SCUOLA INTERNAZIONALE SUPERIORE di STUDI AVANZATI
International School for Advanced Studies

Existence and multiplicity of solutions for problems in Geometrical Analysis

Ph.D. Thesis

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ACADEMIC YEAR 2009/2010

Il presente lavoro costituisce la tesi presentata da Francesca De Marchis, sotto la direzione di ricerca del prof. Andrea Malchiodi, al fine di ottenere l'attestato di ricerca post-universitaria di *Doctor Philosophiae in Analisi Matematica* presso la SISSA di Trieste. Ai sensi dell'art. 18, comma 3, dello Statuto della SISSA pubblicato sulla G.U. n. 62 del 15.03.2001, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, anno accademico 2009/2010.

Contents

Introduction

In this thesis we present some results concerning existence and multiplicity of solutions for mean field equations of Liouville type on compact surfaces and for the prescribed Q-curvature equation on fourth dimensional compact manifolds.

Introduction of the problems

Let (Σ, g) be a compact Riemannian surface (without boundary), $h \in C^2(\Sigma)$ be a positive function and ρ a real number. We consider the equation

$$- \triangle_g u + \frac{\rho}{\int_\Sigma dV_g} = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} \qquad x \in \Sigma,\ u \in H_g^1(\Sigma). \tag*{$(*)_\rho$}$$

where $\triangle_g$ is the Laplace-Beltrami operator on Σ.

The above equation arises in statistical mechanics as a mean field equation for the Euler flow. More precisely, it has been proved in [8] and [50] that, according to Onsager's vortex theory, when the number of vortices is supposed to tend to $+\infty$, the stream function is given by $\frac{u}{\rho}$, where u satisfies $(*)_\rho$ with $h = 1$. In this interpretation the exponential represents the vorticity of the flow and $\rho > 0$ corresponds to negative values of the statistical temperature, a range which is expected to describe the high energy (turbulent) behavior of the flow.

This PDE also concerns the description of self-dual condensates of some Chern-Simon-Higgs model; indeed via its solutions it is possible to describe the asymptotic behavior of a class of condensates (or multivortex) solutions which are relevant in theoretical physics and which were absent in the classical (Maxwell-Higgs) vortex theory (see [73], [82], [86] and references therein).

Another motivation for the study of $(*)_\rho$ is the problem of prescribing the Gauss curvature of a surface via a conformal transformation of the metric. Indeed, setting $\tilde{g} = e^w g$ we have

$$\triangle_{\tilde{g}} = e^{-w} \triangle_g; \qquad\qquad -\triangle_g w + 2K_g = 2K_{\tilde{g}} e^w,$$

where K_g and $K_{\tilde{g}}$ are the Gauss curvature of (Σ, g) and of $(\Sigma, \tilde{g})$. In this context, of particular interest is the classical Uniformization Theorem, which asserts that every

compact surface carries a conformal metric with constant curvature. Motivated by this result, one may ask whether, given a surface with constant curvature, it is possible to obtain conformal metrics for which the Gauss curvature becomes a given function K.
From the analytical point of view, this amounts to ask for which $K \in C(\Sigma)$ the problem

$$-\triangle_g w + 2K_g = 2Ke^w \tag{1}$$

is solvable on Σ. When Σ is the standard sphere, the latter is known as the Kazdan-Warner problem, or as the Nirenberg problem, and it represents the most delicate situation in this analysis (see for example [11], [13] and [49]).

The Gauss–Bonnet theorem gives a necessary condition for the existence of a solution to (1)

$$\int_\Sigma Ke^w dV_g = 2\pi\chi(\Sigma). \tag{2}$$

This formula has played a crucial role in the problem of the solvability of (1) when Σ is a surface of positive genus, which is now completely understood.

Problem $(*)_\rho$ has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \rho \fint_\Sigma u\, dV_g - \rho \log \int_\Sigma h(x)e^u dV_g \qquad u \in H^1_g(\Sigma). \tag{3}$$

Since equation $(*)_\rho$ is invariant when adding constants to u, we can restrict ourselves to the subspace of the functions with zero average

$$\bar{H}^1_g(\Sigma) := \left\{ u \in H^1_g(\Sigma) \,:\, \fint_\Sigma u\, dV_g = 0 \right\}.$$

Because of the Moser-Trudinger inequality one can easily prove the compactness and the coercivity of I_ρ when $\rho < 8\pi$ and thus one can find solutions to $(*)_\rho$ by minimization.

If $\rho = 8\pi$ the situation is more delicate since I_ρ still has a lower bound but it is not coercive anymore; in general when ρ is an integer multiple of 8π, the existence problem of $(*)_\rho$ is much harder (a far from complete list of references on the subject includes works by Chang and Yang [13], Chang, Gursky and Yang [11], Chen and Li [24], Nolasco and Tarantello [73], Ding, Jost, Li and Wang [35] and Lucia [60]).

For $\rho > 8\pi$, as the functional I_ρ is unbounded from below and from above, solutions have to be found as saddle points.

Closely related to problem $(*)_\rho$, considered according to its geometrical interpretation, is the case when we allow the conformal class to contain metrics that introduce conical type singularities on Σ. Let us explain it in more detail.

A conformal metric g_s on Σ is said to have a conical singularity of order $\alpha \in (-1, +\infty)$ (or of angle $\vartheta_\alpha = 2\pi(1+\alpha)$) at a given point $P_0 \in \Sigma$ if there exist local coordinates $z(P) \in \Omega \subset \mathbb{C}$ and $w \in C^0(\Omega) \cap C^2(\Omega \setminus \{P_0\})$ such that $z(P_0) = 0$ and

$$\tilde{g}_s(z) = |z|^{2\alpha} e^w |dz|^2, \quad z \in \Omega,$$

where $\tilde{g}_s$ is the local expression of g_s. The information concerning finitely many conical singularities is encoded in a divisor, which is the formal sum

$$\underline{\alpha}_m = \sum_{j=1}^{m} \alpha_j P_j, \quad m \in \mathbb{N}, \tag{4}$$

of the orders of the singularities $\{\alpha_1, \ldots, \alpha_m\}$ times the singular points $\{P_1, \cdots, P_m\}$. In particular, a metric g_s on Σ is said to represent the divisor $\underline{\alpha}_m$ if it has conical singularities of order α_j at point P_j for any $j \in \{1, \ldots, m\}$. We will denote by $(\Sigma, \underline{\alpha}_m)$ the singular surface.

As for the regular equation, given a Lipschitz function K on Σ we seek a conformal metric $\tilde{g}$ on $(\Sigma, \underline{\alpha}_m)$ whose Gaussian curvature is K. As above, by considering the conformal factor w, we can reduce such a geometrical problem to the solvability of the following differential problem

$$-\triangle_g w + 2K_g = 2Ke^w - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{P_j} \tag{5}$$

see Proposition 0.6 for a detailed proof.

It is evident that (5) contains $(*)_\rho$ as a particular case and this equation, as the previous one, is not only relevant in conformal geometry but is also physically meaningful. Indeed, as it happens for $(*)_\rho$, (5) can be seen as the mean field equation for the two-dimensional Euler flow, where the singularities play the role of k sinks of vorticity $-4\pi\frac{\alpha_j}{\rho}$. More precisely, if we take $K = 1$ and $K_g = 0$, $\Psi = \frac{u}{\rho}$ is the stream function for a Euler flow of vorticity $\frac{e^u}{\int_\Sigma e^u} - 4\pi \sum_{j=1}^{m} \frac{\alpha_j}{\rho} \delta_{P_j}$.
Besides this problem concerns with the search of Abrikosov's vortex–like configurations for the Electroweak theory of Glashow–Salam–Weinberg [51] in a selfdual regime.

In four-dimensional geometry there exists a conformally covariant operator, the Paneitz operator (introduced in [74]), which enjoys analogous properties to the Laplace-Beltrami operator on surfaces, and to which is associated a natural concept of curvature: the Q-curvature (introduced in [7]). Let denote by $\mathrm{P_g}$ this operator and by Q_g the Q-curvature corresponding to a given 4-manifold (M, g). Their expressions in terms of the Ricci tensor Ric_g and of the scalar curvature R_g are as follows

$$\mathrm{P_g}(\varphi) = \Delta_g^2 \varphi + div_g \left(\frac{2}{3} R_g g - 2Ric_g \right) d\varphi, \qquad Q_g = -\frac{1}{12} \left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2 \right),$$

and considering the conformal change of metric $\tilde{g} = e^{2u}g$, $Q_{\tilde{g}}$ is given by

$$\mathrm{P_g}\, u + 2Q_g = 2Q_{\tilde{g}} \frac{e^{4u}}{\int_M e^{4u} dV_g}. \tag{6}$$

Apart from the analogy with the prescribed Gauss curvature equation, there is an extension of the Gauss-Bonnet formula involving the Weyl tensor W and the integral of Q_g, which is a conformal invariant:

$$4\pi^2 \chi(M) = \int_M (Q_g + \frac{1}{8}|W|^2)\, dV_g. \tag{7}$$

We refer to [12], [15] and [46] for details.

As for the Uniformization theorem one can ask whether every four-manifold (M, g) carries a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a constant. Writing $\tilde{g} = e^{2u}g$ the question amounts to solving (6) in u with $Q_{\tilde{g}}$ constant, namely the equation

$$\mathrm{P_g}\, u + 2Q_g = 2k_P \frac{e^{4u}}{\int_M e^{4u} dV_g}. \tag{\#}$$

where $k_P := \int_M Q_g dV_g$.

The regular mean field equation

We start considering the regular case, namely equation $(*)_\rho$, where no Dirac distributions appear. In the following we will refer to $8\mathbb{N}\pi$ as the set of critical values of the parameter ρ.

Regular ranges of the parameter

For this problem, Li and Shafrir, exploiting an earlier work of Brezis and Merle [6], proved an important compactness property. Indeed they showed that if $\rho \notin 8\pi\mathbb{N}$, then solutions of $(*)_\rho$ are bounded in $C^{2,\alpha}(\Sigma)$ for any $\alpha \in (0, 1)$.

The previous result permits to define the global Leray-Schauder degree of $(*)_\rho$. As a consequence of the homotopy invariance of the degree, it turns out that it is independent of h, of the parameter $\rho \in (8k\pi, 8(k+1)\pi)$ for $k \in \mathbb{N}$ and of the metric of Σ. In [52], Y.Y.Li first pointed out that the degree of $(*)_\rho$ only depends on $k \in \mathbb{N}$ (for $\rho \in (8k\pi, 8(k+1)\pi)$) and on the Euler characteristic of Σ, $\chi(\Sigma)$, so we will use $\mathrm{d}(k, \chi(\Sigma))$ to denote it. Extending the results in [36] and [54], Chen and Lin in [18] analyzed the jump values of the degree after crossing the critical thresholds and obtained the following explicit expression for the degree when $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$:

$$\mathrm{d}(k, \chi(\Sigma)) = \binom{k - \chi(\Sigma)}{k} \equiv \begin{cases} \frac{(k-\chi(\Sigma))\dots(2-\chi(\Sigma))(1-\chi(\Sigma))}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{8}$$

In the latter statement we specified what we mean by the binomial coefficient because the upper term, $k-\chi(\Sigma)$, can be negative; clearly this definition extends the usual one.

We would like to remark that the positivity of h is a necessary condition for the Leray-Schauder degree to be counted as in formula (8). If h vanishes somewhere, then the degree formula is different; see [19].

Notice that $\mathrm{d}(0,2)=1$, $\mathrm{d}(1,2)=-1$, $\mathrm{d}(k,2)=0$ for any $k\geq 2$, so if Σ has the homology of a sphere the degree does not suffice to guarantee the existence of a solution; while when Σ has the homology of a torus, being $\mathrm{d}(k,0)=1$ for any $k\geq 0$, we can deduce existence but we have no information about multiplicity.

Finally, Djadli generalized these previous results establishing, for $\rho\notin 8\mathbb{N}\pi$, the existence also in the case of positive Euler characteristic.

To do that, he deeply investigated the topology of low sublevels of I_ρ in order to perform a min-max scheme (already introduced in Djadli and Malchiodi [38]). A crucial observation, as noticed in [25], is that the constant in the Moser-Trudinger inequality (1.1) can be roughly divided by the number of regions where $\frac{e^u}{\int_\Sigma e^u}$ is *supported* (see Lemma 1.2 for details). As a consequence, if $\rho\in(8k\pi,8(k+1)\pi)$ and if I_ρ attains large negative values, $\frac{e^u}{\int_\Sigma e^u}$ has to concentrate near at most k points of Σ, in the sense specified in Lemma 1.4. From these considerations one is led naturally to associate with the probability measure $\frac{e^u}{\int_\Sigma e^u}$ some formal barycenter $\sigma\in\Sigma_k$.
We recall that in literature the set

$$\Sigma_k=\left\{\sum_{i=1}^k t_i\delta_{x_i}\,|\,t_i\geq 0,\,\sum_{i=1}^k t_i=1,\,x_i\in\Sigma\right\},\tag{9}$$

where δ_{x_i} stands for the Dirac mass at x_i, is known as the set of formal barycenters of Σ of order k. It is in fact possible to prove that $\{I_\rho\leq -L\}$ has the same homology of Σ_k for L very large positive [64].

On the other hand a deformation lemma due to Lucia [61] and adapted by Malchiodi permits to show (see [64]) that the high sublevels of I_ρ are contractible.

Taking advantage of the previous analysis Malchiodi provided a clear interpretation of the degree–counting formula. Indeed he recently gave an alternative and direct proof of (8), via a Morse-theoretical approach which relates the degree to the topology of low and high sublevels of the Euler functional. We refer to subsection 1.2.1 for a more detailed exposition of this result.

In the present work we prove generic multiplicity of solutions also in the cases when $\chi(\Sigma)\geq 0$ and we improve significantly for the other surfaces the estimate of the number of solutions which can be derived from the degree formula. Our main result in this direction reads as follows.

Theorem 0.1. *[32] Let $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}^*$. Then, for a generic choice of the metric g and of the function h (namely for (g,h) in an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$)*

$$\#\{\textit{solutions of } (*)_\rho\} \geq \begin{cases} p_k & \textit{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \textit{if } \chi(\Sigma) \leq 0, \end{cases} \tag{10}$$

where $p_0 = 1$, $p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j$ for any $m \in \mathbb{N}^*$.
Moreover the latter estimate holds true also for (g,h) in an open and dense subset of $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

In the above statement $\mathcal{M}^2$ stands for the space of all C^2 Riemannian metrics on Σ equipped with the C^2 norm, while $\mathcal{M}_1^2$ is the subset of $\mathcal{M}^2$ of the metrics g such that $\int_\Sigma dV_g = 1$. In literature it is usually studied the case when $Vol_g(\Sigma) := \int_\Sigma dV_g = 1$, namely when $g \in \mathcal{M}_1^2$. It is for this reason that we specified that the set of (g,h) for which (10) holds true is dense not only in $\mathcal{M}^2 \times C^2(\Sigma)^+$ but also in $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

By direct calculation and an asymptotic formula for the sequence p_r, obtained in [62], we derive the following Corollary.

Corollary 0.2. *[32] Under the hypotheses of Theorem 0.1, for generic $(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+$:*

1. *for any Σ and for any $k \in \mathbb{N}^*$ (except for the case $\chi(\Sigma) = 2$ and $k = 1$)*

   $$\#\{\textit{solutions of } (*)_\rho\} > \mathrm{d}(k, \chi(\Sigma)) \geq 0,$$

   where by $\mathrm{d}(k,\chi(\Sigma))$ we mean the Leray-Shauder degree of the equation $(*)_\rho$ (see (8)).
   When $\chi(\Sigma) = 2$ and $k = 1$ the right hand side of formula (10) is simply equal to $1 = |\,\mathrm{d}(1,2)|$.

2. *for any Σ, as $k \geq k_0$, $k_0 \in \mathbb{N}^*$ (independent of Σ),*

   $$\#\{\textit{solutions of } (*)_\rho\} \geq C\Big(\frac{[\frac{k}{2}]}{\log[\frac{k}{2}]}\Big)^{\frac{1}{2l_2}\log(\frac{[\frac{k}{2}]}{\log[\frac{k}{2}]})+1+\frac{ll_2}{l_2}}[\frac{k}{2}]^{(\frac{1}{l_2}-\frac{1}{2})}, \tag{11}$$

   where by $[\frac{k}{2}]$ we mean the integer part of $\frac{k}{2}$, $l_2 := \log 2$ and $ll_2 =: \log\log 2$; so in particular for any Σ

   $$\#\{\textit{solutions of } (*)_\rho\} \to +\infty \quad \textit{as } k \to +\infty.$$

Moreover points 1 and 2 hold true also for (g,h) in an open and dense subset of $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

Actually it is not surprising that our estimate improves the one obtained with the degree. Indeed we tackle the problem using Morse inequalities and in general Morse theory gives more information about the structure of the critical points compared to degree theory, just because one includes the other as a particular case.

Besides it is worth pointing out that our estimate is not only better than the one given by the degree (point 1 of Corollary 0.2), but improves considerably the order of infinity, as $\rho \to +\infty$, of the number of solutions (point 2 of Corollary 0.2). Indeed for $\chi(\Sigma) \geq 0$ $|\mathrm{d}(k, \chi(\Sigma))| \leq 1$ and for $\chi(\Sigma) < 0$ the degree is just a polynomial in k, more precisely $\mathrm{d}(k, \chi(\Sigma)) = O_k(k^{-\chi(\Sigma)})$, while by means of the rough estimate $\frac{k}{\log(k)} \geq k^{\frac{1}{2}}$ (which holds for any $k \geq 2$) formula (11) implies that

$$\#\{\text{solutions of } (*)_\rho\} \geq C[\frac{k}{2}]^{\frac{1}{8l_2}\log[\frac{k}{2}] + \frac{2+ll_2}{2l_2}}.$$

To prove Theorem 0.1 we first show that we are in position to apply a transversality result due to Saut and Temam, which guarantees that for (g, h) in an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$ all the critical points of I_ρ are non degenerate. Then we derive the estimate (10) under the further assumption that all the critical points of I_ρ are non degenerate, i.e. that I_ρ is a Morse functional. In these hypotheses we can exploit the weak Morse inequalities, which, together with the exactness of the homology of a pair, permit to prove that

$$\#\{\text{solutions of } (*)_\rho\} \geq \sum_{q\geq 0} \dim H_q(\{I_\rho \leq b\}), \{I_\rho \leq -L\}; \mathbb{Z}_2). \tag{12}$$

Actually Morse inequalities require the Palais-Smale condition to hold, which is not known for I_ρ, but a deformation lemma from [64] (see also [61]) allows to overcome the problem. From formula (12) it is clear that the core of the analysis is the understanding of the homology groups of high and low sublevels. In [64] the author proved that for large values of b the sublevel $\{I_\rho \leq b\}$ has the homology of a point, while for dealing with low sublevels we can take advantage of the aforementioned characterization in [64] (see Theorem 1.8).
From these considerations it can be deduced that the problem reduces to the computation of the following sum: $\sum_{q\geq 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$. To get it we use a Theorem due to Kallel and Karoui [48] dealing with the homology of the set of formal barycenters on topological spaces (and so on manifolds), which in particular, combined with results in [70] and [72], permits to have a nice description of the homology of the family of formal barycenters on spheres of any dimension.

Critical thresholds of the parameter

Equation $(*)_\rho$ is much more delicate when $\rho = 8k\pi$ for some positive integer k. For instance, in these *critical* cases the degree depends on h and therefore the search for solutions is much more involved. Only partial results are known.

When $\rho = 8\pi$, extending a formula due to Chang and Yang [13] for counting the topological degree $\mathrm{d}_{8\pi}(h)$ when Σ is the sphere, Chen and Lin derived, from their deep study on the degree contributions of blow-up solutions, a complete degree counting formula. First of all they pointed out that, if Σ has constant Gaussian curvature and $h \in C^2(\Sigma)^+$ satisfies $\triangle h(p) + 8\pi g h(p) \neq 0$ for any critical point p of h, then the Leray–Schauder degree is well defined. Furthermore, if h is a Morse function, then

$$\mathrm{d}_{8\pi}(h) = 1 - \sum_{q \in \Lambda^-} (-1)^{\mathrm{ind}\, q} \tag{13}$$

where $\Lambda^- = \{p : \nabla h(p) = 0,\ \triangle h(p) + 8\pi g h(p) < 0\}$ and $\mathrm{ind}\, q$ stands for the Morse index of h at q.

Actually, there is a situation where it is possible to claim a general existence result and it is given by the flat 2–torus. As already noticed, when $\Sigma = T$ and then $\chi(\Sigma) = 0$, formula (8) implies that, for any $k \in \mathbb{N}$, $\mathrm{d}(k, 0) = 1$. Since the degree admits no-jump when ρ crosses an integer multiple of 8π, then it is reasonable to expect that no solution blows-up, when ρ approaches a critical threshold. This is exactly the case when $\rho \nearrow 8k\pi$, with $k \in \mathbb{N}$, while blow-up phenomena can occur when the parameter approaches a critical value from above. The fact that solutions pass to the limit in the first case follows again by the accurate analysis carried out in [18]. In conclusion, problem $(*)_\rho$ on the flat torus admits a solution for every $\rho > 0$.

Always in [18] it was shown that for any compact Riemannian surface Σ with non positive Euler-characteristic and for any C^2 positive function h, there exists a real number defined by

$$\rho_h = \max_{\Sigma}(2K_g - \triangle \log h), \tag{14}$$

such that equation $(*)_\rho$ possesses a solution provided that $\rho > \rho_h$.

A multiplicity result has been obtained in a simpler case, namely when $h \equiv 1$ and one seeks for 1-dimensional periodic solutions on the torus T. More precisely Ricciardi and Tarantello [76] showed that for $\rho > 4k^2\pi^2$ problem $(*)_\rho$ admits at least k geometrically distinct solutions, i.e. solutions which do not differ one from another just for a shift in the unique variable. Also in the case of axially symmetric solutions on S^2, it has been shown that the number of solutions increases as ρ increases [39].

In the same spirit of Chen and Lin, who obtained the existence of at least one solution for ρ sufficiently large, depending on h (see (14) above), we are able, exploiting our multiplicity estimate (10) holding for the regular values of the parameter, to prove the following Theorem dealing with generic multiplicity for large critical values of ρ.

Theorem 0.3. *[34] For any $\bar{h} \in C^3(\Sigma)^+$ there exist $\delta > 0$ and $n_{\bar{h}} \in \mathbb{N}$ such that for any $k \geq n_{\bar{h}}$ and for a generic choice of $h \in B_\delta(\bar{h}) := \left\{h \in C^3(\Sigma) : \left\|h - \bar{h}\right\|_{C^3} < \delta\right\}$*

(namely for h in an open and dense subset of $B_\delta(\bar h)$)

$$\#\{\text{solutions of } (*)_{8k\pi}\} \geq \begin{cases} p_{k-1} & \text{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^{k-1} \binom{k-r-\chi(\Sigma)}{k-r-1} p_r & \text{if } \chi(\Sigma) \leq 0, \end{cases} \tag{15}$$

The equation on the 2-torus

The case of the flat torus is a relevant situation from the physical point of view, since many vortex-like configurations naturally develop into periodic lattices.

In particular when the cell of the torus is a square, $\left\{(x_1, x_2) \,:\, |x_1| \leq \frac{a}{2}, |x_2| \leq \frac{a}{2}\right\}$, and $h \equiv 1$ the problem looks as follows:

$$-\triangle u + \frac{\rho}{|T|^2} = \rho \frac{e^u}{\int_T e^u dx} \qquad u \in \bar H^1(T). \tag{16}$$

It is plain to see that in this periodic situation, for any value of ρ, the function $u = 0$ solves (16). A uniqueness result has been proved by Lin and Lucia [55]. Indeed they showed that the trivial solution is the only solution to (16) for $\rho \leq 8\pi^2$. The same authors partially answered also to the problem for the general flat torus showing that, if $\rho \leq 8\pi$ and u is a minimizer for the corresponding functional, then u is a one-dimensional solution.

Instead, when $\rho \in (8\pi, 4\pi^2)$ and the fundamental domain of the torus is a square, the trivial solution $u = 0$ turns out to be a strict local minimum for I_ρ, since the second variation in the direction $v \in \bar H^1_g(T)$ can be estimated as follows

$$D^2 I_\rho(0)[v, v] = \|v\|^2 - \rho \int_\Sigma v^2 dx \geq \left(1 - \frac{\rho}{4\pi^2}\right) \|v\|^2 . \tag{17}$$

Under these assumptions Struwe and Tarantello [81] showed that the functional possesses a mountain pass geometry and by thanks to this structure they detected the existence of a saddle point u_ρ of I_ρ satisfying $I_\rho(u_\rho) \geq (1 - \rho/4\pi^2)c_0$, for some constant $c_0 > 0$ independent of ρ.

As g is the flat metric and h is constant, if u is a solution of $(*)_\rho$, the functions $u_{\rho,x_0}(x) := u_\rho(x - x_0)$ still solve $(*)_\rho$, for any $x_0 \in T$; so from the previous result we can deduce the existence of an infinite number of non trivial solutions to $(*)_\rho$.

Perturbing g and h there is still a local minimum, $\bar u$, close to $u = 0$ and the same procedure of [81] ensures the presence of a saddle point, but on the other hand, if u is a non-trivial solution, the criticality of the translated functions u_{x_0} is not anymore guaranteed. In [31] the author improved this result stating that apart from $\bar u$ there are at least two critical points, see Theorem 2.17 in Section 2.2.

The strategy of the proof consists in defining a deformed functional $\tilde I_\rho$, having the same saddle points of I_ρ but a greater topological complexity of its low sublevels, and in estimating from below the number of saddle points of $\tilde I_\rho$ using the notion of Lusternik-Schnirelmann relative category (roughly speaking a natural number measuring how a set is far from being contractible, when a subset is fixed).

Always in [31] the author conjectured that apart from the minimum and the two saddle points another critical point should exist. In fact this turns out to be true and we are now able to prove it.

Theorem 0.4. *[33] If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least three different saddle points.*

In the above statement $S^{2\times 2}$ stands for the symmetric matrices on T. To prove Theorem 0.4 we exploit the following inequality derived in [31]:

$$\#\,\{\text{solutions of } (*)_\rho\} \geq \mathrm{Cat}_{X,\partial X}\, X,$$

where X is the topological cone over T. Next, applying a classical result we are able to estimate from below the previous relative category by one plus the cup–length of the pair $(T\times[0,1], T\times(\{0\}\cup\{1\}))$. The cup–length of a topological pair (Y,Z), denoted by $\mathrm{CL}(Y,Z)$, is the maximum number of elements of the cohomology ring $H^*(Y)$ having positive dimensions and whose cup product do not "annihilate" the ring $H^*(Y,Z)$; we refer to the next chapter for a rigourous definition. Finally, to obtain the thesis, we show that $\mathrm{CL}(T\times[0,1], T\times(\{0\}\cup\{1\})) \geq \mathrm{CL}(T) = 2$.

Since all the arguments only use the presence of a strict local minimum and the fact that X is the topological cone over T, whenever on some (Σ, g) the functional I_ρ possesses a strict local minimum, the theorem holds true, more precisely I_ρ has at least $CL(\Sigma)+1$ critical points other than the minimum.

The mean field equation with singular data

The study of conformal metrics on surfaces with conical singularities dates back at least to Picard [75], and has been widely discussed in the last decades, see for example [22], [27], [25], [28], [26], [42], [57], [59], [68], [83], [84] and the references quoted there. In this thesis we are concerned with the construction of conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities. We refer the reader in particular to [84] where a systematic analysis of this problem was initiated.

In the latter paper the Euler characteristic of the singular surface $(\Sigma, \underline{\alpha}_m)$ is defined by

$$\chi(\Sigma, \underline{\alpha}_m) = \chi(\Sigma) + \sum_{j=1}^m \alpha_j,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

The Trudinger constant of the singular surface $(\Sigma, \underline{\alpha}_m)$ (see [23], [84]) is instead given by

$$\tau(\Sigma, \underline{\alpha}_m) = 2 + 2 \min_{j\in\{1,\dots,m\}} \min\{\alpha_j, 0\}.$$

According to the definitions in [84] the singular surface $(\Sigma,\, \underline{\alpha}_m)$ is said to be

$$\begin{cases} \text{subcritical} & \text{if} \quad \chi(\Sigma,\, \underline{\alpha}_m) < \tau(\Sigma,\, \underline{\alpha}_m), \\ \text{critical} & \text{if} \quad \chi(\Sigma,\, \underline{\alpha}_m) = \tau(\Sigma,\, \underline{\alpha}_m), \\ \text{supercritical} & \text{if} \quad \chi(\Sigma,\, \underline{\alpha}_m) > \tau(\Sigma,\, \underline{\alpha}_m). \end{cases}$$

As far as one is interested in proving the existence of at least one conformal metric on $(\Sigma, \underline{\alpha}_m)$ with prescribed Gaussian curvature, the subcritical case is well understood. This is mainly due to the fact that on subcritical singular surfaces the problem corresponds to minimizing a coercive functional [84]. On the contrary, much less is known concerning critical and supercritical singular surfaces.
We refer the reader to [25], [28], [26], [42], [56], [68], [83], for some positive results in this direction. In the same spirit of [36], Bartolucci and Tarantello [4] obtained a result which implies that: if $(\Sigma, \underline{\alpha}_m)$ is a supercritical singular surface with $\alpha_j > 0$, $j \in \{1, \ldots, m\}$, $\chi(\Sigma) \le 0$ and $4\pi\chi(\Sigma,\, \underline{\alpha}_m) \in (8\pi, 16\pi) \setminus \{8\pi(1+\alpha_j),\, j = 1, \ldots, m\}$, then any positive Lipschitz continuous function K on Σ is the Gaussian curvature of at least one conformal metric on $(\Sigma, \underline{\alpha}_m)$. See also [19] for related issues.

We are going to present a generalization of this result, obtained via a Morse theoretical approach.

Let

$$\Gamma(\underline{\alpha}_m) = \{\mu \in \mathbb{R}^+ \,|\, \mu = 8\pi k + 8\pi \sum_{j=1}^{m} (1+\alpha_j) n_j,\, k \in \mathbb{N}\cup\{0\},\, m \in \mathbb{N}\cup\{0\},\, n_j \in \{0,1\}\}.$$

Our main result is the following

Theorem 0.5. *[3] Let $(\Sigma, \underline{\alpha}_m)$ be a supercritical singular surface with $\alpha_j > 0$, $j \in \{1, \ldots, m\}$, $\chi(\Sigma) \le 0$ and $4\pi\chi(\Sigma,\, \underline{\alpha}_m) \notin \Gamma(\underline{\alpha}_m)$. Then, any positive Lipschitz continuous function K on Σ is the Gaussian curvature of at least one conformal metric on $(\Sigma, \underline{\alpha}_m)$.*

We attack this problem by a variational approach as first proposed in [5] and then pursued by many authors, see for example [2], [25], [49], [84] and the references quoted there. Proposition 0.6 below allows to reduce the problem to a scalar differential equation on Σ. To state it we need to introduce some notation. Let $Q \in \Sigma$ be a given point and $G(P, Q)$ be the solution of (see [2])

$$-\Delta_g G(P, Q) = \delta_Q - \frac{1}{|\Sigma|} \quad \text{in} \quad \Sigma, \qquad \int_\Sigma G(P, Q) dV_g(P) = 0,$$

where δ_Q denotes the Dirac delta with pole Q, Δ_g the Laplace-Beltrami operator associated to g and $|\Sigma|$ the area of Σ with respect to the volume form induced by g. For a given divisor $\underline{\alpha}_m$ we define

$$h_m(P) = 4\pi \sum_{j=1}^{m} \alpha_j G(P, P_j).$$

Let us also denote by K_g the (smooth) Gaussian curvature induced by g. Then we have

Proposition 0.6. *[3] Let $\alpha_j > 0$ for $j = 1, \dots, m$, K a Hölder continuous function on Σ and suppose that $\chi(\Sigma, \underline{\alpha}_m) > 0$. The metric*

$$\tilde{g} = \rho \frac{e^{-h_m} e^u}{\int_\Sigma 2K e^{-h_m} e^u dV_g} g, \quad \textit{with} \quad \rho = 4\pi\chi(\Sigma, \underline{\alpha}_m),$$

is a conformal metric on $(\Sigma, \underline{\alpha}_m)$ with Gaussian curvature K if and only if u is a classical solution to

$$-\Delta_g u + 2K_g = \rho \frac{K e^{-h_m} e^u}{\int_\Sigma K e^{-h_m} e^u dV_g} - \frac{4\pi}{|\Sigma|} \sum_{j=1}^{m} \alpha_j \quad \textit{in} \quad \Sigma. \tag{18}$$

It is immediate to understand that equation (18) is just a reformulation of problem (5), indeed one can pass from one to another by setting $w = u - h_m$.
By using Proposition 0.6, we are reduced to finding a classical solution of (18), that is, by standard elliptic regularity theory, a critical point $u \in \bar{H}^1_g(\Sigma)$ of

$$J_\rho(u) = \int_\Sigma |\nabla u|^2 \, dV_g - \rho \log \left(\int_\Sigma 2K e^{-h_m} e^u \, dV_g \right), \tag{19}$$

where ρ satisfies the Gauss-Bonnet constraint

$$\rho = \int_\Sigma 2K e^{-h_m} e^u dV_g = 4\pi\chi(\Sigma) + 4\pi \sum_{j=1}^{m} \alpha_j = 4\pi\chi(\Sigma, \underline{\alpha}_m). \tag{20}$$

By means of Proposition 0.6, Theorem 0.5 will follow immediately from the next result.

Theorem 0.7. *[3] Let Σ be a closed surface of positive genus, $K_g \in L^s(\Sigma)$ for some $s > 1$ and K any positive Lipschitz function on Σ. Suppose moreover that $\alpha_j \geq 0$ for $j \in \{1, \dots, m\}$. Then, for any $\rho \in (8\pi, +\infty) \setminus \Gamma(\underline{\alpha}_m)$ there exists at least one critical point $u \in \bar{H}^1_g(\Sigma)$ for J_ρ.*

As a consequence of the results in [53] (see also [52]) and in [4], it is straightforward to verify that our proof of Theorem 0.7 works whenever K is positive and Hölder continuous in Σ and Lipschitz continuous in a neighborhood of $\{P_1, \cdots, P_m\}$. We conclude that the result of Theorem 0.5 holds also under these assumptions on K.

We notice that in case $\alpha_j = 0$, $j \in \{1, \dots, m\}$, since $\Gamma(\underline{\alpha}_m) = 8\pi\mathbb{N}$, we come up with another proof of the existence of solutions for the regular mean field equation. In the same spirit of [32], [65], other positive results concerning the existence of solutions for (18) have been derived in [66]. Other results, in the same direction of [18], have been recently announced in [20], see [21].

Furthermore let us observe in particular that if $\chi(\Sigma, \underline{\alpha}_m) \leq 0$, then $(\Sigma, \underline{\alpha}_m)$ is subcritical. Therefore, as far as we are concerned with supercriticality, there is no loss of generality in assuming $\chi(\Sigma, \underline{\alpha}_m) > 0$. We also remark that if $\chi(\Sigma, \underline{\alpha}_m) \leq 0$ a set of much more detailed results concerning the prescribed Gaussian curvature problem are at hand, see [84].

We are also able to prove the following generic multiplicity result, where $\mathcal{M}^2$ stands for the space of all C^2 Riemannian metrics on Σ equipped with the C^2 norm.

Theorem 0.8. *[3] Let $\rho \in (8k\pi, 8(k+1)\pi) \setminus \Gamma(\underline{\alpha}_m)$. Then, under the hypotheses of Theorem 0.7 and (g, K) in an open and dense subset of $\mathcal{M}^2 \times C^{0,1}(\Sigma)$, J_ρ admits at least $\binom{k+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(k+\mathfrak{g}-1)!}{k!(\mathfrak{g}-1)!}$ critical points, where $\mathfrak{g}$ is the genus of Σ.*

We prove Theorems 0.7 and 0.8 using a variational and Morse-theoretical approach, looking at topological changes in the structure of sublevels of J_ρ. For the regular case (without Dirac masses), we saw that for $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$, low sublevels are homotopically equivalent to formal barycenters of Σ of order k, here we use a related argument. Even if we do not completely characterize the topology of low sublevels, we are still able to retrieve some partial information. In particular we embed a *bouquet* of circles, $B^{\mathfrak{g}}$, in Σ which does not intersect the singular points, and we construct a global projection of Σ onto $B^{\mathfrak{g}}$. The latter map induces a projection from the barycenters of Σ onto those of $B^{\mathfrak{g}}$ and we show that the latter set embeds non-trivially into arbitrarily low sublevels of J_ρ. More precisely, we prove that low sublevels are non contractible, yielding Theorem 0.7, and that their Betti numbers are comparable to those of the barycenters of the *bouquet*, which gives Theorem 0.8.

The Q-curvature equation

By the regularity results in [85], it can be seen that the problem of finding a conformal metric of constant Q-curvature admits a variational formulation. Indeed, critical points of the following functional

$$II(u) = \langle P_g u, u\rangle + 4\int_M Q_g u dV_g - k_P \log \int_M e^{4u} dV_g; \qquad u \in H^2(M), \tag{21}$$

which are weak solutions of (#), are also strong solutions. Here, for u, $v \in H^2(M)$, the symbol $\langle P_g u, v\rangle$ stands for

$$\langle P_g u, v\rangle = \int_M \left(\triangle_g u \triangle_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v - 2(Ric_g \nabla_g u, \nabla_g v) \right) dV_g. \tag{22}$$

The existence of a solution to (#) was proved in [14] under the assumptions $\mathrm{P_g} \geq 0$ and $k_P < 8\pi^2$, which are naively the counterpart of $\rho < 8\pi$ for $(*)_\rho$. Also in this case there is a variant of the Moser-Trudinger inequality, the Adams inequality, which makes the problem coercive.

In [38] an extension of this result was obtained for a large class of manifolds, indeed Djadli and Malchiodi only assumed that $k_P \neq 8k\pi^2$, $k \in \mathbb{N}$, and that $\mathrm{P_g}$ has no kernel. The proof relies on a direct min-max method based on the study of the topology of the sublevels of the associated Euler functional II, on some improvement of the Adams inequality and on some compactness results in [64, 41], which are the equivalent of the result by Li and Shafrir [53] for the mean field equation.

Thanks to the boundedness of solutions it is possible to define the Leray-Schauder degree of equation (#) and the following counting formula was obtained in [64]. Let (M,g) be a compact four-manifold such that the Paneitz operator $\mathrm{P_g}$ has $\bar{k}$ negative eigenvalues and only trivial kernel (the constant functions) and such that $k_P := \int_M Q_g dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, for some $k \in \mathbb{N}$. Then the degree of (#) is given by

$$\mathrm{d}(k,\bar{k},\chi(M)) = \begin{cases} (-1)^{\bar{k}} & \text{if } k_P < 8\pi^2; \\ (-1)^{\bar{k}} \frac{(k-\chi(M))\dots(2-\chi(M))(1-\chi(M))}{k!} & \text{if } k_P \in (8k\pi^2, 8(k+1)\pi^2),\ k \geq 1. \end{cases} \tag{23}$$

Notice that under these hypotheses, since formula (7) implies that $\chi(M) \geq 2k$, the degree is always positive.

Concerning the Q-curvature equation, again applying Morse inequalities, we can prove the following multiplicity result.

Theorem 0.9. *[3] Let (M,g) be a compact four-manifold such that the Paneitz operator $\mathrm{P_g}$ has $\bar{k}$ negative eigenvalues and only trivial kernel (the constant functions) and such that $k_P := \int_M Q_g dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, for some $k \in \mathbb{N}^*$. If in addition all the solutions of (#) are non degenerate, then*

$$\#\{\textit{solutions of } (\#)\} \geq \begin{cases} p_k & \textit{if } \chi(M) = 2, \\ p_k + \sum_{r=0}^{k-1} \binom{k-r+\chi(\Sigma)-3}{k-r} p_r & \textit{if } \chi(M) \geq 3, \end{cases} \tag{24}$$

where $p_0 = 1$, $p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j$ for any $m \in \mathbb{N}^*$.

Since, as already pointed out, $2k \leq \chi(M)$, the Euler characteristic of M is always greater or equal than 2 for any $k \geq 1$. Therefore the statement above takes into account all the possible situations which can occur with $k_P \in (8k\pi^2, 8(k+1)\pi^2)$, $k \in \mathbb{N}^*$.

Although in the case of four-manifolds there is no any classification result in terms of the Euler characteristic, the latter result permits to improve the degree estimate, as specified in the following Corollary.

Corollary 0.10. *[3] For any (M,g) satisfying the hypotheses of Theorem 0.9 with $k_P := \int_M Q_g dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, then, except for $\chi(M) = 2$ and $k = 1$,*

$$\#\{\textit{solutions of } (\#)\} > |\,\mathrm{d}_P(k,\bar{k},\chi(M))| > 0.$$

When $\chi(M) = 2$ and $k = 1$ the righ–hand side of formula (10) is just equal to $1 = |\,\mathrm{d}_P(1,\bar{k},2)|$ for any $\bar{k}$.

Actually, exactly as in Corollary 0.2, it can also be proved that under these hypotheses the number of solutions of ($\#$) for k large enough can be estimated from below by the right–hand side of formula (11). But in fact this is not as relevant as for equation $(*)_\rho$ because now k and $\chi(M)$ are related by (7) and so it is not possible to fix M and let k tend to $+\infty$.

Notation

We want to stress that I_ρ (respectively J_ρ) depends on g and h (respectively g and K) and sometimes to emphasize this dependence and to avoid any ambiguity we write $I_{\rho,(g,h)}$ for I_ρ and $J_{\rho,(g,K)}$ for J_ρ.

As already specified we set $\bar{H}^1_g(\Sigma) := \left\{u \in H^1_g(\ \Sigma)\ :\ \fint_\Sigma u dV_g\right\}$ and for the average of a function we fix the following notation $\bar{u} := \fint_\Sigma u dV_g$.

Throughout the thesis the symbol $B_r(p)$ denotes the metric ball of radius r and center p. The genus of Σ is denoted as $\mathfrak{g}(\Sigma)$ or simply $\mathfrak{g}$ and, given two sets A and B, $\mathrm{d}(A, B)$ stands for the distance between them.

For any manifold M, let M_k denote the set of formal barycenters of order k supported in M, namely

$$M_k = \left\{ \sum_{i=0}^{k} t_i \delta_{x_i}\ :\ \sum_{i=0}^{k} t_i = 1,\ t_i \geq 0,\ x_i \in M \right\} \tag{25}$$

endowed with the weak topology of distributions.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula.

Chapter 1

Preliminaries

1.1 Analytical preliminaries

In this section we collect some facts needed in order to obtain our results. First of all we consider some improvements of the Moser-Trudinger inequality which are useful to study the topological structure of the low sublevels of I_ρ. Then we state a deformation lemma, proved in [61], and a compactness property of the set of solutions to $(*)_\rho$, derived in [52]. These last results, for $\rho \neq 8k\pi$, allow us to overcome the possible failure of the Palais Smale condition and to get a counterpart of the classical deformation lemma. Finally we present a result dealing with the topology of high sublevels of I_ρ, which leads directly to the existence of a solution to $(*)_\rho$ in correspondence to regular values of ρ.

1.1.1 The Moser-Trudinger inequality

First of all we recall the well-known Moser-Trudinger inequality on compact surfaces which can be found in [44].

Lemma 1.1 (*Moser-Trudinger inequality*)**.** *There exists a constant C, depending only on (Σ, g) such that for all $u \in H_g^1(\Sigma)$*

$$\int_\Sigma e^{\frac{4\pi(u-\bar{u})^2}{\int_\Sigma |\nabla_g u|^2 dV_g}} dV_g \leq C. \tag{1.1}$$

As a consequence one has that for any $p \geq 0$ and for all $u \in H_g^1(\Sigma)$

$$\log \int_\Sigma e^{p(u-\bar{u})} dV_g \leq \frac{p^2}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C. \tag{1.2}$$

Chen and Li [25] showed from this result that if e^u has integral controlled from below (in terms of $\int_\Sigma e^u dV_g$) into $(l+1)$ distinct regions of Σ, the constant $\frac{1}{16\pi}$ can be basically divided by $(l+1)$, in the sense specified in the following result.

Lemma 1.2. *[25] Let δ_0, γ be positive real numbers, and for a fixed integer l, let $\Omega_1, \dots \Omega_{l+1}$ be subsets of Σ satisfying $\mathrm{d}(\Omega_i, \Omega_j) \geq \delta_0$, for $i \neq j$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(l, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq C + \frac{1}{16(l+1)\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2 dV_g$$

for all the functions $u \in H^1_g(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0 \qquad \text{for every } i \in \{1, \dots, l+1\}. \tag{1.3}$$

Proof. Following [65] we present a modification of the argument in [25], avoiding the use of truncations: this approach has the advantage of being useful for extensions to the case of higher dimensions operators, as the Paneitz operator, see [38].

Assuming without loss of generality that $\bar{u} = 0$, we can find $l+1$ functions $g_1, \dots, g_{l+1}$ satisfying the following properties

$$\begin{cases} g_i(x) \in [0,1], & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \Omega_i,\ i = 1, \dots, l+1; \\ g_i(x) = 0, & \text{if } \mathrm{d}(x, \Omega_i) \geq \frac{\delta_0}{4}; \\ \|g_i\|_{C^4(\Sigma)} \leq C_{\delta_0}, & \end{cases}$$

where C_{δ_0} is a positive constant (depending only on δ_0. By interpolation, see [58], for any $\varepsilon > 0$ there exists C_{ε,δ_0}depending only on ε and δ_0) such that, for any $v \in H^1_g(\Sigma)$ and for any $i \in \{1, \dots, l+1\}$ there holds

$$\int_\Sigma |\nabla_g (g_i v)|^2 dV_g \leq \int_\Sigma g_i^2 |\nabla_g v|^2 dV_g + \varepsilon \int_\Sigma |\nabla_g v|^2 dV_g + C_{\varepsilon,\delta_0} \int_\Sigma v^2 dV_g. \tag{1.4}$$

If we write u as $u = u_1 + u_2$ with $u_1 \in L^\infty(\Sigma)$, then for our assumptions we deduce

$$\int_{\Omega_i} e^{u_2} dV_g \geq e^{-\|u_1\|_{L^\infty(\Sigma)}} \gamma_0 \int_\Sigma e^u dV_g; \qquad i = 1, \dots, l+1. \tag{1.5}$$

Using the properties of g_i, (1.5) and then (1.2) with $p = 1$ we obtain:

$$\begin{aligned} \log \int_\Sigma e^u dV_g &\leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Sigma)} + \log \int_\Sigma e^{g_i u_2} dV_g + C \\ &\leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Sigma)} + C + \frac{1}{16\pi} \int_\Sigma |\nabla_g (g_i u_2)|^2 dV_g + \overline{g_i u_2}, \end{aligned}$$

where C depends only on Σ. We now choose i such that $\int_\Sigma |\nabla_g (g_i u_2)|^2 dV_g \leq \int_\Sigma |\nabla_g (g_j u_2)|^2 dV_g$ for every $j \in \{1, \dots, l+1\}$. Since the functions $g_1, \dots, g_{l+1}$ have disjoint supports, the last formula and (1.4) imply

$$\begin{aligned} \log \int_\Sigma e^u dV_g \leq \log \frac{1}{\gamma_0} + \|u_1\|_{L^\infty(\Sigma)} + C + \left(\frac{1}{16(l+1)\pi} + \varepsilon \right) \int_\Sigma |\nabla_g u_2|^2 dV_g \\ + C_{\varepsilon,\delta_0} \int_\Sigma u_2^2 dV_g + \overline{g_i u_2}. \end{aligned}$$

Next we choose $\lambda_{\varepsilon,\delta_0}$ to be an eigenvalue of $-\triangle_g$ such that $\frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} < \varepsilon$, where C_{ε,δ_0} is given in the last formula, and we set

$$u_1 = P_{V_{\varepsilon,\delta_0}} u; \qquad u_2 = P_{V_{\varepsilon,\delta_0}^\perp} u.$$

Here V_{ε,δ_0} is the direct sum of the eigenspaces of $-\triangle_g$ with eigenvalues less or equal to $\lambda_{\varepsilon,\delta_0}$, and $P_{V_{\varepsilon,\delta_0}}$, $P_{V_{\varepsilon,\delta_0}^\perp}$ denote the projections onto V_{ε,δ_0} and $V_{\varepsilon,\delta_0}^\perp$ respectively. Since $\bar{u} = 0$, the L^2-norm and the L^∞-norm on V_{ε,δ_0} are equivalent (with a proportionality factor which depends on ε and δ_0), and hence by our choice of u_1 and u_2 there holds

$$\|u_1\|^2_{L^\infty(\Sigma)} \leq \hat{C}_{\varepsilon,\delta_0} \int_\Sigma |\nabla_g u_1|^2 dV_g;$$

$$C_{\varepsilon,\delta_0} \int_\Sigma u_2^2 dV_g \leq \frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} \int_\Sigma |\nabla_g u_2|^2 dV_g < \varepsilon \int_\Sigma |\nabla_g u_2|^2 dV_g,$$

where $\hat{C}_{\varepsilon,\delta_0}$ depends on ε and δ_0. Furthermore, by the Poincaré inequality (recall that $\bar{u} = 0$), we have

$$\overline{g_i u_2} \leq C \|u_2\|_{L^2(\Sigma)} \leq C \|u\|_{L^2(\Sigma)} \leq C \int_\Sigma |\nabla_g u|^2 dV_g^{\frac{1}{2}}.$$

Hence the last formulas imply

$$\begin{aligned}
\log \int_\Sigma e^u dV_g \leq\ & \log \frac{1}{\gamma_0} + \hat{C}_{\varepsilon,\delta_0} \left(\int_\Sigma |\nabla_g u_1|^2 dV_g \right)^{\frac{1}{2}} + C + \\
& + \left(\frac{1}{16(l+1)\pi} + \varepsilon \right) \int_\Sigma |\nabla_g u_2|^2 dV_g + \\
& + \varepsilon \int_\Sigma |\nabla_g u_2|^2 dV_g + C \left(\int_\Sigma |\nabla_g u_2|^2 dV_g \right)^{\frac{1}{2}} \\
\leq\ & \left(\frac{1}{16(l+1)\pi} + 3\varepsilon \right) \int_\Sigma |\nabla_g u|^2 dV_g + \bar{C}_{\varepsilon,\delta_0} + C + \log \frac{1}{\gamma_0},
\end{aligned}$$

where $\bar{C}_{\varepsilon,\delta_0}$ depends only on ε and δ_0 (and l, which is fixed). This concludes the proof. ■

In the next lemma we show a criterion which gives sufficient conditions for (1.3) to hold.

Lemma 1.3. *Let l be a given positive integer, and suppose that ε and r are positive numbers. Suppose that for a non-negative function $f \in L^1(\Sigma)$ with $\|f\|_{L^1(\Sigma)} = 1$ there holds*

$$\int_{\cup_{i=1}^l B_r(p_i)} f dV_g < 1 - \varepsilon \qquad \textit{for any } l\textit{-tuple } p_1, \dots, p_l \in \Sigma.$$

Then there exist $\bar{\varepsilon} > 0$ and $\bar{r} > 0$, depending only on ε, r, l and Σ (but not on f), and $l + 1$ points $\bar{p}_1, \dots, \bar{p}_{l+1} \in \Sigma$ (which depend on f) satisfying

$$\int_{B_{\bar{r}}(\bar{p}_1)} f dV_g \geq \bar{\varepsilon}, \dots, \int_{B_{\bar{r}}(\bar{p}_{l+1})} f dV_g \geq \bar{\varepsilon}; \qquad B_{2\bar{r}}(\bar{p}_i) \cap B_{2\bar{r}}(\bar{p}_j) = \emptyset \quad \textit{for } i \neq j.$$

PROOF. Suppose by contradiction that for every $\bar{\varepsilon}, \bar{r} > 0$ there is f satisfying the assumptions and such that for every $(l+1)$-tuple of points $p_1, \dots, p_{l+1}$ in Σ we have the implication

$$\int_{B_{\bar{r}}(p_1)} f dV_g \geq \bar{\varepsilon}, \dots, \int_{B_{\bar{r}}(p_{l+1})} f dV_g \geq \bar{\varepsilon} \quad \Rightarrow \quad B_{2\bar{r}}(p_i) \cap B_{2\bar{r}}(p_j) \neq \emptyset \quad \textit{for some } i \neq j. \tag{1.6}$$

We let $\bar{r} = \frac{r}{8}$, where r is given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \dots, x_h \in \Sigma$ such that Σ is covered by $\cup_{i=1}^h B_{\bar{r}}(x_i)$. For ε given in the statement of the Lemma, we also set $\bar{\varepsilon} = \frac{\varepsilon}{2h}$. We point out that the choice of $\bar{r}$ and $\bar{\varepsilon}$ depends on r, ε, l and Σ only, as required.

Let $\{\tilde{x}_1, \dots, \tilde{x}_j\} \subset \{x_1, \dots, x_h\}$ be the points for which $\int_{B_{\bar{r}}(\tilde{x}_i)} f dV_g \leq \bar{\varepsilon}$. We define $\tilde{x}_{j_1} = \tilde{x}_1$, and let A_1 denote the set

$$A_1 = \{\cup_i B_{\bar{r}}(\tilde{x}_i) \, : \, B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_1}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_1}).$$

If there exists $\tilde{x}_{j_2}$ such that $B_{2\bar{r}}(\tilde{x}_{j_2}) \cap B_{2\bar{r}}(\tilde{x}_{j_1}) = \emptyset$, we define

$$A_2 = \{\cup_i B_{\bar{r}}(\tilde{x}_i) \, : \, B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_2}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we define recursively some points $\tilde{x}_{j_3}, \tilde{x}_{j_4}, \dots, \tilde{x}_{j_s}$ satisfying

$$B_{2\bar{r}}(\tilde{x}_{j_s}) \cap B_{2\bar{r}}(\tilde{x}_{j_a}) = \emptyset \qquad \textit{for any } a \textit{ such that } 1 \leq a < s,$$

and some sets $A_3, \dots, A_s$ by

$$A_s = \{\cup_i B_{\bar{r}}(\tilde{x}_i) \, : \, B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_s}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_s}).$$

By (1.6) the process cannot go further than $\tilde{x}_{j_l}$, and hence $s \leq l$. Using the definition of $\bar{r}$ we obtain

$$\cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i) \subseteq \cup_{i=1}^s A_i \subseteq \cup_{i=1}^s B_{4\bar{r}}(\tilde{x}_{j_i}) \subseteq \cup_{i=1}^s B_r(\tilde{x}_{j_i}). \tag{1.7}$$

Then by our choice of h, $\bar{\varepsilon}$, $\{\tilde{x}_1, \dots, \tilde{x}_j\}$ and by (1.7) there holds

$$\int_{\Sigma \setminus \cup_{i=1}^s B_r(\tilde{x}_{j_i})} f dV_g \leq \int_{\Sigma \setminus \cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i)} f dV_g \leq \int_{(\cup_{i=1}^h B_{\bar{r}}(x_i)) \setminus (\cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i))} f dV_g$$
$$< (h - j)\bar{\varepsilon} \leq \frac{\varepsilon}{2}.$$

Finally, if we chose $p_i = \tilde{x}_{j_i}$, for $i = 1 \dots, s$ and $p_i = \tilde{x}_{j_s}$ for $i = s+1, \dots, l$, we get a contradiction to the assumptions of the lemma. ■

Combining the previous results we obtain that, if $\rho \in (8k\pi, 8(k+1)\pi)$ and $l \geq k$, the functional I_ρ is uniformly bounded below. Therefore, if $I_\rho(u)$ attains large negative values, the measure $\frac{e^u}{\int_\Sigma e^u dV_g}$ has to concentrate near at most k points in the sense specified by the following lemma.

Lemma 1.4. *[36] If $\rho \in (8k\pi, 8(k+1)\pi)$ with $k \geq 1$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that, for every $u \in H^1_g(\Sigma)$ with $I_\rho(u) \leq -L$, there exist k points $p_{1,u}, \dots, p_{k,u} \in \Sigma$ such that*

$$\frac{\int_{\Sigma \setminus \cup_{i=1}^k B_r(p_{i,u})} he^u dV_g}{\int_\Sigma he^u dV_g} < \varepsilon. \tag{1.8}$$

Proof. Suppose by contradiction that the statement is not true, namely that there exist ε, $r > 0$ and $(u_n)_n \subset H^1_g(\Sigma)$ with $I_\rho(u_n) \to -\infty$ and such that for every k-tuple $p_1, \dots, p_k$ in Σ there holds $\int_{\cup_{i=1}^k B_r(p_{i,u})} he^{u_n} dV_g < 1 - \varepsilon$. Recall that without loss of generality, since I_ρ is invariant under translation by constants of the argument, we can assume that for every n there holds $\int_\Sigma he^{u_n} dV_g = 1$. Then we can apply Lemma 1.3 with $l = k$, $f = he^{u_n}$, and in turn Lemma 1.2 with $\delta_0 = 2\bar{r}$, $\Omega_1 = B_{\bar{r}}(\bar{p}_1), \dots, \Omega_{k+1} = B_{\bar{r}}(\bar{p}_{k+1})$ and $\gamma_0 = \bar{\varepsilon}$, where $\bar{\varepsilon}$, $\bar{r}$ and $(\bar{p}_i)_i$ are given by Lemma 1.4. This implies that for a given $\tilde{\varepsilon} > 0$ there exists $C > 0$ depending only on ε, $\tilde{\varepsilon}$ and r such that

$$I_\rho(u_n) \geq \frac{1}{2}\int_\Sigma |\nabla_g u_n|^2 dV_g + \rho \int_\Sigma u_n dV_g - C\rho - \frac{\rho}{8(k+1)\pi - \tilde{\varepsilon}} \frac{1}{2} \int_\Sigma |\nabla_g u_n|^2 dV_g - \rho \bar{u}_n,$$

where C is independent of n. Since $\rho < 8(k+1)\pi$, we can choose $\tilde{\varepsilon} > 0$ so small that $1 - \frac{\rho}{8(k+1)\pi - \tilde{\varepsilon}} := \delta > 0$. Hence using also the Poincaré inequality we deduce

$$\begin{aligned} I_\rho(u_n) &\geq \delta \int_\Sigma |\nabla_g u_n|^2 dV_g + \rho \int_\Sigma (u_n - \bar{u}_n) dV_g - C\rho \geq \\ &\geq \delta \int_\Sigma |\nabla_g u_n|^2 dV_g - C(\frac{1}{2}\int_\Sigma |\nabla_g u_n|^2 dV_g)^{\frac{1}{2}} - C\rho \geq -C. \end{aligned}$$

This violates our contradiction assumption, and concludes the proof. ■

1.1.2 The structure of low sublevels

Lemma 1.4 implies that the unit measure $\frac{he^u}{\int_\Sigma he^u}$ resembles a finite linear combination of Dirac deltas with at most k elements, and hence $\frac{he^u}{\int_\Sigma he^u} \simeq \sum_{i=1}^k t_i \delta_{x_i} = \sigma$, where $t_i \geq 0$, $x_i \in \Sigma$ for every $i \in \{1, \dots, k\}$, $\sum_{i=1}^k t_i = 1$ and where δ_{x_i} stands for the Dirac mass at x_i. Therefore $\frac{he^u}{\int_\Sigma he^u}$ is *close* to some formal barycenter $\sigma \in \Sigma_k$. It was indeed shown in [38] (see also Section 4 in [65] for the specific case of I_ρ) that it is possible to define a continuous and non trivial map Ψ from low sublevels of I_ρ into Σ_k, in the sense specified by Proposition 1.8 below. To state it, we need to introduce the following family of test functions.

For $\delta > 0$ small, let us consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties

$$\begin{cases} \chi_\delta(t) = t, & \text{for } t \in [0, \delta] \\ \chi_\delta(t) = 2\delta & \text{for } t \geq 2\delta \\ \chi_\delta(t) \in [\delta, 2\delta], & \text{for } t \in [\delta, 2\delta]. \\ \|\chi_\delta(t) - t\|_\infty = o(\delta^4), & \text{for } t \in [\delta, 2\delta]. \end{cases}$$

Then given $\sigma \in \Sigma_k$, $\sigma = \sum_{i=1}^k t_i \delta_{x_i}$ ($\sum_{i=1}^k t_i = 1$) and $\lambda > 0$, we define $\varphi_{\lambda,\sigma} : \Sigma \to \mathbb{R}$ by

$$\varphi_{\lambda,\sigma}(y) = \log \sum_{i=1}^{k} t_i \left(\frac{\lambda}{1 + \lambda^2 \chi_\delta^2(\mathrm{d}_i(y))^2} \right) - \log(\pi), \tag{1.9}$$

where we have set

$$\mathrm{d}_i(y) = \mathrm{d}(y, x_i), \qquad x_i, y \in \Sigma.$$

We point out that, since the distance is a Lipschitz function, $\varphi_{\lambda,\sigma}(y)$ is also Lipschitz in y, and hence it belongs to $H_g^1(\Sigma)$. We have then the following result.

Proposition 1.5. *[37] Let $\varphi_{\lambda,\sigma}$ be defined as in* (1.9)*. Then, as $\lambda \to +\infty$ the following properties hold true*

(i) $e^{\varphi_{\lambda,\sigma}} \rightharpoonup \sigma$ weakly in the sense of distributions;

(ii) $I_\rho(\varphi_{\lambda,\sigma}) \to -\infty$ uniformly for $\sigma \in \Sigma_k$.

Proof. To prove (i) we first consider the function

$$\tilde{\varphi}_{\lambda,x}(y) = \left(\frac{\lambda}{1 + \lambda^2 \chi_\delta^2(\mathrm{d}(x,y))^2} \right)^2, \qquad y \in \Sigma,$$

where x is a fixed element of Σ. It is easy to show that $\tilde{\varphi}_{\lambda,x} \to \pi\delta_x$ as $\lambda \to +\infty$. Then (i) follows immediately from the explicit expression of $\varphi_{\lambda,\sigma}$.

In order to prove (ii), we evaluate separately each term of I_ρ, and claim that the following estimates hold

$$\rho \int_\Sigma \varphi_{\lambda,\sigma} dV_g = -2(\rho + o_\lambda(1)) \log \lambda \qquad (o_\lambda(1) \to 0 \text{ as } \lambda \to +\infty); \tag{1.10}$$

$$\log \int_\Sigma h\, e^{\varphi_{\lambda,\sigma}} dV_g = O(1) \qquad (\text{as } \lambda \to +\infty); \tag{1.11}$$

$$\frac{1}{2} \int_\Sigma |\nabla_g u|^2 \, dV_g \leq 16k\pi(1 + o_\lambda(1)) \log \lambda \qquad (\text{as } \lambda \to +\infty). \tag{1.12}$$

If these estimates are proved, then (ii) follows immediately.

Proof of (1.10). We have

$$\varphi_{\lambda,\sigma}(y) = \log \frac{\lambda^2}{(1+4\lambda^2\delta^2)^2} - \log \pi, \qquad \text{for } y \in \Sigma \setminus \cup_i B_{2\delta}(x_i),$$

and

$$\log \lambda^2 (1+4\lambda^2\delta^2)^2 - \log \pi \le \varphi_{\lambda,\sigma}(y) \le \log \lambda^2 - \log \pi, \qquad \text{for } y \in \cup_i B_{2\delta}(x_i).$$

Writing

$$\begin{aligned} \int_\Sigma \varphi_{\lambda,\sigma}(y) dV_g(y) &= \log \frac{\lambda^2}{(1+4\lambda^2\delta^2)^2} \int_\Sigma dV_g \\ &+ \int_\Sigma \left(\varphi_{\lambda,\sigma}(y) - \log \frac{\lambda^2}{(1+4\lambda^2\delta^2)^2} \right) dV_g(y), \end{aligned}$$

from the last three formulas it follows that

$$\int_\Sigma \varphi_{\lambda,\sigma}(y) dV_g(y) = \log \lambda^2 (1+4\lambda^2\delta^2)^2 + O(\delta^2 \log(1+4\lambda^2\delta^2)) + O(1).$$

And this implies immediately (1.10).

Proof of (1.11). By the definition of $\varphi_{\lambda,\sigma}$, there holds

$$\frac{\min_\Sigma h}{\pi} \sum_{i=1}^k \int_\Sigma \frac{\lambda^2}{(1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^2} dV_g(y) \le \int_\Sigma h(y) e^{\varphi_{\lambda,\sigma}(y)} dV_g(y)$$

and

$$\int_\Sigma h(y) e^{\varphi_{\lambda,\sigma}(y)} dV_g(y) \le \frac{\max_\Sigma h}{\pi} \sum_{i=1}^k t_i \int_\Sigma \frac{\lambda^2}{(1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^2} dV_g(y).$$

We divide each of the above integrals into the metric ball $B_\delta(x_i)$ and its complement. By construction of χ_δ, working in normal coordinates centered at x_i, we have (for δ sufficiently small)

$$\begin{aligned} &\int_{B_\delta(x_i)} \int_\Sigma \frac{\lambda^2}{(1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^2} dV_g(y) = \int_{B_\delta^{\mathbb{R}^2}(0)} (1+O(\delta)) \frac{\lambda^2}{(1+\lambda^2|y|^2)^2} dy \\ &= \int_{B_{\lambda\delta}^{\mathbb{R}^2}(0)} (1+O(\delta)) \frac{1}{(1+|y|^2)^2} dy = (1+O(\delta)) \left(\pi + O(\frac{1}{\lambda^2\delta^2}) \right). \end{aligned}$$

On the other hand, for $\mathrm{d}(y,x_i) \ge \delta$ there holds

$$\frac{\lambda^2}{(1+4\lambda^2\delta^2)^2} \le \frac{\lambda^2}{(1+\lambda^2\chi_\delta^2(\mathrm{d}(y,x_i)))^2} \le \frac{\lambda^2}{(1+\lambda^2\delta^2)^2}.$$

From these two formulas we deduce

$$\int_\Sigma h(y) e^{\varphi_{\lambda,x}(y)} dV_g(y) \geq \pi \min_\Sigma h + O(\delta) + O(\frac{1}{\lambda^2\delta^2}) + O(\frac{\lambda^2}{(1+\lambda^2\delta^2)^2}), \tag{1.13}$$

and

$$\int_\Sigma h(y) e^{\varphi_{\lambda,x}(y)} dV_g(y) \leq \pi \max_\Sigma h + O(\delta) + O(\frac{1}{\lambda^2\delta^2}) + O(\frac{\lambda^2}{(1+\lambda^2\delta^2)^2}). \tag{1.14}$$

Finally (1.11) follows immediately from (1.13) and (1.14).

Proof of (1.12). To prove this inequality we need to show two pointwise estimates on the gradient of $\varphi_{\lambda,\sigma}$

$$|\nabla_g \varphi_{\lambda,\sigma}(y)| \leq C\lambda, \qquad \text{for every } y \in \Sigma, \tag{1.15}$$

where C is a constant independent of σ and λ, and

$$|\nabla_g \varphi_{\lambda,\sigma}(y)| \leq \frac{4}{\chi_\delta(\mathrm{d}_{\min}(y))} \qquad \text{where } \mathrm{d}_{\min}(y) = \min_{i=1,\dots,m} \{\mathrm{d}(y, x_i)\}. \tag{1.16}$$

For proving (1.15) we notice that the following inequality holds

$$\frac{\lambda^2 \chi_\delta(\mathrm{d}(y,x_i))}{1+\lambda^2\chi_\delta^2(\mathrm{d}(y,x_i))} \leq C\lambda, \qquad i = 1, \dots, m, \tag{1.17}$$

where C is fixed constant (independent of λ and x_i). Moreover we have

$$\nabla_g \varphi_{\lambda,\sigma}(y) = -2\lambda^2 \frac{\sum_i t_i (1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^{-3} \nabla_y(\chi_\delta^2(\mathrm{d}_i(y)))}{\sum_j t_j (1+\lambda^2\chi_\delta^2(\mathrm{d}_j(y)))^{-2}}. \tag{1.18}$$

Inserting (1.17) into (1.18) we obtain immediately (1.15). Similarly we find

$$\begin{aligned}
|\nabla \varphi_{\lambda,\sigma}(y)| &\leq 4\lambda^2 \frac{\sum_i t_i (1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^{-3} \chi_\delta(\mathrm{d}_i(y))}{\sum_j t_j (1+\lambda^2\chi_\delta^2(\mathrm{d}_j(y)))^{-2}} \\
&\leq 4\lambda^2 \frac{\sum_i t_i (1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^{-2} \frac{\chi_\delta(\mathrm{d}_i(y))}{1+\lambda^2\chi_\delta^2}(\mathrm{d}_i(y))}{\sum_j t_j (1+\lambda^2\chi_\delta^2(\mathrm{d}_j(y)))^{-2}} \\
&\leq 4 \frac{\sum_i t_i (1+\lambda^2\chi_\delta^2(\mathrm{d}_i(y)))^{-2} \frac{1}{\chi_\delta(\mathrm{d}_{\min}(y))}}{\sum_j t_j (1+\lambda^2\chi_\delta^2(\mathrm{d}_j(y)))^{-2}} \leq \frac{4}{\chi_\delta(\mathrm{d}_{\min}(y))},
\end{aligned}$$

which is (1.16).

From (1.15) we then deduce that

$$\int_{\cup_{i=1}^k B_{\frac{1}{\lambda}}(x_i)} |\nabla_g \varphi_{\lambda,\sigma}|^2 \, dV_g \leq Ck \tag{1.19}$$

for some fixed C depending only on Σ. We define now the sets

$$A_i = \left\{ y \in \Sigma \,:\, \mathrm{d}(y, x_i) = \min_{j=1}^{k} \{\mathrm{d}(y, x_j)\} \right\}.$$

Then we have that

$$\begin{aligned} \int_{\Sigma \setminus \cup_{i=1}^{k} B_{\frac{1}{\lambda}}(x_i)} |\nabla_g \varphi_{\lambda,\sigma}|^2 \, dV_g &\leq \sum_{i=1}^{k} \int_{A_i \setminus B_{\frac{1}{\lambda}}(x_i)} |\nabla_g \varphi_{\lambda,\sigma}|^2 \, dV_g \\ &\leq 16k \int_{A_i \setminus B_{\frac{1}{\lambda}}(x_i)} \frac{1}{\chi_\delta^2(\mathrm{d}(y, x_i))} dV_g \\ &\leq 32k\pi(1 + o_\delta(1) + o_\lambda(1)) \log \lambda \end{aligned}$$

as $\lambda \to +\infty$. From (1.19) and the last formula we finally deduce (1.12).

The proof is thereby concluded. ■

Remark 1.6. *The same estimates of Proposition 4.2 hold when the constant ρ in the left–hand side of $(*)_\rho$ is replaced by a smooth function over Σ.*

Thanks to Lemma 1.4 and to the previous proposition the following result (see [37] and [65]) has been proved.

Proposition 1.7. *Suppose $\rho \in (8k\pi, 8\pi(k+1))$ with $k \geq 1$. Then there exists $L > 0$ and a continuous projection $\Psi : \{I_\rho \leq -L\} \to \Sigma_k$ such that for λ large the map $\sigma \mapsto \Psi(\varphi_{\lambda,\sigma})$ is homotopically equivalent to the identity on Σ_k.*

Conversely, given $L > 0$ large, one can construct a homotopy between the identity on $\{I_\rho \leq -L\}$ and the map $u \mapsto \varphi_{\lambda,\Psi(u)}$, for λ sufficiently large (see the Appendix of [64]). The latter facts and the invariance of homology groups under homotopy equivalences imply the following result.

Proposition 1.8. *[64] If $k \geq 1$ and $\rho \in (8k\pi, 8\pi(k+1))$, then $\{I_\rho \leq -L\}$ has the same homology as Σ_k.*

For our purposes it is sufficient to present the proof of Proposition 1.7 in the case $k = 1$ and we refer to [37] for the general case. We just want to stress that for $k > 1$ much more work is required, because Σ_k is not anymore a smooth manifold but only a stratified set, namely union of open manifolds of different dimensions, whose maximal one is $3k - 1$.

Proof of Proposition 1.7 for $k = 1$. Since the functional is invariant under addition of constants to the argument, we can assume that the $H_g^1(\Sigma)$ functions we are dealing with satisfy the volume normalization $\int_\Sigma e^w dV_g = 1$. Whitney's theorem assures that it is possible to embed Σ in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. We will denote by $\Omega : \Sigma \to \mathbb{R}^m$ the diffeomorphism which realizes the embedding and by $\mathcal{M}$ the embedded surface $\Omega(\Sigma)$. First we define the map $\tilde{\Psi} : H^1_g(\Sigma) \to \mathbb{R}^m$ by $\tilde{\Psi}(u) = \int_\Sigma \Omega(x) e^{u(x)} dV_g$, whose continuity ensue from the Moser-Trudinger inequality. The key point is to prove that

$$\text{for any } \delta > 0 \text{ there exists } L_\delta > 0 \text{ such that } I_\rho(u) \leq -L_\delta \text{ implies } \mathrm{d}(\tilde{\Psi}(u), \mathcal{M}) < \delta. \tag{1.20}$$

To prove (1.20) we let $\varepsilon = \frac{\delta}{2} \frac{1}{diam(\mathcal{M})}$, $r = \frac{\delta}{2} \frac{1}{\|d\Omega\|}$ and we apply Lemma 1.4 with these values of ε and r. Then, if $I_\rho \leq -L(\varepsilon, r)$, we obtain a point p_u such that (1.8) holds. By our normalization we can write $\tilde{\Psi}(u) - \Omega(p_u) = \int_{B_r(p_u)} (\Omega(x) - \Omega(p_u)) e^{4u(x)} dV_g(x) + \int_{\Sigma \setminus B_r(p_u)} (\Omega(x) - \Omega(p_u)) e^{4u(x)} dV_g(x)$.
This implies $\left\| \tilde{\Psi}(u) - \Omega(p_u) \right\| \leq r \|d\Omega\| + \varepsilon diam(\mathcal{M}) \leq \delta$, and hence (1.20) follows. Now we fix δ sufficiently small such that there exists a continuous projection P from a δ-neighborhood of $\mathcal{M}$ into $\mathcal{M}$. Therefore it is sufficient to define $L_\delta = L(\varepsilon, r)$ and

$$\Psi(u) := \Omega^{-1} \circ P \circ \tilde{\Psi}(u) \qquad u \in \{I_\rho \leq -L\}. \tag{1.21}$$

Thanks to Proposition 1.5 the proof in the case $k = 1$ is complete. ■

Remark 1.9. It is well known that the set of formal barycenters Σ_k is not contractible since the $(3k-1)$-th homology group of Σ_k with coefficients in $\mathbb{Z}_2$ is non trivial.

In [31] the author showed that, when $\rho \in (8\pi, 16\pi)$, if $u \in \bar{H}^1_g(\Sigma)$ belongs to $\{I_\rho \leq b\}$, for some $b \in \mathbb{R}$, and e^u does not concentrate, then $\|u\|_{\bar{H}^1_g(\Sigma)}$ is bounded by a constant depending only on ρ and b and then the following lemma holds.

Lemma 1.10. *Suppose $\rho \in (8\pi, 16\pi)$. Then, given $b \in \mathbb{R}$, there exists $C_{\rho,b}$ such that it is possible to extend the map Ψ defined in Proposition 1.7 also to $\{I_\rho \leq b\} \setminus \bar{B}_{C_{\rho,b}} \subset \bar{H}^1_g(\Sigma)$.*

Proof. Considering the arguments in the proof of the aforementioned proposition, it is clear that we only need to find a constant $C_{\rho,b}$ such that for any $u \in \{I_\rho \leq b\}$ either $\|u\|_{\bar{H}^1_g(\Sigma)} \leq C_{\rho,b}$, or, given an ε opportunely fixed, there exists a point $p_u \in \Sigma$ where the function e^u concentrates, namely

$$\frac{\int_{\Sigma \setminus B_r(p_u)} e^u dV_g}{\int_\Sigma e^u dV_g} < \varepsilon.$$

Let $u \in \{I_\rho \leq b\}$ such that for any $p \in \Sigma$

$$\frac{\int_{\Sigma \setminus B_r(p_u)} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \varepsilon,$$

then Lemma 1.3 ensures the existence of two positive numbers $\bar{\varepsilon}$ and $\bar{r}$ (independent of u) and two points $\bar{p}_1$ and $\bar{p}_2$ (which instead depend on u) such that

$$\frac{\int_{B_{\bar{r}}(\bar{p}_i)} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \bar{\varepsilon} \qquad \text{for } i = 1, 2 \quad \text{and} \quad B_{2\bar{r}}(\bar{p}_1) \cap B_{2\bar{r}}(\bar{p}_2) = \emptyset.$$

So we can apply Lemma 1.2 with $\delta_0 = 2\bar{r}$, $\Omega_i = B_{\bar{r}}(\bar{p}_i)$ and $\gamma_0 = \min[\bar{\varepsilon}, \frac{1}{3}]$; in particular, choosing $\tilde{\varepsilon}$ such that $\frac{4\pi^2}{32\pi - \tilde{\varepsilon}} < \frac{1}{2}$, we obtain the existence of a constant $K = K(\varepsilon, r)$ such that

$$\log \int_\Sigma e^u dV_g \leq K + \frac{1}{32\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla u|^2 dV_g.$$

Then

$$b \geq I_\rho(u) \geq \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g - \rho K - \frac{\rho}{32\pi - \tilde{\varepsilon}} \int_T |\nabla u|^2 dV_g \geq a \|u\|^2 - \rho K,$$

where $a = \frac{1}{2} - \frac{\rho}{32\pi - \tilde{\varepsilon}} > 0$. At last, as K does not depend on u, taking $C^2_{\rho,b} := \frac{b+\rho K}{a}$ the thesis is proved. ■

1.1.3 A deformation Lemma and a Compactness Result

It is well known that, if $I \in C^1(\bar{H}^1_g(\Sigma), \mathbb{R})$ satisfies the Palais-Smale condition, a classical deformation lemma ensures that we have the following alternative: either

1. $\{I \leq a\}$ is a deformation retract of $\{I \leq b\}$ $(a < b)$, or
2. there is a critical point $\bar{u}$ for the functional I, with $a \leq I(\bar{u}) \leq b$.

This lemma, which is usually employed to derive existence of critical points, can be obtained by considering the pseudo-gradient vector field associated to I.

Unfortunately for our functional I_ρ the (PS)-condition is known to hold only for bounded sequences. Here we recall a result in [61], where is constructed a vector field which deforms suitable sublevels of the functional I_ρ, bypassing the Palais-Smale condition. In [80] it was previously used a related argument, which exploited a monotonicity property in the parameter ρ.

Below we set

$$K(u) = -\log \int_\Sigma h(x) e^u dV_g, \quad x \in \bar{H}^1_g(\Sigma),$$

so we have $I_\rho(u) = \frac{1}{2}\|u\|^2 + \rho K(u)$. The result in [61] we need is the following.

Lemma 1.11. *Given a, $b \in \mathbb{R}$, $a < b$, the following alternative holds: either*

1. $\exists (\rho_n, u_n) \subset \mathbb{R} \times \bar{H}^1_g(\Sigma)$ *satisfying*

$$I'_{\rho_n}(u_n) = 0 \text{ for every } n, \qquad a \leq I_\rho(u_n) \leq b, \qquad \rho_n \to \rho,$$

2. *or the set $\{I_\rho \leq a\}$ is a deformation retract of $\{I_\rho \leq b\}$.*

By deformation retract onto $A \subset X$ we mean a continuous map $\eta : [0,1] \times X \to X$ such that $\eta(t, u_0) = u_0$ for every $(t, u_0) \in [0,1] \times A$ and such that $\eta(1, \cdot)_{|B}$ is contained in A.

To prove the lemma, one argues as follows: assuming the second alternative false, let $\tau > 0$ be such that $I_{\tilde{\rho}}$ has no critical point $\bar{u}$ for $\tilde{\rho} \in (\rho - \tau, \rho)$, with $I_{\tilde{\rho}}(\bar{u}) \in [a, b]$. The strategy of the proof consists in constructing, under these hypotheses, a flow which deforms I_ρ^b onto a subset of I_ρ^a by keeping bounded every integral curve (with bounds depending on the initial datum, a, b and τ). To do this let Z be defined by:

$$Z(u) := -[|\nabla K(u)| \nabla I_\rho(u) + |\nabla I_\rho(u)| \nabla K(u)]. \tag{1.22}$$

Then we choose a smooth non-decreasing cut-off function $\omega_\tau : \mathbb{R} \to [0,1]$ satisfying

$$0 \le \omega_\tau \le 1, \quad \omega_\tau(\zeta) = 0 \;\; \forall \zeta \le \tau, \quad \omega_\tau(\zeta) = 1 \;\; \forall \zeta \ge 2\tau,$$

and we consider the local flow $\eta = \eta(t, u_0)$ defined by the Cauchy problem:

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, t} = -\omega_\tau \left(\frac{|\nabla I_\rho(u)|}{|\nabla K(u)|} \right) \nabla I_\rho(u) + Z(u), \qquad u(0) = u_0, \tag{1.23}$$

where $\omega_\tau \left(\frac{|\nabla I_\rho(u)|}{|\nabla K(u)|} \right)$ is understood to be equal to 1 when $\nabla K(u) = 0$. A key point is to notice that $\langle Z(u), \nabla I_\rho(u) \rangle \le 0$, and that if $\langle Z(u_k), \nabla I_\rho(u_k) \rangle$ tends to zero along some sequence $(u_k)_k$, then $\lim_{k \to \infty} \frac{Z(u_k)}{|\nabla J(u_k)|} = 0$.

This lemma is still too weak because it only guarantees that if sublevels are not homotopically equivalent, then there exists a sequence of solutions of perturbed problems. Nevertheless, if $\rho \neq 8k\pi$, as in our case, a compactness result due to Yan Yan Li, [52], comes to our rescue.

Theorem 1.12. *If $\rho \neq 8k\pi$, $k \in \mathbb{N}$, $\rho_n \to \rho$ and $(u_n)_n \subset H^1_g(\Sigma)$ is a sequence of solutions of $(*)_{\rho_n}$ such that $\int_\Sigma h e^u dV_g = 1$, then $(u_n)_n$ admits a subsequence converging in C^2 to a solution of $(*)_\rho$.*

To establish this result it is crucial a theorem of Brezis-Merle [6], and its completion given by Li-Shafrir [53], concerning the blow up of solutions to

$$-\triangle w_n = V_n(x) e^{w_n} \qquad \text{on } \Omega \subset \mathbb{R}^2.$$

In particular in [53] it is proved that in case of blow up

$$V_n e^{w_n} \rightharpoonup \sum_{i=1}^m 8\pi m_i \delta_{x_i},$$

where $m_i \in \mathbb{N}$ and $x_i \in \Omega$. A similar result holds for compact surfaces and moreover in [52] it is shown that $m_i = 1$ for any i. From these considerations Theorem 1.12 follows immediately.

So, employing together Lemma 1.11 and Theorem 1.12 (just considering the right normalization), it is immediate to establish a strong result concerning our functional I_ρ, through and through analogous to the classical aforementioned deformation lemma.

Corollary 1.13. *If $\rho \neq 8k\pi$ and if I_ρ has no critical levels inside some interval $[a,b]$, then $\{I_\rho \leq a\}$ is a deformation retract of $\{I_\rho \leq b\}$.*

It is useful to recall that in [31] the deformation Lemma has been extended to a slightly more general case, more precisely the following has been proved.

Lemma 1.14. *Consider $c \in \mathbb{R}$ and let $U \subset \bar{H}^1_g(\Sigma)$ be an open neighborhood of Z_c, possibly empty. The following alternative holds: either*

1. *$\exists \delta > 0$ such that $I_\rho^{c+\delta} \setminus U$ can be deformed in $I_\rho^{c-\delta}$ in a way that $I_\rho^{c-2\delta} \setminus U$ holds steady, or*

2. *for any $\delta > 0$ there exists $\rho_n \to \rho$, $\rho_n \leq \rho$, such that I_{ρ_n} admits a critical point $u_n \in \bar{H}^1_g(\Sigma) \setminus U$ and $c - \delta \leq I_\rho(u_n) \leq c + \delta$.*

1.1.4 The structure of high sublevels and an existence result

Since the functional I_ρ stays uniformly bounded on the solutions of $(*)_\rho$ (by Corollary 1.13), the Deformation Lemma 1.11 can be used to prove that it is possible to retract the whole Hilbert space $\bar{H}^1_g(\Sigma)$ onto a high sublevel $\{I_\rho \leq b\}$, $b \gg 0$. More precisely one has:

Proposition 1.15. *[64] If $\rho \notin 8\pi\mathbb{N}$ and if b is sufficiently large positive, the sublevel $\{I_\rho \leq b\}$ is a deformation retract of $\bar{H}^1_g(\Sigma)$, and hence is contractible.*

At last, collecting previous results, we can argue the existence of a solution of $(*)_\rho$, when $\rho \in (8k\pi, 8(k+1)\pi)$. Indeed we know that, when $b \gg 1$, $\{I_\rho \leq b\}$ is contractible, while, if $L \gg 1$, $\{I_\rho \leq -L\}$ has non trivial homology, as pointed out in Remark 1.9. Therefore this difference of topology between high and low sublevels implies that the first alternative of Corollary 1.13 cannot hold and then allows to establish a general existence result.

Theorem 1.16. *[37] If $\rho \in (8k\pi, 8(k+1)\pi)$, there exists a solution of $(*)_\rho$.*

A complete proof of the previous theorem can be found in [37] or [65], but there the approach is quite different. Indeed, in the spirit of [38], Djadli introduces a minmax scheme based on the construction on the topological cone over Σ_k, using the monotonicity trick due to Struwe to find the existence of bounded Palais-Smale sequences.

1.2 Geometrical preliminaries

This section is devoted to collect some classical and more recent results in Morse theory, which will be useful to derive multiplicity of solutions of $(*)_\rho$ (resp. $(\#)$ and (18)) from the topological structure of the sublevels of I_ρ (resp. II and J_ρ). We will also give a short review of basic notions of algebraic topology needed to study the homology groups which come out from Morse inequalities. Finally, we recall the definition of Lusternik-Schnirelman relative category stating also some results relating the category to both the cup-length and the existence of critical points.

1.2.1 Morse theoretical results

First of all, we recall a classical result in Morse theory: Morse inequalities.

Theorem 1.17. *Let N be a Hilbert manifold, $f \in C^2(N;\mathbb{R})$ be a Morse function (i.e. all critical points are non degenerate) satisfying the (PS)-condition. Let a, b $(a < b)$ be regular values for f and*

$$C_q(a,b;G) := \#\left\{\text{critical points of } f \text{ in } \{a \le f \le b\} \text{ with index } q\right\},$$

$$\beta_q(a,b;G) := \operatorname{rank}(H_q(\{f \le b\}, \{f \le a\}; G)), \ \text{where } G \text{ is an abelian group},$$

then

$$\sum_{q=0}^{n}(-1)^{n-q}C_q(a,b;G) \ge \sum_{q=0}^{n}(-1)^{n-q}\beta_q(a,b;G), \qquad n \in \mathbb{N} \quad (\text{strong ineq.})$$

$$C_q(a,b;G) \ge \beta_q(a,b;G), \qquad q \in \mathbb{N} \quad (\text{weak ineq.})$$

$$\text{and} \quad \sum_{q=0}^{\infty}(-1)^{q}C_q(a,b;G) = \sum_{q=0}^{\infty}(-1)^{q}\beta_q(a,b;G).$$

To prove the above inequalities the (PS)-condition is not necessarily needed, it only suffices that appropriate deformation lemmas for f hold true (see for example [10] Theorem 4.3, Lemma 3.2, and Theorem 3.2). Therefore this hypothesis can be replaced by the request that some proper deformation lemmas hold for f. We now want to point out that, despite the (PS)-condition is not known for I_ρ, is still possible to get Theorem 1.17 for $N = \bar{H}^1_g(\Sigma)$ and $f = I_\rho$, under the further assumption that all the critical points of I_ρ are non-degenerate.
In [64] (Proof of Theorem 1.2) Malchiodi defined a new flow $\tilde{W}$, which is nothing but the steepest descent flow in a big ball of $\bar{H}^1_g(\Sigma)$, containing all the critical points of I_ρ (such a ball exists by the compactness of the set of solutions; see Theorem 1.12), and which coincides with the flow W constructed by Lucia outside a bigger ball. More precisely:

$$\tilde{W}(u) := -\theta(u)\nabla I_\rho(u) + (1-\theta(u))W(u) \tag{1.24}$$

where $\theta : \bar{H}^1_g(\Sigma) \to [0,1]$ is a radial cutoff function satisfying

$$\theta(u) = 1 \text{ for } u \in B_R; \qquad \theta(u) = 0 \text{ for } u \in \bar{H}^1_g(\Sigma) \setminus B_{2R}.$$

By means of $\tilde{W}$ it is still possible to get the alternative of Lemma 1.11, but this flow has been defined because, unlike W, it allows to adapt to I_ρ the classical deformation lemmas needed so that Theorem 1.17 can be applied.

To sum up, if I_ρ is a Morse functional and a and b are regular values for I_ρ, then the weak and the strong Morse inequalities are verified.

Moreover from the last formula of Theorem 1.17 one can deduce the Poincaré–Hopf index theorem, which can be found in [10], pages 99-104. Here we adapt the statement to our purposes.

Proposition 1.18. *Let X be a Hilbert space and let $f : X \to \mathbb{R}$ be of class C^2. Suppose that $\nabla f(x)$ is of the form Identity-compact for every $x \in H$, and that f satisfies the Palais-Smale condition. Assume also that, for some a, $b \in \mathbb{R}$, $a < b$, $\{a \le f \le b\}$ is bounded, and that f has no critical points at the levels a, b. Then, one has*

$$\deg_{\mathrm{LS}}(\nabla f, \{a \le f \le b\}, 0) = \chi(\{a \le f \le b\}, \{f = a\}).$$

Applying this result to I_ρ with $a = -L \ll 0$ and $b \gg 0$ (verifying respectively the hypotheses of Proposition 1.8 and Proposition 1.15), Malchiodi [64] obtained a clear interpretation of the degree-formula (8), in terms of the barycenters of Σ.
Since by compactness $\{a \le I_\rho \le b\}$ contains all the solutions of $(*)_\rho$, one can compute the degree as

$$\mathrm{d}(k, \chi(\Sigma)) = \chi(\{I_\rho \le b\}, \{I_\rho \le -L\}) = \chi(\{I_\rho \le b\}) - \chi(\{I_\rho \le -L\}) = 1 - \chi(\Sigma_k).$$

The first inequality is derived excising $\{I_\rho < a\}$, while the second follows from the exactness of the homology sequence and the third from Proposition 1.8 and Proposition 1.15.
Clearly this argument is purely intuitive and heuristic but actually it can be made rigorous. Indeed, even if $\{a \le I_\rho \le b\}$ is not bounded the problem has been tackle using a generalized notion of degree, which extends the classical one; whereas the Palais-Smale condition has been bypassed thanks to the vector field $\tilde{W}$ defined in (1.24).

1.2.2 Some notions in algebraic topology

Let now recall some well known definitions and results in algebraic topology (see [47] and [10] for further details). Throughout, the sign $\simeq$ will refer to homotopy equivalences, while $\cong$ will refer to homeomorphisms between topological spaces or isomorphisms between groups. Given a pair of spaces (X, A) we will denote by $H_q(X, A; G)$ (resp. $H^q(X, A; G)$) the relative q-th homology (resp. cohomology) group and by $\tilde{H}_q(X; G) := H_q(X, x_0; G)$ (resp. $\tilde{H}^q(X; G) := H^q(X, x_0; G)$), for $x_0 \in X$, the reduced homology (resp. cohomology) with coefficients in a group G.
Sometimes we will omit A if it is the empty set and G if it is not worthwhile.
Finally, if (X, Y), (X', Y') are two topological pairs and $f : (X, Y) \to (X', Y')$ is a

continuous function, we will denote by $f^* : H^q(X', Y') \to H^q(X, Y)$, for $q \in \mathbb{N}$, the homomorphism induced by f.

First of all, we recall the Kunneth Theorem for cohomology in a particular case.

Theorem 1.19. *([10], page 8) If $(X \times Y', Y \times X')$ is an excisive couple in $X \times X'$, and $H^*(X, Y; G)$ is of finite type, i.e., $H^q(X, Y; G)$ is finitely generated for each q, and G is a field, then the map*

$$\mu \,:\, H^*(X, Y; G) \otimes H^*(X', Y'; G) \longrightarrow H^*((X, Y) \times (X', Y'); G), \tag{1.25}$$

defined as $\mu(u \otimes v) := u \times v \in H^{p+q}((X, Y) \times (X', Y'); G)$, for any $u \in H^p(X, Y; G)$ and $v \in H^q(X', Y'; G)$, is an isomorphism.

Cup product. We recall that it is possible to endow the direct sum of the cohomology groups, $H^*(X) = \bigoplus_q H^q(X)$, with an associative and graded multiplication, namely the cup product $\bigcup : H^p(X) \times H^q(X) \to H^{p+q}(X)$. This multiplication turns $H^*(X)$ into a ring; in fact it is naturally a $\mathbb{Z}$-graded ring with the integer q serving as degree and the cup product respects this grading. This definition can be extended to topological pairs; in particular, if (Y_1, Y_2) is an excisive couple in X, it is possible to define the cup product

$$\cup \,:\, H^p(X, Y_1; G) \times H^q(X, Y_2; G) \longrightarrow H^{p+q}(X, Y_1 \cup Y_2; G)$$

In de Rham cohomology the cup product of differential forms is also known as the wedge product.

Proposition 1.20. *([79], page 253) Let $(X \times Y', Y \times X')$ be an excisive couple in $X \times X'$, and let $p_1 : (X, Y) \times X' \to (X, Y)$ and $p_2 : X \times (X', Y') \to (X', Y')$ be the projections. Given $u \in H^p(X, Y; G)$ and $v \in H^q(X', Y'; G)$, if G is a field, then in $H^{p+q}((X, Y) \times (X', Y'); G)$ we have*

$$u \times v = p_1^*(u) \cup p_2^*(v).$$

Cup–length. A numerical invariant derived from the cohomology ring is the cup-length, which for a topological space X is defined as follows:

$$\begin{aligned} \mathrm{CL}(X) = \max\{\, l \in \mathbb{N} \,|\, \exists\, c_1, \ldots, c_l \in H^*(X), \text{ with } \dim(c_i) > 0,\ i = 1, 2, \ldots, l, \\ \text{such that } c_1 \cup \ldots \cup c_l \neq 0\}. \end{aligned}$$

For example the cup–length of the 2-torus is equal to 2; too see it one can think to the volume form in de-Rham cohomology.

More generally, we define the cup length for a topological pair (X, Y).

$$\mathrm{CL}(X,Y) = \max\{\ l \in \mathbb{N} \mid \exists\, c_0 \in H^*(X,Y),\ \exists\, c_1, \ldots, c_l \in H^*(X),\ \text{with } \dim(c_i) > 0 \\ \text{for } i = 1, 2, \ldots, l, \text{ such that } c_0 \cup c_1 \cup \ldots \cup c_l \neq 0\}.$$

In the case where $Y = \emptyset$, we just take $c_0 \in H^0(X)$; thus the two definitions are the same.

Join. The join of two spaces X and Y is the space of all segments "joining points" in X to points in Y. It is denoted by $X * Y$ and is the identification space

$$X*Y := X\times[0,1]\times Y/(x,0,y) \sim (x',0,y), (x,1,y) \sim (x,1,y')\ \ \forall\, x,\, x' \in X, \forall\, y,\, y' \in Y.$$

Wedge sum. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ obtained by identifying x_0 and y_0 to a single point. If $\{x_0\}$ (resp. $\{y_0\}$) is a closed subspace of X (resp. Y) that is a deformation retract of some neighborhood in X (resp. Y), then $\tilde{H}_q(X \vee Y) \cong \tilde{H}_q(X) \bigoplus \tilde{H}_q(Y)$, provided that the wedge sum is formed at basepoints x_0 and y_0.

Smash Product. Inside a product space $X \times Y$ there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$. These two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The smash product $X \bigwedge Y$ is then defined to be the quotient $X \times Y/X \vee Y$. For example $S^n \wedge S^m \cong S^{n+m}$.

Suspension. The k-fold (*unreduced*) suspension of X is defined to be $S^{k-1} * X$, while the k-fold *reduced* suspension is the smash product $S^k \wedge X$. A useful property of the reduced suspension is that, for any $q,\, n \geq 0$, $\tilde{H}_q(X) \cong \tilde{H}_{q+n}(S^n \wedge X)$. It is crucial to notice that reduced and unreduced constructions are homotopically equivalent constructions for the spaces we will deal with. In the following we will often use the latter property for replacing in some results of [48] the unreduced suspension by the reduced one.

Reduced symmetric product. We denote by $\overline{SP}^k(X)$ the k-th reduced symmetric product which is the symmetric smash product $X^{(k)}/\mathfrak{S}_k$, where $X^{(k)}$ is the k-fold smash product of X with itself and $\mathfrak{S}_k$ is the permutation group. We set $\overline{SP}^0(X) = S^0$. Let us recall also another characterization of the reduced symmetric product. Write $SP^k(X)$ for the k-th symmetric product of X obtained as the quotient of X^k by the permutation action of $\mathfrak{S}_k$. There is a topological embedding $SP^{k-1}(X) \hookrightarrow SP^k(X)$ which adjoins the basepoint to a configuration in $SP^{k-1}(X)$ and $\overline{SP}^k(X)$ is nothing but the cofiber of this embedding, $\overline{SP}^k(X) \cong SP^k(X)/SP^{k-1}(X)$. So a Theorem by Dold ([40], Theorem 7.2) on the homology of symmetric products of simplicial complexes implies that the homology of reduced symmetric products only depends on the homology of the underlying space. Moreover it has been proved that $\overline{SP}^k(X \vee Y) = \vee_{r+s=k} \overline{SP}^r(X) \wedge \overline{SP}^s(Y)$; finally in the case of the 2-sphere $\overline{SP}^k(S^2) \cong S^{2k}$ (see [48]).

Eilenberg MacLane space. A space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ (where G is a group and $n \in \mathbb{N}$) is called an Eilenberg-MacLane space $K(G,n)$. For any choice of G and n it is possible to build a $K(G,n)$ space such that the homotopy type of a $K(G,n)$ space is uniquely determined by G and n.

Steenrod squares. Steenrod defined some homomorphisms between cohomology groups: $Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ $(i \geq 0)$, where X is any topological space. Properties of those homomorphisms can be found in [78] and references therein. To abbreviate notation we will denote the composition $Sq^{i_1} \circ Sq^{i_2} \circ \ldots \circ Sq^{i_m}$ by Sq^I, where $I = \{i_1, i_2, \ldots, i_m\}$.

Next let us recall a basic result in homology (see [47] Theorem 2.13 and Proposition 2.22).

Theorem 1.21. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\ldots \to \tilde{H}_q(A) \to \tilde{H}_q(X) \to H_q(X,A) \to \tilde{H}_{q-1}(A) \to \ldots \to \tilde{H}_0(X,A) \to 0.$$

1.2.3 Lusternik-Schnirelman relative category

We recall first the definition of Lusternik-Schnirelman category (category, for short); then, following [45], we will introduce a more powerful notion. In fact, to be precise, it is not a notion but rather a family of (Lusternik-Schnirelman) relative categories. In this family we choose only two for their special properties, which are given in Proposition 1.24. We will see that the category is a useful tool in critical point theory to obtain multiplicity results.

Definition 1.22. Let X be a topological space and A a subset of X. The *category of A with respect to X*, denoted by $\mathrm{Cat}_X A$, is the least integer k such that $A \subset A_1 \cup \ldots \cup A_k$, with A_i $(i = 1, \ldots, k)$ closed and contractible in X. We set $\mathrm{Cat}_X \emptyset = 0$ and $\mathrm{Cat}_X A = +\infty$ if there are no integers satisfying the demand.

Definition 1.23. Let X be a topological space and Y a closed subset of X. A closed subset A of X *is of the k-th (strong) category relative to Y* (we write $\mathrm{Cat}_{X,Y} A = k$) if k is the least positive integer such that there exist $A_i \subset A$ closed and $h_i : A_i \times [0,1] \to X$, $i = 0, \ldots, k$, such that

(i) $A = \cup_{i=0}^{k} A_i$,

(ii) $h_i(x, 0) = x \quad \forall\, x \in A_i \;\; 0 \le i \le m$,

(iii) $h_0(x, 1) \in Y \quad \forall x \in A_0$ and $h_0(y, t) = y \quad \forall y \in Y \quad \forall t \in [0, 1]$,

(iv) $\forall\, i \ge 1 \;\; \exists\, x_i \in X$ such that $h_i(x, 1) = x_i$,

(v) $\forall\, i \ge 1 \;\; h_i(A_i \times [0, 1]) \cap Y = \emptyset$.

We say that A *is of the k-th weak category relative to* Y, written $\operatorname{cat}_{X,Y} A = k$, if k is minimal satisfying conditions $(i) - (iv)$.
If one such k does not exist, we set $\operatorname{Cat}_{X,Y} A = +\infty$ (respectively $\operatorname{cat}_{X,Y} A = +\infty$).

Starting from the above definition, it is easy to check that the following properties hold true.

Proposition 1.24. *[45] Let A, B and Y be closed subsets of X:*

1. *if $Y = \emptyset$, then $\operatorname{cat}_{X,\emptyset} A = \operatorname{Cat}_{X,\emptyset} A = \operatorname{Cat}_X A$;*

2. $\operatorname{Cat}_{X,Y} A \ge \operatorname{cat}_{X,Y} A$;

3. *if $A \subset B$, then $\operatorname{Cat}_{X,Y} A \le \operatorname{Cat}_{X,Y} B$;*

4. *if there exists an homeomorphism $\phi : X \to X'$ such that $Y' = \phi(Y)$ and $A' = \phi(A)$, then $\operatorname{Cat}_{X',Y'} A' = \operatorname{Cat}_{X,Y} A$;*

5. *if $X' \supset X \supset A$ and $r : X' \to X$ is a retraction such that $r^{-1}(Y) = Y$ and $r^{-1}(A) \supset A$, then $\operatorname{Cat}_{X',Y} A \ge \operatorname{Cat}_{X,Y} A$.*

Usually, the notion of category is employed to find critical points of a functional I on a manifold X, in connection with the topological structure of X. Moreover a classical theorem by Lusternik-Schnirelman shows that either there are at least $\operatorname{Cat}_X X$ critical points of I on X, or at some critical level of I there is a continuum of critical points (see, for example, [1]).

This result cannot directly help us because, since we look for critical points on $\bar{H}^1_g(T)$, we would take $X = \bar{H}^1_g(T)$ which, clearly, has category equal to 1 (being contractible).

So we will need a generalization of such a theorem which involves relative category of sublevels. In particular a Theorem in [45] can be adapted to our functional.

Theorem 1.25. *If $-\infty < a < b < +\infty$ and a, b are regular value for I_ρ, then*

$$\#\left\{\textit{critical points of } I_\rho \textit{ in } a \le I_\rho \le b\right\} \ge \operatorname{Cat}_{\{I_\rho \le b\},\{I_\rho \le a\}} \{I_\rho \le b\}.$$

In its original formulation the previous statement dealt with C^1 functionals verifying the Palais-Smale condition, but, as pointed out in [31], the (PS)-condition is used in the proof only twice to apply the classical deformation lemma (see for example [29]). Thus, it is not hard to understand that Corollary 1.13 allows to extend the result to I_ρ.

Besides, in a particular case the relative category can be estimated by means of the cup-length of a pair in the following way:

Theorem 1.26. *[10] For any topological space X, if Y is a closed subset of X, then:*

$$\mathrm{cat}_{X,Y} X \geq \mathrm{CL}(X,Y) + 1.$$

1.3 Critical values of ρ

First of all we give a really brief account of the main ideas of the proof of the degree–counting formula (8). That turns out to be helpful in studying $(*)_\rho$ when $\rho \in 8\mathbb{N}\pi$. To compute the degree for regular values of ρ we should calculate the jump $\mathrm{d}(k, \chi(\Sigma)) - \mathrm{d}(k-1, \chi(\Sigma))$ at $\rho = 8k\pi$. We recall an asymptotic estimate which enables under some assumptions to obtain a priori bounds for solutions of $(*)_{8k\pi}$. Besides, the following result gives useful information whether the parameter ρ_n is greater or less than $8k\pi$ when the bubbling phenomenon occurs.

Theorem 1.27. *[17] Let u_n be a sequence of blowing–up solutions of $(*)_{\rho_n}$ and assume $\lim_{n\to+\infty} \rho_n = 8k\pi$. Let p_j, $j = 1, \dots, k$, be blowup points. Then*

$$\rho_n - 8k\pi = \frac{2}{k} \sum_{j=1}^{k} h(p_{n,j})^{-1} [\triangle_g \log h(p_{n,j}) + 8k\pi - 2K_g(p_{n,j}) + o(1)] \lambda_{n,j} e^{-\lambda_{n,j}},$$

where $\lambda_{n,j}$ and $p_{n,j}$ are, respectively, local maxima and local minimum points of u_n near p_j, and K_g denotes the Gaussian curvature of Σ.

By the previous theorem, if we take h satisfying $\triangle_g \log h + 8k\pi - 2K_g > 0$ on Σ (or such that $\triangle_g h(p) \neq 0$ for any critical point p of h, if $k = 1$ and $\Sigma = S^2$), we find that the degree $\mathrm{d}_{8k\pi}(h)$ is well defined. Note that in general $\mathrm{d}_{8k\pi}(h)$ depends on h, as shown for example in (13). Therefore, in order to evaluate $\mathrm{d}(k, \chi(\Sigma))$ by induction on k, we can compute $\mathrm{d}(k, \chi(\Sigma)) - \mathrm{d}_{8k\pi}(h)$ and $\mathrm{d}_{8k\pi}(h) - \mathrm{d}(k-1, \chi(\Sigma))$ for some suitably chosen h. When ρ crosses the critical threshold the Leray-Schauder degree for bounded solutions remains constant, then the value $\mathrm{d}(k, \chi(\Sigma)) - \mathrm{d}_{8k\pi}(h)$ depends only on the degree contributed of blowing–up solutions to $(*)_{8k\pi+\varepsilon}$ as $\varepsilon \searrow 0$. The same happens for $\mathrm{d}_{8k\pi}(h) - \mathrm{d}(k-1, \chi(\Sigma))$ which counts the degree contributed of blowing–up solutions to $(*)_{8k\pi-\varepsilon}$ as $\varepsilon \searrow 0$. In order to compute these values, Chen and Lin constructed all possible k-bubble solutions and calculated their contribution to the degree.

To do that they introduced a function f_h defined on Σ^k, the k times product space of Σ, such that the set of blow up points is a critical point for f_h. The function is the following

$$f_h(x_1,\ldots,x_k)=\sum_{j=1}^{k}[\log h(x_j)+4\pi\varphi(x_j)+\sum_{\ell\neq j}8\pi G(x_j,x_\ell)] \tag{1.26}$$

where $(x_1,\ldots,x_k)\in\Sigma^k$ and $\varphi(x_j)=\tilde{G}(x_j,x_j)$ is the regular part of the Green's function, being $\tilde{G}(x,p)=\frac{1}{2\pi}\log \mathrm{d}(x,p)+G(x,p)$.
After ignoring the permutations of $(x_1,\ldots,x_k)$, it is meaningful to regard f_h as a function on $\Sigma^k\setminus\Gamma_k$, where

$$\Gamma_k=\{(x_1,\ldots,x_k): x_i=x_j \text{ for some } i\neq j\}\,.$$

In [18] the authors introduced approximate blowing–up solutions, $u_{P,\Lambda,A}$, which allow to reduce $(*)_\rho$ to an equation on a finite dimensional space. Roughly speaking $P=(p_1,\ldots,p_k)$ specifies the location of blowup points, $\Lambda=(\lambda_1,\ldots,\lambda_k)$ the heights of the bubbles and $A=(a_1,\ldots,a_k)\simeq(1,\ldots,1)$ is a correction parameter (see Section 3 of [18] for details). Moreover these approximate solutions should fulfill some compatibility conditions, involving also ρ, and are one-to-one corresponding to the critical points of f_h on $\Sigma^k\setminus\Gamma_k$.

To understand the relationship between blowing–up solutions and critical points of f_h, at least very naively, we need to introduce some quantities. For any critical point Q of f_h let us consider

$$l(Q)=\sum_{j=1}^{k}(\triangle_g\log h(q_j)+8k\pi-2K_g(q_j))h(q_j)e^{G_j^*(q_j)}, \tag{1.27}$$

where $G_j^*(q_j)=8\pi(\sum_{\ell\neq j}G(q_j,q_\ell)+\tilde{G}(q_j,q_j))$.
By means of $l(Q)$, for any $j=1,\ldots,k$, we set $\lambda_j(Q)$ to satisfy

$$\rho-8k\pi=\frac{2}{k}\frac{l(Q)}{h^2(q_j)}e^{-G_j^*(q_j)\lambda_j(Q)e^{-\lambda_j(Q)}}. \tag{1.28}$$

Moreover, for any critical point Q of f_h, the set $S_\rho(Q)\subset\bar{H}^1_g(\Sigma)$ will play a crucial role. A rigorous definition of $S_\rho(Q)$ goes beyond the aim of this section and again we refer to [18] the interested reader. Anyhow it is not hard to rough out the elements of this set. Let $Q=(q_1,\ldots,q_k)$ be a critical point of f_h, a function $u\in\bar{H}^1_g(\Sigma)$ belongs to $S_\rho(Q)$ if it is of the form $u_{P,\Lambda,A}+w$, with the H^1-norm of w and the distance between the blowup points p_j and q_j controlled in terms of $\lambda_1(Q)e^{-\lambda_1(Q)}$. Besides $|a_j-1|\leq C\lambda_1^{\frac{1}{2}}(Q)e^{-\lambda_1(Q)}$, the difference $|\lambda_1-\lambda_1(Q)|$ should be less or equal to $C\lambda_1(Q)^{-1}$ and some extra assumptions on w and λ_j, for $j=2,\ldots,k$, are required.

The following deep result shows that any blowing–up solution must be contained in $S_\rho(Q)$ for some critical point Q of f_h, provided that ρ is sufficiently close to $8k\pi$.

Theorem 1.28. *[18] Let h be a C^3 positive function on Σ satisfying the following two conditions*

(c1) the function $f_h(x)$ is a Morse function on $\Sigma^k \backslash \Gamma_k$ with critical points $Q_1, \dots, Q_N$;

(c2) the quantity $l(Q)$ does not vanish for any critical point of f_h.

Then there exist $\varepsilon_k > 0$, $C_\rho > C_k > 0$, with ε_k and C_k depending on k only, C_ρ continuous in ρ, and $\lim_{\rho \to 8k\pi} C_\rho = +\infty$ such that for each solution $u \in \bar{H}^1_g(\Sigma)$ of $(*)_\rho$, with $|\rho - 8k\pi| < \varepsilon_k$, the following hold:

- *$\|u\|_{H^1} < C_\rho$ for $\rho \neq 8k\pi$;*
- *if $\rho = 8k\pi$, then $\|u\|_{H^1} \leq C_k$;*
- *for $\rho \neq 8k\pi$, we have either*
  - *$\|u\|_{H^1} < C_k$ or*
  - *$\|u\|_{H^1} > C_k$ and there exists a critical point Q of f_h such that $u \in S_\rho(Q)$. In this case we have*

$$\left| \|u\|_{H^1} - 16\pi \sum_{j=1}^{k} \lambda_j(Q) \right| \leq c$$

for some constant c independent of ρ, where $\lambda_j(Q)$ is given in (1.28).

We point out that h is assumed C^3 just for technical reasons.

1.4 The singular case

If instead of $(*)_\rho$ we consider the singular equation (18) and the corresponding functional J_ρ, all the results in subsection 1.1.1 still hold. Hence, it is still possible to project the low sublevels of J_ρ into Σ_k.

Lemma 1.29. *If $\rho \in (8k\pi, 8(k+1)\pi)$ with $k \geq 1$, then there exists a continuous projection $\Psi : \{J_\rho \leq -L\} \to \Sigma_k$.*

On the other hand, for what concerns the embedding of the space of formal barycenters Σ_k into arbitrarily low sublevels, the statement of Proposition 1.5 does not apply entirely, indeed the point (ii) should be modified in the following way.

Proposition 1.30. *Suppose $\rho \in (8k\pi, 8(k+1)\pi)$, with $k \geq 1$. Let $\varphi_{\lambda,\sigma}$ be defined as in (1.9) and let K be a compact subset of $\Sigma \setminus \{P_1, \dots, P_m\}$. Then,*

$$\frac{e^{\varphi_{\lambda,\sigma}}}{\int_\Sigma e^{\varphi_{\lambda,\sigma}} dV_g} \rightharpoonup \sigma \text{ and } J_\rho(\varphi_{\lambda,\sigma}) \to -\infty \text{ uniformly for } \sigma \in K_k, \text{ as } \lambda \to +\infty,$$

where K_k is the set of formal barycenters of order k supported in K (see (25)).

To prove this result it is enough to consider the proof of Proposition 1.5, changing slightly the estimates used to obtain (1.11).

With regard to compactness, when we include Dirac measures as inhomogeneous data in (18) the analysis of the corresponding solution-set becomes more involved; in this situation one needs to deal with the additional difficulty of considering solution sequences which become unbounded from above around a pole of the given Dirac measures.
It is worth to illustrate briefly, by means of an example, the main difference with the regular case. Let consider the sequence

$$u_n(x) = \log \frac{\lambda_n |x-P|^{2\alpha}}{\left(1 + \frac{1}{8(1+\lambda)^2}\lambda_n |x-P|^{2(1+\alpha)}\right)^2}, \quad x \in \mathbb{R}^2$$

of solutions to

$$-\triangle u_n = \rho_n \frac{e^{u_n}}{\int_\Omega e^{u_n} dx} - 4\pi\delta_P \qquad \text{on } \Omega,$$

where $\rho_n = \int_\Omega e^{u_n} \to 8\pi(1+\alpha)$, as $n \to \infty$. Note that for any domain $\Omega \subset \mathbb{R}^2$ with $P \in \Omega$ we have that e^{u_n} *concentrates* near P in the sense that

$$e^{u_n} = \rho_n \frac{e^{u_n}}{\int_\Omega e^{u_n} dx} \rightharpoonup 8\pi(1+\alpha)\delta_P \quad \text{ weakly in the sense of measures in } \Omega.$$

Therefore in the singular case we see that the condition $\rho \notin 8\pi\mathbb{N}$ is no longer sufficient to guarantee uniform upper estimates of solutions; in fact the values $\rho = 8\pi(1+\alpha_j)\mathbb{N}$ may be responsible for a possible blow up point at the Dirac pole P_j, $j = 1, \dots, m$, and yield a concentration phenomenon. In this direction Bartolucci and Tarantello proved the following result.

Theorem 1.31. *[4] Let K be a positive Lipschitz function on Σ and let $\tilde{h} = Ke^{-h_m}$. Let u_i solve* (18) *with $\alpha_j > 0$, $p_j \in \Sigma$ and $\rho = \rho_i$, $\rho_i \to \overline{\rho}$. Suppose that $\int_\Sigma \tilde{h}e^{u_i} dV_g \le C_1$ for some fixed $C_1 > 0$. Then along a subsequence u_{i_k} one of the following alternatives holds:*

(i) *u_{i_k} is uniformly bounded from above on Σ;*

(ii) $\max_\Sigma \left(u_{i_k} - \log \int_\Sigma \tilde{h}e^{u_{i_k}} dV_g\right) \to +\infty$ *and there exists a finite blow-up set $S = \{q_1, \dots, q_l\} \subset \Sigma$ such that*

(a) for any $s \in \{1, \dots, l\}$ there exist $x_n^s \to q_s$ such that $u_{i_k}(x_n^s) \to +\infty$ and $u_{i_k} \to -\infty$ uniformly on the compact sets of $\Sigma \setminus S$,

(b) $\rho_{i_k} \frac{\tilde{h}e^{u_{i_k}}}{\int_\Sigma \tilde{h}e^{u_{i_k}} dV_g} \rightharpoonup \sum_{s=1}^l \beta_s \delta_{q_s}$ in the sense of measures, with $\beta_s = 8\pi$ for $q_s \neq \{p_1, \dots, p_m\}$, or $\beta_s = 8\pi(1+\alpha_j)$ if $q_s = p_j$ for some $j = \{1, \dots, m\}$. In particular one has that

$$\overline{\rho} \in \Gamma(\underline{\alpha}_m).$$

From the above result we obtain immediately the following corollary.

Corollary 1.32. *Suppose we are in the above situation, and that $\rho \notin \Gamma(\underline{\alpha}_m)$. Then the solutions of* (18) *stay uniformly bounded in $C^2(\Sigma)$.*

Exactly as for the regular equation, Corollary 1.32 is a compactness criterion useful to bypass the Palais-Smale condition, which is not known for the functional J_ρ. This corollary, combined with the arguments in [61] (see Lemma 1.11 above, which adapts in a straightforward way to the singular case), allows to prove the next alternative.

Lemma 1.33. *If $\rho \notin \Gamma(\underline{\alpha}_m)$ and if J_ρ has no critical levels inside some interval $[a, b]$, then $\{J_\rho \leq a\}$ is a deformation retract of $\{J_\rho \leq b\}$.*

It is then clear that one can derive from the previous lemma the contractibility of high sublevels of the functional J_ρ, namely

Proposition 1.34. *If $\rho \notin \Gamma(\underline{\alpha}_m)$ and if b is sufficiently large positive, the sublevel $\{J_\rho \leq b\}$ is a deformation retract of $\bar{H}^1_g(\Sigma)$ and hence is contractible.*

Remark 1.35. As far as we are concerned with the approach presented in this paper it seems not easy to remove the hypothesis on the positivity of K. The difficulties are inherited by the lack of concentration-compactness-quantization results (in the same spirit of [4], [6], [52]) for solutions of (18) with K possibly changing sign or even just nonnegative. Actually, our analysis relies heavily on Theorem 1.31 (see also results in [6] and [52]) where this hypothesis is required (see [67] for related issues in the regular case).
However the necessary condition imposed by the Gauss-Bonnet constraint (20) just reads

$$\int_\Sigma 2Ke^{-h_m}e^u dV_g = 4\pi\chi(\Sigma, \underline{\alpha}_m),$$

so that in principle there should be no obstructions (as in the regular and subcritical cases [49], [84]) in finding conformal metrics on supercritical singular surfaces of positive genus with Gaussian curvature just assumed to be positive somewhere.

This Remark motivates the following question: is it true that any Lipschitz continuous function on S can be realized as the Gaussian curvature of a conformal metric on a supercritical surface satisfying the hypotheses of Theorem 0.5?

Chapter 2

Proofs of the main theorems

2.1 Generic multiplicity

In order to apply Morse theory to $I_\rho = I_{\rho,(g,h)}$ we need to show that it is a Morse functional for a generic choice of the metric g and of the positive function h.

Proposition 2.1. *Let $\rho \in (8k\pi, 8(k+1)\pi)$. Then*

$$\mathcal{D}(\rho) = \left\{(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+ \,:\, \textit{all critical points of } I_{\rho,(g,h)} \textit{ are non degenerate}\right\}$$

is an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$ and

$$\mathcal{D}_1(\rho) = \left\{(g,h) \in \mathcal{M}_1^2 \times C^2(\Sigma)^+ \,:\, \textit{all critical points of } I_{\rho,(g,h)} \textit{ are non degenerate}\right\}$$

is an open and dense subset of $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

Proof. The main tool of the proof it is an abstract transversality Theorem due to Saut and Temam [77]. In particular we will apply the following scheme performed by Micheletti and Pistoia in [69].

First of all we introduce the space $\mathcal{S}^2$ of all C^2 symmetric matrices on Σ. $\mathcal{S}^2$ is a Banach space endowed with the C^2 norm, defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha\in L}$ of Σ such that the closure of V_α is contained in U_α, where $\{U_\alpha, \psi_\alpha\}$ is the open coordinate neighborhood. If $g \in \mathcal{S}^2$ we denote by g_{ij} the components of g with respect to the coordinates $(x_1, \ldots, x_N)$ on V_α. We define

$$\|g\|_2 := \sum_{\alpha\in L} \sum_{|\beta|\leq 2} \sum_{i,j=1}^{N} \sup_{\psi_\alpha(V_\alpha)} \frac{\partial^2 g_{ij}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}. \tag{2.1}$$

The set $\mathcal{M}^2$ of all C^2 Riemannian metrics on Σ is an open subset of $\mathcal{S}^2$.

We fix now $(\overline{g}, \overline{h}) \in \mathcal{M}^2 \times C^2(\Sigma)^+$.

It is easy to verify that there exists $\delta > 0$ such that if $g \in \mathcal{G}_\delta := \left\{g \in \mathcal{S}^2 \,:\, \|g\|_2 < \delta\right\}$ then $\bar{g}+g$ is a Riemannian metric and the sets $H^1_{\bar{g}+g}(\Sigma)$, $L^2_{\bar{g}+g}(\Sigma)$, $L^1_{\bar{g}+g}(\Sigma)$ coincide respectively with $H^1_{\bar{g}}(\Sigma)$, $L^2_{\bar{g}}(\Sigma)$, $L^1_{\bar{g}}(\Sigma)$ and the two norms are equivalent. Moreover we will choose δ sufficiently small in order to have that $\bar{h} + h \in C^2(\Sigma)^+$ for any $h \in \mathcal{H}_\delta := \left\{h \in C^2(\Sigma) \,:\, \|h\|_\infty < \delta\right\}$.

Definition 2.2. For $g \in \mathcal{G}_\delta$ we set $A(g) := A_g : L^2_{\bar{g}}(\Sigma) \longrightarrow H^1_{\bar{g}}(\Sigma)$ to be the only linear operator such that

$$(A_g u, v)_{H^1_{\bar{g}+g}(\Sigma)} = (u, v)_{L^2_{\bar{g}+g}(\Sigma)} \qquad \forall\, v \in H^1_{\bar{g}}(\Sigma),\ \forall\, u \in L^2_{\bar{g}}(\Sigma). \tag{2.2}$$

Clearly

$$(A_g u, v)_{H^1_{\bar{g}+g}(\Sigma)} = (u, A_g v)_{H^1_{\bar{g}+g}(\Sigma)} \qquad \forall\, u, v \in H^1_{\bar{g}}(\Sigma),$$

moreover A_g is nothing but the adjoint operator $i^*_{\bar{g}+g}$ of the compact embedding $i_{\bar{g}+g} : H^1_{\bar{g}+g}(\Sigma) \to L^2_{\bar{g}+g}(\Sigma)$. Integrating by parts it can be checked that the main term of the explicit expression of A_g is the inverse of the laplacian operator. Let us notice that in the definition of A_g we used the fact that $H^1_{\bar{g}+g}(\Sigma)$ and $H^1_{\bar{g}}(\Sigma)$ (respectively $L^2_{\bar{g}+g}(\Sigma)$ and $L^2_{\bar{g}}(\Sigma)$) are the same as sets and that the two norms are equivalent.

For what concerns the regularity in g of $A(g)$ we have the following result.

Lemma 2.3. *The map $A : \mathcal{G}_\delta \longrightarrow \mathcal{L}(L^{p'}_{\bar{g}}(\Sigma); H^1_{\bar{g}}(\Sigma))$ is of class C^1, where $\mathcal{L}(L^{p'}_{\bar{g}}(\Sigma); H^1_{\bar{g}}(\Sigma))$ stands for the space of linear operators from $L^{p'}_{\bar{g}}(\Sigma)$ to $H^1_{\bar{g}}(\Sigma)$.*

For the proof, see Lemma 2.3 of [69].

Moreover we can assume that δ is sufficiently small such that there exists $\bar{R} > 0$ such that for any $(g_0, h_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta$ all the critical points of $I_{\rho,(\bar{g}+g,\bar{h}+h)}$ are contained in the ball $\mathcal{B} := B_{\bar{R}}(0)$ of $\bar{H}^1_{\bar{g}}(\Sigma)$.
We are finally in position to introduce the map $F : \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B} \longrightarrow \bar{H}^1_{\bar{g}}(\Sigma)$:

$$F(g, h, u) := S_g^{-1}(\tilde{F}_g(h, S_g(u))), \tag{2.3}$$

where

$$\begin{aligned} \tilde{F}_g : \mathcal{H}_\delta \times \bar{H}^1_{\bar{g}+g}(\Sigma) &\longrightarrow \bar{H}^1_{\bar{g}+g}(\Sigma) \\ (h, w) \qquad &\mapsto w - A_g\Big(\rho \frac{(\bar{h}+h)e^w}{\int_\Sigma (\bar{h}+h)e^w dV_{\bar{g}+g}} - \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} + w\Big), \end{aligned}$$

while $S_g : \bar{H}^1_{\bar{g}}(\Sigma) \to \bar{H}^1_{\bar{g}+g}(\Sigma)$ is defined as $S_g(u) := u - \fint_\Sigma u dV_{\bar{g}+g}$. Clearly S_g is linear, invertible and the inverse is given by $S_g^{-1} : \bar{H}^1_{\bar{g}+g}(\Sigma) \to \bar{H}^1_{\bar{g}}(\Sigma)$, $S_g^{-1}(w) := w - \fint_\Sigma w dV_{\bar{g}}$.
By the regularity of the map A, which associates to g the linear operator A_g, (see Lemma 2.3) we get that the map F is of class C^1.

It is easy to see that $(g, h, u) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ are such that $F(g, h, u) = 0$ if and only if u is a critical point of $I_{\rho,(\bar{g}+g,\bar{h}+h)}$. Taking into account this remark, to establish the claim we need the following transversality theorem.

Theorem 2.4. *[77] Let X, Y, Z be three real Banach spaces and let $U \subset X$, $V \subset Y$ be open subsets. Let $F : V \times U \longrightarrow Z$ be a C^k-map with $k \geq 1$ such that*

(i) *for any* $y \in V$, $F(y, \cdot) : x \mapsto F(y, x)$ *is a Fredholm map of index* l *with* $l \leq k$;

(ii) z_0 *is a regular value of* F, *that is the operator* $F'(y_0, x_0) : Y \times X \longrightarrow Z$ *is onto at any point* (y_0, x_0) *such that* $F(y_0, x_0) = z_0$;

(iii) *the set of* $x \in U$ *such that* $F(y_0, x) = z_0$ *with* y_0 *in a compact set of* V *is relatively compact in* U.

Then the set $\{y \in V : z_0$ *is a regular value of* $F(y, \cdot)\}$ *is a dense open subset of* V.

We collect now some technical Lemmas needed to verify that we are in condition to apply Theorem 2.4.

Lemma 2.5. *For any* $(g, h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta$ *the map* $u \mapsto F(g, h, u)$ *with* $u \in \mathcal{B}$ *is Fredholm of index* 0.

Proof. For $(g_0, h_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta$ and $v \in \bar{H}^1_{\tilde{g}}(\Sigma)$ we have

$$\begin{aligned}
&F'_u(g_0, h_0, u_0)[v] = S^{-1}_{g_0}((\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))[S_{g_0}(v)]) = S^{-1}_{g_0}(S_{g_0}(v) + \\
&-A_{g_0}(\rho \frac{\tilde{h}e^{S_{g_0}(u_0)} S_{g_0}(v) \int_\Sigma \tilde{h}e^{S_{g_0}(u_0)} dV_{\tilde{g}} - \tilde{h}e^{S_{g_0}(u_0)} \int_\Sigma \tilde{h}e^{S_{g_0}(u_0)} S_{g_0}(v) dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{S_{g_0}(u_0)} dV_{\tilde{g}})^2} + S_{g_0}(v))) \\
&= v - S^{-1}_{g_0}(A_{g_0}(\rho \frac{\tilde{h}e^{u_0} v \int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}} - \tilde{h}e^{u_0} \int_\Sigma \tilde{h}e^{u_0} v dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}})^2} + v)) \\
&:= v - K(v),
\end{aligned}$$

where $\tilde{g} := \bar{g} + g_0$, $\tilde{h} := \bar{h} + h_0$ and

$$K(v) = S^{-1}_{g_0}(A_{g_0}(\rho \frac{\tilde{h}e^{u_0} v \int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}} - \tilde{h}e^{u_0} \int_\Sigma \tilde{h}e^{u_0} v dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}})^2} + v)).$$

We will verify that $K : \bar{H}^1_{\tilde{g}}(\Sigma) \to \bar{H}^1_{\tilde{g}}(\Sigma)$ is compact and this will end the proof. If v_n is a bounded sequence in $\bar{H}^1_{\tilde{g}}(\Sigma)$, then v_n is also bounded in $\bar{H}^1_{\bar{g}}(\Sigma)$ (because $g_0 \in \mathcal{G}_\delta$). Then up to a subsequence, v_n converges to v in $L^q_{\tilde{g}}(\Sigma)$ for any $q \geq 1$. So, we have

$$\begin{aligned}
&\left(\int_\Sigma \left| \rho \frac{(\int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}})\tilde{h}e^{u_0}(v_n - v) - (\int_\Sigma \tilde{h}e^{u_0}(v_n - v) dV_{\tilde{g}})\tilde{h}e^{u_0}}{(\int_\Sigma \tilde{h}e^{u_0} dV_{\tilde{g}})^2} + (v_n - v) \right|^2 dV_{\tilde{g}} \right)^{\frac{1}{2}} \leq \\
&\leq \rho\left(\frac{\|\tilde{h}e^{u_0}\|_{L^4_{\tilde{g}}} \|v_n - v\|_{L^4_{\tilde{g}}}}{\|\tilde{h}e^{u_0}\|_{L^1_{\tilde{g}}}} + \frac{\|\tilde{h}e^{u_0}\|^2_{L^2_{\tilde{g}}} \|v_n - v\|_{L^2_{\tilde{g}}}}{\|\tilde{h}e^{u_0}\|^2_{L^1_{\tilde{g}}}} \right) + \|v_n - v\|_{L^2_{\tilde{g}}} \longrightarrow 0.
\end{aligned}$$

Therefore, by continuity of A_{g_0} and of $S^{-1}_{g_0}$, $K(v_n - v) \to 0$ in $\bar{H}^1_{\tilde{g}}(\Sigma)$ and so it converges to 0 in $\bar{H}^1_{\tilde{g}}(\Sigma)$. ∎

Remark 2.6. *Arguing exactly in the same way we can also prove that for any* $(g_0, h_0, u_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ *the map* $w \mapsto (\tilde{F}_g)'_w(h, S_{g_0}(u_0))[w]$ *for* $w \in \bar{H}^1_{\bar{g}+g}(\Sigma)$ *is a Fredholm map of index* 0.

Lemma 2.7. *The set*

$$\{u \in \mathcal{B} \,:\, F(g_0, h_0, u) = 0, \quad (g_0, h_0) \text{ belongs to a compact subset of } \mathcal{G}_\delta \times \mathcal{H}_\delta\}$$

is relatively compact in $\mathcal{B} \subset \bar{H}^1_{\bar{g}}(\Sigma)$.

Proof. We show that if $u_n \in \mathcal{B}$ is such that $F(g_n, h_n, u_n) = 0$ with $g_n \to g_0$ and $h_n \to h_0$, then u_n possesses a converging subsequence.
Let us first notice that, thanks to the invertibility of $S_{g_n}^{-1}$ for any n, $F(g_n, h_n, u_n) = 0$ implies $\tilde{F}_{g_n}(h_n, S_{g_n}(u_n)) = 0$, which in turn is equivalent to

$$u_n = A_{g_n}(\rho \frac{\tilde{h}_n e^{u_n}}{\int_\Sigma \tilde{h}_n e^{u_n} dV_{\tilde{g}_n}} - \frac{\rho}{\int_\Sigma dV_{\tilde{g}_n}} + u_n).$$

Since the sequence u_n is bounded in $H^1_{\bar{g}}(\Sigma)$ and also in $H^1_{\tilde{g}}(\Sigma)$ (being $g_0 \in \mathcal{G}_\delta$), u_n (up to a subsequence) converges to a function u in $L^q_{\bar{g}}(\Sigma)$ and in $L^q_{\tilde{g}}(\Sigma)$ for any $q \geq 1$. If we are able to prove that

$$\left\| \rho(\frac{\tilde{h}_n e^{u_n}}{\int_\Sigma \tilde{h}_n e^{u_n} dV_{\tilde{g}_n}} - \frac{\tilde{h} e^u}{\int_\Sigma \tilde{h} e^u dV_{\tilde{g}}}) - \rho(\frac{1}{\int_\Sigma dV_{\tilde{g}_n}} - \frac{1}{\int_\Sigma dV_{\tilde{g}}}) + (u_n - u) \right\|_{L^2_{\tilde{g}}} \longrightarrow 0, \quad (2.4)$$

where $\tilde{g}_n := \bar{g} + g_n$ and $\tilde{h}_n := \bar{h} + h_n$, then we will get the same convergence in $L^2_{\bar{g}}$ and so

$$i^*_{\tilde{g}}(f_n) = A_{g_0}(f_n) \overset{H^1_{\tilde{g}}(\Sigma)}{\longrightarrow} A_{g_0}(\rho \frac{\tilde{h} e^u}{\int_\Sigma \tilde{h} e^u dV_{\tilde{g}}} - \frac{\rho}{\int_\Sigma dV_{\tilde{g}}} + u). \quad (2.5)$$

where $f_n := \rho \frac{\tilde{h}_n e^{u_n}}{\int_\Sigma \tilde{h}_n e^{u_n} dV_{\tilde{g}_n}} - \frac{\rho}{\int_\Sigma dV_{\tilde{g}_n}} + u_n$.
On the other hand by Lemma 2.3 we have that for some $\theta \in (0, 1)$:

$$\begin{aligned}
&\|A_{g_n}(f_n) - A_{g_0}(f_n)\|_{H^1_{\tilde{g}}} = \\
&= \|A'(g_0 + \theta(g_n - g_0))[g_n - g_0](f_n)\|_{H^1_{\tilde{g}}} \leq \\
&\leq \|f_n\|_{L^2_{\tilde{g}}} \|A'(g_0 + \theta(g_n - g_0))[g_n - g_0]\|_{\mathcal{L}(L^2_{\tilde{g}}, H^1_{\tilde{g}})} \leq \\
&\leq \|f_n\|_{L^2_{\tilde{g}}} \|A'(g_0 + \theta(g_n - g_0))\|_{\mathcal{L}(\mathcal{G}_\delta, \mathcal{L}(L^2_{\tilde{g}}, H^1_{\tilde{g}}))} \|g_n - g_0\|_2. \quad (2.6)
\end{aligned}$$

From (2.5) and (2.6) we can deduce that

$$A_{g_n}(f_n) \overset{H^1_{\tilde{g}}(\Sigma)}{\longrightarrow} A_{g_0}(\rho \frac{\tilde{h} e^u}{\int_\Sigma \tilde{h} e^u dV_{\tilde{g}}} - \frac{\rho}{\int_\Sigma dV_{\tilde{g}}} + u).$$

Therefore, since $u_n = A_{g_n}(f_n)$, we get the claim.

Finally to conclude it remains to verify (2.4); as $g_n \to g_0$ in $\|\cdot\|_2$ and $u_n \to u$ in $L^2_{\tilde g}$, it will be enough to prove

$$\left\| (\int_\Sigma \tilde h e^u dV_{\tilde g}) \tilde h_n e^{u_n} - (\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}) \tilde h e^u \right\|_{L^2_{\tilde g}} \longrightarrow 0.$$

Simply manipulating the integrands and using Holder's inequality we have:

$$\begin{aligned}
&\int_\Sigma [(\int_\Sigma \tilde h e^u dV_{\tilde g}) \tilde h_n e^{u_n} - (\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}) \tilde h e^u]^2 dV_{\tilde g} = \\
&= (\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n})^2 \int_\Sigma \tilde h^2 e^{2u} [\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h} e^{(u_n - u)} - 1]^2 dV_{\tilde g} \leq \\
&\leq (\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n})^2 (\int_\Sigma \tilde h^4 e^{4u} dV_{\tilde g})^{\frac12} (\int_\Sigma [\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h} e^{(u_n - u)} - 1]^4 dV_{\tilde g})^{\frac12}.
\end{aligned}$$

The first two terms are bounded according to the Moser-Trudinger inequality (1.2); let us consider the square of the third one and use the simple estimate $|e^x - 1| \leq |x| e^{|x|}$, the triangular inequality and Holder's inequality.

$$\begin{aligned}
&\int_\Sigma \left[e^{\log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}) + (u_n - u)} - 1 \right]^4 dV_{\tilde g} \leq \\
&\leq \int_\Sigma \left[|\log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}) + (u_n - u)| \, e^{|\log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}) + (u_n - u)|} \right]^4 dV_{\tilde g} \leq \\
&\leq \int_\Sigma \left[\left| \log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}) + (u_n - u) \right|^4 \times \right. \\
&\qquad \left. \times \max \left\{ \frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}, \frac{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}}{\int_\Sigma \tilde h e^u dV_{\tilde g}} \frac{\tilde h}{\tilde h_n} \right\}^4 e^{4|u_n - u|} \right] dV_{\tilde g} \leq \\
&\leq \left(\int_\Sigma |\log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}) + (u_n - u)|^{12} dV_{\tilde g} \right)^{\frac13} \times \\
&\qquad \times \left(\int_\Sigma \max\{ \frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}, \frac{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}}{\int_\Sigma \tilde h e^u dV_{\tilde g}} \frac{\tilde h}{\tilde h_n} \}^{12} dV_{\tilde g} \right)^{\frac13} \left(\int_\Sigma e^{12|u_n - u|} dV_{\tilde g} \right)^{\frac13}.
\end{aligned}$$

Again the last two terms can be bounded using (1.2), while the cube of the first one can be controlled by:

$$C[\int_\Sigma (\log(\frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h_n}{\tilde h}))^{12} dV_{\tilde g} + \|u_n - u\|^{12}_{L^{12}_{\tilde g}}]$$

and this sequence converges to 0, as $n \to +\infty$, because $u_n \to u$ in $L^{12}_{\tilde g}(\Sigma)$ and $\left\| \frac{\int_\Sigma \tilde h e^u dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \frac{\tilde h}{\tilde h_n} \right\|_\infty \to 1$. Indeed $\left\| \frac{\tilde h}{\tilde h_n} \right\|_\infty \to 1$ and

$$\begin{aligned}
&\frac{\int_\Sigma \tilde h (e^u - e^{u_n}) dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g_n}} \le \frac{\int_\Sigma \tilde h e^u (1 - e^{(u_n - u)}) dV_{\tilde g}}{C \int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g}} \le \\
&\le C \frac{(\int_\Sigma \tilde h^2 e^{2u} dV_{\tilde g})^{\frac12}}{\int_\Sigma \tilde h_n e^{u_n} dV_{\tilde g}} (\int_\Sigma (1 - e^{(u_n-u)})^2 dV_{\tilde g})^{\frac12} \le C(\int_\Sigma |u_n - u|^2 e^{2|u_n - u|} dV_{\tilde g})^{\frac12} \le \\
&\le C \|u_n - u\|_{L^4_{\tilde g}} \|e^{(u_n-u)}\|_{L^4_{\tilde g}} \longrightarrow 0,
\end{aligned}$$

where we used one more time the Holder's inequality, the estimate $|e^x - 1| \le |x| e^{|x|}$, (1.2) and the fact that $u_n \to u$ in $L^4_{\tilde g}(\Sigma)$. ■

Lemma 2.8. *For any $(g_0, h_0, u_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ it holds that if*

$$w \in \mathrm{Ker}(\tilde F_{g_0})'_w (h_0, S_{g_0}(u_0)) \subset \bar H^1_{\tilde g}(\Sigma)$$

and

$$(S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h]), w)_{H^1_{\tilde g}} = 0 \qquad \forall h \in C^2(\Sigma)$$

then $w = 0$.

Proof. By hypothesis

$$\begin{aligned}
0 &= (S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h]), w)_{H^1_{\tilde g}} = (\tilde F_{g_0})'_h (h_0, S_{g_0}(u_0))[h], w)_{H^1_{\tilde g}} = \\
&= -\frac{\rho}{(\int_\Sigma \tilde h e^{u_0} dV_{\tilde g})^2} (\int_\Sigma h e^{u_0} [w \int_\Sigma \tilde h e^{u_0} dV_{\tilde g} - \int_\Sigma \tilde h e^{u_0} w dV_{\tilde g}] dV_{\tilde g})
\end{aligned}$$

for any $h \in C^2(\Sigma)$. This implies that $w \int_\Sigma \tilde h e^{u_0} dV_{\tilde g} - \int_\Sigma \tilde h e^{u_0} w dV_{\tilde g} = 0$, that is $w \equiv \frac{\int_\Sigma \tilde h e^{u_0} w dV_{\tilde g}}{\int_\Sigma \tilde h e^{u_0} dV_{\tilde g}}$ is constant. Finally by the fact that $w \in \bar H^1_{\tilde g}(\Sigma)$ we deduce $w = 0$. ■

Lemma 2.9. *For any $(g_0, h_0, u_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ such that $F(g_0, h_0, u_0) = 0$ and for any $b \in \bar H^1_{\tilde g}(\Sigma)$ there exists $(g_b, h_b, v_b) \in \mathcal{S}^2 \times C^2(\Sigma) \times \bar H^1_{\tilde g}(\Sigma)$ such that*

$$F'_{(g,h)}(g_0, h_0, u_0)[g_b, h_b] + F'_u(g_0, h_0, u_0)[v_b] = b.$$

Proof. Let us take $b \in \bar H^1_{\tilde g}(\Sigma)$. In the following we will use the notations $\tilde g := \bar g + g_0$ and $\tilde h := \bar h + h_0$.
Since by Remark 2.6 the selfadjoint operator

$$w \mapsto (\tilde F_{g_0})'_w (h_0, S_{g_0}(u_0))[w] = w - A_{g_0}(\rho \frac{(\int_\Sigma \tilde h e^{u_0} dV_{\tilde g}) \tilde h e^{u_0} w - (\int_\Sigma \tilde h e^{u_0} w dV_{\tilde g}) \tilde h e^{u_0}}{(\int_\Sigma \tilde h e^{u_0})^2} + w)$$

is Fredholm of index 0, the following decomposition holds

$$\mathrm{Im}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0)) \bigoplus \mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0)) = \bar{H}^1_{\tilde{g}}(\Sigma).$$

We will denote by P_{Im} and $\mathrm{P}_{\mathrm{Ker}}$ the orthogonal projections from $\bar{H}^1_{\tilde{g}}(\Sigma)$ onto $\mathrm{Im}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ and $\mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ respectively. According to these notations we can decompose b as follows

$$b = S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(S_{g_0}(b))) + S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Ker}}(S_{g_0}(b))).$$

Let us show first that there exists $h_b \in C^2(\Sigma)$ such that

$$\mathrm{P}_{\mathrm{Ker}}(S_{g_0}(b)) = \mathrm{P}_{\mathrm{Ker}}(S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b])). \tag{2.7}$$

Let $\{w_1, \ldots, w_\nu\}$ be a basis of $\mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ and let us consider the linear functionals $f_i : C^2(\Sigma) \to \mathbb{R}$ defined by

$$f_i(h) := (F'_{(g,h)}(g_0, h_0, u_0)[0, h], w_i)_{H^1_{\tilde{g}}} \qquad i = 1, \ldots, \nu.$$

By Lemma 2.8 it follows that the f_i's are independent; then there exist ν linearly independent functions $h_1,\ldots,h_\nu$ in $C^2(\Sigma)$ such that $f_i(h_i) = 1$ for $i = 1 \ldots, \nu$ and so we are able to find $h_b \in C^2(\Sigma)$ verifying (2.7).

At this point we have

$$\begin{aligned} b &= S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Ker}}(S_{g_0}(b))) + S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(S_{g_0}(b))) = \\ &= S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Ker}}(S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]))) + S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(S_{g_0}(b))) = \\ &= S_{g_0}^{-1}(S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b])) + \\ &\quad + S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(-S_{g_0}(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]) + S_{g_0}(b))) = \\ &= F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b))). \end{aligned}$$

Since by definition $\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b)) \in \mathrm{Im}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$, it is possible to find $w_b \in \bar{H}^1_{\tilde{g}}(\Sigma)$ such that

$$\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b)) = (\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))[w_b].$$

Finally, if we set $v_b := S_{g_0}^{-1}(w_b)$, we have

$$\begin{aligned} & S_{g_0}^{-1}(\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b))) = \\ & \qquad = S_{g_0}^{-1}((\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))[w_b]) = F'_u(g_0, h_0, u_0)[v_b]. \end{aligned}$$

Therefore, taking $g_b = 0$, we get $b = F'_{(g,h)}(g_0, h_0, u_0)[g_b, h_b] + F'_u(g_0, h_0, u_0)[v_b]$. The proof is thereby complete. ■

Thanks to Lemma 2.5, Lemma 2.9 and Lemma 2.7 we have that, if we take as F the map defined in (2.3) and we set $X = Z = \bar{H}^1_{\tilde{g}}(\Sigma)$, $Y = \mathcal{S}^2 \times C^2(\Sigma)$,

$V = \mathcal{G}_\delta \times \mathcal{H}_\delta$, $U = \mathcal{B}$ and $z_0 = 0$, all the assumptions of Theorem 2.4 are fulfilled. Applying Theorem 2.4 we get that the following set is an open and dense subset of $\mathcal{G}_\delta \times \mathcal{H}_\delta$

$$\begin{aligned}
&\{(g,h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \ : \ F'_u(g,h,u) : \bar{H}^1_{\bar{g}} \longrightarrow \bar{H}^1_{\bar{g}} \text{ is invertible at any point} \\
&\qquad\qquad (g,h,u) \text{ such that } F(g,h,u) = 0 \text{ with } u \in \mathcal{B}\} = \\
&\{(g,h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \ : \ \text{any } u \in \mathcal{B} \text{ solution of the equation} \\
&\qquad -\triangle_{\bar{g}+g} u + \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} = \rho \frac{(\bar{h}+h)e^u}{\int_\Sigma (\bar{h}+h)e^u dV_{\bar{g}+g}} \quad \text{is non degenerate}\} = \\
&\{(g,h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \ : \ \text{any solution of the equation} \\
&\qquad -\triangle_{\bar{g}+g} u + \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} = \rho \frac{(\bar{h}+h)e^u}{\int_\Sigma (\bar{h}+h)e^u dV_{\bar{g}+g}} \quad \text{is non degenerate}\},
\end{aligned}$$

where the last equality follows from our choice of $\bar{R}$. Finally, since we have this for any $(\bar{g}, \bar{h}) \in \mathcal{M}^2 \times C^2(\Sigma)^+$, the proof of the first part of the claim is complete.

For what concerns $\mathcal{D}_1(\rho)$, the openness in $\mathcal{M}^2_1 \times C^2(\Sigma)^+$ follows immediately from the openness of $\mathcal{D}(\rho)$ in $\mathcal{M}_2 \times C^2(\Sigma)^+$. Actually the previous proof also implies the density; indeed focusing on the statement of Lemma 2.8 it can be understand that we proved that for any $(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+$ there exists $\tilde{h}$ arbitrarily close to h such that $(g, \tilde{h}) \in \mathcal{D}_\rho$. Applying this remark to an element $(g,h) \in \mathcal{M}^2_1 \times C^2(\Sigma)^+$ we get the second part of the claim and this concludes the proof. ■

For what concerns the singular equation, it is worth to notice that exactly the same procedure followed in the previous proof allows to show generic non degeneracy of the critical points of the functional J_ρ. That will be crucial to obtain the multiplicity estimate.

Proposition 2.10. *For $\rho \notin \Gamma(\underline{\alpha}_m)$ and for (g,K) in an open and dense subset of $\mathcal{M}^2 \times C^{0,1}(S)$ $J_{\rho,(g,K)}$ is a Morse functional.*

2.1.1 Regular values of ρ

Proof of Theorem 0.1 We will first compute the $\sum_{q \geq 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$, showing that it equals the right–hand side of formula (10). Finally to conclude, thanks to Proposition 2.1, it will be enough to get the estimate (10) under the further assumption that (g,h) are such that $I_{\rho,(g,h)}$ is a Morse functional.

`Step 1`

We now focus on the homology with coefficients in $\mathbb{Z}_2$ of Σ_k, i.e. the set of formal barycenters of a surface Σ of order k, defined in (9). We will present the main steps of the procedure, performed in [48], to achieve a description of $H_*((S^2)_k; \mathbb{Z}_2)$ and we will derive from that the description in the case of any surface.

Then we will compute the sum of the dimensions of the homology groups of Σ_k, the real aim of this step.

First of all the main theorems in [48], dealing with the space of formal barycenters on topological spaces, imply in particular that

Theorem 2.11. *[48] For any manifold M, let M_k be the set of formal sums defined in* (25)*.*
Then for any $q \geq 0$ $\tilde{H}_q(M_k;\mathbb{Z}_2) \cong H_{q+1}(\overline{SP}^k(S^1 \wedge M);\mathbb{Z}_2)$.

Remark 2.12. *A key point is that, thanks to the isomorphism above, in the case of a surface Σ, the homology of Σ_k only depends on the homology of Σ, in particular on its genus.*

Let us consider two particular situations. When $M \cong S^n$, applying Theorem 2.11 we can immediately describe the reduced homology of the space of formal barycenters by means of the homology of a reduced symmetric product of S^n. With some more work we can also deal with the case when M is a surface of genus $\mathfrak{g}$, reducing again the comprehension of the homology of the formal barycenters to the understanding of the homology of a reduced symmetric product of S^3.

- Let $M \cong S^2$, then for any $q \geq 0$

$$\tilde{H}_q((S^2)_k;\mathbb{Z}_2) \cong H_{q+1}(\overline{SP}^k(S^3);\mathbb{Z}_2). \tag{2.8}$$

- Let $M \cong \Sigma_{\mathfrak{g}}$, a surface of genus $\mathfrak{g}$. Notice that $S^1 \wedge \Sigma_{\mathfrak{g}}$ has the same homology of $S^3 \vee (\bigvee_{j=1}^{2\mathfrak{g}} S^2)$; hence, recalling that the reduced symmetric product of a space only depends on its homology and using, in order, the properties of the reduced symmetric product, those of the homology of the wedge sum, the fact that $\overline{SP}^n(S^2) \cong S^{2n}$ and the properties of the homology of the reduced suspension, we obtain for any $q \geq 0$:

$$\begin{aligned}
\tilde{H}_q((\Sigma_(\mathfrak{g}))_k;\mathbb{Z}_2) &\cong H_{q+1}(\overline{SP}^k(S^1 \wedge \Sigma_{\mathfrak{g}});\mathbb{Z}_2) \\
&\cong H_{q+1}(\overline{SP}^k(S^3 \vee (\bigvee_{j=1}^{2\mathfrak{g}} S^2));\mathbb{Z}_2) \\
&\cong H_{q+1}(\bigvee_{r+s_1+\ldots+s_{2\mathfrak{g}}=k} (\overline{SP}^r S^3 \wedge (\bigwedge_{j=1}^{2\mathfrak{g}} \overline{SP}^{s_j} S^2));\mathbb{Z}_2) \\
&\cong \bigoplus_{r+s_1+\ldots+s_{2\mathfrak{g}}=k} H_{q+1}(\overline{SP}^r S^3 \wedge (\bigwedge_{j=1}^{2\mathfrak{g}} \overline{SP}^{s_j} S^2);\mathbb{Z}_2) \\
&\cong \bigoplus_{r+s_1+\ldots+s_{2\mathfrak{g}}=k} H_{q+1}(\overline{SP}^r S^3 \wedge (\bigwedge_{j=1}^{2\mathfrak{g}} S^{2s_j});\mathbb{Z}_2) \\
&\cong \bigoplus_{r+s_1+\ldots+s_{2\mathfrak{g}}=k} \tilde{H}_{q-2k+2r+1}(\overline{SP}^r(S^3);\mathbb{Z}_2).
\end{aligned} \tag{2.9}$$

In the last line, if $q < \max\{0, 2k-2r-1\}$, we mean $\tilde{H}_{q-2k+2r+1}(\overline{SP}^r(S^3))$ to be 0.

The above examples show that it is really useful to have a description of $\tilde{H}_*(\overline{SP}^r(S^{n+1}); \mathbb{Z}_2)$ for $r \geq 1$, being $\overline{SP}^0(S^{n+1}) = S^0$. Actually what we need is to estimate the dimensions of the homology groups $\tilde{H}_q(\overline{SP}^r(S^{n+1}); \mathbb{Z}_2)$, seen as vector spaces. To do that it will be more convenient, at least for notations, to switch by duality to cohomology; namely to study the dual vector space $\tilde{H}^*(\overline{SP}^r(S^{n+1}); \mathbb{Z}_2)$. In fact at the moment we are just interested in the case $n = 2$, but the general case will be exploited in Subsection 2.1.3.
General facts about symmetric products ([47], page 483) show that

$$\tilde{H}^*(\overline{SP}^r(S^{n+1}); \mathbb{Z}_2) \hookrightarrow \bigotimes_{i \geq 0} H^*(K(\tilde{H}_i(S^n), i+1); \mathbb{Z}_2) = H^*(K(\mathbb{Z}, n+1); \mathbb{Z}_2).$$

Actually we will just summarize how Kallel and Karoui found it, deeply using works of Milgram [70], Nakaoka [72] and Serre [78]. Using the Steenrod splitting it is possible to write:

$$\tilde{H}^*(K(\mathbb{Z}, n); \mathbb{Z}_2) \cong \bigoplus_{j \geq 1} \tilde{H}^*(\overline{SP}^j S^n; \mathbb{Z}_2);$$

therefore, if we are able to filter $\tilde{H}^*(K(\mathbb{Z}, n); \mathbb{Z}_2)$ over the positive integers so that $\tilde{H}^*(\overline{SP}^j S^n; \mathbb{Z}_2)$ corresponds to the class of filtration degree precisely j, we are done. This procedure relies on the following result.

Theorem 2.13. *[78]$H^*((\mathbb{Z}, n); \mathbb{Z}_2)$ is the polynomial algebra with coefficients in $\mathbb{Z}_2$ generated by the iterated Steenrod squares $Sq^I(u_n)$, where u_n is the only generator of $H^n((\mathbb{Z}, n); \mathbb{Z}_2)$ and $I = \{i_1, \ldots, i_r\}$ is admissible, i.e. if I satisfies the conditions below:*

- $i_1 - i_2 - \ldots - i_m < n$,
- $i_k \geq 2i_{k+1}, \;\; k = 1, 2, \ldots, m-1$,
- $i_m > 1$.

Finally the following Theorem leads to the characterization of $\tilde{H}^*(\overline{SP}^r S^{n+1}; \mathbb{Z}_2)$.

Theorem 2.14. *([70], [72]) Set the filtration degree of $Sq^I(u_n)$, $I = \{i_1, i_2, \ldots, i_m\}$, to be 2^m. Then $\tilde{H}^*(\overline{SP}^r S^{n+1}; \mathbb{Z}_2)$ corresponds to elements of exact filtration r in $H^*((\mathbb{Z}, n); \mathbb{Z}_2)$.*
In particular when $n = 3$:

$$\tilde{H}^*(\overline{SP}^r S^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[f_{(3,1)}, f_{(5,2)}, \ldots, f_{(2^{i+1}+1, 2^i)}, \ldots]_r \tag{2.10}$$

where $f_{(3,1)} = u_3$ and, for $i \geq 1$, $f_{(2^{i+1}+1, 2^i)} = Sq^I u_3$ with $I = \{2^i, \ldots, 4, 2\}$.

Since after considering the filtration $H^*(\overline{SP}^r S^3;\mathbb{Z}_2)$ is a bigraded algebra over $\mathbb{Z}_2$, writing $f_{(q,m)}$ we want to emphasize that $f_{(q,m)}$ is an element of cohomological degree q and filtration degree m.

Clearly Theorem 2.11 together with Theorem 2.14 (see also (2.8)) yield by duality to a complete description of $\tilde{H}_*((S^2)_k;\mathbb{Z}_2)$. Notice also that our computations in (2.9) allow to describe $\tilde{H}_*(\Sigma_k;\mathbb{Z}_2)$ for any other Σ.

We can now turn to the estimate of $\sum_{q\geq 0}^{\infty}\dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2)$. By (2.8), (2.9) and using that for any $k\geq 1$ $\overline{SP}^k(S^3)$ is connected while $\overline{SP}^0(S^3)=S^0$, we obtain:

$$\sum_{q\geq 0}\dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2) = \tag{2.11}$$
$$=\begin{cases}\sum_{q\geq 0}\dim(\tilde{H}_q(\overline{SP}^k S^3;\mathbb{Z}_2)) & \text{if } \mathfrak{g}(\Sigma)=0,\\ \sum_{r=0}^{k}\binom{k-r+2\mathfrak{g}-1}{k-r}\sum_{q\geq 0}\dim(\tilde{H}_q(\overline{SP}^r S^3;\mathbb{Z}_2)) & \text{if } \mathfrak{g}=\mathfrak{g}(\Sigma)>0.\end{cases}$$

In the last line the binomial coefficient $\binom{k-r+2g-1}{k-r}$ counts the number of tuples $(s_1,\ldots,s_{2\mathfrak{g}})$ such that $\sum_{j=1}^{2\mathfrak{g}} s_j = k-r$; instead we denote as $\mathfrak{g}(\Sigma)$ the genus of the surface Σ.

Formula (2.18) rewritten in terms of the Euler characteristic of Σ, $\chi(\Sigma)=2-2\mathfrak{g}(\Sigma)$, becomes:

$$\sum_{q\geq 0}\dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2) = \tag{2.12}$$
$$=\begin{cases}\sum_{q\geq 0}\dim(\tilde{H}_q(\overline{SP}^k S^3;\mathbb{Z}_2)) & \text{if } \chi(\Sigma)=2,\\ \sum_{r=0}^{k}\binom{k-r-\chi(\Sigma)+1}{k-r}\sum_{q\geq 0}\dim(\tilde{H}_q(\overline{SP}^r S^3;\mathbb{Z}_2)) & \text{if } \chi(\Sigma)\leq 0.\end{cases}$$

In order to estimate, given $r\geq 1$, the quantity $\sum_{q\geq 0}\dim(\tilde{H}_q(\overline{SP}^r S^3;\mathbb{Z}_2))$, we can first pass to cohomology by duality, being, for any $q\in\mathbb{N}$, $\dim(\tilde{H}_q(\overline{SP}^r S^3;\mathbb{Z}_2)) = \dim(\tilde{H}^q(\overline{SP}^r S^3;\mathbb{Z}_2))$, and then exploit the isomorphism in (2.10) and compute how many elements of filtration degree r there are in $\mathbb{Z}_2[f_{(3,1)},f_{(5,2)},\ldots,f_{(2^{i+1}+1,2^i)},\ldots]$. These elements are of the form

$$F(r,n,a_1,\ldots,a_{i_n}) = f_{(3,1)}^{r-2n} f_{(2^{1+1}+1,2^1)}^{a_1}\cdots f_{(2^{i_n+1}+1,2^{i_n})}^{a_{i_n}}, \tag{2.13}$$

where $n\in\mathbb{N}$ is such that

$$r-2n\geq 0;\quad i_0:=1,\quad i_n=\max\left\{i\,|\,2^i\leq 2n\right\};\quad a_j\in\mathbb{N}\ \text{s.t.}\ \sum_{j=1}^{i_n} a_j 2^j = 2n. \tag{2.14}$$

Since the last condition can be rewritten as $\sum_{j=0}^{i_n-1} a_{j+1}2^j = n$, for any $n\in\{0,\ldots,[\frac{r}{2}]\}$, there are as many i_n-tuples $(a_1,\ldots,a_{i_n})$ as the partition of n into powers of 2.

Finding such number p_n (as a function of n) is a classical problem in combinatorics going back to Euler. Indeed Euler in [43] showed that p_n is described by the following recurrence formula:

$$p_0 = 1, \qquad p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j \quad \forall\, m \in \mathbb{N}.$$

In particular, since in our case n is varying in $\{0, \dots, [\frac{r}{2}]\}$, adding up over n we obtain that there are exactly $\sum_{n=0}^{[\frac{r}{2}]} p_n = p_r$ elements of the form (2.13), which are the generators of $\mathbb{Z}_2[f_{(3,1)}, f_{(5,2)}, \dots, f_{(2^{i_{r,n}+1}+1, 2^{i_{r,n}})}]_r$. At last, this computation together with (2.12) permits to get an explicit formula for the sum in terms of the elements of the sequence $\{p_n\}_n$:

$$\sum_{q\geq 0} \dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2) = \begin{cases} p_k & \text{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \text{if } \chi(\Sigma) \leq 0. \end{cases} \tag{2.15}$$

Step 2

In order to prove Theorem 0.1, thanks to what we proved in Proposition 2.1, we can assume without loss of generality that $g \in \mathcal{M}^2$ and $h \in C^2(\Sigma)^+$ are such that all the critical points of $I_{\rho,(g,h)}$ are non degenerate. Henceforth we will work assuming this property of g and h to hold and we will write I_ρ for $I_{\rho,(g,h)}$.
Let us fix two real positive numbers $b > 0$ and $L > 0$ sufficiently large so that the hypotheses of Proposition 1.8 and of Proposition 1.15 are verified and such that b and $-L$ are regular values of I_ρ. Thanks to the considerations after Theorem 1.17, we can apply weak Morse Inequalities to I_ρ with $G = \mathbb{Z}_2$ and we have that

$$\begin{aligned} \#\{\text{critical points of } I_\rho \text{ in } -L \leq I_\rho \leq b\} &\geq \sum_{q\geq 0} \beta_q(-L, b; \mathbb{Z}_2) \\ &\equiv \sum_{q\geq 0} \dim H_q(\{I_\rho \leq b\}), \{I_\rho \leq -L\}; \mathbb{Z}_2). \end{aligned} \tag{2.16}$$

We point out that whenever the group G is a field, the *rank* of the homology group is nothing but the *dimension* of the homology group seen as vector space. Now, to estimate from below the number of critical points we have to focus on the right hand side of the previous inequality.

Since $-L$ is a regular value, by Corollary 1.13 we have that $\{I_\rho \leq -L\}$ is a deformation retract of some neighborhood in $H^1_g(\Sigma)$ and so we can apply Theorem 1.21 obtaining:

$$\begin{aligned} &\dots \to \tilde{H}_q(\{I_\rho \leq -L\}; \mathbb{Z}_2) \to \tilde{H}_q(\{I_\rho \leq b\}; \mathbb{Z}_2) \to H_q(\{I_\rho \leq b\}, \{I_\rho \leq -L\}; \mathbb{Z}_2) \to \dots \\ &\dots \to \tilde{H}_{q-1}(\{I_\rho \leq -L\}; \mathbb{Z}_2) \to \tilde{H}_{q-1}(\{I_\rho \leq b\}; \mathbb{Z}_2) \to \dots \end{aligned}$$

Then by Proposition 1.8, Proposition 1.15 and from the exactness of the latter homology sequence we get:

$$\begin{cases} H_{q+1}(\{I_\rho \leq b\}, \{I_\rho \leq -L\}; \mathbb{Z}_2) \cong \tilde{H}_q(\Sigma_k; \mathbb{Z}_2) \ q \geq 0 \\ H_0(\{I_\rho \leq b\}, \{I_\rho \leq -L\}; \mathbb{Z}_2) = 0. \end{cases} \tag{2.17}$$

Finally (2.16), (2.17) and (2.15) imply that

$$\begin{aligned}
&\#\ \{\text{solutions of } (*)_\rho\} \geq \#\{\text{critical points of } I_\rho \text{ in } \{-L \leq I_\rho \leq b\}\}\\
&\overset{(2.16)}{\geq} \sum_{q\geq 0} \dim H_q(\{I_\rho \leq b\}\,, \{I_\rho \leq -L\}\,; \mathbb{Z}_2) \overset{(2.17)}{\geq} \sum_{q\geq 0} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)\\
&\overset{(2.15)}{=} \begin{cases} p_k & \text{if } \chi(\Sigma) = 2,\\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \text{if } \chi(\Sigma) \leq 0. \end{cases}
\end{aligned} \tag{2.18}$$

This concludes the proof. ■

Proof of Corollary 0.2 1. Let us denote by $N_{k,\chi(\Sigma)}$ the right–hand side of formula (10). It will be enough to prove that (except in the case $\chi(\Sigma) = 2$ and $k = 1$) $N_{k,\chi(\Sigma)} > \mathrm{d}(k, \chi(\Sigma)) \geq 0$. This is trivial for $\chi(\Sigma) = 2$, while in the remaining cases, since $p_r \geq 1$ for any $r \in \mathbb{N}$, we have:

$$\begin{aligned}
N_{k,\chi(\Sigma)} &= \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r \geq \binom{k-\chi(\Sigma)+1}{k} p_0 =\\
&= \frac{k-\chi(\Sigma)+1}{-\chi(\Sigma)+1}\, \mathrm{d}(k, \chi(\Sigma)) > \mathrm{d}(k, \chi(\Sigma)).
\end{aligned}$$

2. To prove this point we will use a formula on the asymptotic behavior of p_{2n} derived by Mahler (see [62] and also [30]). Let us recall his result in an explicit way:

$$p_{2n} = O_n(1)\Big(\frac{n}{\log n}\Big)^{\frac{1}{2l_2}\log(\frac{n}{\log n})+1+\frac{ll_2}{l_2}}\, n^{(\frac{1}{l_2}-\frac{1}{2})},$$

where $l_2 := \log 2$ and $ll_2 =: \log\log 2$.
Now just combining Theorem 0.1 with the previous asymptotic formula, we obtain estimate (11). ■

2.1.2 Critical values of ρ

We want now to derive from our estimate (10) the multiplicity result for critical values of ρ stated in the Introduction, namely Theorem 0.3. The containts of this subsection are part of a work in progress [34].

We first need a proposition guaranteeing the compactness in $\Sigma^k \setminus \Gamma_k$ of the set of critical points of the function f_h defined in (1.26).

Proposition 2.15. *Let $\bar{h}$ be a C^2 positive function on Σ, then there exists a compact subset $K \subset \Sigma^k \setminus \Gamma_k$ such that all the critical points of $f_{\bar{h}} : \Sigma^k \setminus \Gamma_k \to \mathbb{R}$ are contained in K.*

PROOF. Suppose by contradiction that there exists a sequence of critical points $Q_n = (q_1^n, \ldots, q_k^n)$ of f_h such that $\mathrm{d}(Q_n, \Gamma_k) \overset{n\to+\infty}{\longrightarrow} 0$.
Since Σ^k is compact, up to a subsequence $Q_n \overset{n\to+\infty}{\longrightarrow} \bar{Q} = (\bar{q}_1, \ldots, \bar{q}_k) \in \Gamma_k$. Let denote by $\{\bar{q}_{s_1}, \ldots, \bar{q}_{s_d}\}$, $d < k$, a maximal subset of distinct points between $\{\bar{q}_1, \ldots, \bar{q}_k\}$. Without loss of generality we can assume that $\bar{q}_{s_1} = \bar{q}_1$ and that there exist at least one index $j > 1$ such that $\bar{q}_j = \bar{q}_1$.

Next, we can define $\delta := \frac{1}{2}\min_{i\neq j} \mathrm{d}(\bar{q}_{s_i}, \bar{q}_{s_j})$ and we take $\varepsilon > 0$ smaller than the injectivity radius of Σ. By our contradiction assumption there exists $n := n_\varepsilon$ sufficiently large in order to have that, for any $j \in \{1, \ldots, k\}$, if $q_j^n \to \bar{q}_{s_l}$ then $q_j^n \in B_\varepsilon(\bar{q}_{s_l})$.
Again, the invariance under permutations allows to assume, without loss of generality, that q_1^n is such that $\mathrm{d}(q_1^n, \bar{q}_1) = \max\{\mathrm{d}(q_m^n, \bar{q}_1) \,|\, q_m^n \in B_\varepsilon(\bar{q}_1)\}$.

We choose normal coordinates in $B_\varepsilon(\bar{q}_1)$ and we compute explicitly

$$\nabla f_h(q_1^n, \ldots, q_k^n) \cdot \frac{v^n}{\|v^n\|} \tag{2.19}$$

where the sequence of vectors v^n is defined as follows

$$v^n := \left(\sum_{m\neq 1, q_m^n \in B_\varepsilon(\bar{q}_1)} (q_1^n - q_m^n), 0, \ldots, 0 \right). \tag{2.20}$$

Let notice that by hypothesis (2.19) should be 0.
For $\varepsilon < \delta$, up to bounded terms, (2.19) equals:

$$\begin{aligned}
&8\pi\nabla\Big(\sum_{j=1}^{k}\sum_{l\neq j}\frac{1}{2\pi}\log \mathrm{d}(q_j^n, q_l^n)\Big)\cdot\frac{v^n}{\|v^n\|} = \\
&= 8\left(\sum_{l\neq 1}\frac{\nabla_{q_1}\mathrm{d}(q_1^n, q_l^n)}{\mathrm{d}(q_1^n, q_l^n)}, \ldots, \sum_{l\neq j}\frac{\nabla_{q_j}\mathrm{d}(q_j^n, q_l^n)}{\mathrm{d}(q_j^n, q_l^n)}, \ldots, \sum_{l\neq k}\frac{\nabla_{q_k}\mathrm{d}(q_k^n, q_l^n)}{\mathrm{d}(q_k^n, q_l^n)}\right)\cdot\frac{v^n}{\|v^n\|} = \\
&= 8 \sum_{\substack{l\neq 1 \\ m\neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1)}} \frac{\nabla_{q_1}\mathrm{d}(q_1^n, q_l^n)}{\mathrm{d}(q_1^n, q_l^n)}\cdot\frac{(q_1^n - q_m^n)}{\|v^n\|} = \\
&= 8 \sum_{\substack{l\neq 1,\, q_l^n \in B_\varepsilon(\bar{q}_1) \\ m\neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1)}} \frac{(q_1^n - q_l^n)\cdot(q_1^n - q_m^n)}{(\mathrm{d}(q_1^n, q_l^n))^2\ \|v^n\|} + \\
&\qquad + 8 \sum_{\substack{l\neq 1,\, q_l^n \notin B_\varepsilon(\bar{q}_1) \\ m\neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1)}} \frac{\nabla_{q_1}\mathrm{d}(q_1^n, q_l^n)}{\mathrm{d}(q_1^n, q_l^n)}\cdot\frac{q_1^n - q_m^n}{\|v^n\|}.
\end{aligned} \tag{2.21}$$

Let consider first the absolute value of the second term of (2.21)

$$\left| \sum_{\substack{l \neq 1,\, q_l^n \notin B_\varepsilon(\bar{q}_1) \\ m \neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1)}} \frac{\nabla_{q_1} \mathrm{d}(q_1^n, q_l^n)}{\mathrm{d}(q_1^n, q_l^n)} \cdot \frac{q_1^n - q_m^n}{\|v^n\|} \right| \leq \sum_{\substack{l \neq 1 \\ q_l^n \notin B_\varepsilon(\bar{q}_1)}} \frac{\|\nabla_{q_1} \mathrm{d}(q_1^n, q_l^n)\|}{\mathrm{d}(q_1^n, q_l^n)} \leq C_\delta. \quad (2.22)$$

Next, we focus on the first term of (2.21)

$$\sum_{\substack{l \neq 1,\, q_l^n \in B_\varepsilon(\bar{q}_1) \\ m \neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1)}} \frac{(q_1^n - q_l^n) \cdot (q_1^n - q_m^n)}{(\mathrm{d}(q_1^n, q_l^n))^2 \; \|v^n\|} =$$

$$= \sum_{\substack{l \neq 1,\, q_l^n \in B_\varepsilon(\bar{q}_1) \\ m \neq 1,\, q_m^n \in B_\varepsilon(\bar{q}_1) \\ l \neq m}} \frac{(q_1^n - q_l^n) \cdot (q_1^n - q_m^n)}{(\mathrm{d}(q_1^n, q_l^n))^2 \; \|v^n\|} + \frac{1}{\|v^n\|} \geq \frac{1}{\|v^n\|}. \quad (2.23)$$

The last inequality follows from our choice of q_1^n, indeed either the sum is empty, or, for any q_l^n, $q_m^n \in B_\varepsilon(\bar{q}_1)$, the scalar product $(q_1^n - q_l^n) \cdot (q_1^n - q_m^n)$ is positive, being $\mathrm{d}(q_1^n, \bar{q}_1) = \max\{\mathrm{d}(q_m^n, \bar{q}_1) \,|\, q_m^n \in B_\varepsilon(\bar{q}_1)\}$.
Finally, since, as $\varepsilon \to 0$ and consequently $n \to +\infty$, $\frac{1}{\|v^n\|} \to +\infty$, collecting (2.21), (2.22) and (2.23) we obtain that

$$\nabla f_h(q_1^n, \ldots, q_k^n) \cdot \frac{v^n}{\|v^n\|} \overset{\varepsilon \to 0, n \to +\infty}{\longrightarrow} +\infty,$$

which violates our contradiction assumption, namely the fact that, for any $n \in \mathbb{N}$, $Q^n = (q_1^n, \ldots, q_k^n)$ is a critical point of f_h, and concludes the proof. ■

We are now in position to prove the aforementioned theorem.

Proof of Theorem 0.3 Let us fix $\bar{h} \in C^3(\Sigma)^+$. We will see that

there exists $n_{\bar{h}} \in \mathbb{N}$ and $\delta > 0$ such that for any $k \geq n_{\bar{h}}$ the following holds: there exists an open and dense subset $\mathcal{D}$ of $B_\delta(\bar{h}) \subset C^3(\Sigma)^+$ such that for any $h \in \mathcal{D}$ the function f_h, introduced in (1.26)*, is a Morse function on $\Sigma^k \setminus \Gamma_k$ and the function $\triangle \log h + 8k\pi - 2K_g$ is positive on all its critical points,* (♮)

We assume for the moment that (♮) holds true, postponing its proof, and we fix an integer $k \geq n_{\bar{h}}$.
We remark that for any $h \in \mathcal{D}$, the conditions $(c1)$ and $(c2)$ are fulfilled and then we could apply Theorem 1.28. On the other hand we recall that, as showed in

Proposition 2.1, for a generic choice of h in $C^2(\Sigma)^+$ the functional $I_{\rho,h}$ is Morse. It is immediate to understand that if we replace $C^2(\Sigma)^+$ with $C^3(\Sigma)^+$ the statement remains true. Actually, analyzing the proof of the generic non degeneracy, we can see that the non criticality of ρ was needed only to ensure the boundedness of the set of solutions.
Keeping in mind these considerations and recalling that, thanks to Theorem 1.28 for any $h \in \mathcal{D}$ there exists a positive constant such that all the critical points of $I_{8k\pi}$ are contained in a ball of radius $C_k(h)$, we can easily deduce the existence of an open and dense subset $\tilde{\mathcal{D}}$ of $\mathcal{D}$ such that, for any $h \in \tilde{\mathcal{D}}$, the functional $I_{8k\pi}$ is Morse.

Then, by non degeneracy, for any $h \in \tilde{\mathcal{D}}$ there exists ε_h such that, for any $\rho \in (8k\pi - \varepsilon_h, 8k\pi)$, $I_{\rho,h}$ is still a Morse functional.

Clearly, without loss of generality, we can take $2\varepsilon_h$ to be smaller than the ε_k introduced in the statement of Theorem 1.28, in order to have that for any $\rho \in (8k\pi - \varepsilon_h, 8k\pi)$ all the critical points of $I_{\rho,h}$ are contained in $B_{C_k(h)}(0) \subset \bar{H}^1_g(\Sigma)$.

Finally, for $h \in \tilde{\mathcal{D}}$, which is an open and dense subset of $B_\delta(\bar{h}) \subset C^3(\Sigma)^+$, we are able to estimate from below the number of solutions of $(*)_{8k\pi}$. Indeed, the positivity of $\triangle \log h + 8k\pi - 2K_g$ on the critical points of f_h allow to exclude the existence of blowing up solutions when ρ approaches $8k\pi$ from below (see Theorem 1.28 and (1.28) or Theorem 1.27), and then the non degeneracy and the uniform bound on the set of solutions permit to deduce that:

$$\#\{\text{ solutions to } (*)_{8k\pi,h}\ \} \geq \#\{\text{ critical points of } I_{8k\pi-\varepsilon_h,h}\ \}.$$

In turn the right hand side can be controlled taking advantage of the proof of Theorem 0.1, being $I_{8k\pi-\varepsilon_h,h}$ a Morse functional, and so we obtain the desired estimate

$$\#\{\text{ solutions to } (*)_{8k\pi,h}\ \} \geq \begin{cases} p_{k-1} & \text{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^{k-1} \binom{k-r-\chi(\Sigma)}{k-r-1} p_r & \text{if } \chi(\Sigma) \leq 0, \end{cases}$$

To conclude it remains to verify ($\natural$).
Let $K \subset \Sigma^k \setminus \Gamma_k$ be a compact set containing all the critical points of $f_{\bar{h}}$, whose existence is guaranteed by the previous proposition. Then, let fix $\gamma > 0$ such that $\gamma < \frac{1}{2}\,\mathrm{d}(K, \Gamma_k)$. Now, eventually decreasing γ, it is possible to define an atlas on the tubular neighborhood $K_\gamma = \left\{x \in \Sigma^k : \mathrm{d}(x, K) < \gamma\right\}$ whose charts are $(B_g(\xi,\gamma), \phi^{-1})$, where $\phi : B(0,\gamma) \to B_g(\xi,\gamma)$. Here $B_g(\xi,\gamma) \subset \Sigma^k \setminus \Gamma_k$ is the ball centered at $\xi \in K$ with radius γ given by the metric g and $B(0,\gamma) \subset \mathbb{R}^{2k}$ is the ball centered at 0 with radius γ in the Euclidean space $\mathbb{R}^{2k}$.

Let choose $n_{\bar{h}}$ to be such that $\triangle \log \bar{h} + 8n_{\bar{h}}\pi - 2K_g$ is positive on $\bar{K_\gamma}$ and consider an integer $k \geq n_{\bar{h}}$.

It is clearly possible to find $\delta > 0$ sufficiently small such that if $h \in \mathcal{H}_\delta := \left\{h \in C^3(\Sigma) : \|h\|_{C^3} < \delta\right\}$, then $\bar{h} + h$ is positive, $\triangle \log(\bar{h} + h) + 8k\pi - 2K_g$ is still positive on $\bar{K_\gamma}$ and all the critical points of $f_{\bar{h}+h}$ belong to K_γ.

Given $\xi_0 \in K_\gamma$ and the chart $(B_g(\xi_0,\gamma),\phi^{-1})$ we set

$$\tilde f_{\bar h+h}(x) := f_{\bar h+h}(\phi(x)), \quad x \in B(0,\gamma), \quad h \in \mathcal{H}_\delta.$$

Now we introduce the C^1-map $F : \mathcal{H}_\delta \times B(0,\gamma) \subset C^3(\Sigma)\times\mathbb{R}^{2k} \longrightarrow \mathbb{R}^{2k}$ defined by

$$F(h,x) := \nabla \tilde f_{\bar h+h}(x). \tag{2.24}$$

We shall apply to the map F the transversality Theorem 2.4, in order to obtain the following claim:

the set $\Theta(\delta) = \big\{h \in C^3(\Sigma) \,:\, \text{all critical points of } f_{\bar h+h} \text{ are non degenerate}\big\}$
is an open and dense subset of $\mathcal{H}_\delta$*.*

In this case, using the notations of Theorem 2.4, we have $V = \mathcal{H}_\delta$, $U = B(0,\gamma)$, $X = Z = \mathbb{R}^{2k}$ and $Y = C^3(\Sigma)$. We choose $z_0 = 0$. Since X is a finite dimensional space, it is easy to check that for any $h \in \mathcal{H}_\delta$ the map $x \mapsto F(h,x)$ is Fredholm of index 0 and so assumption (i) of Theorem 2.4 holds. Moreover, assumption (iii) immediately follows again by the fact that X is a finite dimensional space together with the observation that $\overline{(K_\gamma)_\gamma} = \overline{K_2\gamma} \subset\subset \Sigma^k \setminus \Gamma_k$. Assumption (ii) is verified in Lemma 2.16 below.
Finally we are in position to apply the transversality theorem and we get that the set

$$\begin{aligned}\Theta(\xi_0,\delta) :=& \Big\{h \in \mathcal{H}_\delta \,:\, F'_x(h,x) : \mathbb{R}^{2k} \to \mathbb{R}^{2k} \text{ is invertible at any point} \\ &\qquad (h,x) \text{ such that } F(h,x) = 0\big\} \qquad (2.25)\\ =& \ \big\{h \in \mathcal{H}_\delta \,:\, \text{the critical points of } f_{\bar h+h} \text{ in } B_g(\xi_0,\gamma) \text{ are non degenerate}\big\}\end{aligned}$$

is an open and dense subset of $\mathcal{H}_\delta$.

Next we take a finite covering $\{B_g(\xi_i,\gamma)\}_{i=1,\dots,\nu}$ of K_γ, where $\xi_1,\dots,\xi_\nu \in K_\gamma$. For any index i there exists an open and dense subset $\Theta(\xi_i,\delta)$ (see (2.25)) of $\mathcal{H}_\delta$ such that the critical points of $f_{\bar h+h}$ in $B_g(\xi_i,\gamma)$ are non degenerate for any $h \in \Theta(\xi_i,\delta)$. Let $\Theta(\delta) := \cap_{i=1,\dots,\nu}\Theta(\xi_i,\delta)$. It is immediate that $\Theta(\delta)$ is an open and dense subset of $\mathcal{H}_\delta$ such that all the critical points of $f_{\bar h+h}$ are non degenerate for $h \in \Theta(\delta)$.
Finally it is enough to set $\mathcal{D} = \bar h + \Theta(\delta)$ to obtain ($\natural$).
The proof is thereby complete. ∎

Lemma 2.16. *The map* $(h,x) \mapsto F'_h(\tilde h,\tilde x)[h] + F'_x(\tilde h,\tilde x)x$ *is onto on* $\mathbb{R}^{2k}$ *for any* $(\tilde h,\tilde x) \in \mathcal{H}_\delta \times B(0,\gamma)$ *such that* $F(\tilde h,\tilde x) = 0$.

Proof. Let $(\tilde h,\tilde x)$ be such that $F(\tilde h,\tilde x) = 0$. We will prove that the map $F'_h(\tilde h,\tilde x) : C^3(\Sigma) \to \mathbb{R}^{2k}$ is onto. More precisely we are going to show that given $d = (d_1,\dots,d_{2k}) \in \mathbb{R}^{2k}$

$$\text{there exists } h \in C^3(\Sigma) \text{ such that } F'_h(\tilde h,\tilde x)[h] = d; \tag{2.26}$$

the claim will follow immediately.

We point out that the ontoness of the map $h \mapsto F'_h(\tilde{h},\tilde{x})[h]$ is invariant with respect to a change of variable. Then, to show (2.26) we calculate $D_h\partial_i f_{\bar{h}+\tilde{h}}(\phi(\tilde{x}))[h]$ by choosing the normal coordinates.

Let us compute the i-th component of $F'_h(\tilde{h},\tilde{x})[h]$, for $i = 1,\dots,2k$,

$$(F'_h(\tilde{h},\tilde{x})[h])_i = \sum_{j=1}^{k} \frac{\nabla(\bar{h}+h)(\phi_j(\tilde{x}))\cdot\frac{\partial\phi_j(\tilde{x})}{\partial x_i}(\bar{h}+\tilde{h})(\phi_j(\tilde{x})) - \nabla(\bar{h}+\tilde{h})(\phi_j(\tilde{x}))\cdot\frac{\partial\phi_j(\tilde{x})}{\partial x_i}(\bar{h}+h)(\phi_j(\tilde{x}))}{((\bar{h}+\tilde{h})(\phi_j(\tilde{x})))^2},$$

where $\phi(\tilde{x}) = (\phi_1(\tilde{x}),\dots,\phi_k(\tilde{x}))$.

If we restrict ourselves to consider h such that, for any $j = 1,\dots,k$, $(\bar{h}+h)(\phi_j(\tilde{x})) = 1$, then to demonstrate (2.26) it is enough to find a C^3 function h verifying also the following conditions:

$$\sum_{j=1}^{k}\frac{\nabla(\bar{h}+h)(\phi_j(\tilde{x}))}{(\bar{h}+\tilde{h})(\phi_j(\tilde{x}))}\cdot\frac{\partial\phi_j(\tilde{x})}{\partial x_i} = d_i + \sum_{j=1}^{k}\frac{\nabla(\bar{h}+\tilde{h})(\phi_j(\tilde{x}))\cdot\frac{\partial\phi_j(\tilde{x})}{\partial x_i}}{((\bar{h}+\tilde{h})(\phi_j(\tilde{x})))^2} \qquad i = 1,\dots,2k. \tag{2.27}$$

Finally, noticing that for any i the right hand side of (2.27) is constant, using that the Jacobian of ϕ is invertible and that $\phi(\tilde{x}) \in \Sigma^k \setminus \Gamma_k$ (and then its components $\phi_j(\tilde{x})$ are distinct points of Σ), it is not hard to see that it is sufficient to prescribe, according to (2.27), the values of $\nabla(\bar{h}+h)(\phi_j(\tilde{x}))$ to find the desired function h. That concludes the proof. ■

2.1.3 Conformal metrics with constant Q-curvature

This subsection is devoted to obtain generic multiplicity of conformal metrics with constant Q-curvature and to compare this result to the multiplicity estimate which can be deduced by the degree formula (23). More precisely, without following the order in which the results are stated in the Introduction, we are going to prove Theorem 0.9 and Corollary 0.10; the reason is that these results are strictly linked with the others already proved in this section.

Proof of Theorem 0.9 We can reason as in the proof of Theorem 0.1: the main differences are that the presence of negative eigenvalues for $\mathrm{P_g}$ affects the topology of the sublevels of the Euler functional and that in four dimensions we can not classify the manifolds in term of their Euler characteristic.

In [38] was shown that the counterpart of Proposition 1.7 holds true replacing Σ_k with $A_{k,\bar{k}} = M_k \times B_1^{\bar{k}}$, moreover, reasoning exactly as in Morse, one can see that the low sublevels of II are homotopically equivalent to $A_{k,\bar{k}}$.

Here M_k is the set of k-barycenters of M (defined in (25)), $B_1^{\bar{k}}$ the closed unit ball in $\mathbb{R}^{\bar{k}}$ while the equivalence relation $\sim$ means that $M_k \times \partial B_1^{\bar{k}}$ is identified with $\partial B_1^{\bar{k}}$,

namely $M_k \times \{y\}$ for every fixed $y \in \partial B_1^{\bar{k}}$ is collapsed to a single point.
Furthermore the proof of Proposition 1.15 adapts with minor modifications to equation (#), therefore the high sublevels of II turn out to be retractions of $H^2(M)$ and hence contractible sets.

Then, following exactly Step 2 of the proof of Theorem 0.1, we find that, choosing L and b sufficiently large positive real numbers, such that $-L$ and b are regular values for II,

$$\begin{aligned}&\#\{\text{solutions of } (\#)\} \geq \#\{\text{critical points of } II \text{ in } \{-L \leq II \leq b\}\}\\ &\geq \sum_{q\geq 0} \dim H_q(\{II \leq b\}, \{II \leq -L\}; \mathbb{Z}_2) \geq \sum_{q \geq 0} \dim \tilde{H}_q(A_{k,\bar{k}}; \mathbb{Z}_2).\end{aligned} \tag{2.28}$$

To compute the latter sum we can use the Mayer-Vietoris sequence, see for example [47], page 149. We can cover $A_{k,\bar{k}}$ with the two sets

$$\mathcal{A} = M_k \times B^{\bar{k}}_{\frac{3}{4}}, \qquad \mathcal{B} = M_k \times (B_1^{\bar{k}} \setminus B^{\bar{k}}_{\frac{1}{4}}),$$

where $B_r^{\bar{k}}$ stands for the closed ball of radius r in $\mathbb{R}^{\bar{k}}$. Clearly $\mathcal{A}$ has the homology type of M_k, $\mathcal{B}$ that of $S^{\bar{k}-1}$ and $\mathcal{A}\cap\mathcal{B}$ that of $M_k \times S^{\bar{k}-1}$. Therefore, by the exactness of the Mayer-Vietoris sequence and the Kunneth theorem we find the relation

$$\begin{cases} \tilde{H}_{\bar{k}+p}(A_{k,\bar{k}}) \cong \tilde{H}_p(M_k) & \text{for } p \geq 1, \\ \tilde{H}_q(A_{k,\bar{k}}) \cong 0 & \text{for } 0 \leq q \leq \bar{k}, \end{cases}$$

which implies

$$\sum_{q\geq 0} \dim \tilde{H}_q(A_{k,\bar{k}}; \mathbb{Z}_2) = \sum_{q \geq 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2). \tag{2.29}$$

From formula (2.28) and (2.29) we deduce that the problem reduces to the computation of $\sum_{q\geq 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2)$. By Theorem 2.11 we immediately get

$$\sum_{q\geq 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2) = \sum_{q\geq 0} \dim H_{q+1}(\overline{SP}^k(S^1 \wedge M); \mathbb{Z}_2). \tag{2.30}$$

Since $S^1 \wedge M$ is a CW complex with top integral homology group $H_5(M;\mathbb{Z}) = \mathbb{Z}$ and $\operatorname{rank}(H_3(M;\mathbb{Z})) \geq \chi(M) - 2$, it has the homology of $S^5 \vee (\bigvee_{j=1}^{\chi(M)-2} S^3) \vee Y$ for some topological space Y. Thus, as we did in the case of a surface of genus $\mathfrak{g} > 0$, we can apply the properties of the reduced symmetric product and of the homology of the wedge sum to obtain

$$H_{q+1}(\overline{SP}^k(S^1 \wedge M); \mathbb{Z}_2) \cong \bigoplus_{r+s_1+\ldots+s_{\chi(M)-2}+t=k} H_{q+1}(\overline{SP}^r S^5 \wedge (\bigwedge_{j=1}^{\chi(M)-2} \overline{SP}^{s_j} S^3) \wedge \overline{SP}^t Y; \mathbb{Z}_2).$$

Considering now the sum of the dimensions we have

$$\sum_{q\geq 0} \dim H_{q+1}(\overline{SP}^k(S^1\wedge M);\mathbb{Z}_2) \geq \sum_{q\geq 0} \dim \tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2) +$$
$$+\sum_{r=0}^{k-1} \sum_{\sum_{j=1}^{\chi(M)-2} s_j = k-r} \sum_{q\geq 0} \dim H_{q+1}(\overline{SP}^r S^5 \wedge (\bigwedge_{j=1}^{\chi(M)-2} \overline{SP}^{s_j} S^3);\mathbb{Z}_2). \quad (2.31)$$

Recalling that by definition the smash product $X\bigwedge Y$ is the quotient $X\times Y/X\vee Y$ and using the exact sequence for relative homology it is possible to see that for any $(r, s_1,\ldots,s_{\chi(M)-2})$ such that $\sum_{j=1}^{\chi(M)-2} s_j = k-r > 0$

$$H_{5r+3\sum_{j=1}^{\chi(M)-2} s_j}(\overline{SP}^r S^5 \wedge (\bigwedge_{j=1}^{\chi(M)-2} \overline{SP}^{s_j} S^3);\mathbb{Z}_2) \neq 0. \quad (2.32)$$

Clearly for $\chi(M)=2$ we just have

$$H_{q+1}(\overline{SP}^k(S^1\wedge M);\mathbb{Z}_2) \cong \bigoplus_{r+t=k} H_{q+1}(\overline{SP}^r S^5 \wedge \overline{SP}^t Y;\mathbb{Z}_2)$$

and

$$\sum_{q\geq 0} \dim H_{q+1}(\overline{SP}^k(S^1\wedge M);\mathbb{Z}_2) \geq \sum_{q\geq 0} \dim \tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2). \quad (2.33)$$

Next collecting formulas (2.28), (2.29), (2.30), (2.31), (2.32) and (2.33) we get that the number of solutions of (#) can be estimated from below by

$$\begin{cases} \sum_{q\geq 0} \dim \tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2) & \text{if } \chi(M)\geq 2, \\ \sum_{q\geq 0} \dim \tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2) + \sum_{r=0}^{k-1} \binom{k-r+\chi(M)-3}{k-r} & \text{if } \chi(M)\geq 3, \end{cases}$$

where the binomial coefficient $\binom{k-r+\chi(M)-3}{k-r}$ counts the number of tuples $(s_1,\ldots,s_{\chi(M)-2})$ such that $\sum_{j=1}^{\chi(M)-2} s_j = k-r$.
Finally, since all the admissible tuples $\{i_1,\ldots,i_r\}$ for $n=3$ are also admissible for $n=5$, the elements of exact filtration k in $H^*(\overline{SP}^k(S^5))$ are at least as many as the elements of exact filtration k in $H^*(\overline{SP}^k(S^3))$. Then by Theorem 2.14 and duality we have the desired estimate. ■

Proof of Corollary 0.10 This estimate follows immediately from Theorem 0.9, indeed it is sufficient to prove that the right–hand side of formula (24) is greater then $\mathrm{d}(k,\bar{k},\chi(M))$ (except for the case $\chi(M)=2$). But this is trivial because for

$\chi(M) \geq 3$

$$\sum_{r=0}^{k} \binom{k-r+\chi(\Sigma)-3}{k-r} \geq \binom{k+\chi(\Sigma)-3}{k} >$$
$$> \frac{(\chi(M)-k)\dots(\chi(M)-2)(\chi(M)-1)}{k!} = |\,\mathrm{d}(k,\bar{k},\chi(M))|.$$

On the other hand if $\chi(M) = 2$, then k should be 1 and then $p_1 = 1 = |\,\mathrm{d}(1,\bar{k},2)|$.
■

2.2 Multiplicity in presence of a local minimum

Before proving Theorem 0.4 we recall the previous result in [31] and we give an account of its proof.
In this section to simplify the notation, for any functional I and for any $c \in \mathbb{R}$ we will set $I^c := \{I \leq c\}$.

Theorem 2.17. *[31] If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least two different saddle points.*

Let us consider a new functional $\tilde{I}_\rho$ which coincides with I_ρ out of a small neighborhood of its local minimum, $\bar{u}$, and assumes large negative values near $\bar{u}$. To do that let consider an increasing cut-off function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

$$\zeta(x) < -L \text{ if } x < \frac{I_\rho(0) + \inf_{\partial B_r(0)} I_\rho}{2} \text{ and } \zeta(x) = x \text{ if } |x| \geq \inf_{\partial B_r(0)} I_\rho,$$

where L is a large positive constant to be fixed and $r > 0$ is such that $\inf_{\partial B_r(0)} I_\rho > I_\rho(\bar{u})$ and that $D^2 I_{\rho|B_r(0)}$. By means of ζ we define $\tilde{I}_\rho$ as follows:

$$\tilde{I}_\rho(u) := \begin{cases} I_\rho(u) & \text{if } u \in \bar{H}^1_g(T) \setminus B_r(0) \\ \zeta(I_\rho(u)) & \text{if } u \in B_r(0). \end{cases}$$

The choice of $\tilde{I}_\rho$, instead of I_ρ, is convenient because of the greater topological complexity of its low sublevels; in particular we will use that they are disconnected (just for the presence of a strict local minimum). Besides it is crucial to remark that saddle points of $\tilde{I}_\rho$ are also saddle points of I_ρ, hence we can limit ourselves to study $\tilde{I}_\rho$.

Let X denote the contractible cone over T and let ∂X be its boundary; they can be represented as $X = \frac{T\times[0,1]}{T\times\{0\}}$, $\partial X = \frac{T\times(\{0\}\cup\{1\})}{T\times\{0\}}$. To get the thesis it is sufficient to establish the following chain of inequalities:

$$
\begin{aligned}
\#\{\text{critical points of } \tilde{I}_\rho \text{ in } -L \le \tilde{I}_\rho \le b\} &\overset{1}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \tilde{I}_\rho^{-L}} \tilde{I}_\rho^b \overset{2}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \tilde{I}_\rho^b \qquad (2.34)\\
&\overset{3}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \phi(X) \overset{4}{\ge} \mathrm{Cat}_{\phi(X), \phi(\partial X)} \phi(X)\\
&\overset{5}{\ge} \mathrm{Cat}_{X, \partial X} X \overset{6}{\ge} 2,
\end{aligned}
$$

being ϕ is the homeomorphism on the image defined as follows:

$$
\begin{aligned}
\phi : \ X \quad &\longrightarrow \bar{H}_g^1(T)\\
(x,t) &\longmapsto t\,\tilde{\varphi}_{\lambda,x}.
\end{aligned}
$$

In the latter line $\tilde{\varphi}_{\lambda,x} := \varphi_{\lambda,x} - \bar{\varphi}_{\lambda,x}$, where $\varphi_{\lambda,x}$ is the function defined in (1.9) with $\sigma = x$.
Moreover the constants L, b and λ are suitably chosen in such a way that neither $-L$ nor b are critical levels, that Propositions 1.8 and 1.15 hold and so as to have $I_\rho(\tilde{\varphi}_{\lambda,x}) \le -L$, $\min_{x\in T} \|\tilde{\varphi}_{\lambda,x}\| > C_{\rho,b}$ and $\Psi(X) \subset I_\rho^b$ (where Ψ is defined in (1.21)), see [31] for further details. By this choice $\tilde{I}_\rho^b = I_\rho^b$.

The first inequality follows from Theorem 1.25 and the considerations after it; the only important thing to remark is that in the neighborhood of the origin, where $\tilde{I}_\rho$ differs from I_ρ, we can deform along the flux generated by a cutoff of the opposite of the gradient.

To derive the second inequality it is worth pointing out that in the hypotheses of Theorem 2.17 (merely when g is sufficiently close to dx^2 and h to 1), the map Ψ introduced in (1.21) turns out to be a diffeomorphism between $\{\tilde{\varphi}_{\lambda,x} \,|\, x \in \Sigma\}$ and Σ. So we can define a diffeomorphism $\omega : \Sigma \to \Sigma$ such that $\omega(\Psi(\tilde{\varphi}_{\lambda,x})) = x$.

Next, reminding that $\tilde{I}_\rho^{-L}$ is the disjoint union of I_ρ^{-L} and a neighborhood U of the origin, we can consider the following map:

$$
\begin{aligned}
\chi : \tilde{I}_\rho^{-L} &\longrightarrow \phi(\partial X)\\
u &\longmapsto \varphi_{\lambda,\omega(\Psi(u))} \qquad u \in I_\rho^{-L}\\
u &\longmapsto 0 \qquad\qquad\quad\ \ u \in U.
\end{aligned}
$$

Now, our purpose is to find a deformation retract (in $\tilde{I}_\rho^b$) of $\tilde{I}_\rho^{-L}$ onto $\phi(\partial X)$. First of all, let us set

$$
\begin{aligned}
\gamma : \tilde{I}_\rho^{-L} \times [0,1] &\longrightarrow \bar{H}_g^1(T)\\
(u,t) \quad &\longmapsto (1-t)u + t\chi(u).
\end{aligned}
$$

Then, thanks to our choice of b we know that $\tilde{I}_\rho^b \equiv I_\rho^b$ is a deformation retract of $\bar{H}_g^1(T)$, namely there exists a continuous map $\tau : \bar{H}_g^1(T) \to \tilde{I}_\rho^b$ such that $\tau_{|\tilde{I}_\rho^b} = \mathrm{Id}_{|\tilde{I}_\rho^b}$. So composing γ and τ we get the map ($h := \tau \circ \gamma : \tilde{I}_\rho^{-L} \times [0,1] \to \tilde{I}_\rho^b$) we were looking for. Indeed, for any $u \in \tilde{I}_\rho^{-L}$, $h(u,0) = u$ and $h(u,1) = \chi(u) \in \phi(\partial X)$, while, for any $(y,t) \in \phi(\partial X) \times [0,1]$, $h(y,t) = y$ (being $\chi_{|\phi(\partial X)} = \mathrm{Id}_{|\phi(\partial X)}$).

At last, if A_i and h_i ($i = 1, \dots, \mathrm{Cat}_{\tilde{I}^b_\rho, \tilde{I}^{-L}_\rho} \tilde{I}^b_\rho$) fulfill the conditions of the definition of relative category for $\mathrm{Cat}_{\tilde{I}^b_\rho, \tilde{I}^{-L}_\rho} \tilde{I}^b_\rho$, it is easy to prove that A_0, $h * h_0$ and A_i, h_i ($i \geq 1$) verify the definition of category for $\mathrm{Cat}_{\tilde{I}^b_\rho, \phi(\partial X)} \tilde{I}^b_\rho$, where $h * h_0 : A_0 \times [0,1] \to \tilde{I}^b_\rho$ is defined as follows:

$$h * h_0(x,t) := \begin{cases} h(h_0(x,2t),0) & t \leq \frac{1}{2} \\ h(h_0(x,1),2t-1) & t \geq \frac{1}{2}. \end{cases}$$

The third and the fifth inequality are merely applications of Points 3 and 4 of Proposition 1.24, since ϕ is an homeomorphism on the image.

Moreover, thanks to Proposition 1.24, Point 5, if we construct a continuous map $r : \tilde{I}^b_\rho \to \phi(X)$ such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and that $r^{-1}(\phi(\partial X)) = \phi(\partial X)$, we immediately prove the forth inequality. Let us define $C_\lambda := \min_{x\in T} \|\tilde{\varphi}_{\lambda,x}\|$, which is bigger than $C_{\rho,b}$, according to our choice of λ; then we are able to define Ψ and also χ on the set $\{v \in \bar{H}^1_g(T) \,:\, \|v\| \geq C_\lambda\}$.
Therefore the following map is well defined (see Figure 2.1):

$$\begin{aligned} r : \tilde{I}^b_\rho &\longrightarrow \phi(X) \\ 0 &\longmapsto 0 \\ u \in \{\|v\| \geq C_\lambda\} &\longmapsto \eta(\mathrm{dist}_{x\in T}(u, \tilde{\varphi}_{\lambda,x}))\, \chi(u) \\ u \in \{\|v\| \leq C_\lambda\} &\longmapsto \frac{\|u\|}{C_\lambda}\, r\Big(\frac{C_\lambda}{\|u\|}u\Big), \end{aligned}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth strictly decreasing function, such that $\eta(0) = 1$ and $\eta([\frac{1}{2}, +\infty)) = \frac{1}{3}$. Finally it is easy to verify that r is continuous and such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and $r^{-1}(\phi(\partial X)) = \phi(\partial X)$.

At last the sixth inequality has been tackled using a direct topological argument.

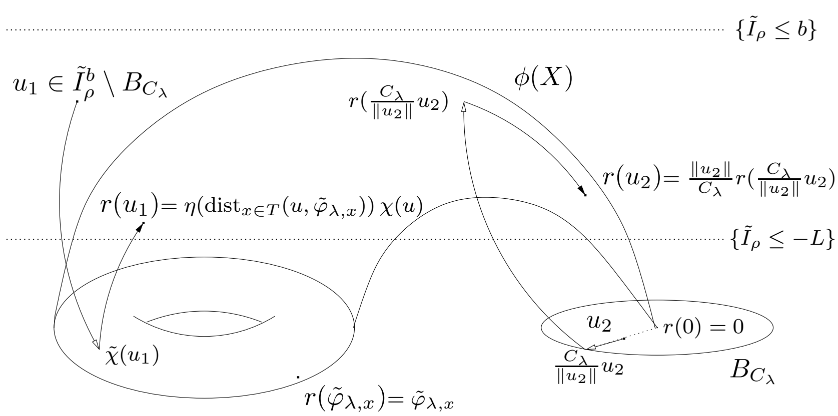

Figure 2.1: Construction of the map $r : \tilde{I}^b_\rho \to \phi(X)$.

PROOF OF THEOREM 0.4 The first five steps of (2.34) give

$$\#\left\{\text{critical points of } I_\rho \text{ different from } \bar{u}\right\} \geq \mathrm{Cat}_{X,\partial X}\, X, \tag{2.35}$$

where, as above, $\bar{u}$ is the strict local minimum of I_ρ near the origin and X is the contractible cone over T.
Thus, if we are able to improve the last inequality of (2.34), proving that $\mathrm{Cat}_{X,\partial X}\, X \geq 3$, the thesis follows. To do that we are going to establish a new chain of inequalities, involving the notion of cup length.

$$\begin{aligned}
\mathrm{Cat}_{X,\partial X}\, X &\overset{a}{\geq} \mathrm{Cat}_{T\times[0,1],T\times(\{0\}\cup\{1\})}(T\times[0,1]) \\
&\overset{b}{\geq} \mathrm{cat}_{T\times[0,1],T\times(\{0\}\cup\{1\})}(T\times[0,1]) \\
&\overset{c}{\geq} \mathrm{CL}(T\times[0,1],T\times(\{0\}\cup\{1\}))+1 \\
&\overset{d}{\geq} \mathrm{CL}(T)+1 \overset{e}{=} 3.
\end{aligned} \tag{2.36}$$

Let us first prove point a. Let consider the A_i and h_i verifying the conditions for $\mathrm{Cat}_{X,\partial X}\, X$.

First of all, in order to show that A_0 is disconnected, let us denote by $X_0 := T\times\{0\}/T\times\{0\}$ and $X_1 := T\times\{1\}/T\times\{0\}$ the two disconnected components of ∂X. By definition we know that $X_0 \cup X_1 = \partial X \subset A_0$ and that there exists $h_0 : A_0\times[0,1] \to X$ continuous with the properties: $h_0(A_0,1)\subset\partial X$ and $h_{0|\partial X\times[0,1]} \equiv \mathrm{Id}_{\partial X}$. Now, if A_0 was connected we would get a contradiction because $h_0(A_0,1)$ would be connected (by continuity of h_0) and disconnected being the union of X_0 and X_1.

Thus we can consider the connected component A_{00} of A_0 containing X_0 and its complementary in A_0, $A_{01} := A_0 \setminus A_{00}$. Then, we define

$$\tilde{A}_{0j} := \{(x,t)\,|\,x\in T,\, t\in[0,1],\, [(x,t)]\in A_{0j})\} \qquad j=0,1,$$

where $[(x,t)]$ stands for the equivalence class of (x,t) in X.
Let us set $\tilde{A}_0 := \tilde{A}_{00}\cup\tilde{A}_{01}$.
Next, we construct a continuous map $\tilde{h}_0 : \tilde{A}_0\times[0,1]\to T\times[0,1]$ in the following way:

$$\tilde{h}_0((x,t),s) := \begin{cases} (x,(1-s)t) & (x,t)\in\tilde{A}_{00} \\ (x,(1-s)t+s) & (x,t)\in\tilde{A}_{01}. \end{cases}$$

Just to be rigorous we also define the sets

$$\tilde{A}_i := \{(x,t)\,|\,x\in T,\, t\in[0,1],\, [(x,t)]\in A_i)\} \qquad i\geq 1,$$

which are nothing but the A_i's seen as subsets of $T\times[0,1]$, without the equivalence relation. Analogously we define the maps

$$\tilde{h}_i((x,t),s) := h_i([(x,t)],s)$$

which turn out to be well defined, being $A_i \cap \partial X = \emptyset$, for any $i \geq 1$ (see point (v) of Definition 1.23).

Now, it is easy to see that the sets $\tilde{A}_i$'s, together with the continuous maps $\tilde{h}_i$'s, satisfy the conditions of Definition 1.23 for $\mathrm{Cat}_{T\times[0,1],T\times(\{0\}\cup\{1\})}(T\times[0,1])$ and this concludes the proof of this first inequality.

Point b follows directly from property 2 of Proposition 1.24, while applying Theorem 1.26 we obtain inequality c.

To get step d, let us denote by k the cup–length of T. By definition there exist $\alpha_1, \ldots, \alpha_k \in H^*(T;\mathbb{R})$, with $\dim(\alpha_i) > 0$ for any $i \in \{1, \ldots, k\}$, such that

$$\alpha_1 \cup \ldots \cup \alpha_k \neq 0.$$

Since $H^1([0,1], \{0\}\cup\{1\}; \mathbb{R}) = \mathbb{R}$, we can also choice $0 \neq \beta \in H^1([0,1], \{0\}\cup\{1\}; \mathbb{R})$. We are now in position to apply Theorem 1.19 with $G = \mathbb{R}$, $X = [0,1]$, $Y = \{0\}\cup\{1\}$, $X' = T$ and $Y' = \emptyset$. By definition of μ, see (1.25), and its injectivity, we obtain

$$\beta \times (\alpha_1 \cup \alpha_k) = \mu(\beta \otimes (\alpha_1 \cup \alpha_k)) \neq 0. \tag{2.37}$$

Consider now the projections $p_1 : T \times ([0,1], \{0\} \cup \{1\}) \to ([0,1], \{0\} \cup \{1\})$ and $p_2 : T \times [0,1] \to T$. Applying Proposition 1.20, we find:

$$\beta \times (\alpha_1 \cup \alpha_k) = p_1^*(\beta) \cup p_2^*(\alpha_1 \cup \alpha_k) = p_1^*(\beta) \cup p_2^*(\alpha_1) \cup \ldots \cup p_2^*(\alpha_k). \tag{2.38}$$

Notice that $p_1^*(\beta) \in H^*(T \times [0,1], T \times (\{0\} \cup \{1\}); \mathbb{R})$ and, for any $i \in \{1, \ldots, k\}$, $p_2^*(\alpha_i) \in H^*(T \times [0,1]; \mathbb{R})$, with $\dim(p_2^*(\alpha_i)) > 0$.

In conclusion, thanks to (2.37) and (2.38), we proved exactly that $\mathrm{CL}(T) \leq \mathrm{CL}(T \times [0,1], T \times (\{0\} \cup \{1\}))$.

Finally, the equality named e is just due to the well known fact that $\mathrm{CL}(T) = 2$. The proof is thereby complete. ■

Remark 2.18. Going back over the previous proof, it is immediate to understand that in the first four steps we did not use that T is the 2 torus. Thus, as anticipated in the Introduction, if on some (Σ, g) the functional I_ρ possesses a strict local minimum, the theorem holds true, more precisely I_ρ has at least $CL(\Sigma)+1$ critical points other than the minimum.

2.3 Supercritical conformal metrics on surfaces with singularities

We postpone to the end of this section the proof of Proposition 0.6, which is rather standard, giving priority to the proofs of Theorem 0.7 and 0.8. To get these existence and multiplicity results we analyze the topology of sublevels of J_ρ in terms of the barycenters of the bouquet $B^{\mathfrak{g}}$ (see Proposition 2.19), whose Betti numbers are computed explicitly (see Proposition 2.20).

We want to point out that in the following, according to the notations introduced in Theorem 1.17, for any couple of real numbers a, b, $\beta_q(a,b,\mathbb{Z})$ will denote the rank of $H_q(\{J_\rho \le b\}, \{J_\rho \le a\}; \mathbb{Z})$. While for any topological space X, $\beta_q(X;\mathbb{Z})$ stands for the q-th Betti number of X, namely the rank of $H_q(X;\mathbb{Z})$. Finally, if X, Y are two topological spaces and $f : X \to Y$ is a continuous function, we will denote by $f_* : H_q(X) \to H_q(Y)$, for $q \in \mathbb{N}$, the homomorphism induced by f.

Proof of Theorem 0.7 and Theorem 0.8 We first make the following claim, whose proof follows from Propositions 2.19 and 2.20 below.

Claim. For $\rho \in (8\pi k, 8\pi(k+1)) \setminus \Gamma(\underline{\alpha}_m)$, choosing L sufficiently large positive one has that
$$\beta_{2k-1}(L,-L;\mathbb{Z}) \ge \binom{k+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(k+\mathfrak{g}-1)!}{k!(\mathfrak{g}-1)!}.$$

Once the claim is proved, the conclusion follows from Lemma 1.33. To prove Theorem 0.8 it is instead sufficient to apply Proposition 2.10 and then Theorem 1.17 (using the observations after it) with $a = -L$ and $b = L$. ■

Proposition 2.19. *There exists $L > 0$ sufficiently large such that, for any $q \in \mathbb{N}$, $\beta_q(L,-L;\mathbb{Z}) \ge \beta_q(B_k^{\mathfrak{g}};\mathbb{Z})$, where $B_k^{\mathfrak{g}}$ is the space of formal barycenters (see* (25)*) on a bouquet of $\mathfrak{g}$ circles, with $\mathfrak{g}$ the genus of Σ.*

We recall that a space $B^{\mathfrak{g}}$ is a bouquet of $\mathfrak{g}$ circles if $B^{\mathfrak{g}} = \cup_{j=1}^{\mathfrak{g}} A_j$, with A_j homeomorphic to S^1 and $A_i \cap A_j = \{P\}$; P is called the center of the bouquet.

Proof. Proposition 1.34 implies that $\{J_\rho \le L\}$ is contractible (for L sufficiently large). Thus, from the exactness of the homology sequence
$$\begin{aligned}&\dots \to \tilde{H}_q(\{J_\rho \le -L\};\mathbb{Z}) \to \tilde{H}_q(\{J_\rho \le b\};\mathbb{Z}) \to H_q(\{J_\rho \le b\}, \{J_\rho \le -L\};\mathbb{Z}) \to \dots\\ &\dots \to \tilde{H}_{q-1}(\{J_\rho \le -L\};\mathbb{Z}) \to \tilde{H}_{q-1}(\{J_\rho \le b\};\mathbb{Z}) \to \dots\end{aligned}$$
we derive that
$$\begin{cases} H_{q+1}(\{J_\rho \le L\}, \{J_\rho \le -L\};\mathbb{Z}) \cong \tilde{H}_q(\{J_\rho \le -L\};\mathbb{Z}),\ q \ge 0; \\ H_0(\{J_\rho \le L\}, \{J_\rho \le -L\};\mathbb{Z}) = 0. \end{cases}$$

Now to obtain the thesis it suffices to construct $j : B_k^{\mathfrak{g}} \to \{J_\rho \le -L\}$ and $f : \{J_\rho \le -L\} \to B_k^{\mathfrak{g}}$ such that $f \circ j$ is homotopically equivalent to the $\mathrm{Id}_{|B_k^{\mathfrak{g}}}$. In fact, if this is true, we have that
$$f_* \circ j_* = Id_{|H_*(B_k^{\mathfrak{g}};\mathbb{Z})},$$

which implies that $\mathrm{rank}(H_q(\{J_\rho \leq -L\};\mathbb{Z})) \geq \mathrm{rank}(H_q(B_k^{\mathfrak{g}};\mathbb{Z})) = \beta_q(B_k^{\mathfrak{g}};\mathbb{Z})$.

In order to build these maps we will regard $B^{\mathfrak{g}}$ as an appropriate subset of Σ: let us understand how.

Since any two differentiable, compact, orientable surfaces with the same genus are homeomorphic, we can consider an embedding Θ from Σ to $\mathbb{R}^3$ (with coordinates z_1, z_2, z_3) such that in any hole passes a line parallel to the z_3 axis and moreover such that the projection on the plane $\{z_3 = 0\}$ is a circle with g rounds holes as in Figure 2.2. Let us denote by ϖ the map projecting $\mathbb{R}^3$ onto the plane $\{z_3 = 0\}$.

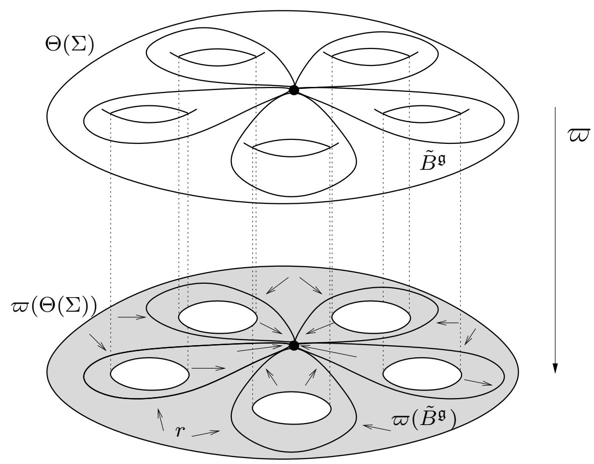

Figure 2.2: $\tilde{B}^{\mathfrak{g}}$ embedded in $\Theta(\Sigma)$ and their projections.

In $\Theta(\Sigma \setminus \{P_1, \ldots, P_m\})$ it is clearly possible to find a bouquet of circles, $\tilde{B}^{\mathfrak{g}}$, verifying:

- $\varpi_{|\tilde{B}^{\mathfrak{g}}}$ is an homeomorphism,
- $\varpi(\tilde{B}^{\mathfrak{g}})$ is a bouquet having a hole of $\varpi(\Theta(\Sigma))$ in each loop,
- $\varpi(\tilde{B}^{\mathfrak{g}}) \cap \varpi(\{P_1, \ldots, P_m\}) = \emptyset$.

Then there exists a retraction $r : \varpi(\Theta(\Sigma)) \to \varpi(\tilde{B}^{\mathfrak{g}})$.

Let us set $B^{\mathfrak{g}} := \Theta^{-1}(\tilde{B}^{\mathfrak{g}})$, which is again a bouquet with $\mathfrak{g}$ loops.
We are at last in position to define the desired maps.

$$\begin{array}{rccc} j: & B_k^{\mathfrak{g}} & \longrightarrow & \{J_\rho \leq -L\} \\ & \sigma = \sum_{i=1}^k t_i \delta_{b_i} \ \ (b_i \in B^{\mathfrak{g}}) & \longmapsto & \varphi_{\mu,\sigma} \end{array} \tag{2.39}$$

$$\begin{array}{rccccc} f: & \{J_\rho \leq -L\} & \xrightarrow{\Psi} & \Sigma_k & \xrightarrow{\Upsilon} & B_k^{\mathfrak{g}} \\ & u & \longmapsto & \Psi(u) = \sum_{i=1}^k t_i \delta_{x_i} & \longmapsto & \sum_{i=1}^k t_i \delta_{\Theta^{-1} \circ \varpi^{-1} \circ r \circ \varpi \circ \Theta(x_i)} \end{array} \tag{2.40}$$

Going back to the construction of Ψ, carried out in the proof of Proposition 1.7, we see that all the arguments used to prove the statement (1.20) hold true also in the singular case. Besides, being $B^{\mathfrak{g}}$ a compact subset of $\Sigma \setminus \{P_1, \ldots, P_m\}$, we are in position to apply Proposition 1.30, with $K = B^{\mathfrak{g}}$. Then, combining (1.20) with the aforementioned Proposition and the uniform continuity of Υ on $B^{\mathfrak{g}}_k$, we obtain easily that $f \circ j$ is homotopically equivalent to the identity on $B^{\mathfrak{g}}_k$. ■

Proposition 2.20. $\beta_{2k-1}(B^{\mathfrak{g}}_k;\mathbb{Z}) = \binom{k+\mathfrak{g}-1}{\mathfrak{g}-1} = \frac{(k+\mathfrak{g}-1)!}{k!(\mathfrak{g}-1)!}$.

Proof. Theorems 1.1 and 1.3 in [48] imply that for any $q \geq 0$

$$\tilde{H}_q(B^{\mathfrak{g}}_k;\mathbb{Z}) \cong H_{q+1}(\overline{SP}^k(S^1 \wedge B^{\mathfrak{g}});\mathbb{Z}).$$

Now notice that $S^1 \wedge B^{\mathfrak{g}}$ has the same homology of $\bigvee_{j=1}^{\mathfrak{g}} S^2$; hence, since the reduced symmetric product of a space only depends on its homology, it follows that for any $q \geq 0$

$$\begin{aligned}
&\tilde{H}_q((B^{\mathfrak{g}})_k;\mathbb{Z}) \cong H_{q+1}(\overline{SP}^k(S^1 \wedge B^{\mathfrak{g}});\mathbb{Z}) \cong \\
&\quad\cong\ H_{q+1}(\overline{SP}^k(\bigvee_{j=1}^{\mathfrak{g}} S^2);\mathbb{Z}) \cong \ \text{[property of the reduced symmetric product]} \\
&\quad\cong\ H_{q+1}(\bigvee_{n_1+\ldots+n_{\mathfrak{g}}=k} (\bigwedge_{j=1}^{\mathfrak{g}} \overline{SP}^{s_j} S^2);\mathbb{Z}) \cong \ \text{[property of the homology of the wedge sum]} \\
&\quad\cong\ \bigoplus_{n_1+\ldots+n_{\mathfrak{g}}=k} H_{q+1}(\bigwedge_{j=1}^{\mathfrak{g}} (\overline{SP}^{s_j} S^2);\mathbb{Z}) \cong \ [\overline{SP}^n(S^2) \cong S^{2n}] \\
&\quad\cong\ \bigoplus_{n_1+\ldots+n_{\mathfrak{g}}=k} H_{q+1}(S^{2k};\mathbb{Z}) \cong \\
&\quad\cong\ \begin{cases} \mathbb{Z}^s, \ q = (2k-1), \\ 0, \quad \text{otherwise} \end{cases}
\end{aligned} \tag{2.41}$$

Here $s = \binom{k+\mathfrak{g}-1}{\mathfrak{g}-1}$ counts the number of tuples $(n_1, \ldots, n_{\mathfrak{g}})$ such that $\sum_{j=1}^{\mathfrak{g}} n_j = k$. The proof is thereby complete. ■

Proof of Proposition 0.6 It is well known ([49], [84]) that $\tilde{g} = e^{2\tilde{w}}g$ is a conformal metric on $(\Sigma, \underline{\alpha}_m)$ with Gaussian curvature K if and only if

$$\begin{cases} -\Delta_g \tilde{w} = Ke^{2\tilde{w}} - K_g \quad \text{in} \quad \Sigma \setminus \{P_1, \cdots, P_m\}, \\[2mm] \dfrac{1}{2\pi}\displaystyle\int_\Sigma Ke^{2\tilde{w}} dV_g = \chi(\Sigma) + \sum_{j=1}^m \alpha_j, \\[2mm] \tilde{w}(\pi_j(z)) = \alpha_j \log|z - z_j| + \mathrm{O}(1), \quad z \in B_r(z_j),\ j \in 1, \ldots, m \end{cases} \tag{2.42}$$

where π_j is a set of local (complex) isothermal coordinates around $z_j = \pi_j^{-1}(P_j)$ (as induced by the g partition of unity construction) and $r > 0$ a suitably chosen positive small enough number. Let us define

$$w(P) = \tilde{w}(P) + 2\pi \sum_{j=1}^{m} \alpha_j G(P, P_j). \tag{2.43}$$

Then w is a distributional solution of the equation

$$-\Delta_g w = K e^{-h_m} e^{2w} - K_g - \frac{2\pi}{|\Sigma|} \sum_{j=1}^{m} \alpha_j \quad \text{in} \quad \Sigma \setminus \{P_1, \cdots, P_m\}, \tag{2.44}$$

which also satisfies

$$\frac{1}{2\pi} \int_{\Sigma} K e^{-h_m} e^{2w} dV_g = \chi(\Sigma) + \sum_{j=1}^{m} \alpha_j, \tag{2.45}$$

and for $z \in B_r(z_j)$, $j \in 1, \ldots, m$,

$$w(\pi_j(z)) = \alpha_j \log |z - z_j| + 2\pi \sum_{\ell=1}^{m} \alpha_\ell G(\pi_j(z), \pi_\ell(z_\ell)) + \mathrm{O}(1).$$

However it is also well known [2] that

$$G(P, P_j) = \frac{1}{2\pi} \log \left(d_g(P, P_j)\right) + O(1), \ P \simeq P_j,$$

where $d_g(\cdot, \cdot)$ is the geodesic distance defined by g. In particular it is not too difficult to verify that

$$G(\pi_j(z), \pi_j(z_j)) = -\frac{1}{2\pi} \log |z - z_j| + O(1), \ z \simeq z_j, \tag{2.46}$$

and we readily conclude that

$$w(\pi_j(z)) = \mathrm{O}(1), \quad z \in B_r(z_j), \ j \in 1, \ldots, m.$$

By standard elliptic theory this condition implies that w is a distributional solution for (2.44) on Σ. In particular, by using (2.46) and the explicit expression of h_m we see that e^{-h_m} is Hölder continuous in Σ, and the standard elliptic regularity theory shows that w is a classical solution to (2.44).
At this point we conclude that if $u = 2w$ then u is a classical solution for

$$-\Delta_g u = 2K e^{-h_m} e^{u} - 2K_g - \frac{4\pi}{|\Sigma|} \sum_{j=1}^{m} \alpha_j \quad \text{in} \quad \Sigma, \tag{2.47}$$

and then setting

$$\rho = 4\pi\left(\chi(\Sigma) + \sum_{j=1}^{m}\alpha_j\right),$$

and by using (2.45) we conclude that u is a classical solution for (18). Therefore, if

$$\tilde{g} = e^{2\tilde{w}}g = e^{-h_m}e^u g \equiv \rho\frac{e^{-h_m}e^u}{\int_\Sigma 2Ke^{-h_m}e^u dV_g}g,$$

is a conformal metric on $(\Sigma, \underline{\alpha}_m)$ with Gaussian curvature K, then u is a classical solution for (18).
On the other side, if u is a classical solution for (18) then (20) holds. Thus, we can define w by

$$2w = u + \log\rho - \log\left(\int_\Sigma 2Ke^{-h_m}e^u dV_g\right),$$

and come up with a classical solution for (2.44) on all Σ. At this point we can use (2.43) to define $\tilde{w}$ and conclude that

$$\rho\frac{e^{-h_m}e^u}{\int_\Sigma 2Ke^{-h_m}e^u dV_g}g = e^{-h_m}e^{2w}g = e^{2\tilde{w}}g$$

is a conformal metric on $(\Sigma, \underline{\alpha}_m)$ with Gaussian curvature K. ■

Acknowledgements

It is a pleasure for me to express deep gratitude to my advisor, Andrea Malchiodi, for having proposed to me the topic of the present thesis and for having helped and supported me many times during the whole time of my PhD.

Moreover I wish to warmly thank him and Daniele Bartolucci for having developed with me part of my PhD research project.

I would like to thank Rod Gover and Luca Martinazzi for the time we spent together collaborating

I am also indebted with Angela Pistoia for her useful suggestions and with Sadok Kallel for kindly answering my questions.

I want to thank SISSA, in particular the sector of Functional Analysis and Applications, and the University of Trieste for having made possible my Degree and my PhD studies. I also thank my collegues and all the friends who shared with me these unforgettable five years in Trieste.

Finally, I have no words to say thank you to my parents.

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