VARIATIONAL RESULTS FOR NEMATIC ELASTOMERS AND SINGULAR PERTURBATIONS OF EVOLUTION PROBLEMS

Ph.D. Thesis

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Contents

Introduction

This thesis collects variational results concerning the modeling of nematic elastomers and some issues regarding the characterization of the limit behavior of solutions to some singularly perturbed evolution problems. A large part of these results, which are the content of Chapters 2–7, has been published in [**1**]–[**4**]. Part of the material composing this thesis is extracted from these papers.

Let us start with a brief introduction to nematic elastomers. Synthesized at the end of the 80's, these materials are rubbery elastic solids made of cross–linked polymeric chains to which rigid rod–like molecules, the nematic mesogens, are attached. In particular, nematic elastomers exhibit large spontaneous deformations, which can be triggered and controlled by temperature, applied electric fields, irradiation by UV light. These properties make them interesting as materials for fast soft actuators and justify the considerable attention that they have attracted in recent years.

Theoretical modeling of the mechanical response of nematic elastomers has concentrated on the occurrence of equilibrium configurations exhibiting fine domain patterns (stripe domains), and the stress plateau associated with rearrangement of stripe domains in stretching experiments (soft elasticity). Starting from the pioneering work of Warner, Terentjev, and their collaborators [**10, 69**], several models have been proposed [**6, 13, 37, 40, 48, 70, 71**]. The model based on the free energy density put forward in [**10**] is particularly worth mentioning, both for its fundamental nature and for its success at reproducing (and even predicting) essential features of experimental observations. In fact, energy minimizing states computed with this model reproduce experimental evidence with a remarkable degree of accuracy. Examples include the highly nontrivial spatially dependent domain structures observed in [**73**] and simulated numerically in [**16, 17**], the existence of a plateau in the stress–strain response in some uniaxial extension experiments [**16, 17, 35**], and the decay of shear moduli in stretching experiments when the imposed stretch reaches the ends of the stress plateau [**9, 32, 55**].

In Chapter 1 we describe in more detail the most important feature of nematic elastomers, namely, the coupling between nematic orientational order and rubber elasticity. Indeed, the nematic mesogens are randomly oriented at high temperature, but, upon cooling through a specific transition temperature, they align along a common direction described by the *nematic director*, which is represented by a unit vector n of $\mathbb{R}^3$. At the same time, the underlying polymer network exhibits the spontaneous elastic distortion described by the tensor

$$L_n := a^{\frac{2}{3}} n \otimes n + a^{-\frac{1}{3}}(I - n \otimes n), \tag{0.1}$$

where $a > 1$ is a non dimensional material parameter. We then introduce the following expression for the energy density of an isotropic and incompressible nematic elastomers:

$$\frac{\mu}{2}\left[\operatorname{tr}(F^T L_n^{-1} F) - 3\right], \qquad \det F = 1. \tag{0.2}$$

Here, F is a 3×3 matrix representing the gradient of a deformation with respect to the reference configuration Ω, chosen as the one the sample would exhibit in the high–temperature phase. This is a classical expression, studied, e.g., in [**16**] and [**31**], and obtained from an earlier proposal by Bladon, Terentjev and Warner [**10**] by an affine change of variables, first introduced in [**29**]. This energy is always nonnegative and it is null precisely when FF^T is of the form (0.1). The fact that these states of deformation are those observed experimentally (under sufficiently small applied loads) is one of the main justifications for the physical soundness of the model.

In Chapter 2, we study the following natural generalization of energy (0.2), in order to handle compressible nematic elastomers:

$$W_n(F) := \frac{\mu}{2}\left[\mathrm{tr}\,(F^T L_n^{-1} F) - 3 - 2\ln(\det F)\right] + \frac{\lambda}{2}(\det F - 1)^2, \qquad \det F > 0, \tag{0.3}$$

Then, we consider the energy

$$W(F) := \min_{|n|=1} W_n(F), \qquad \det F > 0, \tag{0.4}$$

which models the purely mechanical response of the system. We present and discuss in details the linearized versions of (0.3)–(0.4) obtained on the basis of Taylor expansion, in the spirit of [**32**]. To proceed with the presentation of the results of Chapter 2, we briefly comment on the justification via Γ–convergence of linearized theories in elasticity (we discusss this subject in Subsection 1.2.2 in more details).

The energy stored by a homogeneous and hyperelastic body occupying a reference configuration $\Omega \subseteq \mathbb{R}^n$ and subject to a deformation $v : \Omega \to \mathbb{R}^n$ is

$$\int_\Omega f(\nabla v)dx,$$

where ∇v is the deformation gradient and f is the energy density. Suppose that f is minimized at the value 0 by the identity matrix I, and that f is frame indifferent. In these conditions one expects that small external loads $\varepsilon l(x)$ will produce small deformations $v(x) = x + \varepsilon u(x)$. In turn, the total energy will be given by

$$\int_\Omega f(I + \varepsilon\nabla u)dx - \varepsilon^2 \int_\Omega l u dx.$$

Denoting by $e(u)$ the symmetric part of the displacement gradient ∇u, the stored elastic energy of linearized elasticity is

$$\frac{1}{2}\int_\Omega D^2 f(I)[e(u)]^2 dx - \int_\Omega l u dx,$$

and can be obtained by Taylor–expansion from the previous formula rescaled by ε^{-2}. This formal derivation of linear elasticity does not guarantee that the minimizers of the "ε–functionals" (under prescribed boundary conditions) converge to the minimizer of the limit functional (under the same boundary conditions). On the other hand, if one manages to prove a statement of Γ–convergence for the functionals involved, then it is possible to recover information on the convergence of the minimizers. This is one of the most important features of Γ–convergence (see [**23**, Chapter 7]).

In this framework, convergence of minimizers has been established by Dal Maso, Negri and Percivale in [**26**], under the assumption

$$f(F) \geq C d^2(F, SO(n)), \tag{0.5}$$

where $SO(n)$ is the set of rotations of $\mathbb{R}^n$ and $d^2(F, SO(n))$ is the square of the distance of F from $SO(n)$. This result has been extended in [**60**] to a family $\{f_\varepsilon\}$ of stored energy densities whose set of minimizers is of the form

$$SO(n)U_{1,\varepsilon} \cup \dots \cup SO(n)U_{k,\varepsilon}, \tag{0.6}$$

where $U_{i,\varepsilon} = I + \varepsilon\hat{U}_i + o(\varepsilon)$ is a positive definite symmetric matrix. In [**60**], the growth behavior of the f_ε's is again as in (0.5), with the set (0.6) in place of $SO(n)$.

Going back to the expressions (0.3)–(0.4), we see that the energy density W_ε obtained from (0.4) by replacing L_n with

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n \otimes n + (1+\varepsilon)^{-1}(I - n \otimes n),$$

has the following set of minimizers

$$SO(3)\mathscr{U}_\varepsilon, \qquad \text{where} \quad \mathscr{U}_\varepsilon := \bigcup_{|n|=1} L_{n,\varepsilon}^{\frac{1}{2}}.$$

Also, it satisfies

$$W_\varepsilon(F) \geq Cd^2(F, SO(3)\mathscr{U}_\varepsilon), \qquad \text{for every } F. \tag{0.7}$$

A straightforward extension of the Γ–convergence result of [**60**] applies to our family of energy densities $\{W_\varepsilon\}$. We then apply this result and show that, under prescribed boundary conditions, the minimizers of (proper rescalings of) the nonlinear functionals converge to minimizers of the relaxed linearized functional (Theorem 2.4). The linearized functional is given by $\int_\Omega V(e(u))dx$, with the small strain energy density V (*linear limit*) defined on every symmetric matrix E as

$$V(E) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_\varepsilon(I + \varepsilon E).$$

More explicitly, the linearized functional has the expression

$$\int_\Omega \left\{ \mu \min_{|n|=1} \left| (e(u))_d - \hat{U}_n \right|^2 + \frac{k}{2} (\operatorname{tr} \nabla u)^2 \right\} dx,$$

where k is a function of the material parameters appearing in (0.3), and $\hat{U}_n$ comes from the linearization $L_{\varepsilon,n} = I + 2\varepsilon\hat{U}_n + o(\varepsilon)$. An explicit relaxation formula for the linearized functional is available in [**12**]. We conclude Chapter 2 with the analysis of another compressible version of (0.2), alternative to (0.3)–(0.4). This alternative model shares with (0.3)–(0.4) the set of wells $\mathscr{U}_\varepsilon$ as well as the linear limit. What is different is the growth behavior in the regime of large deformations. For this reason, we discuss and justify a modified version of it to which the theory of [**60**] applies and gives Theorem 2.7.

The previous analysis leaves open the question whether the results of [**60**] can be generalized to energies satisfying condition (0.7) only near the wells. In Chapter 3, we give a positive answer for the case of single–well energies. More precisely, we obtain in Theorem 3.2 the same conclusions as Dal Maso, Negri and Percivale under the assumption that (0.5) is satisfied only in a neighbourhood of $SO(n)$, while far away from $SO(n)$ the growth condition can be weakened to

$$f(F) \geq c\, d^p(F, SO(n)), \qquad \text{for some } \ 1 < p < 2. \tag{0.8}$$

Similar results have been obtained in [**59**] assuming also a bound of order p from above. It is worth noticing that the reason for considering energies satisfying (0.8) (without any bound from above) is not purely academic. Indeed, for a large class of compressible rubber–like materials, the growth behavior (0.8) is the appropriate one (see Subsection 3.1.1).

Concerning the strategy of the proof, we observe that in [**26**] the main tool adopted to prove the compactness of the minimizers is the *Geometric Rigidity Lemma* of [**38**]. To obtain the analogous issue when (0.5) holds only near $SO(n)$, while (0.8) holds far from $SO(n)$, see Theorem 3.3, we need a version with two exponents of the Geometric Rigidity Lemma, similar to those used in [**18**], [**51**], and in [**59**].

In proving the Γ–convergence result (Theorem 3.4), our approach is different from the one employed in [**26**], as well as from the further improvements introduced in [**60**]. The main simplification relies on some arguments developed in [**38**] for the rigorous proof of dimension reduction results. Moreover, in place of the weak convergence in $W^{1,2}$ of the minimizers obtained in [**26**], we provide strong convergence in $W^{1,p}$ (see Theorem 3.5).

We hope that all our results can be extended to multi–well energies.

We now describe the results of Chapter 4. Albeit the Warner–Terentjev model has been quite successful at reproducing observed material instabilities (stripe domains and soft elasticity, which are associated with the non convexity of the proposed energies), however it does not predict accurately stress–build–up at large imposed stretches. The reason for that is the Neo–Hookean form of the expression for the free energy density, which results from the assumption of phantom gaussian chains made in its derivation from statistical mechanics. Just as in classical rubber elasticity, stress–strain curves showing the typical hardening response of rubbers at high strains and stresses requires the use of functional forms richer than the Neo–Hookean template. Inspired by the seminal work of Ogden [**54**], we provide Ogden–type extensions of the Warner–Terentjev model to the regime of very high strains, and also include finite compressibility effects.

The main new results contained in Chapter 4 are the following. By exploiting a multiplicative decomposition of the deformation gradient into an elastic and a remanent or spontaneous part, we propose the Ogden–type expressions

$$W_n(F) := \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\operatorname{tr} \left(L_n^{-\frac{1}{2}} F F^T L_n^{-\frac{1}{2}} \right)^{\frac{\gamma_i}{2}} - 3 \right], \qquad \det F = 1,$$

for the free energy density of nematic elastomers, and provide a template for further extensions. Here, $\gamma_i \geq 2$ and $c_i > 0$, for every $i = 1, ..., N$. We compute the geometrically linear version of the new models, which shows the geometric structure of the underlying energy landscape in a very transparent fashion: the energy grows quadratically with the distance from the nonconvex set of spontaneous strains (energy wells). Energies of this type are very common in the theoretical and computational mechanics community, especially in the context of active and phase-transforming materials [**8**]. Our discussion of their relation with a parent fully nonlinear theory may have the additional side benefit of inspiring generalizations in the opposite direction, namely, finite deformation generalizations of existing small strain theories for active materials.

We then derive (Proposition 4.3) the expression of the energies $W := \min_{|n|=1} W_n$ describing the purely mechanical response governed by the new model, which turns out to be

$$W(F) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\left(\frac{\lambda_1(F)}{a^{-\frac{1}{6}}} \right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{a^{-\frac{1}{6}}} \right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{a^{\frac{1}{3}}} \right)^{\gamma_i} - 3 \right], \qquad \det F = 1, \tag{0.9}$$

where $\lambda_1(F) \leq \lambda_2(F) \leq \lambda_3(F)$ are the ordered singular values of F. This expression is "separable" in the sense discussed in [**54**] and justifies the name "Ogden–type" for this new model. Moreover, because of the structure with multiple energy wells, these energies are not quasiconvex. Finally, using the results of [**31**], we provide explicit formulas for their quasiconvex envelopes in Theorem 4.10, and apply them to a simple thought experiment (pure–shear) to demonstrate their use and their potential at reproducing the stiffening behavior at very large imposed strains, that is typical of elastomeric materials.

In Chapter 5, we present two results of attainment of the minimal energy, one for the nonlinear model and the other for the geometrically linear one. These results have been obtained in collaboration with G. Dal Maso and A. DeSimone. We consider first the nonlinear energy density given by

$$W(F) := \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\left(\frac{\lambda_1(F)}{e_1} \right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{e_2} \right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{e_3} \right)^{\gamma_i} - 3 \right], \qquad \det F = 1,$$

where $\gamma_i \geq 2$, $c_i > 0$ for every $i = 1, ..., N$, and $0 < e_1 \leq e_2 \leq e_3$ are three fixed ordered real numbers such that $e_1 e_2 e_3 = 1$. Note that Ogden–type energies for nematic elastomers (0.9) are included in this expression. The function W is minimized at the value zero if $\lambda_i(F) = e_i$, for $i = 1, 2, 3$. Theorem 5.2 states that for every function $v : \Omega \to \mathbb{R}^3$ which is piecewise affine and Lipschitz, if

$$\det \nabla v = 1 \quad \text{a.e. in } \Omega, \qquad \operatorname{ess\,inf}_\Omega |\lambda_1(\nabla v)| > e_1, \qquad \operatorname{ess\,sup}_\Omega |\lambda_3(\nabla v)| < e_3, \tag{0.10}$$

then there exists a dense set of Lipschitz functions $y : \Omega \to \mathbb{R}^3$ such that

$$\int_\Omega W(\nabla y) = 0 \quad \text{a.e. in } \Omega, \qquad y = v \quad \text{on } \partial\Omega. \tag{0.11}$$

The same holds if v is of class $C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3)$, for some $0<\alpha<1$, and satisfy (0.10). This result is an application of the theory developed by Müller and Sverák in [**53**]. In this paper, the authors study the solutions of first order partial differential relations

$$\nabla y \in K \quad \text{a.e. in } \Omega, \qquad y = v \quad \text{on } \partial\Omega, \tag{0.12}$$

where the set K is contained in $\{F : M(F) = t\}$, $M(F)$ is a fixed minor of F, and $t \neq 0$. The case $M(F) = \det F$ and $t = 1$ perfectly applies to our minimum problem (0.11), which can be rewritten as (0.12) with $K = \{3{\times}3 \text{ matrices } F : \lambda_i(F) = e_i\,,\ i = 1,2,3\}$

The case where the set K appearing in (0.12) is contained in $\{F : \operatorname{tr} F = 0\}$ is not explicitly treated in [**53**]. Thus, focusing on the two–dimensional case, we state and prove Theorem 5.11, which is a linear version of the main result of Müller and Sverák, with slightly simplified assumptions. We then apply Theorem 5.11 to obtain the following result (Theorem 5.7). We consider the small strain (incompressible) energy density given by

$$V(E) := \left(|E| - \sqrt{2}\right)^2, \quad \text{for every } 2{\times}2 \text{ symmetric matrix } E \text{ such that } \operatorname{tr} E = 0. \tag{0.13}$$

This expression can be derived by taking the limit, as ε goes to zero, of the ratio $W_\varepsilon(I + \varepsilon E)/\varepsilon^2$, where W_ε is obtained from (0.2) by replacing L_n with $(1+\varepsilon)^2 n \otimes n + (1+\varepsilon)^{-2}(I - n \otimes n)$, and considering n as a unit vector of $\mathbb{R}^2$. We prove that for every piecewise affine Lipschitz map $w : \Omega \to \mathbb{R}^2$ such that

$$\operatorname{div} w = 0 \quad \text{a.e. in } \Omega, \qquad \operatorname{ess\,sup}_\Omega |e(w)| < \sqrt{2}, \tag{0.14}$$

there exists a dense set of Lipschitz functions $u : \Omega \to \mathbb{R}^2$ such that

$$\int_\Omega V(e(u)) = 0 \quad \text{a.e. in } \Omega, \qquad u = w \quad \text{on } \partial\Omega. \tag{0.15}$$

The same holds if w is of class $C^{1,\alpha}(\overline{\Omega};\mathbb{R}^2)$, for some $0<\alpha<1$, and satisfies (0.14). In fact, our minimum problem (0.15) can be rewritten as

$$\nabla u \in K_0 \quad \text{a.e. in } \Omega, \qquad u = w \quad \text{on } \partial\Omega, \tag{0.16}$$

where $K_0 := \left\{2{\times}2 \text{ matrices } F : \operatorname{tr} F = 0,\ |symF| = \sqrt{2}\right\}$.

We also propose another method to solve problem (0.15) in the case where $w = 0$, see Proposition 5.6, without making use of the theory of [**53**]. This method provides solutions of class $W_0^{1,p}$, for every $1 \le p < \infty$, and gives explicit solutions in the case where Ω is a disk.

The motivation for the study of minimal energy's attainment problems is the attempt to understand the dynamic response of nematic elastomers. Concerning the dynamics, we have considered an evolution problem in the simplest situation: the two–dimensional small strain regime. Let Ω be the reference configuration, $\partial_D\Omega$ a Dirichlet part of the boundary $\partial\Omega$ with positive $\mathscr{H}^1$ measure, and let H be defined by

$$H := \{u \in H^1(\Omega;\mathbb{R}^2) \,:\, u = 0 \text{ on } \partial_D\Omega\},$$

where the equality is referred to the traces of the functions. Given $g \in H^{-1/2}(\partial\Omega \setminus \partial_D\Omega;\mathbb{R}^3)$, we consider the functional

$$\mathcal{E}(u) := \int_\Omega V(e(u))dx + \langle g, u\rangle,$$

where V is defined in (0.13), and $\langle g, u\rangle$ is the duality product between g and the trace of u on $\partial\Omega \setminus \partial_D\Omega$. The functional $\mathcal{E}$ is a Lyapunov function for the evolution problem

$$\operatorname{div} S = 0, \quad \operatorname{div} u = 0 \quad \text{on } \Omega, \qquad u(t,\cdot) \in H, \qquad S\nu = g \quad \text{on } \partial\Omega \setminus \partial_D\Omega, \tag{0.17}$$

where

$$S := \frac{\partial V}{\partial E}(e(u)) + e(\dot{u}).$$

While the existence of the solutions is not difficult to prove, up to a regularization of V in a neighborhood of the origin, the crucial question regards the behavior of the solutions at infinity. It is implicit that the aim of this approach is finding a way to select some critical points of $\mathcal{E}$.

This problem is very difficult and, apart from some examples of evolution built by hand, we are unable to produce general results at the moment.

In the last two chapters we consider two other dynamic problems in a more abstract framework. In particular, we deal with singular perturbations of these evolution problems and study the limit behavior of their solutions. Even if not directly related to the modeling of nematic elastomers, they present some connections with problem (0.17).

In Chapter 6 we address the problem of finding a function $t \mapsto u(t)$ satisfying

$$\nabla_x \mathcal{E}(t, u(t)) = 0 \qquad \text{and} \qquad \nabla_x^2 \mathcal{E}(t, u(t)) > 0. \tag{0.18}$$

This problem appears in many areas of applied mathematics, where, usually, the real–valued function $\mathcal{E}(t, x)$ represents a time–dependent energy, defined for $t \in [0, T]$ and $x \in \mathbb{R}^n$. The symbol ∇_x denotes the gradient with respect to x, while ∇_x^2 is the corresponding Hessian. The inequality in (0.18) means that the matrix $\nabla_x^2 \mathcal{E}(t, u(t))$ is positive definite. Therefore, (0.18) says that, for every t, the state $u(t)$ is a stable equilibrium point for the potential $\mathcal{E}(t, \cdot)$.

If we look for a continuous solution $t \mapsto u(t)$, defined only in a neighborhood of a prescribed time, the problem is solved by the Implicit Function Theorem. In many applications, however, we want to obtain a piecewise continuous solution $t \mapsto u(t)$ on the whole interval $[0, T]$. The main problem is, therefore, to extend the solution beyond its maximal interval of continuity. A first possibility is to select, for every t, a global minimizer $u(t)$ of $\mathcal{E}(t, \cdot)$. This choice has some drawbacks, as we shall explain later. Different extension criteria can be proposed, motivated by different interpretations of the problem.

Problem (0.18) can be considered, for instance, as describing the limiting case of a system governed by an overdamped dynamics, as the relaxation time tends to 0. Indeed, one can prove that, when the relaxation time is very small, the state $u(t)$ of the system is always close to a stable equilibrium for the potential $\mathcal{E}(t, \cdot)$, which, in general, is not a global minimizer of $\mathcal{E}(t, \cdot)$. The first general result in this direction has been obtained by Zanini (see [**72**]), who considers (0.18) as limit of the viscous dynamics governed by the gradient flow

$$\varepsilon \dot{u}^\varepsilon(t) + \nabla_x \mathcal{E}(t, u^\varepsilon(t)) = 0. \tag{0.19}$$

She proves that the limit $u(t)$ of the solution $u^\varepsilon(t)$ to problem (0.19) is a piecewise continuous function satisfying (0.18), and describes the trajectories followed by the system at the jump times. Under different and stronger hypotheses, similar vanishing viscosity limits have been studied in finite dimension [**27, 34, 49, 50, 62**], and even in infinite dimension in [**11, 23, 24, 25, 46, 65**].

Simple examples show that the solution $u(t)$ found in [**72**] is, in general, different from the global minimizer. We note that the global minimizer may exhibit abrupt discontinuities at times where it must jump from a potential well to another one with the same energy level. This jump cannot be justified if we interpret (0.18) as limit of a dynamic problem, since the state should overcome a potential barrier during the jump.

We consider (0.18) as the limiting case of a sequence of singular second order evolution problems, namely

$$\varepsilon^2 A \ddot{u}^\varepsilon(t) + \varepsilon B \dot{u}^\varepsilon(t) + \nabla_x \mathcal{E}(t, u^\varepsilon(t)) = 0, \tag{0.20}$$

where A and B are positive definite and symmetric matrices. This describes the evolution of a mechanical system where both inertia and friction are taken into account, encoded in A and B, respectively. We use the same assumptions as in [**72**]. Among these assumptions, a very important one is that

$$\text{the critical points of } \ \mathcal{E}(t, \cdot) \ \text{ are isolated,} \quad \text{for every } \ t \in [0, T]. \tag{0.21}$$

We prove that the solution u^ε of (0.20) is such that $(u^\varepsilon, \varepsilon B \dot{u}^\varepsilon)$ tends to $(u, 0)$, where u is piecewise continuous and satisfies (0.18). Moreover, the trajectories of the system at the jump times are described through suitable autonomous second order systems related to A, B, and $\nabla_x \mathcal{E}$.

Let us explain, in more detail, the procedure that we follow. We first construct a suitable piecewise continuous solution u of problem (0.18) and then show that the solutions $u^\varepsilon(t)$ of (0.20), with the same initial conditions, converge to $u(t)$ at every continuity time t.

The function u is defined in the following way (see Proposition 6.6). We begin with a point $u(0)$ such that $\nabla_x\mathcal{E}(0,u(0)) = 0$ and $\nabla_x^2\mathcal{E}(0,u(0)) > 0$. By the Implicit Function Theorem, we find a continuous solution u of (0.18) up to a certain time $t_1 \le T$ such that $\nabla_x^2\mathcal{E}(t_1,u(t_1^-))$ has only one zero eigenvalue. In a "generic" situation (see Assumption 3 in Section 6.1 and Remark 6.3), certain transversality conditions hold at the point $(t_1,u(t_1^-))$. These conditions imply that a saddle–node bifurcation of the vector field $F(t,\cdot)$, corresponding to the first order autonomous system equivalent to

$$A\ddot{w}(s) + B\dot{w}(s) + \nabla_x\mathcal{E}(t,w(s)) = 0, \tag{0.22}$$

occurs at t_1. Let (t,x) be close enough to $(t_1,u(t_1^-))$. If $t < t_1$, then $F(t,\cdot)$ has two zeros, a saddle and a node. If $t > t_1$, there are no zeros of $F(t,\cdot)$. Under these conditions, it is also possible to prove (see Lemma 6.4) existence and uniqueness, up to time–translations, of a non constant solution to system (0.22), satisfying

$$\lim_{s\to-\infty}(w(s),\dot{w}(s)) = (u(t_1^-),0). \tag{0.23}$$

Moreover, the limit

$$\lim_{s\to+\infty}(w(s),\dot{w}(s)) =: (x_1^r,0) \tag{0.24}$$

exists, and x_1^r is another zero of $\nabla_x\mathcal{E}(t_1,\cdot)$. If $t_1 < T$, we make the "generic" assumption that $\nabla_x^2 f(t_1,x_1^r)$ is positive definite (see Assumption 4 in Section 6.1). This allows us to restart the procedure and, in turn, to find a solution of (0.18) on $[t_1,t_2)$, for a certain $t_2 \le T$, and so on. In this way, we find a piecewise continuous solution u of (0.18), with certain discontinuity times $t_1,...,t_{m-1}$, and, for $j = 1,...,m-1$, a heteroclinic solution w_j of (0.22) with $t = t_j$, which connects a degenerate critical point of $\mathcal{E}(t_j,\cdot)$ at $s = -\infty$ to a non degenerate critical point at $s = +\infty$.

The next step consists in proving that, if $(u^\varepsilon(0),\varepsilon\dot{u}^\varepsilon(0)) \to (u(0),0)$, then $(u^\varepsilon,\varepsilon B\dot{u}^\varepsilon)$ converges to $(u,0)$ uniformly on the compact subsets of $[0,T]\setminus\{t_1,...,t_{m-1}\}$, while a proper rescaling v_j^ε of u^ε is such that $(v_j^\varepsilon,\dot{v}_j^\varepsilon)$ converges uniformly to $(w_j,\dot{w}_j)$ on the compact subsets of $\mathbb{R}$ (see Theorem 6.8 and Remark 6.17). This shows that (0.22) governs the fast dynamics of the system at the jump times. Theorem 6.9 summarizes these convergences in a more geometric statement involving the Hausdorff distance.

We conclude Chapter 6 showing that the same solution u of (0.18) introduced before can be obtained as the limit of a discrete time approximation, which uses only autonomous systems. For every $k \in \mathbb{N}$, we consider the partition $\tau_i^k = \frac{i}{k}T$, $i = 1,...,k$, of the interval $[0,T]$. Let $\{u_i^k\}_i$ recursively defined by $u_0^k = u(0)$ and by

$$u_i^k := \lim_{\sigma\to+\infty} v_i^k(\sigma), \tag{0.25}$$

where v_i^k is the solution to the autonomous system

$$A\ddot{v}_i^k(\sigma) + B\dot{v}_i^k(\sigma) + \nabla_x\mathcal{E}(\tau_i^k,v_i^k(\sigma)) = 0, \tag{0.26}$$

with initial conditions $(v_i^k(0),\dot{v}_i^k(0)) = (u_{i-1}^k,0)$. The existence of the limit in (0.25) is a property of the autonomous system, ensured by Lemma 6.4.

We prove that $u_i^k = u(\tau_i^k)$, unless τ_i^k is close to the discontinuity times $t_1,...,t_{m-1}$ of u. More precisely, given an arbitrary neighborhood U of the set $\{t_1,...,t_{m-1}\}$, we prove that $u_i^k = u(\tau_i^k)$ whenever k is sufficiently large and $\tau_i^k \notin U$ (see Lemma 6.20 and Lemma 6.21). This implies that the piecewise constant and the piecewise affine interpolations of the values u_i^k's converge uniformly to u on the compact subsets of $[0,T]\setminus\{t_1,...,t_{m-1}\}$.

In order to obtain the convergence to the heteroclines w_j's near the jump times, as well as the convergence of the velocity (Proposition 6.25 and Theorem 6.18), we introduce a suitable interpolation of u_i^k based on the solution v_i^k of (0.26) (see the definition in (6.108)).

In Chapter 7, we extend to an infinite dimensional setting the analysis performed in [**72**] about the compactness and the limit behavior of a family of solutions to (0.19). These results have been obtained in collaboration with G. Savaré and R. Rossi.

We work with an energy functional $\mathcal{E} : [0,T] \times X \to \mathbb{R}$, where X is a Hilbert space. Apart from standard and minimal semicontinuity and coerciveness hypotheses on the functional $\mathcal{E}$, all the assumptions of [**72**] have been removed, with the only exception of the crucial requirement (0.21).

For a vanishing sequence $\{\varepsilon_n\}$, we let $\{u_{\varepsilon_n}\}$ be a family of solutions to the gradient flow (0.19) and we prove in Theorem 7.12 that, up to a subsequence, $\{u_{\varepsilon_n}\}$ converges a.e. to a function u satisfying

$$\nabla_x \mathcal{E}(t, u(t)), \qquad \text{for a.e. } t \in (0,T). \tag{0.27}$$

To show this, we start with a detailed analysis of the limit behavior, as $n \to +\infty$, of the integrals

$$\int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, u_{\varepsilon_n}(r))\| \|\dot{u}_{\varepsilon_n}(r)\| dr, \tag{0.28}$$

when the sequences $\{t_1^n\}$ and $\{t_2^n\}$ tend to t, for some $t \in [0,T]$. Using assumption (0.21), it is possible to prove that if the sequences $\{u_{\varepsilon_n}(t_1^n)\}$ and $\{u_{\varepsilon_n}(t_2^n)\}$ converge to two different critical points x_1 and x_2 of $\mathcal{E}(t,\cdot)$, then the integrals (0.28) are bounded below by a single positive integral

$$\int_0^1 \|\nabla_x \mathcal{E}(t, \vartheta(r))\| \|\dot{\vartheta}(r)\| dr,$$

where ϑ is a function in the class

$$\begin{aligned} \mathscr{A}^t_{x_1,x_2} := \Big\{ & \vartheta : [0,1] \to X \text{ continuous} : \text{there exist } 0 = \mathsf{t}_0 < \mathsf{t}_1 < \ldots < \mathsf{t}_j = 1 \text{ s.t.} \\ & \vartheta(0) = x_1,\ \vartheta(1) = x_2, \quad \text{and for every} \quad i = 0, \ldots, j-1, \\ & \vartheta \text{ is locally Lipschitz on } (\mathsf{t}_i, \mathsf{t}_{i+1}),\ \vartheta(\mathsf{t}_i) \neq \vartheta(\mathsf{t}_{i+1}),\ \nabla_x \mathcal{E}(\mathsf{t}_i, \vartheta(\mathsf{t}_i)) = 0 \Big\}. \end{aligned}$$

This is the content of Lemma 7.4. Building on this, we can prove that the following minimum is achieved:

$$c(t; x_1, x_2) := \min \left\{ \int_0^1 \|\nabla_x \mathcal{E}(t, \vartheta(r))\| \|\dot{\vartheta}(r)\| dr \ : \ \vartheta \in \mathscr{A}^t_{x_1,x_2} \right\}.$$

Combining some key properties (Proposition 7.9) of the *cost function* $c(t; x_1, x_2)$ with suitable a priori estimates, we finally prove the a.e. convergence of a family of solutions $\{u_{\varepsilon_n}\}$ to a limit function u satisfying (0.27).

Theorem 7.12 also says that u is continuous on $[0,T] \setminus J$, where the jump set J is a countable set. Moreover, the left and the right limits $u_-(t)$, $u_+(t)$ exist everywhere and are such that

$$\mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) = c(t; u_-(t), u_+(t)), \qquad \text{for every } t \in J.$$

We point out that from this condition and from the definition of the cost function it is possible to deduce that, for every $t \in J$, there exists a function $w \in \mathscr{A}^t_{u_-(t),u_+(t)}$ satisfying the equation

$$\dot{w}(s) + \lambda(s) \nabla_x \mathcal{E}(t, w(s)) = 0, \qquad \text{for a.e. } s \in (0,1),$$

for some function $\lambda(s) > 0$. The existence of this kind of functions at the jump times is the infinite dimensional counterpart of what happen in the finite dimensional case [**72**]. In particular, in [**72**] it is proved that, at any fixed jump time t, there exists a function which satisfies (0.22) with $A = 0$ and $B = I$, and which connects two critical point of $\mathcal{E}(t,\cdot)$.

Notation

- $\mathbb{R}^n$ is the set of real n–dimensional vectors;
- $\mathbb{R}^n_+$ is the subset of $\mathbb{R}^n$ of the vectors whose entries are all nonnegative;
- $\mathbb{M}^{m\times n}$ is the set of $m\times n$ real matrices;
- $\mathbb{M}^{n\times n}_0$ is the set of $n\times n$ traceless (deviatoric) matrices;
- $M^T \in \mathbb{M}^{n\times m}$ is the transpose of the matrix $M \in \mathbb{M}^{m\times n}$;
- $Sym(n) := \{M \in \mathbb{M}^{n\times n} \,:\, M = M^T\}$ is the set of $n\times n$ symmetric matrices;
- $Sym_0(n)$ the set of $n\times n$ symmetric and traceless matrices;
- $Skw(n) := \{M \in \mathbb{M}^{n\times n} \,:\, M = -M^T\}$ is the set of $n\times n$ skew–symmetric matrices;
- $Lin^+(n)$: the set of $n\times n$ invertible matrices with positive determinant;
- $Orth(n) := \{M \in \mathbb{M}^{n\times n} \,:\, M^{-1} = M^T\}$ is the set of $n\times n$ orthogonal matrices;
- $SO(n) := \{M \in \mathbb{M}^{n\times n} \,:\, M^{-1} = M^T, \det M = 1\}$ is the set of rotations of $\mathbb{R}^n$;
- $Psym(n)$ is the set of $n\times n$ positive definite symmetric matrices;
- $symM := \frac{M+M^T}{2}$ is the symmetric part of $M \in \mathbb{M}^{n\times n}$;
- $skwM := \frac{M-M^T}{2}$ is the skew–symmetric part of $M \in \mathbb{M}^{n\times n}$;
- $M_d := \left[M - \frac{1}{n}(\operatorname{tr} M)I\right] \in \mathbb{M}^{n\times n}_0$ is the deviatoric part of $M \in \mathbb{M}^{n\times n}$;
- $\mu_1(M) \leq ... \leq \mu_n(M)$ are the ordered eigenvalues of $M \in Sym(n)$;
- $\lambda_1(M) \leq ... \leq \lambda_n(M)$ are the ordered singular values of $M \in \mathbb{M}^{n\times n}$, where $\lambda_i(M) := \left[\mu_i(M^TM)\right]^{\frac{1}{2}}$;
- $\Lambda(M)$ is the set of the singular values of $M \in \mathbb{M}^{n\times n}$;
- S^n is the unit sphere in $\mathbb{R}^{n+1}$;
- I is the unit matrix;
- $e(u) := sym(\nabla u)$, for a generic deformation or displacement u;
- $\mathscr{L}^n$ is the n–dimensional Lebesgue measure.
- $\mathscr{H}^n$ is the n–dimensional Hausdorff measure.

The measure of a $\mathscr{L}^n$–measurable set $\Omega \subseteq \mathbb{R}^n$ is sometimes denoted by $|\Omega|$. We use the symbol 1_Ω for the characteristic function of Ω, which is defined by

$$1_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout, $d(\cdot,\cdot)$ indicates the Euclidean distance both between two points and between a point and a set, and $d^p(\cdot,\cdot)$ means the p–th power of $d(\cdot,\cdot)$.

We denotes the Euclidean inner product between A and B by $A \cdot B$. Thus,

$$a \cdot b := \sum_{i=1}^{n} a_i b_j \quad \text{if } a, b \in \mathbb{R}^n, \qquad A \cdot B := \operatorname{tr}(A^T B) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij} \quad \text{if } A, B \in \mathbb{M}^{m\times n},$$

where $\operatorname{tr} A$ is the trace of A. The corresponding Euclidean norm is $|\cdot|$. We often use the tensor product $a \otimes b : \mathbb{R}^n \to \mathbb{R}^n$ between vectors a, $b \in \mathbb{R}^n$, which is defined by

$$(a \otimes b)v := (b \cdot v)a \qquad \text{for every } v \in \mathbb{R}^n.$$

We recall that the ij–component of $a \otimes b$ is given by $(a \otimes b)_{ij} = a_i b_j$.

If not otherwise stated, $B(x, r)$ denotes the open ball centered at x with radius r. The ball $B(0, r)$ is sometime denoted by B_r.

For $f : \Omega \to X$, we use the symbols

$$Df(x)[u] \qquad \text{and} \qquad D^2 f(x)[u, w]$$

to denote the differential of f at the point x applied to u and the second differential of f at the point x applied to the pair $[u, w]$.

We use the following classes of functions defined on Ω and taking values in X:

- $C(\Omega; X)$ the class of continuous functions;
- $C^k(\Omega; X)$ the class of k–times differentiable functions;
- $C_c^\infty(\Omega; X)$ the class of smooth functions with compact support in Ω;
- $C^{k,\alpha}(\Omega; X)$ the class of Hölder functions;
- $\mathrm{Lip}(\Omega; X)$ the class of Lipschitz functions;
- $\mathrm{Lip}_{\mathrm{loc}}(\Omega; X)$ the class of locally Lipschitz functions;
- $L^p(\Omega; X)$ the class of Lebesgue functions;
- $W^{1,p}(\Omega; X)$ the class of Sobolev functions.

The space $W^{1,2}(\Omega; X)$ is also denoted by $H^1(\Omega; X)$. The space $W_0^{1,p}(\Omega; X)$, for $1 \le p < \infty$, is the closure of $C_c^\infty(\Omega; X)$ with respect to the topology of $W^{1,p}(\Omega; X)$. For functions defined on an open interval (a, b), we also use the notation $L^p(a, b; X)$, $H^1(a, b; X)$, and so on. The spaces of absolutely continuous functions and of functions with bounded variation on $[a, b]$ are denoted by $\mathrm{AC}([a, b]; X)$ and $\mathrm{BV}([a, b]; X)$, respectively.

We point out that the codomain is sometimes dropped out in the notation, whenever it is clear from the context.

Throughout we denote by C a generic constant whose value may vary from line to line.

CHAPTER 1

Modeling of nematic elastomers

Nematic liquid crystal elastomers are rubbery elastic solids which consist of a polymeric backbone, made of cross–linked polymeric chains, where nematic mesogens are embedded. We limit ourselves to a brief description of these materials, referring the reader to the monograph by Warner and Terentjev [**69**] for a thorough introduction to the chemistry and physics of nematic elastomers, and for an extensive list of references.

The polymeric backbone is made of monomers containing tetra–valent atoms which form long and flexible chains which are connected to each other by means of other flexible chains, the cross–linkers, to form a network. The nematic mesogens are rigid rod–like molecules (containing benzenic rings) which can either be part of the backbone or be attached sideways. These mesogens are randomly oriented at high temperature (isotropic phase), but at a sufficiently low temperature they align along a common average direction, the *nematic director*, and the system becomes anisotropic (nematic phase).

The polymeric backbone experiences reversible distortions as the material is cooled through the isotropic–to–nematic phase transition temperature: a uniaxial elongation occurs parallel to the nematic director as a consequence of the ordering of the mesogenic units that are incorporated into the network.

Given the isotropy of the high–temperature phase, the system is free to choose an arbitrary direction of alignment, so that different parts of the sample may spontaneously deform in different ways. We will focus on monodomain nematic elastomers, that is on the case where, in the nematic phase, there is only one (average) direction of alignment throughout the sample.

We work in the framework of a Frank–type theory, in which the liquid crystal order is supposed to be uniaxial with fixed degree of orientation. To describe the mechanical implication of such an order (in dimension 3), we consider the tensor

$$L_n := a^{\frac{2}{3}} n\otimes n + a^{-\frac{1}{3}}(I - n\otimes n), \qquad n \in S^2. \tag{1.1}$$

Here, n represents the nematic director, and the material parameter $a > 1$ is the step–length anisotropy quantifying the magnitude of the spontaneous stretch along n accompanying the isotropic–to–nematic phase transformation. This spontaneous distortion of the polymer chains induced by the alignment of the nematic mesogens along the direction n is

$$L_n^{\frac{1}{2}} = a^{\frac{1}{3}} n\otimes n + a^{-\frac{1}{6}}(I - n\otimes n).$$

Note that $L_n^{\frac{1}{2}}$ represents a volume–preserving uniaxial stretch. More precisely, $a^{\frac{1}{3}}$ is the elongation along n and $a^{-\frac{1}{6}}$ is the contraction along all the orthogonal directions. We remark that the parameter a is in principle a function of the temperature, but we assume it to be constant, because we will be working at a fixed constant temperature well below the phase transition temperature.

In Section 1.1 we describe the most basic and fundamental expression for the elastic energy density stored by a nematic elastomer. We will refer to it throughout the following chapters. In Section 1.2 we introduce some mathematical tools and results which will be useful later on.

1.1. The classical BTW expression for the energy density

Fixing a reference orientation n_r (e.g., the first basis vector of a given cartesian frame) and focusing on the incompressible case, the expression for the energy density proposed by Bladon, Terentjev and Warner [**10, 69**] stored by a monodomain nematic elastomer in the state $(\overline{F}, n)$ is

$$\overline{W}_n(\overline{F}) = \frac{\mu}{2}\left[\operatorname{tr}\left(L_{n_r}\overline{F}^T L_n^{-1}\overline{F}\right) - 3\right], \qquad \det \overline{F} = 1, \tag{1.2}$$

where $\mu > 0$ is a material parameter controlling the rubber energy scale (shear modulus) and $\overline{F} = \nabla\overline{y}$ is the gradient of the deformation $\overline{y}$ mapping a minimum energy configuration associated with n_r (chosen as reference configuration) into the current configuration. Note that the choice of n_r is arbitrary and, just as the choice of the reference configuration, is only a matter of convenience. If there exists a distinguished orientation in the material, it is natural to use it as a reference one. This is the case, e.g., when treating anisotropic nematic elastomers [**32**].

Following [**29, 30**] (see also the discussion in [**32**, Section 3]), we choose as reference configuration a minimum energy configuration associated with the high–temperature isotropic state, see Figure 1.1.

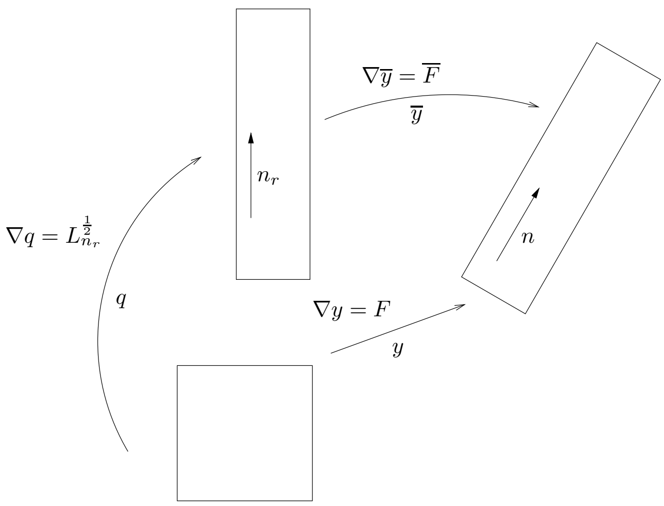

FIGURE 1.1. Schematic diagram illustrating two possible choices of reference configuration (the one for y and the other for $\overline{y}$).

Introducing the affine change of variables q, with $\nabla q = L_{n_r}^{\frac{1}{2}}$, we set

$$y = \overline{y} \circ q,$$

where $\circ$ denotes the composition of the maps $\overline{y}$ and q, and let $F := \nabla y$. We have

$$\overline{F} = F L_{n_r}^{-\frac{1}{2}} \tag{1.3}$$

and we can rewrite energy (1.2) as

$$W_n(F) := \frac{\mu}{2}\left[\operatorname{tr}\left(F^T L_n^{-1} F\right) - 3\right], \qquad \det F = 1. \tag{1.4}$$

Throughout our discussion we will refer to (1.4) as to the BTW model. We remark that the energies $\overline{W}_n$ and W_n, which are related to each other by $\overline{W}_n(\overline{F}) = W_n(F)$, in view of (1.3), are entirely oblivious to the reference orientation n_r. They describe a material for which there exist no distinguished material directions, namely, an isotropic material. In fact, we have that

$$W_n(FQ) = W_n(F), \qquad \text{for every } Q \in SO(3),$$

for every $n \in S^2$. Note that, in checking the isotropy of W_n, we have not applied Q to n. This is because, considering $W_n(F)$ as a function of n, this variable should be interpreted as a Eulerian variable. This is not the case for n_r in formula (1.2): to check that (1.2) governs an isotropic material, we have to show that expression (1.2) does not change if we replace $\overline{F}$ and n_r with $\overline{F}Q$ and Qn_r, respectively, because n_r in (1.2) represents a Lagrangian variable.

While W_n is the energy density modeling the mechanic response of nematic elastomers when n is maintained fixed (e.g., by an applied electric field), the energy density W we are going to introduce models the so called purely mechanical response, that is the mechanical response when the system is free to adjust n at fixed F. Following the notation $0 < \lambda_1(F) \le \lambda_2(F) \le \lambda_3(F)$ for the ordered singular values of F (so that $\lambda_1^2(F) \le \lambda_2^2(F) \le \lambda_3^2(F)$ are the ordered eigenvalues of FF^T), we consider

$$\begin{aligned} W(F) &:= \min_{n\in S^2} W_n(F) \\ &= \frac{\mu}{2}\left[\left(\frac{\lambda_1(F)}{a^{-\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_2(F)}{a^{-\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_3(F)}{a^{\frac{1}{3}}}\right)^2 - 3\right], \qquad \det F = 1. \end{aligned} \tag{1.5}$$

We set $W_n(F) = W(F) = +\infty$, if $\det F \neq 1$. The expression in (1.5) can be obtained observing that

$$\begin{aligned} \min_{n\in S^2} \operatorname{tr}\left(F^T L_n^{-1} F\right) &= \min_{n\in S^2} (FF^T)\cdot L_n^{-1} \\ &= \min_{n\in S^2} (FF^T)\cdot\left[a^{-\frac{2}{3}} n\otimes n + a^{\frac{1}{3}}(I - n\otimes n)\right] \\ &= a^{\frac{1}{3}} \min_{n\in S^2} (FF^T)\cdot\left[I + \left(\frac{1}{a} - 1\right) n\otimes n\right] \\ &= a^{\frac{1}{3}} \min_{n\in S^2}\left[\operatorname{tr}(FF^T) + \left(\frac{1}{a} - 1\right)(FF^T n)\cdot n\right]. \end{aligned} \tag{1.6}$$

Since $a > 1$, the minimum in (1.6) is attained when n is an eigenvector of FF^T corresponding to its maximum eigenvalue $\lambda_3^2(F)$, so that

$$\min_{n\in S^2} \operatorname{tr}\left(F^T L_n^{-1} F\right) = a^{\frac{1}{3}}\left[\operatorname{tr}(FF^T) + \left(\frac{1}{a} - 1\right)\lambda_3^2(F)\right].$$

From this equivalence, expression (1.5) follows.

Proposition 1.1. *Considering W_n and W defined by (1.4) and (1.5), respectively, we have that*

(i) $W_n \ge 0$ and $W_n(F) = 0$ if and only if $FF^T = L_n$;
(ii) $W \ge 0$ and $W(F) = 0$ if and only if $FF^T = L_n$, for some $n \in S^2$.

Note that, by left polar decomposition, the condition $FF^T = L_n$ for some $F \in Lin^+(3)$ is equivalent to

$$F = U_n R \quad \text{for some } R \in SO(3), \qquad \text{where } U_n := L_n^{\frac{1}{2}}.$$

Moreover, Proposition 1.1 (ii) tells us that W attains its minimum value zero on the set of energy wells

$$\mathscr{U} := \bigcup_{n\in S^2} \{U_n R : R \in SO(3)\} = \{Q U_{\hat{n}} R : Q, R \in SO(3)\}, \tag{1.7}$$

where $\hat{n}$ is some fixed unit vector. Equivalently, $F \in \mathscr{U}$ if and only if $\Lambda(F) = \left\{a^{\frac{1}{3}}, a^{-\frac{1}{6}}, a^{-\frac{1}{6}}\right\}$.

Proof. Denoting by ν_1, ν_2 and ν_3 the (positive) eigenvalues of $L_n^{-\frac{1}{2}}FF^TL_n^{-\frac{1}{2}}$, the standard inequality between geometric and arithmetic mean gives

$$\mathrm{tr}\,(F^TL_n^{-1}F) = \sum_{k=1}^{3}\nu_k \geq 3\left(\prod_{k=1}^{3}\nu_k\right)^{\frac{1}{3}} = 3\left[\det(F^TL_n^{-1}F)\right]^{\frac{1}{3}} = 3, \tag{1.8}$$

where we have also used the fact that $\det F = \det L_n^{-1} = 1$. Note the equality holds in (1.8) if and only if $\nu_1 = \nu_2 = \nu_3 = 1$, that is $F^TL_n^{-1}F = I$. This concludes the proof of (i). Property (ii) trivially follows from (i), from the definition of W, and from the fact that, for every $m \in S^2$ and $R \in SO(3)$, $W(RU_m) := \min_{n\in S^2} W_n(RU_m) = 0$. □

1.2. Mathematical tools

In this section we introduce some mathematical tools which will be useful in our treatment.

1.2.1. Notions of convexity. The variational problems arising in the study of nematic elastomers are vectorial problems of the multi–dimensional calculus of variations, whose fundamental convexity condition is quasiconvexity. Together with quasiconvexity, in Chapters 2, 4, and 5 the following notions of convexity will be useful. For completeness, we start with the definition of convexity. For this notion, as well as for the following ones, we allow the functions to take the value $+\infty$: this is standard, when one is interested in applications to elasticity.

Definition 1.2. *A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is* convex *if*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B), \quad \text{for every } \lambda \in [0,1],\ A, B \in \mathbb{M}^{m\times n}. \tag{1.9}$$

In passing, we recall that for a function $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$ the convexity condition is (1.9) adding more $f(A)$, $f(B) < +\infty$.

Definition 1.3. *A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is* polyconvex *if there exists a convex function g which depends on the vector $M(F)$ of all minors of F such that $f(F) = g(M(F))$.*

In particular, if $m = n = 2$, then $f(F) = g(F, \det F)$ with g defined in $\mathbb{R}^5$, and, in the case $m = n = 3$, $f(F) = g(F, \mathrm{cof}F, \det F)$ with g defined in $\mathbb{R}^{19}$.

Definition 1.4. *Let $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ be Borel measurable and bounded below. The function f is called* quasiconvex *if*

$$f(F) \leq \frac{1}{|\Omega|}\int_\Omega f(F + \nabla\varphi)dx, \tag{1.10}$$

for every bounded open set $\Omega \subseteq \mathbb{R}^n$ with $|\partial\Omega| = 0$, for every $F \in \mathbb{M}^{m\times n}$, and every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

Suppose that (1.10) holds for some nonempty bounded open set $\Omega \subseteq \mathbb{R}^n$, for some $F \in \mathbb{M}^{m\times n}$, and for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. Suppose further that

$$f(F) < +\infty.$$

Then, for any other bounded open set $\Omega' \subseteq \mathbb{R}^n$, there exist $\hat{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $\hat{x} + \varepsilon\Omega' \subseteq \Omega$. Therefore, it follows from (1.10) that for every $\varphi \in W_0^{1,\infty}(\Omega';\mathbb{R}^m)$

$$|\Omega| f(F) \leq \int_{\hat{x}+\varepsilon\Omega'} f\left(F + \nabla\varphi\left(\frac{x-\hat{x}}{\varepsilon}\right)\right)dx + |\Omega\setminus(\hat{x}+\varepsilon\Omega')|\,f(F),$$

and in turn that

$$|\Omega'| f(F) \leq \int_{\Omega'} f(F + \nabla\varphi(x))dx.$$

Note that this argument fails when $f(F) = +\infty$. Nevertheless, we have the following proposition whose proof is based on a Vitali covering argument and can be found in [**7**].

Proposition 1.5. *If (1.10) holds for some nonempty bounded open set $\Omega \subseteq \mathbb{R}^n$, for every $F \in \mathbb{M}^{m\times n}$, and every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$, then f is quasiconvex.*

In Section 1.2.2 and in Chapter 2 we will deal also with the following variant of quasiconvexity.

Definition 1.6. *Let $f : Sym(n) \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable and bounded below. The function f is* quasiconvex on linear strains *if*

$$f(E) \leq \frac{1}{|\Omega|} \int_{\Omega} f(E + e(\varphi))dx,$$

for every bounded open set $\Omega \subseteq \mathbb{R}^n$ with $|\partial\Omega| = 0$, for every $E \in Sym(n)$, and every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$.

Here, $e(\varphi) := sym(\nabla\varphi)$ is the linear strain. Observe that f is quasiconvex on linear strains if its extension $F \mapsto f(symF)$ to all of $\mathbb{M}^{n\times n}$ is quasiconvex. Conversely, if f is quasiconvex, this does not imply in general that f restricted to $Sym(n)$ is quasiconvex on linear strains.

Definition 1.7. *A function $f : \mathbb{M}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is* rank–one convex *if*

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B), \tag{1.11}$$

for every $\lambda \in [0, 1]$ and every A, $B \in \mathbb{M}^{m\times n}$ such that $\mathrm{rank}\,(A - B) \leq 1$.

Equivalently, f is rank–one convex if the function $t \mapsto f(F + tR)$ is convex for every F, $R \in \mathbb{M}^{m\times n}$ such that $\mathrm{rank}\, R = 1$.

If $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$, polyconvexity implies both quasiconvexity and rank–one convexity, but quasiconvexity does not imply rank–one convexity. If we restrict our attention to the case of real valued functions, then

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank–one convex}$$

(see [**20**]).

The *polyconvex envelope* of f is the largest polyconvex function less than or equal to f. The *quasiconvex envelope*, the *quasiconvex envelope on linear strains*, and the *rank–one convex envelope* of f are defined analogously and denoted by f^{qc}, f^{qce} and f^{rc}, respectively.

The macroscopic response of nematic elastomers is governed by the quasiconvex envolope W^{qc} of the free energy density W of the system. This is true in general for materials displaying fine internal structures.

The *polyconvex hull* K^{pc}, the *quasiconvex hull* K^{qc}, and the *rank–one convex hull* K^{rc} of a compact set $K \subseteq \mathbb{M}^{m\times n}$ are defined by duality with polyconvex, quasiconvex, and rank–one convex functions, respectively, in the following way:

$$\begin{aligned}
K^{pc} &:= \left\{ F \in \mathbb{M}^{m\times n} : f(F) \leq \sup_{G\in K} f(G) \text{ for every } f : \mathbb{M}^{m\times n} \to \mathbb{R} \text{ polyconvex} \right\}, \\
K^{qc} &:= \left\{ F \in \mathbb{M}^{m\times n} : f(F) \leq \sup_{G\in K} f(G) \text{ for every } f : \mathbb{M}^{m\times n} \to \mathbb{R} \text{ quasiconvex} \right\}, \qquad (1.12) \\
K^{rc} &:= \left\{ F \in \mathbb{M}^{m\times n} : f(F) \leq \sup_{G\in K} f(G) \text{ for every } f : \mathbb{M}^{m\times n} \to \mathbb{R} \text{ rank–one convex} \right\}.
\end{aligned}$$

The last type of convexity we introduce is defined set–theoretically and will be crucial later.

Definition 1.8. *A set $K \subseteq \mathbb{M}^{m\times n}$ is* lamination convex *if*

$$(1 - \lambda)A + \lambda B \in K$$

for every $\lambda \in [0, 1]$ and every A, $B \in K$ such that $\mathrm{rank}(A - B) \leq 1$.

The *lamination convex hull* K^{lc} is defined as the smallest lamination convex set which contains K. The following characterization of the lamination convex hull will be very useful. It states that the lamination convex hull can be obtained by successively adding rank–one segments.

Proposition 1.9. *For every $K \subseteq \mathbb{M}^{m\times n}$, we have that*

$$K^{lc} = \bigcup_i K^i,$$

where $K^0 := K$ *and*

$$K^{i+1} := \{(1-\lambda)A + \lambda B \,:\, A, B \in K^i,\ \lambda \in [0,1],\ \operatorname{rank}(A-B) = 1\}.$$

Moreover, if K *is open, then the sets* K^i*'s are open.*

For the proof of this proposition, see [**52**]. For sake of completeness, we recall the following alternative characterization of K^{lc}.

PROPOSITION 1.10. *For every* $K \in \mathbb{M}^{m\times n}$,

$$K^{lc} = \left\{ F \in \mathbb{M}^{m\times n} : f(F) \le \sup_{G\in E} f(G) \text{ for every } f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\} \text{ rank–one convex} \right\}. \tag{1.13}$$

Note that this characterization is obtained by allowing all $\mathbb{R}\cup\{+\infty\}$ valued rank–one convex functions in the definition of K^{rc}.

PROOF. Let us call $\hat{K}$ the set on the right–hand side of (1.13). We want to prove that $K^{lc} = \hat{K}$. Note that $\hat{K}$ is lamination convex. Indeed, if A, $B \in \hat{K}$ are such that $\operatorname{rank}(A-B) \le 1$ and $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is rank–one convex, then, by definition of $\hat{K}$,

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B) \le \sup_{G\in K} f(G),$$

and in turn $\lambda A + (1-\lambda)B \in \hat{K}$. Since $K \subseteq \hat{K}$, by definition of K^{lc} we have $K^{lc} \subseteq \hat{K}$. On the other hand, let us consider the function $f : \mathbb{M}^{m\times n} \to [0,\infty]$ defined by

$$f(F) := \begin{cases} 0 & \text{if } F \in K^{lc}, \\ \infty & \text{otherwise.} \end{cases}$$

We have that f is rank–one convex. To see this, consider A, $B \in \mathbb{M}^{m\times n}$ such that $\operatorname{rank}(A-B) \le 1$ and $\lambda \in [0,1]$. If $f(A) = \infty$ or $f(B) = \infty$, then (1.11) is trivially satisfied. Otherwise, by definition of f, we have that A, $B \in K^{lc}$ and $f(A) = f(B) = 0$. Thus, $f(\lambda A + (1-\lambda)B) \le 0$, because $\lambda A + (1-\lambda)B \in K^{lc}$. Hence, if $F \in \hat{K}$, by definition of $\hat{K}$ we have that $f(F) \le \sup_{G\in K} f(G) = 0$ and therefore $F \in K^{lc}$. This concludes the proof. □

Definitions (1.12), characterization (1.13), and the relations between the different notions of convexity imply the inclusions

$$K^{lc} \subseteq K^{rc} \subseteq K^{qc} \subseteq K^{pc},$$

for every compact $K \subseteq \mathbb{M}^{m\times n}$.

The next example, which was found independently by several authors (see, e.g., [**64**]) illustrates the difference between lamination convexity (which is defined set–theoretically) and rank–one convexity (which is defined by duality with functions). Let K be the subset of the diagonal matrices in $\mathbb{M}^{2\times 2}$ given by

$$K := \{A_1, A_2, A_3, A_4\}, \tag{1.14}$$

where

$$A_1 := \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3 := \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \quad A_4 := \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}.$$

It turns out that K^{lc} is strictly contained in K^{rc}. Indeed, since K does not contain rank–one connections, we have that $K^{lc} = K$. We can check that $K^{lc} \subseteq K$ (and therefore that $K^{lc} = K$), also using the functional characterization (1.13): it is enough to consider the function $f : \mathbb{M}^{2\times 2} \to [0,\infty]$ defined by

$$f(F) := \begin{cases} 0 & \text{if } F \in K, \\ \infty & \text{otherwise.} \end{cases}$$

This is a rank–one function. To check this, we can equivalently prove that the function $t \mapsto f(A+tR)$ is convex for every A, $R \in \mathbb{M}^{2\times 2}$ such that $\operatorname{rank} R = 1$. This function is clearly nonconvex iff

$$A + t_i R = A_i \qquad \text{and} \qquad A + t_j R = A_j \qquad \text{for some } i \neq j. \tag{1.15}$$

But (1.15) is not possible, because it implies $\operatorname{rank}(A_i - A_j) = 1$, which is not true. Thus, if $F \in K^{lc}$, then $f(F) \leq \sup_{G\in K} f(G) = 0$ and therefore $F \in K$.

On the other hand, if $F_{i,j}$ are the matrices' coordinates, we have that K^{rc} contains the square $Q := \{|F_{11}| \leq 1, |F_{22}| \leq 1\}$ and the segments $[A_k, J_k]$, for $k = 1, ..., 4$, where

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_3 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_4 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the vertexes of Q. To see this, let $f : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be rank–one convex. From the definition of K^{rc}, we have to check that $f(F) \leq \sup_K f$, for every $F \in Q \cup \bigcup_{k=1,...,4}[A_k, J_k]$. Note that if $\sup_K f = \infty$ there is nothing to prove and that, up to consider $f - \sup_K f$, we can suppose that $\sup_K f \leq 0$ and then prove that

$$f(F) \leq 0, \qquad \text{for every } F \in Q \cup \bigcup_{k=1,...,4} [A_k, J_k].$$

Since f is convex along all the lines parallel to $[J_1, J_2]$ and those parallel to $[J_2, J_3]$, it must attain its maximum over Q in one of its vertex, say J_1. If we prove that $f(J_1) \leq 0$, we are done. Note that $[J_1, J_2] \subseteq [A_1, J_2]$. If $f(J_1) > 0$, then convexity along $[A_1, J_2]$ yields the contradiction $f(J_2) > f(J_1)$.

1.2.2. Two known results from Γ–convergence theory. In this section we collect two already established Γ–convergence results which are fundamental for the rest part of our discussion. Even if their statements are not written in terms of Γ–convergence, it is clear that their proofs require Γ–convergence as well as compactness arguments. Both of them take into account energy densities of the form $W(x, F)$, with $(x, F) \in \Omega\times\mathbb{M}^{n\times n}$. For simplicity, we present such results in the homogeneous case where W does not depend on x.

Consider an elastic body occupying a reference configuration $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, subject to some deformation $v : \Omega \to \mathbb{R}^n$. Assuming that the body is homogeneous and hyperelastic, the stored energy can be written as

$$\int_\Omega W(\nabla v)dx,$$

where ∇v is the deformation gradient, and the energy density $W(F) \geq 0$ is defined for every $F \in \mathbb{M}^{n\times n}$ and it is finite only for $\det F > 0$. Assume that the energy density W is minimized at the value 0 by the identity matrix I, which amounts to saying that the reference configuration is stress free. Assume also that W is frame indifferent, i.e., $W(F) = W(RF)$ for every $F \in \mathbb{M}^{n\times n}$ and every R in the space $SO(n)$ of rotations.

Since the deformation $v(x) = x$ is an equilibrium when no external loads are applied, we expect that small external loads $\varepsilon l(x)$ will produce deformations of the form $v(x) = x + \varepsilon u(x)$, so that the total energy is given by

$$\int_\Omega W(I + \varepsilon\nabla u)dx - \varepsilon^2 \int_\Omega lu\, dx. \tag{1.16}$$

In the case ∇u bounded, by Taylor–expanding $W(I+\varepsilon\nabla u)$ around I and rescaling (1.16) by ε^{-2}, we obtain in the limit $\varepsilon \to 0$ the formula

$$\frac{1}{2}\int_\Omega D^2W(I)[\nabla u]^2 dx - \int_\Omega lu\, dx, \tag{1.17}$$

where $D^2W(I)[\nabla u]^2$ is the second differential of W at I applied to the pair $[\nabla u, \nabla u]$. By frame indifference, the first summand in (1.17) depends only on the symmetric part $e(u)$ of the displacement gradient ∇u, i.e.,

$$\frac{1}{2}\int_\Omega D^2W(I)[\nabla u]^2 dx = \frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2 dx.$$

This functional is the linearized elastic energy associated with the displacement u.

This elementary derivation of linear elasticity requires only C^2 regularity of W near I, and hence in a neighbourhood of $SO(n)$, by frame indifference. However, it does not guarantee that the minimizers of the most natural boundary value problems for (1.16) converge to the minimizer of the corresponding problems for the limit functional (1.17), as example (1.12) shows.

Convergence of minimizers has been established by Dal Maso, Negri and Percivale [**26**] in the framework of Γ–convergence, under the assumption

$$W(F) \geq Cd^2(F, SO(n)), \qquad \text{for every } F \in \mathbb{M}^{n\times n}. \tag{1.18}$$

More precisely, let Ω be a bounded domain with Lipschitz boundary and let $\partial\Omega$ be a closed subset of $\partial_D\Omega$ such that $\mathscr{H}^{n-1}(\partial_D\Omega) > 0$. Fixed $h \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, let $H^1_{h,\partial_D\Omega}$ denote the closure in $H^1(\Omega;\mathbb{R}^n)$ of the space of functions $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that $u = h$ on $\partial_D\Omega$, and let $\mathscr{L} : H^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$ be a continuous linear operator. Define $\mathscr{G}_\varepsilon$, $\mathscr{G} : H^1(\Omega;\mathbb{R}^n) \to (-\infty,\infty]$ by

$$\mathscr{G}_\varepsilon(u) := \frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla u)dx - \mathscr{L}(u) \quad \text{and} \quad \mathscr{G}(u) := \frac{1}{2}\int_\Omega D^2W(I)[e(u)]dx - \mathscr{L}(u),$$

if $u \in H^1_{h,\partial_D\Omega}$, and by $\mathscr{G}_\varepsilon(u) = \mathscr{G}(u) = +\infty$ otherwise in $H^1(\Omega;\mathbb{R}^n)$.

The main result in [**26**] is the following theorem.

THEOREM 1.11. *Let $W : \mathbb{M}^{n\times n} \to [0,\infty]$ be $\mathscr{B}$–measurable, where $\mathscr{B}$ is the σ–algebra of the Borel measurable subsets of $\mathbb{M}^{n\times n}$. Suppose that W is frame indifferent, and of class C^2 in some neighborhood of $SO(n)$. Moreover, suppose that $W = 0$ on $SO(n)$ and that (1.18) holds.*

If $\{u_\varepsilon\}$ satisfies

$$\mathscr{G}_\varepsilon(u_\varepsilon) = \inf_{H^1_{g,\partial_D\Omega}} \mathscr{G}_\varepsilon + o(1), \tag{1.19}$$

then $\{u_\varepsilon\}$ converges weakly in $H^1(\Omega;\mathbb{R}^n)$ to the (unique) solution of $\min\limits_{H^1_{g,\partial_D\Omega}} \mathscr{G}$.

The following example shows that, if other energy wells are present, we might loose compactness.

EXAMPLE 1.12. Consider $\Omega := (-1,1)\times(-1,1)$ and $\mathscr{L}(u) := \int_\Omega u\cdot e_1 dx$, where $e_1 := \binom{1}{0}$. Let $w \in H^1_0(\Omega;\mathbb{R}^2)$ be defined by

$$w(x_1,x_2) := \begin{pmatrix} 1-\max\{|x_1|,|x_2|\} \\ 0 \end{pmatrix}, \qquad (x_1,x_2)\in\mathbb{R}^2.$$

Set $w_\varepsilon := w/\varepsilon$ and note that $I + \varepsilon\nabla w_\varepsilon$ takes only four values:

$$F_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad F_3 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_4 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If W is an energy density such that $W(F_i) = 0$ for $i = 1, ..., 4$, we have that

$$\inf_{H^1_0(\Omega;\mathbb{R}^2)} \mathscr{G}_\varepsilon \leq \frac{1}{\varepsilon^2}\int_\Omega W(I+\varepsilon\nabla w_\varepsilon)dx - \int_\Omega w_\varepsilon dx = -\frac{1}{\varepsilon}||w||_{L^1(\Omega;\mathbb{R}^2)}.$$

If $\{u_\varepsilon\}$ is a sequence satisfying (1.19), then

$$-\frac{1}{\varepsilon}||w||_{L^1(\Omega;\mathbb{R}^2)} + o(1) \geq ||u_\varepsilon||_{L^1(\Omega;\mathbb{R}^2)},$$

and therefore $\{||u_\varepsilon||_{L^1(\Omega;\mathbb{R}^2)}\}$ diverges.

In [**60**] Schmidt extends the results of Dal Maso, Negri and Percivale allowing for a family $\{W_\varepsilon\}$ of stored energy densities where, for every $\varepsilon > 0$ arbitrarily small, the set of the energy wells of W_ε is of the form

$$\mathcal{U}_\varepsilon := SO(n)U_1(\varepsilon) \cup ... \cup SO(n)U_k(\varepsilon), \tag{1.20}$$

with

$$U_i(\varepsilon) \in Sym(n) \qquad \text{and} \qquad U_i(\varepsilon) = I + \varepsilon U_i + o(\varepsilon), \tag{1.21}$$

for some $U_i \in \mathbb{M}^{n\times n}$, for every $i = 1, ..., k$. These energies are important when modeling materials with different "variants", i.e. preferred strains represented by the wells $SO(n)U_i(\varepsilon)$. This occurs, e.g., in the martensitic phase of shape memory alloys. In these cases the energies are not quasiconvex and the materials tend to form microstructures in order to assume energetically favorable configurations. As for [**26**], the intent of Schmidt is to understand the limit behaviour of the functionals

$$\mathscr{E}_\varepsilon(u) := \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I + \varepsilon\nabla u)dx - \mathscr{L}(u).$$

In order to derive a geometrically linear model in this multiple–well case, the energy wells have to be sufficiently close to each other. This is why the small parameter ε, in terms of which the typical distance between the energy wells (1.20) is measured, is introduced. Moreover, the physically interesting regime is when the displacements scale with the same parameter ε. Indeed, recalling that v stands for a deformation, if $\nabla v - I$ tends to 0 more slowly than ε, then the corresponding geometrically linear version would result trivialized into the case where $U_1 = ... = U_k = I$ in (1.21). On the other hand, if $|\nabla v - I|$ is much smaller than ε, then one would effectively try to linearize at one particular well and this would lead to a loss of compactness.

Let us specify in more details what are the assumptions on the family $\{W_\varepsilon\}$. For every ε arbitrarily small, the energy density $W_\varepsilon : \mathbb{M}^{n\times n} \to [0,\infty]$ is $\mathscr{B}$–measurable, frame indifferent, and C^0 in an ε–independent neighborhood of $SO(n)$. Moreover, $W_\varepsilon = 0$ on the set $\mathcal{U}_\varepsilon$, defined in (1.20), and

$$W_\varepsilon(F) \geq Cd^2(F, \mathcal{U}_\varepsilon).$$

Let us introduce $V_\varepsilon : Sym(n) \to \mathbb{R}$ defined by

$$V_\varepsilon(E) = \frac{1}{\varepsilon^2}W_\varepsilon(I + \varepsilon E),$$

and suppose that $V_\varepsilon(E)$ tends to the *linear limit* $V(E)$, as $\varepsilon \to 0$, for every $E \in Sym(n)$. Finally, let us define $\mathscr{E}$, $\overline{\mathscr{E}} : H^1_{h,\partial_D\Omega} \to (-\infty,\infty]$ by

$$\mathscr{E}(u) = \int_\Omega V(e(u))dx - \mathscr{L}(u) \quad \text{and} \quad \overline{\mathscr{E}}(u) = \int_\Omega V^{qce}(e(u))dx - \mathscr{L}(u), \tag{1.22}$$

where V^{qce} is the quasiconvexification on linear strains of V (see Subsection 1.2.1 for a definition).

The main result in [**60**] is the following theorem.

Theorem 1.13. *Suppose that $V_\varepsilon \to V$ on the compact subsets of $Sym(n)$ and that there exists $\alpha \in \mathbb{R}$ such that*

$$V(E) \leq \alpha(1 + |E|^2), \qquad \textit{for every } E \in Sym(n). \tag{1.23}$$

Then

$$\lim_{\varepsilon\to 0} \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}_\varepsilon = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E} = \min_{H^1_{h,\partial_D\Omega}} \overline{\mathscr{E}}.$$

Moreover, if $\{u_\varepsilon\}$ satisfies

$$\mathscr{E}_\varepsilon(u_\varepsilon) = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}_\varepsilon + o(1),$$

then

$$\lim_{\varepsilon\to 0} \mathscr{E}(u_\varepsilon) = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}.$$

Furthermore, there exists a subsequence of $\{u_\varepsilon\}$ *which converges weakly in* $H^1(\Omega;\mathbb{R}^n)$ *to some* $u \in H^1_{h,\partial_D\Omega}$ *and* u *is a minimizer of* $\overline{\mathscr{E}}$.

1.2.3. Rigidity estimates. In this section we recall some well known results from geometric rigidity theory, which will be mainly employed in Chapter 3. We begin by recalling the following version of the crucial Korn's inequality, for which we refer, e.g., to [**63**].

THEOREM 1.14. *Let* Ω *be a Lipschitz domain of* $\mathbb{R}^n$, $\Gamma \subseteq \partial\Omega$ *with* $\mathscr{H}^{n-1}(\Gamma) > 0$, *and* $1 < p < \infty$. *Then there exists a constant* $C > 0$, *depending on* Ω, Γ *and* p, *such that*

$$||u||_{W^{1,p}(\Omega;\mathbb{R}^n)} \leq C||e(u)||_{L^p(\Omega;\mathbb{R}^n)},$$

for every $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *with trace null on* Γ.

We recall also this version of Korn's inequality: *for every* $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ *there exists* $A \in Skw(n)$ *such that*

$$\int_\Omega |\nabla u - A|^2 dx \leq C \int_\Omega |e(u)|^2 dx.$$

This can be seen as the linear counterpart of the following inequality due to Friesecke, James and S. Müller [**38**], which is, in turn, a quantitative version of Liouville's Theorem.

THEOREM 1.15. *Let* Ω *be a bounded Lipschitz domain of* $\mathbb{R}^n$. *There exists a constant* $C > 0$, *depending on* Γ, *with the following property. For each* $v \in W^{1,2}(\Omega;\mathbb{R}^n)$ *there is an associated rotation* $R \in SO(n)$ *such that*

$$\int_\Omega |\nabla v - R|^2 dx \leq C \int_\Omega |d(\nabla v, SO(n))|^2 dx.$$

To prove the compactness results of Chapter 3 we will need a variant of Theorem 1.15 with two exponents: Lemma 3.8. As for Theorem 1.15, the proof of Lemma 3.8 hinges on the following auxiliary truncation result.

PROPOSITION 1.16 ([**38**], Proposition A.1). *Let* Ω *be a bounded Lipschitz domain of* $\mathbb{R}^n$, n, $m \geq 1$, *and* $1 \leq p < \infty$. *There exists a constant* C, *depending on* Ω, m *and* p, *with the following property. For each* $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ *and every* $\lambda > 0$, *there exists* $V \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ *such that*

$$\text{(i)} \quad ||\nabla V||_{L^\infty(\Omega;\mathbb{R}^m)} \leq C\lambda,$$

$$\text{(ii)} \quad |\{x \in \Omega : v(x) \neq V(x)\}| \leq \frac{C}{\lambda^p} \int_{\{x\in\Omega\,:\,|\nabla v(x)|>\lambda\}} |\nabla v|^p dx,$$

$$\text{(iii)} \quad ||\nabla v - \nabla V||^p_{L^p(\Omega;\mathbb{R}^m)} \leq C \int_{\{x\in\Omega\,:\,|\nabla v(x)|>\lambda\}} |\nabla v|^p dx.$$

Conti, Dolzmann, and Müller [**19**] have recently proved the following version of Theorem 1.15 with mixed growth conditions. This result was first stated without proof in [**39**]. We will use it to prove strong convergence of minimizers in Section 3.4.

THEOREM 1.17. *Let* Ω *be a bounded Lipschitz domain of* $\mathbb{R}^n$ *and* $1 < p_1 < p_2 < \infty$. *There exists* $C > 0$, *depending on* Ω, p_1, *and* p_2 *with the following property. For every* $v \in W^{1,1}$ *with*

$$d(\nabla v, SO(n)) = f_1 + f_2 \quad \textit{a.e. in } \Omega, \quad \textit{and} \quad f_i \in L^{p_i},\ i = 1,2,$$

there exist $g_i \in L^{p_i}$, $i = 1,2$, *and a constant rotation* $R \in SO(n)$ *such that*

$$\nabla v = R + g_1 + g_2, \quad \textit{a.e. in } \Omega, \quad \textit{with} \quad ||g_i||_{L^{p_i}} \leq C||f_i||_{L^{p_i}},\ i = 1,2.$$

CHAPTER 2

From the nonlinear to the geometrically linear model via Γ–convergence

In this chapter, we present the results of [**3**]. We consider two variational models which describe the mechanical behavior of nematic elastomers either in the fully nonlinear regime or in the framework of a geometrically linear theory. We show that there exists a sequence of minimizers of suitable rescalings of the nonlinear functionals which converges to a minimizer of the relaxed linearized functional. We focus on compressible nematic elastomers and therefore on energy densities which are finite only in $Lin^+(3)$. We use the same notation W_n and W already employed in Section 1.1 for the incompressible model and consider the expression

$$W(F) := \min_{n \in S^2} W_n(F), \qquad F \in \mathbb{M}^{3\times 3}, \tag{2.1}$$

where

$$W_n(F) := \begin{cases} \dfrac{\mu}{2}\left[\mathrm{tr}\,(F^T L_n^{-1} F) - 3 - 2\ln(\det F)\right] + \dfrac{\lambda}{2}(\det F - 1)^2 & \text{if } F \in Lin^+(3), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

and L_n is defined as in (1.1). This is a natural generalization of (1.4). Indeed, observe that for $\det F = 1$ (2.2) reduces to (1.4). Moreover, by Proposition 2.9, W attains its minimum value zero on the set $\mathscr{U}$, defined in (1.7), which is the set of the wells of energy (1.5). The term in square brackets in (2.2) is motivated by Flory's work on polymer elasticity [**36**]. The presence of the term $\frac{\lambda}{2}(\det F - 1)^2$ guarantees that the Taylor expansion at order two coincides with isotropic linear elasticity with two independent natural parameters (shear modulus and bulk modulus, see (2.14) below).

In Section 2.1 we present the linearized version of (2.1)–(2.2) on the basis of Taylor expansion, in the spirit of [**32**]. Then, in Section 2.2 we provide a justification, via Γ–convergence, of the linearized theory, and of its relaxation obtained in [**12**]. In Section 2.3 we use the same approach of Sections 2.1 and 2.2 and deal with another compressible energy density.

2.1. The geometrically linear version of the BTW model

Let us explain why we are interested in a geometrically linear theory for nematic elastomers. In spite of its obvious limitations (see, e.g., [**8**]), the geometrically linear theory is a valuable conceptual tool in the study of phase transforming materials: it is simpler and familiar to a larger group of users, the resulting energy landscape has an easier geometric structure, and rigorous and more complete mathematical results are available for it [**12, 14**]. Furthermore, the linear theory is suitable for the exploration of new model extensions taking into account, e.g., the effects of applied electric fields (see, e.g., [**40**]).

In order to obtain the geometrically linear approximation of energy (2.2), we consider the small strain regime $|\nabla u| = \varepsilon$, where u is the displacement associated with the deformation y through $y(x) = x + u(x)$, and matrices L_n that scale with ε as

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n \otimes n + (1+\varepsilon)^{-1}(I - n \otimes n). \tag{2.3}$$

This scaling is necessary to ensure that the stress–free strains described by the tensors $L_{n,\varepsilon}$'s are reachable within a small strain theory (see the discussion in Subsection 1.2.2 and [**32**, Appendix

B.1]). With the notation introduced in Section 1.1 for L_n, based on the material parameter a, we have that $a^{\frac{1}{3}} = 1 + \varepsilon$. By expanding (2.3) in ε around 0, we obtain

$$L_{n,\varepsilon} = I + \varepsilon \hat{L}_n + o(\varepsilon), \quad \text{with} \quad \hat{L}_n := 3\left(n \otimes n - \frac{1}{3}I\right). \tag{2.4}$$

Similarly, from

$$U_{n,\varepsilon} := L_{n,\varepsilon}^{\frac{1}{2}} = (1+\varepsilon) n \otimes n + (1+\varepsilon)^{-\frac{1}{2}}(I - n \otimes n), \tag{2.5}$$

we have that

$$U_{n,\varepsilon} = I + \varepsilon \hat{U}_n + o(\varepsilon), \quad \text{with} \quad \hat{U}_n = \frac{1}{2}\hat{L}_n. \tag{2.6}$$

Now, we define

$$W_\varepsilon(F) := \min_{n \in S^2} W_{n,\varepsilon}(F), \qquad F \in \mathbb{M}^{3\times 3}, \tag{2.7}$$

where $W_{n,\varepsilon}$ is given by (2.2) with $L_{n,\varepsilon}$ in place of L_n, that is

$$W_{n,\varepsilon}(F) := \begin{cases} \frac{\mu}{2}\left[\mathrm{tr}\,(F^T L_{n,\varepsilon}^{-1} F) - 3 - 2\ln(\det F)\right] + \frac{\lambda}{2}(\det F - 1)^2 & \text{if } F \in Lin^+(3), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

Also, we have that $W_{n,\varepsilon}(F) = \tilde{W}_{n,\varepsilon}(FF^T)$, for every $F \in Lin^+(3)$, where

$$\tilde{W}_{n,\varepsilon}(B) := \frac{\mu}{2}\left[B \cdot L_{n,\varepsilon}^{-1} - 3 - \ln(\det B)\right] + \frac{\lambda}{2}(\sqrt{\det B} - 1)^2, \qquad B \in Psym(3). \tag{2.9}$$

Proposition 2.1. *In the small strain regime $|\nabla u| = \varepsilon$, we have that, modulo terms of order higher than two in ε,*

$$W_{n,\varepsilon}(I + \nabla u) = \mu \left|[e(u)]_d - \varepsilon \hat{U}_n\right|^2 + \frac{k}{2}(\mathrm{tr}\,\nabla u)^2, \tag{2.10}$$

where $\hat{U}_n$ is the traceless matrix defined in (2.6) and $k = \lambda + \frac{2}{3}\mu$.

We can recognize in (2.10) the formula for the energy in the small deformations regime obtained in [**32**].

Proof. In order to derive (2.10), let us define for every $E \in Sym(3)$ the linear limit

$$V_n(E) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} W_{n,\varepsilon}(I + \varepsilon E) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \tilde{W}_{n,\varepsilon}\left((I + \varepsilon E)^2\right).$$

Since $\tilde{W}_{n,\varepsilon}$ is minimized by $L_{n,\varepsilon}$ at the value 0 (see Proposition 2.9), the linear term of the Taylor expansion vanishes and we have

$$\begin{aligned} V_n(E) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left\{\frac{1}{2}D^2\tilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[(I+\varepsilon E)^2 - L_{n,\varepsilon}]^2 + o\left(|(I+\varepsilon E)^2 - L_{n,\varepsilon}|^2\right)\right\} \\ &= \frac{1}{2}D^2\tilde{W}_{n,0}(I)[2E - \hat{L}_n]^2 = 2D^2\tilde{W}_{n,0}(I)[E - \hat{U}_n]^2, \end{aligned} \tag{2.11}$$

where the last two equalities are obtained using (2.4)–(2.6). Simple calculations give

$$D\tilde{W}_{n,\varepsilon}(B)[H] = \frac{\mu}{2}[H \cdot L_{n,\varepsilon}^{-1} - B^{-T} \cdot H] + \frac{\lambda}{2}(\sqrt{\det B} - 1)\sqrt{\det B}B^{-T} \cdot H,$$

and in turn

$$D^2\tilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 = \frac{\mu}{2}\mathrm{tr}(L_{n,\varepsilon}^{-1}H)^2 + \frac{\lambda}{4}\mathrm{tr}^2(L_{n,\varepsilon}^{-1}H), \tag{2.12}$$

for every $B \in Psym(3)$ and $H \in Sym(3)$. Thus, from (2.11) and (2.12) we have that

$$\begin{aligned} V_n(E) &= \frac{\mu}{4}\mathrm{tr}(2E - \hat{L}_n)^2 + \frac{\lambda}{4}\mathrm{tr}^2(2E - \hat{L}_n) \\ &= \mu\,\mathrm{tr}(E - \hat{U}_n)^2 + \lambda\mathrm{tr}^2(E - \hat{U}_n), \end{aligned} \tag{2.13}$$

where in the second identity we used the fact that $\hat{L}_n = 2\hat{U}_n$. Since $\hat{U}_n$ is traceless, we prefer to write the first summand in (2.13) in terms of the deviatoric part E_d. Thus, since

$$|E - \hat{U}_n|^2 = |E_d - \hat{U}_n|^2 + \frac{1}{3}(\operatorname{tr} E)^2,$$

setting $k = \lambda + \frac{2}{3}\mu$ we obtain that

$$V_n(E) = \mu\,|E_d - \hat{U}_n|^2 + \frac{k}{2}(\operatorname{tr} E)^2, \qquad \text{for every } E \in Sym(3). \tag{2.14}$$

Note that

$$\hat{U}_n = \frac{3}{2}\left(n \otimes n - \frac{1}{3}I\right), \qquad \text{for every } n \in S^2. \tag{2.15}$$

It remains to observe that, since $W_{n,\varepsilon}(F)$ can be expressed in terms of FF^T (through $\tilde{W}_{n,\varepsilon}$), it turns out that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} W_{n,\varepsilon}(I + \varepsilon M) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} W_{n,\varepsilon}(I + \varepsilon\, symM) =: V_n(symM),$$

for every $M \in \mathbb{M}^{3\times 3}$. In particular, we have that, modulo terms of order higher than two,

$$W_{n,\varepsilon}\left(I + \varepsilon\frac{\nabla u}{|\nabla u|}\right) = \varepsilon^2 V_n\left(sym\left(\frac{\nabla u}{|\nabla u|}\right)\right) = \varepsilon^2 V_n\left(\frac{e(u)}{|\nabla u|}\right).$$

Thus, considering ∇u with the proper scale $|\nabla u| = \varepsilon$ and using (2.14), we obtain (2.10). □

Remark 2.2. Note that the incompressible version of the large and small strain theories can be obtained by considering the formal limit $\lambda \to +\infty$ and $k \to +\infty$ in (2.2) and in (2.10), respectively: in the large strain regime we obtain energy (1.4), and in the small strain regime we obtain

$$W_{n,\varepsilon}(I + \nabla u) = \mu\left|e(u) - \varepsilon\hat{U}_n\right|^2, \qquad \operatorname{div} u = 0,$$

or, equivalently, the linear limit

$$V_n(E) = \mu\left|E - \hat{U}_n\right|^2, \qquad E \in Sym_0(3).$$

Now, let us consider the smallest energy density achievable by the system, in the small strain regime, if it is allowed to freely adjust n, at fixed $E \in Sym(3)$. Recalling that $\mu_1(E) \le \mu_2(E) \le \mu_3(E)$ are the ordered eigenvalues of E, this is given by

$$V(E) := \min_{n\in S^2} V_n(E) = \mu \min_{n\in S^2} |E_d - \hat{U}_n|^2 + \frac{k}{2}(\operatorname{tr} E)^2 \tag{2.16}$$

$$= \mu\left[\left(\mu_1(E) + \frac{1}{2}\right)^2 + \left(\mu_2(E) + \frac{1}{2}\right)^2 + (\mu_3(E) - 1)^2\right] + \left(\frac{k}{2} - \frac{\mu}{3}\right)(\operatorname{tr} E)^2. \tag{2.17}$$

Expression (2.17) can be obtained by considering that

$$\begin{aligned}\min_{n\in S^2} |E_d - \hat{U}_n|^2 &= \min_{n\in S^2}\left(|E_d|^2 - 2E_d\cdot\hat{U}_n + |\hat{U}_n|^2\right)\\ &= \min_{n\in S^2}\left(|E|^2 - \frac{1}{3}(\operatorname{tr} E)^2 - 3(En)\cdot n + \operatorname{tr} E + \frac{3}{2}\right).\end{aligned}$$

Since the minimum in the last expression is attained when n is an eigenvector corresponding to the maximum eigenvalue $\mu_3(E)$ of E, we have

$$\begin{aligned}\min_{n\in S^2} |E_d - \hat{U}_n|^2 &= |E|^2 + \left[\mu_1(E) + \mu_2(E) - 2\mu_3(E)\right] + \frac{3}{2} - \frac{1}{3}(\operatorname{tr} E)^2\\ &= \left(\mu_1(E) + \frac{1}{2}\right)^2 + \left(\mu_2(E) + \frac{1}{2}\right)^2 + (\mu_3(E) - 1)^2 - \frac{1}{3}(\operatorname{tr} E)^2,\end{aligned}$$

and in turn (2.17).

The following remark will be useful in Chapter 5

REMARK 2.3. Moving from the three dimensional case to the two dimensional case, we consider the nematic director

$$L_n := a\, n\otimes n + a^{-1}(I - n\otimes n), \qquad n \in S^1,$$

where $a > 1$. In the small strain regime $a = (1+\varepsilon)^2$, we define

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n\otimes n + (1+\varepsilon)^{-2}(I - n\otimes n), \qquad n \in S^1,$$

and

$$W_{n,\varepsilon}(F) := \frac{\mu}{2}\left[\mathrm{tr}\,(F^T L_{n,\varepsilon}^{-1} F) - 2 - 2\ln(\det F)\right] + \frac{\lambda}{2}(\det F - 1)^2, \qquad F \in Lin^+(2).$$

From Proposition 2.9 we have that $W_{n,\varepsilon}$ is nonnegative and that $W_{n,\varepsilon}(F) = 0$ if and only if $FF^T = L_{n,\varepsilon}$. In this case, the linear limit $V_n(E) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_{n,\varepsilon}(I + \varepsilon E)$ has the expression

$$V_n(E) = \mu|E_d - \hat{U}_n|^2 + \frac{k}{2}\mathrm{tr}^{\,2} E, \qquad \text{for every } E \in Sym(2),$$

where now $k = \lambda + \mu$, and

$$\hat{U}_n := 2n\otimes n - I, \qquad \text{for every } n \in S^1. \tag{2.18}$$

In Chapter 5 we will consider the incompressible version

$$V_n(E) = \mu\, |E - \hat{U}_n|^2, \qquad E \in Sym_0(2). \tag{2.19}$$

In this case, we set $V_n(E) = +\infty$ for every $E \in Sym(2)$ such that $\mathrm{tr}\, E \neq 0$.

2.2. Justification of the geometrically linear theory via Γ–convergence

To present the following theorem, let us introduce some notation. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain, $\partial_D\Omega$ a subset of $\partial\Omega$ with positive surface measure, $h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ some boundary data, and $\mathscr{L} : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R}$ a continuous linear operator representing the work of the loads. Moreover, let $H^1_{h,\partial_D\Omega}$ be the closure of the set $\left\{v \in W^{1,\infty}(\Omega; \mathbb{R}^3) : v = h \text{ on } \partial_D\Omega\right\}$ in $H^1(\Omega; \mathbb{R}^3)$. Considering the energy W_ε defined by (2.7)–(2.8), we introduce the energy functionals $\mathscr{E}_\varepsilon$ and $\mathscr{E}$ defined on $H^1(\Omega; R^3)$ as

$$\mathscr{E}_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega W_\varepsilon(I + \varepsilon\nabla u)dx - \mathscr{L}(u) & \text{if } u \in H^1_{h,\partial_D\Omega}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathscr{E}(u) := \begin{cases} \int_\Omega V(e(u))dx - \mathscr{L}(u) & \text{if } u \in H^1_{h,\partial_D\Omega}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.20}$$

where the function $V : Sym(3) \to \mathbb{R}$ is given by (2.16). In what follows, $\overline{\mathscr{E}}$ is the relaxation of $\mathscr{E}$ in the weak sequential (briefly, w. s.) topology of H^1, that is

$$\overline{\mathscr{E}} := \sup\{\mathscr{F} : \mathscr{F} \text{ is } H^1(\Omega, \mathbb{R}^3)\text{–w. s. lower semicontinuous, } \mathscr{F} \leq \mathscr{E}\}.$$

THEOREM 2.4. *We have that*

$$\lim_{\varepsilon\to 0} \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}_\varepsilon = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E} = \min_{H^1_{h,\partial_D\Omega}} \overline{\mathscr{E}}, \tag{2.21}$$

with

$$\overline{\mathscr{E}}(u) = \begin{cases} \int_\Omega V^{qce}(e(u))dx - \mathscr{L}(u) & \textit{if } u \in H^1_{h,\partial_D\Omega}, \\ +\infty & \textit{otherwise.} \end{cases} \tag{2.22}$$

Here, V^{qce} is given by

$$V^{qce}(E) = \mu \min_{Q\in\mathscr{Q}} |E_d - Q|^2 + \frac{k}{2}\mathrm{tr}^2 E, \qquad \textit{for every } E \in Sym(3), \tag{2.23}$$

where $k = \lambda + \frac{2}{3}\mu$ *and*

$$\mathscr{Q} := \left\{ M \in Sym_0(3) \text{ with eigenvalues in } \left[-\frac{1}{2}, 1 \right] \right\}. \tag{2.24}$$

Moreover, if $\{u_\varepsilon\}$ *is a sequence of "almost minimizers" for* $\{\mathscr{E}_\varepsilon\}$*, which means*

$$\mathscr{E}_\varepsilon(u_\varepsilon) = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}_\varepsilon + o(1),$$

then $\{u_\varepsilon\}$ *is also a minimizing sequence for* $\mathscr{E}$*, that is*

$$\lim_{\varepsilon\to 0} \mathscr{E}(u_\varepsilon) = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E} \tag{2.25}$$

Furthermore, there exists a subsequence of $\{u_\varepsilon\}$ *which converges weakly in* $H^1(\Omega;\mathbb{R}^3)$ *to some* $u \in H^1_{h,\partial_D\Omega}$ *and* u *is a minimizer of* $\overline{\mathscr{E}}$*.*

We remark that V^{qce} is the quasiconvex envelope on linear strains of V (see Definition 1.6). Expressions (2.16) and (2.23) show that the parameters μ and k have the physical meaning of a shear modulus and a bulk modulus, respectively.

Remark 2.5. In the engineering literature, it is customary to write small strain theories using the leading order term of the deviation of the strain from the identity. In other words, one considers $F = I + \nabla w + o(\varepsilon)$, where $|\nabla w| = \varepsilon$, and then writes the energy as a function of w. This energy is related to (2.20) by a simple scaling (see Proposition 2.1), so that, modulo terms of order higher than two in ε,

$$\mathscr{E}(w) = \int_\Omega \left\{ \mu \min_{n\in S^2} |(e(w))_d - E_0(n)|^2 + \frac{k}{2}(\operatorname{tr} \nabla w)^2 \right\} dx - \mathscr{L}(w), \tag{2.26}$$

where

$$E_0(n) = \frac{3}{2}\varepsilon \left(n \otimes n - \frac{1}{3} I \right).$$

The corresponding relaxation is

$$\overline{\mathscr{E}}(w) = \int_\Omega \left\{ \mu \min_{Q\in\mathscr{Q}} |(e(w))_d - \varepsilon Q|^2 + \frac{k}{2}(\operatorname{tr} \nabla w)^2 \right\} dx - \mathscr{L}(w).$$

This relaxed functional may prove very useful to set up effective numerical schemes in applications where one is interested in the behaviour of global energy minimizers, similarly to what has been done in [**16, 17**]. When, instead, local minimizers or dynamics are studied (see, e.g., [**13**] and [**40**]), (2.26) describes the correct energetics.

Our result is an application of the abstract theory developed by Schmidt in [**60**], where linearized theories are derived from nonlinear elasticity theory for multi–well energies, via Γ–convergence. We have presented the main result of [**60**] in Section 2. We refer the reader to that section for some details which are implicit in the following discussion as well as in the remaining part of this chapter. One class of energy densities to which Schmidt's result applies is of the form

$$\mathsf{W}_\varepsilon(F) = \min_{i=1,\dots k} \mathsf{W}_{i,\varepsilon}(F), \tag{2.27}$$

where, for $i = 1, ..., k$, the function $\mathsf{W}_{i,\varepsilon}$ is a frame indifferent single–well energy minimized and equal to zero on $SO(N)U_{i,\varepsilon}$. Here, $U_{i,\varepsilon} \in Sym(N)$ is of the form $U_i(\varepsilon) = I + \varepsilon U_i + o(\varepsilon)$ and

$$\mathsf{W}_{i,\varepsilon}(F) \geq Cd^2(F, SO(N)U_{i,\varepsilon}).$$

The linear limit of W_ε is

$$\mathsf{V}(E) = \frac{1}{2} \min_{i=1,\dots,k} A_i[E - U_i]^2, \qquad E \in Sym(N),$$

where $A_i := \lim_{\varepsilon\to 0} D^2\mathsf{W}_{\varepsilon,i}(U_i(\varepsilon))$. The double–well case $k = 2$ with $A_1 = A_2$ is of particular interest since, in this case, an explicit formula for the quasiconvex envelope of V is available [**47, 56**].

Now, the family of energy densities $\{W_\varepsilon\}$ that we consider (see (2.7)–(2.8)) can be viewed as an infinite–dimensional analogue of (2.27). To handle this case, an extension of the theory for energy densities with wells which vary on a compact is required. For this purpose, we generalize Schmidt's Theorem 1.13 to the following class of "admissible" energy densities.

DEFINITION 2.6. *We say that $\{W_\varepsilon\}$ is an* admissible *family of energy densities if, for every ε arbitrarily small, the following properties are satisfied:*

(i) $W_\varepsilon : \mathbb{M}^{N\times N} \to [0, +\infty]$ *is frame indifferent;*
(ii) W_ε *is minimized at the value* 0 *on* $SO(N)\mathscr{U}_\varepsilon$*, where*

$$\mathscr{U}_\varepsilon = \{U \in Sym(N) \, : \, U = I + \varepsilon\hat{U} + o(\varepsilon), \; \hat{U} \in \mathscr{M}\}, \tag{2.28}$$

and $\mathscr{M}$ *is a compact in* $\mathbb{M}^{N\times N}$*;*
(iii) W_ε *is measurable and continuous in an* ε*–independent neighbourhood of* I*;*
(iv) there exists a constant C *not depending on* ε *and* F *such that*

$$W_\varepsilon(F) \geq C d^2(F, SO(N)\mathscr{U}_\varepsilon), \qquad \text{for every } F \in \mathbb{M}^{N\times N}. \tag{2.29}$$

The generalization of Theorem 1.13 to this class of energies does not require any change in its proof. In fact, for such a proof, it is sufficient that W_ε satisfies (i), (iii), and

$$W_\varepsilon(F) \geq c\, d^2(F, SO(N)) - C\varepsilon^2,$$

for some c, $C > 0$, see [**60**, Remark 2.9]. Observe that this condition is implied by (2.28) and (2.29). Indeed, let

$$d(F, SO(N)\mathscr{U}_\varepsilon) = |F - RU|,$$

for some $R \in SO(N)$ and $U = I + \varepsilon\hat{U} + o(\varepsilon)$. Since $\hat{U}$ varies in the compact $\mathscr{M}$, we have that

$$|F - R| \leq |F - RU| + |U - I| \leq d(F, SO(N)\mathscr{U}_\varepsilon) + K\varepsilon,$$

and therefore

$$d(F, SO(N)) \leq d(F, SO(N)\mathscr{U}_\varepsilon) + K\varepsilon,$$

for some constant $K > 0$ and every $\varepsilon > 0$ small enough.
We now move to the specific energies for nematic elastomers and focus on the three–dimensional case $N = 3$. Let us introduce the set

$$\mathscr{U}_\varepsilon := \left\{U_{n,\varepsilon} \, : \, n \in S^2\right\}, \tag{2.30}$$

where $U_{n,\varepsilon}$ is defined in (2.5). From (2.6) it is clear that $\mathscr{U}_\varepsilon$ is a class of type (2.28).

PROOF OF THEOREM 2.4. To apply Theorem 1.13 in the generalized version discussed above, we have first to check that $\{W_\varepsilon\}$ is an admissible family of energy densities in the sense of Definition 2.6. Conditions (i) and (iii) trivially hold. To prove (ii), note that, if $\det F > 0$, then

$$W_\varepsilon(F) = \frac{\mu}{2} f_\varepsilon^{opt}(FF^T) + \frac{\lambda}{2}(\det F - 1)^2,$$

where f_ε^{opt} is defined as in (2.51) (specialized to dimension 3). By Proposition 2.9, this is minimal at the value 0 on $SO(3)\mathscr{U}_\varepsilon$. Also, observe that $SO(3)\mathscr{U}_\varepsilon = \mathscr{U}_\varepsilon SO(3)$. To prove (iv), we restrict the attention to the non trivial case $\det F > 0$ and look separately at three regimes: the case F far from $SO(3)$, the case F close to $SO(3)$ and the intermediate regime. Thus, we divide the proof into three steps. In what follows, we use the standard convention and denote by C a generic positive constant whose exact value may change from line to line.
Step 1. We prove that there exist $\alpha > 0$ and $C_1 > 0$ such that, for every ε small enough,

$$\text{if} \quad d(F, SO(3)) \leq \alpha, \quad \text{then} \quad W_\varepsilon(F) \geq C_1 d^2(F, SO(3)\mathscr{U}_\varepsilon).$$

We can write $W_{n,\varepsilon}(F) = \tilde{W}_{n,\varepsilon}(FF^T)$, where $\tilde{W}_{n,\varepsilon}$ is defined on $Psym(3)$ as in (2.9). Let $d(F, SO(3)) \leq \alpha$, with $\alpha > 0$ to be chosen later. Then $|FF^T - I| \leq \alpha^2 + 2\alpha$ and FF^T belongs to the closed ball centered in $L_{n,\varepsilon}$ and with radius $2\alpha^2 + 4\alpha$, for every ε small enough. Thus, for α small enough, we can expand $\tilde{W}_{n,\varepsilon}$ around $L_{n,\varepsilon}$ and obtain

$$\tilde{W}_{n,\varepsilon}(FF^T) = \frac{1}{2}D^2\tilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[FF^T - L_{n,\varepsilon}]^2 + R_\alpha, \tag{2.31}$$

where

$$|R_\alpha| \leq C_\alpha |FF^T - L_{n,\varepsilon}|^3, \tag{2.32}$$

for a certain positive constant C_α, which depends on α but not on ε and n. From (2.12) we note that

$$D^2\tilde{W}_{n,\varepsilon}(L_{n,\varepsilon})[H]^2 \geq \frac{\mu}{2}\mathrm{tr}(L_{n,\varepsilon}^{-1}H)^2 \geq \frac{\mu}{4}|H|^2, \tag{2.33}$$

for every $n \in S^2$, $H \in Sym(3)$, and every ε sufficiently small. Thus, from (2.31), (2.32) and (2.33) it turns out that

$$\begin{aligned} W_{n,\varepsilon}(F) &\geq \frac{\mu}{8}|FF^T - L_{n,\varepsilon}|^2 + R_\alpha \\ &\geq \frac{\mu}{8}|FF^T - L_{n,\varepsilon}|^2 \left(1 - \frac{8C_\alpha}{\mu}|FF^T - L_{n,\varepsilon}|\right), \end{aligned}$$

for every ε small enough. Now, it is possible to choose $\alpha > 0$ such that the parenthesis in the last inequality is arbitrarily close to one and hence

$$W_{n,\varepsilon}(F) \geq C|FF^T - L_{n,\varepsilon}|^2.$$

Therefore, since $|\sqrt{G} - \sqrt{H}| \leq C|G - H|$ for every G, $H \in Psym(3)$, if H is sufficiently near I, then there exists a constant $C_1 > 0$, not depending on F, ε and n, such that

$$W_{n,\varepsilon}(F) \geq C_1|\sqrt{FF^T} - U_{n,\varepsilon}|^2.$$

Then, we can conclude by using the following inequalities:

$$W_\varepsilon(F) := \min_{n\in S^2} W_{n,\varepsilon}(F) \geq C_1 \min_{n\in S^2} |\sqrt{FF^T} - U_{n,\varepsilon}|^2 \geq C_1 d^2(F, SO(3)\mathscr{U}_\varepsilon).$$

Step 2. Let α be the constant found in the Step 1. We now show that there exists $C_2 > 0$ such that, for every ε small enough,

$$\text{if} \quad d(F, SO(3)) > \alpha, \quad \text{then} \quad W_\varepsilon(F) \geq C_2.$$

Recall that, by polar decomposition, $|\sqrt{FF^T} - I| = d(F, SO(3))$ (see [**42**, Ex. 7, p. 17] for more details). Thus, if $d(F, SO(3)) > \alpha$, using Lemma 2.11 with $B = FF^T$ and $d = 3$, there exists $\delta \in (0, 1)$ such that, if

$$\det(FF^T) \in [1 - \delta, 1 + \delta],$$

then $f^\varepsilon_{opt} > \frac{\alpha^2}{2}$ for every ε small enough and therefore

$$W_\varepsilon(F) = \frac{\mu}{2}f^\varepsilon_{opt}(FF^T) + \frac{\lambda}{2}(\det F - 1)^2 > \frac{\mu\alpha^2}{4} > 0.$$

On the other hand, if $\det(FF^T) \in \mathbb{R} \setminus [1 - \delta, 1 + \delta]$, then

$$W_\varepsilon(F) \geq \frac{\lambda}{2}(\det F - 1)^2 \geq \frac{\lambda}{2}\min\{1, \delta^2\} > 0.$$

Step 3. Finally, we prove that there exists β large enough such that,

$$\text{if} \quad d(F, SO(3)) > \beta, \quad \text{then} \quad W_\varepsilon(F) \geq C_3 d^2(F, SO(3)\mathscr{U}_\varepsilon),$$

for some constant $C_3 > 0$ and for every ε small enough.
By using Proposition (2.9) with $d = 3$ and denoting by $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ the ordered singular values of F, we have that

$$W_\varepsilon(F) \geq \frac{\mu}{2}[(1 + \varepsilon)(\lambda_1^2 + \lambda_2^2) + (1 + \varepsilon)^{-2}\lambda_3^2 - 3 - 2\ln(\lambda_1\lambda_2\lambda_3)].$$

Therefore, since $(1+\varepsilon)^{-2}$, $(1+\varepsilon)$ tend to 1 as ε tends to zero, we have that, for ε small enough,

$$W_\varepsilon(F) \geq \frac{\mu}{2}\left[\frac{|F|^2}{2} - 3 - \ln(\lambda_1\lambda_2\lambda_3)^2\right]. \tag{2.34}$$

By using the inequality between arithmetic and geometric mean, we obtain from (2.34)

$$\begin{aligned} W_\varepsilon(F) &\geq \frac{\mu}{2}\left[\frac{|F|^2}{2} - 3 - 3\ln\frac{|F|^2}{3}\right] \\ &> \frac{\mu}{2}\left[\frac{|F|^2}{2} - 3\ln|F|^2\right], \end{aligned}$$

so that, if $|F|$ is sufficiently large, we have that $W_\varepsilon(F) \geq \frac{\mu}{8}|F|^2$. Thus, if β is large enough, we have that

$$W_\varepsilon(F) \geq C|FF^T - I|, \tag{2.35}$$

for a certain constant $C > 0$. Now, observe that

$$|\sqrt{FF^T} - I|^2 = \sum_{i=1}^{3}(\lambda_i - 1)^2 \leq \sqrt{3\sum_{i=1}^{3}(\lambda_i^2 - 1)^2 + 6} = \sqrt{3}|FF^T - I| + 6. \tag{2.36}$$

Thus, if ε is small enough, from (2.6) and (2.36) we have that

$$\begin{aligned} \frac{1}{2}|\sqrt{FF^T} - U_{n,\varepsilon}|^2 &\leq |\sqrt{FF^T} - I|^2 + |I - U_{n,\varepsilon}|^2 \\ &\leq \sqrt{3}|FF^T - I| + 6 + |\varepsilon U_n + o(\varepsilon)|^2 \\ &\leq \sqrt{3}|FF^T - I| + 7\frac{C}{\beta}\frac{\beta}{C} \\ &< \left(\sqrt{3} + 7\frac{C}{\beta}\right)|FF^T - I|, \end{aligned} \tag{2.37}$$

for every $n \in S^2$. From (2.35) and (2.37), by choosing $\beta > 0$ sufficiently large, we can conclude that there exists $C_3 > 0$ such that for every $\varepsilon > 0$ small enough

$$W_\varepsilon(F) \geq C_3|\sqrt{FF^T} - U_{n,\varepsilon}|^2,$$

and in turn

$$W_\varepsilon(F) \geq C_3 d^2(F, SO(3)\mathscr{U}_\varepsilon).$$

The quadratic growths established by Steps 1 and 3, together with the estimate in Step 2, show that we can bound W_ε with a single function, growing with the square of the distance, so that (2.29) holds.

In order to apply Theorem 1.13, it remains to consider the linear limit

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}W_\varepsilon(I + \varepsilon E) = \lim_{\varepsilon\to 0}\min_{n\in S^2}\frac{1}{\varepsilon^2}W_{n,\varepsilon}(I + \varepsilon E). \tag{2.38}$$

Note that this limit coincides with the function V, given by (2.16), which defines the functional $\mathscr{E}$ in (2.20). We have to check that the limit (2.38) is uniform on the compact subsets of $Sym(3)$ and that V satisfies growth condition (1.23). Both these properties are trivially satisfied, in view of the expressions of the functions $W_{n,\varepsilon}$ and V, and of the computations performed in the previous section. Hence, Theorem 1.13 directly gives (2.21), (2.25), and the last sentence of our theorem. Finally, the characterization (2.22)–(2.24) of the relaxed functional $\overline{\mathscr{E}}$ can be obtained by using [**12**, Theorem 1] with $\gamma = 1$, $\mathscr{Q}$ in place of $\mathscr{Q}_B$, and the set

$$\left\{\frac{3}{2}\left(n\otimes n - \frac{1}{3}I\right) \;:\; n \in S^2\right\}$$

of matrices in $Sym_0(3)$ with eigenvalues 1, $-\frac{1}{2}$, $-\frac{1}{2}$, in place of $\mathscr{Q}_{Fr}$. The proof of Theorem 2.4 is thus concluded. □

2.3. An alternative model

It is natural to explore the small strain behavior of another class of model energies, discussed in [**32**], and obtained from the BTW incompressible template (1.4), by a procedure which is quite common in rubber elasticity and computational mechanics. This is based on the additive split of the energy density into a distortional term (invariant under the transformation $F \to \alpha F$, with α a positive scalar), obtained from (1.4) by replacing F with $(\det F)^{-\frac{1}{3}}F$, and a volumetric term (which only depends on $\det F$). The resulting energy is of the form

$$W_{1,\varepsilon}(F) := \min_{n\in S^2} W_{1,n,\varepsilon}(F), \qquad F \in \mathbb{M}^{3\times 3}, \tag{2.39}$$

where

$$W_{1,n,\varepsilon}(F) := \begin{cases} \dfrac{\mu}{2}(\det F)^{-\frac{2}{3}}\operatorname{tr}(F^T L_{n,\varepsilon}^{-1} F) - \dfrac{3}{2}\mu + \dfrac{k}{2}(\det F - 1)^2 & \text{if } F \in Lin^+(3), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.40}$$

and $L_{n,\varepsilon}$ is given by (2.3) for $\varepsilon > 0$ and $n \in S^2$. $W_{1,\varepsilon}$ is again a natural generalization of (1.4) because it coincides with it for $\det F = 1$, it has the same set of energy wells

$$SO(3)\mathscr{U}_\varepsilon = \bigcup_{n\in S^2} \left\{ RL_{n,\varepsilon}^{\frac{1}{2}} \,:\, R \in SO(3) \right\}$$

as (2.7) and the same behavior near the energy wells (same linear limit (2.16) or, equivalently, same Taylor expansion at order two). However, $W_{1,\varepsilon}$ violates the hypothesis of quadratic growth with respect to $d(F, SO(3)\mathscr{U}_\varepsilon)$ (see Remark 2.8). Therefore, we cannot apply to it the abstract theory of [**60**] and the characterization of the Γ–limit of the functionals

$$\mathscr{E}_{1,\varepsilon}(u) = \frac{1}{\varepsilon^2}\int_\Omega W_{1,\varepsilon}(I + \varepsilon\nabla u)dx - \mathscr{L}(u) \tag{2.41}$$

requires an extension of Schmidt's theory. Some results in this direction are given in [**2**] for the single–well case. These results will be described in Chapter 3.

While Schmidt's theory does not apply to (2.39), it does apply to energies with quadratic growth that are obtained from (2.39) by changing its functional form only for matrices F such that, simultaneously, $d(F, SO(3))$ and $\det F$ are large. More in detail, we define, for $\beta > 0$,

$$W_\varepsilon^\beta(F) := \begin{cases} W_{1,\varepsilon}(F) & \text{if either } \; d(F, SO(3)) \le \beta \; \text{ or } \; \det F \le \beta, \\ W_2(F) & \text{otherwise,} \end{cases} \tag{2.42}$$

where W_2 is any frame indifferent function of F such that $W_2(F) \ge Cd^2(F, SO(3)\mathscr{U}_\varepsilon)$ for some constant $C > 0$, whenever $\det F > \beta$ and $d(F, SO(3)) > \beta$. W_ε^β has the same set of energy wells $SO(3)\mathscr{U}_\varepsilon$ of (2.39), for every $\beta > 0$. Moreover, $W_{1,\varepsilon}$ and W_ε^β have the same linear limit (2.14). Since the threshold β can be made arbitrarily large, (2.42) modifies energy (2.39) only in a regime in which $|F|$ and $\det F$ are very large. It is well known from rubber elasticity that, in such extreme regimes, Neohookean–type energies such as (1.4), in which the energy depends linearly on FF^T, are unable to reproduce the experimentally observed behaviour. In fact, expression (1.4) is best regarded as a conceptual tool to explore the behaviour of nematic elastomers under small applied forces, i.e., near the energy wells. The correction W_2 in (2.42) can thus be seen as a technical device with no mechanical significance, since it alters the values of the energy in a regime of deformations where expression (1.4), and hence (2.39), is no longer reliable.

Once the legitimacy of the correction (2.42) is accepted, again using [**60**] (generalized to the admissible energy densities of Definition 2.6) it is possible to compute the small strain Γ–limit of all energies of this type for every β sufficiently large. It is implicit in the following theorem that they all share the same Γ–limit $\overline{\mathscr{E}}$, which is independent of β. This is not surprising since, looking at the proof of Theorem 2.4, it is clear that the important features of the energy densities are their behavior near the energy wells (the only part which is involved in the computation of the linear limit: this is given by (2.16), which is independent of β), and the quadratic growth.

Let us follow the same notation of Section 2.2 and define the energy functionals $\mathscr{E}_\varepsilon^\beta$ on $H^1(\Omega; R^3)$ as

$$\mathscr{E}_\varepsilon^\beta(u) := \begin{cases} \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega W_\varepsilon^\beta(I+\varepsilon\nabla u)dx - \mathscr{L}(u) & \text{if } u \in H^1_{h,\partial_D\Omega}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.43}$$

with W_ε^β defined by (2.39), (2.40) and (2.42) for $\beta > 0$.

Theorem 2.7. *We have that*

$$\lim_{\varepsilon\to 0} \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E}_\varepsilon^\beta = \inf_{H^1_{h,\partial_D\Omega}} \mathscr{E} = \min_{H^1_{h,\partial_D\Omega}} \overline{\mathscr{E}},$$

where $\mathscr{E}$ and $\overline{\mathscr{E}}$ are given by (2.20) and (2.22)–(2.24), respectively. Moreover, if $\{u_\varepsilon\}$ is a sequence of "almost minimizers" for $\{\mathscr{E}_\varepsilon^\beta\}$, then $\{u_\varepsilon\}$ is also a minimizing sequence for $\mathscr{E}$. Furthermore, there exists a subsequence of $\{u_\varepsilon\}$ which converges weakly in $H^1(\Omega;\mathbb{R}^3)$ to some $u \in H^1_{h,\partial_D\Omega}$ and u is a minimizer of $\overline{\mathscr{E}}$.

Proof of Theorem 2.7. Let us verify the admissibility of $\{W_\varepsilon^\beta\}$ in the sense of Definition 2.6. It is clear that conditions (i) and (iii) hold. To prove (ii), consider the non trivial case $\det F > 0$, and notice that, if $d(F, SO(3)) > \beta$ and $\det F > \beta$, $W_\varepsilon^\beta(F)$ is nonnegative, otherwise

$$W_\varepsilon^\beta(F) = \frac{\mu}{2} g_\varepsilon^{opt}(FF^T) + \frac{k}{2}(\det F - 1)^2, \tag{2.44}$$

where g_ε^{opt} is defined as in (2.56) with $d = 3$. By Proposition 2.10, expression (2.44) is minimal at the value 0 on $SO(3)\mathscr{U}_\varepsilon$, where $\mathscr{U}_\varepsilon$ is defined in (2.30). Next, we prove that (iv) holds for W_ε^β, for every β large enough. More precisely, we want to prove that for every β sufficiently large there exists a constant $C_\beta > 0$ such that $W_\varepsilon^\beta(F) \geq C_\beta d^2(f, SO(3)\mathscr{U}_\varepsilon)$ for every F and every $\varepsilon > 0$ small enough. In view of the definition of W_ε^β, it is enough to prove that there exists $\beta_1 > 0$ such that, for every $\beta \geq \beta_1$,

$$W_\varepsilon^1(F) \geq C_\beta d^2(F, SO(3)\mathscr{U}_\varepsilon),$$

whenever $d(F, SO(3)) \leq \beta$ or $\det F \leq \beta$, and $\varepsilon > 0$ is sufficiently small. We divide the proof of this in the following three steps and restrict attention to the non-trivial case $\det F > 0$.
Step 1. We prove that there exist $\alpha > 0$ and $C_1 > 0$ such that, for every ε small enough and for every $F \in \mathbb{R}^{3\times 3}$,

$$\text{if} \quad d(F, SO(3)) \leq \alpha, \quad \text{then} \quad W_{1,\varepsilon}(F) \geq C_1 d^2(F, SO(3)\mathscr{U}_\varepsilon).$$

This can be shown as done in Step 1 of the proof of Theorem 2.4: we use the expansion of $\tilde{W}_{1,n,\varepsilon}$ around $L_{n,\varepsilon}$, where $\tilde{W}_{1,n,\varepsilon}$ is defined in (2.62) and $W_{1,n,\varepsilon}(F) = \tilde{W}_{1,n,\varepsilon}(FF^T)$, and we use Lemma 2.13 to conclude.
Step 2. Let α be the constant found in the Step 1. We want to show that there exists $C_2 > 0$ such that, for every ε small enough,

$$\text{if} \quad d(F, SO(3)) > \alpha, \quad \text{then} \quad W_{1,\varepsilon}(F) \geq C_2.$$

Again, the proof is the same of Step 2 of the proof of Theorem 2.4, by using Lemma 2.12 in place of Lemma 2.11.
Step 3. Finally, we prove that there exists β_1 large enough such that, for every $\beta \geq \beta_1$,

$$\text{if} \quad d(F, SO(3)) > \beta \quad \text{and} \quad \det F \leq \beta, \quad \text{then} \quad W_{1,\varepsilon}(F) \geq C_\beta d^2(F, SO(3)\mathscr{U}_\varepsilon),$$

for every ε small enough.
By using Proposition (2.10) and denoting by $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq$ the ordered singular values of F, we have that

$$\begin{aligned} W_{1,\varepsilon}(F) &\geq \frac{\mu}{2}(\det F)^{-\frac{2}{3}}[(\lambda_1^2+\lambda_2^2)(1+\varepsilon) + \lambda_3^2(1+\varepsilon)^{-2}] - \frac{3}{2}\mu \\ &\geq \frac{\mu}{2\beta^{\frac{2}{3}}}[(\lambda_1^2+\lambda_2^2)(1+\varepsilon) + \lambda_3^2(1+\varepsilon)^{-2}] - \frac{3}{2}\mu. \end{aligned}$$

Therefore, since $(1+\varepsilon)^{-2}$, $(1+\varepsilon)$ tend to 1 as ε tends to zero, we have that, for ε small enough,

$$W_{1,\varepsilon}(F) \geq \frac{\mu}{4\beta^{\frac{2}{3}}}[\lambda_1^2+\lambda_2^2+\lambda_3^2] - \frac{3}{2}\mu = \frac{\mu}{4\beta^{\frac{2}{3}}}|FF^T| - \frac{3}{2}\mu. \tag{2.45}$$

Observe that $\beta < |\sqrt{FF^T} - I| \leq |FF^T - I|$. Thus, if β_1 is large enough, on one hand, from (2.45), we have that for every $\beta \geq \beta_1$

$$W_{1,\varepsilon}(F) \geq \tilde{C}_\beta |FF^T - I|; \tag{2.46}$$

on the other hand, proceeding as in (2.36)–(2.37), we obtain again that, for every ε small enough,

$$\frac{1}{2}|\sqrt{FF^T} - U_{n,\varepsilon}|^2 < \left(\sqrt{3} + 7\frac{C}{\beta}\right)|FF^T - I|, \tag{2.47}$$

for every $n \in S^2$. From (2.46) and (2.47) we can conclude that for every $\beta \geq \beta_1$ and every $\varepsilon > 0$ sufficiently small,

$$W_{1,\varepsilon}(F) \geq C_\beta|\sqrt{FF^T} - U_{n,\varepsilon}|^2,$$

for a certain $C_\beta > 0$, from which

$$W_{1,\varepsilon}(F) \geq C_\beta d^2(F, SO(3)\mathscr{U}_\varepsilon).$$

The quadratic growths established by Steps 1 and 3, together with the estimate in Step 2, show that we can bound $W_{1,\varepsilon}$ with a single function, growing with the square of the distance, in the case $d(F, SO(3)) \leq \beta$ or $\det B \leq \beta$.

Now, let us compute the linear limit

$$\hat{V}(E) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}W_\varepsilon^\beta(I+\varepsilon E), \qquad E \in Sym(3).$$

It is clear that

$$\hat{V}(E) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}W_{1,\varepsilon}(I+\varepsilon E) = \min_{n\in S^2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\tilde{W}_{1,n,\varepsilon}((I+\varepsilon E)(I+\varepsilon E)^T),$$

where $\tilde{W}_{1,n,\varepsilon}$ is defined as in (2.62). Since $\tilde{W}_{1,n,\varepsilon}$ and its gradient vanish at $L_{n,\varepsilon}$, we have, from (2.4), that

$$\hat{V}(E) = \frac{1}{2}\min_{n\in S^2} D^2\tilde{W}_{1,n,0}(I)[2E - \hat{L}_n]^2.$$

From (2.65) and from the fact that $\hat{L}_n = 2\hat{U}_n$, it turns out that

$$\hat{V}(E) = \mu\min_{n\in S^2}|E - \hat{U}_n|^2 + \left(\frac{k}{2} - \frac{\mu}{3}\right)\operatorname{tr}^2 E = V(E), \qquad \text{for every } E \in Sym(3),$$

where V is given by (2.16). This complete the proof, in view of Theorem 1.13. The characterization of $\overline{\mathscr{E}}$ has been already established in the proof of Theorem 2.4 using the relaxation results of [**12**]. □

We remark again that, even if the Γ–limits of (2.43) are all the same, independent of β, this says nothing about the Γ–limit of (2.41). In fact, $W_{1,\varepsilon}(F) \geq Cd^{\frac{3}{2}}(F, SO(3)\mathscr{U}_\varepsilon)$ for $|F|$ large enough (as can be seen using Young's inequality) and $W_{1,\varepsilon}$ violates the hypothesis of quadratic growth on $\mathbb{M}^{3\times 3}$ (see the following remark) required by Schmidt's theory. Characterizing the Γ–limit of (2.41), and establishing whether this coincides with the Γ–limit of (2.43) requires an extension of Schmidt's theory. These are interesting questions, and will be addressed in future work.

Remark 2.8. The function $W_{1,\varepsilon}$ does not have a quadratic growth in $d(F, SO(3)\mathscr{U}_\varepsilon)$ in the regime of large determinant and norm. By Proposition 2.10, we have that

$$W_{1,\varepsilon}(F) = \frac{\mu}{2}(\det F)^{-\frac{2}{3}}[(\lambda_1^2+\lambda_2^2)(1+\varepsilon) + \lambda_3^2(1+\varepsilon)^{-2}] - \frac{3}{2}\mu + \frac{k}{2}(\det F - 1)^2,$$

where $\lambda_i = \lambda_i(F)$. More in general, consider an energy of the form

$$G_\varepsilon(F) = \frac{\mu}{2}(\det F)^{-\frac{2}{3}}[(\lambda_1^2+\lambda_2^2)(1+\varepsilon)+\lambda_3^2(1+\varepsilon)^{-2}] + g(\det F),$$

with g any scalar–valued function which goes to $+\infty$ as $\det F \to +\infty$. We observe that G_ε cannot satisfy

$$G_\varepsilon(F) \geq Cd^2(F, SO(3)\mathscr{U}_\varepsilon) \quad \text{for every } F \in \mathbb{M}^{3\times 3},$$

for a certain $C > 0$ and for any ε small enough. Indeed, G_ε doesn't satisfy this growth condition if F has both norm and determinant arbitrarily large.

In order to prove this, given a fixed arbitrary constant $C > 0$, we have to show that for every $\hat{\varepsilon} > 0$ there exists $\varepsilon < \hat{\varepsilon}$ and $F \in \mathbb{M}^{3\times 3}$ such that

$$G_\varepsilon(F) < Cd^2(F, SO(3)\mathscr{U}_\varepsilon).$$

Consider

$$F := \begin{bmatrix} \lambda g(\lambda)(1+\varepsilon) & 0 & 0 \\ 0 & \frac{(1+\varepsilon)^{-\frac{1}{2}}}{\lambda g(\lambda)} & 0 \\ 0 & 0 & \lambda(1+\varepsilon)^{-\frac{1}{2}} \end{bmatrix}$$

with $\varepsilon < \hat{\varepsilon}$ and $\lambda \in \mathbb{R}$ to be chosen later. It turns out that

$$G_\varepsilon(F) = \frac{\mu}{2}\lambda^{-\frac{2}{3}}\left[\lambda^2 g^2(\lambda) + \frac{1}{\lambda^2 g^2(\lambda)} + \lambda^2\right] + g(\lambda),$$

which behaves as $\lambda^{2-\frac{2}{3}}g^2(\lambda)$ for λ large. At the same time, when $\varepsilon \to 0$,

$$d^2(F, SO(3)\mathscr{U}_\varepsilon) \longrightarrow (\lambda g(\lambda)-1)^2 + \left(\frac{1}{\lambda g(\lambda)} - 1\right)^2 + (\lambda - 1)^2.$$

Thus, for any $\delta > 0$, we can find $\varepsilon < \hat{\varepsilon}$ such that

$$d^2(F, SO(3)\mathscr{U}_\varepsilon) > (\lambda g(\lambda)-1)^2 + \left(\frac{1}{\lambda g(\lambda)} - 1\right)^2 + (\lambda - 1)^2 - \delta. \tag{2.48}$$

Now, if we choose λ large enough such that

$$G_\varepsilon(F) \leq C\left[(\lambda g(\lambda)-1)^2 + \left(\frac{1}{\lambda g(\lambda)} - 1\right)^2 + (\lambda - 1)^2 - \delta\right],$$

we can conclude from (2.48) that

$$G_\varepsilon(F) < Cd^2(F, SO(3)\mathscr{U}_\varepsilon),$$

as claimed.

2.4. Appendix: some results from tensor calculus

We recall that $\mu_1(M) \leq \mu_2(M) \leq ... \leq \mu_d(M)$ are the ordered eigenvalues of the matrix $M \in Sym(d)$. The next proposition is a slight variant of [**32**, Proposition 1].

PROPOSITION 2.9. *Let $B \in Psym(d)$, $L \in \mathbb{M}^{d\times d}$, and consider the scalar–valued function*

$$f(B, L) = B \cdot L^{-1} - d - \ln(\det B). \tag{2.49}$$

The following properties hold:

(i) for every $L \in Psym(d)$ with $\det L = 1$, we have that

$$\min_{B \in Psym(d)} f(B, L) = f(L, L) = 0;$$

(ii) assume that L *is of the form*

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n\otimes n + (1+\varepsilon)^{-\frac{2}{(d-1)}}(I - n\otimes n), \tag{2.50}$$

for $\varepsilon > 0$ *and* n *belonging to the unitary sphere* S^{d-1}*. Then, for every* $B \in Psym(d)$*, we have that*

$$\begin{aligned} f^{\varepsilon}_{opt}(B) &:= \min_{n\in S^{d-1}} f(B, L_{n,\varepsilon}) \\ &= (1+\varepsilon)^{\frac{2}{d-1}}\left[\mathrm{tr}B - \mu_d(B)\right] + (1+\varepsilon)^{-2}\mu_d(B) - d - \ln(\det B); \end{aligned} \tag{2.51}$$

(iii) for every $\varepsilon > 0$,

$$\min_{B\in Psym(d)} f^{\varepsilon}_{opt}(B) = 0$$

and this minimum is obtained by any matrix in $Psym(d)$ *whose largest eigenvalue is* $(1+\varepsilon)^2$ *and whose other eigenvalues are all equal to* $(1+\varepsilon)^{-\frac{2}{(d-1)}}$.

Proof. To prove (i), let $\{b_1, ..., b_d\}$ and $\{l_1, ..., l_d\}$ be the orthonormal bases of eigenvectors of B, $L \in Psym(d)$, respectively. Then

$$\begin{aligned} B\cdot L^{-1} &= \left(\sum_{i=1}^{d}\mu_i(B) b_i\otimes b_i\right)\cdot\left(\sum_{j=1}^{d}\mu_j(L^{-1}) l_j\otimes l_j\right) \\ &= \sum_{i,j=1}^{d}\mu_i(B)\lambda_j(L^{-1})(b_i l_j)^2 \geq \sum_{i=1}^{d}\mu_i(B)\mu_i(L^{-1})(b_i l_i)^2. \end{aligned}$$

Observe that the equality holds if and only if $b_i l_j = 0$ for all $i \neq j$; thus, in order to minimize $f(\cdot, L)$, we restrict our attention to the case in which both B and L are in diagonal form. Then, by using the well-known inequality between arithmetic and geometric mean and the fact that $\det L = 1$, we have that

$$\begin{aligned} f(B,L) &= \sum_{i=1}^{d}\mu_i(B)\mu_i(L^{-1}) - d - \ln(\det B) && (2.52)\\ &\geq d\,(\det BL^{-1})^{\frac{1}{d}} - d - \ln(\det B) && (2.53)\\ &= d\,\psi(\alpha), && (2.54) \end{aligned}$$

where $\psi(\alpha) := \alpha - 1 - \ln\alpha$ and $\alpha := (\det B)^{\frac{1}{d}}$. Since $\psi \geq 0$ and $\psi(\alpha) = 0$ if and only if $\alpha = 1$, we have, from (2.52)-(2.54), that $f(B, L) = 0$ if and only if $\mu_i(B)\mu_i(L^{-1}) = \mu_j(B)\mu_j(L^{-1})$ for every i, $j \in \{1, ..., d\}$ and $\alpha = 1$. These conditions are equivalent to

$$1 = \det B \det L^{-1} = \prod_{i=1}^{d}\mu_i(B)\mu_i(L^{-1}) = \left[\mu_i(B)\mu_i(L^{-1})\right]^d, \quad \text{for every } i = 1, ..., d,$$

which gives $B = L$.

To prove (ii), let us fix $\hat{n} \in S^{d-1}$ and observe that

$$L^{-1}_{\hat{n},\varepsilon} = (1+\varepsilon)^{\frac{2}{d-1}}\left[I - \left(1 - (1+\varepsilon)^{-\frac{2d}{d-1}}\right)\hat{n}\otimes\hat{n}\right].$$

Clearly,

$$f^{\varepsilon}_{opt}(B) = \min_{R\in Orth(d)} f(B, RL_{\hat{n},\varepsilon}R^T),$$

thus

$$\begin{aligned} f^{\varepsilon}_{opt}(B) &= (1+\varepsilon)^{\frac{2}{d-1}}\min_{R\in Orth(d)} B\cdot\left[I - \left(1-(1+\varepsilon)^{-\frac{2d}{d-1}}\right)R\hat{n}\otimes R\hat{n}\right] - d - \ln(\det B) \\ &= (1+\varepsilon)^{\frac{2}{d-1}}\min_{R\in Orth(d)}\left[\mathrm{tr}B - \left(1-(1+\varepsilon)^{-\frac{2d}{d-1}}\right)BR\hat{n}\cdot R\hat{n}\right] - d - \ln(\det B). \end{aligned}$$

From the last equality we deduce that the minimum is attained when R maps $\hat{n}$ onto the maximum eigenvalue of B and thus the thesis follows.
To prove (iii), observe that

$$\begin{aligned}\min_{B\in Psym(d)} f^{\varepsilon}_{opt}(B) &= \min_{n\in S^{d-1}} \min_{B\in Psym(d)} f(B, L_{n,\varepsilon}) \\ &= \min_{n\in S^{d-1}} f(L_{n,\varepsilon}, L_{n,\varepsilon}) = 0,\end{aligned}$$

where the last equality follows from (i). □

We also use the following result, which we state without proof.

PROPOSITION 2.10. *Let $B \in Psym(d)$, $L \in \mathbb{M}^{d\times d}$, and consider the scalar–valued function*

$$g(B, L) = (\det B)^{-\frac{1}{d}} B \cdot L^{-1} - d. \tag{2.55}$$

The following statements hold:

(i) for every $L \in Psym(d)$ with $\det L = 1$, *we have that*

$$\min_{B\in Psym(d)} g(B, L) = g(\alpha L, L) = 0, \quad \text{for every } \alpha > 0;$$

(ii) assume that $L = L_{n,\varepsilon}$, for some $\varepsilon > 0$ and $n \in S^{d-1}$, where $L_{n,\varepsilon}$ is defined in (2.50). Then, for every $B \in Psym(d)$, we have that

$$\begin{aligned} g^{\varepsilon}_{opt}(B) &:= \min_{n\in S^{d-1}} g(B, L_{n,\varepsilon}) \\ &= (\det B)^{-\frac{1}{d}} \left\{ (1+\varepsilon)^{\frac{2}{(d-1)}} [\mathrm{tr}B - \lambda_1(B)] + (1+\varepsilon)^{-2} \mu_d(B) \right\} - d; \end{aligned} \tag{2.56}$$

(iii) for every $\varepsilon > 0$,

$$\min_{B\in Psym(d)} g^{\varepsilon}_{opt}(B) = 0$$

and this minimum is obtained by any matrix in $Psym(d)$ whose largest eigenvalue is $\alpha(1+\varepsilon)^2$ and whose other eigenvalues are all equal to $\alpha(1+\varepsilon)^{-\frac{2}{(d-1)}}$, for some $\alpha > 0$.

We now collect some results from tensor calculus that we used in the Section 2.2 and 2.3.

LEMMA 2.11. *Let $B \in Psym(d)$ and suppose that $|\sqrt{B} - I| > \alpha > 0$. There exists $\delta \in (0, 1)$ such that, if*

$$\det B \in [1-\delta, 1+\delta],$$

then, for every ε small enough,

$$f^{\varepsilon}_{opt}(B) > \frac{\alpha^2}{2} > 0,$$

where f^{ε}_{opt} is the function defined in (2.51).

PROOF. From the expression of f^{ε}_{opt} given in point (ii) of Proposition 2.9, it is clear that for a parameter $\eta \in (0, 1)$ to be chosen and for every ε small enough, we have that

$$f^{\varepsilon}_{opt}(B) > \eta\, \mathrm{tr}B - d - \ln(\det B). \tag{2.57}$$

Now, if we write $\mu_i = \mu_i(\sqrt{B})$, the hypothesis $|\sqrt{B} - I|^2 > \alpha^2$ becomes

$$\sum_{i=1}^{d} (\mu_i - 1)^2 > \alpha^2.$$

Expanding the squares and using again the inequality between arithmetic and geometric mean, we obtain

$$\begin{aligned} \mathrm{tr}B = \sum_{i=1}^{d} \mu_i^2 &> \alpha^2 - d + 2\sum_{i=1}^{d} \mu_i \\ &\geq \alpha^2 - d + 2d(\det B)^{\frac{1}{2d}}. \end{aligned} \tag{2.58}$$

From (2.57) and (2.58) it descends that

$$\begin{aligned} f^{\varepsilon}_{opt}(B) \; &> \; \eta \left[\alpha^2 - d + 2d(\det B)^{\frac{1}{2d}}\right] - d - \ln(\det B) \\ &\geq \; \eta \left[\alpha^2 - d + 2d(1-\delta)^{\frac{1}{2d}}\right] - d - \ln(1+\delta) := K, \end{aligned} \tag{2.59}$$

where in the last inequality we are supposing $\det B$ to vary in $[1-\delta, 1+\delta]$, with $\delta \in (0,1)$ a parameter to be chosen. Finally, since the right hand side of (2.59) tends to α^2 as $\eta \to 1^-$ and $\delta \to 0^+$, we can choose η sufficiently near 1 and δ sufficiently near 0 such that $K \geq \frac{\alpha^2}{2}$ and the thesis follows. □

LEMMA 2.12. *Let $B \in Psym(d)$ and suppose that $|\sqrt{B} - I| > \alpha > 0$. There exists $\delta \in (0,1)$ such that, if*

$$\det B \in [1-\delta, 1+\delta],$$

then, for every ε small enough,

$$g^{\varepsilon}_{opt}(B) > \frac{\alpha^2}{2},$$

where g^{ε}_{opt} is the function defined in (2.56).

PROOF. From the expression of g^{ε}_{opt}, we have that, for a parameter $\eta \in (0,1)$ to be chosen and for any ε small enough,

$$g^{\varepsilon}_{opt}(B) > \eta(\det B)^{-\frac{1}{d}}\mathrm{tr}B - d. \tag{2.60}$$

Now, as in the proof of Lemma 2.11, consider (2.58) (where $\mu_i = \mu_i(\sqrt{B})$), which descends from the hypothesis. From (2.60) and (2.58) we obtain that

$$\begin{aligned} g^{\varepsilon}_{opt}(B) \; &> \; \eta(\det B)^{-\frac{1}{d}} \left[\alpha^2 - d + 2d(\det B)^{\frac{1}{2d}}\right] - d \\ &\geq \; \frac{\eta}{(1+\delta)^{\frac{1}{d}}} \left[\alpha^2 - d + 2d(1-\delta)^{\frac{1}{2d}}\right] - d := K, \end{aligned} \tag{2.61}$$

where in the last inequality we are supposing $\det B$ to vary in $[1-\delta, 1+\delta]$, with $\delta \in (0,1)$ a parameter to be chosen. Since the right hand side of (2.61) tends to α^2 as $\eta \to 1^-$ and $\delta \to 0^+$, we can choose η sufficiently near 1 and δ sufficiently near 0 such that $K \geq \frac{\alpha^2}{2}$ and the thesis follows. □

LEMMA 2.13. *Let μ and k be two positive constants. For $\varepsilon > 0$ and $n \in S^2$, let $\tilde{W}_{1,n,\varepsilon}$ be the scalar–valued function which, to each $B \in Psym(3)$, gives the value*

$$\tilde{W}_{1,n,\varepsilon}(B) = \frac{\mu}{2} g(B, L_{n,\varepsilon}) + \frac{k}{2}(\sqrt{\det B} - 1)^2, \tag{2.62}$$

where g and $L_{n,\varepsilon}$ are defined in (2.55) and (2.50), specialized to dimension 3, respectively. Then, there exists a positive constant C such that

$$D^2\tilde{W}_{1,n,\varepsilon}(L_{n,\varepsilon})[S]^2 \geq C|S|^2$$

for every $n \in S^2$, $S \in Sym(3)$, and for every ε small enough.

PROOF. For $B \in Psym(3)$, let $h_1(B) = (\det B)^{-\frac{1}{3}}B$ and $h_2(B) = (\sqrt{\det B} - 1)^2$. Then, for every $S \in Sym(3)$, we have

$$Dh_1(B)[S] = -\frac{1}{3}(\det B)^{-\frac{1}{3}}(B^{-1} \cdot S)B + (\det B)^{-\frac{1}{3}}S,$$

and

$$Dh_2(B)[S] = (\det B - \sqrt{\det B})B^{-1} \cdot S.$$

By some computations, we obtain:

$$D^2h_1(B)[S,H] = \frac{1}{9}(\det B)^{-\frac{1}{3}}(B^{-1}\cdot S)(B^{-1}\cdot H)B+ \\ + \frac{1}{3}(\det B)^{-\frac{1}{3}}[(B^{-1}HB^{-1})\cdot S]B - \frac{1}{3}(\det B)^{-\frac{1}{3}}(B^{-1}\cdot S)H- \\ - \frac{1}{3}(\det B)^{-\frac{1}{3}}(B^{-1}\cdot H)S,$$

and

$$D^2h_2(B)[S,H] = \left(\det B - \frac{\sqrt{\det B}}{2}\right)(B^{-1}\cdot S)(B^{-1}\cdot H)- \\ - (\det B - \sqrt{\det B})(B^{-1}HB^{-1})\cdot S,$$

for every S, $H \in Sym(3)$. Thus, if $L = L_{n,\varepsilon}$ for some ε and n, we have that

$$D^2h_1(L)[S]^2 = \frac{1}{9}(L^{-1}\cdot S)^2L + \frac{1}{3}[(L^{-1}SL^{-1})\cdot S]L - \frac{2}{3}(L^{-1}\cdot S)S, \tag{2.63}$$

and

$$D^2h_2(L)[S]^2 = \frac{1}{2}(L^{-1}\cdot S)^2, \tag{2.64}$$

for every $S \in Sym(3)$. Since $g(B,L) = h_1(B)\cdot L^{-1} - 3$, by using (2.63) and (2.64) we obtain that

$$\begin{aligned} D^2\tilde{W}_{1,n,\varepsilon}(L)[S]^2 &= \frac{\mu}{2}D^2h_1(L)[S]^2\cdot L^{-1} + \frac{k}{2}D^2h_2(L)[S]^2 \\ &= \frac{\mu}{2}\left[-\frac{1}{3}(L^{-1}\cdot S)^2 + (L^{-1}SL^{-1})\cdot S\right] + \frac{k}{4}(L^{-1}\cdot S)^2, \end{aligned}$$

and therefore, by the fact that $(L^{-1}SL^{-1})\cdot S = \mathrm{tr}(L^{-1}S)^2$, that

$$2D^2\tilde{W}_{1,n,\varepsilon}(L)[S]^2 = \left(\frac{k}{2} - \frac{\mu}{3}\right)\mathrm{tr}^2(L^{-1}S) + \mu\mathrm{tr}(L^{-1}S)^2. \tag{2.65}$$

Now, since for every $H \in Sym(3)$ one has that $\mathrm{tr}^2H \leq 3\mathrm{tr}H^2$, then

$$2D^2\tilde{W}_{1,n,\varepsilon}(L_{n,\varepsilon})[S]^2 \geq \min\left\{\mu, \frac{3}{2}k\right\}\mathrm{tr}(L_{n,\varepsilon}^{-1}S)^2.$$

The conclusion follows from the fact that $\mathrm{tr}(L_{n,\varepsilon}^{-1}S)^2 \geq \frac{1}{4}|S|^2$ for every ε sufficiently small. $\square$

CHAPTER 3

From finite to linear elasticity via Γ–convergence under weak conditions

In this chapter, we present the results of [**2**]. We consider a homogeneous and hyperelastic body occupying a reference configuration $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, subject to a deformation $v : \Omega \to \mathbb{R}^n$, and endowed with a frame indifferent energy density W minimized at the value 0 by the identity matrix I. The linearized elastic energy associated with the displacement $u(x) = v(x) - x$ is given by the formula

$$\frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2 dx. \tag{3.1}$$

We suppose that W satisfies the growth conditions

$$W(F) \geq c\, d^2(F, SO(n)) \text{ around } SO(n), \qquad W(F) \geq C\, d^p(F, SO(n)) \text{ far from } SO(n), \tag{3.2}$$

for some $1 < p \leq 2$. In Theorem 3.2, we essentially show that, under prescribed boundary conditions, the minimizers of the functionals

$$\int_\Omega W(I + \varepsilon \nabla u) dx$$

converge strongly in $W^{1,p}(\Omega;\mathbb{R}^n)$ to the minimizer of the corresponding boundary value problem for the functional (3.1). In the case where $p = 2$ in (3.2), the justification of (3.1) as the small strain Γ–limit of finite elasticity has been already established in [**26**]. We refer the reader to Subsection 1.2.2 in Section 1.2 for an account of the main result of [**26**] showing the crucial role of Γ–convergence for the derivation of linear elasticity.

The proof of Theorem 3.2 hinges on a compactness result and on a Γ–convergence result, as well as on a result of strong convergence of minimizers, which are proved in Sections 3.2, 3.3, and 3.4, respectively. In Section 3.1, we specify the setting of our problem and state the main results of this chapter. Also, we show with some examples that the growth behavior (3.2) is the appropriate one for a large class of compressible rubber–like materials.

3.1. Energy densities with a weak coerciveness property

The reference configuration Ω is a bounded connected open set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Throughout this chapter, the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^n)$ will be denoted by $W^{1,p}$. We will prescribe a Dirichlet condition on a part $\partial_D\Omega$ of $\partial\Omega$ with Lipschitz boundary in $\partial\Omega$, according to the following definition.

Definition 3.1. *Let us define*

$$Q := (-1,1)^n, \qquad Q^+ := (-1,1)^{n-1}\times(0,1),$$
$$Q_0 := (-1,1)^{n-1}\times\{0\}, \qquad Q_0^+ := (-1,1)^{n-2}\times(0,1)\times\{0\}.$$

We say that $E \subseteq \partial\Omega$ has Lipschitz boundary in $\partial\Omega$ *if it is nonempty and for every x in the boundary of E for the relative topology of $\partial\Omega$ there exist an open neighbourhood U of x in $\mathbb{R}^n$ and a bi–Lipschitz homeomorphism $\psi : U \to Q$ such that*

$$\psi(U\cap\Omega) = Q^+, \qquad \psi(U\cap\partial\Omega) = Q_0, \qquad \psi(U\cap E) = Q_0^+.$$

To deal with the Dirichlet boundary condition, for every $h \in W^{1,p}$ we introduce the set

$$W^{1,p}_h := \left\{ u \in W^{1,p} : u = h \quad \mathscr{H}^{n-1}\text{–a.e. on } \partial_D\Omega \right\}, \tag{3.3}$$

where the equality on $\partial_D\Omega$ refers to the traces of the functions on the boundary $\partial\Omega$.

We consider a hyperelastic material with a $\mathscr{L} \times \mathscr{B}$–measurable stored energy density

$$W : \Omega \times \mathbb{M}^{n\times n} \to [0, \infty],$$

where $\mathscr{L}$ and $\mathscr{B}$ are the σ–algebras of the Lebesgue measurable subsets of $\mathbb{R}^n$ and Borel measurable subsets of $\mathbb{M}^{n\times n}$, respectively. We assume that W satisfies the following properties for a.e. $x \in \Omega$:

(i) $W(x, \cdot)$ is frame indifferent;
(ii) $W(x, \cdot)$ is of class C^2 in some neighbourhood of $SO(n)$, independent of x, where the second derivatives are bounded by a constant independent of x;
(iii) $W(x, F) = 0$ if $F \in SO(n)$;
(iv) $W(x, F) \geq g_p(d(F, SO(n)))$, for some $1 < p \leq 2$, where $g_p : [0, \infty) \to \mathbb{R}$ is defined by

$$g_p(t) := \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1, \\ \frac{t^p}{p} + \frac{1}{2} - \frac{1}{p} & \text{if } t > 1. \end{cases} \tag{3.4}$$

Observe that these assumptions are compatible with the condition $W(x, F) = \infty$, if $\det F \leq 0$, which is classical in the context of finite elasticity. Also, observe that g_p is a convex function. By frame indifference, for a.e $x \in \Omega$ we have that

$$D^2W(x, I)[M]^2 = D^2W(x, I)[symM]^2, \quad \text{for every } M \in \mathbb{M}^{n\times n}. \tag{3.5}$$

Together with assumption (iv), this implies that the quadratic form $D^2W(x, I)[\cdot]^2$ is null on $Skw(n)$ and satisfies the coerciveness condition

$$D^2W(x, I)[symM]^2 \geq |symM|^2, \quad \text{for a.e } x \in \Omega \text{ and every } M \in \mathbb{M}^{n\times n}. \tag{3.6}$$

The load is modelled by a continuous linear functional $\mathscr{L} : W^{1,p} \to \mathbb{R}$. If $v \in W^{1,p}$ represents the deformation of the elastic body, the stable equilibria of the elastic body are obtained by minimizing the functional

$$\int_\Omega W(x, \nabla v)dx - \mathscr{L}(v),$$

under the prescribed boundary conditions. We are interested in the case where the load has the form $\varepsilon\mathscr{L}$ and we want to study the behaviour of the solution as ε tends to zero. We write

$$v = x + \varepsilon u$$

and we assume Dirichlet boundary condition of the form

$$v = x + \varepsilon h \quad \mathscr{H}^{n-1}\text{–a.e. on } \partial_D\Omega,$$

with a prescribed $h \in W^{1,\infty}$. The corresponding minimum problem for u becomes

$$\min_{W^{1,p}} \left\{ \int_\Omega W(x, I + \varepsilon\nabla u)dx - \varepsilon\mathscr{L}(\varepsilon u) \right\}, \tag{3.7}$$

where the term $\varepsilon\mathscr{L}(x)$ has been neglected since it does not depend on u. The following theorem is the main result of this chapter. It describes the behavior of the minimizers of (3.7).

Theorem 3.2. *Assume that $W : \Omega \times \mathbb{M}^{n\times n} \to [0, \infty]$ satisfies conditions (i)–(iv) for some $1 < p \leq 2$, and let $h \in W^{1,\infty}$. For every $\varepsilon > 0$ let*

$$m_\varepsilon := \inf_{u \in W^{1,p}_h} \left\{ \frac{1}{\varepsilon^2} \int_\Omega W(x, I + \varepsilon\nabla u)dx - \mathscr{L}(u) \right\}, \tag{3.8}$$

and let $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence such that

$$\frac{1}{\varepsilon^2} \int_\Omega W(x, I + \varepsilon\nabla u_\varepsilon)dx - \mathscr{L}(u_\varepsilon) = m_\varepsilon + o(1). \tag{3.9}$$

Then, $\{u_\varepsilon\}$ converges strongly in $W^{1,p}$ to the unique solution of the problem

$$m := \min_{u \in W_h^{1,2}} \left\{ \frac{1}{2} \int_\Omega D^2 W(x,I)[e(u)]^2 - \mathscr{L}(u) \right\}. \tag{3.10}$$

Moreover, $m_\varepsilon \to m$.

In the case $1 < p < 2$, Theorem 3.2 asserts that a sequence of "almost minimizers" in $W_h^{1,p}$ for the ε–problems converges to a minimizer for the limit problem in a different Sobolev space: indeed, the limit problem is formulated in $W_h^{1,2}$.

In the case $p = 2$, weak convergence of the "almost minimizers" has already been proved in [**26**]. Theorem 3.2 extends this result to the case $1 < p \leq 2$ and provides also strong convergence. The proof is based on the following three results which are proved in Section 3.2, 3.3, and 3.4, respectively. To simplify the exposition, the proofs are given only when W does not depend explicitly on x. The general case requires only minor modifications. Such results involve the functionals $\mathscr{F}_\varepsilon$, $\mathscr{F} : W^{1,p} \to [0,\infty]$ defined by

$$\mathscr{F}_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon^2} \displaystyle\int_\Omega W(x, I + \varepsilon \nabla u) dx & \text{if } u \in W_h^{1,p}, \\ \infty & \text{otherwise}, \end{cases} \tag{3.11}$$

and

$$\mathscr{F}(u) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega D^2 W(x,I)[e(u)]^2 dx & \text{if } u \in W_h^{1,2}, \\ \infty & \text{otherwise}, \end{cases} \tag{3.12}$$

and the functionals $\mathscr{G}_\varepsilon$, $\mathscr{G} : W^{1,p} \to (-\infty, \infty]$ defined by

$$\mathscr{G}_\varepsilon := \mathscr{F}_\varepsilon - \mathscr{L}, \qquad \mathscr{G} := \mathscr{F} - \mathscr{L}. \tag{3.13}$$

Observe that, due to the growth property (iv) of W, the functionals $\mathscr{G}_\varepsilon$ and $\mathscr{G}$ are bounded from below.

Theorem 3.3. *Assume that $W : \Omega \times \mathbb{M}^{n\times n} \to [0,\infty]$ satisfies conditions (i)–(iv) for some $1 \leq p \leq 2$. There exists a constant $C > 0$ depending on Ω, $\partial_D\Omega$, and p such that for every $h \in W^{1,p}$ and every sequence $\{u_\varepsilon\} \subseteq W_h^{1,p}$ we have*

$$\int_\Omega |\nabla u_\varepsilon|^p dx \leq C \left[1 + \mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathscr{H}^{n-1} \right)^2 \right], \tag{3.14}$$

for every $\varepsilon > 0$ sufficiently small.

The previous theorem ensures that, if $\{u_\varepsilon\}$ is a sequence in $W_h^{1,p}$ such that $\{\mathscr{F}_\varepsilon(u_\varepsilon)\}$ is bounded, then $\{u_\varepsilon\}$ is bounded in $W^{1,p}$, hence a subsequence converges weakly in $W^{1,p}$.

Theorem 3.4. *Under the hypotheses of Theorem 3.2, for every $\varepsilon_j \to 0$ we have that*

$$\mathscr{F}_{\varepsilon_j} \xrightarrow{\Gamma} \mathscr{F}, \qquad \text{as } j \to \infty,$$

in the weak topology of $W^{1,p}$.

Theorem 3.4, together with the compactness result provided by Theorem 3.3, implies the convergence of minima and the weak convergence of minimizers, using standard results on Γ–convergence. The next theorem and the previous remarks allow us to obtain the strong convergence of minimizers.

Theorem 3.5. *Under the hypotheses of Theorem 3.2, let $\varepsilon_j \to 0$ and let $\{u_j\}$ be a recovery sequence for $u \in W_h^{1,2}$, that is $u_j \rightharpoonup u$ weakly in $W^{1,p}$ and $\mathscr{F}_{\varepsilon_j}(u_j) \to \mathscr{F}(u)$. Then $\{u_j\}$ converges strongly in $W^{1,p}$.*

REMARK 3.6 (On the condition $\partial_D\Omega \neq \emptyset$). Observe that in Theorem 3.3 the assumption $\partial_D\Omega \neq \emptyset$ is crucial. When $\partial_D\Omega = \emptyset$, inequality (3.14) is false, as the following example shows. Consider the simple case $W(F) := g_p(d(F, SO(n)))$ for every $F \in Lin^+(n)$. For every $\varepsilon > 0$ and some $R \in SO(n) \setminus \{I\}$, set

$$u_\varepsilon(x) := \frac{R - I}{\varepsilon}x, \qquad x \in \Omega.$$

In this case, we have that

$$\int_\Omega |\nabla u_\varepsilon|^p dx = \frac{|\Omega||R - I|^p}{\varepsilon^p} \longrightarrow \infty, \quad \text{as } \varepsilon \to 0^+,$$

whereas

$$\mathscr{F}_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^2}\int_\Omega g_p(d(I + \varepsilon\nabla u_\varepsilon, SO(n)))dx = 0, \quad \text{for every } \varepsilon > 0.$$

REMARK 3.7 (On the condition $h \in W^{1,\infty}$). In Theorems 3.2, 3.4 and 3.5 the hypothesis $h \in W^{1,\infty}$ cannot be replaced by $h \in W^{1,2}$, unless W satisfies suitable bounds from above, which are not natural in the context of finite elasticity. Consider the simple case $\partial_D\Omega = \partial\Omega$, $\mathscr{L} = 0$, and assume that for some $r > 2$ we have

$$W(F) \geq |F|^r \quad \text{for } |F| \text{ large enough.}$$

By well known properties of the images of Sobolev spaces under the trace operator, there exists $h \in W^{1,2}$ such that

$$\{u \in W^{1,r} : u = h \ \mathscr{H}^{n-1}\text{–a.e. on } \partial\Omega\} = \emptyset. \tag{3.15}$$

Let us prove that $\mathscr{F}_\varepsilon(u) = \infty$ for every $u \in W^{1,p}$. Assume by contradiction that there exists $u \in W^{1,p}$ with $\mathscr{F}_\varepsilon(u) < \infty$. By (3.11) we have that $\nabla u \in L^r$, hence $u \in W^{1,r}$, because Ω has Lipschitz boundary. This contradicts (3.15). Therefore $\{\mathscr{F}_\varepsilon\}$ cannot Γ–converge to $\mathscr{F}$, because $\mathscr{F}(h) < \infty$.

3.1.1. Model energy densities. A large class of models where the energy density grows quadratically near the wells and less than quadratically elsewhere is provided by rubber elasticity, when one wishes to take into account the compressibility of the material. We recall that we have formalized this growth behaviour by introducing, as bound from below of our energies, the function

$$g_p(d(\cdot, SO(3))), \qquad \text{for some } 1 < p < 2,$$

where g_p is the function defined in (3.4). For simplicity, we focus on the homogeneous case.

As seen, e.g., in Section 2.3 and 4.1, a common practice to pass from an incompressible model, with associated energy density $\tilde{W}$ defined on $\{F \in \mathbb{M}^{n\times n} : \det F = 1\}$, to a corresponding compressible model W (see also [**45**]) is to define

$$W(F) := \tilde{W}((\det F)^{-1/3}F) + W_{vol}(\det F), \quad \text{for every } F \in Lin^+(3),$$

where W_{vol} is such that

$$W_{vol} \geq 0 \quad \text{and} \quad W_{vol}(t) = 0 \quad \text{if and only if} \quad t = 1.$$

For example, we can take W_{vol} of the form

$$W_{vol}(t) = c\left[t^2 - 1 - 2\log t\right], \quad \text{for every } t > 0,$$

for $c > 0$. Consider first the *Neo-Hookean* incompressible model for hyperelastic materials, where the energy density is of the form

$$\tilde{W}_{\mathscr{N}}(F) := a\left(|F|^2 - 3\right), \quad \text{for every } F \in \mathbb{M}^{n\times n} \text{ with } \det F = 1,$$

for a certain $a > 0$. Following the procedure described above, we consider the corresponding compressible energy density defined for every $F \in Lin^+(3)$ by

$$\begin{aligned} W_{\mathcal{N}}(F) &:= \tilde{W}_{\mathcal{N}}\left(\frac{F}{(\det F)^{1/3}}\right) + W_{vol}(\det F) \\ &= a\left(\frac{|F|^2}{(\det F)^{2/3}} - 3\right) + W_{vol}(\det F). \end{aligned}$$

Let us check that $W_{\mathcal{N}}$ has "g_p–growth". By using the well known inequality between arithmetic and geometric mean, it is easy to see that

$$W_{\mathcal{N}} \geq 0 \quad \text{and} \quad W_{\mathcal{N}}(F) = 0 \quad \text{if and only if} \quad F \in SO(3). \tag{3.16}$$

Moreover, recalling the Green–St. Venant strain tensor $E = \frac{1}{2}(F^T F - I)$ and using simple rules of tensor calculus, it turns out that in the small–strain regime, W has the expression

$$W_{\mathcal{N}}(F) = \mu|E|^2 + \frac{\lambda}{2}\mathrm{tr}^2 E + o(|E|^2), \tag{3.17}$$

where

$$\mu = 2a, \qquad \lambda = 4\left(-\frac{a}{3} + c\right).$$

The parameters μ and $\lambda + \frac{2}{3}\mu$ have the physical meaning of a shear modulus and a bulk modulus, respectively. Since $|E|^2 \geq \frac{1}{3}\mathrm{tr}^2 E$ for every $E \in Sym(3)$, from (3.17) we obtain that

$$W_{\mathcal{N}}(F) \geq \min\{\mu, 6c\}|E|^2 + o(|E|^2),$$

and in turn,

$$W_{\mathcal{N}}(F) \geq \frac{1}{2}\min\{\mu, 6c\}|E|^2, \tag{3.18}$$

if $|E|$ is small enough, that is, if $d(F, SO(3))$ is small enough. Since $|\sqrt{C} - I| \leq |C - I|$ for every $C \in Psym(3)$, from (3.18) we obtain that

$$W_{\mathcal{N}}(F) \geq \frac{1}{8}\min\{\mu, 6c\}|\sqrt{F^T F} - I|^2 = \frac{1}{8}\min\{\mu, 6c\}d^2(F, SO(3)), \tag{3.19}$$

if $d(F, SO(3))$ is sufficiently small. Now, we want to study the growth of W in the regime $|F| \to \infty$. In this case, if $\det F$ is bounded, then

$$W_{\mathcal{N}}(F) \geq C|F|^2 - 3a \geq \tilde{C}d^2(F, SO(3)), \qquad (\det F \text{ bounded}), \tag{3.20}$$

for some C, $\tilde{C} > 0$. In the case $\det F \to \infty$, we have that

$$W_{\mathcal{N}}(F) \geq K\left(\frac{|F|^2}{\det^{2/3} F} + \det^2 F\right),$$

for some $K > 0$. By using Young's inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

with $x = \left(\frac{|F|^3}{\det F}\right)^{1/2}$ and $y = (\det F)^{1/2}$, it is easy to show that

$$W_{\mathcal{N}}(F) \geq K|F|^{3/2} \geq \tilde{K}d^{3/2}(F, SO(3)), \qquad (\det F \to \infty), \tag{3.21}$$

for some $\tilde{K} > 0$. (3.16), (3.19), (3.20) and (3.21) shows that $W_{\mathcal{N}}$ has g_p growth from below with $p = \frac{3}{2}$. It is important to notice that $W_{\mathcal{N}}$ has not quadratic growth everywhere. In particular, $W_{\mathcal{N}}$ has not quadratic growth in the regime $\det F \to \infty$. This can be checked by taking into account deformation gradients of the type

$$F = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{with } \lambda \gg 0. \tag{3.22}$$

In Remark 2.8 a similar example shows that the BTW model for nematic elastomers has not quadratic growth everywhere.

As a second example, we consider the *Mooney–Rivlin* compressible model given, for some a, $b > 0$, by

$$\begin{aligned} W_{\mathscr{M}}(F) &:= a\left(\frac{|F|^2}{(\det F)^{2/3}} - 3\right) + b\left((\det F)^{2/3}|F^{-1}|^2 - 3\right) + W_{vol}(\det F) \\ &= W_{\mathscr{N}}(F) + b\left((\det F)^{2/3}|F^{-1}|^2 - 3\right), \end{aligned} \tag{3.23}$$

for every $F \in Lin^+(3)$, and derived from the corresponding incompressible version as explained before. The inequality between arithmetic and geometric mean implies that the second summand in (3.23) is nonnegative, so that, from (3.16), we have that

$$W_{\mathscr{M}} \geq 0 \quad \text{and} \quad W_{\mathscr{M}}(F) = 0 \quad \text{if and only if} \quad F \in SO(3).$$

The formula for the small strain regime is given by (3.17), with

$$\mu = 2(a+b), \qquad \lambda = 4\left(-\frac{a+b}{3} + c\right).$$

From the fact that $W_{\mathscr{N}}$ has g_p–growth and from the positiveness of the second summand of (3.23) the g_p–growth of $W_{\mathscr{M}}$ trivially follows. Also in this case, deformation gradients of the type (3.22) show that $W_{\mathscr{M}}$ does not grow quadratically everywhere.

Finally, we mention some *Ogden–type* compressible energy densities:

$$W_{\mathscr{O}}(F) := \sum_{i=1}^{m} a_i \left(\frac{\operatorname{tr}\left((F^T F)^{\gamma_i/2}\right)}{(\det F)^{\gamma_i/3}} - 3\right) + W_{vol}(\det F),$$

defined for every $F \in Lin^+(3)$, for some $m \geq 1$ and a_i, $\gamma_i > 0$, $i = 1, ..., m$. The formula for $W_{\mathscr{O}}$ in the small strain regime is again given by (3.17), with

$$\mu = 2\sum_{i=1}^{m} a_i, \qquad \lambda = 4\left(-\frac{1}{3}\sum_{i=1}^{m} a_i + c\right).$$

Arguing similarly to the Neo–Hookean and the Mooney–Rivlin models, we obtain that $W_{\mathscr{O}}$ attains its minimum 0 at $SO(3)$. By using Young's inequality and proper counterexamples, it is possible to show that $W_{\mathscr{O}}$ has g_p growth for some $1 < p < 2$ (p depending on the exponents γ_i), but not a quadratic growth in general, if $0 < \gamma_i < 3$ for every $i = 1, ..., m$ and $\gamma_i > \frac{6}{5}$ for at least one index $i \in \{1, ..., m\}$. The (multi–well) Ogden–type energies for nematic elastomers discussed in Chapter 4 have a similar behavior.

3.2. Compactness

The compactness result requires the following extension of the well known geometric rigidity result of [**38**], where a power of $d(\nabla v, SO(n))$ is replaced by $g_p(d(\nabla v, SO(n)))$.

Lemma 3.8 (Geometric rigidity). *Let g_p be the function defined in (3.4). There exists a constant $C = C(\Omega, p) > 0$ with the following property: for every $v \in W^{1,p}$ there exists a constant rotation $R \in SO(n)$ satisfying*

$$\int_\Omega g_p(|\nabla v - R|)dx \leq C \int_\Omega g_p(d(\nabla v, SO(n)))dx. \tag{3.24}$$

Similar versions of Lemma 3.8 can be found in [**18**], [**51**], and in [**59**]. For sake of completeness, we give the proof in Section 3.5.

We need two more lemmas in order to prove Theorem 3.3.

LEMMA 3.9. *Let $S \subseteq \mathbb{R}^n$ be a bounded $\mathscr{H}^m$–measurable set with $0 < \mathscr{H}^m(S) < \infty$ for some $m > 0$. Then*

$$|F|_S := \min_{\zeta \in \mathbb{R}^n} \int_S |Fx - \zeta| d\mathscr{H}^m$$

is a seminorm on $\mathbb{M}^{n\times n}$. Define

$$S_0 := \{x \in S : \mathscr{H}^m(S \cap B_\rho(x)) > 0 \text{ for every } \rho > 0\},$$

and let $\mathrm{aff}(S_0)$ be the smallest affine space containing S_0. Let $K \subseteq \mathbb{M}^{n\times n}$ be a closed cone such that

$$\dim(\mathrm{Ker}(F)) < \dim(\mathrm{aff}(S_0)), \qquad \text{for every } F \in K \setminus \{0\}. \tag{3.25}$$

Then, there exists a constant $C = C(S) > 0$ such that

$$C|F| \leq |F|_S, \qquad \text{for every } F \in K.$$

PROOF. It is not difficult to check that the minimum which defines $|\cdot|_S$ exists and that $|\cdot|_S$ is a seminorm. The following argument is an adaptation of the the proof of [**26**, Lemma 3.3] to the L^1 norm. Suppose, by contradication, that for every integer k there exists $F_k \in K \setminus \{0\}$ such that

$$\frac{|F_k|}{k} > \min_{\zeta \in \mathbb{R}^n} \int_S |F_k x - \zeta| d\mathscr{H}^m. \tag{3.26}$$

Let $\{\zeta_k\} \subseteq \mathbb{R}^n$ be such that

$$\min_{\zeta \in \mathbb{R}^n} \int_S |F_k x - \zeta| d\mathscr{H}^m = \int_S |F_k x - \zeta_k| d\mathscr{H}^m, \qquad \text{for every } k,$$

and observe that

$$\min_{\zeta \in \mathbb{R}^n} \int_S |F_k x - \zeta| d\mathscr{H}^m = |F_k| \min_{\zeta \in \mathbb{R}^n} \int_S \left| \frac{F_k}{|F_k|} x - \zeta \right| d\mathscr{H}^m,$$

so that in (3.26) we can suppose $|F_k| = 1$ for every k and then write

$$\frac{1}{k} > \int_S |F_k x - \zeta_k| d\mathscr{H}^m, \qquad \text{for every } k. \tag{3.27}$$

The fact that K is closed and $|F_k| = 1$ for every k imply that, up to a subsequence,

$$F_k \to F \in K, \qquad \text{with } |F| = 1. \tag{3.28}$$

(3.27), together with the boundedness of $\{F_k\}$ and of S, implies that $\{\zeta_k\}$ is bounded. Therefore, up to a further subsequence, we can suppose that

$$\zeta_k \to \zeta \in \mathbb{R}^n. \tag{3.29}$$

(3.27), (3.28) and (3.29) imply, in the limit $k \to \infty$, that

$$\int_S |Fx - \zeta| d\mathscr{H}^m = 0, \quad \text{for some } \zeta \in \mathbb{R}^n \quad \text{and} \quad F \in K \setminus \{0\}.$$

From the last equality we deduce that $Fx = \zeta$ for $\mathscr{H}^m$–a.e. $x \in S$ and, in turn, by the continuity of F, for every $x \in S_0$. Finally, the linearity of F implies that $Fx = \zeta$ for every $x \in \mathrm{aff}(S_0)$, so that

$$\dim(\mathrm{Ker}(F)) \geq \dim(\mathrm{aff}(S_0)),$$

against (3.25). □

We will use the next lemma also in the proof of the Γ–convergence result.

LEMMA 3.10. *Let $\varepsilon > 0$ and $u_\varepsilon \in W^{1,p}_h$. Under the hypotheses of Theorem 3.3, let $R_\varepsilon \in SO(n)$ be a constant rotation satisfying (3.24) with $v = x + \varepsilon u_\varepsilon$. Then,*

$$|I - R_\varepsilon|^2 \leq C\varepsilon^2 \left[\mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathscr{H}^{n-1} \right)^2 \right],$$

where C depends only on Ω, $\partial_D\Omega$, and p.

PROOF. Consider the deformation $v_\varepsilon := x + \varepsilon u_\varepsilon$. Lemma 3.8 tells us that there exists a constant rotation $R_\varepsilon \in SO(n)$ such that

$$\int_\Omega g_p(|\nabla v_\varepsilon - R_\varepsilon|)dx \le C \int_\Omega g_p(d(\nabla v_\varepsilon, SO(n)))dx,$$

where C depends only on Ω and p. Then, by assumption (iv) on W, we have that

$$\int_\Omega g_p(|\nabla v_\varepsilon - R_\varepsilon|)dx \le C \int_\Omega W(\nabla v_\varepsilon)dx = C\varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon).$$

Jensen inequality thus implies

$$g_p\left(\frac{1}{|\Omega|}\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx\right) \le C\varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon). \tag{3.30}$$

Poincaré–Wirtinger inequality and the continuity of the trace operator give

$$\int_{\partial_D\Omega} |v_\varepsilon - R_\varepsilon x - \zeta_\varepsilon| d\mathscr{H}^{n-1} \le C\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx,$$

where $\zeta_\varepsilon := \frac{1}{|\Omega|}\int_\Omega (v_\varepsilon - R_\varepsilon x)dx$ and C depends on Ω, so that, since $v_\varepsilon = x + \varepsilon h$ $\mathscr{H}^{n-1}$-a.e. on $\partial_D\Omega$, we obtain

$$\int_{\partial_D\Omega} |(I - R_\varepsilon)x - \zeta_\varepsilon| d\mathscr{H}^{n-1} \le C\left(\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx + \varepsilon \int_{\partial_D\Omega} |h| d\mathscr{H}^{n-1}\right). \tag{3.31}$$

Now, let us use Lemma 3.9 with $S = \partial_D\Omega$ and with K equal to the closed cone generated by $I - SO(n)$. Showing first that every $F \in K$ belongs to the cone generated by $I - SO(n)$ or to $Skw(n)$, it is easy to prove that every $F \in K \setminus \{0\}$ is such that

$$\dim(\mathrm{Ker}(F)) < n - 1.$$

On the other hand, $\partial\Omega$ Lipschitz implies that the right-hand side of (3.25) is equal to $n-1$. Thus, we can apply Lemma 3.9 to $(I - R_\varepsilon) \in K$ and write that

$$C|I - R_\varepsilon| \le \min_{\zeta\in\mathbb{R}^n} \int_{\partial_D\Omega} |(I - R_\varepsilon)x - \zeta| d\mathscr{H}^{n-1}, \tag{3.32}$$

where C depends on $\partial_D\Omega$ and not on ε. From (3.31) and (3.32) we obtain that

$$|I - R_\varepsilon|^2 \le C\left[\left(\frac{1}{|\Omega|}\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx\right)^2 + \varepsilon^2\left(\int_{\partial_D\Omega} |h| d\mathscr{H}^{n-1}\right)^2\right]. \tag{3.33}$$

We conclude the proof by distinguishing two cases. If $\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx \le |\Omega|$, then (3.30) and the definition of g_p tell us that

$$\frac{1}{2}\left(\frac{1}{|\Omega|}\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx\right)^2 \le C\varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon).$$

Using this last inequality in (3.33), it turns out (3.14). If $\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|dx > |\Omega|$, again (3.30) and the definition of g_p tell us that

$$C\varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon) > \frac{1}{2}.$$

This bound from below of $\varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon)$ gives trivially (3.14), in view of the fact that $|I - R_\varepsilon| \le 2\sqrt{n}$. □

For the proof of Theorem 3.3 we will need the following estimate.

$$g_p(s + t) \le C[g_p(s) + t^2], \quad \text{for every } s, t \ge 0, \tag{3.34}$$

for a certain C depending on p. This estimate can be easily deduced from the convexity of g_p and from the growth properties of g_p which give

$$g_p(t) \le \frac{1}{p}\min\{t^p, t^2\} \quad \text{and} \quad g_p(2t) \le C g_p(t), \quad \text{for every } t \ge 0,$$

for some C depending on p.

Proof of Theorem 3.3. Let R_ε be given by Lemma 3.8 for $v_\varepsilon := x + \varepsilon u_\varepsilon$, for every $\varepsilon > 0$. By using (3.34), we have that

$$\begin{aligned}\int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)dx &\le C\int_\Omega \left[g_p(|\nabla v_\varepsilon - R_\varepsilon|) + |I - R_\varepsilon|^2\right]dx\\ &\le C\left[\int_\Omega g_p(d(\nabla v_\varepsilon, SO(n)))dx + |I - R_\varepsilon|^2\right],\end{aligned}$$

where in the last inequality we have used Lemma 3.8. Assumption (iv) on W and Lemma 3.10 then imply that for some C, depending on Ω, $\partial_D\Omega$, and p,

$$\int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)dx \le C\varepsilon^2\left[\mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right]. \tag{3.35}$$

In particular, from (3.35) and from the definition of g_p we obtain

$$\begin{aligned}\int\limits_{\{x\in\Omega\,:\,|\varepsilon\nabla u_\varepsilon(x)|\le 1\}}|\varepsilon\nabla u_\varepsilon|^2dx &\le 2\int_\Omega g_p(|\varepsilon\nabla u_\varepsilon|)dx\\ &\le C\varepsilon^2\left[\mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right],\end{aligned}$$

so that, by Hölder inequality, it turns out

$$\begin{aligned}\int\limits_{\{x\in\Omega\,:\,|\varepsilon\nabla u_\varepsilon(x)|\le 1\}}|\varepsilon\nabla u_\varepsilon|^pdx &\le \left(\int\limits_{\{x\in\Omega\,:\,|\varepsilon\nabla u_\varepsilon(x)|\le 1\}}|\varepsilon\nabla u_\varepsilon|^2dx\right)^{p/2}|\Omega|^{1-(p/2)}\\ &\le C\varepsilon^p\left[\mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right]^{p/2}\\ &\le C\varepsilon^p\left[1 + \mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right].\end{aligned} \tag{3.36}$$

Note that in (3.36) we have used the fact that

$$t^{p/2} \le 1 + t, \quad \text{for every } t \ge 0.$$

On the other hand, from (3.76) and again from (3.35) we obtain that

$$\begin{aligned}\int_{\{x\in\Omega:|\varepsilon\nabla u_\varepsilon(x)|>1\}}|\varepsilon\nabla u_\varepsilon|^pdx &\le C\int_{\{x\in\Omega:|\varepsilon\nabla u_\varepsilon(x)|>1\}}g_p(|\varepsilon\nabla u_\varepsilon|)dx\\ &\le C\varepsilon^2\left[\mathscr{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D\Omega}|h|d\mathscr{H}^{n-1}\right)^2\right].\end{aligned} \tag{3.37}$$

Inequalities (3.36) and (3.37) imply that (3.14) holds. □

In the next remark we construct a counterexample which shows that Theorem 3.3 is not true in general for $p \in (0, 1)$.

Remark 3.11. Let $p \in (0, 1)$ and consider the simple case in which Ω is the open unitary ball $B(0, 1)$ in $\mathbb{R}^2$, $W(F) := g_p(d(F, SO(2)))$ for every $F \in Lin^+(2)$, $h = 0$, and $\mathscr{L} = 0$. For any $\varepsilon > 0$ and some $\alpha > 0$ to be chosen, we introduce the set

$$S_\varepsilon := \left\{x \in \mathbb{R}^2 : \tfrac{1}{2} < |x| < \tfrac{1}{2} + \varepsilon^\alpha\right\}.$$

For every $\varepsilon > 0$ sufficiently small, S_ε is an open annulus strictly included in Ω. We want to define a sequence $\{u_\varepsilon\} \subseteq W_0^{1,p}$ such that the values $\mathscr{F}_\varepsilon(u_\varepsilon)$ are equibounded and $\int_\Omega |\nabla u_\varepsilon|^p dx \to \infty$ as

$\varepsilon \to 0^+$. In order to do this, we consider for every $\varepsilon > 0$ arbitrarily small a function $\varphi_\varepsilon \in C_c^\infty(\Omega, \mathbb{R})$ such that $\text{supp}(\varphi_\varepsilon) \subseteq \overline{B\left(0, \frac{1}{2}\right)} \cup S_\varepsilon$, $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon \equiv 1$ on $B\left(0, \frac{1}{2}\right)$ and

$$|\nabla\varphi_\varepsilon| \le \frac{C}{\varepsilon^\alpha} \quad \text{for some } C \text{ independent of } \varepsilon. \tag{3.38}$$

Then, we choose $R \in SO(2) \setminus \{I\}$ and define the function

$$u_\varepsilon(x) := \varphi_\varepsilon(x)\frac{R - I}{\varepsilon}x, \qquad x \in \Omega,$$

which belongs to C^∞ for every $\varepsilon > 0$ sufficiently small. Observe that

$$\int_\Omega |\nabla u_\varepsilon|^p dx \ge \int_{B\left(0,\frac{1}{2}\right)} |\nabla u_\varepsilon|^p dx = \frac{\pi|R - I|^p}{4\varepsilon^p},$$

so that $\int_\Omega |\nabla u_\varepsilon|^p dx \to \infty$ as $\varepsilon \to 0^+$ (for every choice of $\alpha > 0$). Now, let us compute

$$\nabla u_\varepsilon(x) = \frac{1}{\varepsilon}\left\{\varphi_\varepsilon(x)(R - I) + [(R - I)x] \otimes \nabla\varphi_\varepsilon(x)\right\}$$

and observe that $\nabla u_\varepsilon \equiv 0$ on $\Omega \setminus \left[\overline{B\left(0, \frac{1}{2}\right)} \cup S_\varepsilon\right]$, so that $d(I + \varepsilon\nabla u_\varepsilon, SO(2)) \equiv 0$ on the same set. Thus, recalling that g_p is increasing, it turns out that

$$\begin{aligned} \varepsilon^2 \mathscr{F}_\varepsilon(u_\varepsilon) &\le \int_{B\left(0,\frac{1}{2}\right)\cup S_\varepsilon} g_p(|I + \varepsilon\nabla u_\varepsilon - R|)dx \\ &\le \int_{S_\varepsilon} g_p(|R - I|(1 + |x||\nabla\varphi_\varepsilon|))dx, \end{aligned} \tag{3.39}$$

where in the last inequality we have also used the fact that $\varphi_\varepsilon \equiv 1$ on $B\left(0, \frac{1}{2}\right)$. Therefore, from (3.4) and (3.39) we obtain that

$$\mathscr{F}_\varepsilon(u_\varepsilon) \le \frac{C}{\varepsilon^2}\int_{S_\varepsilon} (1 + |\nabla\varphi_\varepsilon|^p)\, dx, \tag{3.40}$$

for some C independent of ε. Using (3.38) and noticing that $|S_\varepsilon| = \pi\varepsilon^\alpha + o(\varepsilon^\alpha)$, (3.40) implies that

$$\mathscr{F}_\varepsilon(u_\varepsilon) \le \frac{C}{\varepsilon^2}[\pi\varepsilon^\alpha + o(\varepsilon^\alpha)]\left(1 + \frac{1}{\varepsilon^{\alpha p}}\right),$$

so that $\{\mathscr{F}_\varepsilon(u_\varepsilon)\}$ turns out to be bounded whenever $\alpha > \frac{2}{1-p}$.

We end this section with the following corollary.

Corollary 3.12. *Under the hypotheses of Theorem 3.3, the functionals $\mathscr{G}_\varepsilon$ are equicoercive in the weak topology of $W^{1,p}$.*

Proof. Let $t \in \mathbb{R}$ and $\{u_\varepsilon\}$ a sequence with $\mathscr{G}_\varepsilon(u_\varepsilon) \le t$, so that $\{u_\varepsilon\} \subseteq W^{1,p}_h$. Thus, by the definition of $\mathscr{G}_\varepsilon$ (3.13), we have

$$\mathscr{F}_\varepsilon(u_\varepsilon) \le t + \mathscr{L}(u_\varepsilon).$$

Theorem 3.3 implies that for every ε sufficiently small

$$\int_\Omega |\nabla u_\varepsilon|^p dx \le C\left[1 + \mathscr{L}(u_\varepsilon) + \left(\int_{\partial_D\Omega} |h| d\mathscr{H}^{n-1}\right)^2\right],$$

for some C independent of ε. By Poincaré inequality, this gives

$$||u_\varepsilon||^p_{W^{1,p}} \le C\left(||u_\varepsilon||_{W^{1,p}} + 1\right), \tag{3.41}$$

where C now depends also on h and $\mathscr{L}$. Therefore, since $p > 1$, from (3.41) we obtain that $||u_\varepsilon||_{W^{1,p}}$ is bounded. □

Observe that the proofs of Theorem 3.3, Lemma 3.10 and Corollary E_1 do not use the fact that $\partial_D\Omega$ has Lipschitz boundary in $\partial\Omega$ (see Definition 3.1): actually, these results hold under the weaker hypothesis $\mathscr{H}^{n-1}(\partial_D\Omega) > 0$.

3.3. Γ–convergence

Consider a sequence $\varepsilon_j \to 0^+$ as $j \to \infty$. By Theorem 3.3, we can characterize the Γ–limit in the weak topology of $W^{1,p}$ in terms of weakly converging sequences (see [**23**, Proposition 8.10]). In particular, we have that

$$\mathscr{F}'(u) := \Gamma\text{–}\liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u) = \inf\{\liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}\};$$
$$\mathscr{F}''(u) := \Gamma\text{–}\limsup_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u) = \inf\{\limsup_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ weakly in } W^{1,p}\}. \tag{3.42}$$

Thus, in order to prove Theorem 3.4, we will show that $\mathscr{F}(u) \geq \mathscr{F}''(u)$ and $\mathscr{F}(u) \leq \liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j)$, for every $u \in W^{1,p}$ and every $u_j \rightharpoonup u$ weakly in $W^{1,p}$.

Proof of Theorem 3.4.

(I) We want to show that $\mathscr{F}(u) \geq \mathscr{F}''(u)$. Consider the nontrivial case $\mathscr{F}(u) < \infty$, so that $u \in W^{1,2}_h$ and

$$\mathscr{F}(u) = \frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2 dx.$$

Suppose first $u \in W^{1,\infty}$. The boundedness of ∇u and assumption (ii) on W, together with the fact that $W(I) = 0$ and $DW(I) = 0$, imply that

$$\lim_{j\to\infty} \frac{1}{\varepsilon_j^2} W(I + \varepsilon_j \nabla u(x)) = \frac{1}{2} D^2W(I)[\nabla u(x)]^2, \quad \text{for a.e. } x \in \Omega,$$

and that there exists $C > 0$ such that for every $\varepsilon_j > 0$ sufficiently small

$$W(I + \varepsilon_j\nabla u) \leq \varepsilon_j^2 C|\nabla u|^2, \quad \text{a.e. in } \Omega.$$

Then, by dominated convergence and by (3.5), we obtain

$$\lim_{j\to\infty} \frac{1}{\varepsilon_j^2}\int_\Omega W(I + \varepsilon_j\nabla u)dx = \frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2 dx.$$

Therefore, by (3.42),

$$\mathscr{F}(u) = \lim_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u) \geq \mathscr{F}''(u). \tag{3.43}$$

Consider now the general case $u \in W^{1,2}_h$. Since $\partial_D\Omega$ has Lipschitz boundary in $\partial\Omega$, from Proposition 3.15 we have that there exists a sequence $\{u_k\} \subseteq W^{1,\infty}_h$ such that $u_k \to u$ strongly in $W^{1,2}$, as $k \to \infty$. Observe that by (3.43) we have $\mathscr{F}''(u_k) \leq \mathscr{F}(u_k)$ for every k. Thus, by the weak lower semicontinuity of $\mathscr{F}''$ in $W^{1,p}$ and the strong continuity of $\mathscr{F}$ in $W^{1,2}_h$, it turns out that

$$\mathscr{F}(u) = \lim_{k\to\infty} \mathscr{F}(u_k) \geq \liminf_{k\to\infty} \mathscr{F}''(u_k) \geq \mathscr{F}''(u).$$

(II) We want to prove that, if $u_j \rightharpoonup u$ weakly in $W^{1,p}$, then $\mathscr{F}(u) \leq \liminf_j \mathscr{F}_{\varepsilon_j}(u_j)$. Consider the nontrivial case $\liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j) < \infty$ so that, up to a subsequence, we can suppose $\{\mathscr{F}_{\varepsilon_j}(u_j)\}$ bounded and, in particular, $\{u_j\} \subseteq W^{1,p}_h$. Let 1_{B_j} be the characteristic function of B_j, where

$$B_j := \left\{x \in \Omega : |\nabla u_j(x)| \leq \frac{1}{\sqrt{\varepsilon_j}}\right\}. \tag{3.44}$$

Claim 1. We have that $\left\{1_{B_j}\nabla u_j\right\}$ is bounded in L^2.

Proof of Claim 1. By Lemma 3.8 and by the growth hypothesis on W we have that for every j there exists $R_j \in SO(n)$ such that

$$\int_\Omega g_p(|I + \varepsilon_j \nabla u_j(x) - R_j|)dx \le \varepsilon_j^2 C \mathscr{F}_{\varepsilon_j}(u_j) \le C\varepsilon_j^2, \tag{3.45}$$

where the last inequality follows from the boundedness of $\{\mathscr{F}_{\varepsilon_j}(u_j)\}$. Considering the set

$$A_j := \left\{x \in \Omega : |I + \varepsilon_j \nabla u_j(x) - R_j| \le 3\sqrt{n}\right\},$$

it is easy to check that $B_j \subseteq A_j$ for every j large enough, so that

$$\int_{B_j} |\nabla u_j|^2 dx \le \frac{2}{\varepsilon_j^2} \int_{A_j} \left(|\varepsilon_j \nabla u_j + I - R_j|^2 + |I - R_j|^2\right) dx. \tag{3.46}$$

Therefore, by using (3.75) and the definition of A_j, from (3.46) we obtain that

$$\begin{aligned} \int_{B_j} |\nabla u_j|^2 dx &\le \frac{C}{\varepsilon_j^2} \int_{A_j} \left[g_p\left(|\varepsilon_j \nabla u_j + I - R_j|\right) + |I - R_j|^2\right] dx \\ &\le C\left(1 + \frac{|I - R_j|^2}{\varepsilon_j^2}\right), \end{aligned} \tag{3.47}$$

where in the last inequality we have used (3.45) and C depends on Ω and p. Since $\{\mathscr{F}(u_j)\}$ is bounded, Lemma 3.10 tells us that $|I - R_j|^2/\varepsilon_j^2$ is bounded. This fact, together with (3.47), gives the claim.

Claim 2. $\nabla u \in L^2$ and, up to a subsequence, we have that

$$1_{B_j} \nabla u_j \rightharpoonup \nabla u \text{ weakly in } L^2.$$

Proof of Claim 2. By Claim 1, we have that, up to a subsequence,

$$1_{B_j} \nabla u_j \rightharpoonup v \text{ weakly in } L^2, \tag{3.48}$$

for some $v \in L^2$. Let us prove that

$$1_{B_j^c} \nabla u_j \to 0 \text{ strongly in } L^\alpha, \tag{3.49}$$

for every $\alpha \in [1, p)$. We first observe that $|B_j^c| \to 0$, by Chebyshev inequality. Taking into account the boundedness of $\{u_j\}$ in $W^{1,p}$, by Hölder inequality we obtain

$$\int_\Omega \left|1_{B_j^c} \nabla u_j\right|^\alpha dx \le \left(\int_\Omega |\nabla u_j|^p dx\right)^{\alpha/p} |B_j^c|^{(p-\alpha)/p} \le C|B_j^c|^{(p-\alpha)/p} \to 0,$$

which proves (3.49).

The weak convergence of u_j to u in $W^{1,p}$ implies also that $\nabla u_j \rightharpoonup \nabla u$ weakly in L^α, for every $\alpha \in [1, p)$. This fact, together with (3.49), gives that

$$1_{B_j} \nabla u_j = \left(\nabla u_j - 1_{B_j^c} \nabla u_j\right) \rightharpoonup \nabla u \quad \text{weakly in } L^\alpha, \tag{3.50}$$

for every $\alpha \in [1, p)$. By (3.48) and (3.50) we conclude that $\nabla u = v \in L^2$ and Claim 2 follows.

From assumptions (ii) and (iii) on W it is easy to show that

$$W(I + F) \ge \frac{1}{2} D^2 W(I)[F]^2 - \eta(|F|)|F|^2, \quad \text{for every } F \in \mathbb{M}^{n\times n},$$

where η is an increasing function on $[0, \infty)$ such that $\eta(t) \to 0$ as $t \to 0^+$. Therefore, we can write

$$\begin{aligned} \mathscr{F}_{\varepsilon_j}(u_j) &\ge \int_{B_j} \left\{\frac{1}{2} D^2 W(I)[e(u_j)]^2 - \eta(\varepsilon_j |\nabla u_j|)|\nabla u_j|^2\right\} dx \\ &\ge \int_\Omega \left\{\frac{1}{2} D^2 W(I)[1_{B_j} e(u_j)]^2 - \eta(\sqrt{\varepsilon_j}) 1_{B_j} |\nabla u_j|^2\right\} dx, \end{aligned} \tag{3.51}$$

where in the last inequality we have used the definition of B_j and the monotonicity of η. Thus, from (3.51) we obtain that

$$\begin{aligned}
&\liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j) \\
&\quad\geq \frac{1}{2}\liminf_{j\to\infty}\int_\Omega D^2W(I)[1_{B_j}e(u_j)]^2dx - \lim_{j\to\infty}\eta(\sqrt{\varepsilon_j})\int_\Omega 1_{B_j}|\nabla u_j|^2dx \\
&\quad= \frac{1}{2}\liminf_{j\to\infty}\int_\Omega D^2W(I)[1_{B_j}e(u_j)]^2dx \qquad (3.52)\\
&\quad\geq \frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2dx, \qquad (3.53)
\end{aligned}$$

where (3.52) follows from Claim 1 and from the convergence of $\eta(\sqrt{\varepsilon_j})$ to 0, while (3.53) is deduced from Claim 2 and from the lower semicontinuity of

$$w \mapsto \frac{1}{2}\int_\Omega D^2W(I)[w]^2$$

in the weak topology of L^2, which is a consequence of (3.5) and (3.6). In order to conclude the proof, it remains to show that $u \in W^{1,2}_h$, so that from (3.53) we have $\liminf_{j\to\infty}\mathscr{F}_{\varepsilon_j}(u_j) \geq \mathscr{F}(u)$. We already know, from Claim 2, that $\nabla u \in L^2$. Since u is at least in L^1, it is easy to show, by using Sobolev embeddings, that $u \in L^2$. Therefore, $u \in W^{1,2}$. Since $u_j \rightharpoonup u$ weakly in $W^{1,p}$ and $\{u_j\} \subseteq W^{1,p}_h$, we have $u \in W^{1,p}_h$. Thus, $u \in W^{1,p}_h \cap W^{1,2} = W^{1,2}_h$. □

Remark 3.13. In the case $p = 2$, one can prove a slightly different version of Theorems 3.2 and 3.4, assuming only that $\partial_D\Omega$ is a subset of $\partial\Omega$ with $\mathscr{H}^{n-1}(\partial_D\Omega) > 0$, as in [**26**]. In this case, in the definitions of the functionals (3.11)–(3.13) the space $W^{1,2}_h$ has to be replaced by the closure of $W^{1,\infty}_h$ in $W^{1,2}$.

3.4. Convergence of minimizers

Recall that a family $\mathscr{F} := \{f\} \subseteq L^1(\Omega)$ is *equiintegrable* if for every $\eta > 0$ there exists $M_\eta > 0$ such that

$$\int_{\{x\in\Omega\,:\,|f(x)|>M_\eta\}} |f|dx < \eta, \qquad \text{for every } f \in \mathscr{F}. \tag{3.54}$$

Equivalently, $\mathscr{F}$ is equiintegrable if for every $\eta > 0$ there exists $\delta_\eta > 0$ such that, if $A \subseteq \Omega$ and $|A| < \delta_\eta$, then

$$\int_A |f|dx < \eta, \qquad \text{for every } f \in \mathscr{F}. \tag{3.55}$$

The following criterion of equiintegrability will be useful.

Lemma 3.14. *The family $\mathscr{F} := \{f\} \subseteq L^1$ is equiintegrable if and only if for every $\eta > 0$ there exists $M_\eta > 0$ and $p \in (1, \infty]$ such that any $f \in \mathscr{F}$ can be written as*

$$f = g + h, \quad \textit{with} \quad ||g||_{L^1} < \eta \quad \textit{and} \quad ||h||_{L^p} < M_\eta. \tag{3.56}$$

Proof. Suppose $\mathscr{F}$ equiintegrable, so that, for every $\eta > 0$, there exists $M_\eta > 0$ such that (3.54) holds. By setting

$$g := f1_{\{|f|>M_\eta\}} \quad \text{and} \quad h := f1_{\{|f|\leq M_\eta\}},$$

we have that $f = g + h$ and

$$||g||_{L^1} = \int_{\{|f|>M_\eta\}} |f|dx < \eta, \qquad ||h||^p_{L^p} \leq |\Omega|M^p_\eta.$$

Conversely, assume (3.56). We want to prove that, for every $\eta > 0$, there exists $\delta_\eta > 0$ such that (3.55) holds, whenever $|A| < \delta_\eta$. By hypothesis, for every $f \in \mathscr{F}$ there exist g, h, and $p \in (1, \infty]$ such that (3.56) holds with $\frac{\eta}{2}$ in place of η. Thus, by using Hölder inequality, we have that

$$\int_A |f| dx \leq \int_A |g| dx + \int_A |h| dx < \frac{\eta}{2} + M_{\eta/2} |A|^{(p-1)/p},$$

so that, by imposing $\delta_\eta := \left(\frac{\eta}{2M_{\eta/2}}\right)^{p/(p-1)}$, we can conclude. □

In the next proof, we will make use of *Vitali's Convergence Theorem*: if $\{f_j\}$ is a sequence of equiintegrable functions on Ω which converges pointwise to a function f, then

$$f \in L^1 \quad \text{and} \quad f_j \to f \text{ in } L^1.$$

Proof of Theorem 3.5. Let $\{u_j\}$ be a recovery sequence for $u \in W^{1,2}_h$. In order to prove that $\{u_j\}$ converges to u strongly in $W^{1,p}$, we show that

$$\begin{aligned} &\text{(i)} \quad e(u_j)1_{B_j} \to e(u) \text{ strongly in } L^2, \\ &\text{(ii)} \quad \left\{ \frac{d^p(I + \varepsilon_j \nabla u_j, SO(n))}{\varepsilon_j^p} \right\} \text{ is equiintegrable,} \\ &\text{(iii)} \quad \{|\nabla u_j|^p\} \text{ is equiintegrable,} \end{aligned}$$

where B_j is the set defined in (3.44). Once (i) and (iii) are proved ((ii) is an intermediate step to prove (iii)), we can conclude as follows. From (i) we have that, up to a subsequence,

$$e(u_j)1_{B_j} \to e(u) \quad \text{a.e. in } \Omega. \tag{3.57}$$

Moreover, $e(u_j)1_{B_j^c} \to 0$ strongly in L^1 by Hölder inequality:

$$\int_{B_j^c} |e(u_j)| dx \leq ||e(u_j)||_{L^p} |B_j^c|^{(p-1)/p} \to 0, \tag{3.58}$$

where we have used the boundedness of $\{u_j\}$, which implies $|B_j^c| \to 0$ by Chebyshev inequality. Thus, by (3.57) and (3.58), we have that, up to a further subsequence,

$$e(u_j) = e(u_j)1_{B_j} + e(u_j)1_{B_j^c} \to e(u) \quad \text{a.e. in } \Omega. \tag{3.59}$$

Let us apply Vitali's Convergence Theorem to the functions $f_j := |e(u_j) - e(u)|^p$ and $f = 0$. Since $f_j \to f$ a.e. in Ω by (3.59) and $\{f_j\}$ is equiintegrable by (iii), we obtain that

$$e(u_j) \to e(u) \quad \text{in} \quad L^p.$$

Observe that, by the hypothesis $\mathscr{F}_{\varepsilon_j}(u_j) \to \mathscr{F}(u) < \infty$, $u_j = h$ on $\partial_D \Omega$ for every j, thus it is sufficient to apply Korn's inequality 1.14 to deduce that $u_j \to u$ strongly in $W^{1,p}$.

We now prove (i)–(iii). Let us set, for every j,

$$v_j := x + \varepsilon_j u_j, \quad \text{for a.e. } x \in \Omega.$$

Proof of (i). As shown in the proof of Theorem 3.4, the boundedness of $\{\mathscr{F}(u_j)\}$ for every j sufficiently large implies that, up to a subsequence, the sequence $\{1_{B_j} \nabla u_j\}$ converges to ∇u weakly in L^2, and

$$\lim_{j\to\infty} \mathscr{F}_{\varepsilon_j}(u_j) \geq \limsup_{j\to\infty} \frac{1}{\varepsilon_j^2} \int_{B_j} W(\nabla v_j) dx \geq \limsup_{j\to\infty} \int_\Omega \frac{1}{2} D^2 W(I)[e(u_j)1_{B_j}]^2 dx,$$

$$\liminf_{j\to\infty} \frac{1}{\varepsilon_j^2} \int_{B_j} W(\nabla v_j) dx \geq \liminf_{j\to\infty} \int_\Omega \frac{1}{2} D^2 W(I)[e(u_j)1_{B_j}]^2 dx \geq \mathscr{F}(u).$$

Since $\mathscr{F}_{\varepsilon_j}(u_j) \to \mathscr{F}(u)$, it turns out that

$$\frac{1}{\varepsilon_j^2}\int_{B_j} W(\nabla v_j)dx \to \frac{1}{2}\int_\Omega D^2W(I)[e(u)]^2dx, \tag{3.60}$$

$$\int_\Omega D^2W(I)[e(u_j)1_{B_j}]^2dx \to \int_\Omega D^2W(I)[e(u)]^2dx.$$

The latter, together with the positive definiteness of $D^2W(I)$ on symmetric matrices and the weak convergence of $\left\{1_{B_j}e(u_j)\right\}$ to $e(u)$ in L^2, proves (i).

Proof of (ii). Let us write

$$\frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n)) = \frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n))\left(1_{B_j} + 1_{B_j^c}\right), \tag{3.61}$$

and prove that both terms of the sum in (3.61) are equiintegrable. Observe that

$$\begin{aligned} d(\nabla v_j, SO(n)) &\le d(\nabla v_j, I + \varepsilon_j skw(\nabla u_j)) + d(I + \varepsilon_j skw(\nabla u_j), SO(n)) \\ &= \varepsilon_j|e(u_j)| + d(I + \varepsilon_j skw(\nabla u_j), SO(n)). \end{aligned} \tag{3.62}$$

Since $\varepsilon_j skw(\nabla u_j)$ is an element of the tangent space to the C^∞ manifold $SO(n)$ at I, we have that

$$d(I + \varepsilon_j skw(\nabla u_j), SO(n)) \le C\varepsilon_j^2|skw(\nabla u_j)|^2 \le C\varepsilon_j^2|\nabla u_j|^2, \tag{3.63}$$

for every ε_j small enough. Inequalities (3.62) and (3.63) imply that

$$\frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n)) \le 2^p\left\{|e(u_j)|^p + C\varepsilon_j^p|\nabla u_j|^{2p}\right\}. \tag{3.64}$$

Now, by using the definition of B_j and writing

$$|\nabla u_j|^{2p}1_{B_j} = |\nabla u_j|^p|\nabla u_j|^p 1_{B_j} \le \frac{1}{\varepsilon_j^{p/2}}|\nabla u_j|^p 1_{B_j},$$

from (3.64) we obtain that

$$\frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n))1_{B_j} \le 2^p\left\{|e(u_j)1_{B_j}|^p + C\varepsilon_j^{p/2}|\nabla u_j 1_{B_j}|^p\right\}.$$

This last inequality gives that

$$\frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n))1_{B_j} \quad \text{is equiintegrable,}$$

in view of (i) and of the fact that $\{\nabla u_j 1_{B_j}\}$ converges weakly in L^2. It remains to prove that $\left\{\frac{1}{\varepsilon_j^p}d^p(\nabla v_j, SO(n))1_{B_j^c}\right\}$ is equiintegrable. Indeed, it turns out that

$$\frac{1}{\varepsilon_j^p}\int_{B_j^c} d^p(\nabla v_j, SO(n))dx \to 0. \tag{3.65}$$

In order to see this, we use the fact that

$$\frac{1}{\varepsilon_j^2}\int_{B_j^c} W(\nabla v_j)dx \to 0, \tag{3.66}$$

which descends from (3.60) and from the convergence of $\{\mathscr{F}_{\varepsilon_j}(u_j)\}$ to $\mathscr{F}(u)$. By the growth hypothesis on W and by the inequality $t^p \le t^2 + 1$, for $t \ge 0$, it is easy to show that

$$\frac{1}{\varepsilon^p}d^p(I + \varepsilon F, SO(n)) \le \frac{2}{\varepsilon^2}W(I + \varepsilon F) + 1, \quad \text{for every } F \in \mathbb{M}^{n\times n} \text{ and } \varepsilon \in (0,1),$$

so that

$$\frac{1}{\varepsilon_j^p}\int_{B_j^c} d^p(\nabla v_j, SO(n))dx \le \frac{2}{\varepsilon_j^2}\int_{B_j^c} W(\nabla v_j)dx + |B_j^c|.$$

This last inequality, together with (3.66) and the fact that $|B_j^c| \to 0$, implies (3.65).

Proof of (iii). For every $M > 0$ and every j, we set

$$E^j_M := \{x \in \Omega : d^p(\nabla v_j(x), SO(n)) \geq \varepsilon_j^p M\}.$$

Let us fix $q > p$. By using (ii), it is easy to show that for every $\eta > 0$ there exists $M_\eta > 0$ with the following property. If

$$f_1^j := d(\nabla v_j, SO(n))1_{E^j_{M_\eta}} \quad \text{and} \quad f_2^j := d(\nabla v_j, SO(n))1_{\left(E^j_{M_\eta}\right)^c},$$

then $f_1^j \in L^p$, $f_2^j \in L^q$, $d(\nabla v_j, SO(n)) = f_1^j + f_2^j$, and

$$||f_1^j||^p_{L^p} < \eta\varepsilon_j^p, \qquad ||f_2^j||^q_{L^q} \leq |\Omega| M_\eta^{q/p}\varepsilon_j^q. \tag{3.67}$$

Applying Theorem 1.17 due to Conti, Dolzmann, and Müller, it turns out that for every j there exists $R_j \in SO(n)$ such that $\nabla v_j = R_j + g_1^j + g_2^j$ a.e. in Ω, with

$$||g_1^j||_{L^p} \leq C||f_1^j||_{L^p}, \quad ||g_2^j||_{L^q} \leq C||f_2^j||_{L^q}. \tag{3.68}$$

In particular,

$$\frac{1}{\varepsilon_j^p}|\nabla v_j - R_j|^p \leq \left(\frac{2}{\varepsilon_j}\right)^p \left(|g_1^j|^p + |g_2^j|^p\right) \tag{3.69}$$

and, due to (3.67) and (3.68),

$$\frac{1}{\varepsilon_j^p}\int_\Omega |g_1^j|^p dx < C\eta, \quad \frac{1}{\varepsilon_j^p}\left(\int_\Omega |g_2^j|^{p\alpha} dx\right)^{1/\alpha} < CM_\eta, \tag{3.70}$$

for $\alpha = \frac{q}{p} > 1$. Therefore, by considering (3.69) and (3.70), and using Lemma 3.14, we have that

$$\left\{\frac{|\nabla v_j - R_j|^p}{\varepsilon_j^p}\right\} \quad \text{is equiintegrable.} \tag{3.71}$$

Recalling that $v_j = x + \varepsilon_j h$ $\mathscr{H}^{n-1}$–a.e. on $\partial_D\Omega$, it turns out that

$$|I - R_j| \leq C\left(\int_\Omega |\nabla v_j - R_j| dx + \varepsilon_j \int_{\partial_D\Omega} |h| d\mathscr{H}^{n-1}\right), \tag{3.72}$$

where C depends on Ω and $\partial_D\Omega$. This can be shown as done in the proof of Lemma 3.10 by using Poincaré–Wirtinger inequality and Lemma 3.9. From (3.71) follows in particular that $\left\{\frac{|\nabla v_j - R_j|^p}{\varepsilon_j^p}\right\}$ is bounded in L^1 so that, by (3.72), we obtain that

$$\left\{\frac{|I - R_j|}{\varepsilon_j}\right\} \quad \text{is bounded.} \tag{3.73}$$

Finally, observe that for every measurable subset A of Ω

$$\int_A |\nabla u_j|^p dx \leq \frac{2^p}{\varepsilon_j^p}\left\{\int_A |\nabla v_j - R_j|^p dx + |A||I - R_j|^p\right\},$$

for every j. This inequality, together with (3.71) and (3.73), gives (iii). □

Proof of Theorem 3.2. Consider a sequence $\varepsilon_j \to 0$. By using the notation introduced in (3.11)–(3.13), the infima m_{ε_j} and m (see (3.8) and (3.10)) can be rewritten as

$$m_{\varepsilon_j} = \inf_{W^{1,p}} \mathscr{G}_{\varepsilon_j}, \qquad m = \min_{W^{1,p}} \mathscr{G}.$$

It is easy to show that $\mathscr{G}$ has a unique minimizer $u \in W^{1,2}_h$ on $W^{1,p}$. By standard properties of Γ–convergence (see [**23**, Theorem 7.8]), Theorem 3.4 and Corollary E_1 imply that

$$m_{\varepsilon_j} \to m = \mathscr{G}(u)$$

and in turn, by (3.9), that

$$\mathscr{G}_{\varepsilon_j}(u_{\varepsilon_j}) \to \mathscr{G}(u) < \infty, \tag{3.74}$$

when $\{u_{\varepsilon_j}\}$ is a sequence of "almost minimizers". Again by standard arguments, (3.74) and Corollary E_1 imply that

$$u_{\varepsilon_j} \rightharpoonup u \text{ weakly in } W^{1,p} \quad \text{and} \quad \mathscr{F}_{\varepsilon_j}(u_{\varepsilon_j}) \to \mathscr{F}(u).$$

This last result and Theorem 3.5 give that $\{u_{\varepsilon_j}\}$ converges to u strongly in $W^{1,p}$. Since this is true for every $\varepsilon_j \to 0$, the whole sequence $\{u_\varepsilon\}$ converges to u strongly in $W^{1,p}$ (and $m_\varepsilon \to m$). □

3.5. Appendix: the Geometric Rigidity Lemma with two exponents and other tools

We collect here some estimates involving the function g_p, which describes the growth from below of our energy density. We use them mainly in the proof of Lemma 3.8.

For every $K > 0$, there exists C depending on p and K such that

$$t^2 \le C g_p(t), \quad \text{for every } 0 \le t \le K, \tag{3.75}$$

$$t^p \le C g_p(t), \quad \text{for every } t \ge K. \tag{3.76}$$

Moreover, since $g_p(t) \le \frac{1}{2}\min\{t^p, t^2\}$ for every $t \ge 0$ and g_p is convex, there exists C depending on p such that

$$g_p(s+t) \le C(s^p + t^2), \quad \text{for every } s, t \ge 0. \tag{3.77}$$

Proof of Lemma 3.8. For $v \in W^{1,p}$, let $V \in W^{1,\infty}$ be given by Proposition 1.16 (with $\lambda > 0$ to be chosen), and $R \in SO(n)$ arbitrary. Since g_p is nondecreasing, by using (3.77) we have

$$\int_\Omega g_p(|\nabla v - R|)dx \le C \int_\Omega \left(|\nabla v - \nabla V|^p + |\nabla V - R|^2\right) dx, \tag{3.78}$$

where C depends on p. Let $S(x) \in SO(n)$ be such that $|\nabla v - S| = d(\nabla v, SO(n))$ a.e. in Ω. Observe that, in the set where

$$|\nabla v - S| \ge \sqrt{n}, \tag{3.79}$$

we have

$$|\nabla v|^p \le 2^p \left(|\nabla v - S|^p + n^{p/2}\right) \le 2^{p+1} d^p(\nabla v, SO(n)). \tag{3.80}$$

It is clear that (3.79) is satisfied if $|\nabla v| \ge 2\sqrt{n}$. Thus, by using (3.80) and Proposition 1.16 (iii) with $\lambda = 2\sqrt{n}$, we have that

$$\begin{aligned}\int_\Omega |\nabla v - \nabla V|^p dx &\le C \int_{\{x\in\Omega: |\nabla v(x)| > 2\sqrt{n}\}} |\nabla v|^p dx \\ &\le C \int_{\{x\in\Omega: |\nabla v(x)| > 2\sqrt{n}\}} d^p(\nabla v(x), SO(n))dx\end{aligned}$$

and in turn, by using (3.76), that

$$\int_\Omega |\nabla v - \nabla V|^p dx \le C \int_\Omega g_p(d(\nabla v(x), SO(n)))dx. \tag{3.81}$$

In the case $p = 2$, the lemma we are proving is already well known (see [**38**]) and we apply it to V: there exist C independent of V and a constant rotation $R \in SO(n)$ such that

$$\int_\Omega |\nabla V - R|^2 dx \le C \int_\Omega d^2(\nabla V, SO(n))dx. \tag{3.82}$$

By rewriting (3.78) for such an $R \in SO(n)$, from (3.81) and (3.82) we obtain

$$\int_\Omega g_p(|\nabla v - R|)dx \le C \int_\Omega \left\{g_p(d(\nabla v, SO(n))) + d^2(\nabla V, SO(n))\right\} dx, \tag{3.83}$$

where C depends on Ω and p. Next, we prove that

$$d^2(\nabla V, SO(n)) \le C\left\{|\nabla V - \nabla v|^p + g_p(d(\nabla v, SO(n)))\right\} \quad \text{a.e. in } \Omega, \tag{3.84}$$

for some C depending on Ω and p. We use again the matrix $S(x) \in SO(n)$ such that $|\nabla v - S| = d(\nabla v, SO(n))$ a.e. in Ω.

(i) In the set where $|\nabla v - S| \leq 1$, the function $|\nabla V - \nabla v|$ is bounded by a constant independent of V:

$$|\nabla V - \nabla v| \leq |\nabla V| + |S| + 1 \leq C,$$

where in the last inequality we have used Proposition 1.16 (i). Thus, since

$$t^2 \leq K^{2-p} t^p, \quad \text{for every} \;\; t \in [0, K] \;\; \text{and} \;\; K \geq 1, \tag{3.85}$$

we have

$$|\nabla V - \nabla v|^2 \leq C |\nabla V - \nabla v|^p$$

and in turn, using the definition of g_p,

$$\begin{aligned} d^2(\nabla V, SO(n)) &\leq |\nabla V - S|^2 \leq 2|\nabla V - \nabla v|^2 + 2|\nabla v - S|^2 \\ &\leq C \left\{ |\nabla V - \nabla v|^p + g_p(d(\nabla v, SO(n))) \right\}, \end{aligned}$$

which gives (3.84).

(ii) In the set where $|\nabla v - S| > 1$, Proposition 1.16 (i) and (3.85) give that

$$\begin{aligned} d^2(\nabla V, SO(n)) &\leq |\nabla V - S|^2 \leq C |\nabla V - S|^p \\ &\leq C \left\{ |\nabla V - \nabla v|^p + d^p(\nabla v, SO(n)) \right\}. \end{aligned}$$

From this inequality and from (3.76) we obtain (3.84).

Inequalities (3.83) and (3.84) imply that

$$\int_\Omega g_p(|\nabla v - R|) dx \leq C \int_\Omega \left\{ g_p(d(\nabla v, SO(n))) + |\nabla V - \nabla v|^p \right\} dx,$$

and in turn, by considering (3.81), gives the thesis. □

We finish by proving an approximation result for functions in $W^{1,p}_h$, which has been useful in the proof of the Γ–convergence results. We write $x \in \mathbb{R}^n$ in the form $x = (x'', x_{n-1}, x_n)$ and refer the reader to Definition 3.1 and to (3.3) for the notation.

PROPOSITION 3.15. *Suppose that $\partial_D \Omega$ has Lipschitz boundary in $\partial\Omega$, according to Definition 3.1, and let $W^{1,p}_h$ be defined in (3.3).*
If $h \in W^{1,\infty}$ and $1 \leq p < \infty$, then $W^{1,p}_h$ is the closure of $W^{1,\infty}_h$ in $W^{1,p}$.

In order to prove Proposition 3.15, we need the following lemma.

LEMMA 3.16. *For $p \in [1, \infty)$, let $u \in W^{1,p}(Q^+)$ be such that* $\operatorname{supp}(u) \subset\subset Q$ *and $u = 0$ $\mathscr{L}^{n-1}$-a.e. on Q^+_0. Then, for every $\varepsilon > 0$ there exists $u_\varepsilon \in C^\infty(Q)$ such that $u_\varepsilon = 0$ on Q^+_0 and*

$$||u_\varepsilon - u||_{W^{1,p}(Q^+)} < \varepsilon. \tag{3.86}$$

PROOF. Let $u \in W^{1,p}(Q^+)$ satisfy the hypotheses of the lemma. Consider the subset $M := (-1,1)^{n-2} \times (0,1) \times (-1,0]$ of Q and define

$$v := \begin{cases} u, & \text{on } Q^+, \\ 0, & \text{on } M. \end{cases}$$

It turns out that $v \in W^{1,p}(Q^+ \cup M)$. Up to extend v to a function in $W^{1,p}(Q)$ and to multiply it by a function $\zeta \in C^\infty_c(Q)$ such that $\zeta \equiv 1$ on $\operatorname{supp}(u)$, we can suppose that $v \in W^{1,p}(Q)$ and that $\operatorname{supp}(v) \subset\subset Q$. Starting from v, we want to construct a sequence $\{v_k\}$ which approximates u in $W^{1,p}(Q^+)$ and is such that $\operatorname{supp}(v_k) \subset\subset Q \setminus M$. To this end, we define for every k

$$v_k(x) := v\left(x'', x_{n-1} + \tfrac{1}{k}, x_n - \tfrac{1}{k}\right), \quad \text{for every } x \in Q_k,$$

where

$$Q_k := (-1,1)^{n-2} \times \left(-1 - \tfrac{1}{k}, 1 - \tfrac{1}{k}\right) \times \left(-1 + \tfrac{1}{k}, 1 + \tfrac{1}{k}\right).$$

Observe that
$$\mathrm{supp}(v_k) \subset\subset Q \setminus M, \quad \text{for every } k \text{ sufficiently large.} \tag{3.87}$$
Moreover, v and v_k are functions in $W^{1,p}(\mathbb{R}^n)$, up to extend them at 0 out of Q and Q_k, respectively. In this case, it is well known that $v_k \to v$ in $W^{1,p}(\mathbb{R}^n)$. In particular, we have obtained that
$$v_k \to u \quad \text{in} \quad W^{1,p}(Q^+).$$
The last step of the proof consists in choosing k_ε such that
$$||v_{k_\varepsilon} - u||_{W^{1,p}(Q^+)} < \frac{\varepsilon}{2} \tag{3.88}$$
and considering a standard family $\{\rho_m\}_m$ of mollifiers. By (3.87), there exists m_ε such that $u_\varepsilon := v_{k_\varepsilon} * \rho_{m_\varepsilon} \in C_c^\infty(Q \setminus M)$ (thus, $u_\varepsilon \equiv 0$ on Q_0^+) and
$$||u_\varepsilon - v_{k_\varepsilon}||_{W^{1,p}(Q)} < \frac{\varepsilon}{2}. \tag{3.89}$$
Inequalities (3.88) and (3.89) give (3.86). □

Proof of Proposition 3.15. By a standard argument based on a partition of unity subordinate to a finite covering of $\overline{\Omega}$ and on local bi–Lipschitz charts, we can use Lemma 3.16 to prove that $\left\{u \in W^{1,p} : u = 0 \ \ \mathscr{H}^{n-1}\text{–a.e. on } \partial_D\Omega\right\}$ is contained in the closure of $\left\{u \in C^\infty(\overline{\Omega}) : u = 0 \text{ on } \partial_D\Omega\right\}$ in $W^{1,p}$. The opposite inclusion is trivial. The result for a general boundary value $h \in W^{1,\infty}$ is obtained by adding h to both sets. □

CHAPTER 4

Ogden–type energies

In this chapter, we describe the results obtained in [**4**]. Exploiting the multiplicative decomposition of the deformation gradient into an elastic and a spontaneous or remanent part, we propose and analyze Ogden–type extensions of the BTW free energy density currently used to model the mechanical behavior of nematic elastomers (see Section 4.1). Geometrically linear versions of the new models are provided and discussed in Section 4.2, while in Section 4.3 we derive the expression of the energies which govern the purely mechanical response. Since these energies are not quasiconvex, in Section 4.4 we exhibit their quasiconvex envelopes and apply them to compute the stiffening response of a specimen tested in plane strain extension (pure shear). This shows that the proposed Ogden–type models provide a suitable framework to study the regime of high imposed stretches.

4.1. Ogden–type expressions for the energy density

We recall from Section 1.1 the classical formula for the energy density stored by a monodomain nematic elastomer

$$\overline{W}_n(\overline{F}) = \frac{\mu}{2}\left[\operatorname{tr}\left(L_{n_r}\overline{F}^T L_n^{-1}\overline{F}\right) - 3\right], \qquad \det \overline{F} = 1, \tag{4.1}$$

due to Bladon, Terentjev and Warner. Here, n_r is an arbitrarily chosen reference orientation and the reference configuration is a natural state (i.e., a stress–free state) of the material corresponding to $n = n_r$. See (1.1) for the definition of L_n.

In Section 1.1 we have rewritten (4.1) choosing as reference configuration a minimum energy configuration associated with the high–temperature isotropic state. Referring to Figure 4.1, we have introduced the affine change of variables q, with $\nabla q := L_{n_r}^{\frac{1}{2}}$, set $y := \overline{y} \circ q$ and $F := \nabla y$, and obtained

$$W_n(F) := \frac{\mu}{2}\left[\operatorname{tr}\left(F^T L_n^{-1} F\right) - 3\right], \qquad \det F = 1.$$

Equivalently, $W_n(F) = \tilde{W}_n(FF^T)$, where

$$\tilde{W}_n(B) := \frac{\mu}{2}\left[\operatorname{tr}\left(BL_n^{-1}\right) - 3\right]. \tag{4.2}$$

Observe that the schematic graph of Figure 4.1 naturally suggests to introduce the matrices

$$F_n^e := L_n^{-\frac{1}{2}} F, \qquad B_n^e := F_n^e (F_n^e)^T = L_n^{-\frac{1}{2}} F F^T L_n^{-\frac{1}{2}} \tag{4.3}$$

arising from the decomposition

$$F = L_n^{\frac{1}{2}} F_n^e$$

of the deformation gradient F into an elastic part F_n^e and a spontaneous part $L_n^{\frac{1}{2}}$. We recall that the matrix $L_n^{\frac{1}{2}}$ describes the stress–free strain of the material corresponding to the current orientation n of the nematic director. Using (4.3), expression (4.2) assumes the classical Neo–Hookean form

$$\tilde{W}_n(B) = \frac{\mu}{2}\left(\operatorname{tr} B_n^e - 3\right). \tag{4.4}$$

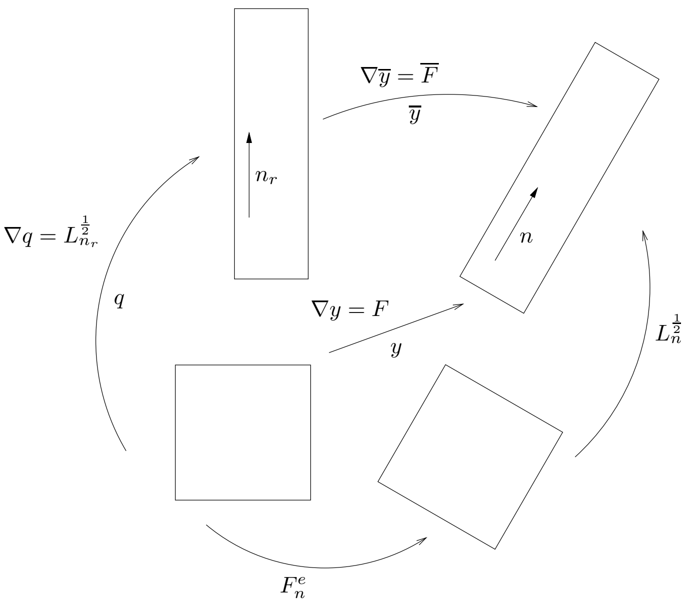

FIGURE 4.1. Schematic diagram illustrating two possible choices of reference configuration (the one for y and the other for $\overline{y}$). and the elastic part F_n^e of the deformation gradient F.

Following Ciarlet [**15**, Chapter 4], we use the same notation W_n, W, and $\tilde{W}_n$ used for the classical expressions of the energies and propose the following natural generalization of (4.4):

$$\tilde{W}_n(B) := \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\operatorname{tr}(B_n^e)^{\frac{\gamma_i}{2}} - 3 \right] + \sum_{j=1}^{M} \frac{d_j}{\delta_j} \left[\operatorname{tr} \operatorname{Cof}(B_n^e)^{\frac{\delta_j}{2}} - 3 \right], \tag{4.5}$$

where¡ c_i, γ_i, d_j and δ_j are constants such that

$$\gamma_i,\ \delta_j \in \mathbb{R} \setminus \{0\}, \qquad \frac{c_i}{\gamma_i}, \frac{d_j}{\delta_j} \geq 0, \qquad i = 1, ..., N, \quad j = 1, ..., M.$$

Then, we set

$$W_n(F) := \tilde{W}_n(FF^T), \qquad \det F = 1. \tag{4.6}$$

We recall that the p–th power A^p of a matrix $A \in Psym(3)$ is well defined by the formula

$$A^p := Q\operatorname{Diag}(\lambda_i^p)Q^T, \qquad p \in \mathbb{R},$$

where $Q \in Orth(3)$ is a matrix which diagonalizes A. Observe that, choosing $N = M = 1$ and $\gamma_1 = \delta_1 = 2$, $\tilde{W}_n$ takes the Mooney–Rivlin form

$$\tilde{W}_n(B) = \frac{c_1}{2} \left(\operatorname{tr} B_n^e - 3 \right) + \frac{d_1}{2} \left(\operatorname{tr} \operatorname{Cof} B_n^e - 3 \right), \qquad \det B = 1, \tag{4.7}$$

and we obtain the Neo–Hookean model (4.4) if $d_1 = 0$ and $c_1 = \mu$. Moreover, if in (4.5) we set $d_j = 0$ for $j = 1, ..., M$ and $\gamma_i \geq 2$ for $i = 1, ..., N$, and take the minimum with respect to $n \in S^2$, we obtain energies in "separable form" of the type discussed by Ogden in [**54**, Chapter 4] (see Section 4.4).

A common practice to pass from an incompressible model, with associated energy density W_{dev} to a corresponding compressible model W^{comp} is to define

$$W^{comp}(F) := W_{dev}((\det F)^{-\frac{1}{3}} F) + W_{vol}(\det F), \qquad F \in Lin^+(3),$$

where W_{vol} is such that

$$W_{vol} \geq 0 \quad \text{and} \quad W_{vol}(t) = 0 \ \text{ if and only if } \ t = 1. \tag{4.8}$$

Here, we choose W_{vol} of the form

$$W_{vol}(t) := c(t^2 - 1) - d \log t.$$

By imposing condition (4.8), we obtain the function

$$W_{vol}(t) = c\left(t^2 - 1 - 2\log t\right), \qquad t > 0. \tag{4.9}$$

This function has also the following properties:

(i) W_{vol} is a convex function;
(ii) $W_{vol}(t) \to +\infty$, as $t \to 0^+$;
(iii) $W_{vol}(t) \to +\infty$, as $t \to +\infty$.

In Section 2.3 we have chosen for $W_{vol}(t)$ the simpler form $\frac{k}{2}(t-1)^2$.

Setting $W_{dev} = W_n$, where W_n is defined by (4.5)–(4.6), we define for $F \in Lin^+(3)$

$$W_n^{comp}(F) := W_n((\det F)^{-\frac{1}{3}} F) + W_{vol}(\det F),$$

so that

$$W_n^{comp}(F) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[(\det F)^{-\frac{\gamma_i}{3}} \operatorname{tr}(L_n^{-\frac{1}{2}} F F^T L_n^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} - 3\right]$$
$$+ \sum_{j=1}^{M} \frac{d_j}{\delta_j}\left[(\det F)^{-\frac{2\delta_j}{3}} \operatorname{tr} \operatorname{Cof}(L_n^{-\frac{1}{2}} F F^T L_n^{-\frac{1}{2}})^{\frac{\delta_j}{2}} - 3\right] + W_{vol}(\det F), \tag{4.10}$$

for every $F \in Lin^+(3)$. We set $W_n^{comp}(F) = +\infty$ for every F such that $\det F \leq 0$. Also in this case it is useful to express the energy density as function of $B = FF^T$ and we have

$$W_n^{comp}(F) = \tilde{W}_n^{comp}(FF^T), \quad \text{for every } F \in Lin^+(3),$$

where

$$\tilde{W}_n^{comp}(B) := \sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[(\det B)^{-\frac{\gamma_i}{6}} \operatorname{tr}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} - 3\right]$$
$$+ \sum_{j=1}^{M} \frac{d_j}{\delta_j}\left[(\det B)^{-\frac{\delta_j}{3}} \operatorname{tr} \operatorname{Cof}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\delta_j}{2}} - 3\right] + W_{vol}(\sqrt{\det B}), \tag{4.11}$$

for every $B \in Psym(3)$.

Proposition 4.1. *W_n^{comp} is a nonnegative function and*

$$W_n^{comp}(F) = 0 \ \text{ if and only if } \ FF^T = L_n.$$

We recall that the condition $FF^T = L_n$, for some $F \in Lin^+(3)$, is equivalent to

$$F = U_n R \quad \text{for some } R \in SO(3), \qquad \text{where } U_n := L_n^{\frac{1}{2}}.$$

Proof. Let $F \in Lin^+(3)$ and let ν_1, ν_2, and ν_3 be the (positive) eigenvalues of $L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}$, where $B = FF^T \in Psym(3)$. Then, by using the standard inequality between geometric and arithmetic mean and the fact that $\det L_n = 1$, for $i = 1, ..., N$ and $j = 1, ..., M$ we have that

$$\begin{aligned}(\det B)^{-\frac{\gamma_i}{6}} \operatorname{tr}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} &= (\det B)^{\frac{-\gamma_j}{6}} \sum_{k=1}^{3} \nu_k^{\frac{\gamma_i}{2}} \geq 3 (\det B)^{\frac{-\gamma_i}{6}} \left(\prod_{k=1}^{3} \nu_k^{\frac{\gamma_i}{2}}\right)^{\frac{1}{3}} \\ &= 3 (\det B)^{\frac{-\gamma_i}{6}} \left[\det\left(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}\right)\right]^{\frac{\gamma_i}{6}} = 3,\end{aligned} \tag{4.12}$$

and

$$(\det B)^{-\frac{\delta_j}{3}} \operatorname{tr}\operatorname{Cof}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\delta_j}{2}} = (\det B)^{\frac{-\delta_j}{3}} \left[(\nu_1\nu_2)^{\frac{\delta_j}{2}} + (\nu_1\nu_3)^{\frac{\delta_j}{2}} + (\nu_2\nu_3)^{\frac{\delta_j}{2}}\right]$$
$$\geq 3(\det B)^{\frac{-\delta_j}{3}} \left[(\nu_1\nu_2)^{\frac{\delta_j}{2}} (\nu_1\nu_3)^{\frac{\delta_j}{2}} (\nu_2\nu_3)^{\frac{\delta_j}{2}}\right]^{\frac{1}{3}} = 3, \quad (4.13)$$

so that, looking at (4.11) and recalling (4.8), W_n^{comp} is nonnegative. The equality holds in (4.12) if and only if $\nu_1 = \nu_2 = \nu_3 = \nu$, that is

$$L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}} = \nu I, \quad \text{for some } \nu > 0, \tag{4.14}$$

and in (4.13) if and only if $\nu_1\nu_2 = \nu_1\nu_3 = \nu_2\nu_3 = \alpha^2$, that is

$$\operatorname{Cof}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}) = \alpha^2 I, \quad \text{for some } \alpha > 0. \tag{4.15}$$

By (4.14) and (4.15) and by property (4.8) of W_{vol}, we obtain that $W_n^{comp}(F) = \tilde{W}_n^{comp}(B) = 0$ if and only if $L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}} = I$. □

4.2. Behavior for small strains

In order to obtain the geometrically linear approximation of the Ogden–type model introduced in the previous section, we consider, as done in Section 2.1 for the classical BTW model, the small strain regime $|\nabla u| = \varepsilon$, where u is the displacement associated with the deformation y through $y(x) = x + u(x)$, and matrices L_n that scale with ε as

$$L_{n,\varepsilon} := (1+\varepsilon)^2 n \otimes n + (1+\varepsilon)^{-1}(I - n \otimes n). \tag{4.16}$$

By expanding (4.16) in ε around 0, we obtain that $L_{n,\varepsilon} = I + \varepsilon \hat{L}_n + o(\varepsilon)$, where $\hat{L}_n$ is defined in (2.4), and $U_{n,\varepsilon} := L_{n,\varepsilon}^{\frac{1}{2}} = I + \varepsilon \hat{U}_n + o(\varepsilon)$, with

$$\hat{U}_n = \frac{1}{2}\hat{L}_n = \frac{3}{2}\left(n \otimes n - \frac{1}{3}I\right). \tag{4.17}$$

Now, we define

$$W_{n,\varepsilon}^{comp}(F) := \tilde{W}_{n,\varepsilon}^{comp}(FF^T), \qquad \text{for every } F \in Lin^+(3),$$

where $\tilde{W}_{n,\varepsilon}^{comp}$ is given by (4.11) with $L_{n,\varepsilon}$ in place of L_n. More explicitly,

$$\tilde{W}_{n,\varepsilon}^{comp}(B) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[(\det B)^{-\frac{\gamma_i}{6}} \operatorname{tr}(L_{n,\varepsilon}^{-\frac{1}{2}} B L_{n,\varepsilon}^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} - 3\right]$$
$$+ \sum_{j=1}^{M} \frac{d_j}{\delta_j} \left[(\det B)^{-\frac{\delta_j}{3}} \operatorname{tr}\operatorname{Cof}(L_{n,\varepsilon}^{-\frac{1}{2}} B L_{n,\varepsilon}^{-\frac{1}{2}})^{\frac{\delta_j}{2}} - 3\right] + W_{vol}(\sqrt{\det B}), \quad (4.18)$$

for every $B \in Psym(3)$.

Proposition 4.2. *In the small strain regime $|\nabla u| = \varepsilon$, we have that, modulo terms of order higher than two in ε,*

$$W_{n,\varepsilon}^{comp}(I + \nabla u) = \mu \left|[e(u)]_d - \varepsilon \hat{U}_n\right|^2 + \frac{k}{2}\operatorname{tr}^2(\nabla u), \tag{4.19}$$

where

$$\mu = \frac{1}{2}\left(\sum_{i=1}^{N} c_i\gamma_i + \sum_{j=1}^{M} d_i\delta_j\right), \qquad k = 4c, \tag{4.20}$$

and c, c_i, γ_i, d_j, and δ_j are the constants appearing in (4.9) and (4.18).

PROOF. In order to obtain (4.19), as in Section 2.1 we define for every $E \in Sym(3)$ the linear limit

$$V(E) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} W^{comp}_{n,\varepsilon}(I+\varepsilon E) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} \tilde{W}^{comp}_{n,\varepsilon}\left((I+\varepsilon E)^2\right).$$

Since 0 is the minimum value attained by $\tilde{W}^{comp}_{n,\varepsilon}$ at $L_{n,\varepsilon}$, see Proposition 4.1, the linear terms of the Taylor expansions vanish and we have

$$V(E) = \frac{1}{2} D^2\tilde{W}^{comp}_{n,0}(I)[2E-\hat{L}_n]^2 = 2D^2\tilde{W}^{comp}_{n,0}(I)[E-\hat{U}_n]^2, \tag{4.21}$$

where the last equality is obtained using (4.17). Note that, for every $B \in Psym(3)$,

$$\tilde{W}^{comp}_{n,0}(B) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[(\det B)^{-\frac{\gamma_i}{6}} \operatorname{tr} B^{\frac{\gamma_i}{2}} - 3\right] + \sum_{j=1}^{M} \frac{d_j}{\delta_j}\left[(\det B)^{-\frac{\delta_j}{3}} \operatorname{tr}\operatorname{Cof} B^{\frac{\delta_j}{2}} - 3\right] + W_{vol}(\sqrt{\det B}).$$

Simple calculations give that, for every symmetric matrix H,

$$D^2\tilde{W}^{comp}_{n,0}(I)[H]^2 = \sum_{i=1}^{N} c_i\gamma_i\left\{-\frac{1}{12}\operatorname{tr}^2 H + \frac{1}{4}|H|^2\right\} + \sum_{j=1}^{M} d_j\delta_j\left\{-\frac{1}{12}\operatorname{tr}^2 H + \frac{1}{4}|H|^2\right\} + c\operatorname{tr}^2 H,$$

so that, from (4.21) and from the fact that $\hat{U}_n$ is traceless, we have

$$V(E) = \frac{1}{2}\left(\sum_{i=1}^{N} c_i\gamma_i + \sum_{j=1}^{M} d_i\delta_j\right)|E-\hat{U}_n|^2 + \left[-\frac{1}{6}\left(\sum_{i=1}^{N} c_i\gamma_i + \sum_{j=1}^{M} d_i\delta_j\right) + 2c\right]\operatorname{tr}^2 E.$$

Writing now $V(E)$ in terms of E_d, since

$$|E-\hat{U}_n|^2 = |E_d-\hat{U}_n|^2 + \frac{1}{3}\operatorname{tr}^2 E,$$

we obtain that

$$V(E) = \frac{1}{2}\left(\sum_{i=1}^{N} c_i\gamma_i + \sum_{j=1}^{M} d_i\delta_j\right)|E_d-\hat{U}_n|^2 + 2c\operatorname{tr}^2 E. \tag{4.22}$$

Finally, arguing as in the proof of Proposition 2.1 and considering ∇u with the proper scale $|\nabla u| = \varepsilon$, we obtain from (4.22)

$$W^{comp}_{n,\varepsilon}(I+\nabla u) = \frac{1}{2}\left(\sum_{i=1}^{N} c_i\gamma_i + \sum_{j=1}^{M} d_i\delta_j\right)\left|[e(u)]_d - \varepsilon\hat{U}_n\right|^2 + 2c\operatorname{tr}^2(\nabla u).$$

□

Energy densities like (4.19) have been used in the study of nematic elastomers in the small strain regime in [**3, 12, 13, 14, 32, 40**]. One reason to derive small strain theories from the fully nonlinear ones is to obtain the expressions for the initial shear and bulk moduli in terms of the constants and exponents of the fully nonlinear models, as done in (4.20). While our main interest here has been to derive the small strain limit of fully nonlinear Ogden–type models, also the opposite path is interesting. In fact, energies of the form (4.19) are quite common in the modeling of active and phase–transforming materials, where geometrically linear theories are often used [**8**]. Our discussion of their relation with parent (fully nonlinear) theories such as

(4.5)–(4.6) provides several templates to generalize these small strain theories to the regime of large deformations.

4.3. Purely mechanical response

In this section we focus on the purely mechanical response of an incompressible material governed by the energy densities W_n introduced in Section 4.1. This means to consider the stored elastic energies obtained, for each fixed F, by minimizing the energy density W_n with respect to n.

Referring to the expressions (4.5)–(4.6), let us restrict the attention to the case $c_i > 0$, $\gamma_i \geq 2$ for $i = 1, ..., N$ and $d_j = 0$ for $j = 1, ..., M$, so that $W_n(F) = \tilde{W}_n(FF^T)$, for every $F \in \mathbb{M}^{3\times3}$ with $\det F = 1$, where

$$\tilde{W}_n(B) := \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\mathrm{tr}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} - 3 \right]. \tag{4.23}$$

In order to minimize (4.23) with respect to $n \in S^2$, we need the following proposition.

PROPOSITION 4.3. *Let $B \in Psym(3)$ and let $0 < \mu_1 \leq \mu_2 \leq \mu_3$ be its ordered eigenvalues. For every $\gamma \geq 2$, we have that*

$$\min_{n\in S^2} \mathrm{tr}\left(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}\right)^{\frac{\gamma}{2}} = a^{\frac{\gamma}{6}} \left[\mu_1^{\frac{\gamma}{2}} + \mu_2^{\frac{\gamma}{2}} + \left(\frac{\mu_3}{a}\right)^{\frac{\gamma}{2}} \right]. \tag{4.24}$$

The minimum is achieved when n is aligned with an eigenvector of B corresponding to μ_3.

We recall that $a > 1$ and that L_n is defined in (1.1). In order to simplify the notation for the proof of Proposition 4.3, let us set

$$\alpha := \frac{\gamma}{2}, \qquad \text{and} \qquad M_n := a^{-\frac{1}{2}} n\otimes n + (I - n\otimes n) = (a^{-\frac{1}{2}} - 1) n\otimes n + I, \tag{4.25}$$

so that

$$L_n^{-\frac{1}{2}} = a^{\frac{1}{6}} M_n. \tag{4.26}$$

By using the positions (4.25)–(4.26), identity (4.24) is equivalent to

$$\min_{n\in S^2} \mathrm{tr}(M_n B M_n)^{\alpha} = \mu_1^{\alpha} + \mu_2^{\alpha} + \left(\frac{\mu_3}{a}\right)^{\alpha}. \tag{4.27}$$

For what follows, we fix an orthonormal basis $\{b_1, b_2, b_3\}$ of eigenvectors of B such that $Bb_i = \mu_i b_i$, $i = 1, 2, 3$. The next lemma will be useful.

LEMMA 4.4. *For every unit vector $n \in \mathbb{R}^3$, the maximum eigenvalue of $M_n B M_n$ is greater than or equal to* $\max\{\mu_2, \mu_3/a\}$.

PROOF OF LEMMA 4.4. By definition of the maximum eigenvalue of $M_n B M_n$, to prove the lemma it is enough to show that

$$(M_n B M_n m) \cdot m \geq \max\left\{\mu_2, \frac{\mu_3}{a}\right\}, \quad \text{for some} \quad m \in S^2.$$

If $\frac{\mu_3}{a} \geq \mu_2$, we define

$$m := \frac{v}{|v|}, \qquad \text{where} \quad v := \frac{1}{\sqrt{a}} M_n^{-1} b_3.$$

With this choice of m, we have that

$$(M_n B M_n m) \cdot m = \frac{1}{|v|^2} (B M_n v) \cdot (M_n v) = \frac{1}{a|v|^2} (B b_3) \cdot b_3 = \frac{1}{|v|^2} \frac{\mu_3}{a}. \tag{4.28}$$

From the definition of M_n it turns out that $|v| \leq |b_3| = 1$, because $a > 1$ and

$$v = \left[n\otimes n + \frac{1}{\sqrt{a}} (I - n\otimes n) \right] b_3.$$

Thus, from (4.28), we obtain $(M_n B M_n m) \cdot m \geq \mu_3/a$.

If $\mu_2 \geq \frac{\mu_3}{a}$, we consider $(\mathrm{Span}\{n\})^\perp$, the orthogonal space to n, and choose a unit vector m in the set $\mathrm{Span}\{b_2, b_3\} \cap (\mathrm{Span}\{n\})^\perp$, which contains at least one line. Thus, the fact that $m \in \mathrm{Span}\{b_2, b_3\}$ implies

$$(Bm) \cdot m \geq \mu_2, \tag{4.29}$$

while $m \in (\mathrm{Span}\{n\})^\perp$ implies that $M_n m = m$. This fact, together with (4.29), gives that

$$(M_n B M_n m) \cdot m = (B M_n m) \cdot (M_n m) \geq \mu_2.$$

□

The proof of Proposition 4.3 hinges on the following crucial technical result, whose proof is postponed at the end of this section.

Lemma 4.5. *Let* $0 < \overline{x} \leq \overline{y} \leq \overline{z}$ *and* $0 < x \leq y \leq z$ *be such that*

$$\text{(i)}\ \ xyz = \overline{x}\,\overline{y}\,\overline{z}, \quad \text{(ii)}\ \ x + y + z \geq \overline{x} + \overline{y} + \overline{z}, \quad \text{(iii)}\ \ z \geq \overline{z}. \tag{4.30}$$

Then, for every $\alpha > 1$ *we have that*

$$x^\alpha + y^\alpha + z^\alpha \geq \overline{x}^\alpha + \overline{y}^\alpha + \overline{z}^\alpha.$$

Proof of Proposition 4.3. Recall that we want to prove (4.27) and that M_n is defined in (4.25). Note that (4.27) is true for $\alpha = 1$: this has been proved in (1.6) (with FF^T in place of B and $a^{-\frac{1}{6}} L_n^{-\frac{1}{2}}$ in place of M_n), so that

$$\min_{n \in S^2} \mathrm{tr}(M_n B M_n) = \mu_1 + \mu_2 + \frac{\mu_3}{a}, \tag{4.31}$$

and the minimum is attained when n is parallel to b_3. Now, by using the definition of the α–power of a positive definite and symmetric matrix, we write our minimum problem as

$$\min_{n \in S^2} \mathrm{tr}(M_n B M_n)^\alpha = \min_{(x,y,z) \in \mathscr{A}} (x^\alpha + y^\alpha + z^\alpha), \tag{4.32}$$

where

$$\mathscr{A} := \big\{(x, y, z) \in \mathbb{R}^3 \ :\ 0 < x \leq y \leq z,\ x, y, z \text{ eigenvalues of } M_n B M_n \text{ for some } n \in S^2\big\}.$$

It is easy to check that μ_1, μ_2 and μ_3/a are eigenvalues of $M_{b_3} B M_{b_3}$, so that, by relabeling them $\overline{x}$, $\overline{y}$ and $\overline{z}$ in such a way that $\overline{x} \leq \overline{y} \leq \overline{z}$, we have that

$$(\overline{x}, \overline{y}, \overline{z}) \in \mathscr{A}, \tag{4.33}$$

with $\overline{z} \in \{\mu_2, \frac{\mu_3}{a}\}$. Finally, observe that for every $(x, y, z) \in \mathscr{A}$,

$$xyz = \det(M_n B M_n) = \det B \det M_n^2 = \frac{\mu_1 \mu_2 \mu_3}{a} = \overline{x}\,\overline{y}\,\overline{z}. \tag{4.34}$$

We now apply Lemma 4.5. Take $(x, y, z) \in \mathscr{A}$: since (4.30) (i) is assured by (4.34), (4.30) (ii) by (4.31), and (4.30) (iii) by Lemma 4.4, we have that

$$x^\alpha + y^\alpha + z^\alpha \geq \overline{x}^\alpha + \overline{y}^\alpha + \overline{z}^\alpha,$$

for every $\alpha > 1$. Thus, by considering also (4.32) and (4.33), we have obtained that

$$\min_{n \in S^2} \mathrm{tr}(M_n B M_n)^\alpha = \overline{x}^\alpha + \overline{y}^\alpha + \overline{z}^\alpha,$$

that is (4.27). □

By considering $\tilde{W}_n$ given by (4.23), we define

$$\tilde{W}(B) := \min_{n \in S^2} \tilde{W}_n(B) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \min_{n \in S^2} \left[\mathrm{tr}(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}})^{\frac{\gamma_i}{2}} - 3 \right], \qquad \det B = 1. \tag{4.35}$$

In view of Proposition 4.3, we have that

$$\tilde{W}(B) = \sum_{i=1}^{N} \frac{c_i a^{\frac{\gamma_i}{6}}}{\gamma_i} \left[\mu_1^{\frac{\gamma_i}{2}} + \mu_2^{\frac{\gamma_i}{2}} + \left(\frac{\mu_3}{a}\right)^{\frac{\gamma_i}{2}} - 3a^{-\frac{\gamma_i}{6}} \right], \qquad \det B = 1,$$

where $0 < \mu_1 \le \mu_2 \le \mu_3$ are the ordered eigenvalues of B, and in turn that

$$\begin{aligned} W(F) &:= \tilde{W}(FF^T) \\ &= \sum_{i=1}^{N} \frac{c_i a^{\frac{\gamma_i}{6}}}{\gamma_i} \left[\lambda_1^{\gamma_i} + \lambda_2^{\gamma_i} + \left(\frac{\lambda_3}{\sqrt{a}}\right)^{\gamma_i} - 3a^{-\frac{\gamma_i}{6}} \right], \qquad \det F = 1 \end{aligned} \tag{4.36}$$

where $0 < \lambda_1 \le \lambda_2 \le \lambda_3$ are the ordered singular values of F. Expression (4.36) tells us that the energy density W is of "Ogden–type", that is separable in the sense discussed in [**54**, Chapter 4]. We remark that in all the terms of the sum in (4.35), the minimum is achieved when n aligned with an eigenvector of B corresponding to its largest eigenvalue μ_3. Therefore, within this model, the nematic director is always aligned with the direction of maximal principal stretch. The following proposition characterize the set of the wells of W.

PROPOSITION 4.6. *Considering the function W defined by (4.36) if $\det F = 1$ and equal to $+\infty$ otherwise in $\mathbb{M}^{3\times 3}$, we have that*

$$W \ge 0 \qquad \textit{and} \qquad W(F) = 0 \quad \textit{if and only if} \quad \lambda_1(F) = \lambda_2(F) = a^{-\frac{1}{6}} \textit{ and } \lambda_3(F) = a^{\frac{1}{3}}.$$

PROOF. It is sufficient to use Proposition 4.1 (specialized to the incompressible case) and argue as in the proof of Proposition 1.1 (ii). □

Consider now

$$W_2(F) := \tilde{W}_2(FF^T), \qquad \det F = 1,$$

where

$$\tilde{W}_2(B) := \min_{n\in S^2} \tilde{W}_n(B) = \min_{n\in S^2} \left\{ \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\mathrm{tr}(B_n^e)^{\frac{\gamma_i}{2}} - 3 \right] + \sum_{j=1}^{M} \frac{d_j}{\delta_j} [\mathrm{tr}\,\mathrm{Cof}\,(B_n^e)^{\frac{\delta_j}{2}} - 3] \right\},$$

and B_n^e is given by (4.3). If $d_j > 0$ for some $j = 1, ..., M$, these expressions may be not separable in the sense discussed in [**54**, Chapter 4], except in the Mooney–Rivlin case $N = M = 1$ and $\gamma_1 = \delta_1 = 2$ (and c_1, $d_1 \ge 0$). In this case, $\tilde{W}_n$ is of the form (4.7) and, recalling that

$$L_n = a^{\frac{2}{3}} n\otimes n + a^{-\frac{1}{3}}(I - n\otimes n) \qquad (a > 1), \tag{4.37}$$

it is easy to show that

$$\begin{aligned} W_2(F) = \frac{c_1}{2} &\left[\left(\frac{\lambda_1}{a^{-\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_2}{a^{-\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_3}{a^{\frac{1}{3}}}\right)^2 - 3 \right] \\ &+ \frac{d_1}{2} \left[\left(\frac{\lambda_2\lambda_3}{a^{\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_1\lambda_3}{a^{\frac{1}{6}}}\right)^2 + \left(\frac{\lambda_1\lambda_2}{a^{-\frac{1}{3}}}\right)^2 - 3 \right]. \end{aligned} \tag{4.38}$$

Again, the minimum which defines $W_2(F)$ is attained when n is aligned with an eigenvector of FF^T corresponding to its largest eigenvalue, just as in the case of (4.36).

PROOF OF LEMMA 4.5. Suppose first that $\overline{x} = \overline{y} = \overline{z}$. In this case, we have to prove that $x^\alpha + y^\alpha + z^\alpha \ge 3\overline{x}^\alpha$. To have this, it is enough to use condition (4.30) (ii), which gives $x + y + z \ge 3\overline{x}$. Indeed:

$$x^\alpha + y^\alpha + z^\alpha \ge 3^{1-\alpha}(x + y + z)^\alpha \ge 3\overline{x}^\alpha,$$

where the first inequality is standard (descending, e. g., from Hölder's inequality). Thus, in the rest part of the proof, we will suppose

$$\overline{x} < \overline{z}. \tag{4.39}$$

We introduce the functions

$$w(x,y,z) := x^\alpha + y^\alpha + z^\alpha, \quad v(x,y,z) := xyz, \quad u(x,y,z) := x+y+z,$$

and the minimum problem

$$\min_{x,y,z>0} w(x,y,z) \tag{4.40}$$

with constraints

$$\text{(i)} \;\; v(x,y,z) = \overline{x}\,\overline{y}\,\overline{z}, \quad \text{(ii)} \;\; u(x,y,z) \geq \overline{x}+\overline{y}+\overline{z}, \quad \text{(iii)} \;\; z \geq \overline{z}. \tag{4.41}$$

By standard arguments it can be proved that the minimum exists. Let (x_0, y_0, z_0) be a minimum point. It is not restrictive to suppose that

$$x_0 \leq y_0 \leq z_0.$$

Claim 1. $x_0 < z_0$.
Suppose, by contradiction, that $x_0 = z_0$. In this case, (4.41) (i) and (ii) would give $x_0^3 = \overline{x}\,\overline{y}\,\overline{z}$ and $3x_0 \geq \overline{x}+\overline{y}+\overline{z}$, respectively. Thus, by the standard inequality between arithmetic and geometric mean, we would obtain

$$x_0 \geq \frac{\overline{x}+\overline{y}+\overline{z}}{3} \geq (\overline{x}\,\overline{y}\,\overline{z})^{\frac{1}{3}} = x_0$$

and in turn $\overline{x} = \overline{z}$, against (4.39).

Claim 1 will be used in the proof of the following claim.
Claim 2. $z_0 = \overline{z}$.

Let us see how the thesis descends from Claim 2 and postpone the proof of the claim. Since $z_0 = \overline{z}$, conditions (4.41) (i) and (ii) become

$$x_0 y_0 = \overline{x}\,\overline{y}, \qquad x_0 + y_0 \geq \overline{x} + \overline{y}. \tag{4.42}$$

This two conditions imply the inequality

$$y_0^2 - (\overline{x}+\overline{y})y_0 + \overline{x}\,\overline{y} \geq 0$$

and in turn that

$$\text{either} \qquad y_0 \leq \overline{x} \qquad \text{or} \qquad y_0 \geq \overline{y}. \tag{4.43}$$

As an intermediate step, we want to prove that

$$x_0^\alpha + y_0^\alpha \geq \overline{x}^\alpha + \overline{y}^\alpha. \tag{4.44}$$

If the contrary were true, by considering also the first condition in (4.42) we would obtain the inequality

$$(y_0^\alpha)^2 - (\overline{x}^\alpha + \overline{y}^\alpha)y_0^\alpha + (\overline{x}\,\overline{y})^\alpha < 0$$

which is true if and only if

$$\overline{x} < y_0 < \overline{y},$$

against (4.43). Thus, (4.44) holds and therefore

$$\overline{x}^\alpha + \overline{y}^\alpha + \overline{z}^\alpha = \overline{x}^\alpha + \overline{y}^\alpha + z_0^\alpha \leq x_0^\alpha + y_0^\alpha + z_0^\alpha.$$

This fact, together with the definition of (x_0, y_0, z_0) as a minimum point of (4.40)–(4.41), gives the thesis.
Proof of Claim 2. Suppose, by contradiction, that

$$z_0 > \overline{z}. \tag{4.45}$$

Constraint (4.41) (ii) tells us that $x_0+y_0+z_0 \geq \overline{x}+\overline{y}+\overline{z}$. If $x_0+y_0+z_0 > \overline{x}+\overline{y}+\overline{z}$, this strict inequality, together with conditions (4.41) (i) and (4.45), gives

$$\nabla v(x_0, y_0, z_0) = \mu \nabla w(x_0, y_0, z_0),$$

for some Lagrange multiplier $\mu \neq 0$. A direct computation shows that this last condition implies $x_0 = z_0$, against Claim 1. Therefore, we must have

$$x_0 + y_0 + z_0 = \overline{x} + \overline{y} + \overline{z}. \tag{4.46}$$

Since $x_0 < z_0$ from Claim 1, we have three possibilities which we treat separately in the following cases (a), (b) and (c). We are going to show that every case leads to a contradiction resulting from (4.45).

(a). Here we suppose that

$$x_0 = y_0 < z_0.$$

Let $\varepsilon > 0$ be such that $x(\varepsilon) := y_0 - \varepsilon > 0$ and let $y = y(\varepsilon)$ and $z = z(\varepsilon) \geq y$ satisfy the conditions

$$\begin{cases} x(\varepsilon) + y + z = 2y_0 + z_0, \\ x(\varepsilon) y z = y_0^2 z_0. \end{cases}$$

Setting

$$a_0 = y_0 + z_0, \qquad b_0 = y_0 z_0,$$

it turns out that

$$\begin{aligned} y(\varepsilon) &= \frac{1}{2}\left\{a_0 + \varepsilon - \sqrt{(a_0+\varepsilon)^2 - 4b_0\left(\frac{y_0}{y_0 - \varepsilon}\right)}\right\}, \\ z(\varepsilon) &= \frac{1}{2}\left\{a_0 + \varepsilon + \sqrt{(a_0+\varepsilon)^2 - 4b_0\left(\frac{y_0}{y_0 - \varepsilon}\right)}\right\}. \end{aligned}$$

It is easy to show that $x(\varepsilon) \leq y(\varepsilon) \leq z(\varepsilon)$ for ε sufficiently small. Moreover, up to a smaller ε, we have that $z(\varepsilon) \geq \overline{z}$, since $z(0) = z_0$ and (4.45) holds. Now, let us introduce the function

$$f(\varepsilon) := x(\varepsilon)^\alpha + y(\varepsilon)^\alpha + z(\varepsilon)^\alpha.$$

Since $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ and $(x(\varepsilon), y(\varepsilon), z(\varepsilon))$ satisfies the constraints (4.41) of the minimum problem (4.40), it follows that

$$f'(0) = 0 \quad \text{and} \quad f''(0) \geq 0. \tag{4.47}$$

Simple computations show that

$$f''(0) = \frac{2\alpha}{y_0(z_0 - y_0)}\left[y_0^\alpha + \alpha y_0^{\alpha-1}(z_0 - y_0) - z_0^\alpha\right]. \tag{4.48}$$

Now, since $y_0 < z_0$, we have that $y_0^\alpha + \alpha y_0^{\alpha-1}(z_0 - y_0) < z_0^\alpha$, in view of the strict convexity of the function $t \mapsto t^\alpha$ $(\alpha > 1)$. Thus, from (4.48) we obtain that $f''(0) < 0$, against (4.47).

(b). Here we suppose that

$$x_0 < y_0 = z_0.$$

In this case, constraints (4.41) (i) and (4.46) give

$$\begin{cases} x_0 z_0^2 = G^2 \overline{z}, \\ x_0 + 2z_0 = 2A + \overline{z}, \end{cases} \tag{4.49}$$

where

$$A := \frac{\overline{x} + \overline{y}}{2}, \qquad G := \sqrt{\overline{x}\,\overline{y}}.$$

From (4.49) we deduce that z_0 solves the third order equation

$$P(t) := 2t^3 - (2A + \overline{z})t^2 + G^2\overline{z} = 0.$$

The function P has a local maximum at $t = 0$ with $P(0) > 0$ and a local minimum at $t = \frac{2A+\overline{z}}{3}$ with $P\left(\frac{2A+\overline{z}}{3}\right) < 0$. Now, from (4.45), from the fact that

$$\overline{z} > \frac{2A + \overline{z}}{3} \tag{4.50}$$

and that z_0 is a zero of P, it is easy to deduce that

$$P(t) < 0 \ \text{ for } \ \frac{2A+\overline{z}}{3} < t < z_0 \quad \text{and } P(t) \geq 0 \ \text{ for } \ t \geq z_0. \tag{4.51}$$

On the other hand,

$$P(\overline{z}) = \overline{z}\left[\overline{z}^2 - (\overline{x} + \overline{y})\overline{z} + \overline{x}\,\overline{y}\right],$$

so that $P(\overline{z}) \geq 0$ in view of the fact that $\overline{z} \geq \overline{y}$. Together with (4.50) and (4.51), this implies that $\overline{z} \geq z_0$, against (4.45).

(c). Finally, we suppose that

$$x_0 < y_0 < z_0.$$

In this case, we consider the matrix whose lines are the gradients $\nabla w(x_0, y_0, z_0)$, $\nabla v(x_0, y_0, z_0)$, $\nabla u(x_0, y_0, z_0)$. Considering (4.40), (4.41) (i), (4.45), and (4.46), it turns out that

$$D := \det \begin{bmatrix} \nabla w(x_0, y_0, z_0) \\ \nabla v(x_0, y_0, z_0) \\ \nabla u(x_0, y_0, z_0) \end{bmatrix} = 0. \tag{4.52}$$

Computing such a determinant gives

$$\begin{aligned} D &= \alpha \left[x_0 \left(y_0^\alpha - z_0^\alpha \right) - y_0 \left(x_0^\alpha - z_0^\alpha \right) + z_0 \left(x_0^\alpha - y_0^\alpha \right) \right] \\ &= -\alpha \left[y_0^\alpha (z_0 - x_0) - z_0^\alpha (y_0 - x_0) - x_0^\alpha (z_0 - y_0) \right]. \end{aligned} \tag{4.53}$$

Setting $\lambda := \frac{y_0 - x_0}{z_0 - x_0} \in (0, 1)$, so that

$$y_0 = \lambda z_0 + (1 - \lambda) x_0, \tag{4.54}$$

from (4.53) we obtain that

$$D = -\alpha (z_0 - x_0) \left[y_0^\alpha - \lambda z_0^\alpha - (1 - \lambda) x_0^\alpha \right].$$

This last equality, together with (4.54) and the strict convexity of the function $t \mapsto t^\alpha$ ($\alpha > 1$), implies that $D > 0$, against (4.52). □

In the following remark we show that in the case where the two largest eigenvalues of $B \in Psym(3)$ are equal it is possible to find the analytical expression of the eigenvalues of $L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}$ and prove Proposition 4.3 in a more direct way.

REMARK 4.7. Let $B \in Psym(3)$ and suppose that $\mu_1 < \mu_2 = \mu_3$ (the case $\mu_1 = \mu_2 = \mu_3$ is trivial), where μ_1, μ_2, μ_3 are the ordered eigenvalues of B. Let b_1, b_2, b_3 be the corresponding orthonormal eigenvectors. For $a > 1$ and a unit vector $n \in \mathbb{R}^3$, consider $L_n = L_n(a)$ defined as in (4.37) and suppose that

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \text{in the orthonormal basis } \{b_1, b_2, b_3\}.$$

Then, up to the multiplicative constant $a^{\frac{1}{3}}$, we have that the spectrum of $L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}}$ is

$$\left\{ \mu_2, \frac{g(n_1^2) + \sqrt{g^2(n_1^2) - 4\frac{\mu_1 \mu_2}{a}}}{2}, \frac{g(n_1^2) - \sqrt{g^2(n_1^2) - 4\frac{\mu_1 \mu_2}{a}}}{2} \right\}, \tag{4.55}$$

where

$$g(t) := (\mu_2 - \mu_1) \left(1 - \frac{1}{a} \right) t + \mu_1 + \frac{\mu_2}{a}, \qquad \text{for every } \ 0 \leq t \leq 1. \tag{4.56}$$

Moreover, we have that

$$\min_{n \in S^2} \left(L_n^{-\frac{1}{2}} B L_n^{-\frac{1}{2}} \right)^{\frac{\gamma}{2}} = a^{\frac{\gamma}{6}} \left[\mu_1^{\frac{\gamma}{2}} + \left(1 + a^{-\frac{\gamma}{2}} \right) \mu_2^{\frac{\gamma}{2}} \right],$$

and the minimum is attained for $n \in (\mathrm{Span}\{b_1\})^\perp$.

In order to prove this, let us use the same position used for the proof of Proposition 4.3: $\alpha = \frac{\gamma}{2}$ and

$$M_n := a^{-\frac{1}{6}} L_n^{-\frac{1}{2}} = \left(\frac{1}{\sqrt{a}} - 1 \right) n \otimes n + I.$$

With this notation, we are going to check that the spectrum of M_nBM_n is (4.55) and that

$$\min_{n\in S^2} \operatorname{tr}(M_nBM_n)^\alpha = \mu_1^\alpha + \left(1+\frac{1}{a^\alpha}\right)\mu_2^\alpha \tag{4.57}$$

with the minimum attained for $n \in (\operatorname{Span}\{b_1\})^\perp$.

We note that, as $\mu_1 < \mu_2 = \mu_3$, we can write B in the following way:

$$B = \mu_1(b_1 \otimes b_1) + \mu_2(I - b_1\otimes b_1) = \mu_2 C, \tag{4.58}$$

where

$$\rho := \frac{\mu_1}{\mu_2} < 1, \qquad C := \rho\, b_1\otimes b_1 + (I - b_1\otimes b_1).$$

We are going to find the eigenvalues of M_nCM_n. Note that M_n^{-1} is an invertible matrix and that there exist $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^3 \setminus \{0\}$ such that $M_nCM_nv = \lambda v$ if and only if $CM_n^2\left(M_n^{-1}v\right) = \lambda\left(M_n^{-1}v\right)$. Therefore, we look for the eigenvalues of the matrix

$$\begin{aligned} CM_n^2 &= [(\rho-1)b_1\otimes b_1 + I]\left[\left(\frac{1}{a}-1\right)n\otimes n + I\right] \\ &= (\rho-1)\left(\frac{1}{a}-1\right)(b_1\cdot n)b_1\otimes n + (\rho-1)b_1\otimes b_1 + \left(\frac{1}{a}-1\right)n\otimes n + I, \end{aligned} \tag{4.59}$$

since in this case there are shorter formulas to handle. Recall that we have fixed the orthonormal basis $\{b_1, b_2, b_3\}$ where

$$b_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{and} \quad n = \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}.$$

Using these expressions we can compute the coefficients of the matrix CM_n^2 and obtain

$$CM_n^2 = \begin{bmatrix} \rho\left[\left(\frac{1}{a}-1\right)n_1^2+1\right] & \rho\left(\frac{1}{a}-1\right)n_1n_2 & \rho\left(\frac{1}{a}-1\right)n_1n_3 \\ \left(\frac{1}{a}-1\right)n_1n_2 & \left(\frac{1}{a}-1\right)n_2^2+1 & \left(\frac{1}{a}-1\right)n_2n_3 \\ \left(\frac{1}{a}-1\right)n_1n_3 & \left(\frac{1}{a}-1\right)n_2n_3 & \left(\frac{1}{a}-1\right)n_3^2+1 \end{bmatrix}$$

It is already clear that 1 is an eigenvalue of CM_n^2. Indeed, using expression (4.59), it turns out that $CM_n^2v = v$ for every vector v in the orthogonal space to $\operatorname{Span}\{b_1, n\}$. In order to find the other eigenvalues of CM_n^2, we use the standard procedure and look for the solutions w of the equation $\det\left(CM_n^2 - wI\right) = 0$. A direct computation gives

$$\begin{aligned} \det\left(CM_n^2 - wI\right) = &\left\{\rho\left[\left(\frac{1}{a}-1\right)n_1^2+1\right] - w\right\}\left[w^2 - (\delta+1)w + \delta\right] \\ &- \rho\left(\frac{1}{a}-1\right)^2 n_1^2\left(n_2^2+n_3^2\right)(1-w), \end{aligned} \tag{4.60}$$

where

$$\delta := \left(\frac{1}{a}-1\right)\left(n_2^2+n_3^2\right)+1.$$

Now, since $\left[w^2-(\delta+1)w+\delta\right] = (w-1)(w-\delta)$, we use the fact that $1 - n_2^2 - n_3^2 = n_1^2$ and rewrite (4.60) as

$$\det\left(CM_n^2 - wI\right) = (w-1)P(w), \tag{4.61}$$

where

$$P(w) = -w^2 + \frac{1}{\mu_2}g(n_1^2)w - \frac{\mu_1}{\mu_2 a},$$

and g is defined in (4.56). The zeros of P are

$$\frac{g(n_1^2) \pm \sqrt{g^2(n_1^2) - 4\frac{\mu_1\mu_2}{a}}}{2\mu_2}$$

and

$$\Delta(t) := g^2(t) - 4\frac{\mu_1\mu_2}{a} = (\mu_2 - \mu_1)^2 \left(1 - \frac{1}{a}\right)^2 t^2 \\ + 2(\mu_2 - \mu_1)\left(1 - \frac{1}{a}\right)\left(\mu_1 + \frac{\mu_2}{a}\right) t + \left(\mu_1 - \frac{\mu_2}{a}\right)^2 \geq 0 \quad \text{for every } 0 \leq t \leq 1. \quad (4.62)$$

Thus, looking at (4.61), we have that the spectrum of CM_n^2 is

$$\left\{1, \frac{g(n_1^2) + \sqrt{\Delta(n_1^2)}}{2\mu_2}, \frac{g(n_1^2) - \sqrt{\Delta(n_1^2)}}{2\mu_2}\right\}.$$

Recalling (4.58), multiplying these eigenvalues by μ_2 gives the spectrum of BM_n^2, which is the same of M_nBM_n.

In order to prove (4.57), let us introduce the function

$$f(t) := \mu_2^\alpha + \left[\frac{g(t) + \sqrt{\Delta(t)}}{2}\right]^\alpha + \left[\frac{g(t) - \sqrt{\Delta(t)}}{2}\right]^\alpha,$$

and observe that $f(n_1^2) = \mathrm{tr}(M_nBM_n)^\alpha$. Now, we differentiate f in $(0, 1)$:

$$f'(t) = \frac{\alpha g'(t)}{\sqrt{\Delta(t)}}\left[\left(\frac{g(t) + \sqrt{\Delta(t)}}{2}\right)^\alpha - \left(\frac{g(t) - \sqrt{\Delta(t)}}{2}\right)^\alpha\right] \quad \text{for every } 0 < t < 1.$$

This tells us that

$$f'(t) > 0 \quad \text{for every} \quad 0 < t < 1,$$

since $g'(t) = (\mu_2 - \mu_1)\left(1 - \frac{1}{a}\right) > 0$ and $\Delta(t) > 0$ for every $t > 0$ (see (4.62)). Thus,

$$f(0) \leq f(t) \quad \text{for every} \quad 0 < t \leq 1,$$

and therefore

$$f(0) = \mu_2^\alpha + \mu_1^\alpha + \left(\frac{\mu_2}{a}\right)^\alpha = \min_{n\in S^2} f(n_1^2) = \min_{n\in S^2} \mathrm{tr}(M_nBM_n)^\alpha. \quad (4.63)$$

Finally, observe that

$$f(0) = \mathrm{tr}(M_nBM_n)^\alpha, \quad \text{where} \quad n = \begin{bmatrix} 0 \\ n_2 \\ n_3 \end{bmatrix} \in (\mathrm{Span}\{b_1\})^\perp. \quad (4.64)$$

Considering (4.63) and (4.64), the proof of (4.57) is completed.

4.4. Stress-strain response through quasiconvex envelopes

Although the free energy density obtained in the previous section minimizing with respect to n is not quasiconvex, nevertheless this notion of convexity plays an essential role in predicting global features of the material response of nematic elastomers: the passage from the mesoscopic to the macroscopic free energy is achieved by quasiconvexification and the quasiconvex hull of the set of the energy wells has a very clear physical interpretation as the set of macroscopic deformation gradients achievable by minimum–energy microstructures. Indeed, minimization with respect to n leads to a loss of stability of homogeneously deformed states with respect to configurations exhibiting shear bands (stripe domains, which are indeed observed experimentally, see Figure 4.2).

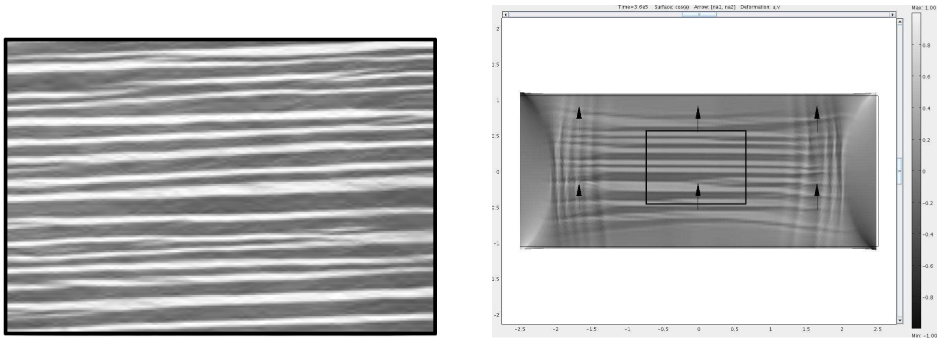

FIGURE 4.2. Comparison between experimental data from [**68**] and numerical results from [**28**]. This is a stretching test on a sample of nematic elastomer material with clamped lateral edges. In the second picture the grey scale represents the cosine of the angle between the nematic director and the horizontal, and the vertical arrows represent the direction of the nematic director at the beginning of the experiment and of the simulation.

Using the arguments of [**31**], we will see that the quasiconvex envelope W^{qc} of the energy density W defined in (4.36) is given by

$$W^{qc}(F) = \inf_{\substack{w\in W^{1,\infty}(\Omega;\mathbb{R}^3)\\ w(x)=Fx \text{ on } \partial\Omega}} \frac{1}{|\Omega|}\int_\Omega W\left(\nabla w(x)\right)dx\,, \tag{4.65}$$

where Ω is an arbitrary bounded open set with $|\partial\Omega| = 0$ and w is an arbitrary Lipschitz continuous displacement field perturbing the affine state $y(x) = Fx$ and vanishing on $\partial\Omega$. Stable materials are characterized by $W^{qc} \equiv W$. If, for some F, $W^{qc}(F) < W(F)$, then the state of homogeneous deformation F is unstable: the material shows an energetic preference to develop spatially modulated deformations with gradient $F + \nabla w(x)$ (typically, shear bands) at fixed average deformation F. The minimal energy cost to maintain the state of average deformation F is $W^{qc}(F)$, rather than $W(F)$, and this is achieved through domain patterns with length scales which are very small compared to the size of the domain Ω.

For energies such as (4.36), an explicit formula for their quasiconvex envelope is available, thanks to [**31**]. Figure 4.3 gives a sketch of the sets L, I_1, and S appearing in the following proposition. To simplify the notation, let us introduce the set

$$\Sigma := \{F \in \mathbb{M}^{3\times3} \,:\, \det F = 1\},$$

which models the incompressibility of the elastomer. Note that

$$F\in\Sigma \quad\Rightarrow\quad \lambda_3(\operatorname{Cof} F) = \frac{1}{\lambda_1(F)}. \tag{4.66}$$

PROPOSITION 4.8. *Let W be the energy density given by (4.36), with $c_i > 0$ and $\gamma_i \geq 2$. Consider the following sets of* $3{\times}3$ *matrices:*

$$L := \left\{F\in\Sigma \,:\, \lambda_3(\operatorname{Cof} F) \leq a^{\frac{1}{6}}\right\}, \tag{4.67}$$

$$I_1 := \left\{F\in\Sigma \,:\, \lambda_3(\operatorname{Cof} F) \geq a^{\frac{1}{6}}, \lambda_3(\operatorname{Cof} F) \geq a^{-\frac{1}{2}}\lambda_3^2(F)\right\}, \tag{4.68}$$

$$S := \left\{F\in\Sigma \,:\, \lambda_3(\operatorname{Cof} F) \leq a^{-\frac{1}{2}}\lambda_3^2(F)\right\}, \tag{4.69}$$

Then, the quasiconvex envelope W^{qc} of W is given by

$$
\begin{aligned}
W^{qc}(F) &= \inf_{\substack{\varphi\in W^{1,\infty}(\Omega;\mathbb{R}^3)\\ \varphi(x)=Fx \text{ on } \partial\Omega}} \frac{1}{|\Omega|}\int_\Omega W(\nabla\varphi(x))dx \\
&= \begin{cases} 0 & \text{if } F\in L \\ \displaystyle\sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(a^{\frac{1}{6}}\lambda_1(F)\right)^{\gamma_i} + 2\left(\frac{1}{a^{\frac{1}{6}}\lambda_1(F)}\right)^{\frac{\gamma_i}{2}} - 3\right] & \text{if } F\in I_1, \\ W(F) & \text{if } F\in S \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}
\tag{4.70}
$$

Note that in the $(\lambda_3(F), \lambda_3(\operatorname{Cof} F))$–plane the set where $\det F = 1$ is the set bounded by the curves $\lambda_3(\operatorname{Cof} F) = \sqrt{\lambda_3(F)}$ and $\lambda_3(\operatorname{Cof} F) = \lambda_3^2(F)$. In view of (4.66), this is due to the fact that

$$F\in\Sigma \quad\Rightarrow\quad \lambda_3(\operatorname{Cof} F)\le\lambda_3^2(F),\quad \lambda_3(\operatorname{Cof} F)\ge\sqrt{\lambda_3(F)}.$$

Also, it is easy to see that $L\cup I_1\cup S=\Sigma$.

Let us postpone the proof of Proposition 4.8 and note that each summand in (4.36) is of the form

$$\frac{c_i}{\gamma_i}\left[\left(\frac{\lambda_1(F)}{e_1}\right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{e_2}\right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{e_3}\right)^{\gamma_i} - 3\right], \qquad \det F = 1, \tag{4.71}$$

where

$$0 < e_1 \le e_2 \le e_3 \quad \text{are such that} \quad e_1e_2e_3 = 1. \tag{4.72}$$

In [**31**] explicit formulas for the quasiconvex envelopes of functions like (4.71) are given. Now, in order to conclude that (4.70) holds, we cannot directly use the result of [**31**], because it is not always true that the quasiconvex envelope of a sum is the sum of the quasiconvex envelopes. Thus, for sake of completeness, in the following Theorem 4.10 we give the results of [**31**] for functions of the type

$$f(F) := \begin{cases} \displaystyle\sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{\lambda_1(F)}{e_1}\right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{e_2}\right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{e_3}\right)^{\gamma_i} - 3\right] & \text{if } F\in\Sigma, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.73}$$

Note that (4.73) includes (4.36) for $e_1 = e_2 = a^{-\frac{1}{6}}$ and $e_3 = a^{\frac{1}{3}}$ and, in particular, includes [BTW] model for $N = 1$ and $\gamma_1 = 2$. By the standard arithmetic–geometric mean inequality, we have that

$$\left(\frac{\lambda_1(F)}{e_1}\right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{e_2}\right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{e_3}\right)^{\gamma_i} \ge 3\left(\frac{\det F}{e_1e_2e_3}\right)^{\frac{\gamma_i}{3}} = 3. \tag{4.74}$$

Thus, f is a nonnegative function and $f(F) = 0$ if and only if

$$F\in K := \{F\in\Sigma : \lambda_i(F) = e_i,\ i = 1,2,3\}.$$

Referring to Subsection 1.2.1 for the following notions of convexities and hulls, the quasiconvex hull of K is given by the following theorem. We recall in particular that the set $K^{(2)}$ is involved in characterization of the lamination convex hull K^{lc} of Proposition 1.9. An important result contained in [**31**] is the following.

THEOREM 4.9.

$$K^{(2)} = K^{lc} = K^{rc} = K^{qc} = K^{pc} = \{F\in\Sigma : \Lambda(F)\subseteq[e_1,e_3]\}.$$

We refer the reader to [**31**] for the proof of this theorem. We remark that the constraint Σ is stable under taking the lamination convex hull. Indeed, consider A, $B\in\Sigma$ such that $\operatorname{rank}(A-B) = 1$. Let $\lambda\in(0,1)$ and note that, up to a change of coordinates, we have that

$$\lambda A + (1-\lambda)B = B + \lambda\begin{pmatrix} r_1 & 0 & 0 \\ r_2 & 0 & 0 \\ r_3 & 0 & 0\end{pmatrix}.$$

Thus, since the determinant is a multilinear function of the columns, we have that $\det(\lambda A+(1-\lambda)B)=\det B=1$.

To prove Proposition 4.8, it is sufficient to apply the following theorem with $e_1=e_2=a^{-\frac{1}{6}}$ and $e_3=a^{\frac{1}{3}}$.

Theorem 4.10. *Let f be given by (4.72) and (4.73). Then, for every $F\in\mathbb{M}^{3\times 3}$ and every bounded open set $\Omega\subseteq\mathbb{R}^3$ with $|\partial\Omega|=0$,*

$$f^{qc}(F) = \inf_{\substack{\varphi\in W^{1,\infty}(\Omega;\mathbb{R}^3)\\ \varphi(x)=Fx \text{ on } \partial\Omega}} \frac{1}{|\Omega|}\int_\Omega f(\nabla\varphi(x))dx \tag{4.75}$$

$$= \begin{cases} 0 & \text{if } F\in L,\\ \sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{\lambda_1(F)}{e_1}\right)^{\gamma_i}+2\left(\frac{e_1}{\lambda_1(F)}\right)^{\frac{\gamma_i}{2}}-3\right] & \text{if } F\in I_1,\\ f(F) & \text{if } F\in S,\\ \sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{\lambda_3(F)}{e_3}\right)^{\gamma_i}+2\left(\frac{e_3}{\lambda_3(F)}\right)^{\frac{\gamma_i}{2}}-3\right] & \text{if } F\in I_3,\\ +\infty & \text{otherwise,}\end{cases} \tag{4.76}$$

where

$$\begin{aligned} L &:= \left\{F\in\Sigma:\ \lambda_3(F)\le e_3,\ \lambda_3(\operatorname{Cof} F)\le\frac{1}{e_1}\right\},\\ I_1 &:= \left\{F\in\Sigma:\lambda_3(\operatorname{Cof} F)\ge\frac{1}{e_1},\ \lambda_3(\operatorname{Cof} F)\ge\gamma^*\lambda_3^2(F)\right\},\\ S &:= \left\{F\in\Sigma:\sqrt{\Gamma^*\lambda_3(F)}\le\lambda_3(\operatorname{Cof} F)\le\gamma^*\lambda_3^2(F)\right\},\\ I_3 &:= \left\{F\in\Sigma:\lambda_3(F)\ge e_3,\ \lambda_3(\operatorname{Cof} F)\le\sqrt{\Gamma^*\lambda_3(F)}\right\},\end{aligned} \tag{4.77}$$

and

$$\gamma^*:=\frac{e_2}{e_3},\qquad \Gamma^*:=\frac{e_2}{e_1}. \tag{4.78}$$

As we will see in the proof, f^{qc} turns out to be polyconvex. Also, note that $L=K^{qc}$, in view of Theorem 4.9. In particular, we see that $f^{qc}=0$ on K^{qc}.

Proof of Theorem 4.10. Following [**31**], the proof proceeds in three steps:

(i) finding an upper bound $\tilde f$ for f^{rc} such that $\tilde f\le f$;
(ii) establishing that $\tilde f$ is polyconvex, so that $f^{rc}=f^{pc}=\tilde f$;
(iii) showing that $f^{qc}\le\hat f\le f^{pc}$, where $\hat f$ is the function on the right hand side of (4.75).

Step (i). Consider the function $\tilde f:\mathbb{M}^{3\times 3}\to[0,\infty]$ defined as the right hand side of (4.76). It is easy to see that $\tilde f\le f$. Moreover, as done in the proof of [**31**, Proposition 1], it can be showed that $f^{rc}\le\tilde f$ and that for every $F\in\Sigma$ there exist $m\in\{1,...,4\}$, $\lambda_1,...,\lambda_m\subseteq[0,1]$, and $F_1,...,F_m\in\Sigma$, such that

$$F=\sum_{i=1}^m\lambda_iF_i\qquad\text{and}\qquad \tilde f(F)=f(F_i),\quad\text{for } i=1,...,m.$$

Step (ii). In order to prove that $\tilde{f}$ is polyconvex, consider the function $\psi : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\psi(s,t) := \begin{cases} 0 & \text{if } (s,t) \in \tilde{L}, \\ \sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{1}{e_1 t}\right)^{\gamma_i} + 2(e_1 t)^{\frac{\gamma_i}{2}} - 3\right] & \text{if } (s,t) \in \tilde{I}_1, \\ \sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{1}{e_1 t}\right)^{\gamma_i} + \left(\frac{t}{e_2 s}\right)^{\gamma_i} + \left(\frac{s}{e_3}\right)^{\gamma_i} - 3\right] & \text{if } (s,t) \in \tilde{S}, \\ \sum_{i=1}^N \frac{c_i}{\gamma_i}\left[\left(\frac{s}{e_3}\right)^{\gamma_i} + 2\left(\frac{e_3}{s}\right)^{\frac{\gamma_i}{2}} - 3\right] & \text{if } (s,t) \in \tilde{I}_3, \end{cases}$$

and

$$\begin{aligned} \tilde{L} &:= [0, e_3] \times \left[0, \frac{1}{e_1}\right], \\ \tilde{I}_1 &:= \left\{(s,t) \in \mathbb{R}^2_+ : t \geq \frac{1}{e_1},\, t \geq \gamma^* s^2\right\}, \\ \tilde{S} &:= \left\{(s,t) \in \mathbb{R}^2_+ : \sqrt{\Gamma^* s} \geq t \leq \gamma^* s^2\right\}, \\ \tilde{I}_3 &:= \left\{(s,t) \in \mathbb{R}^2_+ : s \geq e_3,\, t \leq \sqrt{\Gamma^* s}\right\}, \end{aligned}$$

where γ^* and Γ^* are defined as in (4.78). In [**31**, Proposition 2] it is shown that every summand of ψ is convex and nondecreasing in its arguments. Hence, ψ is convex and nondecreasing in its arguments. The same arguments of the proofs of [**31**, Proposition 3, Theorem 3] then give that $\tilde{f}$ is polyconvex, because $\tilde{f}(F) = \psi(\lambda_3(F), \lambda_3(\operatorname{Cof} F))$, if $F \in \Sigma$.

Step (iii). Note that Theorem 4.10 does not follows automatically from the previous steps, because quasiconvexity does not imply rank–one convexity for functions which take values in $\mathbb{R} \cup \{+\infty\}$. To close this gap, the authors in [**31**] use the following result by Müller and Šverák [**53**, Lemma 4.1] particularized to a subset V of Σ and to the lamination convex hull of V.

LEMMA 4.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, let V be an open set in Σ, and $F \in V^{lc}$. Then, for every $\varepsilon > 0$ there exists a piecewise affine Lipschitz function $y_\varepsilon : \Omega \to \mathbb{R}^m$ such that $\nabla u_\varepsilon \in V^{lc}$ a.e. in Ω and*

$$|\{x \in \Omega \,:\, \nabla u_\varepsilon(x) \notin V\}| \leq \varepsilon|\Omega|, \qquad u_\varepsilon(x) = Fx \quad \text{on } \partial\Omega.$$

Let us introduce the function $\hat{f} : \mathbb{M}^{3\times 3} \to [0, \infty]$ defined by

$$\hat{f}(F) := \inf_{\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^3)} \frac{1}{|\Omega|}\int_\Omega f(F + \nabla\varphi(x))dx, \quad \text{for every } F \in \mathbb{M}^{3\times 3}, \tag{4.79}$$

for some open bounded set $\Omega \subseteq \mathbb{R}^3$ with $|\partial\Omega| = 0$. Firstly, note that, by definition, every quasiconvex function $g : \mathbb{M}^{3\times 3} \to \mathbb{R} \cup \{+\infty\}$ such that $g \leq f$ satisfies also $g \leq \hat{f}$. Indeed, for every $\varphi \in W_0^{1,\infty}$ we have that

$$g(F) \leq \frac{1}{|\Omega|}\int_\Omega g(F + \nabla\varphi)dx \leq \frac{1}{|\Omega|}\int_\Omega f(F + \nabla\varphi)dx \leq \hat{f}(F).$$

In particular, we have that

$$f^{pc} \leq f^{qc} \leq \hat{f}. \tag{4.80}$$

If one shows that

$$\hat{f} \leq f^{pc}, \tag{4.81}$$

this inequality together with (4.80) gives that $f^{qc} = f^{pc} = \hat{f}$ and in turn, in view of the definition of $\tilde{f}$ and of Step (ii), concludes the proof of the theorem. To prove (4.81) the idea is to construct, for every $\delta > 0$ and every $F \in \Sigma$, a function $\varphi_{\delta,F} \in W^{1,\infty}$ such that $\varphi_{\delta,F}(x) = Fx$ on $\partial\Omega$ and

$$\int_\Omega f(\nabla\varphi_{\delta,F})dx \leq |\Omega| f^{pc}(F) + \mathcal{O}(\delta), \tag{4.82}$$

where $\mathcal{O}(\delta) \to 0$, as $\delta \to 0$. This implies (4.81). From Step (ii) we have that $f^{pc} = \tilde{f}$. Thus, using Step (i), we obtain that for every $F \in \Sigma$ there exist $m \in \{1, ..., 4\}$, $\lambda_1, ..., \lambda_m \subseteq [0, 1]$, and $F_1, ..., F_m \in \Sigma$, such that

$$F = \sum_{i=1}^{m} \lambda_i F_i \qquad \text{and} \qquad f^{pc}(F) = f(F_i), \quad \text{for } i = 1, ..., m.$$

Set $\tilde{\Sigma} := \{F_1, ..., F_m\}$ and define, for every $\delta > 0$ arbitrarily small

$$V_\delta := \{F \in \Sigma : d(F, \tilde{\Sigma}) \leq \delta\}, \qquad \omega_\delta := \sup_{G \in V_\delta} f(G) - f^{pc}(F).$$

Since f is continuous on Σ, then $\omega_\delta \to 0$, as $\delta \to 0$. Lemma 4.11 provides a piecewise affine Lipschitz map $\varphi_{\delta,F} : \Omega \to \mathbb{R}^3$ such that $\nabla\varphi_{\delta,F} \in V_\delta^{rc}$ a.e.,

$$|\{x \in \Omega : \nabla\varphi_{\delta,F}(x) \notin V_\delta\}| \leq \delta|\Omega|, \qquad \text{and} \qquad \varphi_{\delta,F}(x) = Fx \ \text{ on } \partial\Omega.$$

Now, $V_\delta^{lc} \subseteq V_1^{lc}$ for every δ arbitrarily small and, since Σ is stable under taking the lamination convex hull, it turns out that V_1^{lc} is a bounded set contained in Σ. Therefore, if M is an upper bound for f on V_1^{lc},

$$\begin{aligned} \int_\Omega f(\nabla\varphi_{\delta,F})dx &\leq |\{x \in \Omega : \nabla\varphi_{\delta,F}(x) \in V_\delta\}|(f^{pc}(F) + \omega_\delta) + \delta|\Omega|M \\ &\leq |\Omega|f^{pc}(F) + |\Omega|(\omega_\delta + \delta M). \end{aligned}$$

Hence, 4.82 is true with $\mathcal{O}(\delta) := |\Omega|(\omega_\delta + \delta M)$ and this concludes the proof of the theorem. □

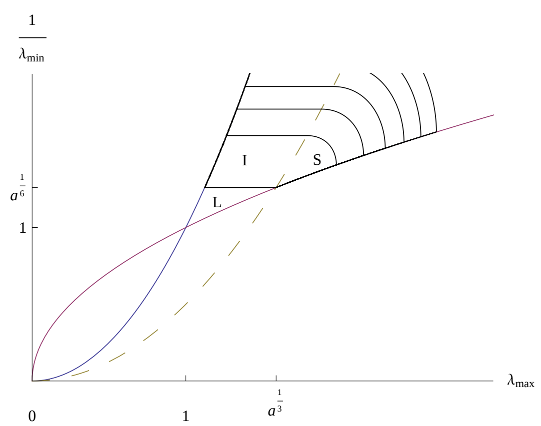

FIGURE 4.3. Level curves of the quasiconvex envelope (4.70) of the Ogden–type energy (4.36) and illustration of the sets L, I_1, and S appearing in its definition ($a = 4$).

From the expression of W^{qc} in (4.70) we obtain that in the region L the response of the system in completely soft and the nematic elastomer behaves essentially like a liquid. On the other hand, in the region S the expression of W^{qc} is equal to W and describes an Ogden–type rubber, so that the material behaves like an elastic solid. In the intermediate region I_1, the energy W^{qc} depends only on the smallest singular value of the deformation gradient and the material response is intermediate between liquid and solid.

Remark 4.12. For sake of completeness, we remark that the quasiconvexification of the Mooney–Rivlin model W_2 given by (4.38) has been obtained by Šilhavý in [**61**]. We have that

$$W_2^{qc}(F) = \begin{cases} 0 & \text{if } F \in L \\ \frac{c_1}{2}\left[\left(a^{\frac{1}{6}}\lambda_1(F)\right)^2 + \frac{2}{a^{\frac{1}{6}}\lambda_1(F)} - 3\right] + \frac{d_1}{2}\left[\left(\frac{1}{a^{\frac{1}{6}}\lambda_1(F)}\right)^2 + 2a^{\frac{1}{6}}\lambda_1(F) - 3\right], & \text{if } F \in I_1, \\ W_2(F) & \text{if } F \in S, \end{cases}$$

where L, I_1, and S are the sets defined in (4.67)–(4.69).

In the rest of this section we will use the knowledge of the quasiconvex envelope W^{qc} (see (4.70)) of W (see (4.36)) to examine the mechanical response of a sample tested in pure shear, through stress–strain curves. Pure shear is a plane strain condition (also called plane strain extension) often used in classical rubber elasticity to assess the performance of constitutive models, see [**54, 66**] and Figure 4.6 for a sketch illustrating these laoding conditions. We are unaware of experimental results on nematic elastomers that exploit this geometry, which we propose for future investigations. In fact, stress–strain curves have been typically obtained, up to now, from uniaxial extension tests on narrow strips of thin films (we refer to [**9, 16, 17**] for theoretical results on uniaxial extension in thin film geometries, and to [**35, 55**] for the corresponding experimental measurements).

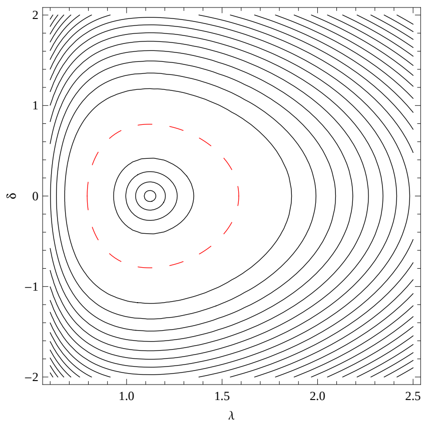

Figure 4.4. Level curves of the Ogden–type energy (4.85) with $a = 4$, $c_1 = 1$, $\gamma_1 = 4$, arbitrary units. The dashed (red) line gives the zero level set describing the spontaneous deformations that minimize the energy density.

Plane strain conditions lead to a simplified expression for $W(F)$ (which becomes a function of $\lambda_3(F)$ alone) and to a very transparent representation of the quasiconvex envelope in (λ, δ)–plane, where λ and δ denote applied stretch and shear, respectively. We start by rewriting the energy given in (4.36) as

$$W(F) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[\left(\frac{\lambda_1(F)}{a^{-\frac{1}{6}}}\right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{a^{-\frac{1}{6}}}\right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{a^{\frac{1}{3}}}\right)^{\gamma_i} - 3\right]. \tag{4.83}$$

Consider the plane strain conditions encoded by the deformation gradient

$$F(\lambda, \delta) = \begin{bmatrix} a^{-\frac{1}{6}} & 0 & 0 \\ 0 & \lambda & \delta \\ 0 & 0 & \frac{a^{\frac{1}{6}}}{\lambda} \end{bmatrix}, \tag{4.84}$$

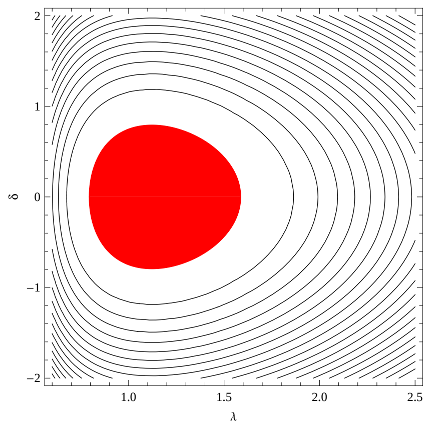

FIGURE 4.5. Level curves of the quasiconvex envelope of the Ogden–type energy (4.85) with $a = 4$, $c_1 = 1$, $\gamma_1 = 4$, arbitrary units. The shaded (red) region gives the set of macroscopic strains that can be accommodated at zero energy.

where $\lambda > 0$. From the expression of $F(\lambda, \delta)$ it is clear that $a^{-\frac{1}{6}}$ is a singular value and that $\lambda_3(F(\lambda, \delta)) \neq a^{-\frac{1}{6}}$, because $F(\lambda, \delta) \in \Sigma$ and $a > 1$. Thus, the expression (4.83) with $F = F(\lambda, \delta)$ simplifies to

$$W(F(\lambda,\delta)) = \sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[1 + \left(\frac{a^{\frac{1}{3}}}{\lambda_3} \right)^{\gamma_i} + \left(\frac{\lambda_3}{a^{\frac{1}{3}}} \right)^{\gamma_i} - 3 \right], \tag{4.85}$$

where $\lambda_3 = \lambda_3(F(\lambda, \delta))$, with equality holding if and only if $\lambda_3 = a^{\frac{1}{3}}$. Referring now to and (4.67)–(4.70), we observe that if $\lambda_3 \leq a^{\frac{1}{3}}$, then $F(\lambda, \delta) \in L \cap I_1$, and if $\lambda_3 \leq a^{\frac{1}{3}}$, then $F(\lambda, \delta) \in S$, so that

$$W^{qc}(F(\lambda,\delta)) = \begin{cases} 0 & \text{if } \lambda_3(F(\lambda,\delta)) \leq a^{\frac{1}{3}}, \\ W(F(\lambda,\delta)) & \text{if } \lambda_3(F(\lambda,\delta)) \geq a^{\frac{1}{3}}. \end{cases} \tag{4.86}$$

Simple computations give

$$\lambda_3^2(F(\lambda,\delta)) = \frac{1}{2}\left(\lambda^2 + \delta^2 + \frac{a^{\frac{1}{3}}}{\lambda^2} \right) + \frac{1}{2}\sqrt{\left(\lambda^2 + \delta^2 + \frac{a^{\frac{1}{3}}}{\lambda^2} \right)^2 - 4a^{\frac{1}{3}}},$$

and $\lambda_3(F(\lambda, \delta)) \leq a^{\frac{1}{3}}$ if and only if $\lambda \in \left[a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right]$ and $|\delta| \leq \delta^*(\lambda)$, where

$$\delta^*(\lambda) := \frac{1}{\lambda}\left(\lambda^2 - \frac{1}{a^{\frac{1}{3}}} \right)^{\frac{1}{2}} \left(a^{\frac{2}{3}} - \lambda^2 \right)^{\frac{1}{2}}. \tag{4.87}$$

This allows us to rewrite (4.86) as

$$W^{qc}(F(\lambda,\delta)) = \begin{cases} 0 & \text{if } \lambda \in \left[a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right] \text{ and } |\delta| \leq \delta^*(\lambda), \\ W(F(\lambda,\delta)) & \text{otherwise.} \end{cases} \tag{4.88}$$

Level curves of energy

$$f(\lambda,\delta) := W(F(\lambda,\delta)).$$

and of (4.88) are shown in Figures 4.4 and 4.5. These plots clearly illustrate that, in fact,

$$W^{qc}(F(\lambda,\delta)) = f^c(\lambda,\delta).$$

Observe that at a macroscopic unsheared ($\delta = 0$) deformation with $\lambda \in \left(a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right)$ the energy $W^{qc}(F(\lambda, 0)) = f^c(\lambda, 0) = 0$ can be obtained by combining the microscopic deformation states $(\lambda, \pm\delta^*(\lambda))$, with alternating equal and opposite shears of magnitude δ^* given by (4.87), in a stripe domain configuration with stripes of equal width and parallel to the direction x_2 of imposed stretch (see [**14, 16, 17, 30**] for further details and Figure 4.6 for a sketch).

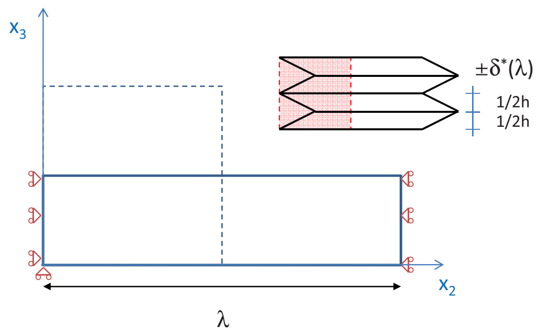

FIGURE 4.6. Sketch of the geometry for the pure shear experiment, and stripe domain patterns with alternating shears $\pm\delta^*(\lambda)$ on stripes of thickness $1/2h$, $h \gg 1$, providing the lowest energy configurations for stretches $\lambda \in \left(a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right)$ in the plateau region.

Since $\frac{\partial}{\partial\delta} f(\lambda, 0) = 0$ (note that f is even in the δ variable), $\delta = 0$ always gives a stationary point for $f(\lambda, \cdot)$. This equilibrium state is, however, unstable if $\lambda \in \left(a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right)$ (the energy plots in Figures 4.4 and 4.7 show a local maximum at $\delta = 0$ along lines with constant λ, leading to a negative shear modulus). Since, as already mentioned, the macroscopic deformation state $(\lambda, \delta = 0)$ can be resolved by a stripe domain pattern alternating the states $(\lambda, \pm\delta^*(\lambda))$ in stripes of equal width at a smaller (in fact, zero) energy cost, we have $W^{qc}(F(\lambda, 0)) = W(F(\lambda, \pm\delta^*(\lambda)))$ (see Figure 4.6), and the quasiconvex envelope W^{qc} can be used to obtain a stable, macroscopically unsheared state of minimal energy for all imposed stretches $\lambda > 0$. The corresponding stresses can be computed from

$$\sigma(\lambda) := \frac{\partial}{\partial\lambda} f^c(\lambda, 0) = \begin{cases} 0 & \text{if } a^{-\frac{1}{6}} \le \lambda \le a^{\frac{1}{3}}, \\ \displaystyle\sum_{i=1}^{N} c_i \left[-\frac{a^{\frac{\gamma_i}{3}}}{\lambda^{\gamma_i+1}} + \frac{\lambda^{\gamma_i - 1}}{a^{\frac{\gamma_i}{3}}} \right] & \text{if } \lambda \ge a^{\frac{1}{3}}. \end{cases} \tag{4.89}$$

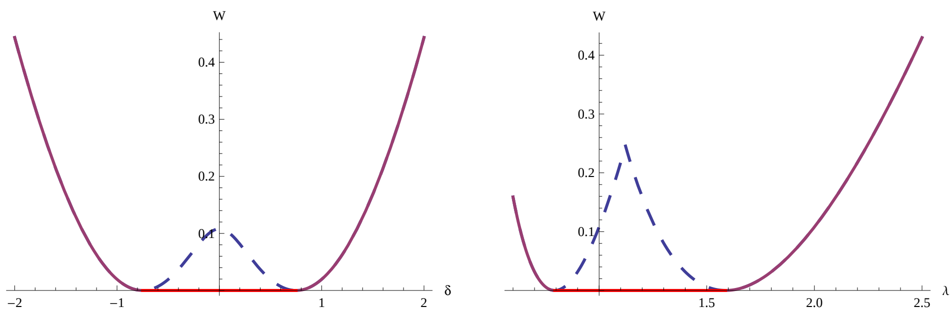

FIGURE 4.7. Sections of the Ogden–type energy (4.85) (dashed line) and of its quasiconvex envelope (4.88) (full line) at constant $\lambda = 1$ (left) and at constant $\delta = 0$ (right). The energy is in arbitrary units, the material parameter values are $a = 4$, $c_1 = 1$, $\gamma_1 = 4$.

In order to obtain the last equality we have used the fact that, since

$$\lambda_3(F(\lambda,0)) = \begin{cases} \dfrac{a^{\frac{1}{6}}}{\lambda} & \text{if } 0 < \lambda \le a^{\frac{1}{12}}, \\ \lambda & \text{if } \lambda \ge a^{\frac{1}{12}}, \end{cases}$$

we also have

$$f^c(\lambda,0) = W^{qc}(F(\lambda,0)) = \begin{cases} \displaystyle\sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[1 + \left(a^{\frac{1}{6}}\lambda\right)^{\gamma_i} + \left(\frac{1}{a^{\frac{1}{6}}\lambda}\right)^{\gamma_i} - 3\right] & \text{if } 0 < \lambda < a^{-\frac{1}{6}}, \\ 0 & \text{if } \lambda \in \left[a^{-\frac{1}{6}}, a^{\frac{1}{3}}\right], \\ \displaystyle\sum_{i=1}^{N} \frac{c_i}{\gamma_i}\left[1 + \left(\frac{a^{\frac{1}{3}}}{\lambda}\right)^{\gamma_i} + \left(\frac{\lambda}{a^{\frac{1}{3}}}\right)^{\gamma_i} - 3\right] & \text{if } \lambda > a^{\frac{1}{3}}. \end{cases}$$

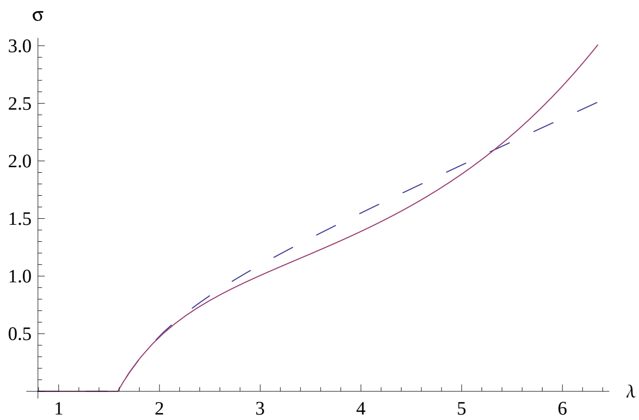

FIGURE 4.8. Stress–strain response in plane strain extension (pure shear). The dashed line comes from the Neo–Hookean expression obtained from (4.89) with $N = 1$, $c_1 = 2$, $\gamma_1 = 2$, the full curve from the Ogden–type expression obtained from (4.89) with $N = 1$, $c_1 = 1$, $\gamma_1 = 4$ ($a = 4$, arbitrary units).

Figure 4.8 shows force–stretch curves for the plane strain extension experiment (4.84) starting from the minimal energy configuration associated with a director uniformly aligned with x_3 (this is given by $\lambda = a^{-\frac{1}{6}}$, $\delta = 0$). Let us focus for simplicity on the Ogden–type expression (4.85) with $N = 1$, that is

$$W(F(\lambda,\delta)) = \frac{c_1}{\gamma_1}\left[1 + \left(\frac{a^{\frac{1}{3}}}{\lambda_3}\right)^{\gamma_1} + \left(\frac{\lambda_3}{a^{\frac{1}{3}}}\right)^{\gamma_1} - 3\right], \tag{4.90}$$

for some $\gamma_1 \ge 2$. We note that while the Neo–Hookean model ($\gamma_1 = 2$) misses the stiffening response at large imposed stretches, this is captured by expression (4.90) as soon as $\gamma_1 > 2$, that is by the simplest expression of an Ogden–type energy density. Indeed, denoting by σ_{NH} the expression given by (4.89) with $N = 1$, $\gamma_1 = 2$, and $c_1 = c$, and by σ_O the same expression with $N = 1$, $\gamma_1 = \gamma > 2$, and $c_1 = d$, we have that for $\lambda \ge a^{\frac{1}{3}}$

$$\sigma_{NH}(\lambda) = c\left[-\frac{a^{\frac{2}{3}}}{\lambda^3} + \frac{\lambda}{a^{\frac{2}{3}}}\right] \qquad \text{and} \qquad \sigma_O(\lambda) = d\left[-\frac{a^{\frac{\gamma}{3}}}{\lambda^{\gamma+1}} + \frac{\lambda^{\gamma-1}}{a^{\frac{\gamma}{3}}}\right]. \tag{4.91}$$

Simple computations show that whereas σ_{NH} is concave for every $\lambda > a^{\frac{1}{3}}$, σ_O changes concavity for some $\lambda > a^{\frac{1}{3}}$. Imposing the conditions $\gamma > 2$ and $\sigma'_{NH}((a^{\frac{1}{3}})^+) = \sigma'_O((a^{\frac{1}{3}})^+)$, the constants in

(4.91) have to fulfill

$$c > d, \qquad \gamma = \frac{2c}{d}.$$

In Figure 4.8, the prediction of σ_{NH} is compared with that of σ_O with $c = 2$, $d = 1$, and $\gamma = 4$. As is well known, the plateau at zero applied stress is unrealistic, and it is possible to add anisotropic corrections to ensure that director reorientation need to be triggered by a nonzero minimum stress level (see, e.g., [**14, 17**]).

CHAPTER 5

Attainment of minimal energy

In this chapter, we collect two attainment results regarding minimum problems for the free energy functionals modeling incompressible nematic elastomers. They have been obtained in collaboration with G. Dal Maso and A. DeSimone. We will consider both the nonlinear model and the geometrically linear one. In Section 5.1 we treat the nonlinear case and consider energy densities of the form

$$W(F) := \begin{cases} \displaystyle\sum_{i=1}^{N} \frac{c_i}{\gamma_i} \left[\left(\frac{\lambda_1(F)}{e_1}\right)^{\gamma_i} + \left(\frac{\lambda_2(F)}{e_2}\right)^{\gamma_i} + \left(\frac{\lambda_3(F)}{e_3}\right)^{\gamma_i} - 3 \right] & \text{if } F \in \Sigma, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\gamma_i \geq 2$, $c_i > 0$ for every $i = 1, ..., N$,

$$\Sigma := \left\{ F \in \mathbb{M}^{3\times 3} \,:\, \det F = 1 \right\},$$

and e_1, e_2, and e_3 are three fixed real numbers such that

$$0 < e_1 \leq e_2 \leq e_3, \qquad\qquad e_1 e_2 e_3 = 1.$$

Recall that $\lambda_1(F) \leq \lambda_2(F) \leq \lambda_3(F)$ are the orered singular values of F. We have seen in Section 4.3 that an energy density of this type governs the purely mechanical response of incompressible nematic elastomers if $e_1 = e_2 = a^{-\frac{1}{6}}$, $e_3 = a^{\frac{1}{3}}$, where a is the material parameter appearing in the definition (1.1) of the nematic tensor L_n. Recalling (4.74), we have that $W \geq 0$ and that $W(F) = 0$ if and only if

$$F \in K := \{ F \in \Sigma \,:\, \lambda_i(F) = e_i,\, i = 1, 2, 3 \}. \tag{5.2}$$

We will prove that for every piecewise affine Lipschitz map $v : \Omega \to \mathbb{R}^3$ such that

$$\nabla v \in \Sigma \quad \text{a.e. in } \Omega, \qquad \operatorname{ess\,inf}_\Omega |\lambda_1(\nabla v)| > e_1, \qquad \operatorname{ess\,sup}_\Omega |\lambda_3(\nabla v)| < e_3, \tag{5.3}$$

there exists a dense set of Lipschitz functions $y : \Omega \to \mathbb{R}^3$ such that

$$W(\nabla y) = 0 \quad \text{a.e. in } \Omega, \qquad y = v \quad \text{on } \partial\Omega.$$

The same result holds if v satisfying (5.3) is of class $C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)$, for some $0 < \alpha < 1$.

In Section 5.2 we will consider the linear case. We will focus on the two dimensional case, because this will allow us to provide simpler proofs of the results. Referring to Remark 2.3 for the derivation of the geometrically linear model in dimension two, and more in general to Chapter 2 for the justification of the small strain theory, we will consider the energy density $V(E) := \min_{n\in S^1} V_n(E)$, for every $E \in Sym(2)$, where V_n is defined in (2.19). Normalizing multiplicative constants, the energy V will thus be defined as

$$V(E) := \begin{cases} \displaystyle\min_{n\in S^1} |E - \hat{U}_n|^2 & \text{if } E \in Sym_0(2), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.4}$$

where $\hat{U}_n$ is given by (2.18). We recall that V governs the purely mechanical response in the small strain limit. Note that $V(E) = 0$ if and only if

$$\begin{aligned} E \in \hat{K}_0 \quad &:= \quad \left\{ E \in Sym_0(2) \,:\, E = 2n\otimes n - I, \text{ for some } n \in S^1 \right\} \\ &= \quad \left\{ E \in Sym_0(2) \,:\, |E| = \sqrt{2} \right\}. \end{aligned} \tag{5.5}$$

In fact, we will mainly use the set

$$K_0 := \left\{ A \in \mathbb{M}_0^{2\times 2} \; : \; symA \in \hat{K}_0 \right\}. \tag{5.6}$$

We will provide a method to find solutions u of class $W_0^{1,p}$, for every $1 \leq p < \infty$, to the problem $V(e(u)) = 0$ a.e. in Ω. This method gives explicit solutions in the case where Ω is a disk.

We will then prove that for every piecewise affine Lipschitz map $w : \Omega \to \mathbb{R}^2$ such that

$$\operatorname{div} w = 0 \quad \text{a.e. in } \Omega, \qquad \operatorname{ess\,sup}_\Omega |e(w)| < \sqrt{2}, \tag{5.7}$$

there exists a dense set of Lipschitz functions $u : \Omega \to \mathbb{R}^2$ such that

$$V(e(u)) = 0 \quad \text{a.e. in } \Omega, \qquad u = w \quad \text{on } \partial\Omega. \tag{5.8}$$

The same result holds if w satisfying (5.7) is of class $C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)$, for some $0 < \alpha < 1$.

The results of this chapter are an application of the theory developed by Müller and Sverák in [**53**]. In that paper, the authors use Gromov's theory of convex integration to study the solutions of first order partial differential relations $\nabla y \in U$ a.e. in Ω, where the unknown $y : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map and U is contained in $\mathscr{S} := \{F \in \mathbb{M}^{m\times n} \; : \; M(F) = t\}$, with $M(F)$ a minor of F, and $t \neq 0$. A crucial step in this theory is the construction of a suitable approximation of U by means of sets U_i relatively open in $\mathscr{S}$ and satisfying some technical assumptions (see Definition 5.3).

In Section 5.1, we provide such an approximation for our set $K \subseteq \mathscr{S}$ with $\mathscr{S} = \Sigma$ and apply the results of [**53**] directly. The same results for the linear constraint $\operatorname{div} u = 0$ (in place of its nonlinear version $\det \nabla y = 1$) are not explicitly stated in [**53**], and, to our knowledge, are not available in the literature. Thus, in Section 5.2 we state without proof Theorem 5.11, which is a linear version of the main result of Müller and Sverák, with slightly simplified assumptions. We then apply Theorem 5.11 to our minimum problem (5.8), where the condition $V(e(u)) = 0$ a.e. in Ω is equivalent to $\nabla u \in K_0$ a.e. in Ω, and K_0 is the set defined in (5.6). In this case, we exhibit a suitable approximation of K_0 by means of sets U_i relatively open in $\mathbb{M}_0^{2\times 2}$ and satisfying some technical assumptions (see Definition 5.10). Finally, in Section 5.3, we prove Theorem 5.11 adapting the approach of [**53**] to the linear constraint $\operatorname{div} u = 0$.

5.1. The nonlinear case

Here and in the following two sections Ω will be a bounded Lipschitz domain of $\mathbb{R}^n$ ($n = 3$ in this section, $n = 2$ in the following ones). Recall that this is a sufficient condition for $W^{1,\infty}(\Omega; \mathbb{R}^m)$ to agree with the class of Lipschitz functions $\mathrm{Lip}(\overline{\Omega}; \mathbb{R}^m)$. The set $W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ is the set of Lipschitz functions which are null on the boundary. It is worth pointing out that this definition differs from the definition of $W_0^{1,p}(\Omega; \mathbb{R}^m)$, for $1 \leq p < \infty$, as the closure of the set $C_c^\infty(\Omega; \mathbb{R}^m)$ in the topology of $W^{1,p}(\Omega; \mathbb{R}^m)$.

The following notion will be crucial in the sequel.

DEFINITION 5.1. *A map $u : \Omega \to \mathbb{R}^m$ is* piecewise affine *if u is continuous and there exist countably many mutually disjoint Lipschitz domains $\Omega_i \subseteq \Omega$ such that*

$$u_{|\Omega_i} \quad \textit{is affine and} \quad \mathscr{L}^n \left(\Omega \setminus \bigcup_i \Omega_i \right) = 0.$$

For every piecewise affine function u, the gradient $\nabla u(x)$ is clearly defined for a.e. x. Note that it may happen that $u \notin W^{1,1}$ even when ∇u is bounded. For instance, in dimension one, the *Cantor–Vitali* function is piecewise affine according to the previous definition.

We consider the following problem: find a minimizer of $\int_\Omega W(\nabla y)dx$, where W is defined in (5.1), under a prescribed boundary condition. This problem is equivalent to the following: given a Dirichlet datum v, find y such that

$$\nabla y \in K \quad \text{a.e. in } \Omega, \qquad y = v \quad \text{on } \partial\Omega,$$

where K, given by (5.2), is the set of the wells of W. We have the following theorem.

THEOREM 5.2. *Consider a piecewise affine Lipschitz map* $v : \Omega \to \mathbb{R}^3$ *such that*

$$\nabla v \in \Sigma \quad a.e. \ in \ \Omega, \qquad \operatorname{ess\,inf}_\Omega |\lambda_1(\nabla v)| > e_1, \qquad \operatorname{ess\,sup}_\Omega |\lambda_3(\nabla v)| < e_3. \tag{5.9}$$

Then, for every $\varepsilon > 0$ *there exists* $y_\varepsilon : \Omega \to \mathbb{R}^3$ *Lipschitz such that*

$$\int_\Omega W(\nabla y_\varepsilon)dx = \min_{y \in v + W_0^{1,2}} \int_\Omega W(\nabla y)dx = 0,$$

and $||y_\varepsilon - v||_\infty \le \varepsilon$. *The same result holds if* $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)$, *for some* $0 < \alpha < 1$, *and satisfies (5.9).*

Note that from Theorem 5.2 we have obtained a dense set of minimizers of the energy functional at the level zero: this makes very interesting the study of a dynamic model which is able to select minimizers. To prove Theorem 5.7, we use the following definition and theorem.

DEFINITION 5.3. *Consider* $K \subseteq \Sigma$. *A sequence of sets* $\{U_i\} \subseteq \Sigma$, *where* U_i *is open in* Σ *for every* i, *is an* in–approximation *of* K *if the following three conditions are satisfied.*

1. $U_i \subseteq U_{i+1}^{lc}$,
2. $\{U_i\}$ *is bounded,*
3. *if* $F_i \in U_i$ *and the sequence* $\{F_i\}$ *converges to* $F \in \mathbb{M}^{3\times 3}$ *as* $i \to \infty$, *then* $F \in K$.

The following theorem is stated and proved in [**53**].

THEOREM 5.4. *Suppose that* $K \subseteq \Sigma$ *admits an in–approximation* $\{U_i\}$ *in the sense of Definition 5.3. Suppose that* $v : \Omega \to \mathbb{R}^3$ *is piecewise affine, Lipschitz, and such that*

$$\nabla v \in U_1 \quad a.e. \ in \ \Omega. \tag{5.10}$$

Then, for every $\varepsilon > 0$ *there exists a Lipschitz map* $y_\varepsilon : \Omega \to \mathbb{R}^3$ *such that*

(i) $\nabla y_\varepsilon \in K$ *a.e. in* Ω,
(ii) $y_\varepsilon = v$ *on* $\partial\Omega$,
(iii) $||u_\varepsilon - v||_{L^\infty} \le \varepsilon$.

The same result holds if $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)$, *for some* $0 < \alpha < 1$, *and satisfies (5.10).*

In Section 5.3 we will prove the analogue of Theorem 5.4 in dimension two with the linear constraint $\operatorname{div} u = 0$ in place of $\det \nabla y = 1$. From that proof, the importance of the approximation by open sets relatively open in the constraint (and endowed with the other technical properties) will emerge.

PROOF OF THEOREM 5.2. As already observed, finding $y : \Omega \to \mathbb{M}^{3\times 3}$ such that $\int_\Omega W(\nabla y)dx = 0$ is equivalent to finding $y : \Omega \to \mathbb{M}^{3\times 3}$ such that $\nabla y \in K$ a.e. in Ω. Thus, to prove the theorem, we can directly apply Theorem 5.4 showing that K admits an in–approximation in the sense of Definition 5.3. The two inequalities in (5.9) imply that there exist two sequences $\{e_1^i\}$, $\{e_3^i\} \subseteq (e_1, e_3)$ such that

$$e_1 < e_1^1 < \operatorname{ess\,inf}_\Omega |\lambda_1(\nabla v)| \le \operatorname{ess\,sup}_\Omega |\lambda_3(\nabla v)| < e_3^1 < e_3, \tag{5.11}$$

and

$$\{e_1^i\} \text{ strictly decreasing }, \ e_1^i \to e_1, \quad \{e_3^i\} \text{ strictly increasing }, \ e_3^i \to e_3.$$

We define

$$U_1 := \left\{F \in \Sigma \,:\, \Lambda(F) \subseteq (e_1^1, e_3^1)\right\} \tag{5.12}$$

and, to define U_i for every $i \ge 2$, we distinguish the following three cases:

(1) if $e_1 = e_2 < e_3$, we define

$$U_i := \left\{F \in \Sigma \,:\, \lambda_1(F), \lambda_2(F) \in (e_1^i, e_1^{i-1}),\ \lambda_3(F) \in (e_3^{i-1}, e_3^i)\right\};$$

(2) if $e_1 < e_2 < e_3$, we consider a strictly increasing sequence $\{e^i_{2,-}\} \subseteq (e_1, e_3)$ such that $e^i_{2,-} \to e_2$ and a strictly decreasing sequence $\{e^i_{2,+}\} \subseteq (e_1, e_3)$ such that $e^i_{2,+} \to e_2$. We can also suppose that $e^i_1 < e^i_{2,-} < e^i_{2,+} < e^i_3$. We define

$$U_i := \left\{F \in \Sigma \,:\, \lambda_1(F) \in (e^i_1, e^{i-1}_1),\ \lambda_2(F) \in (e^i_{2,-}, e^i_{2,+}),\ \lambda_3(F) \in (e^{i-1}_3, e^i_3)\right\};$$

(3) if $e_1 < e_2 = e_3$, we define

$$U_i := \left\{F \in \Sigma \,:\, \lambda_1(F) \in (e^i_1, e^{i-1}_1),\ \lambda_2(F), \lambda_3(F) \in (e^{i-1}_3, e^i_3)\right\}.$$

It is clear that in each of these cases U_i is open in Σ for every $i \geq 1$, that $\{U_i\}_{i\geq 1}$ is bounded, and that if $F_i \in U_i$ and $F_i \to F$ then $\Lambda(F) = \{e_1, e_2, e_3\}$. Now, let us check that $U_i \subseteq U^{lc}_{i+1}$ for every $i \geq 1$. We note that

$$U^{lc}_{i+1} \supseteq \left\{F \in \Sigma \,:\, \Lambda(F) \subseteq (e^{i+1}_1, e^{i+1}_3)\right\}. \tag{5.13}$$

To see this, let us focus on case (1) (in the other cases, inclusion (5.13) can be proved similarly). For every $\alpha_1 > e^{i+1}_1$ arbitrarily close to e^{i+1}_1 and for every $\alpha_3 < e^{i+1}_3$ arbitrarily close to e^{i+1}_3, we have that $e^{i+1}_1 < \alpha_1 < e^i_1 < e^i_3 < \alpha_3 < e^{i+1}_3$, so that,

$$\{F \in \Sigma : \lambda_1(F) = \lambda_2(F) = \alpha_1,\ \lambda_3(F) = \alpha_3\} \subseteq U_{i+1}.$$

Thus,

$$\{F \in \Sigma : \Lambda(F) \subseteq [\alpha_1, \alpha_3]\} = \{F \in \Sigma : \lambda_1(F) = \lambda_2(F) = \alpha_1,\ \lambda_3(F) = \alpha_3\}^{lc} \subseteq U^{lc}_{i+1}, \tag{5.14}$$

where the first equality is guaranteed by Theorem 4.9. Therefore, since (5.14) is true for every $\alpha_1 > e^{i+1}_1$ arbitrarily close to e^{i+1}_1 and for every $\alpha_3 < e^{i+1}_3$ arbitrarily close to e^{i+1}_3, inclusion (5.13) follows. The fact that trivially $U_i \subseteq \left\{F \in \Sigma \,:\, \Lambda(F) \subseteq (e^{i+1}_1, e^{i+1}_3)\right\}$ and (5.13) conclude the proof that condition (1) of Definition 5.3 holds and conclude the proof of the theorem. $\square$

5.2. The geometrically linear case

In this section, we consider the problem of finding a minimizer of the functional $\int_\Omega V(e(u))dx$, where V is defined as in (5.4), under a prescribed boundary condition. This problem is equivalent to the following: given a Dirichlet datum w, find u such that

$$\nabla u \in K_0 \quad \text{a.e. in } \Omega, \qquad u = w \quad \text{on } \partial\Omega, \tag{5.15}$$

where K_0 is defined as in (5.6).

In order to prove the results of this section, it is useful to recall the following fundamental corollary of Vitali's Covering Theorem. We refer the reader to [**21**] for its proof.

Theorem 5.5 (Corollary of Vitali's Covering Theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $G \subseteq \mathbb{R}^n$ a compact set with $|G| > 0$. Let $\mathscr{G}$ be a family of translated and dilated sets of G such that for almost every $x \in \Omega$ and $\varepsilon > 0$ there exists $\hat{G} \in \mathscr{G}$ with $\operatorname{diam} \hat{G} < \varepsilon$ and $x \in \hat{G}$.*

Then, there exists a countable subset $\{G_k\} \subseteq \mathscr{G}$ such that

$$\bigcup_k G_k \subseteq \Omega, \quad G_k \cap G_h = \emptyset \ \text{ for every } k \neq h, \quad \mathscr{L}^n\left(\Omega \setminus \bigcup_k G_k\right) = 0.$$

Note that, chosen $A \in K_0$, we have that the affine function $x \mapsto Ax$ is a trivial solution of (5.15) with $w(x) = Ax$. We address the preliminary question whether there exists a (nontrivial) solution of problem (5.15) with $w \equiv 0$. We denote by (x, y) the coordinates of a generic point ξ of $\mathbb{R}^2$ and write $u = (u_1, u_2)$, so that

$$e(u) = \begin{bmatrix} \partial_x u_1 & \frac{\partial_y u_1 + \partial_x u_2}{2} \\ \frac{\partial_y u_1 + \partial_x u_2}{2} & \partial_y u_2 \end{bmatrix}.$$

Note that $\nabla u \in K_0$ a.e. in Ω is equivalent to

$$\operatorname{div} u = 0 \quad \text{and} \quad |e(u)| = \sqrt{2} \qquad \text{a.e. in } \Omega, \tag{5.16}$$

and these conditions give the following nonlinear system of partial differential equations in Ω:

$$\begin{cases} \partial_x u_1 + \partial_y u_2 = 0, \\ (\partial_x u_1)^2 + \left(\frac{\partial_y u_1 + \partial_x u_2}{2}\right)^2 = 1. \end{cases} \tag{5.17}$$

In order to solve this system, a possible strategy is to choose u as a $\frac{\pi}{2}$–(counterclockwise) rotation of the gradient of a scalar function, that is

$$u = (-\partial_y \varphi, \partial_x \varphi), \qquad \text{for some scalar–valued function } \varphi. \tag{5.18}$$

This gives automatically $\operatorname{div} u = 0$ and the constraint on the norm becomes

$$(\partial^2_{xy}\varphi)^2 + \left(\frac{\partial^2_{xx}\varphi - \partial^2_{yy}\varphi}{2}\right)^2 = 1. \tag{5.19}$$

This a fully nonlinear second–order partial differential equation which does not seem to be studied yet.

Proposition 5.6. *There exists $u \in W^{1,p}_0(\Omega;\mathbb{R}^2)$, for every $1 \le p < \infty$, such that*

$$\int_\Omega V(e(u))dx = \min_{w \in W^{1,2}_0} \int_\Omega V(e(w))dx = 0. \tag{5.20}$$

If $\Omega = B_r$ for some $r > 0$, then

$$u(x,y) = \pm \log\left(\frac{x^2+y^2}{r^2}\right)(-y,x) \tag{5.21}$$

is a solution to (5.20).

We remark that the case of the disk is very lucky, leading to the explicit solution u defined in (5.21). Observe that $u \in C(\overline{B}_r;\mathbb{R}^2)$ and that

$$\nabla u(x,y) = 2\begin{bmatrix} -\frac{xy}{\rho^2} & -\frac{y^2}{\rho^2} - \log\left(\frac{\rho}{r}\right) \\ \frac{x^2}{\rho^2} + \log\left(\frac{\rho}{r}\right) & \frac{xy}{\rho^2} \end{bmatrix},$$

so that $\nabla u \in C^\infty(\overline{B}_r \setminus \{0\};\mathbb{M}^{2\times 2})$. Moreover, $e(u) \in L^\infty(B_r;\mathbb{M}^{2\times 2})$, whereas ∇u is unbounded about the origin. To find a solution when Ω is not a disk, the strategy is to express Ω as a disjoint union of a sequence of disks and a null set. This method, which works for homogeneous boundary conditions, does not provide solutions as explicit as those on disks. Clearly, we cannot exclude that other explicit solutions defined on domains with some special geometries can be found out.

Proof. Let us proceed as anticipated in (5.18)–(5.19) and look for solutions of (5.19) of the form $\varphi(x,y) = \psi(\rho^2)$, where $\rho := \sqrt{x^2+y^2}$. In this case, equation (5.19) becomes an ordinary differential equation in ρ:

$$\begin{aligned} 1 &= (4xy\psi'')^2 + \left(\frac{4x^2\psi'' - 4y^2\psi''}{2}\right)^2 \\ &= 4\rho^4(\psi'')^2, \end{aligned}$$

which gives $\varphi(x,y) = \psi(\rho^2) = \pm\frac{1}{2}\left(\rho^2\log\rho^2 - 1\right) + C_1\rho^2 + C_2$. Setting $u = (-\partial_y\varphi, \partial_x\varphi)$ and imposing $u = 0$ on $\partial B(0,r)$ we obtain (5.21). By Theorem 5.5, there exists a countable collection $\{B_i\}$ of disjoint closed disks in Ω such that

$$\mathscr{L}^2\left(\Omega \setminus \bigcup_i B_i\right) = 0.$$

Let $\xi_i \in \mathbb{R}^2$ and $r_i > 0$ be the center and the radius of the ball B_i, respectively. The function defined by

$$u_i(\xi) := r_i u\left(\frac{\xi - \xi_i}{r_i}\right), \qquad \text{for every } \xi \in B_i,$$

is such that $\operatorname{div} u_i = 0$ and $|e(u_i)| = \sqrt{2}$ a.e. in B_i, and $u_i = 0$ on ∂B_i. Now, define

$$u := \begin{cases} 0 & \text{on } \Omega \setminus \bigcup_i B_i, \\ u_i & \text{on } B_i, \text{ for every } i. \end{cases}$$

This function is a solution to our problem. To see this, let us introduce the functions

$$u^k := \begin{cases} 0 & \text{on } \Omega \setminus \bigcup_{i=1}^k B_i, \\ u_i & \text{on } B_i, \text{ for every } i = 1, ..., k. \end{cases}$$

Extending each u_i at zero outside B_i, we can also write $u = \sum_i u_i 1_{B_i}$ and $u^k = \sum_i^k u_i 1_{B_i}$ and it is clear that

$$u^k(x) \to u(x), \qquad \text{as } k \to \infty, \qquad \text{for every } x \in \Omega. \tag{5.22}$$

Since $|e(u^k)| \leq \sqrt{2}$ a.e. in Ω, we have that $\{u^k\}$ is equibounded in $W_0^{1,2}(\Omega; \mathbb{R}^2)$, by Korn's inequality (see Theorem 1.14). This fact, together with the pointwise convergence (5.22) gives that $u \in W_0^{1,2}(\Omega; \mathbb{R}^2)$. Finally, the fact that $\operatorname{div} u = 0$ and $|e(u)| = \sqrt{2}$ a.e. in Ω comes from the definition of u itself, from the corresponding properties of u_i on B_i, and from the fact that $\mathscr{L}^2 (\Omega \setminus \bigcup_i B_i) = 0$. Another application of Korn's inequality and the fact that $|e(u)| = \sqrt{2}$ a.e. in Ω then implies $u \in W_0^{1,p}(\Omega; \mathbb{R}^2)$, for every $1 \leq p < \infty$. □

In Section 2.2 we have used a relaxation result of [**12**] applied to nematic elastomers' geometrically linear model in three dimension. Clearly, the same result holds in two dimensions and tells us that the relaxation of the functional $\int_\Omega V(e(u))dx$ in the weak sequential topology of $W^{1,2}$ is given by $\int_\Omega V^{qce}(e(u))dx$ (for every u such that $\operatorname{div} u = 0$), where V^{qce} is the quasiconvex envelope on linear strains of V (see Definition 1.6) and it is given by

$$V^{qce}(E) := \begin{cases} \min_{Q \in \mathscr{Q}} |E - Q|^2 & \text{if } A \in Sym_0(2) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathscr{Q} := \{E \in Sym_0(2) \text{ with eigenvalues in } [-1, 1]\}.$$

In particular, for every $v \in W^{1,2}(\Omega; \mathbb{R}^2)$ such that

$$\operatorname{div} v = 0 \qquad \text{and} \qquad |e(v)| \leq \sqrt{2} \qquad \text{a.e. in } \Omega, \tag{5.23}$$

$$\inf_{u \in v + W_0^{1,2}} \int_\Omega V(e(u))dx = \min_{u \in v + W_0^{1,2}} \int_\Omega V^{qce}(e(u))dx = 0. \tag{5.24}$$

The following theorem tells us essentially that if $\operatorname{ess\,sup}_\Omega |e(v)| < \sqrt{2}$, then there exists a minimizer of the unrelaxed functional too.

Theorem 5.7. *Consider a piecewise affine Lipschitz map $v : \Omega \to \mathbb{R}^2$ such that*

$$\operatorname{div} v = 0 \quad \text{a.e. in } \Omega, \qquad \operatorname{ess\,sup}_\Omega |e(v)| < \sqrt{2}. \tag{5.25}$$

Then, for every $\varepsilon > 0$ there exists $u_\varepsilon : \Omega \to \mathbb{R}^2$ Lipschitz such that

$$\int_\Omega V(e(u_\varepsilon))dx = \min_{u \in v + W_0^{1,2}} V(e(u))dx = 0,$$

and $||u_\varepsilon - v||_{L^\infty(\Omega; \mathbb{R}^2)} \leq \varepsilon$. The same result holds if $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)$, for some $0 < \alpha < 1$, and satisfies (5.25).

Condition 5.23 and equalities (5.24) leads to suppose that the result of Theorem 5.7 can be obtained even with $|e(v)| \leq \sqrt{2}$ a.e.. Nevertheless, the proof of Theorem 5.7, based on the following definitions and on Theorem 5.11, strongly relies on the fact that $\operatorname{ess\,sup}_\Omega |e(v)| < \sqrt{2}$.

DEFINITION 5.8. *For every set $U \subseteq \mathbb{M}^{2\times 2}$, we define*

$$\tilde{U} := \{(1-\lambda)A + \lambda B \,:\, A, B \in U,\, 0 \le \lambda \le 1,\, \mathrm{rank}(A-B) = 1\}.$$

REMARK 5.9. Observe that if $U \subseteq \mathbb{M}_0^{2\times 2}$ is open in $\mathbb{M}_0^{2\times 2}$, then $\tilde{U}$ is open in $\mathbb{M}_0^{2\times 2}$. To see this, consider $C \in \tilde{U}$ and suppose that $C + D \in \mathbb{M}_0^{2\times 2}$. We want to show that $C + D \in \tilde{U}$ if $|D|$ is sufficiently small. We have that $C = (1-\lambda)A + \lambda B$ for some $0 \le \lambda \le 1$ and some A, $B \in U$. Note that $A + D$, $B + D \in \mathbb{M}_0^{2\times 2}$ and that $A + D$, $B + D \in U$ if $|D|$ is sufficiently small, in view of the fact that U is open in $\mathbb{M}_0^{2\times 2}$. Thus, $C + D \in \tilde{U}$, because

$$C + D = (1-\lambda)(A + D) + \lambda(B + D),$$

and $\mathrm{rank}[(A + D) - (B + D)] = 1$.

Note that $\tilde{U}$ this is the first set among those which recursively define the lamination convex hull of U, according to Proposition 1.9. Indeed, to prove Theorem 5.7, we will use Theorem 5.11, which is a slightly simplified version of Theorem 5.4 where instead of considering $K \subseteq \Sigma$ endowed with an in–approximation, we use $K_0 \subseteq \mathbb{M}_0^{2\times 2}$ endowed with a strong–in–approximation, according to the following definition.

DEFINITION 5.10. *Consider $K_0 \subseteq \mathbb{M}_0^{2\times 2}$. A sequence of sets $\{U_i\} \subseteq \mathbb{M}_0^{2\times 2}$, where U_i is open in $\mathbb{M}_0^{2\times 2}$ for every i, is a* strong–in–approximation *of K_0 if the following three conditions are satisfied.*

1. *$U_i \subseteq \tilde{U}_{i+1}$,*
2. *$\{U_i\}$ is bounded,*
3. *if $F_i \in U_i$ and the sequence $\{F_i\}_i$ converges to $F \in \mathbb{M}^{2\times 2}$ as $i \to \infty$, then $F \in K_0$.*

THEOREM 5.11. *Suppose that $K_0 \subseteq \mathbb{M}_0^{2\times 2}$ admits a strong–in–approximation $\{U_i\}$ in the sense of Definition 5.10. Suppose that $v : \Omega \to \mathbb{R}^2$ is piecewise affine, Lipschitz, and such that*

$$\nabla v \in U_1 \quad \text{a.e. in } \Omega.$$

Then, for every $\varepsilon > 0$ there exists a Lipschitz map $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that

(i) $\nabla u_\varepsilon \in K_0$ a.e. in Ω,
(ii) $u_\varepsilon = v$ on $\partial\Omega$,
(iii) $||u_\varepsilon - v||_{L^\infty} \le \varepsilon$.

We devote Section 5.3 to the proof of this theorem. Now, let us apply it to prove Theorem 5.7. In order to do this, we have to exhibit a strong–in–approximation of the set K_0 (defined by (5.5)–(5.6)) in the sense of Definition 5.10. Observe that every $E \in Sym_0(2)$ such that $|E| = \sqrt{2}$ can be written as

$$E = \begin{bmatrix} e_1 & e_2 \\ e_2 & -e_1 \end{bmatrix}, \quad \text{with} \quad e_1^2 + e_2^2 = 1.$$

Thus, we can represent every $A \in K_0$ in the following way:

$$A = symA + skwA = \begin{bmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & a_3 \\ -a_3 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 + a_3 \\ a_2 - a_3 & -a_1 \end{bmatrix},$$

where $a_1^2 + a_2^2 = 1$. Denoting $a := (a_1, a_2)$, the set K_0 has the equivalent expression

$$K_0 = \{A \in \mathbb{M}_0^{2\times 2} \,:\, |a| = 1\},$$

and it is easy to verify that, for every A, $B \in K_0$, the condition $\mathrm{rank}\,(A - B) = 1$ is equivalent to

$$(a_3 - b_3)^2 = |a - b|^2. \tag{5.26}$$

Since a strong–in–approximation has to be bounded, we will have to replace K_0 by the bounded set

$$K_0^m := \{A \in K_0 \,:\, |a_3| \le m\},$$

for some $m \ge 2$. Also, we will use the set

$$\mathscr{C}_m := \left\{A \in \mathbb{M}_0^{2\times 2} \,:\, |a_3| \in (m-1, m) \;\text{ and }\; |a| < |a_3| - m + 1\right\}.$$

Proof of Theorem 5.7. Suppose $v : \Omega \to \mathbb{R}^2$ to be piecewise affine and Lipschitz. Since by hypothesis

$$M := \operatorname{ess\,sup}_{\Omega} \frac{|e(v)|}{\sqrt{2}} < 1,$$

by choosing $\max\{1/2, M\} < r_0 < 1$, we have that

$$\nabla v \in U_1 := \left\{A \in \mathbb{M}_0^{2\times 2} \,:\, |a| < r_0\,,\, |a_3| < m\right\} \setminus \overline{\mathscr{C}_m} \quad \text{a.e. in } \Omega, \tag{5.27}$$

for some $m \geq 2$. In order to use Theorem 5.11, we construct a suitable strong–in–approximation of K_0^m starting from U_1. We consider a strictly increasing sequence $\{r_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ such that $r_i \to 1^-$ as $i \to \infty$ and define

$$U_i := \{A \in \mathbb{M}_0^{2\times 2} \,:\, r_{i-1} < |a| < r_i\,,\, |a_3| < m\} \setminus \overline{\mathscr{C}_m}, \quad i = 1, 2, \ldots \tag{5.28}$$

See Figure 5.1 for a sketch of the sets K_0^m and U_i. Observe that $\{U_i\}$ is a bounded sequence of sets which are open in $\mathbb{M}_0^{2\times 2}$. Also, it is clear from the geometry of these sets that whenever $F_i \in U_i$ and $F_i \to F \in \mathbb{M}^{2\times 2}$ as $i \to \infty$, then $F \in K_0^m$. It remains to check that the first condition of Definition 5.10 hold. Consider $C \in U_i$ and suppose for simplicity that $0 \leq c_3 < m$ (the case $-m < c_3 < 0$ can be treated in a similar way). In particular, we have that $|c| < r_i$ and, if $c_3 > m - 1$, then

$$|c| \geq c_3 - m + 1, \tag{5.29}$$

by definition of $\mathscr{C}_m$. We have to prove that there exist A, $B \in U_{i+1}$ such that

$$\operatorname{rank}(A - B) = 1 \quad \text{and} \quad C = (1-\lambda)A + \lambda B, \quad \text{for some } 0 \leq \lambda \leq 1.$$

Recall that the condition $\operatorname{rank}(A-B) = 1$ is equivalent to (5.26). We fix $\tilde{r} \in (r_i, r_{i+1})$ and choose

$$a := \frac{\tilde{r}c}{|c|}, \qquad b := -\frac{\tilde{r}c}{|c|},$$

so that $|a - b| = 2\tilde{r}$ and

$$c = (1-\lambda)a + \lambda b, \qquad \text{with} \quad \lambda := \frac{\tilde{r} - |c|}{2\tilde{r}}.$$

Choosing $a_3 = b_3 + 2$, we have that (5.26) is satisfied and a necessary condition for A, B to be in U_{i+1} is given by

$$-m + 2\tilde{r} < a_3 < m, \qquad -m < b_3 < m - 2\tilde{r}. \tag{5.30}$$

Now, since the condition $c_3 = (1-\lambda)a_3 + \lambda b_3$ must hold, we have

$$b_3 = c_3 - |c| - \tilde{r}, \qquad a_3 = c_3 - |c| + \tilde{r}. \tag{5.31}$$

Finally, the choices which have been made are compatible if the inequalities

$$|c| - m + \tilde{r} < c_3 < |c| + m - \tilde{r}, \tag{5.32}$$

which have been derived comparing (5.30) and (5.31), hold true. If $c_3 \in [0, m-1]$, then (5.32) is true because $|c| < r_i < \tilde{r} < 1$ and in turn

$$[0, m-1] \subseteq (|c| - m + \tilde{r}, |c| + m - \tilde{r}).$$

If $c_3 \in (m-1, m)$, the first inequality in (5.32) is true because $m \geq 2$, and the second inequality comes from (5.29). It remains to check that A, $B \in U_{i+1}$. Let us check that $A \in U_{i+1}$. First, note that $|a| = \tilde{r} \in (r_i, r_{i+1})$. Secondly, observe that, since $\tilde{r} \geq 1/2$, we have $-m + 2\tilde{r} \geq -m + 1$. Thus, in view of (5.30), to prove that $A \in U_{i+1}$ we are left to check that if $m - 1 < a_3 < m$, then $|a| \geq |a_3| - m + 1$. This last inequality is equivalent to (5.29), in view of (5.31), and (5.29) is true if $c_3 \in (m-1, m)$ and trivially true if $c_3 \in [0, m-1]$. This finish the proof of $U_i \subseteq \tilde{U}_{i+1}$. Hence, we have constructed a strong–in–approximation $\{U_i\}$ of $K_0^m \subseteq K_0$ in the sense of Definition 5.10 and, from (5.27), $\nabla v \in U_1$ a.e. in Ω. Applying Theorem 5.11, we obtain the first part of the theorem. It remains to consider the case where $v \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ (and satisfies (5.25)). Proposition 5.12 ensures the existence of a piecewise affine Lipschitz function $v_\delta : \Omega \to \mathbb{R}^2$ such that $\operatorname{div} v_\delta = 0$

a.e. in Ω, $||v_\delta - v||_{W^{1,\infty}} \le \delta$, and $v_\delta = v$ on $\partial\Omega$. If δ is sufficiently small, we have that $\nabla v_\delta \in U_1$ a.e. in Ω, where U_1 is defined in (5.28), and we can proceed as in the first part of the proof. $\square$

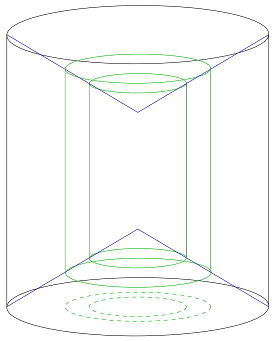

FIGURE 5.1. Illustration of the sets K_0^m and U_i in the (a_1, a_2, a_3)–space. U_i is the set bounded by the green lines and circles.

PROPOSITION 5.12. *Let $u \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ be such that* $\operatorname{div} u = 0$ *in* Ω.
Then, for every $\delta > 0$ there exists a piecewise affine Lipschitz map $u_\delta : \Omega \to \mathbb{R}^2$ such that

$$\begin{aligned} &\operatorname{div} u_\delta = 0 \quad \text{a.e. in } \Omega,\\ &u_\delta = u \quad \text{on } \partial\Omega,\\ &||u_\delta - u||_{W^{1,\infty}} \le \delta. \end{aligned}$$

In order to prove this proposition, we use a procedure already used in [**53**]. The idea is that on a ball $B(a, r)$ where $r^\alpha[\nabla u]_\alpha$ is sufficiently small, u can be replaced by a map with the same boundary values which is affine on $B(a, r/2)$. This replacement can be obtained by introducing first an interpolation between the functions $x \mapsto u(a) + \nabla u(a)(x - a)$ and u in $\overline{B(a, r)} \setminus B(a, r/2)$ and then using the following result of Dacorogna [**20**] to reestablish the constraint.

THEOREM 5.13. *Let $m \ge 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a $C^{m+2,\alpha}$ boundary consisting of finitely many connected components. Set*

$$X := \left\{ f \in C^{m,\alpha}(\overline{\Omega}) \,:\, \int_\Omega f(x)dx = 0 \right\},$$

and

$$Y := \{ u \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \,:\, u = 0 \text{ on } \partial\Omega \}.$$

There exists a bounded linear operator $L : X \to Y$ such that

$$\operatorname{div} L(f) = f,$$

for every $f \in X$.

This theorem tells us that there exists a constant $K = K(m, \alpha, \Omega) > 0$ with the following property: if $f \in C^{m,\alpha}(\overline{\Omega})$ satisfies $\int_\Omega f(x)dx = 0$, then there exists $u \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ verifying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and such that

$$||u||_{C^{m+1,\alpha}} \le K||f||_{C^{m,\alpha}}.$$

The proof of Proposition 5.12 is based on a inductive argument which hinges on the following corollary of Theorem 5.13.

Corollary 5.14. *Let $0 < \alpha < 1$. There exists a constant $C = C(\alpha) > 0$ with the following property. For every $\delta > 0$, $a \in \mathbb{R}^2$, $r > 0$, and every $u \in C^{1,\alpha}(\overline{B(a,r)};\mathbb{R}^2)$ such that*

$$\operatorname{div} u = 0 \quad \text{in } B(a,r) \qquad \text{and} \qquad r^\alpha [\nabla u]_\alpha \leq \delta, \tag{5.33}$$

there exists $\tilde{u} \in C^0(\overline{B(a,r)};\mathbb{R}^2) \cap C^{1,\alpha}(\overline{B(a,r)} \setminus B(a,r/2);\mathbb{R}^2)$ such that

$$\operatorname{div} \tilde{u} = 0 \quad \text{a.e. in } B(a,r), \tag{5.34}$$

$$\nabla \tilde{u}(x) = \nabla u(a) \quad \text{for every } x \in B(a,r/2), \qquad \tilde{u} = u \quad \text{on } \partial B(a,r/2), \tag{5.35}$$

$$r^{-1}||u - \tilde{u}||_\infty + ||\nabla u - \nabla \tilde{u}||_\infty \leq C\delta. \tag{5.36}$$

Note that the function $\tilde{u}$ is affine on $B(a,r/2)$.

Proof. Let us first prove the corollary in the case $a = 0$, $r = 1$, and $u(0) = 0$.

Fix a cut-off function $\varphi \in C_c^\infty(B_1)$ such that $\varphi \equiv 1$ on $B_{\frac{1}{2}}$. For every $u \in C^{1,\alpha}(\overline{B_1};\mathbb{R}^2)$ such that

$$\operatorname{div} u = 0 \quad \text{in } B_1 \qquad \text{and} \qquad [\nabla u]_\alpha \leq \delta, \tag{5.37}$$

we define

$$u_0(x) := \nabla u(0)x, \qquad \text{for every } x \in \overline{B_1},$$

and the interpolation

$$\hat{u} := \varphi u_0 + (1-\varphi)u, \qquad \text{on } \; U := \overline{B_1} \setminus B_{\frac{1}{2}}.$$

Note that

$$f := \operatorname{div} \hat{u} = \nabla \varphi \cdot (u_0 - u), \tag{5.38}$$

that $f \in C^{1,\alpha}(U)$, and that

$$\int_U f(x)dx = \int_{\partial U} \hat{u}(x) \cdot \nu ds(x) = \int_{B_1} \operatorname{div} u(x)dx - \int_{B_{\frac{1}{2}}} \operatorname{tr} \nabla u(0)dx = 0.$$

Thus, by Theorem 5.13, there exists $L(f) \in C^{1,\alpha}(U;\mathbb{R}^2)$ such that $\operatorname{div} L(f) = f$ in U, $L(f) = 0$ on ∂U, and

$$||L(f)||_{C^{1,\alpha}(U;\mathbb{R}^2)} \leq K||f||_{C^{0,\alpha}(U)},$$

where K is a positive constant depending only on α and U. Thus, from (5.38), we have

$$||L(f)||_{C^{1,\alpha}(U;\mathbb{R}^2)} \leq \tilde{K}||u - u_0||_{C^{0,\alpha}(U;\mathbb{R}^2)}, \tag{5.39}$$

where $\tilde{K} > 0$ depends on α, U, and φ. Now, consider the function

$$\tilde{u} := \begin{cases} u_0 & \text{on } \overline{B_{\frac{1}{2}}}, \\ \hat{u} - L(f) & \text{on } U. \end{cases}$$

From the properties of $L(f)$ it turns out that $\tilde{u} \in C^0(\overline{B_1};\mathbb{R}^2) \cap C^{1,\alpha}(U;\mathbb{R}^2)$, that $\operatorname{div} \tilde{u} = 0$ a.e. in B_1, that $\nabla \tilde{u}(x) = \nabla u(0)$ for every $x \in B_{\frac{1}{2}}$, and that $\tilde{u} = u$ on ∂B_1. It remains to check (5.36). Note that

$$||\nabla u - \nabla u_0||_\infty = \sup_{x \in B_1 \setminus \{0\}} \frac{|\nabla u(x) - \nabla u(0)|}{|x|^\alpha}|x|^\alpha \leq [\nabla u]_\alpha,$$

and that

$$||u - u_0||_\infty = \sup_{x \in B_1} \left| \int_0^1 [\nabla u(tx) - \nabla u_0(tx)]\, x dt \right| \leq ||\nabla u - \nabla u_0||_\infty \leq [\nabla u]_\alpha. \tag{5.40}$$

Also, we have that

$$[u - u_0]_\alpha = \sup_{\substack{x,y \in B_1 \\ x \neq y}} \frac{1}{|x-y|^\alpha} \left| \int_0^1 \frac{d}{dt} h(t) dt \right|,$$

where $h(t) := u(tx + (1-t)y) - u_0(tx + (1-t)y)$, for every $0 \le t \le 1$. Thus,

$$\begin{aligned}[u-u_0]_\alpha &\le \sup_{\substack{x,y\in B_1\\ x\neq y}} \int_0^1 \frac{|\nabla u(tx+(1-t)y) - \nabla u(0)|}{|x-y|^\alpha}|x-y|dt \\ &\le \sup_{\substack{x,y\in B_1\\ x\neq y}} \int_0^1 \frac{|\nabla u(tx+(1-t)y) - \nabla u(0)|}{|tx+(1-t)y|^\alpha}|x-y|dt \le 2[\nabla u]_\alpha. \qquad (5.41)\end{aligned}$$

Estimates (5.39) and (5.40)–(5.41) give

$$\begin{aligned}||u-\tilde{u}||_\infty &= ||u-u_0||_{\infty,B_{\frac12}} + ||u-\hat{u}+L(f)||_{\infty,U} \\ &\le ||u-u_0||_\infty + ||u-\hat{u}||_{\infty,U} + ||L(f)||_{\infty,U} \\ &\le [\nabla u]_\alpha + ||\varphi(u-u_0)||_\infty + \tilde{K}||u-u_0||_{C^{0,\alpha}} \\ &= (1+||\varphi||_\infty + 3\tilde{K})[\nabla u]_\alpha, \qquad (5.42)\end{aligned}$$

and

$$\begin{aligned}||\nabla u - \nabla\tilde{u}||_\infty &= ||\nabla(u-\hat{u})||_\infty + ||\nabla L(f)||_\infty \\ &\le ||(u-u_0)\otimes\nabla\varphi||_\infty + ||\varphi\nabla(u-u_0)||_\infty + \tilde{K}||u-u_0||_{C^{0,\alpha}} \\ &\le ||\nabla\varphi||_\infty||u-u_0||_\infty + ||\varphi||_\infty||\nabla u - \nabla u_0||_\infty + \tilde{K}||u-u_0||_{C^{0,\alpha}} \\ &\le 3(||\varphi||_{W^{1,\infty}} + \tilde{K})[\nabla u]_\alpha. \qquad (5.43)\end{aligned}$$

By estimates (5.42) and (5.43), and by (5.37), we obtain

$$||u-\tilde{u}||_{W^{1,\infty}} \le C\delta,$$

where

$$C := 1 + 4||\varphi||_{W^{1,\infty}} + 6\tilde{K}(\alpha, U, \varphi). \qquad (5.44)$$

Now, let us prove the proposition for a generic ball $B(a,r) \subseteq \mathbb{R}^2$ and for every $u \in C^{1,\alpha}(\overline{B(a,r)};\mathbb{R}^2)$ satisfying (5.33). Consider the function $v \in C^{1,\alpha}(\overline{B_1};\mathbb{R}^2)$ defined by

$$v(x) := \frac{u(rx+a) - u(a)}{r}.$$

We have that $v(0) = 0$ and that (5.37) are satisfied with v in place of u. We have seen that the corollary holds in this case, so that there exists $\tilde{v} \in C^0(\overline{B_1};\mathbb{R}^2) \cap C^{1,\alpha}(\overline{B_1}\setminus B_{\frac12};\mathbb{R}^2)$ such that

$$\operatorname{div}\tilde{v} = 0 \quad \text{a.e. in } B_1,$$

$$\nabla\tilde{v}(x) = \nabla v(0) \quad \text{for every } x \in B_{\frac12}, \qquad \tilde{v} = v \quad \text{on } \partial B_{\frac12},$$

and

$$||v-\tilde{v}||_{W^{1,\infty}} \le C\delta,$$

where C is given by (5.44). By these properties of $\tilde{v}$ and by defining

$$\tilde{u}(x) := r\tilde{v}\left(\frac{x-a}{r}\right) + u(a), \quad \text{for every } x \in \overline{B}(a,r),$$

it turns out that $\tilde{u} \in C^0(\overline{B(a,r)};\mathbb{R}^2) \cap C^{1,\alpha}(\overline{B(a,r)}\setminus B(a,r/2);\mathbb{R}^2)$ and satisfies (5.34), (5.35), and (5.36) with C given by (5.44). □

We are now in position to prove Proposition 5.12. The iterative method used in the proof is illustrated in Figure 5.2.

Proof of Proposition 5.12. Fix $\delta > 0$. We are going to construct a decreasing sequence of open sets $\Omega_k \subseteq \Omega$ and a sequence of maps $u^{(k)}$ such that $u^{(0)} = u$, $u^{(k)} \in W^{1,\infty}(\Omega;\mathbb{R}^2)$, and

$$||u^{(k)} - u^{(k+1)}||_{W^{1,\infty}} \leq \tfrac{\delta}{2^{k+1}}, \tag{5.45}$$

$$\operatorname{div} u^{(k)} = 0 \quad \text{a.e. in } \Omega, \tag{5.46}$$

$$u^{(k)} = 0 \quad \text{on } \partial\Omega, \tag{5.47}$$

$$u^{(k+1)} = u^{(k)} \quad \text{on} \quad \bigcup_{i=1}^{n_k} \overline{A_i^{(k)}} \cup N_k = \Omega \setminus \Omega_k, \quad \text{for every } k \geq 1, \tag{5.48}$$

$$|\Omega_{k+1}| \leq \eta |\Omega_k|, \quad \text{for every } k \geq 1, \tag{5.49}$$

where $\eta \in (0,1)$, $u^{(k)}$ is affine on $\overline{A_i^{(k)}}$, and N_k is a null set. This construction implies that there exists a Lipschitz map $v : \Omega \to \mathbb{R}^2$ such that $u^{(k)} \to v$ in $W^{1,\infty}$ (by (5.45)), $\operatorname{div} v = 0$ a.e. on Ω (by (5.46)), and $v = u$ on $\partial\Omega$ (by (5.47)). Moreover, (5.45) implies that

$$||u - u^{(k+1)}||_{W^{1,\infty}} \leq \sum_{i=0}^{k} ||u^{(i)} - u^{(i+1)}||_{W^{1,\infty}} \leq \delta \sum_{i=0}^{k} \frac{1}{2^{i+1}} \leq \delta,$$

for every k, and therefore

$$||u - v||_{W^{1,\infty}} = \lim_{k\to\infty} ||u - u^{(k+1)}||_{W^{1,\infty}} \leq \delta.$$

Finally, (5.49) implies that

$$|\Omega_{k+1}| \leq \eta|\Omega_k| \leq \eta^2 |\Omega_{k-1}| \leq ... \leq \eta^k |\Omega_1|,$$

and in turn that $|\Omega \setminus \Omega_k| \to |\Omega|$, as $k \to \infty$. Since $\Omega \setminus \Omega_k$ is the set where $u^{(k)}$ is piecewise affine, and $u_l = u_k$ on $\Omega \setminus \Omega_k$ for very $l \geq k$ (recall that $\{\Omega \setminus \Omega_k\}$ is an increasing sequence of sets), we have obtained that v is piecewise affine on Ω. Now, let us describe the construction of the sequences $\{\Omega_k\}$ and $\{u^{(k)}\}$. There exist $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ such that $|\Omega''| \geq \frac{1}{2}|\Omega|$. Cover Ω'' by a lattice of n_1 disjoint open squares $C_i^{(1)}$ with half–side $r \leq 1$. If r is sufficiently small, then $\bigcup_{i=1}^{n_1} C_i^{(1)} \subseteq \Omega'$. Also, since Ω' is compactly contained in Ω, there exists a constant $M(\Omega') > 0$ such that

$$[\nabla u]_{\alpha, C_i^{(1)}} \leq M(\Omega'), \qquad \text{for every } i = 1, ..., n_1. \tag{5.50}$$

Let $B_i^{(1)}$ be the open inscribed disk of $C_i^{(1)}$ ($B_i^{(1)}$ has radius r). By (5.50) we have that

$$r^\alpha [\nabla u]_{\alpha, B_i^{(1)}} \leq \frac{\delta}{2} \qquad \text{(if } r \text{ is small enough).}$$

Note that the hypotheses of Corollary 5.14 are satisfied by $u_{|B_i^{(1)}} \in C^{1,\alpha}(B_i^{(1)};\mathbb{R}^2)$. Hence, denoting by $A_i^{(1)}$ the open disk with the same center as $B_i^{(1)}$ and with radius $r/2$, it turns out that there exists $u_i^{(1)} \in C^0\left(\overline{B_i^{(1)}};\mathbb{R}^2\right) \cap C^{1,\alpha}\left(\overline{B_i^{(1)}} \setminus A_i^{(1)};\mathbb{R}^2\right)$ such that

$$\operatorname{div} u_i^{(1)} = 0 \qquad \text{a.e. in } \; B_i^{(1)},$$

$$u_i^{(1)} \text{ is affine in } A_i^{(1)}, \qquad u_i^{(1)} = u \quad \text{on } \partial B_i^{(1)},$$

and

$$||u - u_i^{(1)}||_{W^{1,\infty}(B_i^{(1)};\mathbb{R}^2)} \leq r^{-1} ||u - u_i^{(1)}||_{L^\infty(B_i^{(1)};\mathbb{R}^2)} + ||\nabla u - \nabla u_i^{(1)}||_{L^\infty(B_i^{(1)};\mathbb{M}^{2\times 2})} \leq \frac{c\delta}{2},$$

where the constant $c > 0$ depends only on α. Now, we define

$$u^{(1)} := \begin{cases} u_i^{(1)} & \text{on } B_i^{(1)} \; (i = 1, ..., n_1), \\ u & \text{on } \Omega \setminus \bigcup_{i=1}^{n_1} B_i^{(1)}, \end{cases}$$

and set

$$\Omega_1 := \Omega \setminus \bigcup_{i=1}^{n_1} \left(\overline{A_i^{(1)}} \cup \partial B_i^{(1)} \right).$$

Note that, since the ratio between the area of a disk and the area of a circumscribed square is a constant $\lambda \in (0,1)$, we have that

$$\sum_{i=1}^{n_1} \left| A_i^{(1)} \right| = \lambda \sum_{i=1}^{n_1} \left| C_i^{(1)} \right| \geq \lambda |\Omega''| \geq \frac{\lambda}{2} |\Omega|, \tag{5.51}$$

and in turn

$$|\Omega_1| \leq \eta |\Omega|, \quad 0 < \eta := 1 - \frac{\lambda}{2} < 1. \tag{5.52}$$

From the definition of $u^{(1)}$ we deduce that $u^{(1)}$ is piecewise affine in $\Omega \setminus \Omega_1$, that $\Omega \setminus \Omega_1$ is a finite union of disjoint disks (up to a null set), that $u^{(1)} \in W^{1,\infty}(\Omega;\mathbb{R}^2) \cap C^{1,\alpha}_{loc}(\Omega_1;\mathbb{R}^2)$, that $u^{(1)} = u$ on $\partial\Omega$, that $\operatorname{div} u^{(1)} = 0$ a.e. in Ω, and that

$$||u - u^{(1)}||_{W^{1,\infty}(\Omega;\mathbb{R}^2)} = \max_{i \in \{1,\dots,n_1\}} ||u - u_i^{(1)}||_{W^{1,\infty}(B_i^{(1)};\mathbb{R}^2)} \leq \frac{c(\alpha)\delta}{2}.$$

Now, suppose to have defined $\Omega_1, \dots, \Omega_k$ and $u^{(1)}, \dots, u^{(k)}$ $(k \geq 1)$, and let us see how to construct Ω_{k+1} and $u^{(k)}$. There exist $\Omega_k'' \subset\subset \Omega_k' \subset \Omega_k$ such that $|\Omega_k''| \geq \frac{1}{2}|\Omega_k|$. Cover Ω_k'' by a lattice of n_{k+1} disjoint open squares $C_i^{(k+1)}$ with half-side $r \leq 1$. If r is sufficiently small, then $\bigcup_{i=1}^{n_{k+1}} C_i^{(k+1)} \subseteq \Omega_k'$. Also, since Ω_k' is compactly contained in Ω_k and $u^{(k)} \in C^{1,\alpha}_{loc}(\Omega_k;\mathbb{R}^2)$, there exists a constant $M(\Omega_k') > 0$ such that

$$[\nabla u^{(k)}]_{\alpha, C_i^{(k+1)}} \leq M(\Omega_k'), \quad \text{for every } i = 1, \dots, n_{k+1}. \tag{5.53}$$

Let $B_i^{(k+1)}$ be the open inscribed disk of $C_i^{(k+1)}$. By (5.53), we have that

$$r^\alpha [\nabla u^{(k)}]_{\alpha, B_i^{(k+1)}} \leq \frac{\delta}{2^{k+1}}.$$

Since the hypotheses of Corollary 5.14 are satisfied by $u^{(k)}_{|B_i^{(k+1)}} \in C^{1,\alpha}(B_i^{(k+1)};\mathbb{R}^2)$, labeling by $A_i^{(k+1)}$ the open disk with the same center as $B_i^{(k+1)}$ and with radius $r/2$, there exists $u_i^{(k+1)} \in C^0(\overline{B_i^{(k+1)}};\mathbb{R}^2) \cap C^{1,\alpha}(\overline{B_i^{(k+1)}} \setminus A_i^{(k+1)};\mathbb{R}^2)$ such that

$$\operatorname{div} u_i^{(k+1)} = 0 \qquad \text{a.e. in } B_i^{(k+1)},$$

$$u_i^{(k+1)} \text{ is affine in } A_i^{(k+1)}, \qquad u_i^{(k+1)} = u^{(k)} \text{ on } \partial B_i^{(k+1)},$$

and

$$\begin{aligned} &||u^{(k)} - u_i^{(k+1)}||_{W^{1,\infty}(B_i^{(k+1)};\mathbb{R}^2)} \\ &\qquad \leq r^{-1} ||u^{(k)} - u_i^{(k+1)}||_{L^\infty(B_i^{(k+1)};\mathbb{R}^2)} + ||\nabla u^{(k)} - \nabla u_i^{(k+1)}||_{L^\infty(B_i^{(k+1)};\mathbb{M}^{2\times 2})} \leq \frac{c\delta}{2^{k+1}}, \end{aligned}$$

where the constant $c > 0$ depends only on α. Now, define

$$u^{(k+1)} := \begin{cases} u_i^{(k+1)} & \text{on } B_i^{(k+1)} \ (i = 1, \dots, n_{k+1}), \\ u^{(k)} & \text{on } \Omega \setminus \bigcup_{i=1}^{n_{k+1}} B_i^{(k+1)}, \end{cases}$$

and set

$$\Omega_{k+1} := \Omega_k \setminus \bigcup_{i=1}^{n_{k+1}} \left(\overline{A_i^{(k+1)}} \cup \partial B_i^{(k+1)} \right).$$

Again, we have that $\sum_{i=1}^{n_{k+1}} \left|A_i^{(k+1)}\right| = \lambda \sum_{i=1}^{n_{k+1}} \left|C_i^{(k+1)}\right| \geq \lambda|\Omega_k''| \geq \frac{\lambda}{2}|\Omega_k|$, where λ is the same as in (5.51), and in turn $|\Omega_{k+1}| \leq \eta|\Omega_k|$ where $0 < \eta < 1$ has been defined in (5.52). From the definition of $u^{(k+1)}$ we deduce that $u^{(k+1)}$ is piecewise affine in $\Omega \setminus \Omega_{k+1} \supseteq \Omega \setminus \Omega_k$, that $\Omega \setminus \Omega_{k+1}$ is a finite union of disjoint disks (up to a null set), that $u^{(k+1)} \in W^{1,\infty}(\Omega; \mathbb{R}^2) \cap C^{1,\alpha}_{loc}(\Omega_{k+1}; \mathbb{R}^2)$, that $u^{(k+1)} = u$ on $\partial\Omega$, that $\operatorname{div} u^{(k+1)} = 0$ a.e. in Ω, and that

$$||u^{(k)} - u^{(k+1)}||_{W^{1,\infty}(\Omega;\mathbb{R}^2)} = \max_{i\in\{1,\dots,n_{k+1}\}} ||u^{(k)} - u_i^{(k+1)}||_{W^{1,\infty}(B_i^{(k+1)};\mathbb{R}^2)} \leq \frac{c(\alpha)\delta}{2^{k+1}}.$$

Also, observe that $u^{(k+1)} = u^{(k)}$ on $\Omega \setminus \bigcup_{i=1}^{n_{k+1}} B_i^{(k+1)}$, thus, in particular, $u^{(k+1)} = u^{(k)}$ (piecewise affine) on $\Omega \setminus \Omega_k$. This finishes the construction of the sequences $\{\Omega_k\}$ and $\{u^{(k)}\}$ endowed with the properties stated at the beginning of the proof. $\square$

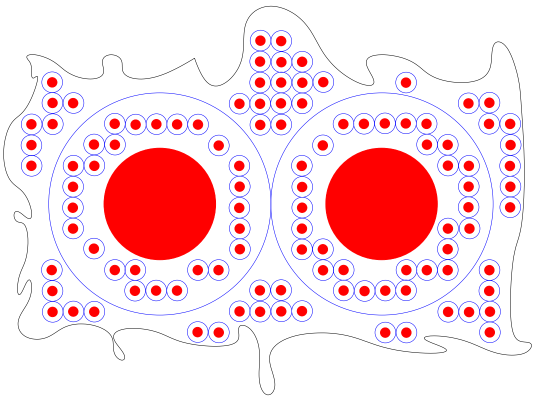

FIGURE 5.2. Schematic picture illustrating the first two step of the iterative procedure used in the proof of Proposition 5.12. The big and the small blue circles are the sets $\partial B_i^{(1)}$ and $\partial B_i^{(2)}$, respectively; the big and the small red disks are the sets $A_i^{(1)}$ and $A_i^{(2)}$, respectively.

5.3. Convex integration for divergence free vector fields

In this section we prove Theorem 5.11, adapting the procedure used in [**53**] to our linear constraint $\operatorname{div} u = 0$. We denote by $[A, B]$ the segment between the matrices A and B.

The proof of Theorem 5.11, postponed at the end of this section, is the last step of an approximation process which passes through some preliminary results: Lemma 5.15, Lemma 5.17, and Theorem 5.18. In Lemma 5.15 we solve the following problem: given two matrices A and B, with rank–one difference, and given $C = (1-\lambda)A + \lambda B$ for some $\lambda \in (0, 1)$, we construct a map u which satisfies the constraint and the boundary condition $u(x) = Cx$, and whose gradient lies in a sufficiently small neighborhood of $[A, B]$. In the next step we consider U relatively open in $\mathbb{M}_0^{2\times 2}$ and $\tilde{U}$ obtained by adding rank–one segments with end points in U (see Definition 5.8). Lemma 5.17 states that for every affine boundary data $x \mapsto Cx$ with $C \in \tilde{U}$, there exists a piecewise affine and Lipschitz map u whose gradient is always in $\tilde{U}$ and most of the time in U. Then, the same iterative method used in the proof of Lemma 5.17 makes it possible to remove step by step the set where $\nabla u \notin U$ and allows for boundary data v such that $\nabla v \in \tilde{U}$ a.e.: this is the content of Theorem 5.18. Finally, the set U relatively open in $\mathbb{M}_0^{2\times 2}$ is replaced by a set $K_0 \subseteq \mathbb{M}_0^{2\times 2}$ satisfying Definition 5.10 (see Theorem 5.11). This last step requires another more subtle iteration process.

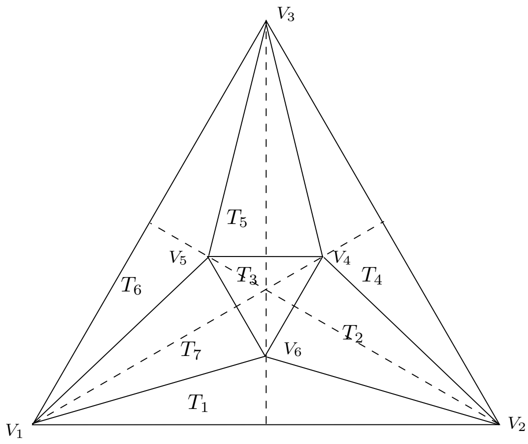

FIGURE 5.3. Triangle T.

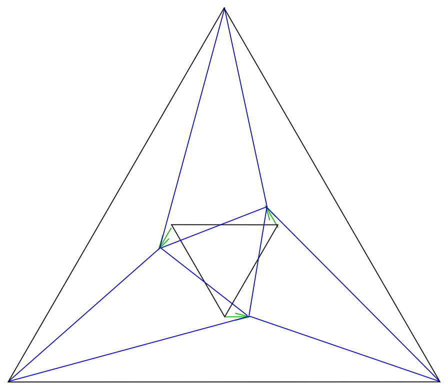

FIGURE 5.4. A prototype of piecewise affine deformation u such that $\operatorname{div} u = 0$ and $u = 0$ on ∂T.

LEMMA 5.15 (Basic construction). *Let A, $B \in \mathbb{M}_0^{2\times 2}$ be such that* $\operatorname{rank}(A - B) = 1$ *and set*

$$C = (1-\lambda)A + \lambda B, \qquad \text{for some } \lambda \in (0,1). \tag{5.54}$$

Then, for every $\varepsilon > 0$ arbitrarily small there exists a piecewise affine Lipschitz map $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that

$$\nabla u_\varepsilon \in \mathbb{M}_0^{2\times 2} \quad \text{a.e. in } \Omega, \tag{5.55}$$

$$u_\varepsilon(x) = Cx, \quad \text{for every } x \in \partial\Omega, \tag{5.56}$$

$$d(\nabla u_\varepsilon, [A,B]) < \varepsilon \quad \text{a.e. in } \Omega, \tag{5.57}$$

$$|\{x \in \Omega : d(\nabla u_\varepsilon, \{A,B\}) \geq \varepsilon\}| \leq c|\Omega|, \tag{5.58}$$

$$\sup_{x\in\Omega} |u_\varepsilon(x) - Cx| < \varepsilon. \tag{5.59}$$

The constant c appearing in (5.58) *is such that $0 < c < 1$ and it does not depend neither on ε nor on Ω.*

To prove Lemma 5.15, we use the following lemma and construct a particular piecewise affine function u with $\operatorname{div} u = 0$ a.e. on an equilateral triangle T and such that $u = 0$ on ∂T.

LEMMA 5.16. *Consider a triangle $T \subseteq \mathbb{R}^2$ with vertices V_1, V_2, and V_3, and an affine function $u : T \to \mathbb{R}^2$ such that $u(V_1) = u(V_2) = 0$. Then,*

$$\operatorname{div} u = 0 \quad \textit{if and only if} \quad u(V_3) \textit{ is parallel to } V_1 - V_2.$$

PROOF. Suppose for simplicity that $V_1 - V_2$ is parallel to the first vector of the canonical basis of $\mathbb{R}^2$. Let ν_1, ν_2, and ν_3 be the outer unit normals on the sides $[V_1, V_2]$, $[V_2, V_3]$, and $[V_3, V_1]$, respectively, so that

$$\nu_1 = (0, a), \qquad \text{with } a \in \{+1, -1\}. \tag{5.60}$$

By the Divergence Theorem and by the fact that $\nabla u(x)$ is constant, we have that

$$|T| \operatorname{tr} \nabla u(x) = u(V_3) \cdot (\nu_2 |V_2 - V_3| + \nu_3 |V_3 - V_1|). \tag{5.61}$$

Since the relation

$$\nu_1 |V_1 - V_2| + \nu_2 |V_2 - V_3| + \nu_3 |V_3 - V_1| = 0$$

holds in general, from (5.61) we obtain that $|T| \operatorname{tr} \nabla u(x) = -|V_1 - V_2| u(V_3) \cdot \nu_1$. From this last expression, considering also (5.60), it turns out that $\operatorname{div} u(x) = 0$ if and only if the second component of $u(V_3)$ is zero. □

For the following construction, we refer the reader to Figures 5.3 and 5.4. Let T be the equilateral triangle with vertices

$$V_1 = \left(-1, -\frac{1}{\sqrt{3}}\right), \quad V_2 = \left(1, -\frac{1}{\sqrt{3}}\right), \quad V_3 = \left(0, \frac{2}{\sqrt{3}}\right).$$

Let V_4, V_5, and V_6 be the middle points of the segments bounded by the center O of T and the middle points of $[V_2, V_3]$, $[V_3, V_1]$, and $[V_1, V_2]$, respectively, that is

$$V_4 = \left(\frac{1}{4}, \frac{1}{4\sqrt{3}}\right), \quad V_5 = \left(-\frac{1}{4}, \frac{1}{4\sqrt{3}}\right), \quad V_6 = \left(0, -\frac{1}{2\sqrt{3}}\right).$$

We divide T into the seven triangles

$$T_1 := \triangle^{V_1V_2V_6}, \quad T_2 := \triangle^{V_2V_4V_6}, \quad T_3 := \triangle^{V_4V_5V_6}, \quad T_4 := \triangle^{V_2V_3V_4}$$
$$T_5 := \triangle^{V_3V_4V_5}, \quad T_6 := \triangle^{V_1V_3V_5}, \quad T_7 := \triangle^{V_1V_5V_6}.$$

It turns out that

$$|T_1| = |T_4| = |T_6| = \frac{|T|}{6}, \quad |T_2| = |T_5| = |T_7| = \frac{7}{48}|T|, \quad |T_3| = \frac{|T|}{16}. \tag{5.62}$$

Consider the following vectors representing displacements which will be applied at the points V_4, V_5, V_6, respectively:

$$u_4^\delta := \frac{\delta}{2}(-1, \sqrt{3}), \qquad u_5^\delta := -\frac{\delta}{2}(1, \sqrt{3}), \qquad u_6^\delta := \delta(1, 0).$$

These three vectors have the same length δ and are chosen in such a way that u_4^δ has the same direction as $(V_3 - V_2)$, u_5^δ the same direction as $(V_1 - V_3)$, and u_6^δ the same direction as $(V_2 - V_1)$. Finally, we define u as the piecewise affine function defined by

$$u(V_1) = u(V_2) = u(V_3) = 0, \qquad u(V_i) = u_i^\delta, \quad i = 4, 5, 6. \tag{5.63}$$

It is clear that $u = 0$ on ∂T. To check that $\operatorname{div} u = 0$ a.e. in T, let us prove that $u_{|T_i}$ is divergence–free for $i = 1, 2, 5$ (for the other triangles the arguments are the same). In view of Lemma 5.16, $\operatorname{div} u = 0$ on T_1, because $u(V_1) = u(V_2) = 0$ and $u(V_6) = u_6^\delta$ is parallel to $V_1 - V_2$. For what concerns u on T_3, note that it can be written as the sum of three function, u_4, u_5, and u_6, where $u_i(V_i) = u_i^\delta$ and $u_i(V_j) = 0$ for $j \in \{4, 5, 6\} \setminus \{i\}$, for $i = 4, 5, 6$. Again using Lemma 5.16, from the definition of u_i^δ we obtain that u_i has divergence–free on T_3, for $i = 4, 5, 6$, and in turn $\operatorname{div} u = 0$ on T_3. To check that $\operatorname{div} u = 0$ on T_5, we use the Divergence Theorem. Note that $|V_3 - V_4| = |V_3 - V_5|$ and that, if ν_1 and ν_2 are the first and the second component of the outer normal on ∂T_5, we have that $\nu_{1|[V_3,V_5]} = -\nu_{1|[V_3,V_4]}$ and $\nu_{2|[V_3,V_5]} = \nu_{2|[V_3,V_4]}$. Moreover, $\nu_{|[V_4,V_5]} = (-1,0)$. Thus,

$$\begin{aligned}
|T_5| \operatorname{tr} \nabla u_{|T_5}(x) &= \int_{\partial T_5} u\nu \, ds \\
&= \frac{|V_3 - V_4|}{2}(u_4^\delta \cdot \nu_{|[V_3,V_4]}) + u_5^\delta \cdot \nu_{|[V_3,V_5]}) + \frac{|V_4 - V_5|}{2}(u_4^\delta + u_5^\delta) \cdot \nu_{|[V_4,V_5]} \\
&= \frac{\delta|V_3 - V_4|}{4}[(-1,\sqrt{3}) \cdot \nu_{|[V_3,V_4]}) - (1,\sqrt{3}) \cdot \nu_{|[V_3,V_5]}] + \frac{\delta|V_4 - V_5|}{2}(-1,0) \cdot \nu_{|[V_4,V_5]} \\
&= 0.
\end{aligned}$$

We write down here the explicit expression of u:

$$u_{|T_1}(x) = \begin{pmatrix} 0 & 2\sqrt{3}\delta \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 2\delta \\ 0 \end{pmatrix}, \quad u_{|T_2}(x) = \begin{pmatrix} -\frac{12}{7}\delta & -\frac{10\sqrt{3}}{7}\delta \\ \frac{6}{7\sqrt{3}}\delta & \frac{12}{7}\delta \end{pmatrix} x + \begin{pmatrix} \frac{2}{7}\delta \\ \frac{2\sqrt{3}}{7}\delta \end{pmatrix},$$

$$u_{|T_3}(x) = \begin{pmatrix} 0 & -2\sqrt{3}\delta \\ 2\sqrt{3}\delta & 0 \end{pmatrix} x, \quad u_{|T_4}(x) = \begin{pmatrix} \frac{3}{2}\delta & \frac{\sqrt{3}}{2}\delta \\ -\frac{3\sqrt{3}}{2}\delta & -\frac{3}{2}\delta \end{pmatrix} x + \begin{pmatrix} -\delta \\ \sqrt{3}\delta \end{pmatrix},$$

$$u_{|T_5}(x) = \begin{pmatrix} 0 & \frac{2\sqrt{3}}{7}\delta \\ 2\sqrt{3}\delta & 0 \end{pmatrix} x + \begin{pmatrix} -\frac{4}{7}\delta \\ 0 \end{pmatrix}, \quad u_{|T_6}(x) = \begin{pmatrix} -\frac{3}{2}\delta & \frac{\sqrt{3}}{2}\delta \\ -\frac{3\sqrt{3}}{2}\delta & \frac{3}{2}\delta \end{pmatrix} x + \begin{pmatrix} -\delta \\ -\sqrt{3}\delta \end{pmatrix},$$

$$u_{|T_7}(x) = \begin{pmatrix} \frac{12}{7}\delta & -\frac{10\sqrt{3}}{7}\delta \\ \frac{6}{7\sqrt{3}}\delta & -\frac{12}{7}\delta \end{pmatrix} x + \begin{pmatrix} \frac{2}{7}\delta \\ -\frac{2\sqrt{3}}{7}\delta \end{pmatrix}.$$

We can now prove Lemma 5.15, following the lines of the proof of [**57**, Proposition 2.6].

Proof of Lemma 5.15. Here, we use the notation (x, y) or (ξ, η) in place of x for a generic point of $\mathbb{R}^2$. We will suppose $\mathbb{M}^{2\times 2}$ to be endowed with the l_∞ norm, which will be denoted by $|\cdot|_\infty$, and d to be the distance corresponding to such norm. This assumption is not restrictive for the proof of the statement, due to the equivalence of all the norms in a finite dimensional vector space. The proof is divided in three cases.

Case 1. Consider the matrix $E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and suppose that $A - B = E$ and that $C = 0$. This fact, together with (5.54), gives that

$$A = \lambda E, \qquad B = (\lambda - 1)E.$$

In this case,

$$\begin{aligned}
d(M, [A,B]) &= \min_{0\le\mu\le 1} |M - (1-\mu)A - \mu B|_\infty \\
&= \min_{0\le\mu\le 1} |M + (\mu - \lambda)E|_\infty,
\end{aligned}$$

for every $M \in \mathbb{M}^{2\times 2}$. From the definitions of E and of $|\cdot|_\infty$, we have in particular that

$$d(M, [A,B]) = \max\{|M_{11}|, |M_{21}|, |M_{22}|\}, \quad \text{if } \lambda - 1 \le M_{12} \le \lambda. \tag{5.64}$$

We are going to construct a piecewise affine function w_ε which satisfies (5.55)–(5.59) on a compact set T_ε with $|T_\varepsilon| > 0$. We will then conclude the proof of Case 1 applying Theorem 5.5. Since $\varepsilon > 0$ is an arbitrarily small parameter, it is not restrictive to assume that

$$\varepsilon^3 < \min\{\lambda, 1 - \lambda\}. \tag{5.65}$$

This will be useful later. Consider the piecewise affine function u of components (u_1, u_2) defined by (5.63) on the triangle T and, from the explicit expression of u, note that

$$||\nabla u||_{L^\infty(T;\mathbb{M}^{2\times 2})} = \operatorname{ess\,sup}_{(x,y)\in T} |\nabla u(x,y)|_\infty = \frac{\partial u_1}{\partial y}(x,y), \quad \text{for every } (x,y) \in T_1,$$

where $\frac{\partial u_1}{\partial y}(x,y) = 2\sqrt{3}\delta$ for every $(x,y) \in T_1$. Choosing $\delta = \frac{\varepsilon^3}{2\sqrt{3}}$ and relabeling u by u^ε, we obtain that

$$\varepsilon^3 = \frac{\partial u_1^\varepsilon}{\partial y} \quad \text{on} \quad T_1, \tag{5.66}$$

and that

$$||\nabla u^\varepsilon||_{L^\infty(T;\mathbb{M}^{2\times 2})} = \varepsilon^3 < \varepsilon \quad \text{and} \quad ||u^\varepsilon||_{L^\infty(T;\mathbb{R}^2)} \leq \varepsilon^3 \hat{c}, \tag{5.67}$$

for some constant $\hat{c} > 0$ which does not depend on ε. Also, again from the explicit expression of $u = u^\varepsilon$ we see that

$$\sup_T \frac{\partial u_1^\varepsilon}{\partial y} = \sup \left\{ \left| \frac{\partial u_1^\varepsilon}{\partial y} \right| \,:\, \frac{\partial u_1^\varepsilon}{\partial y} \leq 0 \right\} = \varepsilon^3. \tag{5.68}$$

Set $m_\varepsilon := \varepsilon^3 \max\{1/\lambda, 1/(1-\lambda)\}$, so that

$$0 < m_\varepsilon < 1, \tag{5.69}$$

in view of (5.65). Then, define

$$S_\varepsilon := \begin{pmatrix} \sqrt{m_\varepsilon} & 0 \\ 0 & \frac{1}{\sqrt{m_\varepsilon}} \end{pmatrix} \quad \text{and} \quad T_\varepsilon := S_\varepsilon^{-1}(T).$$

The function

$$w^\varepsilon(\xi,\eta) := S_\varepsilon^{-1} u^\varepsilon \left(S \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right), \qquad \text{for every } (\xi,\eta) \in T_\varepsilon.$$

satisfies conditions (5.55)–(5.59). Indeed, (5.55) and (5.56) trivially follow from the fact that $\operatorname{div} u^\varepsilon = 0$ a.e. on T and $u^\varepsilon = 0$ on ∂T. Note that

$$\nabla w^\varepsilon(\xi,\eta) = \begin{pmatrix} \frac{\partial u_1^\varepsilon}{\partial x} & \frac{1}{m_\varepsilon}\frac{\partial u_1^\varepsilon}{\partial y} \\ m_\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} & \frac{\partial u_2^\varepsilon}{\partial y} \end{pmatrix}_{|\left(\sqrt{m_\varepsilon}\xi, \frac{\eta}{\sqrt{m_\varepsilon}}\right)}, \qquad \text{for every } (\xi,\eta) \in T_\varepsilon.$$

Thus,

$$\left| \frac{\partial w_1^\varepsilon}{\partial \xi} \right|, \left| \frac{\partial w_2^\varepsilon}{\partial \xi} \right|, \left| \frac{\partial w_2^\varepsilon}{\partial \eta} \right| < \varepsilon, \tag{5.70}$$

in view of (5.67) and (5.69). Moreover, (5.68) and the definition of m_ε give that $\lambda - 1 \leq \frac{\partial w_1^\varepsilon}{\partial \eta} \leq \lambda$. This fact, together with (5.64) and (5.70) give (5.57), that is $d\left(\nabla w^\varepsilon, [A,B]\right) < \varepsilon$ a.e. in T_ε. Also, equivalence (5.66) implies that, for every $(\xi,\eta) \in S_\varepsilon^{-1}(T_1) \subseteq T_\varepsilon$, $d(\nabla w^\varepsilon(\xi,\eta), \{A,B\}) \leq \varepsilon$, and in turn that

$$|\{(\xi,\eta) \in T_\varepsilon \,:\, d(\nabla w^\varepsilon(\xi,\eta), \{A,B\}) > \varepsilon\}| \leq |T_\varepsilon \setminus S_\varepsilon^{-1}(T_1)|.$$

This inequality, together with (5.62) and the fact that areas are invariant under S_ε^{-1}, gives that $|\{(\xi,\eta) \in T_\varepsilon \,:\, d(\nabla w^\varepsilon(\xi,\eta), \{A,B\}) \geq \varepsilon\}| \leq \frac{16}{17}|T_\varepsilon|$ and in turn (5.58). Finally, (5.67) and the definition of m_ε implies that

$$||w_\varepsilon||_{L^\infty(T_\varepsilon;\mathbb{R}^2)} \leq \frac{||u^\varepsilon||_{L^\infty(T;\mathbb{R}^2)}}{\sqrt{m_\varepsilon}} \leq \frac{\varepsilon^{\frac{3}{2}}\hat{c}}{\max\{\lambda, 1-\lambda\}},$$

so that $||w_\varepsilon||_{L^\infty(T_\varepsilon;\mathbb{R}^2)} < \varepsilon$, if $\varepsilon > 0$ is sufficiently small. Thus, we have constructed a piecewise affine Lipschitz map $w^\varepsilon : T_\varepsilon \to \mathbb{M}^{2\times 2}$ which satisfies (5.55)–(5.59) on T_ε. It remains to note that the function $(\xi,\eta) \mapsto \lambda w^\varepsilon(\xi/\lambda, \eta/\lambda)$ satisfies (5.55)–(5.59) on the dilated set λT_ε for every $\lambda > 0$, and that the function $(\xi,\eta) \mapsto w^\varepsilon(\xi - \xi_\alpha, \eta - \eta_\alpha)$ satisfies (5.55)–(5.59) on the translated set $T_\varepsilon + (\xi_\alpha, \eta_\alpha)$. Thus, using Theorem 5.5, there exists a disjoint numerable union $\bigcup_i \mathcal{T}_\varepsilon^i \subseteq \Omega$ of dilated and translated sets of T_ε such that

$$\mathscr{L}^2\left(\Omega \setminus \bigcup_i \mathcal{T}_\varepsilon^i \right) = 0,$$

and piecewise affine Lipschitz maps $w_i^\varepsilon : \mathcal{T}_\varepsilon^i \to \mathbb{M}^{2\times 2}$ satisfying (5.55)–(5.59) on $\mathcal{T}_\varepsilon^i$. Arguing as in the proof of Proposition 5.6, it is possible to prove, starting from the functions w_i^ε's, the existence of a piecewise affine and Lipschitz function $u : \Omega \to \mathbb{R}^2$ satisfying (5.55)–(5.59) on Ω.

Case 2. Here, suppose $C = 0$ and A, B arbitrary in $\mathbb{M}_0^{2\times 2}$. Since $\det(A - B) = 0$, 0 is an eigenvalue of $A - B$ which may have algebraic multiplicity equal either to 1 or 2. The Jordan Decomposition Theorem tells us that, in the first case, there exists an invertible matrix L and $\mu \in \mathbb{R} \setminus \{0\}$ such that $A - B = L^{-1} \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} L$, but this is not possible, because $A - B \in \mathbb{M}_0^{2\times 2}$. In the second case, we have that $A - B = L^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L$, for some invertible matrix L. Let w be given by Case 1 and satisfying conditions (5.55)–(5.59) on a rectangle R for $\hat{A} := LAL^{-1}$ and $\hat{B} := LBL^{-1}$ (note that $\hat{A} - \hat{B} = E$ and $(1-\lambda)\hat{A} + \lambda\hat{B} = 0$). It is easy to verify that $u(\xi, \eta) := L^{-1} \left(v \left(L \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix} \right) \right) \right)$ satisfies conditions (5.55)–(5.59) on $L^{-1}(R)$. Using again Theorem 5.5 and covering Ω by dilated and translated copies of $L^{-1}(R)$, we obtain a function satisfying conditions (5.55)–(5.59) on Ω.

Case 3. Here, suppose C, A, and B to be as in the hypotheses. The matrices $\hat{A} := A - C$ and $\hat{B} := B - C$ are such that $(1-\lambda)\hat{A} + \lambda\hat{B} = 0$. Thus, from Case 2, there exists $w : \Omega \to \mathbb{R}^2$ piecewise affine and Lipschitz satisfying (5.55)–(5.59) with $\hat{A}$, $\hat{B}$, and 0 in place of A, B, and C, respectively. It is easy to verify that $u(x, y) := w(x, y) + C \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ satisfies (5.55)–(5.55) on Ω. $\square$

For the following lemma, we recall that the set $\tilde{U}$ is obtained from the set U by Definition 5.8. Observe that if $U \subseteq \mathbb{M}_0^{2\times 2}$, then $\tilde{U} \subseteq \mathbb{M}_0^{2\times 2}$.

LEMMA 5.17. *Let $U \subseteq \mathbb{M}_0^{2\times 2}$ be bounded and open in $\mathbb{M}_0^{2\times 2}$ and let $C \in \tilde{U}$. Then, for every $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that*

$$\nabla u_\varepsilon \in \tilde{U} \quad \text{a.e. in } \Omega, \tag{5.71}$$

$$u_\varepsilon(x) = Cx, \quad \text{for every } x \in \partial\Omega, \tag{5.72}$$

$$|\{x \in \Omega : \nabla u_\varepsilon(x) \notin U\}| < \varepsilon|\Omega|, \tag{5.73}$$

$$\sup_{x\in\Omega} |u_\varepsilon(x) - Cx| < \varepsilon. \tag{5.74}$$

PROOF. Let $C \in \tilde{U}$. By definition of $\tilde{U}$, there exists $0 \le \lambda \le 1$ such that $C = (1-\lambda)A + \lambda B$ for some A, $B \in U$. Consider the nontrivial case $0 < \lambda < 1$. By Lemma 5.15, for every $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that conditions (5.55)–(5.59) are satisfied. In particular, conditions (5.56) and (5.59) give directly (5.72) and (5.74), respectively, and (5.55) and (5.57) give (5.71), because $\tilde{U}$ is open in $\mathbb{M}_0^{2\times 2}$ (see Remark 5.9). Now, observe that (5.58) implies

$$|\{x \in \Omega : \nabla u_\varepsilon(x) \notin U\}| \le c|\Omega|, \tag{5.75}$$

where c is a constant such that $0 < c < 1$ and does not depend neither on ε nor on Ω. Indeed, since A, $B \in U$ and $U \subseteq \mathbb{M}_0^{2\times 2}$ is relatively open, and since $\nabla u_\varepsilon \in \tilde{U} \subseteq \mathbb{M}_0^{2\times 2}$ a.e., we have that $d(\nabla u_\varepsilon(x), A, B) \ge \varepsilon$ for a.e. $x \in \{x \in \Omega : \nabla u_\varepsilon(x) \notin U\}$. In turn, from (5.58), we obtain (5.75). Let $w^{(1)}$ be a piecewise affine Lipschitz function (depending on ε) which satisfies (5.71), (5.72), (5.74) with $\frac{\varepsilon}{2}$ in place of ε, that is

$$\sup_{x\in\Omega} |w^{(1)}(x) - Cx| < \frac{\varepsilon}{2}, \tag{5.76}$$

and (5.75). Starting from $w^{(1)}$, we are going to construct a sequence of functions $\{w^{(k)}\}$ which at the end will give a map u_ε (piecewise affine and Lipschitz) satisfying (5.71)–(5.74). Since $w^{(1)}$ is piecewise affine, there exist countably many mutually disjoint Lipschitz domains $\Omega_k \subseteq \Omega$ such that

$$w_k^{(1)} := w^{(1)}_{|\Omega_k} \quad \text{is affine and} \quad \mathscr{L}^2 \left(\Omega \setminus \bigcup_k \Omega_k \right) = 0.$$

Let $\left\{\Omega_k^{(1)}\right\}_k \subseteq \{\Omega_k\}$ be the sequence of the sets where $\nabla w^{(1)} \notin U$. Thus,

$$\sum_k \left|\Omega_k^{(1)}\right| = \left|\left\{x \in \Omega \,:\, \nabla w^{(1)}(x) \notin U\right\}\right| \leq c|\Omega|. \tag{5.77}$$

Applying again Lemma 5.15 on each $\Omega_k^{(1)}$ (where $w^{(1)}$ is affine), now with $\frac{\varepsilon}{4}$ in place of ε, we find $w_k^{(2)} : \Omega_k^{(1)} \to \mathbb{R}^2$ piecewise affine and Lipschitz such that $\nabla w_k^{(2)} \in \tilde{U}$ a.e. in $\Omega_k^{(1)}$, $w_k^{(2)} = w^{(1)}$ on $\partial\Omega_k^{(1)}$,

$$\left|\left\{x \in \Omega_k^{(1)} \,:\, \nabla w_k^{(2)}(x) \notin U\right\}\right| \leq c|\Omega_k^{(1)}|, \tag{5.78}$$

and

$$\sup_{x \in \Omega_k^{(1)}} |w_k^{(2)}(x) - w^{(1)}(x)| < \frac{\varepsilon}{4}. \tag{5.79}$$

Define $w^{(2)} : \Omega \to \mathbb{R}^2$ in the following way:

$$w^{(2)} = \begin{cases} w^{(1)} & \text{on } \Omega \setminus \bigcup_k \Omega_k^{(1)}, \\ w_k^{(2)} & \text{on } \Omega_k^{(1)}. \end{cases}$$

It turns out that $w^{(2)}$ is piecewise affine and Lipschitz continuous, because it can be seen as the limit of a sequence of Lipschitz functions. Moreover, $\nabla w^{(2)} \in \tilde{U}$ a.e. in Ω and $w^{(2)}(x) = Cx$ for every $x \in \partial\Omega$. Also, in view of (5.77) and (5.78),

$$\begin{aligned} \left|\left\{x \in \Omega \,:\, \nabla w^{(2)}(x) \notin U\right\}\right| &= \sum_k \left|\left\{x \in \Omega_k^{(1)} \,:\, \nabla w_k^{(2)}(x) \notin U\right\}\right| \\ &\leq \sum_k c\left|\Omega_k^{(1)}\right| \leq c^2|\Omega|, \end{aligned}$$

and, in view of (5.76) and (5.79),

$$\sup_{x\in\Omega}\left|w^{(2)}(x) - Cx\right| \leq \sup_{x\in\Omega}\left\{\left|w^{(2)}(x) - w^{(1)}(x)\right| + \left|w^{(1)}(x) - Cx\right|\right\} < \frac{\varepsilon}{2}\left(1 + \frac{1}{2}\right).$$

By iterating this procedure we find out that for every $m \in \mathbb{N} \setminus \{0\}$ there exists a piecewise affine Lipschitz map $w^{(m)} : \Omega \to \mathbb{R}^2$ such that $\nabla w^{(m)} \in \tilde{U}$ a.e. in Ω, $w^{(m)}(x) = Cx$ for every $x \in \partial\Omega$,

$$\left|\left\{x \in \Omega \,:\, \nabla w^{(m)}(x) \notin U\right\}\right| \leq c^m|\Omega|,$$

and

$$\sup_{x\in\Omega}|w^{(m)}(x) - Cx| < \frac{\varepsilon}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{2^{m-1}}\right) < \varepsilon.$$

Since $0 < c < 1$, for m sufficiently large $c^m < \varepsilon$. Setting $u_\varepsilon := w^{(m)}$ for such a big m, we have obtained that u_ε satisfies (5.71)–(5.74). □

The same iterative method used in the proof of Lemma 5.17 allows to remove step by step the set where $\nabla u \notin U$ and obtain the following theorem.

THEOREM 5.18. *Let $U \subseteq \mathbb{M}_0^{2\times 2}$ be open in $\mathbb{M}_0^{2\times 2}$ and bounded. Suppose that $v : \Omega \to \mathbb{R}^2$ is piecewise affine, Lipschitz, and such that*

$$\nabla v \in \tilde{U} \quad \text{a.e. in } \Omega.$$

Then, for every $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $u_\varepsilon : \Omega \to \mathbb{R}^2$ such that

$$\nabla u_\varepsilon \in U \quad \text{a.e. in } \Omega, \tag{5.80}$$

$$u_\varepsilon = v \quad \text{on } \partial\Omega, \tag{5.81}$$

$$||u_\varepsilon - v||_{L^\infty} < \varepsilon. \tag{5.82}$$

PROOF. Consider first the case where v is affine, so that $\nabla v(x) = Cx$ for every $x \in \Omega$, for some $C \in \tilde{U}$. Fixed $\varepsilon > 0$, by Lemma 5.17 there exists a piecewise affine Lipschitz map $u^{(1)} : \Omega \to \mathbb{R}^2$ such that $\nabla u^{(1)} \in \tilde{U}$ a.e. in Ω and $u^{(1)} = v$ on $\partial\Omega$. Thus, there exist countably many mutually disjoint Lipschitz domains $\Omega_i \subseteq \Omega$ such that

$$u_i^{(1)} := u^{(1)}_{|\Omega_i} \quad \text{is affine and} \quad \mathscr{L}^2\left(\Omega \setminus \bigcup_i \Omega_i\right) = 0.$$

In particular, we can write

$$\Omega = \bigcup_{i\in\mathcal{A}^{(1)}} \Omega_i^{(1)} \cup \bigcup_{i\in\mathcal{B}^{(1)}} \Omega_i^{(1)} \cup N^{(1)},$$

where

$$\mathcal{A}^{(1)} := \left\{ i \in \mathbb{N} : \nabla u_i^{(1)} \in U \right\}, \quad \mathcal{B}^{(1)} := \left\{ i \in \mathbb{N} : \nabla u_i^{(1)} \notin U \right\}, \quad |N^{(1)}| = 0.$$

Moreover, $u^{(1)}$ can be chosen in such a way that conditions (5.73) and (5.74) are satisfied with $\frac{\varepsilon}{2}$ in place of ε, so that, setting $M_1 := \bigcup_{i\in\mathcal{B}^{(1)}} \Omega_i^{(1)}$,

$$|M^{(1)}| < \varepsilon|\Omega|, \qquad ||u^{(1)} - v||_{L^\infty(\Omega;\mathbb{R}^2)} < \frac{\varepsilon}{2}. \tag{5.83}$$

Applying again Lemma 5.17, with $\frac{\varepsilon}{4}$ in place of ε, on each $\Omega_i^{(1)}$ with $i \in \mathcal{B}^{(1)}$, we find $u_i^{(2)} : \Omega_i^{(1)} \to \mathbb{R}^2$ piecewise affine and Lipschitz such that $\nabla u_i^{(2)} \in \tilde{U}$, $u_i^{(2)} = u^{(1)}$ on $\partial\Omega_i^{(1)}$, and

$$||u_i^{(2)} - u^{(1)}||_{L^\infty(\Omega_i^{(1)};\mathbb{R}^2)} < \frac{\varepsilon}{4}, \tag{5.84}$$

for every $i \in \mathcal{B}^{(1)}$. Now, define $u^{(2)} : \Omega \to \mathbb{R}^2$ by

$$u^{(2)} = \begin{cases} u^{(1)} & \text{on } \displaystyle\bigcup_{i\in\mathcal{A}^{(1)}} \Omega_i^{(1)} \cup N^{(1)}, \\ u_i^{(2)} & \text{on } \Omega_i^{(1)}, \quad \text{for every } i \in \mathcal{B}^{(1)} \end{cases}$$

As done before for Ω, we can write $M^{(1)} = \bigcup_{i\in\mathcal{A}^{(2)}} \Omega_i^{(2)} \cup \bigcup_{i\in\mathcal{B}^{(2)}} \Omega_i^{(2)} \cup N^{(2)}$, where $u_i^{(2)}$ is affine on each $\Omega_i^{(2)}$ and

$$\mathcal{A}^{(2)} := \left\{ i \in \mathbb{N} : \nabla u_i^{(2)} \in U \right\}, \quad \mathcal{B}^{(2)} := \left\{ i \in \mathbb{N} : \nabla u_i^{(2)} \notin U \right\}, \quad |N^{(2)}| = 0.$$

Setting $M^{(2)} := \bigcup_{i\in\mathcal{B}^{(2)}} \Omega_i^{(2)}$, we obtain that

$$|M^{(2)}| = |\{x \in \Omega : \nabla u^{(2)} \notin U\}| \le \varepsilon|M^{(1)}| \le \varepsilon^2|\Omega|, \tag{5.85}$$

that $u^{(2)}$ is a piecewise affine Lipschitz function such that $\nabla u^{(2)} \in \tilde{U}$ a.e. in Ω, that $u^{(2)} = v$ on $\partial\Omega$, and that

$$||u^{(2)} - v||_{L^\infty(\Omega;\mathbb{R}^2)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4}.$$

Note that $u^{(2)} = u^{(1)}$ on $\Omega \setminus M^{(1)}$. By iterating this procedure, we find

$$u^{(m)} = \begin{cases} u^{(1)} & \text{on } \displaystyle\bigcup_{i\in\mathcal{A}^{(1)}} \Omega_i^{(1)} \cup N^{(1)}, \\ u^{(2)} & \text{on } \displaystyle\bigcup_{i\in\mathcal{A}^{(2)}} \Omega_i^{(2)} \cup N^{(2)}, \\ \vdots & \\ u^{(m-1)} & \text{on } \displaystyle\bigcup_{i\in\mathcal{A}^{(m-1)}} \Omega_i^{(m-1)} \cup N^{(m-1)}, \\ u_i^{(m)} & \text{on } \Omega_i^{(m-1)}, \quad \text{for every } i \in \mathcal{B}^{(m-1)}, \end{cases}$$

and write

$$\begin{aligned} M^{(m-1)} &:= \bigcup_{i\in\mathcal{B}^{(m-1)}} \Omega_i^{(m-1)} \\ &= \bigcup_{i\in\mathcal{A}^{(m)}} \Omega_i^{(m)} \cup \bigcup_{i\in\mathcal{B}^{(m)}} \Omega_i^{(m)} \cup N^{(m)}, \end{aligned}$$

where $u_i^{(m)}$ is affine on each $\Omega_i^{(m)}$ and

$$\mathcal{A}^{(m)} := \left\{ i\in\mathbb{N} : \nabla u_i^{(m)} \in U \right\}, \quad \mathcal{B}^{(m)} := \left\{ i\in\mathbb{N} : \nabla u_i^{(m)} \notin U \right\}, \quad |N^{(m)}| = 0.$$

We remark that, for every $i \in \mathcal{B}^{(m-1)}$, $u_i^{(m)} : \Omega_i^{(m-1)} \to \mathbb{R}^2$ is piecewise affine and Lipschitz and have been obtained from $u^{(m-1)}$ by applying Lemma 5.17 with $\frac{\varepsilon}{2^m}$ in place of ε, so that $\nabla u_i^{(m)} \in \tilde{U}$, $u_i^{(m)} = u^{(m-1)}$ on $\partial\Omega_i^{(m-1)}$, and

$$||u_i^{(m)} - u^{(m-1)}||_{L^\infty(\Omega_i^{(m-1)};\mathbb{R}^2)} < \frac{\varepsilon}{2^m}. \tag{5.86}$$

Setting $M^{(m)} := \bigcup_{i\in\mathcal{B}^{(m)}} \Omega_i^{(m)}$, it turns out that

$$|M^{(m)}| = |\{x\in\Omega : \nabla u^{(m)} \notin U\}| \le \varepsilon^m |\Omega|, \tag{5.87}$$

that $u^{(m)}$ is a piecewise affine Lipschitz function such that $\nabla u^{(m)} \in \tilde{U}$ a.e. in Ω, that

$$u^{(m)} = v \quad \text{on } \partial\Omega, \tag{5.88}$$

and that

$$||u^{(m)} - v||_{L^\infty(\Omega;\mathbb{R}^2)} < \frac{\varepsilon}{2} + ... + \frac{\varepsilon}{2^m} < \varepsilon. \tag{5.89}$$

Note that $u^{(m)} = u^{(m-1)}$ on $\Omega \setminus M^{(m-1)}$, and that $M^{(1)} \supseteq M^{(2)} \supseteq ....$ Finally, consider the function $u : \Omega \to \mathbb{R}^2$ defined by

$$u := \begin{cases} u^{(1)} & \text{on } \Omega\setminus\overline{M}^{(1)}, \\ u^{(2)} & \text{on } M^{(1)}\setminus\overline{M}^{(2)}, \\ \vdots & \\ u^{(m)} & \text{on } M^{(m-1)}\setminus\overline{M}^{(m)}, \\ \vdots & \end{cases} \tag{5.90}$$

Observe that u is defined on Ω up to a set of null measure, because $|\Omega\cap\partial M_m| = 0$ for every m (equivalently, $|\overline{M}^{(m)}\cap\Omega| = |M^{(m)}|$). Note that $u^{(m)} = u^{(1)}$ on $\Omega\setminus\overline{M}^{(1)}$ for every $m\ge 1$, and that $u^{(m)} = u^{(k)}$ on $M^{(k-1)}\setminus\overline{M}^{(k)}$ for every $m \ge k$ and $k \ge 2$. Since $\{u^{(m)}\}$ is a bounded sequence in $W^{1,\infty}$, we have in particular that $u^{(m)} \rightharpoonup^* u$ in $W^{1,\infty}$, and in turn that u is a Lipschitz function on Ω such that $u = v$ on $\partial\Omega$, from (5.88). Moreover, by definition (5.90), and by definition of $\{M^{(m)}\}$, u turns out to be piecewise affine. Estimate (5.82) is given by (5.89).

If v is piecewise affine, it is enough to apply the previous argument to each region where v is affine. □

We are now in position to prove Theorem 5.11, where a set $K_0 \subseteq \mathbb{M}_0^{2\times 2}$ not necessarily open in $\mathbb{M}_0^{2\times 2}$ is considered. The idea of the proof is to approximate K_0 by sets U_i open in $\mathbb{M}_0^{2\times 2}$. This leads to approximated solutions u_i which satisfy $\nabla u_i \in U_i$. It turns out that by a careful choice of u_i one can obtain strong or a.e. convergence of the sequence $\{u_i\}$, despite the fact that the functions u_i's develop increasingly faster spatial oscillations. The idea is to superimpose at each step oscillations which are much faster than the ones of the previous step. The sets U_i have to approximate K_0 in the sense of Definition 5.10, which is motivated by the fact that Theorem 5.18 is used to obtain the approximation u_{i+1}, with $\nabla u_{i+1} \in U_{i+1}$ a.e., from u_i, which is such that $\nabla u_i \in U_i \subseteq \tilde{U}_{i+1}$ a.e..

Proof of Theorem 5.11. As in the proof of Theorem 5.18, it is not restrictive to suppose v affine. Fix $\varepsilon > 0$. Since $\nabla v \in U_1 \subseteq \tilde{U}_2$, by Theorem 5.18 there exists a piecewise affine Lipschitz map $u_1 : \Omega \to \mathbb{R}^2$ such that $\nabla u_1 \in U_2$ a.e. in Ω, $u_1 = v$ on $\partial\Omega$, and $||u_1 - v||_{L^\infty(\Omega;\mathbb{R}^2)} < \varepsilon/2$. Let $\{\rho_\delta\}$ be a family of mollifiers and set

$$\Omega_1 := \left\{ x \in \Omega \,:\, d(x, \partial\Omega) > \frac{1}{2} \right\}.$$

Not that Ω_1 is a nonempty set, up to replacing $1/2$ by some smaller positive constant. Let $0 < \delta_1 \leq 1/2$ be such that $||\rho_{\delta_1} * \nabla u_1 - \nabla u_1||_{L^1(\Omega_1;\mathbb{M}^{2\times 2})} < 1/2$ and set

$$\varepsilon_1 := \delta_1 \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4}.$$

Since $\nabla u_1 \in U_2 \subseteq \tilde{U}_3$, again by Theorem 5.18 there exists a piecewise affine Lipschitz map $u_2 : \Omega \to \mathbb{R}^2$ such that $\nabla u_2 \in U_3$ a.e. in Ω, $u_2 = u_1$ on $\partial\Omega$, and $||u_2 - u_1||_{L^\infty(\Omega;\mathbb{R}^2)} < \varepsilon_1$. Set

$$\Omega_2 := \left\{ x \in \Omega \,:\, d(x, \partial\Omega) > \frac{1}{4} \right\},$$

let $0 < \delta_2 \leq \min\{\delta_1, 1/4\}$ be such that $||\rho_{\delta_2} * \nabla u_2 - \nabla u_2||_{L^1(\Omega_2;\mathbb{M}^{2\times 2})} < 1/4$, and define

$$\varepsilon_2 := \delta_2 \varepsilon_1 \leq \frac{\varepsilon_1}{2} \leq \frac{\varepsilon}{8}.$$

We have that $||u_2 - v||_{L^\infty(\Omega;\mathbb{R}^2)} \leq ||u_2 - u_1||_{L^\infty(\Omega;\mathbb{R}^2)} + ||u_1 - v||_{L^\infty(\Omega;\mathbb{R}^2)} < \frac{\varepsilon}{2}\left(1 + \frac{1}{2}\right)$. Now, for $i = 2, 3, ...$, suppose to have a piecewise affine Lipschitz map $u_i : \Omega \to \mathbb{R}^2$ such that

$$\nabla u_i \in U_{i+1} \quad \text{a.e. in } \Omega, \tag{5.91}$$

$u_i = u_{i-1}$ on $\partial\Omega$, and $||u_i - u_{i-1}||_{L^\infty(\Omega;\mathbb{R}^2)} < \varepsilon_{i-1}$, set

$$\Omega_i := \left\{ x \in \Omega \,:\, d(x, \partial\Omega) > \frac{1}{2^i} \right\},$$

let $0 < \delta_i \leq \min\{\delta_{i-1}, 1/2^i\}$ be such that

$$||\rho_{\delta_i} * \nabla u_i - \nabla u_i||_{L^1(\Omega_i,\mathbb{M}^{2\times 2})} < 1/2^i, \tag{5.92}$$

and define

$$\varepsilon_i := \delta_i \varepsilon_{i-1} \leq \frac{\varepsilon_{i-1}}{2} \leq ... \leq \frac{\varepsilon_1}{2^{i-1}} \leq \frac{\varepsilon}{2^i}.$$

Since $\nabla u_i \in U_{i+1} \subseteq \tilde{U}_{i+2}$, by Theorem 5.18 there exists a piecewise affine Lipschitz map $u_{i+1} : \Omega \to \mathbb{R}^2$ such that $\nabla u_{i+1} \in U_{i+2}$ a.e. in Ω, $u_{i+1} = u_i$ on $\partial\Omega$, and

$$||u_{i+1} - u_i||_{L^\infty(\Omega;\mathbb{R}^2)} < \varepsilon_i. \tag{5.93}$$

Thus,

$$\begin{aligned} ||u_{i+1} - v||_\infty &\leq ||u_{i+1} - u_i||_{L^\infty(\Omega;\mathbb{R}^2)} + ... + ||u_2 - u_1||_{L^\infty(\Omega;\mathbb{R}^2)} + ||u_1 - v||_{L^\infty(\Omega;\mathbb{R}^2)} \\ &< \frac{\varepsilon}{2} + \varepsilon_1 + ... + \varepsilon_i \leq \frac{\varepsilon}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{2^i}\right) < \varepsilon. \end{aligned} \tag{5.94}$$

Since

$$||u_{i+1} - u_i||_{L^\infty(\Omega;\mathbb{R}^2)} < \varepsilon_i \to 0, \quad \text{as } i \to \infty,$$

and $\{u_i\}_i$ is bounded in $W^{1,\infty}$, we have that $u_i \to \hat{u}$ uniformly, as $i \to \infty$, for some $\hat{u} \in W^{1,\infty}$. By this convergence, from the fact that $v = u_i$ on $\partial\Omega$ for every i, and from (5.94), we obtain conditions (ii) and (iii) (with $\hat{u}$ in place of u_ε), respectively. It remains to show that $\nabla\hat{u} \in K_0$

a.e. in Ω. Since $||\nabla\rho_{\delta_i}||_{L^1(\Omega;\mathbb{R}^2)} \leq \frac{C}{\delta_i}$ for some constant $C > 0$ independent of δ_i, from (5.93) we have that

$$\begin{aligned} ||\rho_{\delta_i}*(\nabla u_i - \nabla\hat{u})||_{L^1(\Omega_i;\mathbb{M}^{2\times 2})} &\leq ||\nabla\rho_{\delta_i}||_{L^1(\Omega;\mathbb{R}^2)}||u_i - \hat{u}||_{L^\infty(\Omega;\mathbb{R}^2)} \\ &\leq \frac{C}{\delta_i}\sum_{l=i}^{+\infty}||u_{i+1} - u_i||_{L^\infty(\Omega;\mathbb{R}^2)} \\ &< \frac{C}{\delta_i}\sum_{l=i}^{+\infty}\delta_l\varepsilon_{l-1} \leq C\sum_{l=i}^{\infty}\varepsilon_{l-1} < 2C\varepsilon_{i-1}. \end{aligned}$$

Taking into account (5.92) and writing L^1 in place of $L^1(\Omega_i;\mathbb{M}^{2\times 2})$, we have that

$$\begin{aligned} ||\nabla u_i - \nabla\hat{u}||_{L^1(\Omega;\mathbb{M}^{2\times 2})} &\leq ||\nabla u_i - \nabla\hat{u}||_{L^1} + ||\nabla u_i - \nabla\hat{u}||_{L^1(\Omega\setminus\Omega_i;\mathbb{M}^{2\times 2})} \\ &\leq ||\nabla u_i - \rho_{\delta_i}*\nabla u_i||_{L^1} + ||\rho_{\delta_i}*(\nabla u_i - \nabla\hat{u})||_{L^1} \\ &\qquad + ||\rho_{\delta_i}*\nabla\hat{u} - \nabla\hat{u}||_{L^1} + ||\nabla u_i - \nabla\hat{u}||_{L^1(\Omega\setminus\Omega_i;\mathbb{M}^{2\times 2})} \\ &\leq \frac{1}{2^i} + 2C\varepsilon_{i-1} + ||\rho_{\delta_i}*\nabla\hat{u} - \nabla\hat{u}||_{L^1} + ||\nabla u_i - \nabla\hat{u}||_{L^1(\Omega\setminus\Omega_i;\mathbb{M}^{2\times 2})}. \end{aligned}$$

Since $\delta_i \to 0$, as $i \to \infty$, and since u_i and $\hat{u}$ are equibounded in $W^{1\infty}(\Omega,\mathbb{R}^2)$, from the previous inequalities we deduce that $\nabla u_i \to \nabla\hat{u}$ in $L^1(\Omega,\mathbb{M}^{2\times 2})$, as $i \to \infty$. In particular, we have that, up to a subsequence, $\nabla u_i \to \nabla\hat{u}$ a.e. in Ω and in turn, in view of (5.91) and of Definition 5.10, that $\nabla\hat{u} \in K_0$. $\square$

CHAPTER 6

Singular perturbations of second order evolution problems in finite dimension

In this chapter, we describe the results of [**1**]. We study the limit, as ε goes to zero, of a particular solution $u^\varepsilon : [0,T] \to \mathbb{R}^n$ to the equation

$$\varepsilon^2 A\ddot{u}^\varepsilon(t) + \varepsilon B\dot{u}^\varepsilon(t) + \nabla_x \mathcal{E}(t, u^\varepsilon(t)) = 0, \tag{6.1}$$

where $\mathcal{E} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is an energy functional satisfying suitable coerciveness conditions, A and B are positive definite symmetric matrices of $\mathbb{M}^{n\times n}$, and the symbol ∇_x denotes the gradient with respect to $x \in \mathbb{R}^n$. In Section 6.1 we specify the assumptions on the potential $\mathcal{E}$ and describe the limit solution u, which is a piecewise continuous function satisfying

$$\nabla_x \mathcal{E}(t, u(t)) = 0 \qquad \text{and} \qquad \nabla_x^2 \mathcal{E}(t, u(t)) > 0, \tag{6.2}$$

where ∇_x^2 stands for the Hessian matrix with respect to the variable x and the inequality means that the matrix $\nabla_x^2 \mathcal{E}(t, u(t))$ is positive definite. Moreover, we show that certain jump conditions characterize the behavior of $u(t)$ at the discontinuity times t_i's using a heteroclinic solution to the second order autonomous equation

$$A\ddot{w}(s) + B\dot{w}(s) + \nabla_x \mathcal{E}(t_i, w(s)) = 0, \tag{6.3}$$

satisfying $w(-\infty) = \lim_{t\to t_i^-} u(t)$ (see Proposition 6.6). In Section 6.2 we prove that u^ε converges to u in a suitable sense (see Theorem 6.8 and Theorem 6.9). In Section 6.3, the same limit behavior is obtained by considering a different approximation scheme based on time discretization as well as on the solutions of suitable autonomous systems (see Theorem 6.18).

Section 6.4 contains the proof of the existence and uniqueness, up to time–traslations of a heteroclinic solution w to equation (6.3) satisfying $w(-\infty) = \xi$, when certain transversality conditions at the zero ξ of the vector field are satisfied.

6.1. Setting of the problem and preliminaries

In this section we formulate four assumptions we will refer to in this chapter, and give some preliminary results. We will use the following terminology: $x \in \mathbb{R}^n$ is a *critical* point of $\mathcal{E}(t,\cdot)$ if $\nabla_x \mathcal{E}(t,x) = 0$. A critical point x of $\mathcal{E}(t,\cdot)$ is *degenerate* if $\det \nabla_x^2 \mathcal{E}(t,x) = 0$. It is useful to recall that if D_x, D_x^2, ∇_x, ∇_x^2 stand for the first differential, the second differential, the gradient, and the Hessian matrix, with respect to x, respectively, then $D_x\mathcal{E}(t,x)[l] = \nabla_x \mathcal{E}(t,x)\cdot l$ and $D_x^2 \mathcal{E}(t,x)[l,l] = \big(\nabla_x^2 \mathcal{E}(t,x) l\big)\cdot l$, for every $l \in \mathbb{R}^n$.

Assumption 1. The energy $\mathcal{E} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ is a C^3 function satisfying, for every $(t,x) \in [0,T]\times\mathbb{R}^n$, the properties:

(i) $\nabla_x \mathcal{E}(t,x)\cdot x \geq b|x|^2 - a$, for some $a \geq 0$ and $b > 0$,
(ii) $\partial_t \mathcal{E}(t,x) \leq d|x|^2 + c$, for some d, $c \geq 0$,

where the symbol ∂_t denotes the partial derivative with respect to the variable t.

Observe that, from Assumption 1 (i), it descends that there exist $\tilde{a} \geq 0$ and $\tilde{b} > 0$ such that

$$\mathcal{E}(t,x) \geq \tilde{b}|x|^2 - \tilde{a}, \qquad \text{for every } (t,x) \in [0,T]\times\mathbb{R}^n. \tag{6.4}$$

Moreover, Assumption 1 (i) implies that, for every $t \in [0, T]$, all the critical points of $\mathcal{E}(t, \cdot)$ belong to the closed ball $\overline{B}$ centered at zero and with radius $\sqrt{\frac{a}{b}}$. Since the function $\mathcal{E}(t, \cdot)$ has minimum and maximum on $\overline{B}$, it has at least one critical point and it belongs to $\overline{B}$.

Assumption 2. For every $t \in [0, T]$, the set

$$\{\xi \in \mathbb{R}^n \,:\, \xi \text{ degenerate critical point of } \mathcal{E}(t, \cdot)\} \quad \text{is discrete.}$$

REMARK 6.1. Assumptions 1–2 imply that, for every $t \in [0, T]$, the set of the critical points of $\mathcal{E}(t, \cdot)$ is discrete. Indeed, by Assumption 2, the set of the degenerate critical points of $\mathcal{E}(t, \cdot)$ is discrete, while the set of the nondegenerate critical points of $\mathcal{E}(t, \cdot)$ is discrete by the Implicit Function Theorem.

For simplicity, in the sequel we will suppose that there are no critical points of $\mathcal{E}(T, \cdot)$.

DEFINITION 6.2. *We say that* $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ *is a* degenerate approximable critical pair *if* ξ *is a degenerate critical point of* $\mathcal{E}(\tau, \cdot)$ *and there exist two sequences* $t_n \to \tau^-$ *and* $\xi_n \to \xi$ *with* $\nabla_x \mathcal{E}(t_n, \xi_n) = 0$ *and* $\nabla_x^2 \mathcal{E}(t_n, \xi_n) > 0$ *for every* n.

Observe that if (τ, ξ) is a degenerate approximable critical pair, then $\nabla_x \mathcal{E}(\tau, \xi)$ is positive semidefinite. From now on, A and B will be two given symmetric and positive definite matrices of $\mathbb{M}^{n \times n}$, unless differently specified. λ_{min}^A and λ_{min}^B will denote the minimum eigenvalue of A and B, respectively.

Assumption 3. If $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ is a degenerate approximable critical pair, then there exists $l \in \mathbb{R}^n \setminus \{0\}$ such that

(i) $\ker \nabla_x^2 \mathcal{E}(\tau, \xi) = \mathrm{span}(l)$,
(ii) $(A^{-T} B l) \cdot \nabla_x (\partial_t \mathcal{E})(\tau, \xi) \neq 0$,
(iii) $(A^{-T} B l) \cdot D_x^3 \mathcal{E}(\tau, \xi)[l, l] \neq 0$,

where A^{-T} is the transpose of the inverse matrix A^{-1} and D_x^3 denotes the third differential with respect to the variable x, so that $D^3 \mathcal{E}(\tau, \xi)[l, l]$ is the vector of $\mathbb{R}^n$ obtained by taking the third differential of $\mathcal{E}(\tau, \cdot)$ at ξ and applying it to the pair $[l, l]$.

In the sequel, we will consider the equation $A \ddot{w}(s) + B \dot{w}(s) + \nabla_x \mathcal{E}(\tau, w(s)) = 0$, which is equivalent to the system

$$\begin{bmatrix} \dot{w}(s) \\ \dot{v}(s) \end{bmatrix} = F\left(\tau, \begin{bmatrix} w(s) \\ v(s) \end{bmatrix}\right), \tag{6.5}$$

where $F : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} B^{-1} y \\ -B A^{-1} \left(y + \nabla_x \mathcal{E}(t, x)\right) \end{bmatrix}, \qquad t \in [0, T],\ x, y \in \mathbb{R}^n. \tag{6.6}$$

For this reason, we collect in the following remark some properties of the function F which descend from Assumption 3 and which will prove useful.

Throughout this chapter, we will use both the notation $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ or (x, y) for a point of $\mathbb{R}^{2n}$.

REMARK 6.3. Let Assumption 3 hold for some (τ, ξ) degenerate approximable critical pair and some $l \in \mathbb{R}^n$. Setting $\eta := \left[\begin{smallmatrix} \xi \\ 0 \end{smallmatrix}\right] \in \mathbb{R}^{2n}$, observe first that

$$F(\tau, \eta) = 0.$$

Since

$$\nabla_\eta F(\tau, \eta) = \begin{bmatrix} 0 & B^{-1} \\ -B A^{-1} \nabla_x^2 \mathcal{E}(\tau, \xi) & -B A^{-1} \end{bmatrix},$$

where ∇_η denotes $\frac{\partial}{\partial(x, y)}$, from Assumption 3 (i) it turns out that

$$\ker \nabla_\eta F(\tau, \eta) = \mathrm{span}(\omega), \qquad \ker \nabla_\eta F(\tau, \eta)^T = \mathrm{span}(\nu), \tag{6.7}$$

where

$$\omega := \begin{bmatrix} l \\ 0 \end{bmatrix}, \qquad \nu := \begin{bmatrix} B^2 A^{-1} l \\ l \end{bmatrix},$$

Moreover, simple calculations give that

$$\partial_t F(\tau,\eta) = \begin{bmatrix} 0 \\ -BA^{-1}\nabla_x(\partial_t \mathcal{E})(\tau,\xi) \end{bmatrix}, \qquad D^2_\eta F(\tau,\eta)[\omega,\omega] = \begin{bmatrix} 0 \\ -BA^{-1}D^3_x \mathcal{E}(\tau,\xi)[l,l] \end{bmatrix},$$

so that, from Assumption 3 (ii) and (iii), we obtain that

$$\nu\cdot\partial_t F(\tau,\eta) \neq 0, \qquad \nu\cdot D^2_\eta F(\tau,\eta)[\omega,\omega] \neq 0. \tag{6.8}$$

Observe that $\lambda \in \mathbb{C}$ is an eigenvalue of $\nabla_\eta F(\tau,\eta)$ if and only if there exists $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ such that

$$\begin{cases} y = \lambda B x, \\ \nabla^2_x \mathcal{E}(\tau,\xi)x = -\lambda(B+\lambda A)x. \end{cases} \tag{6.9}$$

Let us show that

$$\text{the algebraic multiplicity of the null eigenvalue of } \nabla_\eta F(\tau,\eta) \text{ is } 1. \tag{6.10}$$

It is well known that such multiplicity corresponds to the dimension of the generalized eigenspace associated to the null eigenvalue, that is $\ker(\nabla_\eta F(\tau,\eta))^k$, where k is the smallest integer k such that $\ker(\nabla_\eta F(\tau,\eta))^k = \ker(\nabla_\eta F(\tau,\eta))^{k+1}$. Thus, in order to prove (6.10), it is enough to show that $\ker(\nabla_\eta F(\tau,\eta))^2 \subseteq \ker(\nabla_\eta F(\tau,\eta))$. If $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker(\nabla_\eta F(\tau,\eta))^2$, then, in view of (6.7),

$$\nabla_\eta F(\tau,\eta)\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} l \\ 0 \end{bmatrix}, \qquad \text{for some } \alpha \in \mathbb{C}. \tag{6.11}$$

If $\alpha \neq 0$, (6.11) implies that

$$\begin{cases} y = \alpha B l, \\ \nabla^2_x \mathcal{E}(\tau,\xi)x = y, \end{cases}$$

and, in turn, that $0 = x\cdot\left(\nabla^2_x \mathcal{E}(\tau,\xi)l\right) = \alpha Bl\cdot l \neq 0$, which is an absurd. Thus, $\alpha = 0$ in (6.11), so that $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker(\nabla_\eta F(\tau,\eta))$. This concludes the proof of (6.10). Now, we want to show that every eigenvalue λ of $\nabla_\eta F(\tau,\eta)$ is such that:

$$\mathrm{Re}(\lambda) < 0, \quad \text{for every eigenvalue} \quad \lambda \neq 0. \tag{6.12}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector associated to the eigenvalue $\lambda \neq 0$ and write $x \in \mathbb{C}^n \setminus \{0\}$ as $x = a + ib$, for some a, $b \in \mathbb{R}^n$. In the case a, $b \in \mathrm{span}(l)$, from the second equation in (6.9) we obtain that $(B+\lambda A)l = 0$. The scalar product of this equality with l gives

$$\lambda = -\frac{Bl\cdot l}{Al\cdot l} \leq -\frac{\lambda^B_{min}}{|A|} < 0.$$

In the case $\{a,b\} \nsubseteq \mathrm{span}(l)$, we consider the hermitian product of the second equation of (6.9) with x, which gives

$$C = -\lambda(C_A\lambda + C_B), \tag{6.13}$$

where

$$C := \left(\nabla^2_x \mathcal{E}(\tau,\xi)a\cdot a + \nabla^2_x \mathcal{E}(\tau,\xi)b\cdot b\right), \qquad C_A := Aa\cdot a + Ab\cdot b, \qquad C_B := Ba\cdot a + Bb\cdot b.$$

Now, by setting $\lambda = \lambda_1 + i\lambda_2$ for some λ_1, $\lambda_2 \in \mathbb{R}$, from (6.13) we obtain

$$\lambda_2(C_B + 2C_A\lambda_1) = 0, \tag{6.14}$$

and

$$C_A\lambda_1^2 + C_B\lambda_1 - C_A\lambda_2^2 + C = 0. \tag{6.15}$$

We want to prove that $\lambda_1 < 0$. If $\lambda_2 \neq 0$, from (6.14) it is easy to deduce that

$$\lambda_1 \leq -\frac{\lambda^B_{min}}{2|A|} < 0.$$

In the case where $\lambda_2 = 0$, we can suppose $b = 0$ and from (6.15) we obtain that $C_B^2 - 4CC_A \geq 0$ and that

$$\lambda_1 \leq \frac{-Ba{\cdot}a + \sqrt{(Ba{\cdot}a)^2 - 4(\nabla_x^2 \mathcal{E}(\tau,\xi)a{\cdot}a)(Aa{\cdot}a)}}{2Aa{\cdot}a}.$$

Since $a \notin \operatorname{span}(l) = \ker \nabla_x^2 \mathcal{E}(\tau,\xi)$ and $\nabla_x^2 \mathcal{E}(\tau,\xi) \geq 0$, we have that $\nabla_x^2 \mathcal{E}(\tau,\xi)a{\cdot}a \geq \lambda_\tau |a|^2$, where $\lambda_\tau > 0$ is the smallest eigenvalue of $\nabla_x^2 \mathcal{E}(\tau,\xi)$ different from 0. By using this fact, together with the hypotheses on A and B, we can easily prove, by rationalization, that

$$\frac{-Ba{\cdot}a + \sqrt{(Ba{\cdot}a)^2 - 4(\nabla_x^2 \mathcal{E}(\tau,\xi)a{\cdot}a)(Aa{\cdot}a)}}{2Aa{\cdot}a} \leq -\frac{\lambda_\tau \lambda_{min}^A}{|A||B|} < 0.$$

This concludes the proof of (6.12). Let us collect together properties (6.7), (6.8), (6.10), and (6.12). We obtain that $F : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, defined as is (6.6), is a C^2 function such that $F(\tau,\eta) = 0$ and satisfies:

(TC1) 0 is an eigenvalue of $\nabla_\eta F(\tau,\eta)$ with algebraic multiplicity 1, and there exist ω, $\nu \in \mathbb{R}^m$ such that $\omega{\cdot}\nu \neq 0$ and $\ker \nabla_\eta F(\tau,\eta) = \operatorname{span}(\omega)$, $\ker \nabla_\eta F(\tau,\eta)^T = \operatorname{span}(\nu)$. Moreover, $\operatorname{Re}(\lambda) < 0$ for every eigenvalue $\lambda \neq 0$;

(TC2) $\nu{\cdot}\partial_t F(\tau,\eta) \neq 0$;

(TC3) $\nu{\cdot}D_\eta^2 F(\tau,\eta)[\omega,\omega] \neq 0$.

We remark that, by [**41**, Theorem 3.4.1], the set of the vector fields satisfying (TC1)–(TC3) is open and dense in the space of C^∞ one–parameter families of vector fields with an equilibrium at (τ,ξ) with a zero eigenvalue. In this sense, we can say that our Assumption 3 is “generic”.

With the next lemma we introduce the heterocline which will allow us to connect, at a specific time τ, a degenerate critical point of $\mathcal{E}(\tau,\cdot)$ to another suitable critical point of $\mathcal{E}(\tau,\cdot)$.

LEMMA 6.4. *Let $(\tau,\xi) \in [0,T] \times \mathbb{R}^n$ be a degenerate approximable critical pair. Suppose that Assumption 1 and 2 and Assumption 3 (i) and (iii) hold. Excluding the constant solution ξ, there exists a unique solution, up to time–translations, to the problem*

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla_x \mathcal{E}(\tau, w(s)) = 0, & s \in (-\infty, 0] \\ \lim_{s\to-\infty} w(s) = \xi, \\ \lim_{s\to-\infty} \dot{w}(s) = 0. \end{cases} \tag{6.16}$$

The solution w is defined on all $\mathbb{R}$, there exists $\lim_{s\to+\infty} w(s) =: \zeta \in \mathbb{R}^n$, with ζ critical point of $\mathcal{E}(\tau,\cdot)$, and there exists $\lim_{s\to+\infty} \dot{w}(s) = 0$.

PROOF. Writing the equation in (6.16) in the equivalent form (6.5) and using the properties of the function F derived in Remark 6.3, we can apply Proposition 6.27 (with $m = 2n$ and $F(\tau,\cdot)$ in place of F), and we obtain existence and uniqueness (up to time–translations) of the nontrivial solution to (6.16). The other properties of such a solution con be proved using Lemma 6.5. □

LEMMA 6.5. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that*

$$g(x) \geq C_1|x|^2 - C_2, \qquad \text{for every } x \in \mathbb{R}^n, \tag{6.17}$$

for some constants $C_1 > 0$ and $C_2 \geq 0$. Suppose that the set of the critical points of g is discrete. Let w be the (unique) solution of the Cauchy problem associated to

$$A\ddot{w} + B\dot{w} + \nabla g(w) = 0, \tag{6.18}$$

with initial conditions at some $s_0 \in \mathbb{R}$.

Then, $(w,\dot{w})$ is bounded and defined on $[s_0,+\infty)$ and there exists the limit

$$\lim_{s\to+\infty} (w(s), \dot{w}(s)) =: (\zeta, 0), \tag{6.19}$$

where ζ is a critical point of g. Moreover, if $(w,\dot{w})$ is bounded on its maximal interval of existence, then $(w,\dot{w})$ is bounded and defined on all $\mathbb{R}$ and there exists the limit

$$\lim_{s\to-\infty} (w(s), \dot{w}(s)) =: (\xi, 0),$$

where ξ is a critical point of g.

Proof. Let us denote by (s_0^-, s_0^+) the maximal interval of existence of w. Consider, for $x, y \in \mathbb{R}^n$, the function

$$V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \frac{1}{2} Ay \cdot y + g(x),$$

and observe that, by multiplying (6.18) by $\dot{w}$, we obtain that

$$\frac{d}{ds} V\left(\begin{bmatrix} w(s) \\ \dot{w}(s) \end{bmatrix}\right) = -B\dot{w}(s) \cdot \dot{w}(s) \leq 0.$$

Thus, for every $s \in [s_0, s_0^+)$,

$$\frac{1}{2}\lambda_{min}^A |\dot{w}(s)|^2 + g(w(s)) \leq \frac{1}{2} A\dot{w}(s) \cdot \dot{w}(s) + g(w(s)) \leq \frac{1}{2} A\dot{w}(s_0) \cdot \dot{w}(s_0) + g(w(s_0)).$$

Therefore, by using (6.17), we deduce that the positive semiorbit of $(w, \dot{w})$ is bounded and therefore defined on $[s_0, +\infty)$. This fact, together with the monotonicity of $V\left(\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]\right)$ on $[s_0, +\infty)$, implies the existence of the limit

$$\lim_{s \to +\infty} V\left(\begin{bmatrix} w(s) \\ \dot{w}(s) \end{bmatrix}\right) =: L \in \mathbb{R}. \tag{6.20}$$

Let $\left[\begin{smallmatrix} \overline{x} \\ \overline{y} \end{smallmatrix}\right]$ be a point of the ω–limit set associated to $(w, \dot{w})$ (which is nonempty because of the boundedness of the positive semiorbit of $(w, \dot{w})$), and consider the solution φ to the problem

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla g(w(s)) = 0, & s \in [s_0, +\infty) \\ w(s_0) = \overline{x}, \\ \dot{w}(s_0) = \overline{y}. \end{cases}$$

Since, from (6.20), $V\left(\left[\begin{smallmatrix} \overline{x} \\ \overline{y} \end{smallmatrix}\right]\right) = L$, and the ω–limit sets are invariant sets, we obtain that $V\left(\left[\begin{smallmatrix} \varphi(s) \\ \dot{\varphi}(s) \end{smallmatrix}\right]\right) = L$ for every $s \geq s_0$. Thus,

$$\frac{d}{ds} V\left(\begin{bmatrix} \varphi(s) \\ \dot{\varphi}(s) \end{bmatrix}\right) = -B\dot{\varphi}(s) \cdot \dot{\varphi}(s) = 0, \qquad \text{for every } s \geq s_0,$$

so that $\overline{y} = 0$ and $\ddot{\varphi}(s) = 0$ for every $s \geq s_0$. Considering also (6.18), it turns out that $\nabla g(\overline{x}) = 0$. In this way, we have proved that the ω–limit set is contained in the set $Z := \{(\zeta, 0) \in \mathbb{R}^{2n} : \zeta \text{ critical point of } g\}$, which is, by assumption, discrete. Therefore, the ω–limit set, that is connected, is reduced to one point of Z, and this proves (6.19). The proof of the remaining part of the lemma can be done in a similar way, by using the boundedness of $(w, \dot{w})$ on $(s_0^-, +\infty)$ and again the monotonicity of $V\left(\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]\right)$. □

Assumption 4. For every degenerate approximable critical pair $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$, let w be the unique solution (up to time–translation) of (6.16). We assume that

$$\nabla_x^2 \mathcal{E}(\tau, w(+\infty)) \text{ is positive definite.}$$

With the following proposition we construct a piecewise continuous solution to problem (6.2) on the interval $[0, T]$. We will then prove, in Section 6.2, that this solution is suitably approximated by a solution of equation (6.1).

Proposition 6.6. *Under Assumptions 1–4, let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x \mathcal{E}(0, x_0^r) = 0$ and $\nabla_x^2 \mathcal{E}(0, x_0^r) > 0$.*

There exists a partition $0 = t_0 < ... < t_m = T$ of the interval $[0, T]$ and, for every $j \in \{1, ..., m-1\}$, two distinct points x_j^r, $x_j^s \in \mathbb{R}^n$ with the following properties:

(1) *for every $j \in \{1, ..., m\}$, there exists a unique function $u_j : [t_{j-1}, t_j) \to \mathbb{R}^n$ of class C^2 such that $u_j(t_{j-1}) = x_{j-1}^r$ and*

$$\nabla_x \mathcal{E}(t, u_j(t)) = 0 \quad \text{and} \quad \nabla_x^2 \mathcal{E}(t, u_j(t)) > 0, \qquad \text{for every } t \in [t_{j-1}, t_j);$$

(2) for every $j \in \{1, ..., m-1\}$, $x_j^s = \lim_{t \to t_j^-} u_j(t)$, (t_j, x_j^s) *is a degenerate approximable critical pair, and there exists a unique (up to time–translation) function* $w_j : \mathbb{R} \to \mathbb{R}^n$ *of class* C^2 *satisfying*

$$A\ddot{w}_j(s) + B\dot{w}_j(s) + \nabla_x \mathcal{E}(t_j, w_j(s)) = 0, \qquad s \in \mathbb{R}, \tag{6.21}$$

and such that

$$\lim_{s \to -\infty} w_j(s) = x_j^s, \qquad \lim_{s \to +\infty} w_j(s) = x_j^r.$$

The proof of Proposition 6.6 is similar to the proof of [**72**, Proposition 1]. The only difference is in the choice of the heterocline which connects x_i^s to x_i^r: in [**72**], it is the solution to the equation $\dot{w}_j(s) = \nabla_x \mathcal{E}(t_j, w_j(s))$, while, here, equation (6.21) is considered. The scheme of the proof is the following: starting from x_0^r, we find a unique solution u_1 of (6.2) on the maximal interval of existence $[0, t_1)$ such that $u_1(0) = x_0^r$. If $t_1 < T$, then there exists the limit $x_1^s := \lim_{t \to t_1^-} u_1(t)$ (the index s stands for "singular") and (t_1, x_1^s) is a degenerate approximable critical pair. Thus, Assumption 3 holds for (t_1, x_1^s). In particular, Lemma 6.4 tells us that Assumption 3 (i) and (iii) (together with Assumption 1 and 2) ensure the existence and uniqueness, up to time–translations, of the solution w_1 to (6.21) with $j = 1$, satisfying $w_1(-\infty) = x_1^s$. Moreover, there exists the limit $\lim_{s \to +\infty} w_1(s) =: x_1^r$ (the index r stands for "regular") and x_1^r is a critical point of $\mathcal{E}(t_1, \cdot)$. At this point, using Assumption 4, we have that $\nabla_x^2 \mathcal{E}(t_1, x_1^r) > 0$ and we can repeat the same argument with (t_1, x_1^r) in place of $(0, x_0^r)$. In this way, we find the solution u_2 of (6.2), defined on the maximal (on the right) interval of existence $[t_1, t_2)$, and such that $u_2(t_1) = x_1^r$, and so on. Observe that, by Assumption 2, $\nabla_x^2 \mathcal{E}(T, u_m(T))$ is positive definite. The functions $u_1, ..., u_m$ give us a piecewise continuous solution u to problem (6.2), according to the next definition.

DEFINITION 6.7. *Under Assumptions 1–4, we define* $u : [0, T] \to \mathbb{R}^n$ *by:*

$$u(t) := u_j(t), \qquad \text{for every } t \in [t_{j-1}, t_j), \quad j \in \{1, ..., m\}, \qquad u(T) := u_m(t),$$

where $0 = t_0 < ... < t_m = T$ *and* $u_1, ..., u_m$ *are the partition and the functions obtained in Proposition 6.6.*

Since (t_j, x_j^s) is an approximable critical pair for every $j \in \{1, ..., m-1\}$, Assumption 3 implies that the transversality conditions (TC1)–(TC3) listed in Remark 6.3 hold for F (see (6.6) for a definition) at $\left(t_j, \begin{bmatrix} x_j^s \\ 0 \end{bmatrix}\right)$ for $j = 1, ..., m-1$, as shown in the same remark. Such *transversality conditions* ensure (see [**41**, Theorem 3.4.1]) the existence of a smooth curve of equilibria $\left(t(\cdot), \begin{bmatrix} x \\ y \end{bmatrix}(\cdot)\right)$ passing through $\left(t_j, \begin{bmatrix} x_j^s \\ 0 \end{bmatrix}\right)$, tangent to the hyperplane $\{t_j\} \times \mathbb{R}^{2n}$. In particular, in a left neighborhood of t_j there are two regular branches of solutions to $F(t, \cdot) = 0$, a saddles' branch and a nodes' branch, while in a right neighborhood of t_j there are no solutions. The nodes' branch is the already defined $u_j(t)$ (more precisely, it is $\left(t, \begin{bmatrix} u_j(t) \\ 0 \end{bmatrix}\right)$). For every $j = 1, ..., m-1$, we denote the other branch, which is defined starting from some $t_j^* \in [t_{j-1}, t_j)$, by

$$\overline{u}_j : [t_j^*, t_j) \to \mathbb{R}^n. \tag{6.22}$$

Note that, by Assumption 2, for every $\delta > 0$ sufficiently small we have that

$$x \in \overline{B}(x_j^s, \delta) \text{ satisfies } \nabla_x \mathcal{E}(t_j, x) = 0 \text{ if and only if } x = x_j^s. \tag{6.23}$$

Moreover, in view of the behavior of the vector field F at the point $\left(t_j, \begin{bmatrix} x_j^s \\ 0 \end{bmatrix}\right)$, we can introduce the specific times

$$t_{j-1} < t_j^* < t_j^\delta < t_j < t_j^{**} \tag{6.24}$$

endowed with the following properties:

$$\text{for every } t \in [t_j^\delta, t_j),\ \nabla_x \mathcal{E}(t,x) = 0 \text{ for some } x \in \overline{B}\left(x_j^s, \frac{\delta}{4}\right) \text{ if and only if } x \in \{u_j(t), \overline{u}_j(t)\}, \tag{6.25}$$

$$|F(\cdot,\cdot)| > 0 \ \text{ on } (t_j, t_j^{**}] \times \overline{B}\big((x_j^s, 0), \delta\big). \tag{6.26}$$

6.2. Singular perturbations of second order

In this section, we consider the equation

$$\varepsilon^2 A\ddot{u}^\varepsilon(t) + \varepsilon B\dot{u}^\varepsilon(t) + \nabla_x \mathcal{E}(t, u^\varepsilon(t)) = 0, \qquad t \in [0,T], \tag{6.27}$$

where $\varepsilon > 0$ is an arbitrarily small parameter, and we show that a solution to this equation, satisfying suitable initial conditions, approximates the solution to problem (6.2) constructed in the previous section in a sense that we are going to specify. Note that equation (6.27) can be seen as a singular perturbation of the evolution problem

$$A\ddot{u}(t) + B\dot{u}(t) + \nabla_x \mathcal{E}(t, u(t)) = 0, \qquad t \in [0,T].$$

In the present section and also in Section 6.3 we will take into account the following objects. Let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x \mathcal{E}(0, x_0^r) = 0$ and $\nabla_x^2 \mathcal{E}(0, x_0^r)$ is positive definite. We consider a point $(x_0, y_0) \in \mathbb{R}^{2n}$ such that v_0 is the solution to the autonomous problem

$$\begin{cases} A\ddot{v}_0(\sigma) + B\dot{v}_0(\sigma) + \nabla_x \mathcal{E}(0, v_0(\sigma)) = 0, & \sigma \in [0, +\infty) \\ v_0(0) = x_0, \\ \dot{v}_0(0) = y_0, \end{cases} \tag{6.28}$$

and

$$\lim_{\sigma \to +\infty} v_0(\sigma) = x_0^r. \tag{6.29}$$

Under Assumptions 1 and 2, Lemma 6.5 ensures the existence of the solution to problem (6.28) and of the limit in (6.29). Also, it tells us that $v_0(+\infty)$ is a critical point of $\mathcal{E}(0,\cdot)$ and that $\dot{v}_0(+\infty) = 0$. The main results of this section are given by the following two theorems, which describe how the function u of Definition 6.7 and the trajectories of the heteroclines w_j's at the jump times t_j's are approximated by suitable solutions u^ε of (6.27).

THEOREM 6.8. *Under Assumptions 1–4, let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x \mathcal{E}(0, x_0^r) = 0$ and $\nabla_x^2 \mathcal{E}(0, x_0^r)$ is positive definite. Let $u : [0,T] \to \mathbb{R}^n$, with $u(0) = x_0^r$, be given by Definition 6.7 and $u^\varepsilon : [0,T] \to \mathbb{R}^n$ a solution to (6.27) such that*

$$(u^\varepsilon(0), \varepsilon \dot{u}^\varepsilon(0)) \to (x_0, y_0), \qquad \text{as } \varepsilon \to 0^+, \tag{6.30}$$

where (x_0, y_0) satisfies (6.28) and (6.29). Then, we have that

(1) $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ converges uniformly to $(u, 0)$ on the compact subsets of $(0,T] \setminus \{t_1, ..., t_{m-1}\}$;
(2) for every $j \in \{1, ..., m-1\}$, there exists a sequence $\{a_j^\varepsilon\}$, with $a_j^\varepsilon \to t_j$, and a heteroclinic solution w_j of

$$\begin{cases} A\ddot{w}_j(s) + B\dot{w}_j(s) + \nabla_x \mathcal{E}(t_j, w_j(s)) = 0, \\ \lim_{s\to-\infty} w_j(s) = x_j^s, \\ \lim_{s\to-\infty} \dot{w}_j(s) = 0, \end{cases} \tag{6.31}$$

such that

$$(v_j^\varepsilon, \dot{v}_j^\varepsilon) \to (w_j, \dot{w}_j) \ \ \text{uniformly on the compact subsets of } \mathbb{R},$$

where

$$v_j^\varepsilon(s) := u^\varepsilon(a_j^\varepsilon + \varepsilon s), \qquad \text{for every } s \in \left[-\frac{a_j^\varepsilon}{\varepsilon}, \frac{T - a_j^\varepsilon}{\varepsilon}\right].$$

The next theorem can be viewed as a corollary of Theorem 6.8 and gives a geometric interpretation of how $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ approximates $(u, 0)$ and the trajectory of $(w_j, B\dot{w}_j)$, for $j = 1, ..., m-1$. It deals with the following sets. Recall the heteroclines given by Proposition 6.6 and the function v_0 previously introduced. We define

$$\mathscr{I}_0 := \{(v_0(s), B\dot{v}_0(s)),\quad s \geq 0\} \qquad \text{and} \qquad \mathscr{I}_j := \{(w_j(s), B\dot{w}_j(s)),\quad s \in \mathbb{R}\}, \tag{6.32}$$

for $j = 1, ..., m-1$, and set

$$\mathscr{S}^\varepsilon := \{(t, u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)) :\ t \in [0,T]\}, \qquad \mathscr{S} := \mathscr{S}_{reg} \cup \mathscr{S}_{sing}, \tag{6.33}$$

where

$$\mathscr{S}_{reg} := \{(t, u(t), 0)\ :\ t \in [0,T]\}, \tag{6.34}$$

and

$$\mathscr{S}_{sing} := [\{0\} \times \mathscr{I}_0] \cup \bigcup_{j=1}^{m-1} \{t_j\} \times [\mathscr{I}_j \cup \{(x_j^s, 0)\}]. \tag{6.35}$$

Observe that the set $\mathscr{S}_{sing}$ does not change if we replace some w_j's by some of their time–translated. We recall that $d(\cdot,\cdot)$ stands for the euclidean distance either between two points or between a point and a set, and denote by d_H the *Haudorff distance*. If K_1 and K_2 are two compact subsets of a compact metric space, the Hausdorff distance between K_1 and K_2 is defined as

$$d_H(K_1, K_2) := \sup_{x\in K_1} d(x, K_2) + \sup_{x \in K_2} d(x, K_1).$$

THEOREM 6.9. *Under the hypotheses of Theorem 6.8, we have that*

$$d_H(\mathscr{S}^\varepsilon, \mathscr{S}) \to 0, \quad \text{as } \varepsilon \to 0^+.$$

In order to prove Theorem 6.8 and Theorem 6.9, we need some preliminary results. First, we state a property of uniform boundedness of the solutions to equation (6.27).

LEMMA 6.10. *Let Assumption 1 hold and let $\{t^\varepsilon\}$ be a sequence converging to some $\tilde{t} \in [0,T]$, as $\varepsilon \to 0^+$. Then, there exists a unique $u^\varepsilon : [t^\varepsilon, T] \to \mathbb{R}^n$ of class C^2, solution of the Dirichlet problem associated to (6.27) with initial condition at t^ε. Moreover, if $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot{u}^\varepsilon(t^\varepsilon)$ are uniformly bounded, then $u^\varepsilon(t)$ and $\varepsilon\dot{u}^\varepsilon(t)$ are uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε.*

PROOF. The standard theory of ordinary differential equations tells us that there exists locally a unique solution u^ε of the Cauchy problem associated to (6.27). Multiplying equation (6.27) by $\dot{u}^\varepsilon(t)$, it turns out the equation

$$\frac{\varepsilon^2}{2}\frac{d}{dt}A\dot{u}^\varepsilon\cdot\dot{u}^\varepsilon + \varepsilon B\dot{u}^\varepsilon\cdot\dot{u}^\varepsilon + \frac{d}{dt}\mathcal{E}(t, u^\varepsilon) - \partial_t\mathcal{E}(t, u^\varepsilon) = 0,$$

which, by integration between t^ε and $t \in [t^\varepsilon, T]$ and by the positive definiteness of A and B, gives

$$\frac{\varepsilon^2}{2}\lambda^A_{min}|\dot{u}^\varepsilon(t)|^2 + \mathcal{E}(t, u^\varepsilon(t)) \leq \frac{\varepsilon^2}{2}A\dot{u}^\varepsilon(t^\varepsilon)\cdot\dot{u}^\varepsilon(t^\varepsilon) + \mathcal{E}(t^\varepsilon, u^\varepsilon(t^\varepsilon)) + \int_{t^\varepsilon}^t \partial_t\mathcal{E}(\tau, u^\varepsilon(\tau))d\tau. \tag{6.36}$$

Then, by using Assumption 1 and (6.4), we have that

$$|u^\varepsilon(t)|^2 \leq K_1^\varepsilon + K_2\int_0^t |u^\varepsilon(\tau)|^2 d\tau, \qquad \text{for every } t \in [0,T],$$

where

$$K_1^\varepsilon = \frac{1}{\tilde{b}}\left[\frac{\varepsilon^2}{2}A\dot{u}^\varepsilon(t^\varepsilon)\cdot\dot{u}^\varepsilon(t^\varepsilon) + \mathcal{E}(t^\varepsilon, u^\varepsilon(t^\varepsilon)) + c(T - t^\varepsilon) + \tilde{a}\right], \qquad K_2 := \frac{d}{\tilde{b}}. \tag{6.37}$$

By differential inequalities (see, e.g., [**43**]), we obtain that

$$|u^\varepsilon(t)|^2 \leq K_1^\varepsilon e^{K_2(T-t^\varepsilon)}, \qquad \text{for every } t \in [0,T],$$

so that, by hypothesis and by (6.37), $u^\varepsilon(t)$ is uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε. This fact, together with (6.36), gives that also $\varepsilon\dot{u}^\varepsilon$ is uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε. This in particular implies that u^ε and $\dot{u}^\varepsilon$ are defined on $[t^\varepsilon, T]$ and completes the proof. $\square$

The following technical proposition will play a crucial role in the proof of the main results of this section. To handle equation (6.27), we will use the function $F : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined in (6.6), so that (6.27) is equivalent to

$$\varepsilon \begin{bmatrix} \dot{u}^\varepsilon \\ \dot{v}^\varepsilon \end{bmatrix} = F\left(t, \begin{bmatrix} u^\varepsilon \\ v^\varepsilon \end{bmatrix}\right).$$

Also, we will consider a function $u \in C([\bar{t},\hat{t}];\mathbb{R}^n)$ for some $0 \le \bar{t} < \hat{t} \le T$ and we will use Lemma 6.10 to say that if $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot{u}^\varepsilon(t^\varepsilon)$ are uniformly bounded for a certain sequence $\{t^\varepsilon\} \subseteq [\bar{t},\hat{t}\,)$, then there exists a compact $K \subseteq \mathbb{R}^{2n}$ such that

$$\left\{(su^\varepsilon(t) + (1-s)u(t), \varepsilon s B\dot{u}^\varepsilon(t)) : (s,t) \in [0,1]\times[t^\varepsilon,\hat{t}\,]\right\} \subseteq K, \tag{6.38}$$

for every $\varepsilon > 0$.

PROPOSITION 6.11. *Let $\mathcal{E} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $u \in C([\bar{t},\hat{t}];\mathbb{R}^n)$ be a solution of (6.2) on $[\bar{t},\hat{t}]$, for some $0 \le \bar{t} < \hat{t} \le T$. Let $\{t^\varepsilon\} \subseteq [\bar{t},\hat{t}\,)$ be such that*

$$t^\varepsilon \to \tilde{t}, \qquad \text{for some } \tilde{t} \in [\bar{t},\hat{t}\,),$$

and let u^ε be a C^2 solution of (6.27) on $[t^\varepsilon,T]$ such that $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot{u}^\varepsilon(t^\varepsilon)$ are uniformly bounded. Finally, let K be a compact of $\mathbb{R}^{2n}$ such that (6.38) hold and let ω be the modulus of continuity on K (uniform with respect to $t \in [0,T]$) of the function $\nabla_\eta F(t,\cdot)$, where F is defined in (6.6).

There exists a positive constant $C = C(\mathcal{E},u)$ such that, if $r \in (0,C)$ and

$$\limsup_{\varepsilon\to0^+} \left|\left(u^\varepsilon(t^\varepsilon) - u(\tilde{t}), \varepsilon B\dot{u}^\varepsilon(t^\varepsilon)\right)\right| < \min\{r, r\omega(2r)\}, \tag{6.39}$$

then

$$\limsup_{\varepsilon\to0^+} \sup_{t\in[t^\varepsilon,\hat{t}\,]} \left|\left(u^\varepsilon(t) - u(t), \varepsilon B\dot{u}^\varepsilon(t)\right)\right| \le r. \tag{6.40}$$

The proof of Proposition 6.11 requires two lemmas.

LEMMA 6.12. *Let $A \in \mathbb{M}^{n\times n}$ be such that*

$$\mathrm{Re}(\lambda) \le -\alpha, \qquad \text{for every } \lambda \text{ eigenvalue of } A, \quad \text{for some } \alpha > 0.$$

There exists a constant C_A, depending on A, such that

$$\left|e^{tA}\right| \le C_A e^{-\frac{\alpha}{2}t}, \quad \text{for every } t \ge 0.$$

The proof of Lemma 6.12 is straightforward, once A is written in Jordan canonical form. In Section 6.4 we will use more general estimates of this kind (see (6.173)–(6.174)). With the following remark we underline the fact that the constant C_A of the previous lemma is not universal, but generally depending on A.

REMARK 6.13. For $a \in \mathbb{R}$, consider the matrix $A = \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}$, whose spectrum is $\{-1\}$. Since A is the sum of the matrices $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, which commute, it is easy to compute

$$e^{tA} = e^{-t} \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}.$$

The norm of e^{tA} is $e^{-t}\sqrt{2 + a^2t^2}$. Therefore, a constant C not depending on A and such that $\left|e^{tA}\right| \le Ce^{-\frac{t}{2}}$ should satisfy $\sqrt{2 + a^2t^2} \le Ce^{\frac{t}{2}}$ for every $a \in \mathbb{R}$, but this is impossible.

LEMMA 6.14. *Let $A \in \mathbb{M}^{n\times n}$ be such that*

$$\left|e^{tA}\right| \le Ce^{-\gamma t}, \qquad \text{for every } t \ge 0, \quad \text{for some } C,\gamma > 0.$$

There exist two positive constants δ and b, depending only on C and γ, such that, if $B \in \mathbb{M}^{n\times n}$ and $|B| \le \delta$, then

$$\left|e^{t(A+B)}\right| \le be^{-\frac{\gamma}{2}t}, \qquad \text{for every } t \ge 0.$$

PROOF. Observe that when A and B commute the proof is straightforward. Otherwise, for $x \in \mathbb{R}^n$, let us consider the solution v^x of the problem

$$\begin{cases} \dot{v}(t) = (A+B)v(t), & t > 0 \\ v(0) = x. \end{cases} \tag{6.41}$$

Since

$$\left|e^{t(A+B)}\right| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|v^x(t)|}{|x|},$$

the thesis follows if we prove that there exist δ, $b > 0$, depending only on C and γ, such that, if $|B| \leq \delta$, then

$$|v^x(t)| \leq b e^{-\frac{\gamma}{2}t}|x|, \qquad \text{for every } t \geq 0 \text{ and } x \in \mathbb{R}^n. \tag{6.42}$$

For certain constants δ, $b > 0$ to be chosen later, let us fix a function $z \in C([0,+\infty);\mathbb{R}^n)$ such that $|z(t)| \leq b e^{-\frac{\gamma}{2}t}|x|$ for all $t \geq 0$, and consider, for $|B| \leq \delta$, the problem

$$\begin{cases} \dot{v}(t) = Av(t) + Bz(t), & t > 0 \\ z(0) = x. \end{cases} \tag{6.43}$$

The solution of (6.43) can be represented by the variation of constants formula and estimated in the following way:

$$|v(t)| \leq C\left(e^{-\gamma t}|x| + \int_0^t e^{-\gamma(t-s)}|B||z(s)|ds\right) \leq C|x|e^{-\frac{\gamma}{2}t}\left(1 + \frac{2b\delta}{\gamma}\right). \tag{6.44}$$

In order to obtain (6.42), we want $C\left(1+\frac{2b\delta}{\gamma}\right) \leq b$ so that we choose

$$\delta < \frac{\gamma}{2C}, \qquad b \geq \frac{\gamma C}{\gamma - 2\delta C}. \tag{6.45}$$

Now, we define the space

$$X := \left\{ v \in C([0,+\infty),\mathbb{R}^n) \ : \ v(0) = x \text{ and } \sup_{t\in[0,+\infty)} v(t)e^{\frac{\gamma}{2}t} < \infty \right\},$$

which is a Banach space endowed with the norm $\|v\|_X := \sup_{t\in[0,+\infty)} v(t)e^{\frac{\gamma}{2}t}$, and the subset

$$\Omega := \{v \in X \ : \ \|v\|_X \leq |x|b\}.$$

From (6.44) and thanks to the choice (6.45), we have obtained that the operator

$$G : \Omega \to \Omega,$$

that to each $z \in \Omega$ associates the solution of (6.43), is well defined. If we prove that G is a contraction from Ω to Ω, we will prove that the solution v of (6.41) satisfies (6.42), that is our aim. Let z_1, $z_2 \in \Omega$ and suppose $|B| \leq \delta$. Then, we have that

$$\|G(z_1) - G(z_2)\|_X = \sup_{t\geq 0} e^{\frac{\gamma}{2}t}\left|\int_0^t e^{(t-s)A}B[z_1(s) - z_2(s)]ds\right| \leq \frac{2C\delta}{\gamma}\|z_1 - z_2\|_X.$$

From (6.45), it descends that $\frac{2C\delta}{\gamma} < 1$, so that G is a contraction from Ω to Ω. □

PROOF OF PROPOSITION 6.11. Note that, defining $W := \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]$ and $W_\varepsilon := \left[\begin{smallmatrix} u^\varepsilon \\ v^\varepsilon \end{smallmatrix}\right] - W$, equation (6.27) is equivalent to

$$\varepsilon \dot{W}_\varepsilon = F(t, W + W_\varepsilon) - \varepsilon \dot{W}, \tag{6.46}$$

where F, already defined in (6.6), is given by

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} B^{-1}y \\ -BA^{-1}\left(y + \nabla_x \mathcal{E}(t,x)\right) \end{bmatrix}, \qquad t \in [0,T], \quad x, y \in \mathbb{R}^n.$$

Set

$$M(t) := \nabla_\eta F(t, W(t)), \qquad t \in [\bar{t}, \hat{t}].$$

The regularity assumptions on $\mathcal{E}$ and u imply that $M \in C([\bar{t}, \hat{t}]; \mathbb{M}^{2n\times 2n})$ and that

$$\nabla_x^2 \mathcal{E}(t, u(t)) \geq \alpha, \qquad \text{for every } t \in [\bar{t}, \hat{t}], \quad \text{for some } \alpha > 0. \tag{6.47}$$

First, let us explain how we find the constant C of the statement. As done in Remark 6.3, we can prove using (6.47) that there exists $\beta > 0$ such that $\mathrm{Re}(\lambda) \leq -\beta$ for every λ eigenvalue of $M(s)$, for every $s \in [\bar{t}, \hat{t}]$. Therefore, from Lemma 6.12 and Lemma 6.14, it turns out that there exists $b > 0$ such that

$$\left| e^{tM(s)} \right| \leq b e^{-\frac{\beta}{4}t}, \quad \text{for every } t \geq 0 \text{ and } s \in [\bar{t}, \hat{t}]. \tag{6.48}$$

Indeed, from Lemma 6.12, we have that, for every $t \geq 0$,

$$\left| e^{tM(s)} \right| \leq C_{M(s)} e^{-\frac{\beta}{2}t}, \tag{6.49}$$

with $C_{M(s)} > 0$ a constant depending on $M(s)$ for every $s \in [\bar{t}, \hat{t}]$. Considering (6.49) for a certain $s_0 \in [\bar{t}, \hat{t}]$, let δ_0, $b_0 > 0$, depending on $C_{M(s_0)}$ and $\frac{\beta}{2}$, be given by Lemma 6.14. By the uniform continuity of M on $[\bar{t}, \hat{t}]$, there exists $\sigma_0 > 0$ and a finite number of s_i in $[\bar{t}, \hat{t}]$ such that, if $s \in [\bar{t}, \hat{t}]$, then $|s - s_i| < \sigma_0$ for some i and $|M(s) - M(s_i)| \leq \delta_0$, so that, by Lemma 6.14,

$$\left| e^{tM(s)} \right| = \left| e^{t(M(s_i) + M(s) - M(s_i))} \right| \leq b_0 e^{-\frac{\beta}{4}t}, \qquad \text{for every } t \geq 0.$$

Now, let $C > 0$ be a constant (depending on b and β and, in turn, on f and u) such that, if $0 < r < C$, then

$$\omega(2r) \leq \frac{1}{2b} \left(1 + \frac{10}{\beta} \max\{1, 2b\} \right)^{-1}. \tag{6.50}$$

The reason why the estimate (6.50) is needed will be clear at the end of the proof. By now, let $0 < r < C$ and suppose that (6.39) holds true for a certain $t^\varepsilon \to \tilde{t} \in [\bar{t}, \hat{t})$. Then, there exists $\varepsilon_r > 0$ such that

$$\left| (u^\varepsilon(t^\varepsilon) - u(\tilde{t}), \varepsilon B \dot{u}^\varepsilon(t^\varepsilon)) \right| \leq \min\{r, r\omega(2r)\}, \qquad \text{for every } \varepsilon \in (0, \varepsilon_r). \tag{6.51}$$

Since $t^\varepsilon \to \tilde{t}$, it is easy to check that (6.51) implies, up to a smaller ε_r, that

$$|W_\varepsilon(t^\varepsilon)| \leq 2 \min\{r, r\omega(2r)\}, \quad \text{for every } \varepsilon \in (0, \varepsilon_r). \tag{6.52}$$

Therefore, it makes sense to define, for $\varepsilon \in (0, \varepsilon_r)$,

$$\hat{t}^\varepsilon := \inf\{t \in [t^\varepsilon, \hat{t}] \; : \; |W_\varepsilon(t)| > 2r\},$$

with the convention $\inf \emptyset = \hat{t}$, so that $\sup_{[t^\varepsilon, \hat{t}^\varepsilon]} |W_\varepsilon| \leq 2r$ for every $\varepsilon \in (0, \varepsilon_r)$.
Claim. There exists $\tilde{\varepsilon}_r \in (0, \varepsilon_r]$ such that

$$\sup_{t \in [t^\varepsilon, \hat{t}^\varepsilon]} |W_\varepsilon(t)| \leq r, \qquad \text{for every } \varepsilon \in (0, \tilde{\varepsilon}_r).$$

Observe that the claim implies that $\hat{t}^\varepsilon = \hat{t}$ and, in turn, that $\sup_{[t^\varepsilon, \hat{t}]} |W_\varepsilon| \leq r$ for every $\varepsilon \in (0, \tilde{\varepsilon}_r)$, that is (6.40).
Proof of the claim. Using again the uniform continuity of M on $[\bar{t}, \hat{t}]$, let $\sigma > 0$ be such that $|M(t) - M(s)| \leq \omega(2r)$ if $|s - t| < \sigma$, and define

$$\tau_i = \tau_i(\varepsilon) := t^\varepsilon + i\sigma, \quad \text{for} \quad i = 0, ..., k_\varepsilon, \quad \text{where} \quad k_\varepsilon := \left\lfloor \frac{\hat{t}^\varepsilon - t^\varepsilon}{\sigma} \right\rfloor,$$

and

$$M_\varepsilon(t) := \begin{cases} M(t^\varepsilon), & t \in [t^\varepsilon, \tau_1) \\ M(\tau_1), & t \in [\tau_1 \tau_2) \\ \vdots & \\ M(\tau_{k_\varepsilon}), & t \in [\tau_{k_\varepsilon}, \hat{t}^\varepsilon]. \end{cases}$$

Observe that $M_\varepsilon(t) = M\left(t^\varepsilon + \left\lfloor \frac{t-t^\varepsilon}{\sigma} \right\rfloor\right)$. With such definitions, we obtain that

$$\sup_{t\in[t^\varepsilon,\hat{t}^\varepsilon]} |M_\varepsilon(t) - M(t)| \le \omega(2r). \tag{6.53}$$

Let us write equation (6.46) on $[t^\varepsilon, \hat{t}^\varepsilon]$ in the following equivalent way:

$$\varepsilon \dot{W}_\varepsilon = M_\varepsilon W_\varepsilon + H_\varepsilon,$$

where

$$H_\varepsilon := (M - M_\varepsilon)W_\varepsilon + [F(t, W + W_\varepsilon) - MW_\varepsilon] - \varepsilon \dot{W}.$$

Clearly, there exists $\tilde{\varepsilon}_r \in (0, \varepsilon_r]$ such that

$$\sup_{t\in[t^\varepsilon,\hat{t}^\varepsilon]} |\varepsilon \dot{W}(t)| \le r\omega(2r), \qquad \text{for every } \varepsilon \in (0, \tilde{\varepsilon}_r). \tag{6.54}$$

Since $F(t, W(t)) = 0$ for every $t \in [\bar{t}, \hat{t}\,]$, it turns out that

$$\begin{aligned}\sup_{t\in[t^\varepsilon,\hat{t}^\varepsilon]} &|F(t, W(t) + W_\varepsilon(t)) - M(t)W_\varepsilon(t)| \\ &\le 2r \sup\left\{|\nabla_\eta F(t, W(t) + sW_\varepsilon(t)) - M(t)| : (s,t) \in [0,1] \times [t^\varepsilon, \hat{t}^\varepsilon]\right\} \le 2r\omega(2r). \end{aligned}\tag{6.55}$$

Inequalities (6.53), (6.54) and (6.55) imply that

$$\sup_{[t^\varepsilon,\hat{t}^\varepsilon]} |H_\varepsilon(t)| \le 5r\omega(2r). \tag{6.56}$$

By setting $Z_\varepsilon(t) := W_\varepsilon(\varepsilon t)$, let us consider another equation equivalent to (6.46) on $[t^\varepsilon, \hat{t}^\varepsilon]$:

$$\dot{Z}_\varepsilon = M_\varepsilon(\varepsilon t) Z_\varepsilon + H_\varepsilon(\varepsilon t), \qquad t \in \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\hat{t}^\varepsilon}{\varepsilon}\right]. \tag{6.57}$$

If $k_\varepsilon = 0$, that is $\hat{t}^\varepsilon - t^\varepsilon < \sigma$, the solution of (6.57) is

$$Z_\varepsilon(t) = e^{\left(t - \frac{t^\varepsilon}{\varepsilon}\right)M(t^\varepsilon)} Z_\varepsilon\left(\frac{t^\varepsilon}{\varepsilon}\right) + \int_{\frac{t^\varepsilon}{\varepsilon}}^{t} e^{(t-\tau)M(t^\varepsilon)} H_\varepsilon(\varepsilon\tau)d\tau.$$

Then, by using (6.48) and (6.56), we have that

$$\sup_{t\in[t^\varepsilon,\hat{t}^\varepsilon]} |W_\varepsilon(t)| = \sup_{t\in\left[\frac{t^\varepsilon}{\varepsilon},\frac{\hat{t}^\varepsilon}{\varepsilon}\right]} |Z_\varepsilon(t)| \le b\left(|W_\varepsilon(t^\varepsilon)| + \frac{20}{\beta} r\omega(2r)\right) \le 2b\left(1 + \frac{10}{\beta}\right) r\omega(2r),$$

where the last inequality is due to (6.52). Then, the thesis follows from (6.50).

If $k_\varepsilon \neq 0$, we define Z_ε^0 as the solution of equation (6.57) in $[0, \frac{\tau_1}{\varepsilon})$ and, for $i = 1, ..., k_\varepsilon$, we define Z_ε^i as the solution of equation (6.57) in $\left[\frac{\tau_i}{\varepsilon}, \min\left\{\frac{\tau_{i+1}}{\varepsilon}, \frac{\hat{t}^\varepsilon}{\varepsilon}\right\}\right)$ with $Z_\varepsilon^{(i-1)}\left(\frac{\tau_i}{\varepsilon}-\right)$ as initial condition at $\frac{\tau_i}{\varepsilon}$. By using the variation of constants formula, it turns out that

$$|Z_\varepsilon^0| \le R_\varepsilon^0 \quad \text{on } \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\tau_1}{\varepsilon}\right), \tag{6.58}$$

where

$$R_\varepsilon^0(t) := b\left(e^{-\frac{\beta}{4}\left(t - \frac{t^\varepsilon}{\varepsilon}\right)}|W_\varepsilon(t^\varepsilon)| + \frac{20}{\beta} r\omega(2r)\right), \quad t \in \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\tau_1}{\varepsilon}\right],$$

and

$$|Z_\varepsilon^i| \le R_\varepsilon^i \quad \text{on } \left[\frac{\tau_i}{\varepsilon}, \frac{\tau_{i+1}}{\varepsilon}\right), \quad \text{for } i = 1, ..., k_\varepsilon - 1, \tag{6.59}$$

$$|Z_\varepsilon^{k_\varepsilon}| \le R_\varepsilon^{k_\varepsilon} \quad \text{on } \left[\frac{\tau_{k_\varepsilon}}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}\right], \tag{6.60}$$

where

$$R_\varepsilon^i(t) := b\left(e^{-\frac{\beta}{4}\left(t - \frac{\tau_i}{\varepsilon}\right)} R_\varepsilon^{i-1}\left(\frac{\tau_i}{\varepsilon}\right) + \frac{4}{\beta}\omega(r)r\right), \quad t \in \left[\frac{\tau_i}{\varepsilon}, \frac{\tau_{i+1}}{\varepsilon}\right], \quad \text{for } i = 1, ..., k_\varepsilon. \tag{6.61}$$

From this definition it is easy to check that, up to a smaller $\tilde{\varepsilon}_r$ (such that $b\exp\left(-\frac{\beta\sigma}{4\varepsilon}\right)\le\frac{1}{2}$ for every $\varepsilon\in(0,\tilde{\varepsilon}_r)$), $R^i_\varepsilon\left(\frac{\tau_{i+1}}{\varepsilon}\right)\le 2r\omega(2r)\left(1+\frac{20b}{\beta}\right)$, for $i=0,...,k_\varepsilon-1$. In turn, again from (6.61), we have that

$$R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\le 2br\omega(2r)\left(11+\frac{20b}{\beta}\right),$$

so that, from the choice made in (6.50), $R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\le r$, for $i=0,...,k_\varepsilon$. Thus, since R^i_ε is decreasing in t, from (6.58)–(6.60) we obtain that

$$\begin{aligned}\sup_{t\in[t^\varepsilon,\hat{t}^\varepsilon]}|W_\varepsilon(t)| &\le \max\left\{\max_{i\in\{0,\dots,k_\varepsilon-1\}}\sup_{t\in\left[\frac{\tau_i}{\varepsilon},\frac{\tau_{i+1}}{\varepsilon}\right)}|Z^i_\varepsilon(t)|,\ \sup_{t\in\left[\frac{\tau_{k_\varepsilon}}{\varepsilon},\frac{t_\varepsilon}{\varepsilon}\right]}|Z^{k_\varepsilon}_\varepsilon(t)|\right\}\\ &\le \max_{i\in\{0,\dots,k_\varepsilon\}}R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\le r, \qquad \text{for every } \varepsilon\in(0,\tilde{\varepsilon}_r).\end{aligned}$$

□

Proposition 6.11 allows us to prove a first part of Theorem 6.8.

Proof of Theorem 6.8 restricted to $(0,t_1)$. We begin the proof of Theorem 6.8 by showing that

$$(u^\varepsilon,\varepsilon B\dot{u}^\varepsilon)\to(u,0)\ \text{ uniformly on the compact subsets of } (0,t_1). \tag{6.62}$$

Consider $[t^*,\hat{t}]\subseteq(0,t_1)$ and let $\delta>0$ be sufficiently small in order to apply Proposition 6.11 with $r=\delta$. Observe that the function

$$v_0^\varepsilon(s):=u^\varepsilon(\varepsilon s),\qquad s\in\left[0,\frac{T}{\varepsilon}\right], \tag{6.63}$$

satisfies the problem

$$\begin{cases}A\ddot{v}_0^\varepsilon(s)+B\dot{v}_0^\varepsilon(s)+\nabla_x\mathcal{E}(\varepsilon s,v_0^\varepsilon(s))=0, & s\in\left[0,\frac{T}{\varepsilon}\right]\\ v_0^\varepsilon(0)=u^\varepsilon(0),\\ \dot{v}_0^\varepsilon(0)=\varepsilon\dot{u}^\varepsilon(0),\end{cases}$$

so that, by (6.30),

$$(v_0^\varepsilon,\dot{v}_0^\varepsilon)\to(v_0,\dot{v}_0)\ \text{ uniformly on the compact subsets of } [0,+\infty), \tag{6.64}$$

where v_0 satisfies (6.28) and (6.29). This convergence, the limit in (6.29) and the fact that $\dot{v}_0(+\infty)=0$ imply that there exists $s_0^\delta>0$ such that

$$\left|\left(v_0(s)-x_0^r,B\dot{v}_0(s)\right)\right|\le\frac{1}{2}\min\{\delta,\delta\omega(2\delta)\},\qquad \text{for every } s\ge s_0^\delta, \tag{6.65}$$

and

$$\limsup_{\varepsilon\to0^+}\left|\left(u^\varepsilon(\varepsilon s_0^\delta)-x_0^r,\varepsilon B\dot{u}^\varepsilon(\varepsilon s_0^\delta)\right)\right|<\min\{\delta,\delta\omega(2\delta)\}, \tag{6.66}$$

where ω is defined in Proposition 6.11. Using the same proposition with $\overline{t}=\tilde{t}=0$ and u_1 in place of u, and

$$b_0^\varepsilon:=\varepsilon s_0^\delta \tag{6.67}$$

in place of t^ε, from (6.66) we obtain that

$$\limsup_{\varepsilon\to0^+}\sup_{t\in[t^*,\hat{t}]}\left|\left(u^\varepsilon(t)-u(t),\varepsilon B\dot{u}^\varepsilon(t)\right)\right|\le\limsup_{\varepsilon\to0^+}\sup_{t\in[b_0^\varepsilon,\hat{t}]}\left|\left(u^\varepsilon(t)-u(t),\varepsilon B\dot{u}^\varepsilon(t)\right)\right|\le\delta, \tag{6.68}$$

and, in turn, (6.62). □

Note that convergence (6.62), together with the fact that $\lim_{t\to t_1^-} u(t) = x_1^s$ and the definition of $t_1^\delta < t_1$ (see (6.25)), implies that

$$\big|\big(u^\varepsilon(t_1^\delta) - x_1^s, \varepsilon B\dot u^\varepsilon(t_1^\delta)\big)\big| \le \frac{\delta}{2},$$

for every ε sufficiently small. To prove Theorem 6.8 (2) for $j = 1$, we need to introduce the first time larger that t_1^δ in which $(u^\varepsilon(t), \varepsilon B\dot u^\varepsilon(t))$ escapes from $\overline{B}\big((x_1^s,0),\delta\big)$, that is

$$a_1^\varepsilon := \max\left\{\overline{t} \in [t_1^\delta, t_1^{**}] : \big(u^\varepsilon(t), \varepsilon B\dot u^\varepsilon(t)\big) \in \overline{B}\big((x_1^s,0),\delta\big) \text{ for every } t \in [t_1^\delta, \overline{t}\,]\right\}, \tag{6.69}$$

where $t_1^{**} > t_1$ is defined in (6.26). Observe that, for every ε small enough, a_1^ε is well defined, because the maximum is taken over a nonempty set, and it depends on δ. Also, note that if $a_1^\varepsilon < t_1^{**}$ then $(u^\varepsilon(a_1^\varepsilon), \varepsilon B\dot u^\varepsilon(a_1^\varepsilon)) \in \partial B\big((x_1^s,0),\delta\big)$.

Lemma 6.15.

$$a_1^\varepsilon \to t_1, \qquad \textit{as } \varepsilon \to 0^+.$$

Proof. We divide the proof in two steps.
(i) Let $\tau_k \ge t_1^\delta$ be a sequence approaching t_1 from the left, as $k \to +\infty$. From (6.62) we have that, for every k, there exists ε_k such that $\sup_{[t_1^\delta,\tau_k]}\big|(u^\varepsilon - u, \varepsilon B\dot u^\varepsilon)\big| \le \frac{\delta}{2}$ for all $\varepsilon \in (0,\varepsilon_k)$. Thus, also in view of the definition of t_1^δ,

$$(u^\varepsilon(t), \varepsilon B\dot u^\varepsilon(t)) \in \overline{B}\big((x_1^s,0),\delta\big), \qquad \text{for every } t \in [t_1^\delta, \tau_k],\ \varepsilon \in (0,\varepsilon_k).$$

In turn, from the definition of a_1^ε, we obtain that $a_1^\varepsilon \ge \tau_k$ for every $\varepsilon \in (0,\varepsilon_k)$ and every k, so that

$$\liminf_{\varepsilon\to 0} a_1^\varepsilon \ge t_1.$$

(ii) Here, we want to prove that

$$\limsup_{\varepsilon\to 0} a_1^\varepsilon \le t_1.$$

Suppose, by contradiction, that there exists a sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0$, such that $\{a_1^{\varepsilon_k}\} \subseteq [\hat t, t_1^{**}]$, for some $t_1 < \hat t < t_1^{**}$. Then, up to a subsequence,

$$a_1^{\varepsilon_k} \to \tilde t, \qquad \text{for some } \tilde t \in [\hat t, t_1^{**}]. \tag{6.70}$$

Note that the function $v_1^\varepsilon := u^\varepsilon(a_1^\varepsilon + \varepsilon s)$ satisfies the problem

$$\begin{cases} A\ddot v_1^\varepsilon(s) + B\dot v_1^\varepsilon(s) + \nabla_x \mathcal{E}(a_1^\varepsilon + \varepsilon s, v_1^\varepsilon(s)) = 0, & s \in \left[-\frac{a_1^\varepsilon}{\varepsilon}, \frac{T-a_1^\varepsilon}{\varepsilon}\right] \\ v_1^\varepsilon(0) = u^\varepsilon(a_1^\varepsilon), \\ \dot v_1^\varepsilon(0) = \varepsilon\dot u^\varepsilon(a_1^\varepsilon). \end{cases}$$

From the definition of a_1^ε, we have that $(v_1^\varepsilon(0), B\dot v_1^\varepsilon(0)) \in \overline{B}\big((x_1^s,0),\delta\big)$, and, in turn, up to a further subsequence, that

$$(v_1^{\varepsilon_k}(0), B\dot v_1^{\varepsilon_k}(0)) \to (z,\dot z), \qquad \text{for some } (z,\dot z) \in \overline{B}\big((x_1^s,0),\delta\big). \tag{6.71}$$

The limits (6.70) and (6.71) imply that, if w is the solution to the problem

$$\begin{cases} A\ddot w(s) + B\dot w(s) + \nabla_x \mathcal{E}(\tilde t, w(s)) = 0, \\ w(0) = z, \\ B\dot w(0) = \dot z, \end{cases} \tag{6.72}$$

then $(v_1^{\varepsilon_k}, \dot v_1^{\varepsilon_k}) \to (w,\dot w)$ uniformly on the the compact subsets of a common interval of existence. From this convergence, using Lemma 6.5 and the definition of a_1^ε, it is easy to check that w and $\dot w$ are defined on all $\mathbb{R}$ and that $(w(s), B\dot w(s)) \in \overline{B}\big((x_1^s,0),\delta\big)$ for every $s \in (-\infty,0]$. Moreover, by Lemma 6.5, there exist the limits

$$\lim_{s\to-\infty} w(s) =: w(-\infty), \qquad \lim_{s\to-\infty} \dot w(s) =: \dot w(-\infty),$$

and satisfy $F\left(\tilde{t}, \begin{bmatrix} w(-\infty) \\ \dot{w}(-\infty) \end{bmatrix}\right) = F\left(\tilde{t}, \begin{bmatrix} w(-\infty) \\ 0 \end{bmatrix}\right) = 0$. At the same time, $(w(+\infty), 0) \in \overline{B}\big((x_1^s, 0), \delta\big)$. These facts give a contradiction, because $\tilde{t} \in (t_1, t_1^{**}]$ and (6.26) holds. □

By Lemma 6.4, any solution of problem (6.31) differs from any other solution by time–translation, so that the trajectories $\mathscr{I}_j$'s (defined in (6.32)) are uniquely defined. By using Morse–Sard Theorem (see, e.g., [**44**, Theorem 1.3 ch. 3]) applied to the function

$$t \mapsto |(w_j(t) - x_j^s, B\dot{w}_j(t))|^2,$$

it is easy to check that the set

$$E_j := \{\delta > 0 \,:\, \mathscr{I}_j \text{ is tangent to } \partial B((x_j^s, 0), \delta) \text{ at a point of intersection}\} \tag{6.73}$$

has zero measure. The reason why we introduce the set E_j, for $j = 1, ..., m-1$, will be clear in the next proof.

Proof of Theorem 6.8, complete. Let δ be sufficiently small. First, let us prove statement (2) in the case $j = 1$. Consider an arbitrary sequence $\varepsilon_k \to 0$ and the function

$$v_1^\varepsilon(s) := u^\varepsilon(a_1^\varepsilon + \varepsilon s), \qquad s \in \left[-\frac{a_1^\varepsilon}{\varepsilon}, \frac{T - a_1^\varepsilon}{\varepsilon}\right], \tag{6.74}$$

with a_1^ε given by (6.69). Observe that v_1^ε depends on δ. By using Lemma 6.15 and arguing similarly to its proof, we can show that, up to a subsequence,

$$(v_1^{\varepsilon_k}(0), B\dot{v}_1^{\varepsilon_k}(0)) \to (z, \dot{z}), \qquad \text{for some } (z, \dot{z}) \in \partial B((x_1^s, 0), \delta),$$

and that $(v_1^{\varepsilon_k}, \dot{v}_1^{\varepsilon_k}) \to (w_1, \dot{w}_1)$ uniformly on the compact subsets of $\mathbb{R}$, where w_1 is the solution of problem (6.72), with t_1 in place of $\tilde{t}$, and satisfies

$$w_1(-\infty) = x_1^s, \qquad \dot{w}_1(-\infty) = 0. \tag{6.75}$$

The first condition in (6.75) is due to the fact that $w_1(-\infty) \in \overline{B}(x_1^s, \delta)$ must be a critical point of $\mathcal{E}(t_1, \cdot)$ and, since we are supposing δ small enough, the unique critical point of $\mathcal{E}(t_1, \cdot)$ in $\overline{B}(x_1^s, \delta)$ is x_1^s (see (6.23)). Observe that w_1 depends on δ. To conclude the proof, it remains to show that, given any other sequence $\varepsilon_h \to 0$, $(v_1^{\varepsilon_h}, \dot{v}_1^{\varepsilon_h})$ converges (up to a subsequence) to $(w_1, \dot{w}_1)$, as $(v_1^{\varepsilon_k}, \dot{v}_1^{\varepsilon_k})$ does. By repeating the same arguments above, we have that, up to a subsequence, $(v_1^{\varepsilon_h}, \dot{v}_1^{\varepsilon_h}) \to (\tilde{w}_1, \dot{\tilde{w}}_1)$ uniformly on the compact subsets of $\mathbb{R}$, where $(\tilde{w}_1, \dot{\tilde{w}}_1)$ satisfies the same system that $(w_1, \dot{w}_1)$ satisfies, and the conditions in (6.75). Therefore, by Lemma 6.4, we have that

$$\tilde{w}_1(s) = w_1(s + s_0), \qquad s \in \mathbb{R}, \tag{6.76}$$

for a certain constant s_0, which we can assume to be nonnegative. We want to prove that, indeed, $s_0 = 0$. Let us suppose, by contradiction, that $s_0 > 0$. By (6.76) and the definition of a_1^ε, we have, on one hand, that

$$(w_1(s), B\dot{w}_1(s)) \in \overline{B}((x_1^s, 0), \delta), \qquad \text{for every } s \leq s_0; \tag{6.77}$$

on the other hand, since E_1 has measure 0 (see (6.73)), it is not restrictive to assume $\delta \notin E_1$, so that there exists $\sigma > 0$ such that $(w_1(s), B\dot{w}_1(s)) \notin \overline{B}((x_1^s, 0), \delta)$ for every $s \in (0, \sigma)$, against (6.77). Therefore, it has to be $s_0 = 0$ and, in turn, $w_1 = \tilde{w}_1$. Thus, we have proved that

$$(v_1^\varepsilon, \dot{v}_1^\varepsilon) \to (w_1, \dot{w}_1) \text{ uniformly on the compact subsets of } \mathbb{R}, \tag{6.78}$$

where, among the solutions of the problem

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla_x \mathcal{E}(t_1, w(s)) = 0, \\ \lim_{s \to -\infty} w(s) = x_1^s, \end{cases}$$

w_1 is the one such that $(w_1(0), B\dot{w}_1(0)) = (z, \dot{z})$ (being $(u^\varepsilon(a_1^\varepsilon), \varepsilon B\dot{u}^\varepsilon(a_1^\varepsilon)) \to (z, \dot{z}) \in \partial B((x_1^s, 0), \delta)$). Moreover,

$$(w_1(s), B\dot{w}_1(s)) \in \overline{B}((x_1^s, 0), \delta), \qquad \text{for every } s \leq 0. \tag{6.79}$$

Now, recall that, by Proposition 6.6, the limit of w_1 at $+\infty$ selects a point which allows us to find, as done for $[0,t_1)$, a solution u_2 of problem (6.2) on $[t_1,t_2)$ starting from $w_1(+\infty)$. More precisely:

$$\lim_{s\to+\infty}(w_1(s),\dot w_1(s)) = (x_1^r,0), \qquad u_2(t_1)=x_1^r,$$

and

$$\nabla_x\mathcal{E}(t,u_2(t))=0, \qquad \nabla_x^2\mathcal{E}(t,u_2(t))>0, \quad \text{for every } t\in[t_1,t_2).$$

In particular, there exists $s_1^\delta>0$ such that

$$|(w_1(s)-x_1^r,B\dot w_1(s))|\le\frac{\delta}{2} \qquad \text{for every } s\ge s_1^\delta. \tag{6.80}$$

Moreover, due to (6.78) and to the definition of v_1^ε, there exists $\varepsilon_\delta>0$ such that

$$|(u^\varepsilon(b_1^\varepsilon)-w_1(s_1^\delta),\varepsilon B\dot u^\varepsilon(b_1^\varepsilon)-B\dot w_1(s_1^\delta))|<\frac{\delta}{2}, \qquad \text{for every } \varepsilon\in(0,\varepsilon_\delta), \tag{6.81}$$

where

$$b_1^\varepsilon=b_1^\varepsilon(\delta):=a_1^\varepsilon+\varepsilon s_1^\delta,$$

so that

$$|(u^\varepsilon(b_1^\varepsilon)-x_1^r,\varepsilon B\dot u^\varepsilon(b_1^\varepsilon)|<\delta, \qquad \text{for every } \varepsilon\in(0,\varepsilon_\delta). \tag{6.82}$$

By using (6.82) and Proposition 6.11 with $\tilde t=t_1$, b_1^ε in place of t^ε, u_2 in place of u (since it can be $b_1^\varepsilon<t_1$, note that u_2 is defined in a left neighbourhood of t_1, also) and δ in place of $\min\{r,r\omega(2r)\}$, we can prove statement (1) of the theorem restricted to (t_1,t_2). In turn, we can define a_2^ε and b_2^ε, corresponding to the jump point t_2, and prove statement (2) of the theorem for $j=2$. Repeating the same argument for all the other intervals (t_{j-1},t_j) and taking into account the quantities a_j^ε and b_j^ε, according to the following definition, completes the proof of the theorem. ☐

For $j=1,...,m-1$ and $\delta>0$ sufficiently small, let $t_j^{**}\in(t_j,t_{j+1})$ and $t_j^\delta\in(t_{j-1},t_j)$ be defined as in (6.25) and (6.26), respectively. From Theorem 6.8 we have that

$$|(u^\varepsilon(t_j^\delta)-x_j^s,\varepsilon B\dot u^\varepsilon(t_j^\delta))|\le\frac{\delta}{2}, \qquad \text{for every } j=1,...,m-1,$$

for every ε sufficiently small (depending on δ). For every such a small ε, we give the following definition.

Definition 6.16. *For $\delta>0$ sufficiently small and $j=1,...,m-1$, we define*

$$a_j^\varepsilon:=\max\big\{\bar t\in[t_j^\delta,t_j^{**}]:\big(u^\varepsilon(t),\varepsilon B\dot u^\varepsilon(t)\big)\in\overline{B}\big((x_j^s,0),\delta\big)\ \textit{for every}\ t\in[t_j^\delta,\bar t\,]\big\}$$

and

$$b_j^\varepsilon:=a_j^\varepsilon+\varepsilon s_j^\delta,$$

where $s_j^\delta>0$ is such that

$$\big|\big(w_j(s)-x_j^r,B\dot w_j(s)\big)\big|\le\frac{\delta}{2}, \qquad \textit{for every } s\ge s_j^\delta.$$

Remark 6.17. In the case where

$$\big(u^\varepsilon(0),\varepsilon\dot u^\varepsilon(0)\big)\to(x_0^r,0),$$

then

$$(u^\varepsilon,\varepsilon B\dot u^\varepsilon)\to(u,0) \text{ on the compact subsets of } [0,T]\setminus\{t_1,...,t_{m-1}\}.$$

To check this on a compact $[0,\hat t]$ of $[0,t_1)$, it is enough to apply Proposition 6.11 with $\bar t=\tilde t=0$ and $t^\varepsilon=0$.

We can now prove the last result of this section. We recall that $\mathscr{S}$ and $\mathscr{S}^\varepsilon$ are defined in (6.32)–(6.35).

Proof of Theorem 6.9. Chosen $\delta > 0$ small enough and such that

$$\delta \notin \bigcup_{j=1}^{m-1} E_j,$$

where E_j, for $j = 1, ..., m-1$, is defined in (6.73) (recall that $\bigcup_{j=1}^{m-1} E_j$ has zero measure), we suppose to work with the particular heteroclinic solutions depending on δ found in the proof of Theorem 6.8 (see (6.79)). Due to the definition of the Hausdorff distance, we divide the proof in two parts.
(a) Here, we show that there exists $\varepsilon_\delta > 0$ such that

$$\sup_{\mathscr{S}^\varepsilon} d(\cdot, \mathscr{S}) \leq 2\delta, \quad \text{for every } \varepsilon \in (0, \varepsilon_\delta). \tag{6.83}$$

Set

$$d_\varepsilon(t) := d\big((t, u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)), \mathscr{S}\big), \qquad t \in [0, T].$$

By referring to (6.24)–(6.25), to (6.65)–(6.67) and to Definition 6.16 for the notation, and in view of the fact that

$$b_0^\varepsilon \to 0, \qquad a_j^\varepsilon, b_j^\varepsilon \to t_j, \qquad \text{for } j = 1, ..., m-1, \tag{6.84}$$

we consider, for every ε small enough, the partition

$$0 < b_0^\varepsilon < t_1^\delta < a_1^\varepsilon < b_1^\varepsilon < ... < b_{m-1}^\varepsilon < T.$$

In order to prove (6.83), it is enough to give a proper estimate of d_ε on $[0, b_1^\varepsilon)$, since we can proceed in a similar way on the remaining part of the interval $[0, T]$. By looking at the definition of v_0^ε (see (6.63)), observe that

$$\begin{aligned} \sup_{t\in[0,b_0^\varepsilon)} d_\varepsilon(t) &\leq \sup_{s\in[0,s_0^\delta]} \big[\varepsilon s + d\big((v_0^\varepsilon(s), B\dot{v}_0^\varepsilon(s)), \mathscr{I}_0\big)\big] \\ &\leq b_0^\varepsilon + \sup_{s\in[0,s_0^\delta]} \big|\big(v_0^\varepsilon(s) - v_0(s), B\dot{v}_0^\varepsilon(s) - B\dot{v}_0(s)\big)\big|, \end{aligned} \tag{6.85}$$

while, by using (6.68) with t_1^δ in place of $\hat{t}$, it turns out that

$$\sup_{t\in[b_0^\varepsilon,t_1^\delta)} d_\varepsilon(t) \leq \sup_{t\in[b_0^\varepsilon,t_1^\delta]} \big|\big(u^\varepsilon(t) - u(t), \varepsilon B\dot{u}^\varepsilon(t)\big)\big| \leq \delta. \tag{6.86}$$

Now, observe that we can suppose

$$t_1 - t_1^\delta \leq \frac{\delta}{2}. \tag{6.87}$$

This fact, together with the definition of a_1^ε, implies that

$$\begin{aligned} \sup_{t\in[t_1^\delta,a_1^\varepsilon)} d_\varepsilon(t) &\leq \sup_{t\in[t_1^\delta,a_1^\varepsilon)} \Big[|t - t_1| + \big|\big(u^\varepsilon(t) - x_1^s, \varepsilon B\dot{u}^\varepsilon(t)\big)\big|\Big] \\ &\leq \max\left\{|t_1 - a_1^\varepsilon|, \frac{\delta}{2}\right\} + \delta \end{aligned} \tag{6.88}$$

Finally, consider that

$$\begin{aligned} \sup_{t\in[a_1^\varepsilon,b_1^\varepsilon)} d_\varepsilon(t) &\leq \sup_{t\in[a_1^\varepsilon,b_1^\varepsilon)} \Big[|t - t_1| + d\big((u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)), \mathscr{I}_1\big)\Big] \\ &\leq \varepsilon s_1^\delta + |t_1 - a_1^\varepsilon| + \sup_{s\in[0,s_1^\delta]} \big|\big(v_1^\varepsilon(s) - w_1(s), B\dot{v}_1^\varepsilon(s) - B\dot{w}_1(s)\big)\big|. \end{aligned} \tag{6.89}$$

Inequalities (6.85)–(6.86) and (6.88)–(6.89), together with (6.64), Theorem 6.8 (2), the convergences in (6.84) and the convergence of εs_1^δ to 0, imply that there exists $\varepsilon_\delta > 0$ such that

$$\sup_{t\in[0,b_1^\varepsilon)} d_\varepsilon(t) \leq 2\delta, \qquad \text{for every } \varepsilon \in (0, \varepsilon_\delta),$$

and, in turn, imply (6.83).

(b) Here, we show that there exists $\tilde{\varepsilon}_\delta > 0$ such that

$$\sup_{\mathscr{S}} d(\cdot, \mathscr{S}^\varepsilon) \le 2\delta, \qquad \text{for every } \varepsilon \in (0, \tilde{\varepsilon}_\delta). \tag{6.90}$$

By the definition of $\mathscr{S}$ and by the fact that $(x_1^s, 0) \in \overline{\mathscr{I}}_1$, it is sufficient to analyze

$$\sup_{\{0\}\times\mathscr{I}_0} d(\cdot, \mathscr{S}^\varepsilon), \qquad \sup_{t\in[0,t_1)} d\big((t, u_1(t), 0), \mathscr{S}^\varepsilon\big), \qquad \sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot, \mathscr{S}^\varepsilon).$$

The other cases can be treated in a similar way. Let us consider separately $s \in [0, s_0^\delta]$ and $s > s_0^\delta$, and write

$$\begin{aligned} \sup_{s\in[0,s_0^\delta]} d((0, v_0(s), B\dot{v}_0(s)), \mathscr{S}^\varepsilon) &\le \sup_{s\in[0,s_0^\delta]} \Big[\varepsilon s + d\big((v_0(s), B\dot{v}_0(s)), (u^\varepsilon(\varepsilon s), \varepsilon B\dot{u}^\varepsilon(\varepsilon s))\big)\Big] \\ &\le b_0^\varepsilon + \sup_{s\in[0,s_0^\delta]} \big|\big(v_0(s) - v_0^\varepsilon(s), B\dot{v}_0(s) - B\dot{v}_0^\varepsilon(s)\big)\big|, \end{aligned} \tag{6.91}$$

and, in view of (6.65),

$$\begin{aligned} \sup_{s>s_0^\delta} d\big((0, v_0(s), \dot{v}_0(s)), \mathscr{S}^\varepsilon\big) &\le b_0^\varepsilon + \sup_{s>s_0^\delta} d\big((v_0(s), B\dot{v}_0(s)), (u^\varepsilon(b_0^\varepsilon), \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon))\big) \\ &\le b_0^\varepsilon + \frac{\delta}{2} + \big|\big(u^\varepsilon(b_0^\varepsilon) - x_0^r, \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon)\big)\big|. \end{aligned} \tag{6.92}$$

Now, to carry out a proper estimate of $\sup_{t\in[0,t_1)} d\big((t, u_1(t), 0), \mathscr{S}^\varepsilon\big)$, we divide $[0, t_1)$ in $[0, b_0^\varepsilon)$, $[b_0^\varepsilon, t_1^\delta)$ and $[t_1^\delta, t_1)$. It turns out that

$$\begin{aligned} \sup_{t\in[0,b_0^\varepsilon)} d\big((t, u_1(t), 0), \mathscr{S}^\varepsilon\big) &\le b_0^\varepsilon + \sup_{t\in[0,b_0^\varepsilon)} d\big((u_1(t), 0), (u^\varepsilon(b_0^\varepsilon), \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon))\big) \\ &\le b_0^\varepsilon + \omega_{u_1}(b_0^\varepsilon) + \big|\big(u^\varepsilon(b_0^\varepsilon) - x_0^r, \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon)\big)\big|, \end{aligned} \tag{6.93}$$

where ω_{u_1} is the modulus of continuity of u_1 on $[0, t_1/2]$. Moreover, we have that

$$\sup_{t\in[b_0^\varepsilon, t_1^\delta)} d\big((t, u_1(t), 0), \mathscr{S}^\varepsilon\big) \le \sup_{t\in[b_0^\varepsilon, t_1^\delta]} \big|\big(u^\varepsilon(t) - u_1(t), \varepsilon B\dot{u}^\varepsilon(t)\big)\big|, \tag{6.94}$$

and, in view of (6.87) and (6.25), that

$$\begin{aligned} &\sup_{t\in[t_\delta^1, t_1)} d((t, u_1(t), 0), \mathscr{S}^\varepsilon) \\ &\qquad \le \sup_{t\in[t_\delta^1, t_1)} d\big((t, u_1(t), 0), (t_1^\delta, u^\varepsilon(t_1^\delta), \varepsilon B\dot{u}^\varepsilon(t_1^\delta))\big) \\ &\qquad \le \frac{\delta}{2} + \big|\big(u^\varepsilon(t_1^\delta) - u_1(t_1^\delta), \varepsilon B\dot{u}^\varepsilon(t_1^\delta)\big)\big| + \sup_{t\in[t_1^\delta, t_1)} |u_1(t) - u_1(t_\delta^1)| \\ &\qquad \le \delta + \big|\big(u^\varepsilon(t_1^\delta) - u_1(t_1^\delta), \varepsilon B\dot{u}^\varepsilon(t_1^\delta)\big)\big|. \end{aligned} \tag{6.95}$$

Finally, consider $\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot, \mathscr{S}^\varepsilon)$. Observe that

$$\begin{aligned} d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^\varepsilon\big) \le |t_1 - b_1^\varepsilon| &+ \big|\big(w_1(s) - w_1(s_1^\delta), B\dot{w}_1(s) - B\dot{w}_1(s_1^\delta)\big)\big| \\ &+ \big|\big(u^\varepsilon(b_1^\varepsilon) - w_1(s_1^\delta), \varepsilon B\dot{u}^\varepsilon(b_1^\varepsilon) - B\dot{w}_1(s_1^\delta)\big)\big|, \end{aligned}$$

so that, from (6.80)–(6.81), we obtain

$$\sup_{s>s_1^\delta} d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^\varepsilon\big) \le |t_1 - b_1^\varepsilon| + \frac{3}{2}\delta. \tag{6.96}$$

Now, similarly to what is done in (6.80)–(6.82), we can define c_1^ε in the following way. Since $(w_1(-\infty), \dot{w}_1(-\infty)) = (x_1^s, 0)$, there exists $\overline{s}_1^\delta < 0$ such that

$$\big|\big(w_1(s) - x_1^s, B\dot{w}_1(s)\big)\big| \le \frac{\delta}{2}, \qquad \text{for every } s \le \overline{s}_1^\delta. \tag{6.97}$$

Moreover, due to (6.74) and (6.78), there exists $\tilde{\varepsilon}_\delta > 0$ such that

$$\big|\big(u^\varepsilon(a_1^\varepsilon + \varepsilon\overline{s}_1^\delta) - w_1(s_1^\delta), \varepsilon B\dot{u}^\varepsilon(a_1^\varepsilon + \varepsilon\overline{s}_1^\delta) - B\dot{w}_1(s_1^\delta)\big)\big| \le \frac{\delta}{2}, \qquad \text{for every } \varepsilon \in (0,\tilde{\varepsilon}_\delta). \tag{6.98}$$

Let us define

$$c_1^\varepsilon = c_1^\varepsilon(\delta) := a_1^\varepsilon + \varepsilon\overline{s}_1^\delta < a_1^\varepsilon,$$

and observe that $c_1^\varepsilon \to t_1$. We have that

$$\sup_{s\in[\overline{s}_1^\delta, s_1^\delta]} d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^\varepsilon\big) \le |t_1 - a_1^\varepsilon| + \varepsilon \max\{|\overline{s}_1^\delta|, s_1^\delta\} + \sup_{s\in[\overline{s}_1^\delta, s_1^\delta]} \big|\big(v_1^\varepsilon(s) - w_1(s), B\dot{v}_1^\varepsilon(s) - B\dot{w}_1(s)\big)\big|, \tag{6.99}$$

while, since

$$\sup_{s<\overline{s}_1^\delta} d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^\varepsilon\big) \le |t_1 - c_1^\varepsilon| + \big|\big(w_1(s) - w_1(s_1^\delta), B\dot{w}_1(s) - B\dot{w}_1(s_1^\delta)\big)\big| + \big|\big(w_1(s_1^\delta) - u^\varepsilon(c_1^\varepsilon), B\dot{w}_1(s_1^\delta) - \varepsilon B\dot{u}^\varepsilon(c_1^\varepsilon)\big)\big|,$$

from (6.97) and (6.98) it turns out that

$$\sup_{s<\overline{s}_1^\delta} d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^\varepsilon\big) \le |t_1 - c_1^\varepsilon| + \frac{3}{2}\delta. \tag{6.100}$$

Inequalities (6.91)–(6.96) and (6.99)–(6.100), together with (6.64), (6.66), (6.62), and (6.78), give that, up to a smaller $\tilde{\varepsilon}_\delta$,

$$\sup_{\{0\}\times\mathscr{I}_0} d(\cdot, \mathscr{S}^\varepsilon), \ \sup_{t\in[0,t_1)} d\big((t, u_1(t), 0), \mathscr{S}^\varepsilon\big), \ \sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot, \mathscr{S}^\varepsilon) \le 2\delta,$$

for every $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, and, in turn, give (6.90). □

6.3. An alternative approach: time discretization

In this section, we study a second order, discrete–time approximation of the same limit problem constructed in Section 6.1 and approximated in Section 6.2 by second order singular perturbations. The present approximation process is modelled on the following idea. We consider a partition $0 = \tau_0^k < \tau_1^k < ... < \tau_{k-1}^k < \tau_k^k = T$ of the interval $[0,T]$ such that

$$\rho_k := \max_{0\le i\le k-1} (\tau_{i+1}^k - \tau_i^k) \to 0, \qquad \text{as } k \to +\infty, \tag{6.101}$$

and suppose to have defined u_{i-1}^k as the approximation of the function u given by Definition 6.7 on the interval $[\tau_{i-1}^k, \tau_i^k)$. Since $u(\tau_i^k)$ is a critical point of $\mathcal{E}(\tau_i^k, \cdot)$, we find the next approximating point u_i^k by considering the solution v_i^k of the autonomous problem

$$\begin{cases} A\ddot{v}_i^k(\sigma) + B\dot{v}_i^k(\sigma) + \nabla_x\mathcal{E}(\tau_i^k, v_i^k(\sigma)) = 0, & \sigma \in [0,+\infty) \\ v_i^k(0) = u_{i-1}^k, \\ \dot{v}_i^k(0) = 0, \end{cases} \tag{6.102}$$

and setting

$$u_i^k := \lim_{\sigma\to+\infty} v_i^k(\sigma), \qquad i = 2, ..., k. \tag{6.103}$$

Consider a point $x_0^r \in \mathbb{R}^n$ such that $\nabla_x\mathcal{E}(0, x_0^r) = 0$ and $\nabla_x^2\mathcal{E}(0, x_0^r)$ is positive definite. Clearly, the first approximating point of this process could be defined as the limit at $+\infty$ of the solution of (6.102) with τ_1^k and $x_0^r = u(0)$ in place of τ_i^k and u_{i-1}^k, respectively. Actually, it does not cost much more effort to define

$$u_1^k := \lim_{\sigma\to+\infty} v_1^k(\sigma), \tag{6.104}$$

and v_1^k as the solution of

$$\begin{cases} A\ddot{v}_1^k(\sigma) + B\dot{v}_1^k(\sigma) + \nabla_x \mathcal{E}(\tau_1^k, v_1^k(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ v_1^k(0) = x_k, \\ \dot{v}_1^k(0) = y_k. \end{cases} \tag{6.105}$$

Here,

$$(x_k, y_k) \to (x_0, y_0), \qquad \text{as } k \to +\infty,$$

and (x_0, y_0) lies in the basin of attraction of $(x_0^r, 0)$ for the autonomous problem at time 0, that is (x_0, y_0) satisfies (6.28) and (6.29). In order to uniform the notation, we set $u_0^k = x_k$. Note that Lemma 6.5 ensures the existence of the solutions of problems (6.102), (6.105), and (6.28), and of the limits (6.103), (6.104), and (6.29). Also, Lemma 6.5 tells us that u_i^k is a critical point of $\mathcal{E}(\tau_i^k, \cdot)$ and that $\dot{v}_0(+\infty) = \dot{v}_k^i(+\infty) = 0$, for $i = 1, ..., k$.

Let $\mathscr{S}$ be the same set defined in (6.33)–(6.35). In order to define a suitable set $\mathscr{S}^k$ approximating $\mathscr{S}$, we choose arbitrarily some

$$\alpha_i^k \in (\tau_{i-1}^k, \tau_i^k), \qquad \text{for } i = 1, ..., k,$$

and introduce a function u^k which has, on every $[\tau_{i-1}^k, \tau_i^k]$, the following features. On $[\tau_{i-1}^k, \alpha_i^k]$, it is a suitable reparametrization of v_i^k from a certain big interval $[0, a_i^k]$ to $[\tau_{i-1}^k, \alpha_i^k]$, and, on $[\alpha_i^k, \tau_i^k]$, it is a convex combination of $v_i^k(a_i^k)$ taken in α_i^k and u_i^k taken in τ_i^k. More precisely, we fix a sequence $\delta_k \to 0$ and a constant $C > 0$, and, for $i = 1, ..., k$, we consider a value $a_i^k > 0$ with the following properties:

$$\min_{i \in \{1,...,k\}} a_i^k \to +\infty, \qquad \text{as } k \to +\infty, \tag{6.106}$$

and, for every k,

$$\max_{i \in \{1,...,k\}} \frac{|v_i^k(a_i^k) - u_i^k|}{\tau_i^k - \alpha_i^k} \le C, \qquad \max_{i \in \{1,...,k\}} |\dot{v}_i^k(a_i^k))| \le \delta_k. \tag{6.107}$$

It is clear that such values exist, in view of Lemma 6.5. We can now define the function $u^k \in C([0,T]; \mathbb{R}^n)$ by

$$u^k(t) := \begin{cases} v_i^k\left(\dfrac{t - \tau_{i-1}^k}{\alpha_i^k - \tau_{i-1}^k} a_i^k\right), & t \in [\tau_{i-1}^k, \alpha_i^k], \\ \dfrac{(\tau_i^k - t)v_i^k(a_i^k) + (t - \alpha_i^k)u_i^k}{\tau_i^k - \alpha_i^k}, & t \in [\alpha_i^k, \tau_i^k]. \end{cases} \tag{6.108}$$

Observe that

$$u^k(0) = v_1^k(0) = x_k, \qquad u^k(\tau_{i-1}^k) = v_i^k(0) = u_{i-1}^k, \qquad \text{for } i = 2, ..., k,$$
$$u^k(\alpha_i^k) = v_i^k(a_i^k), \qquad \text{for } i = 1, ..., k,$$

and that

$$\{u^k(t) : \, t \in [\tau_{i-1}^k, \alpha_i^k]\} = \{v_i^k(\sigma) : \, \sigma \in [0, a_i^k]\},$$

while, on $[\alpha_i^k, \tau_i^k]$, $u^k(t)$ is an affine function connecting $v_i^k(a_i^k)$ to u_i^k. Moreover, u^k can be not differentiable at $\alpha_1^k, \tau_1^k, \alpha_2^k, \tau_2^k, ..., \alpha_k^k$. Thus, with abuse of notation, we set

$$\dot{u}^k(\tau_i^k) := \lim_{\tau \to (\tau_i^k)^+} \dot{u}^k(\tau), \qquad \text{for } i = 0, ..., k-1,$$

$$\dot{u}^k(\alpha_i^k) := \lim_{\tau \to (\alpha_i^k)^+} \dot{u}^k(\tau), \qquad \text{for } \; i = 1, ..., k,$$

and $\dot{u}^k(T) := \lim_{\tau \to T^-} \dot{u}^k(\tau)$, so that

$$\dot{u}^k(t) := \begin{cases} \dfrac{a_i^k}{\alpha_i^k - \tau_{i-1}^k} \dot{v}_i^k\left(\dfrac{t - \tau_{i-1}^k}{\alpha_i^k - \tau_{i-1}^k} a_i^k\right), & t \in [\tau_{i-1}^k, \alpha_i^k), \\ \dfrac{u_i^k - v_i^k(a_i^k)}{\tau_i^k - \alpha_i^k}, & t \in [\alpha_i^k, \tau_i^k), \end{cases} \tag{6.109}$$

and $\dot{u}^k(T) := \frac{u_k^k - v_k^k(a_k^k)}{T-\alpha_k^k}$. Note that, for $i = 2, ..., k$, $\dot{u}^k(\tau_{i-1}^k) = \frac{a_i^k}{\alpha_i^k - \tau_{i-1}^k}\dot{v}_i^k(0) = 0$, while $\dot{u}^k(0) = \frac{a_1^k}{\alpha_1^k} y_k$. Finally, we need some coefficients which have, in the present analysis, the same role played by ε in Section 6.2. To this aim, we define

$$h_k(t) := \sum_{i=1}^{k} \frac{\alpha_i^k - \tau_{i-1}^k}{a_i^k} \chi_{[\tau_{i-1}^k, \tau_i^k)}(t), \qquad t \in [0, T), \tag{6.110}$$

with $h_k(T) = \frac{\alpha_k^k - \tau_{k-1}^k}{a_k^k}$, and, in turn,

$$\mathscr{S}^k := \overline{\{(t, u^k(t), h_k(t) B \dot{u}^k(t)) \,:\, t \in [0, T]\}}. \tag{6.111}$$

By referring to Section 6.1 for Assumptions 1–4, we are in position to state the main result of this section.

THEOREM 6.18. *Under the hypotheses of Theorem 6.8, we have that*

$$d_H(\mathscr{S}^k, \mathscr{S}) \to 0, \qquad \text{as} \;\; k \to +\infty.$$

To prove Theorem 6.18, we need some preliminary results. Under Assumptions 1 and 2, fix $\tau \in [0, T]$ and let $\tilde{x}$, $\tilde{y} \in \mathbb{R}^n$ be such that, if v is the solution to the problem

$$\begin{cases} A\ddot{v}(\sigma) + B\dot{v}(\sigma) + \nabla_x \mathcal{E}(\tau, v(\sigma)) = 0, & \sigma \in [0, +\infty) \\ v(0) = \tilde{x}, \\ \dot{v}(0) = \tilde{y}, \end{cases}$$

and $v_\infty := \lim_{\sigma \to +\infty} v(\sigma)$, then $\nabla_x^2 \mathcal{E}(\tau, v_\infty)$ is positive definite. By the Implicit Function Theorem, there exist a connected neighbourhood U of τ in [0,T], a neighbourhood V of v_∞ in $\mathbb{R}^n$ and a C^2 function $u : U \to \mathbb{R}^n$ such that $u(\tau) = v_\infty$ and, if $(t, x) \in U \times V$, then $\nabla_x \mathcal{E}(t, x) = 0$ if and only if $x = u(t)$. Moreover, $\nabla_x^2 \mathcal{E}(t, u(t))$ is positive definite on U.

Consider three sequences $x_k \to \tilde{x}$, $y_k \to \tilde{y}$ and $\tau_k \in [0, T]$ such that $\tau_k \to \tau$, and denote by v_k the solution to the problem

$$\begin{cases} A\ddot{v}_k(\sigma) + B\dot{v}_k(\sigma) + \nabla_x \mathcal{E}(\tau_k, v_k(\sigma)) = 0, & \sigma \in [0, +\infty) \\ v_k(0) = x_k, \\ \dot{v}_k(0) = y_k. \end{cases}$$

By continuous dependence, we have that $(v_k, \dot{v}_k) \to (v, \dot{v})$ uniformly on the compact subsets of $[0, +\infty)$, and, by Lemma 6.5, we know that $v_k(+\infty)$ is a critical point of $\mathcal{E}(\tau_k, \cdot)$ and $\dot{v}_k(+\infty) = 0$. The following lemma tells us that, if k is sufficiently large, $v_k(+\infty) = u(\tau_k)$. Moreover, this convergence is uniform with respect to k.

LEMMA 6.19. *Under Assumptions 1 and 2, let u and v_k be defined as above. Then, there exists k_0 such that*

$$\lim_{\sigma \to +\infty} (v_k(\sigma), \dot{v}_k(\sigma)) = (u(\tau_k), 0), \qquad \text{for every } k \geq k_0. \tag{6.112}$$

Moreover, for every $\delta > 0$, there exists k_δ, $\sigma_\delta > 0$ such that

$$(v_k(\sigma), B\dot{v}_k(\sigma)) \in \overline{B}((u(\tau_k), 0), \delta), \qquad \text{for every } \sigma \geq \sigma_\delta, \; k \geq k_\delta. \tag{6.113}$$

PROOF. Let us refer to the previous paragraph for the notation. For every $t \in U$ and every $x \in \mathbb{R}^n$, there exists $\alpha \in [0, 1]$, depending on x and $u(t)$, such that

$$\mathcal{E}(t, x) = \mathcal{E}(t, u(t)) + \nabla_x^2 \mathcal{E}(t, u(t) + \alpha(x - u(t)))(x - u(t), x - u(t)). \tag{6.114}$$

Let $\overline{\tau} < \tau < \hat{\tau}$ be such that $[\overline{\tau}, \hat{\tau}] \subseteq U$. Since $\nabla_x^2 \mathcal{E}(\cdot, u(\cdot))$ is positive definite on $[\overline{\tau}, \hat{\tau}]$, there exists $R > 0$, depending on $[\overline{\tau}, \hat{\tau}]$, such that, if $\delta \in (0, R)$, then

$$\min\left\{\lambda \,:\, \lambda \text{ is an eigenvalue of } \nabla_x^2 \mathcal{E}(t, u(t) + z), \; |z| \leq \delta, \; t \in [\overline{\tau}, \hat{\tau}]\right\} =: \beta_{2\delta} > 0. \tag{6.115}$$

Choose $\delta \in (0, R)$. From (6.114) and (6.115), we obtain that

$$\min_{|x-u(t)|=\frac{\delta}{2}} \mathcal{E}(t,x) \geq \mathcal{E}(t,u(t)) + \beta_\delta \frac{\delta^2}{4}, \qquad \text{for every } t \in [\overline{\tau}, \hat{\tau}], \tag{6.116}$$

while the uniform continuity of $\mathcal{E}(\cdot, u(\cdot))$ on $[\overline{\tau}, \hat{\tau}]$ implies that

$$\max_{|x-u(t)|\leq r} \mathcal{E}(t,x) \leq \mathcal{E}(t,u(t)) + \beta_\delta \frac{\delta^2}{8}, \qquad \text{for every } t \in [\overline{\tau}, \hat{\tau}], \tag{6.117}$$

for a certain $r \in \left(0, \frac{\delta}{2}\right)$. Since $(v(\sigma), \dot{v}(\sigma)) \to (v_\infty, 0)$, as $\sigma \to +\infty$, we can find $\sigma_\delta > 0$ such that

$$|v(\sigma) - v_\infty| \leq \frac{r}{3}, \qquad |\dot{v}(\sigma)| \leq \frac{\delta}{2} \min\left\{\frac{1}{2}\sqrt{\frac{\beta_\delta}{2|A|}}, \frac{1}{|B|}\right\}, \qquad \text{for every } \sigma \geq \sigma_\delta. \tag{6.118}$$

By the uniform convergence of $(v_k, \dot{v}_k)$ to $(v, \dot{v})$ on the compact subsets of $[0, +\infty)$, there exists k_δ such that, for every $k \geq k_\delta$,

$$|v_k(\sigma_\delta) - v(\sigma_\delta)| \leq \frac{r}{3}, \qquad |\dot{v}_k(\sigma_\delta) - \dot{v}(\sigma_\delta)| \leq \frac{\delta}{4}\sqrt{\frac{\beta_\delta}{2|A|}}. \tag{6.119}$$

Also, we can suppose that

$$|u(\tau_k) - v_\infty| \leq \frac{r}{3}, \qquad \text{for every } k \geq k_\delta. \tag{6.120}$$

Let $\sigma \geq \sigma_\delta$ and $k \geq k_\delta$. By arguing as in the proof of Lemma 6.5, we obtain that

$$\mathcal{E}(\tau_k, v_k(\sigma)) \leq \frac{1}{2} A\dot{v}_k(\sigma_\delta)\cdot\dot{v}_k(\sigma_\delta) + \mathcal{E}(\tau_k, v_k(\sigma_\delta)), \tag{6.121}$$

and, by using (6.118)–(6.120), we have that

$$|v_k(\sigma_\delta) - u(\tau_k)| \leq |v_k(\sigma_\delta) - v(\sigma_\delta)| + |v(\sigma_\delta) - v_\infty| + |v_\infty - u(\tau_k)| \leq r. \tag{6.122}$$

Since $\tau_k \in [\overline{\tau}, \hat{\tau}]$ for every k sufficiently large, (6.117), (6.121) and (6.122) imply that

$$\begin{aligned} \mathcal{E}(\tau_k, v_k(\sigma)) &\leq \frac{|A|}{2}|\dot{v}_k(\sigma_\delta)|^2 + \mathcal{E}(\tau_k, u(\tau_k)) + \beta_\delta\frac{\delta^2}{8} \\ &\leq \mathcal{E}(\tau_k, u(\tau_k)) + \frac{3}{16}\beta_\delta\delta^2, \end{aligned} \tag{6.123}$$

where in the last inequality we have used also the second estimates in (6.118) and in (6.119). From (6.116) and (6.123), we obtain that $v_k(\sigma) \in B\left(u(\tau_k), \frac{\delta}{2}\right)$ for all $\sigma \geq \sigma_\delta$ and $k \geq k_\delta$. This fact, together with the second estimate of (6.118), gives (6.113). In particular, let us fix $\delta_0 > 0$ such that $\overline{B}\left(u(\tau_k), \frac{\delta_0}{2}\right) \subseteq V$ for every $k \geq k_0$, for some $k_0 > 0$. Then, by Lemma 6.5 and by the fact that the unique critical point of $\mathcal{E}(\tau_k, \cdot)$ in $\overline{B}\left(u(\tau_k), \frac{\delta_0}{2}\right)$ is $u(\tau_k)$, (6.112) is proved. □

For the following lemma, observe that, for $j = 1, ..., m-1$, the function u_{j+1}, defined in Proposition 6.6, is more generally defined on $[\overline{t}_j, t_{j+1})$, for a certain $\overline{t}_j < t_j$ sufficiently close to t_j such that

$$\nabla_x \mathcal{E}(t, u_{j+1}(t)) = 0, \qquad \nabla_x^2 \mathcal{E}(t, u_{j+1}(t)) > 0, \qquad \text{for every } t \in [\overline{t}_j, t_{j+1}). \tag{6.124}$$

Since the notation is unavoidably heavy, be careful to distinguish the functions u_j's from the functions u^k's defined in (6.108) proceeding from the points u_i^k's, defined in (6.103) and (6.104). The next lemma tells us essentially that, for k large enough, the points u_i^k are indeed values approximating u_1 on the compact subsets of $(0, t_1)$.

LEMMA 6.20. *Choose $\hat{t} \in (0, t_1)$ and $\delta > 0$. There exists $\hat{k}_\delta$, $\sigma_\delta > 0$ such that, for every $k \geq \hat{k}_\delta$, we have that*

$$(v_1^k(\sigma), B\dot{v}_1^k(\sigma)) \in \overline{B}\big((u_1(\tau_1^k), 0), \delta\big), \qquad \textit{for every } \sigma \geq \sigma_\delta, \tag{6.125}$$

and, if $\tau_i^k \in [\tau_2^k, \hat{t}]$*, then*

$$(v_i^k(\sigma), B\dot{v}_i^k(\sigma)) \in \overline{B}\big((u_1(\tau_i^k), 0), \delta\big), \qquad \textit{for every } \sigma \geq 0. \tag{6.126}$$

In particular, there exists $\hat{k}$ *such that*

$$u_i^k = u_1(\tau_i^k), \qquad \textit{for every } \tau_i^k \in [\tau_1^k, \hat{t}],\ k \geq \hat{k}. \tag{6.127}$$

To show that (6.125) and (6.127) hold for $i = 1$, we can use Lemma 6.19 with $\tau = 0$, $\tilde{x} = x_0$, $\tilde{y} = y_0$, v_0 in place of v, u_1 in place of u, $v_\infty = x_0^r$, and τ_1^k, v_1^k in place of τ_k, v_k, respectively. The proof of the remaining part of Lemma 6.20 can be done by induction and by using essentially the same argument of the proof of Lemma 6.19.

While Lemma 6.20 takes into account the approximating points u_i^k on the compact subsets of $(0, t_1)$, the following lemma, whose proof is similar to the previous one, deals with $[\overline{t}_j, t_{j+1})$, which is a slight modification of $[t_j, t_{j+1})$ in the sense of (6.124), for $j = 1, ..., m-1$.

LEMMA 6.21. *For* $j = 1, ..., m-1$*, let* $\overline{t}_j < t_j$ *be sufficiently close to* t_j *so that (6.124) holds . For* $j = 1, ..., m-2$*, choose* $\hat{t}_j \in [t_j, t_{j+1})$*, and set* $\hat{t}_{m-1} = T$*. For every* $\delta > 0$*, there exists* $\hat{k}_\delta > 0$ *such that, if*

$$\tau_l^k, \tau_{l+1}^k \in [\overline{t}_j, \hat{t}_j] \qquad \textit{and} \qquad u_l^k = u_{j+1}(\tau_l^k),$$

for some $j \in \{1, ..., m-1\}$*, then*

$$(v_i^k(\sigma), B\dot{v}_i^k(\sigma)) \in \overline{B}\big((u_{j+1}(\tau_i^k), 0), \delta\big), \qquad \textit{for every } \sigma \geq 0,$$

for every $\tau_i^k \in [\tau_{l+1}^k, \hat{t}_j]$*, and* $k \geq \hat{k}_\delta$*. In particular, there exists* $\hat{k} > 0$ *such that*

$$u_i^k = u_{j+1}(\tau_i^k), \qquad \textit{for every } \tau_i^k \in [\tau_{l+1}^k, \hat{t}_j],\ k \geq \hat{k}.$$

In order to prove Theorem 6.18, we need to select some special indices among $i = 0, ..., k$ and show certain properties of those. Lemma (6.20) and (6.21) suggest that we can expect that there exist some indices o_k^j which mark a transition around t_j from the approximation of u_j to the approximation of u_{j+1}, that is $u_i^k = u_j(\tau_i^k)$ for every $\tau^k_{o_k^{j-1}} < i \leq \tau^k_{o_k^j}$. Unluckily, it is not really like this, since, as we will see, it may happen that, if $\tau_i^k \leq t_j$ is too much close to t_j, $u_i^k \in \{u_j(\tau_i^k), \overline{u}_j(\tau_i^k)\}$ (see (6.22) for a definition). We will show later that the indices introduced by the following definition, which depends on a small parameter δ estimating the distance from x_j^s, are those responsible for the transition.

DEFINITION 6.22. *Let* $\delta > 0$ *be small enough. For every* $j \in \{1, ..., m-1\}$*, we define*

$$o_k^j = o_k^j(\delta) := \min A_k^j,$$

where $A_k^j = A_k^j(\delta)$ *is the set of the indices* $i \in \{0, ..., k-1\}$ *such that*

$$\tau_i^k \leq t_j, \qquad u_i^k \in \overline{B}\left(x_j^s, \frac{\delta}{2}\right),$$

and

$$(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B\big((x_j^s, 0), \delta\big), \qquad \textit{for some } \sigma > 0,$$

where x_j^s*, for* $j = 1, ..., m-1$*, is defined in Proposition 6.6.*

REMARK 6.23. Observe that, for k sufficiently large, the definition of o_k^j is well posed, since $A_k^j \neq \emptyset$. Let us check this fact in the case $j = 1$. For $j = 2, ..., m-1$, the proof can be conducted in a similar way, by using also the next lemma. Choose $\tilde{t} \in (t_1^\delta, t_1)$, where t_1^δ is defined in (6.25). By Lemma 6.20, for every k sufficiently large, there exists at least one index i such that $\tau_i^k \in [t_1^\delta, \tilde{t}\,]$ and $u_i^k = u_1(\tau_i^k) \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$. Now, there are two possibilities:

(1) $\tau_{i+1}^k > t_1$: in this case, we can suppose, up to bigger k's, that $\tau_{i+1}^k \leq t_1^{**}$, where t_1^{**} is defined in (6.24) and (6.26). Recalling that $u_{i+1}^k = v_{i+1}^k(+\infty)$ is a critical point of $\mathcal{E}(\tau_{i+1}^k, \cdot)$, from the

definition of t_1^{**} we have that $u_{i+1}^k \notin \overline{B}(x_1^s, \delta)$. Therefore, since $u_i^k = v_{i+1}^k(0) \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$ and $\dot{v}_{i+1}^k(0) = 0$, it turns out that $(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B\big((x_1^s, 0), \delta\big)$ for some $\sigma > 0$, so that $i \in A_k^1$; (2) $\tau_{i+1}^k \le t_1$: in this case, if $(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B\big((x_1^s, 0), \delta\big)$ for some $\sigma > 0$, then $i \in A_k^1$; otherwise, $\lim_{\sigma \to +\infty}(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) = (u_{i+1}^k, 0) \in \overline{B}\big((x_1^s, 0), \delta\big)$. But $u_{i+1}^k \in \{u_j(\tau_{i+1}^k), \overline{u}_j(\tau_{i+1}^k)\}$ and $t_1^\delta \le \tau_i^k < \tau_{i+1}^k \le t_1$, therefore $u_{i+1}^k \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$. At this point, we begin again by considering τ_{i+2}^k and, in turn, case (1) or (2).
By this procedure, in a finite number of steps we find some $i \in A_k^1$.

It is useful to underline two facts which emerge from Remark 6.23. We have that

$$u_{o_k^j}^k \in \left\{u_j\left(\tau_{o_k^j}^k\right), \overline{u}_j\left(\tau_{o_k^j}^k\right)\right\}, \tag{6.128}$$

and we cannot determine whether $\tau_{o_k^j+1}^k > t_j$ or $\tau_{o_k^j+1}^k \le t_j$. The following lemma will be useful to prove the main result of this section and tells us (see point (3)) that the index o_k^j marks the transition from the branches u_j and $\overline{u}_j$ to u_{j+1}, as it was expected.

Lemma 6.24. *For every $j \in \{1, ..., m-1\}$ and $\delta > 0$ small enough, the following properties hold:*

(1) $\tau_{o_k^j}^k \to t_j^-$;
(2) $u_{o_k^j}^k \to x_j^s$;
(3) for every k large enough, $u_{o_k^j+1}^k = u_{j+1}(\tau_{o_k^j+1}^k)$, hence $u_{o_k^j+1}^k \to x_j^r$.

Proof. Let us begin with the case $j = 1$ and write, in order to simplify the notation, $o_k = o_k(\delta)$ in place of o_k^1. We will use the point t_1^δ, which is defined in (6.25).
(1) Observe that, from Definition 6.22, $\tau_{o_k}^k \le t_1$. Let $\overline{t} < t_1$ be arbitrarily close to t_1. We want to show that there exists $\overline{k}$ such that $\tau_{o_k}^k \in (\overline{t}, t_1]$, for every $k \ge \overline{k}$. We can suppose that $\overline{t} \ge t_1^\delta$. Observe that, if x, $y \in \mathbb{R}^n$ vary in a compact, by uniform continuity there exists $\rho = \rho(\delta) > 0$ such that

$$|\mathcal{E}(t,x) - \mathcal{E}(t,y)| < \frac{\delta^2 \lambda_{min}^A}{32|B|^2}, \qquad \text{for every } t \in [0,T] \text{ and } |x-y| \le \rho. \tag{6.129}$$

Choose $\tilde{t} \in (\overline{t}, t_1)$ and set $\tilde{\delta} := \frac{1}{2}\min\{\rho, \delta\}$. Lemma 6.20 tells us that, for every k large enough (depending on $\tilde{\delta}$ and on $[\overline{t}, \tilde{t}]$), there exists an index $i \ge 1$ such that

$$t_1^\delta \le \overline{t} \le \tau_i^k < \tau_{i+1}^k \le \tilde{t}, \tag{6.130}$$

$$u_i^k = u_1(\tau_i^k), \tag{6.131}$$

and

$$v_{i+1}^k(\sigma) \in \overline{B}(u_1(\tau_{i+1}^k), \tilde{\delta}), \qquad \text{for every } \sigma \ge 0. \tag{6.132}$$

Thus, from (6.132) and the definition of $\tilde{\delta}$ and of t_1^δ, we obtain that

$$|v_{i+1}^k(\sigma) - x_1^s| \le |v_{i+1}^k(\sigma) - u_1(\tau_{i+1}^k)| + |u_1(\tau_{i+1}^k) - x_1^s| \le \frac{3}{4}\delta. \tag{6.133}$$

Also, recall that

$$\begin{aligned}\frac{\lambda_{min}^A}{2}|\dot{v}_{i+1}^k(\sigma)|^2 + \mathcal{E}(\tau_{i+1}^k, v_{i+1}^k(\sigma)) &\le \frac{1}{2}A\dot{v}_{i+1}^k(\sigma)\cdot\dot{v}_{i+1}^k(\sigma) + \mathcal{E}(\tau_{i+1}^k, v_{i+1}^k(\sigma)) \\ &\le \mathcal{E}(\tau_{i+1}^k, u_i^k),\end{aligned} \tag{6.134}$$

for every $\sigma \ge 0$, and observe that, by (6.101) and (6.131)–(6.132), we can suppose, up to greater k's, that

$$\begin{aligned}|u_i^k - v_{i+1}^k(\sigma)| &= |u_1(\tau_i^k) - v_{i+1}^k(\sigma)| \\ &\le |u_1(\tau_i^k) - u_1(\tau_{i+1}^k)| + |u_1(\tau_{i+1}^k) - v_{i+1}^k(\sigma)| \le \rho,\end{aligned}$$

for every $\sigma \geq 0$. Thus, from (6.129) and (6.134), it descends that

$$|B\dot{v}_{i+1}^k(\sigma)| \leq |B|\sqrt{\frac{2}{\lambda_{min}^A}}\left[\mathcal{E}(\tau_{i+1}^k, u_i^k) - \mathcal{E}(\tau_{i+1}^k, v_{i+1}^k(\sigma))\right]^{\frac{1}{2}} < \frac{\delta}{4}, \qquad \text{for every } \sigma \geq 0. \tag{6.135}$$

By using (6.130), (6.131), (6.133), and (6.135) we have that

$$u_i^k \in \overline{B}\left(x_1^s, \frac{\delta}{2}\right) \quad \text{and} \quad (v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in B\big((x_1^s, 0), \delta\big), \quad \text{for every } \sigma \geq 0. \tag{6.136}$$

By the same argument just used, we can prove that, whenever $l < i$ is such that $u_l^k \in \overline{B}\left(x_1^s, \frac{\delta}{2}\right)$, we have that $(v_k^{l+1}(\sigma), B\dot{v}_k^{l+1}(\sigma)) \in B\big((x_1^s, 0), \delta\big)$, for every $\sigma \geq 0$. This fact, together with (6.136) and the definition of o_k, implies that $o_k > i$ and therefore $\tau_{o_k}^k > \tau_i^k \geq \overline{t}$.
(2) This limit follows from property (1) and from (6.128).
(3) To further simplify the notation, let us write v_k instead of $v_{o_k+1}^k$, so that v_k is the solution to the problem

$$\begin{cases} A\ddot{v}_k(\sigma) + B\dot{v}_k(\sigma) + \nabla_x\mathcal{E}(\tau_{o_k+1}^k, v_k(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ v_k(0) = u_{o_k}^k, \\ \dot{v}_k(0) = 0. \end{cases}$$

By Definition 6.22, the following parameter is well defined for every k sufficiently large:

$$\sigma_k := \min\{\sigma > 0 : \ (v_k(\sigma), B\dot{v}_k(\sigma)) \in \partial B((x_1^s, 0), \delta)\}. \tag{6.137}$$

The compactness of $\partial B((x_1^s, 0), \delta)$ implies that, up to a subsequence,

$$(v_k(\sigma_k), B\dot{v}_k(\sigma_k)) \to (z, \dot{z}), \qquad \text{for some } (z, \dot{z}) \in \partial B((x_1^s, 0), \delta). \tag{6.138}$$

We claim that

$$\sigma_k \to +\infty, \qquad \text{as } k \to +\infty. \tag{6.139}$$

Suppose, by contradiction, that, up to a subsequence, $\{\sigma_k\} \subseteq [0, M]$ and $\sigma_k \to \hat{\sigma}$, for some $M > 0$ and $\hat{\sigma} \in [0, M]$. By properties (1) and (2), $(v_k, \dot{v}_k)$ is uniformly convergent to $(v, \dot{v})$ on $[0, M]$, where v is the solution to the problem

$$\begin{cases} A\ddot{v}(\sigma) + B\dot{v}(\sigma) + \nabla_x\mathcal{E}(t_1, v(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ v(0) = x_1^s, \\ \dot{v}(0) = 0. \end{cases}$$

In particular, $(v_k(\sigma_k), \dot{v}_k(\sigma_k)) \to (v(\hat{\sigma}), \dot{v}(\hat{\sigma}))$ and, in turn, by (6.137), $(v(\hat{\sigma}), B\dot{v}(\hat{\sigma})) \in \partial B\big((x_1^s, 0), \delta\big)$. This is an absurd, because $v \equiv x_1^s$, and therefore (6.139) is proved.
Now, let us define

$$\tilde{v}_k(\sigma) := v_k(\sigma + \sigma_k), \qquad \sigma \in [-\sigma_k, +\infty),$$

which satisfies the system $A\ddot{\tilde{v}}_k(\sigma) + B\dot{\tilde{v}}_k(\sigma) + \nabla_x\mathcal{E}(\tau_{o_k+1}^k, \tilde{v}_k(\sigma)) = 0$ for every $\sigma \in [-\sigma_k, +\infty)$, and the conditions $\tilde{v}_k(0) = v_k(\sigma_k)$, $\dot{\tilde{v}}_k(0) = \dot{v}_k(\sigma_k)$, and let w_1 be the solution to the problem

$$\begin{cases} A\ddot{w}(\sigma) + B\dot{w}(\sigma) + \nabla_x\mathcal{E}(t_1, w(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ w(0) = z, \\ B\dot{w}(0) = \dot{z}. \end{cases}$$

By property (1) and convergences (6.138) and (6.139), we have that $(\tilde{v}_k, \dot{\tilde{v}}_k) \to (w_1, \dot{w}_1)$ uniformly on the compact subsets of any common interval of existence. By using this fact together with the definition of σ_k, it is easy to show that

$$\{(w_1(\sigma), B\dot{w}_1(\sigma)) \, : \, \sigma \leq 0\} \subseteq \overline{B}\big((x_1^s, 0), \delta\big). \tag{6.140}$$

Thus, by Lemma 6.4 together with (6.23) and Proposition 6.6, we have that

$$\lim_{s \to -\infty}(w_1(s), \dot{w}_1(s)) = (x_1^s, 0), \qquad \lim_{s \to +\infty}(w_1(s), \dot{w}_1(s)) = (x_1^r, 0), \tag{6.141}$$

and that

$$(\tilde{v}_k, \dot{\tilde{v}}_k) \to (w_1, \dot{w}_1) \ \text{ uniformly on the compact subsets of } \mathbb{R}. \tag{6.142}$$

Observing that, by definition, $u^k_{o_k+1} = \lim_{\sigma\to+\infty}\tilde{v}_k(\sigma)$, it is enough to apply Lemma 6.19 with t_1, z, $B^{-1}\dot{z}$, x^r_1, u_2, $v_k(\sigma_k)$, $\dot{v}_k(\sigma_k)$, $\tau^k_{o_k+1}$ and $\tilde{v}_k$ in place of τ, $\tilde{x}$, $\tilde{y}$, v_∞, u, x_k, y_k, τ_k and v_k, respectively, to conclude that $u^k_{o_k+1} = u_2(\tau^k_{o_k+1})$ for every k large enough, and, in turn, that $u^k_{o_k+1} \to x^r_1$.
The proof of the cases $j = 2, ..., m-1$ can be done in a similar way and by using, more, the case $j = 1$. □

Lemma 6.20 and Lemma 6.24 allow us to state and prove a result of approximation of u on the compact subsets of $(0,T]\setminus\{t_1,...,t_{m-1}\}$. Since the jump times t_j's are not so far considered, the heteroclines w_j's appear in the statement just because they are involved in the definition of u through their limit points x^s_j and x^r_j at $-\infty$ and $+\infty$, respectively (see Definition 6.7). Notice that Proposition 6.25, by including the uniform convergence to 0 of the "modified" velocity $h_kB\dot{u}^k$ on the compact subsets of $(0,T]\setminus\{t_1,...,t_{m-1}\}$, recovers all the information collected in Theorem 6.8 (1) by using a different approach. We refer the reader to (6.106)–(6.110) for the notation. In the remaining part of this section, we will sometimes use the symbol ω_{u_1} to denote the modulus of continuity of u_1 on $[0, t_1/2]$.

Proposition 6.25. *Under the hypotheses of Theorem 6.8, we have that*

$$(u^k, h_kB\dot{u}^k) \to (u,0), \text{ uniformly on the compact subsets of } (0,T]\setminus\{t_1,...,t_{m-1}\}.$$

Proof. Let us consider the interval $(0,t_1)$. The proof for the other intervals (t_{j-1},t_j) can be done in a similar way, by using, more, Lemma 6.24 (3). Choose $\bar{t}$ and $\tilde{t}$ such that $0<\bar{t}<\tilde{t}<t_1$ and $\delta>0$ arbitrarily small. Observe that, for k sufficiently large, there exist i and m such that $i-m\geq 2$ and

$$\tau^k_{i-m-1} \leq \bar{t} < \alpha^k_{i-m} < ... < \alpha^k_i \leq \tilde{t} < \tau^k_i \leq \frac{\tilde{t}+t_1}{2},$$

so that it is sufficient to analyze the following two model cases.
(i) If $t\in[\bar{t},\alpha^k_{i-m})$, then $u^k(t) = v^k_{i-m}(\sigma)$ and $h_k(t)\dot{u}^k(t) = \dot{v}^k_{i-m}(\sigma)$ for some $\sigma\in[0,a^k_{i-m})$. Thus, since

$$\big|\big(u^k(t)-u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| \leq \big|\big(v^k_{i-m}(\sigma)-u_1(\tau^k_{i-m}), B\dot{v}^k_{i-m}(\sigma)\big)\big| + |u_1(\tau^k_{i-m})-u_1(t)|,$$

we have that, for every k large enough,

$$\sup_{t\in[\bar{t},\alpha^k_{i-m})} \big|\big(u^k(t)-u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| \leq \delta + \omega_{u_1}(\rho_k),$$

in view of Lemma 6.20.
(ii) If $t\in[\alpha^k_i,\tilde{t}]$ then $u^k(t) = \frac{(\tau^k_i-t)v^k_i(a^k_i)+(t-\alpha^k_i)u^k_i}{\tau^k_i-\alpha^k_i}$, so that, by using Lemma 6.20 ($u^k_i = u_1(\tau^k_i)$ for every k large enough, since $\tau^k_1<\tau^k_i\leq\frac{\tilde{t}+t_1}{2}$),

$$\big|\big(u^k(t)-u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| \leq$$
$$|v^k_i(a^k_i)-u^k_i| + |B|\frac{(\alpha^k_i-\tau^k_{i-1})}{a^k_i}\frac{|u^k_i-v^k_i(a^k_i)|}{\tau^k_i-\alpha^k_i} + |u_1(\tau^k_i)-u_1(t)|. \quad (6.143)$$

Assumption (6.107) and inequality (6.143) give that

$$\begin{aligned}\sup_{t\in[\alpha^k_i,\tilde{t}]} \big|\big(u^k(t)-u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| &\leq C\left(\tau^k_i-\alpha^k_i+|B|\frac{\alpha^k_i-\tau^k_{i-1}}{a^k_i}\right)+\omega_{u_1}(\rho_k)\\ &\leq C\rho_k\left(1+\frac{|B|}{a^k_i}\right)+\omega_{u_1}(\rho_k).\end{aligned}$$

Since $\rho_k\to 0$, cases (i) and (ii) together with (6.106) tell us that

$$\sup_{t\in[\bar{t},\tilde{t}]} \big|\big(u^k(t)-u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| \leq 2\delta, \qquad \text{for every } k \text{ large enough.}$$

□

From Proposition 6.25, one can easily deduce an approximation result related to the piecewise constant and the piecewise affine interpolations of the points u_i^k, seen as the piecewise constant and the piecewise affine approximations of u^k, respectively. To be precise, let us set

$$\tilde{u}^k(t) := \begin{cases} x_k, & t \in [0, \tau_1^k) \\ u_1^k, & t \in [\tau_1^k \tau_2^k) \\ \vdots \\ u_{k-1}^k, & t \in [\tau_{k-1}^k, T], \end{cases} \qquad \hat{u}^k(t) := \begin{cases} \dfrac{(\tau_1^k - t)x_k + tu_1^k}{\tau_1^k}, & t \in [0, \tau_1^k) \\ \dfrac{(\tau_2^k - t)u_1^k + (t - \tau_1^k)u_2^k}{\tau_2^k - \tau_1^k}, & t \in [\tau_1^k \tau_2^k) \\ \vdots \\ \dfrac{(T - t)u_{k-1}^k + (t - \tau_{k-1}^k)u_k^k}{T - \tau_{k-1}^k}, & t \in [\tau_{k-1}^k, T]. \end{cases}$$

We have that

$$\tilde{u}^k, \hat{u}^k \to u \ \text{ uniformly on the compact subsets of } \ (0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

Since these convergences do not take into account the velocities, it is clear that their proof requires much less effort than the proof of Proposition 6.25.

REMARK 6.26. Observe that, if $x_0 = x_0^r$ and $y_0 = 0$ (recall (6.28), (6.29) and (6.105)), we obtain that $v_0 \equiv x_0^r$. In this case, it turns out that

$$(u^k, h_k B\dot{u}^k) \to (u, 0) \ \text{ uniformly on the compact subsets of } \ [0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

To see this, choose $\hat{t} \in (0, t_1)$ and $\delta > 0$. In view of the proof of Proposition 6.25, it is enough to consider the case $t \in [0, \alpha_1^k) \subseteq [0, \hat{t}]$, so that

$$\big|\big(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| = \big|\big(v_1^k(\sigma), B\dot{v}_1^k(\sigma)\big)\big|, \qquad \text{for some } \sigma \in [0, a_1^k). \tag{6.144}$$

Let σ_δ and $\hat{k}_\delta$ be given by Lemma 6.20. Then, from (6.144), we have that, if $\sigma \geq \sigma_\delta$ and $k \geq \hat{k}_\delta$,

$$\begin{aligned} \big|\big(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| &\leq \big|\big(v_1^k(\sigma) - u_1(\tau_1^k), B\dot{v}_1^k(\sigma)\big)\big| + |u_1(\tau_1^k) - u_1(t)| \\ &\leq \delta + \omega_{u_1}(\rho_k); \end{aligned}$$

if $\sigma \in [0, \sigma_\delta)$,

$$\begin{aligned} \big|\big(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t)\big)\big| &\leq \big|\big(v_1^k(\sigma) - x_0^r, B\dot{v}_1^k(\sigma)\big)\big| + |x_0^r - u_1(t)| \\ &\leq \sup_{\sigma \in [0, \sigma_\delta]} \big|\big(v_1^k(\sigma) - x_0^r, B\dot{v}_1^k(\sigma)\big)\big| + \omega_{u_1}(\rho_k), \end{aligned}$$

and we can conclude by using the fact that $(v_1^k, \dot{v}_1^k) \to (x_0^r, 0)$ uniformly on the compact subsets of $[0, +\infty)$ and (6.101).

What it remains to do now is an accurate study at time 0 and at the jump times t_j's. This is done in the proof of the main result of this section. We refer to (6.32)–(6.35) and to (6.111) for the definitions of the sets $\mathscr{S}$ and $\mathscr{S}_k$.

PROOF OF THEOREM 6.18. We follow the position already used in the proof of Lemma 6.24: we write o_k and v_k in place of o_k^1 and $v_{o_k+1}^k$, respectively. In the sequel, whenever $\delta > 0$ is arbitrarily chosen, it will be implicit that the following objects, which depend on δ and have been defined in the proof of Lemma 6.24, are involved: the sequence $\{\sigma_k\}$ such that $\sigma_k \to +\infty$ and the functions $\tilde{v}_k(\sigma) := v_k(\sigma + \sigma_k)$ and w_1 such that (6.140)–(6.142) hold and

$$\big\{\big(\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma)\big) \,:\, \sigma \in [-\sigma_k, 0]\big\} \subseteq \overline{B}\big((x_1^s, 0), \delta\big).$$

Choose $\delta > 0$ arbitrarily small. In order to prove the theorem, we are going to show that, for every k large enough,

$$d_H(\mathscr{S}^k, \mathscr{S}) = \sup_{\mathscr{S}^k} d(\cdot, \mathscr{S}) + \sup_{\mathscr{S}} d(\cdot, \mathscr{S}^k) \leq 2\delta. \tag{6.145}$$

We can suppose that

$$t_1 - t_1^\delta < \delta, \tag{6.146}$$

where t_1^δ is defined in (6.25), and that, for some k_δ,

$$\tau^k_{o_k} \in \left(\frac{t_1^\delta + t_1}{2}, t_1\right] \quad \text{and} \quad u_i^k = u_1(\tau_i^k) \quad \text{for every } \tau_i^k \in \left[\tau_1^k, \frac{t_1^\delta + t_1}{2}\right], \tag{6.147}$$

for every $k \geq k_\delta$, in view of Lemma 6.20.

We divide the proof in two parts, in view of the definition of the Hausdorff distance.
(a) Here, consider $\sup_{\mathscr{S}^k} d(\cdot, \mathscr{S})$ and set

$$d_k(t) := d\big((t, u^k(t), h_k(t)B\dot{u}^k(t)), \mathscr{S}\big), \qquad t \in [0,T].$$

By considering the partition $0 < \tau^k_{o_k} < \tau^k_{o_k+1} < \ldots < T$, which depends on δ (see Definition 6.22), it is clear that it is sufficient to analyze d_k in the model cases $t \in \big[0, \tau^k_{o_k}\big)$ and $t \in \big[\tau^k_{o_k}, \tau^k_{o_k+1}\big)$, since the other cases $j \neq 1$ can be treated in a similar way. Let us divide part (a) in two subparts.
(a1) Consider first $d_k(t)$ for $t \in [\tau^k_{o_k}, \tau^k_{o_k+1})$. Using Lemma 6.19, since convergence (6.142) hold and $w_1(+\infty) = x_1^r$ with $\nabla_x \mathcal{E}(t_1, x_1^r) = 0$ and $\nabla_x^2 \mathcal{E}(t_1, x_1^r) > 0$, there exists $\tilde{\sigma}_\delta$ such that, up to a greater k_δ,

$$(\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma)) \in \overline{B}\big((u_2(\tau^k_{o_k+1}), 0), \delta\big), \qquad \text{for every } \sigma \geq \tilde{\sigma}_\delta,\ k \geq k_\delta. \tag{6.148}$$

If $t \in [\tau^k_{o_k}, \alpha^k_{o_k+1})$, we have that

$$d_k(t) = d\big((t, v_k(\sigma), B\dot{v}_k(\sigma)), \mathscr{S}\big), \qquad \text{for some } \sigma \in [0, a^k_{o_k+1}).$$

Recall that $\tilde{v}_k(\sigma) = v_k(\sigma + \sigma_k)$, with $\sigma_k \to +\infty$. Therefore, if $\sigma - \sigma_k \geq \tilde{\sigma}_\delta$, we use (6.148) and obtain

$$\begin{aligned} d_k(t) &\leq |t - t_1| + d\big((\tilde{v}_k(\sigma - \sigma_k), B\dot{\tilde{v}}_k(\sigma - \sigma_k)), (x_1^r, 0)\big) + |u_2(\tau^k_{o_k+1}) - x_1^r| \\ &\leq (t_1 - \tau^k_{o_k}) + \rho_k + \delta + |u_2(\tau^k_{o_k+1}) - x_1^r|, \end{aligned} \tag{6.149}$$

for every $k \geq k_\delta$. If $0 \leq \sigma - \sigma_k < \tilde{\sigma}_\delta$, then

$$\begin{aligned} d_k(t) &\leq |t - t_1| + d\big((\tilde{v}_k(\sigma - \sigma_k), B\dot{\tilde{v}}_k(\sigma - \sigma_k)), \mathscr{I}_1\big) \\ &\leq (t_1 - \tau^k_{o_k}) + \rho_k + \sup_{\sigma \in [0, \tilde{\sigma}_\delta]} \big|\big(\tilde{v}_k(\sigma) - w_1(\sigma), B\dot{\tilde{v}}_k(\sigma) - B\dot{w}_1(\sigma)\big)\big|. \end{aligned} \tag{6.150}$$

In the case $\sigma < \sigma_k$, by the definition of σ_k we have that $\big(v_k(\sigma), B\dot{v}_k(\sigma)\big) \in \overline{B}\big((x_1^s, 0), \delta\big)$. This fact, together with (6.149), (6.150), (6.142), and Lemma 6.24, gives that

$$d_k(t) \leq 2\delta, \qquad \text{for every } t \in [\tau^k_{o_k}, \alpha^k_{o_k+1}), \tag{6.151}$$

for every k sufficiently large.

In the remaining case $t \in [\alpha^k_{o_k+1}, \tau^k_{o_k+1})$, we use (6.107) and Lemma 6.24 (3), so that

$$d_k(t) \leq (t_1 - \tau^k_{o_k}) + \rho_k + C(\tau^k_{o_k+1} - \alpha^k_{o_k+1}) + |u_2(\tau^k_{o_k+1}) - x_1^r| + |B|C\frac{\alpha^k_{o_k+1} - \tau^k_{o_k}}{a^k_{o_k+1}}. \tag{6.152}$$

Since $\tau^k_{o^k} \to t_1^-$ by Lemma 6.24 (1), $(\tau^k_{o_k+1} - \alpha^k_{o_k+1}) \to 0$ from (6.101), $\tau^k_{o_k+1} \to t_1$ and therefore $u_2(\tau^k_{o_k+1}) \to x_1^r$, and $a^k_{o_k+1} \to +\infty$ from (6.106), inequalities (6.152) and (6.151) give

$$\sup_{\left[\tau^k_{o_k}, \tau^k_{o_k+1}\right)} d_k \leq 2\delta, \tag{6.153}$$

for every k large enough.
(a2) Now, let us consider $d_k(t)$ for $t \in [0, \tau^k_{o_k})$. We have to distinguish the case $t \in \left[0, \frac{t_1^\delta + t_1}{2}\right)$ from the case $t \in \left[\frac{t_1^\delta + t_1}{2}, \tau^k_{o_k}\right)$. Suppose $t \in [\tau_i^k, \tau_{i+1}^k)$ for some $i \geq 1$. The case $t \in [0, \tau_1^k)$ can be handled similarly to the case (a1), but more easy, since in this case we have to use the uniform convergence of $(v_1^k, \dot{v}_1^k)$ to $(v_0, \dot{v}_0)$ on the compact subsets of $[0, +\infty)$, instead of the uniform convergence of $(\tilde{v}_k, \dot{\tilde{v}}_k)$ to $(w_1, \dot{w}_1)$ on the compact subsets of $\mathbb{R}$.

If $t \in \left[0, \frac{t_1^\delta+t_1}{2}\right) \cap [\tau_i^k, \alpha_{i+1}^k)$, we have that $d_k(t) = d\big((t, v_{i+1}^k(\sigma), B\dot v_{i+1}^k(\sigma)), \mathscr{S}\big)$, for some $\sigma \in [0, a_{i+1}^k)$. Observe that, up to a bigger k_δ,

$$\tau_{i+1}^k \le \frac{t_1^\delta + t_1}{2}, \qquad \text{for every } k \ge k_\delta,$$

so that, by using (6.126), we obtain

$$u_{i+1}^k = u_1(\tau_{i+1}^k), \tag{6.154}$$

and

$$d_k(t) \le (\tau_{i+1}^k - t) + \big|\big(v_{i+1}^k(\sigma) - u_1(\tau_{i+1}^k), B\dot v_{i+1}^k(\sigma)\big)\big| \le \rho_k + \delta, \qquad \text{for every } \sigma \ge 0. \tag{6.155}$$

If $t \in \left[0, \frac{t_1^\delta+t_1}{2}\right) \cap [\alpha_{i+1}^k, \tau_{i+1}^k)$, from (6.154) it turns out that

$$\begin{aligned} d_k(t) &\le \tau_{i+1}^k - t + |v_{i+1}^k(a_{i+1}^k) - u_1(\tau_{i+1}^k)| + |B|\frac{(\alpha_{i+1}^k - \tau_i^k)}{a_{i+1}^k}\frac{|u_1(\tau_{i+1}^k) - v_{i+1}^k(a_{i+1}^k)|}{\tau_{i+1}^k - \alpha_{i+1}^k} \\ &\le \rho_k + C|\tau_{i+1}^k - \alpha_{i+1}^k| + |B|C\frac{(\alpha_{i+1}^k - \tau_i^k)}{a_{i+1}^k}, \end{aligned} \tag{6.156}$$

where the last inequality is due to (6.107).
In the case $t \in \left[\frac{t_1^\delta+t_1}{2}, \tau_{o_k}^k\right) \cap [\tau_i^k, \tau_{i+1}^k)$, observe first that we can suppose, for larger k's, that $\tau_i^k \ge t_1^\delta$. Thus, since $u_i^k \in \{u_1(\tau_i^k), \overline{u}_1(\tau_i^k)\}$, we have that

$$u_i^k \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right).$$

This inclusion, together with the fact that $i < o_k$ and the definition of o_k, gives that

$$\big(v_{i+1}^k(\sigma), B\dot v_{i+1}^k(\sigma)\big) \in B\big((x_1^s, 0), \delta\big), \qquad \text{for every } \sigma \ge 0. \tag{6.157}$$

Thus, if $t \in [\tau_i^k, \alpha_{i+1}^k)$, so that $d_k(t) = d\big((t, v_{i+1}^k(\sigma), B\dot v_{i+1}^k(\sigma)), \mathscr{S}\big)$ for some $\sigma \in [0, a_{i+1}^k)$, from (6.146) and (6.157) it turns out that

$$\begin{aligned} d_k(t) &\le d\big((t, v_{i+1}^k(\sigma), B\dot v_{i+1}^k(\sigma)), (t_1, x_1^s, 0)\big) \\ &\le \frac{t_1 - t_1^\delta}{2} + \big|\big(v_{i+1}^k(\sigma) - x_1^s, B\dot v_{i+1}^k(\sigma)\big)\big| \le 2\delta. \end{aligned} \tag{6.158}$$

Otherwise, if $t \in [\alpha_{i+1}^k, \tau_{i+1}^k)$, we have

$$\begin{aligned} d_k(t) &\le \rho_k + |v_{i+1}^k(a_{i+1}^k) - u_1(\tau_{i+1}^k)| + |B|\frac{(\alpha_{i+1}^k - \tau_i^k)}{a_{i+1}^k}\frac{|u_{i+1}^k - v_{i+1}^k(a_{i+1}^k)|}{\tau_{i+1}^k - \alpha_{i+1}^k} \\ &\le \rho_k + C|\tau_{i+1}^k - \alpha_{i+1}^k| + |u_{i+1}^k - x_1^s| + |x_1^s - u_1(\tau_{i+1}^k)| + |B|C\frac{\alpha_{i+1}^k - \tau_i^k}{a_{i+1}^k}, \end{aligned}$$

where, in the last inequality, we have used (6.107). Then, by using (6.101), the definition of t_1^δ and (6.157) (which gives $u_{i+1}^k \in \overline{B}(x_1^s, \delta)$), we have that, for every k large enough,

$$d_k(t) \le 2\delta, \qquad \text{for every } t \in \left[\frac{t_1^\delta + t_1}{2}, \tau_{o_k}^k\right) \cap [\alpha_{i+1}^k, \tau_{i+1}^k). \tag{6.159}$$

Inequalities (6.155), (6.156), (6.158) and (6.159) imply that, for every k large enough,

$$\sup_{[0, \tau_{o_k}^k)} d_k \le 2\delta. \tag{6.160}$$

(b) Here, we consider $\sup_{\mathscr{S}} d(\cdot,\mathscr{S}^k)$. By definition of $\mathscr{S}$ and by the fact that $(x_1^s,0)\in\overline{\mathscr{I}}_1$, it is sufficient to consider

$$\sup_{t\in[0,t_1)} d((t,u_1(t),0),\mathscr{S}^k),\qquad \sup_{\{0\}\times\mathscr{I}_0} d(\cdot,\mathscr{S}^k),\qquad \sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\mathscr{S}^k).$$

The other cases can be treated in a similar way. Let us divide part (b) in three subparts.
(b1) If $t\in[0,t_1^\delta)$, suppose $t\in[\tau_i^k,\tau_{i+1}^k)$ for a certain index i, so that, up to a bigger k_δ, $\tau_{i+1}^k\le t_1^\delta$ and, in turn, $u_{i+1}^k=u_1(\tau_{i+1}^k)$, for $k\ge k_\delta$. Therefore, by recalling that $u^k(\tau_{i+1}^k)=u_{i+1}^k$ and $\dot u^k(\tau_{i+1}^k)=0$, we obtain that

$$d\big((t,u_1(t),0),\mathscr{S}^k\big)\le(\tau_{i+1}^k-t)+|u_1(t)-u_1(\tau_{i+1}^k)|\le\rho_k+\omega_{u_1}(\rho_k), \tag{6.161}$$

for every $k\ge k_\delta$. For $t\in[t_1^\delta,t_1)$, we write, in view of (6.146) and (6.147),

$$\begin{aligned} d\big((t,u_1(t),0),\mathscr{S}^k\big) &\le d\big((t,u_1(t),0),(\tau_{o_k}^k,u^k(\tau_{o_k}^k),h_k(\tau_{o_k}^k)B\dot u^k(\tau_{o_k}^k)\big)\\ &\le (t_1-t_1^\delta)+|u_1(t)-x_1^s|+|x_1^s-u_{o_k}^k|<2\delta, \end{aligned}\tag{6.162}$$

for every $k\ge k_\delta$. (6.161) and (6.162), together with (6.101), give, up to a bigger k_δ,

$$\sup_{t\in[0,t_1)} d((t,u_1(t),0),\mathscr{S}^k)<2\delta,\qquad \text{for every } \ k\ge k_\delta. \tag{6.163}$$

(b2) Consider $(0,v_0(\sigma),B\dot v_0(\sigma))\in\{0\}\times\mathscr{I}_0$ and let $s_0^\delta>0$ be defined as in (6.65). Since $a_1^k>s_0^\delta$ for every k large enough, it turns out that

$$d\big((0,v_0(\sigma),B\dot v_0(\sigma)),\mathscr{S}^k\big)\le\alpha_1^k+\min_{s\in[0,\sigma_\delta]} d\big((v_0(\sigma),B\dot v_0(\sigma)),(v_1^k(s),B\dot v_1^k(s))\big).$$

Then, if $\sigma\in[0,s_0^\delta]$, we have that

$$d\big((0,v_0(\sigma),B\dot v_0(\sigma)),\mathscr{S}^k\big)\le\alpha_1^k+\sup_{\sigma\in[0,\sigma_\delta]}\big|\big(v_0(\sigma)-v_1^k(\sigma),B\dot v_0(\sigma)-B\dot v_1^k(\sigma)\big)\big|, \tag{6.164}$$

while, if $\sigma>s_0^\delta$,

$$\begin{aligned} d\big((0,v_0(\sigma),B\dot v_0(\sigma)),\mathscr{S}^k\big)&\le\tau_1^k+d\big((v_0(\sigma),B\dot v_0(\sigma)),(u^k(\tau_1^k),h_k(\tau_1^k)B\dot u^k(\tau_1^k))\big)\\ &\le\tau_1^k+d\big((v_0(\sigma),B\dot v_0(\sigma)),(x_0^r,0)\big)+|x_0^r-u_1(\tau_1^k)|. \end{aligned}\tag{6.165}$$

Inequalities (6.164) and (6.165), together with (6.101) and the uniform convergence of $(v_1^k,\dot v_1^k)$ to $(v_0,\dot v_0)$ on the compact subsets of $[0,+\infty)$, give

$$\sup_{\{0\}\times\mathscr{I}_0} d(\cdot,\mathscr{S}^k)\le\delta,\qquad \text{for every } k \text{ large enough.} \tag{6.166}$$

(b3) Finally, let us consider $\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\mathscr{S}^k)$. By recalling (6.140), we obtain that

$$\begin{aligned} d\big((t_1,w_1(s),B\dot w_1(s)),\mathscr{S}^k\big) &\le (t_1-\tau_{o_k}^k)+d\big((w_1(s),B\dot w_1(s)),(u_{o_k}^k,0)\big)\\ &\le (t_1-\tau_{o_k}^k)+\delta+|x_1^s-u_{o_k}^k|, \end{aligned}\tag{6.167}$$

for every $s<0$. Similarly, if $s_1^\delta>0$ is defined as in (6.80), for every $s>s_1^\delta$ we can write

$$\begin{aligned} d\big((t_1,w_1(s),B\dot w_1(s)),\mathscr{S}^k\big) &\le |t_1-\tau_{o_k+1}^k|+d\big((w_1(s),B\dot w_1(s)),(u_{o_k+1}^k,0)\big)\\ &\le |t_1-\tau_{o_k+1}^k|+\frac{\delta}{2}+|x_1^r-u_{o_k+1}^k|. \end{aligned}\tag{6.168}$$

To finish the proof we need the following claim, whose proof is postponed. See (6.137) for the definition of σ_k.
Claim. For every $\hat s\ge0$ and $\delta>0$ sufficiently small, there exists $k_{\hat s,\delta}>0$ such that

$$a_{o_k+1}^k-\sigma_k>\hat s,\qquad \text{for every } k\ge k_{\hat s,\delta}.$$

It remains to consider $s \in [0, s_1^\delta]$. In this case,

$$\begin{aligned} &d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^k\big) \\ &\qquad \le \inf_{t\in[\tau^k_{o_k}, \alpha^k_{o_k+1})} d\big((t_1, w_1(s), B\dot{w}_1(s)), (t, u^k(t), h_k(t)B\dot{u}^k(t))\big) \\ &\qquad \le (t_1 - \tau^k_{o_k}) + \rho_k + \inf_{\sigma\in[0, a^k_{o_k+1})} d\big((w_1(s), B\dot{w}_1(s)), (v_k(\sigma), B\dot{v}_k(\sigma))\big) \\ &\qquad = (t_1 - \tau^k_{o_k}) + \rho_k + \inf_{\sigma\in[-\sigma_k, a^k_{o_k+1}-\sigma_k)} d\big((w_1(s), B\dot{w}_1(s)), (\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma))\big). \end{aligned}$$

Thus, since $\sigma_k \to +\infty$, in view of Lemma 6.24 (1), of the claim and of (6.142) we have that, up to a bigger k_δ,

$$\begin{aligned} &d\big((t_1, w_1(s), B\dot{w}_1(s)), \mathscr{S}^k\big) \\ &\qquad \le (t_1 - \tau^k_{o_k}) + \rho_k + \inf_{\sigma\in[0, s_1^\delta]} d\big((w_1(s), B\dot{w}_1(s)), (\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma))\big) \\ &\qquad \le (t_1 - \tau^k_{o_k}) + \rho_k + \sup_{\sigma\in[0, s_1^\delta]} \big|\big(w_1(\sigma) - \tilde{v}_k(\sigma), B\dot{w}_1(\sigma) - B\dot{\tilde{v}}_k(\sigma)\big)\big| \\ &\qquad \le \delta + \rho_k, \end{aligned} \tag{6.169}$$

for every $k \ge k_\delta$. Inequalities (6.167), (6.168) and (6.169), together with (6.101), imply that, up to a bigger k_δ,

$$\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot, \mathscr{S}^k) \le 2\delta, \qquad \text{for every } k \ge k_\delta. \tag{6.170}$$

By considering together the estimates in (6.153), (6.160), (6.163), (6.166) and (6.170), which hold also for generic j's in place of $j = 1$, we obtain (6.145). □

Proof of the claim. Suppose, by contradiction, that, for a certain $\hat{s} \ge 0$, $\delta > 0$ and up to a subsequence, $a^k_{o_k+1} \le \hat{s} + \sigma_k$ for every k. Then, by definition of σ_k,

$$\big(v_k(a^k_{o_k+1} - \hat{s}), B\dot{v}_k(a^k_{o_k+1} - \hat{s})\big) \in \overline{B}\big((x_1^s, 0), \delta\big)$$

so that, up to a subsequence, $\big(v_k(a^k_{o_k+1} - \hat{s}), B\dot{v}_k(a^k_{o_k+1} - \hat{s})\big) \to (p, \dot{p})$, for some $(p, \dot{p}) \in \overline{B}\big((x_1^s, 0), \delta\big)$. Consider

$$\hat{v}_k(\sigma) := v_k(\sigma + a^k_{o_k+1} - \hat{s}), \qquad \text{for } \sigma \ge \hat{s} - a^k_{o_k+1}.$$

From Lemma 6.24, from the definition of v_k and from the fact that $a^k_{o_k+1} \to +\infty$, it is clear that $(\hat{v}_k, \dot{\hat{v}}_k)$ converges uniformly on the compact subsets of $\mathbb{R}$ to $(\hat{w}_1, \dot{\hat{w}}_1)$, where $\hat{w}_1$ is the solution to the problem

$$\begin{cases} A\ddot{\hat{w}}_1(s) + B\dot{\hat{w}}_1(s) + \nabla_x \mathcal{E}(t_1, \hat{w}_1(s)) = 0, \\ \hat{w}_1(0) = p, \\ B\dot{\hat{w}}_1(0) = \dot{p}. \end{cases}$$

Observe that

$$\big(v_k(a^k_{o_k+1}), B\dot{v}_k(a^k_{o_k+1})\big) = (\hat{v}_k(\hat{s}), B\dot{\hat{v}}_k(\hat{s})) \to (\hat{w}_1(\hat{s}), B\dot{\hat{w}}_1(\hat{s})), \tag{6.171}$$

and $(\hat{w}_1(\hat{s}), B\dot{\hat{w}}_1(\hat{s})) \ne (x_1^r, 0)$, otherwise it would be $(\hat{w}_1, B\dot{\hat{w}}_1) \equiv (x_1^r, 0)$, so that $(x_1^r, 0) = (\hat{w}_1(0), B\dot{\hat{w}}_1(0)) = (p, \dot{p}) \in \overline{B}\big((x_1^s, 0), \delta\big)$, which is not true if δ is small enough. But convergence (6.171) and $(\hat{w}_1(\hat{s}), B\dot{\hat{w}}_1(\hat{s})) \ne (x_1^r, 0)$ give a contradiction, because by Lemma 6.24 (3) and by (6.107), for every k large enough,

$$\begin{aligned} \big|\big(v_k(a^k_{o_k+1}) - x_1^r, B\dot{v}_k(a^k_{o_k+1})\big)\big| &\le \big|\big(v_k(a^k_{o_k+1}) - u^k_{o_k+1}, B\dot{v}_k(a^k_{o_k+1})\big)\big| + |u^k_{o_k+1} - x_1^r| \\ &\le C(\tau^k_{o_k+1} - \alpha^k_{o_k+1}) + |B|\delta_k + |u^k_{o_k+1} - x_1^r|, \end{aligned}$$

so that

$$\big(\hat{v}_k(\hat{s}), B\dot{\hat{v}}_k(\hat{s})\big) = \big(v_k(a^k_{o_k+1}), B\dot{v}_k(a^k_{o_k+1})\big) \to (x_1^r, 0).$$

□

6.4. Appendix: existence and uniqueness of the heteroclinic solution

For sake of completeness and since we could not find in the literature a satisfying proof, we state and prove here a result of existence and uniqueness, up to translations, of the solution of a first order autonomous system, issuing from a zero of the vector field where suitable transversality conditions are satisfied.

PROPOSITION 6.27. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a C^2 function such that $F(\eta) = 0$. Let the following two conditions be satisfied:*

(i) 0 is an eigenvalue of $\nabla F(\eta)$ with algebraic multiplicity 1. This implies that there exist ω, $\nu \in \mathbb{R}^m$ such that $\omega{\cdot}\nu \neq 0$, $\ker \nabla F(\eta) = \mathrm{span}(\omega)$, and $\ker \nabla F(\eta)^T = \mathrm{span}(\nu)$. Moreover, suppose that $\mathrm{Re}(\lambda) < 0$ for every eigenvalue $\lambda \neq 0$;

(ii) $\nu{\cdot}D^2F(\eta)[\omega,\omega] \neq 0$.

Excluding the constant solution η, there are infinitely many solutions to the problem

$$\begin{cases} \dot{x}(t) = F(x(t)), & t \le 0 \\ \lim_{t\to-\infty} x(t) = \eta, \end{cases} \tag{6.172}$$

and they differ from each other by time–translations.

From assumption (i) of Proposition 6.27 it descends that $\mathbb{R}^m$ can be decomposed as

$$\mathbb{R}^m = X_1 \oplus X_2, \qquad \text{with} \quad X_1 := \mathrm{span}(\omega) \quad \text{and} \quad X_2 := \{\nu\}^\perp.$$

We denote by π_i the projection on X_i, $i = 1, 2$, so that every $x \in \mathbb{R}^m$ can be uniquely written as $x = x_1 + x_2$, where $x_i = \pi_i(x)$. Observe that

$$\pi_1(x) = x_\omega \omega, \qquad \text{where} \quad x_\omega := \frac{x{\cdot}\nu}{\omega{\cdot}\nu}.$$

For every $\mathbb{R}^m$–valued function g, we use the notation

$$g_\omega := (g(\cdot))_\omega, \qquad g_i := (g(\cdot))_i, \quad i = 1, 2.$$

To further simplify the notation, we write A in place of $\nabla F(\eta)$ and denote by β the *spectral gap* of A, that is

$$\beta := \min\{|\mathrm{Re}(\lambda)| \,:\, \lambda \text{ is eigenvalue of } A \text{ and } \mathrm{Re}(\lambda) \neq 0\}.$$

It is well known that for every $\varepsilon \in (0, \beta)$, there exists $C_\varepsilon > 0$ (also depending on A) such that the following fundamental estimates hold for every $x \in \mathbb{R}^m$:

$$\left|e^{tA}\pi_1(x)\right| \le C_\varepsilon e^{\varepsilon|t|}|x|, \qquad \text{for every } t \in \mathbb{R}, \tag{6.173}$$

$$\left|e^{tA}\pi_2(x)\right| \le C_\varepsilon e^{-(\beta-\varepsilon)t}|x|, \qquad \text{for every } t \ge 0. \tag{6.174}$$

Remember that both π_1 and π_2 commute with A and hence with e^{tA}. The proof of Proposition 6.27 requires the following lemma.

LEMMA 6.28. *Under the same assumptions of Proposition 6.27, for every $a > 0$ sufficiently large there exists a unique solution of the problem*

$$\begin{cases} \dot{x}(t) = F(x(t)), & t \le 0 \\ x_\omega(0) = \eta_\omega + \frac{1}{a}, \\ \lim_{t\to-\infty} x(t) = \eta, \end{cases} \tag{6.175}$$

in the space

$$Y^a := \left\{ y : (-\infty, 0] \to \mathbb{R}^m \;:\; \|y_1\|_{Y_1^a} < \infty,\ \|y_2\|_{Y_2^a} < \infty \right\},$$

where

$$\|y_1\|_{Y_1^a} := \sup_{t\le 0} |t-a||y_1(t)|, \qquad \|y_2\|_{Y_2^a} := \sup_{t\le 0} |t-a|^{\frac{3}{2}}|y_2(t)|.$$

We remark that, throughout, the symbol $|\cdot|$ denotes the euclidean norm in the appropriate finite dimensional euclidean space.

PROOF. First, observe that Y^a is a Banach space with the norm

$$\|y\|_{Y^a} := \|y_1\|_{Y_1^a} + \|y_2\|_{Y_2^a}.$$

Note that we can suppose $\eta = 0$ and $|\omega|$, $|\nu| = 1$. Also, we can suppose

$$\nu\cdot D^2F(0)[\omega,\omega] = 2(\omega\cdot\nu). \tag{6.176}$$

Using the expansion

$$F(x(t)) = \nabla F(0)x(t) + \frac{1}{2}D^2F(0)[x(t)]^2 + o(|x(t)|^2),$$

where $D^2F(0)[x(t)]^2$ stands for $D^2F(0)[x(t),x(t)]$, by assumptions (i) and (ii) and by (6.176), $F_\omega(x(t))$ has the following expression:

$$F_\omega(x(t)) = x_\omega^2(t) + \frac{\nu}{\omega\cdot\nu}\cdot\left\{D^2F(0)[x_1(t),x_2(t)] + \frac{1}{2}D^2F(0)[x_2(t)]^2 + o(|x(t)|^2)\right\}. \tag{6.177}$$

By assumption (i) and since $\mathrm{Rank}(\nabla F(0)) \subseteq X_2$, we have that

$$F_2(x(t)) = \nabla F(0)x_2(t) + \pi_2\left(\frac{1}{2}D^2F(0)[x(t)]^2 + o(|x(t)|^2)\right). \tag{6.178}$$

For $y \in Y^a$, with $a > 0$ to be chosen, we define for $t \le 0$ the following functions:

$$h_1^y(t) := F_\omega(y(t)) - y_\omega^2(t), \tag{6.179}$$

$$h_2^y(t) := F_2(y(t)) - \nabla F(0)y_2(t). \tag{6.180}$$

Let y vary in $B_R := \{y \in Y^a : \|y\|_{Y^a} \le R\}$ for a certain $R > 0$ to be chosen later and observe that from the definition of $\|\cdot\|_{Y^a}$ easily follows that $\|y\|_{Y^a} \ge \min\{a, a^{\frac{3}{2}}\}\sup_{t\le 0}|y(t)|$. Therefore, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, if

$$\sup_{t\le 0}|y(t)| \le \frac{R}{\min\{a, a^{\frac{3}{2}}\}} \le \delta(\varepsilon), \tag{6.181}$$

the following estimates, which descend from (6.177) and (6.179) and from (6.178) and (6.180), respectively, hold for every $t \le 0$:

$$\begin{aligned}
|h_1^y(t)| &\le \frac{1}{|\omega\cdot\nu|}\left[|\nabla^2F(0)|\left(|y_1(t)||y_2(t)| + \frac{|y_2(t)|^2}{2}\right) + \varepsilon|y(t)|^2\right] \\
&\le \frac{R^2}{(t-a)^2}\frac{1}{|\omega\cdot\nu|}\left[|\nabla^2F(0)|\left(\frac{1}{\sqrt{a}} + \frac{1}{2a}\right) + 2\varepsilon\left(1 + \frac{1}{a}\right)\right]; \\
|h_2^y(t)| &\le \frac{1}{2}|\nabla^2F(0)||y(t)|^2 + \varepsilon|y(t)|^2 \\
&\le \frac{R^2}{|t-a|^{\frac{3}{2}}}(|\nabla^2F(0)| + 2\varepsilon)\left(\frac{1}{\sqrt{a}} + \frac{1}{a^{\frac{3}{2}}}\right).
\end{aligned}$$

We can briefly write, for $t \le 0$,

$$|h_1^y(t)| \le \frac{R^2}{(t-a)^2}M(a,\varepsilon), \qquad \text{with } M(a,\varepsilon) \to 0, \text{ as } a \to +\infty, \ \varepsilon \to 0^+, \tag{6.182}$$

$$|h_2^y(t)| \le \frac{R^2}{|t-a|^{\frac{3}{2}}}\tilde{M}(a,\varepsilon), \qquad \text{with } \tilde{M}(a,\varepsilon) \to 0, \text{ as } a \to +\infty. \tag{6.183}$$

We consider the auxiliary problems

$$\begin{cases} \dot{x}_\omega(t) - x_\omega^2(t) = h_1^y(t), \quad t \le 0 \\ x_\omega(0) = \dfrac{1}{a}, \end{cases} \tag{6.184}$$

$$\begin{cases} \dot{x}_2(t) - \nabla F(0) x_2(t) = h_2^y(t), \quad t \le 0 \\ \lim\limits_{t \to -\infty} x_2(t) = 0. \end{cases} \tag{6.185}$$

We are going to prove, in Step 1 and Step 2, that problems (6.184) and (6.185) have unique solutions and that the solution of problem (6.184) tends to 0 as $t \to -\infty$. Therefore, if $x = y$, problems (6.184)–(6.185) are equivalent to (6.175).

Step 1. If $y \in B_R$ and (6.181) holds, (6.182) implies that there exists a real–valued function H_1^y with $\sup_{t \le 0} |H_1^y(t)| \le R^2 M(a, \varepsilon)$ such that

$$h_1^y(t) = \frac{H_1^y}{(t-a)^2}, \qquad \text{for every } t \le 0.$$

Now, by observing that the equation in (6.184) is a particular Riccati equation and by setting $x_\omega = \frac{u}{(t-a)}$, we have that problem (6.184) is equivalent to

$$\begin{cases} \dot{u}(t) = \dfrac{u^2(t) + u(t) + H_1^y(t)}{(t-a)}, \quad t \le 0 \\ u(0) = -1, \end{cases} \tag{6.186}$$

Let w be the solution of (6.186) with $-R^2 M(a, \varepsilon)$ in place of H_1^y and v the solution of (6.186) with $R^2 M(a, \varepsilon)$ in place of H_1^y. It is easy to check that, if

$$M(a, \varepsilon) < \frac{1}{4R^2}, \tag{6.187}$$

then $\frac{-1-\sqrt{1+4R^2 M(a,\varepsilon)}}{2} < w \le -1 \le v < \frac{-1-\sqrt{1-4R^2 M(a,\varepsilon)}}{2}$. Therefore, by differential inequalities (see, e.g., [**43**]), we obtain that for every $t \le 0$

$$\frac{-1-\sqrt{1+4R^2 M(a,\varepsilon)}}{2} < u(t) = (t-a) x_\omega(t) < \frac{-1-\sqrt{1-4R^2 M(a,\varepsilon)}}{2}, \tag{6.188}$$

and in turn, from (6.187), that

$$\sup_{t \le 0} |t-a||x_\omega(t)| < \frac{1+\sqrt{1+4R^2 M(a,\varepsilon)}}{2} < \frac{1+\sqrt{2}}{2}.$$

Step 2. By the variation of constants formula, we can write a solution of the equation in (6.185) as

$$x_2(t) = e^{(t-t_0)\nabla F(0)} x_2(t_0) + \int_{t_0}^t e^{(t-\tau)\nabla F(0)} h_2^y(\tau) d\tau.$$

By using (6.174), we have that there exists a constant $C_\beta > 0$, depending on the spectral gap $\beta > 0$ of $\nabla F(0)$, such that

$$\lim_{t_0 \to -\infty} \left| e^{(t-t_0)\nabla F(0)} x_2(t_0) \right| \le \lim_{t_0 \to -\infty} C_\beta e^{-\frac{\beta}{2}(t-t_0)} |x_2(t_0)| = 0.$$

Therefore, the solution to problem (6.185) is

$$x_2(t) = \int_{-\infty}^t e^{(t-\tau)\nabla F(0)} h_2^y(\tau) d\tau, \qquad \text{for every } t \le 0.$$

Now, if $y \in B_R$ and (6.181) holds, it is easy to check, by using (6.174) and (6.183), that

$$\|x_2\|_{Y_2^a} \le \frac{2}{\beta} C_\beta R^2 \tilde{M}(a, \varepsilon).$$

Observe that $\|x_2\|_{Y_2^a} \leq \frac{R}{2}$ if

$$\tilde{M}(a,\varepsilon) \leq \frac{\beta}{4C_\beta R}. \tag{6.189}$$

From Step 1 and Step 2 we have obtained that, if

$$R := 1 + \sqrt{2},$$

and a is large enough and ε small enough such that (6.181), (6.187) and (6.189) are satisfied, then the operator

$$\Lambda : B_R \to B_R$$

which associates to $y \in B_R$ the function $x = x_\omega \omega + x_2$, with x_ω and x_2 the solutions of (6.184) and (6.185) respectively, is well defined.

To conclude the proof, it remains to show that Λ is a contraction. Given y, $y^* \in B_R$, set

$$\Lambda(y) = x = x_1 + x_2, \qquad \Lambda(y^*) = x^* = x_1^* + x_2^*.$$

Let us handle the first component and the second one separately, by proceeding in two steps. The following estimates can be obtained similarly to (6.182) and (6.183). They hold if $R/\min\{a, a^{\frac{3}{2}}\} \leq \tilde{\delta}(\varepsilon)$ for some $\tilde{\delta}(\varepsilon) > 0$ which can be supposed to be equal to $\delta(\varepsilon)$ (see (6.181)).

$$|h_1^y(t) - h_1^{y^*}(t)| \leq \frac{R}{(t-a)^2} N(a,\varepsilon) \|y - y^*\|_{Y^a}, \qquad \text{for every } t \leq 0, \tag{6.190}$$

where

$$N(a,\varepsilon) := \frac{1}{|\omega\cdot\nu|}\left[|\nabla^2 F(0)|\left(\frac{1}{\sqrt{a}} + \frac{1}{a}\right) + 4\varepsilon\left(1 + \frac{1}{a}\right)\right],$$

and

$$|h_2^y(t) - h_2^{y^*}(t)| \leq \frac{R}{|t-a|^{\frac{3}{2}}} \tilde{N}(a,\varepsilon) \|y - y^*\|_{Y^a}, \qquad \text{for every } t \leq 0, \tag{6.191}$$

where

$$\tilde{N}(a,\varepsilon) := \frac{1}{\sqrt{a}}\left[|\nabla^2 F(0)|\left(1 + \frac{1}{a} + \frac{2}{a}\right) + 4\varepsilon\left(1 + \frac{1}{a}\right)\right].$$

Step 3. As already done in Step 1, let us set $x_\omega = \frac{u}{(t-a)}$ and $x_\omega^* = \frac{u^*}{(t-a)}$. From (6.188) we deduce that, for a large enough and ε small enough,

$$\alpha := u + u^* \quad \text{ is such that } \quad 1 + \alpha(t) < -\frac{1}{2}, \quad \text{for every } t \leq 0. \tag{6.192}$$

Observe that the function $z := u - u^*$ satisfies the equation

$$\dot{z}(t) = \frac{1}{(t-a)}\left\{[1 + \alpha(t)]z(t) + H_1^y(t) - H_1^{y^*}(t)\right\}, \qquad \text{for every } t \leq 0,$$

and the condition $z(0) = 0$. Therefore, by the variation of constants formula, $z(t)$ satisfies the following estimate for every $t \leq 0$:

$$|z(t)| \leq \sup_{s\leq 0} |H_1^y(s) - H_1^{y^*}(s)| \int_t^0 \frac{\exp\left(\int_\tau^t \frac{1+\alpha(s)}{s-a})ds\right)}{a-\tau} d\tau. \tag{6.193}$$

From (6.192), it turns out that

$$\int_t^0 \frac{\exp\left(\int_\tau^t \frac{1+\alpha(s)}{s-a})ds\right)}{a-\tau} d\tau \leq \int_t^0 \frac{\exp\left(-\frac{1}{2}\int_t^\tau \frac{ds}{a-s}\right)}{a-\tau} d\tau$$
$$= |t-a|^{-\frac{1}{2}} \int_t^0 (a-\tau)^{-\frac{1}{2}} d\tau \leq 2. \tag{6.194}$$

Thus, since $\sup_{t\le 0}|H_1^y(t)-H_1^{y^*}(t)| = \sup_{t\le 0}(t-a)^2|h_1^y(t)-h_1^{y^*}(t)|$, from (6.193), (6.194) and (6.190) we obtain that

$$\|x_1 - x_1^*\|_{Y_1^a} = \sup_{t\le 0}|u(t)-u^*(t)| \le 2RN(a,\varepsilon)\|y-y^*\|_{Y^a}. \tag{6.195}$$

Step 4. Since

$$|x_2(t)-x_2^*(t)| \le C_\beta \int_{-\infty}^t e^{-\frac{\beta}{2}(t-\tau)}|h_2^y(\tau)-h_2^{y^*}(\tau)|d\tau,$$

from (6.191) we have that

$$\begin{aligned}\|x_2 - x_2^*\|_{Y_2^a} &\le C_\beta R\tilde{N}(a,\varepsilon)\|y-y^*\|_{Y^a}\int_{-\infty}^t \frac{e^{-\frac{\beta}{2}(t-\tau)}}{|t-a|^{\frac{3}{2}}}d\tau \\ &\le \frac{2}{\beta}C_\beta R\tilde{N}(a,\varepsilon)\|y-y^*\|_{Y^a}. \end{aligned}\tag{6.196}$$

Finally, if we choose $a>0$ sufficiently large and $\varepsilon>0$ sufficiently small such that, for $R=1+\sqrt{2}$, (6.181), (6.187), (6.189) and (6.192) hold together with

$$N(a,\varepsilon)\le \frac{1}{8R}, \qquad \tilde{N}(a,\varepsilon)\le \frac{\beta}{8C_\beta R},$$

from (6.195) and (6.196) we obtain that

$$\|\Lambda(y)-\Lambda(y^*)\|_{Y^a} = \|x_1-x_1^*\|_{Y_1^a} + \|x_2-x_2^*\|_{Y_2^a} \le \frac{1}{2}\|y-y^*\|_{Y^a},$$

that is Λ is a contraction from B_R to B_R. □

Proof of Proposition 6.27. The existence of a solution of problem (6.172), different from the constant η, is given by Lemma 6.28. It remains to show the uniqueness of such a solution, up to time–translations. Clearly, we can suppose $\eta=0$ in (6.172). The idea is to show that for every solution x to problem (6.172), different from the constant solution, there exists a sequence $t_n\to-\infty$ such that $x_\omega(t_n)>0$ (and $x_\omega(t_n)\to 0$). In this way, it is possible to prove that the projections of the trajectories on X_1 intersect, and conclude by using Lemma 6.28.

Let x be a solution of (6.172). As shown in Lemma 6.28, the system $\dot{x}(t)=F(x(t))$ is equivalent to

$$\dot{x}_\omega(t) = x_\omega^2(t) + \frac{\nu}{\omega\cdot\nu}\cdot\left\{D^2F(0)[x_1(t),x_2(t)] + \frac{1}{2}D^2F(0)[x_2(t)]^2 + o(|x(t)|^2)\right\}, \tag{6.197}$$

$$\dot{x}_2(t) = \nabla F(0)x_2(t) + h_2^x(t), \tag{6.198}$$

where $h_2^x(t) := \pi_2\left(\frac{1}{2}D^2F(0)[x(t)]^2 + o(|x(t)|^2)\right)$, for every $t\le 0$. Observe that for every $\delta>0$ small enough, if $\sup_{t\le 0}|x(t)|\le\delta$, then

$$|h_2^x(t)| \le \frac{1}{2}(|\nabla^2F(0)|+1)|x(t)|^2, \qquad \text{for every } t\le 0. \tag{6.199}$$

Since $x(t)\to 0$ as $t\to-\infty$, there exists $t_0=t_0(\delta)$ such that $|x(t)|\le\delta$ for every $t\le t_0$. Therefore, up to change x with $y(t):=x(t+t_0)$, we can suppose $t_0=0$ and then $\sup_{t\le 0}|x(t)|\le\delta$. This assumption, together with (6.199), gives that

$$|h_2^x(t)| \le (|\nabla^2F(0)|+1)(|x_\omega(t)|^2+\delta|x_2(t)|), \qquad \text{for every } t\le 0. \tag{6.200}$$

From equation (6.198) and estimates (6.174) and (6.200), we obtain the following inequalities for every $t \leq \hat{t} \leq 0$:

$$\begin{aligned} |x_2(t)| &= \left| \int_{-\infty}^{t} e^{(t-\tau)\nabla F(0)} h_2^x(\tau) d\tau \right| \\ &\leq C_\beta \int_{-\infty}^{t} e^{-\frac{\beta}{2}(t-\tau)} |h_2^x(\tau)| d\tau \\ &\leq \frac{2}{\beta} C_\beta (|\nabla^2 F(0)| + 1) \left(\sup_{\tau \leq \hat{t}} |x_\omega(\tau)|^2 + \delta \sup_{\tau \leq \hat{t}} |x_2(\tau)| \right). \end{aligned}$$

Choosing δ such that $\delta \frac{2}{\beta} C_\beta (|\nabla^2 F(0)| + 1) < 1$, we obtain that

$$\sup_{\tau \leq \hat{t}} |x_2(\tau)| \leq K \sup_{\tau \leq \hat{t}} x_\omega^2(\tau), \qquad \text{for every } \hat{t} \leq 0, \tag{6.201}$$

for some $K > 0$. Now, note that it is possible to construct a sequence $\{t_n\}$ such that $t_n \to -\infty$ as $n \to \infty$ and

$$|x_\omega(t_n)| = \max_{t \leq t_n} |x_\omega(t)|. \tag{6.202}$$

Thus, from (6.201), we have that

$$|x_2(t_n)| \leq K x_\omega^2(t_n), \qquad \text{for every } n.$$

From this inequality and from (6.197), up to a smaller δ depending on some $\varepsilon > 0$ such that $\frac{2\varepsilon}{|\omega \cdot \nu|} < 1$, it descends that

$$\begin{aligned} \dot{x}_\omega(t_n) &\geq x_\omega^2(t_n) - \frac{1}{|\omega \cdot \nu|} \left[|\nabla^2 F(0)| \left(|x_\omega(t_n)||x_2(t_n)| + \frac{|x_2(t_n)|^2}{2} \right) + \varepsilon |x(t_n)|^2 \right] \\ &\geq \left(1 - \frac{2\varepsilon}{|\omega \cdot \nu|} \right) x_\omega^2(t_n) - K \frac{|x_\omega(t_n)|^3}{|\omega \cdot \nu|} \left[|\nabla^2 F(0)| \left(1 + \frac{K}{2} |x_\omega(t_n)| \right) + 2\varepsilon K |x_\omega(t_n)| \right]. \end{aligned} \tag{6.203}$$

Now, if $x_\omega(t_n) = 0$ for some n, then $x \equiv 0$, in view of (6.202) and (6.201). Otherwise, from (6.203) we have that $\dot{x}_\omega(t_n) > 0$ for every n, and this implies, from the definition of t_n, that

$$x_\omega(t_n) > 0, \qquad \text{for every } n. \tag{6.204}$$

Let x and x^* be solutions of (6.172) (with $\eta = 0$). The above argument show that (6.204) hold for x_ω and x_ω^* on some sequences $\{t_n\}$ and $\{t_n^*\}$, respectively. We conclude by considering two cases:

(i) if there exist n and m such that $x_\omega(t_n) = x_\omega^*(t_m^*)$, we define $y(t) := x(t + t_n)$ and $y^*(t) := x^*(t + t_m^*)$. y and y^* satisfy problem (6.175) (with $\eta = 0$) with $a = \frac{1}{x_\omega(t_n)}$ sufficiently large. Therefore, y and y^* coincide and, in turn, x and x^* coincide up to time–translations, in view of Lemma 6.28.

(ii) if $x_\omega(t_n) \neq x_\omega^*(t_m^*)$ for every n and m, there exist n and $k > m$ such that $x_\omega(t_k) < x_\omega^*(t_n^*) < x_\omega(t_m)$. Thus, there exists $\bar{t} \in (t_k, t_m)$ such that $x_\omega(\bar{t}) = x_\omega^*(t_n^*)$. By defining $y(t) := x(t + \bar{t})$ and $y^*(t) := x^*(t + t_n^*)$, we conclude as in (i).

□

CHAPTER 7

Singular perturbations of gradient flow problems in infinite dimension

In this chapter, we present some partial results obtained in collaboration with G. Savaré and R. Rossi. We want to study the limit behavior, as ε goes to zero, of a solution to the gradient flow

$$\varepsilon \dot{u}_\varepsilon(t) + \nabla_x \mathcal{E}(t, u_\varepsilon(t)) = 0, \tag{7.1}$$

where $\mathcal{E}$ is an energy functional defined on $[0,T] \times X$. Here, X is a Hilbert space and $\nabla_x \mathcal{E}$ is the gradient of $\mathcal{E}$ with respect to the variable $x \in X$. Note that equation (7.1) can be viewed as a singular perturbation of the evolution problem

$$\dot{u}(t) + \nabla_x \mathcal{E}(t, u(t)) = 0.$$

Consider a vanishing sequence $\{\varepsilon_n\}$. Under very weak hypotheses on the energy functional (see Section 7.1), in Section 7.2 we perform a detailed analysis, for every $t \in [0,T]$ and all sequences $t_1^n \leq t_2^n$ converging to t, of the integral quantities

$$\int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, u_{\varepsilon_n}(r))\| \|\dot{u}_{\varepsilon_n}(r)\| dr.$$

We show that these integrals are bounded below by a strictly positive *cost function* $c(t; x_1, x_2)$, whenever $u_{\varepsilon_n}(t_1^n)$ and $u_{\varepsilon_n}(t_2^n)$ converge to two different critical points x_1 and x_2 of $\mathcal{E}(t,\cdot)$. Some key properties of the cost function are listed and proved in Section 7.3. Taking advantage of these properties, we are able to prove in Section 7.4 a.e. convergence on $[0,T]$ of a family $\{u_{\varepsilon_n}\}$ of solutions to (7.1). Moreover, we show that the limit solution u is continuous on $[0,T] \setminus J$, where the jump set J is a numerable set, and satisfies $\nabla_x \mathcal{E}(t, u(t)) = 0$ for a.e. $t \in (0,T)$. Finally, the left and the right limits $u_-(t)$, $u_+(t)$ exist everywhere and are such that

$$\mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) = c(t; u_-(t), u_+(t)), \qquad \text{for every } t \in J. \tag{7.2}$$

7.1. Setting of the problem

To simplify the notation, throughout this chapter we suppose that X is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and with the associated norm $\|\cdot\|$. All the results we are going to show can be easily extended to the case where X is a separable Banach space endowed with the Radon–Nikodym property (see [**33**, Chapter 3]).

We consider an energy functional $\mathcal{E} : [0,T] \times X \to \mathbb{R}$ satisfying the following properties.

For every $t \in [0,T]$, the map $\mathcal{E}(t,\cdot)$ is lower semicontinuous and Gâteaux–differentiable. $\quad (\mathrm{E}_0)$

We recall that the Gâteaux–differentiability is not a sufficient condition to guarantee semicontinuity, even in the case where X has finite dimension. The gradient of $\mathcal{E}(t,\cdot)$ at $x \in X$, denoted by $\nabla_x \mathcal{E}(t,x)$, is the element of X such that

$$\langle \nabla_x \mathcal{E}(t,x), h \rangle = \langle d_G \mathcal{E}(t,x), h \rangle_*,$$

where $d_G \mathcal{E}(t,x)$ is the Gâteaux–differential of $\mathcal{E}(t,\cdot)$ at x, and $\langle \cdot, \cdot \rangle_*$ is the duality pairing between X^* and X.

Hereafter, we will use the function

$$\mathcal{G}(x) := \sup_{t\in[0,T]} \mathcal{E}(t,x), \qquad \text{for every } x \in X.$$

Coercivity:

$$\text{the function } \mathcal{G} \text{ has compact sublevels.} \tag{E$_1$}$$

Note that, by assumptions (E_0) and (E_1), up to additive constants, we can suppose that

$$\mathcal{E}(t,x) \geq 0, \qquad \text{for every } (t,x) \in [0,T]\times X.$$

Time-dependence:

$$\begin{aligned}&\text{for every } x\in X, \text{ the function } \mathcal{E}(\cdot,x) \text{ is differentiable on } (0,T) \text{ with derivative } \partial_t\mathcal{E}(\cdot,x);\\ &\text{there exists } C_1>0 \text{ such that } |\partial_t\mathcal{E}(t,x)| \leq C_1\mathcal{E}(t,x), \text{ for every } (t,x)\in[0,T]\times X.\end{aligned} \tag{E$_2$}$$

REMARK 7.1. Using the Gronwall Lemma, it is not difficult to deduce from (E_2) that

$$\mathcal{G}(x) \leq \exp(C_1 T) \inf_{t\in[0,T]} \mathcal{E}(t,x), \qquad \text{for every } x\in X. \tag{7.3}$$

Closedness: for every $t\in[0,T]$ and every sequence $\{x_n\}_n \subseteq X$, we have the following condition:

$$\begin{aligned}&\text{if } (t_n,x_n)\to(t,x) \text{ in } [0,T]\times X, \quad \sup_n \|\nabla_x\mathcal{E}(t_n,x_n)\| \leq C, \quad \mathcal{E}(t_n,x_n)\to\mathscr{E}<\infty,\\ &\text{then} \quad \liminf_n \|\nabla_x\mathcal{E}(t_n,x_n)\| \geq \|\nabla_x\mathcal{E}(t,x)\|, \quad \limsup_n \partial_t\mathcal{E}(t_n,x_n) \leq \partial_t\mathcal{E}(t,x), \quad \mathscr{E}=\mathcal{E}(t,x).\end{aligned} \tag{E$_3$}$$

Note that in the first line of (E_3) we can replace $x_n\to x$ with $x_n\rightharpoonup x$. Indeed, condition $\mathcal{E}(t_n,x_n)\to\mathscr{E}<\infty$ implies that, up to subsequences, $\sup_n \mathcal{G}(x_n)<\infty$ (see Remark 7.1) and in turn implies strong convergence, in view of (E_1).

Chain rule: For every $u\in \mathrm{AC}([0,T];X)$ such that

$$\sup_{t\in[0,T]} \mathcal{E}(t,u(t)) < +\infty, \qquad \int_0^T \|\nabla_x\mathcal{E}(t,u(t))\|\|\dot{u}(t)\|dt < +\infty, \tag{7.4}$$

the map $t\mapsto\mathcal{E}(t,u(t))$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) = \langle\nabla_x\mathcal{E}(t,u(t)),\dot{u}(t)\rangle + \partial_t\mathcal{E}(t,u(t)), \qquad \text{for a.e. } t\in(0,T)\,. \tag{E$_4$}$$

Note that in the case where $\mathcal{E}\in C^1((0,T)\times X;\mathbb{R})$, conditions ($E_3$) and ($E_4$) are trivially satisfied. We will follow the same terminology used in Chapter 6: $x\in X$ is a *critical* point of $\mathcal{E}(t,\cdot)$ if $\nabla_x\mathcal{E}(t,x)=0$. We will also use the notation, for any $t\in[0,T]$,

$$\mathcal{C}(t) := \{x\in X \,:\, \nabla_x\mathcal{E}(t,x)=0\}. \tag{7.5}$$

Our last assumption on $\mathcal{E}$ is that

$$\text{for every } t\in[0,T], \text{ the set } \mathcal{C}(t) \text{ is discrete.} \tag{E$_5$}$$

7.2. Properties of integral functionals

In what follows, for a fixed $t\in[0,T]$, we consider two sequences $\{t_1^n\}_n$, $\{t_2^n\}_n$ fulfilling

$$0\leq t_1^n \leq t_2^n \leq T, \quad \text{for every } n\in\mathbb{N}, \qquad t_1^n, t_2^n \to t, \quad \text{as } n\to\infty, \tag{7.6}$$

and a sequence $\{\vartheta_n\}_n \subseteq \mathrm{AC}([t_1^n,t_2^n];X)$ such that

$$\vartheta_n(t_1^n)\to x_1, \qquad \vartheta_n(t_2^n)\to x_2, \qquad \sup_n \sup_{r\in[t_1^n,t_2^n]} \mathcal{G}(\vartheta_n(r)) < \infty, \tag{7.7}$$

for some x_1, $x_2\in X$. We have the following preliminary lemma.

LEMMA 7.2. *Assume* (E_0), (E_1), (E_3), *and* (E_5). *If* $x_1 \neq x_2$, *there exists* $\eta = \eta(t, x_1, x_2) > 0$ *such that*

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr \geq \eta. \tag{7.8}$$

PROOF. It follows from (7.7) and from assumption (E_1) that there exists a compact sublevel K of the functional $\mathcal{G}$ such that $\vartheta_n(r) \in K$ for every $r \in [t_1^n, t_2^n]$, $n \in \mathbb{N}$. Assumption (E_5) implies that the set $\mathcal{C}(t) \cap K$ is finite and there exists $\hat{\rho} = \hat{\rho}(t, x_1, x_2)$ such that

$$B(x, 2\hat{\rho}) \cap B(y, 2\hat{\rho}) = \emptyset, \qquad \text{for every } x, y \in \mathcal{C}(t) \cup \{x_1, x_2\} \text{ with } x \neq y.$$

Note that it may happen $x_1 \in \mathcal{C}(t)$ or not, the same for x_2. Let us introduce the compact set $\hat{K} = \hat{K}(t, x_1, x_2)$ defined by

$$\hat{K} := K \setminus \bigcup_{x \in \mathcal{C}(t) \cup \{x_1, x_2\}} B(x, \hat{\rho})$$

and observe that $\min_{x \in \hat{K}} \|\nabla_x \mathcal{E}(t, x)\| > 0$. It follows from Lemma 7.3 below that for some $\alpha = \alpha(t, x_1, x_2) > 0$

$$\hat{\eta} := \min_{x \in \hat{K}, r \in [t-\alpha, t+\alpha]} \|\nabla_x \mathcal{E}(r, x)\| > 0. \tag{7.9}$$

Moreover, from (7.7) and from the definition of $\hat{K}$ we obtain that $\{r \in [t_1^n, t_2^n] : \vartheta_n(r) \in \hat{K}\} \neq \emptyset$ for every n large enough, and that $\vartheta_n(r_1) \in \partial B(x_1, \hat{\rho})$, $\vartheta_n(r_2) \in \partial B(x_2, \hat{\rho})$ for some $r_1 \neq r_2$ in the set $\{r \in [t_1^n, t_2^n] : \vartheta_n(r) \in \hat{K}\}$. Thus, by (7.9), since $[t_1^n, t_2^n] \subseteq [t-\alpha, t+\alpha]$ for every n sufficiently large,

$$\begin{aligned}
\int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr &\geq \int_{\{r \in [t_1^n, t_2^n] : \vartheta_n(r) \in \hat{K}\}} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr \\
&\geq \hat{\eta} \int_{\{r \in [t_1^n, t_2^n] : \vartheta_n(r) \in \hat{K}\}} \|\dot{\vartheta}_n(r)\| dr, \\
&\geq \hat{\eta} \min_{x, y \in \mathcal{C}(t) \cup \{x_1, x_2\}} (|x - y| - 2\hat{\rho}).
\end{aligned} \tag{7.10}$$

Thus, (7.8) holds with η defined as the right hand side of (7.10). Observe that η is positive in view of (7.9) and of the definition of $\hat{\rho}$, and depends only on t if x_1, $x_2 \in \mathcal{C}(t)$. ☐

LEMMA 7.3. *Assume* (E_0), (E_1), *and* (E_3). *Let* $K \subseteq X$ *be a closed set such that*

$$K \subset \{x \in X : \mathcal{G}(x) \leq r\}, \qquad \textit{for some } r > 0, \tag{7.11}$$

and suppose that

$$\inf_{x \in K} \|\nabla_x \mathcal{E}(t, x)\| > 0, \qquad \textit{for some } t \in (0, T). \tag{7.12}$$

Then, the inf *in* (7.12) *is attained, and there exists* $\alpha = \alpha(K, t) > 0$ *such that*

$$\min_{x \in K, s \in [t-\alpha, t+\alpha]} \|\nabla_x \mathcal{E}(s, x)\| > 0\,. \tag{7.13}$$

PROOF. Since the closed set K fulfills (7.11), it follows from (E_1) that it is compact and from (E_3) that $\inf_{x \in K} \|\nabla_x \mathcal{E}(s, x)\|$ is attained for every $s \in [0, T]$. Moreover, the function $s \mapsto \min_{x \in K} \|\nabla_x \mathcal{E}(s, x)\|$ is lower semicontinuous. Combining this fact with (7.12), we conclude (7.13). ☐

A crucial result for the sequel is the following.

LEMMA 7.4. *Assume* (E_0), (E_1), (E_3), *and* (E_5). *Let* $t \in [0, T]$ *and* x_1, $x_2 \in X$ *be fixed. Let* $\{t_1^n\}_n$, $\{t_2^n\}_n$ *and* $\{\vartheta_n\}_n \subseteq \mathrm{AC}([t_1^n, t_2^n]; X)$ *fulfill* (7.6) *and* (7.7), *respectively. We have that*

(1) if

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr = 0, \tag{7.14}$$

then $x_1 = x_2$.

(2) If

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr > 0, \tag{7.15}$$

then there exist $-\infty \le a < b \le +\infty$ *and a curve*

$$\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X) := \Big\{ \vartheta \in \mathrm{C}([a,b];X) \,:\, \textit{there exist } j \in \mathbb{N} \textit{ and } a = \mathsf{t}_0 < \mathsf{t}_1 < \ldots < \mathsf{t}_j = b \ \textit{ s.t.} \\ \vartheta(a) = x_1,\ \vartheta(b) = x_2, \ \ \textit{for every} \ \ i = 0, \ldots, j-1 \\ \vartheta|_{(\mathsf{t}_i,\mathsf{t}_{i+1})} \in \mathrm{Lip}_{\mathrm{loc}}(\mathsf{t}_i, \mathsf{t}_{i+1}; X),\ \vartheta(\mathsf{t}_i) \ne \vartheta(\mathsf{t}_{i+1}), \\ \textit{and} \ \ \vartheta(\mathsf{t}_i) \in \mathcal{C}(t) \ \ \textit{for every} \ \ i = 1, \ldots, j-1 \Big\} \tag{7.16}$$

such that

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr \ge \int_a^b \|\nabla_x \mathcal{E}(t, \vartheta(s))\| \|\dot\vartheta(s)\| ds\,. \tag{7.17}$$

Proof. For $t \in [0,T]$ and $x_1,\ x_2 \in X$ fixed, let $\{t_1^n\}_n$, $\{t_2^n\}_n$ and $\{\vartheta_n\}_n$ be as in the statement.

Ad (1): assume (7.14). In view of Lemma 7.2, $x_1 = x_2$.

Ad (2): suppose that $x_1 \ne x_2$, and that the $\liminf$ in (7.15) is finite (otherwise (7.17) is trivial). By Lemma 7.2, we can suppose that, up to a subsequence, there exists

$$\lim_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr =: L_t > 0,$$

and

$$\sup_{n\in\mathbb{N}} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr \le C < \infty. \tag{7.18}$$

We split the proof in several steps.

Step 1: reparameterization. Let us define, for every $r \in [t_1^n, t_2^n]$,

$$s_n(r) := r + \int_{t_1^n}^{r} \|\nabla_x \mathcal{E}(\tau, \vartheta_n(\tau))\| \|\dot\vartheta_n(\tau)\| d\tau.$$

Also, we set

$$s_1^n := s_n(t_1^n) = t_1^n, \qquad\qquad s_2^n := s_n(t_2^n),$$

and note that

$$s_1^n \to t, \qquad\qquad s_2^n \to (t + L_t) > t.$$

Since $s_n' > 0$, we define

$$r_n(s) := s_n^{-1}(s) \qquad \text{and} \qquad \tilde\vartheta_n(s) := \vartheta_n(r_n(s)), \qquad \text{for every } s \in [s_1^n, s_2^n].$$

Observe that

$$\tilde\vartheta_n(s_1^n) \to x_1, \qquad\qquad \tilde\vartheta_n(s_2^n) \to x_2, \tag{7.19}$$

that the functions r_n's are equi–Lipschitz, and that

$$\|\nabla_x \mathcal{E}(r_n(s), \tilde\vartheta_n(s))\| \|\dot{\tilde\vartheta}_n(s)\| = 1 - \frac{1}{1 + \|\nabla_x \mathcal{E}(r_n(s), \tilde\vartheta_n(s))\| \|\dot\vartheta_n(r_n(s))\|} \le 1, \quad \text{for a.e. } s \in (s_1^n, s_2^n). \tag{7.20}$$

By using the rescaling map r_n, we obtain

$$\int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr = \int_{s_1^n}^{s_2^n} \|\nabla_x \mathcal{E}(r_n(s), \tilde{\vartheta}_n(s))\| \|\dot{\tilde{\vartheta}}_n(s)\| ds. \tag{7.21}$$

Step 2: localization and equicontinuity estimates. Assumptions (7.7) and (E_1) give that there exists a compact $K \subseteq X$ such that

$$\vartheta_n(t) \in K, \qquad \text{for every } t \in [t_1^n, t_2^n],\ n \in \mathbb{N}, \tag{7.22}$$

and that $\mathcal{C}(t) \cap K$ consists of a finite number of points, where $\mathcal{C}(t)$ is defined in (7.5). As in the proof of Lemma 7.2, we observe that there exists $\overline{\delta} = \overline{\delta}(t, x_1, x_2) > 0$ such that, for $0 < \delta \leq \overline{\delta}$,

$$B(x, 2\delta) \cap B(y, 2\delta) = \emptyset, \quad \text{for every } x, y \in \mathcal{C}(t) \cup \{x_1, x_2\} \text{ with } x \neq y. \tag{7.23}$$

We now introduce the compact set $K_\delta = K_\delta(t, x_1, x_2)$ defined by

$$K_\delta := K \setminus \bigcup_{x \in \mathcal{C}(t) \cup \{x_1, x_2\}} B(x, \delta) \tag{7.24}$$

and use Lemma 7.3 to obtain that

$$\min_{x \in K_\delta, s \in [t-\alpha, t+\alpha]} \|\nabla_x \mathcal{E}(s, x)\| =: e_\delta > 0, \tag{7.25}$$

for some $\alpha = \alpha(t, \delta, x_1, x_2) > 0$. Observe that $r_n(s) \in [t - \alpha, t + \alpha]$, for every $s \in (s_1^n, s_2^n)$ and every n large enough. Defining the open set

$$A_n^\delta := \left\{ s \in (s_1^n, s_2^n) \, : \, \tilde{\vartheta}_n(s) \in \mathring{K}_\delta \right\},$$

we obtain that $A_n^\delta \neq \emptyset$ for every n sufficiently large, in view of the definition of K_δ and of (7.19). We write A_n^δ as the countable union of its connected components

$$A_n^\delta = \bigcup_{k=1}^{\infty} (a_{n,k}^\delta, b_{n,k}^\delta), \tag{7.26}$$

where we suppose $b_{n,k}^\delta \leq a_{n,k+1}^\delta$, for every k. Inequality (7.20), the minimum (7.25), and the definition of $a_{n,k}^\delta$ and $b_{n,k}^\delta$ imply that

$$e_\delta \|\dot{\tilde{\vartheta}}_n(s)\| \leq \|\nabla_x \mathcal{E}(r_n(s), \tilde{\vartheta}_n(s))\| \|\dot{\tilde{\vartheta}}_n(s)\| \leq 1, \qquad \text{for a.e. } s \in \left(a_{n,k}^\delta, b_{n,k}^\delta\right). \tag{7.27}$$

Furthermore, it is clear that

$$\tilde{\vartheta}_n(a_{n,k}^\delta) \in \partial B(x, \delta), \quad \tilde{\vartheta}_n(b_{n,k}^\delta) \in \partial B(y, \delta), \qquad \text{for some } x, y \in \mathcal{C}(t) \cup \{x_1, x_2\}. \tag{7.28}$$

Note that it may happen $x = y$. In the following lines we show that there is a *finite* number of intervals $(a_{n,k}^\delta, b_{n,k}^\delta)$ bringing $\tilde{\vartheta}$ from one ball to a different one: this will allow us to find the function ϑ of the statement consisting of a finite number of $\mathrm{Lip}_{\mathrm{loc}}$–pieces. Thus, we focus on the cases where $x \neq y$ in (7.28) and introduce the set

$$\begin{gathered} B_n^\delta := \bigcup_{(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta} (a_{n,k}^\delta, b_{n,k}^\delta) \qquad \text{with} \\ \mathfrak{B}_n^\delta = \left\{ (a_{n,k}^\delta, b_{n,k}^\delta) \subseteq A_n^\delta \, : \, \tilde{\vartheta}_n(a_{n,k}^\delta) \in \partial B(x, \delta),\ \tilde{\vartheta}_n(b_{n,k}^\delta) \in \partial B(y, \delta),\ x \neq y \right\}. \end{gathered} \tag{7.29}$$

From (7.21), (7.27), and the definition of A_n^δ and B_n^δ, we obtain that there exists n_δ such that for every $n \geq n_\delta$

$$\begin{aligned}\int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr &\geq \int_{A_n^\delta} \|\nabla_x \mathcal{E}(r_n(s), \tilde{\vartheta}_n(s))\| \|\dot{\tilde{\vartheta}}_n(s)\| ds \\ &\geq e_\delta \int_{B_n^\delta} \|\dot{\tilde{\vartheta}}_n(s)\| ds \\ &\geq e_\delta \sum_{(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta} \min_{\substack{x,y \in \mathcal{C}(t) \cup \{x_1, x_2\} \\ x \neq y}} (|x-y| - 2\delta) \\ &\geq e_\delta \sum_{(a_{n,k}^\delta, b_{n,k}^\delta) \in \mathfrak{B}_n^\delta} \overline{m}, \end{aligned} \tag{7.30}$$

where $0 < \overline{m} := \min_{\substack{x,y \in \mathcal{C}(t) \cup \{x_1, x_2\} \\ x \neq y}} (|x-y| - 2\overline{\delta})$. Inequality (7.30), together with (7.18), implies that B_n^δ has a finite number $N(n, \delta)$ of components, more precisely

$$N(n, \delta) \leq \frac{C}{e_\delta \overline{m}}, \qquad \text{for every } 0 < \delta \leq \overline{\delta} \text{ and } n \geq n_\delta, \tag{7.31}$$

with C as in (7.18). In what follows, we will show that $N(n, \delta)$ has a uniform bound with respect to n and δ.

Step 3: compactness. We claim that

there exist a sequence $(n_j, \delta_{m_j})_j$, such that

$$N(n_j, \delta_{m_j}) = N, \qquad \textit{for every } j \in \mathbb{N}, \tag{7.32}$$

a partition

$$t \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_N < \beta_N \leq t + L_t \qquad \textit{of} \quad [t, t+L_t], \tag{7.33}$$

and $\vartheta \in \mathrm{Lip}_{\mathrm{loc}} \left(\bigcup_{k=1}^N (\alpha_k, \beta_k); X \right)$, such that, in the limit $j \to \infty$,

$$\begin{aligned} &\tilde{\vartheta}_{n_j} \to \vartheta \quad \textit{uniformly on the compact subsets of } \bigcup_{k=1}^N (\alpha_k, \beta_k), \\ &\dot{\tilde{\vartheta}}_{n_j} \rightharpoonup^* \dot{\vartheta} \quad \textit{in } L^\infty(\alpha_k + \rho, \beta_k - \rho), \quad \textit{for every } \rho > 0 \textit{ and } k = 1, \ldots, N. \end{aligned} \tag{7.34}$$

To prove this claim, let us observe first that, since $\{N(n, \delta)\}_{n \geq n_\delta}$ is a bounded sequence by (7.31), there exists a subsequence $\{n_l^\delta\}_l$ and an integer $N(\delta)$ such that

$$N(n_l^\delta, \delta) \to N(\delta), \qquad \text{as } l \to \infty. \tag{7.35}$$

Clearly, since $x_1 \neq x_2$, taking (7.19) into account we see that $N(\delta) \geq 1$ for every $0 < \delta \leq \overline{\delta}$. Also, we have that

$$N(n, \delta_1) \geq N(n, \delta_2), \qquad \text{if } \delta_1 \geq \delta_2, \tag{7.36}$$

for every $n \geq \max\{n_{\delta_1}, n_{\delta_2}\}$. Indeed, since $\delta_1 \geq \delta_2$, then $K_{\delta_1} \subseteq K_{\delta_2}$ and in turn $B_n^{\delta_1} \subseteq B_n^{\delta_2}$. This means that, for every $k \in \{1, ..., N(n, \delta_1)\}$,

$$(a_{n,k}^{\delta_1}, b_{n,k}^{\delta_1}) \subseteq (a_{n,j_k}^{\delta_2}, b_{n,j_k}^{\delta_2}), \qquad \text{for some } j_k \in \{1, ..., N(n, \delta_2)\}. \tag{7.37}$$

At the same time, $(a_{n,k}^{\delta_i}, b_{n,k}^{\delta_i}) \subseteq [s_1^n, s_2^n]$, for $i = 1, 2$, and $[s_1^n, s_2^n] \to [t, t + L_t]$, as $n \to \infty$. Therefore, there exists $\rho > 0$ such that $(a_{n,k}^{\delta_i}, b_{n,k}^{\delta_i}) \subseteq [t - \rho, t + L_t + \rho]$, for $i = 1, 2$. This fact, together with (7.37), gives (7.36).

In order to prove (7.32), we develop the following diagonal argument. Consider a sequence $\{\delta_m\}_m \subseteq (0, \overline{\delta}]$ such that $\delta_m \to 0$, as $m \to \infty$. Using (7.35) and (7.36), it is possible to construct for each $m \in \mathbb{N}$ a subsequence $\{n_l^m\}_l$, where n_l^m is a short–hand notation for $n_l^{\delta_m}$, such that

$$N(n_l^m, \delta_m) \to N(\delta_m), \qquad \text{as } l \to \infty, \quad \text{for every } m \in \mathbb{N}, \tag{7.38}$$

$$N(\delta_1) \geq N(\delta_2) \geq ... \geq N(\delta_m) \geq ... \geq 1, \tag{7.39}$$

and

$$a^{\delta_m}_{n^m_l,k} \to \alpha^m_k, \quad b^{\delta_m}_{n^m_l,k} \to \beta^m_k, \quad \text{as } l \to \infty, \quad \text{for all } k = 1, ..., N(\delta_m) \text{ and all } m \in \mathbb{N}. \tag{7.40}$$

Now, since $\{N(\delta_m)\}_m$ is a decreasing sequence of integers, up to a subsequence we may suppose that

$$\text{there exists } \ N \in \mathbb{N} \ \text{ such that } \ N(\delta_m) = N, \quad \text{for every } m \in \mathbb{N},$$

and that there exist the limits

$$\alpha_k := \lim_{m\to\infty} \alpha^m_k, \qquad \beta_k := \lim_{m\to\infty} \beta^m_k, \qquad \text{for every } k = 1, ..., N. \tag{7.41}$$

Observe that the points α_k, β_k satisfy (7.33). Now, choose $k \in \{1, ..., N\}$ and observe that for every $j \in \mathbb{N}$ arbitrarily large there exists m_j and l_j such that

$$\left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right] \subseteq \left(a^{\delta_{m_j}}_{n^{m_j}_l,k}, b^{\delta_{m_j}}_{n^{m_j}_l,k}\right), \qquad \text{for every } l \geq l_j, \tag{7.42}$$

and

$$\left|a^{\delta_{m_j}}_{n^{m_j}_l,k} - \alpha^{m_j}_k\right| + \left|b^{\delta_{m_j}}_{n^{m_j}_l,k} - \beta^{m_j}_k\right| \leq \frac{1}{m_j}, \qquad \text{for every } l \geq l_j. \tag{7.43}$$

Moreover, we can suppose that m_j, $l_j \to \infty$, as $j \to \infty$. Combining (7.42) with estimate (7.27), we have that, up to a subsequence,

$$\begin{aligned} &\tilde{\vartheta}_{n^{m_j}_l} \to \vartheta \quad \text{uniformly on} \quad \left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right], \quad \text{as } l \to \infty \\ &\dot{\tilde{\vartheta}}_{n^{m_j}_l} \rightharpoonup^* \dot{\vartheta} \quad \text{in } L^\infty\left(\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right), \quad \text{as } l \to \infty, \end{aligned} \tag{7.44}$$

for some $\vartheta \in \mathrm{Lip}\left(\left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right]; X\right)$. If at each step j we extract a subsequence from the previous one, we obtain a sequence $\{n^{m_j}_{l_j}\}_j$, which we relabel by $\{n_j\}_j$, such taht $N(n_j, \delta_{m_j}) = N$ for every j, and a unique $\vartheta \in \mathrm{Lip}_{\mathrm{loc}}\left(\bigcup_{k=1}^N (\alpha_k, \beta_k); X\right)$ such that, for all $k \in \{1, ..., N\}$,

$$\left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right] \subseteq \left(\tilde{a}_{j,k}, \tilde{b}_{j,k}\right), \quad \text{where} \quad \tilde{a}_{j,k} := a^{\delta_{m_j}}_{n_j,k}, \quad \tilde{b}_{j,k} := b^{\delta_{m_j}}_{n_j,k}, \tag{7.45}$$

$$|\tilde{\vartheta}_{n_j}(s) - \vartheta(s)| < \frac{1}{j}, \qquad \text{for every } s \in \left[\alpha_k + \frac{1}{j}, \beta_k - \frac{1}{j}\right]. \tag{7.46}$$

Therefore, we have proved (7.32)–(7.34). From (7.41) and (7.43) we obtain also that

$$\tilde{a}_{j,k} \to \alpha_k, \qquad \tilde{b}_{j,k} \to \beta_k, \qquad \text{as } j \to \infty, \tag{7.47}$$

where $\tilde{a}_{j,k}$, $\tilde{b}_{j,k}$ are defined in (7.45). These limits will be useful in Step 5.

Before proceeding with the proof, some comments are in order. Recall that $N = N(n_j, \delta_{m_j})$ is the number of the pieces of the trajectory of $\tilde{\vartheta}_{n_j}$ which go from $\partial B(x, \delta_{m_j})$ to $\partial B(y, \delta_{m_j})$, for some x, $y \in \mathcal{C}(t) \cup \{x_1, x_2\}$ with $x \neq y$. Thus, we have so far excluded that, for example, on $(\tilde{a}_{j,k}, \tilde{b}_{j,k})$ the trajectory of $\tilde{\vartheta}_{n_j}$ runs from $\partial B(x, \delta_{m_j})$ to $\partial B(x, \delta_{m_j})$. At the same time, the following situation is allowed: on $(\tilde{a}_{j,k}, \tilde{b}_{j,k})$ the trajectory of $\tilde{\vartheta}_{n_j}$ goes from $\partial B(x, \delta_{m_j})$ to $\partial B(y, \delta_{m_j})$ and on $(\tilde{a}_{j,k+1}, \tilde{b}_{j,k+1})$ goes from $\partial B(y, \delta_{m_j})$ to $\partial B(x, \delta_{m_j})$. Moreover, so far we have overlooked what happens to the trajectory of $\tilde{\vartheta}_{n_j}$ on the interval $\left[\tilde{b}_{j,k}, \tilde{a}_{j,k+1}\right]$. It is not difficult to imagine that, if $\beta_k < \alpha_{k+1}$ some "loops" around a certain point $x \in \mathcal{C}(t) \cup \{x_1, x_2\}$ have been created by the trajectory of $\tilde{\vartheta}_{n_j}$ on $\left[\tilde{b}_{j,k}, \tilde{a}_{j,k+1}\right]$, as $j \to \infty$. Note that we cannot deduce that the number of these loops is definitely bounded as we have done for $N(n_j, \delta_{m_j})$.

Step 4: passage to the limit. In order to take the limit of the integral term in (7.17), we observe that

$$\begin{aligned}\int_{t_1^{n_j}}^{t_2^{n_j}} \|\nabla_x \mathcal{E}(r, \vartheta_{n_j}(r))\| \|\dot{\vartheta}_{n_j}(r)\| dr &\geq \sum_{k=1}^{N} \int_{\tilde{a}_{j,k}}^{\tilde{b}_{j,k}} \|\nabla_x \mathcal{E}(r_{n_j}(s), \tilde{\vartheta}_{n_j}(s))\| \|\dot{\tilde{\vartheta}}_{n_j}(s)\| ds \\ &\geq \sum_{k=1}^{N} \int_{\alpha_k+1/j}^{\beta_k-1/j} \|\nabla_x \mathcal{E}(r_{n_j}(s), \tilde{\vartheta}_{n_j}(s))\| \|\dot{\tilde{\vartheta}}_{n_j}(s)\| ds,\end{aligned} \tag{7.48}$$

where we have used (7.45). We now pass to the limit as $j \to \infty$ in (7.48). Observe that, since $\{r_{n_j}(s)\}_j \subseteq [t_1^{n_j}, t_2^{n_j}]$ for every $s \in [s_1^{n_j}, s_2^{n_j}]$, then $r_{n_j}(s) \to t$ as $j \to \infty$ thanks to (7.6). Furthermore, the first of (7.34), joint with the bound in (7.7) and the closedness condition (E_3) yield

$$\begin{aligned}&\liminf_{j\to\infty} \|\nabla_x \mathcal{E}(r_{n_j}(s), \tilde{\vartheta}_{n_j}(s))\| \geq \|\nabla_x \mathcal{E}(t, \vartheta(s))\|, \\ &\qquad\qquad \text{for every } s \in [\alpha_k + \rho, \beta_k - \rho],\ \rho > 0,\ k = 1, \ldots, N.\end{aligned} \tag{7.49}$$

Combining (7.49) with the second of (7.34), applying an infinite dimensional version of Ioffe's theorem (see, e.g., [**67**, Theorem 21]), and arguing as in the proof of [**49**, Lemma 3.1], we have that

$$\begin{aligned}\liminf_{j\to\infty} \int_{\alpha_k+1/j}^{\beta_k-1/j} \|\nabla_x \mathcal{E}(r_{n_j}(s), \tilde{\vartheta}_{n_j}(s))\| \|\dot{\tilde{\vartheta}}_{n_j}(s)\| ds \geq \int_{\alpha_k+\rho}^{\beta_k-\rho} \|\nabla_x \mathcal{E}(t, \vartheta(s))\| \|\dot{\vartheta}(s)\| ds, \\ \text{for every } \rho > 0,\ k = 1, \ldots, N.\end{aligned}$$

Ultimately, we have proved that

$$\lim_{j\to\infty} \int_{t_1^{n_j}}^{t_2^{n_j}} \|\nabla_x \mathcal{E}(r, \vartheta_{n_j}(r))\| \|\dot{\vartheta}_{n_j}(r)\| dr \geq \sum_{k=1}^{N} \int_{\alpha_k}^{\beta_k} \|\nabla_x \mathcal{E}(t, \vartheta(s))\| \|\dot{\vartheta}(s)\| ds. \tag{7.50}$$

Step 5: conclusion. We claim that

$$\lim_{s\to\alpha_1^+} \vartheta(s) = x_1, \qquad \lim_{s\to\beta_N^-} \vartheta(s) = x_2, \tag{7.51}$$

$$\lim_{s\to\beta_k^-} \vartheta(s) = \lim_{s\to\alpha_{k+1}^+} \vartheta(s) = x, \qquad \textit{for some } x \in \mathcal{C}(t) \cup \{x_1, x_2\}, \tag{7.52}$$

for every $k = 1, ..., N-1$.

Let us check the first limit in (7.51) only, since the other limits can be verified in a similar way. Let $\{s_i\}_i \subseteq (\alpha_1, \beta_1)$ be a sequence such that $s_i \to \alpha_1^+$. We want to prove that

$$\vartheta(s_i) \to x_1, \qquad \text{as } i \to \infty. \tag{7.53}$$

The first of (7.34) gives that $\tilde{\vartheta}_{n_j}(s_i) \to \vartheta(s_i)$, as $j \to \infty$, for every i. In particular, there exists a strictly increasing sequence $\{j_i\}_i$ such that

$$|\tilde{\vartheta}_{n_{j_i}}(s_i) - \vartheta(s_i)| \leq \frac{1}{i}, \qquad \text{for every } i. \tag{7.54}$$

Note that $\tilde{a}_{j_i,1} \to \alpha_1$, as $i \to \infty$, in view of (7.47). Moreover, from the definition of $\tilde{a}_{j_i,1}$ (7.45) it follows that

$$\tilde{\vartheta}_{n_{j_i}}(\tilde{a}_{j_i,1}) \to x_1, \qquad \text{as } i \to \infty. \tag{7.55}$$

Now, note that from (7.20) and from the fact that s_i, $\tilde{a}_{j_i,1} \to \alpha_1$, as $i \to \infty$, we have that

$$\left| \int_{s_i}^{\tilde{a}_{j_i,1}} \|\nabla_x \mathcal{E}(r_{n_{j_i}}(s), \tilde{\vartheta}_{n_{j_i}}(s))\| \|\dot{\tilde{\vartheta}}_{n_{j_i}}(s)\| ds \right| \leq |s_i - \tilde{a}_{j_i,1}| \to 0, \qquad \text{as } i \to \infty.$$

Also, we have that

$$\left| \int_{s_i}^{\tilde{a}_{j_i,1}} \|\nabla_x \mathcal{E}(r_{n_{j_i}}(s), \tilde{\vartheta}_{n_{j_i}}(s))\| \|\dot{\tilde{\vartheta}}_{n_{j_i}}(s)\| ds \right| = \left| \int_{r_i}^{\tilde{r}_i} \|\nabla_x \mathcal{E}(r, \vartheta_{n_{j_i}}(r))\| \|\dot{\vartheta}_{n_{j_i}}(r)\| dr \right|,$$

for some $\{r_i\}_i$, $\{\tilde{r}_i\}_i \subseteq \big[t_1^{n_{j_i}}, t_2^{n_{j_i}}\big]$, where

$$\vartheta_{n_{j_i}}(r_i) = \tilde{\vartheta}_{n_{j_i}}(s_i), \qquad \vartheta_{n_{j_i}}(\tilde{r}_i) = \tilde{\vartheta}_{n_{j_i}}(\tilde{a}_{j_i,1}), \qquad \text{for every } i. \tag{7.56}$$

Also, we can suppose that, up to a subsequence, $r_i \leq \tilde{r}_i$ for every i, and that

$$\vartheta_{n_{j_i}}(r_i) \to \hat{x}, \qquad \text{for some } \hat{x} \in X.$$

This fact, together with the limit

$$\vartheta_{n_{j_i}}(\tilde{r}_i) \to x_1, \qquad \text{as } i \to \infty$$

which comes from (7.55) and the second of (7.56), gives that

$$\tilde{\vartheta}_{n_{j_i}}(s_i) = \vartheta_{n_{j_i}}(r_i) \to x_1, \qquad \text{as } i \to \infty, \tag{7.57}$$

as a consequence of Lemma 7.4 (1). Inequality (7.54) and convergence (7.57) imply (7.53).

By the limits in (7.51) and (7.52) we can trivially extend ϑ on the whole interval $[\alpha_1, \beta_N]$ and obtain, from (7.50), that

$$\liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(r, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr \geq \int_{\alpha_1}^{\beta_N} \|\nabla_x \mathcal{E}(t, \vartheta(s))\| \|\dot{\vartheta}(s)\| ds.$$

Thus, we have deduced (7.17) with $[a,b] = [\alpha_1, \beta_N]$. Note that we can choose an arbitrary $[a,b]$, reparametrize ϑ on it and obtain again (7.17), by scaling invariance. Let γ be the smallest point in $\{\beta_1, \alpha_2, \beta_2, ..., \alpha_N, \beta_N\}$ such that $\vartheta(\gamma) = x_2$. Relabelling $\alpha_1 < ... < \gamma$ by $\mathsf{t}_0 < ... < \mathsf{t}_j$, we have that $\vartheta(\mathsf{t}_0) = x_1$, $\vartheta(\mathsf{t}_j) = x_2$, and that $\vartheta(\mathsf{t}_i) \in \mathcal{C}(t)$ for every $i = 1, ..., j-1$. This last remark gives that $\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$ and concludes the proof. □

7.3. The cost function

In this section we introduce the cost function $c(t; x_1, x_2)$ (see Definition 7.8 below) which will prove useful to obtain the convergence of a sequence of solutions $\{u_{\varepsilon_n}\}$ of (7.1) to a limit function u satisfying the jump condition (7.2) (see Theorem 7.12). Since the definition of the cost function is based on the class $\mathscr{A}^t_{x_1,x_2}([a,b];X)$ introduced in Lemma 7.4, we list some properties of such class in the following remark. From now on, we will refer to the curves $\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$, for some $t \in [0,T]$, x_1, $x_2 \in X$, and some interval $[a,b]$, as to *admissible* curves connecting x_1 and x_2 at the time t.

Remark 7.5. The class $\mathscr{A}^t_{x_1,x_2}([a,b];X)$ is endowed with the following properties:

(1) up to a reparameterization, every absolutely continuous curve $\vartheta \in \mathrm{AC}([a,b];X)$ such that there exists $a = \mathsf{t}_0 < ... < \mathsf{t}_j = b$ with $\vartheta(\mathsf{t}_i) \neq \vartheta(\mathsf{t}_{i+1})$ $(i = 0, ..., j-1)$ and $\vartheta(\mathsf{t}_i) \in \mathcal{C}(t)$ $(i = 1, ..., j-1)$ is an admissible curve;

(2) if the energy functional $\mathcal{E}$ complies with the chain rule (E$_4$) along any absolutely continuous curve $u \in \mathrm{AC}([0,T];X)$ fulfilling (7.4), then the chain–rule identity (E$_4$) also holds along any *admissible* curve u fulfilling (7.4);

(3) it is possible to extend the statement of Lemma 7.2 to curves $\vartheta_n \in \mathscr{A}^t_{x_1^n,x_2^n}([a,b];X)$ in the following sense: let x_1, $x_2 \in X$ with $x_1 \neq x_2$. There exists $\eta > 0$ with the following property. For all sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ such that

$$x_1^n \to x_1, \qquad x_2^n \to x_2, \tag{7.58}$$

and every $\{\vartheta_n\}_n \subseteq \mathscr{A}^t_{x_1^n,x_2^n}([a,b];X)$ satisfying

$$\sup_n \sup_{s\in[a,b]} \mathcal{G}(\vartheta_n(s)) < \infty, \tag{7.59}$$

we have

$$\liminf_{n\to\infty} \int_a^b \|\nabla_x \mathcal{E}(t, \vartheta_n(r))\| \|\dot{\vartheta}_n(r)\| dr \geq \eta.$$

The following lemma is a variant of Lemma 7.4 where $\vartheta_n \in \mathscr{A}^t_{x_1^n,x_2^n}([a,b];X)$ and the integrands $\|\nabla_x \mathcal{E}(s,\vartheta_n(s))\|\|\dot\vartheta_n(s)\|$ are replaced by $\|\nabla_x \mathcal{E}(t,\vartheta_n(s))\|\|\dot\vartheta_n(s)\|$. This result will be used to deduce key properties of the cost function in Proposition 7.9.

LEMMA 7.6. *Assume* (E_0), (E_1), (E_3), *and* (E_5). *Let* $t \in [0,T]$, x_1, $x_2 \in X$, *let* $\{x_1^n\}_n$, $\{x_2^n\}_n$ *be two sequences satisfying (7.58), and* $\{\vartheta_n\}_n \subseteq \mathscr{A}^t_{x_1^n,x_2^n}([a,b];X)$ *fulfilling (7.59). We have that*

$$\liminf_{n\to\infty} \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta_n(s))\|\|\dot\vartheta_n(s)\|ds = 0 \quad \Rightarrow \quad x_1 = x_2, \tag{7.60}$$

and there exists $\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$ *such that*

$$\liminf_{n\to\infty} \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta_n(s))\|\|\dot\vartheta_n(s)\|ds \geq \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta(s))\|\|\dot\vartheta(s)\|ds. \tag{7.61}$$

REMARK 7.7. Lemma 7.6 holds in the more general case where $\vartheta_n \in \mathscr{A}^{t_n}_{x_1^n,x_2^n}([t_1^n,t_2^n];X)$ for some $\{t_n\} \subseteq [0,T]$ such that $t_n \to t$, and gives that

$$\liminf_{n\to\infty} \int_a^b \|\nabla_x \mathcal{E}(t_n,\vartheta_n(s))\|\|\dot\vartheta_n(s)\|ds \geq \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta(s))\|\|\dot\vartheta(s)\|ds,$$

for some $\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$.

PROOF. While implication (7.60) immediately follows from property (3) of Remark 7.5, we only sketch the argument for (7.61), dwelling on the differences with the proof of Lemma 7.4. It follows from (7.59) that there exists a compact set $K \subseteq X$ such that

$$\vartheta_n(t) \in K, \qquad \text{for every } t \in [a,b],\ n \in \mathbb{N}.$$

Starting from K, we define K_δ as in (7.24). By definition of $\mathscr{A}^t_{x_1^n,x_2^n}([a,b];X)$, we have that $\vartheta_n \in \mathrm{C}([a,b];X)$ and there exists a partition $a = \tau_0^n < \tau_1^n < \ldots < \tau_{M_n}^n = b$ such that $\vartheta_n|_{(\tau_i^n,\tau_{i+1}^n)} \in \mathrm{Lip}_{\mathrm{loc}}(\tau_i^n,\tau_{i+1}^n;X)$, for every $i = 0,\ldots,M_n$, and $\vartheta_n(a) = x_1^n$, $\vartheta_n(b) = x_2^n$. We use the same rescaling r_n introduced in the proof of Lemma 7.4 (with a and b in place of t_1^n and t_2^n, respectively) and obtain $\tilde\vartheta_n(s) := \vartheta_n(r_n(s))$, for every $s \in \left[\tilde a_n, \tilde b_n\right]$, where $\tilde a_n := r_n^{-1}(a) = a$ and $\tilde b_n := r_n^{-1}(b)$. It turns out that $\tilde\vartheta_n \in \mathrm{C}\left(\left[\tilde a_n,\tilde b_n\right];X\right)$ and there exists a partition

$$\tilde a_n = \sigma_0^n < \sigma_1^n < \ldots < \sigma_{M_n}^n = \tilde b_n$$

such that $\tilde\vartheta_n|_{(\sigma_i^n,\sigma_{i+1}^n)} \in \mathrm{Lip}_{\mathrm{loc}}(\sigma_i^n,\sigma_{i+1}^n;X)$, for every $i = 0,\ldots,M_n$, and $\tilde\vartheta_n(\tilde a_n) = x_1^n$, $\tilde\vartheta_n(\tilde b_n) = x_2^n$. Moreover,

$$\|\nabla_x \mathcal{E}(r_n(s),\tilde\vartheta_n(s))\|\|\dot{\tilde\vartheta}_n(s)\| \leq 1, \qquad \text{for a.e. } s \in (\tilde a_n,\tilde b_n),$$

and

$$\int_a^b \|\nabla_x \mathcal{E}(r,\vartheta_n(r))\|\|\dot\vartheta_n(r)\|dr = \int_{\tilde a_n}^{\tilde b_n} \|\nabla_x \mathcal{E}(r_n(s),\tilde\vartheta_n(s))\|\|\dot{\tilde\vartheta}_n(s)\|ds.$$

We now define, for every $i = 0,\ldots,M_n - 1$, the sets

$$A_n^{i,\delta} := \left\{ s \in (\sigma_i^n,\sigma_{i+1}^n) \,:\, \tilde\vartheta_n(s) \in \overset{\circ}{K}_\delta \right\},$$

and write $A_n^{i,\delta}$ as the countable union of its connected components

$$A_n^{i,\delta} = \bigcup_{k=1}^{\infty} (a_{n,k}^{i,\delta}, b_{n,k}^{i,\delta}).$$

Similarly, we consider the analogues of the sets B_n^δ (7.29), that is

$$
\begin{aligned}
B_n^{i,\delta} &:= \bigcup_{(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\in\mathfrak{B}_n^{i,\delta}} (a_{n,k}^{i,\delta}, b_{n,k}^{i,\delta}) \quad \text{with} \\
\mathfrak{B}_n^{i,\delta} &= \Big\{ (a_{n,k}^{i,\delta}, b_{n,k}^{i,\delta}) \subseteq A_n^{i,\delta} \,:\, \tilde\vartheta(a_{n,k}^{i,\delta}) \in \partial B(x,\delta),\ \tilde\vartheta(b_{n,k}^{i,\delta}) \in \partial B(y,\delta),\ x \neq y \Big\},
\end{aligned}
\tag{7.62}
$$

for $i = 1, .., M_n - 1$. We denote by $N(i,n,\delta)$ the cardinality of the set $\mathfrak{B}_n^{i,\delta}$. Then, we have, up to a subsequence,

$$
\begin{aligned}
C \geq \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr &\geq e_\delta \sum_{i=1}^{M_n} \sum_{(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\in\mathfrak{B}_n^{i,\delta}} \min_{\substack{x,y\in\mathcal{C}(t)\cup\{x_1,x_2\} \\ x\neq y}} (|x-y| - 2\delta) \\
&\geq e_\delta M_n \sum_{(a_{n,k}^{i,\delta},b_{n,k}^{i,\delta})\in\mathfrak{B}_n^{i,\delta}} \overline{m},
\end{aligned}
$$

where $0 < \overline{m} := \min_{\substack{x,y\in\mathcal{C}(t)\cap K \\ x\neq y}} (|x-y| - 2\overline{\delta})$ and $e_\delta > 0$ is the minimum defined in (7.25). Therefore, we conclude the estimate

$$
M_n N(i,n,\delta) \leq \frac{C}{e_\delta \overline{m}}, \qquad \text{for every } 0 < \delta \leq \overline{\delta} \text{ and } n \in \mathbb{N},
$$

which is a bound for both $\{M_n\}_n$ and $\{N(i,n,\delta) \,:\, i = 1, ..., M_n - 1,\ n \in \mathbb{N}\}$. The proof can be then carried on by suitably adapting the argument for Lemma 7.4 (2). □

We are now in position to define the cost function.

Definition 7.8. *Let $t \in [0,T]$, x_1, $x_2 \in X$, and $-\infty \leq a \leq b \leq +\infty$ be fixed. We define the cost function*

$$
c(t; x_1, x_2) := \inf \left\{ \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta(s))\| \|\dot\vartheta(s)\| ds \,:\, \vartheta \in \mathscr{A}_{x_1,x_2}^t([a,b];X) \right\} \tag{7.63}
$$

Note that this definition does not depend on the interval $[a,b]$ on which the admissible curves ϑ are defined, due to the scaling invariance of the integral $\int_a^b \|\nabla_x \mathcal{E}(t,\vartheta(s))\| \|\dot\vartheta(s)\| ds$. The following result collects the properties of the cost function.

Proposition 7.9. *Assume* (E_0), (E_1), (E_3), *and* (E_5). *For every $t \in [0,T]$ and x_1, $x_2 \in X$ we have that:*

(1) $x_1 \neq x_2 \Leftrightarrow c(t; x_1, x_2) > 0$;
(2) there exists an optimal curve $\vartheta \in \mathscr{A}_{x_1,x_2}^t([a,b];X)$ attaining the inf *in (7.63);*
(3) we have

$$
\begin{aligned}
c(t; x_1, x_2) \leq \inf \Big\{ \liminf_{n\to\infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(s,\vartheta_n(s))\| \|\dot\vartheta_n(s)\| ds \,:\, \\
\vartheta_n \in \mathrm{AC}([t_1^n, t_2^n]; X),\ t_i^n \to t,\ \vartheta_n(t_i^n) \to x_i \text{ for } i = 1,2 \Big\};
\end{aligned}
\tag{7.64}
$$

(4) the following lower semicontinuity property holds

$$
\Big(t_n \to t,\ x_1^n \rightharpoonup x_1,\ x_2^n \rightharpoonup x_2,\ \sup_n (\mathcal{G}(x_1^n) + \mathcal{G}(x_2^n)) \Big) < \infty \quad \Rightarrow \liminf_{n\to\infty} c(t_n; x_1^n, x_2^n) \geq c(t; x_1, x_2).
$$

PROOF. **Ad (2):** we use the direct method of the calculus of variations: let $\vartheta_n \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$ be a minimizing sequence for $c(t;x_1,x_2) < \infty$. Up to extracting a subsequence, we may suppose that $\sup_n \int_a^b \|\nabla_x \mathcal{E}(t,\vartheta_n(s))\| \|\dot\vartheta_n(s)\| ds \le C$. Applying the chain rule (E_4) yields for all $s \in [a,b]$

$$\mathcal{E}(t,\vartheta_n(s)) \le \mathcal{E}(t,\vartheta_n(a)) + \int_a^s \|\nabla_x \mathcal{E}(t,\vartheta_n(r))\| \|\dot\vartheta_n(r)\| dr \le \mathcal{E}(t,x_1) + C.$$

Hence, by Remark 7.1, there exists $C > 0$ such that

$$\mathcal{G}(\vartheta_n(s)) \le C, \qquad \text{for every } n \in \mathbb{N},\ s \in [a,b]. \tag{7.65}$$

Therefore, we may apply Lemma 7.6 to the curves $\vartheta_n \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$ and conclude.
Ad(3): is a direct consequence of Lemma 7.4.
Ad (4): for every $n \in \mathbb{N}$, let ϑ_n be an optimal curve for $c(t_n; x_1^n, x_2^n)$. Arguing as in the above lines, we prove estimate (7.65). Therefore we apply Lemma 7.6 and conclude.
Ad (1): suppose that $c(t;x_1,x_2) = 0$. It follows from (2) that $\int_a^b \|\nabla_x \mathcal{E}(t,\vartheta(s))\| \|\dot\vartheta(s)\| ds = 0$, for some $\vartheta \in \mathscr{A}^t_{x_1,x_2}([a,b];X)$. Using Lemma 7.6 with $\vartheta_n = \vartheta$ and $x_i^n = x_i$, we obtain that $x_1 = x_2$. The converse implication can be trivially checked. ☐

7.4. Compactness

With the following proposition some a priori estimates involving a family of functions $\{u_\varepsilon\}$ satisfying equation (7.1) are derived. They will be useful to obtain the next compactness results.

PROPOSITION 7.10. *Assume* (E_0), (E_2), *and* (E_4). *Let* $\{u^0_\varepsilon\}_\varepsilon \subseteq X$ *be such that, in the limit* $\varepsilon \to 0^+$,

$$\text{there exists } u_0 \in X: \qquad u^0_\varepsilon \to u_0 \qquad \text{and} \qquad \mathcal{E}(0,u^0_\varepsilon) \to \mathcal{E}(0,u_0), \tag{7.66}$$

and let $\{u_\varepsilon\}_\varepsilon \subseteq H^1(0,T;X)$ *be a family of solutions to the gradient flow (7.1) satisfying the initial condition* $u_\varepsilon(0) = u^0_\varepsilon$. *Then, the functions* u_ε*'s fulfill for every* $0 \le s \le t \le T$ *the energy identity*

$$\int_s^t \left(\frac{\varepsilon}{2}\|\dot u_\varepsilon(r)\|^2 + \frac{\|\nabla_x \mathcal{E}(r,u_\varepsilon(r))\|^2}{2\varepsilon} \right) dr + \mathcal{E}(t,u_\varepsilon(t)) = \mathcal{E}(s,u_\varepsilon(s)) + \int_s^t \partial_t \mathcal{E}(r,u_\varepsilon(r)) dr. \tag{7.67}$$

Furthermore, there exists a constant $C > 0$ *such that for every* $\varepsilon > 0$ *arbitrarily small the following estimates hold*

$$\sup_{t\in[0,T]} \mathcal{G}(u_\varepsilon(t)) + \sup_{t\in[0,T]} |\partial_t \mathcal{E}(t,u_\varepsilon(t))| \le C, \tag{7.68}$$

$$\int_s^t \left(\frac{\varepsilon}{2}\|\dot u_\varepsilon(r)\|^2 + \frac{1}{2\varepsilon}\|\nabla_x \mathcal{E}(r,u_\varepsilon(r))\|^2 \right) dr \le C, \qquad \text{for every } 0 \le s \le t \le T. \tag{7.69}$$

PROOF. The energy identity (7.67) follows from the chain rule (E_4), using that u_ε fulfills (7.1). Now, taking into account (E_2), we obtain from (7.67) that

$$\mathcal{E}(t,u_\varepsilon(t)) + \int_0^t \left(\frac{\varepsilon}{2}\|\dot u_\varepsilon(s)\|^2 + \frac{1}{2\varepsilon}\|\nabla_x \mathcal{E}(s,u_\varepsilon(s))\|^2 \right) ds \le \mathcal{E}(0,u^0_\varepsilon) + C_1 \int_0^t \mathcal{E}(s,u_\varepsilon(s)) ds. \tag{7.70}$$

Hence, convergences (7.66), the Gronwall Lemma and (7.3) yield $\sup_{t\in[0,T]} \mathcal{G}(u_\varepsilon(t)) \le C$, which in turn implies (7.68), in view of (E_2). Therefore, we also conclude (7.69). ☐

We are now in the position to state our first compactness result. From now on, we will use the place–holder $P_t(u)$ for $\partial_t \mathcal{E}(t,u)$.

PROPOSITION 7.11. *Assume* (E_0), (E_2), *and* (E_4). *Let* $\{u^0_\varepsilon\}_\varepsilon \subseteq X$ *fulfill* (7.66), *and let* $\{u_\varepsilon\}_\varepsilon \subseteq H^1(0,T;X)$ *be a family of solutions to* (7.1), *fulfilling* $u_\varepsilon(0) = u^\varepsilon_0$. *Then, for every vanishing sequence* $\{\varepsilon_n\}_n$, *considering the sequence of measures*

$$\mu_n := \left(\frac{\varepsilon_n}{2}\|\dot u_{\varepsilon_n}(\cdot)\|^2 + \frac{1}{2\varepsilon_n}\|\nabla_x \mathcal{E}(\cdot,u_{\varepsilon_n}(\cdot))\|^2 \right) \mathscr{L}^1 \tag{7.71}$$

(with $\mathscr{L}^1$ the Lebesgue measure on $(0,T)$), there exist a Radon measure $\mu \in \mathrm{M}(0,T)$ and functions $\mathscr{E} \in \mathrm{BV}([0,T])$ and $\mathscr{P} \in L^\infty(0,T)$ such that, along a not relabeled subsequence, the following convergences hold as $n \to \infty$:

$$\mu_n \rightharpoonup^* \mu \quad in \quad \mathrm{M}(0,T), \tag{7.72}$$

$$\mathcal{E}(t, u_{\varepsilon_n}(t)) \to \mathscr{E}(t), \qquad for\ every\ t \in [0,T], \tag{7.73}$$

$$P_t(u_{\varepsilon_n}(t)) \rightharpoonup^* \mathscr{P} \quad in \quad L^\infty(0,T). \tag{7.74}$$

Moreover, denoting by $\mathscr{E}(t_-)$ and $\mathscr{E}(t_+)$ the left and right limits of $\mathscr{E}$ in t, respectively, and by $\mathrm{d}\mathscr{E}$ the distributional derivative of $\mathscr{E}$, the following identities hold

$$\mathscr{E}(t_-) - \mathscr{E}(t_+) = \mu(\{t\}), \qquad for\ every\ t \in (0,T), \tag{7.75}$$

$$\mathrm{d}\mathscr{E} + \mu = \mathscr{P}\mathscr{L}^1. \tag{7.76}$$

Proof. It follows from (7.69) in Proposition 7.10 that the measures μ_n's have uniformly bounded variation, so that (7.72) follows. As for (7.73), we observe that, by (7.67), the map $t \mapsto \eta_n(t) := \mathcal{E}(t, u_{\varepsilon_n}(t)) - \int_0^t P_s(u_{\varepsilon_n}(s))ds$ is nonincreasing on $[0,T]$. Therefore, by Helly's Compactness Theorem (see, e.g. [**5**, Lemma 3.3.3]) there exists $\eta \in \mathrm{BV}([0,T])$ such that, up to a subsequence, $\eta_n(t) \to \eta(t)$ for every $t \in [0,T]$. On the other hand, (7.68) yields (7.74), up to a subsequence. Therefore, (7.73) follows with

$$\mathscr{E}(t) = \eta(t) + \int_0^t \mathscr{P}(s)ds.$$

Finally, passing to the limit as $n \to \infty$ in the energy identity (7.67) and taking into account convergences (7.72)–(7.74), it turns out that

$$\mu([s,t]) + \mathscr{E}(t) = \mathscr{E}(s) + \int_s^t \mathscr{P}(r)dr, \qquad \text{for every } t \in [0,T],\ s \in [0,t]. \tag{7.77}$$

In particular,

$$\mathscr{E}(t-\rho) - \mathscr{E}(t+\rho) + \int_{t-\rho}^{t+\rho} \mathscr{P}(s)ds = \mu([t-\rho, t+\rho]),$$

for every $t \in (0,T)$ and $\rho > 0$ arbitrarily small. Observe that, since $\mathscr{E} \in \mathrm{BV}([0,T])$, the left and right limits $\mathscr{E}(t-)$ and $\mathscr{E}(t_+)$ exist for every $t \in [0,T]$. Therefore, taking the limit as $\rho \to 0^+$ in the above identity gives (7.75). Identity (7.76) trivially follows from (7.77). □

Hereafter, we will denote by J the set where the measure μ is atomic, that is

$$J := \{t \in (0,T) \,:\, \mu(\{t\}) > 0\}.$$

Furthermore, we will denote by B the set

$$B = \{t \in (0,T) \,:\, \nabla_x \mathcal{E}(t, u_{\varepsilon_n}(t)) \to 0\}$$

where $\{u_{\varepsilon_n}\}_n$ is the sequence for which convergences (7.72)–(7.74) hold. Observe that (7.69) gives that $\lim_{n\to\infty} \int_0^T \|\nabla_x \mathcal{E}(r, u_{\varepsilon_n}(r))\|^2 dr = 0$, hence, up to a subsequence, the set B has full Lebesgue measure.

Finally, we can state our main compactness result. We recall that the set $\mathcal{C}(t)$ has been defined in (7.5).

Theorem 7.12. *Assume* (E_0)–(E_5). *Let $\{u^0_\varepsilon\}_\varepsilon \subseteq X$ fulfill (7.66) with $u_0 \in \mathcal{C}(0)$ and let $\{u_{\varepsilon_n}\}_n \subseteq H^1(0,T;X)$ be a family of solutions to (7.1) satisfying $u_{\varepsilon_n}(0) = u^0_{\varepsilon_n}$ and such that convergences (7.72)–(7.74) hold. Then, there exists a function $u : [0,T] \to X$, with $u \in L^\infty(0,T;X)$, such that, along a not relabeled subsequence, the following convergences hold:*

$$u_{\varepsilon_n}(t) \to u(t) \quad for\ every\ t \in [0,T], \qquad with \qquad u(t) \in \mathcal{C}(t) \quad for\ every\ t \in [0,T) \setminus J, \tag{7.78}$$

$$u_{\varepsilon_n} \rightharpoonup^* u \quad in \quad L^\infty(0,T;X), \qquad u_{\varepsilon_n} \to u \quad in \quad L^p(0,T;X) \quad for\ every \quad 1 \le p < \infty, \tag{7.79}$$

and

$$\mathscr{E}(t) \geq \mathcal{E}(t, u(t)) \quad \text{for every } t \in [0, T], \qquad \mathscr{E}(t) = \mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T), \tag{7.80}$$

$$\mathscr{P}(t) \leq \partial_t \mathcal{E}(t, u(t)) \quad \text{for a.e. } t \in (0, T). \tag{7.81}$$

Furthermore, the left and right limits

$$u_-(t) := \lim_{s \uparrow t} u(s), \qquad u_+(t) := \lim_{s \downarrow t} u(s) \qquad \text{exist for every } t \in [0, T], \tag{7.82}$$

fulfilling

$$u_-(t),\, u_+(t) \in \mathcal{C}(t), \tag{7.83}$$

$$J = \{t \in (0, T) \,:\, u_-(t) \neq u_+(t)\}, \tag{7.84}$$

the set J consists of at most countably many points, (7.85)

$$c(t; u_-(t), u_+(t)) = \mu(\{t\}) = \mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)), \qquad \text{for every } t \in J. \tag{7.86}$$

REMARK 7.13. Observe that from (7.82) it follows that u is a *regulated* function. As a consequence of (7.78) and (7.85), we may conclude that

$$u(t) \in \mathcal{C}(t), \qquad \text{for a.e. } t \in (0, T). \tag{7.87}$$

The proof of Theorem 7.12 hinges on the following result.

LEMMA 7.14. *Assume* (E_0)–(E_5). *Let* $\{u_{\varepsilon_n}\}_n \subseteq H^1(0, T; X)$ *be a family of solutions to* (7.1) *such that*

$$\sup_n \sup_{t \in [0,T]} \mathcal{G}(u_{\varepsilon_n}(t)) < \infty,$$

and convergence (7.72) *to a measure* $\mu \in \mathrm{M}(0, T)$ *holds. Then, for every* $t \in [0, T]$ *and for all sequences* $\{t_1^n\}_n$, $\{t_2^n\}_n$ *fulfilling* (7.6) *with limit* t *and such that, as* $n \to \infty$,

$$u_{\varepsilon_n}(t_1^n) \to u_1, \qquad u_{\varepsilon_n}(t_2^n) \to u_2, \qquad \text{for some } u_1,\, u_2 \in X, \tag{7.88}$$

there holds

$$\mu(\{t\}) \geq c(t, u_1, u_2). \tag{7.89}$$

In particular, for every $t \in [0, T] \setminus J$ *we have that* $u_1 = u_2$.

PROOF. Observe that for every $\rho > 0$ there holds

$$\begin{aligned} \mu([t - \rho, t + \rho]) &\geq \limsup_{n \to \infty} \mu_n([t_1^n, t_2^n]) \\ &= \limsup_{n \to \infty} \int_{t_1^n}^{t_2^n} \left(\frac{\varepsilon_n}{2} \|\dot{u}_{\varepsilon_n}(s)\|^2 + \frac{1}{2\varepsilon_n} \|\nabla_x \mathcal{E}(s, u_{\varepsilon_n}(s))\|^2 \right) ds \\ &\geq \limsup_{n \to \infty} \int_{t_1^n}^{t_2^n} \|\nabla_x \mathcal{E}(s, u_{\varepsilon_n}(s))\| \|\dot{u}_{\varepsilon_n}(s)\| ds \geq c(t; u_1, u_2), \end{aligned} \tag{7.90}$$

where the first inequality is due to (7.72), the second one to (7.71), the third one to the Young inequality, and the last one to (7.64) in Proposition 7.9. Since ρ is arbitrary, we conclude (7.89). In particular, if $\mu(\{t\}) = 0$ then by (1) in Proposition 7.9 we deduce that $u_1 = u_2$. □

PROOF OF THEOREM 7.12. We split the argument in several points.

Ad (7.85)**:** this property follows from the fact that $\mu(\Omega) < \infty$ by standard arguments. E.g., one can use Lemma 7.15 below.

Ad (7.78)–(7.79)**:** let us consider the set

$$I := J \cup A \cup \{0\} \text{ with } A \subset (B \backslash J) \text{ dense in } [0, T] \text{ and consisting of countably many points.} \tag{7.91}$$

It follows from (7.68) and (E_1) that there exists a compact $K \subseteq X$ such that

$$u_{\varepsilon_n}(t) \in K, \qquad \text{for every } n \in N,\ t \in [0, T]. \tag{7.92}$$

Therefore, since I has countably many points, with a diagonal argument it is possible to extract from $\{u_{\varepsilon_n}\}_n$ a (not relabeled) subsequence such that there exists $\hat{u} : I \to X$, with

$$u_{\varepsilon_n}(t) \to \hat{u}(t), \qquad \text{for every } t \in I, \tag{7.93}$$

and with $\hat{u}(0) = u_0$, thanks to (7.66). It follows from estimate (7.68) and the lower semicontinuity (E_0) of $u \mapsto \mathcal{E}(t,u)$ that

$$\mathcal{G}(\hat{u}(t)) \leq C, \qquad \text{for every } t \in I, \text{ for some } C > 0\,. \tag{7.94}$$

Moreover, by the construction (7.91) of I, by (E_3), and by the fact that $u_0 \in \mathcal{C}(0)$, we conclude that

$$\hat{u}(t) \in \mathcal{C}(t), \qquad \text{for every } t \in I \setminus J. \tag{7.95}$$

We now prove (7.78). From now on, for notational simplicity we will write u_n in place of u_{ε_n}. Firstly, we show that

$$\begin{gathered}\text{for every } t \in (0,T] \setminus I, \quad \tilde{u}(t) := \lim_{k\to\infty} \hat{u}(t_k) \ \text{ is uniquely defined for every } \ \{t_k\}_k \in \mathfrak{S}(t) \\ \text{with} \quad \mathfrak{S}(t) := \left\{ \{t_k\}_k \subseteq A \,:\, t_k \to t \text{ and there exists } \lim_{k\to\infty} \hat{u}(t_k) \right\}.\end{gathered} \tag{7.96}$$

Observe that $\mathfrak{S}(t) \neq \emptyset$, because $\hat{u} \in L^\infty(I;X)$, in view of (7.94) and (E_1). To see (7.96), let $\{t_1^k\}_k$, $\{t_2^k\}_k \in \mathfrak{S}(t)$ be such that

$$\lim_{k\to\infty} \hat{u}(t_1^k) =: u_1 \qquad \text{and} \qquad \lim_{k\to\infty} \hat{u}(t_2^k) =: u_2.$$

We want to show that $u_1 = u_2$. Note that $\hat{u}(t_1^k) = \lim_{n\to\infty} u_n(t_1^k)$ and $\hat{u}(t_2^k) = \lim_{n\to\infty} u_n(t_2^k)$ for every k, because of (7.93). Now, since t_1^k, $t_2^k \in A \subseteq B$ for every $k \in \mathbb{N}$, by (E_3) there holds $\nabla_x \mathcal{E}(t_1^k, \hat{u}(t_1^k)) = \nabla_x \mathcal{E}(t_2^k, \hat{u}(t_2^k)) = 0$, for every $k \in \mathbb{N}$. Therefore, again by (E_3) (which we can apply thanks to (7.94)), we conclude that u_1, $u_2 \in \mathcal{C}(t)$. Furthermore, with a diagonal procedure we can extract a subsequence $\{n_k\}_k$ such that

$$u_1 = \lim_{k\to\infty} u_{n_k}(t_1^k) \qquad \text{and} \qquad u_2 = \lim_{k\to\infty} u_{n_k}(t_2^k).$$

Thus, we are in the position to apply Lemma 7.14, so that, since $t \notin J$, we have that $u_1 = u_2$. This concludes the proof of (7.96).

Now, we define the (candidate) limit function u everywhere on $[0,T]$ by

$$u(t) := \begin{cases} \hat{u}(t) & \text{if } t \in I \\ \tilde{u}(t) & \text{if } t \in (0,T] \setminus I, \end{cases} \tag{7.97}$$

and prove (7.78) with u given by (7.97). We already know that, for every $t \in I$, $\lim_{n\to\infty} u_n(t) = \hat{u}(t) = u(t)$. Thus, consider $t \in (0,T] \setminus I$. We want to show that any subsequence of $\{u_n(t)\}_n$ admits a further subsequence converging to $u(t) = \tilde{u}(t)$. Let us fix a (not relabeled) subsequence $\{u_n(t)\}_n$ and consider a sequence $\{t_k\}_k \subseteq A$ such that $t_k \uparrow t$ and $u(t) = \lim_{k\to\infty} \hat{u}(t_k)$. Arguing as in the above lines, we see again that $\hat{u}(t_k) \in \mathcal{C}(t_k)$ for all $k \in \mathbb{N}$ hence $u(t) \in \mathcal{C}(t)$. Moreover, as seen before, there exists a subsequence $\{n_k\}_k$ such that

$$u(t) = \lim_{k\to\infty} u_{n_k}(t_k). \tag{7.98}$$

Now, considering $\{u_{n_k}(t)\}_k$, in view of (7.92) there exists a subsequence $\{n_{k_j}\}_j$ such that

$$u_{n_{k_j}}(t) \to \tilde{u}, \qquad \text{for some } \tilde{u} \in X.$$

At the same time, from (7.98), we have in particular that

$$u_{n_{k_j}}(t_{k_j}) \to u(t).$$

From the last two convergences, since $t \notin J$, an application of Lemma 7.14 with t_{k_j}, t, $u_{n_{k_j}}$, $u(t)$, and $\tilde{u}$ in place of t_1^n, t_2^n, u_{ε_n}, u_1, and u_2, respectively, gives $\tilde{u} = u(t)$. Therefore, (7.78) holds.

Observe that (7.78) implies (7.79), in view of (7.92). The latter and assumption (E_0) also imply that

$$\mathcal{G}(u(t)) \leq C, \qquad \text{for every } t \in [0,T], \text{ for some } C > 0\,. \tag{7.99}$$

Ad (7.80)–(7.81)**:** the first of (7.80) follows from (7.78) and from assumption (E_0). The second of (7.80) and (7.81) follow from (E_3), which we can apply thanks to estimates (7.68) and (7.69).

Ad (7.82)–(7.83)**:** consider $\{t_1^k\}_k$, $\{t_2^k\}_k \subseteq [0,T]$ such that $t_1^k \leq t_2^k$ for every k , t_1^k, $t_2^k \to t$ for some $t \in [0,T]$, and there exist

$$\lim_{k\to\infty} u(t_1^k) =: u_1 \qquad \lim_{k\to\infty} u(t_2^k) =: u_2.$$

Note that, up to subsequences, we can suppose that $\{t_1^k\}_k$ approaches t either from the left or from the right, and that the same holds for $\{t_2^k\}_k$. In particular, there exist $\lim_{k\to\infty} \mathscr{E}(t_1^k)$ and $\lim_{k\to\infty} \mathscr{E}(t_2^k)$, because $\mathscr{E} \in \mathrm{BV}([0,T])$. Now, (7.73) gives that $\mathscr{E}(t_i^k) = \lim_{n\to\infty} \mathcal{E}(t_i^k, u_n(t_i^k))$ for every k, for $i = 1, 2$. Hence, there exists $n_k \to \infty$ such that

$$\left|\mathcal{E}(t_1^k, u_{n_k}(t_1^k)) - \mathscr{E}(t_1^k)\right| \leq \frac{1}{k}, \qquad \left|\mathcal{E}(t_2^k, u_{n_k}(t_2^k)) - \mathscr{E}(t_2^k)\right| \leq \frac{1}{k},$$

for every k, so that

$$\lim_{k\to\infty} \mathcal{E}(t_1^k, u_{n_k}(t_1^k)) = \lim_{k\to\infty} \mathscr{E}(t_1^k), \qquad \lim_{k\to\infty} \mathcal{E}(t_2^k, u_{n_k}(t_2^k)) = \lim_{k\to\infty} \mathscr{E}(t_2^k). \tag{7.100}$$

Arguing as previously done, we can also suppose that, up to a subsequence,

$$u_1 = \lim_{k\to\infty} u_{n_k}(t_1^k), \qquad u_2 = \lim_{k\to\infty} u_{n_k}(t_2^k). \tag{7.101}$$

Now, recalling definition 7.71 of μ_n, the energy identity (7.67) with t_1^k, t_2^k, u_{n_k} in place of s, t, and u_ε, respectively, gives

$$\mathcal{E}(t_1^k, u_{n_k}(t_1^k)) - \mathcal{E}(t_2^k, u_{n_k}(t_2^k)) + \int_{t_1^k}^{t_2^k} P_r(u_{n_k}(r))dr = \mu_n([t_1^k, t_2^k]).$$

This equality, together with (7.100), (7.101), and with (7.64) in Proposition 7.9, implies that

$$\lim_{k\to\infty} \mathscr{E}(t_1^k) - \lim_{k\to\infty} \mathscr{E}(t_2^k) \geq \liminf_{k\to\infty} \mu_{n_k}([t_1^k, t_2^k]) \geq c(t; u_1, u_2) \tag{7.102}$$

(note that we have also used (7.74)). Since $\mathscr{E} \in \mathrm{BV}([0,T])$, the left and right limits of $\mathscr{E}$ exist on all $[0,T]$. Thus, considering t_1^k, $t_2^k \uparrow t$ or t_1^k, $t_2^k \downarrow t$, inequality (7.102) immediately tells us that $c(t; u_1, u_2) = 0$, so that $u_1 = u_2$, in view of Proposition 7.9 (1). In this way, we have proved (7.82). Property (7.83) follows from (7.78) and (7.85).

Now, let us make some comments which will be useful later. Consider $t \in [0,T] \setminus J$ and suppose that $t_1^k \uparrow t$ and $t_2^k \downarrow t$, so that $u_1 = u(t_-)$ and $u_2 = u(t_+)$ in (7.102). Denoting by $\mathscr{E}(t_-)$ and $\mathscr{E}(t_+)$ the left and the right limits of $\mathscr{E}$ at t, from (7.102) we obtain

$$\mathscr{E}(t_-) - \mathscr{E}(t_+) \geq c(t; u(t_-), u(t_+)). \tag{7.103}$$

This inequality, together with (7.75), gives that $\mu(\{t\}) \geq c(t; u(t_-), u(t_+))$. Since $t \notin J$ we have $c(t; u(t_-), u(t_+)) = 0$, hence $u(t_-) = u(t_+)$. Thus, we have proved that (7.84) holds with $\supseteq$ in place of $=$.

Ad (7.84)**,** (7.86)**:** from the previous step we have obtained in particular that for every $t \in J$ there exist $u_-(t)$ and $u_+(t)$. Now, let us choose $\{t_1^k\}_k$, $\{t_2^k\}_k \subseteq [0,T] \setminus J$ such that $t_1^k \uparrow t$, $t_2^k \downarrow t$, the second of (7.80) holds on such sequences, and

$$u_-(t) = \lim_{k\to\infty} u(t_1^k), \qquad u_+(t) = \lim_{k\to\infty} u(t_2^k).$$

Due to (7.80), we also have

$$\mathscr{E}(t_-) = \lim_{k\to\infty} \mathscr{E}(t_1^k) = \lim_{k\to\infty} \mathcal{E}(t_1^k, u(t_1^k)) = \mathcal{E}(t, u_-(t)),$$

where the last equality follows from (E_3). Similarly, we have that $\mathscr{E}(t_+) = \mathcal{E}(t, u_+(t))$. Thus, from (7.103) it descends that

$$\mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) \geq c(t; u_-(t), u_+(t)), \qquad \text{for every } t \in J. \tag{7.104}$$

To prove the converse inequality, let $\vartheta \in \mathscr{A}^t_{u_-(t),u_+(t)}([a,b]; X)$ be a minimizing curve for the cost $c(t; u_-(t), u_+(t))$, whose existence is guaranteed by Proposition 7.9 (2). Then, by the chain rule (E_4) (see also Remark 7.5 (2)),

$$\begin{aligned} c(t; u_-(t), u_+(t)) &= \int_a^b \|\nabla_x \mathcal{E}(t, \vartheta(s))\| \|\dot{\vartheta}(s)\| ds \\ &\geq -\big[\mathcal{E}(t, \vartheta(b)) - \mathcal{E}(t, \vartheta(a))\big] = \mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)). \end{aligned} \tag{7.105}$$

Combining (7.75), (7.104), and (7.105) it turns out the jump condition (7.86). Furthermore, taking into account (1) in Proposition 7.9, from (7.86) it is immediate to deduce that J coincides with the jump set of u, that is (7.84). This concludes the proof. □

Lemma 7.15. *Let I be a continuous family of indices and let $\{\alpha_i\}_{i\in I} \subseteq [0, \infty)$ be such that $\sum_{i\in I} \alpha_i < \infty$. Then, there exists $J \subseteq I$ at most countable such that $\alpha_i = 0$ for every $i \in I \setminus J$.*

Proof. Suppose that $\sum_{i\in I} \alpha_i = C < \infty$. For every $k \in \mathbb{N} \setminus \{0\}$, define

$$I_k := \left\{ i \in I \,:\, \alpha_i \geq \frac{1}{k} \right\}.$$

It is clear that

$$\sum_{i\in I} \alpha_i = \sum_{\{i \in I: \alpha_i > 0\}} \alpha_i = \sum_{k \in \mathbb{N}\setminus\{0\}} \sum_{i \in I_k} \alpha_i. \tag{7.106}$$

Note that, since $\frac{1}{k} \sharp I_k \leq \sum_{i\in I_k} \alpha_i \leq C < \infty$, then

$$\sharp I_k \quad \text{is finite for every } k. \tag{7.107}$$

From (7.106) and (7.107) the thesis follows. □

Bibliography

[1] V. Agostiniani, *Second order approximations of quasistatic evolution problems in finite dimension*, Discrete Contin. Dyn. Syst. A **32** n. 4, 2012, 1125–1167.

[2] V. Agostiniani, G. Dal Maso, A. DeSimone, *Linear elasticity obtained from finite elasticity by Γ-convergence under weak coerciveness conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, http://dx.doi.org/10.1016/j.anihpc.2012.04.001 .

[3] V. Agostiniani, A. DeSimone, *Γ-convergence of energies for nematic elastomers in the small strain limit*, Cont. Mech. Thermodyn. **23** n. 3, 2011, 257–274.

[4] V. Agostiniani, A. DeSimone, *Ogden-type energies for nematic elastomers*, International Journal of Non-Linear Mechanics **47**, 2012, 402–412.

[5] L. Ambrosio, N. Gigli, G. Savaré, "Gradient flows in metric spaces and in the space of probability measures," Birkhäuser, Basel, 2005.

[6] D. R. Anderson, D. E. Carlson, E. Fried, *A continuum-mechanical theory for nematic elastomers*, J. Elasticity **56**, 1999, 33–58.

[7] J. M. Ball, F. Murat, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58**, 1984, 225–253.

[8] K. Bhattacharya, *Microstructure of martensite*, Oxford University Press, Oxford, 2003.

[9] J.S. Biggins, E.M. Terentjev, M. Warner, *Semisoft elastic response of nematic elastomers to complex deformations*, Phys. Rev. E **78**, 2008, 041704.1–9.

[10] P. Bladon, E. M. Terentjev, M. Warner, *Transitions and instabilities in liquid-crystal elastomers*, Phys. Rev. E **47**, 1993, R3838–R3840.

[11] F. Cagnetti, *A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path*, Math. Models Methods Appl. Sci., **18** (2008), 1027–1071.

[12] P. Cesana, *Relaxation of multi-well energies in linearized elasticity and applications to nematic elastomers*, Arch. Rat. Mech. Anal. **197** n. 3, 2010, 903–923.

[13] P. Cesana, A. DeSimone, *Strain-order coupling in nematic elastomers: equilibrium configurations*, Math. Models Methods Appl. Sci. **19**, 2009, 601–630.

[14] P. Cesana, A. DeSimone, *Quasiconvex envelopes of energies for nematic elastomers in the small strain regime and applications*, J. Mech. Phys. Solids **59** n. 4, 2011, 787-803.

[15] P. G. Ciarlet, "Mathematical elasticity. Volume I:three dimensional elasticity," North Holland, Amsterdam, 1988.

[16] S. Conti, A. DeSimone, G. Dolzmann, *Soft elastic response of stretched sheets of nematic elastomers: a numerical study*, J. Mech. Phys. Solids **50**, 2002, 1431–1451.

[17] S. Conti, A. DeSimone, G. Dolzmann, *Semi-soft elasticity and director reorientation in stretched sheets of nematic elastomers*, Phys. Rev. E **60**, 2002, 61710-1–8.

[18] S. Conti, G. Dolzmann, *Γ-convergence for incompressible elastic plates*, Calc. Var. Partial Differential Equations **34** n. 4, 2009, 531–551.

[19] S. Conti, G. Dolzmann, S. Müller, *Korn's second inequality and geometric rigidity with mixed growth conditions*, arXiv:1203.1138.

[20] B. Dacorogna, "Direct methods in the calculus of variations," second ed. Springer, 2007.

[21] B. Dacorogna, P. Marcellini, "Implicit partial differential equations," Birkhäuser, Boston, 1999.

[22] G. Dal Maso, "An introduction to Γ-convergence," Birchäuser, Boston, 1993.

[23] G. Dal Maso, A. DeSimone, M. G. Mora, M. Morini, *A vanishing viscosity approach to quasistatic evolution in plasticity with softening*, Arch. Ration. Mech. Anal., **189**, 2008, 469–544.

[24] G. Dal Maso, A. DeSimone, F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling*, Calc. Var. Partial Differential Equations, **40**, 2011, 125–181.

[25] G. Dal Maso, A. DeSimone, F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: properties of the viscosity solution*, Calc. Var. Partial Differential Equations, in press.

[26] G. Dal Maso, M. Negri, D. Percivale, *Linearized elasticity as Γ-limit of finite elasticity*, Set-Valued Anal. **10**, 2002, 165–183.

[27] G. Dal Maso, F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: the spatially homogeneous case*, Netw. Heterog. Media, **5**, 2010, 97–132.

[28] M. de Luca, A. DeSimone, *Mathematical and numerical modeling of liquid crystal elastomer phase transition and deformation*, MRS Proceedings **1403**, http://dx.doi.org/10.1557/opl.2012.249 .

[29] A. DeSimone, *Energetics of fine domain structures*, Ferroelectrics **222**, 1999, 275–284.

[30] A. DeSimone, G. Dolzmann, *Material instabilities in nematic elastomers*, Physica D **136**, 2000, 175–191.

[31] A. DeSimone, G. Dolzmann, *Macroscopic response of nematic elastomers via relaxation of a class of $SO(3)$-invariant energies*, Arch. Rat. Mech. Anal. **161**, 2002, 181–204.

[32] A. DeSimone, L. Teresi, *Elastic energies for nematic elastomers*, Eur. Phys. J. E **29**, 2009, 191–204.

[33] GJ. Diestel, J. J. Hul, "Vector measures," American Mathematical Society, Providence, 1977.

[34] M. A. Efendiev, A. Mielke, *On the rate-independent limit of systems with dry friction and small viscosity*, J. Convex Anal. **13**, 2006, 151–167.

[35] J. Küpfer, H. Finkelmann, *Nematic liquid single-crystal elastomers*, Makromol. Chem. Rapid Commun. **12**, 1991, 717-726.

[36] P. J. Flory, "Principles of polymer chemistry," Cornell University Press, London, 1953.

[37] E. Fried, V., Korchagin, *Striping of nematic elastomers*, Int. J. Solids Structures **39**, 2002, 3451–3467.

[38] G. Friesecke, R. D. James, S. Müller, *A theorem on geometric rigidity and the derivation on nonlinear plate theory from three-dimensional elasticity*, Commun. Pure Appl. Math. **55**, 2002, 1461–1506.

[39] G. Friesecke, R. D. James, S. Müller, *A hierarchy of plate models derived from nonlinear elasticity by Γ-convergence*, Arch. Ration. Mech. Anal. **180**, 2006, 183–236.

[40] A. Fukunaga, K. Urayama, T. Takigawa, A. DeSimone, L. Teresi, *Dynamics of electro-opto-mechanical effects in swollen nematic elastomers*, Macromolecules **41**, 2008, 9389–9396.

[41] J. Guckenheimer, P. Holme, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations on Vector Fields," Springer-Verlag, New York, 1983.

[42] M. E. Gurtin, "An introduction to continuum mechanics," Academic Press, New York, 1981.

[43] J. K. Hale, "Ordinary Differential Equations," Krieger, Florida, 1980.

[44] M. W. Hirsch, "Differential Topology," Springer-Verlag, New York, 1976.

[45] G. A. Holzapfel, "Nonlinear solid mechanics: a continuum approach for engineering," Wiley, Chichester, 2000.

[46] D. Knees, A. Mielke, C. Zanini, *Crack growth in polyconvex materials*, Phys. D **239**, 2010, 1470–1484.

[47] R. V. Kohn, *The relaxation of a double-well energy*, Continuum Mech. Thermodyn. **3** n. 3, 1991, 193–236.

[48] A. Menzel, H. Pleiner, H. Brand, *Nonlinear relative rotations in liquid crystalline elastomers*, J. Chem. Phys. **126**, 2009, 234901–1–9.

[49] A. Mielke, R. Rossi, G. Savaré, *Modeling solutions with jumps for rate-independent systems on metric spaces*, Discrete Contin. Dyn. Syst. **25**, 2009, 585–615.

[50] A. Mielke, R. Rossi, G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var. **18**, 2011, 36–80.

[51] S. Müller, M. Palombaro, *Derivation of a rod theory for biphase materials with dislocations at the interface*, arXiv:1201.4290v1.

[52] S. Müller, V. Šverák, *Attainment results for the two-well problem by convex integration*, in Geometric analysis and the calculus of variations, Internat. Press, Cambridge, 1996, 239–251.

[53] S. Müller, V. Šverák, *Convex integration with constraints and applications to phase transitions and partial differential equations*, J. Eur. Math. Soc. **1**, 1999, 393–442.

[54] R. W. Ogden, "Non-linear elastic deformations," Dover, Mineola (N. Y.), 1997.

[55] A. Petelin, M. Copic, *Observation of a soft mode of elastic instability in liquid crystal elastomers*, Phys. Rev. Lett. **103**, 2009, 077801–1–4.

[56] A.C. Pipkin, *Elastic materials with two preferred states*, Quart. J. Mech. Appl. Math. **44** n. 1, 1991, 1–15.

[57] W. Pompe, *Explicit construction of piecewise affine mappings with constraints*, Bulletin of the Polish Academy of Sciences Mathematics **58** n. 3, 2010, 209–220.

[58] W. Rudin, "Real and complex analysis," McGrw-Hill, New York, 1974.

[59] L. Scardia, C. I. Zeppieri, *Line-tension model for plasticity as the Γ-limit of a nonlinear dislocation energy*, SIAM J. Math. Anal. **44**, 2012, 2372–2400.

[60] B. Schmidt, *Linear Γ-limits of multiwell energies in nonlinear elasticity theory*, Continuum Mech. Thermodyn. **20** n. 6, 2008, 375–396.

[61] M. Šilhavý, *Ideally soft nematic elastomers*, Netw. Heterog. Media **2** n. 2, 2007, 279–311.

[62] F. Solombrino, *Quasistatic evolution for plasticity with softening: the spatially homogeneous case*, Discrete Contin. Dyn. Syst. **27**, 2010, 1189–1217.

[63] R. Temam, "Problèmes mathématiques en plasticité," Gauthiers-Villars, Paris, 1983.

[64] L. Tartar, *Some remarks on separately convex functions*, in Microstructures and phase transitions, Springer, New York, 1993, 191–204.

[65] R. Toader, C. Zanini, *An artificial viscosity approach to quasistatic crack growth*, Boll. Unione Mat. Ital. (9) **2**, 2009, 1–35.

[66] L.R.G. Treloar, "The Physics of Rubber Elasticity," 3rd ed., Oxford University Press, 1975.
[67] M. Valadier, *Young measures*, in Methods of nonconvex analysis, Springer-Verlag, Berlin, 1990, 152–188.
[68] G. C. Verwey, M. Warner, E. M. Terentjev, *Elastic instability and stripe domains in liquid crystalline elastomers*, J. Phys. II France **34**, 1996, 1273–1290.
[69] M. Warner, E. M. Terentjev, "Liquid crystal elastomers," Clarendon Press, Oxford, 2003.
[70] J. Weilepp, H. R. Brand, *Director reorientation in nematic-liquid-single-crystal elastomers by external mechanical stress*, Europhys. Lett. **34**, 1996, 495–500.
[71] F. Ye, R. Mukhopadhyay, O. Stenull, T.C. Lubensky, *Semisoft nematic elastomers and nematics in crossed electric and magnetic fields*, Phys. Rev. Lett. **98**, 2007, 147801.
[72] C. Zanini, *Singular perturbations of finite dimensional gradient flows*, Discrete Contin. Dyn. Syst., **18**, 2007, 657–675.
[73] E. R. Zubarev, S. A. Kuptsov, T. I. Yuranova, R. V. Talroze, H. Finkelmann, *Monodomain liquid crystalline networks: reorientation mechanism from uniform to stripe domains*, Liquid Crystals **26**, 1999, 1531–1540.