamma}^{(0,2)} = 0$, it satisfies

$$\dot{\Gamma}_k\,[g_{\mu\nu}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)}\,[g_{\mu\nu}] + \mathcal{R}_k\,[g]\right)^{-1} \dot{\mathcal{R}}_k\,[g] \tag{3.166}$$

that is, in form, analog to the ERGE.

From the point of view of an actual calculation, the approximation $\hat{\Gamma}^{(0,2)} = 0$ forces $\int \frac{1}{2} h_{\mu\nu} \mathit{\Gamma}_k^{(0,2)} \left[g_{\mu\nu}, 0\right]^{\mu\nu}_{\alpha\beta} h_{\alpha\beta}$ to be exactly the second order expansion of the single metric functional $\Gamma_k \left[g_{\mu\nu} + h_{\mu\nu}\right]$ with argument displaced by $h_{\mu\nu}$. In this sense this approximation can be seen as a certain parametrization of the couplings of the full action $\mathit{\Gamma}_k \left[g_{\mu\nu}, h_{\mu\nu}\right]$, in which corresponding couplings at order zero and two in $h_{\mu\nu}$ are set equal.

The main difference of the flow equation for $\Gamma_k \left[g_{\mu\nu}\right]$ with the ERGE we derived in chapter one, is that it has a built-in dependence of the cutoff kernel $\mathcal{R}_k \left[g\right]$ on the metric $g_{\mu\nu}$. This feature is not generally present in the exact RG equation. The result of the integration of this flow is gauge invariant provided the initial conditions are. This gauge invariance is a direct consequence of the background gauge invariance, we kept from the beginning (we always maintained $\bar{g}_{\mu\nu}$ gauge invariance and not we set $\bar{g}_{\mu\nu} = g_{\mu\nu}$). It was crucial before to preserve background gauge invariance to have $\mathcal{R}_k = \mathcal{R}_k \left[\bar{g}\right]$. Now that the cutoff depends on $\bar{g}_{\mu\nu} = g_{\mu\nu}$, it is crucial that $\mathcal{R}_k \left[g\right]$ depends on the metric to maintain gauge $g_{\mu\nu}$ invariance.

Chapter 4

Inclusion of gravity.

In this chapter we want to couple gravitational degrees of freedom to the simple Yukawa system we considered before. The first two calculation we are going to present involve a direct computation of the flow of $V[\phi]$ and $H[\phi]$. We will extract those runnings evaluating the effective action at a certain proper configuration for the fields. Then we will address the computation of the gravitationally corrected anomalous dimensions, using momentum space diagrammatic technique. Finally we will use heat kernel techniques to obtain the flow of Z_g.

4.1 Coupling of spinors in brief.

The truncated action we choose is essentially the one of the flat euclidean case, but minimally coupled to the metric $g_{\mu\nu}$. This means that a volume form $\sqrt{g}$ is added to each term to correctly densitize it and derivatives are replaced by covariant ones. Further a Einstein term is added encoding the dynamic of the metric, while the cosmological constant is already present, in principle, in the potential. The action turns out to be

$$\begin{aligned}\Gamma_k\left[\phi,\bar{\psi},\psi,g_{\mu\nu}\right] &= \int\sqrt{g}\Big[Z_g R\left[g_{\mu\nu}\right]+\frac{Z_\phi}{2}\left(\partial_\mu\phi\right)^2+Z_\psi\bar{\psi}i\not{D}\psi \\ &\quad +H\left[\phi\right]\bar{\psi}\psi+V\left[\phi\right]\Big]\end{aligned} \tag{4.1}$$

We will consider this action in four dimensions in the following.

Some care is required when coupling spinor fields to gravity. The general procedure involves increasing the number of gravitational degrees of freedom by introducing a vierbein $e^a{}_\mu$, that is a set of one-forms such that

$$g_{\mu\nu}=e^a{}_\mu e^b{}_\nu\delta_{ab} \tag{4.2}$$

We also require the vierbein to have an inverse, so a set of vectors $e^\mu{}_a$ such that

$$e^a{}_\mu e^\mu{}_b = \delta^a_b \tag{4.3}$$

$$e^\mu{}_a e^a{}_\nu = \delta^\mu_\nu \tag{4.4}$$

The vierbein are sometimes called tetrad. These are useful because when adopting them as a basis the metric gets trivialized. The number of degrees of freedom of the vierbein is bigger than that of the metric and it is easy to see it. The metric is a symmetric tensor that in four dimensions has ten independent components, while $e^a{}_\mu$ has sixteen components. A tetrad is therefore a general linear transformation. One is allowed to think of it as a basis transformation on the tangent bundle. Alternatively it is seen as an endomorphism between the tangent bundle and another bundle with flat metric tensor.

Let $\mathcal{M}$ be the euclidean spacetime manifold and $\mathcal{TM}$ its tangent bundle with metric connection

$$\Gamma_\mu{}^\nu{}_\sigma = \frac{1}{2} g^{\nu\rho} \left(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\mu\sigma} \right)$$

Further, we call V a vector bundle with fiber being $\mathbb{R}^4$. In this new bundle we can construct a connection $A_\mu{}^a{}_b$ requiring

$$\nabla_\mu e^a{}_\nu = 0 \tag{4.5}$$

where ∇ is the total covariant derivative on $\mathcal{TM} \oplus V$. This gives its relation to the Christoffel connection on $\mathcal{TM}$

$$A_\mu{}^a{}_b = \Gamma_\mu{}^\alpha{}_\beta e^a{}_\alpha e^\beta{}_b + e^\alpha{}_b \partial_\mu e^a{}_\alpha \tag{4.6}$$

This new connection is particularly important because it can be used to construct a spinor covariant derivative

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2} A_{\mu,ab} J^{ab} \psi \tag{4.7}$$

where the generators J^{ab} of $O(4)$ rotations in V are obtained from the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$

$$J^{ab} = \frac{1}{4} \left[\gamma^a, \gamma^b \right] \tag{4.8}$$

The $O(4)$ gauge invariance accounts for the additional degrees of freedom of the tetrad and can be gauge fixed in such a way that the vielbeins are symmetric. With this choice they essentially maintain only the degrees of freedom of the metric. Finally we can construct the spinor field action. The conjugate spinor is introduced as usual $\bar{\psi} = \psi^\dagger \gamma^0$ and the lagrangian is

$$i\bar{\psi}\gamma^a e^\mu{}_a D_\mu \psi = \bar{\psi} i \not{D} \psi \tag{4.9}$$

It is easily seen that the metric degrees of freedom will enter in three ways in the spinor action, namely in the volume element, the covariant derivative and the inverse tetrad that couples the covariant derivative. This makes the treatment of spinor field in gravitational settings more complicated than that of a scalar.

4.2 Non-Diagonal cutoff computation.

In this section we present a computation with a particular kind of cutoff the form of which will be explained in the following [19]. We will also temporarily stick to the approximation of zero anomalous dimensions for the matter fields. Let us review step by step the lagrangian pieces we are interested in.

The theory contains a single scalar field with Lagrangian

$$L_b = \sqrt{g}(\tfrac{1}{2}Z_\phi \nabla^\mu \phi \nabla_\mu \phi + V(\phi)) \ . \tag{4.10}$$

As in the previous chapter we choose the potential V to be even in ϕ. Spinor degrees of freedom are added in the form of N_f Dirac fermions ψ with $U(N_f)$-symmetric Lagrangian

$$L_f = \sqrt{g}(\tfrac{i}{2}Z_\psi(\bar{\psi}\gamma^\mu D_\mu \psi - D_\mu \bar{\psi}\gamma^\mu \psi) + i\, H(\phi)\, \bar{\psi}\, \psi) \ . \tag{4.11}$$

We explicitly symmetrized it in terms of ψ and its conjugate using integration by parts. This is a convenient procedure that ensures that the second order expansion kernel will be an adjoint operator. As previously mentioned, the $O(4)$ gauge is chosen such that the vierbein is symmetric. In particular it means that all vierbein fluctuations can be written in terms of the metric fluctuations and there are no $O(4)$ ghosts [41]. The function $H(\phi)$ is kept general as in the flat case and represents the Yukawa sector, that in curved space couples also with the determinant of the metric. In order to neglect anomalous dimensions we will set $Z_\phi = Z_\psi = 1$.

Gravity is included through an Einstein term

$$L_g = -Z\sqrt{g}R\,[g_{\mu\nu}] \tag{4.12}$$

where $Z = 1/(16\pi G)$. Gravity possesses a gauge invariance, namely diffeomorphism invariance, so we shall work with the background-field method. We refer to the dedicated section for more details. The procedure ensures us that the final result will be a gauge invariant functional.

For the purpose of calculating the flows of V and H it is sufficient to expand around constant backgrounds. We still denote

$$g_{\mu\nu} = \delta_{\mu\nu}, \phi, \psi, \bar{\psi} \tag{4.13}$$

the background fields and

$$h_{\mu\nu}, \varphi, \chi, \bar{\chi} \tag{4.14}$$

the corresponding fluctuations. In particular the gravitational background is flat. This has to be stressed because a flat choice of background forbids the calculation of the running of the Newton constant that is hidden in Z. The calculations of this section are going to give the gravitational corrections to the running of V and H. Moreover, the backround is off-shell because the constant ϕ is not necessarily a solution of the equations of motion.

Diffeomorphism invariance is fixed by a covariant background gauge. The gauge fixing term is rather standard in gravitational applications

$$L_{GF} = \frac{Z}{2\alpha}\delta^{\mu\nu}F_\mu F_\nu \ ; \ F_\mu = \left(\delta^\beta_\mu \partial^\alpha - \frac{1+\beta}{4}\delta^{\alpha\beta}\partial_\mu\right) g_{\alpha\beta} \tag{4.15}$$

and represents a two parameters generalization of de Donder gauge [1] . From the gauge fixing term the ghost action is easily derived using the Fadeev-Popov

[1] In general d-dimensional applications it is sometimes convenient to replace $\frac{1+\beta}{4}$ with $\frac{1+\beta}{d}$. It ensures that the choice $\beta = 1$ still removes completely the gauge spin-1 degree of freedom in any dimensionality. However, we will always study the case $d = 4$ only, where the two coincides.

trick and exponentiating the functional determinant. It is given by

$$L_{gh} = \bar{c}_\mu \left(-\delta^{\mu\nu}\partial^2 + \frac{\beta - 1}{2}\partial^\mu\partial^\nu \right) c_\nu \ . \tag{4.16}$$

It is useful to manage gravitational degrees of freedom in a slightly different way when using a general cutoff choice. The off-shell symmetric tensor $h_{\mu\nu}$ contains various spins in its decomposition. We will analyze the structure of the fluctuation in more details in the next sections. For the moment it is sufficient to know that it admits the tensor decomposition

$$h^\perp_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + \left(\partial_\mu\partial_\nu\sigma - \tfrac{1}{4}\delta_{\mu\nu}\partial^2\sigma\right) + \tfrac{1}{4}\delta_{\mu\nu}h \tag{4.17}$$

The spin-2 $h^\perp_{\mu\nu}$ is transverse and traceless $\partial^\mu h^\perp_{\mu\nu} = \eta^{\mu\nu}h^\perp_{\mu\nu} = 0$, the spin-1 is also transverse $\partial^\mu v_\mu = 0$. Finally there also are two scalar degrees of freedom, h, that is the trace of the metric fluctuation $h = \delta^{\mu\nu}h_{\mu\nu}$, and σ.

The application of the exact RG equation requires the second order expansion of the entire lagrangian, including the gauge fixing term. We shall give it in terms of the fluctuations and for the constant background we employed. In particular the choice of a constant background greatly simplify the calculation. It is given by

$$\begin{aligned}
\mathcal{L}^{(2)} = \quad & -\frac{1}{4}h^\perp_{\mu\nu}\left(Z\partial^2 + V + iH\bar\psi\psi\right)h^{\perp\,\mu\nu} - \frac{i}{16}h^{T\lambda}_\mu\partial_\rho h^T_{\lambda\nu}\bar\psi\gamma^{\mu\nu\rho}\psi \\
& + \frac{1}{2}v_\mu\left(\frac{Z}{\alpha}\partial^2 + V + iH\bar\psi\psi\right)\partial^2 v^\mu + \frac{i}{16}v_\mu\partial_\rho\partial^2 v_\nu\bar\psi\gamma^{\mu\nu\rho}\psi \\
& + \frac{3}{32}\partial^2\sigma\left(\frac{\alpha - 3}{\alpha}Z\partial^2 - 2V - 2iH\bar\psi\psi\right)\partial^2\sigma + 3\frac{\beta - \alpha}{16\alpha}Z\partial^2\sigma\partial^2 h \\
& - \frac{1}{32}h\left(\frac{\beta^2 - 3\alpha}{\alpha}Z\partial^2 - 2V - 2iH\bar\psi\psi\right)h + \frac{1}{2}(V' + iH'\bar\psi\psi)h\,\varphi \\
& + \frac{1}{2}\varphi\,(-\partial^2 + V'' + iH''\bar\psi\psi)\varphi - \frac{1}{2}\bar{c}_\mu\partial^2 c^\mu + \frac{i}{2}\left(\bar\chi\gamma^\mu\partial_\mu\chi - \partial_\mu\bar\chi\gamma^\mu\chi\right) \\
& + iH\bar\chi\chi + iH'\varphi(\bar\psi\chi + \bar\chi\psi) + \frac{i}{2}Hh(\bar\psi\chi + \bar\chi\psi) \\
& + \left(\frac{i}{4}\partial^2 v_\nu + \frac{3i}{16}\partial_\nu\partial^2\sigma - \frac{3i}{16}\partial_\nu h\right)(\bar\psi\gamma^\nu\chi - \bar\chi\gamma^\nu\psi)
\end{aligned} \tag{4.18}$$

As usual primes denote derivatives with respect to ϕ. We also used some property of Clifford algebra. The tensor $\gamma^{\mu\nu\rho}$ is defined by $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ and it has the useful property $\gamma^{\mu\nu\rho} = \{J^{\mu\nu}, \gamma^\rho\}$.

As is well known, changing field variables generally produces functional determinants, which one has to take into account. In particular when moving to irreducible spin components for the metric fluctuation some determinant is appearing. However it is easy to see that these determinants are precisely canceled if we also use the redefinitions $-\partial^2\sigma \to \sigma$ and $\sqrt{-\partial^2}v_\mu \to v_\mu$. In this way we remove the Jacobians arising from the tensor decomposition. We wrote all these relations in flat space background, but are easily generalized to curved one. We again refer to future sections for a more detailed analysis.

We want to write the RG flow for our system using the exact equation. It is convenient to introduce a quite compact notation in the form of a supermultiplet

$\Upsilon^T = (h^\perp_{\mu\nu}, v_\mu, c_\mu, \bar{c}_\mu, \sigma, h, \varphi, \chi^T, \bar{\chi})$ that contains all the field fluctuations of the system. Using this notation the second functional derivative of the action is a supermatrix $\Gamma_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta\Upsilon^T}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\Upsilon}$. Notice that the ghosts are included in the super-multiplet Υ. Their action (4.16) must be added to the second order expansion (4.18). It is already quadratic, so no further manipulation is needed.

We define the cutoff requiring that $\Gamma_k^{(2)} + R_k$ is like $\Gamma_k^{(2)}$, once the replacement

$$i\partial_\mu \quad \to \quad \sqrt{P_k(-\partial^2)/(-\partial^2)}\,i\partial_\mu \tag{4.19}$$

is performed. We therefore define the cutoff so that

$$\Gamma_k^{(2)} + R_k \quad = \quad \Gamma_k^{(2)}\Big|_{i\partial_\mu \to \sqrt{P_k(-\partial^2)/(-\partial^2)}\,i\partial_\mu} \tag{4.20}$$

It is very easy to determine its explicit form

$$R_k \quad = \quad \Gamma_k^{(2)}\Big|_{i\partial \to \sqrt{P_k/(-\partial^2)}\,i\partial} - \Gamma_k^{(2)} \tag{4.21}$$

The infrared cutoff term is

$$\Delta S_k = \frac{1}{2}\int \Upsilon^T R_k \Upsilon \tag{4.22}$$

The function $P_k(z)$ contains the feature of the cutoff profile and plays the role of modified scalar propagator. In the optimized case it is given by

$$P_k(z) = z + R_k(z) = z\theta\left(z - k^2\right) + k^2\theta\left(k^2 - z\right) \tag{4.23}$$

The cutoff defined from the requirement (4.20) is non-diagonal in field space. There are in fact elements in the cutoff ΔS_k defined in (4.22) that mix the spinor fluctuations χ and $\bar{\chi}$ with the spin-0, -1 and -2[2].

4.2.1 Beta Functions for the non-diagonal cutoff.

First of all we define the dimensionless field $\bar{\phi} = \phi/k$. It coincides with the dimensionless renormalized one in the approximation of zero anomalous dimensions, because the wavefunction renormalization constants are set to one. In this approximation also the dimensionless potential $v(\bar{\phi}) = V(k\bar{\phi})/k^4$ and dimensionless function $h(\bar{\phi}) = H(k\bar{\phi})/k$, coincide with their dimensionless renormalized partners. This slightly changes the canonical scaling of the flow, which can easily be obtained from the anomalous dimension corrected one, by setting $\eta_\phi = \eta_\psi = 0$

$$\dot{v} \quad = \quad -4v + \bar{\phi}v' + k^{-4}\dot{V} \tag{4.24}$$
$$\dot{h} \quad = \quad -h + \bar{\phi}h' + k^{-1}\dot{H} \tag{4.25}$$

The constant background is appropriate for our calculation because it is possible to obtain the flow of V and H using directly the ERGE

$$\dot{\Gamma} \sim \int d^4x\left(\dot{V} + i\dot{H}\bar{\psi}\psi\right) \tag{4.26}$$

[2]One can observe, however, that all these mixings are proportional to the backgrounds ψ and $\bar{\psi}$. Later in this chapter, we will use a diagonal cutoff obtained with the further requirement that $\psi = \bar{\psi} = 0$.

and hence that of their dimensionless partners.

We have all the ingredients to perform the calculation in the given background. Some special care is needed because we are working with supermatrices. In particular inversion and product of supermatrices are needed. The calculation was done in the general two parameter gauge and without any perturbative assumption on the gravitational coupling G. However the beta functionals for v and h are generally very complicated functions of the parameters and nonlinear in the dimensionless partner of the Newton constant $\bar{G} = k^2 G$. For the sake of presentation we give the beta functionals for v and h, in the gauge $\beta = 1$ and expanding to first order in the dimensionless Newton constant $\bar{G} = k^2 G$:

$$\begin{aligned} \dot{v} &= -4v + \bar{\phi}v' - \frac{N_f}{8\pi^2\,(1+h^2)} + \frac{3+2v''}{32\pi^2\,(1+v'')} \\ &\quad -\bar{G}\frac{(3-\alpha)v'^2\,(2+v'')}{2\pi\,(1+v'')^2} + \bar{G}\frac{v(3+2\alpha)}{\pi} + O(\bar{G}^2)\,, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \dot{h} &= -h + \bar{\phi}h' - \frac{h''}{32\pi^2\,(1+v'')^2} + \frac{hh'^2\left(2+h^2+v''\right)}{16\pi^2\,(1+h^2)^2\,(1+v'')^2} \\ &\quad +\bar{G}\frac{(3-\alpha)v'^2}{\pi\,(1+v'')^3}\left(\frac{1}{2}h''\,(3+v'') - \frac{hh'^2\left(4+3h^2+\left(2+h^2\right)v''\right)}{(1+h^2)^2}\right) \\ &\quad +\bar{G}h'v'\frac{4\alpha-6-(3-2\alpha)v''+h^2(15-4\alpha)+2h^2(3-\alpha)\left(\left(2+h^2\right)v''+2h^2\right)}{2\pi\,(1+h^2)^2\,(1+v'')^2} \\ &\quad +\bar{G}h\frac{27+\alpha\left(29+96h^2+48h^4\right)}{16\pi\,(1+h^2)^2} + O(\bar{G}^2). \end{aligned} \tag{4.28}$$

These results give the gravitational dressing of the matter beta functions in the low energy regime, where gravity is supposed to be weak $k^2/M^2_{\rm Planck} \equiv \bar{G} \ll 1$. It is known from perturbation theory, that the anomalous dimension of the Dirac field is not really negligible. The beta functionals we calculated here are useful for a first qualitative look at the coupled system. We will see in the following that the qualitative form of the gravitational dressing they give remains true also when taking into account the anomalous dimensions.

Our next task is to fix the form of the functions v and h and expand around an appropriate basis of operators. In this way one may find the running of any coupling of interest. In particular we will consider in the following local power-law potentials. There are two cases of interest, corresponding to expansions around a zero VEV $\langle\bar{\phi}\rangle = 0$ or a non-zero one $\langle\bar{\phi}\rangle = \sqrt{\bar{\kappa}}$. In the first case $\langle\bar{\phi}\rangle = 0$ we have a $\mathbb{Z}_2$ symmetric phase, because the ground state is $\mathbb{Z}_2$ symmetric like the potential. Instead, in the second case we expand around a non-zero VEV $\langle\bar{\phi}\rangle = \sqrt{\bar{\kappa}} \neq 0$ and therefore we describe a phase in which the $\mathbb{Z}_2$ symmetry is broken. Concerning the function h, from now on to the rest of this section, we will restrict ourselves to a simple Yukawa interaction $h = \bar{y}\bar{\phi}$.

4.2.2 Expansion around $\langle\bar{\phi}\rangle = 0$.

The first case we face is that of a symmetric potential. We parametrize it through a polynomial expansion in even powers of the scalar field. For the time being we do not have any real idea on how far we should push the polynomial expansion to capture the flow under a good approximation, so the beta functions we are going to give are mainly illustrative. We decide to give them in a truncation in which the potential is at most quartic. The simple reason is that the quartic interaction behavior is interesting because should give an understanding of the gravitational dressing of the standard model Higgs self-coupling. We therefore truncate

$$v(\bar{\phi}) \quad = \quad \bar{\lambda}_0 + \bar{\lambda}_2\bar{\phi}^2 + \bar{\lambda}_4\bar{\phi}^4\,, \tag{4.29}$$

We now insert this parametrization in (4.27) and expand $\dot{v}$ in powers so that we can extract the beta functions of the powerlaw couplings. For the time being $\bar{\lambda}_2$ is assumed to be positive. To extract the flow of $\bar{y}$ it is sufficient to take the linear part in $\bar{\phi}$ of $\dot{h}$. We also employ some further simplification, namely we set $\alpha = 0$, while still we work in the gauge $\beta = 1$. Finally, the results are given in the limit $\lambda_0 = 0$. All these simplifications make the beta functions accessible, while still giving a hint of their complexity

$$\begin{aligned}
\dot{\lambda}_0 &= \frac{3+4\lambda_2}{32\pi^2\,(1+2\lambda_2)} - \frac{N_f}{8\pi^2}\,,\\
\dot{\lambda}_2 &= -2\lambda_2 + \frac{N_f y^2}{8\pi^2} - \frac{3\lambda_4}{8\pi^2\,(1+2\lambda_2)^{\,2}} + \frac{3\tilde{G}\lambda_2}{\pi\,(1+2\lambda_2)^2}\,,\\
\dot{\lambda}_4 &= \frac{9\lambda_4^2}{2\pi^2\,(1+2\lambda_2)^3} - \frac{N_f y^4}{8\pi^2}\\
&\quad + 3\,\tilde{G}\lambda_4 \frac{1-10\lambda_2+36\lambda_2^2+24\lambda_2^3}{\pi\,(1+2\lambda_2)^3} + O(\tilde{G}^2)\,,\\
\dot{y} &= \frac{y^3\,(1+\lambda_2)}{8\pi^2\,(1+2\lambda_2)^2} + \tilde{G}y\frac{27+12\lambda_2\,(1+\lambda_2)}{16\pi\,(1+2\lambda_2)^2}\,.
\end{aligned} \tag{4.30}$$

It is important to stress that, in general, the beta functions would depend nonpolynomially on λ_0 and $\tilde{G}$. This is a typical feature of exact RG calculations because we resum many orders in perturbation theory. As an example, it is evident from the denominator that appear, $(1+2\lambda_2)$, which also works as a threshold for the flow of the couplings. We decided to set $\lambda_0 = 0$, in the assumption it is negligible. These beta functions give a qualitative picture of how gravity dresses the flow of typical standard model couplings. A further technical remark is that in the approximation $\lambda_0 = 0$, $\bar{G}$ appears only polynomially: the highest power of $\tilde{G}$ occurs in $\dot{\lambda}_4$ and is 2. In all other terms $\tilde{G}$ appears at most linearly.

For completeness we can exhibit also the corrections appearing when $\alpha \neq 0$. These are linear in α, so it is sufficient to add to the previous system the following correction terms:

$$\begin{aligned}
\Delta\dot{y} &= \alpha\tilde{G}y\frac{29+180\lambda_2\,(1+\lambda_2)}{16\pi\,(1+2\lambda_2)^2}\,,\\
\Delta\dot{\lambda}_2 &= 2\alpha\tilde{G}\lambda_2\frac{1+6\lambda_2\,(1+\lambda_2)}{\pi\,(1+2\lambda_2)^2}\,,\\
\Delta\dot{\lambda}_4 &= 2\alpha\tilde{G}\lambda_4\frac{1+14\lambda_2}{\pi\,(1+2\lambda_2)^3}\,.
\end{aligned} \tag{4.31}$$

Let us discuss the content of the system of beta functions we obtained. We know from (3.115) that the leading term of the beta function of the Yukawa coupling is $\dot{y} = \frac{5y^3}{16\pi^2} + \ldots$. On the other hand, if we neglect $\bar{G}$ and λ_2 in the Yukawa beta function in (4.30) we remain with $\dot{y} = \frac{y^3}{8\pi^2} + \ldots$. The difference is due to the fact that here we neglect the anomalous dimensions of ϕ and ψ. In particular, it is known that these contribute consistently to the beta function, although leaving it positive. Anyway, our results should still give a reasonable qualitative picture of the gravitational corrections. At least, as long as the validity of the LPA approximation holds, which we assume. It is important to also stress that even though this is a simple toy model, the leading one loop gravitational correction applies also to realistic theories. The typical 1-loop leading contribution, that the beta function of any matter coupling λ has from gravity, is of the form $\Delta\beta_\lambda \sim \bar{G}\lambda$. In the realistic standard model case the Yukawa couplings form a matrix y_{ij}. Every beta function $\dot{y}_{ij}$ will receive the same correction $(27/16\pi)\tilde{G}y_{ij}$.

Switching off the gravitational corrections, our results are in agreement with those of [18], when the anomalous dimensions are neglected. Furthermore, the results for $\dot{\lambda}_i$ in (4.30) are also in agreement with those of [42]. In (4.30) appears also the beta function of the vacuum energy λ_0. One can see the leading contribution, proportional to $(3 - 4N_f)$, the difference between the number of bosonic and fermionic degrees of freedom.

Having used an expansion around flat space, gravity is off-shell [3] . This is the cause of the dependence of the results on the gauge parameter α (and β, the dependence on which we have computed but not reported for simplicity). We note that the sign of the leading corrections does not change as long as $\alpha > 0$; we can also check that it remains the same at least for $0 \leq \beta \leq 1.8$, which comprises the most popular gauge choices. Furthermore, there are arguments showing that if α is allowed to run, $\alpha = 0$ would correspond to a nonperturbative fixed point [43]. This suggests that the results obtained for $\alpha = 0$ are probably the most reliable.

The procedure also generically depends on the choice of cutoff scheme, and in particular on the cutoff profile function. The leading terms in the beta functions of λ_4 and y turn out to be independent on this choice, but not the gravitational corrections, which depend on a dimensionful coupling. In the results presented above we only used the cutoff $r(y) = (1-y)\theta(1-y)$, so the scheme dependence is not manifest, but the numerical coefficients of the gravitational correction would change if we used another cutoff function. We have checked that the leading gravitational correction is proportional to a single integral involving the profile function, so that the leading correction terms in (4.30) and (4.31) is independent of it. Furthermore, the sign of the gravitational correction would be the same for any choice of the cutoff profile, that satisfies the boundary and monotonicity conditions to be a good cutoff.

The system (4.30) has a gaussian fixed point when $\lambda_2 = \lambda_4 = y = 0$. Without gravity both λ_4 and y are marginal, but the gravitational corrections make them irrelevant. In fact the gravitationally dressed critical exponents are $2 - (3+2\alpha)\tilde{G}/\pi$, $-(3+2\alpha)\tilde{G}/\pi$ and $-(27+26\alpha)\tilde{G}/16\pi$, corresponding to the eigenvectors $\lambda_2 - 3\lambda_4/16\pi^2$, λ_4 and y respectively. It is important to note that the gravitational corrections depend on α but are always negative. This is a remarkable result, because in the standard model these couplings are free

[3] We actually gave the system of beta functions (4.30) in the limit $\lambda_0 = 0$, that corresponds to on-shell, but we computed them in their full nonlinearity.

parameters, to be determined by experiment, whereas here they are predicted to be zero at high energy. Any value they have at low energy is due to the nonlinearity of the RG flow. It is worth citing that the result may change in the presence of other matter fields: it was shown in [42] that a certain number of minimally coupled matter fields can change the sign of the critical exponent, making λ_4 relevant. Then its value at low energy would be a free parameter, while at high energy it would be asymptotically free.

4.2.3 Expansion around a nonzero VEV.

In this subsection we apply the alternate expansion we introduced in the previous chapter. It is important to notice that, depending on the sign of $\bar{\lambda}_2$, the potential is either in a symmetric phase or in a symmetry broken phase. The powerlaw expansion around zero we gave previously works well as long as $\bar{\lambda}_2$ is positive. When it is negative it is convenient to expand v around the VEV $\langle\bar{\phi}\rangle = \sqrt{\kappa}$. The VEV is by definition a minimum

$$v'(\sqrt{\kappa}) = 0\,. \tag{4.32}$$

Again we restrict ourselves to fourth order polynomials. We parametrize v in the form

$$v(\bar{\phi}) \quad = \quad \theta_0 + \theta_4(\bar{\phi}^2 - \kappa)^2\ . \tag{4.33}$$

The new couplings of the broken $\mathbb{Z}_2$ phase are related to those in (4.29) by a simple algebraic transformation $\theta_4 = \lambda_4$, $\kappa = -\lambda_2/2\lambda_4$, $\theta_0 = \lambda_0 - \lambda_2^2/4\lambda_4$. Therefore if we derive the beta functions of the new couplings using these relations nothing nontrivial will happen. In particular the beta functions also will be related by an algebraic transformation and the features of the flow will be unaltered, but simply seen from a different parametrization. There is however an alternate procedure that resums some order in perturbation theory. The key idea it that one can obtain the running of κ by deriving (4.32), which yields

$$\dot{\kappa} = -2\sqrt{\kappa}\,\dot{v}'(\sqrt{\kappa})/v''(\sqrt{\kappa})\ . \tag{4.34}$$

and forces κ to be a minimum for all k. Again we will give the results in a simplified version for illustrative purposes. In the broken phase, using (4.27) and retaining only terms up to first order in $\bar{G}$, we then obtain

$$\begin{aligned}
\dot{\theta}_0 &= -4\theta_0 + \frac{3+16\kappa\theta_4}{32\pi^2\left(1+8\kappa\theta_4\right)} - \frac{N_f}{8\pi^2\left(1+\kappa y^2\right)} + \frac{3\tilde{G}\theta_0}{\pi}\,, \\
\dot{\kappa} &= -2\kappa + \frac{3}{16\pi^2\left(1+8\theta_4\kappa\right)^2} - \frac{N_f y^2}{16\pi^2(1+\kappa y^2)^2}\,, \\
\dot{\theta}_4 &= \frac{9\theta_4^2}{2\pi^2\left(1+8\kappa\theta_4\right)^3} - \frac{N_f y^4}{8\pi^2\left(1+\kappa y^2\right)^3} + \frac{3\tilde{G}\theta_4}{\pi\left(1+8\kappa\theta_4\right)^2}\,, \\
\dot{y} &= \frac{y^3}{16\pi^2\left(1+\kappa y^2\right)^3\left(1+8\kappa\theta_4\right)^3}\Big[2-16\kappa\theta_4\left(3+8\kappa\theta_4\right) \\
&\quad -3\kappa y^2\left(1+8\kappa\theta_4\left(7+16\kappa\theta_4\right)\right) - \kappa^2 y^4\left(1+56\kappa\theta_4\right)\Big] \\
&\quad + \frac{3\tilde{G}y}{16\pi\left(1+y^2\kappa\right)^3\left(1+8\theta_4\kappa\right)^2}\Big[9+16\theta_4\kappa\left(1+4\theta_4\kappa\right) \\
&\quad -3y^2\kappa\left(1+8\theta_4\kappa\right)\left(9+8\theta_4\kappa\right) + 192y^4\theta_4\kappa^3\left(3+16\theta_4\kappa\right) \\
&\quad +256y^6\theta_4\kappa^4\left(1+4\theta_4\kappa\right)\Big]\,.
\end{aligned} \tag{4.35}$$

This time we do not give the $O(\alpha)$ corrections to these formulas.

There is some difference with the previous expansion which is worth mentioning. Unlike in the expansion around $\langle\bar{\phi}\rangle = 0$, here θ_0 appears only in its own beta function. Up to order $\tilde{G}$, there is no approximation involved in setting $\theta_0 = 0$ in the beta functions of κ, θ_4 and y, as is natural in an expansion around flat space.

A remarkable fact is that the beta function of κ does not receive any gravitational correction, as was already noted in [44] for the potential (4.33) with $\theta_0 = 0$, even taking into account the scalar field anomalous dimension. It could be expected because the property of being a minimum is equivalent to an on-shell condition, as long as the field is constant. We can show this feature in general. For any scalar potential v, we use the general flow of the VEV (4.34) and the functional beta function for v (4.27) and obtain

$$\dot{\kappa} = -2\kappa + \frac{\sqrt{\kappa}v'''}{16\pi^2 v''\left(1+v''\right)^2} - \left.\frac{hN_f\sqrt{\kappa}h'}{2\left(1+h^2\right)^2\pi^2 v''}\right|_{\bar{\phi}=\sqrt{\kappa}}\,. \tag{4.36}$$

What happened is that in this flow any $\bar{G}$ contribution couples with at least one first derivative of v, that goes to zero when evaluating in the background. We will see that when adding the anomalous dimensions, this property will remain.

In the final part of this chapter we will also compute the flow of the Newton constant. For the moment, however, we can keep it as a free parameter we can tune. It is known that in the Einstein-Hilbert truncation gravity has a nontrivial fixed point, also in the presence of minimally coupled matter fields. Therefore we may assume it reaches a fixed point also here, despite we do not know its real value. Since the Yukawa system has a Gaussian fixed point, one can conclude that the theory of gravity coupled to scalars and fermions also has a fixed point, which we may argue a "Gaussian matter" fixed point. However, it is clear that to study the properties of this fixed point, in particular the critical exponents, it is necessary to calculate also the beta function of $\tilde{G}$.

There is also the possibility that the matter sector exhibits a non trivial fixed point [18], although N_f has to be set less than one ($N_f \lesssim 0.2 - 0.3$). If $\bar{G}$ is kept sufficiently small, the fixed point of [18] must be present also in our system. Indeed we can show that, within the appriximations adopted and as long as $\tilde{G}_* \lesssim 1$ and $N_f \lesssim 0.8$, the nontrivial fixed point exists. This confirms the expectation of [18] that adding new bosonic degrees of freedom (the gravitational ones), the number of fermionic flavors N_f can increase.

4.3 Diagonal cutoff computation.

In this section we will essentially go through the same steps of the previous section with only a small modification to the cutoff kernel. It is worth remembering that the previous cutoff choice was obtained by requiring that the cutoff kernel, when added to the 2-point function, cut-off any derivative appearing on the sum. When this is performed at a general $\psi =$ const. background, a ψ dependence in the cutoff is introduced and non-diagonal terms in field space are present. We will use a different cutoff in field space that greatly simplify the calculation. With this simplification, we will also be able to take into account the corrections given by the anomalous dimensions.

It is sufficient to note, at this point, that what makes non-diagonal the previous cutoff is the presence of constant ψ. A diagonal cutoff kernel is therefore easily obtained by

$$R_k = \Gamma_k^{(2)}\Big|_{i\partial\to\sqrt{P_k/(-\partial^2)}i\partial,\psi\to 0} - \Gamma_k^{(2)}\Big|_{\psi\to 0} \tag{4.37}$$

that is worth comparing with (4.20). The infrared cutoff term is defined as in (4.22). The function $P_k(z)$ again contains the feature of the cutoff profile and plays the role of modified scalar propagator. We still will use the optimized shape of (4.23).

4.3.1 Beta functions for the diagonal cutoff.

As we already said, to calculate the functional beta functions, it is sufficient to go through all the steps of the previous computation with the new cutoff. In this section we will also take into account the anomalous dimensions. We give them in their full nonlinearity, but in the gauge $\alpha = 0$ and $\beta = 1$. We start with the flow of the potential

$$\begin{aligned}
\dot{v} \;=\; & -4v + \frac{1}{2}\phi\, v'\,(\eta_\phi + 2) + \frac{N_f\,(\eta_\psi - 5)}{40\pi^2\,(1+h^2)} - \frac{5Z\left(8+\frac{\dot{Z}}{Z}\right)}{192\pi^2(v-Z)} + \frac{8+\frac{\dot{Z}}{Z}}{64\pi^2} - \frac{1}{4\pi^2} \\
& + \frac{(v-2Z)\,(v''+1)\left(8+\frac{\dot{Z}}{Z}\right) - 3\left((v')^2\left(8+\frac{\dot{Z}}{Z}\right) + 2(Z-v)\right)}{192\pi^2\left((v-Z)\,(1+v'') - 3\,(v')^2\right)} \\
& + \frac{(Z-v)\eta_\phi}{192\pi^2\left((v-Z)\,(1+v'') - 3\,(v')^2\right)}
\end{aligned} \tag{4.38}$$

where $Z = 1/16\pi\bar{G}$. We can recognize where each term is coming from. The first line contains, in order the canonical scalings, the contributions of spinors, spin-2 modes, spin-1 modes and ghosts. In the second and third line we have instead the mixed contributions of the spin-0 φ, h and σ.

The expression for $\dot{h}$ is slightly more complicated. We obtain

$$
\begin{aligned}
\dot{h} \;=\;& -(1-\eta_\psi)\,h + \phi h'\left(\frac{\eta_\phi}{2}+1\right) + \frac{5hZ\left(8+\frac{\dot{Z}}{Z}\right)}{192\pi^2(Z-v)^2} \\
&+\frac{(\eta_\phi-6)\left(h''(v-Z)^2+6h'(Z-v)v'+3h\,(v')^2\right)}{192\pi^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&+\frac{Z\left(8+\frac{\dot{Z}}{Z}\right)\left(3h''\,(v')^2-6h'v'\,(1+v'')+h\,(1+v'')^2\right)}{192\pi^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&+\frac{h'(v-Z)v'\,(\eta_\phi-6)+Zh'v'\,(1+v'')\left(8+\frac{\dot{Z}}{Z}\right)}{16\pi^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&-\frac{h'v'\left((226+87\frac{\dot{Z}}{Z})Z\,(1+v'')+(29\eta_\phi-168)(v-Z)\right)}{560\pi^2\,(1+h^2)\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&+3h\frac{-\dot{Z}\,(1+v'')^2\left(\left(1120h^2+349\right)-8960h^2-3106\right)}{35840\pi^2\,(1+h^2)\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&+9h\frac{(v')^2\left(-\left(1120h^2+349\right)\eta_\phi+6720h^2+2408\right)}{35840\pi^2\,(1+h^2)\left((v-Z)\,(1+v'')-3\,(v')^2\right)^2} \\
&+\frac{h\,(h')^2\,(v-Z)\,(5-\eta_\psi)}{80\pi^2\,(1+h^2)^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)} \\
&+\frac{h\,(1+v'')\left(19+96h^2-\left(20h^2+\frac{23}{7}\right)\eta_\psi\right)}{640\pi^2\,(1+h^2)^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)} \\
&+\frac{2h'v'\left((1+11h^2)\eta_\psi-3(1+9h^2)\right)}{320\pi^2\,(1+h^2)^2\left((v-Z)\,(1+v'')-3\,(v')^2\right)}
\end{aligned}
\tag{4.39}
$$

The first line contains the canonical scaling. The other contributions are coming from the loops of the spin-2, -1 and -0. Additionally there are the contributions coming from the non diagonal terms of the second order expansion of the effective action (4.18). In this case in the loop is running the spin-1/2 and one among the spin-2, -1 or -0.

4.4 The anomalous dimensions in the diagonal cutoff.

In this section we want to complete the results of the previous computation with the diagonal cutoff, calculating the gravitationally induced contributions to the anomalous dimensions. The same comments of Section 3.3 are valid here. If we want to compute the anomalous dimensions using the flow of the effective action (1.69) and the background field technique, it is necessary to use a non-constant background. This procedure is expected to lead to much more involved computations. Instead, as we did for the flat space case (see again Section 3.3), it is easier to systematically address the problem of calculating the flow of the 2-point functions, using the momentum space technique. It should be clear that in order to do so, we need a simple way to work with the graviton propagator. A hint, coming from the previous computations of this chapter is to use the irreducible decomposition in which the metric perturbations is decomposed in

$$h_{\mu\nu} = \left(h^T_{\mu\nu}, \xi_\mu, \sigma, h\right) \tag{4.40}$$

Some considerations are useful at this point. The first thing we have to note is that the gravitational degrees of freedom mix with the scalar field. The reason is simple, the metric couples in a non-linear way with both the potential and the kinetic term of the scalar. In particular the potential interacts with the volume element via

$$\int \sqrt{g} V\left[\phi\right] \tag{4.41}$$

As long as the background configuration for ϕ is chosen to be constant and off-shell, we have in general that

$$V'\left[\phi\right] \neq 0 \tag{4.42}$$

and this implies that the perturbation φ of the scalar field mixes with the trace $h = h_\mu{}^\mu$ through the second order expansion

$$\int \frac{1}{2} h V'\left[\phi\right] \varphi \tag{4.43}$$

Since we are interested in the values of the anomalous dimensions η_ϕ and η_ψ at the VEV, we can put $V'\left[\phi\right] = 0$. In other words we will calculate them on-shell.

As a consistency check we will also extract the on-shell flow of $V\left[\phi\right]$, $V''\left[\phi\right]$ and $H\left[\phi\right]$ from the flow of 0- and 2-point functions. These coincide with those calculated in Subsection 4.4.1 provided $V'\left[\phi\right]$ is set to zero.

We just greatly simplified the gravi-scalar propagator, decoupling it into a gravitational and a scalar ones. The prescriptions on how the scalar propagator is constructed are the same, in momentum space and with flat background metric, of the previous chapter. We are left with the gravitational degrees of freedom that will be discussed in the next Subsection.

4.4.1 Coarse-graining of the graviton propagator.

We dedicate Appendix A to the construction of a set of irreducible spin-projectors, which allow to compactly include all the propagations of gravitational modes and still keep them separated. We will extensively use the results of Appendix A in this Subsection.

We recall from definitions (4.10), (4.11), (4.12) (4.15) and (4.16), that the action is truncated to be

$$\Gamma_k\left[\phi, \psi, g_{\mu\nu}\right] = \int \sqrt{g}\left(L_b + L_f + L_g + L_{gf}\right) + \text{ghost} \tag{4.44}$$

In order to apply our momentum space rules to it we have to take functional derivatives with respect to the fields around a certain background. The technique allows us to set the simple background for which $\phi = \text{const.}$, $\psi = 0$ and $g_{\mu\nu} = \delta_{\mu\nu}$. The momentum space propagators appearing in the flow of the n-point correlators are functions of their incoming momentum. This subsection, in particular, is dedicated to the study of the graviton propagator and its coarse graining.

Now the question is, what are the terms contributing to it? We see that ϕ appears quadratically in its kinetic term and has a potential

$$L_b = \frac{Z_\phi}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V[\phi] \tag{4.45}$$

The kinetic term, as long as the background is constant, cannot contribute, because it will always contain a $\partial_\mu \phi$ that goes to zero. Conversely, the potential term will through a second variation of the volume element. Due to the fact that $\phi = \text{const.}$ it will act like a sort of generalized cosmological constant. The spinor lagrangian, instead, will not contribute to the graviton propagator in the limit, because it is always proportional to the couple $\bar{\psi}$ times ψ, that goes to zero in the limit. Finally, the Einstein-Hilbert action for gravity will provide through second variation a kernel for the propagation of the graviton modes

$$\frac{\delta^2}{\delta h_{\mu\nu;q} \delta h_{\alpha\beta;-q}} \int \sqrt{g} L_g \tag{4.46}$$

Gauge invariance imply it will have zero modes corresponding to the transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\nabla_{(\mu} v_{\nu)}$ These are zero modes because the matrix of second variations gives zero when applied to them

$$\left(\frac{\delta^2}{\delta h_{\mu\nu;q} \delta h_{\alpha\beta;-q}} \int \sqrt{g} L_g \right) q_{(\mu} v_{\nu)} = 0 \tag{4.47}$$

This property is shared by any diffeomorphism invariant action. The issue of gauge invariance is solved by the gauge fixing term (4.15). In this way the non-trivial zero modes disappears. In the future calculations we will also consistently add a ghost action to take into account the induced functional determinant. The gauge fixing will depend on the two gauge parameters α and β.

We introduce now a complete set of spin-projectors P_i with $i = 2, 1, S, \sigma, S\sigma$ that are defined in Appendix A and in particular in (A.13). We can write down the inverse graviton propagator in terms of the spin-projectors

$$\frac{\delta^2}{\delta h_{\mu\nu;q} \delta h_{\alpha\beta;-q}} \Gamma_k = \sum_{i=2,1,S,\sigma,S\sigma} \Upsilon_i P_{i\mu\nu\alpha\beta} \tag{4.48}$$

The components Υ_i refer to the propagator components in the chosen basis, they depend on q^2 because any tensor structure is absorbed in the projectors. The projectors therefore depend on momentum q_μ and for the moment we neglect to write it for notational simplicity. We give a list of the actual value of the components in the chosen truncation and in general d dimensions

$$\begin{aligned}
\Upsilon_2 &= -\frac{1}{2}V[\phi] + \frac{1}{2}Zq^2 \\
\Upsilon_1 &= -\frac{1}{2}V[\phi] + \frac{1}{2\alpha}Zq^2 \\
\Upsilon_S &= \frac{d-3}{4}V[\phi] + \left(\frac{d-1}{16\alpha}(1+\beta)^2 + \frac{2-d}{2}\right)Zq^2 \\
\Upsilon_\sigma &= -\frac{1}{4}V[\phi] + \frac{(\beta-3)^2}{16\alpha}Zq^2 \\
\Upsilon_{S\sigma} &= \frac{\sqrt{d-1}}{4}V[\phi] + \frac{\sqrt{d-1}(\beta-3)(1+\beta)}{16\alpha}Zq^2
\end{aligned} \tag{4.49}$$

The other ingredient we need for average action computations in the coarse-graining in the form of an infrared cutoff ΔS_k. In particular, this term will contain a quadratic kernel $\mathcal{R}_k$ that kills the propagation of infrared modes. If we want to perform a computation that agrees with that of $\dot{V}[\phi]$ and $\dot{H}[\phi]$, we will follow the same strategy of the previous sections of this chapter. The key idea was to ask that the modified inverse propagator, so those in which the cutoff is added, is equal to the original inverse propagator apart for the replacement

$$q^2 \to P_k\left[q^2\right] = q^2\theta\left(q^2-k^2\right) + k^2\theta\left(k^2-q^2\right) \tag{4.50}$$

that realize the same of (4.37).

We express the cut-off procedure as we previously did by

$$q_\mu \to \hat{q}_\mu\sqrt{P_k\left[q^2\right]} \tag{4.51}$$

The nice feature of replacement (4.50,4.51) is that it leaves the spin-projectors of Appendix A invariant. This is easily seen by their building blocks P_L and P_T defined in (A.2). For example P_L goes to

$$P_{L\mu\nu} = \frac{q_\mu q_\nu}{q^2} \to \hat{q}_\mu\hat{q}_\nu\frac{\sqrt{P_k}^2}{P_k} = P_{L\mu\nu} \tag{4.52}$$

and similarly for P_T.

Using the same notation of the previous chapters we define the modified inverse propagator for the gravitational degrees of freedom

$$G^{h\,-1} = \Gamma_k^{(2)} + \mathcal{R}_k \tag{4.53}$$

In components it is

$$G^{h\,-1}{}_{\mu\nu\alpha\beta} = \sum_{i=2,1,S,\sigma,S\sigma}\bar{\Upsilon}_i P_{i\mu\nu\alpha\beta} \tag{4.54}$$

and the single terms are

$$\begin{aligned}
\bar{\Upsilon}_2 &= -\frac{1}{2}V[\phi] + \frac{1}{2}Zq^2\theta\left(q^2-k^2\right) + \frac{1}{2}Zk^2\theta\left(k^2-q^2\right) \\
\bar{\Upsilon}_1 &= -\frac{1}{2}V[\phi] + \frac{1}{2\alpha}Zq^2\theta\left(q^2-k^2\right) + \frac{1}{2\alpha}Zk^2\theta\left(k^2-q^2\right) \\
\bar{\Upsilon}_S &= \frac{d-3}{4}V[\phi] + \left(\frac{d-1}{16\alpha}(1+\beta)^2 + \frac{2-d}{2}\right)Zq^2\theta\left(q^2-k^2\right) \\
&\quad + \left(\frac{d-1}{16\alpha}(1+\beta)^2 + \frac{2-d}{2}\right)Zk^2\theta\left(k^2-q^2\right) \\
\bar{\Upsilon}_\sigma &= -\frac{1}{4}V[\phi] + \frac{(\beta-3)^2}{16\alpha}Zq^2\theta\left(q^2-k^2\right) + \frac{(\beta-3)^2}{16\alpha}Zk^2\theta\left(k^2-q^2\right) \\
\bar{\Upsilon}_{S\sigma} &= \frac{\sqrt{d-1}}{4}V[\phi] + \frac{\sqrt{d-1}(\beta-3)(1+\beta)}{16\alpha}Zq^2\theta\left(q^2-k^2\right) \\
&\quad + \frac{\sqrt{d-1}(\beta-3)(1+\beta)}{16\alpha}Zk^2\theta\left(k^2-q^2\right)
\end{aligned} \tag{4.55}$$

The cutoff that realizes this structure can be determined backward by solving the definition of G^h for it. We get $\mathcal{R}_k = G^{h\,-1} - \Gamma_k^{(2)}$ and in components it means

$$\mathcal{R}_{k\ \mu\nu\alpha\beta} = \sum_{i=2,1,S,\sigma,S\sigma} \left(\bar{\Upsilon}_i - \Upsilon_i\right) P_{i\mu\nu\alpha\beta} \tag{4.56}$$

The cutoff explicit components are easily obtained by subtracting the two sets of components (4.55) and (4.49). Note the important property that the cutoff do not depend on the background constant scalar field, but only on the background metric $\delta_{\mu\nu}$.

Finally, one is actually interested in the modified propagator rather than the modified inverse propagator, when computing with the average effective action. To calculate the inverse we have to resort to the rules we derived in the previous subsection. The spin-2 and spin-1 modes are easily inverted, while the scalar sector corresponds to the inversion of a 2×2 matrix. The coefficients of $G^{h\,-1}$ are actually functions of theta functions. We therefore have first to algebrically invert each coefficient and then support-wise calculate it. We define

$$G^h{}_{\mu\nu\alpha\beta} = \sum_{i=2,1,S,\sigma,S\sigma} \tilde{\Upsilon}_i P_{i\mu\nu\alpha\beta} \tag{4.57}$$

For $i=2,1$ we previously proved that $\tilde{\Upsilon}_i = \bar{\Upsilon}_i^{-1}$. For example

$$\bar{\Upsilon}_2 = \frac{1}{2}\left(Zq^2 - V[\phi]\right)\theta\left(q^2-k^2\right) + \frac{1}{2}\left(Zk^2 - V[\phi]\right)\theta\left(k^2-q^2\right) \tag{4.58}$$

Now that we wrote it disentangling the two supports, we easily calculate the inverse

$$\begin{aligned}
\tilde{\Upsilon}_2 &= \bar{\Upsilon}_2^{-1} \\
&= 2\left(Zq^2 - V[\phi]\right)^{-1}\theta\left(q^2-k^2\right) + 2\left(Zk^2 - V[\phi]\right)^{-1}\theta\left(k^2-q^2\right)
\end{aligned} \tag{4.59}$$

Similarly we can perform the same steps for the other coefficients. The only additional complication for the scalar sector are more involved functions of the theta functions, but that can unambiguously be inverted.

We conclude this subsection with a couple of comments. A crucial simplification was that we could take $V'[\phi] = 0$, given the purpose of it was to calculate the anomalous dimensions in the vacuum expectation value of the potential. For this reason our construction is on-shell, but it is all we need to be consistent with the powerlaw expansions we will later introduce in the potential. One may wonder how to completely stay off-shell, by means of a general expansion around a given ϕ configuration that is not a VEV. A straightforward way to do that is to introduce new spin-projectors in the algebra of the P_i for $i = 2, 1, S, \sigma, S\sigma$. A new projector, let us call it P_ϕ, would take into account the propagation of ϕ "polarizations" of the scalar subsector and naturally would mix with P_S and P_σ. This easily imply that the actual basis is completed adding two analogues of $P_{S\sigma}$, that are $P_{S\phi}$ and $P_{\sigma\phi}$. One can show that there is a consistent way to do that and that, once represented, the set P_i for $i = S, \sigma, \phi, S\sigma, S\phi, \sigma\phi$ realizes the same 3×3 matrices appearing as scalar sector of [22]. We ultimately plan to give the full off-shell computation somewhere.

4.4.2 Consistency checks from 0- and 2-point functions.

We constructed the coarse-grained graviton propagator. We now refer to Appendix B for the vertices involving interactions of gravity and matter. We therefore have all the ingredients to evaluate the anomalous dimensions. However before doing that it is worth considering some computation in which we know we can check explicitly the consistency of the momentum space method, with the "super-matrix" technique used in the beginning of the chapter.

There are essentially three checks we can perform: the 0-point function, the 2-point function of the scalar and the 2-point function of the spinor fields. From the flow of the 0-point correlator, the action, we can obtain $\dot{V}[\phi]$. Since our momentum space rules are on-shell, we can only obtain

$$\dot{V}[\phi]\Big|_{V'[\phi]=0} \tag{4.60}$$

and compare it with the result of the super-matrix technique with a diagonal cutoff. There are four graphs contributing to this result, each of them corresponding to a loop with a cutoff insertion. The modes that will run in the loop are obviously the scalar, spinor, graviton and ghosts ones.

We already computed the scalar and spinor loops in the previous chapter, with the same momentum space rules, so we can concentrate our attention on the new degrees of freedom introduced in this section. We denote $\Delta_h \dot{V}$ and $\Delta_{\rm gh} \dot{V}$ the contributions to the running of the potential due to graviton and ghosts respectively. We postpone to the end of the subsection the discussion of the ghost loop. The graviton loop corresponds to the integration of the graviton modified propagator and its cutoff (Fig. 4.1). The projectors are useful in this situation, because the argument of the integration reduces to traces of products of them. The spin-2 and spin-1 projectors decouple because they commute with the rest, while the spin-0 projectors can be represented using the 2×2 matrices we introduced. Using the notation we introduced in section 4.5.1, the loop

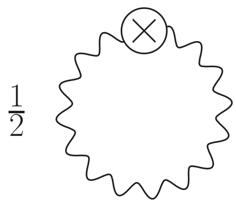

Figure 4.1: The graviton loop contributing to $\dot{\Gamma}_k$ at $V'[\phi] = 0$.

integral is

$$
\begin{aligned}
\Delta^h \dot{V} &= \frac{1}{2}\int_q G^h{}_{\mu\nu}{}^{\alpha\beta}\dot{\mathcal{R}}_{k\alpha\beta}{}^{\mu\nu} \\
&= \frac{1}{2}\int_q \sum_{i,j=2,1,S,\sigma,S\sigma} \tilde{\Upsilon}_i \partial_t \left(\bar{\Upsilon}_j - \Upsilon_j\right) \mathrm{tr} P_i P_j \\
&= \frac{1}{2}\int_q \tilde{\Upsilon}_2 \left(\bar{\Upsilon}_2 - \Upsilon_2\right) + \frac{1}{2}\int_q \tilde{\Upsilon}_1 \left(\bar{\Upsilon}_1 - \Upsilon_1\right) \\
&\quad + \frac{1}{2}\int_q \sum_{i,j=S,\sigma,S\sigma} \tilde{\Upsilon}_i \partial_t \left(\bar{\Upsilon}_j - \Upsilon_j\right) \mathrm{tr} P_i P_j
\end{aligned} \tag{4.61}
$$

The traces of the projectors, in particular the spin-0 sector that we need, can be derived easily. It is now an algebraic task to perform them. Once this is done we can proceed through the integration.

We give the result in the gauge $\beta = 1$ for simplicity, while α is left arbitrary. In terms of the renormalized quantities, the contribution to the renormalized potential running is

$$
\Delta^h \dot{\bar{v}}_R = \frac{(Z + \frac{\dot{Z}}{8})(5Z - (2+3\alpha)\bar{v}_R)}{12\pi^2 (Z - \bar{v}_R)(Z - \alpha \bar{v}_R)} \tag{4.62}
$$

In order to check the consistency with the previous calculation we have to add (3.106) in the on-shell limit $v' = 0$ to (4.62) and compare it with formula (4.38) in the same limit. When this is done, it is easy to show that the matrix-method and the momentum space give indeed the same result.

A completely similar discussion can be done by checking the flow of the second derivative of the potential $\Delta^h \dot{V}''$, through the flow of the scalar correlator. There are three possible graphs contributing to this flow. Two contain a graviton emitted and absorbed by the scalar and are different because the cutoff insertion is either on the scalar (Fig. (4.2)) or on the graviton (Fig. (4.3)). The third is a tadpole-like graph, where in the loop runs the graviton (Fig. (4.4)). All the graviton interactions are derivative interaction, but fortunately the calculation is strongly simplified by the fact that we are, for the moment, interested only in the zero external momentum limit. The result of the sum of these three graphs can be compared with the second derivative of (4.38) in the limit $\bar{v}'_R \to 0$. It is important to perform the limit after deriving the expression of the flow, otherwise some terms will be missed. For this reason, this check gives different informations from the previous one.

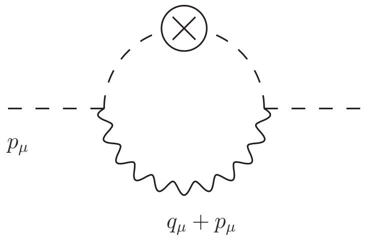

Figure 4.2: One of the graviton contribution to the scalar 2-point function. In this contribution the cutoff is inserted in the scalar propagator.

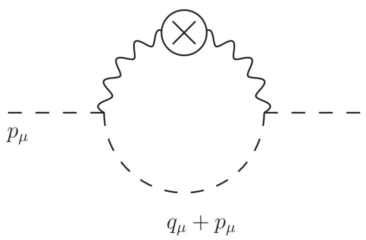

Figure 4.3: One of the graviton contribution to the scalar 2-point function. In this contribution the cutoff is inserted in the graviton propagator.

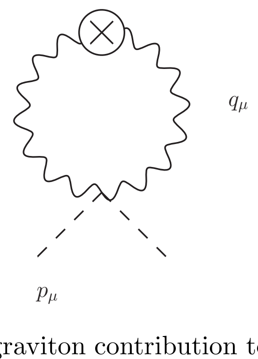

Figure 4.4: Tadpole-like graviton contribution to the scalar 2-point function.

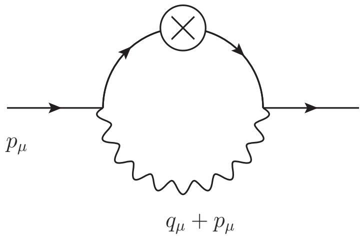

Figure 4.5: One of the graviton contribution to the spinor 2-point function. In this contribution the cutoff is inserted in the spinor propagator.

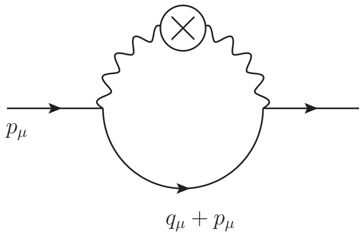

Figure 4.6: One of the graviton contribution to the spinor 2-point function. In this contribution the cutoff is inserted in the graviton propagator.

The third and final check is analogous to the second, but involves the spinor degrees of freedom rather that the scalar ones. From the two point function of the spinor field, calculated using our momentum space rules, we can determine the gravitational contribution to the flow of the generalized Yukawa interaction $\Delta^h \dot{H}$ evaluated at $V' \to 0$. Then we can compare it with the corresponding flow obtained with the matrix-technique. Again there are three diagrams contributing to the flow and they are of the same form of the previous check, once the scalar field is replaced by the spinor one. The tensor structure of the spinor vertices is quite more involved than that of the scalar, but still we can set zero external momentum that simplify to some extent the calculation. The diagrams involved in the calculation are drawn in Figs. (4.5), (4.6) and (4.7).

Some words concerning the ghost loop contribution $\Delta^{\mathrm{gh}} \dot{V}$ is needed before

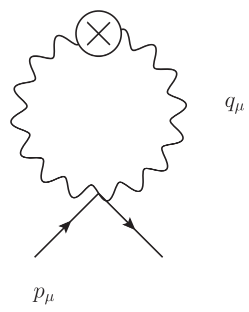

Figure 4.7: Tadpole-like graviton contribution to the spinor 2-point function.

concluding the subsection. If the ghost fields are introduced in a functional RG scheme, it is natural to assume that they may run as well, with the rest of the degrees of freedom. Once their action is written, one has to decide how to truncate it. There are some natural choices. In the matrix-technique we always fixed the wavefunction renormalization of the ghost $Z_{\rm gh} = 1$ Alternatively, we could decide to give them a separate wavefunction renormalization $Z_{\rm gh}$. This is currently under study in the literature [45, 46]. Or, in order to keep things as simple as possible, we can fix their scaling by requiring that it is the same as the graviton one $Z_{\rm gh} = Z$. In functional RG studies, this situation is different from $Z_{\rm gh} = 1$. Although a wavefunction renormalization can be parametrized away by a rescaling of the fields, this must be done at each scale. In fact the ghost loop is different in the two situations. This is easily seen because, if $Z = \frac{1}{16\pi G}$ is present in front of the ghost propagator, the derivative of the ghost cutoff will contain $\eta_Z = -\frac{\dot{G}}{G}$ that introduce some further nonlinearity in the flow. For consistency with the super-matrix technique we decided to set $Z_{\rm gh} = 1$ and it is trivial to show that the momentum space technique gives the same result. However, when explicitly using the beta functions for the purpose of finding fixed points, we always checked that the possibilities $Z_{\rm gh} = Z$ and $Z_{\rm gh} = 1$ do give very similar values for FP positions and critical exponents. Given that the ghosts do not couple with the (background) matter fields, this is also the last time we will actually use the ghost loop, because it does not contribute to the two point functions running of ϕ and ψ.

4.4.3 The flow of the scalar 2-point function.

In this subsection we finally calculate the gravitational corrections to the scalar anomalous dimension. We refer for the diagrammatic part to the graphs introduced in the previous section, namely Figs. (4.2), (4.3) and (4.4). The computation of the anomalous dimension of the scalar is quite more involved than that of the flow of the potential and its second derivative. The reason is that we obtain it, as usual, from the coefficient of the p^2 term in the flow of the 2-point function with incoming momentum p_μ

$$\dot{Z}_\phi p^2 + \dot{V}''[\phi] \tag{4.63}$$

It means that in Fig. (4.2) we have to expand the graviton propagator to order p^2.

The expansion of the graviton propagator imply that we have to simultaneously expand the coefficients of the projectors and the projectors themselves. The propagator in momentum $q_\mu + p_\mu$ is

$$G^h_{q+p\ \mu\nu\alpha\beta} = \sum_{i=2,1,S,\sigma,S\sigma} \tilde{\Upsilon}_i\left[q+p\right] P_i\left[q+p\right]_{\mu\nu\alpha\beta} \tag{4.64}$$

In this formula we simply introduced the explicit dependence on the incoming momentum. The expansion of the coefficients $\tilde{\Upsilon}_i\left[q+p\right]$ resembles that of the modified inverse scalar propagator $G_{k,q+p}$. They are functions of the momentum square $\left(q+p\right)^2$ through the support functions $\theta\left(k^2-\left(q+p\right)^2\right)$ and $\theta\left(\left(q+p\right)^2-k^2\right)$. Their general structure is

$$\tilde{\Upsilon}_i\left[q\right] \;\; = \;\; F_i\left(q^2\right)\theta\left(q^2-k^2\right) + F_i\left(k^2\right)\theta\left(k^2-q^2\right) \tag{4.65}$$

This kind of momentum dependence slightly simplifies the expansion, because, at first order, we get only terms that are out of the support $q^2 \leq k^2$ set by the cutoff. The general expansion up to quadratic powers of p_μ is

$$\begin{aligned}\tilde{\Upsilon}_i\left[q+p\right] &= \tilde{\Upsilon}_i\left[q\right] + 2\left(q\cdot p\right)F_i'\left(q^2\right)\theta\left(q^2-k^2\right)\\ &\quad +2\left(q\cdot p\right)^2\left(F_i'\left(k^2\right)\delta\left(q^2-k^2\right)+F_i''\left(q^2\right)\theta\left(q^2-k^2\right)\right)\\ &\quad +p^2F_i'\left(q^2\right)\theta\left(q^2-k^2\right)\end{aligned} \tag{4.66}$$

Like in the third chapter, we are not really interested in the tangent space indices structure of the result, so we can adopt the approximated diagonal expansion where

$$p_\mu p_\nu \quad \rightarrow \quad \frac{p^2}{4}\delta_{\mu\nu} \tag{4.67}$$

and actually replace $\tilde{\Upsilon}_i\left[q+p\right]$ of (4.66) by

$$\begin{aligned}&\tilde{\Upsilon}_i\left[q\right] + 2\left(q\cdot p\right)F_i'\left(q^2\right)\theta\left(q^2-k^2\right)\\ &+p^2\frac{q^2}{2}\left(F_i'\left(k^2\right)\delta\left(q^2-k^2\right)+F_i''\left(q^2\right)\theta\left(q^2-k^2\right)\right)\\ &+2p^2F_i'\left(q^2\right)\theta\left(q^2-k^2\right)\end{aligned} \tag{4.68}$$

that slightly simplifies computations.

The expansion of the projectors requires quite more effort and is structurally more complicated. We are not going to give the full expansion of the projectors now, because it is long and not that instructive. We just want to stress that it still simplifies, but the resulting expressions are again very long and involved. Further, as an additional tensor structure to take into account, there is the fact that the vertices where matter couples to gravity are derivative vertices. In summary, there are two components of any graph that do contribute to the term p^2 and are the propagator displaced to momentum $q_\mu + p_\mu$, that itself contains projectors and coefficients to be expanded, and the vertices. p^2 terms are formed also by their mixing, so it is an hard task to find all of them because we have to combine all the long expansion together.

The situation is simpler when evaluating Fig. (4.3) and Fig. (4.4). In the first case Fig. (4.3) we just have to expand the scalar propagator, because we always parametrize the graphs to have the integrated momentum as argument of the cutoff. The gravitational tensor structure, that is the product of two propagators with a cutoff inserted in the middle, may be strongly simplified using the projectors products. It is a general fact, that any polynomial of the projectors can be reduced to a sum of products of one or two of them. In the second case Fig. (4.4), the calculation is even simpler, because the only object that carry a p_μ dependence in the graph is the 4-vertex and therefore it is sufficient to find its p^2 contribution and isolate it.

Now we just want to give the results associated to each graph. We call $\Delta_{h,1}\eta_\phi$ that coming from Fig. (4.2), $\Delta_{h,2}\eta_\phi$ from Fig. (4.3) and $\Delta_{h,3}\eta_\phi$ from

Fig. (4.4). We obtain in the gauge $\alpha = 0$ and $\beta = 1$

$$\begin{aligned}
\Delta_{h,1}\eta_\phi &= \frac{v''\left(Z\left(-3v''+\eta_\phi-6\right)-(\eta_\phi-6)v\right)}{32\pi^2(Z-v)^2\left(1+v''\right)^2} \\
\Delta_{h,2}\eta_\phi &= -\frac{Zv''\left((\frac{\dot{Z}}{Z}+11)v''+\frac{\dot{Z}}{Z}+8\right)}{32\pi^2(Z-v)^2\left(1+v''\right)^2} \\
\Delta_{h,3}\eta_\phi &= 0
\end{aligned} \tag{4.69}$$

Notice that we have $\Delta_{h,3}\eta_\phi = 0$ independently of the gauge choice. It is zero because we are in four dimensions. We explicitly checked that for general $d \neq 4$ it is proportional to $d-4$. Concerning the entire system (4.69) we want to stress that the complete form with α, β and eventually d dependence is quite complicated. We decided to present it in the gauge $\alpha = 0$ and $\beta = 1$ mainly for illustrative purposes.

4.4.4 The flow of the spinor 2-point function.

This subsection completes the results of the previous one giving the gravitational contributions to the spinor anomalous dimension η_ψ. Much of the comments we made previously are valid, especially those on how to expand the graviton propagator. The calculation of the spinor anomalous dimension is slightly simplified, if compared to that of η_ϕ, because we just need the first order in p_μ expansion given the flow of the spinor propagator is

$$\dot{Z}_\psi \not{p} + \dot{H}\left[\phi\right] \tag{4.70}$$

It means that it is sufficient to expand everything at first order and a lot of tensor complications, that we previously mentioned, do not even appear. However, the Clifford algebra structure of the vertices and of the propagators do appear and makes the calculation of η_ψ much more involved than that of η_ϕ.

We refer to the graphs we drew before Figs. (4.5), (4.6) and (4.7). These contain all the anomalous dimension contributions. We will call $\Delta_{h,1}\eta_\psi$ that coming from Fig. (4.5), similarly $\Delta_{h,2}\eta_\phi$ from Fig. (4.6) and $\Delta_{h,3}\eta_\phi$ from Fig. (4.7). The result is

$$\begin{aligned}
\Delta_{h,1}\eta_\psi &= \frac{(66-13\eta_\psi)h^2+\eta_\psi-6}{320\pi^2\left(1+h^2\right)^2(Z-v)} \\
\Delta_{h,2}\eta_\psi &= \frac{3Z\left(4(\frac{\dot{Z}}{Z}+12)h^2+3(\frac{\dot{Z}}{Z}+9)\right)}{1280\pi^2\left(1+h^2\right)(Z-v)^2} \\
\Delta_{h,3}\eta_\psi &= -\frac{9Z(\frac{\dot{Z}}{Z}+8)}{512\pi^2(Z-v)^2}
\end{aligned} \tag{4.71}$$

The same remarks of system (4.69) are valid here. Namely, in a general gauge (4.71) is a huge and complicated system and we gave the result in the gauge $\alpha = 0$ and $\beta = 1$ for brevity. We want also to remember that, for consistency with the rest, the anomalous dimensions have to be evaluated in a fixed constant on-shell configuration.

4.5 The flow of the Newton constant.

In this section we wish to calculate the flow of the Newton constant G, that in our parametrization of the couplings is inside $Z = \frac{1}{16\pi G}$ [2, 47]. However, we will apply a different method than the momentum space rules, that is actually closer to the super-matrix technique we adopted at the beginning of the chapter. The reason is that the calculation is a lot simpler if heat kernel methods are applied. Before going into the details of the calculation, let us explain why this happens.

We want to remind that we introduced the background field method in the context of functional RG in the third chapter. Using that technique one generally has a bimetric functional $\Gamma_k [\bar{g}_{\mu\nu}, g_{\mu\nu}]$, where $\bar{g}_{\mu\nu}$ is the background metric and $g_{\mu\nu}$ the classical one (3.158). The essence of the method is that background gauge invariance is kept through the construction of the average effective action, thanks to the fact that the cutoff is background gauge covariant. It has therefore a kernel that is background dependent $\mathcal{R}_k = \mathcal{R}_k [\bar{g}]$. A bimetric functional is quite a complicated object to study, so we adopted another functional that depends on a single metric (3.164) defining $\Gamma_k [g_{\mu\nu}] = \Gamma_k [g_{\mu\nu}, g_{\mu\nu}]$. This functional was shown to have a flow equation analog to the ERGE (3.166), but by construction it has a metric dependence in the cutoff. This dependence is fundamental to obtain a gauge invariant result, because it ensures background gauge invariance, that in the single metric functional becomes the gauge invariance itself. However, it also makes the diagrammatic computations we extensively used before a lot more complex.

As an example of this feature, suppose we want to calculate the running of the Newton constant, using the 2-point function of the metric $g_{\mu\nu}$. We therefore have to act with a couple of functional derivatives on the flow equation, but, unlike the cases we studied up to now, this functional derivative will also act on the cutoff function. This function is present in two places of the flow equation. It is at the denominator and its derivative, in this case, will contribute modifying the 3-vertex. It is also at the numerator and in that case the derivative will give a new genuine vertex contributing to the flow. For the discussion on the gauge invariances, these new contributions have to sum up in such a way that the result is gauge invariant. However, from the point of view of the calculation, it is quite nontrivial to obtain the correct result, because it requires a lot more effort in the construction of the new vertices, and in fact it is the subject of another Ph.D. defense [48]. We checked by hand that, if the metric dependence of the cutoff is neglected, the results are not anymore gauge invariant. For example, each spin component of the graviton receives a different wavefunction renormalization.

Instead, we decided to use a different strategy. We will try now to reformulate all the traces involved in the computation of the flow equation, in terms of functional traces that can be easily evaluated by heat kernel method [24, 47]. The clear advantage of this procedure is that heat kernel coefficients have already been evaluated in the literature for many classes of operators.

We first notice that we are interested in finding out curvature terms from the flow equation, now regarded as the functional trace of a certain set of operators that we have to specify. We are free to choose a preferred background space to evaluate the traces. The background must contain enough information to distinguish the Ricci scalar from the volume element. We decided to take a sphere of radius r as a background. The limit $r \to \infty$ gives the possibility to compare our findings with those of the rest of the section. In fact, in the Seeley-deWitt expansion of the heat kernel, the curvature term will appear through $R \sim 1/r^2$ and one easily sees that it is sent to zero in the limit.

We can now define the background for the matter fields. We can safely set $\psi = \bar{\psi} = 0$ as background condition. Further, we decided to set $\phi = \text{const.}$ as in all previous calculations. The ϕ dependence of the running of Z will be treated with the same spirit of Section 3.3. It will, for example, give a different running of Z when considering a zero or a nonzero VEV. This time, however, ϕ is not defined to be a minimum $V'[\phi] = 0$ of V like in the momentum space technique, but it will be general like in the super-matrix application. The result will be therefore completely off-shell.

We will also employ the spin decomposition of the modes of the fluctuation, this time expressed in covariant form for any background

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \nabla^2 \sigma + \frac{1}{4} g_{\mu\nu} h \tag{4.72}$$

The scalar modes of the graviton will mix with the scalar matter field. After the second order expansion of our action, there are in general further mixing of the spinor fluctuation fields χ and $\bar{\chi}$, with the spin-0, -1 and -2. However, due to Lorentz invariance these are expected to be proportional to the background fields ψ and $\bar{\psi}$. The simple reason is that there are spinor indices to be saturated in the expression. These mixing disappear in the chosen background. We also rescale the fields σ and ξ_μ to $\hat{\sigma}$ and $\hat{\xi}_\mu$ using

$$\hat{\sigma} = \sqrt{-\nabla^2} \sqrt{-\nabla^2 - \frac{R}{3}} \sigma \tag{4.73}$$

$$\hat{\xi}_\mu = \sqrt{-\nabla^2 - \frac{R}{4}} \xi_\mu \tag{4.74}$$

This rescaling produces a Jacobian in the path integral, that cancels the Jacobian produced by (4.72).

The final result for the second order expansion is an expansion that is diagonal in the spin-sectors of the theory, rather than in field space. We get

$$\begin{aligned} \mathcal{L}^{(2)} = & \frac{1}{2} h^{T\mu\nu} \left(-\frac{Z}{2} \nabla^2 + \frac{Z}{3} R - \frac{1}{2} V \right) h^T_{\mu\nu} \\ & + \frac{1}{2} \hat{\xi}^\mu \left(-\frac{Z}{\alpha} \nabla^2 - \frac{Z(1-2\alpha)}{4\alpha} R - V \right) \hat{\xi}_\mu \\ & + \frac{1}{2} \begin{pmatrix} \hat{\sigma} & h & \varphi \end{pmatrix} S\left[\nabla^2, R\right] \begin{pmatrix} \hat{\sigma} \\ h \\ \varphi \end{pmatrix} \\ & + \begin{pmatrix} \chi & \bar{\chi}^T \end{pmatrix} Y\left[\nabla^2, R\right] \begin{pmatrix} \chi^T \\ \bar{\chi} \end{pmatrix} \end{aligned} \tag{4.75}$$

where $R = R[g]$ is the curvature scalar of the sphere metric. We employed the matrix notation

$$S\left[\nabla^2,R\right]= \tag{4.76}$$
$$\begin{pmatrix} \left(\frac{3(\alpha-3)Z}{16\alpha}\right)\nabla^2+\frac{3(\alpha-1)R}{16\alpha}-\frac{3}{8}V & \frac{3Z(\beta-\alpha)}{16\alpha}\sqrt{-\nabla^2}\sqrt{-\nabla^2-\frac{R}{3}} & 0 \\ \frac{3Z(\beta-\alpha)}{16\alpha}\sqrt{-\nabla^2}\sqrt{-\nabla^2-\frac{R}{3}} & \frac{Z\left(3\alpha-\beta^2\right)}{16\alpha}\nabla^2+\frac{1}{8}V & \frac{1}{2}V' \\ 0 & \frac{1}{2}V' & -Z_\phi\nabla^2+V'' \end{pmatrix}$$

$$Y\left[\nabla^2,R\right] = \begin{pmatrix} 0 & iZ_\psi\gamma^\mu\nabla_\mu-H[\phi] \\ iZ_\psi\gamma^{T\mu}\nabla_\mu+H[\phi] & 0 \end{pmatrix} \tag{4.77}$$

for brevity.

We write also the ghost sector. We decompose the ghost fields c_μ and $\bar{c}_\mu$ in irreducible representations $c_\mu=c_{1\mu}+\nabla_\mu c_0$ and $\bar{c}_\mu=\bar{c}_{1\mu}+\nabla_\mu\bar{c}_0$. We also rescale the scalar modes $\hat{c}_0=\sqrt{-\nabla^2}c_0$ and $\hat{\bar{c}}_0=\sqrt{-\nabla^2}\bar{c}_0$. The combination of change of coordinates and rescaling has unit Jacobian in the path integral. The ghost action becomes

$$S_{\text{gh}} = \bar{c}_1^\mu\left(Z\nabla^2+\frac{R}{4}\right)c_{1\mu} \tag{4.78}$$
$$+\hat{\bar{c}}_0\left(-\frac{3-\beta}{2}Z\nabla^2-\frac{3-\beta}{4}R\right)\hat{c}_0 \tag{4.79}$$

The spin sectors are decoupled thanks to the background choice. Further also the ghosts are decoupled. The total decoupling involves therefore both spin and statistics. This is a useful feature, that we want to preserve after adding the cutoff structure.

Let us concentrate first on the second order expansions that contain $-\nabla^2$, so all the spin blocks apart for the spinor one. For those blocks, we choose the cutoff matrix R_k such that

$$\Gamma^{(2)}+R_k=\Gamma^{(2)}\Big|_{-\nabla^2\to P_k[-\nabla^2]} \tag{4.80}$$

It is understood that spinors are left apart for a moment. This allows to determine backwards the structure of the cutoff, so we regard it as its definition.

The spinors play a distinguished role, because their quadratic kernel is a function of $\not\nabla=\gamma^\mu\nabla_\mu$ and not of $-\nabla^2$. We know that the square of their free propagator is

$$(i\not\nabla)^2 = -\nabla^2+\frac{R}{4} \tag{4.81}$$

This means that in curved space this square is displaced by the simple $-\nabla^2$ by a curvature term. In order to take this into account, we wish that the spinor cut-off r realizes a similar relation

$$\left(i\not\nabla+r\left(\not\nabla\right)\right)^2 = P_k\left[-\nabla^2\right]+\frac{R}{4} \tag{4.82}$$

We determine backward the spinor cutoff from this relation and we choose the prescription for the determination of the entire cutoff matrix to be

$$\Gamma^{(2)}+R_k = \Gamma^{(2)}\Big|_{-\nabla^2\to P_k[-\nabla^2],\, i\not\nabla\to i\not\nabla\frac{\sqrt{P_k[-\nabla^2]+\frac{R}{4}}}{\sqrt{-\nabla^2+\frac{R}{4}}}} \tag{4.83}$$

The profile of the cut-off propagator P_k is as usual the optimized one (4.23).

Before going on, it is easy to see that these cutoff, once the flat limit is taken, do indeed coincide with the coarse-graining schemes we used for flat backgrounds. It is sufficient to perform the limit of infinite radius of the sphere, so that $R \to 0$ and $\nabla_\mu \to \partial_\mu$. In this limit the cutoff definition we just give coincides with that of the diagonal cutoff computation. In this way we are sure that the next calculations are consistent with those performed in flat background.

Due to the diagonal structure of the second order expansion and the cutoff, the trace of the exact renormalization group equation will reduce to a sum of traces in each spin-statistic subsector. If we further trace over the 3×3 matrix structure of the spin-0, all the subsectors apart from the spinor one, give rise to traces of functions of $-\nabla^2$, which can be easily computed with the aid of heat kernel method (see Appendix C). This actually happens also for the spinor subsector once the trace over the Clifford algebra, that we denote by tr, is performed. First, notice that the Clifford algebra structure appears only through the operator $\not\nabla$. Any trace of any number of copies of $\not\nabla$, will reduce to functions of ∇^2 thanks to the properties of the traces of γ-matrices. Given there are at most two elements of the algebra in the flow equation, we actually just need the single traces $\not\nabla = 0$ and any time $(\not\nabla)^2$ appears it is sufficient to use (4.81).

The final result of these manipulations, is that it is possible to write in our background ansatz the flow of the effective action as

$$\dot{\Gamma}_k \;=\; \sum_{j=0,\frac{1}{2},1,2,0_{\mathrm{gh}},1_{\mathrm{gh}}} \mathrm{STr}\, f_j\left[-\nabla^2, R\right] \tag{4.84}$$

The functions f_j contain all the information about the flow, including both the profile of the cutoff function and the structures emerging from the trace over 3×3 spin-0 and Dirac indices. We introduced the notation 0_{gh} and 1_{gh}, referring to the spin-0 and -1 of the ghost field, to stress that they are decoupled from the rest and to remember that they have a different statistic. Additionally, we made explicit a curvature dependence of the functions f_j. This dependence refers to every R appearing in the flow equation, before acting with the functional traces. The super-trace appearing in (4.84) is now simply a sum of functional traces over the operator $-\nabla^2$, each acting on a different vector space of spin-j.

We are interested in the terms of this flow that are linear in the curvature R. It is useful to expand the functions f_j in powers of the scalar R, before acting with the trace. We get

$$\begin{aligned}\dot{\Gamma}_k \;&=\; \sum_j \mathrm{STr} f_j\left[-\nabla^2, 0\right]\\ &\quad + R \sum_j \mathrm{STr} f_j^{(0,1)}\left[-\nabla^2, 0\right] + \mathcal{O}\left(R^2\right)\end{aligned} \tag{4.85}$$

where it is understood that j runs over $j = 0, 1/2, 1, 2, 0_{\mathrm{gh}}, 1_{\mathrm{gh}}$. The further terms appearing in the expansion are beyond our curvature truncation.

We evaluate the remaining two traces using heat kernel techniques, in particular using the Seeley-deWitt expansion (C.20). In this case, the Seeley-deWitt expansion, coincides with an expansion in powers of R, due to the structure of the background. In four dimensions we have

$$\begin{aligned}\dot{\Gamma}_k &= \sum_j (-)^j B_{0,j} Q_2 (f_j [z,0]) + \sum_j (-)^j B_{2,j} Q_1 (f_j [z,0]) \\ &\quad + R \sum_j (-)^j B_{0,j} Q_2 \left(f_j^{(0,1)} [z,0] \right) + \mathcal{O}\left(R^2\right) \end{aligned} \tag{4.86}$$

We introduced the heat kernel coefficients $B_{n,j}$ of the Laplacian operator in the spin-j space (C.15). They are

$$\begin{array}{ll} B_{0,0} = 1 & B_{0,1/2} = 4 \\ B_{0,1} = 3 & B_{0,2} = 5 \\ B_{2,0} = \dfrac{R}{6} & B_{2,1/2} = \dfrac{2R}{3} \\ B_{2,1} = \dfrac{1}{2} & B_{2,2} = \dfrac{5}{6} \end{array}$$

The functions Q_m are certain integrals of their argument, that contain the information about the shape of the functions we are tracing (C.21,C.22). We give them also here

$$\begin{aligned} Q_2 (g [z]) &= \int dz\, z\, g(z) \\ Q_1 (g [z]) &= \int dz\, g(z) \end{aligned}$$

Finally, we added $(-)^j$ to the traces, referring that anticommuting fields will have an additional minus sign. Among all the heat kernel coefficients, we know that $B_{0,j} \sim R^0$ and $B_{2,j} \sim R$, while any other term in the Seeley-deWitt expansion will contain higher powers of R.

We therefore know that the terms linear in R of (4.86) are

$$\dot{\Gamma}_k \Big|_R = \sum_j (-)^j B_{2,j} Q_1 (f_j) + R \sum_j (-)^j B_{0,j} Q_2 \left(f_j^{(0,1)} [z,0] \right) \tag{4.87}$$

but we also know, that, according to our truncation $\dot{\Gamma}_k \Big|_R = -\dot{Z} \int \sqrt{g} R$. Once the integrals are performed, we are able to write the beta function of the gravitational coupling.

We give the running of the coupling Z in terms of its dimensionless partner $\bar{Z} = k^{-2} Z$. Further, we split it into the contributions for each single spin that we denote $\Delta_j \dot{\bar{Z}}$. The complete result has the canonical scaling term and the sum of all contributions

$$\dot{\bar{Z}} = -2\bar{Z} + \sum_{j=0,\frac{1}{2},1,2,0_{\mathrm{gh}},1_{\mathrm{gh}}} \Delta_j \dot{\bar{Z}} \tag{4.88}$$

First we give the single spin terms except for the spin-0

$$\begin{aligned}
\Delta_{\frac{1}{2}}\dot{\bar{Z}} &= N_f \frac{15-8\eta_\psi+(60-17\eta_\psi)h^2}{1440\pi^2\left(1+h^2\right)^2}\\
\Delta_1\dot{\bar{Z}} &= \bar{Z}\frac{3v\alpha+\bar{Z}(8\alpha-7)}{128\pi^2(\bar{Z}-v\alpha)^2}-\dot{\bar{Z}}\frac{v\alpha+2\bar{Z}(\alpha-1)}{256\pi^2(\bar{Z}-v\alpha)^2}\\
\Delta_2\dot{\bar{Z}} &= -5\bar{Z}\frac{9v-25\bar{Z}}{576\pi^2(\bar{Z}-v)^2}-5\dot{\bar{Z}}\frac{7\bar{Z}-3v}{1152\pi^2(\bar{Z}-v)^2}\\
\Delta_{0_{\rm gh}}\dot{\bar{Z}} &= \frac{5}{96\pi^2}\\
\Delta_{1_{\rm gh}}\dot{\bar{Z}} &= \frac{5}{64\pi^2}
\end{aligned} \tag{4.89}$$

Additionally, we see that a spinor flavor number N_f appeared, indicating that we are actually considering N_f Dirac spinors coupled in a symmetric way.

We still have to give the spin-0 contribution, coming from the 3×3 scalar subsector. The full expression can be written, but it is very long. Therefore, for space reasons, we decided to give it in the gauge $\alpha=0$ and $\beta=1$. Further we set it on-shell so we give it when evaluated in a configuration $V'[\phi]=0$. It is

$$\begin{aligned}
\Delta_0\dot{\bar{Z}} &= -\frac{2\bar{Z}\left(6+5v''\right)-v\left(9+7v''\right)}{192\pi^2(Z-v)\left(1+v''\right)}\\
&\quad+\frac{\eta_\phi}{384\pi^2\left(1+v''\right)}-\frac{\dot{\bar{Z}}(2v-3Z)}{384\pi^2(Z-v)}
\end{aligned} \tag{4.90}$$

The complete result for the running of $\dot{Z}$, in the gauge $\alpha=0$ and $\beta=1$ and with the on-shell condition $v'=0$, comes from the sum of (4.89) and (4.90)

$$\begin{aligned}
\dot{\bar{Z}} &= -2\bar{Z}-5\bar{Z}\frac{9v-25\bar{Z}}{576\pi^2(\bar{Z}-v)^2}-5\dot{\bar{Z}}\frac{7\bar{Z}-3v}{1152\pi^2(\bar{Z}-v)^2}\\
&\quad+\frac{\dot{\bar{Z}}}{128\pi^2(\bar{Z})}-\frac{2\bar{Z}\left(6+5v''\right)-v\left(9+7v''\right)}{192\pi^2(Z-v)\left(1+v''\right)}\\
&\quad+\frac{\eta_\phi}{384\pi^2\left(1+v''\right)}-\frac{\dot{\bar{Z}}(2v-3Z)}{384\pi^2(Z-v)}\\
&\quad+N_f\frac{15-8\eta_\psi+(60-17\eta_\psi)h^2}{1440\pi^2\left(1+h^2\right)^2}\\
&\quad+\frac{5}{96\pi^2}+\frac{5}{64\pi^2}-\frac{7}{128\pi^2}
\end{aligned} \tag{4.91}$$

It is worth remembering that the ϕ dependence in v is the dependence that $\dot{\bar{Z}}$ has in the configuration we use to renormalize and not a genuine dependence (the same that happened for η_ϕ and η_ψ). By non genuine we mean that in the truncation we give $\bar{Z}$ is ϕ independent, so the dependence on the constant ϕ configuration that arises has to be fixed, much like we did for the matter anomalous dimensions.

We realize at this point that some of the formulas in (4.89) and (4.90), as well as (4.91), do depend on $\dot{\bar{Z}}$ on the right hand side. Therefore, we see that (4.88) is actually a linear algebraic equation we can solve in terms of $\dot{\bar{Z}}$ to obtain the actual beta function of the coupling. The final expression is very complicated, because further nonlinearities are added after the solution is obtained.

4.5.1 A simple example for the running of Z.

In order to have a flavor of the behavior of the beta function of Z, we calculate it in a very simple truncation where all the matter couplings are set to zero, including the cosmological constant that in our parametrization belongs to the potential. Further, we set the number of flavors to be zero so that we end in a theory in which there is just a scalar, minimally coupled to gravity through its kinetic term. In these settings, we can safely set $\eta_\phi = 0$ due to the lack of scalar self interactions. In these limits we have that (4.88) reduces to

$$\dot{\bar{Z}} = -2\bar{Z} + \frac{265}{1152\pi^2} - \frac{17\dot{\bar{Z}}}{1152\pi^2\bar{Z}} \tag{4.92}$$

We can solve it in terms of $\dot{\bar{Z}}$ and get

$$\dot{\bar{Z}} = \bar{Z}\,\frac{265 - 2304\pi^2\bar{Z}}{17 - 1152\pi^2\bar{Z}} \tag{4.93}$$

This equation has a fixed point located at

$$\bar{Z}^\star = \frac{265}{2304\pi^2} \simeq 0.0116 \tag{4.94}$$

$$\bar{G}^\star = \frac{1}{16\pi\bar{Z}^\star} = \frac{144\pi}{265} \simeq 1.707 \tag{4.95}$$

To obtain the critical exponent with which this fixed point is approached by the dimensionless Newton constant it is sufficient to note that $\dot{\bar{G}} = -\dot{\bar{Z}}/16\pi\bar{Z}^2$. So we can use (4.93) to calculate $\dot{\bar{G}}$. For the critical exponent we obtain

$$-\frac{530}{231} \simeq -2.29437 \tag{4.96}$$

indicating the existence, in this approximation, of a nontrivial UV-attractive phase where the gravitational coupling can be asymptotically safe [2].

We can also verify the existence of the gaussian FP of the theory, by simply noticing that $\bar{G} = 0$ is a zero of its beta function. The critical exponent at the gaussian fixed point is the negative of the canonical mass dimension of the coupling, thus 2. This shows that around the gaussian fixed point the theory is not perturbatively renormalizable.

Chapter 5

Fixed points of the Yukawa system.

This chapter is dedicated to the study of the flow of the system of beta functions for v (4.38), h (4.39) and $\bar{Z}$ (4.91). We will also take into account the corrections coming from the anomalous dimension η_ϕ (3.108) and η_ψ (3.109) including the gravitational corrections (4.69) and (4.71). The anomalous dimensions and the beta function for $\bar{Z}$ have to be evaluated in a physical configuration $\langle\phi\rangle$ that realizes $V'[\langle\phi\rangle] = 0$. In particular, η_ϕ and η_ψ will be evaluated on the VEV of the potential, that can either be zero ($\mathbb{Z}_2$-symmetric phase) or non-zero (symmetry breaking case). We will mainly work in the gauge $\alpha = 0$, $\beta = 1$. It is worth to remind that $\alpha = 0$ is a FP of the flow [43] and for this reason we consider this choice the most reliable.

Before going into the details of our findings, we want to stress that they strongly depend on the choice of the cutoff made in Section 4.5 and in particular on (4.83). Using the terminology of [24], we choose a "type-I" cutoff. This choice is crucial because it gives a different screening than [25] for the gravitational constant when the number of fermion flavors N_f increases. We expect this situation to hold in more realistic settings, for example those in which the matter sector is that of a GUT theory and has a large number of scalars $N_s \gg N_f$ in the Higgs multiplet. In such a case, no matter what cutoff is chosen, the large-N behavior of the Newton constant is the same we have. Using again the terminology from [24], it is independent of the choice of type-I or -II cutoff.

5.1 General features of the flow.

5.1.1 The gaussian fixed point.

The gaussian fixed point (GFP) is by definition the fixed point for which all the coupling are zero. It is generally expected to be always present in a theory. We shall consider the dimensionless couplings. We know that the linearized flow around the GFP is determined by a matrix whose eigenvalues are the negative of the canonical dimensions of the (dimensonful) couplings.

We ultimately want to analyze the behavior of the anomalous dimension at the GFP. This can be done quite generally, by evaluating

$$\eta_i|_{v=h=0} \tag{5.1}$$

where v and h are set to zero as a condition of gaussianity, the mass is assumed to be zero and $i = \phi, \psi$. The condition $Z \to \infty$ that ensures that also gravity reaches its GFP is missing and we will add it in a moment. We obtain for a general gauge

$$\begin{aligned} \eta_\phi|_{v=h=0} &= -\frac{18\left(\alpha\left(3\beta^2-18\beta+31\right)-3(\beta-1)^2\right)}{768\pi^2 Z(\beta-3)^2+\alpha\left(-3\beta^2+18\beta-31\right)+3(\beta-1)^2} \\ \eta_\psi|_{v=h=0} &= \frac{\alpha\left(-631\beta^2+3786\beta-6799\right)-697\beta^2+6190\beta-8937}{7(\beta-3)\left(1280\pi^2 Z(\beta-3)+\alpha(\beta-3)+2(\beta+3)\right)} \end{aligned} \tag{5.2}$$

We see that, independently of the gauge choice (left apart $\beta = 2$), the limit $Z \to \infty$ gives the expected result $\eta_\phi = \eta_\psi = 0$ at the gaussian fixed point. Formula (5.2) is seen also as the gravitational dressing to the anomalous scaling of the matter fields around the GFP. For simplicity, we now set $\beta = 1$ leaving only α as a parameter. In proximity of the GFP we can think perturbatively, so we expand (5.2) in powers of the Newton's constant $G = 1/16\pi Z$ and obtain

$$\begin{aligned} \eta_\phi|_{v=h=0} &= -\frac{3\alpha}{2\pi}G+\mathcal{O}\left(G^2\right) \\ \eta_\psi|_{v=h=0} &= -\frac{(911\alpha+861)}{560\pi}G+\mathcal{O}\left(G^2\right) \end{aligned} \tag{5.3}$$

These results depend on the gauge parameter α, but do not change sign as a function of it because $\alpha \geq 0$ are the only admissible gauges. Further, the limit $\alpha \to \infty$ must not be taken, because it corresponds to the limit in which the gauge fixing term disappear. Using the gauge $\alpha = 0$ in (5.3), we see that η_ϕ goes to zero, while η_ψ has a negative dressing due to gravity.

We can actually use (5.3) as a further argument for the use of the gauge $\alpha = 0$. The α dependence is due to the coupling to gravity. We may want to find an alternative cutoff such that the anomalous dimensions are minimized. There exists for sure a cutoff such that the anomalous dimension of the scalar is $\eta_\phi = 0$, corresponding to the choice $\alpha = 0$. We further argue an important consequence, namely, if we minimize η_ϕ, we still have a nonzero negative anomalous dimension for the Dirac fields. This result makes us expect that the inclusion of anomalous dimensions in a Yukawa system is important to understand completely the features of the flow.

5.1.2 The gaussian-matter fixed point.

In [22] it was conjectured that in systems describing the interaction of gravity with matter a gaussian-matter fixed point (GMFP) is always present. It is defined to be a fixed point such that the gravitational sector of the action approaches a non-trivial FP value, while the matter couplings are asymptotically zero. It is an interesting fixed point because it represents a minimal generalization of the nontrivial fixed point of the gravitational sector.

The GMFP of a scalar theory coupled to gravity was shown to exist and studied in detail in [22]. In [19] it was shown to exist if also a number N_f

of spinors is added. What these two aforementioned works have in common is that the neglected the anomalous dimensions of the matter fields. In the present work we were able to show that the GMFP exists even if anomalous dimensions are added. At the GMFP η_ψ and η_ϕ take the same values as (5.2), but for $Z = Z^\star$. One can insert their value in the beta function of the Newton constant and see if the non-gaussian fixed point for Z resist the inclusion of η_ψ and η_ϕ. It turns out that it does resist, that imply that a GMFP is present. The difference of our GMFP from the one calculated in [19] at zero anomalous dimensions is negligible. We agree with [22] and think that the GMFP is a general feature of any flow coupling gravity and matter. It is an important test for all the works like [22, 47], where the anomalous dimensions have been neglected, to see if the GMFP persists in more complex truncations.

For illustrative purposes, we give now the GMFP for Z as a function of the anomalous dimensions and in the approximation of zero cosmological constant. Note that the anomalous dimensions do depend on $Z^\star$ itself for the reasons explained above and through (5.2)

$$Z^\star = \frac{(60 - 32\eta_\psi)\, N_f + 15\eta_\phi + 1325}{11520\pi^2} \tag{5.4}$$

It is easy to see that for small N_f the numerator of (5.4) is dominated by the factor 1352. Within the range of values of interest for $Z^\star$, the anomalous dimensions are small. We conclude that the GMFP value, for small N_f is approximately $Z^\star \approx 1325/11520$.

5.1.3 Leading corrections to v and h.

We now analyze quite generally the leading contributions of the function h, that will allow us to understand the leading contributions of their couplings. We will neglect v. It is calculated through (4.39)

$$\dot{h}\Big|_{\text{leading}} = \dot{h}\Big|_{h'=h''=0,v=0} \tag{5.5}$$

We evaluate it with arbitrary α and $\beta = 1$ and obtain

$$\begin{aligned} \dot{h}\Big|_{\text{leading}} &= -(1-\eta_\psi)h + \frac{(5\alpha+23)\eta_\psi h}{4480\pi^2 Z} \\ &\quad - \frac{(2135\alpha+2313)\dot{Z}h}{107520\pi^2 Z^2} + \frac{(1603\alpha+1437)h}{10752\pi^2 Z} \end{aligned} \tag{5.6}$$

Now, the terms proportional to η_ψ and η_ϕ in (5.6) are already proportional to h. We are interested in contributions of η_ψ and η_ϕ which are constant. The constant contributions are those calculated in the previous subsection in (5.2) and (5.3).

It is of particular interest to analyze the behavior of h. In order to understand better what happens to the leading term of the generalized Yukawa interaction h, we substitute (5.2) for $\beta = 1$ in (5.6) and then expand at first order in $G = 1/16\pi Z$. We also assume we sit in proximity of a FP, so that $\dot{Z} = 0$ We obtain

$$\dot{h}\Big|_{\text{leading}} = -h + \frac{(2549\alpha+2019)}{3360\pi} hG + \mathcal{O}\left(G^2\right) \tag{5.7}$$

Apart for the usual canonical scaling $\dot{h} \sim -h$, this shows that the leading contribution of gravity to any powerlike coupling inside the function h *does not change sign with the gauge* α and is always positive. In other words, the Yukawa coupling seems not to posses a FP of the kind advocated by [49]. This is important, because it forbids in the model of [49] the predictivity of the top-Yukawa coupling. However, the study of the leading contributions (5.7) trivializes too much the structure of the flow. It needs to be studied in its full nonlinearity and, indeed, something non-trivial will appear. In particular in the following, some non-trivial FP will be presented.

5.2 FPs in Symmetric phase.

We studied the symmetric phase using the powerlaw ansatz for the potential

$$v = \lambda_0 + \lambda_2\phi^2 + \lambda_4\phi^4 \tag{5.8}$$

while the Yukawa function is assumed to contain just the Yukawa coupling

$$h = y\phi \tag{5.9}$$

In particular, we analyzed the beta functions of the set $\{\lambda_0, \lambda_2, \lambda_4, y, Z\}$ and used numerical tools to evaluate the FP of their flow. The condition $\lambda_2 > 0$ is assumed, since $\lambda_2 < 0$ is generally regarded as an indication that we should change the parametrization of the potential. Further, we ask to a physical FP to have $Z > 0$, while λ_0 is allowed to be both positive and negative. When a possible physical FP was found, we always tested its stability under the inclusion of further couplings in (5.8) like λ_6 and λ_8.

In the following, we will often refer to fixed points as"branches", because we decided to study them as functions of the number of fermion flavors N_f. A branch will therefore be a set of fixed points that depends on N_f. Note that N_f is a natural number, but sometimes it is useful to extend it to the whole (positive) real axis.

5.2.1 Nontrivial FP branch in symmetric phase.

In the symmetric phase a very interesting non-trivial FP was found, which we will call "SYM branch". It does not appear for small values of N_f, but it exists from $N_f \gtrsim 2.7$ on. For N_f close to this lower bound we have $\lambda_4 < 0$. The only natural value of N_f that realizes this condition is $N_f = 3$. This generally signals that to ensure that the potential is bounded, we should study the inclusion of further couplings. However, for the moment, we are interested to its behavior when increasing N_f. λ_4 has a maximum around $N_f = 5$ where it is close to 1. Then it starts decreasing for increasing N_f. This feature is shared by all other couplings except for λ_0 ($|\lambda_0|$ is expected to increase for increasing N_f, being the vacuum energy). This behavior signals some interesting feature of the branch in the large-N_f limit and we shall discuss it in the next section. Another very important feature of this branch is that the anomalous dimensions of the matter fields are small and becomes smaller for increasing N_f. This is telling us that we can trust the LPA approximation we employed.

Further, we tested that the branch is stable under the addition of further power coupling like λ_6 and λ_8 (see Fig. (5.1)). In particular, it is more stable for increasing values of N_f. The corrections coming from λ_6 inclusion do increase for increasing values of N_f and for the couplings λ_0, λ_2 and λ_4. However the relative correction remains always negligible.

We give a table for the values of SYM branch, as functions of N_f, that we found in the symmetric phase. In the table we also show the values of the anomalous dimensions.

N_f	λ_0	λ_2	λ_4	y	Z	η_ψ	η_ϕ
3	-0.0094	0.49	-11.	3.9	0.0058	-0.31	0.51
4	-0.013	0.24	0.83	2.7	0.0056	-0.22	0.40
5	-0.016	0.17	1.0	2.1	0.0058	-0.17	0.33
6	-0.020	0.13	0.77	1.7	0.0062	-0.14	0.29
7	-0.023	0.10	0.58	1.4	0.0066	-0.12	0.25
8	-0.026	0.087	0.44	1.2	0.0070	-0.10	0.23
9	-0.029	0.075	0.34	1.1	0.0075	-0.091	0.20
10	-0.032	0.066	0.27	0.98	0.0079	-0.082	0.18
20	-0.064	0.029	0.054	0.47	0.013	-0.041	0.096
30	-0.096	0.018	0.022	0.31	0.018	-0.027	0.065
40	-0.13	0.013	0.012	0.23	0.023	-0.020	0.049
50	-0.16	0.011	0.0072	0.18	0.029	-0.016	0.039
60	-0.19	0.0095	0.0061	0.16	0.034	-0.015	0.035
70	-0.22	0.0084	0.0050	0.14	0.039	-0.013	0.031
80	-0.25	0.0073	0.0039	0.13	0.045	-0.011	0.027
90	-0.29	0.0062	0.0028	0.11	0.050	-0.0098	0.024
100	-0.32	0.0051	0.0017	0.089	0.055	-0.0082	0.020
1000	-3.2	0.00050	0.000016	0.0089	0.53	-0.00082	0.0020

We complete the analysis of SYM branch, looking at the critical properties of the flow in its neighbour. We computed the stability matrix of the system $\{\lambda_0, \lambda_2, \lambda_4, y, Z\}$ at the FP and calculated its eigenvalues. It turns out that the number of attractive directions, so the dimension of the critical surface, is 3 for every N_f. This is very important for the predictivity properties of the theory. It is telling us that we have to perform three experiments to locate our position in theory space. All the couplings are determined by these three experiments. In the following table we give the critical exponents of the stability matrix (that are the negative of the eigenvalues) in the symmetric branch under consideration. The second column indicates the dimension of the critical surface, that is always three. There are two operators that become marginal in the limit $N_f \to \infty$ and they correspond to λ_4 and y.

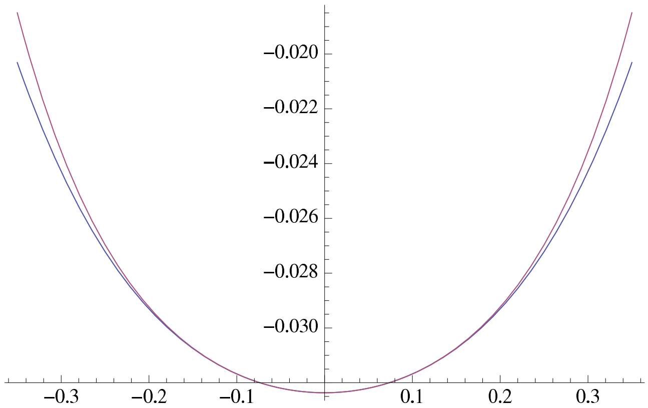

Figure 5.1: Comparison of two fixed point potentials, one in the truncation up to λ_4 and the other in the truncation up to λ_6. The values of λ_2 and λ_4 change of order 10^{-2} after λ_6 inclusion. λ_0 and G change is negligible.

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
3	3	-1.1	-0.36	1.1	1.3	3.8
4	3	-1.0	-0.57	1.3	1.4	3.9
5	3	-0.94	-0.42	1.4	1.5	3.9
6	3	-0.81	-0.33	1.5	1.6	3.9
7	3	-0.70	-0.28	1.6	1.6	3.9
8	3	-0.61	-0.24	1.7	1.7	4.0
9	3	-0.54	-0.21	1.7	1.7	4.0
10	3	-0.48	-0.19	1.7	1.7	4.0
20	3	-0.24	-0.094	1.9	1.9	4.0
30	3	-0.16	-0.063	1.9	1.9	4.0
40	3	-0.12	-0.048	1.9	1.9	4.0
50	3	-0.096	-0.038	1.9	1.9	4.0
60	3	-0.086	-0.035	1.9	2.0	4.0
70	3	-0.077	-0.031	2.0	2.0	4.0
80	3	-0.067	-0.027	2.0	2.0	4.0
90	3	-0.058	-0.023	2.0	2.0	4.0
100	3	-0.049	-0.019	2.0	2.0	4.0
1000	3	-0.0049	-0.0020	2.0	2.0	4.0

5.2.2 Large-N_f behavior.

Here we want to give the analytic structure of the asymptotic limit for large-N_f of SYM branch [50]. It is convenient to think about the values of the fixed points in the branch as functions of N_f. We therefore have

$$\{\lambda_0\left(N_f\right), \lambda_2\left(N_f\right), \lambda_4\left(N_f\right), y\left(N_f\right), Z\left(N_f\right)\} \tag{5.10}$$

We want to determine the asymptotic expansion of these in terms of N_f. We found convenient to parametrize them as

$$\lambda_0 = -\frac{N_f}{32\pi^2} + \lambda_{0,\infty} \tag{5.11}$$
$$\lambda_2 = \frac{\lambda_{2,\infty}}{N_f} \tag{5.12}$$
$$\lambda_4 = \frac{\lambda_{4,\infty}}{N_f^2} \tag{5.13}$$
$$y = \frac{y_\infty}{N_f} \tag{5.14}$$
$$Z = Z_\infty + \frac{N_f}{192\pi^2} \tag{5.15}$$

where the values of the constants we introduced is

$$\lambda_{0,\infty} = -\frac{541}{58800\pi^2} \simeq -0.001 \tag{5.16}$$
$$\lambda_{2,\infty} = \frac{2721}{5438} \simeq 1.2 \tag{5.17}$$
$$\lambda_{4,\infty} = -\frac{822649\pi^2}{517244} \simeq 15.7 \tag{5.18}$$
$$y_\infty = \frac{1}{7}\sqrt{\frac{2721}{7}}\pi \simeq 8.85 \tag{5.19}$$
$$Z_\infty = \frac{63433}{2822400\pi^2} \simeq 0.002 \tag{5.20}$$

The asymptotic expansion of the anomalous dimensions is

$$\eta_\psi = \frac{\eta_{\psi,\infty}}{N_f} \tag{5.21}$$
$$\eta_\phi = \frac{\eta_{\phi,\infty}}{N_f} \tag{5.22}$$

The values of the constants we introduced is

$$\eta_{\psi,\infty} = -\frac{801}{980} \simeq -0.8 \tag{5.23}$$
$$\eta_{\phi,\infty} = \frac{2721}{1372} \simeq 2. \tag{5.24}$$

When working in large-N_f limit, it is possible to study the renormalizability of gravity in powers of $1/N_f$ [50]. The idea is to take the limit $N_f \to \infty$, while keeping Z/N_f fixed. This is precisely what is realized by our infinite N_f limit, where the ratio Z/N_f takes the value $\frac{1}{192\pi^2}$. The intriguing fact of this branch of fixed points is that it joins a nontrivial FP at small N_f, with a gaussian-matter FP at large N_f. By gaussian-matter FP we mean a FP which is nontrivial for gravity, but gaussian for the matter sector. There are indications that gravity can be renormalized in the large-N_f limit, at least in a perturbative sense, and we think that this FP is a first step towards the understanding of the large-N_f behavior in a functional RG scheme.

5.3 Broken symmetry phase.

We want to study also situations in which the potential is $\mathbb{Z}_2$-symmetric, but the ground state is not. The symmetry breaking regime is signalled also by $\lambda_2 < 0$. When this happens it is convenient to change the basis of operators with which we truncated v. To explore the symmetry breaking regime we truncate the potential to

$$v = \theta_0 + \theta_4 \left(\phi^2 - \kappa\right)^2 \tag{5.25}$$

The Yukawa function, instead, is parametrized with the Yukawa coupling, like in the symmetric case

$$h = y\phi \tag{5.26}$$

This parametrization is known to be efficient in resumming some order of perturbation theory, due to the properties of the flow of the VEV κ.

The running of the VEV can be computed in quite generality, before resorting to any truncation of the potential.

$$\dot{\bar{\kappa}}_R = -\left(2+\eta_\phi\right)\bar{\kappa}_R + \frac{\sqrt{\bar{\kappa}_R}\bar{v}'''_R\left(1-\frac{\eta_\phi}{6}\right)}{16\pi^2\bar{v}''_R\left(1+\bar{v}''_R\right)^2} - \left.\frac{hN_f\sqrt{\bar{\kappa}_R}\bar{h}'_R\left(1-\frac{\eta_\psi}{5}\right)}{2\left(1+\bar{h}^2_R\right)^2\pi^2\bar{v}''_R}\right|_{\bar{\phi}_R=\sqrt{\bar{\kappa}_R}} \tag{5.27}$$

This formula has to be compared with the analog one in which gravity is not present (3.140). Indeed, (3.140) and (5.27) are equal in form. The only difference is in the fact that in (5.27) the anomalous dimensions have a gravitational correction. We can say that the VEV, being a quantity intrinsically on-shell, couples weakly to gravity and in particular only through the anomalous dimensions. An argument for the fact that a VEV generally couples weakly to gravity goes as follows. The leading order gravitational correction to the flow of the potential is $\Delta\dot{\bar{v}} \sim \bar{G}\bar{v}$. If we use formula (4.34), that is an on-shell condition, it is clear that $\Delta\dot{\bar{v}}$ is not going to contribute any correction to the VEV $\Delta\dot{\bar{\kappa}}_R = 0$. We believe this argument to be more general and to extend beyond the leading orders, as hinted by (5.27).

We require that $\kappa > 0$ and $Z > 0$, in agreement with the ansatz we made. It turns out that the flow in the symmetry breaking regime is far more complex than in the symmetric one. Also, the results we are going to show are not as stable as those we showed in the symmetric case. We found five branches of fixed point, however they are less reliable that the symmetric one because anomalous dimensions tend to be big and they are less stable under the inclusion of further couplings (like θ_6 and θ_8).

We list the branches we found giving some hint of their properties:

- SB branch 1: Has big η_ψ and η_ϕ is increasing with N_f.
- SB branch 2: Exists for $N_f \gtrsim 1$, has big η_ψ and η_ϕ is increasing with N_f. Has a critical surface of dimension 2.
- SB branch 3: Exists only for the (natural) values $N_f = 1, 2$. It is totally repulsive. It has acceptable anomalous dimensions.
- SB branch 4: Exists for $N_f \gtrsim 1$. It has $y = \theta_4 = 0$. The anomalous dimensions are small and decrease with N_f.

- SB branch 5: Exists for $N_f \gtrsim 1$. Has a critical surface of dimension 1, that is interesting, but has a very big η_ϕ.

We also studied the asymptotic behavior for increasing N_f of the expansion around a VEV. It is a general feature of the flow that at large-N_f, within symmetry breaking expansion, one finds either non-physical fixed points or fixed points with anomalous dimensions increasing in absolute value with N_f. In the first case, the asymptotic fixed points have negative VEV, a feature that is forbidden by the very basic assumptions of the broken phase. In the second case, the LPA cannot be trusted anymore. Therefore, there will not be an analog discussion to that given in Section 5.2.2.

In the following we give tables for the values of the couplings and the anomalous dimensions at the fixed points for each branch. We alternate these tables with those containing the critical exponents of the corresponding stability matrices.

SB branch 1 values (symmetry breaking phase):

N_f	θ_0	κ	θ_4	y	Z	η_ψ	η_ϕ
1	0.0074	0.00037	75.	31.	0.022	3.3	-1.4
2	0.0071	0.00037	76.	28.	0.021	3.7	-3.6
3	0.0069	0.00036	72.	25.	0.020	3.8	-5.2
4	0.0066	0.00035	68.	23.	0.020	4.0	-6.6
5	0.0064	0.00035	65.	22.	0.019	4.0	-7.8
6	0.0062	0.00034	62.	21.	0.018	4.1	-8.9
7	0.0060	0.00033	60.	20.	0.018	4.1	-9.8
8	0.0059	0.00032	58.	20.	0.017	4.2	-11.
9	0.0057	0.00031	56.	19.	0.016	4.2	-12.
10	0.0056	0.00030	54.	18.	0.016	4.3	-12.
20	0.0050	0.00022	44.	16.	0.012	4.4	-20.
30	0.0050	0.00017	40.	15.	0.010	4.5	-26.
50	0.0051	0.00012	37.	14.	0.0087	4.5	-40.

Critical exponents in SB branch 1:

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
1	4	-14.	2.5-1.8 i	2.5+1.8 i	38.	1100.
2	3	-300.	-20.	2.5-1.8 i	2.5+1.8 i	37.
3	3	-240.	-24.	2.5-1.8 i	2.5+1.8 i	41.
4	3	-220.	-28.	2.5-1.9 i	2.5+1.9 i	44.
5	3	-210.	-31.	2.5-1.9 i	2.5+1.9 i	48.
6	3	-210.	-34.	2.4-2.0 i	2.4+2.0 i	51.
7	3	-210.	-37.	2.4-2.0 i	2.4+2.0 i	54.
8	3	-200.	-39.	2.4-2.1 i	2.4+2.1 i	57.
9	3	-200.	-42.	2.4-2.2 i	2.4+2.2 i	60.
10	3	-200.	-44.	2.4-2.3 i	2.4+2.3 i	63.
20	3	-220.	-64.	3.0-3.1 i	3.0+3.1 i	93.
30	3	-240.	-81.	4.8-3.3 i	4.8+3.3 i	120.
50	3	-320.	-120.	3.5	24.	180.

SB branch 2 values (symmetry breaking phase):

N_f	θ_0	κ	θ_4	y	Z	η_ψ	η_ϕ
3	-0.0019	0.00032	66.	23.	0.00032	3.80	-5.
4	-0.0011	0.00029	64.	22.	0.00049	3.86	-7.
5	-0.0007	0.00026	63.	22.	0.00049	3.9	-10.
6	-0.0004	0.00023	62.	22.	0.00042	3.86	-14.
7	-0.00022	0.00020	64.	22.	0.00032	3.8	-20.
8	-0.00012	0.00018	66.	23.	0.00026	3.6	-29.
9	-0.000074	0.00017	67.	23.	0.00024	3.5	-36.
10	-0.000047	0.00016	67.	23.	0.00024	3.4	-42.
20	0.0001	0.0002	60.	20.	0.0004	3.7	-72.
30	0.00024	0.00017	60.	21.	0.00059	3.90	-85.
40	0.0004	0.00017	56.	20.	0.00085	4.07	-92.
50	0.0007	0.00018	53.	19.	0.0012	4.2	-95.

Critical exponents in SB branch 2:

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
3	2	-126.	-25.	-3.	3.4	63.
4	2	-117.	-32.	-8.	3.0	96.
5	2	-120.	-41.	-15.	2.9	140.
6	2	-124.	-53.	-24.	2.8	219.
7	2	-130.	-75.	-43.	3.2	390.
8	2	-150.	-110.	-67.	5.4	680.
9	2	-160.	-130.	-85.	9.7	960.
10	2	-180.	-150.	-95.	15.	1200.
20	2	-290.	-250.	-81.	66.	1600.
30	2	-379.	-284.	-51.	115.	1348.
40	2	-463.	-290.	-31.	166.	1053.
50	2	-540.	-280.	-18.	220.	770.

SB branch 3 values (symmetry breaking phase):

N_f	θ_0	κ	θ_4	y	Z	η_ψ	η_ϕ
1	0.0022	0.011	4.51	0.	0.015	-0.85	-1.0
2	-0.0047	0.006	13.8	0.	0.007	-0.56	-0.9

Critical exponents in SB branch 3:

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
1	5	0.69	1.18-0.62 i	1.18+0.62 i	2.34-0.85 i	2.34+0.85 i
2	5	0.22	0.79	0.88	2.74	3.48

SB branch 4 values (symmetry breaking phase):

N_f	θ_0	κ	θ_4	y	Z	η_ψ	η_ϕ
1	0.0024	0.0095	0	0	0.016	-0.86	0
2	-0.0043	0.0095	0	0	0.008	-0.59	0
3	-0.0091	0.0095	0	0	0.0055	-0.34	0
4	-0.013	0.0095	0	0	0.0054	-0.24	0
5	-0.016	0.0095	0	0	0.0056	-0.18	0
6	-0.019	0.0095	0	0	0.0060	-0.16	0
7	-0.023	0.0095	0	0	0.0064	-0.13	0
8	-0.026	0.0095	0	0	0.0069	-0.11	0
9	-0.029	0.0095	0	0	0.0074	-0.096	0
10	-0.032	0.0095	0	0	0.0079	-0.086	0
20	-0.06	0.009	0	0	0.01	-0.042	0
30	-0.1	0.009	0	0	0.02	-0.033	0
40	-0.1	0.009	0	0	0.02	-0.025	0
50	-0.16	0.0095	0	0	0.029	-0.017	0

Critical exponents in SB branch 4:

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
1	3	-2.24	-0.35	2.	2.32-0.72 i	2.32+0.72 i
2	3	-1.34	-0.016	0.77	2.	3.6
3	4	-0.66	0.13	1.1	2.0	3.8
4	4	-0.42	0.14	1.3	2.0	3.9
5	4	-0.30	0.13	1.5	2.0	3.9
6	4	-0.25	0.12	1.5	2.0	3.9
7	4	-0.19	0.11	1.6	2.0	4.0
8	4	-0.17	0.097	1.7	2.0	4.0
9	4	-0.14	0.088	1.7	2.0	4.0
10	4	-0.12	0.082	1.7	2.0	4.0
20	4	-0.055	0.045	1.9	2.0	4.0
30	4	-0.044	0.036	1.9	2.0	4.0
40	4	-0.032	0.028	1.9	2.0	4.0
50	4	-0.021	0.019	1.9	2.0	4.0

SB branch 5 values (symmetry breaking phase):

N_f	θ_0	κ	θ_4	y	Z	η_ψ	η_ϕ
1	-0.00002	0.020	0.033	6.0	0.	4.6	-9.0
2	-0.00004	0.039	0.030	5.6	0.	4.4	-8.9
3	-0.00005	0.055	0.028	5.3	0.	4.2	-9.3
4	-0.00006	0.067	0.026	5.2	0.	4.1	-10.
5	-0.00007	0.077	0.025	5.1	0.000012	4.0	-11.
6	-0.00007	0.084	0.025	5.1	0.000013	3.8	-12.
7	-0.00007	0.090	0.024	5.1	0.000015	3.7	-14.
8	-0.00006	0.095	0.023	5.1	0.000016	3.6	-15.
9	-0.00006	0.10	0.022	5.1	0.000017	3.6	-17.
10	-0.00006	0.10	0.021	5.1	0.000018	3.5	-18.
20	-0.00003	0.1	0.01	5.	0.00002	2.6	-41.
30	-0.00002	0.2	0.007	5.	0.00002	1.8	-78.
40	0.	0.2	0.005	4.	0.00002	0.90	-130.
50	0.	0.28	0.003	3.8	0.000019	-0.028	-200.

Critical exponents in SB branch 5:

N_f	dim$\mathcal{C}$	c_1	c_2	c_3	c_4	c_5
1	1	-419.-394. i	-419.+394. i	-42.	-8.7	20.5
2	1	-257.-2. i	-257.+2. i	-26.	-7.5	21.
3	1	-220.-130. i	-220.+130. i	-24.	-8.5	22.
4	1	-220.-90. i	-220.+90. i	-24.	-11.	24.
5	1	-230.-40. i	-230.+40. i	-24.	-14.	27.
6	1	-310.-0.35 i	-190.-0.13 i	-21.	-20.	31.
7	1	-390.	-190.	-22.-4. i	-22.+4. i	35.
8	1	-460.	-180.	-24.-8. i	-24.+8. i	40.
9	1	-550.	-180.	-26.-10. i	-26.+10. i	45.
10	1	-640.	-190.	-27.-12. i	-27.+12. i	51.
20	1	-3100.	-310.	-56.-26. i	-56.+26. i	130.
30	1	-12000.	-480.	-106.-33. i	-106.+33. i	250.
40	1	-40000.	-710.	-180.-20. i	-180.+20. i	410.
50	1	-110000.	-970.	-330.-0.36 i	-230.-0.22 i	620.

5.4 Discussion.

In this chapter we numerically analyzed the possibility that the system, describing the interaction of a scalar field and N_f symmetric spinor fields with gravity, possesses a nontrivial ultraviolet fixed point. In the truncation of the effective action we included a general potential for the scalar field as well as a generalized Yukawa interaction. We also took into account the anomalous dimensions of the matter fields.

We approached the study of the fixed points within two approximation schemes. In one case we expanded the potential in a powerlaw form around a trivial VEV $\langle\phi\rangle = 0$. In the other case we still relied on a powerlaw expansion, but expanded around a nonzero VEV. These two expansions capture all the essential features of a $\mathbb{Z}_2$-symmetric potential, namely both the case in which the ground state is symmetric and the case in which it is not.

In the case in which the ground state is symmetric, that we called symmetric phase, our numerical analysis was able to find a nontrivial fixed point as a function of the number of spinor flavors N_f. This fixed point is interesting because the dimension of its critical surface is three, meaning that, once the the cosmological and the Newton constants and the mass of the scalar field are determined experimentally at some scale, one should be able to predict the values of the ϕ^4 and Yukawa couplings at that scale. This is of course not an easy task because it requires to integrate with good precision the flow emanated by the fixed point, while it is known that going in the direction of infrared scales nonlocal effects become important. The symmetric case fixed point is interesting also because it has a large-N_f behavior that we could explicitly parametrize and study. It is seen that the large-N_f limit do not alter the essential features of the fixed point (the dimension of the critical surface), but rather joins it to a fixed point ($N_f \to \infty$) where the matter is gaussian and gravity is nontrivial (a gaussian matter fixed point GMFP in the sense of [22]). We found these features very interesting and feel that they deserve some further study in the future. In particular, we think that a more realistic structure deserve attention in which the matter multiplets are closer to those of the standard model.

We now want to discuss briefly the situation in which the expansion is around a nonzero VEV, that we called symmetry breaking phase. When the VEV is nonzero, an higher complexity is expected in the flow and that is precisely what happened. We adopted the point of view of [18] and for consistency we expanded every coupling around the aforementioned VEV. This results in a more involved structure for the beta functions of the Yukawa coupling and the Newton constant, not to mention the flow of the potential. This complexity translated in a richer fixed point structure when numerically analyzing the flow. We were able to parametrize five branches of fixed points as functions of N_f. Some of them extend to large values of N_f, some are limited only to a certain interval $N_{f,\min} \leq N_f \leq N_{f,\max}$. Every possible fixed point should represent, in principle, a different ultraviolet limit. It is also important to note that each fixed point has its own unique features of attractivity, thus some branch is interesting because it could lower or increase the predictivity features of the theory under consideration.

However, one should take into account that these fixed points have often very large anomalous dimension. This may be a symptom that the adopted LPA approximation fails, so we think that the existence of all these fixed points should be tested against a truncation that goes beyond the local potential one. We think this should be the main step in the direction of studying the symmetry breaking phase in presence of gravity.

We also found that there are fixed points in which the Yukawa coupling goes to zero. This triggered the study of the simpler system of a scalar field coupled to gravity. The main difference to [22] is that we are including the anomalous dimensions and we want to study the system in the symmetry breaking phase. In this study we decided to find all the fixed point potentials for increasing number of powerlaw couplings in the truncation, starting from θ_4 (the ϕ^4 coupling), up to θ_{12}. We refer to Fig. 5.2 for the plot of all the fixed point potentials of all the truncations from θ_4 to θ_{12}. From the figure one can see that these potentials are, in principle, very different. Compared to the symmetric case (Fig. 5.1), here the fixed point potentials are not stable under the inclusion of further powerlaw couplings. We want however to give now an argument against this

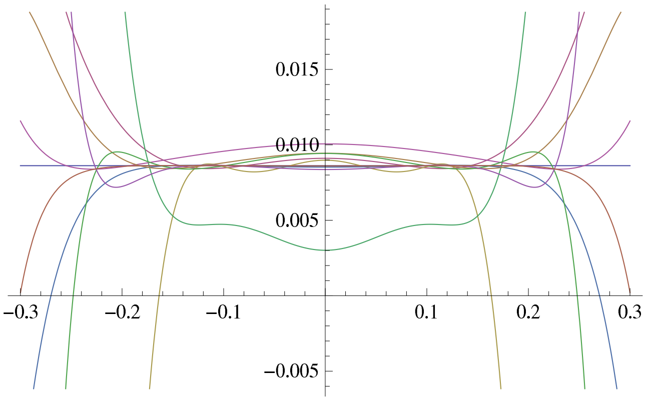

Figure 5.2: Plot of all the fixed point in all the truncations from θ_4 to θ_{12}. The convergence radius is approximately $\bar{\phi}_{R,\max} \approx 2.3$. Its relative to the VEV value is approximately $\bar{\phi}_{R,\max}/\sqrt{\bar{\kappa}_R} \approx 1.2$.

statement. The actual flow of the potential v is (4.38) (at $N_f = 0$ in this case). The powerlaw expansion "breaks" the denominators of (4.38) that include the functions v'' and v itself. It thus means that, when expanding, we are introducing a convergence radius of the expansion that we can evaluate. It turns out that for all the potentials drawn in Fig. 5.2 the convergence radius is very similar and slightly bigger than the VEV value. If we compare all the potentials within that radius of convergence, one can see that, apart for some isolated case, they all agree numerically. The question we think that should be addressed in the future is, are these fixed points really corresponding to different UV limits? We think there are good reasons that the answer is not. Maybe, we see many fixed point potentials only because the beta functions symmetry breaking expansion are more involved than the symmetric expansion. Our present approximations are still too limited to address this question properly, but we think that the evaluation of the radius of convergence in the powerlaw approximations is a tool that must be added from now on in this kind of investigations.

Chapter 6

Higher derivative nonlinear sigma models.

6.1 Introduction.

In the study of quantum gravity one encounters many technical complications, and it is often desirable to test one's ideas and tools in a simpler setting. The nonlinear sigma models (NLSMs) have striking similarities to gravity: they are nonpolynomially interacting theories, and from the point of view of power counting, they have exactly the same structure as gravity. On the other hand, they lack the complications due to gauge invariance. They are therefore a good theoretical laboratory where one can study various technical aspects of the renormalization of gravity without having to consider the complications due to gauge fixing, and with the certainty that one's results are not gauge artifacts. There exists the possibility that the NLSM shares with gravity also the property of being asymptotically safe. In any case understanding the UV behavior of the NLSM may shed some light on the analogous issue for gravity.

The NLSMs also play an important role in particle physics phenomenology: they are used as low energy effective field theories both for strong and weak interactions. In the former case the scalar fields are identified with the light mesons [51], in the latter with the three Goldstone degrees of freedom of the complex Higgs doublet [52]. These effective field theories are usually thought to break down at some cutoff scale, of the order of the GeV in the strong case and of the TeV in the weak case. It is an interesting question in itself, and one that may have some relevance also for particle physics, whether some of these NLSM's might actually be asymptotically safe. Old work on the epsilon expansion and $1/N$ expansion suggests that a fixed point with the right properties may exist [53, 54, 55, 56]. More recently, the beta functions of the NLSM were recalculated using a two-derivative truncation of an exact renormalization group (RG) equation, and it was found in the case of the $O(N)$ models that they have a nontrivial UV FP [28]. In the present chapter we begin addressing the issue of asymptotic safety in the NLSM taking into account also four-derivative interactions. The beta functions of four-derivative NLSM were considered before in [57, 58]. The former reference uses a formalism that applies only to group-valued models; the latter uses dimensional regularization and therefore cannot properly compute the running of the two-derivative terms, which is necessary to establish asymptotic safety. In this chapter we present the results of [29] and partly correct the results of these earlier papers. The comparison with [57] is given in appendix. For a general discussion about effective field theories with higher derivatives we refer to [59]. The renormalization of the higher derivative chiral model at order p^4 and p^6 was considered in [60, 61]. It is also worth noting that the parallelism of the two derivative NLSM and gravity in any dimension has been established explicitly [62] when gravity is described by a pure Ricci scalar term. In particular the critical exponents are shown to coincide.

6.2 The theory.

6.2.1 Geometry and action.

In general the NLSM is a field theory whose configurations are maps from $\varphi : X \to Y$, where X is a d-dimensional manifold interpreted as spacetime and Y is some n-dimensional internal manifold. We will always take X to be four dimensional and to have a fixed flat Euclidean metric, and we will call h a Riemannian metric on Y. Given a map φ, one calls the "vectorfield along φ" a rule that assigns to each point x of X a vector tangent to Y at $\varphi(x)$. [1] For example, given a fixed vector v tangent to X at x, the image of v under the tangent map $T\varphi$ is a vectorfield along φ. Its components are $v^\mu \partial_\mu \varphi^\alpha$. Thus we can view the matrix $\partial_\mu \varphi^\alpha$ as the components of four vectorfields along φ.

The Levi-Civita connection of the metric h in TY can be used to define the covariant derivative of vectorfields along φ. Let $\Gamma_\alpha{}^\beta{}_\gamma$ be the Christoffel symbols of h

$$\Gamma_\alpha{}^\beta{}_\gamma = \frac{1}{2} h^{\beta\delta} \left(\partial_\alpha h_{\delta\gamma} + \partial_\gamma h_{\alpha\delta} - \partial_\delta h_{\alpha\gamma} \right)$$

and $R_{\alpha\beta}{}^\gamma{}_\delta = \partial_\alpha \Gamma_\beta{}^\gamma{}_\delta - \partial_\beta \Gamma_\alpha{}^\gamma{}_\delta + \Gamma_\alpha{}^\gamma{}_\epsilon \Gamma_\beta{}^\epsilon{}_\delta - \Gamma_\beta{}^\gamma{}_\epsilon \Gamma_\alpha{}^\epsilon{}_\delta$ its Riemann tensor. The covariant derivative of a vectorfields along φ is

$$\nabla_\mu \xi^\alpha = \partial_\mu \xi^\alpha + \partial_\mu \varphi^\gamma \Gamma_\gamma{}^\alpha{}_\beta \xi^\beta \tag{6.1}$$

A diffeomorphism f of Y can be represented in coordinates by $y' = f(y)$. It maps vectorfields along φ to vectorfields along $\varphi' = f \circ \varphi$. One can check explicitly using the transformation properties

$$\xi'^\alpha = \frac{\partial \varphi'^\alpha}{\partial \varphi^\beta} \xi^\beta \ ; \qquad \Gamma'_\gamma{}^\alpha{}_\beta = \frac{\partial \varphi^\eta}{\partial \varphi'^\gamma} \frac{\partial \varphi'^\alpha}{\partial \varphi^\delta} \frac{\partial \varphi^\epsilon}{\partial \varphi'^\beta} \Gamma_\eta{}^\delta{}_\epsilon + \frac{\partial \varphi'^\alpha}{\partial \varphi^\delta} \frac{\partial^2 \varphi^\delta}{\partial \varphi'^\gamma \partial \varphi'^\beta} \tag{6.2}$$

that the covariant derivative transforms in the same way as ξ under diffeomorphisms of Y.

We also note for future reference that the curvature of the pullback connection is the pullback of the curvature of the Levi-Civita connection:

$$[\nabla_\mu, \nabla_\nu] \xi^\gamma \equiv \Omega_{\mu\nu}{}^\gamma{}_\delta \xi^\delta = \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta R_{\alpha\beta}{}^\gamma{}_\delta \xi^\delta \tag{6.3}$$

We can now discuss the dynamics of the NLSM. Since the ordinary derivatives of φ^α are the components of vectorfields along φ, the second covariant

[1] The vectorfields along φ should be thought of, in geometrical terms, as sections of the pullback bundle $\varphi^* TY$.

derivatives of the scalars are given by

$$\nabla_\mu \partial_\nu \varphi^\alpha = \partial_\mu \partial_\nu \varphi^\alpha + \partial_\mu \varphi^\beta \Gamma_\beta{}^\alpha{}_\gamma \partial_\nu \varphi^\gamma \ . \tag{6.4}$$

Note that due to the symmetry of the Christoffel symbols $\nabla_\mu \partial_\nu \varphi^\alpha = \nabla_\nu \partial_\mu \varphi^\alpha$. We also define $\Box\varphi^\alpha = \nabla^\mu \partial_\mu \varphi^\alpha$. After these preliminaries, the most general Lorentz– and parity–invariant NLSM with up to four derivatives has an action of the form:

$$\begin{aligned} \tfrac{1}{2}\int d^4x \Big[\ & \partial_\mu\varphi^\alpha \partial^\mu \varphi^\beta h^{(2)}_{\alpha\beta}(\varphi) + \Box\varphi^\alpha \Box\varphi^\beta h^{(4)}_{\alpha\beta}(\varphi) \\ & + \nabla_\mu \partial_\nu \varphi^\alpha \partial^\mu \varphi^\beta \partial^\nu \varphi^\gamma A_{\alpha\beta\gamma}(\varphi) + \partial_\mu \varphi^\alpha \partial^\mu \varphi^\beta \partial_\nu \varphi^\gamma \partial^\nu \varphi^\delta T_{\alpha\beta\gamma\delta}(\varphi)\Big] \ . \end{aligned} \tag{6.5}$$

Here we defined parity to correspond to the reflection $\varphi^\alpha(x_1, x_2, x_3, x_4) \mapsto \varphi^\alpha(-x_1, x_2, x_3, x_4)$. This is the only parity operation one can define in full generality. We will discuss below other "parities" that can be defined on special manifolds. At the classical level, $h^{(2)}$, $h^{(4)}$ A, and T are fixed tensorfields on Y. They represent, in general, an infinite number of interaction terms. In the quantum theory these tensors will be subject to RG flow. The tensors $h^{(2)}$, $h^{(4)}$ are assumed to be positive definite metrics. In the present chapter we will always use $h^{(4)}$ to raise and lower indices, while $h^{(2)}$ is treated as any tensor. Of course nothing ultimately can depend on this convention. The tensor A can be assumed to be totally symmetric without loss of generality. The tensor T must have the following symmetry properties:

$$T_{\alpha\beta\gamma\delta} = T_{\beta\alpha\gamma\delta} = T_{\alpha\beta\delta\gamma} = T_{\gamma\delta\alpha\beta} \, . \tag{6.6}$$

In (6.5) we have not considered (parity violating) terms that involve the ϵ tensor, of the form

$$c \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta \partial_\rho \varphi^\gamma \partial_\sigma \varphi^\delta B_{\alpha\beta\gamma\delta}(\varphi) \, , \tag{6.7}$$

where B is some four-form on Y. These could be called "Wess-Zumino-Witten terms" in a generalized sense. A proper Wess-Zumino-Witten term is one for which the four form B is not defined everywhere on Y, but the five-form $H = dB$ is. Then H defines a nontrivial fifth-cohomology class and the coefficient c has to obey a quantization condition. The original Wess-Zumino term corresponds to the case $Y = SU(N)$ and $H = \mathrm{tr}(g^{-1}dg)^5$. We will briefly return to these terms in the discussion.

We observe that since the field φ appears nonpolynomially in the action, it must be dimensionless. Then, $h^{(2)}$ must have dimension of mass squared, whereas the other tensors are dimensionless. Later on we will find it convenient to split off a dimensionful coupling from the dimensionful tensors, so that all the tensors are dimensionless.

We will be especially interested in cases in which the theory has some global symmetries. Let Φ be a diffeomorphism of Y that leaves the tensors $h^{(2)}$, $h^{(4)}$, A, T invariant, for example

$$T_{\alpha\beta\gamma\delta}(y) = \frac{\partial \Phi^{\alpha'}}{\partial y^\alpha} \frac{\partial \Phi^{\beta'}}{\partial y^\beta} \frac{\partial \Phi^{\gamma'}}{\partial y^\gamma} \frac{\partial \Phi^{\delta'}}{\partial y^\delta} T_{\alpha'\beta'\gamma'\delta'}(\Phi(y)) \ . \tag{6.8}$$

In particular, Φ is an isometry of $h^{(4)}$. Then the action is invariant under the transformation $\varphi \mapsto \Phi \circ \varphi$. Such global symmetries may be discrete, or they may form a continuous group G. In the latter case there exist vector fields K_a on Y (with $a = 1 \dots \mathrm{dim}G$) whose Lie brackets form an algebra isomorphic to the Lie algebra of G, and such that $h^{(2)}$, $h^{(4)}$, A, T are invariant under G:

$$\mathcal{L}_{K_a} h^{(2)} = 0 \ ; \quad \mathcal{L}_{K_a} h^{(4)} = 0 \ ; \quad \mathcal{L}_{K_a} A = 0 \ ; \quad \mathcal{L}_{K_a} T = 0 \ . \tag{6.9}$$

In particular, K_a are Killing vectors for the metric $h^{(4)}$: $\nabla_\alpha K_{a\beta} + \nabla_\beta K_{a\alpha} = 0$. Then, the action (6.5) is invariant under the infinitesimal transformation $\delta_\epsilon \varphi^\alpha = \epsilon^a K_a^\alpha(\varphi)$.

Discrete isometries may appear in the definition of parity or time reversal. In linear scalar theories one can define the operation $\phi \mapsto -\phi$. For example the pions transform as $(P\pi)^a(x_1, x_2, x_3, x_4) = -\pi^a(-x_1, x_2, x_3, x_4)$ under parity. In a general NLSM the transformation $\varphi^\alpha \mapsto -\varphi^\alpha$ has no intrinsic meaning. However, suppose that every point $y \in Y$ is the fixed point of an involutive isometry Φ_y. Such a manifold is said to be a symmetric space [63]. We can then define a new parity operation, let us call it "Parity" with capital P, by $(P\varphi)^\alpha(x_1, x_2, x_3, x_4) = \Phi_0 \circ \varphi(-x_1, x_2, x_3, x_4)$, where Φ_0 is the involutive isometry of the vacuum. The transformation properties of the action under this new definition of parity are different than under the previous definition. In particular, if $A_{\alpha\beta\gamma}(\Phi_0(y)) = A_{\alpha\beta\gamma}(y)$, then the A-term will not be Parity–invariant. On the other hand if $B_{\alpha\beta\gamma\delta}(\Phi_0(y)) = -B_{\alpha\beta\gamma\delta}(y)$, then the Wess-Zumino-Witten term is Parity–invariant [64].

6.2.2 Background field expansion.

We use the background field techniques developed in [65, 66, 67, 68]. We review here some of the main points. Having chosen a (not necessarily constant) background $\bar\varphi$, any other field φ in an open neighborhood of $\bar\varphi$ can be written $\varphi^\alpha = \bar\varphi^\alpha + \eta^\alpha$. In principle one could work with the quantum fields η^α, but this is not convenient because, as differences of coordinates, they do not have nice transformation properties. It is therefore convenient to proceed as follows. For each x one can find a unique vector $\xi(x)$ tangent to $\bar\varphi(x)$ such that $\varphi(x)$ is the point on the geodesic passing through $\bar\varphi(x)$ and tangent to $\xi(x)$, the distance between $\varphi(x)$ and $\bar\varphi(x)$ being equal to $|\xi(x)|$. We can thus write $\varphi(x) = Exp_{\bar\varphi(x)}\xi(x)$, where Exp is the exponential map. The field, $\xi^\alpha(x)$ is a vectorfield along $\bar\varphi$, and its covariant derivative is defined as in (6.1).

In principle, then, the action φ can be rewritten as $S(\varphi) = \bar S(\bar\varphi, \xi)$. In practice one can compute the first few terms in an expansion $\bar S(\bar\varphi, \xi) = \bar S^{(0)}(\bar\varphi, \xi) + \bar S^{(1)}(\bar\varphi, \xi) + \bar S^{(2)}(\bar\varphi, \xi) + \dots$, where $\bar S^{(n)}$ contains n powers of ξ. The first term is clearly $\bar S^{(0)}(\bar\varphi, \xi) = \bar S(\bar\varphi, 0) = S(\bar\varphi)$. To compute the next terms we use the following formulas (whose derivation can be found in [65]):

$$\partial_\mu \varphi^\alpha = \partial_\mu \bar\varphi^\alpha + \bar\nabla_\mu \xi^\alpha - \frac{1}{3} \partial_\mu \bar\varphi^\gamma \bar R_{\gamma\epsilon}{}^\alpha{}_\eta \xi^\epsilon \xi^\eta + \dots \tag{6.10}$$

$$\begin{aligned} t_{\alpha\beta\dots}(\varphi) &= t_{\alpha\beta\dots}(\bar\varphi) + \xi^\epsilon \bar\nabla_\epsilon t_{\alpha\beta\dots}(\bar\varphi) + \frac{1}{2} \xi^\epsilon \xi^\eta \bar\nabla_\epsilon \bar\nabla_\eta t_{\alpha\beta\dots}(\bar\varphi) \\ &\quad - \frac{1}{6} \xi^\epsilon \xi^\eta \bar R^\gamma{}_{\epsilon\alpha\eta} t_{\gamma\beta\dots}(\bar\varphi) - \frac{1}{6} \xi^\epsilon \xi^\eta \bar R^\gamma{}_{\epsilon\beta\eta} t_{\alpha\gamma\dots}(\bar\varphi) + \dots \end{aligned} \tag{6.11}$$

A bar over the derivatives and the curvatures indicates that they have to be computed with the background field $\bar{\varphi}$. In particular for the metric g we have

$$g_{\alpha\beta}(\varphi) = g_{\alpha\beta}(\bar{\varphi}) - \frac{1}{3}\bar{R}_{\alpha\epsilon\beta\eta}\xi^\epsilon\xi^\eta + \ldots \tag{6.12}$$

Inserting in (6.5), with $A = 0$, and keeping terms of second order in ξ we obtain

$$\begin{aligned}
\frac{1}{2}\int d^4x \Big[& h^{(2)}_{\alpha\beta}\nabla_\mu\xi^\alpha\nabla^\mu\xi^\beta - \xi^\alpha\xi^\beta R_{\alpha\gamma\beta}{}^\epsilon h^{(2)}_{\epsilon\delta}\partial_\mu\varphi^\gamma\partial_\mu\varphi^\delta + 2\xi^\alpha\nabla_\mu\xi^\beta\nabla_\alpha h^{(2)}_{\beta\gamma}\partial_\mu\varphi^\gamma \\
&+\frac{1}{2}\xi^\alpha\xi^\beta\nabla_\alpha\nabla_\beta h^{(2)}_{\gamma\delta}\partial_\mu\varphi^\gamma\partial_\mu\varphi^\delta + h^{(4)}_{\alpha\beta}\Box\xi^\alpha\Box\xi^\beta + 2\xi^\alpha\Box\xi^\beta R_{\alpha\gamma\beta\delta}\partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta \\
&-4\xi^\alpha\nabla_\mu\xi^\beta R_{\alpha\gamma\beta\delta}\partial^\mu\varphi^\gamma\Box\varphi^\delta - \xi^\alpha\xi^\beta R_{\alpha\gamma\beta\delta}\Box\varphi^\gamma\Box\varphi^\delta \\
&+\xi^\alpha\xi^\beta\left(\nabla_\alpha R_{\epsilon\gamma\beta\delta} + \nabla_\gamma R_{\epsilon\alpha\beta\delta}\right)\partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta\Box\varphi^\epsilon \\
&+\xi^\alpha\xi^\beta R_{\phi\gamma\delta\alpha}R^\phi{}_{\epsilon\eta\beta}\partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta\partial_\nu\varphi^\epsilon\partial^\nu\varphi^\eta + 2\nabla_\mu\xi^\alpha\nabla^\mu\xi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta T_{\alpha\beta\gamma\delta} \\
&+4\nabla_\mu\xi^\alpha\nabla_\nu\xi^\beta\partial^\mu\varphi^\gamma\partial^\nu\varphi^\delta T_{\alpha\gamma\beta\delta} - 2\xi^\alpha\xi^\beta R^\phi{}_{\alpha\gamma\beta}T_{\phi\delta\epsilon\eta}\partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta\partial_\nu\varphi^\epsilon\partial^\nu\varphi^\eta \\
&+4\xi^\alpha\nabla_\mu\xi^\beta\nabla_\alpha T_{\beta\gamma\delta\epsilon}\partial^\mu\varphi^\gamma\partial_\nu\varphi^\delta\partial^\nu\varphi^\epsilon \\
&+\frac{1}{2}\xi^\alpha\xi^\beta\nabla_\alpha\nabla_\beta T_{\gamma\delta\epsilon\eta}\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta\partial_\mu\varphi^\epsilon\partial^\mu\varphi^\eta\Big] \ .
\end{aligned} \tag{6.13}$$

For notational simplicity here and in the following we drop the bars over φ, ∇ and R, but it is always understood that they are computed at the background field. The terms have been kept in the order in which they appear in (6.5), namely the first four terms come from the variation of the two-derivative term, the next five come from the variation of the term containing $h^{(4)}$, and the others come from the variation of the term containing T.

6.2.3 The running NLSM action.

Our procedure for calculating the beta functions is to implement the 1-loop approximation of ERGE. We define formally an "effective average action" Γ_k by implementing an *infrared* cutoff k in the functional integral over the quantum field ξ. If $\bar{S}(\varphi, \xi)$ is the bare action of the theory, the IR cutoff can be implemented by adding to $\bar{S}$ the term $\Delta S_k(\varphi, \xi)$ which in Fourier space would have the general structure:

$$\Delta S_k(\varphi, \xi) = \int d^4q\, \xi^\alpha(-q) R_{k\alpha\beta}(q^2)\xi^\beta(q) \ . \tag{6.14}$$

We further decided to write the cutoff in terms of the eigenvalues of some covariant operator, such as the Laplacian constructed with the background field $-\nabla^2$. This is the choice that was used in [28]. In this chapter we will find it expedient to use instead of $-\nabla^2$ the full covariant fourth order operator $\Delta = \frac{\delta^2 S}{\delta\varphi\delta\varphi}$:

$$\Delta S_k(\varphi, \xi) = \frac{1}{2}\int d^4x\, \xi^\alpha h^{(4)}_{\alpha\beta}(\varphi) R_k(\Delta)\xi^\beta \ . \tag{6.15}$$

Because Δ depends only on the background field, and not on the quantum fields, this cutoff is still quadratic in the quantum fields, as required.

As usual, the effective average action is defined as

$$\begin{aligned}\Gamma_k(\varphi,\bar\xi) &= -\log\int D\xi^\alpha \exp\left(-\bar S(\varphi,\xi)-\Delta S_k(\varphi,\xi)-\int j_\alpha\xi^\alpha\right)\\ &\quad -\int j_\alpha\bar\xi^\alpha-\Delta S_k(\varphi,\bar\xi) \end{aligned}\tag{6.16}$$

and it tends to the full EA in the limit $k\to 0$.

6.2.4 The one loop beta functional.

At one loop one can evaluate the functional Γ_k:

$$\Gamma^{(1)}_k = S+\frac{1}{2}\mathrm{Tr}\log\left(\frac{\delta^2 S}{\delta\varphi\delta\varphi}+R_k\right)\ . \tag{6.17}$$

Note that ΔS_k has canceled out. The only remaining dependence on k is in R_k, so

$$k\frac{d\Gamma^{(1)}_k}{dk}=\frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\varphi\delta\varphi}+R_k\right)^{-1}k\frac{dR_k}{dk}\ . \tag{6.18}$$

The right-hand side can be regarded as the one loop beta functional of the theory. The individual beta functions can be read off by isolating the coefficients of various operators. Although in the present chapter we shall restrict ourselves to the one loop approximation, the formalism is ready for the calculation of the beta functions based on a truncation of the exact RG equation, which amount to resumming infinitely many orders of perturbation theory. A final, important point is that experience with other systems shows that this procedure gives exactly the same results as any other procedure for the universal (scheme-independent) one loop beta functions. We will see in Section 6.4.2 that, to the extent that a comparison is possible, this expectation will be confirmed also in this case.

6.2.5 Global symmetries.

If there are any (global) symmetries, one can define the RG flow so as to preserve them. To see this, let Φ be an internal symmetry, as in Section 6.2.1. Since it is an isometry of $h^{(4)}$, it also leaves the connection invariant, so it maps the geodesic through y tangent to ξ to the geodesic through $\Phi(y)$ tangent to $T\Phi(\xi)$ [69]:

$$\Phi(Exp_y(\xi)) = Exp_{\Phi(y)}(T\Phi(\xi))\ . \tag{6.19}$$

We call $\varphi'=\Phi\circ\varphi$ and $\xi' = T\Phi(\xi)$ the transform of φ and ξ under Φ. Then $\varphi'=\Phi(Exp_{\bar\varphi}\xi)=Exp_{\Phi(\bar\varphi)}(T\Phi(\xi))=Exp_{\bar\varphi'}\xi'$. There follows that

$$\bar S(\bar\varphi',\xi')=S(\varphi')=S(\varphi)=\bar S(\bar\varphi,\xi)\,, \tag{6.20}$$

i.e. the background field action $\bar S$ is G-invariant provided both background and quantum field are transformed. The operator Δ is covariant, so $\Delta'(\xi')=T\Phi(\Delta(\xi))$ or abstractly $\Delta' = T\Phi\circ(\Delta)\circ T\Phi^{-1}$, so also the cutoff term (6.15) is invariant:

$$\Delta S_k(\varphi',\xi')=\Delta S_k(\varphi,\xi)\,, \tag{6.21}$$

One can formally choose the measure in the functional integral (6.16) to be invariant under Φ. Since both measure and integrand are invariant, the effective action Γ_k will also be invariant, for all k.

Somewhat less formally, one can arrive at the same conclusion as follows: observe that the cutoff as defined in (6.15) is a suppression term that depends on the eigenvalue of the operator Δ on the normal modes of the field. From the transformation properties of Δ one sees that if ξ is an eigenvector of Δ with eigenvalue λ, then ξ' is an eigenvector of Δ' with the same eigenvalue. Therefore the spectrum of Δ is invariant. Equation (6.18) gives the (one loop) scale variation of $\Gamma_k(\varphi)$ as a sum of terms, each term being a fixed function evaluated on an eigenvalues of Δ. Since the eigenvalues are invariant, the sum is also invariant, so it follows that $\partial_t \Gamma_k(\varphi)$ is invariant. This implies that if the starting action $\Gamma_{k_0}(\varphi)$ is invariant, also the action at any other k is. This argument is mathematically more meaningful, because unlike the one based on the path integral, it involves only statements about finite expressions.

The previous argument can be applied both to discrete and continuous symmetries. For example in the case of discrete symmetries, it implies that the flow preserves Parity. If the A term violates Parity, it must be set to zero in order to have a Parity-invariant theory. The flow will preserve this property, so the beta function of A will be zero. In other words the condition $A = 0$ will be "protected by Parity". We will see this in an explicit calculation in Section 6.3.2.

6.3 Evaluation of beta functions.

The one loop RG flow Eq. (6.18) can be approximated by resorting to a truncation, which means keeping only a finite number of terms in Γ_k, inserting this ansatz in the flow equation and deriving from it the beta functions of the couplings that enter in the ansatz. The best way of truncating Γ_k is to do so consistently with a derivative expansion, *i.e.* to keep all the terms with a given number of derivatives. In this chapter we will approximate Γ_k by a functional of the form (6.5), where the tensors $h^{(2)}$, $h^{(4)}$, and T are k-dependent, and $A = 0$. In general this is still a functional flow, because the tensors actually contain infinitely many couplings. We will be able to say more in the case when a global symmetry restricts the possible form of these tensors, so that only finitely many couplings remain. In this chapter we will explicitly compute the beta functions in the case when Y is a sphere or a special unitary group. Since these are symmetric spaces, it will be consistent to neglect the A terms altogether.

6.3.1 The inverse propagator.

Integrating by parts one can rewrite (6.13) in the form $\bar{S}^{(2)}(\varphi, \xi) = \frac{1}{2}(\xi, \Delta\xi)$, where the inner product of vectorfields along φ is $(\xi, \zeta) = \int d^4x\, h^{(4)}_{\alpha\beta} \xi^\alpha \zeta^\beta$ and Δ is a self-adjoint operator of the form:

$$\Delta_{\alpha\beta} = h^{(4)}_{\alpha\beta} \Box^2 + \mathcal{B}^{\mu\nu}_{\alpha\beta} \nabla_\mu \nabla_\nu + \mathcal{C}^{\mu}_{\alpha\beta} \nabla_\mu + \mathcal{D}_{\alpha\beta} \,. \tag{6.22}$$

Self-adjointness means that $(\xi, \Delta\zeta) = (\Delta\xi, \zeta)$ and implies the properties:

$$\begin{aligned}
\mathcal{B}^{\mu\nu}_{\alpha\beta} &= \mathcal{B}^{\nu\mu}_{\beta\alpha} \ , && (6.23)\\
\mathcal{C}^{\mu}_{\alpha\beta} &= -\mathcal{C}^{\mu}_{\beta\alpha} + \nabla_\nu \mathcal{B}^{\mu\nu}_{\beta\alpha} + \nabla_\nu \mathcal{B}^{\nu\mu}_{\beta\alpha} \ , && (6.24)\\
\mathcal{D}_{\alpha\beta} &= \mathcal{D}_{\beta\alpha} + \nabla_\nu \mathcal{C}^{\mu}_{\beta\alpha} + \nabla_\nu \nabla_\nu \mathcal{B}^{\nu\mu}_{\beta\alpha} \ . && (6.25)
\end{aligned}$$

In addition by commuting derivatives we can arrange the operator so that $\mathcal{B}^{\mu\nu}_{\alpha\beta} = \mathcal{B}^{\nu\mu}_{\alpha\beta}$. In order to arrive at the operator Δ we proceed in two steps. First we put all the derivatives of (6.13) on one of the ξ's, so that $\bar{S}^{(2)}(\varphi, \xi) = \frac{1}{2}(\xi, \tilde{\Delta}\xi)$, where $\tilde{\Delta}$ is of the form (6.22), with

$$\begin{aligned}
\tilde{\mathcal{B}}^{\mu\nu}_{\alpha\beta} &= \delta^{\mu\nu}(-h^{(2)}_{\alpha\beta} + 2\partial_\rho\varphi^\gamma\partial^\rho\varphi^\delta(R_{\alpha\gamma\beta\delta} - T_{\alpha\beta\gamma\delta})) - 4\partial_\mu\varphi^\gamma\partial_\nu\varphi^\delta T_{\alpha\gamma\beta\delta}\\
\tilde{\mathcal{D}}_{\alpha\beta} &= \partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta\left(\frac{1}{2}\nabla_\alpha\nabla_\beta h^{(2)}_{\gamma\delta} - h^{(2)}_{\gamma\epsilon}R^\epsilon{}_{\beta\delta\alpha}\right)\\
&\quad -\Box\varphi^\gamma\Box\varphi^\delta R_{\alpha\gamma\beta\delta} - 2\partial_\rho\varphi^\gamma\partial^\rho\varphi^\delta\Box\varphi^\epsilon\nabla_{(\delta}R_{\alpha)\epsilon\beta\gamma}\\
&\quad +\partial_\rho\varphi^\gamma\partial^\rho\varphi^\delta\partial_\sigma\varphi^\epsilon\partial^\sigma\varphi^\eta \times\\
&\quad \times\left(R_{\alpha\gamma\delta\phi}R_{\beta\epsilon\eta}{}^\phi + \frac{1}{2}\nabla_\alpha\nabla_\beta T_{\gamma\delta\epsilon\eta} + 2R_{\phi\alpha\beta\epsilon}T^\phi{}_{\eta\gamma\delta}\right) \ . \qquad (6.26)
\end{aligned}$$

We do not display the form of $\tilde{\mathcal{C}}^\mu_{\alpha\beta}$, since it does not contribute to the expressions we want to calculate, as will become clear in due course. This operator $\tilde{\Delta}$ is not self-adjoint, and we define $\Delta = \frac{1}{2}(\tilde{\Delta} + \tilde{\Delta}^\dagger)$. Its coefficients are

$$\begin{aligned}
\mathcal{B}^{\mu\nu}_{\alpha\beta} &= \frac{1}{2}\left(\tilde{\mathcal{B}}^{\mu\nu}_{\alpha\beta} + \tilde{\mathcal{B}}^{\nu\mu}_{\beta\alpha}\right) \ ,\\
\mathcal{C}^{\mu}_{\alpha\beta} &= \frac{1}{2}\left(\tilde{\mathcal{C}}^{\mu}_{\alpha\beta} - \tilde{\mathcal{C}}^{\mu}_{\beta\alpha} + \nabla_\nu\tilde{\mathcal{B}}^{\mu\nu}_{\beta\alpha} + \nabla_\nu\tilde{\mathcal{B}}^{\nu\mu}_{\beta\alpha}\right) \ ,\\
\mathcal{D}_{\alpha\beta} &= \frac{1}{2}\left(\tilde{\mathcal{D}}_{\alpha\beta} + \tilde{\mathcal{D}}_{\beta\alpha} - \nabla_\mu\tilde{\mathcal{C}}^{\mu}_{\beta\alpha} + \nabla_\mu\nabla_\nu\tilde{\mathcal{B}}^{\nu\mu}_{\beta\alpha}\right) \ . \qquad (6.27)
\end{aligned}$$

Note that the last two terms in $\mathcal{C}^\mu_{\alpha\beta}$ and $\mathcal{D}^\mu_{\alpha\beta}$ are total derivatives, and will not contribute to our final formulas because they will be integrated over spacetime. Finally we symmetrize $\mathcal{B}^{\mu\nu}_{\alpha\beta}$ in μ, ν at the cost of generating a commutator term that contributes to $\mathcal{D}_{\alpha\beta}$. The final form of the operator Δ is (6.22), with

$$\begin{aligned}
\mathcal{B}^{\mu\nu}_{\alpha\beta} &= \delta^{\mu\nu}(-h^{(2)}_{\alpha\beta} + 2\partial_\rho\varphi^\gamma\partial^\rho\varphi^\delta(R_{\alpha\gamma\beta\delta} - T_{\alpha\beta\gamma\delta}))\\
&\quad -2\partial_\mu\varphi^\gamma\partial_\nu\varphi^\delta(T_{\alpha\gamma\beta\delta} + T_{\alpha\delta\beta\gamma}) \ , \qquad (6.28)\\
\mathcal{D}_{\alpha\beta} &= \frac{1}{2}\left(\tilde{\mathcal{D}}_{\alpha\beta} + \tilde{\mathcal{D}}_{\beta\alpha}\right) - \partial_\rho\varphi^\gamma\partial^\rho\varphi^\delta\partial_\sigma\varphi^\epsilon\partial^\sigma\varphi^\eta(T_{\alpha\gamma\epsilon\phi}R_{\delta\eta\beta}{}^\phi + T_{\beta\gamma\epsilon\phi}R_{\delta\eta\alpha}{}^\phi) \ ,
\end{aligned}$$

where total derivatives have been omitted. Again we do not give $\mathcal{C}^\mu_{\alpha\beta}$, because it will not contribute to the beta functions. These formulas agree with (Eqs. 3.17-21) in [58], except for a factor 2 in the coefficient of the first term containing $T_{\alpha\beta\gamma\delta}$ in Eq. (6.26).

6.3.2 Beta functionals.

We begin by discussing the general case of the action (6.5) with arbitrary $h^{(2)}$, $h^{(4)}$ and T, and $A = 0$. We evaluate the trace in (6.18) by heat kernel methods.

The advantage of this procedure is that pieces of the calculation are readily available in the literature. Given a differential operator Δ of order p, and some function W, we have

$$TrW(\Delta) = \frac{1}{(4\pi)^2}\Big[Q_{\frac{4}{p}}(W)B_0(\Delta)+Q_{\frac{2}{p}}(W)B_2(\Delta)+Q_0(W)B_4(\Delta)+\ldots\Big] . \quad (6.29)$$

The heat kernel coefficients are defined by the asymptotic expansion

$$\mathrm{Tr}(e^{-s\Delta}) = \frac{1}{(4\pi)^2}\Big[B_0 s^{-4/p} + B_2 s^{-2/p} + B_0 + \ldots\Big] , \quad (6.30)$$

with $B_n = \int d^4x \mathrm{tr} b_n$; b_n are matrices with indices α, β and tr denotes the trace over such indices. The matrices b_n that pertain to a fourth order operator of the form (6.22) can be found in [70]. The quantities $Q_n(W)$ in (6.29) are given by $Q_n(W) = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}W(z)$ for $n > 0$ and $Q_0(W) = W(0)$. We do not need any higher coefficients. In order to be able to evaluate the integrals in closed form we choose the "optimized" cutoff function $R_k(z) = (k^4 - z)\theta(k^4 - z)$ [31]. The scale derivative of the cutoff is $k\frac{dR_k}{dk} = 4k^4\theta(k^4-z)$, and the modified inverse propagator $P_k(z) = z+R_k(z)$ is equal to k^4 for $z < k^4$. Then the function to be traced in the ERGE is just a step function: $W(z) = \frac{1}{2}\frac{1}{P_k}k\frac{dR_k}{dk} = 2\theta(1 - z/k^4)$, and the integrals are very simple:

$$Q_1 = 2k^4 , \qquad Q_{\frac{1}{2}} = \frac{4}{\sqrt{\pi}}k^2 , \qquad Q_0 = 2 . \quad (6.31)$$

The first term in (6.29) is field independent and will be omitted. Putting together the remaining pieces:

$$k\frac{d\Gamma_k}{dk} = \frac{1}{(4\pi)^2}\int d^4x \left(\frac{1}{4}k^2\mathcal{B}^\alpha_\alpha + \frac{1}{6}\Omega^{\alpha\beta}_{\mu\nu}\Omega^{\mu\nu}_{\beta\alpha} + \frac{1}{24}\mathcal{B}^{\alpha\beta}_{\mu\nu}\mathcal{B}^{\mu\nu}_{\beta\alpha} + \frac{1}{48}\mathcal{B}_{\alpha\beta}\mathcal{B}^{\beta\alpha} - \mathcal{D}^\alpha_\alpha\right) \quad (6.32)$$

where Ω is defined as in (6.3) and $\mathcal{B} = \mathcal{B}^\mu_\mu$. The first term comes from B_2, the others from B_4. For brevity we define

$$\mathcal{B}^2 = \mathcal{B}^{\alpha\beta}_{\mu\nu}\mathcal{B}^{\mu\nu}_{\beta\alpha} + \frac{1}{2}\mathcal{B}_{\alpha\beta}\mathcal{B}^{\beta\alpha} \quad (6.33)$$

One finds

$$\begin{aligned}
\frac{1}{4}\mathcal{B}^\alpha_\alpha &= \partial_\mu\varphi^\gamma\partial^\mu\varphi^\delta\left(2R_{\gamma\delta}-2T^\alpha{}_{\alpha\gamma\delta}-T^\alpha{}_{\gamma\alpha\delta}\right) && (6.34)\\
\frac{1}{6}\Omega^{\alpha\beta}_{\mu\nu}\Omega^{\mu\nu}_{\beta\alpha} &= -\frac{1}{6}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta R_{\alpha\gamma\epsilon\eta}R_{\beta\delta}{}^{\epsilon\eta} && (6.35)\\
\frac{1}{24}\mathcal{B}^2 &= \frac{1}{2}h^{(2)}_{\alpha\beta}h^{(2)\alpha\beta}+\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\left(T_\alpha{}^\gamma{}_\beta{}^\delta+2T_{\alpha\beta}{}^{\gamma\delta}-2R_\alpha{}^\gamma{}_\beta{}^\delta\right)h^{(2)}_{\gamma\delta}\\
&\quad+\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta\times\\
&\quad\Big[\frac{2}{3}T_{\alpha\epsilon\gamma\eta}T_\beta{}^{(\epsilon}{}_\delta{}^{\eta)}+\frac{1}{3}T_{\alpha\epsilon\beta\eta}T_\gamma{}^{(\eta}{}_\delta{}^{\epsilon)}+4T_{\alpha(\epsilon\gamma)\eta}T_\beta{}^{(\epsilon}{}_\delta{}^{\eta)}\\
&\quad-4R_{\alpha\epsilon\gamma\eta}T_\beta{}^{(\epsilon}{}_\delta{}^{\eta)}-2R_{\alpha\epsilon\beta\eta}T_\gamma{}^{(\eta}{}_\delta{}^{\epsilon)}+2R_{\alpha\epsilon\beta\eta}R_\gamma{}^{(\eta}{}_\delta{}^{\epsilon)}\Big] && (6.36)\\
-\mathcal{D}^\alpha_\alpha &= \Box\varphi^\alpha\Box\varphi^\beta R_{\alpha\beta}+\Box\varphi^\alpha\partial^\mu\varphi^\beta\partial_\mu\varphi^\gamma(2\nabla_\gamma R_{\alpha\beta}-\nabla_\alpha R_{\beta\gamma})\\
&\quad+\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\left(h^{(2)}_{\alpha\gamma}R^\gamma{}_\beta-\frac{1}{2}\nabla_\gamma\nabla^\gamma h^{(2)}_{\alpha\beta}\right)\\
&\quad+\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta\times && (6.37)\\
&\quad\left(2R_\alpha{}^\epsilon T_{\epsilon\beta\gamma\delta}+2R_{\beta\delta}{}^{\epsilon\eta}T_{\alpha\eta\gamma\epsilon}-\frac{1}{2}\nabla_\epsilon\nabla^\epsilon T_{\alpha\beta\gamma\delta}-R_{\alpha\epsilon\beta\eta}R_\gamma{}^\epsilon{}_\delta{}^\eta\right)
\end{aligned}$$

From here one can read off the beta functionals of $h^{(2)}$, A, T as the coefficients of terms containing two, three and four powers of $\partial_\mu\varphi^\alpha$, respectively. We do not give these general formulae, but just make some observations. The only term proportional to $\Box\varphi^\alpha\Box\varphi^\beta$ is contained in $-\mathcal{D}^\alpha_\alpha$, so the beta functional of $h^{(4)}$ is easily obtained:

$$k\frac{d}{dk}h^{(4)}_{\alpha\beta}=\frac{1}{8\pi^2}R_{\alpha\beta}\,. \qquad (6.38)$$

This is very similar to the result for the two-derivative truncation. In order to compare results obtained with the same type of cutoff, we should repeat the calculation of [28] using a cutoff constructed with the full inverse propagator $\Delta_{\alpha\beta}=-h^{(2)}_{\alpha\beta}\nabla^2-\partial_\mu\varphi^\gamma\partial_\mu\varphi^\delta R_{\alpha\gamma\beta\delta}$. This is a cutoff of type III in the terminology used in [24]. In this case the general beta function of the metric is

$$k\frac{d}{dk}h^{(2)}_{\alpha\beta}=\frac{1}{(4\pi)^2}Q_1\left(\frac{\dot R_k}{P_k}\right)R_{\alpha\beta}=\frac{1}{8\pi^2}k^2R_{\alpha\beta}\ , \qquad (6.39)$$

where R denotes now the curvature of $h^{(2)}_{\alpha\beta}$. As a side remark, this little calculation is also useful to test the scheme dependence of the results: with the type I cutoff used in [28] the result was

$$k\frac{d}{dk}h^{(2)}_{\alpha\beta}=\frac{1}{(4\pi)^2}Q_2\left(\frac{\dot R_k}{P_k^2}\right)R_{\alpha\beta}=\frac{1}{16\pi^2}k^2R_{\alpha\beta}\ , \qquad (6.40)$$

which differs by a factor 2.

Another fact that follows from (6.37) is that the beta function of A (coming from the coefficient of $\Box\varphi^\alpha\partial^\mu\varphi^\beta\partial_\mu\varphi^\gamma$) is generated and proportional to covariant derivatives of the Ricci tensor. For symmetric spaces the covariant derivative of the curvature vanishes and therefore on such spaces it is consistent to set $A=0$ [58]. In [58] it was also noted that there is a larger class of manifolds for which it is consistent to choose $A = 0$. The general statement made in Section 6.2.5 is confirmed. The particular models that we shall consider in the following are symmetric spaces.

6.3.3 The spherical models.

We now consider the class of models for which the target space Y is the sphere S^n. Such models are often called the $O(N)$ models, with $N = n+1$, because they have global symmetry $O(N)$. There is only one $O(n+1)$-invariant nonvanishing rank two tensor on the sphere, there is no invariant rank three tensor and there are only two invariant rank four tensors with the desired index symmetries, up to overall constant factors. If we regard S^n as embedded in $\mathbf{R}^{n+1}$, we call $h_{\alpha\beta}$ the metric of the sphere of unit radius. Its Riemann and Ricci tensors are given by

$$R_{\alpha\beta\gamma\delta} = h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma} \; ; \quad R_{\alpha\beta} = (n-1)h_{\alpha\beta} \; ; \quad R = n(n-1) \; . \tag{6.41}$$

Therefore both $h^{(2)}$ and $h^{(4)}$ must be proportional to h, and T is a combination of h's:

$$\begin{aligned} h^{(2)}_{\alpha\beta} &= \frac{1}{g^2}h_{\alpha\beta} \; ; \qquad h^{(4)}_{\alpha\beta} = \frac{1}{\lambda}h_{\alpha\beta} \; ; \\ T_{\alpha\beta\gamma\delta} &= \frac{\ell_1}{2}\left(h_{\alpha\gamma}h_{\beta\delta} + h_{\alpha\delta}h_{\beta\gamma}\right) + \ell_2 h_{\alpha\beta}h_{\gamma\delta} \; . \end{aligned} \tag{6.42}$$

Here g^2 has mass dimension 2, while λ, ℓ_1, ℓ_2 are dimensionless [2]. It is convenient to regard $\frac{1}{\lambda}$ as the overall factor of the fourth order terms; then we define the ratios between the three coefficients of the four-derivative terms as $f_1 = \lambda\ell_1$ and $f_2 = \lambda\ell_2$. For the reader's convenience we rewrite the action of the S^n models:

$$\begin{aligned} \int d^4x \Bigg[&\frac{1}{2g^2}h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta + \frac{1}{2\lambda}h_{\alpha\beta}\Box\varphi^\alpha\Box\varphi^\beta \\ &+\frac{1}{2\lambda}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta(f_1 h_{\alpha\gamma}h_{\beta\delta} + f_2 h_{\alpha\beta}h_{\gamma\delta})\Bigg] \end{aligned} \tag{6.43}$$

One then finds the following beta functions:

$$\beta_\lambda = -\frac{n-1}{8\pi^2}\lambda^2 \tag{6.44}$$

$$\beta_{f_1} = \frac{\lambda}{48\pi^2}\left((n+21)f_1^2 + 20f_2f_1 + 4f_2^2 + 6(n+3)f_1 + 24f_2 + 8\right) \tag{6.45}$$

$$\begin{aligned} \beta_{f_2} = \frac{\lambda}{8\pi^2}\Bigg(&\frac{n+15}{12}f_1^2 + \frac{3n+17}{3}f_1f_2 + \frac{6n+7}{3}f_2^2 \\ &-(n+3)f_1 - (3n+1)f_2 + n - \frac{7}{3}\Bigg) \end{aligned} \tag{6.46}$$

$$\begin{aligned} \beta_{\tilde g^2} = {} & 2\tilde g^2 + \frac{\tilde g^4}{16\pi^2}\left((5+n)f_1 + (2+4n)f_2 + 4(1-n)\right) \\ & -\frac{\lambda\tilde g^2}{16\pi^2}\left((5+n)f_1 + (2+4n)f_2 + 2(1-n)\right) \end{aligned} \tag{6.47}$$

[2] The names ℓ_1 and ℓ_2 are used commonly in chiral perturbation theory [51].

Equations (6.45) and (6.46) differ in a significant way from Eq. (5.11) in [58]. This is due to the already mentioned factor 2 in a term in (6.26). Unfortunately, this changes completely the picture of the fixed points.

It will be instructive to compare the results of this four-derivative truncation with those of the simpler two-derivative truncation discussed in [28]. If we specialize (6.39) to $Y = S^n$, it gives

$$k\frac{d\tilde{g}^2}{dk} = 2\tilde{g}^2 - \frac{n-1}{8\pi^2}\tilde{g}^4\,, \tag{6.48}$$

whereas from (6.47), setting for simplicity $\ell_1 = \ell_2 = 0$ and in the limit $\lambda \to 0$ one gets

$$k\frac{d\tilde{g}^2}{dk} = 2\tilde{g}^2 - \frac{n-1}{4\pi^2}\tilde{g}^4\,. \tag{6.49}$$

The difference is just a factor 2, which is within the range of variation due to the scheme dependence. It is quite remarkable that the beta function is so similar in spite of the very different dynamics. We shall see in Section 6.4.1 that this fact is quite general.

6.3.4 The chiral models.

Next we consider the case where Y is the group $SU(N)$. In this case it is customary to denote $U(x)$ the matrix (in the fundamental representation) that corresponds to the coordinates φ^α. We demand that the theory be invariant under left and right multiplications $U(x) \mapsto g_L^{-1}U(x)g_R$, forming the group $SU(N)_L \times SU(N)_R$ ("chiral symmetry"). Further we demand that the theory be invariant under the discrete symmetries $U(x) \mapsto U^T(x)$, which corresponds physically to charge conjugation, to the simple parity $x_1 \mapsto -x_1$, to the involutive isometry $\Phi_0 : U \to U^{-1}$ and hence to Parity $U(x_1, x_2, x_3, x_4) \mapsto U^{-1}(-x_1, x_2, x_3, x_4)$. More details on the translation between the tensor and the matrix formalism are given in appendix.

Let e_a be a basis of the Lie algebra, with $a = 1 \dots n^2 - 1$. We denote T_a the corresponding matrices in the fundamental representation; they are a set of Hermitian, traceless $N \times N$ matrices. We fix the normalization of the basis by the equation

$$T_aT_b = \frac{1}{2N}\delta_{ab} + \frac{1}{2}\left(d_{abc} + if_{abc}\right)T_c\,. \tag{6.50}$$

(In the case of $SU(3)$ these matrices are one half the Gell-Mann λ matrices.)

A tensor on $SU(N)$ which is invariant under $SU(N)_L \times SU(N)_R$ is said to be "bi-invariant." There is a one to one correspondence between bi-invariant tensors on $SU(N)$ and Ad-invariant tensors in the Lie algebra of $SU(N)$, where Ad is the adjoint representation. Given an Ad-invariant tensor $t_{ab\dots}{}^{cd\dots}$ on the algebra, the corresponding biinvariant tensorfield on the group is

$$t_{\alpha\beta\dots}{}^{\gamma\delta\dots} = t_{ab\dots}{}^{cd\dots}L^a_\alpha L^b_\beta \dots L^\gamma_c L^\delta_d \dots \tag{6.51}$$

where L^a_α are the components of the left-invariant Maurer Cartan form $L = U^{-1}dU = L^a_\alpha dy^\alpha(-iT_a)$ and L^α_a are the components of the left-invariant vectorfields on $SU(N)$. The matrix L^α_a is the inverse of L^a_α. (In this construction we could use equivalently right-invariant objects.)

Up to rescalings, there is a unique Ad-invariant inner product in the Lie algebra, which we choose as $h_{ab} = 2\mathrm{Tr}T_aT_b = \delta_{ab}$ [3]. Then the corresponding biinvariant metric is

$$h_{\alpha\beta} = L^a_\alpha L^b_\beta \delta_{ab} \ , \tag{6.52}$$

so that the left-invariant vectorfields L_a can also be regarded as a vierbein. The Riemann and Ricci tensors and the Ricci scalar of h are given by

$$R_{\alpha\beta\gamma\delta} = \frac{1}{4} L^a_\alpha L^b_\beta L^c_\gamma L^d_\delta f_{ab}{}^e f_{ecd} \ ; \qquad R_{\alpha\beta} = \frac{1}{4} N h_{\alpha\beta} \ ; \qquad R = \frac{1}{4} N(N^2-1) \ . \tag{6.53}$$

As with the sphere, we define $h^{(2)}_{\alpha\beta} = \frac{1}{g^2} h_{\alpha\beta}$, $h^{(4)}_{\alpha\beta} = \frac{1}{\lambda} h_{\alpha\beta}$. The tensors d_{abc} and f_{abc} are a totally symmetric and a totally antisymmetric Ad-invariant three tensor in the algebra. In principle chiral invariance would permit a term in the action with $A_{\alpha\beta\gamma} = L^a_\alpha L^b_\beta L^c_\gamma d_{abc}$; however using $L^a_\alpha(\Phi_0(y)) = R^a_\alpha(y)$, $L_a^\alpha(y) R^b_\alpha(y) = Ad(g(y))^b{}_a$ and the Ad-invariance of d_{abc}, one sees that $A_{\alpha\beta\gamma}(\Phi_0(y)) = A_{\alpha\beta\gamma}(y)$, so this term violates Parity.

For T we have the following Ad-invariant four-tensors in the algebra with the correct symmetries:

$$\begin{aligned}
T^{(1)}_{abcd} &= \frac{1}{2}\left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \ ; \qquad T^{(2)}_{abcd} = \delta_{ab}\delta_{cd} \ ; \\
T^{(3)}_{abcd} &= \frac{1}{2}\left(f_{ace} f_{bd}{}^e + f_{ade} f_{bc}{}^e\right) \ ; \\
T^{(4)}_{abcd} &= \frac{1}{2}\left(d_{ace} d_{bd}{}^e + d_{ade} d_{bc}{}^e\right) \ ; \qquad T^{(5)}_{abcd} = d_{abe} d_{cd}{}^e \ .
\end{aligned} \tag{6.54}$$

They are not all independent, however. The identity (2.10) of [71] implies that

$$\frac{2}{N} T^{(1)} - \frac{2}{N} T^{(2)} + T^{(3)} + T^{(4)} - T^{(5)} = 0 \ , \tag{6.55}$$

so that $T^{(5)}$ can be eliminated. In the case $N = 3$ the identity (2.23) of [71], together with the preceding relation, further implies

$$T^{(2)} - T^{(3)} - 3T^{(4)} = 0 \ , \tag{6.56}$$

so that we can also eliminate $T^{(4)}$. Finally in the case $N = 2$ the tensor d_{abc} is identically zero, so we can keep only $T^{(1)}$ and $T^{(2)}$ as independent combinations, and use $T^{(3)} = T^{(2)} - T^{(1)}$.

The action of the generic $SU(N)$ models can then be written in the form

$$\int d^4x \left[\frac{1}{2g^2} h_{\alpha\beta} \partial_\mu\varphi^\alpha \partial^\mu\varphi^\beta + \frac{1}{2\lambda} h_{\alpha\beta} \Box\varphi^\alpha \Box\varphi^\beta + \frac{1}{2} \partial_\mu\varphi^\alpha \partial^\mu\varphi^\beta \partial_\nu\varphi^\gamma \partial^\nu\varphi^\delta \sum_{i=1}^{4} \ell_i T^{(i)}_{\alpha\beta\gamma\delta} \right] \tag{6.57}$$

and the sum stops at $i = 3$ and $i = 2$ for $N = 3$ and $N = 2$, respectively. As in (6.43), it will be convenient to use instead of the couplings ℓ_i the combinations $f_i = \lambda\ell_i$.

[3] Here the matrices are in the fundamental representation. The Cartan-Killing form just differs by a constant: $B_{ab} = \mathrm{Tr}(Ad(T_a)Ad(T_b)) = N\delta_{ab}$.

Making repeated use of traces given in [72] one finds the following beta functions:

$$
\begin{aligned}
\beta_\lambda &= -\frac{N}{32\pi^2}\lambda^2 && (6.58)\\
\beta_{f_1} &= \frac{\lambda}{768\pi^2N^2}\Big[16N^2(N^2+20)f_1^2+64N^2f_2^2+180N^2f_3^2\\
&\quad +4(149N^2-1280)f_4^2+320N^2f_1f_2-32N^3f_1f_3\\
&\quad +32N(N^2+4)f_1f_4+128Nf_2f_4-120N^2f_3f_4+24N^3f_1\\
&\quad -108N^2f_3+36N^2f_4+9N^2\Big] && (6.59)\\
\beta_{f_2} &= \frac{\lambda}{768\pi^2N^2}\Big[8N^2(N^2+14)f_1^2+32N^2(6N^2+1)f_2^2+60N^2f_3^2\\
&\quad +4(7N^2+656)f_4^2+32N^2(3N^2+14)f_1f_2+80N^3f_1f_3\\
&\quad +16N(7N^2-44)f_1f_4+288N^3f_2f_3+32N(15N^2-64)f_2f_4\\
&\quad +120N^2f_3f_4-24N^3(f_1+3f_2)-36N^2(f_3+f_4)+3N^2\Big] && (6.60)\\
\beta_{f_3} &= \frac{\lambda}{1536\pi^2N}\Big[52N^2f_3^2+12(23N^2-320)f_4^2+768Nf_1f_3\\
&\quad +256Nf_1f_4+384Nf_2f_3+128Nf_2f_4+24(11N^2-64)f_3f_4\\
&\quad -192N(f_1+f_2)-60N^2(f_3+f_4)+384f_4+N^2\Big] && (6.61)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{f_4} &= \frac{\lambda}{1536\pi^2N}\Big[60N^2f_3^2+4(87N^2-1728)f_4^2+1536Nf_1f_4\\
&\quad +768Nf_2f_4+216N^2f_3f_4-36N^2(f_3+f_4)+3N^2\Big] && (6.62)\\
\beta_{\tilde g^2} &= 2\tilde g^2+\frac{\tilde g^4}{16N\pi^2}\Big(N(N^2+4)f_1+2N(2N^2-1)f_2+3N^2f_3\\
&\quad +5(N^2-4)f_4-N^2\Big)\\
&\quad -\frac{\lambda\tilde g^2}{16N\pi^2}\Big(N(N^2+4)f_1+2N(2N^2-1)f_2+3N^2f_3\\
&\quad +5(N^2-4)f_4-N^2/2\Big) && (6.63)
\end{aligned}
$$

In appendix we establish the dictionary between our notation and that used in [57]. When the beta functions are compared, we find perfect agreement, except for one small difference: the very last term in the first line of $\beta_{\tilde g^2}$ would be $N^2/2$ according to [57], *i.e.* $\tilde g^4$ and $\lambda\tilde g^2$ would have the same coefficients. This is the same difference that we observed between (6.39) (type III cutoff) and (6.40) (type I cutoff), so, effectively the calculation in [57] is equivalent to a type I cutoff. Given that the calculation in [57] was done using completely different techniques, this agreement confirms that the one loop beta functions of the dimensionless couplings (which in a calculation of the effective action would correspond to logarithmic divergences) is scheme independent.

The cases $N=3$ and $N=2$ have to be treated separately, because in these cases only three, respectively two, of the couplings f_i are independent. In the case $N=3$ one can eliminate f_4 in favor of the other three couplings. Then using (6.56) one can obtain the beta functions of f_1, f_2 and f_3 from the ones given above by

$$
\begin{aligned}
\beta_{f_1}\Big|_{N=3} &= \beta_{f_1}\Big|_{N=3,f_4=0} \qquad (6.64)\\
&= \frac{\lambda}{768\pi^2}\Big[464f_1^2+64f_2^2+180f_3^2+320f_1f_2-96f_1f_3\\
&\quad +72f_1-108f_3+9\Big]\\
\beta_{f_2}\Big|_{N=3} &= \beta_{f_2}+\frac{1}{3}\beta_{f_4}\Big|_{N=3,f_4=0} \qquad (6.65)\\
&= \frac{\lambda}{1536\pi^2}\Big[368f_1^2+3520f_2^2+180f_3^2+2624f_1f_2\\
&\quad +480f_1f_3+1728f_2f_3-144f_1-432f_2-108f_3+9\Big]\\
\beta_{f_3}\Big|_{N=3} &= \beta_{f_3}-\frac{1}{3}\beta_{f_4}\Big|_{N=3,f_4=0} \qquad (6.66)\\
&= \frac{\lambda}{32\pi^2}\Big[2f_3^2+16f_1f_3+8f_2f_3-4f_1-4f_2-3f_3\Big]
\end{aligned}
$$

In the case $N=2$ we can set $f_4=0$, because $T^{(4)}=0$ identically, and we can eliminate f_3. One can obtain the beta functions of f_1, f_2 from the ones given above by

$$
\begin{aligned}
\beta_{f_1}\Big|_{N=2} &= \beta_{f_1}-\beta_{f_3}\Big|_{N=2,f_3=0,f_4=0} \qquad (6.67)\\
&= \frac{\lambda}{96\pi^2}\left[48f_1^2+8f_2^2+40f_1f_2+18f_1+12f_2+1\right]\\
\beta_{f_2}\Big|_{N=2} &= \beta_{f_2}+\beta_{f_3}\Big|_{N=2,f_3=0,f_4=0} \qquad (6.68)\\
&= \frac{\lambda}{192\pi^2}\left[36f_1^2+200f_2^2+208f_1f_2-36f_1-60f_2+1\right]
\end{aligned}
$$

The latter result can be used to check also our beta functions for the spherical sigma model. In fact there is exactly one manifold which is simultaneously a sphere and a special unitary group: it is $SU(2)=S^3$. Thus the beta functions should agree in this case. Before comparing, a little point needs to be addressed. In Section 6.3.3 we chose the metric $h_{\alpha\beta}$ to be that of a sphere of unit radius. In this section we have fixed the metric by the conditions (6.50), (6.52). It turns out that in the case $N=2$ this normalization corresponds to a sphere of radius two. This can be seen, for example, from Eq. (6.53), specialized to $N=2$, with $f_{abc}=\varepsilon_{abc}$. In order to compare the beta functions of S^3 with those for $SU(2)$ we therefore have to redefine $\lambda\to\lambda/4$, $f_1\to 4f_1$, $f_2\to 4f_2$, $g^2\to g^2/4$. With these redefinitions, the beta functions do indeed agree.

6.4 Fixed points.

6.4.1 The spherical models.

We now discuss solutions of the RG flow equations. The beta function of λ depends only on λ and the solution is

$$
\lambda(t)=\frac{\lambda_0}{1+\lambda_0\frac{n-1}{8\pi^2}(t-t_0)}\ , \qquad (6.69)
$$

where $\lambda_0 = \lambda(t_0)$. We assume $\lambda_0 > 0$, thus λ is asymptotically free. The beta functions of f_1 and f_2 do not depend on g, so their flow can be studied independently. Here we do not discuss general solutions but merely look for fixed points. The overall factor λ in these beta functions can be eliminated by a simple redefinition $t = t(\tilde{t})$ of the parameter along the RG trajectories:

$$\frac{d}{d\tilde{t}} = \frac{1}{\lambda}\frac{d}{dt}\,. \tag{6.70}$$

Since $\tilde{t}$ is a monotonic function of t, the FPs for f_1 and f_2 are the zeroes of the modified beta functions

$$\tilde{\beta}_{f_i} = \frac{df_i}{d\tilde{t}} = \frac{1}{\lambda}\beta_{f_i}\,. \tag{6.71}$$

They are just polynomials in f_1 and f_2. The model has no real FP for $n = 2$, but there are FPs for all $n > 2$. For $n = 3, \ldots, 8$ they are given in the fifth and sixth column in Table 6.1. One can then insert the FP values of f_1 and f_2 in $\beta_{\tilde{g}^2}$ and look for FP of $\tilde{g}^2$. In each case there are two solutions, one at $\tilde{g}^2 = 0$, the other at some nonzero value. These solutions are reported in the fourth column in Table 6.1, for $n = 3, \ldots, 8$. The first solution describes the Gaussian FP (GFP), where all the couplings $\tilde{g}^2$, λ, $1/\ell_1$, $1/\ell_2$ are zero, the others are non-Gaussian FP's (NFP), where $\tilde{g}^2$ has finite limits instead. Each FP can be approached only from specific directions in the space parametrized by λ, ℓ_1, ℓ_2, *i.e.* the ratios f_1 and f_2 take specific values. For each NFP these values are unique, while for the GFP there may be several possible values: two if $n = 3, 4, 5$ and four if $n = 6, 7, 8$.

When one considers the linearized flow around any of the GFPs, one finds as expected that the critical exponents of the matrix $\frac{\partial \beta_i}{\partial g_j}$, are -2,0,0,0, corresponding to the canonical dimensions of the couplings. The critical exponents at the NGP are instead 2,0,0,0. Thus the dimensionless couplings are marginal, and of the two FPs, the trivial one is IR attractive and the nontrivial one UV attractive for $\tilde{g}$. For λ it is clear that the FP is UV attractive (if we had chosen $\lambda < 0$ it would be IR attractive). In order to establish the attractive or repulsive character of f_1 and f_2, one can look at the linearized flow in the variable $\tilde{t}$, which is described by the 2×2 matrix

$$\frac{\partial \tilde{\beta}_{f_i}}{\partial f_j}\,. \tag{6.72}$$

We define the "critical exponents" $\theta_{1,2}$ to be minus the eigenvalues of this matrix (notice that these are critical exponents relative to the new scaling $\tilde{t}$ we introduced). They are reported in the last two columns of Table 6.1, for $n = 3, \ldots 8$. It is important to realize that even for the GFP the eigenvectors of the stability matrix are not the operators that appear in the action but mixings thereof. We do not report the eigenvectors here.

Beyond the values given in Table I, we have checked numerically the existence of the FP up to $n = 200$. For large n one can study the theory analytically, to some extent. There are four FPs for the system of the f_i's, which are $f_1 = 0$, $f_2 = 1$ with critical exponents $\theta_1 = 6$, $\theta_2 = 12$, $f_1 = 0$, $f_2 = 1/2$ with critical exponents $\theta_1 = 6$, $\theta_2 = -12$, $f_1 = -6$, $f_2 = 5/2$ with critical exponents $\theta_1 = -6$, $\theta_2 = 12$, $f_1 = -6$, $f_2 = 2$ with critical exponents $\theta_1 = -6$, $\theta_2 = -12$.

n	$\tilde{g}_*^{(III)}$	FP	$\tilde{g}_*$	f_{1*}	f_{2*}	θ_1	θ_2
3	8.886	NFP1	6.626	-0.693	0.453	0.094	-0.0121
3		NFP2	6.390	-1.042	0.615	0.103	0.0119
3		GFP1	0	-0.693	0.453	0.094	-0.0121
3		GFP2	0	-1.042	0.615	0.103	0.0119
4	7.255	NFP1	5.877	-0.479	0.398	0.105	-0.0412
4		NFP2	5.442	-1.555	0.852	0.132	0.0392
4		GFP1	0	-0.479	0.398	0.105	-0.0412
4		GFP2	0	-1.555	0.852	0.132	0.0392
5	6.283	NFP1	5.310	-0.400	0.400	0.118	-0.0608
5		NFP2	4.924	-1.875	0.988	0.154	0.0567
5		GFP1	0	-0.400	0.400	0.118	-0.0608
5		GFP2	0	-1.875	0.988	0.154	0.0567
6	5.620	NFP1	4.883	-0.350	0.408	0.131	-0.0780
6		NFP2	4.577	-2.131	1.091	0.171	-0.0717
6		GFP1	0	-0.350	0.408	0.131	-0.0780
6		GFP2	0	-2.131	1.091	0.171	0.0717
6		GFP3	0	-0.814	1.369	-0.161	-0.0539
6		GFP4	0	-2.363	2.091	-0.164	-0.0617
7	5.130	NFP1	4.548	-0.314	0.417	0.143	-0.0939
7		NFP2	4.322	-2.347	1.175	0.185	0.0851
7		GFP1	0	-0.314	0.417	0.143	-0.0939
7		GFP2	0	-2.347	1.175	0.185	0.0851
7		GFP3	0	-2.790	2.130	-0.181	-0.0647
7		GFP4	0	-0.598	1.241	-0.174	-0.0716
8	4.750	NFP1	4.274	-0.286	0.424	0.156	-0.1092
8		NFP2	4.125	-2.535	1.247	0.197	0.0976
8		GFP1	0	-0.286	0.424	0.156	-0.1092
8		GFP2	0	-2.535	1.247	0.197	0.0976
8		GFP3	0	-2.790	2.131	-0.180	0.1023
8		GFP4	0	-0.598	1.247	-0.187	-0.0872

Table 6.1: Gaussian and non-Gaussian fixed points of the S^n model at one loop. The first column gives the dimension n. The second column gives the position of the NGFP in the two-derivative truncation, using a type III cutoff. The rest of the table refers to the four-derivative truncation, also using a type III cutoff. The third column gives the name of the FP. Columns 4,5,6 give the position of the NGFP. Columns 7,8 give the critical exponents, as defined in the text. The coupling λ, not listed, goes to zero and is marginal in this approximation.

The numerical values at finite n do indeed tend towards these limits for growing n.

6.4.2 The chiral models.

The chiral model with $N = 2$ is equivalent to the spherical model with $n = 3$ (up to the redefinition of the couplings mentioned in the end of Section 6.3.4) so we need not discuss this case further. For ease of comparison we just report the properties of its nontrivial FPs in the parametrization we used for the chiral models:

$$\begin{array}{llll} NFP1: & f_{1*} = -0.173 \ ; & f_{2*} = 0.113 \ ; & \tilde{g} = 13.25 \\ NFP2: & f_{1*} = -0.261 \ ; & f_{2*} = 0.154 \ ; & \tilde{g} = 12.78 \end{array}$$

The critical exponents do not depend on the definition of the couplings and therefore are the same as in Table 6.1; they do however depend on the choice of RG parameter and they differ from those given in [57] by a factor $4\pi^2$, which is due to the definition of the parameter x there.

In the case $N = 3$ the system of the f_i's has two FPs at

$$\begin{array}{llll} FP1: & f_{1*} = -0.154 \ ; & f_{2*} = 0.050 \ ; & f_{3*} = 0.085 \ ; \\ FP2: & f_{1*} = -0.108 \ ; & f_{2*} = 0.043 \ ; & f_{3*} = 0.061 \ . \end{array}$$

The attractivity properties in the space spanned by the f_i's are given, as in the spherical case, by studying the modified flow with parameter $\tilde{t}$. The critical exponents at FP1 are 0.0303 with eigenvector (0.411, 0.630, 0.658); 0.0123 with eigenvector (0.515, -0.570, 0.640); 0.00289 with eigenvector (0.869, -0.148, -0.473), whereas at FP2 they are 0.0280 with eigenvector (0.366, 0.618, 0.695); 0.0108 with eigenvector (0.513, -0.575, 0.638) and -0.00293 with eigenvector (0.887, -0.125, -0.445). Therefore FP1 is attractive in all three directions, while FP2 is attractive in two directions. For each of these two FP's, the beta function of $\tilde{g}$ has two FP's: the trivial FP, which has always critical exponents -2, and a nontrivial FP, which is located at $\tilde{g} = 11.17$ for NFP1 or 11.50 for NFP2, and having critical exponent 2 in both cases.

We have found no FP's for $N > 3$: the system of equations $\tilde{\beta}_{f_i} = 0$ for $i = 1, 2, 3, 4$ only has complex solutions. To cover all of theory space we have checked this statement also in the parametrization of the ℓ_i and in the parametrization of $u_i = 1/\ell_i$. This is true also in the large N limit. If we keep only the leading terms (of order N^2 for f_1 and f_2 and of order N for f_3 and f_4), again the resulting polynomials do not have any real zero.

6.5 Discussion.

We have calculated the one loop beta functionals of the NLSM with values in any manifold, in the presence of a very general class of four-derivative terms. We have then specialized our results to two infinite families of models: the $O(N)$ models, with values in spheres, and the chiral models with values in the groups $SU(N)$. Such calculations had been done before, but since the results are rather complicated, it is useful to have independent verifications. Our approach is calculationally very similar to [58], but after correcting some small errors at the general level, we find that the FP structure of the $O(N)$ models is completely different from their findings. On the other hand our results for the chiral models agree completely with [57] for what concerns the dimensionless couplings, even though the calculation was done using very different techniques. Since $SU(2) = S^3$, this provides a check also for our results for the spheres.

In the view of establishing asymptotic safety, or lack thereof, it is important for us to have also the beta functions of the dimensionful coupling g, which in the chiral models is the inverse of the pion decay constant. This had not been considered at all in [58], but it had been calculated in [57] for the chiral models. Again we have agreement with the result of [57], up to a single factor 2 in one term; as discussed before, since this beta function is scheme dependent, we believe that this is not an error on either side, but the result of the different way in which the calculation was done. This difference results in a shift of the FP value of $\tilde{g}$; for example in the case of $SU(2)$ one would find $\tilde{g} = 19.88$ instead of 13.25 for NFP1 and 18.39 instead of 12.78 for NFP2. Such variations by a factor of order 2 are to be expected.

A motivation for studing the asymptotic behaviour of NLSMs is that it could be a toy model for gravity. From this point of view we have a perfect correspondence of results. If we use the $1/p^2$ propagator that comes from the two-derivative term, both theories are perturbatively unitary but nonrenormalizable; if on the other hand we use the $1/p^4$ propagator that comes from the four-derivative terms both theories are renormalizable (see [73] for gravity and [74] for the NLSM) but contain ghosts (the states with negative norm in Hilbert space). In the latter case it had also been established (see [75, 76, 77, 78] for gravity and [57, 58] for the NLSM) that the four-derivative terms, whose couplings are dimensionless, are asymptotically free. Actually the analogy works even in greater detail. The coefficient of the square of the Weyl tensor (for gravity) and the square of $\Box\varphi^\alpha$ (for the NLSM) have at one loop a beta function that is constant. These coefficients diverge logarithmically in the UV, so their inverses, which are the perturbative couplings, are asymptotically free. The coefficients of the other four-derivative terms have more complicated beta functions, but overall there is asymptotic freedom, provided the Gaussian FP is approached from some special direction. There have been many attempts to avoid the effects of the ghosts; see [75, 80] for gravity and [57] for the NLSM. In any case, the existence of the ghosts is only established at tree level. Whether they exist in the full quantum theory is a deep dynamical question whose answer is not known.

All these “old” works on higher derivatives theories concentrated on the behavior of the couplings that multiply the four-derivative terms; much less attention, if any, was paid to the coefficient of the two-derivative term, which has dimension of square of a mass: the inverse of Newton’s constant in gravity and the square of the pion decay constant in the chiral NLSM. In several papers this issue was ignored, or incorrect results were given, because of the use of dimensional regularization. The correct RG flow of these couplings is quadratic in k, and is best seen when a momentum cutoff is used.

It is somewhat gratifying to see that the FP does not always exist for all NLSM: in particular we have seen that within the one loop approximation, adding the higher derivative terms destroys the FP that is present in the two-derivative truncation for the sphere S^2 and for the chiral models with $N > 3$. If there was any doubt, this shows that the existence of the FP is not “built into the formalism" but is a genuine property of the theory. This is somewhat analogous to the situation when one adds minimally coupled matter fields to gravity [22].

The next step will be to replace the one loop functional RG equation (6.18) by its *exact* counterpart, which only differs in the replacement of the bare action S by Γ_k in the right-hand side [5]. There are at least two good reasons to do this calculation. One of the points of [28] that needed further clarification was the value of the lowest critical exponent. In the two-derivative truncation at one loop it was always 2 at the nontrivial FP. Thus the critical exponent ν that governs the rate at which the correlation length diverges was given by

$$\nu = -\left(\frac{d\beta}{dt}\bigg|_* \right)^{-1} = \frac{1}{\theta} = \frac{1}{2} \,, \tag{6.73}$$

which is the value of mean field theory. Using the "exact" RG truncated at two derivatives gave $\nu = 3/8$ for the $O(N)$ models, independent of N. One would like to understand what effect the higher derivative terms have on this exponent. Since here we restricted ourselves to one loop, we found again $\nu = 1/2$, so the calculations of this chapter are of no use in this respect. Another motivation comes from recent calculations in higher derivative gravity [79] that go beyond one loop and find that the theory is not asymptotically free, but rather all couplings reach nonzero values at the UV FP. It would be interesting to see similar behavior in (some) NLSM.

Concerning possible direct phenomenological applications of the NLSM, regarded as an effective field theory, it is interesting to ask what relation, if any, the UV properties of the NLSM may have to the properties of the underlying fundamental theory. Regarding the chiral NLSM as the low energy approximations of a QCD-like theory, one may note that there is rough agreement between the range of existence of the NLSM FP and the "conformal window" for the existence of an IR FP in the case when the quarks are in the adjoint or in the symmetric tensor representation [82]. One could get a better understanding of this issue if the beta functions of the NLSM depended on the number of "colors" of the underlying theory, which in the effective theory are reflected in the coefficient of the Wess-Zumino-Witten term [64]. The one loop beta function of the Wess-Zumino-Witten term is zero [58, 83]; this is consistent with the quantization of the coefficient c. Unfortunately the beta functions of the remaining couplings are completely independent of this coefficient, so the low energy theory seems to be insensitive to this parameter.

Another possible application is to electroweak chiral perturbation theory [52]. If the NLSM turned out to be asymptotically safe in the presence of gauge fields and fermions, then one may envisage a Higgsless standard model up to very high energies. This will also require a separate investigation. A related application of asymptotic safety to the standard model has been discussed recently in [18].

To summarize, we believe that the NLSM are interesting theoretical laboratories in which one may test various theoretical ideas, and they have also important phenomenological applications. The question whether some NLSM could be asymptotically safe seems to us to be a particularly important one, and to deserve more attention. As P. Hasenfratz wrote in [57], "We do not know of any a priori reason, which would imply that these theories are doomed to fail.

The problems are practically untouched." These words are still valid twenty years later.

Appendix A

The spin-projectors.

We refer to the appendices of [44, 73] for a complete account of the spin-projectors we are going to introduce. To fix ideas we have to think about the gravitational inverse propagator in flat space limit and in momentum space. It turns out that its tensor ingredients are the flat metric $\delta_{\mu\nu}$ and the incoming momentum q_μ. With these tensors we have to construct a "matrix" that maps the symmetric 2-tensors space into itself, namely the inverse propagator. It is clear that this tensor structure has to do with the irreducible representation of the graviton modes. It is possible, in fact, to find a basis of spin-projectors that parametrize the space of all inverse propagators and that projects $h_{\mu\nu}$ modes onto $\left(h^T_{\mu\nu}, \xi_\mu, \sigma, h\right)$. The basis is called of spin-projectors because four of them actually projects to the spin modes, while one of them takes into account their mixing. We will also call them simply projectors.

Before giving their definition it is convenient to introduce the transverse end longitudinal projectors on vectors. These decompose any vector v^μ into spin-1 and spin-0 components. They are simply

$$
\begin{aligned}
{P_T}_\mu{}^\nu &= \delta^\nu_\mu - \frac{q_\mu q^\nu}{q^2} && \text{(A.1)}\\
{P_L}_\mu{}^\nu &= \frac{q_\mu q^\nu}{q^2} && \text{(A.2)}
\end{aligned}
$$

These implicitly depend on a momentum q_μ, although no dependence is written explicitly. In case more than a momentum is involved we will remove the ambiguity writing them as $P_i = P_i\,[q]$ for $i = T, L$. These are projectors in the usual sense, so

$$
\begin{aligned}
P_i^2 &= P_i && \text{(A.3)}\\
& && \text{(A.4)}
\end{aligned}
$$

for $i = T, L$. Their traces are easily evaluated in general d-dimensional space to give ${P_T}_\mu{}^\mu = d - 1$ and ${P_L}_\mu{}^\mu = 1$, reflecting the fact that they projects on irreducible vector and scalar subspace. It is also convenient to define out of P_T the matrix acting on symmetric tensors

$$\bar{\delta}_{\mu\nu}{}^{\alpha\beta} = {P_T}_{(\mu}{}^{(\alpha} {P_T}_{\nu)}{}^{\beta)} \quad \text{(A.5)}$$

Parentheses indicate symmetrization. It differs from the identity in the symmetric 2-tensor space, because for each index the longitudinal component is projected out. However $\bar{\delta}$ is already a projector, so it may be what we are looking for or lead to it. To understand $\bar{\delta}$ better, we may apply it to the fourier q_μ component of $h_{\mu\nu}$.

The general definition of the irreducible decomposition in momentum space and flat background is given by

$$h_{\mu\nu} = h^T_{\mu\nu} + 2iq_{(\mu}\xi_{\nu)} - \left(q_\mu q_\nu \sigma - \frac{1}{d}\delta_{\mu\nu}q^2\sigma\right) + \frac{1}{d}\delta_{\mu\nu}h \tag{A.6}$$

and it is easy to see that

$$\bar{\delta}\cdot h_{\mu\nu} = h^T_{\mu\nu} + \frac{1}{d}P_{T\,\mu\nu}\left(h + q^2\sigma\right) \tag{A.7}$$

One easily understands that $\bar{\delta}$ correctly projected out the longitudinal components, but it was unable to remove the "traces" h and σ. The correct projector for the spin-2 irreducible component is instead

$$P_{2\mu\nu}{}^{\alpha\beta} \;\; = \;\; \bar{\delta}_{\mu\nu}{}^{\alpha\beta} - \frac{1}{d-1}P_{T\,\mu\nu}P_T{}^{\alpha\beta} \tag{A.8}$$

A byproduct of its construction is that we introduced another projector

$$P_{S\mu\nu}{}^{\alpha\beta} \;\; = \;\; \frac{1}{d-1}P_{T\,\mu\nu}P_T{}^{\alpha\beta} \tag{A.9}$$

that is interesting because it projects $h_{\mu\nu}$ over the quantity $S \sim h+q^2\sigma$, modulo a normalization factor. In general curved space $S = \frac{1}{d}\left(h - \Box\sigma\right)$ is a gauge invariant quantity.

Other two projectors can easily obtained. First, we note that the two we just constructed are of the form $P_T \otimes P_T$. Along this line we are still missing the structures of the form $P_T \otimes P_L \oplus P_L \otimes P_T$ and $P_L \otimes P_L$ (please note that the notation does not imply any real tensor product but is only a mnemonic device, because for example P_S mixes the indices spaces of $h_{\mu\nu}$). In particular, the structure involving both the projectors may be useful to select the spin-1 component, having nonzero combination over it. In fact, it exists a combination that projects over ξ_μ and therefore cancels all the rest. It is

$$P_{1\mu\nu}{}^{\alpha\beta} \;\; = \;\; \frac{1}{2}P_{T\,(\mu}{}^{(\alpha}P_{L\,\nu)}{}^{\beta)} \tag{A.10}$$

and it does the job of projecting $P_1 \cdot h_{\mu\nu} = 2iq_{(\mu}\xi_{\nu)}$ over the term of $h_{\mu\nu}$ built with the spin-1 term. The final projector is easily found, either by noting that there is only one projector with structure $P_L \otimes P_L$ that projects over the σ sector of $h_{\mu\nu}$, or by requiring that it sums with the others up to the symmetric identity. In any case the result is

$$P_{\sigma\,\mu\nu}{}^{\alpha\beta} \;\; = \;\; P_{L\,\mu\nu}P_L{}^{\alpha\beta} \tag{A.11}$$

Now, as we already said these projectors are not actually a basis of projectors. Each is a projector on its own, but one further element is missing because they are not enough to parametrize any possible structure emerging from a combination of $\delta_{\mu\nu}$ and q_μ. The last ingredient we define to be

$$P_{S\sigma\,\mu\nu}{}^{\alpha\beta} \quad = \quad \frac{1}{\sqrt{d-1}}\left(P_{T\,\mu\nu}P_L{}^{\alpha\beta} + P_{L\,\mu\nu}P_T{}^{\alpha\beta}\right) \tag{A.12}$$

This completes the basis, but spoils the fact that it is a projector basis because it does not commute with P_S and P_σ. Further, its square is not the projector itself but rather a combination of P_S and P_σ. We will give these properties, together with analogues for the other elements soon.

First we argue it is the case to recup all the definitions we gave. The spin-projectors basis is defined to be (in terms of transverse and longitudinal two-indexed ones) [44, 73]

$$\begin{aligned}
P_{2\,\mu\nu}{}^{\alpha\beta} &= P_{T\,(\mu}{}^{(\alpha}P_{T\,\nu)}{}^{\beta)} - \frac{1}{d-1}P_{T\,\mu\nu}P_T{}^{\alpha\beta} \\
P_{1\,\mu\nu}{}^{\alpha\beta} &= \frac{1}{2}P_{T\,(\mu}{}^{(\alpha}P_{L\,\nu)}{}^{\beta)} \\
P_{S\,\mu\nu}{}^{\alpha\beta} &= \frac{1}{d-1}P_{T\,\mu\nu}P_T{}^{\alpha\beta} \\
P_{\sigma\,\mu\nu}{}^{\alpha\beta} &= P_{L\,\mu\nu}P_L{}^{\alpha\beta} \\
P_{S\sigma\,\mu\nu}{}^{\alpha\beta} &= \frac{1}{\sqrt{d-1}}\left(P_{T\,\mu\nu}P_L{}^{\alpha\beta} + P_{L\,\mu\nu}P_T{}^{\alpha\beta}\right)
\end{aligned} \tag{A.13}$$

As promised we list some of their properties. The first four are projectors, as we already said, while the square of the fifth represents the mixing of the scalar degrees of freedom. For $i = 2, 1, S, \sigma$ we have

$$P_i^2 = P_i \tag{A.14}$$

$$P_{S\sigma}^2 = P_S + P_\sigma \tag{A.15}$$

All the other nonzero products of the projectors are in the next list or can be derived from it

$$P_S \cdot P_{S\sigma} = P_{S\sigma} \cdot P_\sigma \tag{A.16}$$

$$P_{S\sigma} \cdot P_S = P_\sigma \cdot P_{S\sigma} \tag{A.17}$$

$$P_{S\sigma} \cdot (P_S + P_\sigma) = 1 \tag{A.18}$$

$$(P_S + P_\sigma) \cdot P_{S\sigma} = 1 \tag{A.19}$$

It should be evident that the sum $P_S + P_\sigma$ acts as an "identity" in the scalars subspace.

It is interesting to note that the subset P_i with $i = S, \sigma, S\sigma$ generates a subalgebra of the projectors on its own. In particular we may find useful to represent it using 2×2 matrices in the form

$$P_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{A.20}$$

$$P_\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.21}$$

$$P_{S\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.22}$$

With this representation in mind we will be able to invert any combination of the scalar subsector matrices, by simply inverting a 2×2 matrix. Obviously, many possible representations are possible but one easily sees that all will give the same results upon matrix inversion. To see this let us restrict attention to solely the scalar subspace in which a combination of them

$$\sum_{i=S,\sigma,S\sigma} \lambda_i P_i \quad = \quad \begin{pmatrix} \lambda_S & \lambda_{S\sigma} \\ \lambda_{S\sigma} & \lambda_\sigma \end{pmatrix} \tag{A.23}$$

is invertible (in the general space also components along $P_{2,1}$ are needed to have a finite expression, in the full inverse propagator these will be present). The inverse is

$$\begin{aligned} \begin{pmatrix} \lambda_S & \lambda_{S\sigma} \\ \lambda_{S\sigma} & \lambda_\sigma \end{pmatrix}^{-1} &= \frac{1}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2} \begin{pmatrix} \lambda_\sigma & -\lambda_{S\sigma} \\ -\lambda_{S\sigma} & \lambda_S \end{pmatrix} \\ &= \frac{\lambda_\sigma}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2} P_S + \frac{\lambda_S}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2} P_\sigma - \frac{\lambda_{S\sigma}}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2} P_{S\sigma} \end{aligned} \tag{A.24}$$

As expected the expression is $S \leftrightarrow \sigma$ symmetric. Further, one could represent $P_{S\sigma}$ with the negative of the matrix we actually used. This ambiguity is expressed in terms of $\lambda_{S\sigma} \to -\lambda_{S\sigma}$. This means, essentially, that $\lambda_{S\sigma}$ either appears squared (as it does in the denominators) or multiplied by $P_{S\sigma}$ making $\lambda_{S\sigma}P_{S\sigma}$ invariant under the redefinition of the representation. The general inverse, including also $P_{2,1}$ degrees of freedom is a straightforward generalization of the former formula. What emerges from this picture is that any inverse gravitational propagator can be written in irreducible components notation using these projectors and easily inverted. The hard task of calculating the inverse will be moved to the easy one of inverting a 2×2 matrix. Further, the tabulated algebra of operators, simplify strongly expressions involving products of consecutive propagators. Again, such products will reduce to simple products of numbers and square matrices, upon correct projection of the elements.

We end this section giving some further properties of the set of projectors. One has to remember that these are going to be traced along graphs, hitting, in particular, gravi-matter vertices. The effect of the vertices will be to trace the projectors, so it is useful to study the possible ways in which the projectors are traced. There are two possible ways to trace them. We call the "straight" trace of a projector P_i the natural vector space trace $\mathrm{tr}P_i = P_{i\mu\nu}{}^{\mu\nu}$, while the "cross" trace is defined as $\mathrm{tr}_X P_i = P_{i\mu\nu}{}^{\alpha\beta}\delta^{\mu\nu}\delta_{\alpha\beta}$. A list of these traces in general d-dimensions is given

$$\mathrm{tr}P_2 = \frac{d^2-d-2}{2} \qquad \mathrm{tr}_X P_2 = 0 \tag{A.25}$$

$$\mathrm{tr}P_1 = d-1 \qquad \mathrm{tr}_X P_1 = 0 \tag{A.26}$$

$$\mathrm{tr}P_S = 1 \qquad \mathrm{tr}_X P_S = d-1 \tag{A.27}$$

$$\mathrm{tr}P_\sigma = 1 \qquad \mathrm{tr}_X P_\sigma = 1 \tag{A.28}$$

$$\mathrm{tr}P_{S\sigma} = 0 \qquad \mathrm{tr}_X P_{S\sigma} = 2\sqrt{d-1} \tag{A.29}$$

We see that the straight traces actually agree with the expectations we have on tensor degrees of freedom. For example the spin-2 projector counts in $d=4$ five components that is the number of components of a symmetric transverse and traceless tensor. The spin-1 correctly counts three, so those of a transverse vector.

The final properties we are going to give involve a general systematic way to translate expressions involving $\delta_{\mu\nu}$ and q_μ, to expressions in the set P_i. The identity in symmetric space morphisms is easily resolved using the completeness of the projectors

$$\delta_{\mu\nu}^{\alpha\beta} = \sum_{i=2,1,S,\sigma} P_{i\,\mu\nu}{}^{\alpha\beta} \tag{A.30}$$

If an expression of the form $\delta_{\mu\nu}\delta^{\alpha\beta}$ is encountered we can use the fact that

$$\delta_{\mu\nu}\delta^{\alpha\beta} = (d-1)\,P_{S\,\mu\nu}{}^{\alpha\beta} + P_{\sigma\,\mu\nu}{}^{\alpha\beta} + \sqrt{d-1}P_{S\sigma\,\mu\nu}{}^{\alpha\beta} \tag{A.31}$$

The last ingredient we have to manage are general expressions of the form $\delta_{\mu\nu}q_\alpha q_\beta \sim \delta_{\mu\nu}P_{L\,\alpha\beta}$. These generally appear only in a limited number of ways, determined by the symmetry properties of the space. In particular, the useful combinations are

$$\delta_{\mu\nu}P_L{}^{\alpha\beta} + \delta^{\alpha\beta}P_{L\,\mu\nu} = 2P_{\sigma\,\mu\nu}{}^{\alpha\beta} + \sqrt{d-1}P_{S\sigma\,\mu\nu}{}^{\alpha\beta} \tag{A.32}$$

$$\delta_{(\mu}^{(\alpha}P_L{}_{\nu)}^{\beta)} = \frac{1}{2}P_{1\,\mu\nu}{}^{\alpha\beta} + P_{\sigma\,\mu\nu}{}^{\alpha\beta} \tag{A.33}$$

and will be the only ones appearing in second variations of gravitational perturbations. This completes all the properties we will need of the projector basis.

Appendix B

Gravi-matter vertices.

For the computations of the momentum space graphs we need the vertices in momentum space representation. We will write them in full generality for the moment, without resorting to the limit $V'[\phi] = 0$. Further, we need the vertices having at most two external graviton lines. The reason is, the graphs we are going to calculate have always two matter external lines, so the vertices are attached to at most two gravitons inside the loop. The structure of the flow of the inverse propagator involves at most vertices with four lines. Altogether, this imply a finite and small amount of vertices we are interested in. We will omit to write the vertices that do not include graviton lines, which have been already used in flat space case, because they are trivially obtained.

The normalization of the fourier components we choose are

$$\phi(x) = \int_q \phi_q e^{-iqx} \tag{B.1}$$

$$\psi(x) = \int_q \psi_q e^{-iqx} \tag{B.2}$$

$$h(x)_{\mu\nu} = \int_q h_{\mu\nu;q} e^{-iqx}. \tag{B.3}$$

Using these definitions we easily express any derivative of any field in terms of momentum space components, for example

$$\partial_\mu \phi(x) = i \int_q q_\mu \phi_q e^{-iqx} \tag{B.4}$$

$$-\partial^2 \phi(x) = \int_q q^2 \phi_q e^{-iqx} \tag{B.5}$$

and so on. The exponential factors produce a delta conservation in momentum space, which we will not write and left understood. Actually, it is precisely the term in front of that delta function that represents the momentum space correlator we are interested in. The results we are going to present are already the correlators we will use. All the momenta are understood to be incoming and it is sufficient to take their negative if one wants the vertex with some outgoing momenta. Finally, conservation is always assumed. If the vertex depends on n incoming momenta k_1,...,k_n we will have $\sum_{i=1,...,n} k_i = 0$. We will give the results without expressing one of the momenta in terms of the others, because the vertices will appear in a more symmetric fashion.

We start giving the vertices that do involve at least one scalar field. As previously stressed, the scalar is somehow special because it forms a vertex with a graviton and no other external lines. We get in the constant background limit

$$\frac{\delta}{\delta\phi_{k_1}}\frac{\delta}{\delta h_{\mu\nu,k_2}}\Gamma_k = \frac{1}{2}\delta^{\mu\nu}V' \tag{B.6}$$

This is precisely telling us that scalars do mix and their propagation should be considered together. However it will not enter in our calculation of η_ϕ and η_ψ, because we assume the on-shell condition $V' = 0$. Note also that is is a non-derivative vertex because it is coming from a non-derivative interaction.

Similarly, a vertex with another graviton line is formed. A short calculation gives us that

$$\frac{\delta}{\delta\phi_{k_1}}\frac{\delta}{\delta h_{\mu\nu,k_2}}\frac{\delta}{\delta h_{\alpha\beta,k_3}}\Gamma_k = -\frac{1}{2}\left(\delta^{\mu\nu,\alpha\beta} - \frac{1}{2}\delta^{\mu\nu}\delta^{\alpha\beta}\right)V' \tag{B.7}$$

It shares the same properties of the previous one and again it will not contribute to the anomalous dimensions flow, being set to zero thanks to the properties of the expectation value.

We continue introducing the vertices with at least two external scalar lines. It is possible to have one external graviton (see Fig. (B.1))

$$\frac{\delta}{\delta\phi_{k_1}}\frac{\delta}{\delta\phi_{k_2}}\frac{\delta}{\delta h_{\mu\nu,k_3}}\Gamma_k = Z_\phi {k_1}^{(\mu}{k_2}^{\nu)} - \frac{1}{2}\delta^{\alpha\beta}\left(Z_\phi k_1\cdot k_2 - V''\right) \tag{B.8}$$

and two external gravitons (Fig. (B.2))

$$\begin{aligned}\frac{\delta}{\delta\phi_{k_1}}\frac{\delta}{\delta\phi_{k_2}}\frac{\delta}{\delta h_{\mu\nu,k_3}}\frac{\delta}{\delta h_{\alpha\beta,k_4}}\Gamma_k = & -Z_\phi {k_1}^{(\mu}\delta^{\nu)(\alpha}{k_2}^{\beta)} - Z_\phi {k_2}^{(\mu}\delta^{\nu)(\alpha}{k_1}^{\beta)} \\ & +\frac{1}{2}Z_\phi {k_1}^{(\mu}{k_2}^{\nu)}\delta^{\alpha\beta} + \frac{1}{2}Z_\phi {k_1}^{(\alpha}{k_2}^{\beta)}\delta^{\mu\nu} \\ & +\frac{1}{2}\left(Z_\phi k_1\cdot k_2 - V''\right)\left(\delta^{\mu\nu,\alpha\beta} - \frac{1}{2}\delta^{\mu\nu}\delta^{\alpha\beta}\right)\end{aligned} \tag{B.9}$$

These vertices turns out to be fundamental when calculating the scalar anomalous dimension. One notices that, thanks to the double functional derivative with respect to the scalar field, the scalar kinetic term comes into play.

This situation is further stressed in the case of the vertices possessing external spinor lines. The number of external spinors is fixed by charge conjugation and the fact that their background is $\psi = 0$. Essentially we have to perform a derivative with respect $\bar{\psi}$, for each derivative in ψ. The anomalous dimensions in our interpretation have to be evaluated in a physical configuration to remove the spurious field dependence, it is worth noting that $\langle\psi\rangle = 0$ in any sensible physical configuration. The calculations performed with these momentum rules will be consistent only with the diagonal-cutoff computation, because in the non-diagonal one the cutoff do depend on the background ψ. There is also the possibility that $\left\langle\bar{\psi}\psi\right\rangle = C$. It easy to realize that it would simply correspond to a shift redefinition of the potential $V[\phi] \to \bar{V}[\phi] = V[\phi] + H[\phi]\,C$.

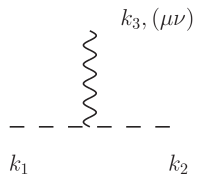

Figure B.1: Vertex including the interaction of two scalar modes with a graviton one. The momenta are all incoming.

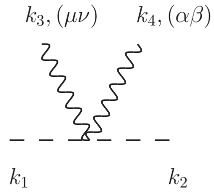

Figure B.2: Vertex including the interaction of two scalar modes and two graviton one. The momenta are all incoming.

Using the results of the appendix involving the second order expansion of the spinor kinetic action, we can calculate the vertex with two spinor lines and one graviton. We obtain (Fig. (B.3))

$$\begin{aligned}\frac{\delta}{\delta h_{\mu\nu,k_1}}\frac{\delta}{\delta\psi_{k_2}}\frac{\delta}{\delta\bar{\psi}_{k_3}}\Gamma_k &= -\frac{Z_\psi}{4}\left(\gamma^{(\mu}\left(k_2-k_3\right)^{\nu)}\right)\\ &+\frac{1}{2}\delta^{\mu\nu}\left(H+\frac{Z_\psi}{2}\gamma^{\alpha}\left(k_2-k_3\right)_\alpha\right)\end{aligned} \tag{B.10}$$

Note that the vertex is an element of the Clifford algebra, even if we do not write its indices explicitly. Where no gamma matrices appear, it is understood that there is an identity in the algebra. Similarly to the scalar case the second functional derivatives increases in complexity. With two external gravitons we obtain (Fig. (B.4))

$$\begin{aligned}\frac{\delta}{\delta h_{\mu\nu,k_1}}\frac{\delta}{\delta h_{\alpha\beta,k_2}}\frac{\delta}{\delta\psi_{k_3}}\frac{\delta}{\delta\bar{\psi}_{k_4}}\Gamma_k &= -\frac{Z_\psi}{8}\delta^{\mu\nu}\gamma^{(\alpha}\left(k_3-k_4\right)^{\beta)}-\frac{Z_\psi}{8}\delta^{\alpha\beta}\gamma^{(\mu}\left(k_3-k_4\right)^{\nu)}\\ &+\frac{Z_\psi}{16}\gamma^{\rho}\left(k_1-k_2\right)_\rho\times\\ &\times\left[\gamma^{(\mu}\delta^{\nu)(\alpha}\gamma^{\beta)}-\frac{1}{2}\delta^{\mu\alpha}\delta^{\nu\beta}-\frac{1}{2}\delta^{\mu\beta}\delta^{\nu\alpha}\right]\\ &+\frac{Z_\psi}{8}\gamma^{(\mu}\delta^{\nu)(\alpha}\left(k_1+2k_3-k_4\right)^{\beta)}\\ &-\frac{Z_\psi}{8}\left(k_1+2k_4-k_3\right)^{(\mu}\delta^{\nu)(\alpha}\gamma^{\beta)}\\ &-\frac{1}{2}\left(\delta^{\mu\nu,\alpha\beta}-\frac{1}{2}\delta^{\mu\nu}\delta^{\alpha\beta}\right)\left(\frac{Z_\psi}{2}\gamma_\rho\left(k_3-k_4\right)^\rho+H\right)\end{aligned} \tag{B.11}$$

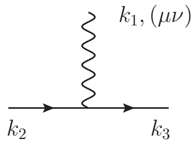

Figure B.3: Vertex including the interaction of the spinor modes with one graviton. The momenta are all incoming. Internal Clifford-algebra indices are understood.

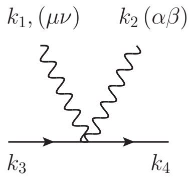

Figure B.4: Vertex including the interaction of the spinor modes with two gravitos. The momenta are all incoming. Internal Clifford-algebra indices are understood.

Those written in this subsection are all the vertices we need for our computations of the gravitational corrections of η_ϕ and η_ψ. Infinitely many more interactions, obviously, exist thanks to the nonpolynomiality of gravity. The construction of such vertices is generally needed for further computations involving the flow of n-point correlators with $n \geq 2$.

Appendix C

Heat kernel expansion.

C.1 Rank-2 operators.

Here we are going to give a very brief account of the heat kernel expansion we used in chapter 5. The heat kernel (HK) is the study of the flow

$$(\partial_s + \Delta) K(s) = 0 \tag{C.1}$$

where Δ is a differential operator acting on an unspecified bundle. Let the base manifold be a d-dimensional riemaniann manifold with metric $g_{\mu\nu}$. For the moment we shall restrict our attention to the case in which Δ is of rank 2 and in particular of the form

$$\Delta = \Box + E = -\nabla^2 + E \tag{C.2}$$

We introduced a covariant derivative ∇, which contains the connection of the bundle, and the box operator defined as $\Box = -\nabla^2 = -g^{\mu\nu}\nabla_\mu\nabla_\nu$. E is a general endomorphism of the bundle. An operator of the form (C.2) is sometimes called "generalized laplacian". The connection itself will be a sum of the connection of the bundle and the Christoffel connection of the base manifold.

The formal solution of (C.1) is

$$K(s) = e^{-s\Delta} \tag{C.3}$$

It is useful to evaluate the functional trace of the heat kernel

$$\mathrm{Tr}K(s) = \mathrm{tr}\int dx K(s; x, x) \tag{C.4}$$

where tr is the trace over the indices of the bundle and $K(s; x, x)$ solves the differential equation

$$(\partial_s + \Delta_x) K(s; y, x) = 0 \tag{C.5}$$

with boundary condition $K(0; y, x) = \delta(x - y)$ (the Dirac delta is for the measure $dx = \sqrt{g}d^d x$).

It turns out that the object $\mathrm{Tr}K(s)$ possesses an expansion in powers of s, that relates to a local expansion in curvatures and powers of the endomorphism

E. This expansion is called Seeley-deWitt expansion and it reads

$$\mathrm{Tr}K\left(s\right) = \frac{1}{\left(4\pi s\right)^{d/2}} \sum_{n\geq 0} s^n B_{2n}\left[\Delta\right] \tag{C.6}$$

The coefficients $B_{2n}\left[\Delta\right]$ can be written in terms of other coefficients $b_{2n}\left[\Delta\right]$ defined as

$$B_{2n}\left[\Delta\right] = \mathrm{tr} \int dx b_{2n}\left[\Delta\right] \tag{C.7}$$

The new coefficients take values in the bundle.

The first two bs are

$$b_0\left[\Delta\right] = 1 \tag{C.8}$$

$$b_2\left[\Delta\right] = E + \frac{R}{6} \tag{C.9}$$

R is the curvature scalar. All the terms in these two expansion, apart for E, are proportional to the identity in the bundle. Therefore, once traced with "tr", will give a result proportional to the dimensionality of the bundle itself, let it be D.

Suppose for a moment that the endomorphism is zero. Suppose also that the space we are working on is a 4-sphere of volume V, so we can relate any curvature invariant to powers of the scalar curvature. The operator will be simply

$$\Delta = \Box \tag{C.10}$$

and (C.9) reduces to

$$b_0\left[\Delta\right] = 1 \tag{C.11}$$

$$b_2\left[\Delta\right] = \frac{1}{6}R \tag{C.12}$$

In such a situation the HK expansion (C.6) becomes an expansion in powers of R. After taking the trace of (C.12) we obtain

$$B_0\left[\Delta\right] = VD \tag{C.13}$$

$$B_2\left[\Delta\right] = \frac{1}{6}DRV \tag{C.14}$$

In the expansion (4.75) we found that each spin-j mode, possesses a second order expansion with kernel $\Delta_j = \Box_j + \ldots$. We are now interested in the laplacian operator of the single spin mode $\Box_j$ that lives in a vector space of dimension D_j. We define

$$B_{2n,j} = B_{2n}\left[\Box_j\right]/V \tag{C.15}$$

In particular, we used $B_{0,j}$ and $B_{2,j}$. It is very easy to determine them once one remembers that $D_j = 1, 4, 3, 5$ for $j = 0, 1/2, 1, 2$ (notice that we are considering Dirac fermions in the 1/2, the result for Majorana is $D_{1/2} = 2$).

C.2 Rank-r operators.

In chapter 6 we needed the study of (C.1) equation for a rank-4 operator of the form

$$\Delta = \Box^2 + B^{\mu\nu}\nabla_\mu\nabla_\nu + C^\mu\nabla_\mu + D \tag{C.16}$$

As an additional condition we required $B_{\mu\nu} = B_{\nu\mu}$.

Suppose now we are considering a general operator of rank-r of the form $\Delta = \Box^{r/2} + \ldots$. The HK is again defined as $\mathrm{Tr}K(s) = \mathrm{Tr}\,\mathrm{Exp}(-s\Delta)$. The general rank-$r$ operator possesses a similar expansion to (C.6) and it reads in $d = 4$

$$\frac{1}{(4\pi)^2}\frac{1}{s^{4/r}}\left(B_0[\Delta] + s^{2/r}B_2[\Delta] + s^{4/r}B_4[\Delta] + \ldots\right) \tag{C.17}$$

The detailed form of the coefficients for $r = 2$ can be found in [70].

C.3 Functional traces using the HK.

The expansions (C.6) and (C.17) can be used to calculate the trace of any function of an operator Δ. Suppose now we are interested in calculating the general trace

$$\mathrm{Tr}\, f[\Delta] \tag{C.18}$$

for arbitrary f.

It is convenient to rewrite f in terms of its Laplace transform $\tilde{f}$ and insert it in the trace

$$\mathrm{Tr}\, f[\Delta] = \mathrm{Tr}\int_0^\infty ds \tilde{f}[s]\, e^{-s\Delta} = \int_0^\infty ds \tilde{f}[s]\,\mathrm{Tr}\, K(s) \tag{C.19}$$

which shows how to relate any trace with that of the HK.

If we substitute in (C.19) the expansion (C.6) or the general (C.17), we obtain an expansion with which evaluate the functional trace (C.18). For the general rank-r case of (C.17) it reads in $d = 4$

$$\mathrm{Tr}\, f[\Delta] = \frac{1}{(4\pi)^2}\left(Q_{4/r}(f)\, B_0[\Delta] + Q_{2/r}(f)\, B_2[\Delta] + Q_0(f)\, B_4[\Delta] + \ldots\right) \tag{C.20}$$

The case $r = 2$ of (C.6) is easily obtained by setting the limit.

The functions $Q_m(f)$ are obtained by integrals or derivatives of the function f. Their general form is, for $m \geq 0$ integer

$$Q_{-m}(f) = (-1)^m \left.\frac{\partial^m f[z]}{\partial z^m}\right|_{z=0} \tag{C.21}$$

while for $m \geq 0$

$$Q_m(f) = \frac{1}{\Gamma(m)}\int_0^\infty dz\, z^{m-1} f[z] \tag{C.22}$$

Appendix D

$SU(N)$ model.

In [57] the action for the chiral $SU(N)$ model is written in the form:

$$\frac{1}{f^2}\int d^4x\,\Big[c_0\mathrm{Tr}L_\mu L^\mu+\frac{1}{2}\mathrm{Tr}(\partial_\mu L^\mu\partial_\nu L^\nu+\partial_\mu L_\nu\partial^\mu L^\nu)$$
$$-\frac{1}{2}c_2\mathrm{Tr}(\partial_\mu L^\mu\partial_\nu L^\nu-\partial_\mu L_\nu\partial^\mu L^\nu)-\frac{1}{2}c_3\mathrm{Tr}(L_\mu L^\mu L_\nu L^\nu+L_\mu L_\nu L^\mu L^\nu)$$
$$-c_4\mathrm{Tr}(L_\mu L^\mu)\mathrm{Tr}(L_\mu L^\mu)-c_5\mathrm{Tr}(L_\mu L^\mu)\mathrm{Tr}(L_\mu L^\mu)\Big]\,. \tag{D.1}$$

where $L_\mu=U^{-1}\partial_\mu U$. We want to translate this action into the form (6.57). Deriving the equation $L_\mu=\partial_\mu\varphi^\alpha L^a_\alpha(-iT_a)$ we obtain

$$\partial_\mu L_\nu=-iT_a(\nabla_\mu\partial_\nu\varphi^\alpha L^a_\alpha-\partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta\nabla_\alpha L^a_\beta) \tag{D.2}$$

The antisymmetric part of this equation is

$$\partial_\mu L_\nu-\partial_\nu L_\mu=-[L_\mu,L_\nu] \tag{D.3}$$

whereas using Killing's equation, the symmetric part is

$$\partial_{(\mu}L_{\nu)}=-iT_a\nabla_\mu\partial_\nu\varphi^\alpha L^a_\alpha \tag{D.4}$$

The terms of (D.1) have the following translation into our tensorial language:

$$
\begin{aligned}
\int d^4x\, \mathrm{Tr} L_\mu L^\mu &= -\frac{1}{2}\int d^4x\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta h_{\alpha\beta} \\
\int d^4x\, \mathrm{Tr}\partial_\mu L^\mu \partial_\nu L^\nu &= -\frac{1}{2}\int d^4x\, \Box\varphi^\alpha\Box\varphi^\beta h_{\alpha\beta} \\
\int d^4x\, \mathrm{Tr}\partial_\mu L_\nu \partial^\mu L^\nu &= -\frac{1}{2}\int d^4x \Big(\nabla^\mu\partial^\nu\varphi^\alpha\nabla_\mu\partial_\nu\varphi^\beta h_{\alpha\beta} \\
&\quad +\frac{1}{4}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta T^{(3)}_{\alpha\beta\gamma\delta}\Big) \\
\int d^4x\, \mathrm{Tr} L_\mu L^\mu L_\nu L^\nu &= \int d^4x\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta \times \\
&\quad \left(\frac{1}{4N}T^{(2)}_{\alpha\beta\gamma\delta} + \frac{1}{8}T^{(5)}_{\alpha\beta\gamma\delta}\right) \\
\int d^4x\, \mathrm{Tr} L_\mu L_\nu L^\mu L^\nu &= \int d^4x\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta \times \\
&\quad \left(\frac{1}{4N}T^{(1)}_{\alpha\beta\gamma\delta} - \frac{1}{8}T^{(3)}_{\alpha\beta\gamma\delta} + \frac{1}{8}T^{(4)}_{\alpha\beta\gamma\delta}\right) \\
\int d^4x\, \mathrm{Tr}(L_\mu L^\mu)\mathrm{Tr}(L_\nu L^\nu) &= \frac{1}{4}\int d^4x\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta T^{(2)}_{\alpha\beta\gamma\delta} \\
\int d^4x\, \mathrm{Tr}(L_\mu L_\nu)\mathrm{Tr}(L^\mu L^\nu) &= \frac{1}{4}\int d^4x\, \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta T^{(1)}_{\alpha\beta\gamma\delta}
\end{aligned} \tag{D.5}
$$

where $\Box\varphi^\alpha = \nabla^\mu\partial_\mu\varphi^\alpha$. One can further manipulate the third term integrating by parts and commuting covariant derivatives. One finds

$$
\int d^4x\, \nabla^\mu\partial^\nu\varphi^\alpha\nabla_\mu\partial_\nu\varphi^\beta h_{\alpha\beta} = \int d^4x\, \left(\Box\varphi^\alpha\Box\varphi^\beta h_{\alpha\beta} + \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\partial_\nu\varphi^\gamma\partial^\nu\varphi^\delta R_{\alpha\beta\gamma\delta}\right) \tag{D.6}
$$

and using (6.53) one can further substitute the Riemann tensor by $T^{(3)}$. In the fourth term one can eliminate $T^{(5)}$.

One has to note that Hasenfratz's action has to be compared to *minus* our action. This is because it appears with the positive sign in the exponent of the functional integral (this is consistent with the fact that the $(\Box\varphi)^2$ term has a negative coefficient in (D.1)). It is then straightforward to calculate the following relations between the couplings used in [57] and our couplings:

$$
\begin{aligned}
g^2 = \frac{f^2}{c_0}\ ; \qquad \lambda = f^2\ ; \qquad f_1 = \frac{c_3}{2N} + \frac{c_5}{2}\ ; \\
f_2 = \frac{c_4}{2}\ ; \qquad f_3 = \frac{1+c_2}{4}\ ; \qquad f_4 = \frac{c_3}{4}\ ;
\end{aligned} \tag{D.7}
$$

With these relations, one can translate his beta functions and one finds that they agree with those given in Section 6.3.4, with a single exception: the term proportional to $\tilde{g}^4$ and containing no f_i in $\beta_{\tilde{g}^2}$. We observe that the two polynomials in the c's in Eq. (39) in [57] are the same, up to an overall factor 2. As a consequence, when one extracts the beta function of $c_0/f^2 = 1/g^2$ and rewrites it in terms of the f_i's, the coefficients of $\tilde{g}^4$ and $\tilde{g}^2\lambda$ are exactly the same. This differs from the beta function given in (6.63), where the two coefficients differ in the last term. We believe that this difference can be attributed to the different cutoff scheme.

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Bibliography

[1] S. Weinberg, In *General Relativity: An Einstein centenary survey*, ed. S. W. Hawking and W. Israel, pp.790–831, Cambridge University Press (1979).

[2] M. Reuter, Phys. Rev. **D57**, 971 (1998).

[3] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971);
K. G. Wilson, Phys. Rev. B **4**, 3184 (1971);
K. G. Wilson and J. B. Kogut, Phys. Rept. **12** (1974) 75;
K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).

[4] S. B. Liao, Phys. Rev. D **53**, 2020 (1996) [arXiv:hep-th/9501124].

[5] C. Wetterich, Phys. Lett. B **301**, 90 (1993).

[6] D. F. Litim, J. M. Pawlowski and L. Vergara, arXiv:hep-th/0602140.

[7] C. Bagnuls and C. Bervillier, Phys. Rept. **348**, 91 (2001) [arXiv:hep-th/0002034].

[8] B. Delamotte, arXiv:cond-mat/0702365.

[9] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).

[10] T. Papenbrock and C. Wetterich, Z. Phys. C **65**, 519 (1995) [arXiv:hep-th/9403164];
D. F. Litim and J. M. Pawlowski, Phys. Rev. D **65**, 081701 (2002) [arXiv:hep-th/0111191];
D. F. Litim and J. M. Pawlowski, Phys. Rev. D **66**, 025030 (2002) [arXiv:hep-th/0202188].

[11] M. Niedermaier and M. Reuter, Living Rev. Relativity 9, (2006), 5. M. Niedermaier, Class. Quant. Grav. **24** (2007) R171;
R. Percacci, in "Approaches to Quantum Gravity" ed. D. Oriti, Cambridge University Press (2009); arXiv:0709.3851 [hep-th];
D.F. Litim, arXiv:0810.3675 [hep-th].

[12] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973);
H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

[13] J. C. Gaite, J. Phys. A **37**, 10409 (2004) [arXiv:hep-th/0404212].

[14] T. R. Morris, Phys. Lett. B **345**, 139 (1995) [arXiv:hep-th/9410141];
T. R. Morris and M. D. Turner, Nucl. Phys. B **509**, 637 (1998) [arXiv:hep-th/9704202].

[15] T. R. Morris, Phys. Lett. B **329**, 241 (1994) [arXiv:hep-ph/9403340]; T. R. Morris, Phys. Lett. B **334**, 355 (1994) [arXiv:hep-th/9405190].

[16] L. Rosa, P. Vitale and C. Wetterich, Phys. Rev. Lett. **86**, 958 (2001) [arXiv:hep-th/0007093].

[17] F. Hofling, C. Nowak and C. Wetterich, Phys. Rev. B **66**, 205111 (2002) [arXiv:cond-mat/0203588].

[18] H. Gies and M. M. Scherer, arXiv:0901.2459 [hep-th].

[19] O. Zanusso, L. Zambelli, G. P. Vacca and R. Percacci, Phys. Lett. B **689**, 90 (2010) [arXiv:0904.0938 [hep-th]].

[20] S.P. Robinson, F. Wilczek Phys. Rev. Lett. **96** 231601 (2006); hep-th/0509050;
A.R. Pietrykowski, Phys. Rev. Lett. **98** 061801 (2007); hep-th/0606208;
D.J. Toms Phys. Rev. **D76** 045015 (2007); arXiv:0708.2990 [hep-th];
D. Ebert, J. Plefka, A. Rodigast, Phys. Lett. B **660** (2008) 579;
Yong Tang, Yue-Liang Wu (2008) arXiv:0807.0331 [hep-ph];
D.J. Toms (2008) Phys. Rev. Lett. **101** (2008) 131301;
A. Rodigast and T. Schuster, Phys. Rev. Lett. **104**, 081301 (2010) [arXiv:0908.2422 [hep-th]].

[21] G. P. Vacca and O. Zanusso, [arXiv:1009.1735 [hep-th]]

[22] R. Percacci and D.Perini, Phys. Rev. **D67**, 081503 (2003) hep-th/0207033;

[23] P. F. Machado and F. Saueressig, Phys. Rev. D **77**, 124045 (2008) [arXiv:0712.0445 [hep-th]].

[24] A. Codello, R. Percacci and C. Rahmede, Int. J. Mod. Phys. A **23** 14 (2008) arXiv:0705.1769 [hep-th];
A. Codello, R. Percacci and C. Rahmede, Ann. of Phys. **324** 414-469 (2009), arXiv:0805.2909 [hep-th].

[25] X. Calmet, S.D.H. Hsu, D. Reeb, Phys. Rev. Lett. **101** 171802 (2008) arXiv:0805.0145 [hep-ph]

[26] A. Codello and R. Percacci, Phys. Rev. Lett. **97** 221301 (2006), [arXiv:hep-th/0607128].

[27] A. Codello, Annals Phys. **325**, 1727 (2010) [arXiv:1004.2171 [hep-th]];
A. Satz, A. Codello and F. D. Mazzitelli, arXiv:1006.3808 [hep-th].

[28] A. Codello and R. Percacci, Phys. Lett. **B 672** 280-283 (2009), arXiv:0810.0715 [hep-th].

[29] R. Percacci and O. Zanusso, Phys. Rev. D **81**, 065012 (2010) [arXiv:0910.0851 [hep-th]].

[30] A. Tonero, R. Percacci and O. Zanusso, in preparation.

[31] D. F. Litim, Phys. Rev. D **64** (2001) 105007.

[32] D. F. Litim, Nucl. Phys. B **631**, 128 (2002) [arXiv:hep-th/0203006].

[33] C. Wetterich, Z. Phys. C **57**, 451 (1993).

[34] H. Gies, S. Rechenberger and M. M. Scherer, Eur. Phys. J. C **66** (2010) 403 [arXiv:0907.0327 [hep-th]];
M. M. Scherer, H. Gies and S. Rechenberger, arXiv:0910.0395 [hep-th];
H. Gies, L. Janssen, S. Rechenberger and M. M. Scherer, Phys. Rev. D **81**, 025009 (2010) [arXiv:0910.0764 [hep-th]].

[35] M. M. Scherer, Ph.D. Thesis, Jena University (2010)

[36] L. F. Abbott, Nucl. Phys. B **185**, 189 (1981).

[37] D. F. Litim and J. M. Pawlowski, Nucl. Phys. Proc. Suppl. **74**, 325 (1999) [arXiv:hep-th/9809020];
F. Freire, D. F. Litim and J. M. Pawlowski, Phys. Lett. B **495**, 256 (2000) [arXiv:hep-th/0009110];
F. Freire, D. F. Litim and J. M. Pawlowski, Int. J. Mod. Phys. A **16**, 2035 (2001) [arXiv:hep-th/0101108];
D. F. Litim and J. M. Pawlowski, Phys. Lett. B **546**, 279 (2002) [arXiv:hep-th/0208216].

[38] F. Freire, D. F. Litim and J. M. Pawlowski, Int. J. Mod. Phys. A **16**, 2035 (2001) [arXiv:hep-th/0101108].

[39] T. R. Morris, Nucl. Phys. B **573**, 97 (2000) [arXiv:hep-th/9910058];
T. R. Morris, JHEP **0012**, 012 (2000) [arXiv:hep-th/0006064].

[40] E. Manrique and M. Reuter, arXiv:0905.4220 [hep-th];
E. Manrique, M. Reuter and F. Saueressig, arXiv:1003.5129 [hep-th].

[41] P. van Nieuwenhuizen, Phys. Rev. **D 24** (1981) 3315; R. P. Woodard, Phys. Lett. B **148** (1984) 440.

[42] R. Percacci and D.Perini, Phys. Rev. **D68**,044018 (2003); hep-th/0304222.

[43] D. F. Litim and J. M. Pawlowski, Phys. Lett. B **435** (1998) 181 [arXiv:hep-th/9802064].

[44] L. Griguolo, R. Percacci Phys. Rev. **D52** 5787 (1995). hep-th/9504092

[45] A. Eichhorn, H. Gies and M. M. Scherer, Phys. Rev. D **80**, 104003 (2009) [arXiv:0907.1828 [hep-th]].

[46] K. Groh and F. Saueressig, J. Phys. A **43**, 365403 (2010) [arXiv:1001.5032 [hep-th]].

[47] D. Dou and R. Percacci, Class.Quant.Grav. **15** 3449 (1998); hep-th/9707239.

[48] A. Codello, Ph.D. Thesis, Mainz University (2010)

[49] M. Shaposhnikov and C. Wetterich, Phys. Lett. B **683**, 196 (2010) [arXiv:0912.0208 [hep-th]].

[50] L. Smolin, Nucl. Phys. B **208**, 439 (1982).

[51] J. Gasser, H. Leutwyler, Annals Phys. **158** 142 (1984).

[52] T. Appelquist, C.W. Bernard, Phys.Rev. D22:200, (1980);
A.C. Longhitano, Phys. Rev. D22, 1166 (1980);
M.J. Herrero, E. Ruiz Morales, Nucl. Phys. B418 431-455 (1994), [arXiv:hep-ph/9308276v1].

[53] A.M. Polyakov, Phys. Lett. **B59** 79-81 (1975).

[54] E. Brezin and J. Zinn–Justin, Phys. Rev. Lett. **36** 691 (1976).

[55] W.A. Bardeen, B.W. Lee and R. Shrock, Phys. Rev. **D 14** 985 (1976).

[56] I.Ya. Arefeva, Ann. Phys. **117** 393-406 (1979);
I.Ya. Arefeva, S.I. Azakov, Nucl. Phys. **B162** 298-310 (1980).

[57] P. Hasenfratz, Nucl. Phys. **B 321** 139-162 (1989).

[58] I.L. Buchbinder and S.V. Ketov, Teor. i Matem. Fiz. **77** 42-50 (1988); Fortsch. Phys. **39** 1-20 (1991).

[59] S. Weinberg, Phys. Rev. D **77**, 123541 (2008) [arXiv:0804.4291 [hep-th]].

[60] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).

[61] J. Bijnens, G. Colangelo and G. Ecker, JHEP **9902**, 020 (1999) [arXiv:hep-ph/9902437];
J. Bijnens, G. Colangelo and G. Ecker, Annals Phys. **280**, 100 (2000) [arXiv:hep-ph/9907333].

[62] P. Fischer and D. F. Litim, AIP Conf. Proc. **861**, 336 (2006) [arXiv:hep-th/0606135].

[63] S. Helgason, “Differential geometry, Lie groups and and symmetric spaces”, Academic Press (1978).

[64] E. Witten, Nucl. Phys. **B223** 422-432 (1983).

[65] J. Honerkamp, Nucl. Phys. **B36** 130-140 (1972).

[66] L. Alvarez-Gaume, D.Z. Freedman, S. Mukhi, Annals Phys. **134** 85 (1981).

[67] D.G. Boulware and L.S. Brown, Annals Phys. **138** 392-433 (1982).

[68] P.S. Howe, G. Papadopoulos, K.S. Stelle, Nucl. Phys. **B296** 26 (1988).

[69] S. Kobayashi and K. Nomizu, “Foundations of differential geometry”, Wiley Interscience (1963).

[70] N.H. Barth, J. Phys. **A 20** 875-888 (1987);
H.W. Lee, P.Y. Pac and H.K. Shin, Phys. Rev. **D 35** 2440 (1987).

[71] A.J. Macfarlane, A. Sudbery and P.H. Weisz, Comm. Math. Phys. **11** 77-90 (1968).

[72] J.A. de Azcarraga, A.J. Macfarlane, A.J. Mountain and J. C. Perez Bueno, Nucl. Phys. **B510** 657-687 (1998) [arXiv:math-ph/9706006].

[73] K.S. Stelle, Phys. Rev. **D 16** 953 (1977).

[74] A.A. Slavnov, Nucl. Phys. **B31** 301 (1971).

[75] J. Julve, M. Tonin, Nuovo Cim. **46B**, 137 (1978).

[76] E.S. Fradkin, A.A. Tseytlin, Phys. Lett. **104 B**, 377 (1981); Nucl. Phys. **B 201**, 469 (1982).

[77] I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).

[78] G. de Berredo-Peixoto and I. L. Shapiro, Phys. Rev. D **70** (2004) 044024; G. de Berredo–Peixoto and I. Shapiro, Phys.Rev. **D71** 064005 (2005).

[79] D. Benedetti, P.F. Machado and F. Saueressig, arXiv:0901.2984 [hep-th], arXiv:0902.4630 [hep-th].

[80] A. Salam and J. A. Strathdee, Phys. Rev. D **18** (1978) 4480.

[81] M. Niedermaier, in proceedings of *Workshop on Continuum and Lattice Approaches to Quantum Gravity*, PoS(CLAQG08)005; Phys. Rev. Lett. **103**, 101303 (2009).

[82] D.D. Dietrich and F. Sannino, Phys. Rev. **D75** 085018 (2007).

[83] E. Braaten, T.L. Curtright, C.K. Zachos, Nucl. Phys. **B260** 630 (1985).