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PhD Thesis:

Flavour Models, R-symmetries and non-Abelian Vortices in Supersymmetric Theories

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17 September 2008

International School for Advanced Studies
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Acknowledgments

I am grateful to many people for their support, help and advices during these years.
First, I would like to thank Andrea Romanino for all the physics I learned from him and for all the support that I received in my doctoral studies. He tried to share his clear understanding of particle physics with me, and I would like to thank him for his patience in doing so.
I would like to thank all my collaborators during these years and all the people who discussed with me and teached me a lot of physics (and good places to eat, and much else!). Among the many: Francesco Benini, Lorenzo Calibbi, Michele Cortelezzi, Stefano Cremonesi, Jarah Evslin, Marcello Mamino, Giacomo Marmorini, Marco Nardecchia, Walter Vinci, Luca Vecchi, Robert Ziegler. I hope to meet them again in the future, in front of a pint of beer, a glass of wine or a dish of mlinci.
I am also grateful to Dave, Vale, Fra, Cla, Matteo, Jerry, Davide, Beppe, Antonio, Simone, Claudio, Claudia, Martina, Marietta, Valeria, Caterina, Andrea, Federico, Francesco, Giulia, Gipo, Carletto, Riccardo and many other friends I can't remember just now. We are scattered around the world but I hope that our roads will cross again in the future.
I am deeply grateful to my parents and grandparents and to my sister Monica for their constant support and their love.
Finally my thoughts and my thanks go to Alice, with whom I would like to share my world and my little need to sleep but to dream…

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Flavour without family symmetries

Chapter 1

Introduction

The Standard Model of particle physics, based on spontaneously broken gauge theories, is a very successful model and can account for almost all aspects of high energy physics up to the energy range of the present experiments. When combined with supersymmetry, the Standard Model can be easily extended to unified theories valid up to scales that are only a few orders of magnitude below the Planck mass.

However the SM cannot account for the values of a large number of parameters. Most of these constants are related to the masses and mixings of the three families of the SM and are encoded in the Yukawa couplings $Y_{ij}\psi_i^c\psi_j h$ that break the global $U(3)^5$ symmetry of the model. These couplings show a clear pattern with a hierarchy of quark and charged lepton masses and small mixing angles between the three families in the quark sector. Data from neutrino oscillations also show that there is a large flavour mixing in the lepton sector.

Understanding this pattern of fermion masses and mixings is a non-trivial task, despite the apparent structure of the Yukawa couplings. We do not have information about the scale of the physics that generates this flavour structure: the origin of the observed pattern could be a few orders of magnitude above the electroweak scale or could be at the unification scale. We also cannot test directly the Higgs sector of the theory, which could play a relevant role in flavour physics; this situation should change in some years, when LHC will hopefully improve our understanding of Higgs physics. However many other high energy experiments can give us precious information about flavour physics. In fact flavour violation in the SM is suppressed or strongly constrained by its flavour structure, therefore there are many processes where new physics could show up with a relevant contribution, detectable with precision experiments.

Flavour violation in the MSSM appears only in the Cabibbo-Kobayashi-

Maskawa mixing matrix that enters the charged current $J^{\pm}$ for $W^{\pm}$ mediated weak processes, and in the Pontecorvo-Maki-Nakagawa-Sakata matrix. The CKM matrix has small mixing angles, the weak flavor-violating processes are suppressed by the W mass, and for neutral-current processes the GIM mechanism enhance the CKM suppression, therefore hadronic flavour-changing neutral-current processes are very suppressed or negligible. Physical processes that violate lepton flavour through the PMNS matrix require a neutrino mass insertion and are therefore strongly suppressed. This means that FCNC processes represent clean channels where to look for contributions induced by new flavour physics.

Supersymmetric extensions of the Standard Model are particularly interesting for flavour physics, because of the presence of new flavoured particles (squarks and sleptons) at the TeV scale. The soft SUSY-breaking couplings (masses and A-terms) of these particles have a flavour structure that receives contributions from all energy scales from the electroweak scale to the one where SUSY-breaking operators involving MSSM fields are generated, therefore they are potentially sensitive to flavour physics at very high scales. At low energy, these particles contribute to loop processes giving rise to measurable FCNC interactions. Their flavour structure is already strongly constrained by the present data on FCNC processes, but there is still plenty of room for deviations from a flavour-blind structure. LHC ad future colliders should also be able to test directly the masses and hierarchies of these particles, giving independent information beyond the one obtained by flavour physics experiments. LHC and flavour factories should become in some years a good ground test for theoretical models of flavour.

Theoretical approaches to the origin of flavour are mainly based on the idea of family symmetries. We review the basic idea in the next section. In chapter 2 we will build a model of flavour without exact family simmetry: the hierarchy will be related to an “accidental” symmetry that originates from the breaking of the Pati-Salam gauge group. In chapter 3 we will show how gauge coupling unification can be preserved in the presence of incomplete GUT multiplets at intermediate scales. In chapter 4 we will upgrade the model to an $SO(10)$ GUT on a 5D orbifold. Finally in chapter 5 we will discuss the phenomenological predictions of the model.

1.1 Family symmetries

The structure of quark and lepton masses can be roughly described as follows. The quark Yukawa matrices are almost diagonal and hierarchical, as can be seen from the mass ratios (evaluated at M_Z)

$$\frac{m_s}{m_b} \simeq 2 \cdot 10^{-2} \ , \ \frac{m_d}{m_s} \simeq 5 \cdot 10^{-2} \ , \ \frac{m_c}{m_t} \simeq 4 \cdot 10^{-3} \ , \ \frac{m_u}{m_c} \simeq 2 \cdot 10^{-3} \tag{1.1}$$

and mixing angles

$$V_{us} \simeq 0.23 \ , \ V_{cb} \simeq 0.04 \ , \ V_{ub} \simeq 0.004 \ . \tag{1.2}$$

The charged lepton masses are also hierarchical

$$\frac{m_\mu}{m_\tau} \simeq 6 \cdot 10^{-2} \ , \ \frac{m_e}{m_\mu} \simeq 5 \cdot 10^{-3} \tag{1.3}$$

while the data we know about neutrino masses are

$$m_\nu \lesssim 1 \text{ eV} \ , \ \Delta m^2_{solar} \sim 8 \cdot 10^{-5} \text{eV}^2 \ , \ \Delta m^2_{atm} \sim 2 \cdot 10^{-3} \text{eV}^2 \tag{1.4}$$

and about their mixing angles

$$\theta_{solar} \sim \frac{\pi}{6} \ , \ \theta_{atm} \sim \frac{\pi}{4} \ , \ \theta_{13} \lesssim 0.2 \tag{1.5}$$

There are many interesting "numerological" relations that can be guessed from this structure and from the precise values of particle masses. For example, we can find relations for the mixing angles $V_{us} \sim \sqrt{\frac{m_d}{m_s}}$, $V_{cb} \sim \frac{m_s}{m_b}$ and for the mass ratios $\left(\frac{m_\tau}{m_b}\right)_{GUT} \sim 1$, $\left(\frac{m_\mu}{m_s}\right)_{GUT} \sim 3$.

A possible way to account for the structure of quark and lepton masses and mixings is given by family symmetries. In fact the clear distinction between the three families for quarks and charged leptons suggests to treat them as separated entities and to distinguish them by assigning different charges to the different families. To generate the small but nonvanishing Yukawa couplings and therefore the small mixing between the three families, these charges should be related to a spontaneously broken symmetry. The small parameter that controls the mixing is the ratio of the symmetry breaking vev and the mass scale where the flavour structure is generated.

These family symmetries (also called horizontal or flavour symmetries) act on the family index $i = 1, 2, 3$ of the SM fields $q_i, l_i, u^c_i, d^c_i, e^c_i$. These symmetries can be abelian [1] or non-abelian [2, 3, 4] (in this last case they are generally subgroups of $SU(3)$), discrete or continuous. As an example, consider the abelian charges under a $U(1)_H$ family symmetry:

$$Q_H(h, q_3, u^c_3, d^c_3) = 0 \ , \ Q_H(q_1, d^c_1)/2 = Q_H(q_2, d^c_2) = H \ , \ Q_H(u^c_1)/2 = Q_H(u^c_2) = H' \tag{1.6}$$

The Yukawas couplings of the first two families are forbidden by $U(1)_H$ quantum numbers of SM fields, but it is possible to have other couplings between SM fields and heavy fields at scale M_H that result in non-renormalizable operators at low energy. Assuming that $U(1)_H$ is broken by the vev of a gauge singlet $\langle S\rangle$ (usually called flavon) with $Q_H(S) = -1$ and $\epsilon = \frac{\langle S\rangle}{M_H} < 1$. Non-renormalizable operators of the form $c^k_{ij}\frac{S^k}{M^k_H}\psi^c_i\psi_j h$ correspond to low energy Yukawas $Y^U_{ij}u^c_i q_j h_u + Y^D_{ij}d^c_i q_j h_d$ of the form

$$Y^U_{ij} \sim \epsilon^{Q_H(h_u)+Q_H(q_j)+Q_H(u^c_i)} \quad , \quad Y^D_{ij} \sim \epsilon^{Q_H(h_d)+Q_H(q_j)+Q_H(d^c_i)} \tag{1.7}$$

and in the explicit example above:

$$Y^D \sim \begin{pmatrix} \epsilon^{4H} & \epsilon^{3H} & \epsilon^{2H} \\ \epsilon^{3H} & \epsilon^{2H} & \epsilon^{H} \\ \epsilon^{2H} & \epsilon^{H} & 1 \end{pmatrix}, Y^U \sim \begin{pmatrix} \epsilon^{2H+2H'} & \epsilon^{H+2H'} & \epsilon^{2H'} \\ \epsilon^{2H+H'} & \epsilon^{H+H'} & \epsilon^{H'} \\ \epsilon^{2H} & \epsilon^{H} & 1 \end{pmatrix}$$

apart from $O(1)$ factors. In this way a mass hierarchy is induced by the same small parameter that controls the mixing between families. In the case of abelian family symmetries the predictive power of the theory is small because of the many possible inequivalent choices of $U(1)$ charges. Non-abelian symmetries are usually more predictive because the possible representations of the flavour group are more restricted.

A simple example of the heavy physics giving rise to the non-renormalizable operators $S^k\psi^c_i\psi_j h$ is given by the Froggatt-Nielsen mechanism. At the scale M_H there are heavy vectorlike fermions Q_i, U^c_i, D^c_i and $\bar{Q}_i, \bar{U}^c_i, \bar{D}^c_i$ with the same gauge quantum numbers of the SM fermions. Their charges Q_H allow them to couple to the light ones through $\langle S\rangle, \langle h\rangle$:

$$\mathcal{L} = M^Q_{ij}Q_i\bar{Q}_j + \eta_{ij}SQ_i\bar{Q}_j + \lambda_{ij}Q_i u^c_j h_u + \alpha_{ij}Sq_i\bar{Q}_j + \ldots \tag{1.8}$$

Integrating out these fermions, we get the non-renormalizable operators and then the usual Yukawa terms after the breaking of the $U(1)_H$ symmetry.

Chapter 2

A model of flavour

The origin of the peculiar pattern of fermion masses and mixing might appear more or less transparent at low scale depending on the degree of understanding of the full theory it requires. As discussed above, most approaches to the problem rely on the possibility that a full understanding is not required and the pattern of fermion masses and mixings follows from a "factorizable" dynamical principle associated to the "horizontal" family indices. In this chapter we discuss the possibility that not even such a dynamics needs to be known, or exists at all, and the peculiar fermion mass pattern we observe simply follows from the fact that one heavy vectorlike family of fields turns out to be lighter than the rest of the heavy fields. The couplings of this lighter heavy family with the light families will not be constrained by any symmetry or alternative mechanism imposed on the theory. They will instead all be of order one, perhaps determined by some fundamental theory we do not need to know, and the charged fermion hierarchy will follow from the hierarchy in the breaking of the vertical gauge structure of the theory, in particular from the breaking of the Pati-Salam (PS) gauge group [5, 6]. Chiral symmetries acting on family indices protecting the masses of the first two fermion families emerge in this context as accidental symmetries.

In section 2.1 we motivate the structure of the model and in particular the choice of the left-right (LR) symmetric and Pati-Salam (PS) gauge groups. In section 2.2 we define in detail the model and sistematically analyze it.

All the flavour models presented in these chapters are intended to be supersymmetric.

2.1 A bottom-up approach to flavour from accidental symmetries

2.1.1 Messenger dominance

Let $\psi_i = q_i, u^c_i, d^c_i, l_i, n^c_i, e^c_i$, $i = 1, 2, 3$ denote the three light SM families in Weyl notations, including three singlet neutrinos, and let $h = h_u, h_d$ denote the light Higgs. As usual, the lightness of the three SM families (except possibly the singlet neutrinos) is guaranteed by their chirality with respect to the SM group, while additional degrees of freedom are allowed to be much heavier because they come in vectorlike representations of the SM group. As anticipated in the introduction, the pattern of fermion masses arises in our model from the existence of a single relatively light vectorlike family of "messengers" $\Psi + \overline{\Psi}$, with $\Psi = Q, U^c, D^c, L, N^c, E^c$, and from the breaking pattern of the gauge group. We also consider the possibility of heavy Higgs messenger fields $H = H_u, H_d$.

Since Ψ has the same SM quantum numbers as ψ_i, we use a discrete $\mathbf{Z}_2$ symmetry to tell the light families from the heavy one. The light fields ψ_i, h are $\mathbf{Z}_2$-odd, while the messengers are even. In the unbroken limit, the light families are massless, while the messengers fields $\Psi, \overline{\Psi}, H$ are allowed to be superheavy[1]. Yukawa couplings for the light fields are forbidden by the $\mathbf{Z}_2$ symmetry. In order to break it, we then also include a SM-singlet $\mathbf{Z}_2$-odd chiral field ϕ. Its scalar component will get a vacuum expectation value (vev) at a heavy scale not far from the messenger scale. Needless to say, the $\mathbf{Z}_2$ symmetry is not a family symmetry, as it does not tell the three families apart, all being odd under it. This is similar to what done in [7, 8, 9, 10], where the hierarchical pattern of fermion masses was also addressed without the use of family symmetries.

Once ϕ gets a vev, the light and heavy fermions mix, which gives rise to the SM Yukawa couplings. In the limit in which the vev is smaller than the mass of the heavy messengers, $\langle\phi\rangle \ll M$, the Yukawa couplings of the light fermions can be seen to arise from higher dimensional operators in the effective theory below the scale M. This limit does not always hold in our model, as we will see, but it is useful for illustrative purposes and will be used in this Section. The exact treatment is postponed to Section 2.2. At the lowest order, the relevant operators are in the form $(\phi/M)\psi_i\psi_j h$ and they arise from the three diagrams in Fig. 2.1.

If the three contributions in Fig. 2.1 are comparable and if the couplings

[1] The SM Higgs h is of course in principle also allowed to be heavy. We do not address this μ-problem here.

Figure 2.1: Messenger exchanges contributing to the operator $(\phi/M)\psi_i\psi_j h$ in the effective theory below the messenger scale. F, f refer to electroweak doublets, while F^c, f^c refer to electroweak singlets.

involved are uncorrelated, we expect the fermion masses of the three families to be comparable. On the other hand, in the limit in which one of the three exchanges dominates (because the corresponding messenger is lighter) one family turns out to be heavier and a hierarchy is generated. This mechanism has several interesting features. The "horizontal" hierarchy among different families follows from a "vertical" hierarchy among messengers belonging to the same family, as in [7, 8, 9, 10]. As a consequence, the interfamily hierarchy can be attributed to the breaking pattern of the gauge group. Moreover, we will see that a two step breaking of the gauge group below the cutoff of the theory is sufficient to account for the complex hierarchical structure of charged fermions. We will also see that in spite of the absence of small coefficients, the CKM mixing angles will turn out to be small, while in the neutrino sector an attractive mechanism is available to give rise to a naturally large atmospheric mixing between normal hierarchical neutrinos.

Let us see how this works in greater detail. Let us concentrate on the two heavier families and let us also neglect for the time being the Higgs exchanges in Fig. 2.1. We will discuss their role in connection to the first family masses in Section 2.2. In compact notations, the most general renormalizable superpotential is (we illegally use the same notation for the chiral superfield and its "R_P-even" component)

$$W = M\bar{\Psi}\Psi + \alpha_i\bar{\Psi}\psi_i\phi + \lambda_i\Psi\psi_i h, \tag{2.1}$$

where

$$\begin{gathered}
M\bar{\Psi}\Psi \equiv M_Q\bar{Q}Q + M_U\bar{U}^cU^c + M_D\bar{D}^cD^c + M_L\bar{L}L + M_N\bar{N}^cN^c + M_E\bar{E}^cE^c \\
\alpha_i\bar{\Psi}\psi_i\phi \equiv \alpha_i^Q\bar{Q}q_i\phi + \alpha_i^U\bar{U}^cu_i^c\phi + \alpha_i^D\bar{D}^cd_i^c\phi + \alpha_i^L\bar{L}l_i\phi + \alpha_i^N\bar{N}^cn_i^c\phi + \alpha_i^E\bar{E}^ce_i^c\phi \\
\begin{aligned}\lambda_i\Psi\psi_i h \equiv{}& \lambda_i^{Qu}Qu_i^ch_u + \lambda_i^{Uq}U^cq_ih_u + \lambda_i^{Qd}Qd_i^ch_d + \lambda_i^{Dq}D^cq_ih_d + \\ & \lambda_i^{Ln}Ln_i^ch_u + \lambda_i^{Nl}N^cl_ih_u + \lambda_i^{Le}Le_i^ch_d + \lambda_i^{El}E^cl_ih_d\end{aligned} .
\end{gathered} \tag{2.2}$$

No family symmetry or other dynamical constraint is imposed on the couplings. As a consequence, the dimensionless parameters in eq. (2.1.1) are all assumed to be $\mathcal{O}(1)$ and uncorrelated. When ϕ gets a vev, the heavy and light fermions mix, which gives rise to the quark Yukawa matrices Y^U and Y^D. In the limit $\langle\phi\rangle \ll M$ (and in the RL convention for the Yukawas)

$$-Y^U_{ij} = \lambda^{Qu}_i \alpha^Q_j \frac{\langle\phi\rangle}{M_Q} + \alpha^U_i \lambda^{Uq}_j \frac{\langle\phi\rangle}{M_U} \tag{2.3a}$$

$$-Y^D_{ij} = \lambda^{Qd}_i \alpha^Q_j \frac{\langle\phi\rangle}{M_Q} + \alpha^D_i \lambda^{Dq}_j \frac{\langle\phi\rangle}{M_D}. \tag{2.3b}$$

Let us first consider the matrix Y^U. The up quark is massless, since Y^U has rank two. If $M_Q \sim M_U$, the charm mass is expected to be of the same order of the top quark mass. This is because no horizontal hierarchy nor alignment is forced among the family dependent parameters α^Q_i, α^U_i, λ^{Qu}_i, λ^{Uq}_i. However, in the limit in which one of the terms in eq. (2.3a) dominates, the charm mass gets suppressed, as one messenger cannot give a mass to more than one family. A small V_{cb} angle is only guaranteed if the Q exchange is dominant in both the up and down quark sectors[2]. We refer to this hypothesis as "left-handed dominance". We have then generated an inter-family hierarchy in terms of order parameters associated to the intra-family messenger structure, $M_Q/M_U, M_Q/M_D \ll 1$. The mechanism at work behind the explicit discussion above has to do with accidental flavour symmetries emerging in specific limits. First of all the discussion above holds in the limit in which the first family is massless. Such a limit, which will be defined in Section 2.2, implies the presence of an accidental chiral symmetry protecting the first family. Moreover, a second accidental symmetry protecting the masses of the of the second family fermions emerges in the limit in which M_U, M_D become heavy.

A closer look to the textures obtained shows that in this framework the features of the fermion masses and mixings are best interpreted in the context of a Pati-Salam extension of the standard model group, as we now see.

2.1.2 V_{us} and SU(2)$_R$ symmetry

In order to write the Yukawa matrices in a simple form, we note that it is possible to choose a basis in the q_i, u^c_i, d^c_i flavour space such that $\alpha^Q_{1,2} = \lambda^{Qu}_{1,2} = \lambda^{Qd}_{1,2} = 0$. We can then also rotate the "1,2" fields to set $\alpha^U_1 = \alpha^D_1 = \lambda^{Uq}_1 = 0$. If the dimensionless coefficients were of the same order and uncorrelated in the initial basis, we expect the non-vanishing coefficient to be still of the same order and uncorrelated in the new basis. The quark Yukawa matrices

[2]This is true unless appropriate correlations are forced between the U and D coefficients, see below.

can now be written as

$$Y^U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_2^U a_2^U \epsilon_U & r_3^U a_2^U \epsilon_U \\ 0 & r_2^U a_3^U \epsilon_U & 1 \end{pmatrix} \alpha_3^Q \lambda_3^{Qu} \frac{\langle \phi \rangle}{M_Q}, \tag{2.4a}$$

$$Y^D = \begin{pmatrix} 0 & 0 & 0 \\ r_1^D a_2^D \epsilon_D & r_2^D a_2^D \epsilon_D & r_3^D a_2^D \epsilon_D \\ r_1^D a_3^D \epsilon_D & r_2^D a_3^D \epsilon_D & 1 \end{pmatrix} \alpha_3^Q \lambda_3^{Qd} \frac{\langle \phi \rangle}{M_Q}, \tag{2.4b}$$

where $\epsilon_U = M_Q/M_U$, $\epsilon_D = M_Q/M_D \ll 1$, while $r_i^U = \lambda_i^{Uq}/\lambda_3^{Qu}$, $r_i^D = \lambda_i^{Dq}/\lambda_3^{Qd}$, $a_i^U = \alpha_i^U/\alpha_3^Q$, $a_i^D = \alpha_i^D/\alpha_3^Q \sim \mathcal{O}(1)$ or vanishing.

A few remarks are in order. First of all, we note that eqs. (2.4) give

$$\frac{m_s}{m_b} \approx r_2^D a_2^D \epsilon_D \sim r_2^D a_3^D \epsilon_D \approx |V_{cb}|, \tag{2.5}$$

in agreement with data. In contrast, flavour symmetries often give $m_s/m_b \sim |V_{cb}|^2$, unless non-abelian symmetries [11, 12, 13, 3, 14] or asymmetric textures [15, 16] are considered. Eqs. (2.4) also show that the top and bottom Yukawa couplings are of the same order, i.e. $\tan\beta$ is large. This is a prediction of the left-handed dominance scenario, which holds in the absence of significant Higgs mixing. Note also that the simplest way to account for the more pronounced hierarchy in the up quark sector, $m_c/m_t \ll m_s/m_b$ is to have $\epsilon_U \ll \epsilon_D$ and therefore a double hierarchy $M_Q \ll M_D \ll M_U$. We will see below that $m_c/m_t \ll m_s/m_b$ can actually be explained without introducing a third scale.

The textures in Eqs. (2.4) also have an unpleasant feature. Although the masses of the first family fermions have still to be generated, the Cabibbo angle does not vanish and ends up being typically large:

$$\tan\theta_C = \left| \frac{r_1^D}{r_2^D} \right| \sim 1. \tag{2.6}$$

While the actual value of the Cabibbo angle is not very small and could be accomodated by e.g. an accidental cancellation, we prefer to consider its smallness as the indication of a non-accidental correlation between the $\lambda_{1,2}^{qU}$ and $\lambda_{1,2}^{qD}$ coefficients in the initial basis. In turn, such a correlation points at an $\mathrm{SU}(2)_R$ gauge symmetry [17, 18, 19] forcing

$$\lambda_i^{Qu} = \lambda_i^{Qd} \qquad \lambda_i^{Ln} = \lambda_i^{Le} \qquad \alpha_i^U = \alpha_i^D \tag{2.7a}$$

$$\lambda_i^{Uq} = \lambda_i^{Dq} \qquad \lambda_i^{Nl} = \lambda_i^{El} \qquad \alpha_i^N = \alpha_i^E. \tag{2.7b}$$

We are therefore lead to a $G_{LR} = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(3)_c \times \mathrm{U}(1)_{B-L}$ extension of the SM gauge group[3]. Eqs. (2.7) lead to $\lambda_1^{Dq} = 0$, $r_1^D = 0$, and therefore $V_{us} = 0$, as anticipated. A non-vanishing value of V_{us} will be generated by the breaking of the $\mathrm{SU}(2)_R$ symmetry, which is anyway needed. The standard way to break G_{LR} to G_{SM} is through the vev of the scalar component $\tilde{L}'_c$ ($\tilde{\bar{L}}'_c$) of a ($\mathbf{Z}_2$-even in our case) chiral right-handed doublet L'_c ($\bar{L}'_c$) transforming as $L^c = (N^c, E^c)^T$ ($\bar{L}^c = (\bar{N}^c, \bar{E}^c)^T$).

With the basis choice above, all the first family $\mathbf{Z}_2$-odd fermions have the same charge under the accidental chiral U(1) symmetry protecting the first family, whereas all the other fields are invariant. While a non-vanishing V_{us} will need the breaking of the $\mathrm{SU}(2)_R$ symmetry, a non vanishing mass for the first family will need the breaking of that accidental chiral U(1). The accidental family symmetry protecting the second family emerges in the limit in which U^c, D^c become heavy so that they can be integrated out. All the second family fermions have the same charge under it.

2.1.3 Neutrino masses and mixing

We have seen above that small mixing angles are easily obtained in the quark mass sector. At the same time, large mixing angles naturally appear in the neutrino sector provided that the right-handed neutrino messengers N^c, $\bar{N}^c$ dominate the see-saw. This is closely related to the peculiar features of our setting, as we now see.

As in the quark sector, it is convenient to consider a basis in which $\alpha^L_{1,2} = \lambda^{Ln}_{1,2} = \lambda^{Le}_{1,2} = 0$ and $\alpha^N_1 = \alpha^E_1 = \lambda^{Nl}_1 = \lambda^{El}_1 = 0$. Because of the left-handed dominance hypothesis, this choice makes in fact the charged lepton Yukawa matrix approximately diagonal. On the other hand, the couplings $\lambda^{Nl}_{2,3}$ of N^c to l_2 and l_3 are expected to be comparable. We have in fact already used our freedom to redefine l_2, l_3 to make the mixings small in the charged lepton sector. As the charged leptons are approximately diagonal, this means that the singlet neutrino N^c has similar $\mathcal{O}(1)$ couplings to ν_μ and ν_τ. If N^c dominates the see-saw, this is precisely the condition needed to obtain a

[3]Note that in the presence of an $\mathrm{SU}(2)_R$ symmetry the possibility of right-handed dominance also opens up. In fact, the argument leading to left-handed dominance holds under the assumption that the couplings in different sectors, in particular in the right-handed up and down sectors, are uncorrelated. On the other hand, we just saw that the $\mathrm{SU}(2)_R$ symmetry does correlate quantities involving right handed up and down quarks and leptons. As a consequence, the possibility that the $Q+\bar{Q}$ exchange be subdominant to the $Q^c+\bar{Q}^c$ exchange opens up. In this context, one finds $\lambda_c \sim \lambda_s$ and therefore $\tan\beta \sim m_c/m_s$. The Q and Q_c dominance scenarios are therefore characterized by different predictions for $\tan\beta$. We do not pursue this possibility further in this paper.

large atmospheric mixing angle and normal hierarchical neutrino masses in a natural way [20, 21, 22, 23]. We will see in the next section that all the heavy singlet neutrino masses will be approximately at the same scale, but the "$N^c N^c$" entry of the inverse heavy Majorana mass can still dominate the see-saw mechanism. Note that this is an example of see-saw dominated by a singlet neutrino that is not a Pati-Salam (or SO(10)) partner of the light lepton doublets.

2.1.4 The charm quark Yukawa and Pati-Salam

Since the fields U^c and D^c are unified in a right-handed doublet $Q^c = (U^c, D^c)^T$, an unwanted consequence of the $\mathrm{SU}(2)_R$ symmetry is $M_U = M_D = M_{Q^c}$, which gives $m_c/m_t \approx m_s/m_b$. The $\mathrm{SU}(2)_R$ symmetry must therefore on the one hand protect V_{us}, on the other be badly broken in order to differentiate the charm and strange Yukawas. This apparent problem turns out to provide additional insight on the structure of the model.

It turns out that an indirect coupling of the available source of $\mathrm{SU}(2)_R$ breaking (the scalar fields $\tilde{L}'_c$, $\bar{\tilde{L}}'_c$) to the fermions Q^c, $\bar{Q}^c$ is the simplest and most natural way to achieve the hierarchy $m_c/m_t \ll m_s/m_b$. Coupling $(\tilde{L}'_c, \bar{\tilde{L}}'_c)$ to $(Q^c, \bar{Q}^c)$ at the renormalizable level needs the introduction of new fields. There are only two possibilities. The one we are interested in is a vectorlike pair of fermion fields $T+\bar{T}$ transforming as $(1,1,3,4/3)+(1,1,\bar{3},-4/3)$ under G_{LR} (the last entry denotes the value of $B-L$). Such fields couple to the $(\tilde{L}'_c, \bar{\tilde{L}}'_c)$ and $(Q^c, \bar{Q}^c)$ fields through the interaction $TQ^c\bar{\tilde{L}}'_c$ and $\bar{T}\bar{Q}^c\tilde{L}'_c$. Once the scalar doublets get a vev, the latter interactions contributes to the masses in the up sector and allows to suppress the charm mass, as we will see in Section 2.2.2. The second possibility[4] does not suppress the charm mass, as it only affects the down quark sector. It can play a role in the case of right-handed dominance.

The introduction of fermions with the quantum numbers of $T+\bar{T}$ might look at first sight quite "ad hoc". On the other hand, such fermions automatically arise with the Pati-Salam extension of the G_{LR} group, $G_{PS} = \mathrm{SU}(4)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The quantum numbers of $T+\bar{T}$ appear in fact in the decomposition under G_{LR} of the $\mathrm{SU}(4)_c$ adjoint and their interactions follow from the standard coupling of the adjoint to the fundamental of $\mathrm{SU}(4)_c$. In particular, fields with the quantum numbers of $T+\bar{T}$ can certainly be found among the $\mathrm{SU}(4)_c$ gauginos[5]. Unfortunately the simplest implementa-

[4] A vectorlike pair $S+\bar{S}$ transforming as $(1,1,3,-2/3)+(1,1,\bar{3},2/3)$ and coupling through $SQ^c\tilde{L}'_c$ and $\bar{S}\bar{Q}^c\bar{\tilde{L}}'_c$.

[5] Note that such $T+\bar{T}$ gauginos automatically get a heavy mass and are thus splitted

tion of the economical interpretation in which the $T+\bar{T}$ fields are gauginos and $L' = L$ leads to problems in the Higgs sector. In order to avoid those problem we will make sure that R-parity is not broken, which requires $T+\bar{T}$ and $\tilde{L}'$, $\tilde{\bar{L}}'_c$ to be associated to new chiral fields.

2.2 A model of flavour from accidental symmetries

2.2.1 Definition of the model

The chiral superfield content of the model and the quantum numbers under G_{PS} and $\mathbf{Z}_2$ are specified in Table 2.1. The first block contains the $\mathbf{Z}_2$-odd fields: the 3 light (in the unbroken $\mathbf{Z}_2$ limit) families (f_i, f_i^c), $i = 1, 2, 3$, the light Higgs h and the $\mathbf{Z}_2$-breaking field ϕ. The latter is in the adjoint representation of $\mathrm{SU}(4)_c$ as this provides the Georgi-Jarlskog factor 3 needed to account for the μ–s mass relation. The second block contains the messengers, in a single vectorlike family $(F, F_c) + (\bar{F}, \bar{F}_c)$. A Higgs messenger is also included, corresponding to Fig. 2.1c. The third block contains the fields $F'_c + \bar{F}'_c$ breaking the Pati-Salam group (including the $\mathrm{SU}(2)_R$ subgroup) and an $\mathbf{Z}_2$-even $\mathrm{SU}(4)_c$ adjoint Σ providing the fields $T + \bar{T}$ discussed in Section 2.1. SO(10) partners $F' + \bar{F}'$ of $F'_c + \bar{F}'_c$ are also included. The last block contains two sources of Pati-Salam breaking. They contain the two possible SM invariant directions in the Pati-Salam adjoint. Table 2.1 also shows the R-parity associated to each field. R-parity plays a role in preventing the economical identification of the primed fields with F^c and $\bar{F}^c$ and of Σ with the $\mathrm{SU}(4)_c$ gauginos. When discussing the neutrino sector we will also introduce Pati-Salam singlets.

Our hypothesis is that the Pati-Salam gauge structure and the fields in Table 2.1 happen to be the only relatively light fields surviving below the cutoff Λ of our theory, which will not be very far from $10^{16}\,\text{GeV}$. We implement this hypothesis by linking the mass of the heavy fields to Pati-Salam breaking. We do not address the origin of this assumption here. No dynamics related to the family indices is required. On the contrary, we will assume that the dimensionless coefficients in the superpotential are $\mathcal{O}(1)$ and uncorrelated.

---

from the lighter gluinos by the $\mathrm{SU}(4)_c \to \mathrm{SU}(3)_c$ spontaneous breaking. Note also that the required coupling with $Q^c\tilde{\bar{L}}'$ is also automatically present in the form of a supersymmetric gauge interaction, provided that $\tilde{L}'$ is the partner of L.

	f_i	f_i^c	h	ϕ	F	$\bar{F}$	F^c	$\bar{F}^c$	H	F'	$\bar{F}'$	F'_c	$\bar{F}'_c$	Σ	X	X_c
$\mathrm{SU}(2)_L$	2	1	2	1	2	2	1	1	2	2	2	1	1	1	1	1
$\mathrm{SU}(2)_R$	1	2	2	1	1	1	2	2	2	1	1	2	2	1	1	3
$\mathrm{SU}(4)_c$	4	$\bar{4}$	1	15	4	$\bar{4}$	$\bar{4}$	4	1	4	$\bar{4}$	$\bar{4}$	4	15	15	1
$\mathbf{Z}_2$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
R_P	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$

Table 2.1: Field content of the model and quantum numbers under G_{PS} and $\mathbf{Z}_2$

The renormalizable part of the superpotential is

$$\begin{aligned} W^{\mathrm{ren}} = {} & \lambda_i f_i^c F h + \lambda_i^c f_i F^c h + \alpha_i \phi f_i \bar{F} + \alpha_i^c \phi f_i^c \bar{F}^c + X \bar{F} F + X_c \bar{F}^c F^c \\ & + \bar{\sigma}_c \bar{F}'_c \Sigma F^c + \sigma_c \bar{F}^c \Sigma F'_c + \bar{\sigma} \bar{F}' \Sigma F + \sigma \bar{F} \Sigma F' + \gamma X \Sigma^2 \\ & + \lambda_{ij}^H f_i^c f_j H + \eta F^c F H + \bar{\eta} \bar{F}^c \bar{F} H + \eta' F'_c F' H + \bar{\eta}' \bar{F}'_c \bar{F}' H. \end{aligned} \quad (2.8)$$

We have included all terms compatible with our hypotheses except a mass term for the Higgses h and H. We have not shown the part of the superpotential involving the primed fields and all other fields getting a vev. An irrelevant term $X\bar{F}^c F^c$ is also omitted. As anticipated, the messenger fields and Σ only get a mass through the Pati-Salam breaking fields. Besides X, X_c, the fields getting a vev are ϕ, F'_c, $\bar{F}'_c$ (R_P is thus preserved). The hierarchy of fermion masses originates from the assumption that the Pati-Salam breakings along the T_{3R} and $N'_c, \bar{N}'_c$ directions, $\langle X_c \rangle = M_R(2T_{3R})$ and $\langle F'_c \rangle = (V_c, 0)^T$, $\langle \bar{F}'_c \rangle = (\bar{V}_c, 0)^T$ respectively, both take place at a scale $M_R \sim V_c$ much higher scale than the scale $M_L \sim v$ of the breaking along the $B-L$ direction, $\langle X \rangle = M_L T_{B-L}$, $\langle \phi \rangle = v T_{B-L}$.[6] The horizontal fermion hierarchy therefore follows from the vertical structure of the theory. The vev of ϕ breaks the $\mathbf{Z}_2$ symmetry and mixes light and heavy fields, thus giving rise to the Yukawa couplings of light fields. The vevs of F'_c and $\bar{F}'_c$ are responsible for the full

[6]One example for the superpotential involving the primed fields and X_c, X, ϕ only is (neglecting F', $\bar{F}'$, including mass terms)

$$W' = (M_R - \delta_c X_c)\bar{F}'_c F'_c + \frac{M_{X_c}}{2} X_c^2 + \frac{M_X}{2} X^2 + \frac{M_\phi}{2} \phi^2 + \rho_1 X^3 + \rho_2 X \phi^2.$$

This it the most general renormalizable potential except for the $X\bar{F}'_c F'_c$ coupling, which is assumed to vanish. One solution of the F-term equations is (up to an $\mathrm{SU}(2)_R$ rotation) $\delta_c \langle X \rangle = M_R(2T_{3R})$, $(\delta_c/2)^2 \langle \bar{N}'_c N'_c \rangle = M_{X_c}^2$, $\langle \phi \rangle = 0$, $\langle X \rangle = 0$. Both the breaking along the T_{3R} and $N'_c, \bar{N}'_c$ directions take place at the same scale M_R, while the breaking along the $B-L$ direction is suppressed (zero at the renormalizable level).

breaking of the Pati-Salam to the SM group, they generate a mixing between $\mathrm{SU}(3)_c$ triplets which suppresses the charm quark Yukawa, and they make H heavy.

It is convenient to choose a basis in flavour space such that $\lambda_{1,2} = \alpha_{1,2} = 0$, $\lambda^c_1 = \alpha^c_1 = 0$. Moreover, λ_3, α_3, $\lambda^c_{2,3}$, $\alpha^c_{2,3}$, γ, M_L, M_R, $\bar{\sigma}_c$, $\langle\phi\rangle$, $V_c = \bar{V}_c$, can all be taken positive. We therefore see that the effective theory in which H is integrated out possesses an accidental chiral $\mathrm{U}(1)_1$ flavour symmetry protecting the first family Yukawas: $f_1 \to e^{i\alpha} f_1$, $f^c_1 \to e^{i\alpha} f^c_1$. In the limit in which the heavier messengers $F^c, \bar{F}^c$ are also integrated out, an additional accidental flavour symmetry $\mathrm{U}(1)_2$ protects the second family Yukawas: $f_2 \to e^{i\beta} f_2$, $f^c_2 \to e^{i\beta} f^c_2$. The hierarchy between the third and the first two fermion family masses can be seen as a consequence of the above flavour symmetries. The stronger suppression of the first fermion family mass is due to the fact that the heavy Higgs H does not mix with h at the renormalizable level. This is because the coupling $\phi H h$ is not allowed by the $\mathrm{SU}(4)_c$ symmetry. The suppression of the first family masses is therefore obtained for free, as it is a consequence of the Pati-Salam quantum numbers of ϕ, which are independently motivated by the m_μ/m_s ratio.

2.2.2 The fermion spectrum at the renormalizable level

Since R-parity is not broken, we can confine ourselves to the R_P-odd fields. Let us denote by A_Σ, T_Σ, $\bar{T}_\Sigma$, G_Σ the (properly normalized) SM components of Σ. Under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_w \times \mathrm{U}(1)_Y$, A is a singlet, $T \sim (3, 1, 2/3)$ is a color triplet, $\bar{T} \sim (\bar{3}, 1, -2/3)$ is an antitriplet, $G \sim (8, 1, 1)$ is an octet. With standard notations for the SM components of the fields in Table 2.1, the mass terms are

$$
\begin{gathered}
-\bar{L}\left[M_L L + \alpha_3 v l_3\right] - \bar{E}^c\left[M_R E^c + v(\alpha^c_3 e^c_3 + \alpha^c_2 e^c_2)\right] \\
+\frac{1}{3}\bar{Q}\left[M_L Q + \alpha_3 v q_3\right] - \bar{D}^c\left[M_R D^c - \frac{v}{3}(\alpha^c_3 d^c_3 + \alpha^c_2 d^c_2)\right] \\
+\bar{U}^c\left[M_R U^c + \frac{\sigma_c}{\sqrt{2}} V_c \bar{T}_\Sigma + \frac{v}{3}(\alpha^c_3 u^c_3 + \alpha^c_2 u^c_2)\right] + T_\Sigma\left[M_\Sigma \bar{T}_\Sigma + \frac{\bar{\sigma}_c}{\sqrt{2}} V_c U^c\right] \\
+\bar{N}^c\left[M_R N^c - v(\alpha^c_3 n^c_3 + \alpha^c_2 n^c_2)\right] - \sqrt{\frac{3}{8}}\sigma_c V^c \bar{N}^c A_\Sigma - \sqrt{\frac{3}{8}}\bar{\sigma}_c V_c N^c A_\Sigma + M_\Sigma A^2_\Sigma \\
+\eta' V_c L' H_u + \bar{\eta}' V_c \bar{L}' H_d - \frac{M_\Sigma}{2} G^2_\Sigma,
\end{gathered}
\tag{2.9}
$$

where $M_\Sigma = -(2/3)\gamma M_L$. The charged fermion Yukawas are obtained by identifying the massless combinations and expressing the Yukawa lagrangian

$$\lambda^c_i U^c q_i h_u + \lambda^c_i D^c q_i h_d + \lambda^c_i N^c l_i h_u + \lambda^c_i E^c l_i h_d + \lambda_i u^c_i Q h_u + \lambda_i d^c_i Q h_d + \lambda_i n^c_i L h_u + \lambda_i e^c_i L h_d \tag{2.10}$$

in terms of them. We then obtain, at the scale M and at the leading order in ϵ,

$$Y^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^c_2\lambda^c_2\epsilon/3 & \alpha^c_2\lambda^c_3 c\,\epsilon/3 \\ 0 & \alpha^c_3\lambda^c_2\epsilon/3 & -s\lambda_3 \end{pmatrix} \qquad Y^E = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^c_2\lambda^c_2\epsilon & \alpha^c_2\lambda^c_3 c\,\epsilon \\ 0 & \alpha^c_3\lambda^c_2\epsilon & s\lambda_3 \end{pmatrix}, \tag{2.11}$$

where $c = \cos\theta$, $s = \sin\theta$, $\tan\theta \equiv \alpha_3 v/M_L = \mathcal{O}(1)$, $\epsilon \equiv v/M_R \ll 1$. The numerical value of ϵ turns out to be $\epsilon \approx 0.06\,(s\lambda_3)/(\alpha^c_2\lambda^c_2)$.

The up quark sector deserves some additional comments. The situation is different than in the down quark and charged lepton sector, as the triplet $\bar{T}_\Sigma$ has the same SM quantum numbers as u^c_i and U^c and mixes as well. The charm quark Yukawa arises from the interaction $\lambda^c_i U^c q_i h_U$ when U^c is replaced by its light component. The light component must be orthogonal to both the combinations in squared brackets in the third line of eq. (2.9). As a consequence, the charm Yukawa turns out to be suppressed twice. The light component of U^c vanishes in fact both in the $v \to 0$ limit ($\mathbf{Z}_2$ is not broken, u^c_i do not mix with $U^c, \bar{T}_\Sigma$) and in the $M_\Sigma \to 0$ limit (the light component must in this case be orthogonal to U^c). This explains the factors ϵ^2 in

$$Y^U = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & (4/9)\alpha^c_2\lambda^c_2\rho_u\epsilon^2 & (4/9)\alpha^c_2\lambda^c_3 c\rho_u\epsilon^2 \\ 0 & (4/9)\alpha^c_3\lambda^c_2\rho_u\epsilon^2 & s\lambda_3 \end{pmatrix}. \tag{2.12}$$

In the equation above, $\rho_u = (\gamma\alpha_3)/(\sigma_c\bar{\sigma}_c t_\theta)(M_R/V_c)^2$, which turns out to be close to one as it should, as $\rho_u\,\epsilon \approx 0.07$–$0.08$.

The Yukawas of the first family vanish at the renormalizable level, as anticipated. We will see below how they are generated at the non-renormalizable level. For the time being, let us comment about some interesting features of eqs. (2.11,2.12). We have assumed that i) the $\mathbf{Z}_2$-breaking field ϕ is in the adjoint of $\mathrm{SU}(4)_c$ and ii) the masses of the messenger fields and Σ are linked to Pati-Salam breaking, with the breaking along the $B-L$ direction taking place at a much smaller scale than the breaking in the T_{3R} and singlet neutrino directions. As a consequence, we find i) $m_s \ll m_b$ and $m_\mu \ll m_\tau$, ii) $|V_{cb}| \sim m_s/m_b$, iii) $(m_\tau/m_b)_M \approx 1$ iv) $(m_\mu/m_s)_M \approx 3$, v) $m_c/m_t \ll m_s/m_b$. We also predict the suppression of the first family fermion masses. Note in particular that two different hierarchies in the down quark/charged lepton sectors and in the up quark sector are obtained in terms of a single hierarchy between the two scales of the theory M_R and M. Note also that the relation $|V_{cb}| \sim m_s/m_b$ is a direct consequence of the principles of our approach. As usual in the presence of a single Higgs multiplet, one also obtains λ_τ–λ_b–λ_t unification.

Let us now consider the neutrino sector. The (R_P-odd) SM singlet neutrino fields in the model are $n^c_{1,2,3}$, N^c, $\bar{N}^c$, A_Σ. Eq. (2.9) shows that $\alpha^c_3 n^c_3 + \alpha^c_2 n^c_2$, N^c, $\bar{N}^c$, A_Σ get a heavy mass, while $\alpha^c_2 n^c_3 - \alpha^c_3 n^c_2$ and n^c_1 are massless at the renormalizable level. This is clearly a problem, as it implies a Dirac mass to the tau neutrino at the electroweak scale. A possible solution is to invoke (small) non-renormalizable contributions to the massesï¿½ of $\alpha^c_2 n^c_3 - \alpha^c_3 n^c_2$ and n^c_1. However, this would make the latter fields dominate the see-saw, while we saw in the previous section that we prefer N^c to dominate. We therefore couple the SM singlets n^c_i to 3 Pati-Salam singlets $s_i \sim (1, 1, 1, -, -)$ through the Dirac mass term provided by the interaction $\eta^s_{ki} s_k f^c_i \bar{F}'_c$. This raises the fields n^c_i and s_k at the higher of the two scales of our model. Note that it is always possible to choose a basis for the s_k's such that the coupling η^s_{ki} and the Dirac mass term are diagonal.

The fields n^c_i and s_k constitute a pseudo-Dirac system. That is because a Pati-Salam invariant Majorana mass term for the Pati-Salam singlets s_k cannot be written at the renormalizable level, according to our hypothesis stating that the mass terms should originate from PS breaking. The only correction to the pure Dirac limit therefore comes from the mixing of the s_k's with A_Σ, which is however suppressed by $v/M_R = \epsilon$. Since the coupling of the pseudo-Dirac pair (n^c_3, s_3), to the light lepton doublets, $\lambda_3 n^c_3 L h_u$, only involves n^c_3, the contribution to the see-saw of the (n^c_i, s_i) fields is negligible. In fact, that contribution vanishes in the pure Dirac limit. This can be seen for example by diagonalizing the Dirac pairs in terms of two Majorana mass eigenstates with opposite mass. As in the Dirac limit n_3 contains the two eigenstates with exactly the same weight, the two contributions to the see-saw exactly cancel[7]. Taking into account the small corrections to the pure Dirac limit, the contribution of (n^c_i, s_i) to the see-saw turns out to be suppressed by ϵ. More precisely, the contribution to the atmospheric angle is suppressed by ϵ and the contribution to m_2/m_3 by ϵ^2. We can then safely neglect the fields n^c_i and s_k for our purposes. This can also be verified by using the full 9×9 singlet neutrino mass matrix in the see-saw formula.

We are then left with 3 SM singlet (right-handed) neutrinos N^c, $\bar{N}^c$, A_Σ

[7]An alternative way to verify that the Dirac system does not contribute to the see-saw is to observe that its contribution is proportional to $(M_D^{-1})_{n^c_3 n^c_3}$, where M_D is the Dirac mass term for the two Weyl spinors n^c_3, s_3 with vanishing diagonal entries. As the inverse of a Dirac mass matrix is still in the Dirac form, $(M_D^{-1})_{n^c_3 n^c_3} = 0$

with mass terms

$$M_R \bar{N}^c N^c - \sqrt{\frac{3}{8}} V_c A_\Sigma (\sigma_c \bar{N}^c + \bar{\sigma}_c N^c) + M_\Sigma A_\Sigma^2 \tag{2.13}$$

entering the see-saw through the Yukawa interaction $N^c(\lambda_3^c l_3 + \lambda_2^c l_2) h_u$. The following effective $D = 5$ left-handed neutrino mass operator is then generated

$$\frac{1}{4} \frac{\sigma_c}{\bar{\sigma}_c} \frac{1}{M_R} (c\lambda_3^c l'_3 + \lambda_2^c l'_2)^2 h_u^2, \tag{2.14}$$

where $l'_3 = cl_3 - sL$, $l'_2 = l_2$ are the light lepton doublets. We have therefore obtained a normal hierarchy and a large atmospheric mixing angle θ_{23} in a natural way,

$$\tan\theta_{23} = \frac{\lambda_2^c}{c\lambda_3^c}, \qquad m_3 = \rho_\nu \frac{v_{\text{EW}}^2}{2 s_{23}^2 M_R}, \qquad m_{1,2} \approx 0, \tag{2.15}$$

where $v_{\text{EW}} \approx 174\,\text{GeV}$ is the electroweak breaking scale, $s_{23} = \sin\theta_{23}$, and $\rho_\nu = (\sigma_c/\bar{\sigma}_c)(\lambda_2^c)^2 \sim 1$. Eq. (2.15) determines the scale M_R of our model, $M_R \approx 0.6 \cdot 10^{15}\,\text{GeV}\rho_\nu$. The solar mixing angle and mass difference are generated at the non-renormalizable level together with the masses of the first charged fermion masses.

2.2.3 The first family

As discussed, the first family fermion masses are protected by an accidental $\text{U}(1)_1$ family symmetry. That symmetry is actually broken by the coupling of the first family with the heavy Higgs messenger H. However, H does not mix with the light Higgs h at the renormalizable level, which means that for our purposes it is effectively decoupled. The $\text{U}(1)_1$ symmetry can therefore be broken by non-renormalizable interactions either because the interactions directly involve the first family or because they induce a H-h mixing. Here we will consider the second possibility. In both cases, the first family mass will be further suppressed with respect to the other families by the heavy cutoff scale Λ.

Not all the non-renormalizable operators are suitable to give a mass to the first family. For example, the operator $f_i^c f_j \phi h$ gives the same contribution to the Yukawas of the up and down quarks (in this $\lambda_t \approx \lambda_b$ scenario the up quark mass Yukawa needs to be suppressed by a factor of about 200). The operator $F'_c F' \phi h$ is also dangerous, as it indirectly contributes to the up quark mass only. We therefore need to make an assumption on the operators generated by the physics above the cutoff Λ. A simple assumption is that the the heavy physics only couples ϕ to the barred $\bar{F}'$, $\bar{F}'_c$ (but not to F', F'_c). This would still allow an operator in the form

$$\frac{a}{\Lambda}\bar{F}'_c\bar{F}'\phi h, \tag{2.16}$$

which turns out to give mass to the electron and the down quark, but not to the up quark, as desired. The reason is that the operator above induces a mixing in the down Higgs sector but not in the up Higgs sector. As mentioned in Section 2.2.1, H_d and H_u get a mass term, $\eta' V_c L' H_u + \bar{\eta}' V_c \bar{L}' H_d$, from the vev of $\bar{F}'_c$ through the renormalizable interactions in eq. (2.8). In addition, the operator in eq. (2.16) gives a mass term $-a(V_c v/\Lambda)\bar{L}' h_d$, which induces a mixing between the two down Higgses H_d and h_d. This in turn communicates the $\mathrm{U}(1)_1$ breaking provided by $\lambda^H_{ij} f^c_i f_j H$ to the down quark and charged lepton sector. When H_d is expressed in terms of the exact Higgs mass eigestates H'_d and h'_d, the latter operator induces in fact a contribution to the down and charged lepton Yukawas matrices Y^D_{ij} and Y^E_{ij} given by $\epsilon' \rho_h \lambda^H_{ij}$ (up to the L-l'_3 mixing), where

$$\epsilon' = \frac{v}{\Lambda} = \epsilon\frac{M_R}{\Lambda} \tag{2.17}$$

and $\rho_h = a/\bar{\eta}' \sim 1$. The small ratio M_R/Λ explains the further suppression of the first fermion family. We then obtain, at leading order,

$$Y^D = \begin{pmatrix} \rho_h\lambda^H_{11}\epsilon' & \rho_h\lambda^H_{12}\epsilon' & \rho_h\lambda^H_{13}c\,\epsilon' \\ \rho_h\lambda^H_{21}\epsilon' & \alpha^c_2\lambda^c_2\epsilon/3 & \alpha^c_2\lambda^c_3 c\,\epsilon/3 \\ \rho_h\lambda^H_{31}\epsilon' & \alpha^c_3\lambda^c_2\epsilon/3 & -s\lambda_3 \end{pmatrix} \qquad Y^E = \begin{pmatrix} \rho_h\lambda^H_{11}\epsilon' & \rho_h\lambda^H_{12}\epsilon' & \rho_h\lambda^H_{13}c\,\epsilon' \\ \rho_h\lambda^H_{21}\epsilon' & -\alpha^c_2\lambda^c_2\epsilon & -\alpha^c_2\lambda^c_3 c\,\epsilon \\ \rho_h\lambda^H_{31}\epsilon' & -\alpha^c_3\lambda^c_2\epsilon & -s\lambda_3 \end{pmatrix}. \tag{2.18}$$

The up Higgs does not mix, which explains the smallness of the up quark Yukawa. The latter will be eventually generated by Planck scale effects. For example an operator $(c/M_{\mathrm{pl}})f^c_i f_j \phi h$ would provide a up quark Yukawa of the correct order of magnitude for $c \sim 1$. The latter argument also provides an independent estimate (an upper bound in the general case) of the scale M_R, which happens to coincide with our estimate from neutrino physics.

Eq. (2.18) shows that the electron and down quark masses are expected to be similar, while the correct relation is $m_e \sim m_d/3$ at the heavy scale. In order to avoid the wrong relation, λ^H_{11} should be sufficiently suppressed in the basis in flavour space which identifies the first family. Quantitatively, the requirement is $\lambda^H_{11}/\lambda^H_{12,21} < \sqrt{m_d/m_s}/3 \sim 0.08$. This suppression could for example accidentally arise when rotating the fields to go in the basis in which eqs. (2.11,2.18) are written. In this case one obtains $m_e \sim m_d/3$ and

$V_{us} \sim \sqrt{m_d/m_s}$, as observed, at the price of a fine-tuning of at least $\mathcal{O}(10)$[8].

The full CKM matrix can be obtained by diagonalizing the up and down Yukawa matrices. V_{ub}/V_{cb} and V_{td}/V_{ts} both get a contribution from Y^D_{31}. On top of that, V_{td}/V_{ts} also gets a contribution from the commutation of the "12" rotation used to diagonalize Y^D and the relative 23 rotation(V_{cb}). In formulas,

$$\frac{V_{ub}}{V_{cb}} = \frac{\alpha^c_2 \lambda^H_{31}}{\alpha^c_3 \lambda^H_{21}} V_{us}, \qquad \delta = \arg\left[\frac{\alpha^c_2 \lambda^H_{31}}{\alpha^c_3 \lambda^H_{21}}\right], \qquad \left|\frac{V_{td}}{V_{ts}}\right| = \left| |V_{us}| - \left|\frac{V_{ub}}{V_{cb}}\right| e^{i\delta} \right|, \tag{2.19}$$

where δ is the CKM phase in the standard parameterization. The present SM CKM fits give [24] $|(\alpha^c_2 \lambda^H_{31})/(\alpha^c_3 \lambda^H_{21})| \approx 0.4$.

A comment on V_{us} is in order. As we saw, the physics giving rise to the Yukawas of the first family will typically also generate a contribution to V_{us}. V_{us} and the first family are however in principle independent issues. In fact, V_{us} is related to the breaking of the LR symmetry, while the first family requires the breaking of the corresponding accidental flavour symmetry. Indeed, the reason why the mechanism generating first family Yukawas also typically generates V_{us} is that in order to make $m_d/m_b \gg m_u/m_t$ the LR symmetry must be broken. On the other hand, it is possible to generate a contribution to V_{us} without inducing a corresponding contribution to the first family mass. The operator $b_i X_c F^c f_i h/\Lambda$, involving the $SU(2)_R$ breaking field X_c, gives for example a contribution $2(b_1/\lambda^c_2)(M_R/\Lambda)$ to V_{us} without breaking $U(1)_1$ (it also modifies eq. (2.19)). From the previous argument and from eq. (2.18) we expect

$$\frac{M_R}{\Lambda} \sim \frac{|V_{us}|}{2} \sim 0.1. \tag{2.20}$$

Finally, let us go back to neutrino masses. By using the renormalizable interactions, we succeeded in giving a mass to the heaviest neutrino ν_3 and in generating a large atmospheric neutrino angle θ_{23}. We still need to generate a mass for the intermediate neutrino m_2 and a corresponding large solar angle θ_{12}. As we show in section 2.2.3, non-renormalisable interactions involving the fields introduced so far can generate a mass term for m_2 at the correct level together with a non-vanishing θ_{13} close to the current experimental limit, but not a large solar angle θ_{12}. However, a large solar angle can be induced by

[8]One could make at this point the totally disinterested observation that our model involves more than $\mathcal{O}(10)$ relations among $\mathcal{O}(1)$ coefficients, so that accidental cancellation of leaving less than one part out of 10 is expected to occur somewhere. In fact, from this point of view, the distribution of the absolute values of our $\mathcal{O}(1)$ coefficients turns out to be rather peaked on 1.

a Pati-Salam singlet $S \sim (1, 1, 1, +, -)$ coupling at the non-renormalizable level only[9]. Its mass term will be in the form $d'(V_c^2/\Lambda)S^2$. Its Yukawa coupling to the lepton doublets comes from the operator $e_i F_c' S f_i h_u/\Lambda$. Its mixing with the other SM singlets is negligible. Therefore, its contribution to the neutrino mass operator is simply given by

$$-\frac{1}{4d'}\frac{1}{\Lambda}(e_3 c\, l_3' + e_2 l_2' + e_1 l_1')^2 h_u^2. \tag{2.21}$$

We then get an additional contribution to θ_{13}, $\theta_{13}^e = -s_{23}^2 \rho_{12} e_1 (c\, c_{23} e_3 + s_{23} e_2)(M_R/\Lambda)$, where $\rho_{12} = 1/(\rho_\nu d')$. Moreover, in the limit in which only eq. (2.21) adds to the leading term in eq. (2.14), the lighter neutrino masses m_1 and m_2, together with the solar mixing angle, are given by the diagonalization of the "12" mass matrix

$$-s_{23}^2 \rho_{12} m_3 \frac{M_R}{\Lambda} \begin{pmatrix} e_1^2 & e_1(c_{23}e_2 - c\, s_{23}e_3) \\ e_1(c_{23}e_2 - c\, s_{23}e_3) & (c_{23}e_2 - c\, s_{23}e_3)^2 \end{pmatrix}. \tag{2.22}$$

Neutrino mixing, solar angle and θ_{13}

In this section we show that in the absence of S non-renormalizable contributions to the superpotential generate a non-vanishing m_2 and a sizable contribution to θ_{13}, but no large solar mixing angle. In general, the latter contributions can affect the see-saw either through the singlet neutrino mass matrix or through the Yukawa interactions with the light SM lepton doublets. The leading order operators contributing to the singlet neutrino mass matrix are $\bar{F}_c'\bar{F}_c' f_i^c f_j^c$, $\bar{F}_c'\bar{F}_c' F^c F^c$, $F_c' F_c' \bar{F}^c \bar{F}^c$, $\bar{F}_c' F_c' s_k s_h$, $X_c^2 s_k s_h$. Only the two operators involving s_k affect the see-saw in a significant way. Let $d_{ij}(V_c^2/\Lambda)s_i s_j$ be the Majorana mass term induced by those operators. If M_s is the singlet neutrino mass matrix, the $s_3 s_3$ mass term gives $(M_s^{-1})_{n_3^c n_3^c} \approx -2(d_{33}/\eta_3^{s2})/\Lambda$. In turn, through the Yukawa interaction $\lambda_3 n_3^c L h_u$ and the see-saw mechanism, the latter gives a contribution

$$\frac{d}{\eta_3^{s2}}\frac{1}{\Lambda}(s\lambda_3 l_3')^2 h_u^2 \tag{2.23}$$

to the dimension 5 neutrino mass operator, which adds to the leading order contribution in eq. (2.14). By diagonalizing the resulting light neutrino mass matrix we then get

$$\frac{m_2}{m_3} \approx 4\rho_{23} \sin^4\theta_{23} \frac{M_R}{\Lambda}, \tag{2.24}$$

[9]This is an important assumption as renormalizable interactions $S\bar{F}_c' F^c$, $S\bar{F}^c F_c'$ would in principle be allowed by the symmetries of the theory.

where $\rho_{23} = (s\lambda_3/\lambda^c_2)^2(\bar{\sigma}_c d)/(\sigma^c \eta^{s2}_3) \sim 1$ and θ_{23} is the atmospheric mixing angle. The ratio m_2/m_3 turns out to be of the correct order of magnitude given the estimate in eq. (2.20).

We also have non-renormalizable contributions to the Yukawa interactions with the light SM lepton doublets. The relevant operators are $b_i X_c F^c f_i h/\Lambda$ and $b'_i \Sigma F'_c f_i h_u/\Lambda$, other possibilities leading to a higher ϵ suppression. Both operators lead to a contribution to θ_{13} without inducing a significant solar mixing angle or m_2/m_3. We have already discussed the first operator in connection to $\mathrm{SU}(2)_R$ breaking and V_{us}. In the lepton sector its role is again to misalign the Yukawa couplings of N^c and E^c to the lepton doublets l_i. In a basis in which E^c has no Yukawa interaction with l_1, the Yukawa interaction of N^c becomes $N^c[\lambda^c_3 l_3 + \lambda^c_2 l_2 + 2b_1(M_R/\Lambda)l_1]h_u$ and eq. (2.14) becomes

$$\frac{1}{4}\frac{\sigma_c}{\bar{\sigma}_c}\frac{1}{M_R}\left(c\lambda^c_3 l'_3 + \lambda^c_2 l'_2 + 2b_1\frac{M_R}{\Lambda}l'_1\right)^2 h^2_u. \tag{2.25}$$

The second operator $b'_i \Sigma F'_c f_i h_u/\Lambda$ gives rise to a Yukawa interaction for the singlet A_Σ, $-\sqrt{3/8}b'_i(V_c/\Lambda)A_\Sigma l_i h_u$, which induces new contributions to the see-saw. In terms of the inverse mass matrix M^{-1}_s of the singlet neutrinos N^c, $\bar{N}^c$, A_Σ, and in the limit in which the n^c_i contribution is neglected, the neutrino mass operator is in fact now given by

$$\frac{1}{2}\left[(M^{-1}_s)_{N^cN^c}(\lambda^c_i l_i)^2 + (M^{-1}_s)_{A^\Sigma A^\Sigma}\left(\sqrt{\frac{3}{8}}b'_i\frac{V^c}{\Lambda}l_i\right)^2 - 2(M^{-1}_s)_{A^\Sigma N^c}\left(\sqrt{\frac{3}{8}}b'_i\frac{V^c}{\Lambda}l_i\right)(\lambda^c_i l_i)\right]h^2_u.$$

Since the determinant of the inverse matrix elements vanishes, $(M^{-1}_s)_{A^\Sigma A^\Sigma}(M^{-1}_s)_{N^cN^c} - (M^{-1}_s)^2_{A^\Sigma N^c} = (M_s)_{\bar{N}^c\bar{N}^c}/\det(M_s) = 0$, the equation above gives again a contribution to θ_{13} but not to θ_{12} or m_2/m_3. The neutrino mass operator can be rewritten in fact as

$$\frac{1}{4}\frac{\sigma_c}{\bar{\sigma}_c}\frac{1}{M_R}\left(c\lambda^c_3 l'_3 + \lambda^c_2 l'_2 + \frac{b'_1}{\sigma_c}\frac{M_R}{\Lambda}l'_1\right)^2 h^2_u. \tag{2.26}$$

In the presence of both M_R/Λ corrections in eqs. (2.25,2.26), the total contribution to θ_{13} is

$$\theta_{13} \supset \theta^b_{13} = 2\sin\theta_{23}\frac{b_1 + b'_1/(2\sigma_c)}{\lambda^c_2}\frac{M_R}{\Lambda}, \tag{2.27}$$

close to the experimental limit.

2.2.4 Summary

In this section we have discussed a new approach to fermion masses an mixings in which the dominance of a single family of messengers accounts for the lightness of the first family, and the further dominance of the left-handed doublet messengers accounts for the lightness of the second family. With only these assumptions we are able to account for the fermion mass hierarchy, as well as the successful mass relation $m_s/m_b \approx |V_{cb}|$. In order to naturally acount for a small Cabibbo angle, and the correct charm quark mass, we were then led to consider a broken Pati-Salam gauge structure.

The hypothesis underlying our setting is that the Pati-Salam gauge structure, the three SM families, and a relatively small set of heavy fields happen to be the only structure surviving below the cutoff $\Lambda \sim 10^{16-17}$ GeV of our model. The flavour structure of the SM fermions essentially only follows from this hypothesis, with no dynamics related to the family indices or detailed knowledge of the theory above the cutoff required.

This framework has several interesting features. The horizontal hierarchy among different families follows from a vertical hierarchy among messengers belonging to the same family. The latter is in turn related to the breaking pattern of the Pati-Salam group, with the breaking along the T_{3R} and singlet neutrino directions taking place at a higher scale than the breaking along the $B-L$ direction. In spite of the absence of small coefficients, the CKM mixing angles turn out to be small. At the same time, a large atmospheric mixing appears in the neutrino sector between normal hierarchical neutrinos in a natural way. This is obtained through a see-saw mechanism dominated by a singlet neutrino N^c which is not unified with the light lepton doublets, as it belongs to the messenger families. The final scheme has N^c as the dominant singlet, with S as the leading subdominant singlet as in sequential dominance. The relation $|V_{cb}| \sim m_s/m_b$ is a direct consequence of the principles of our approach. The two different mass hierarchies in the down quark/charged lepton sectors on one side and in the up quark sector on the other are obtained in terms of a single hierarchy between the two scales of the theory M_R and M. The suppression of the first fermion family masses also does not need a new scale for the messenger fields. It is actually a prediction of the model, as it again follows from the gauge structure of the model, which forbids the relevant coupling of the Higgs messenger field. As usual in the presence of a single Higgs multiplet, one also obtains λ_τ–λ_b–λ_t unification.

The precise structure of the masses and mixings of the first fermion family requires an assumption on the operators generated by the physics above the cutoff Λ and relies on an accidental cancellation corresponding to a fine-tuning of at least 10. In the neutrino sector, a large solar mixing angle is obtained together with $\theta_{13} = \mathcal{O}\,(m_2/m_3)$, close to the present experimental limit.

Chapter 3

Magic fields and unification

In this chapter we discuss exact 1-loop unification with "magic" fields in incomplete GUT multiplets, which will be used extensively in Chapter 4 and has an intrinsic interest, as alternative way to achieve unification, in particular in the context of theories with GUT extra dimensions.

In the MSSM gauge coupling unification takes place at 1-loop level, to a very good approximation and at a scale $M^0_{GUT} \simeq 2 \cdot 10^{16}$ GeV. This remarkable coincidence is one of the cornerstones of SUSY GUTs.

In a SUSY GUT framework, it is usually assumed that only complete GUT multiplets can be added at an intermediate scale between M_Z and M^0_{GUT}. Complete GUT multiplets give a contribution $\Delta b_1 = \Delta b_2 = \Delta b_3$ to the 1-loop β function and obviously do not spoil the unification of gauge couplings. However there are also examples of sets of fields which do not form complete GUT multiplets but have $\Delta b_1 = \Delta b_2 = \Delta b_3$. One example can be found in [25].

The condition $\Delta b_1 = \Delta b_2 = \Delta b_3$ is sufficient but not necessary in order to achieve gauge coupling unification at 1-loop. This was discussed by Martin and Ramond in [26]. This work belongs to a large amount of literature (see [27]) addressing the possibility of enhancing the unification scale, mainly in the context of string theory phenomenology where the unification scale is more than one order of magnitude higher than M^0_{GUT}. In that paper it was noticed that if the extra matter satisfies the condition

$$\frac{\Delta b_3 - \Delta b_2}{\Delta b_2 - \Delta b_1} = \frac{5}{7} \tag{3.1}$$

then the 1-loop unification of the MSSM is exactly preserved. However the focus of that work was on the possibility that MSSM unification was not exact and the extra matter could cure a wrong $\alpha_3(M_Z)$ prediction.

In this chapter we consider the possibility of adding extra matter that satisfies the condition (3.1) in order to preserve gauge coupling unification

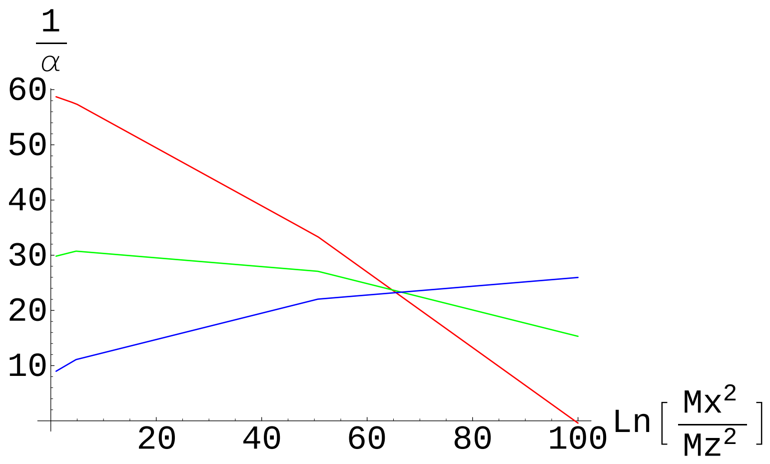

Figure 3.1: Running of the gauge couplings induced by the matter fields at scale $M_L \sim 10^{13} - 10^{14}$ GeV.

with new non-unified physics at intermediate scales. This condition could appear fine-tuned, but such a matter content can appear naturally in some models.

As an example, we consider the model of the previous chapter. This flavour model contains heavy fields at a scale of order 10^{14} GeV, much lower than the unification scale. These heavy field contribute to the beta functions of the gauge couplings above M. They do not lie in complete representations of a unified gauge group, therefore they modify the running of the SM gauge coupling constants between $4 \cdot 10^{13}$ GeV and $2 \cdot 10^{16}$ GeV, generally spoiling the usual MSSM unification.

Now we discuss the effect of these new fields on the running. The matter content at scale M_R is not fully determined in the model above, because it depends on the details of Pati-Salam breaking, so we concentrate instead on the matter content at the lighter scale M. Neglecting mixings, the chiral matter at this scale is $Q, \bar{Q}, L, \bar{L}, G, T_\Sigma, \bar{T}_\Sigma$ and its contribution to the beta functions (b_1, b_2, b_3) is

$$\Delta b = (12/5, 4, 6)$$

The modified running of the gauge couplings is shown in figure 3.1.

Quite surprisingly, the gauge couplings unify even with the new fields, but the unification scale is modified. The reason is that the matter content is very similar to $(Q, \bar{Q}, G) + (L, \bar{L}, D^c, \bar{D}^c)$ except for the hypercharge of the last two fields. The last four fields come from full multiplets of $SU(5)$ while the first three do not fit into complete multiplets. However the contributions of all these fields to the beta functions are

$$\Delta b = (6/5, 4, 6)$$

which satisfy the condition (3.1) and preserve unification at 1-loop, but at an energy scale higher than the usual GUT scale. The effect of the difference in hypercharge of the fields $T_\Sigma, \bar{T}_\Sigma$ with respect to $D^c, \bar{D}^c$ is small for scales larger than 10^{14} GeV and results in a lower prediction for $\alpha_3(M_Z)$ at 1-loop, which cancels with the 2-loop enhancement.

In the rest of the chapter we show the field contents which preserve 1-loop unification and their effects on the unification scale and the unified coupling constant. We also present some applications to 5D models, intermediate scale models and gauge mediation.

3.1 Magic fields

We consider the MSSM field content with additional matter fields at a scale Q_0. Denoting the contribution of these new fields to the beta functions by b_i^N and the contribution of the MSSM by b_i^0, the 1-loop running of the gauge couplings is given by

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M_Z) - \frac{b_i^0}{2\pi}\log\left(\frac{\mu}{M_Z}\right) - \frac{b_i^N}{2\pi}\log\left(\frac{\mu}{Q_0}\right). \qquad (3.2)$$

If we assume 1-loop unification in the MSSM at scale M_{GUT}^0 with unified coupling α_U^0, the condition for preserving gauge coupling unification at scale M_{GUT} with the new field content turns out to be

$$\alpha_U^{-1} = (\alpha_U^0)^{-1} - \frac{b_i^N}{2\pi}log\left(\frac{M_{GUT}^0}{Q_0}\right) - \frac{b_i}{2\pi}log\left(\frac{M_{GUT}}{M_{GUT}^0}\right) \qquad (3.3)$$

with $b_i = b_i^0 + b_i^N$. Eliminating gauge couplings and scales we get the "magic condition"

$$\frac{b_i^N - b_j^N}{b_j^N - b_k^N} = \frac{b_i^0 - b_j^0}{b_j^0 - b_k^0} \qquad (3.4)$$

In the MSSM it can be written as

$$\frac{b_3^N - b_2^N}{b_2^N - b_1^N} = \frac{b_3^0 - b_2^0}{b_2^0 - b_1^0} = \frac{5}{7}. \qquad (3.5)$$

With this condition the 1-loop unification is preserved independently on the value of Q_0.

Generally unification can take place at a different scale $M_{\mathrm{GUT}}^{\mathrm{new}}$ given by

$$M_{\mathrm{GUT}}^{\mathrm{new}} = M_{\mathrm{GUT}}^{0} \left(\frac{Q_0}{M_{\mathrm{GUT}}^{0}} \right)^{r} \tag{3.6}$$

where

$$r = \frac{b_3^N - b_2^N}{b_3 - b_2} \tag{3.7}$$

The unified gauge coupling is

$$\alpha_U^{-1} = (\alpha_U^0)^{-1} - \frac{(1-r)b_i^N - r b_i^0}{2\pi} \log \left(\frac{M_{GUT}^0}{Q_0} \right) \tag{3.8}$$

The scale Q_0 is almost arbitrary, but two mild bounds come from the requirement that $M_{\mathrm{GUT}}^{\mathrm{new}} < M_{\mathrm{Planck}}$ and that at $M_{\mathrm{GUT}}^{\mathrm{new}}$ the gauge coupling is still in the perturbative regime, i.e. $\alpha^{-1}(M_{\mathrm{GUT}}^{\mathrm{new}}) > 1/(4\pi)$

A trivial possibility to preserve unification is to add complete GUT multiplets with $b_3^N = b_2^N = b_1^N$, but there might exist also other field contents which satisfy the condition (3.5) and to which we refer as "magic" sets of fields in the following. Most remarkably, these magic sets can lead to unification at a new GUT scale.

The effect of the magic fields on the running of the gauge couplings is described by the parameter r given in in eq. (3.7). This parameter determines the relative order of the three scales Q, M_{GUT}^{0} and $M_{\mathrm{GUT}}^{\mathrm{new}}$. There are five different scenarios depending on the value of the parameter r.

- $r = 0 \Rightarrow Q < M_{\mathrm{GUT}}^{0} = M_{\mathrm{GUT}}^{\mathrm{new}}$: **Usual unification**
  This corresponds to $b_3^N = b_2^N = b_1^N$ and the GUT scale is unchanged, but the unified coupling changes accordingly to (3.8). The magic fields can be complete GUT multiplets, but not necessarily.

- $-\infty < r < 0 \Rightarrow Q < M_{\mathrm{GUT}}^{0} < M_{\mathrm{GUT}}^{\mathrm{new}}$: **Retarded unification**
  In this scenario the magic fields slow the running of the gauge couplings. It resembles the usual picture of unification with a higher GUT scale.

  The simplest example of retarded unification is $\left(Q + \bar{Q}\right) + G$ or $(3,2)_{1/6} + (\bar{3},2)_{-1/6} + (8,1)_0$, which gives $(b_3^N, b_2^N, b_1^N) = (5, 3, 1/5)$ and $r = -1$.

- $r = \pm\infty \Rightarrow Q = M_{\mathrm{GUT}}^{0} < M_{\mathrm{GUT}}^{\mathrm{new}}$: **Fake unification**
  This curious case corresponds to $b_3 = b_2 = b_1$, which means that the three gauge couplings run parallel above Q_0. This means that the condition for gauge coupling unification at $M_{\mathrm{GUT}}^{\mathrm{new}} > Q_0$ is to identify $Q_0 = M_{\mathrm{GUT}}^{0}$. In this way the gauge couplings unify at the usual scale

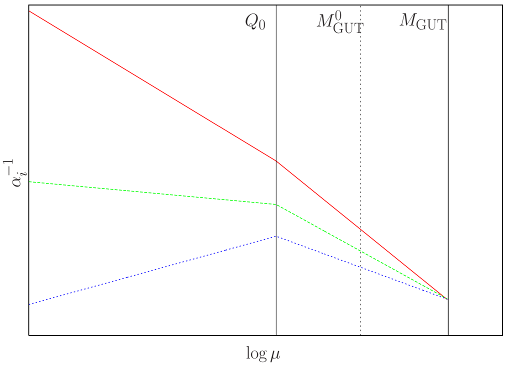

Figure 3.2: Retarded unification

$M^0_{\rm GUT}$, and then run together up to the scale $M^{\rm new}_{\rm GUT}$ where the unified gauge group is broken. Note that in this case we have a prediction for Q_0, while $M^{\rm new}_{\rm GUT}$ is completely undetermined.

From a high-energy point of view, $Q_0 = M^0_{\rm GUT}$ is quite natural. In fact the scenario is as follows: the high-energy gauge group breaks down at a given $M^{\rm new}_{\rm GUT}$, but there are magic fields of mass $Q_0 < M^{\rm new}_{\rm GUT}$ which remain light. Because of these fields, the GUT symmetry is broken but the couplings run together from $M^{\rm new}_{\rm GUT}$ down to Q_0. At this scale the couplings are still unified but start diverging below it because the magic fields decouple, therefore a low-energy observer would define this scale to be the unification scale $M^0_{\rm GUT}$ even if the unified group is broken at an higher scale.

A simple example of fake unification can be obtained with only one multiplet $(6,2)_{-1/6}$ + c.c. which has $(b_3, b_2, b_1) = (10, 6, 2/5)$. This example was noticed in [28].

- $1 < r < +\infty \Rightarrow M^0_{\rm GUT} < Q < M^{\rm new}_{\rm GUT}$: **Hoax unification**
In this scenario the magic field content flips the convergence/divergence of the running. Therefore if this content is added at a scale smaller than $M^0_{\rm GUT}$, the gauge couplings diverge above Q_0. However there is the possibility that unification is preserved if the magic fields have mass above $M^0_{\rm GUT}$. Then the couplings run apart between $M^0_{\rm GUT}$ and Q, start to converge above Q and finally unify again at $M^{\rm new}_{\rm GUT}$, the scale where the unified group is broken.

  Because of the linearity of the condition (3.5), hoax unification can be easily obtained by componing matter contents belonging to the previous cases, but there are also other possibilities: for example, $(1,3)_0 + 2\,\times$

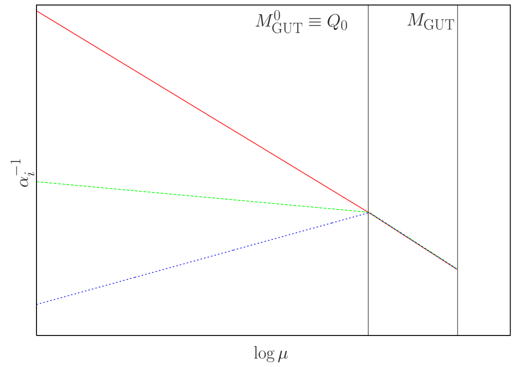

Figure 3.3: Fake unification

$\left((8,2)_{1/2} + \text{c.c.}\right)$ has $r = 3$.

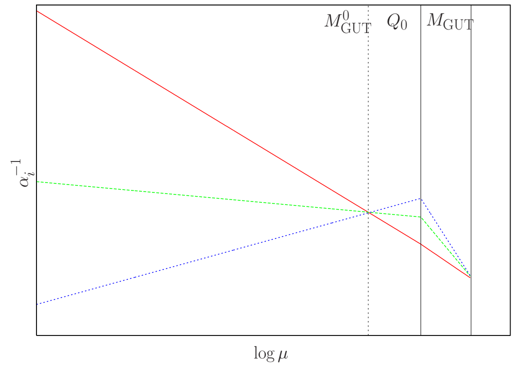

Figure 3.4: Hoax unification

- $0 < r < 1 \Rightarrow Q < M^{\rm new}_{\rm GUT} < M^0_{\rm GUT}$: **Anticipated unification**
  The magic content accelerate the running and unification takes place below the usual GUT scale. Note that with a lower GUT scale there can be some tension with bounds from proton decay searches.

If we do not introduce exotic representations, the magic condition requires $b_3^N - b_2^N$ to be even and $b_2^N - b_1^N$ to be a multiple of 14/5, therefore in the retarded case the only possibility is $b_3^N - b_2^N = 2$ which corresponds to $r = -1$ [26]. In this case the relation (3.6) between the mass scales becomes particularly simple: $M^{\rm new}_{\rm GUT}/M^0_{\rm GUT} = M^0_{\rm GUT}/Q_0$. From this we can see that Q_0 cannot be lower than $10^{13} - 10^{14}$ GeV.

3.1.1 Examples in $SO(10)$ GUT

In this section we give some simple examples of $SO(10)$ superpotentials providing a magic content of light fields.

- The simplest example of retarded unification is $\left(Q+\bar{Q}\right)+G$ which belong to a $16+\overline{16}+45$. This can be obtained by considering for example the superpotential

$$W = 16\,45_H\overline{16}+16_H\,16\,10+\overline{16}_H\,\overline{16}\,10+45_H\,45\,54+16_H\,45\,\overline{16}+\overline{16}_H\,45\,16+M\,10\,10+M\,54\,54 \tag{3.9}$$

  with all trilinear couplings of order 1 and $M \sim M_{GUT}$. The vev of 45_H is taken in the T_{3R} direction and is of order M_{GUT}, therefore this superpotential gives a mass of order M_{GUT} to all fields except $Q, \bar{Q}, G$ which are assumed to get a mass at scale Q_0.

  A two-loop analysis of this case shows that for $Q_0 = 10^{15}$ the unification scale is $M_{\rm GUT}^{\rm new} = 4\cdot 10^{17}$ GeV and $\alpha_U = 0.046$. The corresponding prediction for $\alpha_s(M_Z) = 0.127(3)$ evaluated for a typical SUSY spectrum does not differ significantly from the usual 2-loop MSSM analysis (and around 2σ above the present experimental value [29]).

- An example of fake unification (beyond the simple doubling of the above solution) is

$$2L\bar{L}+2G+2W+2E\bar{E}+\left((8,2)_{1/2}+c.c\right)$$

  which belongs to $2\times 45+120$. This light field content can be obtained by taking $\langle 45_H\rangle = \mathcal{O}(M_{GUT})T_{B-L}$ in the following superpotential

$$W = 45\;45_H\;45' + 120\;45_H\;120' + M\;120'\;120' \tag{3.10}$$

  with all trilinear couplings of order 1 and $M \sim M_{GUT}$.

- A simple example of hoax unification (beyond any combination of retarded and fake solutions, e.g. $3\times\left(Q+\bar{Q}+G\right)$) is given by $4L\bar{L}+2\left((8,2)_{1/2}+c.c\right)$ which belongs to $120+2\times 126\;\overline{126}$. The corresponding superpotential is

$$W = 126\;45_H\;\overline{126}+126'\;45_H\;\overline{126}'+120\;45_H\;120'+M\;120'\;120' \tag{3.11}$$

  with trilinear couplings ~ 1, $M\sim M_{GUT}$ and $\langle 45_H\rangle = \mathcal{O}(M_{GUT})T_{B-L}$.

3.2 Magic content in 2-step breaking of $SO(10)$

If we consider a 2-step breaking of $SO(10)$ with Pati-Salam $SU(4)\times SU(2)_L\times SU(2)_R$ as the unbroken gauge group in the intermediate region, the magic condition is modified and becomes non-linear (because of the contribution of the PS/SM gauge bosons, which would spoil MSSM 1-loop unification). The new condition is

$$\frac{b_4 - b_L}{b_L - b_R} = \frac{1}{3} \tag{3.12}$$

where the contribution of MSSM fields and PS gauge bosons is $(b_4^0, b_L^0, b_R^0) = (-6, 1, 1)$. Note that Pati-Salam couplings do not unify without extra matter, that however should be present in order to break Pati-Salam to the SM[1].

If the field content below the PS scale is the MSSM one, the classification given in section 3.1 can be mantained in these models simply by considering a different definition for r:

$$r = \frac{b_4^N - 3 - b_L^N}{b_4 - b_L} \tag{3.13}$$

and the formula (3.6) is still valid provided that we use (3.13) for r. A more general expression for the new unification scale is:

$$\ln\frac{M_{\rm GUT}^{\rm new}}{M_{\rm GUT}^0} = \left(\frac{b_3 - b_2}{b_4 - b_L} - 1\right)\ln\frac{M_{\rm GUT}^0}{M_{PS}} \tag{3.14}$$

where b_2, b_3 are the SM coefficients just below the PS scale.

If we do not consider the contribution of the fields which break Pati-Salam (e.g. because they appear in complete $SO(10)$ representations), there are simple examples of magic sets of fields which take their mass from PS-breaking vevs. One example of retarded unification is given by $(4,1,2) + (\bar{4},1,2) + (1,2,2) + (1,1,3) + (10,2,2) + (\bar{10},2,2)$ which can take their masses from a $(15,1,1)$ vev proportional to $B-L$. Another example of fake unification is simply $(6,1,1) + (10,1,1) + (\bar{10},1,1)$ which can take their masses from a $(1,1,3)$ vev proportional to T_{3R}.

A complete example of fake unification, which also provides the Pati-Salam breaking, can be constructed with the following fields:

$$A(6,1,1) + W_4(15,1,1) +$$
$$[S(10,1,1) + S_L(10,3,1) + S_R(10,1,3) + F(4,2,1) + F^c(\bar{4},1,2) + c.c.]$$

[1]Even if unification would not occur without extra matter, it is possible to restore the usual unification simply by adding a $(6,1,3)$ field at the PS breaking scale, since the contribution of this field exactly cancels the one of the massive PS gauge bosons

with the superpotential

$$W = \bar{F}^c W_4 F^c + S_R F^c F^c + \bar{S}_R \bar{F}^c \bar{F}^c + M_F \bar{F}^c F^c + M_S \bar{S}_R S_R + \\ + \frac{M_W}{2} W_4 W_4 + W_4 AA + \bar{S} W_4 S + \bar{S}_L W_4 S_L + \bar{F} W_4 F \quad (3.15)$$

where all the trilinear couplings are of order 1 and the masses are of order $M_{\rm GUT}^0 = M_{PS}$. The couplings in the first line break PS to SM with nonzero vevs of S, F^c, W_4 and their conjugates, while those in the second line give mass to all the other fields. Note that this case is interesting because the Pati-Salam breaking scale corresponds to the gauge couplings unification scale, while $SO(10)$ is broken at an higher scale $M_{\rm GUT}^{\rm new}$ which is undetermined.

We can also consider the case of an intermediate left-right gauge group $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. In this case the magic condition becomes

$$\frac{b_3 - b_{2L}}{b_{2L} - \frac{3}{5} b_{2R} - \frac{16}{15} b_{B-L}} = \frac{5}{7}. \quad (3.16)$$

where $(b_3^0, b_{2L}^0, b_{2R}^0, b_{B-L}^0) = (-3, 1, 1, 16)$ while the expression for r is the same as in the MSSM (3.7).

3.3 The magic tower

An interesting possibility appears in theories with extra dimensions. These theories usually have threshold effects near unification, coming from the tower of KK states, which can spoil 1-loop unification of the MSSM. These effects can be highly reduced if the KK states form magic multiplets.

As an example, we take 5D unified models on $S^1/(Z_2 \times Z_2')$ [30]. We consider an $SO(10)$ model with a Pati-Salam brane and an $\mathcal{N} = 1$ brane. The gauge fields live in the bulk together with a chiral hypermultiplet in the adjoint of $SO(10)$, while the SM matter fields and Higgses and other fields live on the branes. The bulk fields are:

(V, Φ)	(Φ_1, Φ_2)	
V_{++}, Φ_{--}	Φ_{1++}, Φ_{2--}	PS adjoints
V_{+-}, Φ_{-+}	Φ_{1+-}, Φ_{2-+}	$SO(10)$/PS adjoints

Their zero-modes are given by gauge fields V_{++} and an adjoint field Φ_{1++}. The odd KK levels contain the fields in the $SO(10)$/PS adjoint representation, while the even KK states contain those in the PS adjoint.

To obtain a magic tower, both odd and even KK states should form a magic set of fields. But this is verified due to the presence of the chiral adjoint field (Φ_1, Φ_2), because together with the gauge fields it forms an

$\mathcal{N} = 4$ hypermultiplet which does not contribute to the β function of the theory (the contribution of three chiral multiplets Φ, Φ_1, Φ_2 cancels exactly the one of the gauge fields V). Therefore both the even and the odd levels of the KK towers do not spoil unification.

The zero-mode Φ_{1++} cannot be light. It has a mass at some intermediate scale M_Φ which can be identified with the PS breaking scale. The content of the theory below this scale is the usual MSSM, while above this scale we have a PS theory. In order to mantain unification it is sufficient to add some fields of mass M_Φ on the PS brane which form a magic field content together with Φ_{1++}: an example is (4,1,2)+(6,1,1)+(1,1,3). Now the threshold effects are of order M_Φ/M_{KK} because the Φ_{1++} tower is shifted with respect to the other KK towers. This effect is small and decreasing for higher KK levels.

3.4 Applications

3.4.1 Intermediate scale models

We briefly discuss an example of a model with multiple intermediate scales and a magic content of fields at all scales. This is a flavor model based on the Pati-Salam gauge group and is a modified version of the model of the previous chapter. Here we only present the field content of the model and its mass scales. The full model unifies at $SO(10)$ on a 5D orbifold and a similar model will be discussed in the next chapter.

The quantum numbers of the chiral supermultiplets of the model are:

	f_i	f_i^c	h	ϕ	F	$\bar{F}$	F^c	$\bar{F}^c$	F_c'	$\bar{F}_c'$	X_c	Φ	H	ϕ_L	ϕ_R
$\mathrm{SU}(2)_L$	2	1	2	1	2	2	1	1	1	1	1	1	2	3	1
$\mathrm{SU}(2)_R$	1	2	2	1	1	1	2	2	2	2	3	1	2	1	3
$\mathrm{SU}(4)_c$	4	$\bar{4}$	1	15	4	$\bar{4}$	$\bar{4}$	4	$\bar{4}$	4	1	15	1	1	1

Table 3.1: Quantum numbers of the fields of the Pati-Salam model.

where $f_i = (l_i, q_i), f_i^c = (n_i^c, e_i^c, u_i^c, d_i^c), h = (h_u, h_d)$ contain the MSSM fields and $F + \bar{F}, F_c + \bar{F}^c$ is an heavy vector-like copy of one SM generation. We call $A_\Phi, T_\Phi, \bar{T}_\Phi, G_\Phi$ the SM components of the $SU(4)$ adjoint field Φ.

The scales of the model are M_L, M_R and satisfy $M_L \sim M_R^2/\Lambda$ where Λ is the cutoff of the theory. The Pati-Salam gauge symmetry is broken at M_R to the SM. The matter content at different mass scales is:

- $E < M_L$: we have the usual MSSM field content;

- $M_L < E < M_R$: beyond the MSSM fields, we have the left-handed heavy fields $F + \bar{F}$, ϕ and the color octet G_Φ. If we want a magic field content we can simply add the fields H, ϕ_L, ϕ_R in the last column of the table.

- $E > M_R$: all the fields in the table are present above M_R. The full field content (including PS/SM gauge bosons) still preserves unification.

The field content at M_L corresponds to a retarded solution and therefore the unification scale increases with respect to M^0_{GUT}:

$$M_{\rm GUT}^{\rm new} = M^0_{GUT} \frac{M^0_{GUT}}{M_L} \tag{3.17}$$

and the extra matter at M_R does not modify the GUT scale.

We can embed this model in a 5D GUT theory on $S^1/Z_2 \times Z_2'$ orbifold. The fields f_i, f_i^c live on the $SO(10)$ brane, $F_c', \bar{F}_c', X_c$ on the PS brane and all the other fields in the bulk. In this setup we obtain that all the KK levels are magic. This is a nontrivial example of the magic KK towers discussed in the previous section.

3.4.2 Gauge mediation

We consider the case of SUSY breaking mediated by a messenger sector consisting of magic fields and communicated to the MSSM through gauge interactions. We assume the usual superpotential

$$W = S\bar{\Psi}_i\Psi_i + M\bar{\Psi}_i\Psi_i \tag{3.18}$$

where $\Psi_i, \bar{\Psi}_i$ are the magic fields and S is the spurion with $\langle F_S \rangle \neq 0$.

Gauge mediation with incomplete GUT multiplets was studied in [25], and many of the conclusions apply also to this case. However the requirement of gauge coupling unification gives additional constraints on the sparticle spectrum.

The gaugino masses at scale Q are given by

$$M_a(Q) = \frac{\alpha_a(Q)}{4\pi} b_a^N \frac{F}{M} \tag{3.19}$$

The scalar masses are

$$\tilde{m}_i^2(Q) = \sum_a 2 \left(\frac{\alpha_a(Q)}{4\pi}\right)^2 C_a^i b_a^N \left[\frac{\alpha_a^2(Q_0)}{\alpha_a^2(Q)} - \frac{b_a^N}{b_a^0}\left(1 - \frac{\alpha_a^2(Q_0)}{\alpha_a^2(Q)}\right)\right] \left|\frac{F}{M}\right|^2 \tag{3.20}$$

The usual sum rules of gauge mediation are still valid. Interestingly, we obtain a sum rule for gaugino masses valid at all scales:

$$7\frac{M_3}{\alpha_3} - 2\frac{M_2}{\alpha_2} + 5\frac{M_1}{\alpha_1} = 0 \tag{3.21}$$

Generally, gaugino and scalar mass hierarchies are more pronounced than in the usual scenario. For instance, if the messenger sector is given by $Q+\bar{Q}+G$, the ratio between gaugino masses is very peculiar, $M_1 : M_2 : M_3 = 1 : 30 : 200$, and also the scalar masses result quite splitted: $m_{\tilde{e}^c}/m_{\tilde{q}} \sim 1/20$. For a less peculiar scenario such as $Q\bar{Q}+G+U^c\bar{U}^c+D^c\bar{D}^c+W$, we get $M_1 : M_2 : M_3 = 1 : 5 : 20$ and $m_{\tilde{e}^c}/m_{\tilde{q}} \sim 1/15$. For solutions with $b_1 = b_2 = b_3$ there should be no difference with respect to the usual gauge-mediation spectrum. Here we present some rough estimates of the masses for the two retarded solutions above, with the selectron mass around the present experimental limit:

	M_1	M_2	M_3	$m_{\tilde{e}^c}$	$m_{\tilde{q}}$
$Q\bar{Q}+G$	25 GeV	750 GeV	5 TeV	100 GeV	2 TeV
$Q\bar{Q}+G+U^c\bar{U}^c+D^c\bar{D}^c+W$	75 GeV	400 GeV	1.5 TeV	100 GeV	1.5 TeV

3.5 Some magic field contents

We present some examples of magic content for the case of SM gauge group. The magic condition (3.5) is linear, therefore we can add magic contents together, obtaining again a magic set of fields. Adding a magic content with $r = 0$ (like complete multiplets) does not modify the type of unification; adding two retarded solutions gives a fake solution, and adding a retarded and a fake (or two fake) solution gives an hoax[2].

This table contains the simplest irreducible magic sets that can be built from fields belonging to $SO(10)$ representations up to **210**. We do not write complete GUT multiplets or anticipated solutions.

[2] Note that the classification based on r can be rewritten in terms of $q = b_3^N - b_2^N$ introduced by [26]: anticipated unification corresponds to $q < 0$, usual to $q = 0$, retarded to $q = 2$, fake to $q = 4$ and hoax to $q > 4$. The q of a combination of magic fields is the sum of the qs of the sets, so one could simply read the type of unification from this.

n	field content	b_1^N	b_2^N	b_3^N	r	type
1	$(6,2)_{-1/6}+\text{c.c.}$	2/5	6	10	∞	fake
2	$\big(Q+\bar{Q}\big)+G$	1/5	3	5	-1	retarded
3	$\big(U^c+\bar{U}^c\big)+\big(D^c+\bar{D}^c\big)+W$	2	2	2	0	usual
3	$\big(D^c+\bar{D}^c\big)+G+((1,3)_1+\text{c.c.})$	4	4	4	0	usual
3	$\big(Q+\bar{Q}\big)+\big((6,1)_{-2/3}+(1,3)_1+\text{c.c.}\big)$	7	7	7	0	usual
3	$\big(L+\bar{L}\big)+\big((6,1)_{1/3}+\text{c.c.}\big)$	5	5	5	0	usual
3	$\big(Q+\bar{Q}\big)+\big(D^c+\bar{D}^c\big)+\big((8,2)_{1/2}+\text{c.c.}\big)$	27/5	11	15	∞	fake
3	$W+2\big((8,2)_{1/2}+\text{c.c.}\big)$	48/5	18	24	3	hoax
3	$W+\big((6,2)_{-1/6}+\text{c.c.}\big)+((1,1)_2+\text{c.c.})$	26/5	8	10	-1	retarded
3	$(3,3)_{2/3}+(6,2)_{-1/6}+\big((6,1)_{4/3}+\text{c.c.}\big)$	18	18	18	0	usual
3	$2W+\big((6,2)_{5/6}+\text{c.c.}\big)$	10	10	10	0	usual
3	$\big((3,3)_{2/3}+(6,2)_{5/6}+(6,1)_{-2/3}+\text{c.c.}\big)$	18	18	18	0	usual
3	$\big((8,1)_1+(\bar{3},1)_{4/3}+\text{c.c.}\big)+(8,3)_0$	16	16	16	0	usual
3	$\big((8,1)_1+(6,1)_{1/3}+\text{c.c.}\big)+(8,3)_0$	52/5	16	20	∞	fake

This table shows the simplest irreducible magic sets which provide retarded unification.

n	field content	b_1^N	b_2^N	b_3^N	r
2	$\big(Q+\bar{Q}\big)+G$	1/5	3	5	-1
5	$\big(E^c+\bar{E}^c\big)+2W+2G$	6/5	4	6	-1
5	$2\big(L+\bar{L}\big)+W+2G$	6/5	4	6	-1
5	$\big(Q+\bar{Q}\big)+\big(U^c+\bar{U}^c\big)+\big(D^c+\bar{D}^c\big)+W+G$	11/5	5	7	-1
6	$3\big(D^c+\bar{D}^c\big)+2W+G$	6/5	4	6	-1
6	$\big(U^c+\bar{U}^c\big)+\big(L+\bar{L}\big)+2W+2G$	11/5	5	7	-1
6	$\big(Q+\bar{Q}\big)+2\big(D^c+\bar{D}^c\big)+\big(E^c+\bar{E}^c\big)+W+G$	11/5	5	7	-1
6	$2\big(Q+\bar{Q}\big)+\big(D^c+\bar{D}^c\big)+2\big(E^c+\bar{E}^c\big)+G$	16/5	6	8	-1
6	$2\big(Q+\bar{Q}\big)+\big(U^c+\bar{U}^c\big)+3\big(D^c+\bar{D}^c\big)$	16/5	6	8	-1
6	$2\big(Q+\bar{Q}\big)+2\big(U^c+\bar{U}^c\big)+\big(L+\bar{L}\big)+G$	21/5	7	9	-1
6	$2\big(Q+\bar{Q}\big)+2\big(D^c+\bar{D}^c\big)+G+\big(V+\bar{V}\big)$	31/5	9	11	-1

This table shows the simplest irreducible magic contents for the Pati-Salam case. We write only fields belonging to representations of $SO(10)$ up to **210**.

n	field content	b_4^N	b_L^N	b_R^N	r
1	$(6,1,3)$	3	0	12	0
2	$(1,2,2)+((20',1,1)+\text{c.c.})$	8	1	1	∞
2	$(6,1,1)+((10,1,1)+\text{c.c.})$	7	0	0	∞
2	$((10,1,1)+\text{c.c.})+(15,2,2)$	22	15	15	∞
3	$(1,2,2)+2(15,1,1)$	8	1	1	∞
3	$(6,1,1)+(6,2,2)+((20',1,1)+\text{c.c.})$	13	6	6	∞
3	$(6,1,1)+(6,1,3)+(1,2,2)$	4	1	13	0
3	$((4,1,2)+(4,2,1)\text{c.c.})+(6,1,3)$	7	4	16	0
3	$(1,3,3)+((10,1,1)+\text{c.c.})+(6,1,3)$	9	6	18	0
3	$(6,2,2)+((20',1,1)+\text{c.c.})+(15,2,2)$	28	21	21	∞
3	$(1,2,2)+(6,1,3)+(15,2,2)$	19	16	28	0
3	$(1,1,3)+(6,1,3)+((20,2,1)+\text{c.c.})$	29	20	14	3
3	$(6,1,3)+((4,2,3)+(20,2,1)+\text{c.c.})$	35	32	44	0
3	$(6,1,3)+((4,3,2)+(20,1,2)+\text{c.c.})$	35	32	44	0
3	$(6,2,2)+(6,3,1)+(15,1,3)$	19	18	36	-1/3
3	$(1,2,2)+(15,1,1)+((10,2,2)+\text{c.c.})$	28	21	21	∞
3	$(1,2,2)+2\,((10,2,2)+\text{c.c.})$	48	41	41	∞

Chapter 4

Flavour and $SO(10)$ GUT on a 5D orbifold

4.1 Pati-Salam model upgrade

As a first step towards the $SO(10)$ model, we consider a slightly modified version of the Pati-Salam model of chapter 2. The symmetries of the theory include the gauge group $G_{\rm PS} = {\rm SU}(2)_L \times {\rm SU}(2)_R \times {\rm SU}(4)_c$ and the discrete symmetries $\mathbf{Z}_2$, R_P. The model has a minimal chiral superfield content and quantum numbers as in Table 4.1. $G_{\rm PS}$ is broken to the SM at a scale M_R. SO(10) grand unification is achieved at a higher scale $M_{\rm GUT} \gg M_R$, which we consider as the cutoff of our model, $\Lambda \equiv M_{\rm GUT}$. We will later add another few PS fields in order to preserve gauge coupling unification above the scale M_R and to take care of singlet neutrino masses. As we will see, the MSSM 1-loop unification will be exactly preserved.

The first block contains the $\mathbf{Z}_2$-odd fields: the 3 light (in the unbroken $\mathbf{Z}_2$ limit) families (f_i, f^c_i), $i = 1, 2, 3$, the light Higgs h and the $\mathbf{Z}_2$-breaking field ϕ. The latter is assumed to be in the adjoint representation of ${\rm SU}(4)_c$ as this provides the Georgi-Jarlskog factor of 3 needed to account for the μ–s mass relation. The second block contains the messengers, in a single vectorlike family $(F, F_c) + (\bar{F}, \bar{F}_c)$. The third block contains the fields $F'_c + \bar{F}'_c$ and X_c breaking the Pati-Salam group at the scale M_R. Finally, the last column corresponds to $\mathbf{Z}_2$-odd ${\rm SU}(4)_c$-adjoint Φ', which is needed to communicate the ${\rm SU}(2)_R$ breaking provided by $F'_c + \bar{F}'_c$ to the messengers $F_c + \bar{F}_c$ (it was called Σ in chapter 2). The up and down components of those messengers need in fact to be different in order to account for $m_c/m_t \ll m_s/m_b$.

	f_i	f_i^c	h	ϕ	F	$\bar{F}$	F^c	$\bar{F}^c$	F_c'	$\bar{F}_c'$	X_c	Φ'
$\mathrm{SU}(2)_L$	2	1	2	1	2	2	1	1	1	1	1	1
$\mathrm{SU}(2)_R$	1	2	2	1	1	1	2	2	2	2	3	1
$\mathrm{SU}(4)_c$	4	$\bar{4}$	1	15	4	$\bar{4}$	$\bar{4}$	4	$\bar{4}$	4	1	15
$\mathbf{Z}_2$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
R_P	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$

Table 4.1: Field content of the model and quantum numbers under G_{PS} and $\mathbf{Z}_2$

4.1.1 Superpotential

Up to explicit mass terms, assumed to be absent, the most general renormalizable superpotential for the fields in Table 4.1 is

$$W = \lambda_i f_i^c F h + \lambda_i^c f_i F^c h + \alpha_i \phi f_i \bar{F} + \alpha_i^c \phi f_i^c \bar{F}^c + a \bar{F}^c X_c F^c + \bar{\sigma}_c \bar{F}_c' \Phi' F^c + \sigma_c \bar{F}^c \Phi' F_c' + W'(F_c', \bar{F}_c', X_c, h) + W''(\phi), \quad (4.1)$$

All the couplings are assumed to be $\mathcal{O}(1)$ and uncorrelated. The terms in $W' + W''$ provide the vevs of the fields F_c', $\bar{F}_c'$, X_c, ϕ along the SM invariant directions. The $\mathbf{Z}_2$ conserving vevs lie near a single scale, M_R, which turns out to be the scale of the mass of the right-handed messengers F^c, $\bar{F}^c$. The $\mathbf{Z}_2$-breaking vev of ϕ lies at a smaller scale M_L, which turns out to be the scale of the left-handed messengers F, $\bar{F}$. The hierarchy of SM fermion masses originates from $M_L \ll M_R$. We assume that $M_L \sim M_R^2/\Lambda$, where Λ is the cutoff of the model.

Non renormalizable terms in the superpotential could give rise (or not) to mass terms of order $M_R^2/\Lambda \sim M_L$. The latter could be relevant for the left-handed messengers F, $\bar{F}$ and for the field Φ'. As we will see below, a mass term $m_\Phi \Phi^2/2$ with $m_\Phi \sim M_L$ is indeed necessary to generate a Yukawa coupling for the charm quark. On the contrary, a mass term for the left-handed messengers could be dangerous, if not in the B-L direction. Here we assume such a mass term vanishes at the order M_R^2/Λ.

We denote $F = (L, Q)$, $\bar{F} = (\bar{L}, \bar{Q})$, $F^c = (L^c, Q^c)$, $\bar{F}^c = (\bar{L}^c, \bar{Q}^c)$, $L^c = (N^c, E^c)$, $Q^c = (U^c, D^c)$, $\bar{L}^c = (\bar{N}^c, \bar{E}^c)$, $\bar{Q}^c = (\bar{U}^c, \bar{D}^c)$ and analogously for the other fields with the same quantum numbers under PS. We also denote by A_Φ, T_Φ, $\bar{T}_\Phi$, G_Φ the (properly normalized) SM components of Φ'. Under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, A is a singlet, $T \sim (3, 1, 2/3)$ is a color triplet, $\bar{T} \sim (\bar{3}, 1, -2/3)$ is an antitriplet, $G \sim (8, 1, 0)$ is an octet. Analogously for the other fields with the same quantum numbers under PS.

4.1.2 Spectrum

In order to identify the massless (in the unbroken EW symmetry limit) fields, forming the MSSM spectrum, we plug the vevs in the $W - W' - W''$ part of the superpotential. Since R_P is not broken, the R_P-even and R_P-odd fields do not mix and we can confine our analysis to the R_P-odd fields.

We denote $a\langle X_c\rangle = M_R\,\sigma_3$, $\langle N_c'\rangle = V_c$, $\langle \bar{N}_c'\rangle = \bar{V}_c$ ($|V_c| = |\bar{V}_c|$ from the D-term conditions), $\langle\phi\rangle = v\,T_{B-L}$. $V_c \sim M_R \gg v \sim M_L$. We choose a basis in flavour space such that $\lambda_{1,2} = \alpha_{1,2} = 0$, $\lambda_1^c = \alpha_1^c = 0$. λ_3, α_3, $\lambda_{2,3}^c$, $\alpha_{2,3}^c$, M_R, v, $V_c = \bar{V}_c$ can all be taken positive. The mass terms are

$$-\bar{E}^c\left[M_R E^c - v(\alpha_3^c e_3^c + \alpha_2^c e_2^c)\right] - \alpha_3 v \bar{L} l_3 \tag{4.2a}$$

$$-\bar{D}^c\left[M_R D^c + \frac{v}{3}(\alpha_3^c d_3^c + \alpha_2^c d_2^c)\right] + \alpha_3\frac{v}{3}\bar{Q} q_3 \tag{4.2b}$$

$$+\bar{U}^c\left[M_R U^c - \frac{\sigma_c}{\sqrt{2}}V_c\bar{T}_\Phi - \frac{v}{3}(\alpha_3^c u_3^c + \alpha_2^c u_2^c)\right] + T_\Phi\left[m_\Phi\bar{T}_\Phi - \frac{\bar{\sigma}_c}{\sqrt{2}}\bar{V}_c U^c\right] \tag{4.2c}$$

$$+\bar{N}^c\left[M_R N_c + v(\alpha_3^c n_3^c + \alpha_2^c n_2^c)\right] + \sqrt{\frac{3}{8}}\sigma_c V_c \bar{N}^c A_\Phi + \sqrt{\frac{3}{8}}\bar{\sigma}_c\bar{V}_c N^c A_\Phi + \frac{m_\Phi}{2}A_\Phi^2 \tag{4.2d}$$

$$+\frac{m_\Phi}{2}G_\Phi^2. \tag{4.2e}$$

Because of the absence of $\bar{L}L$, $\bar{Q}Q$ mass terms, L and Q are massless, while l_3 and q_3 get a mass together with $\bar{L}$ and $\bar{Q}$ from the vev of ϕ. The light lepton and quark doublets are therefore $l_3' = L$, $l_{1,2}' = l_{1,2}$, $q_3' = Q$, $q_{1,2}' = q_{1,2}$ and the heavy ones are $L' = l_3$, $Q' = q_3$, $\bar{L}' = \bar{L}$, $\bar{Q}' = \bar{Q}$. The light $SU(2)_L$ singlets are

$$\begin{aligned}
&(e^c)_3' = e_3^c + \alpha_3^c\epsilon E^c \quad (d^c)_3' = d_3^c - \alpha_3^c\frac{\epsilon}{3}D^c \quad (u^c)_3' = u_3^c - \frac{\sqrt{2}\alpha_3^c}{3\sigma_c^*}\frac{M_R}{V_c}\epsilon\,\bar{T}_\Phi - \frac{2}{3}\frac{\alpha_3^c}{\sigma_c^*\bar{\sigma}_c^*}\frac{m_\Phi}{v}\frac{M_R^2}{V_c^2}\epsilon^2 U^c\\
&(e^c)_2' = e_2^c + \alpha_2^c\epsilon E^c \quad (d^c)_2' = d_2^c - \alpha_2^c\frac{\epsilon}{3}D^c \quad (u^c)_2' = u_2^c - \frac{\sqrt{2}\alpha_2^c}{3\sigma_c^*}\frac{M_R}{V_c}\epsilon\,\bar{T}_\Phi - \frac{2}{3}\frac{\alpha_2^c}{\sigma_c^*\bar{\sigma}_c^*}\frac{m_\Phi}{v}\frac{M_R^2}{V_c^2}\epsilon^2 U^c\\
&(e^c)_1' = e_1^c \quad (d^c)_1' = d_1^c \quad (u^c)_1' = u_1^c
\end{aligned} \tag{4.3}$$

up to $\mathcal{O}(\epsilon^2)$ terms and higher order corrections to the coefficients, where $\epsilon \equiv v/M_R \ll 1$ and $\alpha^c = ((\alpha_3^c)^2 + (\alpha_2^c)^2)^{1/2}$. Note the double ϵ suppression of the U^c component of the light fields, or equivalently the double suppression of the light component of the U^c field, accounting, as we will see, for $m_c/m_t \ll m_s/m_b$. The reason for the double suppression is that a light component in U^c requires both m_Φ and v to be non-vanishing.

Since m_c/m_t arises at the ϵ^2 level, neglecting $\mathcal{O}(\epsilon^2)$ terms in the expressions for the $(u^c)'_i$ in eq. (4.3) is not appropriate. The $\mathcal{O}(\epsilon^2)$ term needed to make $(u^c)'_3$ and $(u^c)'_2$ orthogonal can be added to $(u^c)'_3$, to $(u^c)'_2$, or both. We add it to $(u^c)'_3$. This choice preserves $Y^U_{23} = 0$ at $\mathcal{O}(\epsilon^2)$[1]:

$$\begin{aligned}
(u^c)'_3 &= u^c_3 - \frac{\sqrt{2}\alpha^c_3}{3\sigma^*_c}\frac{M_R}{V_c}\epsilon\bar{T}_\Phi - \frac{2}{3}\frac{\alpha^c_3}{\sigma^*_c\bar{\sigma}^*_c}\frac{m_\Phi}{v}\frac{M^2_R}{V^2_c}\epsilon^2 U^c - \frac{2}{9}\frac{\alpha^c_3\alpha^c_2}{|\sigma_c|^2}\frac{M^2_R}{V^2_c}\epsilon^2 u^c_2 \\
(u^c)'_2 &= u^c_2 - \frac{\sqrt{2}\alpha^c_2}{3\sigma^*_c}\frac{M_R}{V_c}\epsilon\bar{T}_\Phi - \frac{2}{3}\frac{\alpha^c_2}{\sigma^*_c\bar{\sigma}^*_c}\frac{m_\Phi}{v}\frac{M^2_R}{V^2_c}\epsilon^2 U^c \qquad (4.4) \\
(u^c)'_1 &= u^c_1
\end{aligned}$$

We assume that W' is such that the Higgs field h is also light.

4.1.3 SM Yukawas

The light fermion Yukawa matrices Y^D, Y^E, Y^U (in right-left convention) at the scale M are easily determined expressing the superpotential in terms of

[1]The heavy space is spanned by

$$\frac{\sigma_c}{\sqrt{2}}V_c\bar{T}_\Phi + \frac{v}{3}(\alpha^c_3 u^c_3 + \alpha^c_2 u^c_2) - M_R U^c \quad \text{and} \quad m_\Phi\bar{T}_\Phi - \frac{\bar{\sigma}_c}{\sqrt{2}}\bar{V}_c U^c.$$

Let us define $u^c \equiv (\alpha^c_3 u^c_3 + \alpha^c_2 u^c_2)/\alpha^c$ and $\hat{u}^c \equiv (\alpha^c_3 u^c_2 - \alpha^c_2 u^c_3)/\alpha^c$. The light space, orthogonal to the heavy space, is then spanned by the orthonormal fields

$$(u^c)' \equiv \frac{1}{N}\left[u^c - \frac{\sqrt{2}\alpha^c}{3\sigma^*_c}\frac{M_R}{V_c}\frac{\epsilon}{1-\epsilon_1}\bar{T}_\Phi - \frac{2}{3}\frac{\alpha^c}{\sigma^*_c\bar{\sigma}^*_c}\frac{m_\Phi}{v}\frac{M^2_R}{V^2_c}\frac{\epsilon^2}{1-\epsilon_1}U^c\right], \ \hat{u}^c, \text{ and } u^c_1,$$

where N is the norm of the square bracket and $\epsilon_1 = 2(m_\Phi M_R)/(V_c\bar{V}_c)/(\sigma^*_c\bar{\sigma}^*_c) \sim \epsilon$. While $\{(u^c)', \hat{u}^c, u^c_1\}$ would be a totally decent basis for the light fields, we prefer to write the light Yukawas in a basis for the light fields that is closer to the original fields $u^c_{3,2,1}$. We therefore define the following alternative orthonormal set:

$$\begin{aligned}
(u^c)'_3 &\equiv \frac{\alpha^c_3(u^c)' - \alpha^c_2 N\hat{u}^c}{\sqrt{(\alpha^c_3)^2 + N^2(\alpha^c_2)^2}} \\
(u^c)'_2 &\equiv \frac{N\alpha^c_2(u^c)' + \alpha^c_3\hat{u}^c}{\sqrt{(\alpha^c_3)^2 + N^2(\alpha^c_2)^2}} \\
(u^c)'_1 &= u^c_1,
\end{aligned}$$

which gives eqs. (4.4), up to higher order corrections to the coefficients.

the massless fields. At the leading order in ϵ we find

$$Y^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha_2^c\lambda_2^c\epsilon/3 & 0 \\ 0 & -\alpha_3^c\lambda_2^c\epsilon/3 & \lambda_3 \end{pmatrix}, \quad Y^E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2^c\lambda_2^c\epsilon & 0 \\ 0 & \alpha_3^c\lambda_2^c\epsilon & \lambda_3 \end{pmatrix},$$
$$Y^U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(2/3)\alpha_2^c\lambda_2^c\rho_u\epsilon^2 & 0 \\ 0 & -(2/3)\alpha_3^c\lambda_2^c\rho_u\epsilon^2 & \lambda_3 \end{pmatrix}, \tag{4.5}$$

where $\rho_u = (\sigma_c\bar{\sigma}_c)^{-1}(m_\Phi/v)(M_R/V_c)^2$ is an order one coefficient. The numerical value of ϵ turns out to be $\epsilon \approx 0.06\,\lambda_3/(\alpha_2^c\lambda_2^c)$, which implies (using MSSM RGEs however and for $\tan\beta = 10$) $\rho_u \approx 0.9\,(\alpha_2^c\lambda_2^c/\lambda_3)$, indeed of order one.

The model predicts the first family to be massless in the limit in which non-renormalizable corrections to W are neglected, and to be further suppressed by $M_{R,L}/\Lambda$ once those corrections are taken into account. A discussion of the first family masses requires an investigation of the physics at the cutoff. An example, carried out in the context of the effective theory below Λ, can be found in chapter 2. Here, we only consider the physics giving rise to the third and second families of fermion masses.

4.2 Unified theory on $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2$

$SO(10)$ unification is a natural step towards a complete model of flavour based on the Pati-Salam theory of the previous section. We are interested in embedding this theory into the framework of a 5D supersymmetric GUT theory. The extra dimension is compactified on an orbifold $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2$ and the gauge symmetry is broken by the boundary conditions of the gauge fields in the extra dimension.

Starting from a circle S^1 with coordinate y, $0 \le y \le 2\pi R$ where $1/R = M_c$, the orbifold $S^1/Z_2 \times Z_2'$ is obtained by imposing the following identifications

$$P: \quad y \sim -y \qquad\qquad P': \quad y' \sim -y' \tag{4.6}$$

with $y' = y + \pi R/2$. Under these identifications, there are 2 fixed points at $y = 0$ and $y = \pi R/2$, referred to as "branes" in the following. These fixed points are four-dimensional branes where the SM fermions live. The extra dimension can be truncated to the physically irreducible interval $y \in [0, \pi R/2]$. In field space, the action of the identifications is given by

$$P: \quad \Phi(x,y) \sim P\Phi(x,-y) \qquad\qquad P': \quad \Phi(x,y') \sim P'\Phi(x,-y') \tag{4.7}$$

where on the right hand side the P, P' are matrix representations of the Z_2 reflections, with $P^2 = 1$, and can one choose a basis where they are diagonal, with eigenvalues ± 1. In this basis each field can be classified by its eigenvalues $(\pm 1, \pm 1)$. The expansion in KK modes is the following:

$$\Phi_{++}(x,y) = \sqrt{\frac{4}{\pi R}} \sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^{\delta_{n,0}}} \Phi_{++}^{(2n)}(x) \cos \frac{2ny}{R}, \tag{4.8}$$

$$\Phi_{+-}(x,y) = \sqrt{\frac{4}{\pi R}} \sum_{n=0}^{\infty} \Phi_{+-}^{(2n+1)}(x) \cos \frac{(2n+1)y}{R}, \tag{4.9}$$

$$\Phi_{-+}(x,y) = \sqrt{\frac{4}{\pi R}} \sum_{n=0}^{\infty} \Phi_{-+}^{(2n+1)}(x) \sin \frac{(2n+1)y}{R}, \tag{4.10}$$

$$\Phi_{--}(x,y) = \sqrt{\frac{4}{\pi R}} \sum_{n=0}^{\infty} \Phi_{--}^{(2n+2)}(x) \sin \frac{(2n+2)y}{R}. \tag{4.11}$$

Here the normalization is chosen such that the induced 4D kinetic terms for the KK modes are canonical. Notice that on the brane at $y = 0$ only Φ_{++} and Φ_{+-} are non-vanishing, while at the brane at $y = \pi R/2$ only Φ_{++} and Φ_{-+} are non-vanishing. Only Φ_{++} has massless zero modes.

4.2.1 5D bulk action

$\mathcal{N} = 1$ SUSY in 5D is equivalent to $\mathcal{N} = 2$ SUSY in 4D, which can be formulated in $\mathcal{N} = 1$ superfield language. The $\mathcal{N} = 2$ vector multiplet consists of a $\mathcal{N} = 1$ vector V and chiral multiplet Φ, the $\mathcal{N} = 2$ hypermultiplet decomposes in two $\mathcal{N} = 1$ chiral multiplets in conjugate representations H and $\hat{H}$.

In the model under consideration, beside the $SO(10)$ vector multiplet (V, Φ) we have 4 copies of messengers hypermultiplets $(\Psi, \hat{\Psi})$, $(\bar{\Psi}^c, \hat{\overline{\Psi^c}})$ in $(16, \overline{16})$ and $(\Psi^c, \hat{\Psi}^c)$,$(\overline{\Psi}, \hat{\overline{\Psi}})$ in $(\overline{16}, 16)$. The orbifold parities of the vector multiplet components can be chosen as (by choosing appropriate matrices P and P')

$$V = V_{++}^{PS} + V_{+-}^{SO(10)/PS} \tag{4.12}$$

$$\Phi = \Phi_{--}^{PS} + \Phi_{-+}^{SO(10)/PS}. \tag{4.13}$$

Looking at the non-vanishing components on the branes, one recognizes that on both branes $\mathcal{N} = 2$ SUSY is broken to $\mathcal{N} = 1$, $SO(10)$ is unbroken at $y = 0$ (thus referred to as the $SO(10)$-brane) and broken at $y = \pi R/2$ (referred to as PS-brane).

The choice of P and P' on the vectors dictates the relative orbifold parities of the matter PS components: in fact invariance of the bulk action requires them to have the same parity under P and the opposite parity under P'. The overall signs are chosen in the following way

$$
\begin{aligned}
(10,-,+) : h &= h^4_{++} + h^6_{+-} \\
\hat{h} &= \hat{h}^4_{--} + \hat{h}^6_{-+} \\
(16,+,-) : \Psi &= F_{++} + F^c_{+-} \\
\hat{\overline{\Psi}} &= \hat{\overline{F}}_{--} + \hat{\overline{F^c}}_{-+} \\
\Psi^c &= F^c_{++} + F_{+-} \\
\hat{\overline{\Psi^c}} &= \hat{\overline{F^c}}_{--} + \hat{\overline{F}}_{-+} \\
(\overline{16},+,-) : \hat{\Psi} &= \hat{F}_{--} + \hat{F^c}_{-+} \\
\overline{\Psi} &= \overline{F}_{++} + \overline{F^c}_{+-} \\
\hat{\Psi}^c &= \hat{F^c}_{--} + \hat{F}_{-+} \\
\bar{\Psi}^c &= \overline{F^c}_{++} + \overline{F}_{+-} \\
(45,-,+) : \phi &= \phi^{PS}_{++} + \phi^{SO(10)/PS}_{+-} \\
\hat{\phi} &= \hat{\phi}^{PS}_{--} + \hat{\phi}^{SO(10)/PS}_{-+}
\end{aligned}
\tag{4.14}
$$

where $(R, \pm, \pm)$ denotes the quantum numbers under $(SO(10), Z_2, R_P)$. For our model, the relevant (mass generating) part of the 5D bulk action is the superpotential term

$$
W_{bulk} = \int d^4x \int_0^{\pi R/2} dy \int d^2\theta \left(\Psi\partial_5\hat{\Psi} + \overline{\Psi}\partial_5\hat{\overline{\Psi}} + \Psi^c\partial_5\hat{\Psi}^c + \bar{\Psi}^c\partial_5\hat{\overline{\Psi^c}} + \phi\partial_5\hat{\phi} + h\partial_5\hat{h} \right) .
\tag{4.15}
$$

Inserting the KK decomposition and integrating over y will give mass to all bulk field modes except to zero modes of $(++)$ fields. For our model, only the low-energy dynamics is relevant and all the Kaluza-Klein states are practically decoupled. Only the zero-modes play a role in the flavour structure.

4.3 $SO(10)$ model of flavour

4.3.1 Field content

We consider a 5D, SO(10) supersymmetric model with the fifth dimension compactified in a $S^1/Z_2 \times Z_2'$ orbifold broken to G_{PS} on the $R = \pi R/2$ brane and to $\mathcal{N} = 1$ 4D susy on both branes. A bulk hypermultiplet $\mathbf{\Phi} = (\Phi, \hat{\Phi})$ is denoted by the symbol that would be used for its zero mode component. The non-gauge interactions are described by 4D superpotentials on the two branes, W_{PS} and $W_{\text{SO(10)}}$. Each of them has a part directly associated to the origin of the flavour structure and a part that accounts for the vevs used in the flavour part, $W = W_{\text{flavour}} + W_{\text{vevs}} + W_{\text{scales}}$. The flavour part involves the fields listed in Table 4.2,4.3.

	ψ_i	ψ'	$\bar{\psi}'$	F	F_c	$\overline{F}$	$\overline{F}_c$	h_4	ϕ	S_j
Localization	SO(10)			bulk						
Gauge repr	16	16	$\overline{16}$	16	16	$\overline{16}$	$\overline{16}$	10	45	1
$\text{U}(1)_R$	1	0	0	1	1	1	1	0	0	1
Z_{24}	5	5	-7	-6	-6	-6	-6	1	1	2

Table 4.2: $SO(10)$-brane and bulk fields. $i = 1, 2, 3$, $j = 1, 2, 3$

	F_c'	$\overline{F'}_c$	X_c	Σ
Localization	PS			
Gauge repr	$(1, 2, \bar{4})$	$(1, 2, 4)$	$(1, 3, 1)$	$(1, 1, 15)$
$\text{U}(1)_R$	0	0	0	1
Z_{24}	2	2	12	4

Table 4.3: PS-brane fields.

W_{vevs} involves a number of additional fields, listed in Table 4.4. Some of them are needed to generate the necessary vevs (essentially the singlets), some to get a magic field content all the way up to the unification scale (H_6 and the L and R triplets), some to avoid unwanted Goldstones and to set each field at the appropriate scale.

Strong coupling regime, natural units, and the order parameter

We assume the theory approaches a strongly interacting regime at the cutoff scale Λ, where the gauge couplings unify.

	Φ	Y_{10}	Y'_{10}	H_6	$\theta^\pm$	$\Theta^\pm$	$Y_{\rm PS}$	$Y'_{\rm PS}$	x_c	X'_c	x	X
Localization	SO(10)			bulk			PS					
Gauge repr	45	1	1	10	1	1	1	1	$(1,3,1)$	$(1,3,1)$	$(3,1,1)$	$(3,1,1)$
$U(1)_R$	2	2	2	2	0	0	2	2	2	1	1	1
Z_{24}	2	2	0	-2	±3	∓2	-4	0	-7			

Table 4.4:

Naïve dimensional analysis (NDA) suggests to write the action in terms of normalized derivatives $\hat{\partial} = \partial/\Lambda$ and of properly normalized ("natural units") dimensionless chiral and vector superfields $\hat{\phi}$, $\hat{V}$, related to the canonically normalized fields ϕ, V by

$$\phi_4 = \hat{\phi}_4 \left(\frac{\Lambda^2}{l_4}\right)^{1/2}, \quad \phi_5 = \hat{\phi}_5 \left(\frac{\Lambda^3}{l_5}\right)^{1/2}, \quad V_4 = \hat{V}_4 \left(\frac{\Lambda^2}{l_4^V}\right)^{1/2}, \quad V_5 = \hat{V}_5 \left(\frac{\Lambda^3}{l_5^V}\right)^{1/2}, \tag{4.16}$$

where the index 4 (5) denotes brane (bulk) fields. When expressed in terms of the dimensionless fields, the brane superpotential acquires the form

$$W_{\rm brane}(\phi_i) = \frac{\Lambda^3}{l_4} \hat{W}(\hat{\phi}_i), \tag{4.17}$$

where $\hat{W}$ does not contain dimensionful parameters and its expansion involves $\mathcal{O}(1)$ coefficients[2]. The nice thing is that this is independent of whether the fields on which $\hat{W}$ depend are bulk or brane fields[3] [31]

The values of the dimensionless coefficients $l_{4,5}^{(V)}$ leading to an "$\mathcal{O}(1)$" $\hat{W}$ are of course themselves defined up to $\mathcal{O}(1)$ factors, as the statement "becomes strongly interacting" itself. Moreover, we might want to consider a regime in which the couplings are e.g. a factor of 2 smaller than the strong regime estimate. Finally, as the coefficients of the loop expansion depend on the theory under consideration, $l_{4,5}^{(V)}$ also do, and they may also be different

[2] The general 5d Lagrangian at the scale Λ of strong coupling is given by

$$\mathcal{L}_\Lambda = \frac{1}{l_5}\left(\Lambda^3 \hat{K}_{bulk} + \Lambda^4 \hat{W}_{bulk}\right) + \frac{1}{l_4}\delta^4(0,\pi R)\left(\Lambda^2 \hat{K}_{brane} + \Lambda^3 \hat{W}_{brane}\right) \tag{4.18}$$

where all fields and derivatives in $\hat{K}$ and $\hat{W}$ are made dimensionless by rescaling with Λ and all coefficients are dimensionless and $O(1)$. The above Langrangian ensures that every loop contributes with the same strength to the amplitudes. This can be easily understood by the fact that the loop factors appear in the same position as $\hbar$ in the path integral.

[3] Note however that the derivation in the paper assumes an infinite extra-dimension.

for different fields. Having said that, the guideline provided by NDA is that l_D is just the loop factor in D dimensions:

$$l_D = (4\pi)^{D/2}\Gamma(D/2). \tag{4.19}$$

We will use the same factor l_5 (l_4) for all the chiral brane (bulk) superfields (superpotential couplings), while we keep the possibility of having a different normalization for the vector fields (gauge couplings). This is because the gauge couplings are qualitatively different in that the coefficients of the gauge loop expansion grow with the number of charged matter fields. With the field content in the Tables 4.2 and 4.4, we expect in fact l_V to be smaller by a factor $\mathcal{O}(5)$ [32, 33].

We are now in the position of estimating the couplings, vevs, and mass terms of canonically normalized brane and zero-mode bulk fields. A term in $\hat{W}$ involving n_B bulk fields and n_b brane fields will give rise to a coupling of order

$$\left(\frac{2l_5}{\pi R\Lambda}\right)^{n_B/2} l_4^{n_b/2-1} \tag{4.20}$$

at the scale Λ. Let us call $\lambda_g(\Lambda)$ the generic coupling arising from a brane-bulk-bulk interaction. Our model requires $\lambda_g \sim 1$ at the lower scale, the prototypical example being the top Yukawa coupling. This allows to relate ΛR to $l_{4,5}$. The situation can then be summarized as follows.

The most relevant parameter is

$$\lambda \equiv (\lambda_g^2(\Lambda)/l_4)^{1/4} \approx 0.24. \tag{4.21}$$

All the hierarchies will be expressed in terms of this order parameter. The parameters $\lambda_g(\Lambda)$ and l_4 are assumed to be such that λ is close to 0.22, which is the typical order parameter in models with flavour symmetries.

In natural units, the vevs of brane and bulk superfields are all expected to be $\mathcal{O}(1)$ (barring the presence of small numbers in $\hat{W}$, as we will see). This statement can be trivially translated into an expectation for the vevs of canonically normalized fields. However, it turns out that it is more convenient to use directly the vevs of the fields expressed in natural units.

Let us now consider a mass term. When written in terms of natural units fields, the mass terms will be a dimensionless number, say ϵ. For example, ϵ could be the vev of a field (brane or bulk) in natural units. Suppose that the mass term involves n_B bulk fields ($n_B = 0, 1, 2$). In terms of canonically normalized brane and zero-mode bulk fields, the mass term is

$$M = \epsilon\,\lambda^{n_B}\Lambda. \tag{4.22}$$

Note: if ϵ represents the vev of a field in natural units, the size of the corresponding mass term does not depend on the nature (bulk or brane) of the field getting a vev; it only depends on the nature of the fields getting a mass.

Finally, the separation ΛR between the scales Λ and $M_c \equiv 1/R$ and the size of the gauge couplings at the unification scale are determined by l_5 and l_5^V/l_5 respectively. We assume that l_5 is such that $(2l_5/\pi/\lambda_g^2(\Lambda)) \sim 100$, so that $\Lambda R \sim 5$, and that $l_5/l_5^V \sim 5$, so that $g_{4D}^2(\Lambda) \sim l_5^V/(\lambda^2 l_5) \sim 3.5$ (only an estimate, anyway compatible with a radiative enhancement of g_{4D}^2).

Superpotential, F-term equations, vevs

The superpotentials on the two branes (in natural units) are

$$\hat{W}_{\mathrm{SO(10),PS}} = \hat{W}^{\mathrm{flav}}_{\mathrm{SO(10),PS}} + \hat{W}^{\mathrm{vevs}}_{\mathrm{SO(10),PS}} + \hat{W}^{\mathrm{mass}}_{\mathrm{SO(10),PS}}. \tag{4.23}$$

The $W^{\mathrm{flav}}_{\mathrm{SO(10),PS}}$ potentials are directly involved in generating the SM flavour structure and are given by

$$\hat{W}^{\mathrm{flav}}_{\mathrm{SO(10)}} = \lambda_i \hat{\psi}_i \hat{F} \hat{h}_4 + \lambda_i^c \hat{\psi}_i \hat{F}_c \hat{h}_4 + \alpha_i \hat{\psi}_i \hat{\bar{F}} \hat{\phi} + \alpha_i^c \hat{\psi}_i \hat{\bar{F}}_c \hat{\phi} + a_{ij} \hat{\bar{\psi}}' \hat{S}_i \hat{\psi}_j \tag{4.24a}$$

$$\hat{W}^{\mathrm{flav}}_{\mathrm{PS}} = \hat{\bar{F}}_c \hat{X}_c \hat{F}_c + \hat{\bar{F}}'_c \hat{\Sigma} \hat{F}_c + \hat{\bar{F}}_c \hat{\Sigma} \hat{F}'_c + \frac{\hat{F}'_c \hat{X}_c \hat{F}'_c}{2} \hat{\Sigma}^2 + \hat{\bar{F}}'_c \hat{S}_i \hat{F}_c \hat{\Theta}_{\sigma_-} \tag{4.24b}$$

with hopefully self-explanatory notations. The last terms in $W^{\mathrm{flav}}_{\mathrm{SO(10),PS}}$ affect the singlet neutrino mass matrix. All couplings are supposed to be $\mathcal{O}(1)$. Flavour-independent indexes are omitted.

Some of the $R = 0$ fields in the superpotentials above get vev due to $W^{\mathrm{vevs}}_{\mathrm{SO(10),PS}}$, given by

$$\hat{W}^{\mathrm{vevs}}_{\mathrm{SO(10)}} = \hat{Y}_{10}(\hat{\bar{\psi}}' \psi' - \hat{\theta}^2_{\sigma_-} \hat{\Theta}^2_{\sigma_-}) + \hat{Y}'_{10}(\hat{\theta}_+ \hat{\theta}_- - \epsilon_{10}^2) + \hat{\bar{\psi}}' \hat{\Phi}\, \hat{\psi}' + \hat{\theta}_{\sigma_-} \hat{\phi}\, \hat{\Phi} \tag{4.25a}$$

$$\hat{W}^{\mathrm{vevs}}_{\mathrm{PS}} = \hat{Y}_{\mathrm{PS}}(\hat{\bar{F}}'_c \hat{F}'_c - \hat{\Theta}^2_{\sigma_-}) + \hat{Y}'_{\mathrm{PS}}(\hat{\Theta}_+ \hat{\Theta}_- - \epsilon_{\mathrm{PS}}^2) + \hat{\theta}_{\sigma_+} \hat{\bar{F}}'_c\, \hat{x}_c \hat{F}'_c + \hat{\theta}_{\sigma_-} \hat{\Theta}_+ \hat{x}_c\, \hat{X}_c. \tag{4.25b}$$

The seeds of all small vevs are the two anomalously small coefficients $\epsilon_{\mathrm{PS}} \sim \lambda$ and $\epsilon_{10} \sim \lambda^2$ characterizing the PS and SO(10) branes respectively. We do not investigate the origin of those numbers. In the limit in which the R-symmetry is not broken, the only relevant F-term equations are those associated to the $R = 2$ fields containing SM singlets. The Y_{PS}, Y'_{PS}, Y_{10}, Y'_{10} equations give (assuming again that supersymmetry breaking does not make the individual vevs too different)

$$\langle \hat{\Theta}_\pm \rangle \sim \langle \hat{F}'_c \rangle \sim \langle \hat{\bar{F}}'_c \rangle \sim \lambda \qquad \langle \hat{\theta}_\pm \rangle \sim \lambda^2 \qquad \langle \hat{\psi}' \rangle \sim \langle \hat{\bar{\psi}}' \rangle \sim \lambda^3. \tag{4.26}$$

Moreover, the x_c and Φ eqs force X_c and ϕ to be $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^4)$ respectively. The H_6 eqs are not relevant because H_6 does not contain a SM singlet. Note that both the (1,1,15) and (1,3,1) PS components of ϕ get a vev at present. This is a problem because the m_μ/m_s ratio requires the mixing induced by ϕ to be along the (1,1,15) direction only. There are two possible solutions to this problem: i) the $\bar{F}_c \phi_R f^c_i$ interaction is forbidden or suppressed, or ii) the (1,1,15) component of the vev vanishes.

Note that the $\mathcal{O}(\lambda)$ vevs in eq. (4.26) breaks Z_{24} to the group $\mathbf{Z}_2$ of the previous chapters, which is then broken by the higher order vevs.

At this point we have the vevs we need but we still have to take care of the spectrum, which by the way so far includes unwanted light fields, the D'_c component of F'_c for example, which is not an eaten Goldstone. This can also be seen as follows. The interactions in eqs. (4.24) do not affect the masses of F'_c, $\bar{F}'_c$. The interaction with Y_{PS} in eqs. (4.25) is invariant under an SU(8) transformation of the 8 components of the fields F'_c and $\bar{F}'_c$. Their vevs break SU(8) down to SU(7), which leaves 15 massless Goldstones: all components of F'_c and $\bar{F}'_c$ except a linear combination of the SM-singlet fields. The only relevant interaction left is the one with X'_c, which only gives a mass to some of the SU(2)$_L$-singlet charged lepton fields. Analogous considerations hold for other fields. In short, we need to provide mass terms for several otherwise light fields, or unification will be spoiled. The mass terms are provided by

$$\hat{W}^{\text{mass}}_{\text{PS}} = \hat{\Theta}_{\sigma_+} \hat{\bar{F}}'_c \hat{\bar{F}}'_c \hat{H}_6 + \hat{\Theta}_{\sigma_+} \hat{F}'_c \hat{F}'_c \hat{H}_6 + \frac{\hat{\Theta}^3_+}{2}(\hat{x}^2 + \hat{X}^2) \tag{4.26a}$$

The superpotentials are then invariant under the U(1)$_R$ R-symmetry and the Z_{24} discrete symmetry in the Tables.

Let us now discuss the spectrum in more detail. First of all, we have to determine the size of the mass terms generated be the vevs above. In order to do that, we make extensive use of eq. (4.22). Taking $\Lambda = 0.9 \cdot 10^{17}$ GeV, the right-handed messengers in F_c, $\bar{F}_c$ and Σ get a mass

$$\mathcal{O}\left(\lambda^3 \Lambda\right) \sim 10^{15}\,\text{GeV} \equiv M_R. \tag{4.27}$$

The up quark sector also contains a mixed mass term enhanced by a factor $1/\lambda$, $V_c \sim M_R/\lambda$, which accounts for the smallness of m_c/m_t and will give the only threshold correction to the magic running. Once $\mathbf{Z}_2$ is broken by the vev of ϕ, the messengers and the would be light families get mixed by a mass term

$$\mathcal{O}\left(\lambda^5 \Lambda\right) \sim 0.7 \cdot 10^{14}\,\text{GeV} \equiv M_L, \tag{4.28}$$

so that $\epsilon \equiv M_L/M_R \approx 0.06$. The two mass terms mixing the singlets S_i with N_c and n^c_i are both $\mathcal{O}(\lambda M_R)$, half way between M_L and M_R.

Above M_R and below $1/R$ the spectrum is the one showed in the Tables (neglecting the threshold effect mentioned above). This is a "retarded" PS magic field content with $((b_3 - b_2)/(b_4 - b_L) - 1) = 2$. Let us see which fields survive at scales lower than M_R (we only consider non-gauge-singlet fields). Let us first consider the limit in which only the $\mathcal{O}(\lambda)$ vevs are switched on ($\mathbf{Z}_2$ unbroken). In this limit the light fields are: the SM fields; the left-messengers $\bar{L}L + \bar{Q}Q$; the decomposition $\bar{E}^c_X E^c_X$ of X'_c; $\Phi_{\bar{V}}\Phi_V$; some linear combinations $(\phi_{\bar{U}} + \psi'_{\bar{U}})(\phi_U + \psi'_U)$ and $(\phi_{\bar{E}} + \psi'_{\bar{E}})(\phi_E + \psi'_E)$ (the orthogonal combinations are uneaten Goldstones in this limit); the $\bar{5}'_{\text{SU}(5)}5'_{\text{SU}(5)}$ components of $\bar{\psi}'\psi'$. The $\bar{U}'U'$ components of $\bar{F}'_c F'_c$ are mostly eaten Goldstones. This is an "anticipated" SM magic field content with $-(b_1^{\text{new}} - b_2^{\text{new}})/(b_1 - b_2) = -1/3$. All the previous fields get a mass term $\mathcal{O}(M_L)$. This is easily arranged for $\bar{e}_c e_c$, while it is less obvious for the remaining $R = 0$ fields (we could also make x_c a $R = 0$ field at this point). In particular, we need to generate an $\mathcal{O}(M_L)$ mass term for $\Phi_V\Phi_V$ and a mass term for $\bar{\psi}'\psi'$, both breaking R by two units. We do not go into details of how to achieve this. It is not difficult to arrange a perturbative superpotential involving R-charged singlets getting vevs at the level $\mathcal{O}(M_L) = \mathcal{O}(\lambda^5\Lambda)$ and perhaps breaking supersymmetry at the level $\lambda^{10}\Lambda \sim 10^{10}\,\text{GeV}$. However, chances are that this breaking arises non-perturbatively, so that it is not worth spelling all details out.

To summarize, we have the following scales: $\Lambda \approx M^0_{\text{GUT}} \approx 10^{17}\,\text{GeV}$, $M_c = 1/R \approx 2\cdot 10^{16}\,\text{GeV}$, $M_R \approx \lambda^3\Lambda \approx 10^{15}\,\text{GeV}$, $M_L \approx \lambda^5\Lambda \approx 0.7\cdot 10^{14}\,\text{GeV}$ and a magic field set from Λ down to the electroweak scale except for a small threshold.

4.3.2 Gauge coupling unification

Let's analyze the threshold effect. Let α_3^0 be the prediction at low scale, neglecting the threshold given by the two $U^c\bar{U}^c$ lying at M_R/λ instead of M_R. Taking into account the threshold gives

$$\frac{1}{\alpha_3} = \frac{1}{\alpha_3^0} - \left[(b_3^U - b_2^U) + \frac{5}{7}(b_1^U - b_2^U)\right]\frac{\log(V_c/M_R)}{2\pi} \tag{4.29}$$

where $b^U = (16/5, 0, 2)$. For $\alpha_3^0 = 0.118$ we get $\alpha_3 = 0.124$. The threshold effect is not so small. Note that unlike most 5D unification models, here the threshold effect does not come from KK towers as the incomplete floors are magic.

The complete beta coefficients for the running are: $(b_L, b_R, b_4) = (10, 10, 10)$ above $1/R$; (23,29,21) above M_R (neglecting the threshold); $(b_1, b_2, b_3) =$

$(87/5, 9, 3)$ above M_L. The unification scale is given by

$$M_{\rm GUT} = \epsilon \left(\frac{V_c}{M_R}\right)^{8/7} \frac{(M^0_{\rm GUT})^2}{M_R} \ . \tag{4.30}$$

There are also brane corrections. This intrinsic uncertainty is due to the contribution of brane YM terms. Assuming strong coupling regime, those terms give a correction to $g^2(\Lambda)$ of order $1/l_4^V$ expected to be a few percent.

4.3.3 Proton decay

In 4D GUT theories the leading contributions to proton decay come usually from d=5 operators (via Higgs triplet exchange), while d=6 operators (via extra gauge boson exchange) are subleading. In 5D Orbifold GUT models one generally finds the reverse picture: d=5 operators are strongly suppressed due to the $U(1)_R$ symmetry, while d=6 operators are more important because the mass of the extra gauge bosons is smaller than in 4D GUTS ($M_c < M^0_{GUT}$). This general picture holds also in our model. The dominant contribution to proton decay is SO(10)/PS gauge boson exchange, leading to a lower bound on the compactification scale $M_c > 8.8 \times 10^{15}$GeV. Before deriving this bound, we show that d=5 operators are indeed subleading.

d=5 Operators

The low energy effective theory below M_L is the MSSM with R-parity. This allows only for two d=5 operators that induce proton decay [34]

$$\text{LLLL} = QQQL|_{\theta^2} \tag{4.31}$$
$$\text{RRRR} = D^c U^c U^c E^c|_{\theta^2}. \tag{4.32}$$

These operators arise from exchange of fields with standard model quantum numbers

$$T_1 = (3,1)_{-1/3} \to \text{LLLL, RRRR}$$
$$T_4 = (3,1)_{-4/3} \to \text{RRRR}.$$

While there are plenty of fields with T_1 quantum numbers around in the model, there are not dangerous because the above d=5 operators violate the $U(1)_R$ symmetry. Without going into details of SUSY breaking, one can make a simple estimate of the suppression scale assuming a similar mechanism to be at work as in [32, 33]. In this case supersymmetry is broken together with $U(1)_R$ by an F-term VEV of some gauge singlet superfield. The $U(1)_R$

breaking scale is $M_s = \sqrt{F}$, which is related to the soft SUSY breaking scale by $M_{soft} \sim M_s^2 M_c/\Lambda^2$ and typically $M_s \sim 10^{11}$GeV. One can take the mass of the above triplets to be $M_R \sim 10^{15}$, because the triplets at M_L (from ψ') do not couple directly to the MSSM light fields. The suppression scale of the above d=5 operators is then at least M_R^2/M_s, which is of the order of M_{Pl}.

d=6 Operators

Two d=6 proton decay inducing operators are present in the MSSM

$$\begin{aligned} \mathrm{QQ} &= QQ(U^c)^\dagger (E^c)^\dagger |_{\theta^4} \\ &= u_L d_L \bar{u}_R^c \bar{e}_R^c \qquad (4.33) \\ \mathrm{QL} &= QL(U^c)^\dagger (D^c)^\dagger |_{\theta^4} \\ &= u_L e_L \bar{u}_R^c \bar{d}_R^c \qquad (4.34) \end{aligned}$$

These operators come from an exchange of gauge fields with quantum numbers

$$\begin{aligned} X, Y &= (3,2)_{(5/6)} \to \mathrm{QL}, \mathrm{QQ} \\ X', Y' &= (3,2)_{(1/6)} \to \mathrm{QL} \end{aligned}$$

which are the SO(10)/PS gauge bosons $V_{+-}^{(226)}$. Therefore the suppression of the d=6 operators is given by $1/M_c^2$, the numerical coefficients can be calculated in close analogy to the SU(5) case [35].

Calculation of QL and QQ coefficients

We start with the relevant part of the 5D action

$$S_5 = \int d^4x \int_0^L dy \left(\frac{1}{2g_5^2} \mathrm{Tr} \left(W^2|_{\theta^2} + \mathrm{h.c.} + 4\partial_5 V \partial_5 V|_{\theta^4} \right) + \delta(y) \psi_i^\dagger e^{2V} \psi_i |_{\theta^4} \right), \tag{4.35}$$

which fixes the normalization in the KK expansion of $V_{+-}^{(226)}$

$$V_{+-}(x,y) = \sqrt{2} \sum_{n=0}^{\infty} V_{+-}^{(2n+1)}(x) \cos \frac{(2n+1)y}{R}. \tag{4.36}$$

The 4D effective Lagrangian is given by

$$\mathcal{L}_4 = \frac{2}{g_4^2} \sum_{n=0}^{\infty} \left(\frac{2n+1}{R} \right)^2 \mathrm{Tr} \left(V_{+-}^{(2n+1)} V_{+-}^{(2n+1)} \right) + 2\sqrt{2} \left(\psi_i^\dagger \sum_{n=0}^{\infty} V_{+-}^{(2n+1)} \psi_i \right) |_{\theta^4} \tag{4.37}$$

where we have used $\frac{1}{g_4^2} = \frac{L}{g_5^2}$.
In component form, using 2-component Weyl notation with the conventions of [36],

$$\mathcal{L}_4 = -\frac{1}{g_4^2}\sum_{n=0}^{\infty}\left(\frac{2n+1}{R}\right)^2 \mathrm{Tr}\left(A_{+-}^{(2n+1)}A_{+-}^{(2n+1)}\right) - \sqrt{2}\sum_{n=0}^{\infty}\left(\psi_i^\dagger\sigma^\mu\psi_i\right)(A_{+-}^{(2n+1)})_\mu \tag{4.38}$$

Decomposing under SM [37] gives

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{g_4^2}\sum_{n=0}^{\infty}\left(\frac{2n+1}{R}\right)^2\left(X\bar{X}+Y\bar{Y}+X'\bar{X}'+Y'\bar{Y}'\right)\\ &\quad+\sum_{n=0}^{\infty}\left(\bar{X}\mathcal{O}_X+\bar{Y}\mathcal{O}_Y+\bar{X}'\mathcal{O}_{X'}+\bar{Y}'\mathcal{O}_{Y'}+\text{h.c.}\right)\end{aligned} \tag{4.39}$$

with

$$\mathcal{O}_X = e_L^c\sigma^\mu\bar{d}_R + e_R^c\sigma^\mu\bar{d}_L + u_L\sigma^\mu\bar{u}_R^c \tag{4.40}$$
$$\mathcal{O}_Y = -e_R^c\sigma^\mu\bar{u}_L + d_L\sigma^\mu\bar{u}_R^c \tag{4.41}$$
$$\mathcal{O}_{X'} = -d_R^c\sigma^\mu\bar{d}_L \tag{4.42}$$
$$\mathcal{O}_{Y'} = d_R^c\sigma^\mu\bar{u}_L - u_R\sigma^\mu\bar{e}_L^c \tag{4.43}$$

Integrating out the gauge bosons gives

$$\begin{aligned}\mathcal{L}_4 &= \frac{g_4^2}{M_c^2}\frac{\pi^2}{8}\left(\mathcal{O}_X\bar{\mathcal{O}}_X+\mathcal{O}_X\bar{\mathcal{O}}_X+\mathcal{O}_{X'}\bar{\mathcal{O}}_{X'}+\mathcal{O}_{Y'}\bar{\mathcal{O}}_{Y'}+\text{h.c.}\right)\\ &= \frac{g_4^2}{M_c^2}\frac{\pi^2}{4}\left[(1+1+0+0)u_ld_l\bar{u}_R^c\bar{e}_R^c+(1+0+0+1)u_le_l\bar{u}_R^c\bar{d}_R^c\right]\end{aligned} \tag{4.44}$$

where the relative contribution of (X, Y, X', Y') is made explicit. The factor $\pi^2/4$ originates from summing KK states, and up to this factor, the result is the same as of a 4D GUT with gauge boson mass M_c. The final result for the coefficients of the QQ and QL operators is

$$\mathcal{L}_4 = \frac{g_4^2}{M_c^2}\frac{\pi^2}{4}\left(2u_ld_l\bar{u}_R^c\bar{e}_R^c+2u_le_l\bar{u}_R^c\bar{d}_R^c\right) \tag{4.45}$$

Proton lifetime

According to [35, 38], the proton decay rate due to the QL and QQ operators is given by

$$\Gamma(p\to\pi^0e^+) = 8\left(\frac{\pi^2}{4}\right)^2\alpha_H^2\left(\frac{g_4(M_c)^2A_R}{M_c^2}\right)^2\frac{m_p}{64\pi f_\pi^2}(1+D+F)^2 \tag{4.46}$$

Note that $\Gamma_{SO(10)} = 8/5 \times \Gamma_{SU(5)}$, so as a rule of thumb proton decay is roughly twice as fast in $SO(10)$ than in $SU(5)$. Using the hadronic parameter $\alpha_H = 0.015$ GeV^3, the pion decay constant $f_\pi = 0.13$ GeV and the chiral perturbation theory parameters $D = 0.80$ and $F = 0.47$, the partial lifetime is

$$1/\Gamma(p \to \pi^0 e^+) = 2.7 \times 10^{33} \left(\frac{\alpha_4(M_c)}{1/14}\right)^{-2} \left(\frac{A_R}{2.5}\right)^{-2} \left(\frac{M_c}{10^{16}\text{GeV}}\right)^4 \text{years} \tag{4.47}$$

with the typical values for (almost) unified coupling at M_c and renormalization coefficient A_R in the model. Nothe that we obtain roughly the same renormalization coefficient as in the simple orbifold GUT models. Comparing to the Particle Data Group bound [29] on the partial lifetime

$$1/\Gamma(p \to \pi^0 e^+) > 1.6 \times 10^{33}\text{years} \tag{4.48}$$

gives finally for the compactification scale

$$M_c > 8.8 \times 10^{15}\text{GeV} \left(\frac{\alpha_4(M_c)}{1/14}\right)^{1/2} \left(\frac{A_R}{2.5}\right)^{1/2}. \tag{4.49}$$

In the case of $M_c \sim 1.4 \times 10^{16}$ GeV, the lifetime is

$$1/\Gamma(p \to \pi^0 e^+) \sim 1.0 \times 10^{34}\text{years}. \tag{4.50}$$

4.3.4 Neutrino spectrum

The light neutrino mass matrix originates from the NR operator $h_{ij}(l'_i h_u)(l'_j h_u)/(2\Lambda_L)$, where $l'_{1,2,3}$ are the three light lepton doublet mass eigenstates: $m^\nu_{ij} = h_{ij} v_u^2/\Lambda_L$. The coefficients h_{ij}/Λ_L are obtained by integrating out the R_P-odd heavy singlet neutrinos.

We aim at obtaining a large atmospheric angle θ_{23}, the atmospheric squared mass difference Δm^2_{23} at the correct scale, and the suppression of the solar squared mass difference Δm^2_{12} (in the context of normal hierarchical neutrinos) and of the θ_{13} angle. In the previous version of the model, the large atmospheric angle and the $\Delta m^2_{12}/\Delta m^2_{23}$ suppression were obtained essentially through the single right-handed neutrino dominance mechanism [20, 21, 22, 23]. In fact, the whole idea underlying this flavour model, based on the exchange of a single family of flavour messengers, can be considered as an extension of that mechanism. In order to reproduce the single right-handed neutrino dominance mechanism, the left-handed messengers should have a mass term at the M_L scale (along the $B - L$ direction). Here, we prefer to consider the more economical option in which such term only arises at a more suppressed level. This is interesting also because the large atmospheric mixing arises in a different, unusual mechanism, as we are now going to see.

In our model, the singlet neutrinos taking part to the see-saw are actually more than the usual 3. There are 9 R_P-odd singlet neutrino fields in the model. We have the usual 3 "right-handed" neutrinos n^c_i, $i = 1, 2, 3$, $SU(2)_R$ partners of the SM right-handed charged fermions e^c_i, $i = 1, 2, 3$. We also have N^c, $\bar{N}^c$, A_Φ, and three or more gauge singlets S_i [4] (there are also other singlet neutrinos but they have different R_P, but they do not mix with the previous ones, and are not relevant for light neutrino masses. Furthermore, it is sufficient to consider only the KK 0-modes of N^c and $\bar{N}^c$, because their mass terms arise purely from the PS brane. That means that higher KK mode pairs $(++, --)_{n>0}$ and $(+-, -+)_{n\geq 0}$ decouple from the other fields, because one member of these pairs vanishes at the PS brane and has therefore only a heavy massterm with its partner). At the renormalizable level, their masses are given in eq. (4.2d). We see that $\alpha^c \hat{n}^c \equiv \alpha^c_2 n^c_3 - \alpha^c_3 n^c_2$ and n^c_1 are massless at this level[5]. We expect the latter fields to get a mass at a lower scale $M_L \sim M_R^2/\Lambda$ from non-renormalizable operators. The heavy singlet neutrino mass term is then $-(N^c, \bar{N}^c, A_\Phi, n^c_i, S_k)^T M_s (N^c, \bar{N}^c, A_\Phi, n^c_j, S_h)/2$, where

$$M_s = \begin{pmatrix} 0 & M_R & \sqrt{\frac{3}{8}}\bar{\sigma}_c \bar{V}_c & 0 & b_h M_{SN} \\ M_R & 0 & \sqrt{\frac{3}{8}}\sigma_c V_c & \alpha^c_j v & c_h M_{SN} \\ \sqrt{\frac{3}{8}}\bar{\sigma}_c \bar{V}_c & \sqrt{\frac{3}{8}}\sigma_c V^c & M_\Sigma & 0 & 0 \\ 0 & \alpha^c_i v & 0 & 0 & a_{ih} M_{Sn} \\ b_k M_{SN} & c_k M_{SN} & 0 & a_{kj} M_{Sn} & 0 \end{pmatrix} \tag{4.51}$$

and the light neutrino mass operator is

$$\frac{h_{ij}}{2\Lambda}(l'_i h_u)(l'_j h_u) = \frac{1}{2}\left[(M_s^{-1})_{N^c N^c}(\lambda^c_2 l'_2)^2 + (M_s^{-1})_{n^c_3 n^c_3}(\lambda_3 l'_3)^2 + 2(M_s^{-1})_{N^c n^c_3}(\lambda^c_2 l'_2)(\lambda_3 l'_3)\right] h_u^2, \tag{4.52}$$

so that

$$m_\nu = v_u^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\lambda^c_2)^2 (M_s^{-1})_{N^c N^c} & \lambda^c_2 \lambda_3 (M_s^{-1})_{N^c n^c_3} \\ 0 & \lambda^c_2 \lambda_3 (M_s^{-1})_{N^c n^c_3} & \lambda_3^2 (M_s^{-1})_{n^c_3 n^c_3} \end{pmatrix}. \tag{4.53}$$

[4]In general, the presence of additional singlet neutrino at high scales might not be so unlikely, given the different possible sources (flavour, GUT, strings...).

[5]Which would not be tolerable, as the tau neutrino would get an electroweak scale mass from its Yukawa coupling to $\hat{n}^c$.

The entries in the first row and column, accounting for the solar and θ_{13} mixing angles, will be generated, as in the case of charged fermion masses, at the NR level by the physics at the cutoff scale. In eq. (4.51) the entries set to zero arise at a negligible level.

In order to get a large atmospheric mixing angle from eq. (4.53), we need $(M_s^{-1})_{N^cN^c} \sim (M_s^{-1})_{N^cn^c} \sim (M_s^{-1})_{n^cn^c}$ and in order to obtain the (mild) hierarchy between the solar and atmospheric squared mass differences, we need the determinant $(M_s^{-1})_{N^cN^c}(M_s^{-1})_{n^cn^c} - (M_s^{-1})^2_{N^cn^c}$ to be suppressed. This is indeed what happens provided that $M_{SN} \sim M_{Sn} > M_L$, in which case

$$(M_s^{-1})_{N^cN^c} \sim (M_s^{-1})_{N^cn_3^c} \sim (M_s^{-1})_{n_3^cn_3^c} \sim \frac{1}{2M_R} \tag{4.55}$$

$$(M_s^{-1})_{N^cN^c}(M_s^{-1})_{n_3^cn_3^c} - (M_s^{-1})^2_{N^cn_3^c} \sim \frac{M_R^2}{V_c^2}(M_s^{-1})^2_{N^cN^c}. \tag{4.56}$$

In the model under consideration, $M_{SN} \sim M_{Sn} \sim \lambda M_R > M_L \sim \lambda^2 M_R$ and $M_R/V_c \sim \lambda < 1$.

Taking into account all $\mathcal{O}(1)$ coefficients we finally obtain for the light neutrino masses and the atmospheric mixing

$$m_3 = \frac{v_h^2}{M_R}\frac{A}{2\sin^2\theta_{23}} \tag{4.57a}$$

$$\frac{m_2}{m_3} = \frac{4\lambda^2}{3}\sin^2 2\theta_{23} B \tag{4.57b}$$

$$\tan\theta_{23} = C, \tag{4.57c}$$

where

$$A = \frac{(\lambda_2^c)^2\sigma_c}{\bar{\sigma}_c} \tag{4.58a}$$

$$B = \frac{\bar{\sigma}_c x^2}{\sigma_c y^2} \tag{4.58b}$$

$$C = \frac{\lambda_2^c \sigma_c \det a}{\lambda_3 y} \tag{4.58c}$$

$$x = c_2\,(a_{12}a_{31} - a_{11}a_{32}) + c_3\,(a_{11}a_{22} - a_{12}a_{21}) \tag{4.58d}$$

$$y = \bar{\sigma}_c x - \sigma_c b_3\,(a_{11}a_{22} - a_{12}a_{21})\,. \tag{4.58e}$$

In order to agree with the experimental values $m_2/m_3 \approx \sqrt{\Delta m^2_{12}/\Delta m^2_{23}} \simeq 0.2$ and $\tan\theta_{23} \simeq 1$, one has to require that the above functions of $\mathcal{O}(1)$ coefficients take the (not unreasonable) values $B \approx 3$ and $C \approx 1$. The atmospheric squared mass difference provides an experimental determination of $m_3 \approx \sqrt{\Delta m^2_{23}}$, which translates into a determination of the scale M_R, given by

$$M_R = \frac{v_h^2 A}{2\cos^2\theta_{23}\sqrt{\Delta m^2_{23}}} \sim A \times 6 \times 10^{14}\text{GeV}. \tag{4.59}$$

To achieve agreement with the numerical determination of the various scales provided by gauge coupling unification, we have to require that $A \approx 3$.

Chapter 5

Phenomenology

In this chapter we present a preliminary analysis of the phenomenology of the unified model that we presented in the previous chapter. We consider universal soft terms defined at $1/R$: a gaugino mass $M_{1/2}$ and a scalar mass M_0, plus a soft Higgs mass M_{h0}. The A-terms are absent at high scale and are generated through the running.

5.1 Renormalization group equations

We outline here some features of the running of the soft terms in this model. We are only interested in effects in the 2-3 sector, because effects involving the first family are not taken into account by the model of the previous chapter.

The off-diagonal terms in the soft mass matrices receive a large contribution from the high scale Yukawa couplings. These couplings are all $O(1)$ and generates off-diagonal components in $m^2_f, m^2_{f^c}$ through the running between M_c and M_R. However the effects in the LL and RR sectors are quite different because the light states in the left sector are f_2, F and therefore mixed terms cannot be generated above the Z_2 breaking scale M_L, while a mixed term for the light states f^c_2, f^c_3 in the right sector is generated already at M_c. Therefore we expect small or negligible δ_{LL} mass insertions, while δ_{RR} insertions are of order $O(10^{-2})$.

Large Yukawa couplings have also a strong effect on the diagonal terms: both the second and the third family masses receive significant contributions. This effect makes the second family lighter than the first, and the third family lighter than the other diagonal masses.

Note also that in the absence of Higgs mixing, the model naturally predicts a value of $\tan\beta$ of order 50. The problem is that in this regime

$\lambda_\tau, \lambda_b \sim O(1)$ and, given the fact that the RGEs for $m^2_{h_D}$ and $m^2_{h_U}$ are similar and the two Higgs doublets come from the same SO(10) multiplet, it is difficult to reconcile the two requirements that $m^2_{h_U} + \mu^2$ must be negative to break the electro-weak gauge symmetry, while $m^2_{h_D} + \mu^2$ must be positive enough so that the squared pseudoscalar Higgs mass is positive. Moreover, the effect of the large Yukawas above M_R gives a suppression of the unified Higgs mass m_h^2 at that scale. To reconcile the two requirements it is possible to move to the "hard fine tuning" region where the GUT scale gaugino mass is larger than the scalar masses.

We report the full RGEs of the model in the next sections.

RGEs above M_R

The relevant superpotential above M_R is:

$$W_{\rm PS} = \lambda_i f_i^c F h + \lambda_i^c f_i F^c h + \alpha_i \phi f_i \bar{F} + \alpha_i^c \phi f_i^c \bar{F}^c + a \bar{F}^c X_c F^c + \bar{\sigma}_c \bar{F}_c' \Phi F^c + \sigma_c \bar{F}^c \Phi F_c' \tag{5.1}$$

The Yukawa RGEs are:

$$(4\pi)^2 \frac{d}{dt}\lambda_i = \left(8|\vec{\lambda}|^2 + 4|\vec{\lambda^c}|^2\right)\lambda_i + \frac{15}{8}\vec{\alpha^c}\cdot\vec{\lambda}\,\alpha_i^c - \left(\frac{15}{2}g_4^2 + 3g_L^2 + 3g_R^2\right)\lambda_i$$

$$\begin{aligned}(4\pi)^2 \frac{d}{dt}\lambda_i^c &= \left(8|\vec{\lambda^c}|^2 + 4|\vec{\lambda}|^2 + \frac{15}{8}\bar{\sigma}_c^2 + \frac{3}{4}a^2\right)\lambda_i^c + \frac{15}{8}\vec{\alpha}\cdot\vec{\lambda^c}\,\alpha_i \\ &\quad - \left(\frac{15}{2}g_4^2 + 3g_L^2 + 3g_R^2\right)\lambda_i^c\end{aligned}$$

$$(4\pi)^2 \frac{d}{dt}\alpha_i = \left(\frac{19}{4}|\vec{\alpha}|^2 + |\vec{\alpha^c}|^2\right)\alpha_i + 2\vec{\lambda^c}\cdot\vec{\alpha}\,\lambda_i^c - \left(\frac{31}{2}g_4^2 + 3g_L^2\right)\alpha_i$$

$$(4\pi)^2 \frac{d}{dt}\alpha_i^c = \left(\frac{19}{4}|\vec{\alpha^c}|^2 + |\vec{\alpha}|^2 + \frac{15}{8}\sigma_c^2 + \frac{3}{4}a^2\right)\alpha_i^c + 2\vec{\lambda}\cdot\vec{\alpha^c}\,\lambda_i - \left(\frac{31}{2}g_4^2 + 3g_R^2\right)\alpha_i^c$$

$$(4\pi)^2 \frac{d}{dt}a = \left(\frac{7}{2}a^2 + \frac{15}{8}(\sigma_c^2 + \bar{\sigma}_c^2) + \frac{15}{8}|\vec{\alpha^c}|^2 + 2|\vec{\lambda^c}|^2\right)a - \left(\frac{15}{2}g_4^2 + 7g_R^2\right)a$$

$$(4\pi)^2\frac{d}{dt}\sigma_c = \left(\frac{19}{4}\sigma_c^2+\bar{\sigma}_c^2+\frac{15}{8}|\vec{\alpha^c}|^2+\frac{3}{4}a^2\right)\sigma_c-\left(\frac{31}{2}g_4^2+3g_R^2\right)\sigma_c$$

$$(4\pi)^2\frac{d}{dt}\bar{\sigma}_c = \left(\frac{19}{4}\bar{\sigma}_c^2+\sigma_c^2+2|\vec{\lambda^c}|^2+\frac{3}{4}a^2\right)\bar{\sigma}_c-\left(\frac{31}{2}g_4^2+3g_R^2\right)\bar{\sigma}_c$$

The SUSY breaking potential is:

$$\begin{aligned}V_{\rm SSB} &= (m_f^2)_{ij}\tilde{f}_i^*\tilde{f}_j+(m_{f^c}^2)_{ij}\tilde{f}_i^{c*}\tilde{f}_j^c+m_h^2\tilde{h}^*\tilde{h}+m_\phi^2\tilde{\phi}^*\tilde{\phi}+m_F^2\tilde{F}^*\tilde{F}+m_{\bar{F}}^2\tilde{\bar{F}}^*\tilde{\bar{F}}\\ &+m_{F^c}^2\tilde{F}^{c*}\tilde{F}^c+m_{\bar{F}^c}^2\tilde{\bar{F}}^{c*}\tilde{\bar{F}}^c+m_{F_c'}^2\tilde{F}_c'^*\tilde{F}_c'+m_{\bar{F}_c'}^2\tilde{\bar{F}}_c'^*\tilde{\bar{F}}_c'\\ &+m_\Phi^2\tilde{\Phi}^*\tilde{\Phi}+m_X^2\tilde{X}^*\tilde{X}+m_{X^c}^2\tilde{X}^{c*}\tilde{X}^c+m_H^2\tilde{H}^*\tilde{H}\\ &+\Big(A_i^\lambda\tilde{f^c}_i\tilde{F}\tilde{h}+A_i^{\lambda^c}\tilde{f}_i\tilde{F}^c\tilde{h}+A_i^\alpha\tilde{\phi}\tilde{f}_i\tilde{\bar{F}}+A_i^{\alpha^c}\tilde{\phi}\tilde{f^c}_i\tilde{\bar{F}}^c\\ &+A^a\tilde{X}^c\tilde{\bar{F}}^c\tilde{F}^c+A^{\sigma_c}\tilde{\bar{F}}^c\tilde{\Phi}\tilde{F}_c'+A^{\bar{\sigma}_c}\tilde{\bar{F}}_c'\tilde{\Phi}\tilde{F}^c+\text{h.c.}\Big)\\ &+\frac{1}{2}\left(M_4\tilde{W}_4\tilde{W}_4+M_L\tilde{W}_L\tilde{W}_L+M_R\tilde{W}_R\tilde{W}_R+\text{h.c.}\right)\end{aligned} \tag{5.2}$$

The A-term RGEs are:

$$\begin{aligned}(4\pi)^2\frac{d}{dt}A_i^\lambda &= \left(6|\vec{\lambda}|^2+4|\vec{\lambda^c}|^2\right)A_i^\lambda+\left(18\vec{\lambda}\cdot\vec{A^\lambda}+8\vec{\lambda^c}\cdot\vec{A}^{\lambda^c}\right)\lambda_i\\ &+\left(\frac{15}{8}\vec{\alpha^c}\cdot\vec{A^\lambda}+\frac{15}{4}\vec{\lambda}\cdot\vec{A}^{\alpha^c}\right)\alpha_i^c\\ &+\frac{15}{2}g_4^2\left(2M_4\lambda_i-A_i^\lambda\right)+3g_L^2\left(2M_L\lambda_i-A_i^\lambda\right)+3g_R^2\left(2M_R\lambda_i-A_i^\lambda\right)\end{aligned}$$

$$\begin{aligned}(4\pi)^2\frac{d}{dt}A_i^{\lambda^c} &= \left(6|\vec{\lambda^c}|^2+4|\vec{\lambda}|^2+\frac{15}{8}\bar{\sigma}_c^2+\frac{3}{4}a^2\right)A_i^{\lambda^c}\\ &+\left(18\vec{\lambda^c}\cdot\vec{A}^{\lambda^c}+8\vec{\lambda}\cdot\vec{A^\lambda}+\frac{15}{4}\bar{\sigma}_cA^{\bar{\sigma}_c}+\frac{3}{2}aA_a\right)\lambda_i^c\\ &+\left(\frac{15}{8}\vec{\alpha}\cdot\vec{A}^{\lambda^c}+\frac{15}{4}\vec{\lambda^c}\cdot\vec{A^\alpha}\right)\alpha_i\\ &+\frac{15}{2}g_4^2\left(2M_4\lambda_i^c-A_i^{\lambda^c}\right)+3g_L^2\left(2M_L\lambda_i^c-A_i^{\lambda^c}\right)+3g_R^2\left(2M_R\lambda_i^c-A_i^{\lambda^c}\right)\end{aligned}$$

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}A_i^{\alpha} &= \left(\frac{23}{8}|\vec{\alpha}|^2+|\vec{\alpha^c}|^2\right)A_i^{\alpha}+\left(\frac{91}{8}\vec{\alpha}\cdot\vec{A^{\alpha}}+2\vec{\alpha^c}\cdot\vec{A^{\alpha^c}}\right)\alpha_i+\left(2\vec{\lambda^c}\cdot\vec{A^{\alpha}}+4\vec{\alpha}\cdot\vec{A^{\lambda^c}}\right)\lambda_i^c \\
&\quad +\frac{31}{2}g_4^2\left(2M_4\alpha_i-A_i^{\alpha}\right)+3g_L^2\left(2M_L\alpha_i-A_i^{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}A_i^{\alpha^c} &= \left(\frac{23}{8}|\vec{\alpha^c}|^2+|\vec{\alpha}|^2+\frac{15}{8}\sigma_c^2+\frac{3}{4}a^2\right)A_i^{\alpha^c} \\
&\quad +\left(\frac{91}{8}\vec{\alpha^c}\cdot\vec{A^{\alpha^c}}+2\vec{\alpha}\cdot\vec{A^{\alpha}}+\frac{15}{4}\sigma_c A^{\sigma_c}+\frac{3}{2}aA_a\right)\alpha_i^c \\
&\quad +\left(2\vec{\lambda}\cdot\vec{A^{\alpha^c}}+4\vec{\alpha^c}\cdot\vec{A^{\lambda}}\right)\lambda_i \\
&\quad +\frac{31}{2}g_4^2\left(2M_4\alpha_i^c-A_i^{\alpha^c}\right)+3g_R^2\left(2M_R\alpha_i^c-A_i^{\alpha^c}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}A^{a} &= \left(\frac{21}{2}a^2+\frac{15}{8}(\sigma_c^2+\bar{\sigma}_c^2)+\frac{15}{8}|\vec{\alpha^c}|^2+2|\vec{\lambda^c}|^2\right)A^{a} \\
&\quad +\left(\frac{15}{4}(\sigma_c A^{\sigma_c}+\bar{\sigma}_c A^{\bar{\sigma}_c})+\frac{15}{4}\vec{\alpha^c}\cdot\vec{A^{\alpha^c}}+4\vec{\lambda^c}\cdot\vec{A^{\lambda^c}}\right)a \\
&\quad +\frac{15}{2}g_4^2\left(2M_4 a-A^{a}\right)+7g_R^2\left(2M_R a-A^{a}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}A^{\sigma_c} &= \left(\frac{57}{4}\sigma_c^2+\bar{\sigma}_c^2+\frac{15}{8}|\vec{\alpha^c}|^2+\frac{3}{4}a^2\right)A^{\sigma_c}+\left(2\bar{\sigma}_c A^{\bar{\sigma}_c}+\frac{15}{4}\vec{\alpha^c}\cdot\vec{A^{\alpha^c}}+\frac{3}{2}aA^{a}\right)\sigma_c \\
&\quad +\frac{31}{2}g_4^2\left(2M_4\sigma_c-A^{\sigma_c}\right)+3g_R^2\left(2M_R\sigma_c-A^{\sigma_c}\right)
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}A^{\bar{\sigma}_c} &= \left(\frac{57}{4}\bar{\sigma}_c^2+\sigma_c^2+2|\vec{\lambda^c}|^2+\frac{3}{4}a^2\right)A^{\bar{\sigma}_c}+\left(2\sigma_c A^{\sigma_c}+4\vec{\lambda^c}\cdot\vec{A^{\lambda^c}}+\frac{3}{2}aA^{a}\right)\bar{\sigma}_c \\
&\quad +\frac{31}{2}g_4^2\left(2M_4\bar{\sigma}_c-A^{\bar{\sigma}_c}\right)+3g_R^2\left(2M_R\bar{\sigma}_c-A^{\bar{\sigma}_c}\right)
\end{aligned}
$$

The RGEs for the soft masses are:

$$
\begin{aligned}
(4\pi)^2\frac{d}{dt}(m_f^2)_{ij} &= \left(2\lambda_i^c\lambda_k^c+\frac{15}{8}\alpha_i\alpha_k\right)(m_f^2)_{kj}+(m_f^2)_{ik}\left(2\lambda_k^c\lambda_j^c+\frac{15}{8}\alpha_k\alpha_j\right) \\
&\quad +4\left(m_{F^c}^2+m_h^2\right)\lambda_i^c\lambda_j^c+\frac{15}{4}\left(m_{\bar{F}}^2+m_{\phi}^2\right)\alpha_i\alpha_j \\
&\quad +4A_i^{\lambda^c}A_j^{\lambda^c}+\frac{15}{4}A_i^{\alpha}A_j^{\alpha}-15g_4^2M_4^2-6g_L^2M_L^2
\end{aligned}
$$

$$(4\pi)^2 \frac{d}{dt}(m^2_{f^c})_{ij} = \left(2\lambda_i\lambda_k + \frac{15}{8}\alpha^c_i\alpha^c_k\right)(m^2_{f^c})_{kj} + (m^2_{f^c})_{ik}\left(2\lambda_k\lambda_j + \frac{15}{8}\alpha^c_k\alpha^c_j\right)$$
$$+4\left(m^2_F + m^2_h\right)\lambda_i\lambda_j + \frac{15}{4}\left(m^2_{\bar{F}^c} + m^2_\phi\right)\alpha^c_i\alpha^c_j$$
$$+4A^\lambda_i A^\lambda_j + \frac{15}{8}A^{\alpha^c}_i A^{\alpha^c}_j - 15g^2_4 M^2_4 - 6g^2_R M^2_R$$

$$(4\pi)^2 \frac{d}{dt}m^2_h = 8\Big((|\vec{\lambda}|^2 + |\vec{\lambda^c}|^2)m^2_h + (m^2_{f^c})_{ij}\lambda_i\lambda_j + (m^2_f)_{ij}\lambda^c_i\lambda^c_j + m^2_F|\vec{\lambda}|^2 + m^2_{F^c}|\vec{\lambda^c}|^2$$
$$+|\vec{A^\lambda}|^2 + |\vec{A}^{\lambda^c}|^2\Big) - 6g^2_L M^2_L - 6g^2_R M^2_R$$

$$(4\pi)^2 \frac{d}{dt}m^2_\phi = \frac{15}{8}\Big(\sum_i(\alpha^2_i + \alpha^{c2}_i)m^2_\phi + (m^2_f)_{ij}\alpha_i\alpha_j + (m^2_{f^c})_{ij}\alpha^c_i\alpha^c_j + m^2_{\bar{F}}|\vec{\alpha}|^2 + m^2_{\bar{F}^c}|\vec{\alpha^c}|^2$$
$$+|\vec{A^\alpha}|^2 + |\vec{A}^{\alpha^c}|^2)\Big) - 32g^2_4 M^2_4$$

$$(4\pi)^2 \frac{d}{dt}m^2_F = 4\left(|\vec{\lambda}|^2 m^2_F + (m^2_{f^c})_{ij}\lambda_i\lambda_j + m^2_h|\vec{\lambda}|^2 + |\vec{A^\lambda}|^2\right) - 15g^2_4 M^2_4 - 6g^2_L M^2_L$$

$$(4\pi)^2 \frac{d}{dt}m^2_{\bar{F}} = \frac{15}{4}\left(|\vec{\alpha}|^2 m^2_{\bar{F}} + (m^2_f)_{ij}\alpha_i\alpha_j + m^2_\phi|\vec{\alpha}|^2 + |\vec{A^\alpha}|^2\right) - 15g^2_4 M^2_4 - 6g^2_L M^2_L$$

$$(4\pi)^2 \frac{d}{dt}m^2_{F^c} = \left(4|\vec{\lambda^c}|^2 + \frac{3}{2}a^2 + \frac{15}{4}\bar{\sigma_c}^2\right)m^2_{F^c} + 4(m^2_f)_{ij}\lambda^c_i\lambda^c_j + 4m^2_h|\vec{\lambda^c}|^2 + \frac{3}{2}(m^2_{X^c} + m^2_{\bar{F}^c})a^2$$
$$+\frac{15}{4}(m^2_\Phi + m^2_{\bar{F_c}'})\bar{\sigma_c}^2 + 4|\vec{A}^{\lambda^c}|^2 + \frac{3}{2}A^{a2} + \frac{15}{4}A^{\bar{\sigma}_c 2} - 15g^2_4 M^2_4 - 6g^2_R M^2_R$$

$$(4\pi)^2 \frac{d}{dt}m^2_{\bar{F}_c} = \left(\frac{15}{4}|\vec{\alpha^c}|^2 + \frac{3}{2}a^2 + \frac{15}{4}\sigma^2_c\right)m^2_{\bar{F}^c} + \frac{15}{4}(m^2_{f^c})_{ij}\alpha^c_i\alpha^c_j + \frac{15}{4}m^2_\phi|\vec{\alpha}^c|^2$$
$$+\frac{3}{2}(m^2_{X^c} + m^2_{F^c})a^2 + \frac{15}{4}(m^2_\Phi + m^2_{F'_c})\bar{\sigma_c}^2 + \frac{15}{4}|\vec{A}^{\alpha^c}|^2 + \frac{3}{2}A^{a2} + \frac{15}{4}A^{\sigma^c 2}$$
$$-15g^2_4 M^2_4 - 6g^2_R M^2_R$$

$$(4\pi)^2 \frac{d}{dt} m^2_{F'_c} = \frac{15}{4}\left(\sigma_c^2 m^2_{F'_c} + (m^2_\Phi + m^2_{\bar{F}_c})\sigma_c^2 + A^{\sigma_c 2}\right) - 15 g_4^2 M_4^2 - 6 g_R^2 M_R^2$$

$$(4\pi)^2 \frac{d}{dt} m^2_{\bar{F}'_c} = \frac{15}{4}\left(\bar{\sigma}_c{}^2 m^2_{\bar{F}'_c} + (m^2_\Phi + m^2_{F^c})\bar{\sigma}_c{}^2 + A^{\bar{\sigma}_c 2}\right) - 15 g_4^2 M_4^2 - 6 g_R^2 M_R^2$$

$$(4\pi)^2 \frac{d}{dt} m^2_{X^c} = \frac{3}{2}\left(a^2 m^2_{X^c} + (m^2_{F^c} + m^2_{\bar{F}^c})a^2 + A^{a2}\right) - 16 g_R^2 M_R^2$$

$$\begin{aligned}(4\pi)^2 \frac{d}{dt} m^2_\Phi = {} & \frac{15}{8}\left((\sigma_c^2 + \bar{\sigma}_c{}^2) m^2_\Phi + (m^2_{\bar{F}^c} + m^2_{F'_c})\sigma_c^2 + (m^2_{F^c} + m^2_{\bar{F}_c{}'})\bar{\sigma}_c{}^2 + A^{\sigma_c 2} + A^{\bar{\sigma}_c 2}\right) \\ & -32 g_4^2 M_4^2\end{aligned}$$

RGEs between M_L and M_R

The superpotential above M_L is:

$$\begin{aligned} W = {} & (\alpha_q^A)_i\, q_i \bar{Q} A_\phi + (\alpha_l^A)_i\, l_i \bar{L} A_\phi + (\alpha_q^T)_i\, q_i \bar{L} \bar{T}_\phi + (\alpha_l^T)_i\, l_i \bar{Q} T_\phi + (\alpha_q^G)_i\, G_\phi q_i \bar{Q} \\ & + \lambda_i^u\, u_i^c Q h_u + \lambda_i^d\, d_i^c Q h_d + \lambda_i^e\, e_i^c L h_d \end{aligned} \quad (5.3)$$

The boundary conditions at M_R are:

$$\lambda^u = \lambda^d = \lambda^e = \lambda$$

$$\sqrt{24}\,\alpha_q^A = -\frac{\sqrt{24}}{3}\alpha_l^A = \sqrt{2}\,\alpha_q^T = \sqrt{2}\,\alpha_l^T = \alpha_q^G = \alpha \quad (5.4)$$

The first and the second line of Eq. (5.3) are decoupled: in particular the RGEs for the second line Yukawas and soft masses are MSSM-like.

5.2 FCNC predictions

We discuss some results for FCNCs in the model in two different regimes:

- High tanβ: this is the natural regime for the model because of λ_t-λ_b unification.

- Moderate/low tanβ: this regime can be obtained if for example we include Higgs mixing in the model. Mixing in the Higgs sector can lower tanβ and we obtain predictions dominated by the off-diagonal soft mass insertions.

We report here a typical spectrum for the two cases:

	$\tan\beta \simeq 54$	$\tan\beta \simeq 15$
$M_{1/2}$	2300 GeV	550 GeV
M_0	200 GeV	100 GeV
$M_{\tilde{g}}$	3350 GeV	800 GeV
$M_{\tilde{t}_R}$	2350 GeV	560 GeV
$M_{\tilde{\tau}_1}$	650 GeV	170 GeV
M_A	480 GeV	125 GeV

5.2.1 High tanβ

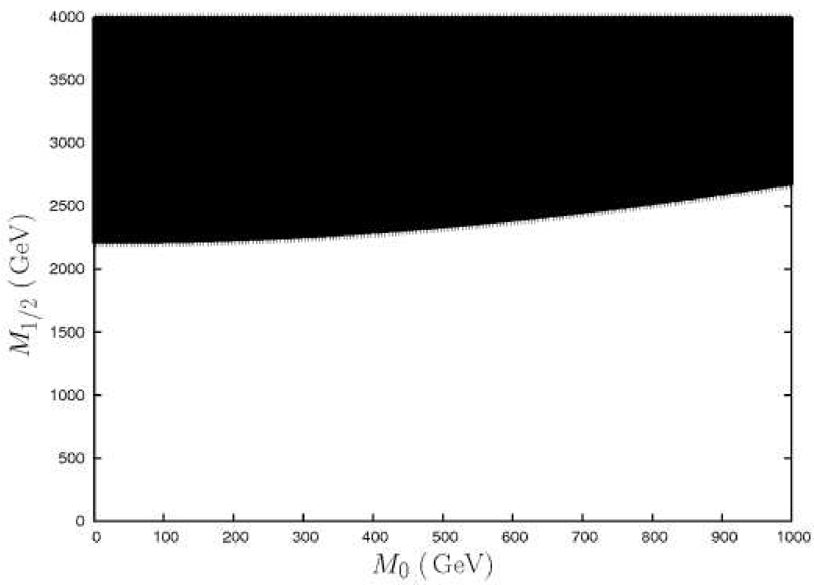

Figure 5.1: Allowed region for the high-scale universal soft masses in the high tanβ regime.

In this regime the allowed spectrum of squarks and gluinos turns out to be heavy, and the heavy Higgses give important tanβ-enhanced contributions to $B \to \mu\mu$ observables. These contributions give strong constraints on the parameter space ($M_0 = M_{h0}, M_{1/2}$). In figure 5.1 we plot the part of the

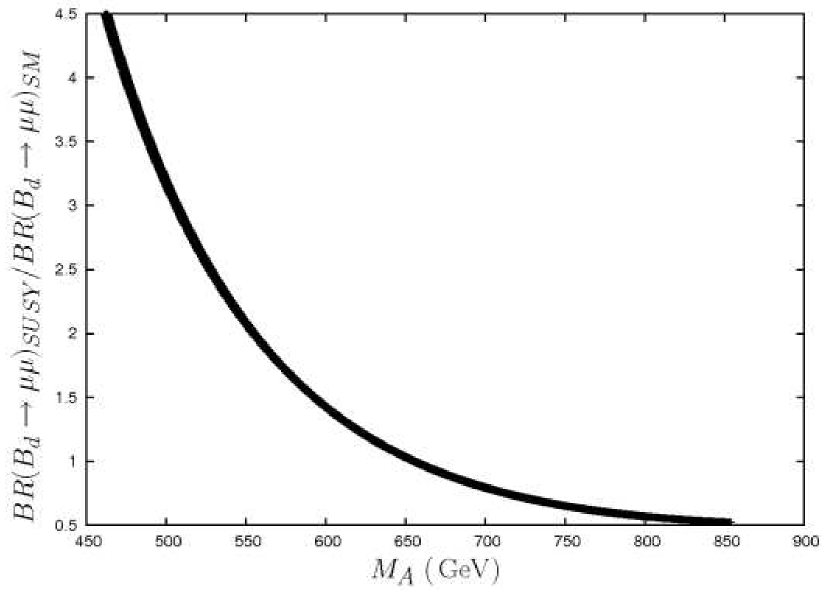

Figure 5.2: Branching ratio for Higgs-mediated $B_d \to \mu\mu$ as a function of the CP-odd Higgs mass in the high tanβ regime.

parameter space that is allowed by the current limits on $BR(B_d \to \mu^+\mu^-)$, that is the observable that give the strongest constraints.

The fact that the light mass of the heavy Higgses is responsible for this bound can be seen from 5.2, where we show the scatter plot of $BR(B_d \to \mu^+\mu^-)_{SUSY}/BR(B_d \to \mu^+\mu^-)_{SM}$ and the mass M_A of the pseudoscalar Higgs (note that we plot only the Higgs-mediated contribution to the process).

The effects coming from the off-diagonal entries of the soft mass matrices are suppressed because of the heavy spectrum of the model. For example, gaugino-mediated $b \to s\gamma$ processes are negligible (but other contributions could be important and deserve future investigations).

However tanβ enhanced contributions can be relevant for processes like $\tau \to \mu\gamma$. In figure 5.3 we plot the branching ratio of this process as a function of the $\tilde{\tau}_1$ mass. The amplitude for this process is large enough to be observed at a future super flavour factory [39], which should be able to set a limit on its branching ratio of order $BR(\tau \to \mu\gamma) \sim 10^{-9}$.

5.2.2 Moderate tanβ

In this regime there are interesting effects coming from the off-diagonal mass insertions in the lepton sector, contributing to processes like $\tau \to \mu\,\gamma$. The

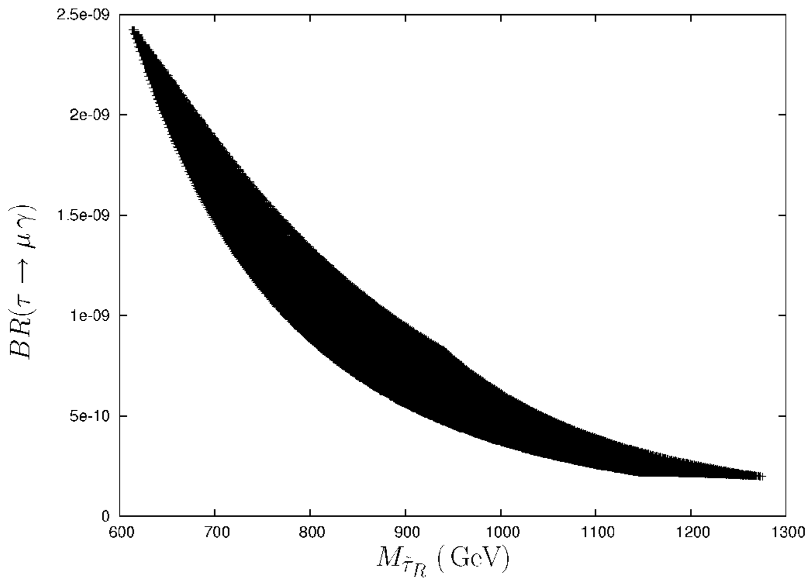

Figure 5.3: Branching ratio for $\tau \to \mu\,\gamma$ as a function of the lighter slepton mass in the high tanβ regime.

region of the plane $(M_0, M_{1/2})$ plotted in figure 5.4 is allowed both by $\tau \to \mu\gamma$ and $B \to \mu\mu$ processes.

The figure 5.5 shows the scatter plot of $BR(B_d \to \mu^+\mu^-)_{SUSY}/BR(B_d \to \mu^+\mu^-)_{SM}$ as a function of M_A.

The figure 5.6 shows the interesting range for the process $\tau \to \mu\gamma$. The effect of the off-diagonal terms is relevant in this regime and the predictions lie in a region that should be explored by superB factories.

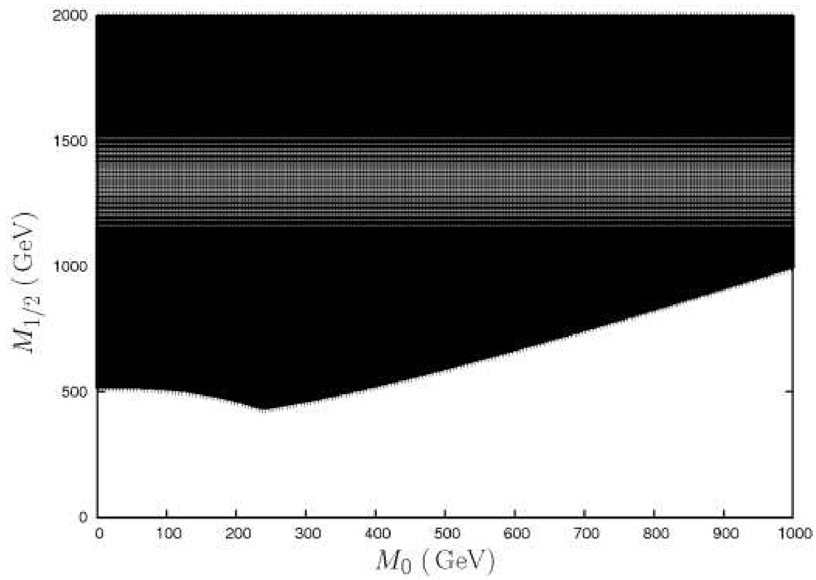

Figure 5.4: Allowed region for the high-scale universal soft masses in the moderate tanβ regime.

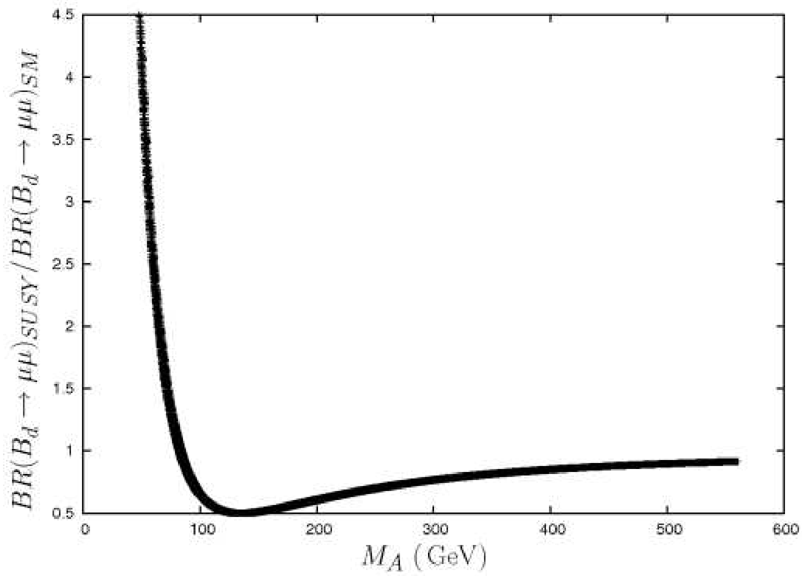

Figure 5.5: Branching ratio for Higgs-mediated $B_d \to \mu\mu$ as a function of the CP-odd Higgs mass in the moderate tanβ regime.

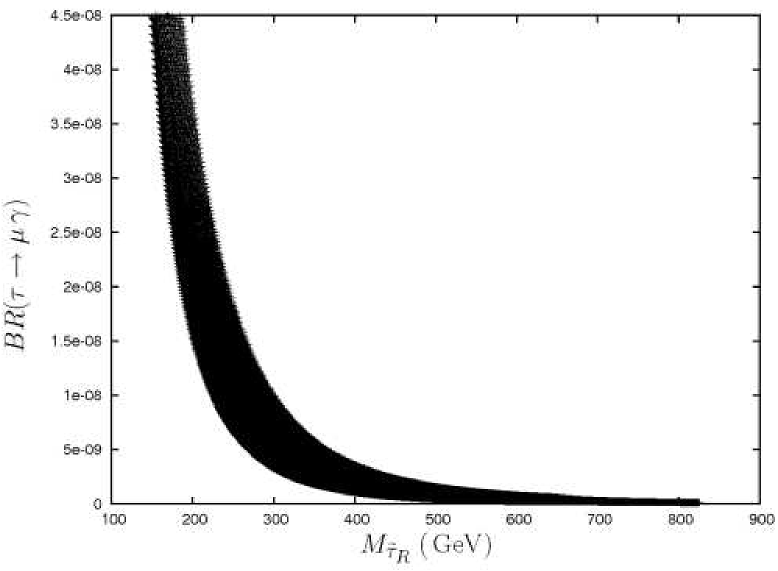

Figure 5.6: Branching ratio for $\tau \to \mu\,\gamma$ as a function of the lighter slepton mass in the moderate tanβ regime.

Part II

Supersymmetry breaking in metastable vacua

Chapter 6

Introduction

Supersymmetry is one of the most interesting possibilities that experiments probing the physics beyond the Standard Model could discover. There are convincing (although indirect) clues that our world could be described by a SUSY GUT at high energies. Supersymmetry stabilizes theories of particle physics in such a way that these theories could be valid up to very high energies and therefore, if it is realized in the world, provides a good window into high energy physics and allows us to probe new physics at scales far from the electroweak one, even if in an indirect way.

However our low energy world is apparently not supersymmetric, therefore if we assume that SUSY is an ingredient of our physical world, it should be spontaneously broken. Many fundamental problems of supersymmetry are related to its breaking. For example, most of the physics at the TeV scale is determined by the structure of the supersymmetry breaking terms of the MSSM Lagrangian. Building a model of SUSY breaking (and its communication to the MSSM) is therefore quite relevant for our understanding of physics beyond the Standard Model. Here we will concentrate on models for SUSY-breaking sectors. In supersymmetric model building, the supersymmetry-breaking sector is the most elusive one: finding generic and natural models of supersymmetry breaking has been a theoretical challenge for many years.

There are many models of SUSY breaking. The simplest ones were developed soon after the discovery of supersymmetry These models work at tree level, giving VEV to F terms (O'Raifeartaigh [40]) or D terms (Fayet-Iliopoulos[41]) and therefore breaking supersymmetry. The scale of SUSY breaking in these models is simply related to some dimensional parameters in the classical Lagrangian. However the scale of these parameters is expected be of the same order of the cutoff scale because of naturalness arguments and if we identify the fundamental scale with the Planck scale we have to explain the huge hierarchy between the Planck scale and the scales entering the dynamics of the SUSY-breaking sector. Tipically the scale of supersymmetry-breaking terms in the MSSM, that is around the TeV scale, is of order $\langle F\rangle/M_{mess}$ where M_{mess} is the scale of the messenger sector that communicate the breaking to the MSSM, therefore $\sqrt{\langle F\rangle} \lesssim 10^{11}$ GeV that is much smaller than the Planck scale.

The attention focused then on models of dynamical SUSY breaking, where supersymmetry is broken by strong coupling dynamics in the low-energy phase of some gauge theory and the corresponding scales are dynamically generated. These models are not easy to obtain and they are generally quite involved, requiring chiral theories or massless particles [42, 43, 44, 45] . One of the main obstacles is the fact that simple gauge theories like SQCD with massive flavours cannot have SUSY-breaking vacua. This can be understood as a consequence of the non-zero Witten index of pure super-Yang-Mills theories [46] , that ensures that supersymmetry is unbroken. This remains valid also for massive SQCD, because SYM can be recovered from it going to the infinite mass limit with a continuous deformations that should not change the index.

The situation changed completely with the work of Intriligator, Seiberg and Shih [47] that discovered a metastable SUSY-breaking vacuum in one of the simplest strongly coupled theories, namely $\mathcal{N}=1$ SQCD with N_f massive flavours and $N_c < N_f < \frac{3}{2}N_c$. SUSY is broken near the origin of field space, while the gauge dynamics restores the usual supersymmetric vacua as implied by Witten index, but they are far away in field space and separated enough from the non-supersymmetric vacua, therefore these non-SUSY vacua are metastable but parametrically long-lived. The ISS model is not the first realization of metastable supersymmetry breaking, see for example [48], but the simplicity of the model suggests that metastable SUSY-breaking vacua are common both in field theories and in string theories. In fact the discovery triggered a long series of papers looking for properties of metastable vacua in field theory and string theory and their applications to gauge mediation and moduli stabilization.

The metastable vacua of the ISS model can be seen in the low-energy effective theory as vacua of an O'Raifeartaigh-type model. This is not uncommon, because strongly coupled gauge theories often have a low-energy description in terms of (weakly gauged) Wess-Zumino model where the effective degrees of freedom are gauge invariant polynomials of the fundamental fields. This description can be very useful because these models are usually perturbative and calculable and therefore their properties can be reliably studied even if SUSY is spontaneously broken. The modern point of view is that O'Raifeartaigh models are interesting because they can be effective theories that encode the dynamics of strongly-coupled gauge theories.

R-symmetry plays a crucial role in O'Raifeartaigh models: in fact the Nelson-Seiberg argument [49] states that the existence of an R-symmetry is a necessary condition for supersymmetry breaking in theories with generic superpotentials. However, the R-symmetry must be broken in order to give an high mass (greater than about 100 GeV) to gauginos. It is possible to break the R-symmetry explicitly, but this implies that the SUSY-breaking vacua are metastable even without including gauge effects, and depending on the model there could be some tension between the requirement of long-lived vacua and the bounds on gaugino masses. Because of this tension, spontaneous breaking of the R-symmetry seems to be an interesting possibility, if we neglect the problem of the existence of a massless Goldstone boson (R-axion). Spontaneous R-symmetry breaking can be driven by gauge interactions as in [50, 51], but this mechanism works only for a small window of the parameter space. Here we are interested in models where the breaking is triggered by the perturbative dynamics of the O'Raifeartaigh superpotential.

Most of the O'Raifeartaigh models featured in the literature have fields whose R-charges can be chosen to be either 2 or 0. In these models R-symmetry does not seem to break spontaneously. Shih noted that this fact is related to the choice of R-charges of these models and that spontaneous breaking often occurs in models containing fields with $R \neq 2, 0$ [52]. The simplest O'Raifeartaigh model that breaks R-symmetry spontaneously for some values of its parameters is:

$$W = fX + \lambda X \phi_{(1)}\phi_{(-1)} + m_1 \phi_{(3)}\phi_{(-1)} + \frac{1}{2} m_2 \phi_{(1)}^2 \tag{6.1}$$

where $R(X) = 2$ and $R(\phi_{(k)}) = k$. The flat direction parametrized by X is lifted by quantum corrections and R-symmetry is broken in a region of the parameter space where the resulting vacuum has $\langle X \rangle \neq 0$. An interesting observation is that the above vacuum is metastable because of the existence of a runaway direction:

$$\phi_{(1)} = -\frac{f}{\lambda \phi_{(-1)}}, \; X = \frac{m_2 f}{\lambda^2 \phi_{(-1)}^2}, \; \phi_{(3)} = \frac{m_2 f^2}{m_1 \lambda^2 \phi_{(-1)}^3}, \; \phi_{(-1)} \to 0 \tag{6.2}$$

A natural question is if these interesting results correspond to general properties of models with general R-charges, or if they depend on the choice of the above model. In these chapters we discuss the properties of these models and show that these features occur in many generalized O'Raifeartaigh models.

In section 6.1 we review the O'Raifeartaigh model and the class of models with R-charges $R = 2, 0$. In section 6.2 we introduce the ISS model and in section 6.3 we discuss the relation between R-symmetry and SUSY breaking.

In section 7.1 we discuss generalized O'Raifeartaigh models where R-charges can be different from 2 or 0. We consider generalizations of the model (7.1) with generic R-charge assignments and a superpotential

$$W = fX + \frac{1}{2}(M^{ij} + N^{ij}X + Q_a^{ij}Y_a)\phi_i\phi_j \tag{6.3}$$

where M, N, Q_a are generic symmetric complex matrices. The vacua of these models break the R-symmetry for a wide range of parameters of the superpotential. In section 7.2 we show that most of the Wess-Zumino models which contain fields with $R \neq 2, 0$ and break SUSY have runaway directions. The runaway vacuum can be supersymmetric or non-supersymmetric. We explain the relation between this runaway behaviour and the R-symmetry of the theory.

6.1 Usual O'Raifeartaigh models

We discuss the physics of the original O'Raifeartaigh model and similar models of SUSY breaking. These models are Wess-Zumino models of chiral fields with a renormalizable superpotential. Supersymmetry breaking is related by the Nelson-Seiberg argument to the existence of an R-symmetry, which transforms fields with R-charge 2 or 0.

The original model contains three chiral superfields X, $\phi_{(2)}$, $\phi_{(0)}$ of R-charge $R(X) = R(\phi_{(2)}) = 2$, $R(\phi_{(0)}) = 0$, a canonical Kahler potential and a superpotential $W = fX + nX\phi_{(0)}^2 + m\phi_{(2)}\phi_{(0)}$. The SUSY vacuum equations are

$$f + n\phi_{(0)}^2 = 0 \tag{6.4}$$
$$m\phi_{(0)} = 0 \tag{6.5}$$
$$2nX\phi_{(0)} + m\phi_{(2)} = 0 \tag{6.6}$$

and there is no solution to this system, therefore SUSY is broken. The non-supersymmetric minimum is obtained by minimizing the potential with respect to $\phi_{(0)}$ and there is a flat direction of minima parametrized by $\langle X \rangle$. This flat direction is lifted by the 1-loop Coleman-Weinberg potential and this quantum correction forces the vev of X and $\phi_{(2)}$ to 0. The global R-symmetry remains unbroken.

This situation is generic for all the models with fields of R-charge 2 or 0. In fact the most general O'Raifeartaigh model of this kind contains n_X fields X_n with $R(X_n) = 2$ and n_ϕ fields ϕ_i with $R(\phi_i) = 0$, has a canonical Kahler term and a superpotential

$$W = \sum_{k=1}^{n_X} X_k \, g_k(\phi_i) \ . \tag{6.7}$$

The SUSY vacuum equations for this model are

$$g_k(\phi_i) = 0 \tag{6.8}$$

$$\sum_k X_k \, \partial_j g_k(\phi_i) = 0 \ . \tag{6.9}$$

The subset (6.8) of these equations is a system of n_X equations in n_ϕ variables and therefore it cannot be solved for general functions $g_k(\phi)$ if $n_X > n_\phi$. If the last condition is satisfied supersymmetry is spontaneously broken. and the minimum of the potential is

$$V_{min} = \sum_{k=1}^{n_X} |g_k(\langle\phi_i\rangle)|^2 \tag{6.10}$$

where $\langle\phi_i\rangle$ satisfy the equations $\sum_k g_k^*(\langle\phi_i\rangle)\partial_j g_k(\langle\phi_i\rangle) = 0$.

The equations (6.9) for X_k form a linear system of n_ϕ equations in n_X variables and can be generally solved for all values of $\langle\phi_i\rangle$, therefore the minima of the potential form an $(n_X - n_\phi)$-dimensional linear space of flat directions parametrized by $X_1 \dots X_{n_X - n_\phi}$. This structure is partly dictated by the R-symmetry. In fact complexified R-symmetry acts as a dilatation[1] on this space: $X_n \to e^{2i\alpha} X_n, \ \alpha \in \mathbb{C}$. The potential in the vacuum contains only squares of F-terms with R-charge 0 and therefore is invariant under complex R-symmetry transformations. This means that the space of vacua $\mathcal{M}$ must be composed of complex rays in the vector space generated by the X_ks: $\langle X_k\rangle \in \mathcal{M} \Rightarrow \langle X'_k\rangle = c\,\langle X_k\rangle \in \mathcal{M}, \ \forall c \in \mathbb{C}$.

This degeneracy of vacua is removed when we take into account the Coleman-Weinberg 1-loop effective potential [53] that chooses a true minimum at $X_n = 0$, as we will show in section 7.1.

6.2 Metastable vacua in simple theories

The Intriligator-Seiberg-Shih model of metastable dynamical SUSY breaking [47] is a very simple theory, namely $\mathcal{N} = 1$ SQCD with gauge group $SU(N_c)$.

[1] Note that this is reminiscent of the usual argument for flat directions in supersymmetric theories. These directions are often present because both the superpotential and the vacuum equations are invariant under complexified symmetry transformations. The difference is that the usual argument applies to SUSY vacua, while the argument in the text applies to SUSY-breaking vacua.

We consider this theory with a matter content of massive fields $Q_k, \tilde{Q}_k$ with $k = 1 \dots N_f$. The mass term has the form $m_{ij} Q_i \tilde{Q}_j$. These theories are asymptotically free for $N_f < 3N_c$ and their low-energy dynamics have been explained by Seiberg and collaborators [54, 55, 56, 43] . Here we are interested in the free magnetic regime, which corresponds to $N_c < N_f < \frac{3}{2}N_c$. For these values of N_c and N_f, the low-energy theory admits a weakly coupled description in terms of the meson field $M_{ij} = Q_i \tilde{Q}_j$ and of N_f flavours of dual "magnetic" quarks $q_i, \tilde{q}_i$ charged under a "magnetic" gauge group $SU(N)$ with $N = N_f - N_c$. The dual description is weakly coupled, has a Kähler potential of the form $K = \alpha \bar{M} M / \Lambda_m^2 + \beta \bar{q} q + \beta \bar{\tilde{q}} \tilde{q}$ and a superpotential

$$W = \frac{1}{\Lambda_m} \tilde{q}_i M_{ij} q_j + m_{ij} M_{ij} \tag{6.11}$$

where Λ_m is the Landau pole of the magnetic gauge coupling. This theory can be only a low-energy effective description because it is infrared free and goes to strong coupling above Λ_m.

Let's study the vacua of this theory. We consider the case of equal masses $m_{ij} = m\delta_{ij}$ when the theory has a flavour $U(N_f)$ symmetry. We start neglecting the effects of the gauge group, because it is at weak coupling. The theory then describes three chiral fields[2] with quantum numbers $M = (\mathbf{1}, \mathbf{N_f^2 - 1}) + (\mathbf{1}, \mathbf{1})$, $q = (\mathbf{N}, \mathbf{N_f})$, $\tilde{q} = (\bar{\mathbf{N}}, \bar{\mathbf{N}}_\mathbf{f})$ under $SU(N) \times SU(N_f)$ and superpotential

$$W = h \tilde{q}_i^\alpha M_{ij} q_j^\alpha - h\mu^2 M_{ii} \tag{6.12}$$

Note that this theory has an R-symmetry with charges $R(M) = 2$,$R(q) = R(\tilde{q}) = 0$ and therefore is an O'Raifeartaigh model like the ones discussed in the previous section. The superpotential is not completely generic so we cannot apply naive counting to understand if SUSY is broken, but we have to look at the vacuum equations

$$h\tilde{q}_i^\alpha q_j^\alpha - h\mu^2 \delta_{ij} = 0 \tag{6.13}$$
$$hM_{ij} q_j^\alpha = 0 \tag{6.14}$$
$$h\tilde{q}_i^\alpha M_{ij} = 0 \tag{6.15}$$
$$\tag{6.16}$$

The first equation cannot be solved if $N_f > N$ because the product $\tilde{q}_i q_j$ has rank N and therefore cannot cancel with δ_{ij}. This means that supersymmetry

[2]We consider an implicit redefinition of these fields in order to obtain a canonical Kähler potential for all of them. With this redefinition M_{ij} becomes a chiral field of mass dimension 1.

is broken by the "rank condition" and there is a minimum of the potential $V = (N_f - N)|h^2\mu^4|$ where the vevs are

$$M = 0, \quad q = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}, \quad \tilde{q}^T = \begin{pmatrix} 0 \\ \tilde{\phi}_0^T \end{pmatrix} \qquad \tilde{\phi}_0\phi_0 = \mu^2 \mathbf{1}_{N\times N} \tag{6.17}$$

up to symmetry rotations. All the pseudo-flat directions are lifted by the Coleman-Weinberg potential and the masses of the fields in this vacuum are of order μ (except for the Goldstone bosons coming from flavour symmetries). The low-energy parameter μ is related to the high-energy parameters as $\mu \sim \sqrt{-m\Lambda}$.

The supersymmetric vacua of the high-energy theory can be seen also in the low-energy description if we switch on the gauge interaction. If we consider points in fields space where $\langle M \rangle$ is large, then the mass of $q, \tilde{q}$ is large and they decouple from the low-energy theory, that contains only M and the gauge fields of $SU(N)$. The gauge dynamics induces a gaugino condensation that depends on the quark masses, i.e. on M, and the low-energy superpotential for M is of the form

$$W = N\left(h^{N_f}\Lambda_m^{3N-N_f}\det(M)\right)^{1/N} - h\mu^2\mathrm{Tr}(M) \tag{6.18}$$

that has $N_f - N$ supersymmetric vacua with $\langle M \rangle \sim \epsilon^{-(N_f-3N)/(N_f-N)}\mu/h$ where $\epsilon = \mu/\Lambda$. If $\epsilon \ll 1$, these vacua are far away from the region of radius μ near the origin of field space where the metastable vacua live, therefore the tunneling amplitude from the metastable vacua to these vacua is parametrically suppressed. This means that SQCD in the free magnetic phase and with small quark masses has long-lived SUSY-breaking vacua.

6.3 SUSY breaking, R-symmetry and metastability

The models of supersymmetry breaking discussed in the previous sections are based on Wess-Zumino theories with exact R-symmetries (or approximate, if we consider the small breaking due to the gauge dynamics). It is easy to show that a strong, generic breaking of these R-symmetries generates a dangerous supersymmetric vacuum in addition to the stable (or metastable) non-supersymmetric vacua of these models. In fact there is a strong connection between R-symmetry and supersymmetry breaking that has been discussed by Nelson and Seiberg [49] .

Their argument goes as follows. Consider a Wess-Zumino model with k chiral fields, a canonical Kahler potential and a generic superpotential

$W(\varphi_i)$. Suppose that the superpotential does not respect an R-symmetry. Then, if there are no other symmetries, the vacuum equations $\partial W/\partial\varphi_i = 0$ are k equations in k variables and they can generally be solved. If there are global symmetries that are not R-symmetries, then the potential is a function $W(I_j)$ of a set of k' independent invariant combinations $I_j(\varphi_i)$ and the vacuum equations have the form $(\partial W/\partial I_j)(\partial I_j/\partial\varphi_i) = 0$, therefore they are linear combinations of the equations $(\partial W/\partial I_j) = 0$ that are k' equations in k' variables ($k' < k$) and can generally be solved. Therefore, without an R-symmetry, supersymmetry is generally unbroken.

Suppose instead that there is an R-symmetry, but it is spontaneously broken by the vev of an R-charged field φ_1. Then we can write the superpotential as $W = \varphi_1^{2/R_1} f\left(\varphi_i/\varphi_1^{R_i/R_1}\right)$ and the vacuum equations are

$$\varphi_1^{(2-R_i)/R_1}\partial_i f = 0 \quad , \quad \frac{2}{R_1}\varphi_1^{(2-R_1)/R_1} f - \frac{R_i + R_1}{R_1}\varphi_1^{(2-R_i-R_1)/R_1}\varphi_i\partial_i f = 0 \tag{6.19}$$

which reduce to $f = 0, \partial_i f = 0$ that are k equations in $k-1$ variables $\varphi_2/\varphi_1^{R_2/R_1}, \dots \varphi_k/\varphi_1^{R_k/R_1}$. This system cannot generally be solved and therefore supersymmetry is generally broken.

The above argument shows that for a generic Wess-Zumino model, R-symmetry is a necessary condition for SUSY breaking and spontaneously broken R-symmetry is a sufficient one. Note that this argument considers only vacua at finite distance in field space, i.e. does not apply to runaway directions.

If the R-symmetry is only approximate, then there are supersymmetric vacua in the theory. Suppose that the coupling in front of the R-symmetry breaking term in the superpotential is ϵ. Then the supersymmetric vacua are not present in the theory for $\epsilon = 0$ and appear only when ϵ is turned on. The potential depends on ϵ in a continuous way in any compact region of the space of fields, therefore the supersymmetric vacua must come in from infinity. The vevs in the SUSY vacua are or order $1/\epsilon^n$ and the non-supersymmetric vacuum is metastable but parametrically long-lived for $\epsilon \ll 1$.

As argued in [51], metastability is a general feature of realistic theories of supersymmetry. In fact, neglecting possible R-symmetry breaking effects of gravity, an R-symmetry should exist at least as an approximate symmetry in order to have SUSY breaking, and it has to be broken explicitly or spontaneously to give mass to gauginos. However, spontaneously broken or not, it must be approximate because it should be broken explicitly at a fundamental level to avoid an exactly massless R-axion. This means that there can generally appear supersymmetric vacua far away in field space and we probably live in a non-supersymmetric metastable vacuum.

In the next chapter we will see models where the R-symmetry is exact but there are non-SUSY vacua where it is spontaneously broken by quantum effects. As we will see, even if the R-symmetry is exact, these vacua are metastable because of runaway directions related to the R-symmetry itself.

Chapter 7

Generalized O'Raifeartaigh models

7.1 Spontaneous R-symmetry breaking

The simplest model that breaks R-symmetry spontaneously for some values of its parameters is [52]:

$$W = fX + \lambda X\phi_{(1)}\phi_{(-1)} + m_1\phi_{(3)}\phi_{(-1)} + \frac{1}{2}m_2\phi_{(1)}^2 \tag{7.1}$$

where $R(X) = 2$ and $R(\phi_{(k)}) = k$. Classically this model has a flat direction of local extrema given by $\phi_{(3)} = \phi_{(1)} = \phi_{(-1)} = 0$; this direction is parametrized by X with potential $V(X) = |f|^2$ and is a local minimum for $|X| < \frac{m_1^2 m_2}{2\lambda^2 f} - \frac{f}{2m_2}$. Quantum corrections modify the tree-level potential as $V(X) = |f|^2 + m_X^2|X|^2 + \ldots$ and if $m_X^2 < 0$ in some region of the space of couplings, then the potential $V(X)$ can have a (local) minimum away from the origin and the R-symmetry is broken in this vacuum.

In the paper [52] a class of models that are natural generalizations of the model (7.1) has been considered. These models consist of a chiral superfield X with $R(X) = 2$ and n_ϕ chiral superfields ϕ_i. All these fields have a canonical Kähler potential and a superpotential

$$W = fX + \frac{1}{2}(M^{ij} + N^{ij}X)\phi_i\phi_j \tag{7.2}$$

where M, N are symmetric complex matrices with $\det(M) \neq 0$. Note that the last condition constrains both the R-charges and the field content of the model; for example, it implies that the number of ϕ fields with $R = r$ is the same as the number of fields with $R = 2 - r$. Moreover, R-symmetry constrains the possible nonzero entries in these matrices:

$$M^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\phi_j) = 2 \quad , \quad N^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\phi_j) = 0 \tag{7.3}$$

Apart from these restrictions and those coming from other symmetries, we consider M, N to be generic.

It is possible to prove that a necessary condition for $m_X^2 < 0$ in these models is given by the existence of fields with R-charge different from $R = 2$ and $R = 0$ [52]. We generalize the analysis to include models with more pseudomoduli Y_a with $R(Y_a) = 2$ [57] coupled as in the superpotential

$$W = fX + \frac{1}{2}(M^{ij} + N^{ij}X + Q_a^{ij}Y_a)\phi_i\phi_j \tag{7.4}$$

This model has a linear space of extrema near the origin, given by $\phi_i = 0$, X, Y_a arbitrary. Supersymmetry is broken along these flat directions, therefore they can be lifted by the Coleman-Weinberg potential

$$V_{eff}^{(1-loop)} = \frac{1}{64\pi^2}\mathrm{Tr}\left(\mathcal{M}_B^4 \ln\frac{\mathcal{M}_B^2}{\Lambda^2} - \mathcal{M}_F^4 \ln\frac{\mathcal{M}_F^2}{\Lambda^2}\right) \tag{7.5}$$

. The trick used in [52] is to rewrite this potential as

$$V_{eff}^{(1-loop)} = -\frac{1}{32\pi^2}\int_0^\infty dv\ v^5 \left(\frac{1}{v^2+\mathcal{M}_B^2} - \frac{1}{v^2+\mathcal{M}_F^2}\right) \tag{7.6}$$

The terms in the Coleman-Weinberg potential that are quadratic in X, Y_a can be written as

$$\begin{aligned} V_{quad} = \frac{1}{16\pi^2}\mathrm{Tr}\int_0^\infty dv\ v^3 \Bigg[&\frac{1}{v^2+\hat{M}^2+f\hat{N}}\left(\hat{Y}^2 - \frac{1}{2}\{\hat{M},\hat{Y}\}\frac{1}{v^2+\hat{M}^2+f\hat{N}}\{\hat{M},\hat{Y}\}\right) + \\ &- \frac{1}{v^2+\hat{M}^2}\left(\hat{Y}^2 - \frac{1}{2}\{\hat{M},\hat{Y}\}\frac{1}{v^2+\hat{M}^2}\{\hat{M},\hat{Y}\}\right)\Bigg] \end{aligned} \tag{7.7}$$

where

$$\hat{M} = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}, \hat{N} = \begin{pmatrix} 0 & N^\dagger \\ N & 0 \end{pmatrix}, \hat{Y} = \begin{pmatrix} 0 & (NX+Q^aY_a)^\dagger \\ NX+Q^aY_a & 0 \end{pmatrix} \tag{7.8}$$

We consider the case of $f \ll N^{-1}M$, because in this limit we can neglect the possibility of tachyonic directions of ϕ fields in a large range of values of X, Y_a around the origin of the flat directions. Then at the lowest nonzero order in $|\hat{M}^{-2}f\hat{N}|$ this expression reduces to

$$V_{quad} = \frac{f^2}{32\pi^2}\mathrm{Tr}\int_0^\infty dv\ v^3 \left[\mathcal{M}_1(v)\mathcal{M}_1^\dagger(v) - \mathcal{M}_2(v)\mathcal{M}_2^\dagger(v)\right] \tag{7.9}$$

with

$$\mathcal{M}_1(v) = \frac{1}{\sqrt{v^2+\hat{M}^2}} \left(\hat{N} \frac{\sqrt{2}v}{v^2+\hat{M}^2} \hat{Y} \right) \frac{1}{\sqrt{v^2+\hat{M}^2}} \tag{7.10}$$

$$\mathcal{M}_2(v) = \frac{1}{\sqrt{v^2+\hat{M}^2}} \left(\hat{N} \frac{\hat{M}}{v^2+\hat{M}^2} \hat{Y} + \hat{Y} \frac{\hat{M}}{v^2+\hat{M}^2} \hat{N} \right) \frac{1}{\sqrt{v^2+\hat{M}^2}} \tag{7.11}$$

after eliminating some terms that do not contribute to the trace. The two terms are generally of the same order, but the contribution of the first term is always positive, while the second term always gives a negative contribution.

If there are only fields with $R = 2, 0$ then the form of the matrices is forced by R-symmetry constraints to be

$$M = \begin{pmatrix} 0 & M_1 \\ M_1^T & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 \\ 0 & N_1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} \tag{7.12}$$

and it is easy to see that in this case $\hat{N}\hat{M}^{2k+1}\hat{Y} = 0$ and therefore $\mathcal{M}_2 = 0$. This means that V_{quad} is positive definite: the quantum corrections lift the flat directions and choose a (local) vacuum with unbroken R-symmetry.

In the general case V_{quad} has no definite sign. If the expression (7.9) is negative for some choice of $(X, Y_a) = (x, y_a)$ then the classical vacuum $X = 0, Y_a = 0$ is unstable because the linear combination $\bar{x}X + \bar{y}_a Y_a$ of these fields has negative m^2. In this case there can be an R-symmetry breaking vacuum along one of these tachyonic directions, stabilyzed by the quartic contributions to the Coleman-Weinberg potential. This is what happens even in simple models with a single pseudomodulus X. We will see an explicit example in the next section.

It is also clear that in models with many pseudomoduli Y_a the range of parameters for spontaneous R-symmetry breaking is much bigger than in models with a single pseudomodulus. In fact there are many directions in field space X, Y_a that can be tachyonic, including the original one $X \neq 0, Y_a = 0$.

7.1.1 Global symmetries

In this section we are interested in studying spontaneous R-symmetry breaking in models with non-abelian global flavor symmetries. Global symmetries are interesting because they can play an important role in mediating supersymmetry breaking: for example, they can be gauged and communicate SUSY breaking directly through gauge interactions, as in [58, 59] or through a messenger sector, as in [60, 61, 62].Non-abelian global symmetries can also be useful when looking for an ultraviolet completion of these models, if we consider them as effective theories of strongly-coupled gauge theories, as in [47].

The models discussed in the previous sections can have non-abelian global symmetries. Starting from the Shih model (7.1), it is easy to write a model with real representations, for example $SO(N)$ fundamentals, or to add other fields that interact only with X and play no role in breaking SUSY:

$$\Delta W = \lambda' X \bar{\varphi}_\alpha \varphi^\alpha \tag{7.13}$$

However it would be useful to include also complex representations in our models. In this section we will study O'Raifeartaigh models with spontaneous R-symmetry breaking where the SUSY-breaking sector contains fields in real or complex representations of a flavour symmetry.

The models discussed in the previous sections can have non-abelian global symmetries. However models that necessarily have a field with $R = 0, 1$ can have only fields in real representations. As an example, the model (7.1) cannot be extended with global symmetries under which the fields transform as complex representations. In fact the mass term for $\phi_{(1)}$ requires that the representations $\mathcal{R}(\phi_{(1)}) \otimes_{simm} \mathcal{R}(\phi_{(1)}) \supset \mathbf{1}$, therefore $\mathcal{R}(\phi_{(1)})$ cannot be an irreducible complex representation and the same is true for the other fields, because $\mathcal{R}(\phi_{(-1)}) \otimes \mathcal{R}(\phi_{(1)}) \supset \mathbf{1}$ and $\mathcal{R}(\phi_{(3)}) \otimes \mathcal{R}(\phi_{(1)}) \supset \mathbf{1}$.

An example of a model with real representations of a non-abelian symmetry is this small modification of the original Shih model (7.1) where $\phi_{(-1)}, \phi_{(1)}, \phi_{(3)}$ are $SO(N)$ fundamentals:

$$W = fX + \lambda X \phi^\alpha_{(1)} \phi^\alpha_{(-1)} + m_1 \phi^\alpha_{(3)} \phi^\alpha_{(-1)} + \frac{1}{2} m_2 \phi^\alpha_{(1)} \phi^\alpha_{(1)} \tag{7.14}$$

By looking at the Coleman-Weinberg formula

$$V_{eff}^{(1-loop)} = \frac{1}{64\pi^2} \mathrm{Tr} \left(\mathcal{M}_B^4 \ln \frac{\mathcal{M}_B^2}{\Lambda^2} - \mathcal{M}_F^4 \ln \frac{\mathcal{M}_F^2}{\Lambda^2} \right) \tag{7.15}$$

it is easy to see that the effective potential is related to that of the original Shih model by[1] $V_{eff}^{(1-loop)}(X)_{SO(N)} = N V_{eff}^{(1-loop)}(X)$. Then the analysis in [52] goes unchanged (except for the height of the potential barrier for the metastable vacuum, which is not relevant) and the model shows spontaneous non-hierarchical R-symmetry breaking in a metastable vacuum for a wide range of parameters. The flavour symmetry is unbroken in the metastable vacuum.

[1]For a generic representation $\mathcal{R}$ of a group G, the only modification is $V_{eff}^{(1-loop)}(X)_{\mathcal{R}(G)} = \dim(\mathcal{R}(G)) V_{eff}^{(1-loop)}(X)$.

If we wish to introduce complex representations, we must consider models without $R = 0, 1$ fields. The simplest example is

$$\begin{aligned} W &= fX + XN_5\phi^{\alpha}_{(5)}\phi_{(-5)\alpha} + XN_3\phi^{\alpha}_{(3)}\phi_{(-3)\alpha} + \\ &\quad +M_7\phi^{\alpha}_{(7)}\phi_{(-5)\alpha} + M_5\phi^{\alpha}_{(5)}\phi_{(-3)\alpha} + M_3\phi^{\alpha}_{(3)}\phi_{(-1)\alpha} \end{aligned} \tag{7.16}$$

where $\phi_{(7)}, \phi_{(5)}, \phi_{(3)}$ are fields in the fundamental representation of a $U(N)$ flavour symmetry and $\phi_{(-5)}, \phi_{(-3)}, \phi_{(-1)}$ are in the antifundamental. Also in this case we have $V^{(1-loop)}_{eff}(X)_{U(N)} = NV^{(1-loop)}_{eff}(X)$, therefore all relevant properties can be found from the model without the flavour symmetry:

$$\begin{aligned} W &= fX + XN_5\phi_{(5)}\phi_{(-5)} + XN_3\phi_{(3)}\phi_{(-3)} + \\ &\quad +M_7\phi_{(7)}\phi_{(-5)} + M_5\phi_{(5)}\phi_{(-3)} + M_3\phi_{(3)}\phi_{(-1)} \end{aligned} \tag{7.17}$$

Now we have to study R-symmetry breaking in this model. All parameters can be chosen real and positive. The condition $|M^{-2}fN| \ll 1$ is generally sufficient to avoid tachyonic directions for small X, so we choose f/M_5^2 to be small.

Numerical minimization of the Coleman-Weinberg potential for the model (7.17) shows that there is spontaneous R-symmetry breaking in some region of the parameter space, in particular for $N_3 \sim N_5$ and $M_3, M_7 < M_5$, as can be seen in figure 7.1.1,7.1.1.

It is possible to show analytically that R-symmetry breaking occurs in this region. It is possible to expand the Coleman-Weinberg potential at lowest order in $|\hat{M}^{-2}f\hat{N}|$ and X and confirm the numerical results. The potential has the form $V(X) = V_0 + m_X^2|X|^2 + \lambda_X|X|^4 + O(|X|^6)$. In figure 7.3,7.4 we plot the expressions found for $m_X^2M_5^2/f^2, \lambda_XM_5^4/f^2$ as functions of M_3/M_5 in the case $M_3 = M_7, N_3 = N_5 = 1$ and $f/M_5^2 \ll 1$.

Note that the results for this model reduce to the Shih model if $N_3 = N_5$ and $M_3 = M_7$.

We have studied the simplest model with complex representations, but we can also consider models with more fields. The results coming from numerical minimization are the same: these models have metastable quantum vacua that break R-symmetry for some range of parameters.

In models with more pseudomoduli the range of parameters for spontaneous R-symmetry breaking becomes wider, because a linear combination of X and Y_a that acquires a negative m^2 is a sufficient condition for R-symmetry breaking. Numerical studies indicate that there are stable vacua that break R-symmetry in a large fraction of the parameter space for parameters $N_{ij}, M_{ij}/M$ of order $O(1)$ and small f/M [63]. Non-hierarchical spontaneous R-symmetry breaking seems therefore a common feature of these models: this opens interesting possibilities for realistic model building.

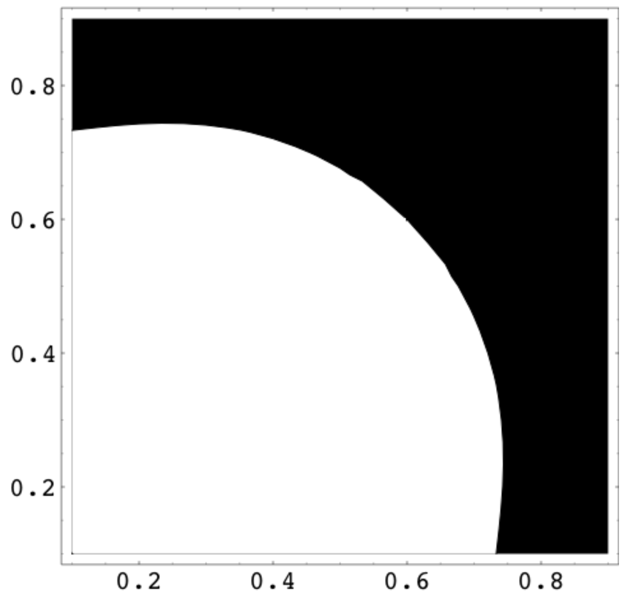

Figure 7.1: The white area is the region of the plane $(M_7/M_5, M_3/M_5)$ where there is spontaneous R-symmetry breaking for $N_3 = N_5 = 1$ and $f/M_5^2 = 0.001$.

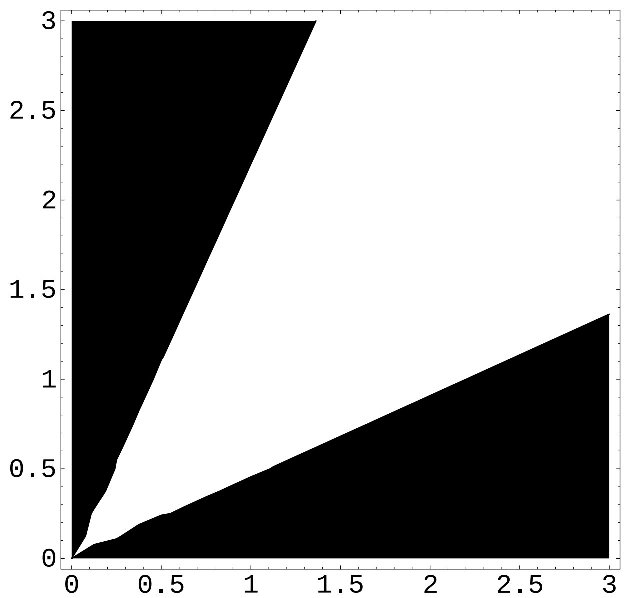

Figure 7.2: The white area is the region of the plane (N_5, N_3) where there is spontaneous R-symmetry breaking for $M_3/M_5 = M_7/M_5 = 0.25$ and $f/M_5^2 = 0.001$.

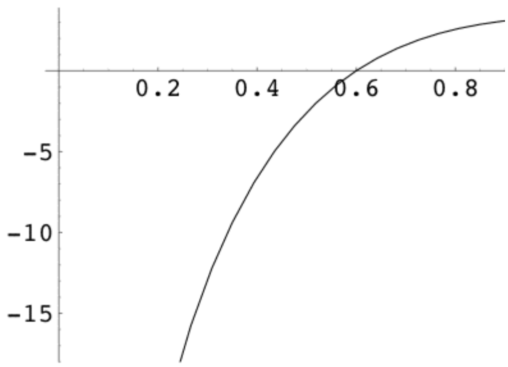

Figure 7.3: Plot of $m_X^2 M_5^2/f^2$ as a function of M_3/M_5.

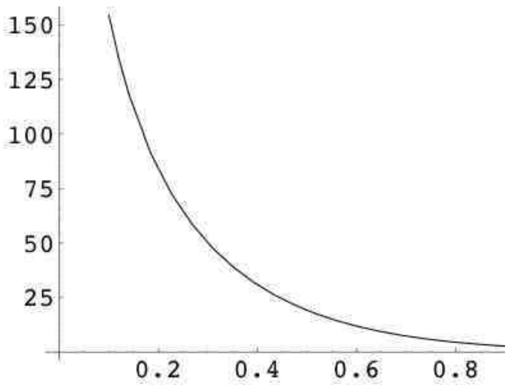

Figure 7.4: Plot of $\lambda_X M_5^4/f^2$ as a function of M_3/M_5.

7.2 Runaway directions

7.2.1 Models with a single pseudomodulus

The SUSY-breaking vacuum of the model (7.1) is metastable because of the existence of a runaway direction [52]:

$$\phi_{(1)} = -\frac{f}{\lambda\phi_{(-1)}}, \; X = \frac{m_2 f}{\lambda^2\phi_{(-1)}^2}, \; \phi_{(3)} = \frac{m_2 f^2}{m_1\lambda^2\phi_{(-1)}^3}, \; \phi_{(-1)} \to 0 \tag{7.18}$$

that is a direction along which the potential goes to a minimum. The runaway direction of the Shih model is "supersymmetric" because $V \to 0$ and the runaway vacuum at infinity has unbroken supersymmetry. It is interesting to note that the runaway direction can be seen as a rescaling of fields

$$\varphi(\epsilon) = \epsilon^{-R(\varphi)}\varphi(0) \quad , \quad \epsilon \to 0 \tag{7.19}$$

, which is a complexified R-symmetry transformation. This feature is not related to the particular model (7.1): in fact most of the Wess-Zumino models with an R-symmetry and generic R-charge assignment have runaway directions [57].

We begin the analysis of runaway directions from the simple class of models (7.2). The superpotential of these models has the form

$$W = fX + \frac{1}{2}(M^{ij} + N^{ij}X)\phi_i\phi_j \tag{7.20}$$

where M, N are generic symmetric complex matrices with $\det(M) \neq 0$. These models form the subset of the models (7.4) without the pseudomoduli Y_a.

According to general arguments, R-symmetry implies that this superpotential can break SUSY [49]. In fact, it is shown in [52] that SUSY is always broken in these models. Let's review the argument for SUSY breaking. The equations for a SUSY vacuum $\partial_a W = 0$ are

$$f + \frac{1}{2}N^{ij}\phi_i\phi_j = 0 \tag{7.21}$$

$$(M^{ij} + N^{ij}X)\phi_j = 0 \tag{7.22}$$

and cannot be solved simultaneously. To prove this it is sufficient to note that if $\det(M + NX) \neq 0$ the only solution for (7.22) is $\phi_i = 0$ that cannot satisfy (7.21). It can be shown that $\det(M+NX) = \det(M)\exp(\mathrm{Tr}\log(M^{-1}NX)) = \det(M)$ if R-symmetry is required, because the traces $\mathrm{Tr}\left((M^{-1}N)^k\right)$ disappear. SUSY is therefore broken in all models with $\det(M) \neq 0$. However, this argument only refers to finite values of the fields and does not exclude a supersymmetric runaway vacuum.

To obtain a SUSY runaway vacuum, we classify the equations (7.22) according to their R-charge:

$$(M^{ij} + N^{ij}X)\phi_j = 0 \quad , \quad R(\phi_i) < 2 \tag{7.23}$$
$$(M^{kj} + N^{kj}X)\phi_j = 0 \quad , \quad R(\phi_k) = 2 \tag{7.24}$$
$$(M^{mj} + N^{mj}X)\phi_j = 0 \quad , \quad R(\phi_m) > 2 \tag{7.25}$$

The equations (7.23),(7.24),(7.25) have positive, zero and negative R-charges respectively. As we have seen, there is no solution for the system of equations (7.21),(7.23),(7.24),(7.25). This can also be seen from the fact the equations (7.21),(7.24),(7.25) are not compatible, because (7.21) requires at least one field with non-positive R-charge to be nonzero, while equations (7.24),(7.25) force all fields with non-positive R-charge to zero. However there could be a field configuration X', ϕ'_i that solves the subsystem (7.21),(7.23),(7.24). If this is the case, the potential of these fields is

$$V = \sum_{R(\phi_m)>2} |(M^{mj} + N^{mj}X')\phi'_j|^2 \tag{7.26}$$

and it goes to zero along the direction parametrized by ϵ in (7.19):

$$\phi_i(\epsilon) = \epsilon^{-R(\phi_i)}\phi'_i \quad , \quad X(\epsilon) = \epsilon^{-2}X' \quad , \quad \epsilon \to 0 \tag{7.27}$$

This means that the theory cannot have a lower ground state, and there is a runaway direction parametrized by non-unitary R-symmetry transformations (7.27).

In the next section we prove that in this class of models it is always possible to solve (7.21),(7.23),(7.24) at the same time if there are fields with[2] $R \neq 0, 1, 2$. For the models (7.2) that satisfy this condition, this result implies that local minima of the potential always correspond to metastable vacua, and that the potential shows a runaway behavior. The properties of these models are therefore very different from usual O'Raifeartaigh models.

Many models in this class have metastable R-breaking vacua. In fact the presence of fields with $R \neq 0, 1, 2$ in these models corresponds both to the necessary condition for spontaneous R-symmetry breaking and to the sufficient condition for runaway behavior. An interesting consequence is that for this class of models, spontaneous R-symmetry breaking implies metastability.

[2] This is not completely correct, because R-charge is defined only up to addition of other $U(1)$ charges. So a more correct formulation is: we can always solve (7.21),(7.23),(7.24) at the same time if for every choice of R-charges there is at least a field with $R \neq 0, 1, 2$.

Proof of solvability of $R \geq 0$ equations

In this section we prove that it is always possible to solve the system of equations (7.21),(7.23),(7.24).

First of all, note that if there is a solution ϕ'_i, X' to (7.23),(7.24) that satisfies $N^{ij}\phi'_i\phi'_j \neq 0$, the equation (7.21) can be solved by rescaling all fields $\phi'_i \to \rho\phi'_i$ by a factor $\rho = (-f/N^{ij}\phi'_i\phi'_j)^{1/2}$. Therefore we only have to prove that (7.23),(7.24) can be solved with $N^{ij}\phi'_i\phi'_j \neq 0$.

The set of fields ϕ_i of a given model (7.2) can be decomposed into minimal subsets in such a way that two fields belonging to different subsets cannot appear in the same equation or in the same term of the superpotential[3]. Each field $\phi_{(r)}$ interacts with X and with fields $\phi_{(2-r)j}, \phi_{(-r)j}$ only and each equation has the form

$$N^{ij}_{(r,-r)}X\phi_{(r)j} + M^{ij}_{(2+r,-r)}\phi_{(2+r)j} = 0 \tag{7.28}$$

involving X and two fields whose R-charges differ by 2. Different subsets give different systems of equations with no fields in common, so we will work with fields belonging to a minimal subset only, and we will neglect all the fields belonging to other subsets.

Let's prove the theorem for the case in which R-charges can be chosen in such a way that no field has $R = 0$ or $R = 1$. (We can always redefine R-charges by adding charges of $U(1)$ global symmetries.) First of all, note that it is always possible to choose an R-charge assignment so that all fields have integer R-charge. In fact if R-charges are not integer it is sufficient to consider the highest one R_{max} and redefine them in the following way: $R(\varphi) \to \lceil R(\varphi) \rceil$ if $R(\varphi) - R_{max}$ is an even integer, $R(\varphi) \to \lfloor R(\varphi) \rfloor$ otherwise. A field with $R(\varphi) - R_{max}$ even is coupled only with fields with $R(\varphi) - R_{max}$ not even, therefore this defines a consistent R-charge assignment with only integer R-charges.

If there are no fields with $R = 0$ or $R = 1$, we have a set of fields of $2m$ different R-charges $\phi_{(k)j}$, $\phi_{(2+k)j} \ldots \phi_{(2m+k)j}$ and $\phi_{(2-k)j}$, $\phi_{(-k)j} \ldots \phi_{(2-2m-k)j}$ with integers k, m satisfying $k > 2, m > 1$. Every term in the superpotential couples fields with R-charges of opposite sign, therefore there is an accidental U(1) symmetry whose charge is $S(\phi_i) = \text{sign}(R(\phi_i))$. Using this symmetry, we redefine the R-symmetry to obtain $\phi^+_{(-1)j}$, $\phi^+_{(1)j} \ldots \phi^+_{(2m-1)j}$ and $\phi^-_{(3)j}$, $\phi^-_{(1)j}$

[3] For example, fields with even and odd R-charge belong to different subsets.

$\dots\phi^-_{(-2m+3)j}$ and the equations (7.23),(7.24) become as follow:

$$
\begin{aligned}
&N^{ij}_{(-2m+3,2m-3)}X\phi^+_{(2m-3)j} + M^{ij}_{(-2m+3,2m-1)}\phi^+_{(2m-1)j} = 0\\
&N^{ij}_{(-2m+5,2m-5)}X\phi^+_{(2m-5)j} + M^{ij}_{(-2m+5,2m-3)}\phi^+_{(2m-3)j} = 0\\
&\dots\\
&N^{ij}_{(1,-1)}X\phi^+_{(-1)j} + M^{ij}_{(1,1)}\phi^+_{(1)j} = 0\\
&N^{ji}_{(1,-1)}X\phi^-_{(1)j} + M^{ji}_{(3,-1)}\phi^-_{(3)j} = 0\\
&N^{ji}_{(-1,1)}X\phi^-_{(-1)j} + M^{ji}_{(1,1)}\phi^-_{(1)j} = 0
\end{aligned}
\tag{7.29}
$$

where $N^{ij}_{k,k'}$ couples $\phi^-_{(k)i}$ and $\phi^+_{(k')j}$ and the same happens for $M^{ij}_{k,k'}$.

We have two systems of equations containing ϕ^+ and ϕ^- fields respectively. For each fixed value of $X, \phi^-_{(-1)j}, \phi^+_{(-1)j}$ we have two linear systems of n^+, n^- equations in n^+, n^- variables, which can always be solved provided that the related linear operators have nonzero determinants. This condition is verified because these determinants are products of $\det(M_{(2-k,k)})$ and these cannot be zero because $\det(M) = \prod_k \det(M_{(2-k,k)}) \neq 0$. If we choose $\phi^-_{(-1)j}, \phi^+_{(-1)j}$ to be different from zero[4] , then also $\phi^-_{(1)j}, \phi^+_{(1)j}$ are nonzero and generically $N^{ij}\phi_i\phi_j \neq 0$. This completes the proof of this case.

Now we will prove the theorem for the case with $\phi_{(1)}$. The equations (7.23),(7.24) become:

$$
\begin{aligned}
&N^{ij}_{(2m-3,-2m+3)}X\phi_{(2m-3)j} + M^{ij}_{(2m-1,-2m+3)}\phi_{(2m-1)j} = 0\\
&N^{ij}_{(2m-5,-2m+5)}X\phi_{(2m-5)j} + M^{ij}_{(2m-3,-2m+5)}\phi_{(2m-3)j} = 0\\
&\dots\\
&N^{ij}_{(-1,1)}X\phi_{(-1)j} + M^{ij}_{(1,1)}\phi_{(1)j} = 0
\end{aligned}
\tag{7.30}
$$

and, applying the same argument we used above, choosing $\phi_{(-1)j} \neq 0$ is a sufficient condition. The case with $\phi_{(0)}$ is very similar, with equations:

$$
\begin{aligned}
&N^{ij}_{(2m-2,-2m+2)}X\phi_{(2m-2)j} + M^{ij}_{(2m,-2m+2)}\phi_{(2m)j} = 0\\
&N^{ij}_{(2m-4,-2m+4)}X\phi_{(2m-4)j} + M^{ij}_{(2m-2,-2m+4)}\phi_{(2m-2)j} = 0\\
&\dots\\
&N^{ij}_{(0,0)}X\phi_{(0)j} + M^{ij}_{(2,0)}\phi_{(2)j} = 0\\
&N^{ij}_{(0,-2)}X\phi_{(-2)j} + M^{ij}_{(2,0)}\phi_{(0)j} = 0
\end{aligned}
\tag{7.31}
$$

[4]The requirements here and in the other cases should be stated more precisely. For example, these fields have to be chosen such that they do not belong to the kernel of the matrices $N_{(-1,1)}, N_{(1,-1)}$ respectively. However similar conditions are easily satisfied for generic nonzero fields.

and choosing $\phi_{(-2)j} \neq 0$ is enough.

To complete the proof, we must discuss what happens when there are abelian or non-abelian symmetries that constrain the form of M, N. The only difference is that now the equations are classified not only by their R-charge, but also by other charges. However this has no effect on the above arguments, provided that we consider systems of equations of the same charge[5]. This completes the proof.

7.2.2 Models with more pseudomoduli

To understand what can happen in more general models, we add to the previous models a set of fields Y_a with $R(Y_a) = 2$, canonical Kähler potential and superpotential

$$W = fX + \frac{1}{2}(M^{ij} + N^{ij}X + Q_a^{ij}Y_a)\phi_i\phi_j \tag{7.32}$$

where Q_a are generic symmetric complex matrices with

$$Q_a^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\phi_j) = 0 \tag{7.33}$$

Similarly to the previous case, these models break SUSY. The proof is identical to the previous one if we substitute NX with $NX + Q_aY_a$, because it depends only on the properties (7.3),(7.33).

The analysis of runaway directions is different from the case with a single pseudomodulus. To see the difference, we analyze some simple examples[6]:

- This is a simple modification of the Shih model (7.1) with a Y field:

  $$W = fX + (\lambda X + \eta Y)\phi_{(1)}\phi_{(-1)} + m_1\phi_{(3)}\phi_{(-1)} + \frac{1}{2}m_2\phi_{(1)}^2 \tag{7.34}$$

  Classically this model has flat directions of SUSY-breaking vacua with $\phi_{(3)} = \phi_{(1)} = \phi_{(-1)} = 0$ for some range of parameters. These flat directions are parametrized by X, Y and are lifted by quantum effects. As in the original model, the quantum vacuum can break the R-symmetry, depending on the choice of parameters.

  Here the equations $\partial_X W = 0$, $\partial_Y W = 0$ have $R = 0$ but cannot be solved at the same time. This means that there are no SUSY runaway

[5] From another point of view, two fields whose charges are not equal or complex conjugate belong to different minimal subsets.

[6] Note that throughout this chapter the indices in parentheses correspond to the R-charges of the fields.

vacua. However there is a runaway direction

$$\phi_{(1)} = -\frac{f}{\lambda'\phi_{(-1)}}, \ X + \frac{\eta}{\lambda}Y = \frac{m_2 f}{\lambda'^2\phi_{(-1)}^2}, \ \phi_{(3)} = \frac{m_2 f^2}{m_1\lambda'^2\phi_{(-1)}^3}, \ \phi_{(-1)} \to 0 \tag{7.35}$$

with $\lambda' = (|\lambda|^2 + |\eta|^2)/\bar{\lambda}$. This non-SUSY runaway vacuum minimizes the potential and the other vacua are therefore metastable.

- This simple model has a $U(1)$ symmetry $\phi^{\pm}_{(k)} \to e^{\pm i\theta}\phi^{\pm}_{(k)}$ and shows a different behavior:

$$W = fX + (\lambda_+ X + \eta_+ Y)\phi^+_{(1)}\phi^-_{(-1)} + (\lambda_- X + \eta_- Y)\phi^+_{(-1)}\phi^-_{(1)} + \\ + m_3\phi^+_{(3)}\phi^-_{(-1)} + m_1\phi^+_{(1)}\phi^-_{(1)} + m_{-1}\phi^+_{(-1)}\phi^-_{(3)} \tag{7.36}$$

Here we can solve all the equations with $R > 0$ in terms of $\phi^+_{(-1)}$,$\phi^-_{(-1)}$,X,Y as in the models of section 7.2.1, obtaining $\phi^{\pm}_{(1)} = -(\lambda_{\mp}X + \eta_{\mp}Y)\phi^{\pm}_{(-1)}/m_1$. The equations with $R = 0$ become

$$fm_1 - [2\lambda_+\lambda_- X + (\lambda_+\eta_- + \lambda_-\eta_+)Y]\,\phi^+_{(-1)}\phi^-_{(-1)} = 0 \tag{7.37}$$

$$[2\eta_+\eta_- Y + (\lambda_+\eta_- + \lambda_-\eta_+)X]\,\phi^+_{(-1)}\phi^-_{(-1)} = 0 \tag{7.38}$$

and can be easily solved with $\phi^+_{(-1)}\phi^-_{(-1)} \neq 0$. Then there is a SUSY runaway vacuum that corresponds to a field rescaling $\phi^+_{(-1)}, \phi^-_{(-1)} \to 0$.

Let's analyze the general case. The equations for a SUSY vacuum are:

$$f + \frac{1}{2}N^{ij}\phi_i\phi_j = 0 \tag{7.39}$$

$$\frac{1}{2}Q^{ij}_a\phi_i\phi_j = 0 \tag{7.40}$$

$$(M^{ij} + N^{ij}X + Q^{ij}_a Y_a)\phi_j = 0 \quad , \quad R(\phi_i) < 2 \tag{7.41}$$

$$(M^{kj} + N^{kj}X + Q^{kj}_a Y_a)\phi_j = 0 \quad , \quad R(\phi_k) = 2 \tag{7.42}$$

$$(M^{mj} + N^{mj}X + Q^{mj}_a Y_a)\phi_j = 0 \quad , \quad R(\phi_m) > 2 \tag{7.43}$$

As in the case with a single pseudomodulus, the equations (7.39),(7.42),(7.43) are not compatible. Then there are three cases:

(a) If we can solve all the equations with non-negative R-charge (7.39),(7.40),(7.41),(7.42) at the same time, we can then rescale the solution as in (7.19) and obtain a runaway direction. The runaway vacuum is supersymmetric, and therefore all other vacua, if any, are metastable.

This is what happens in model (7.36). This case often happens for small n_Y.

(b) If it is not possible to solve the equations (7.39),(7.40),(7.41),(7.42) for any choice of R-charges, we look for absolute minima φ_a^{min} of the potential $V_{min}(\varphi) = \min(V_+(\varphi), V_-(\varphi))$ with respect to all fields and all choices of R-symmetries, where V_+ and V_- are

$$V_+ = \sum_{R(\varphi_a)\leq 2} |\partial_{\varphi_a} W|^2 \quad , \quad V_- = \sum_{R(\varphi_a)\geq 2} |\partial_{\varphi_a} W|^2 \tag{7.44}$$

Now there are two possibilities:

(b1) If there are φ_a^{min} that solve both (7.41) and (7.43), these are the true vacua of the model, with a flat direction parametrized by R-charge rescalings.

This is what happens in original O'Raifeartaigh model and in all models with R=0,2.

(b2) Suppose that the absolute minimum is at $V_+(\varphi_a^{min})$. If there are no field configurations φ_a^{min} that solve (7.41),(7.43) but there is a φ_a^{min} that only solves (7.41), we can then rescale this solution as in (7.19) and obtain a runaway direction. The runaway vacuum is not supersymmetric but it corresponds to the true vacuum of the system and therefore all other vacua, if any, are metastable. The same if we exchange (7.41) with (7.43) and V_+ with V_-.

This is what happens in model (7.34). This case often happens for large n_Y.

(c) The last possibility is that absolute minima of V_{min} do not solve (7.41) nor (7.43). In this case there are no general results, but there can be non-SUSY stable vacua or runaway directions, depending on the details of the models.

A model can belong to one or another of the above cases, depending on its parameters and field content.

It is possible to find sufficient conditions for the existence of runaway directions that consider only the field content of the model. If Y_a have no flavor charges then, roughly speaking, there are runaway directions if $n_Y \gtrsim n_\phi/2$ and there are SUSY runaway vacua if $n_Y \lesssim n_\phi/2$. There is a (small) window of models without runaway vacua, but these conditions imply that most of these models have runaway directions. The precise conditions and their proofs can be found in the next section.

Conditions for runaway

We discuss some conditions for the existence of runaway directions. We consider only the case of minimal subsets, generic couplings and no symmetries. We denote the number of ϕ fields with n_ϕ, the number of Y fields with n_Y and the number of ϕ fields of R-charge j with $n_{(j)}$ (or $n^{\pm}_{(j)}$ for $\phi^{\pm}$).

If it is possible to solve all the equations with $R > 0$ for a generic choice of the fields ϕ_i that appear in $V_{R=0}$, then it is always possible to minimize V_+ (or V_-). If the minimum is zero, there is a SUSY runaway direction, otherwise there is a non-SUSY runaway vacuum.

In models with no fields with $R = 0, 1$, this is possible if $n_Y \geq \frac{n_\phi}{2} + n^-_{(1)} - n^+_{(2m-1)} - 1$. To prove this, we consider the R-charge choice of appendix 7.2.1. We can see that the equations with $R > 0$ are $\frac{n_\phi}{2} + n^-_{(1)}$ generic linear equations in $n_Y + 1 + n^+_{(2m-1)}$ variables and they can be solved if the above condition is satisfied.

In models with a field with $R = 1$ it is possible to repeat the above argument and obtain the condition $n_Y \geq \frac{n_\phi}{2} + \frac{n_{(1)}}{2} - n_{(2m-1)} - 1$.

In models with a field with $R = 0$ the argument is slightly different, because in this case we need to solve also equations with $R = 0$ that contain X, Y_a. Considering also these equations, we obtain the condition $n_Y \geq \frac{n_\phi}{2} + n_{(0)} - n_{(2m)} - 1$.

The above conditions imply SUSY or (generally) non-SUSY runaway vacua. To obtain conditions that imply SUSY runaway vacua, we need to solve all the equations with $R \geq 0$. Consider the case with no fields with $R = 0, 1$. Solving all the equations with $R > 0$, we end with a set of $n_Y + 1$ equations with $R = 0$. The first n_Y are of the form $\sum_{k<3} \phi^+_{(-1)} P^a_{(k)} \phi^-_{(k)} = 0$ where $P^a_{(k)}$ are generic matrices that have a polynomial dependence on X, Y_a. These equation have a nonzero solution (choosing a generic nonzero $\phi^+_{(-1)}$) if $\frac{n_\phi}{2} - n^-_{(3)} \geq n_Y + 1$, so the condition is $n_Y \leq \frac{n_\phi}{2} - n^-_{(3)} - 1$. The remaining equation has the form $\sum_{k<3} \phi^+_{(-1)} P_{(k)} \phi^-_{(k)} = -f$ and can be solved by rescaling all ϕs.

Similar conditions can be found for the other cases. If there are fields with $R = 1$ the condition is $n_Y \leq \frac{n_\phi}{2} - \frac{n_{(1)}}{2} - n_{(-1)} - 1$, while if there are fields with $R = 0$ the condition is $n_Y \leq \frac{n_\phi}{2} - n_{(0)} - 1$.

7.2.3 General models

The interesting result of the previous sections is that many O'Raifeartaigh models have a runaway behavior. In this section we argue that this behavior is quite common in O'Raifeartaigh models with general R-charge assignments.

We briefly review the usual O'Raifeartaigh models in our approach. (For more details about these models, see also the lectures [64].) The superpotential is

$$W = \sum_n X_n g_n(\phi_i) \quad (n = 1 \dots n_X, i = 1 \dots n_0) \tag{7.45}$$

where $R(X_n) = 2$, $R(\phi_i) = 0$. These models break SUSY because the conditions $g_n(\phi_i) = 0$ are generally not compatible if $n_X > n_0$. The fields ϕ_i are determined by minimization of $V = \sum_n |g_n(\phi_i)|^2$; this means that the equations $\sum_n X_n \partial_j g_n = 0$ have at least a nonzero solution $X_n = \bar{g}_n(\bar{\phi}_i)$. Rescaling this solution with respect to the R-charges, we obtain a flat direction of minima[7].

When there are fields with $R \neq 0, 1, 2$ the picture changes completely. In fact most of the Wess-Zumino models with an R-symmetry and generic R-charge assignment have runaway directions [57]. We can understand this if we note that the vacuum equations (and the F-terms) can be classified by their R-charge:

$$\partial_i W = 0,\ R(\varphi_i) < 2 \qquad R > 0 \tag{7.46}$$

$$\partial_i W = 0,\ R(\varphi_i) = 2 \qquad R = 0 \tag{7.47}$$

$$\partial_i W = 0,\ R(\varphi_i) > 2 \qquad R < 0 \tag{7.48}$$

Because of the Nelson-Seiberg argument, it is not possible to solve all these equations at the same time. However, it can be possible to solve a subset of these equations. We can look at two common possibilities:

- In some cases it is possible to solve all the equations with $R \geq 0$ (or $R \leq 0$). In this case it is sufficient to rescale all fields by a factor $\epsilon^{-R(\varphi)}$ (or $\epsilon^{R(\varphi)}$) as in (7.19). Then we send $\epsilon \to 0$ to solve also the equations with $R < 0$ (or $R > 0$) and obtain a supersymmetric runaway vacuum: $V \to 0$ as $\epsilon \to 0$. This is the case of model (7.1). This case often occurs when there are a few equations with $R = 0$.

  If we consider a generic (possibly non-renormalizable) superpotential, the number of equations with $R \geq 0$ is usually smaller than the number of fields on which these equations depend, so they can be often solved. This means that runaway directions are common in these models, and that SUSY-breaking vacua of these models are generally metastable. We have seen in section 7.2.1 an interesting class of models that show this behavior.

[7] Actually there is a $(n_X - n_0)$-dimensional space of solutions. R-symmetry rescaling acts as a dilatation in this space.

It can also happen that only equations with $R \leq 0$ can be solved. This is not common in the models studied in the previous sections, but can happen in general models. An example that appeared early in the literature is the runaway model of [65], that will be discussed in the examples.

- In other cases it is not possible to solve the equations with $R = 0$. This situation is common when there are many equations with $R = 0$. In these cases it is often possible to minimize $V_{R=0} = \sum_{R(\varphi_j)=2} |\partial_j W|^2$ with respect to all the fields and solve the equations with $R > 0$ (or $R < 0$) at the same time. Then the rescaling (7.19) by a factor $\epsilon^{-R(\varphi)}$ (or $\epsilon^{R(\varphi)}$) parametrizes a runaway direction with $V \to V_\infty > 0$ that corresponds to a non-supersymmetric runaway vacuum. We have seen examples of this behaviour in section 7.2.2.

Most of the models with generic R-charges realize one of these two possibilities. Other models can have stable SUSY-breaking vacua or flat directions, as the usual O'Raifeartaigh models.

It is interesting that a relation often exists between R-symmetry breaking and metastability. In [51] it is argued that metastability is a general feature of realistic models of SUSY breaking. In fact R-symmetry must be a good symmetry for the theory to break SUSY, but a small explicit R-symmetry breaking interaction is needed to give mass to the R-axion; this explicit breaking generically restores supersymmetry in vacua far away from the origin of field space. Near the origin, R-symmetry is an approximate symmetry and SUSY is spontaneously broken in a metastable vacuum. It is not clear if metastability in the models of [51] and in our models are related. Some hints in this direction are discussed in the next section, where it is shown that runaway directions are often remnants of supersymmetric vacua generated by (small) explicit R-breaking terms in the superpotential.

7.2.4 Runaway vacua as remnants of SUSY vacua

The existence of an R-symmetry is a sufficient condition for SUSY breaking in the models discussed in sections 7.2.1,7.2.2. More generally, R-symmetry is a necessary condition for SUSY breaking under some hypothesis of genericity of the superpotential.

Consider a superpotential $W(\varphi_a)$ that has an R-symmetry and breaks SUSY spontaneously, and additional terms $W^r_{\not R}(\varphi_a)$ that does not have R-charge 2. An immediate consequence of the statements above is that the theory defined by

$$W_\nu = W + W^{\not R} = W + \sum_r \nu_r W_r^{\not R} \qquad \nu = (\nu_1, \nu_2 \ldots) \tag{7.49}$$

generally has supersymmetric vacua $\langle \varphi_a \rangle = \tilde{\varphi}_a(\nu)$ that satisfy

$$\partial_b W_\nu(\tilde{\varphi}_a(\nu)) = \partial_b W(\tilde{\varphi}_a(\nu)) + \sum_r \nu_r \partial_b W_r^{\not R}(\tilde{\varphi}_a(\nu)) = 0 \tag{7.50}$$

so the SUSY-breaking vacua that survive for $\nu_r \ll 1$ are metastable. However, in the limit $\nu_r \to 0$ the SUSY vacua are pushed to infinity [51].

The potential of the original $\nu_r = 0$ theory along the direction of the SUSY vacua is

$$V(\tilde{\varphi}_a(\nu)) = \sum_b |\partial_b W(\tilde{\varphi}_a(\nu))|^2 = \sum_b \left| \sum_r \nu_r \partial_b W_r^{\not R}(\tilde{\varphi}_a(\nu)) \right|^2 \tag{7.51}$$

Usually this potential doesn't vanish for $\nu_r \to 0$ because the contribution of $\partial_b W_r^{\not R}(\tilde{\varphi}_a(\nu))$ can grow as $1/\nu_r$ or faster, so the theory with $\nu_r = 0$ has no memory of SUSY vacua when they are pushed to infinity.

However there is an interesting exception. If the condition

$$\text{sign}(R(\nu_r)) = \text{sign}(R(\nu_{r'})) = \text{sign}(2 - R(\varphi_b)) \equiv \sigma \quad \forall r, r' \text{ and } \forall \varphi_b \in W^{\not R} \tag{7.52}$$

is satisfied, then the limit $\nu_r \to 0$ can be interpreted as a rescaling with respect to the R-charges $\varphi_a \to \epsilon^{\sigma R(\varphi_a)} \varphi_a, \nu_r \to \epsilon^{\sigma R(\nu_r)} \nu_r$ with $\epsilon \to 0$. In this case metastability of the R-symmetric superpotential can be easily explained, because the runaway vacuum is exactly the SUSY vacuum pushed to infinity as $\nu_r \to 0$, and the runaway direction can be found following the positions of SUSY vacua $\tilde{\varphi}_a(\nu)$ for $\nu_r \neq 0$. In fact we can parametrize these vacua as $\tilde{\varphi}_a(\nu(\epsilon))$ where $\nu_r(\epsilon) = \epsilon^{\sigma R(\nu_r)} \nu_r(0)$ and the potential along the direction parametrized by ϵ is

$$V(\tilde{\varphi}_a(\nu)) \sim \sum_{\varphi_b \in W^{\not R}} |\epsilon|^{2\sigma(2 - R(\varphi_b))} \tag{7.53}$$

whose minimum corresponds to $\epsilon \to 0$ and $|\tilde{\varphi}_a(\nu(\epsilon))| \to \infty$. The above argument means that the metastability of vacua near the origin for $\nu_r = 0$ is a remnant of their metastability for $\nu_r \neq 0$.

For the models with a single pseudomodulus there is a simple R-breaking perturbation that explains the metastability of vacua with $\phi = 0$:

$$W^{\not R} = \sum_{R(\phi_j) > 2} \nu_j \phi_j \tag{7.54}$$

This perturbation satisfies the above conditions (7.52) and in fact it generates a SUSY vacuum with $|\phi| \sim 1/\nu^k$ that becomes a runaway vacuum when $\nu \to 0$. Similar perturbations explain also the metastability of many vacua in models with more pseudomoduli.

7.3 Examples and applications

Affleck-Dine-Seiberg superpotential

The most famous example of runaway is the effective superpotential for the meson field in SQCD with N_f massless quarks, $N_f < N_c$. The superpotential for $M_{ij} = Q_i\tilde{Q}_j$ is constrained by symmetries and holomorphy to have the form [56, 43] :

$$W = (N_c - N_F)\left(\frac{\Lambda^{3N_c-N_F}}{\det(M)}\right)^{\frac{1}{N_c-N_f}} \tag{7.55}$$

It is possible to define an R-symmetry $R(M) = -2(N_c - N_f)/N_f$ and the fact that all the vacuum equations have $R > 0$ shows clearly the existence of runaway directions, given by the complexified R-symmetry transformation $M_{ij}(\alpha) = e^{2(N_c-N_f)\alpha}M_{ij}(0)$, $\alpha \to \infty$.

Witten runaway model

This is a simple model where only equations with $R \leq 0$ can be solved. It appeared early in the literature as a runaway model [65]. Its superpotential is

$$W = fX + \alpha X^2\phi \tag{7.56}$$

with $R(X) = 2$, $R(\phi) = -2$. In this model there are no equations with $R < 0$, so if we solve the $R = 0$ equation $f + 2\alpha X\phi = 0$ and then rescale the fields as $\phi \to \epsilon^{-2}\phi, X \to \epsilon^2 X$ we find a runaway direction with $V(X,\phi) \to 0$ as $\epsilon \to 0$. This runaway vacuum is the only vacuum of this model.

The model of Essig, Sinha and Torroba

This interesting model [66] is an extension of the Intriligator-Seiberg-Shih model, obtained by coupling two SQCD theories through singlets. One SQCD is in the free magnetic phase $N_c < N_f < \frac{3}{2}N_c$ and the other is in the ADS phase $N'_f < N'_c$. The effective potential for this model has the form (neglecting the coupling constants)

$$W = \mathrm{Tr} qM\tilde{q} + \Phi\mathrm{Tr}M + \Phi\mathrm{Tr}P\bar{P} + \left(\det P\bar{P}\right)^{-\frac{1}{N'_c-N'_f}} \tag{7.57}$$

where Φ is a singlet and $P, \bar{P}$ are fundamentals and antifundamentals of $SU(N_c') \times SU(N_f')$.

This superpotential admits an R-symmetry with R-charges $R(P\bar{P}) = R(M) = -2(N_c' - N_f')/N_f'$, $R(\Phi) = R(q\tilde{q}) = 2N_c'/N_f'$. Supersymmetry is broken because of this R-symmetry (the absence of a linear term in the superpotential is compensated by the presence of a meromorphic term that forbids the trivial solution with all fields of zero vev). Choosing $R(q) > 2N_c'/N_f'$, we can solve all the vacuum equations with $R > 0$ or all the vacuum equations with $R > 0$, so we find more than one runaway direction and the runaway vacua are supersymmetric.

The interesting feature of the model is the fact that 1-loop effects can stabilize these runaway directions near to the origin (pseudo-runaway) because of the relative flatness of the runaway direction with respect to the Coleman-Weinber potential, obtaining metastable vacua with spontaneously broken R-symmetry.

The model of Abel, Durnford, Jaeckel and Khoze

This model [67] is an example of another interesting extension of the Intriligator-Seiberg-Shih model that breaks R-symmetry spontaneously on a runaway direction, but the pseudo-runaway vacuum appears because of a different mechanism.

The model is the usual SQCD in the free magnetic phase but with a baryonic operator in the superpotential

$$W = m_{ij} Q_i \tilde{Q}_j + \Lambda_{cutoff}^{3-N_f} Q^{N_f} \tag{7.58}$$

For the case $N_c = 5, N_f = 7$ the flavour symmetry is $SU(5)_F \times SU(2)_F$ and the dual operator appearing in the low-energy superpotential is relevant:

$$W = \tilde{q}_{i\alpha} M_{ij} q_j^\alpha + \mu^2 M_{ii} + \tilde{m} \epsilon^{rs} \epsilon^{\alpha\beta} q_{r\alpha} q_{s\beta} \tag{7.59}$$

where q decomposes into a multiplet of $SU(5)_F$ and a doublet charged under $SU(2)_F$. The low-energy coupling is $\tilde{m} = (\Lambda_m/\Lambda_{cutoff})^4$.

Consider the interesting case of a mass matrix $\mu^2 = \begin{pmatrix} \mu_5^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix}$ with $\mu_2 > \mu_5$. This superpotential admits an R-symmetry[8] and we can choose the R-charges as $R(q) = -R(\tilde{q}) = 1, R(M) = 2$. Here we cannot choose $R(q) = 0$

[8] The absence of an antibaryon operator in the superpotential is crucial to have this nontrivial R-symmetry. An antibaryon would break explicitly the R-symmetry of this model and modify completely the vacuum structure of the model.

because of the baryon term and therefore we expect to find a runaway direction. Note that the vacuum equations with $R = 0$ are the same as in the ISS model, therefore SUSY is spontaneously broken and there are no supersymmetric vacua (and also no supersymmetric runaway vacua). However the usual ISS vacuum does not exists, because the potential coming from the $R = 0$ equations is minimized by taking the vev of the doublet component and this is not compatible with the equations $M_{rj}q_j^\alpha + \tilde{m}\epsilon^{rs}\epsilon^{\alpha\beta}q_{s\alpha} = 0, \tilde{q}_{j\alpha}M_{jr} = 0$.

In the absence of the ISS vacuum, we look for runaway directions in this model. The vacuum equations have $R = 0, 1, 3$ and it is easy to solve the $R = 1, 3$ equations at infinity by rescaling the fields with complexified R-symmetry transformations. In fact we can find a runaway direction if we take the vevs that minimize the $R = 0$ equations and we rescale them as $\varphi(\epsilon) = \epsilon^{-R(\varphi)}\varphi(0), \epsilon \to 0$. This direction goes to a non-supersymmetric minimum at infinity, therefore SUSY is broken along this direction. Away from the origin, this runaway direction resembles a flat direction and the Coleman-Weinberg potential lifts it, leaving only a vacuum along this direction that breaks R-symmetry spontaneously.

Extra-ordinary gauge mediation

Cheung, Fitzpatrick and Shih studied gauge mediation considering the most general messenger sector with an R-symmetry [68]:

$$W = \lambda^{ij} X \phi_i \tilde{\phi}_j + m^{ij} \phi_i \tilde{\phi}_j \tag{7.60}$$

Conditions: $R(W) = R(X) = 2, \quad m^{ij}, \lambda^{ij}$ complex matrices with $m^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\tilde{\phi}_j) = 2$, $\lambda^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\tilde{\phi}_j) = 0$ This model can be completed with a linear term $\Delta W = FX$ to a model of direct (gauge) mediation. In this way it resembles our model (7.16). It can be easily used for direct mediation taking the fields in the $\mathbf{5}\bar{\mathbf{5}}$ of $SU(5) \supset SU(3) \times SU(2) \times U(1)$. This case is interesting because the coupling constants of the triplet and the doublet can be different and we have doublet-triplet splitting, leading to interesting phenomenology.

The phenomenology of the model is related to this splitting and to the R-symmetry of the superpotential[9]. It includes peculiar features such as gauge coupling unification even with doublet-triplet splitting, gaugino mass relations $\frac{\tilde{m}_1}{\alpha_1} = \frac{\tilde{m}_2}{\alpha_2} = \frac{\tilde{m}_3}{\alpha_3}$, modified sfermion masses (peculiar squark/slepton mass ratios), effective messenger number $N_{eff} < 1$, small μ and Higgsino NLSP in some part of the parameter space.

[9] It should be noted that the absence of other pseudomoduli Y_a with $R = 2$ is crucial for some of these results.

Part III

Non-abelian vortices in $\mathcal{N} = 2$ theories

Chapter 8

Introduction

Vortices play a relevant role in many areas of physics, from condensed matter and phase transitions to cosmology. In theoretical physics, vortices are interesting objects that can arise as solitons in quantum field theories and particularly in gauge theories.

Abrikosov-Nielsen-Olesen vortex solitons are well-known to exist in $U(1)$ gauge theories in the Higgs phase. They are an important ingredient in the usual picture of confinement of magnetic charges in abelian superconductors, while their role in the confining phase of gauge theories with an unbroken non-abelian gauge group is still not clear.

In the basic mechanism proposed by 't Hooft and Mandelstam, confinement of electric charges at strong coupling can be understood in terms of confinement of magnetic charges in the Higgs phase of a dual theory. The chromoelectric string between a quark-antiquark pair should correspond to a magnetic vortex on the dual side. An important example of this mechanism is at work in $\mathcal{N} = 2$ SYM, where confinement is driven by condensation of dual quarks in the Seiberg-Witten dual theory. However in this theory dynamical abelianization takes place and the flux of BPS vortices of the dual theory is essentially abelian.

Non-abelian vortices have been introduced some years ago in the Higgs phase of $\mathcal{N} = 2$ theories with gauge group $U(N)$ [69] and $SU(N+1)$ [70]. Confined monopoles in this phase exist as string junctions between vortices of different orientation and can be seen as confined kinks in the worldsheet theory, explaining the correspondence between BPS spectra in 2d and 4d theories [71, 72].

Here we are interested in a different (although similar) system. We consider $\mathcal{N} = 2$ theories softly broken to $\mathcal{N} = 1$. In these theories it is possible to have a hierarchical symmetry breaking

$$G \xrightarrow{\langle\phi\rangle} H \xrightarrow{\langle q\rangle} 1 \qquad \langle q\rangle \ll \langle\phi\rangle \tag{8.1}$$

In this system the heavy regular monopoles of mass $\langle\phi\rangle/g^2$ that live in the theory above $\langle q\rangle$ are not topologically stable in the full theory, because of the complete breaking of the gauge group at $\langle q\rangle$. The system is equivalent to a non-abelian superconductor and the flux of these monopoles is confined in flux tubes that are vortex solutions arising at low energy from the complete gauge symmetry breaking. Both vortices and monopoles carry a non-abelian magnetic flux. This line of thought reveals a correspondence between monopoles in the high-energy theory and vortices in the low-energy theory, that will be discussed in these chapters.

In this chapter we briefly recall the physics of Abrikosov-Nielsen-Olesen vortices and confinement of magnetic charges in superconductors. Then we discuss regular Goddard-Nuyts-Olive monopoles, that are the generalization of the usual $SU(2)$ 't Hooft-Polyakov monopole to other gauge groups, and we present a picture of topological solitons in $\mathcal{N}=2$ gauge theories and their properties.

In the next chapter we present non-abelian vortices. In section 9.1 we discuss non-abelian vortex solutions in the simplest case of $SU(N)\times U(1)$ theories. In section 9.2 we study non-abelian vortex solutions in $SO(N)\times U(1)$ theories. These are the simplest example of vortices in theories with gauge group different from $SU(N)$. Then in section 9.3 we discuss the relation between non-abelian vortices and heavy monopoles and its implication for the moduli space of non-abelian vortices. We also show non-trivial examples of this correspondence. In appendices A and B the reader interested in technical aspects can find the details contained in the papers [73] and [74].

8.1 Abrikosov-Nielsen-Olesen vortices and superconductivity

The simplest system where vortex solutions with finite tension exist is the abelian Higgs model described by the Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{4g_1^2}F^{0\mu\nu}F^0_{\mu\nu} + |\mathcal{D}_\mu q|^2 - \frac{g_1^2}{2}\left|q^\dagger q - 2\xi\right|^2 \;. \tag{8.2}$$

This is simply a $U(1)$ gauge theory coupled to a complex scalar field of charge 1. The potential for this field has a minimum away from the origin, triggering gauge symmetry breaking. The photon is massive and electric and magnetic field cannot enter this system more than a length of order $1/\langle q\rangle$. This lagrangian can be thought as an effective model of the second order transition to low-temperature superconductivity.

We discuss the physics of this system. When electric charges are inserted into a superconductor like the one in (8.2), their electric field is screened by the condensate $\langle q\rangle$, while magnetic charges cannot be screened in this way and their magnetic flux is conserved. This flux inside the superconductor is in an unstable configuration of higher energy. To pass to a configuration of lower energy, the flux can be expelled from the superconductor or shrinked into a flux tube: this is the Meissner effect. In the latter case, there is a thin cylindrical region of space where the background condensate goes to zero and the magnetic field can penetrate the superconductor while being confined there. These flux tubes can be seen as vortices in the above model [75, 76].

Vortex solutions are time-independent and z-independent solutions that can be obtained by minimizing the tension $T = E/L_z$ where L_z is the lenght of the vortex. They carry a nonzero magnetic flux concentrated in the center of the solution where the scalar field vanishes. These solutions are not always stable: depending on the ratio of the gauge and the quartic coupling of the model, vortices can attract or repel each other, and a vortex solution of high flux can break up in vortices with lower flux. The coupling constants of the model (8.2) are fine tuned in a regime where solutions are stable for any choice of winding and relative position: this will be explained in the next section as a consequence of supersymmetry. With this choice, we can write the tension as a sum of squares plus a boundary term. This form is called Bogomol'ny form and gives immediately a bound on the tension. For this model the tension is

$$T = \int d^2x \left\{ \left| \frac{1}{2\,g_1} F^0_{ij} \pm \frac{g_1}{2} \varepsilon_{ij} \left(q^\dagger q - 2\xi\right) \right|^2 + \frac{1}{2} \left|\mathcal{D}_i q \pm i\varepsilon_{ij} \mathcal{D}_j q\right|^2 \pm \varepsilon_{ij}\, \xi F^0_{ij} \right\} \tag{8.3}$$

and the Bogomol'ny-Prasad-Sommerfield (BPS) bound on the tension is [77, 78]

$$T \geq \left| \xi \int d^2x\, \varepsilon_{ij}\, F^0_{ij} \right| \ . \tag{8.4}$$

We can look for solutions of the equations of motion that saturate this bound. These solutions satisfy first (instead of second) order equations:

$$\frac{1}{2\,g_1} F^0_{ij} + \eta \frac{g_1}{2} \varepsilon_{ij} \left(q^\dagger q - 2\xi\right) = 0 \tag{8.5}$$

$$\mathcal{D}_i q + \eta i \varepsilon_{ij} \mathcal{D}_j q = 0 \quad , \quad \eta = \pm 1 \tag{8.6}$$

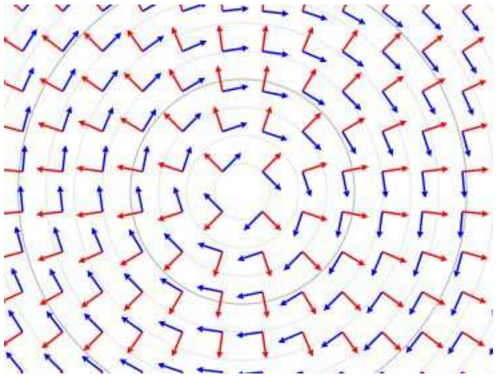

Figure 8.1: The orientation of the fields in the plane x, y for a vortex of minimal winding $n = 1$. The Higgs field q is represented with red arrows (radial), while blue arrows (tangent to the circles) correspond to the gauge field A_i.

Note that at large distance, the natural ansatz for the scalar field is

$$q(r,\theta) \simeq \sqrt{2}\xi e^{in\theta} \quad , \quad n \in \mathbb{Z} \tag{8.7}$$

and the vanishing of the covariant derivative at infinity implies that the form of the gauge field is

$$A_i^0 \simeq n\varepsilon_{ij}\frac{r_j}{r^2} \ . \tag{8.8}$$

From this we can see that the flux is quantized in terms of[1] 2π and read the value of the tension

$$T = 2\xi \left| \int_{\mathbb{R}^2} \vec{B}^0 \cdot d\vec{S} \right| = 4\pi\xi|n| \ . \tag{8.9}$$

So the flux of the vortex obeys a quantization condition of topological nature, related to the homotopy group $\pi_1(U(1)) = \mathbb{Z}$ which describes the possible windings $n \in \mathbb{Z}$ of the scalar field at infinity. This quantization condition corresponds to the analogous condition for Dirac monopoles in this theory. The link between these two solitons appears in more general theories, as we will see later.

[1]With the canonical definition of the gauge coupling the flux is quantized in multiples of $2\pi/g_1$, obtaining the usual Dirac quantization condition of electric and magnetic charges $q_{mag}g_1 = 2\pi$.

8.2 Topological solitons in $\mathcal{N} = 2$ gauge theories

In this section we briefly introduce some of the topological properties of solitons appearing in gauge theories. We also review the features of solitons and their moduli spaces coming from $\mathcal{N} = 2$ extended supersymmetry.

8.2.1 Topology

Solitons are extended field configurations classified by their codimension, defined as the number of spacetime dimensions on which these fields depend. In the hyperplane spanned by these dimensions, their energy density is tipically concentrated in a region of finite size. If we consider this size to be negligible, we can simply read their codimension from the difference between the dimensionality of the spacetime where they live and the dimensionality of their worldspace. In four dimensional theories, domain walls are membrane-like objects having worldspace dimension 3 and codimension 1, vortices are string-like object with worldsheet dimension 2 and codimension 2, monopoles are particle-like objects with worldline dimension 1 and codimension 3, instantons have dimension 0 and codimension 4. In this work we will mainly discuss vortices and monopoles.

We are interested in solitons appearing in the Higgs phase of non-abelian gauge theories. Consider the case of an Higgs field ϕ in some representation of the gauge symmetry and a potential inducing a symmetry breaking pattern

$$G \xrightarrow{\langle\phi\rangle} H \tag{8.10}$$

The moduli space of the theory, i.e. the space of possible vevs of the field ϕ, is described by the coset space G/H, that is a quotient of the space of transformations $U_G \in G$ of $\langle\phi\rangle$ over the group of transformations $U_H \in H \subset G$ leaving $\langle\phi\rangle$ invariant. In fact all the possible vevs are related by transformations of G, then to obtain all the vevs from a given $\langle\phi\rangle$ we transform it as $\langle\phi\rangle' = U_G \langle\phi\rangle$. However these transformations do not always give different vevs: in particular U_G and $U_G U_H$ give the same result because $U_H \langle\phi\rangle = \langle\phi\rangle$.

Consider first solitons of codimension 2, i.e. vortices. These are solutions of the equations of motion which do not depend on z, t. Their energy is proportional to their lenght and is therefore infinite, but we require that their tension $T = E/L_z$ is finite. Then the vev of the Higgs field at infinity in the x, y plane should lie in G/H. The boundary of the plane x, y is the circle at infinity S^1_∞, then the Higgs field describes a map from S^1 to G/H. If we consider two of these maps Π, Π' that cannot be continuously deformed one into the other, the same is true for the corresponding physical configurations of the Higgs field. This means that the theory contains separated sectors that are classified by the topologically inequivalent maps $\Pi : S^1 \to G/H$. The space of these maps is the fundamental homotopy group $\pi_1(G/H)$ and vortices are classified by non-trivial elements of this group.

Solitons of codimension 3, i.e. monopoles, can be classified in a similar way. The Higgs field describes a map from the surface at infinity S^2_∞ to the space of vacua G/H, and inequivalent maps $\Pi : S^2 \to G/H$ correspond to elements of the homotopy group $\pi_2(G/H)$. Monopoles are classified by non-trivial elements of this group.

If G and H are Lie groups and $H \subset G$, there is an important topological relation between the homotopy groups discussed above [79, 80, 81] :

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G) \tag{8.11}$$

This means that there is a one-to-one correspondence between the elements of $\pi_2(G/H)$ and the elements of $\pi_1(H)$ that correspond to the trivial element of $\pi_1(G)$, which means that they can be contracted to a trivial loop when H is embedded in G.

Domain walls and instantons can also be classified using the homotopy groups $\pi_0(G/H)$ and $\pi_3(G)$, but they will not be discussed here.

Solitonic solutions are often part of a multi-parameter family of solutions with the same energy. The parameters labeling these solutions are called collective coordinates, and the space of solutions characterized by the same energy and topological charge is called the moduli space of the soliton. The fluctuations of the collective coordinates correspond to zero-modes in the soliton background. There is a natural metric on the moduli space coming from the overlapping of these zero-modes, but this metric will not play any role in these chapters.

8.2.2 $\mathcal{N} = 2$ supersymmetry

We will study vortices and monopoles in the framework of $\mathcal{N} = 2$ supersymmetric gauge theories. Extended supersymmetry provides interesting features that can simplify the theoretical analysis. The most interesting point is related to the role of the BPS bound in supersymmetric theories. The $\mathcal{N} = 2$

algebra takes the form

$$\{Q^i_\alpha, \bar{Q}^j_{\bar{\alpha}}\} \propto P_{\alpha\bar{\alpha}}\delta_{ij} \tag{8.12}$$

$$\{Q^i_\alpha, Q^j_\beta\} \propto \varepsilon_{\alpha\beta}\varepsilon_{ij}Z \tag{8.13}$$

$$\{\bar{Q}^i_{\bar{\alpha}}, \bar{Q}^j_{\bar{\beta}}\} \propto \varepsilon_{\bar{\alpha}\bar{\beta}}\varepsilon_{ij}Z^* \tag{8.14}$$

$$\tag{8.15}$$

where Z is called a central charge. Considering this algebra in the rest frame we can get a bound for the mass of the system $M \geq const \cdot |Z|$. When this bound is saturated, half of the supersymmetry acts trivially on the representation and the above algebra admits representations with 1/2 of the usual states for a massive $\mathcal{N} = 2$ multiplet, hence they are called short BPS multiplets. It is possible to evaluate Z for a quantum field theory with spontaneously broken gauge symmetry with $\langle\phi\rangle = v$, obtaining as a result [82]

$$Z = v(Q_e + iQ_m) \tag{8.16}$$

where Q_e and Q_m are the electric and magnetic charge of the system. This means a bound on the mass of fields in this theory:

$$M \geq const \cdot |v|\sqrt{|Q_e|^2 + |Q_m|^2} \tag{8.17}$$

This bound applies also to solitonic states like monopoles and vortices and implies that their masses and tensions are protected against quantum corrections, because quantum corrections cannot modify the number of states and therefore cannot turn a short BPS multiplet into a long one. The BPS equations can be also derived asking that the soliton solution is annihilated by half of the supersymmetries, obtaining first-order equations for the background fields.

Extended supersymmetry also has implications on the structure and the metric of the moduli spaces of vortices and monopoles. Moduli spaces of BPS monopoles are hyperKähler manifolds while moduli spaces of BPS vortices are Kähler manifolds.

8.3 Goddard-Nuyts-Olive magnetic monopoles

Magnetic monopoles appear in abelian gauge theories as singular objects. Instead, in some broken gauge theories there are solitonic solutions of codimension 3 that carry magnetic flux. These magnetic monopoles, discovered by 't Hooft and Polyakov [83, 84] , are interesting and important objects.

The original 't Hooft-Polyakov monopole appears in an SU(2) theory with a field ϕ in the adjoint representation. The vev $\langle\phi\rangle$ triggers spontaneous symmetry breaking

$$SU(2) \xrightarrow{\langle\phi\rangle} U(1) \tag{8.18}$$

The Higgs field describes a map from the sphere at infinity S^2_∞ to the quotient $SU(2)/U(1) \sim S^2$, therefore monopoles are classified by their winding $n \in \mathbb{Z} = \pi_2(SU(2)/U(1))$. The configuration of minimal winding can be written in the form (that is usually called an hedgehog)

$$A_i^a(\mathbf{r}) = \varepsilon_{aij}\frac{r^j}{r^2}h(r) \qquad \phi^a(\mathbf{r}) = \frac{r^a}{r}\varphi(r) \quad , \quad \varphi(\infty) = v \quad h(\infty) = -1 \tag{8.19}$$

where the functions $h(r)$ and $\varphi(r)$ satisfy the equations of motion and their behaviour at infinity is determined by the requirement of finite energy. In $\mathcal{N} = 2$ theories, the monopole satisfies a BPS bound $M \geq v\left|\Phi(\vec{B})\right|$ and the equations for $h(r), \varphi(r)$ are first-order differential equations.

The magnetic field $B_i^a = \varepsilon_{ijk}\partial_j A_k^a$ has a nonzero flux on the surface at infinity that is equal to $\Phi(\vec{B}) = 4\pi$. By definition (and by Stokes theorem) the magnetic charge Q_m contained in a region of space equals the flux of the magnetic field on its surface, therefore the charge of the monopole is $q_m = \Phi(\vec{B}) = 4\pi$. We can recover the Dirac quantization condition redefining the gauge fields in the canonical way, obtaining in this way $q_m = 4\pi/g_2$ where g_2 is the $SU(2)$ gauge coupling, and observing that the minimal electric charge present in the theory is $q_e = g_2/2$ and therefore obtaining the usual quantization condition $q_m q_e = 2\pi$.

When we consider more general groups, we find many different embeddings of this solution. These Goddard-Nuyts-Olive-Weinberg nonabelian monopoles [80, 85, 86, 81] appear in systems with the gauge symmetry breaking

$$G \xrightarrow{\langle\phi\rangle} H \tag{8.20}$$

where the homotopy group $\pi_2(G/H)$ is nontrivial and the unbroken gauge group H is nonabelian. We use $\sum_\alpha \alpha_i\alpha_j = \delta_{ij}$ for the metric of the root space and $\mathrm{Tr}(t^a t^b) = \delta_{ab}/2$ for the normalization of the generators.

The vev of the Higgs field has the form

$$\langle\phi\rangle = \mathbf{h}\cdot\mathbf{H}, \tag{8.21}$$

where $H_1, \ldots H_r$ are the generators in the Cartan subalgebra of G, r is the rank of G and the root vectors orthogonal to $\mathbf{h}$ belong to the unbroken subgroup H. The monopole solutions are essentially 't Hooft-Polyakov monopoles embedded in $SU(2)$ subgroups of G that are broken by $\langle\phi\rangle$. The corresponding generators are

$$S_1 = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}) \qquad S_2 = -\frac{i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}) \qquad S_3 = \alpha^* \cdot \mathbf{H} \quad (8.22)$$

where α is the root vector of the broken subgroup and α^* is its dual root, defined by

$$\alpha^* = \frac{\alpha}{\alpha \cdot \alpha}. \quad (8.23)$$

The subgroup generated by the generators (8.22) has a symmetry breaking $SU(2) \to U(1)$, therefore we can simply embed the 't Hooft-Polyakov monopole solution in this subgroup and add a constant term to ϕ so that it goes to the correct vev at infinity:

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h}\cdot\alpha)S_a \qquad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h}\cdot\alpha)S_a + (\mathbf{h} - (\mathbf{h}\cdot\alpha)\alpha^*)\cdot\mathbf{H}, \quad (8.24)$$

where

$$A_i^a(\mathbf{r}) = \epsilon_{aij}\frac{r^j}{r^2}h(r) \qquad \chi^a(\mathbf{r}) = \frac{r^a}{r}\varphi(r) \qquad \varphi(\infty) = \mathbf{h}\cdot\alpha \quad (8.25)$$

is the 't Hooft-Polyakov solution. The flux $\Phi(\vec{B})$ is nonabelian and is given by $\Phi(\vec{B}) = 4\pi\alpha^* \cdot \mathbf{H}$ in the gauge where the Higgs field is constant.

For some gauge groups there are other monopole solutions [86]that are not embeddings of 't Hooft-Polyakov monopoles. These solutions are characterized by nonabelian gauge fields that live only in a finite radius from the center of the monopole, while abelian fields go to infinity, and the solution appears as a purely abelian monopole far away from its core. These solutions are more complicated and will play a role in our discussion of the relation between vortices and monopoles.

Chapter 9

Non-abelian vortices and monopoles

9.1 Non-abelian vortices in $SU(N)$

We study an $\mathcal{N}=2$ gauge theory with gauge group $SU(N+1)$ and $N_f > N$ hypermultiplets of mass m in the fundamental representation of the gauge group. There are interesting nonperturbative results for the low-energy regime of these theories [87, 88] , but here we analyze the system in a semi-classical regime. We add to the theory a small term $\Delta\mathcal{L}_{\mathcal{N}=1} = \int d^2\theta\mu\phi^2$, $\mu \ll m$ that breaks softly $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$. This induces a pattern of symmetry breaking

$$SU(N+1)\xrightarrow{\langle\phi\rangle}SU(N)\times U(1)\xrightarrow{\langle q\rangle}1 \tag{9.1}$$

as can be seen from the vacuum equations

$$\phi^\dagger T^a\phi = 0 \tag{9.2}$$

$$q_A^\dagger t^a q^A - \tilde{q}_A t^a \tilde{q}^{A\dagger} = 0 \tag{9.3}$$

$$\sqrt{2}\phi^a t^{a*}\tilde{q}_A + m\tilde{q}_A = 0 \quad \rightarrow \quad \langle\phi\rangle \sim m \text{ if } \langle q\rangle \neq 0 \tag{9.4}$$

$$\sqrt{2}\phi^a t^a q^A + m q^A = 0 \tag{9.5}$$

$$\sqrt{2}\tilde{q}^A t^a q_A + \mu\phi^a = 0 \quad \rightarrow \quad \langle q\rangle^2 \sim \mu m\ . \tag{9.6}$$

The vev that breaks the high-energy group is

$$\langle\phi\rangle = \frac{m}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -N \end{pmatrix} \tag{9.7}$$

and the low-energy Lagrangian that describes the system at energies lower than $\langle\phi\rangle$ is

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4g_1^2}F^{0\mu\nu}F^0_{\mu\nu}-\frac{1}{4g_N^2}F^{b\mu\nu}F^b_{\mu\nu}+|\mathcal{D}_\mu q_A|^2+\left|\mathcal{D}_\mu\tilde{q}_A^\dagger\right|^2 \\ & -\frac{g_N^2}{2}\left|q_A^\dagger t^b q_A-\tilde{q}_A t^b\tilde{q}_A^\dagger\right|^2-2g_N^2\left|\tilde{q}_A t^b q_A\right|^2 \\ & -\frac{g_1^2}{2}\left|q_A^\dagger q_A-\tilde{q}_A\tilde{q}_A^\dagger\right|^2-2g_1^2\left|\tilde{q}_A q_A-2N(N+1)\mu m\right|^2+\cdots\end{aligned} \tag{9.8}$$

where we are neglecting fluctuations of ϕ and higher orders in μ/m. The Higgs vacuum of this low-energy theory is given by

$$\langle q\rangle=\sqrt{|2N(N+1)\mu m|}\begin{pmatrix}1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\cdots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix}\quad,\qquad|\langle\tilde{q}\rangle|=|\langle q^\dagger\rangle| \tag{9.9}$$

where each column represents the color components of a single flavor. Note that this color-flavor locked form of the vev preserves a global symmetry $SU(N)_{C+F}\subset SU(N)\times SU(N_f)$ acting on the matrix form of q as $q'=UqU^\dagger$.

The term $\xi=|2N(N+1)\mu m|$ plays the role of a Fayet-Iliopoulos term and does not break $\mathcal{N}=2$ supersymmetry, therefore we can find a Bogomol'ny form for the tension

$$\begin{aligned}T=\int d^2x\,\Bigg\{&\left|\frac{1}{2\,g_N}\,F^b_{ij}\pm g_N\varepsilon_{ij}\tilde{q}_A t^b q_A\right|^2+\left|\frac{1}{2\,g_1}F^0_{ij}\pm g_1\varepsilon_{ij}\left(\tilde{q}_A q_A-\xi\right)\right|^2 \\ &+\frac{1}{2}\left|\mathcal{D}_i q_A\pm i\varepsilon_{ij}\mathcal{D}_j\tilde{q}_A^\dagger\right|^2+\frac{1}{2}\left|\mathcal{D}_i\tilde{q}_A^\dagger\pm i\varepsilon_{ij}\mathcal{D}_j q_A\right|^2 \\ &+\frac{g_N^2}{2}\left|q_A^\dagger t^b q_A-\tilde{q}_A t^b\tilde{q}_A^\dagger\right|^2+\frac{g_1^2}{2}\left|q_A^\dagger q_A-\tilde{q}_A\tilde{q}_A^\dagger\right|^2\pm\varepsilon_{ij}\,\xi F^0_{ij}\Bigg\}\,.\end{aligned} \tag{9.10}$$

and obtain the nonabelian BPS equations

$$\frac{1}{2\,g_1}F^0_{ij}+\eta\,g_1\,\varepsilon_{ij}\,\left(q_A^\dagger q_A-\xi\right)=0\,, \tag{9.11}$$

$$\frac{1}{2\,g_N}\,F^b_{ij}+\eta\,g_N\,\varepsilon_{ij}\,q_A^\dagger t^b q_A=0\,, \tag{9.12}$$

$$\mathcal{D}_i q_A+i\,\eta\,\varepsilon_{ij}\,\mathcal{D}_j q_A=0\,,\qquad\eta=\pm1\,, \tag{9.13}$$

and the corresponding BPS bound on the tension

$$T=\eta\int d^2x\,\xi\,\varepsilon_{ij}\,F^0_{ij}\,. \tag{9.14}$$

We can work out the explicit structure of the vortex solutions choosing the ansatz

$$q = \tilde{q}^{\dagger} \tag{9.15}$$

and looking for squarks of the form

$$q(r,\vartheta) = \begin{pmatrix} e^{in_1\vartheta}\varphi_1(r) & 0 & 0 & \cdots \\ 0 & e^{in_2\vartheta}\varphi_2(r) & 0 & \cdots \\ 0 & 0 & e^{in_3\vartheta}\varphi_3(r) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{9.16}$$

and gauge fields of the form

$$A^i_{\alpha\beta} = \varepsilon_{ij}\frac{r_j}{r^2}h_\alpha(r)\delta_{\alpha\beta} \ . \tag{9.17}$$

The asymptotic conditions can be obtained from the requirement of finite energy: $\phi_i(\infty) = \sqrt{\xi}$, $h_\alpha(\infty) = -n_\alpha$. Regularity of the squark fields at the origin requires that all n_α have the same sign, and the tension of the vortex string can be obtained from the bound (9.14):

$$T = \frac{4\pi\xi}{N}\left|n_1 + n_2 + \cdots + n_N\right| \ . \tag{9.18}$$

The vortex solutions obtained with this ansatz are described by N positive (or negative) integers n_α. Vortices of minimal flux are classified by $(1,0,0\ldots),(0,1,0\ldots),\ldots$. The $SU(N)_{C+F}$ transformations interpolate between all these configurations. To find the moduli space of these vortices, we notice that each solution breaks $SU(N)_{C+F}$ to $U(N-1)$ and then the internal collective coordinates are coordinates on the coset space $SU(N)/U(N-1) \sim \mathbb{C}P^{N-1}$. There are also two translational coordinates describing the position of the center of the vortex in the plane, therefore the full moduli space is $\mathbb{C}P^{N-1} \times \mathbb{C}$. In the rest of the chapter we will discuss only the internal moduli space of vortices.

The winding around the $U(1)$ subgroup is given by $\sum_{\alpha=1}^{N} n_\alpha/N$. The reason for this non-integer winding is the topology of the low-energy gauge group, that is not $SU(N)\times U(1)$, but $\frac{SU(N)\times U(1)}{\mathbb{Z}_N}$, because the elements $e^{2\pi i k/N}$ are contained both in $SU(N)$ and in $U(1)$. Vortices are classified by the homotopy group $\pi_1(\frac{SU(N)\times U(1)}{\mathbb{Z}_N}) = \mathbb{Z}$ where the minimal element of this homotopy group $\mathbb{Z}$ is a loop going from 1 to $e^{2\pi i/N}$ in $U(1)$ and then going back to 1 through $SU(N)$.

9.2 Non-abelian vortices in $SO(N)$

We consider the bosonic part of an $\mathcal{N}=2$ theory with gauge group $SO(2N)\times U(1)$ and $N_f=2N$ hypermultiplets $(q_A,\tilde{q}_A^\dagger)$ in the $(\underline{2N},+1)$ representation of the gauge group. We include a Fayet-Iliopoulos term ξ to break the gauge symmetry. The Higgs vacuum of the theory has a $SO(2N)_{C+F}$ color-flavor global symmetry. Using the ansatz $\tilde{q}^\dagger=\phi=0$, the tension can be written in the Bogomol'ny form

$$T=\int d^2x\left\{\left|\frac{1}{2\,g_{2N}}\,F_{ij}^b\mp g_{2N}\varepsilon_{ij}q_A^\dagger t^b q_A\right|^2+\left|\frac{1}{2\,g_1}F_{ij}^0\pm\frac{g_1}{\sqrt{2}}\varepsilon_{ij}(q_A^\dagger q_A-\xi)\right|^2+\right.$$
$$\left.+\left|\mathcal{D}_i q_A\mp i\varepsilon_{ij}\mathcal{D}_j q_A\right|^2\pm\frac{\xi}{\sqrt{2}}\varepsilon_{ij}F_{ij}^0\right\} \tag{9.19}$$

and the ansatz for the solution [74] is $A_i=h_a(r)\,t^a\,\varepsilon_{ij}\,\frac{r_j}{r^2}$ for the gauge fields and

$$q_{iA}(r,\vartheta)=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{in_1^+\vartheta}\varphi_1^+(r) & e^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots\\ ie^{in_1^+\vartheta}\varphi_1^+(r) & -ie^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots\\ 0 & 0 & e^{in_2^+\vartheta}\varphi_2^+(r) & e^{in_2^-\vartheta}\varphi_2^-(r) & \cdots\\ 0 & 0 & ie^{in_2^+\vartheta}\varphi_2^+(r) & -ie^{in_2^-\vartheta}\varphi_2^-(r) & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{9.20}$$

for the squark fields, with the finite-energy conditions

$$\varphi_a^\pm(\infty)=\sqrt{\frac{\xi}{2N}}\qquad n_a^\pm=\frac{1}{\sqrt{2}}\,(h_0(\infty)\mp h_a(\infty)) \tag{9.21}$$

Vortex solutions obtained with this ansatz are therefore classified by $2N+1$ integers N_0,$n_a^\pm$ which satisfy the following conditions:

$$n_a^+ + n_a^- = N_0\quad,\quad \mathrm{sign}(n_a^+)=\mathrm{sign}(n_a^-)=\mathrm{sign}(N_0)\qquad a=1\ldots 2N \tag{9.22}$$

N_0 is related to the winding around the $U(1)$ part of the gauge group. The $U(1)$ factor is needed to stabilize the BPS solutions, therefore N_0 enters also the tension $T=2\pi\xi|N_0|$.

The minimal solutions $N_0=1$ are classified by

$$\begin{pmatrix} n_1^+ & \cdots & n_N^+\\ n_1^- & \cdots & n_N^- \end{pmatrix}=\begin{pmatrix} 1 & 1 & \cdots & 1 & 1\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix},\begin{pmatrix} 1 & 1 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix},\ldots \tag{9.23}$$

We can always apply $SO(2N)_{C+F}$ tranformations to these solutions, obtaining new solutions. However solutions of the form (9.20) belong to the same orbit if they are connected by an $SO(2N)$ transformation. All the solutions with $N_0 = 1$ lie in two different orbits of $SO(2N)_{C+F}$, generated from solutions of the form (9.20) with $\sum_a n_a^+$ even or odd. Therefore the moduli space corresponds to two copies of the quotient space $\mathcal{M} = SO(2N)/U(N)$.

More generally, two solutions which differ only by the exchange $(n_i^+, n_j^+) \leftrightarrow (n_i^-, n_j^-)$ or $(n_i^+, n_i^-) \leftrightarrow (n_j^+, n_j^-)$ for some i,j, belong to the same orbit. Note that it is possible that the moduli space for vortices of higher winding cannot be obtained simply by $SO(2N)_{C+F}$ and the details of its structure are not known.

The topology of these vortex solitons follows the general arguments of section 8.2.1 but the construction of the homotopy groups $\pi_1(G/H)$ is nontrivial. The relevant homotopy group is $\pi_1\left(\frac{SO(2N)\times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z} \times \mathbb{Z}_2$ and vortices are classified by these windings. The details can be found in appendix B.

There is an interesting interpretation of these results. If we consider an high-energy $\mathcal{N} = 2$ theory with gauge group $SO(2N+2)$, $N_f = 2N$ hypermultiplets of mass m in the adjoint representation and a soft $\mathcal{N} = 2$-breaking term $\mu\phi^2$, and then $\langle\phi\rangle \sim m$ breaks the gauge group to $SO(2N) \times U(1)$, the low-energy theory below m corresponds to the one considered above [74]. In fact it is easy to get the tension (9.19) if we use the ansatz $q = \tilde{q}^\dagger$ and define $\xi = \mu m$.

Note that the high-energy theory at scale m contains heavy almost-BPS monopoles coming from the breaking pattern $SO(2N+2) \to SO(2N) \times U(1)$, while the low-energy theory at scale $\sqrt{\mu m}$ contains the vortices studied in this section, which are almost stable (they are unstable under creation of monopole-antimonopole pairs, but this process is heavily suppressed if $\mu \ll m$). Monopoles and vortices also appear with other gauge groups and symmetry breaking patterns in a similar way. In all these systems there is an apparent relation between monopoles and vortices, which will be discussed in the next section.

9.3 Monopole-vortex correspondence

We consider $\mathcal{N} = 2$ gauge theories with gauge group G and N_f matter hypermultiplets $(q, \tilde{q})$ of mass m, usually in the fundamental representation (the model discussed above is an exception). We also add a small mass term $\mu\phi^2$ for the chiral superfield in the vector hypermultiplet which breaks softly $\mathcal{N} = 2$ to $\mathcal{N} = 1$. When $\mu \ll m$ there are vacua in the Higgs phase with a hierarchical pattern of symmetry breaking

$$G \xrightarrow{\langle\phi\rangle} H \xrightarrow{\langle q\rangle} 1 \tag{9.24}$$

where $\langle q\rangle \sim \sqrt{\mu m} \ll m \sim \langle\phi\rangle$. The simplest case is $G = SU(N+1)$ and $H = U(N)$, but examples of such patterns include $G = SO(2N)$ and $H = U(N)$ or $SO(2N-2)\times U(1)$, $G = SO(2N+1)$ and $H = U(N)$ or $SO(2N-1)\times U(1)$, $G = USp(2N)$ and $H = U(N)$ or $USp(2N-2)\times U(1)$.

In the high-energy theory at scale m there are regular monopoles coming from the symmetry breaking $G \to H$, which are not BPS because of $\sqrt{\mu/m}$ corrections. In the low-energy theory below scale m there are regular vortices which come from the breaking $H \to 1$ and are stable in the limit $m \to \infty$ with $\sqrt{\mu m}$ fixed.

We are interested in the case when H is non-abelian. In these systems the regular monopoles are Goddard-Nuyts-Olive non-abelian monopoles [81] which transform under the dual group $\tilde{H}$, while the vortices of the low-energy theory are non-abelian vortices in a theory with gauge group H and Fayet-Iliopoulos parameter $\xi \sim \sqrt{\mu m}$. There is an interesting relation between monopoles and vortices which has been discussed in [70, 89, 73].

We explain this relation starting from the topological classification of solitons in these theories. The relevant homotopy group for regular GNO monopoles is $\pi_2(G/H)$, while singular Dirac monopoles are classified by $\pi_1(G)$. After the breaking $H \to 1$, the only regular monopoles which are topologically stable are those classified by $\pi_2(G)$, which is trivial.

The fate of monopoles classified by a nontrivial element of $\pi_2(G/H)$ is related to the vortices coming from the breaking $H \to 1$. In fact the monopole magnetic flux cannot disappear, but it shrinks into a flux tube of width $1/\sqrt{\mu m}$ which is precisely a vortex of the low-energy theory. So for each monopole in the high-energy theory there should exist a vortex of the low-energy theory which carries the same flux.

This correspondence can be seen from a topological point of view. Vortex solutions are classified by $\pi_1(H)$. The topological relation

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G) \tag{9.25}$$

has a simple interpretation if G is simply connected: in this case the homotopy groups for monopoles and vortices are the same. When $\pi_1(G)$ is nontrivial, the relation (9.25) states that regular monopoles are sources for vortices which correspond to trivial elements of $\pi_1(G)$.

A simple way, when possible, to establish this correspondence is flux matching [89]: the magnetic flux integrated over a plane orthogonal to the axis of the vortex should match the magnetic flux integrated over a sphere surrounding the corresponding monopole, as in figure 9.3. Obviously, the abelian magnetic flux coming from the monopole and the flux carried by the vortex must match precisely, but this is only a check, because the $U(1)$ flux cannot determine the non-abelian orientation of the soliton.

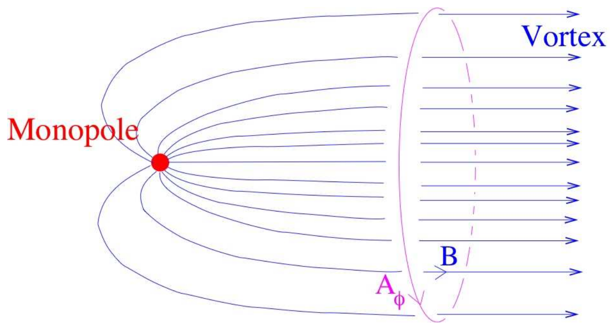

Figure 9.1: A monopole-vortex configuration, illustrating the idea of flux matching.

Matching non-abelian fluxes is an effective way to match a monopole with the corresponding vortex. The problem with non-abelian flux is that it does not obey a conservation law as the abelian flux because of the term $i[A_j, A_k]$ in the magnetic field. Non-abelian flux matching could be reliable only for monopole and vortex solutions which satisfy $[A_j, A_k] = 0$. This is the case for solutions obtained using an ansatz like (9.20) and the corresponding monopoles. All vortices of minimal winding belong to this case.

Unfortunately $[A_j, A_k] = 0$ is generally not true for vortices of higher winding, as can be seen from the explicit expression for vortex solutions of double winding in $U(2)$ [166]. The same problem occurs for the corresponding monopoles, whose explicit expression was discovered by E.Weinberg [91]. In this case flux matching can only be established in an approximate, thus not very useful, way.

The general claim of this section is that for each monopole in the high-energy theory there is a corresponding vortex in the low-energy theory and that their topological classification and fluxes should match. This claim leads to an interesting corollary about the moduli spaces of monopoles and vortices: in a theory where monopoles correspond to vortices of winding k, the internal moduli space of coaxial vortices of winding k should contain a subspace which has the same structure of the internal space of degenerate monopoles. This is an interesting point because the moduli space of non-abelian monopoles is not well-defined due to the non-normalizability of zero-modes, but it can be matched with the moduli space of vortices, which only have normalizable zero-modes.

In the next sections we will discuss some explicit examples of symmetry breaking patterns to check the correspondence discussed above.

9.3.1 $SU(N+1) \to U(N)$

This theory contains matter multiplets in the fundamental representation of $SU(N+1)$. The vacuum is invariant under a $SU(N)_{C+F}$ global symmetry. Non-abelian vortex solutions can be constructed with an ansatz similar to (9.20) and are classified by a set of positive integers $(n_1, n_2 \ldots n_N)$ where $\sum_i n_i$ corresponds to the winding of the vortex. All the solutions with minimal winding belong to the same orbit of $SU(N)_{C+F}$. Monopoles are simply embeddings of 't Hooft-Polyakov monopoles in various $SU(2)$ subgroups. Vortices and monopoles in this theory are both classified by $\pi_2(SU(N+1)/U(N)) = \pi_1(U(N)) = \mathbb{Z}$, so fundamental monopoles correspond to vortices of minimal winding classified by $(1, 0 \ldots 0)$ etc. The moduli space of these vortices is simply $\mathbb{C}P^{N-1}$ with $SU(N)$ isometry and Fubini-Study metric and it corresponds to the configuration space of monopoles. Flux matching can be easily checked and the correspondence works perfectly.

9.3.2 $SO(2N) \to U(N)$

This theory contains matter multiplets in the fundamental representation of $SO(2N)$. The vacuum respects a $SU(N)_{C+F}$ global symmetry. Vortex solutions are identical to the previous case, while monopoles are embeddings of 't Hooft-Polyakov ones in $SU(2) \subset SO(4)$ subgroups. The fact that $\pi_1(SO(2N)) = \mathbb{Z}_2$ and $\pi_2(SO(2N)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}_2$ implies that fundamental monopoles correspond to vortices of winding 2, while flux matching calculations suggest that they correspond precisely to the vortices classified by $(2, 0 \ldots 0)$ etc. and their $SU(N)_{C+F}$ orbit. Both these vortices and the corresponding monopoles have a configuration space which is $\mathbb{C}P^{N-1}$ with $SU(N)$ isometry. However the moduli space of $k = 2$ vortices is much bigger [92]. In the simplest case $N = 2$ the moduli space is the weighted projective space $W\mathbb{C}P^2_{(2,1,1)}$ with $SU(2)$ isometry, which contains a $\mathbb{C}P^1$ corresponding to the vortices discussed above. So in this case the correspondence works correctly, but it seems unable to explain the presence of a bigger moduli space of vortices.

9.3.3 $SO(2N+1) \to U(N)$

This is the same theory as the previous case but with gauge group $SO(2N+1)$. They differ mainly because some of the monopoles are embeddings of 't Hooft-Polyakov monopoles in $SO(3)$ and $SU(2)$ subgroups, while others form a continuous family of solutions interpolating between these two embeddings [91]. In this case flux matching is only partially useful because

$[A_j, A_k] \neq 0$ for the interpolating solutions. Topological arguments suggest that monopoles correspond to $k = 2$ vortices, because $\pi_1(SO(2N+1)) = \mathbb{Z}_2$ and $\pi_2(SO(2N+1)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}_2$ as in the previous case. Fluxes of $(2, 0 \ldots 0)$ vortices agree with those of monopoles embedded in $SU(2)$ subgroups, while fluxes of $(1, 1 \ldots 0)$ vortices agree with those of monopoles embedded in $SO(3)$ subgroups.

This case is an interesting check of the correspondence as both moduli spaces of monopoles and vortices are known in the $N = 2$ case. The moduli space of vortices is $W\mathbb{C}P^2_{(2,1,1)}$, which is a $\mathbb{C}P^2$ with a conical singularity. It contains the $\mathbb{C}P^1$ discussed in the previous case and the rest of the moduli space corresponds to $\mathbb{C}^2/\mathbb{Z}^2$. The metric of this moduli space is unknown. The moduli spaces of monopoles and its metric have been found in [93]: it has the topological structure of $\mathbb{C}^2/\mathbb{Z}^2$, with a separated $\mathbb{C}P^1$ which represents monopoles with long-range magnetic fields. Therefore the correspondence seems to work also for this case and the existence of vortices which do not belong to $\mathbb{C}P^1$ finds a natural explanation in the existence of a large class of monopoles in this theory.

Flux matching

Here we review briefly the non-Abelian flux matching [89, 73] for the system $SO(5) \to U(2) \to 1$. We use the notation S_i, $\tilde{S}_i$ for the group generators:

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad S_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_3 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.26}$$

$$\hat{S}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{S}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{S}_3 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.27}$$

The monopole flux can be obtained in the gauge where $\phi = const$:

$$\int_{S^2} d\vec{S} \cdot \vec{B} = \int_{S^2} d\vec{S} \cdot \frac{\vec{r} S_3^{(m)}}{r^3} = 4\pi S_3^{(m)}$$

where $S_3^{(1)} = S_3$ for the singlet monopole, $S_3^{(2)} = S_3 + \tilde{S}_3$ for the doublet[1].

The vortex flux can be obtained from the explicit solution in [166], using the expressions for the gauge fields $A_\mu = A_\mu^i \tilde{S}_i + A_\mu^0 S_3$:

$$\int_{R^2} d\vec{S} \cdot \vec{B} = \int_{R^2} dS (\vec{\partial} \wedge \vec{A} - i[\vec{A}, \vec{A}])_3 = 4\pi S_3 + 2\pi (1 + \cos\alpha) \tilde{S}_3 + 2\pi \sin^2\alpha \, I_{NA} \tilde{S}_3$$

where $I_{NA} = \int_0^\infty dr(1-g)h/r$ is a (generally small) contribution from the commutator $[A^1, A^2]$ which is not relevant for $\alpha = 0, \pi$ because of the $\sin^2\alpha$ factor. For $\alpha = \pi$ (corresponding to the singlet monopole) the flux is $4\pi S_3$, while for $\alpha = 0$ (corresponding to the doublet) the flux is $4\pi(S_3 + \tilde{S}_3)$, in perfect agreement.

9.3.4 $SO(2N+2) \to SO(2N) \times U(1)$

This case has been discussed at the end of section 9.2. This theory contains matter hypermultiplets in the adjoint representation, but the only components which become massless after the breaking $SO(2N+2) \to SO(2N)$ transform in the fundamental representation of $SO(2N)$, so we end up with a low-energy theory containing squarks in the $(\underline{2N}, +1)$ representation. Non-abelian monopoles are embeddings of 't Hooft-Polyakov ones in $SU(2) \subset SO(4)$ subgroups. The topological structure of the groups in this breaking pattern is $SO(2N+2)/\mathbb{Z}_2 \to (SO(2N) \times U(1))/\mathbb{Z}_2 \to 1$ and therefore the relation $\pi_2(SO(2N+2)/(SO(2N) \times U(1))) = \pi_1((SO(2N) \times U(1))/\mathbb{Z}_2)/\pi_1(SO(2N+2)/\mathbb{Z}_2)$ implies that monopoles should correspond to vortices of winding $N_0 = 2$. Flux matching suggests that monopoles correspond to vortices in the $SO(2N)_{C+F}$ orbit of

$$\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix} \tag{9.28}$$

[1]The exact evaluation of non-Abelian flux for interpolating solutions [91] is not possible without some knowledge of the monopole-vortex junction, because non-Abelian fields contribute to the flux only in a region of finite radius around the monopole.

This orbit corresponds to the complex quadric surface $SO(2N)/(SO(2N-2) \times U(1)) = Q^{2N-2}(C)$ [74]. However, the whole moduli space of $N_0 = 2$ vortices is much bigger and its structure is not known. It should be possible to obtain it using the techniques of [94] .

Appendix A

Non-abelian duality from vortex moduli

In this appendix it is argued that the dual transformation of non-Abelian monopoles occurring in a system with gauge symmetry breaking $G \longrightarrow H$ is to be defined by setting the low-energy H system in Higgs phase, so that the dual system is in confinement phase. The transformation law of the monopoles follows from that of monopole-vortex mixed configurations in the system (with a large hierarchy of energy scales, $v_1 \gg v_2$)

$$G \stackrel{v_1}{\longrightarrow} H \stackrel{v_2}{\longrightarrow} 1,$$

under an unbroken, exact color-flavor diagonal symmetry $H_{C+F} \sim \tilde{H}$. The transformation property among the regular monopoles characterized by $\pi_2(G/H)$, follows from that among the non-Abelian *vortices* with flux quantized according to $\pi_1(H)$, via the isomorphism $\pi_1(G) \sim \pi_1(H)/\pi_2(G/H)$. Our idea is tested against the concrete models – softly-broken $\mathcal{N} = 2$ supersymmetric $SU(N)$, $SO(N)$ and $USp(2N)$ theories, with appropriate number of flavors. The results obtained in the semiclassical regime (at $v_1 \gg v_2 \gg \Lambda$) of these models are consistent with those inferred from the fully quantum-mechanical low-energy effective action of the systems (at $v_1, v_2 \sim \Lambda$).

A.1 Introduction and discussion

A system in which the gauge symmetry is spontaneously broken

$$G \stackrel{\langle \phi_1 \rangle \neq 0}{\longrightarrow} H \tag{A.1}$$

where H is some non-Abelian subgroup of G, possesses a set of regular magnetic monopole solutions in the semi-classical approximation, which are natural generalizations of the 't Hooft-Polyakov monopoles [101] found in the system $G = SO(3)$, $H = U(1)$. A straightforward generalization of the Dirac's quantization condition leads to the GNOW (Goddard-Nuyts-Olive-E.Weinberg) conjecture, *i.e.*, that they form a multiplet of the group $\tilde{H}$, dual of H. The group $\tilde{H}$ is generated by the dual root vectors

$$\alpha^* = \frac{\alpha}{\alpha \cdot \alpha}, \tag{A.2}$$

where α are the non-vanishing roots of H [151]-[153]. There are however well-known difficulties in such an interpretation. The first concerns the topological obstruction discussed in [155]: in the presence of the classical monopole background, it is not possible to define a globally well-defined set of generators isomorphic to H. As a consequence, no "colored dyons" exist. In the simplest example of a system with the symmetry breaking,

$$SU(3) \stackrel{\langle \phi_1 \rangle \neq 0}{\longrightarrow} SU(2) \times U(1), \tag{A.3}$$

this means that no monopoles exist which carry the quantum number, *e.g.*,

$$(\underline{2}, 1^*) \tag{A.4}$$

where the asterisk indicates the dual, magnetic $U(1)$ charge.

The second can be regarded as the infinitesimal version of the same difficulty: certain bosonic zero-modes around the monopole solution, corresponding to the H gauge transformations, are non-normalizable (behaving as $r^{-1/2}$ asymptotically). Thus the standard procedure of semiclassical quantization leading to the H multiplet of the monopoles does not work. Some progress on the check of GNO duality along this orthodox approach has been reported nevertheless in [156] for $\mathcal{N} = 4$ supersymmetric gauge theories, which however requires the consideration of particular multi-monopole systems neutral with respect to the non-Abelian group (more precisely, non-Abelian part of) H.

Both of these difficulties concern the transformation properties of the monopoles under the subgroup H, while the truly relevant question is how they transform under the dual group, $\tilde{H}$. As field transformation groups, H and $\tilde{H}$ are relatively non-local; the latter should look like a non-local transformation group in the original, electric description.

Another related question concerns the multiplicity of the monopoles; take again the case of the system with breaking pattern Eq. (A.3). One might argue that there is only one monopole, as all the degenerate solutions are related by the unbroken *gauge* group $H = SU(2)$.[1] Or one might say that

[1] This interpretation however encounters the difficulties mentioned above. Also there are cases in which degenerate monopoles occur, which are not simply related by the group H, see below.

there are two monopoles as, according to the semiclassical GNO classification, they are supposed to belong to a doublet of the dual $SU(2)$ group. Or, perhaps, one should conclude that there are infinitely many, continuously related solutions, as the two solutions obtained by embedding the 't Hooft solutions in $(1,3)$ and $(2,3)$ subspaces, are clearly part of the continuous set of (*i.e.*, moduli of) solutions. In short, what is the multiplicity (#) of the monopoles:

$$\# \;= 1, \quad 2, \quad \text{or} \quad \infty\, ? \tag{A.5}$$

Clearly the very concept of the *dual gauge group* or *dual gauge transformation* must be better understood. In attempting to gain such an improved insight on the nature of these objects, we are naturally led to several general considerations.

The first is the fact when H and $\tilde{H}$ groups are non-Abelian the dynamics of the system should enter the problem in an essential way. It should not be surprising if the understanding of the concept of non-Abelian duality required a full quantum mechanical treatment of the system.

For instance, the non-Abelian H interactions can become strongly-coupled at low energies and can break itself dynamically. This indeed occurs in pure $\mathcal{N}=2$ super Yang-Mills theories (*i.e.*, theories without quark hypermultiplets), where the exact quantum mechanical result is known in terms of the Seiberg-Witten curves [107]. Consider for instance, a pure $\mathcal{N}=2$, $SU(N+1)$ gauge theory. Even though partial breaking, *e.g.*, $SU(N+1) \to SU(N) \times U(1)$ looks perfectly possible semi-classically, in an appropriate region of classical degenerate vacua, no such vacua exist quantum mechanically. In *all* vacua the light monopoles are Abelian, the effective, magnetic gauge group being $U(1)^N$.

Generally speaking, the concept of a dual group multiplet is well-defined only when $\tilde{H}$ interactions are weak (or, at worst, conformal). This however means that one must study the original, electric theory in the regime of strong coupling, which would usually make the task of finding out what happens in the system at low energies exceedingly difficult. Fortunately, in $\mathcal{N}=2$ supersymmetric gauge theories, the exact Seiberg-Witten curves describe the fully quantum mechanical consequences of the strong-interaction dynamics in terms of weakly-coupled dual magnetic variables. And this is how we know that the non-Abelian monopoles do exist in fully quantum theories [108]: in the so-called r-vacua of softly broken $\mathcal{N}=2$ SQCD, the light monopoles interact as a point-like particle in a fundamental multiplet $\underline{r}$ of the effective, dual $SU(r)$ gauge group. In the system of the type Eq. (A.3) with appropriate number of quark multiplets ($N_f \geq 4$), we know that light magnetic monopoles carrying the non-Abelian quantum number

$$(\underline{2}^*, 1^*) \tag{A.6}$$

under the dual $SU(2) \times U(1)$ appear in the low-energy effective action (*cfr.* Eq. (A.4)). The distinction between H and $\tilde{H}$ is crucial here.

In general $\mathcal{N} = 2$ SQCD with N_f flavors, light non-Abelian monopoles with $SU(r)$ dual gauge group appear for $r \leq \frac{N_f}{2}$ only. Such a limit clearly reflects the dynamical properties of the soliton monopoles under renormalization group: the effective low-energy gauge group must be either infrared free or conformal invariant, in order for the monopoles to emerge as recognizable low-energy degrees of freedom [160]-[163].

A closely related point concerns the phase of the system. Even if there is an ample evidence for the non-Abelian monopoles, as explained above, we might still wish to understand them in terms of something more familiar, such as semiclassical 't Hooft-Polyakov solitons. An analogous question can be (and should be) asked about the Seiberg's "dual quarks" in $\mathcal{N} = 1$ SQCD [113]. Actually, the latter can be interpreted as the GNOW monopoles becoming light due to the dynamics, at least in $SU(N)$ theories [114]. For $SO(N)$ or in $USp(2N)$ theories the relation between Seiberg duals and GNOW monopoles are less clear [114]. For instructive discussions on the relation between Seiberg duals and semiclassical monopoles in a class of $\mathcal{N} = 1$ $SO(N)$ models with matter fields in vector and spinor representations, see Strassler [177].

Dynamics of the system is thus a crucial ingredient: if the dual group were in Higgs phase, the multiplet structure among the monopoles would get lost, generally. Therefore one must study the dual ($\tilde{H}$) system in confinement phase.[2] *But then, according to the standard electromagnetic duality argument, one must analyze the electric system in Higgs phase.* The monopoles will appear confined by the confining strings which are nothing but the vortices in the H system in Higgs phase.

We are thus led to study the system with a hierarchical symmetry breaking,

$$G \overset{\langle\phi_1\rangle\neq 0}{\longrightarrow} H \overset{\langle\phi_2\rangle\neq 0}{\longrightarrow} 1, \tag{A.7}$$

where

$$|\langle\phi_1\rangle| \gg |\langle\phi_2\rangle|, \tag{A.8}$$

[2]The non-Abelian monopoles in the Coulomb phase suffer from the difficulties already discussed.

instead of the original system Eq. (A.1). The smaller VEV breaks H completely. Also, in order for the degeneracy among the monopoles not to be broken by the breaking at the scale $|\langle\phi_2\rangle|$, we assume that some global color-flavor diagonal group

$$H_{C+F} \subset H_{color} \otimes G_F \tag{A.9}$$

remains unbroken.

It is hardly possible to emphasize the importance of the role of the massless flavors too much. This manifests in several different aspects.

(i) In order that H must be non-asymptotically free, there must be sufficient number of massless flavors: otherwise, H interactions would become strong at low energies and H group can break itself dynamically;

(ii) The physics of the r vacua [160, 162] indeed shows that the non-Abelian dual group $SU(r)$ appear only for $r \leq \frac{N_f}{2}$. This limit can be understood from the renormalization group: in order for a non-trivial r vacuum to exist, there must be at least $2\,r$ massless flavors in the fundamental theory;

(iii) Non-Abelian vortices [168, 169], which as we shall see are closely related to the concept of non-Abelian monopoles, require a flavor group. The non-Abelian flux moduli arise as a result of an exact, unbroken color-flavor diagonal symmetry of the system, broken by individual soliton vortex.

The idea that the dual group transformations among the monopoles at the end of the vortices follow from those among the vortices (monopole-vortex flux matching, etc.), has been discussed in several occasions, in particular in [171]. The main aim of the present work is to enforce this argument, by showing that the degenerate monopoles do indeed transform as a definite multiplet under a group transformation, which is non-local in the original, electric variables, and involves flavor non-trivially, even though this is not too obvious in the usual semiclassical treatment. The flavor dependence enters through the infrared regulator. The resulting, exact transformation group is *defined* to be the dual group of the monopoles.

A.2 $SU(N+1)$ model with hierarchical symmetry breaking

Our aim is to show that all the difficulties about the non-Abelian monopole moduli discussed in the Introduction are eliminated by reducing the problem to that of the vortex moduli, related to the former by the topology and symmetry argument.

A.2.1 $U(N)$ model with Fayet-Iliopoulos term

The model frequently considered in the recent literature in the discussion of various solitons [172]-[127], is a $U(N)$ theory with gauge fields W_μ, an adjoint (complex) scalar ϕ, and $N_f = N$ scalar fields in the fundamental representation of $SU(N)$, with the Lagrangian,

$$\begin{aligned}\mathcal{L} &= \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{2}{g^2}\mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\phi - \mathcal{D}_\mu\,H\,\mathcal{D}^\mu H^\dagger - \lambda\left(c\,\mathbf{1}_N - H\,H^\dagger\right)^2\right] \\ &+ \mathrm{Tr}\left[\,(H^\dagger\phi - M\,H^\dagger)(\phi\,H - H\,M)\,\right] \end{aligned} \tag{A.10}$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + i\,[W_\mu, W_\nu]$ and $\mathcal{D}_\mu H = (\partial_\mu + i\,W_\mu)\,H$, and H represents the fields in the fundamental representation of $SU(N)$, written in a color-flavor $N \times N$ matrix form, $(H)^i_\alpha \equiv q^i_\alpha$, and M is a $N \times N$ mass matrix. Here, g is the $U(N)_{\mathrm{G}}$ gauge coupling, λ is a scalar coupling. For

$$\lambda = \frac{g^2}{4} \tag{A.11}$$

the system is BPS saturated. For such a choice, the model can be regarded as a truncation

$$(H)^i_\alpha \equiv q^i_\alpha, \qquad \tilde{q}^\alpha_i \equiv 0 \tag{A.12}$$

of the bosonic sector of an $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory. In the supersymmetric context the parameter c is the Fayet-Iliopoulos parameter. In the following we set $c > 0$ so that the system be in Higgs phase, and so as to allow stable vortex configurations. For generic, unequal quark masses,

$$M = diag\,(m_1, m_2, \ldots, m_N), \tag{A.13}$$

the adjoint scalar VEV takes the form,

$$\langle\phi\rangle = M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_N \end{pmatrix}, \tag{A.14}$$

which breaks the gauge group to $U(1)^N$. In the equal mass case,

$$M = diag\,(m, m, \ldots, m), \tag{A.15}$$

the adjoint and squark fields have the vacuum expectation value (VEV)

$$\langle\phi\rangle = m\,\mathbf{1}_N, \qquad \langle H\rangle = \sqrt{c}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.16}$$

The squark VEV breaks the gauge symmetry completely, while leaving an unbroken $SU(N)_{C+F}$ color-flavor diagonal symmetry (remember that the flavor group acts on H from the right while the $U(N)_{\rm G}$ gauge symmetry acts on H from the left). The BPS vortex equations are

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, H = 0, \quad F_{12} + \frac{g^2}{2}\left(c\,\mathbf{1}_N - H\,H^\dagger\right) = 0. \tag{A.17}$$

The matter equation can be solved by use of the $N\times N$ moduli matrix $H_0(z)$ whose components are holomorphic functions of the complex coordinate $z = x^1 + ix^2$, [173, 124, 125]

$$H = S^{-1}(z,\bar z)\,H_0(z), \quad W_1 + i\,W_2 = -2\,i\,S^{-1}(z,\bar z)\,\bar\partial_z S(z,\bar z). \tag{A.18}$$

The gauge field equations then take the simple form ("master equation") [173, 124, 125]

$$\partial_z\,(\Omega^{-1}\partial_{\bar z}\,\Omega) = \frac{g^2}{4}\,(c\,\mathbf{1}_N - \Omega^{-1}\,H_0\,H_0^\dagger). \tag{A.19}$$

The moduli matrix and S are defined up to a redefinition,

$$H_0(z) \to V(z)\,H_0(z), \qquad S(z,\bar z) \to V(z)\,S(z,\bar z), \tag{A.20}$$

where $V(z)$ is any non-singular $N\times N$ matrix which is holomorphic in z.

A.2.2 The Model

Actually the model we are interested here is not exactly this model, but is a model which contains it as a low-energy approximation. We take as our model the standard $\mathcal{N}=2$ SQCD with N_f quark hypermultiplets, with a larger gauge symmetry, *e.g.*, $SU(N+1)$, which is broken at a much larger mass scale as

$$SU(N+1) \overset{v_1\neq 0}{\longrightarrow} \frac{SU(N)\times U(1)}{\mathbb{Z}_N}. \tag{A.21}$$

The unbroken gauge symmetry is completely broken at a lower mass scale, as in Eq. (A.16).

Clearly one can attempt a similar embedding of the model Eq. (A.10) in a larger gauge group broken at some higher mass scale, in the context of a non-supersymmetric model, even though in such a case the potential must be judiciously chosen and the dynamical stability of the scenario would have to be carefully monitored. Here we choose to study the softly broken $\mathcal{N}=2$ SQCD for concreteness, and above all because the dynamical properties of this model are well understood: this will provide us with a non-trivial check of our results. Another motivation is purely of convenience: it gives a definite potential with desired properties.[3]

The underlying theory is thus

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\, S_{cl}\left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} W W\right] + \mathcal{L}^{(\mathrm{quarks})} + \int d^2\theta\, \mu\, \mathrm{Tr}\Phi^2 + h.c. \tag{A.22}$$

$$\mathcal{L}^{(\mathrm{quarks})} = \sum_i \left[\int d^4\theta\, \{Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger\} + \int d^2\theta\, \{\sqrt{2}\tilde{Q}_i \Phi Q_i + m_i\, \tilde{Q}_i\, Q_i\} + h.c.\right] \tag{A.23}$$

where m is the bare mass of the quarks and we have defined the complex coupling constant

$$S_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \tag{A.24}$$

We also added the parameter μ, the mass of the adjoint chiral multiplet, which softly breaks the supersymmetry to $\mathcal{N}=1$. The bosonic sector of this model is described, after elimination of the auxiliary fields, by

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2}|\mathcal{D}_\mu \Phi|^2 + |\mathcal{D}_\mu Q|^2 + \left|\mathcal{D}_\mu \bar{\tilde{Q}}\right|^2 - V_1 - V_2, \tag{A.25}$$

where

$$V_1 = \frac{1}{8}\sum_A \left(t_{ij}^A\, [\, \frac{1}{g^2}(-2)\, [\Phi^\dagger, \Phi]_{ji} + Q_j^\dagger Q_i - \tilde{Q}_j \tilde{Q}_i^\dagger\,]\right)^2 ; \tag{A.26}$$

$$\begin{aligned} V_2 \;\; &= \;\; g^2 |\mu\, \Phi^A + \sqrt{2}\, \tilde{Q}\, t^A Q|^2 + \tilde{Q}\, [m + \sqrt{2}\Phi]\, [m + \sqrt{2}\Phi]^\dagger\, \tilde{Q}^\dagger \\ &+ \;\; Q^\dagger\, [m + \sqrt{2}\Phi]^\dagger\, [m + \sqrt{2}\Phi]\, Q. \end{aligned} \tag{A.27}$$

[3]Recent developments [128, 129] allow us actually to consider systems of this sort within a much wider class of $\mathcal{N}=1$ supersymmetric models, whose infrared properties are very much under control. We stick ourselves to the standard $\mathcal{N}=2$ SQCD, however, for concreteness.

In the construction of the approximate monopole and vortex solutions we shall consider only the VEVs and fluctuations around them which satisfy

$$[\Phi^\dagger, \Phi] = 0, \qquad Q_i = \tilde{Q}_i^\dagger, \tag{A.28}$$

and hence the D-term potential V_1 can be set identically to zero throughout.

In order to keep the hierarchy of the gauge symmetry breaking scales, Eq. (A.8), we choose the masses such that

$$m_1 = \ldots = m_{N_f} = m, \tag{A.29}$$

$$m \gg \mu \gg \Lambda. \tag{A.30}$$

Although the theory described by the above Lagrangian has many degenerate vacua, we are interested in the vacuum where (see [162] for the detail)

$$\langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & m & 0 \\ 0 & \ldots & 0 & -N\,m \end{pmatrix}; \tag{A.31}$$

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} d & 0 & 0 & 0 & \ldots \\ 0 & \ddots & 0 & \vdots & \ldots \\ 0 & 0 & d & 0 & \ldots \\ 0 & \ldots & 0 & 0 & \ldots \end{pmatrix}, \qquad d = \sqrt{(N+1)\,\mu\, m}. \tag{A.32}$$

This is a particular case of the so-called r vacuum, with $r = N$. Although such a vacuum certainly exists classically, the existence of the quantum $r = N$ vacuum in this theory requires $N_f \geq 2\,N$, which we shall assume.[4]

To start with, ignore the smaller squark VEV, Eq. (A.32). As $\pi_2(G/H) \sim \pi_1(H) = \pi_1(SU(N)\times U(1)) = \mathbb{Z}$, the symmetry breaking Eq. (A.31) gives rise to regular magnetic monopoles with mass of order of $O(\frac{v_1}{g})$, whose continuous transformation property is our main concern here. The semiclassical formulas for their mass and fluxes are well known [153, 157] and will not be repeated here.

[4]This might appear to be a rather tight condition as the original theory loses asymptotic freedom for $N_f \geq 2\,N+2$. This is not so. An analogous discussion can be made by considering the breaking $SU(N) \to SU(r) \times U(1)^{N-r}$. In this case the condition for the quantum non-Abelian vacuum is $2\,N > N_f \geq 2\,r$, which is a much looser condition. Also, although the corresponding $U(N)$ theory Eq. (A.10) with such a number of flavor has semilocal strings [130, 125, 127], these moduli are not directly related to the derivation of the dual gauge symmetry, which is our interest in this work. We shall come back to these questions elsewhere.

A.2.3 Low-energy approximation

At scales much lower than $v_1 = m$ but still neglecting the smaller squark VEV $v_2 = d = \sqrt{(N+1)\,\mu\, m} \ll v_1$, the theory reduces to an $SU(N)\times U(1)$ gauge theory with N_f light quarks $q_i, \tilde{q}^i$ (the first N components of the original quark multiplets $Q_i, \tilde{Q}^i$). By integrating out the massive fields, the effective Lagrangian valid between the two mass scales has the form,

$$\mathcal{L} = \frac{1}{4g_N^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F^0_{\mu\nu})^2 + \frac{1}{g_N^2}|\mathcal{D}_\mu\phi^a|^2 + \frac{1}{g_1^2}|\mathcal{D}_\mu\phi^0|^2 + |\mathcal{D}_\mu q|^2 + |\mathcal{D}_\mu\bar{\tilde{q}}|^2$$
$$- \; g_1^2\left| -\mu\, m\sqrt{N(N+1)} + \frac{\tilde{q}\, q}{\sqrt{N(N+1)}}\right|^2 - g_N^2|\sqrt{2}\,\tilde{q}\, t^a q\,|^2 + \ldots \quad \text{(A.33)}$$

where $a = 1, 2, \ldots N^2-1$ labels the $SU(N)$ generators, t^a; the index 0 refers to the $U(1)$ generator $t^0 = \frac{1}{\sqrt{2N(N+1)}}\, diag(1,\ldots,1,-N)$. We have taken into account the fact that the $SU(N)$ and $U(1)$ coupling constants (g_N and g_1) get renormalized differently towards the infrared.

The adjoint scalars are fixed to its VEV, Eq. (A.31), with small fluctuations around it,

$$\Phi = \langle\Phi\rangle(1 + \langle\Phi\rangle^{-1}\,\tilde{\Phi}), \qquad |\tilde{\Phi}| \ll m. \quad \text{(A.34)}$$

In the consideration of the vortices of the low-energy theory, they will be in fact replaced by the constant VEV. The presence of the small terms Eq. (A.34), however, makes the low-energy vortices not strictly BPS (and this will be important in the consideration of their stability below).[5]

The quark fields are replaced, consistently with Eq. (A.28), as

$$\tilde{q} \equiv q^\dagger, \qquad q \to \frac{1}{\sqrt{2}}\, q, \quad \text{(A.35)}$$

where the second replacement brings back the kinetic term to the standard form.

We further replace the singlet coupling constant and the $U(1)$ gauge field as

$$e \equiv \frac{g_1}{\sqrt{2N(N+1)}}; \qquad \tilde{A}_\mu \equiv \frac{A_\mu}{\sqrt{2N(N+1)}}, \qquad \tilde{\phi}^0 \equiv \frac{\phi^0}{\sqrt{2N(N+1)}} \text{(A.36)}$$

[5]In the terminology used in Davis et al. [132] in the discussion of the Abelian vortices in supersymmetric models, our model corresponds to an F model while the models of [172, 170, 124] correspond to a D model. In the approximation of replacing Φ with a constant, the two models are equivalent: they are related by an $SU_R(2)$ transformation [133, 134].

The net effect is

$$\mathcal{L} = \frac{1}{4g_N^2}(F_{\mu\nu}^a)^2 + \frac{1}{4e^2}(\tilde{F}_{\mu\nu})^2 + |\mathcal{D}_\mu q|^2 - \frac{e^2}{2} \,|\, q^\dagger \, q - c\,\mathbf{1}\,|^2 - \frac{1}{2}\, g_N^2 \,|\, q^\dagger \, t^a q \,|^2 \tag{A.37}$$

$$c = 2N(N+1)\,\mu\, m. \tag{A.38}$$

Neglecting the small terms left implicit, this is identical to the $U(N)$ model Eq. (A.10), except for the fact that $e \neq g_N$ here. The transformation property of the vortices can be determined from the moduli matrix, as was done in [167]. Indeed, the system possesses BPS saturated vortices described by the linearized equations

$$(\mathcal{D}_1 + i\mathcal{D}_2)\; q = 0, \tag{A.39}$$

$$F_{12}^{(0)} + \frac{e^2}{2}\left(c\,\mathbf{1}_N - q\, q^\dagger\right) = 0; \qquad F_{12}^{(a)} + \frac{g_N^2}{2}\, q_i^\dagger\, t^a\, q_i = 0. \tag{A.40}$$

The matter equation can be solved exactly as in [173, 124, 125] ($z = x^1 + ix^2$) by setting

$$q = S^{-1}(z,\bar{z})\, H_0(z), \quad A_1 + i\, A_2 = -2\, i\, S^{-1}(z,\bar{z})\, \bar{\partial}_z S(z,\bar{z}), \tag{A.41}$$

where S is an $N \times N$ invertible matrix over whole of the z plane, and H_0 is the moduli matrix, holomorphic in z.

The gauge field equations take a slightly more complicated form than in the $U(N)$ model Eq. (A.10):

$$\partial_z\,(\Omega^{-1}\partial_{\bar{z}}\,\Omega) = -\frac{g_N^2}{2}\,\mathrm{Tr}\,(\, t^a\,\Omega^{-1}\, q\, q^\dagger)\, t^a - \frac{e^2}{4N}\,\mathrm{Tr}\,(\,\Omega^{-1} q\, q^\dagger - \mathbf{1}), \;\; \Omega = S\, S^\dagger \tag{A.42}$$

The last equation reduces to the master equation Eq. (A.19) in the $U(N)$ limit, $g_N = e$.

The advantage of the moduli matrix formalism is that all the moduli parameters appear in the holomorphic, moduli matrix $H_0(z)$. Especially, the transformation property of the vortices under the color-flavor diagonal group can be studied by studying the behavior of the moduli matrix.

A.3 Topological stability, vortex-monopole complex and confinement

The fact that there must be a continuous set of monopoles, which transform under the color-flavor G_{C+F} group, follows from the following exact homotopy sequence

$$\cdots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \xrightarrow{f} \pi_1(G) \to \cdots , \tag{A.43}$$

applied to our systems with a hierarchical symmetry breaking, Eq. (A.7), with an exact unbroken symmetry, Eq. (A.9). $\pi_2(G) = 1$ for any Lie group, and $\pi_1(G)$ depends on the group considered. Eq. (A.43) was earlier used to obtain the relation between the regular, soliton monopoles (represented by $\pi_2(G/H)$) and the singular Dirac monopoles, present if $\pi_1(G)$ is non-trivial. The isomorphism

$$\pi_1(G) \sim \pi_1(H)/\pi_2(G/H) \tag{A.44}$$

implied by Eq. (A.43) shows that among the magnetic monopole configurations $A_i^a(x)$ classified according to $\pi_1(H)$ [136], the regular monopoles correspond to the kernel of the map $f : \pi_1(H) \to \pi_1(G)$ [154].

When the homotopy sequence Eq. (A.43) is applied to a system with hierarchical breaking, in which H is completely broken at low energies,

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1,$$

it allows an interesting re-interpretation. $\pi_1(H)$ classifies the quantized flux of the vortices in the low-energy H theory in Higgs phase. Vice versa, the high-energy theory (in which the small VEV is negligible) has 't Hooft-Polyakov monopoles quantized according to $\pi_2(G/H)$. However, there is something of a puzzle: when the small VEV's are taken into account, which break the "unbroken" gauge group completely, these monopoles must disappear somehow. A related puzzle is that the low-energy vortices with $\pi_1(H)$ flux, would have to disappear in a theory where $\pi_1(G)$ is trivial.

What happens is that the massive monopoles are confined by the vortices and disappear from the spectrum; on the other hand, the vortices of the low-energy theory end at the heavy monopoles once the latter are taken into account, having mass large but not infinite (Fig. A.2). The low-energy vortices become unstable also through heavy monopole pair productions which break the vortices in the middle (albeit with small, tunneling rates [138]), which is really the same thing. Note that, even if the effect of such string breaking is neglected, a monopole-vortex-antimonopole configuration is not topologically stable anyway: its energy would become smaller if the string becomes shorter (so such a composite, generally, *will* get shorter and shorter and eventually disappear).

In the case $G = SU(N+1)$, $H = \frac{SU(N)\times U(1)}{\mathbb{Z}_N}$ we have a trivial $\pi_1(G)$, so

$$\pi_2\left(\frac{SU(N+1)}{U(N)}\right) = \pi_2(\mathbf{C}P^N) \sim \pi_1(U(N)) = \mathbb{Z} : \tag{A.45}$$

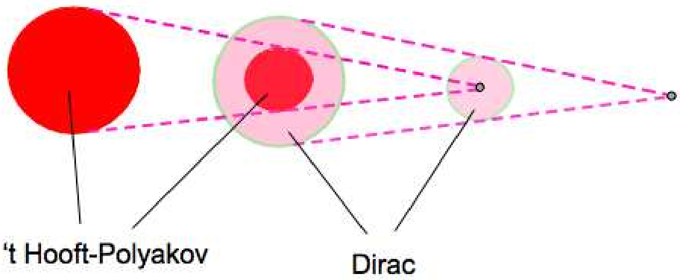

Figure A.1: A pictorial representation of the exact homotopy sequence, Eq. (A.43), with the leftmost figure corresponding to $\pi_2(G/H)$.

each non-trivial element of $\pi_1(U(N))$ is associated with a non-trivial element of $\pi_2(\frac{SU(N+1)}{U(N)})$. Each vortex confines a regular monopole. The monopole transformation properties follow from those of the vortices, as will be more concretely studied in the next section.

In theories with a non-trivial $\pi_1(G)$ such as $SO(N)$, the application of these ideas is slightly subtle: these points will be discussed in Section A.5.

In all cases, as long as the group H is completely broken at low energies and because $\pi_2(G) = 1$ always, none of the vortices (if $\pi_1(G) = 1$) and monopoles are truly stable, as static configurations. They can be only approximately so, in an effective theory valid in respective regions ($v_1 \simeq \infty$ or $v_2 \simeq 0$).

However, this does not mean that, for instance, a monopole-vortex-antimonopole composite configuration cannot be *dynamically* stabilized, or that they are not relevant as a physical configuration. A rotation can stabilize easily such a configuration dynamically, except that it will have a small non-vanishing probability for decay through a monopole-pair production, if such a decay is allowed kinematically.

After all, we believe that the real-world mesons are quark-string-antiquark bound states of this sort, the endpoints rotating almost with a speed of light! An excited meson can and indeed do decay through quark pair productions into two lighter mesons (or sometimes to a baryon-antibaryon pair, if allowed kinematically and by quantum numbers). Only the lightest mesons are truly stable. The same occurs with our monopole-vortex-antimonopole configurations. The lightest such systems, after the rotation modes are appropriately quantized, are truly stable bound states of solitons, even though they might not be stable as static, semiclassical configurations.

Our model is thus a reasonably faithful (dual) model of the quark con-

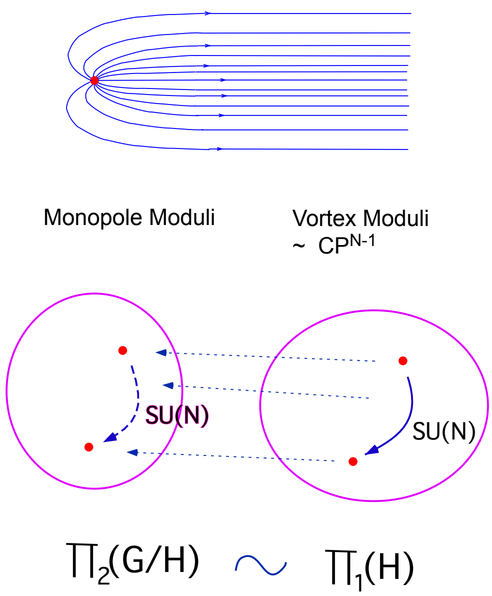

Figure A.2: The non-trivial vortex moduli implies a corresponding moduli of monopoles.

finement in QCD.

A related point, more specific to the supersymmetric models we consider here as a concrete testing ground, is the fact that monopoles in the high-energy theory and vortices in the low-energy theory, are both BPS saturated. It is crucial in our argument that they are both BPS only approximately; *they are almost BPS but not exactly.*[6] They are unstable in the full theory. But the fact that there exists a limit (of a large ratio of the mass scales, $\frac{v_1}{v_2} \to \infty$) in which these solitons become exactly BPS and stable, means that the magnetic flux through the surface of a small sphere surrounding the monopole and the vortex magnetic flux through a plane perpendicular to the vortex axis, must match exactly. These questions (the flux matching) have been discussed extensively already in [171].

Our argument, applied to the simplest case, $G = SO(3)$, and $H = U(1)$, is precisely the one adopted by 't Hooft in his pioneering paper [101] to

[6]The importance of almost BPS soliton configurations have also been emphasized by Strassler [177].

argue that there must be a regular monopole of charge two (with respect to the Dirac's minimum unit): as the vortex of winding number $k = 2$ must be trivial in the full theory ($\pi_1(SO(3)) = \mathbb{Z}_2$), such a vortex must end at a regular monopole. What is new here, as compared to the case discussed by 't Hooft [101] is that now the unbroken group H is non-Abelian and that the low-energy vortices carry continuous, non-Abelian flux moduli. The monopoles appearing as the endpoints of such vortices must carry the same continuous moduli (Fig. A.2).

The fact that the vortices of the low-energy theory are BPS saturated (which allows us to analyze their moduli and transformation properties elegantly, as discussed in the next section), while in the full theory there are corrections which make them non BPS (and unstable), could cause some concern. Actually, the rigor of our argument is not affected by those terms which can be treated as perturbation. The attributes characterized by integers such as the transformation property of certain configurations as a multiplet of a non-Abelian group which is an *exact symmetry group* of the full theory, cannot receive renormalization. This is similar to the current algebra relations of Gell-Mann which are not renormalized. Conserved vector current (CVC) of Feynman and Gell-Mann [139] also hinges upon an analogous situation.[7] The results obtained in the BPS limit (in the limit $v_2/v_1 \to 0$) are thus valid at any finite values of v_2/v_1.

A.4 Dual gauge transformation among the monopoles

The concepts such as the low-energy BPS vortices or the high-energy BPS monopole solutions are thus only approximate: their explicit forms are valid only in the lowest-order approximation, in the respective kinematical regions. Nevertheless, there is a property of the system which is exact and does not depend on any approximation: the full system has an exact, global $SU(N)_{C+F}$ symmetry, which is neither broken by the interactions nor by both sets of VEVs, v_1 and v_2. This symmetry is broken by individual soliton vortex, endowing the latter with non-Abelian orientational moduli, analogous to the translational zero-modes of a kink. Note that the vortex breaks the color-flavor symmetry as

$$SU(N)_{C+F} \to SU(N-1) \times U(1), \tag{A.46}$$

[7] The absence of "colored dyons" [155] mentioned earlier can also be interpreted in this manner.

leading to the moduli space of the minimum vortices which is

$$\mathcal{M} \simeq \mathbf{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)}. \tag{A.47}$$

The fact that this moduli coincides with the moduli of the quantum states of an N-state quantum mechanical system, is a first hint that the monopoles appearing at the endpoint of a vortex, transform as a fundamental multiplet $\underline{N}$ of a group $SU(N)$.

The moduli space of the vortices is described by the moduli matrix (we consider here the vortices of minimal winding, $k = 1$)

$$H_0(z) \simeq \begin{pmatrix} 1 & 0 & 0 & -a_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -a_{N-1} \\ 0 & \dots & 0 & z \end{pmatrix}, \tag{A.48}$$

where the constants a_i, $i = 1, 2, \dots, N-1$ are the coordinates of $\mathbf{C}P^{N-1}$. Under $SU(N)_{C+F}$ transformation, the squark fields transform as

$$q \to U^{-1}\, q\, U, \tag{A.49}$$

but as the moduli matrix is defined *modulo* holomorphic redefinition Eq. (A.20), it is sufficient to consider

$$H_0(z) \to H_0(z)\, U. \tag{A.50}$$

Now, for an infinitesimal $SU(N)$ transformation acting on a matrix of the form Eq. (A.48), U can be taken in the form,

$$U = \mathbf{1} + X, \qquad X = \begin{pmatrix} \mathbf{0} & \vec{\xi} \\ -(\vec{\xi})^\dagger & 0 \end{pmatrix}, \tag{A.51}$$

where $\vec{\xi}$ is a small $N-1$ component constant vector. Computing $H_0\, X$ and making a V transformation from the left to bring back H_0 to the original form, we find

$$\delta a_i = -\xi_i - a_i\, (\vec{\xi})^\dagger \cdot \vec{a}, \tag{A.52}$$

which shows that a_i's indeed transform as the inhomogeneous coordinates of $\mathbf{C}P^{N-1}$. In other words, the vortex represented by the moduli matrix Eq. (A.48) transforms as a fundamental multiplet of $SU(N)$.[8]

[8]Note that, if a $\underline{N}$ vector $\vec{c}$ transforms as $\vec{c} \to (\mathbf{1}+X)\,\vec{c}$, the inhomogeneous coordinates $a_i = c_i/c_N$ transform as in Eq. (A.52).

As an illustration consider the simplest case of $SU(2)$ theory. In this case the moduli matrix is simply [140]

$$H_0^{(1,0)} \simeq \begin{pmatrix} z-z_0 & 0 \\ -b_0 & 1 \end{pmatrix}; \qquad H_0^{(0,1)} \simeq \begin{pmatrix} 1 & -a_0 \\ 0 & z-z_0 \end{pmatrix}. \tag{A.53}$$

with the transition function between the two patches:

$$b_0 = \frac{1}{a_0}. \tag{A.54}$$

The points on this $\mathbf{C}P^1$ represent all possible $k = 1$ vortices. Note that points on the space of a quantum mechanical two-state system,

$$|\Psi\rangle = a_1|\psi_1\rangle + a_2\,|\psi_2\rangle, \qquad (a_1, a_2) \sim \lambda\,(a_1, a_2), \quad \lambda \in \mathbf{C}, \tag{A.55}$$

can be put in one-to-one correspondence with the inhomogeneous coordinate of a $\mathbf{C}P^1$,

$$a_0 = \frac{a_1}{a_2}, \qquad b_0 = \frac{a_2}{a_1}. \tag{A.56}$$

In order to make this correspondence manifest, note that the minimal vortex Eq. (A.53) transforms under the $SU(2)_{C+F}$ transformation, as

$$H_0 \to V\,H_0\,U^\dagger, \qquad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{A.57}$$

where the factor $U^\dagger$ from the right represents a flavor transformation, V is a holomorphic matrix which brings H_0 to the original triangular form [167]. The action of this transformation on the moduli parameter, for instance, a_0, can be found to be

$$a_0 \longrightarrow \frac{\alpha\,a_0 + \beta}{\alpha^* - \beta^*\,a_0}. \tag{A.58}$$

But this is precisely the way a doublet state Eq. (A.55) transforms under $SU(2)$,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{A.59}$$

The fact that the vortices (seen as solitons of the low-energy approximation) transform as in the $\underline{N}$ representation of $SU(N)_{C+F}$, implies that there exist a set of monopoles which transform accordingly, as $\underline{N}$. The existence of such a set follows from the exact $SU(N)_{C+F}$ symmetry of the theory, broken by the individual monopole-vortex configuration. This answers questions such as Eq. (A.5) unambiguously.

Note that in our derivation of continuous transformations of the monopoles, the explicit, semiclassical form of the latter is not utilized.

A subtle point is that in the high-energy approximation, and to lowest order of such an approximation, the semiclassical monopoles are just certain non-trivial field configurations involving $\phi(x)$ and $A_i(x)$ fields, and therefore apparently transform under the color part of $SU(N)_{C+F}$ only. When the full monopole-vortex configuration $\phi(x), A_i(x), q(x)$ (Fig. A.2) are considered, however, only the combined color-flavor diagonal transformations keep the energy of the configuration invariant. In other words, the monopole transformations must be regarded as part of more complicated transformations involving flavor, when higher order effects in $O(\frac{v_1}{v_2})$ are taken into account.[9]

And this means that the transformations are among physically distinct states, as the vortex moduli describe obviously physically distinct vortices [169].

A.4.1 $SU(N)$ gauge symmetry breaking and Abelian monopole-vortex systems

Recently there has been considerable amount of research activity [168],[172]-[127], on systems closely related to ours. As the terminology used and concepts involved are often similar but physically distinct, a confusion might possibly arise.

As should be clear from what we said so far, it is crucial that the color-flavor diagonal symmetry $SU(N)$ remains exactly conserved, for the emergence of non-Abelian dual gauge group. Consider, instead, the cases in which the gauge $U(N)$ (or $SU(N) \times U(1)$) symmetry is broken to Abelian subgroup $U(1)^N$, either by small quark mass differences (*cfr.* Eq. (A.14) and Eq. (A.16)) or dynamically, as in the $\mathcal{N} = 2$ models with $N_f < 2\,N$ [120, 170]. From the breaking of various $SU(2)$ subgroups to $U(1)$ there appear light 't Hooft-Polyakov monopoles of mass $O(\frac{\Delta m}{g})$ (in the case of an explicit breaking) or $O(\Lambda)$ (in the case of dynamical breaking). As the $U(1)^N$ gauge group is further broken by the squark VEVs, the system develops ANO vortices.

[9] Another independent effect due to the massless flavors is that of Jackiw-Rebbi [141]: due to the normalizable zero-modes of the fermions, the semi-classical monopole is converted to some irreducible multiplet of monopoles in the *flavor* group $SU(N_f)$. The "clouds" of the fermion fields surrounding the monopole have an extension of $O(\frac{1}{v_1})$, which is much smaller than the distance scales associated with the infrared effects discussed here and should be regarded as distinct effects.

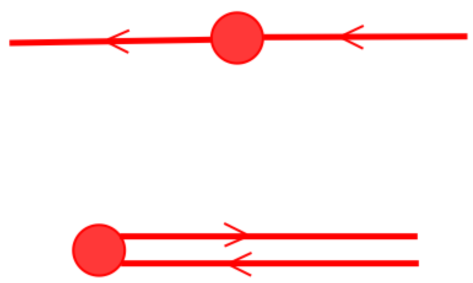

Figure A.3: Monopoles in $U(N)$ systems with abelianization are confined by two Abelian vortices.

The light magnetic monopoles, carrying magnetic charges of two different $U(1)$ factors, look confined by the two vortices (Fig. A.3). These cases have been discussed extensively, within the context of $U(N)$ model of Subsection A.2.1, in [168],[172]-[124]. In Hanany et al. [172, 120] and Shifman et al. [170, 122], furthermore, the dynamics of the fluctuation of the orientational modes along the vortex, described as a two-dimensional $\mathbf{C}P^{N-1}$ model, is studied. It is shown that the kinks of the two-dimensional sigma model precisely correspond to these light monopoles, to be expected in the underlying $4D$ gauge theory. In particular, it was noted that there is an elegant matching between the dynamics of two-dimensional sigma model (describing the dynamics of the vortex orientational modes in the Higgs phase of the $4D$ theory) and the dynamics of the $4D$ gauge theory in the Coulomb phase [142, 172, 120, 170].

Note that this is also a reasonably close (dual) model of *what would occur* in QCD if the color $SU(3)$ symmetry were to dynamically break itself to $U(1)^2$, *i.e.*, with generators $Q^1 = diag\,(1,-1,0)$, $Q^2 = diag\,(0,1,-1)$, respectively. Confinement would be described in this case by the condensation of magnetic monopoles carrying the Abelian charges Q^1, or Q^2, and the resulting ANO vortices will be of two types, 1 and 2 carrying the related fluxes. The quark q_1 will be confined by the vortex 1, the quark q_2 by the composite of the vortices $\bar{1}$ and 2 (just as the light monopoles discussed above – Fig. A.3) and the quark q_3 by the vortex $\bar{2}$.

A.4.2 Non-Abelian duality requires an exact flavor symmetry

In the $\mathcal{N}=2$ supersymmetric QCD, the presence of massless flavor and the exact color-flavor diagonal symmetry is fundamental for the emergence of the dual (non-Abelian) gauge transformations. It is well known in fact that the continuous non-Abelian vortex flux moduli - hence the non-Abelian vortex - disappear as soon as non-zero mass differences $m_i - m_j$ are introduced.[10] Also in order for the $SU(N)_{C+F}$ color-flavor symmetry not to be destroyed by the gauge dynamics itself, it is necessary to have the number of flavors such that $N_f \geq 2\,N$. These points have been emphasized already in the first paper on the subject [169].

It is illuminating that the same phenomenon can be seen in the fully quantum behavior of the theory of Section A.2.2, in another regime,

$$\mu, m_i \sim \Lambda \tag{A.60}$$

(*cfr.* Eq. (A.30)). Indeed, this model was analyzed thoroughly in this regime in [162]. The so-called r vacua with the low-energy effective $SU(r) \times U(1)^{N+1-r}$ gauge symmetry emerges in the equal mass limit $m_i \to m$ in which the global symmetry group $SU(N_f) \times U(1)$ of the underlying theory become exact. When the bare quark masses are almost equal but distinct, the theory possesses a group of $\binom{N_f}{r}$ nearby vacua, each of which is an Abelian $U(1)^N$ theory, with N massless Abelian magnetic monopole pairs. The jump from the $U(1)^N$ to $SU(r) \times U(1)^{N+1-r}$ theory in the exact $SU(N_f)$ limit might appear a discontinuous change of physics, but is not so. What happens is that the range of validity of Abelian description in each Abelian vacuum, neglecting the light monopoles and gauge bosons (including massless particles of the neighboring vacua, and other light particles which fill up a larger gauge multiplet in the limit the vacua coalesce), gradually tends to zero as the vacua collide. The non-Abelian, enhanced gauge symmetry of course only emerges in the strictly degenerate limit, in which the underlying theory has an exact $SU(N_f)$ global symmetry.

[10] Such an alignment of the vacuum with the bare mass parameters is characteristic of supersymmetric theories, familiar also in the $\mathcal{N}=1$ SQCD [143]. In real QCD we do not expect such a strict alignment.

A.5 $SO(2N+1) \to SU(r) \times U(1)^{N-r-1} \to 1$

Let us now test our ideas about duality transformations against another class of theories,

$$SO(2N+1) \stackrel{\langle\phi_1\rangle\neq 0}{\longrightarrow} SU(r) \times U(1)^{N-r+1} \stackrel{\langle\phi_2\rangle\neq 0}{\longrightarrow} 1. \tag{A.61}$$

One of the reasons why this case is interesting is that the semiclassical monopoles arising from the symmetry breaking $SO(2N+1) \stackrel{\langle\phi_1\rangle\neq 0}{\longrightarrow} U(N)$ appear to belong to the second-rank symmetric tensor representation of $SU(N)$ [156, 157]. Another, related reason is the fact that since $\pi_1(G) = \pi_1(SO(2N+1)) = \mathbb{Z}_2$, the homotopy map Eq. (A.43) is less trivial in this case. Thirdly, according to the detailed analysis of the softly-broken $\mathcal{N}=2$ theories with $SO(N)$ gauge group [111] the quantum mechanical behavior of the monopoles is different for $r = N$ and for $r < N$. Non-Abelian monopoles belonging to the fundamental representation of the dual $SU(r)$ group appears only for $r \leq N_f/2$, and because of the requirement of asymptotic freedom of the original theory ($N_f < 2N-1$), this is possible only for $r < N$. It is very encouraging that such a difference in the behavior of non-Abelian monopoles indeed follows, as we shall see, from the way we define the dual group though the transformation properties of mixed monopole-vortex configurations and homotopy map.

A.5.1 Maximal SU factor; $SO(5) \to U(2) \to 1$

Let us first consider the case the $SU(N)$ factor has the maximum rank,

$$SO(2N+1) \stackrel{\langle\phi_1\rangle\neq 0}{\longrightarrow} U(N).$$

To be concrete, let us consider the case of an $SO(5)$ theory, where a scalar VEV of the form

$$\langle\Phi\rangle = \begin{pmatrix} 0 & i\,v & 0 & 0 & 0 \\ -i\,v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,v & 0 \\ 0 & 0 & -i\,v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.62}$$

breaking the gauge group as $SO(5) \to H = SU(2) \times U(1)/\mathbb{Z}_2 = U(2)$. We assume that at lower energies some other scalar VEVs break H completely, leaving however a color-flavor diagonal $SU(2)$ group unbroken. This model arises semiclassically in softly broken $\mathcal{N}=2$ supersymmetric $SO(5)$ gauge theory with large, equal bare quark masses, m, and with a small adjoint scalar mass μ, with scalar VEVs given by $v = m/\sqrt{2}$ in Eq. (A.62) and

$$Q = \tilde{Q}^{\dagger} = \sqrt{\frac{\mu m}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ i & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & i & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix} . \qquad \text{(A.63)}$$

(See Appendix A.7, also the Section 2 of [111], for more details).

The $SO(4) \sim SU(2) \times SU(2)$ subgroup living on the upper-left corner is broken to $SU(2) \times U(1)$, giving rise to a single 't Hooft-Polyakov monopole. On the other hand, by embedding the 't Hooft-Polyakov monopole in the two $SO(3)$ subgroups (in the (125) and (345) subspaces), one finds two more monopoles. All three of them are degenerate. Actually, E. Weinberg [144] has found a continuous set of degenerate monopole solutions interpolating these, and noted that the transformations among them are not simply related to the unbroken $SU(2)$ group.[11]

From the point of view of stability argument, Eq. (A.43), this case is very similar to the case considered by 't Hooft, as $\pi_1(SO(5)) = \mathbb{Z}_2$: a singular $\mathbb{Z}_2$ Dirac monopole can be introduced in the theory. The minimal vortex of the low-energy theory is truly stable in this case, as a minimal non-trivial element of $\pi_1(H)$ represents also a non-trivial element of $\pi_1(G)$. This can be seen as follows. A minimum element of $\pi_1(H) = \pi_1(U(2)) \sim \mathbb{Z}$ corresponds to simultaneous rotations of angle π in the (12) and (34) planes (which is a half circle of $U(1)$), which brings the origin to the $\mathbb{Z}_2$ element of $SU(2)$, $diag\,(-1,-1,-1,-1,1)$, followed by an $SU(2)$ transformation back to the origin, an angle $-\pi$ rotation in the (12) plane and an angle π rotation around (34) plane. The net effect is a 2π rotation in the (34) plane, which is indeed a non-trivial element of $\pi_1(SO(5)) = \mathbb{Z}_2$. Such a vortex would confine the singular Dirac monopole, if introduced into the theory (See Fig. A.1).

On the other hand, there are classes of vortices which appear to be stable in the low-energy approximation, but are not so in the full theory. In fact non-minimal $k = 2$ elements of $\pi_1(H) = \pi_1(SU(2) \times U(1)/\mathbb{Z}_2) \sim \mathbb{Z}$ are actually trivial in the full theory. This means that the $k = 2$ vortices must end at a regular monopole. Vice versa, as $\pi_2(SO(5)) = 1$, the regular 't Hooft Polyakov monopoles of high-energy theory must be confined by these non-minimal vortices and disappear from the spectrum.

The transformation property of $k = 2$ vortices has been studied recently in [145, 166], and in particular, in [167]. It turns out that the moduli space

[11] This and similar cases are sometimes referred to as "accidentally degenerate case" in the literature.

of the $k = 2$ vortices is a $\mathbf{C}P^2$ with a conic singularity. It was shown that the generic $k = 2$ vortices transform under the $SU(2)_{C+F}$ group as a *triplet*. At a particular point of the moduli - an orbifold singularity - the vortex is Abelian: it is a *singlet* of $SU(2)_{C+F}$. [12]

As the full theory has an exact, unbroken $SU(2)_{C+F}$ symmetry, it follows from the homotopy-group argument of Section A.3 that *the monopoles in the high-energy $SO(5) \to U(2)$ theory have components transforming as a triplet and a singlet of $SU(2)_{C+F}$.*

Note that it is not easy to see this result – and is somewhat misleading to attempt to do so – based solely on the semi-classical construction of the monopoles or on the zero-mode analysis around such solutions, where the unbroken color-flavor symmetry is not appropriately taken into account. Generically, the "unbroken" color $SU(2)$ group suffers from the topological obstruction [155] (or perturbatively, from the pathology of non-normalizable gauge zero-modes [155, 156]), as we noted already.

Nevertheless, there are indications that the findings by E. Weinberg [144] are consistent with the properties of the $k = 2$ vortices. In the standard way to embed $SU(2)$ subgroups through the Cartan decomposition (we follow here the notation of [144]),

$$t_1(\nu) = \frac{1}{(2\,\nu^2)^{-1/2}}\,(E_\nu + E_{-\nu});\;\; t_2(\nu) = \frac{-i}{(2\,\nu^2)^{-1/2}}\,(E_\nu - E_{-\nu});\;\; t_3 = (\nu^2)^{-1}\,\nu_j\,T_j \quad \text{(A.64)}$$

where ν denotes the non-vanishing root vectors of $SO(5)$ (Fig. A.4), the unbroken $SU(2)$ group is generated by γ. The monopole associated with the root vector β and the (equivalent) one given by μ naturally form a doublet of the "unbroken" $SU(2)$, while the monopole with the α charges is a singlet. The continuous set of monopoles interpolating among these monopoles found by Weinberg are analogous to the continuous set of vortices we found, which form the points of the $\mathbf{C}P^2$, which transform as a triplet. (See the Fig. A.5 taken from [167]).

An even more concrete hint of consistency comes from the structure of the moduli space of the monopoles. The moduli metric found in [144] is

$$ds^2 = M\,d\mathbf{x}^2 + \frac{16\pi^2}{M}d\chi^2 + k\left[\frac{db^2}{b} + b(d\alpha^2 + \sin^2\alpha\,d\beta^2 + (d\gamma + \cos\alpha\,d\beta)^2)\right] \quad \text{(A.65)}$$

By performing a simple change of coordinate, $B \equiv 2\sqrt{b}$, it becomes evident

[12]In another complex codimension-one subspace, they appear to transform as a *doublet*. However quantum states of *any* triplet of $SU(2)$ contains such an orbit. The state of maximum S_z, $|1,1\rangle$, transforms under $SU(2)$ as an $SO(3)$ vector, staying on a subspace $S^2 \sim CP^1 \subset CP^2$.

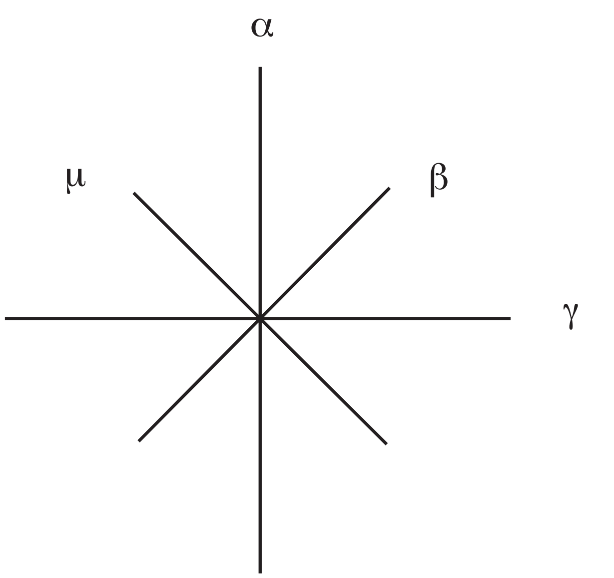

Figure A.4: Non-zero root vectors of $SO(5)$

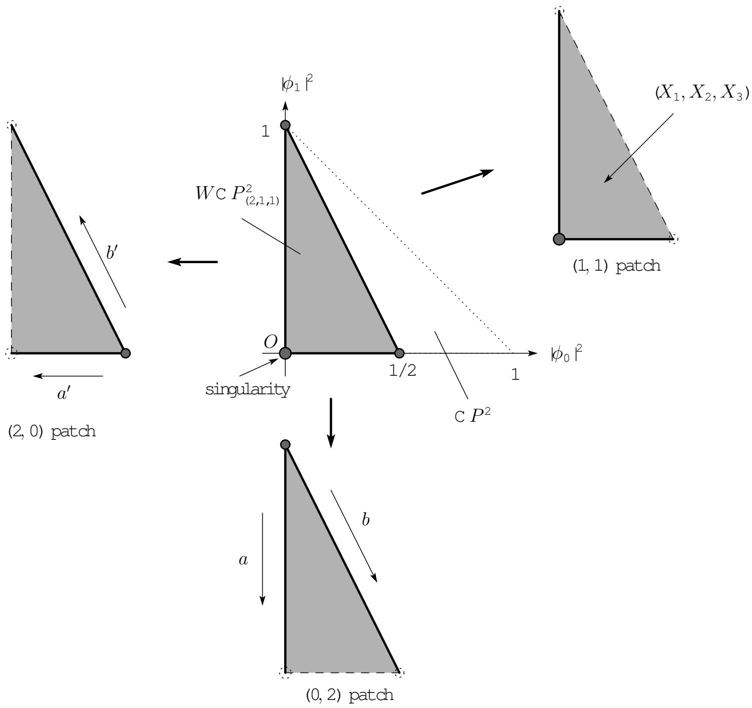

Figure A.5: Moduli space of $k = 2$ vortices of $U(2)$ theory. See [167] for more details.

that the moduli space has the structure

$$\mathbf{C}^2/\mathbb{Z}_2, \tag{A.66}$$

apart from the irrelevant factor $\mathbf{R}^3$ (the position of the monopole) and S^1 ($U(1)$ phase).[13] Eq. (A.66) coincides with the moduli space of the $k=2$ co-axial *vortices*, seen in the central $(1,1)$ patch [167].

These considerations strengthen our conclusion that the continuous set of monopoles found in [144] belongs to a singlet and a triplet representations of the dual $SU(2)$ group. Although the detailed properties of the moduli spaces for monopoles and vortices are different[14], this could be related to the fact that one should ultimately consider a smooth monopole-vortex mixed configurations in the full theory, not each of them separately. Also, related to this point, there remains the fact that the dual group which is exact and under which monopoles transform, is *not* the original $SU(2)$ subgroup but involves the flavor group essentially.

Note that our conclusion is based on the exact symmetry, and should be reliable. However, the degeneracy among all the vortices (or the monopoles) lying in the entire moduli space $\mathbf{C}P^2/\mathbb{Z}_2$ found in the BPS limits, is an artifact of the lowest-order approximation. Only the degeneracy among the vortices (or among the monopoles) belonging to the same multiplet is expected to survive quantum mechanically. $\underline{1}$ and $\underline{3}$ vortex tensions (monopole masses) will split. Which of the multiplets ($\underline{1}$ or $\underline{3}$) will remain stable, after quantum corrections are taken into account, is a question just lying beyond the power of semiclassical considerations.

In the context of asymptotically-free $\mathcal{N}=2$ supersymmetric models, there are no indications that the triplet monopoles of $SO(5) \to U(2)$ theory survive quantum mechanically. This result can be actually understood by a simple renormalization-group argument:

- In a $SO(2N+1)$ theories with $\mathcal{N}=2,1$ supersymmetries, the condition for the original theory to be asymptotic-free (N_f less than $2N-1$, $\frac{3(2N-2)}{2}$, respectively)[15] is not compatible with the low-energy $SU(N)$ theory being non-asymptotic-free ($N_f \geq 2N$ and $N_f \geq 3N$, respectively.)

[13]The monopole modulus due to the unbroken $U(1) \subset U(2)$ is not present in the full system, where the gauge group is completely broken.

[14]The first is known to be hyper-Kähler and the second Kähler – indeed $\mathbf{C}P^2/\mathbb{Z}_2$ does not admit hyper-Kähler structure.

[15]The counting is made for the appropriate supersymmetry multiplets, N_f hypermultiplets for $\mathcal{N}=2$; N_f chiral multiplets for $\mathcal{N}=1$ supersymmetric $SO(N)$ theory.

The problem would not arise if the rank of the unbroken $SU(r)$ were smaller. That such a "sign-flip" of the beta function is a necessary condition for the emergence of low-energy non-Abelian monopoles has been pointed out some time ago by one of the authors [147], even though the validity of such an argument for non-supersymmetric theories is perhaps not obvious.

If the condition of asymptotic freedom of the ultraviolet theory is dropped, then there are no such constraints, and it makes sense to consider symmetry breaking patterns such as $SO(2N+1) \to U(N)$. Our conclusion that the monopoles of $SO(5) \to U(2)$ system transform as a triplet or a singlet would apply under such conditions. Analogously, we expect the monopoles in the system $SO(2N+1) \to U(N)$ to transform as a second-rank symmetric or antisymmetric representation.

A.5.2 $SO(2N+1) \to SU(r) \times U(1)^{N-r-1} \to 1$ $(r < N)$

Consider now the cases in which the unbroken $SU(r)$ factor has a smaller rank, $SO(2N+1) \to SU(r) \times U(1)^{N-r+1} \to 1$, where $r < N$. For concreteness, let us discuss the case of an $SO(7)$ theory,

$$SO(7) \overset{\langle\phi_1\rangle\neq 0}{\longrightarrow} U(2) \times U(1) \overset{\langle\phi_2\rangle\neq 0}{\longrightarrow} 1. \tag{A.67}$$

As we are interested in a concrete dynamical realization of this, we consider the softly broken $\mathcal{N}=2$ theory, with $N_f = 4$ quark hypermultiplets. Such a number of flavors ensures both the original $SO(7)$ theory being asymptotically free and the $SU(2)$ subgroup being non-asymptotically free. The low-energy gauge group $U(2) \times U(1)$ is completely broken by the squark VEV's similar to Eq. (A.63). The large VEV $\langle\phi_1\rangle$ has the form:

$$\langle\phi_1\rangle = \begin{pmatrix} 0 & iv_0 & 0 & 0 & 0 & 0 & 0 \\ -iv_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & iv_0 & 0 & 0 & 0 \\ 0 & 0 & -iv_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & iv_1 & 0 \\ 0 & 0 & 0 & 0 & -iv_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_1 \neq v_0. \tag{A.68}$$

The "unbroken" $U(2)$ lies in $SO(4)_{1234} \sim SU(2) \times SU(2)$ while the $U(1)$ factor corresponds to the rotations in the 56 plane (see Appendix A.7). The semiclassical monopoles of high-energy theory are [16]

[16]Within the softly broken $\mathcal{N}=2$ theory, the quantum mechanical vacua with $SU(2) \times U(1)^2$ gauge symmetry, in the limit $m_i = m \simeq \Lambda$, appears to arise from the semiclassical

(i) a triplet of degenerate monopoles of mass $2\,|v_0|/g$ (they arise as in the $SO(5)$ theory discussed above);

(ii) a doublet of degenerate monopoles of mass $|v_0 - v_1|/g$: they arise from the breaking of $SU_+(2) \subset SO(4)_{1256}$ and $SU_+(2) \subset SO(4)_{3456}$ (see Appendix A.7);

(iii) a doublet of degenerate monopoles of mass $|v_0 + v_1|/g$: they also arise from the breaking of $SU_-(2) \subset SO(4)_{1256}$ and $SU_-(2) \subset SO(4)_{3456}$;

(iv) a singlet monopole of mass $2\,|v_1|/g$ arising from the breaking of $SO(3)_{567}$.

Which of these semiclassical monopoles are the lightest and which of them are stable against decay into lighter monopole pairs, depend on the various VEVs. It is possible that the monopoles (ii) or (iii) are the lightest of all. Of course more detailed issues such as which of the degeneracies survives quantum effects, are questions which go beyond the semiclassical approximations.

In fact, when $v_0, v_1 \sim \Lambda$ the standard semi-classical reasoning fails to give any reliable answer: a fully quantum-mechanical analysis is needed. Fortunately, in the softly broken $\mathcal{N} = 2$ theory such analyses have been performed [111] and we do know that the light monopoles in the fundamental representation ($\underline{2}$) of $SU(2)$ appear in an appropriate vacuum.

Knowing this, we might try to understand how such a result may follow from our definition of the dual group. At low energies the gauge group $U(2) \times U(1)$ is completely broken, leaving a color-flavor diagonal $SU(2)_{C+F}$ symmetry unbroken. The theory possesses vortices of

$$\pi_1(U(2) \times U(1)) = \mathbb{Z} \times \mathbb{Z}. \tag{A.69}$$

The minimal vortices corresponding to $\pi_1(U(2)) = \mathbb{Z}$ transform as a $\underline{2}$ of $SU(2)_{C+F}$.

A minimum element of $\pi_1(U(2) \times U(1))$ such as an angle 2π rotation in the $U(1)_{56}$ factor, or the minimal $U(2)$ loop, corresponds to vortices stable in the full theory. They would confine Dirac monopoles associated with $\pi_1(SO(7)) = \mathbb{Z}_2$, if the latter were introduced in the theory.

---

vacua of the form of Eq. (A.68), with $v_0 = m/\sqrt{2} \gg \Lambda$, $v_1 = 0$, with classical symmetry $SU(2) \times U(1) \times SO(3)_{567}$. The $SO(3)_{567}$ gauge sector (pure $\mathcal{N} = 2$ theory) becomes strongly-coupled at low energies and breaks itself to $U(1)$. Thus it would be more correct to say $v_1 \sim \Lambda$, but then the discussion about semiclassical monopole masses $\sim v_1/g$, etc., should not be taken too literally. If one wishes, one could consider a larger gauge group, *e.g.*, $SO(9)$, to do a straightforward semiclassical analysis for an unbroken $SU(2)$ group. In general, the relation between the classical vacua and the fully quantum mechanical vacua is a rather subtle issue. See for instance the discussions in [114].

The regular monopoles in which we are interested in, are instead confined by some non-minimal ($k = 2$) vortices of the low-energy theory. However, in contrast to the $SO(5)$ theory discussed in the preceding subsection, this *does not* necessarily imply a second-rank tensor representation of $SU(2)_{C+F}$ of these monopoles. In fact, the monopoles of the (ii) group, for instance, carry the minimum charge of $U(2)$ and an unit charge of $U(1)$. Therefore, the relevant $k = 2$ vortex corresponds to the minimum element both of $\pi_1(U(2))$ and of $\pi_1(U(1))$, generated by a 2π rotation in the 56 plane together with a minimal loop of $\pi_1(U(2))$, analogous to the one discussed in the preceding subsection. As a consequence the monopoles confined by such vortices, by our discussion of Section 3, transform as a *doublet* of the dual group $\widetilde{SU}(2) \sim SU(2)_{C+F}$.

This discussion naturally generalizes to all other cases with symmetry breaking, $SO(2N+1) \to SU(r) \times U(1)^{N-r+1} \to 1$, $r < N$. The dual magnetic $SU(r)$ group observed in the low-energy effective theory [111], under which the light monopoles transform as a fundamental multiplet, thus matches nicely with the properties of the dual $\widetilde{SU}(r) \sim SU(r)_{C+F}$ group.

The cases of $SO(2N) \to SU(r) \times U(1)^{N-r+1} \to 1$, $r < N - 1$ are similar. We expect that there is a qualitative difference between the breaking with the maximum (or next to the maximum) rank SU factor and smaller $SU(r)$ unbroken groups. Such a difference is indeed observed in the fully quantum mechanical analysis of $SO(N)$ theory [111].

The behavior of monopoles in asymptotic-free $USp(2N)$ theories ($N_f < 2N+2$) is more similar to those appearing in the $SU(N)$ theories, because of the property, $\pi_1(USp(2N)) = 1$. All monopoles are regular monopoles due to the partial symmetry breaking, $USp(2N) \to SU(r) \times U(1)^{N-r+1}$, $r \leq N$. The transformation property of these monopoles, in the theory with exact unbroken $SU(r)_{C+F}$ global symmetry, is deduced from the transformation properties among the non-Abelian vortices of the low-energy system $SU(r) \times U(1)^{N-r+1} \to 1$: they transform as $\underline{r}$ of $SU(r)_{C+F}$. Such a result is consistent dynamically, as long as $r \leq N_f/2$. It is comfortable that these are precisely what is found from the quantum mechanical analysis [162].

A.5.3 Other symmetry breaking patterns and GNOW duality

Before concluding this section, let us add a few remarks on other symmetry breaking patterns such as $SO(2N+3) \to SO(2N+1) \times U(1)$ and $USp(2N+2) \to USp(2N) \times U(1)$, and the resulting GNOW monopoles. These cases might be interesting as the GNOW dual groups are different from the original one: the dual of $SO(2N+1)$ is $USp(2N)$ and vice versa. It is possible to analyze these systems, again setting up models so that the "unbroken group" is completely broken at a much lower mass scales by the set of squark VEVs. Such a preliminary study has been made in [174].

However, the quantum fate of these GNOW dual monopoles is unclear. More precisely, within the concrete $\mathcal{N}=2$ models we are working on where the exact quantum fate of the semiclassical monopoles is known from the analyses made at small m, μ [111], we *know* that these GNOW monopoles do not survive quantum effects. Only the monopoles carrying the quantum numbers of the $SU(r)$ subgroups discussed in the previous subsection appear. On the other hand, there is clearly a reason why the GNOW monopoles cannot appear at low energies in these cases: the low-energy effective action would have a wrong global symmetry. GNOW monopoles are not always relevant quantum mechanically [17]. These and other peculiar (but consistent) quantum properties of non-Abelian monopoles have been recently discussed in [114].

A.6 Conclusion

In this appendix we have examined an idea about the "non-Abelian monopoles", put forward some time ago [171], more systematically and by using some recent results on the non-Abelian *vortices*. According to this idea, the dual transformation of non-Abelian monopoles occurring in a system with gauge symmetry breaking $G \longrightarrow H$ is to be defined by setting the low-energy H system in Higgs phase, so that the dual system is in confinement phase. The transformation law of the monopoles follows from that of monopole-vortex mixed configurations in the system

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1, \qquad (v_1 \gg v_2)$$

under an unbroken, exact color-flavor diagonal symmetry $H_{C+F} \sim \tilde{H}$. The transformation properties of the regular monopoles (classified by $\pi_2(G/H)$) follow from those among the non-Abelian vortices (classified by $\pi_1(H)$), via the isomorphism $\pi_1(G) \sim \pi_1(H)/\pi_2(G/H)$. Our results, obtained in the semiclassical approximation (reliable at $v_1 \gg v_2 \gg \Lambda$) of softly-broken $\mathcal{N}=2$ supersymmetric $SU(N)$ and $SO(N)$ theories, are – very non-trivially – found to be consistent with the fully quantum-mechanical low-energy effective action description (valid at $v_1, v_2 \sim \Lambda$), available in these theories.

[17] Seiberg duals of $\mathcal{N}=1$ supersymmetric theories with various matter contents, provide us with more than enough evidence for it.

For $G = SU(N+1)$, $H = U(N)$, $G_F = SU(N_f)$, $N_f \geq 2\,N$, this argument proves that the monopoles induced by the G/H breaking transform as $\underline{N}$ of $\tilde{H} = SU(N)$. Analogous result holds for $G = SU(N+1)$, $H = U(r)$, $G_F = SU(N_f)$, $r \leq N_f/2$, where the semi-classical monopoles transform as in the fundamental multiplets ($\underline{r}$) (as well as some singlets) of $SU(r)$. These results are in agreement with what was found in the fully quantum mechanical treatment of the system [160, 162].

For $G = SO(2N+1)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ (with $r \leq N_f/2$, $r < N$) we find monopoles which transform in the fundamental representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$ group. This result is again consistent with the fully quantum mechanical analysis of $\mathcal{N} = 2$ supersymmetric $SO(N)$ models [111] and in agreement with the universality of certain superconformal theories discovered in this context by Eguchi et. al. [149].

In the case of maximal-rank SU subgroup, such as $G = SO(5)$, $H = U(2)$, there is a qualitative difference both in our duality argument and in the full quantum results. For instance the set of monopoles found earlier by E. Weinberg is shown to belong to a singlet and a triplet representations of the dual $SU(2)$ group, but their quantum fate is not known. In supersymmetric models a renormalization-group argument suggests (and the explicit analysis of softly broken $\mathcal{N} = 2$ theory shows) that the triplet does not survive the quantum effects, as long as the underlying $SO(5)$ theory is asymptotically free.

For $G = SO(2N)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ the situation is similar. When $r < N-1$, $r \leq N_f/2$ we find monopoles transforming in the $\underline{r}$ representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$, whereas the maximal and next-to-maximal cases, $r = N, N-1$, encounter the same renormalization-group constraint as in $SO(2N+1)$.

Finally for $G = USp(2N)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ the picture is very much like in $SU(N+1)$. We have monopoles in the fundamental representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$ as long as $N_f \geq 2\,r$.

Summarizing, in the context of softly-broken $\mathcal{N} = 2$ supersymmetric gauge theories with SU, SO and USp groups, where fully quantum mechanical results are available by combining the various knowledges such as the Seiberg-Witten curves, decoupling theorem, Nambu-Goldstone theorem, non-renormalization of Higgs branches, $\mathcal{N} = 1$ ADS instanton superpotential, vacuum counting, universality of conformal theories, etc., our idea on non-Abelian monopoles is in agreement with these known exact results. Although such an agreement is comfortable, our arguments, based on the homotopy-map-stability argument on almost BPS solitons and on some exact symmetries, should be of more general validity.

A.7 Appendix: Monopoles in $SO(N)$ theories

Here are some formulae useful for the discussion of Section A.5. The minimal $SU(2)$ embeddings (*i.e.*, with the smallest Dynkin index, $\mathrm{Tr}\, T^a T^b$) in $SO(N)$ groups are obtained through various $SO(4) \subset SO(N)$ subgroups. For instance the $SU(2) \times SU(2) \subset SO(5)$ subgroups are generated by

$$T_1^{\pm} = -\frac{i}{2}(\Sigma_{23} \pm \Sigma_{41}) \ , \ T_2^{\pm} = -\frac{i}{2}(\Sigma_{31} \pm \Sigma_{42}) \ , \ T_3^{\pm} = -\frac{i}{2}(\Sigma_{12} \pm \Sigma_{43}) , \tag{A.70}$$

where *e.g.*

$$\Sigma_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a rotation in the 23 plane. Non-minimal embeddings correspond to various $SO(3)$ subgroups, acting in 125 and 345 subspaces, for instance, in the $SO(5)$ example.

The VEV Eq. (A.62) is proportional to T_3^+: it leaves $SU_-(2) \times U_+(1)$ unbroken. An $SO(5)$ solution can be obtained [152, 153] by embedding the 't Hooft-Polyakov monopoles [101] in the broken $SU(2)$ as ($S_a \equiv T_a^+$)

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, S_a; \qquad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h}\cdot\alpha)\, S_a + [\, \mathbf{h} - (\mathbf{h}\cdot\alpha)\, \alpha^*] \cdot \mathbf{H}, \tag{A.71}$$

where

$$A_i^a(\mathbf{r}) = \epsilon_{aij}\frac{r^j}{r^2}A(r); \qquad \chi^a(\mathbf{r}) = \frac{r^a}{r}\chi(r), \qquad \chi(\infty) = \mathbf{h}\cdot\alpha. \tag{A.72}$$

Note that $\phi(\mathbf{r} = (0,0,\infty)) = \phi_0$. In the above formula the Higgs field vacuum expectation value (VEV) has been parametrized in the form

$$\phi_0 = \mathbf{h}\cdot\mathbf{H}, \tag{A.73}$$

where $\mathbf{h} = (h_1, \ldots, h_{\mathrm{rank}(G)})$ is a constant vector representing the VEV. The root vectors orthogonal to $\mathbf{h}$ ($\propto \alpha$ in Fig. A.4) belong to the unbroken subgroup H (γ in Fig. A.4).

The above consideration is basically group-theoretic and is valid in any types of theories, supersymmetric or not. Now we specialize to the concrete dynamical models we are working on: $\mathcal{N} = 2$ supersymmetric gauge theories. Under the symmetry breaking $SO(5) \to U(2)$ the quark superfields Q and $\tilde{Q}$ in the first four components of the vector representation rearrange themselves as follows. Recall that the relevant superpotential terms have the form, $Q(m\mathbf{1} + \sqrt{2}\Phi)\tilde{Q}$, summed over diagonal flavor indices, $A = 1, 2, \ldots, N_f$ left implicit. For each flavor, the adjoint scalar VEV of the form Eq. (A.62), with $v = m/\sqrt{2}$, gives rise to a 2×2 block-diagonal mass matrix

$$m\mathbf{1} + \sqrt{2}\Phi = \begin{pmatrix} \mathbf{V} & 0 \\ 0 & \mathbf{V} \end{pmatrix} \qquad \mathbf{V} = m \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{A.74}$$

in color. $\mathbf{V}$ has one vanishing and one massive eigenvalues. Thus the four fields

$$\hat{Q}^1 = \frac{1}{\sqrt{2}}(Q^1 + iQ^2), \hat{Q}^3 = \frac{1}{\sqrt{2}}(Q^3 + iQ^4), \hat{\tilde{Q}}^1 = \frac{1}{\sqrt{2}}(\tilde{Q}^1 + i\tilde{Q}^2), \hat{\tilde{Q}}^3 = \frac{1}{\sqrt{2}}(\tilde{Q}^3 + i\tilde{Q}^4), \tag{A.75}$$

are massless. The orthogonal combinations such as $\frac{1}{\sqrt{2}}(Q^1 - iQ^2)$ become massive and decouple from the low-energy theory.

The massless quark superfields of the low-energy $SU(2)$ theory are the combinations

$$q^1 = \frac{1}{\sqrt{2}}(\hat{Q}^1 + i\hat{Q}^3); \qquad q^2 = \frac{1}{\sqrt{2}}(i\hat{Q}^1 + \hat{Q}^3), \tag{A.76}$$

which form a $\underline{2}$, and

$$\tilde{q}^1 = \frac{1}{\sqrt{2}}(\hat{\tilde{Q}}^1 - i\hat{\tilde{Q}}^3); \qquad \tilde{q}^2 = \frac{1}{\sqrt{2}}(-i\hat{\tilde{Q}}^1 + \hat{\tilde{Q}}^3), \tag{A.77}$$

which form a $\underline{2}^*$. [18]

It is straightforward to generalize the above construction to $SO(2N+1) \to SU(r) \times U(1)^{N-r+1}$, $r < N$. N_f quark hypermultiplets in the $SO(2N+1)$ vector representation yield precisely N_f flavors of massless quarks in $\underline{r}$ of $SU(r)$ plus a number of singlets.

[18] For a general change of basis vectors from $SO(2N)$ to $U(N)$ see the Appendix A of [162].

Appendix B

Non-abelian vortices in $SO(N)$ theories

In this appendix we show how non-Abelian BPS vortex solutions can be constructed in $\mathcal{N} = 2$ theories with gauge groups $SO(N) \times U(1)$. The model has N_f flavors of chiral multiplets in the vector representation of $SO(N)$, and we consider a color-flavor locked vacuum in which the gauge symmetry is completely broken, leaving a global $SO(N)_{C+F}$ diagonal symmetry unbroken. Individual vortices break this symmetry, acquiring continuous non-Abelian orientational moduli. By embedding this model in high-energy theories with a hierarchical symmetry breaking pattern such as $SO(N + 2) \to SO(N) \times U(1) \to 1$, the correspondence between non-Abelian monopoles and vortices can be established through homotopy maps and flux matching, generalizing the known results in $SU(N)$ theories. We find some interesting hints about the dual (non-Abelian) transformation properties among the monopoles.

B.1 Introduction

Recently some significant steps have been made in understanding the non-Abelian monopoles [150, 151, 152, 153, 154, 155, 156, 157], occurring in spontaneously broken gauge field theories [158, 159]. The basic observation is that the regular 't Hooft-Polyakov-like magnetic monopoles occurring in a system

$$G \xrightarrow{v_1} H \; , \tag{B.1}$$

where H is a non-Abelian "unbroken" gauge group, are not objects which transform among themselves under the unbroken group H, but which transform, if any, under the *magnetic dual* of H, namely $\tilde{H}$. As field transformation groups, H and $\tilde{H}$ are relatively non-local, thus a local transformation in the magnetic group $\tilde{H}$ would look like a non-local transformation in the electric theory. Although this was implicit in the work by Goddard-Nuyts-Olive [151] and others [152, 153], the lack of the concrete knowledge on how $\tilde{H}$ acts on semiclassical monopoles has led to long-standing puzzles and apparent difficulties [155, 156].

Detailed study of gauge theories with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry and quark multiplets, on the other hand, shows that light monopoles transforming as multiplets of non-Abelian magnetic gauge group $\tilde{H}$ do occur quite regularly in full quantum systems [160, 161, 162, 163]. They occur under certain conditions, e.g., that there is a sufficiently large exact flavor symmetry group in the underlying theory, which dresses the monopoles with flavor quantum numbers, preventing them from interacting too strongly. Also, the symmetry requirement (i.e. the symmetry of the low-energy effective theory describing the light monopoles be the correct symmetry of the underlying theory) seems to play an important role in determining the low-energy degrees of freedom in each system [164]. There are subtle, but perfectly clear, logical reasons behind these quantum mechanical realizations of dual gauge symmetries in supersymmetric models. Since there are free parameters in these supersymmetric theories which allow us to move from the fully dynamical regime to semiclassical regions, without qualitatively changing any physics, it must be possible to understand these light degrees of freedom in terms of more familiar soliton-like objects, e.g., semiclassical monopoles.

This line of thought has led us to study the system (B.1), in a regime of hierarchically broken gauge symmetries

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1 \,, \qquad v_1 \gg v_2 \,, \tag{B.2}$$

namely, in a phase in which the "unbroken" H gauge system is completely broken at much lower energies (Higgs phase), so that one expects $-$ based on the standard electromagnetic duality argument $-$ the $\tilde{H}$ system to be in confinement phase. The "elementary monopoles" confined by the confining strings in $\tilde{H}$ theory should look like 't Hooft-Polyakov monopoles embedded in a larger picture where their magnetic fluxes are frisked away by a magnetic vortex of the H theory in Higgs phase.

Indeed, in the context of softly broken $\mathcal{N} = 2$ models, this kind of systems can be realized concretely, by tuning certain free parameters in the models, typically, by taking the bare quark masses m (which fix the adjoint scalar VEVs, $\langle\phi||=\rangle v_1 \sim m$) much larger than the bare adjoint scalar mass μ (which sets the scale for the squark VEVs, $\langle q||=\rangle v_2 \sim \sqrt{\mu m}$). In a high-energy approximation, where v_2 is negligible, one has a system, (B.1), with a set of 't Hooft-Polyakov monopoles. In the class of supersymmetric models considered, these monopoles are BPS, and their (semiclassical) properties are well understood. In the low-energy approximation (where the massive monopoles are integrated out and v_1 is regarded as infinitely large) one has the H theory in Higgs phase, with BPS vortices whose properties can also be studied in great detail.

When the full theory is considered, with "small" corrections which involves factors of $\frac{v_2}{v_1}$, there is an important qualitative change to be taken into account at the two sides of the mass scales (high-energy and low-energy). Neither monopoles of the high-energy approximation nor the vortices of the low-energy theory, are BPS saturated any longer. They are no longer topologically stable. This indeed follows from the fact that $\pi_2(G)$ is trivial for any Lie group (no regular monopoles if H is completely broken) or if $\pi_1(G) = 1$ (there cannot be vortices). If $\pi_1(G) \neq 1$ there may be some stable vortices left, but still there will be much fewer stable vortices as compared to what is expected in the low-energy theory (which "sees" only $\pi_1(H)$). As the two effective theories must be, in some sense, good approximations as long as $\frac{v_2}{v_1} \ll 1$, one faces an apparent paradox.

The resolution of this paradox is both natural and useful. The regular monopoles are actually sources (or sinks) of the vortices seen as stable solitons in the low-energy theory; vice versa, the vortices "which should not be there" in the full theory, simply end at a regular monopole. They both disappear from the spectrum of the respective effective theories. This connection, however, establishes one-to-one correspondence between a regular monopole solution of the high-energy theory and the appropriate vortex of the low-energy theory. As the vortex moduli and non-Abelian transformation properties among the vortices, really depend on the exact global symmetry of the full theory (and its breaking by the solitons), such a correspondence provides us with a precious hint about the nature of the non-Abelian monopoles. In other words, the idea is to make use of the better understood non-Abelian *vortices* to infer precise conclusions about the non-Abelian *monopoles*, bypassing the difficulties associated with the latter as mentioned earlier.

A quantitative formulation of these ideas requires a concrete knowledge of the vortex moduli space and the transformation properties among the vortices [165, 166, 167]. This problem has been largely clarified, thanks to our generally improved understanding of non-Abelian vortices [168, 169, 170, 171, 172], and in particular to the technique of the "moduli matrix" [173], especially in the context of $SU(N)$ gauge theories. Also, some puzzles related to the systems with symmetry breaking $SO(2N) \to U(N)$, or $SO(2N) \to U(r) \times U(1)^{N-r}$, have found natural solutions [158].

In this article, we wish to extend these analyses to the cases involving vortices of $SO(N)$ theories. In [174] the first attempts have been made in this direction, where softly broken $\mathcal{N} = 2$ models with $SO(N)$ gauge groups and with a set of quark matter in the vector representation, have been analyzed. In the case of $SO(2N+3)$ theory broken to $SO(2N+1)\times U(1)$ (with the latter completely broken at lower energies) one observes some hints how the dual, $USp(2N)$ group, might emerge. In the model considered in [174], however, the construction of the system in which the gauge symmetry is completely broken, leaving a maximum exact color-flavor symmetry (the color-flavor locking), required an ad hoc addition of an $\mathcal{N}=1$ superpotential, in contrast to $SU(N)$ theories where, due to the vacuum alignment with bare quark masses familiar from $\mathcal{N}=1$ SQCD, the color-flavor locked vacuum appears quite automatically.

In this article we therefore turn to a slightly different class of $SO(N)$ models. The underlying theory is an $SO(N+2)$ gauge theory with matter hypermultiplets in the adjoint representation, with the gauge group broken partially at a mass scale v_1. The analysis is slightly more complicated than the models considered in [174], but in the present model the color-flavor locked vacua occur naturally. Also, these models have a richer spectrum of vortices and monopoles than in the case of [174], providing us with a finer testing ground for duality and confinement.

At scales much lower than v_1, the model reduces to an $SO(N)\times U(1)$ theory with quarks in the *vector* representation. Non-Abelian vortices arising in the color-flavor locked vacuum of this theory transform non-trivially under the $SO(N)_{C+F}$ symmetry. We are interested in their role in the dynamics of gauge theories, but these solitons also play a role in cosmology and condensed matter physics, so the results of sections B.3 and B.4 of this work could be of more general interest (for example they can be useful for cosmic strings, see [178]).

In section B.2 of this article, we present the high-energy model with gauge group $SO(2N+2)$. In section B.3 we study its low-energy effective theory and present the vortex solutions. In section B.4 we study the model with gauge group $SO(2N+3)$. Finally, in section B.5 we discuss the correspondence between monopoles and vortices.

B.2 The model

We shall first discuss the $SO(2N+2)$ theory; the case of $SO(2N+3)$ group will be considered separately later. We wish to study the properties of monopoles and vortices occurring in the system

$$SO(2N+2) \xrightarrow{v_1} SO(2N)\times U(1) \xrightarrow{v_2} 1 \ . \tag{B.3}$$

To study the consequences of such a breaking, we take a concrete example of an $\mathcal{N}=2$ supersymmetric theory with gauge group $SO(2N+2)$ and N_f matter hypermultiplets in the adjoint representation. All the matter fields have a common mass m, so the theory has a global $U(N_f)$ flavor symmetry. We also add a small superpotential term $\mu\phi^2$ in the Lagrangian, which breaks softly $\mathcal{N}=2$ to $\mathcal{N}=1$. For the purpose of considering hierarchical symmetry breaking (B.3), we take

$$m \gg \mu \ . \tag{B.4}$$

The theory is infrared-free for $N_f > 1$, but one may consider it as an effective low-energy theory of some underlying theory, valid at mass scales below a given ultraviolet cutoff. In any case, our analysis will focus on the questions how the properties of the semiclassical monopoles arising from the intermediate-scale can be understood through the moduli of the non-Abelian vortices arising when the low-energy, $SO(2N)$ theory is put in the Higgs phase.

The superpotential of the theory has the form,

$$W = \sqrt{2}\sum_A \mathrm{Tr}\,\tilde{\zeta}_A\,[\phi,\zeta_A] + m\sum_A \mathrm{Tr}\,\tilde{\zeta}_A\zeta_A + \frac{\mu}{2}\mathrm{Tr}\,\phi^2 \ . \tag{B.5}$$

In order to minimize the misunderstanding, we use here the notation of ζ_A, $\tilde{\zeta}_A$ for the quark hypermultiplets in the *adjoint* representation of the high-energy gauge group $SO(2N+2)$ (or $SO(2N+3)$), with $A = 1,2,\ldots,N_f$ standing for the flavor index. We shall reserve the symbols $q_A, \tilde{q}_A$ for the light supermultiplets of the low-energy theory, which transform as the *vector* representation of the gauge group $SO(2N)$ (or $SO(2N+1)$). The vacuum equations for this theory therefore take the form

$$\left[\phi,\phi^\dagger\right] = 0 \ , \tag{B.6}$$

$$\sum_A \left[\zeta_A,\zeta_A^\dagger\right] = \sum_A \left[\tilde{\zeta}_A^\dagger,\tilde{\zeta}_A\right] \ , \tag{B.7}$$

$$\sum_A \sqrt{2}\left[\zeta_A,\tilde{\zeta}_A\right] + \mu\,\phi = 0 \ , \tag{B.8}$$

$$\sqrt{2}\,[\phi,\zeta_A] + m\,\zeta_A = 0 \ , \tag{B.9}$$

$$-\sqrt{2}\left[\phi,\tilde{\zeta}_A\right] + m\,\tilde{\zeta}_A = 0 \ . \tag{B.10}$$

We shall choose a vacuum in which ϕ takes the vacuum expectation value

(VEV)

$$\langle \phi \rangle = \begin{pmatrix} 0 & -iv & 0 & \cdots & 0 \\ iv & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \tag{B.11}$$

which breaks $SO(2N+2)$ to $SO(2N)\times U(1)$ and is consistent with Eq. (B.6).

We are interested in the Higgs phase of the theory. In order for the $SO(2N)\times U(1)$ symmetry to be broken at energies much lower than $v_1 \equiv v$, we have to find non-vanishing VEVs of the squarks which satisfy Eqs. (B.9),(B.10). This means that $v \sim \mathcal{O}(m)$. The magnitude of squark VEVs is then fixed by Eq. (B.8) to be of the order of $(\mu\, m)^{1/2} \ll m$ and defining $v_2 \equiv |\langle q|||\rangle = \mathcal{O}(\sqrt{\mu m})$ we obtain the hierarchical breaking of the gauge group (B.3). The D-term condition (B.8) can be satisfied by the ansatz

$$\zeta = \tilde{\zeta}^{\dagger} . \tag{B.12}$$

One must also determine the components of the fields $\zeta, \tilde{\zeta}$ which do not get a mass of the order of $\mathcal{O}(v) \simeq \mathcal{O}(m)$. We see from Eq. (B.5) that the light squarks are precisely those for which Eqs. (B.9),(B.10) are satisfied non-trivially, i.e., by non-vanishing "eigenvectors" ζ, $\tilde{\zeta}$. The conditions (B.9),(B.10) require that the light components correspond to the generators of $SO(2N)$ which are lowering and raising operators for $\langle \varphi \rangle$. This condition implies also

$$v = \frac{m}{\sqrt{2}} . \tag{B.13}$$

To find the light components of $\zeta, \tilde{\zeta}$, we note that for a single flavor, Eqs. (B.8)-(B.10) together have the form of an $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$ algebra, T_1, T_2, T_3,

$$\phi \propto T_3 , \qquad \zeta_A \propto T_- = T_1 - iT_2 , \qquad \tilde{\zeta}_A \propto T_+ = T_1 + iT_2 , \tag{B.14}$$

with appropriate constants.

The simplest way to proceed is to consider the various $SO(3)$ subgroups, $SO(3)_{12j}$, lying in the $(12j)$ three-dimensional subspaces ($j = 3, 4, 5, \ldots$), with

$$T_3 = H^{(0)} = -i\Sigma_{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{12j} , \tag{B.15}$$

$$T_- = T_1 - iT_2 = L_{j,-} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}_{12j} , \qquad T_+ = T_1 + iT_2 = L_{j,-}^{\dagger} . \tag{B.16}$$

The light fields which remain massless can then be expanded as

$$\zeta_A(x) = \sum_{j=3,4,5,\ldots} \frac{1}{2}\, q_{jA}(x)\, L_{j,-} \ , \qquad \tilde{\zeta}_A(x) = \sum_{j=3,4,5,\ldots} \frac{1}{2}\, \tilde{q}_{Aj}(x)\, L_{j,+} \qquad \text{(B.17)}$$

for each flavor $A = 1, 2, \ldots, N_f$. Written as a full $SO(2N)$ matrix, $L_{j,-}$ looks like

$$L_{j,-} = \begin{pmatrix} 0 & 0 & \ldots & 1 & \ldots \\ 0 & 0 & & -i & \vdots \\ \vdots & & \ddots & & \\ -1 & i & & & \vdots \\ \vdots & \ldots & & \ldots & 0 \end{pmatrix} , \qquad L_{j,+} = L^{\dagger}_{j,-} \ . \qquad \text{(B.18)}$$

In $L_{j,-}$ the only non-zero elements (1 and $-i$) in the first two rows appear in the $(2+j)$-th column; the only two non-zero elements in the first two columns (-1 and i) appear in the $(2+j)$-th row.

An alternative way to find the combinations which do not get mass from $\langle\phi|\rangle$ is to use the independent $SU(2)$ subgroups contained in various $SO(4)$ subgroups living in the subspaces $(1, 2, j, j+1)$, $j = 3, 5, \ldots, 2N-1$. As is well known, the $\mathfrak{so}(4)$ algebra factorizes into two *commuting* $\mathfrak{su}(2)$ algebras,

$$\mathfrak{so}(4) \sim \mathfrak{su}(2) \times \widehat{\mathfrak{su}}(2) \ , \qquad \text{(B.19)}$$

where for instance for $SO(4)_{1234}$ one has

$$S_1 = -\frac{i}{2}(\Sigma_{23}+\Sigma_{41}) \ , \quad S_2 = -\frac{i}{2}(\Sigma_{31}+\Sigma_{42}) \ , \quad S_3 = -\frac{i}{2}(\Sigma_{12}+\Sigma_{43}) \ , \ \text{(B.20)}$$

$$\hat{S}_1 = -\frac{i}{2}(\Sigma_{23}-\Sigma_{41}) \ , \quad \hat{S}_2 = -\frac{i}{2}(\Sigma_{31}-\Sigma_{42}) \ , \quad \hat{S}_3 = -\frac{i}{2}(\Sigma_{12}-\Sigma_{43}) \ , \ \text{(B.21)}$$

where

$$\Sigma_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{23} ,$$

is (up to a phase) the rotation generator in the 23 plane, etc.

Since

$$\frac{\sqrt{2}}{m}\, \langle\phi||=\rangle H^{(0)} = -i\,\Sigma_{12} = S_3 + \hat{S}_3 \ , \qquad \text{(B.22)}$$

it follows from the standard $\mathfrak{su}(2)$ algebra that *both* $S_- = S_1 - iS_2$ and $\hat{S}_- = \hat{S}_1 - i\hat{S}_2$ satisfy the relation,

$$\left[\frac{\sqrt{2}}{m}\langle\phi||,\rangle S_-\right] = -S_- \ , \qquad \left[\frac{\sqrt{2}}{m}\langle\phi||,\rangle \hat{S}_-\right] = -\hat{S}_- \ . \qquad \text{(B.23)}$$

One can choose the two combinations

$$L_- = S_- + \hat{S}_- \; ; \qquad L'_- = S_- - \hat{S}_- \; , \tag{B.24}$$

which satisfy the required relation,

$$\left[\frac{\sqrt{2}}{m}\langle \phi ||,\rangle L_- \right] = -L_- \; , \qquad \left[\frac{\sqrt{2}}{m}\langle \phi ||,\rangle L'_- \right] = -L'_- \; . \tag{B.25}$$

These constructions can be done in all $\mathfrak{su}(2)$ subalgebras living in $SO(4)_{1,2,j,j+1}$, $j = 3, 5, \ldots, 2N-1$.

Explicitly, $S_{j\,-}$, $\hat{S}_{j\,-}$, and $L_{j,-}$, $L'_{j,-}$ have the form ($j = 3, 5, \ldots$)

$$S_{j\,-} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} , \qquad \hat{S}_{j\,-} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} ; \tag{B.26}$$

$$L_{j,-} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} , \qquad L'_{i,-} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} . \tag{B.27}$$

Clearly, one can write

$$L'_{j,-} = i\, L_{j+1,-} \; ; \tag{B.28}$$

and use the first of Eq. (B.27) to define $L_{j,-}$ for all $j = 3, 4, 5, \ldots$, j even or odd. With this definition, $L_{j,-}$ coincide with those introduced in Eq. (B.16) by using various $SO(3)$ subgroups.

Eqs. (B.5),(B.23),(B.25) show that the light fields (those which do not get mass of order m) are the ones appearing in the expansion (B.17). Alternatively, the basis of light fields can be taken as

$$\zeta_A(x) = \frac{1}{\sqrt{2}} \sum_{i=3,5,\ldots} \left[Q_{iA}(x)\, S_{i,-} + \hat{Q}_{iA}(x) \hat{S}_{i,-} \right] , \qquad \tilde{\zeta}_A = \frac{1}{\sqrt{2}} \sum_{i=3,5,\ldots} \left[\tilde{Q}_{Ai}(x)\, S_{i,+} + \hat{\tilde{Q}}_{Ai}(x) \hat{S} \right. \tag{B.29}$$

The relation between the $q_{iA}(x)$ and $Q_{iA}(x)$ fields is ($i = 3, 5, \ldots$):

$$Q_{iA}(x) = \frac{q_{iA}(x) + i\, q_{i+1,A}(x)}{\sqrt{2}} \; ; \qquad \hat{Q}_{iA}(x) = \frac{q_{A,i}(x) - i\, q_{A,i+1}(x)}{\sqrt{2}} = Q_{i+1,A}(x) \, . \tag{B.30}$$

All other components get a mass of order m. There are thus precisely $2N$ light quark fields (color components) q_{iA} ($i=1,2,\ldots,2N$) for each flavor. These are the light hypermultiplets of the theory.

Each of the two bases $\{q_{iA}\}$ or $\{Q_{iA}\}$ has some advantages. Clearly the basis q_{iA} ($i=1,2,\ldots,2N$) corresponds to the usual basis of the fundamental (vector) representation of the $SO(M)$ group ($M=2N$), appearing in the decomposition of an adjoint representation of $SO(M+2)$ into the irreps of $SO(M)$:

$$\frac{(M+2)(M+1)}{2}=\frac{M(M-1)}{2}\oplus M\oplus M\oplus 1\ . \tag{B.31}$$

The low-energy effective Lagrangian can be most easily written down in terms of these fields, and the symmetry property of the vacuum is manifest here.

On the other hand, the basis $(Q_{jA},\hat{Q}_{jA})$, $j=3,5,7,\ldots$, is made of pairs of eigenstates of the $(a\equiv(j-1)/2)$-th Cartan subalgebra generator,

$$H^{(a)}=-i\,\Sigma_{j,j+1}=S_{j,3}-\hat{S}_{j,3}\ ,\qquad a=\frac{j-1}{2}=1,2,\ldots,N\ , \tag{B.32}$$

(see Eqs. (B.20),(B.21),(B.22)), with eigenvalues ±1, so that the vortex equations can be better formulated, and the symmetry maintained by individual vortex solutions can be seen explicitly in this basis. Q_{iA}, ($i=3,5,\ldots$), form an $\underline{\mathbf{N}}$ of $SU(N)\subset SO(2N)$; $\hat{Q}_{iA}$, ($i=3,5,\ldots$), form an $\underline{\bar{\mathbf{N}}}$. In other words, it represents the decomposition of a $\underline{\mathbf{2N}}$ of $SO(2N)$ into $\underline{\mathbf{N}}+\underline{\bar{\mathbf{N}}}$ of $SU(N)\subset SO(2N)$. The change of basis from the vector basis (q) and $U(N)$ basis $(Q,\hat{Q})$ is discussed more extensively in Appendix A.

B.3 Vortices in the $SO(2N)\times U(1)$ theory

B.3.1 The vacuum and BPS vortices

The low-energy Lagrangian for the theory with gauge group $SO(2N)\times U(1)$ and squarks q_A,$\tilde{q}_A$ in the fundamental representation of $SO(2N)$ is

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4g_1^2}F^{0\mu\nu}F^0_{\mu\nu}-\frac{1}{4g_{2N}^2}F^{b\mu\nu}F^b_{\mu\nu}+\left|\mathcal{D}_\mu q_A\right|^2+\left|\mathcal{D}_\mu\tilde{q}_A^\dagger\right|^2 \\ & -\frac{g_{2N}^2}{2}\left|q_A^\dagger t^b q_A-\tilde{q}_A t^b\tilde{q}_A^\dagger\right|^2-2g_{2N}^2\left|\tilde{q}_A t^b q_A\right|^2 \\ & -\frac{g_1^2}{2}\left|q_A^\dagger q_A-\tilde{q}_A\tilde{q}_A^\dagger\right|^2-2g_1^2\left|\tilde{q}_A q_A+\frac{\mu m}{\sqrt{2}}\right|^2+\cdots\end{aligned} \tag{B.33}$$

where the dots denote higher orders in μ/m and terms involving $\delta\phi=\phi-\langle\phi\rangle$. Note that to this order, the only modification is a Fayet-Iliopoulos term which does not break $\mathcal{N}=2$ SUSY. The covariant derivative acts as

$$\mathcal{D}_\mu q_A = \partial_\mu q_A - iA^0_\mu\, q_A - iA^b_\mu t^b q_A \ , \tag{B.34}$$

where t^a is normalized as

$$\mathrm{Tr}\ (t^a)^2 = 1 \ , \tag{B.35}$$

and

$$t^a = \frac{1}{\sqrt{2}} H^{(a)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2} \ , \tag{B.36}$$

where $H^{(a)}$ is the a-th Cartan generator of $SO(2N)$, $a = 1, 2, \ldots, N$, which we take simply as

$$H^{(a)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2} . \tag{B.37}$$

As we have seen already, each light field carries unit charge with respect to $H^{(0)}$; the pair $(Q_{A,j}, \hat{Q}_{A,j})$, $j = 3, 5, 7, \ldots$, furthermore carries the charge ± 1 with respect to $H^{(a)}$ $(a = (j-1)/2)$ and zero charge with respect to other Cartan generators.

Let us define

$$\xi = \frac{\mu\, m}{2} \ , \tag{B.38}$$

which is the only relevant dimensional parameter in the Lagrangian. We set $N_f = 2N$, which is enough for our purposes[1]. By writing q_{iA}, $\tilde{q}_{Ai}$ as color-flavor mixed matrices q, $\tilde{q}$, the vacuum equations are now cast into the form

$$\mathrm{Tr}\big(qq^\dagger\big) = \mathrm{Tr}\big(\tilde{q}^\dagger \tilde{q}\big) \ , \tag{B.39}$$

$$qq^\dagger - \big(qq^\dagger\big)^T = \tilde{q}^\dagger \tilde{q} - \big(\tilde{q}^\dagger \tilde{q}\big)^T \ , \tag{B.40}$$

$$\mathrm{Tr}\big(q\tilde{q}\big) = \xi \ , \tag{B.41}$$

$$\mathrm{Tr}\ \big(t^b q\tilde{q}\big) = 0 \ . \tag{B.42}$$

The vacuum we choose to study is characterized by the color-flavor locked phase

$$\langle q_{A,j}\rangle = \left\langle \tilde{q}^\dagger_{A,j} \right\rangle = \delta_{A,j}\, v_2 \ , \qquad v_2 = \sqrt{\frac{\xi}{2N}} \ , \tag{B.43}$$

or

$$\langle q\rangle = \left\langle \tilde{q}^\dagger \right\rangle = v_2\, 1 = v_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{B.44}$$

[1]Higher N_f are interesting because of semilocal vortex configurations arising in these theories. These solutions will be discussed elsewhere.

which clearly satisfies all the equations above. The gauge (O) and flavor (U) transformations act on them as

$$q \to O\, q\, U^T \quad , \quad \tilde{q} \to U^*\, \tilde{q}\, O^T \qquad O \in SO(2N)\times U(1)\ , \quad U \in U(2N)\ : \tag{B.45}$$

the gauge group is completely broken, while a global $SO(2N)_{C+F}\times U(1)_{C+F}$ group ($U = O$) is left unbroken.

When looking for vortex solutions, one suppresses time and z dependence of the fields and retains only the component F_{xy} of the field strength. The vortex tension can be cast in the Bogomol'nyi form

$$\begin{aligned} T = \int d^2x \,\Bigg\{ & \left|\frac{1}{2\,g_{2N}}\,F^b_{ij} \pm g_{2N}\varepsilon_{ij}\tilde{q}_A t^b q_A\right|^2 + \left|\frac{1}{2\,g_1}F^0_{ij} \pm g_1\varepsilon_{ij}\left(\tilde{q}_A q_A - \xi\right)\right|^2 \\ & + \frac{1}{2}\left|\mathcal{D}_i q_A \pm i\varepsilon_{ij}\mathcal{D}_j \tilde{q}^\dagger_A\right|^2 + \frac{1}{2}\left|\mathcal{D}_i \tilde{q}^\dagger_A \pm i\varepsilon_{ij}\mathcal{D}_j q_A\right|^2 \\ & + \frac{g_{2N}^2}{2}\left|q^\dagger_A t^b q_A - \tilde{q}_A t^b \tilde{q}^\dagger_A\right|^2 + \frac{g_1^2}{2}\left|q^\dagger_A q_A - \tilde{q}_A \tilde{q}^\dagger_A\right|^2 \pm \varepsilon_{ij}\,\xi F^0_{ij}\Bigg\}\ . \end{aligned} \tag{B.46}$$

The terms with the square brackets in the last line of Eq. (B.46) automatically vanish with the ansatz [169]

$$q_{iA} = \tilde{q}^\dagger_{iA}\ : \tag{B.47}$$

thus we shall use this ansatz for the vortex configurations. The resulting BPS equations are

$$\frac{1}{2\,g_1}F^0_{ij} + \eta\, g_1\, \varepsilon_{ij}\,\left(q^\dagger_A q_A - \xi\right) = 0\ , \tag{B.48}$$

$$\frac{1}{2\,g_{2N}}\,F^b_{ij} + \eta\, g_{2N}\,\varepsilon_{ij}\,q^\dagger_A t^b q_A = 0\ , \tag{B.49}$$

$$\mathcal{D}_i q_A + i\,\eta\,\varepsilon_{ij}\,\mathcal{D}_j q_A = 0\ , \qquad \eta = \pm 1\ , \tag{B.50}$$

where we have used the ansatz (B.47). The tension for a BPS solution is

$$T = \eta \int d^2x\, \varepsilon_{ij}\,\xi\, F^0_{ij}\ . \tag{B.51}$$

To obtain a solution of these equations, we need an ansatz for the squark fields. It is convenient to perform a $U(2N)_F$ transformation (B.30), where the vacuum takes the block-diagonal form

$$\langle Q\rangle = \langle \tilde{Q}^\dagger\rangle = \sqrt{\frac{\xi}{2N}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ i & -i & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & i & -i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\ , \tag{B.52}$$

In this basis, the ansatz is:

$$A_i = h_a(r)\, t^a\, \varepsilon_{ij}\, \frac{r_j}{r^2}\ ; \qquad t^0 \equiv \frac{1}{\sqrt{2}}\ , \quad t^a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2}\ ; \tag{B.53}$$

$$Q(r,\vartheta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{in_1^+\vartheta}\varphi_1^+(r) & e^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ ie^{in_1^+\vartheta}\varphi_1^+(r) & -ie^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ 0 & 0 & e^{in_2^+\vartheta}\varphi_2^+(r) & e^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ 0 & 0 & ie^{in_2^+\vartheta}\varphi_2^+(r) & -ie^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{B.54}$$

where t^as are the generators of the Cartan subalgebra of $SO(2N)$. The conditions for the fields at $r \to \infty$ are fixed by the requirement of finite energy configurations:

$$\varphi_a^{\pm}(\infty) = \sqrt{\frac{\xi}{2N}}\ , \tag{B.55}$$

$$n_a^{\pm} = n^{(0)} \mp n^{(a)}\ , \qquad n^{(0)} \equiv \frac{1}{\sqrt{2}}\, h_0(\infty)\ ; \quad n^{(a)} \equiv \frac{1}{\sqrt{2}}\, h_a(\infty)\ , \tag{B.56}$$

where $n^{(0)}$ and $n^{(a)}$ are the winding numbers with respect to the $U(1)$ and to the a-th Cartan $U(1) \in SO(2N)$ defined in Eq. (B.37).

Clearly

$$N_0 \equiv n_a^+ + n_a^- = 2\, n^{(0)}\ , \tag{B.57}$$

is independent of a. The regularity of the fields requires that the Q_As come back to their original value after a 2π rotation, and this yields the quantization condition,

$$n_a^{\pm} \in \mathbb{Z}\ , \qquad \forall a\ , \tag{B.58}$$

implying that *the $U(1)$ winding numbers $n^{(0)}$ and $n^{(a)}$ are quantized in half integer units,* consistently with considerations based on the fundamental groups (see Appendix B.8 and below).

We need only the information contained in Eqs. (B.53),(B.56) to evaluate the tension for a BPS solution:

$$T = 2\,\eta\,\xi \lim_{r\to\infty} \int d\vartheta\, r\, A^0_\vartheta(r) = 2\sqrt{2}\,\pi\,\eta\,\xi\, h_0(\infty) = 2\pi\,\eta\,\xi\, N_0 = 2\pi\,\xi\,|N_0|\ . \tag{B.59}$$

The last equality comes from the requirement for the tension to be positive, so $\eta = \mathrm{sign}(N_0)$. Note that the tension depends only on $|N_0|$, which is twice the $U(1)$ winding.

From the BPS equations we obtain the differential equations for the profile functions h_0, h_a, $\varphi_a^{\pm}$:

$$\frac{dh_0}{dr} = -2\sqrt{2}\,\eta\, g_1^2\, r \left(\sum_a \left(|\varphi_a^+|^2 + |\varphi_a^-|^2 \right) - \xi \right) , \tag{B.60}$$

$$\frac{dh_a}{dr} = 2\sqrt{2}\,\eta\, g_{2N}^2\, r \left(|\varphi_a^+|^2 - |\varphi_a^-|^2 \right) , \tag{B.61}$$

$$\frac{d\varphi_a^{\pm}}{dr} = \eta \left(n_a^{\pm} - \frac{h_0 \mp h_a}{\sqrt{2}} \right) \frac{\varphi_a^{\pm}}{r} . \tag{B.62}$$

In order to cast them in a simple form, we define $f_0 = h_0 - \frac{N_0}{\sqrt{2}}$ and $f_a = h_a + \frac{n_a^+ - n_a^-}{\sqrt{2}}$ and obtain

$$\frac{df_0}{dr} = -2\sqrt{2}\,\eta\, g_1^2 r \left(\sum_a \left(|\varphi_a^+|^2 + |\varphi_a^-|^2 \right) - \xi \right) , \tag{B.63}$$

$$\frac{df_a}{dr} = 2\sqrt{2}\,\eta\, g_{2N}^2 r \left(|\varphi_a^+|^2 - |\varphi_a^-|^2 \right) , \tag{B.64}$$

$$\frac{d\varphi_a^{\pm}}{dr} = -\eta \left(\frac{f_0 \mp f_a}{\sqrt{2}} \right) \frac{\varphi_a^{\pm}}{r} . \tag{B.65}$$

The boundary conditions at $r \to \infty$ are

$$\varphi_a^{\pm}(\infty) = \sqrt{\frac{\xi}{2N}} , \quad f_0(\infty) = f_a(\infty) = 0 , \tag{B.66}$$

There are also regularity conditions at $r = 0$ for the gauge fields $h_0(0) = h_a(0) = 0$ which are

$$f_0(0) = -\frac{N_0}{\sqrt{2}} , \quad f_a(0) = \frac{n_a^+ - n_a^-}{\sqrt{2}} , \tag{B.67}$$

Solving Eq. (B.65) for small r with the conditions (B.67), we obtain $\varphi_a^{\pm} \sim r^{n_a^{\pm}\eta}$. To avoid a singular behavior for these profile functions we need

$$\text{sign}\left(n_a^{\pm} \right) = \eta . \tag{B.68}$$

This condition is consistent with $\eta = \text{sign}(N_0)$. With this condition there are no singularities at $r = 0$ and the equations (B.63),(B.64),(B.65) can be solved numerically with boundary conditions (B.66),(B.67).

The profile functions for the simplest vortex $N_0 = 1,\ n_1^+ = 1,\ n_1^- = 0$ in the $SO(2) \times U(1)$ theory are shown in Figure B.1, B.2. The profile functions

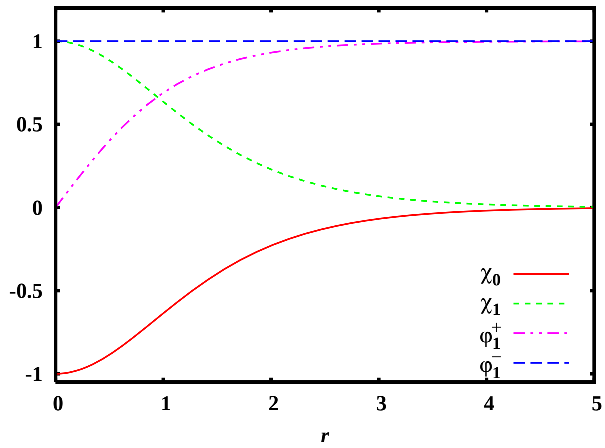

Figure B.1: Numerically integrated minimum vortex solution with $N_0 = 1$, where we have taken the couplings to be $4g_1^2 = 4g_{2N}^2 = 1$. ($\chi_i \equiv \sqrt{2} f_i$).

$(f_0, f_a, \varphi_a^+, \varphi_a^-)$ for the minimal vortex $N_0 = 1$, $n_i^+ = 1$, $n_i^- = 0$ in the $SO(2N) \times U(1)$ theory can be obtained by rescaling $g_{2N}^2 \to g_{2N}^2/N$ and then taking all $\varphi_a^\pm$ equal to the profile functions shown above rescaled by a factor $1/\sqrt{N}$. Similarly, solutions corresponding to the exchange $(n_a^+, n_a^-) = (1, 0) \leftrightarrow (0, 1)$ can be obtained by exchanging $f_a \leftrightarrow -f_a$ and $\varphi_a^+ \leftrightarrow \varphi_a^-$. The typical length scale of the profile functions is $1/\sqrt{\xi}$, which is the only dimensional parameter in the Bogomol'nyi equations.

B.3.2 Vortex moduli space

To study the space of solutions of the BPS equations we have obtained above, it is convenient to rewrite the ansatz (B.54) for the squark fields in the original basis:

$$q(r,\vartheta) = \begin{pmatrix} \mathbf{M}_1(r,\vartheta) & 0 & 0 & \cdots \\ 0 & \mathbf{M}_2(r,\vartheta) & 0 & \cdots \\ 0 & 0 & \mathbf{M}_3(r,\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{B.69}$$

$$\mathbf{M}_a(r,\vartheta) = \frac{1}{2} \begin{pmatrix} e^{in_a^+\vartheta}\varphi_a^+(r) + e^{in_a^-\vartheta}\varphi_a^-(r) & -i\left(e^{in_a^+\vartheta}\varphi_a^+(r) - e^{in_a^-\vartheta}\varphi_a^-(r)\right) \\ i\left(e^{in_a^+\vartheta}\varphi_a^+(r) - e^{in_a^-\vartheta}\varphi_a^-(r)\right) & e^{in_a^+\vartheta}\varphi_a^+(r) + e^{in_a^-\vartheta}\varphi_a^-(r) \end{pmatrix}.$$

In this basis the action of the $SO(2N)_{C+F}$ transformations on squark fields is simply $q' = O\, q\, O^T$. The first observation is that if $\hat{q}(r,\vartheta)$ is a solution to the BPS equations, $O\,\hat{q}(r,\vartheta)\,O^T$ is also a solution. Note also that these solutions

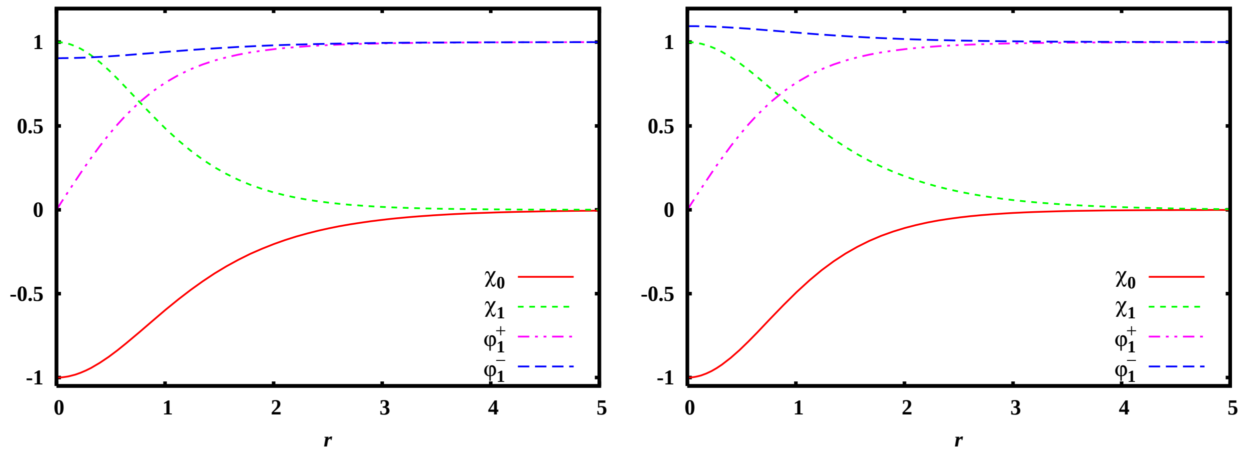

Figure B.2: Numerically integrated minimum vortex solution with $N_0 = 1$, where we have taken the couplings to be $4g_1^2 = 1$ and $4g_{2N}^2 = 2$ for the left panel and $4g_1^2 = 2$ and $4g_{2N}^2 = 1$ for the right panel. ($\chi_i \equiv \sqrt{2} f_i$).

are physically distinct because they are related by a global symmetry. In this way, from a single solution of the form (B.54), we can obtain a whole continuous $SO(2N)$ orbit of solutions. Any given vortex solution is a point in the moduli space and $SO(2N)_{C+F}$ acts as an isometry on this space.

From Eqs. (B.56) and (B.68), we see that regular solutions are described by a set of $2N + 1$ integers N_0,$n_a^{\pm}$ which satisfy the following conditions:

$$n_a^+ + n_a^- = N_0 \ , \quad \forall a \ , \tag{B.70}$$

$$\text{sign}(n_a^+) = \text{sign}(n_a^-) = \text{sign}(N_0) \ , \quad \forall a \ , \tag{B.71}$$

where $N_0 \in \mathbb{Z}$ is related to the winding around the $U(1)$ and is the only parameter of the solution which enters the tension $T = 2\pi\xi|N_0|$.

Let us study the solutions with the minimum tension. Minimal vortices have $N_0 = \pm 1$ and $T = 2\pi\xi$. Note that solutions with $N_0 < 0$ can be obtained by taking the complex conjugate of solutions with $N_0 > 0$, so from now on we will consider only solutions with positive N_0. These vortices can be divided into two groups, the first has 2^{N-1} representative (basis) vortices which are

$$N_0 = 1, \qquad \begin{pmatrix} n_1^+ & n_1^- \\ n_2^+ & n_2^- \\ \vdots & \vdots \\ n_{N-1}^+ & n_{N-1}^- \\ n_N^+ & n_N^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix}, \ldots, \tag{B.72}$$

which all have an even number of n_i^-'s equal to 1; and the second set is represented by 2^{N-1} vortices, characterized by the integers

$$N_0 = 1, \qquad \begin{pmatrix} n_1^+ & n_1^- \\ n_2^+ & n_2^- \\ \vdots & \vdots \\ n_{N-1}^+ & n_{N-1}^- \\ n_N^+ & n_N^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \tag{B.73}$$

with an odd number of n_i^-'s equal to 1.

These two sets belong to two distinct orbits of $SO(2N)_{C+F}$. To see this one must study the way they transform under $SO(2N)_{C+F}$. Consider for instance the case of $N = 2$: the $SO(4)_{C+F}$ transformations $\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ and $\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ exchange $(n_1^+, n_2^+) \leftrightarrow (n_1^-, n_2^-)$ and $(n_1^+, n_1^-) \leftrightarrow (n_2^+, n_2^-)$, respectively. In the general $SO(2N)$ case, two solutions differing by the exchange $(n_i^+, n_j^+) \leftrightarrow (n_i^-, n_j^-)$ or $(n_i^+, n_i^-) \leftrightarrow (n_j^+, n_j^-)$ for some i,j, therefore belong to the same orbit of $SO(2N)_{C+F}$. The vortices in the set (B.72) belong to a continuously degenerate set of minimal vortices; the set (B.73) form the "basis" of another, degenerate set. The two sets do not mix under the $SO(2N)$ transformations.

In order to see better what these two sets might represent, and to see how each vortex transforms under $SO(2N)_{C+F}$, let us assign the two "states", $|\uparrow\rangle_j$, $|\downarrow\rangle_j$ of a j-th $(\frac{1}{2})$ spin, $j = 1, 2, \ldots, N$, to the pair of vortex winding numbers $(n_j^+, n_j^-) = (0,1), (1,0)$. Each of the 2^N minimum vortices (Eqs. (B.72),(B.73)) can then be represented by the 2^N spin state,

$$|s_1|\otimes\rangle|s_2|\otimes\rangle \cdots |s_N|\ \rangle, \qquad |s_j|{=}\rangle|\uparrow|{=}\rangle(0,1)\ , \quad \text{or} \quad |\downarrow|{=}\rangle(1,0)\ . \tag{B.74}$$

For instance the first vortex of Eq. (B.72) corresponds to the state, $|\downarrow\downarrow\ldots\downarrow\rangle$.

Introduce now the "gamma matrices" as direct products of N Pauli matrices acting as

$$\gamma_j \equiv \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{j-1} \otimes \tau_1 \otimes 1 \otimes \cdots \otimes 1\ , \qquad (j = 1, 2, \ldots, N)\ ; \tag{B.75}$$

$$\gamma_{N+j} \equiv \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{j-1} \otimes \tau_2 \otimes 1 \otimes \cdots \otimes 1\ , \qquad (j = 1, 2, \ldots, N)\ . \tag{B.76}$$

γ_k, $k = 1, 2, \ldots, 2N$ satisfy the Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2\,\eta_{ij}\ , \qquad i, j = 1, 2, \ldots, 2N\ ,$$

and the $SO(2N)$ generators can accordingly be constructed by $\Sigma_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j]$. $SO(2N)$ transformations (including finite transformations) among the vortex solutions can thus be represented by the transformations among the N-spin states, (B.74).

As each of Σ_{ij} $(i \neq j)$ flips exactly two spins, the two sets (B.72) and (B.73) clearly belong to two distinct orbits of $SO(2N)$. In fact, a "chirality" operator

$$\Gamma_5 \equiv P \prod_{j=1}^{2N} \gamma_j \ , \qquad \{\Gamma_5, \gamma_j\} = 0 \ , \quad j = 1, 2, \ldots, 2N \ , \tag{B.77}$$

anticommutes with all γ_j's, where $P = 1$ (N even) or $P = i$ (N odd), hence commutes with $SO(2N)$. The two sets Eq. (B.72), Eq. (B.73) of minimal vortices thus are seen to transform as two spinor representations of definite chirality, 1 and -1, respectively (with multiplicity 2^{N-1} each).

Every minimal solution is invariant under a $U(N)$ group embedded in $SO(2N)_{C+F}$. This can be seen from the form of the first solution in (B.72) in the basis (B.69):

$$q_{(1)} = f_+(r, \vartheta) \begin{pmatrix} \mathbf{1} & & \\ & \ddots & \\ & & \mathbf{1} \end{pmatrix} + f_-(r, \vartheta) \begin{pmatrix} \sigma_2 & & \\ & \ddots & \\ & & \sigma_2 \end{pmatrix} . \tag{B.78}$$

This solution is invariant under the subgroup $U(N) \subset SO(2N)$ acting as $U\, q_{(1)}\, U^T$, where $U \in U(N)$ commutes with the second matrix in (B.78).

In the N-spin state representation above, the vortex (B.78) corresponds to the state with all spins down, $|\downarrow\downarrow \ldots \downarrow\rangle$. In order to see how the N-spin states transform under $SU(N) \subset SO(2N)$, construct the creation and annihilation operators

$$a_j = \frac{1}{2}(\gamma_j - i\,\gamma_{N+j}) \ ; \qquad a_j^\dagger = \frac{1}{2}(\gamma_j + i\,\gamma_{N+j}) \ ,$$

satisfying the algebra,

$$\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \ , \qquad \{a_j, a_k^\dagger\} = \delta_{jk} \ .$$

$SU(N)$ generators acting on the spinor representation, can be constructed as [176]

$$T^a = \sum_{j,k} a_j^\dagger\, (t^a)_{jk}\, a_k \ ,$$

where t^a are the standard $N \times N$ $SU(N)$ generators in the fundamental representation. The state $|\downarrow\downarrow \ldots \downarrow\rangle$ is clearly annihilated by all T^a, as it is annihilated by all

$$a_k = \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{k-1} \otimes \tau_- \otimes 1 \otimes \cdots \otimes 1 \ , \qquad k = 1, 2, \ldots N :$$

thus, the vortex (B.78) leaves $U(N)$ invariant.

All other solutions can be obtained as $R\, q_{(1)}\, R^T$ with $R \in O(2N)$, so each solution is invariant under an appropriate $U(N)$ subgroup $R\, U\, R^T$. This means that the moduli space contains two copies of the coset space

$$\mathcal{M} = SO(2N)/U(N) \ . \tag{B.79}$$

The points in each coset space transform according to a spinor representation of definite chirality, each with dimension 2^{N-1}. When discussing the topological properties of vortices, we will see that these disconnected parts correspond to different elements of the homotopy group.

Vortices of higher windings are described by $N_0 > 1$. In the simplest non-minimal case, the vortices are described by:

$$N_0 = 2 \ , \qquad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{B.80}$$

These orbits correspond to parts of the moduli space whose structure corresponds to the coset spaces $SO(2N)/U(N-k) \times SO(2k)$, where k is the number of $(1,1)$ pairs. Analogously vortices with $N_0 \geq 3$ can be constructed.

The argument that the minimum vortices transform as two spinor representations implies that the $N_0 = 2$ vortices (B.80) transform as various irreducible antisymmetric tensor representations of $SO(2N)_{C+F}$, appearing in the decomposition of products of two spinor representations: e.g.

$$2^{N-1} \otimes 2^{N-1} \text{ or } 2^{N-1} \otimes \overline{2^{N-1}} \ , \tag{B.81}$$

Although all these vortices are degenerate in the semi-classical approximation, non-BPS corrections will lift the degeneracy, leaving only the degeneracy among the vortices transforming as an irreducible multiplet of the group $SO(2N)_{C+F}$. For instance the last vortex $n_a^+ = n_a^- = 1$, for all a, carries only the unit $U(1)$ winding and is a singlet, the second last vortex and analogous ones belong to a $\underline{\mathbf{2N}}$, and so on.

Due to the fact that the tension depends only on $N_0 = 2\, n^{(0)}$ (twice the $U(1)$ winding) the degeneracy pattern of the vortices does not simply reflect

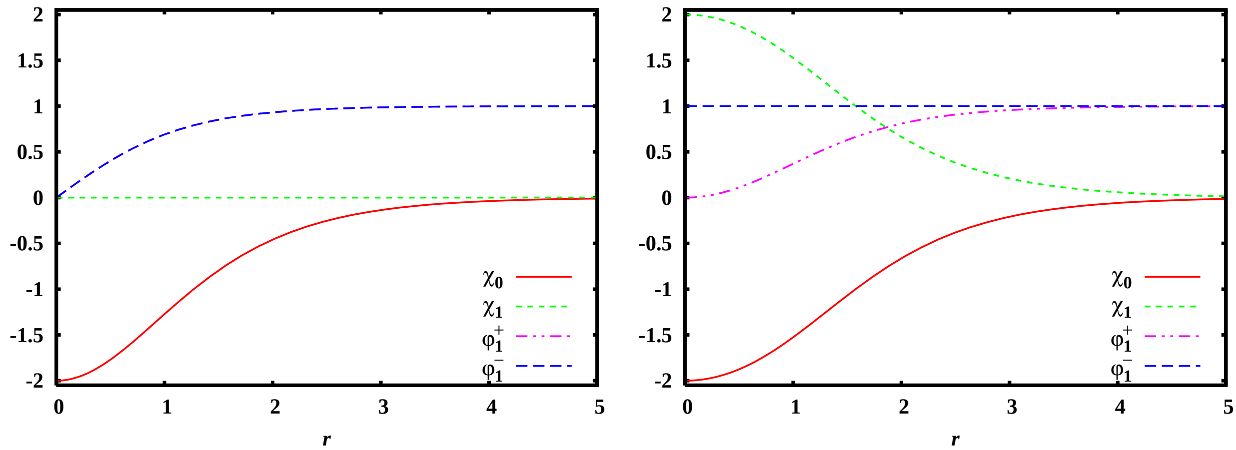

Figure B.3: Numerically integrated minimum vortex solution with $N_0 = 2$, where we have taken the couplings to be $4g_1^2 = 4g_{2N}^2 = 1$. In the left panel we have shown the element $(n^+, n^-) = (1,1)$ and in the right panel the element $(n^+, n^-) = (2,0)$. The dependence of the couplings turns out to be similar to the case of the minimal vortex $(n^+, n^-) = (1,0)$. $(\chi_i \equiv \sqrt{2} f_i)$.

the homotopy map which relates the vortices to the massive monopoles. The monopole-vortex correspondence will be discussed in Section B.5 below.

The profile functions $(f_0, f_a, \varphi_a^+, \varphi_a^-)$ for the simplest non-minimal vortex, $N_0 = 2$ are illustrated in Figure B.3. In the figure is just considered the two simplest elements $(n^+, n^-) = (1,1)$ and $(n^+, n^-) = (2,0)$. Adding elements of the same type corresponds just to a rescaling of the coupling g_{2N}^2 and of the functions $\varphi_a^\pm$ as in the minimal vortex case ($N_0 = 1$). Adding elements of different types ($(2,0)$ or $(1,1)$) does not induce new behavior.

B.4 Vortices in $SO(2N+1)$ theories

Consider now the case of a theory with symmetry breaking

$$SO(2N+3) \xrightarrow{v_1} SO(2N+1) \times U(1) \xrightarrow{v_2} 1 . \tag{B.82}$$

The fields which remain massless after the first symmetry breaking can be found exactly as in the even SO theories by use of various $SO(3)$ groups, leading to Eq. (B.17), with $A = 1, 2, \ldots, N_f$ where we now take $N_f = 2N+1$. The light quarks can get color-flavor locked VEVs as in Eq. (B.44), leading to a vacuum with global $SO(2N+1)_{C+F}$ symmetry.

The ansatz (B.69) must be modified as follows

$$q(r,\vartheta) = \begin{pmatrix} \mathbf{M}_1(r,\vartheta) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \mathbf{M}_N(r,\vartheta) & 0 \\ 0 & \cdots & 0 & e^{i\hat{n}\vartheta}\hat{\varphi}(r) \end{pmatrix} , \tag{B.83}$$

introducing a new integer $\hat{n}$ and a new profile function $\hat{\varphi}(r)$. The equation (B.63) becomes

$$\frac{df_0}{dr} = -2\sqrt{2}\,\eta\, g_1^2 r \left(\sum_a \left(|\varphi_a^+|^2 + |\varphi_a^-|^2 \right) + |\hat{\varphi}|^2 - \xi \right) , \tag{B.84}$$

while the condition of finite energy gives

$$\hat{\varphi}(\infty) = \sqrt{\frac{\xi}{2N+1}} , \tag{B.85}$$

$$\hat{n} = \frac{h_0(\infty)}{\sqrt{2}} = \frac{N_0}{2} , \tag{B.86}$$

and the equation for $\hat{\varphi}(r)$ is

$$\frac{d\hat{\varphi}}{dr} = \eta \left(\hat{n} - \frac{h_0}{\sqrt{2}} \right) \frac{\hat{\varphi}}{r} = -\eta \frac{f_0}{\sqrt{2}} \frac{\hat{\varphi}}{r} . \tag{B.87}$$

Note that the condition (B.86) fixes $\hat{n}$ in terms of N_0: as $\hat{n}$ must be an integer, this theory contains only vortices with even N_0. This can be traced to the different structure of the gauge groups. In fact, $SO(2N+3)$ has no center, so the pattern of symmetry breaking is

$$SO(2N+3) \to SO(2N+1) \times U(1) \to 1 , \tag{B.88}$$

and there are no vortices with half-integer winding around the $U(1)$, or around any other Cartan $U(1)$ subgroups.

The vortices are classified by the same integers $n_a^{\pm}$ as before, but now there are $SO(2N+1)_{C+F}$ transformations which exchange $n_a^+ \leftrightarrow n_a^-$ singly. The minimal vortices are labeled by

$$(n_a^+, n_a^-) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \hat{n} = 1 . \tag{B.89}$$

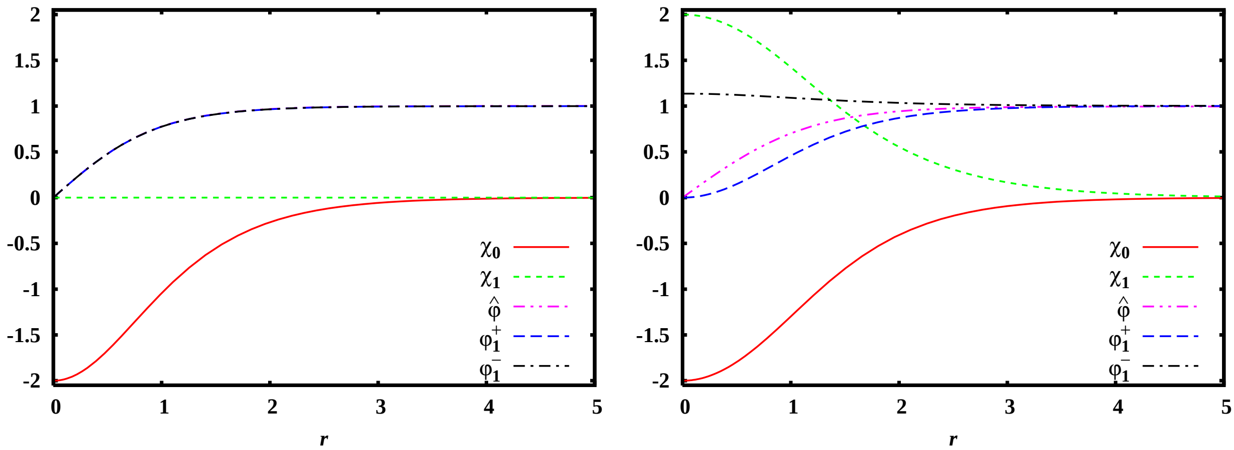

Figure B.4: Numerically integrated minimum vortex solution of the $SO(2N+1)$ theory, with $N_0 = 2$ and we take the couplings to be $4g_1^2 = 4g_{2N}^2 = 1$. In the left panel we have $(n_1^+, n_1^-) = (1,1)$ and in the right panel $(n_1^+, n_1^-) = (2,0)$. The dependence of the couplings turns out to be analogous to the case of the $(n_1^+, n_1^-) = (1,0)$ vortex. ($\chi_i \equiv \sqrt{2} f_i$).

The moduli space contains subspaces corresponding to these orbits, whose structure is that of the coset spaces $SO(2N+1) \big/ (U(N-k) \times SO(2k+1))$ where k is the number of $(1,1)$ pairs.

The vortex profile functions are shown in Figure B.4.

B.5 Monopoles, vortices, topology and confinement

B.5.1 Homotopy map

The multiplicity of vortex solutions depends on the particular topology of the symmetry-breaking pattern of our model.

Usually, in systems with a gauge Lie group G and a symmetry-breaking pattern

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 1 \; , \tag{B.90}$$

there are:

- Stable Dirac monopoles, classified by $\pi_1(G)$;
- Regular monopoles, classified by $\pi_2(G/H)$; topologically stable only in the limit $v_2 \to 0$;

- Vortices, classified by $\pi_1(H)$; if they correspond to a non-trivial element of $\pi_1(G)$, they are topologically stable; otherwise they are topologically stable only in the limit $v_1 \to \infty$.

Monopoles and vortices are related by the topological correspondence [157]

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G) \ , \tag{B.91}$$

so regular monopoles correspond to vortices which are trivial with respect to $\pi_1(G)$, while vortices which are non-trivial with respect to $\pi_1(G)$ correspond to Dirac monopoles.

In our theories of type D_N, however, the center $C_G = \mathbb{Z}_2$ acts trivially on all fields and the breaking pattern is

$$G \xrightarrow{v_1} H \xrightarrow{v_2} C_G \ , \tag{B.92}$$

and the topological relation (B.91) is not directly useful. In fact, vortices are classified by $\pi_1(H/C_G)$, which is a richer homotopy group than $\pi_1(H) \sim \pi_2(G/H) \times \pi_1(G)$. In our example the relevant group is

$$\pi_1 \left(\frac{SO(2N) \times U(1)}{\mathbb{Z}_2} \right) = \mathbb{Z} \times \mathbb{Z}_2 \ . \tag{B.93}$$

The failure of (B.91) would mean that the correspondence between monopoles and vortices is lost.

Actually, it is better to formulate the problem as follows. The theory contains only fields in the adjoint representation, so we can neglect the center C_G from the beginning and consider the gauge group as $G' = G/C_G$. In our example, the gauge group of the high-energy theory can be taken as $G' = SO(2N+2)/\mathbb{Z}_2$, broken to $H' = (SO(2N) \times U(1))/\mathbb{Z}_2$ at scale v_1 and then completely broken at scale v_2:

$$G' \xrightarrow{v_1} H' \xrightarrow{v_2} 1 \ . \tag{B.94}$$

instead of Eq. (B.92). Then the relation (B.91) reads

$$\pi_2 \left(\frac{SO(2N+2)}{SO(2N) \times U(1)} \right) = \frac{\pi_1 \left(\frac{SO(2N) \times U(1)}{\mathbb{Z}_2} \right)}{\pi_1 \left(\frac{SO(2N+2)}{\mathbb{Z}_2} \right)} \ . \tag{B.95}$$

Regular monopoles are classified by the same homotopy group as before, because

$$\frac{SO(2N+2)/\mathbb{Z}_2}{(SO(2N) \times U(1))/\mathbb{Z}_2} = \frac{SO(2N+2)}{SO(2N) \times U(1)} \ , \tag{B.96}$$

while for Dirac monopoles the situation is different: the relevant homotopy group is not $\pi_1\left(SO(2N+2)\right)$, but the larger group $\pi_1\left(SO(2N+2)/\mathbb{Z}_2\right)$ (see Appendix B.8)

$$\pi_1\left(\frac{SO(4J)}{\mathbb{Z}_2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2 \ , \tag{B.97}$$

while

$$\pi_1\left(\frac{SO(4J+2)}{\mathbb{Z}_2}\right) = \mathbb{Z}_4 \ , \tag{B.98}$$

so that the Dirac monopoles have quantized $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ charges.

This means that the theory has a larger set of monopoles, and the correspondence between monopoles and vortices (which confine them) is rather subtle[2].

In appendix B.8 we briefly review the structure of the homotopy groups which are relevant for this analysis.

Finally, for the groups of type B_N, the situation is slightly simpler as there is no non-trivial center. The non-trival element of $\pi_1\left(SO(2N+3)\right) = \mathbb{Z}_2$ represents the (unique type of) Dirac monopoles; the elements of $\pi_1\left(SO(2N+1) \times U(1)\right) = \mathbb{Z}_2 \times \mathbb{Z}$ label the vortices of the low-energy theory. The vortices whose (non-trivial) winding in the group $SO(2N+1)\times U(1)$ corresponds to a contractible loop in the parent theory, confine the regular monopoles.

B.5.2 Flux matching

To establish the matching between regular GNO monopoles and low-energy vortices, we use the topological correspondence discussed in the previous section. Dirac monopoles are classified by $\pi_1\left(SO(2N+2)/\mathbb{Z}_2\right)$ or by $\pi_1\left(SO(2N+3)\right)$ depending on the gauge group, but regular monopoles are classified by $\pi_2\left(\frac{SO(2N+2)}{SO(2N)\times U(1)}\right)$ or by $\pi_2\left(\frac{SO(2N+3)}{SO(2N+1)\times U(1)}\right)$, i.e. homotopically non-trivial paths in the low-energy gauge group, which are trivial in the high-energy gauge group. Regular monopoles can be sources for the vortices corresponding to these paths.

The vortices of the lowest tension which satisfy this requirement are those with $N_0 = \pm 2$ and $\sum_a (n_a^+ - n_a^-)/2$ odd, so vortices corresponding to minimal GNO monopoles belong to the $SO(2N)_{C+F}$ orbits classified by (B.80) with an odd number of $(\pm 2, 0)$ pairs.

For a better understanding of this correspondence, we can also use flux matching between vortices and monopoles [171]. There are $2N$ GNO monopoles

[2]Note that the Lagrangian and fields for the two theories with gauge group $SO(2N+2)$ and $SO(2N+2)/\mathbb{Z}_2$ are the same. The set of vortices is the same for both theories and has a topological correspondence with the larger set of monopoles.

obtained by different embeddings of broken $SU(2) \subset SO(4)$ in $SO(2N+2)$. In a gauge where ϕ is constant, their fluxes are

$$\int_{\mathcal{S}^2} d\vec{S} \cdot \vec{B}^a t^a = 2\sqrt{2}\pi(t_0 \pm t_i) \ , \tag{B.99}$$

where $t_0 \pm t_i$ is the unbroken generator of the broken $SU(2)$ subgroup. In the same gauge, the flux of a vortex is

$$\int_{\mathbb{R}^2} d^2x B_z^a t^a = -N_0\sqrt{2}\pi t_0 + \left(n_j^+ - n_j^-\right)\sqrt{2}\pi t_j \ , \tag{B.100}$$

so the fluxes agree for $N_0 = -2$, $n_j^+ - n_j^- = \pm 2\delta_{ij}$. The antimonopoles correspond to the opposite sign $N_0 = 2$.

B.5.3 Monopole confinement: the $SO(2N)$ theory

We have now all the tools needed to analyze the duality in the SO theories at hand. The general scheme for mapping the monopoles and vortices has been set up in Section B.5.1. An important point to keep in mind is that, while the vortex tension depends only on the $U(1)$ flux in our particular model (Eq. (B.59)), the classification of vortices according to the first homotopy group reflects the other Cartan charges (windings in $SO(2N)$ or $SO(2N+1)$). It is necessary to keep track of these to see how the vortices in the low-energy theory are associated with the monopoles of the high-energy system.

First consider the theories of type D_N, with the symmetry breaking

$$SO(2N+2) \xrightarrow{v_1} SO(2N) \times U(1) \xrightarrow{v_2} 1 \ . \tag{B.101}$$

studied in detail in the preceding sections. The vortices with minimum winding, $N_0 = 1$, of Eqs. (B.72), (B.73), correspond to the minimum non-trivial element of $\pi_1\left((SO(2N) \times U(1))/\mathbb{Z}_2\right)$, which represent also the minimal elements of $\pi_1\left(SO(2N+2)/\mathbb{Z}_2\right)$. This last fact means that they are stable in the full theory. They would confine Dirac monopoles of the minimum charge in the underlying theory, 1 of $\mathbb{Z}_4$ or $(1,0)$ or $(0,1)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$, see Appendix B.8.2.

Consider now the vortices Eq. (B.80) with $N_0 = 2$. As the fundamental group of the underlying theory is given by either Eq. (B.97) or Eq. (B.98), some of the vortices will correspond to non-contractible loops in the underlying gauge group: they would be related to the Dirac monopoles and not to the regular monopoles. Indeed, consider the last of Eq. (B.80):

$$\begin{pmatrix} n_a^- \\ n_a^+ \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} . \tag{B.102}$$

It is characterized by the windings $n^{(0)} = 1$, $n^{(a)} = 0$ for all a. Thus it is an ANO vortex of the $U(1)$ theory, with no flux in the $SO(2N)$ part. It corresponds to a 2π rotation in (12) plane in the original $SO(2N+2)$ group – the path P in Appendix B.8.1: it is to be associated with a Dirac monopole of charge 2.

The vortices of the type

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 2 & 1 & 1 & \dots & 1 \end{pmatrix} , \tag{B.103}$$

and analogous ones (with $(2,0)$ or $(0,2)$ appearing in different positions) are characterized by the two $U(1)$ windings only: a flux $n^{(0)} = 1$ and one of the Cartan flux of $SO(2N)$, e.g., $n^{(1)} = 1$ ($n^{(a)} = 0$, $a \neq 1$). They correspond to a simultaneous 2π rotations in (12) and in (34) planes in the gauge group and it represents a contractible loop in the high-energy gauge group. They confine regular monopoles, as can be seen also by the flux matching argument discussed in section B.5.2.

Part of the continuous moduli of these vortex solutions include

$$SO(2N)\big/U(1) \times SO(2N-2) \ , \tag{B.104}$$

as the individual soliton breaks $SO(2N)_{C+F}$ symmetry of the system. This space corresponds to the complex quadric surface $Q^{2N-2}(C)$. As these vortices are not elementary but composite of the minimal vortices, determining their correct moduli space structure is not a simple task.

Nevertheless, there are some indications that these correspond to a vector representation $\underline{\mathbf{2N}}$ of $SO(2N)_{C+F}$, appearing in the decomposition of the product of two spinor representations, Eq. (B.81). In fact, the vortex Eq. (B.103) arises as a product

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} : \tag{B.105}$$

i.e., a product of two spinors of the same chirality if N is odd; *vice versa*, of spinors of opposite chirality if N is even. This corresponds precisely to the known decomposition rules in $SO(4m+2)$ and $SO(4m)$ groups (see e.g., [176], Eq. (23.40)).

In order to establish that these vortices indeed transform under the $SO(2N)_{C+F}$ as a $\underline{\mathbf{2N}}$ one needs to construct the moduli matrix [173] for these, and study explicitly how the points in the moduli space transform. This problem will be studied elsewhere.

It is interesting to note that there seems to be a relation between the transformation properties of monopoles under the dual GNO group $\widetilde{SO}(2N)$

and the transformation properties of the corresponding vortices under the $SO(2N)_{C+F}$ group. In fact, vortices transforming as a vector of $SO(2N)_{C+F}$ have precisely the net magnetic flux of regular monopoles in **2N** of $\widetilde{SO}(2N)$, as classified by the GNO criterion.

Other vortices in Eq. (B.80) correspond to various Dirac (singular) or regular monopoles in different representations of $SO(2N)_{C+F}$.

B.5.4 Monopole confinement: the $SO(2N+1)$ theory

In the B_N theories with the symmetry breaking

$$SO(2N+3) \xrightarrow{v_1} SO(2N+1) \times U(1) \xrightarrow{v_2} 1 \; . \tag{B.106}$$

the minimal vortices of the low-energy theory have $N_0 = 2$. Reflecting the difference of π_1 group of the underlying theory as compared to the D_N cases ($\mathbb{Z}_2$ as compared to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$), the $N_0 = 1$ vortices (with half winding in $U(1)$ and $SO(2N)$) are absent here.

The minimal vortices (B.89) again correspond to different homotopic types and to various $SO(2N+1)$ representations. The vortex

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}, \qquad \hat{n} = 1 \; , \tag{B.107}$$

has the $U(1)$ charge $n^{(0)} = 1$ and no charge with respect to $SO(2N+1)$. It is associated to the non-trivial element of $\pi_1\left(SO(2N+3)\right) = \mathbb{Z}_2$: it is stable in the full theory. Its flux would match that of a Dirac monopole. This is a singlet of $SO(2N+1)_{C+F}$ (its moduli space consists of a point).

Consider instead the vortices

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 2 & 1 & 1 & \dots & 1 \end{pmatrix}, \qquad \hat{n} = 1 \; , \tag{B.108}$$

and analogous ones, having the winding numbers $n^{(0)} = 1$, $n^{(a)} = \pm 1$, $n^{(b)} = 0$, $b \neq a$, and $\hat{n} = 1$. These would correspond to regular monopoles which, according to GNO classification, are supposed to belong to a **2N** representation of the dual group $USp(2N)$. Again, though it is not a trivial task to establish that these vortices do transform as **2N** of such a group, there are some hints they indeed do so. It is crucial that the symmetry group (broken by individual soliton vortices) is $SO(2N+1)$: it is in fact possible to identify the **2N** generators constructed out of those of $SO(2N+1)$, that transform them appropriately (Appendix). Secondly, the flux matching argument of Section B.5.2 do connect these vortices to the minimum, regular monopoles appearing in the semiclassical analysis. As in the D_N theories these observations should be considered at best as a modest hint that dual group structure as suggested by the monopole-vortex correspondence is consistent with the GNO conjecture.

B.6 Conclusions

In this appendix we have explicitly constructed BPS, non-Abelian vortices of a class of $SO(N) \times U(1)$ gauge theories in the Higgs phase. The models considered here can be regarded as the bosonic part of softly broken $\mathcal{N} = 2$ gauge theories with N_f quark matter fields. The vortices considered here represent non-trivial generalizations of the non-Abelian vortices in $U(N)$ models widely studied in recent literature.

The systems are constructed so that they arise as low-energy approximations to theories in which gauge symmetry suffers from a hierarchical breaking

$$SO(N+2) \xrightarrow{v_1} SO(N) \times U(1) \xrightarrow{v_2} 1 \, , \qquad v_1 \gg v_2 \, , \tag{B.109}$$

leaving an exact, unbroken global $(SO(N) \times U(1))_{C+F}$ symmetry. Even though the low-energy $SO(N) \times U(1)$ model with symmetry breaking

$$SO(N) \times U(1) \xrightarrow{v_2} 1 \, , \tag{B.110}$$

can be studied on its own right, without ever referring to the high-energy $SO(N+2)$ theory, consideration of the system with hierarchical symmetry breaking is interesting as it forces us to try (and hopefully allows us) to understand the properties of the non-Abelian *monopoles* in the high-energy approximate system with $SO(N+2) \xrightarrow{v_1} SO(N) \times U(1)$ and their confinement by the vortices – language adequate in the dual variables – from the properties of the vortices via homotopy map and symmetry argument. Note that in this argument, the fact that the monopoles in the high-energy theory and the vortices in the low-energy theory are both almost BPS but not exactly so, is of fundamental importance [158, 177].

In the models based on $SU(N)$ gauge symmetry, the efforts along this line of thought seem to be starting to give fruits, giving some hints on the nature of non-Abelian duality and confinement. Although the results of this work are a only a small step toward a better and systematic understanding of these questions in a more general class of gauge systems, they provide a concrete starting point for further studies.

B.7 Appendix: $SO(2N)$, $USp(2N)$, $SO(2N+1)$

The change of basis to the one where a vector multiplet $\underline{\mathbf{2N}}$ of $SO(2N)$ naturally breaks to $\underline{\mathbf{N}} + \underline{\bar{\mathbf{N}}}$ under $U(N)$, is given by (see Eq. (B.30))

$$\begin{pmatrix} \hat{Q}_3 \\ \vdots \\ \hat{Q}_{2N+1} \\ -iQ_3 \\ \vdots \\ -iQ_{2N+1} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -i1/\sqrt{2} \\ -i1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} q_3 \\ \vdots \\ q_{2N+1} \\ q_4 \\ \vdots \\ q_{2N+2} \end{pmatrix} . \tag{B.111}$$

The $SO(2N)$ generators,

$$\begin{pmatrix} E & F \\ -\,{}^tF & D \end{pmatrix}, \tag{B.112}$$

where D, E, F are all pure imaginary $N \times N$ matrices, with the constraints ${}^tE = -E$, ${}^tD = -D$, are accordingly transformed as

$$\begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} E & F \\ -\,{}^tF & D \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} (E+D) + i(F + {}^tF) & i(E-D) + (F - {}^tF) \\ -i(E-D) + (F - {}^tF) & (E+D) - i(F + {}^tF) \end{pmatrix} . \tag{B.113}$$

Since both E, D are anti-symmetric, $(E+D)$ in the 1st block is the most general anti-symmetric imaginary matrix, while $i(F + {}^tF)$ is the most general symmetric real matrix. Their sum gives the most general $N \times N$ hermitian matrix, which corresponds to generators of $U(N)$. In other words, the subgroup $U(N) \subset SO(2N)$ is generated by those elements with $E = D$, $F = {}^tF$.

On the other hand, the generators of $USp(2N)$ group have the form

$$\begin{pmatrix} B & A \\ C & -\,{}^tB \end{pmatrix}, \tag{B.114}$$

with the constraints, ${}^tA = A$, ${}^tC = C$, $A^* = C$, $B^\dagger = B$. The fact that A is symmetric while the non-diagonal blocks in Eq. (B.113) are antisymmetric, means that there is no further overlap between the two groups, that is, the maximal common subgroup between $SO(2N)$ and $USp(2N)$ is $U(N)$.

It is possible to get a hint on how $USp(2N)$ groups can appear as transformation group of the vortices. In order to see transformations among the vortices $(\hat{Q}, Q)$ under which the latter could transform as $\underline{\mathbf{2N}}$, it is necessary to embed the system in a larger group, such as $SO(2N+1)$ model considered in Section B.4. The idea is to build a map [3] between the $SO(2N+1)$ generators (antisymmetric matrices) and the $USp(2N)$ generators which have the form, Eq. (B.114). The ith $SO(4) \sim SU(2) \times SU(2)$ subgroup is generated by (with a simplified notation $(1,2,3,4) \equiv (1,2,2i+1,2i+2)$)

$$T_1^{\pm} = -\frac{i}{2}\left(\Sigma_{23} \pm \Sigma_{41}\right) , \quad T_2^{\pm} = -\frac{i}{2}\left(\Sigma_{31} \pm \Sigma_{42}\right) , \quad T_3^{\pm} = -\frac{i}{2}\left(\Sigma_{12} \pm \Sigma_{43}\right) . \tag{B.115}$$

The two vortices living in this $SO(4)$ group are taken to be i-th and $(N+i)$-th components of the fundamental representation of $USp(2N)$. The pairs can be transformed to each other by rotations in the $(2i+2, 2N+3)$ plane ($\subset SO(2N+1)$), thus

$$A_{i,i} = -i\, \Sigma_{2i+2,2N+3} . \tag{B.116}$$

On the other hand, the two vortices associated with subgroups $T^{\pm}$ living in the $(1,2,2i+1,2i+2)$ subspace and those living in the $(1,2,2j+1,2j+2)$ subspace, $j \neq i$, are transformed into each other by rotations in the $(2i+1, 2i+2, 2j+1, 2j+2)$ space: they transform in $SO(2N)$ (in the subspace $i = 3, 4, \ldots, 2N+2$). We have already seen that they actually do transform as a pair of $U(N)$ representations, in the basis Eq. (B.111). As the $U(N)$ elements are generated by the $SO(2N)$ infinitesimal transformations with $E = D$, $F = {}^tF$, one finds the map,

$$B_{i,j} = -i\left(\Sigma_{2i,2j} + \Sigma_{2i+1,2j+1}\right) + \left(\Sigma_{2i,2j+1} - \Sigma_{2i+1,2j}\right) . \tag{B.117}$$

Non-diagonal elements A_{ij}, $i \neq j$, can be generated by commuting the actions of (B.116) and (B.117).

B.8 Appendix: Fundamental groups

Let's briefly discuss the (first) homotopy groups relevant to us:

B.8.1 $SO(2N+2)$

There is only one non-trivial closed path P in this case, the rotation from 0 to 2π around any axis. The rotation from 0 to 4π is homotopically equivalent to the trivial path, so $P^2 = 1$ and the homotopy group is

$$\pi_1\left(SO(2N+2)\right) = \mathbb{Z}_2 , \tag{B.118}$$

[3]This correspondence can be applied equally well to the minimal regular monopoles constructed semi-classically, and has been discussed in this context in [174].

B.8.2 $SO(2N+2)/\mathbb{Z}_2$

Actually, in the model discussed in this appendix, all the fields are in the adjoint representation of $SO(2N+2)$: the gauge group effectively corresponds to $SO(2N+2)$ modulo identification $-1 = 1$. The path P is again non-trivial, but now there are also two inequivalent closed paths P_+ and P_- going from 1 to -1, defined as $P_+ P_-^{-1} = P$. Explicitly, they can be taken as simultaneous rotations in $N+1$ planes

$$P_+ : \quad e^{i\beta_{12}\Sigma_{12}} \prod_{i=3,5,\dots,N-1} e^{i\beta_{i\,i+1}\Sigma_{i,i+1}} \; ; \qquad \beta_{12} : 0 \to \pi \; , \qquad \beta_{i,i+1} : 0 \to \pi \; . \tag{B.119}$$

$$P_- : \quad e^{i\beta_{12}\Sigma_{12}} \prod_{i=3,5,\dots,N-1} e^{i\beta_{i\,i+1}\Sigma_{i,i+1}} \; ; \qquad \beta_{12} : 0 \to -\pi \; , \quad \beta_{i,i+1} : 0 \to \pi \; . \tag{B.120}$$

When $N+1$ is even, $P_+^2 = P_-^2 = 1$ and $P_+ P_- = P$. The homotopy group is generated by P_+, P_-:

$$\pi_1 \left(\frac{SO(4N)}{\mathbb{Z}_2} \right) = \mathbb{Z}_2 \times \mathbb{Z}_2 \; , \tag{B.121}$$

When $N+1$ is odd, $P_+^2 = P_-^2 = P$ and $P_+ P_- = 1$, so the homotopy group is generated by P_+ only, and is of cyclic order four

$$\pi_1 \left(\frac{SO(4N+2)}{\mathbb{Z}_2} \right) = \mathbb{Z}_4 \; . \tag{B.122}$$

B.8.3 $(SO(2N) \times U(1))/\mathbb{Z}_2$

After the symmetry breaking at the higher mass scale v_1, the theory reduces to an $(SO(2N) \times U(1))/\mathbb{Z}_2$ theory. The division by $\mathbb{Z}_2$ corresponds to the identification $(-1,-1) = (1,1)$, inherited from the underlying theory. From the point of view of the low-energy effective theory, it is due to the fact that all the light matter fields $q_{A,j}, \tilde{q}_{A,j}$ are in the vector representation of $SO(2N)$ but they carry at the same time the unit charge with respect to $U(1)$.

The non-trivial paths of $SO(2N) \times U(1)$ are combinations of Q (a 2π rotation in any plane in $SO(2N)$) and the paths R_n winding n times around the $U(1)$. The simplest non-trivial closed paths that arise after the $\mathbb{Z}_2$ quotient are $P_{+,\frac{1}{2}}$, $P_{+,-\frac{1}{2}}$, $P_{-,\frac{1}{2}}$, $P_{-,-\frac{1}{2}}$ going from $(1,1)$ to $(-1,-1)$ with a half winding around $U(1)$. By taking $U(1)$ to act in the (12) plane, $SO(2N)$ in the $(34\dots N)$ space, they can be explicitly chosen as simultaneous rotations in (12), (34), (56)... planes

$$e^{i\gamma_{12}\Sigma_{12}}\, e^{i\beta_{34}\Sigma_{34}} \prod_{i=5,7,\dots,N-1} e^{i\beta_{i\,i+1}\Sigma_{i,i+1}} \; ; \tag{B.123}$$

with

$$P_{+,\frac{1}{2}} : \quad \gamma_{12} : 0 \to \pi \ , \qquad \beta_{34} : 0 \to \pi \ , \qquad \beta_{i,i+1} : 0 \to \pi \ . \tag{B.124}$$

$$P_{+,-\frac{1}{2}} : \quad \gamma_{12} : 0 \to -\pi \ , \qquad \beta_{34} : 0 \to \pi \ , \qquad \beta_{i,i+1} : 0 \to \pi \ . \tag{B.125}$$

$$P_{-,\frac{1}{2}} : \quad \gamma_{12} : 0 \to \pi \ , \qquad \beta_{34} : 0 \to -\pi \ , \qquad \beta_{i,i+1} : 0 \to \pi \ . \tag{B.126}$$

$$P_{-,-\frac{1}{2}} : \quad \gamma_{12} : 0 \to -\pi \ , \qquad \beta_{34} : 0 \to -\pi \ , \qquad \beta_{i,i+1} : 0 \to \pi \ . \tag{B.127}$$

Note that $P_{+,\frac{1}{2}}$ and $P_{+,-\frac{1}{2}}$ correspond respectively to the P_+ and P_- paths in the $SO(2N+2)$ theory.

When N is even, $P_{+,a}P_{+,b} = P_{-,a}P_{-,b} = R_{a+b}$ and $P_{+,a}P_{-,b} = Q\,R_{a+b}$, so every group element can be written as $(P_{+,1/2})^k\, Q^\delta$ with $k \in \mathbb{Z}$, $\delta = \{0,1\}$. The homotopy group is

$$\pi_1\left(\frac{SO(2N)\times U(1)}{\mathbb{Z}_2}\right) = \mathrm{Z}\times\mathbb{Z}_2 \ , \quad N \text{ even} \ , \tag{B.128}$$

When N is odd, $P_{+,a}P_{+,b} = P_{-,a}P_{-,b} = Q\,R_{a+b}$ and $P_{+,a}P_{-,b} = R_{a+b}$, and every group element can again be written as $(P_{+,1/2})^k\, Q^\delta$ with $k \in \mathbb{Z}$, $\delta = \{0,1\}$, as in the N even case. The homotopy group is

$$\pi_1\left(\frac{SO(2N)\times U(1)}{\mathbb{Z}_2}\right) = \mathrm{Z}\times\mathbb{Z}_2 \quad , \quad N \text{ odd} \ , \tag{B.129}$$

Even though the homotopy group is the same for the two cases (N even or odd), its embedding in $\pi_1\left(SO(2N)\times U(1)\right) = \mathrm{Z}\times\mathbb{Z}_2$ is different: R_n corresponds to $k = 2n, \delta = 0$ for N even and to $k = 2n, \delta = 1$ for N odd. In other words

$$R_1 = (P_{+,1/2})^2\, Q \ , \quad (N \,\text{odd}) \ ; \qquad R_1 = (P_{+,1/2})^2 \quad (N\,\text{even}) \ . \tag{B.130}$$

B.8.4 Relation between the smallest elements of the high-energy and low-energy fundamental groups

There are simple relations among the smallest elements of the groups $\pi_1\left(\frac{SO(2N+2)}{\mathbb{Z}_2}\right)$ and $\pi_1\left(\frac{SO(2N)\times U(1)}{\mathbb{Z}_2}\right)$. From the above explicit constructions one sees that

$$P_+ = P_{+,\frac{1}{2}} \ ; \quad P_- = P_{+,-\frac{1}{2}} = R_{-1}\, P_{+,\frac{1}{2}} \ ; \tag{B.131}$$

and by using Eq. (B.130), one has

$$P_{+,-\frac{1}{2}} = \begin{cases} (P_{+,\frac{1}{2}})^{-1} Q \ , & \text{odd } N \ , \\ (P_{+,\frac{1}{2}})^{-1} \ , & \text{even } N \ . \end{cases} \tag{B.132}$$

B.8.5 $SO(2N+3)$

The fundamental group is $\mathbb{Z}_2$ as in the $SO(2N+2)$ cases, and the smallest closed path being

$$P : \quad e^{i\beta_{ij}\Sigma_{ij}} : \quad \beta_{ij} = 0 \to 2\pi \ , \tag{B.133}$$

in any plane (ij). $P^2 = 1$ and the homotopy group is

$$\pi_1 \left(SO(2N+3) \right) = \mathbb{Z}_2 \ . \tag{B.134}$$

B.8.6 $SO(2N+1) \times U(1)$

At the mass scales below v_1 the theory reduces to an $SO(2N+1)\times U(1)$ theory with matter in the fundamental representation, q and $\tilde{q}$ carrying charges ± 1 with respect to $U(1)$. The fundamental group is

$$\pi_1 \left(SO(2N+1) \times U(1) \right) = \mathbb{Z}_2 \times \mathbb{Z} \ , \tag{B.135}$$

where $\mathbb{Z}$ represents the number of winding (charge) in the $U(1)$ part and $\mathbb{Z}_2$ a 2π rotation in any plane in $SO(2N+1)$.

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