



Scuola Internazionale Superiore di Studi Avanzati - Trieste

**Area of Mathematics
Ph.D. in Mathematical Physics**

Thesis

Real Topological String Theory

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Thesis submitted in partial fulfillment of the requirements for the
degree of Philosophiæ Doctor

academic year 2015-2016

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Thesis (to be) defended in front of a Board of Examiners composed by
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on September 19th, 2016.

Dedicated to my Family

Abstract

The Thesis comprises work done at SISSA (Trieste) and UAM (Madrid) under supervision of A. Uranga during academic years 2013-2016 and published in the following works.

- In the first one [89] we describe the type IIA physical realization of the unoriented topological string introduced by Walcher, describe its M-theory lift, and show that it allows to compute the open and unoriented topological amplitude in terms of one-loop diagram of BPS M2-brane states. This confirms and allows to generalize the conjectured BPS integer expansion of the topological amplitude. The M-theory lift of the orientifold is freely acting on the M-theory circle, so that integer multiplicities are a weighted version of the (equivariant subsector of the) original closed oriented Gopakumar-Vafa invariants. The M-theory lift also provides new perspective on the topological tadpole cancellation conditions. We finally comment on the M-theory version of other unoriented topological strings, and clarify certain misidentifications in earlier discussions in the literature.
- In the second [47] we consider the real topological string on certain non-compact toric Calabi-Yau three-folds \mathbb{X} , in its physical realization describing an orientifold of type IIA on \mathbb{X} with an O4-plane and a single D4-brane stuck on top. The orientifold can be regarded as a new kind of surface operator on the gauge theory with 8 supercharges arising from the singular geometry. We use the M-theory lift of this system to compute the real Gopakumar-Vafa invariants (describing wrapped M2-brane BPS states) for diverse geometries. We show that the real topological string amplitudes pick up certain signs across flop transitions, in a well-defined pattern consistent with continuity of the real BPS invariants. We further give some preliminary proposals of an intrinsically gauge theoretical description of the effect of the surface operator in the gauge theory partition function.
- In the third [82], which is in preparation, we focus on target space physics related to real topological strings, namely we discuss the physical superstring correlation functions in type I theory (or equivalently type II with orientifold) that compute real topological string amplitudes. As it turns out that direct computation presents a problem, which also affects the standard case, we consider the correlator corresponding to *holomorphic* derivative of the real topological amplitude \mathcal{G}_χ , at fixed worldsheet Euler character χ . This corresponds in the low-energy effective action

to $\mathcal{N} = 2$ Weyl tensor, appropriately reduced to the orientifold invariant part, and raised to power $g' = -\chi + 1$. In this case, we are able to perform computation, and show that appropriate insertions in the physical string correlator give precisely the holomorphic derivative of topological amplitude. Finally, we apply this method to the standard closed oriented case as well, and prove a similar statement for the topological amplitude \mathcal{F}_g , which solves a small issue affecting that computation.

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Acknowledgements

I am grateful to Angel Uranga for his useful advice and stimulating conversations. I am indebted to Kumar Narain for sharing his thoughts and views on physics. I would like to thank them both for giving me the opportunity of growing as a physicist.

I thank the mathematical physics and geometry group at SISSA, in particular my co-supervisor Alessandro Tanzini for his support during this joint project, Ugo Bruzzo, and Barbara Fantechi for fruitful discussions; the IFT in Madrid for warm hospitality and lively scientific environment; the ICTP in Trieste for providing a unique opportunity to interact with top scientists; the physical mathematics group of Johannes Walcher in Heidelberg for many stimulating discussions, and for hospitality during preparation of part of this Thesis.

I finally want to mention some of the teachers, scientists and friends whom I interacted with during these years: F. Feruglio, G. Benettin, P.A. Marchetti, K. Lechner, H. Hayashi, L. Tizzano, M. Young, P. Longhi, M. Fazzi, F. Apruzzi, R. Gopakumar, N. Nekrasov, A. Tomasiello, Y. Tachikawa, S. Gukov, S. Pasquetti, J. Maldacena, P. Georgieva, B. Mares, M. Ronzani, S. Benvenuti, A. Sciarappa, A. Bawane, J. Scalise, G. Cotti, B. Lima de Souza, A. Retolaza, M. Montero, I. Brivio, F. Carta, F. Torrentí, M. Caorsi, and A. Maiorana.

I also thank the secretaries at various places for their kind support, in particular R. Iancer and the students' secretariat at SISSA, and I. Perez at IFT.

Chapter 1

Introduction

1.1 General setup

The topic of the Thesis lies at the interface between theoretical high energy physics and geometry. Broadly speaking, we focus on topological string theory and its connections with supersymmetric gauge theories in four dimensions, in particular the counting of BPS states.

Recently this subfield of theoretical physics has seen many breakthroughs, both on the string side where topological vertex techniques have been developed to compute all-genera partition functions essentially via Feynman diagram tools, and on the SUSY side, where mathematical-solid arguments have been proposed by N. Nekrasov together with an explicit form (in terms of sums of Young diagrams) for the partition function of some twisted SUSY gauge theories: the development of such localization techniques allowed many exact results to be obtained, e.g. AGT correspondence [6, 105] relating Nekrasov instanton partition function on \mathbb{R}^4 for so-called class-S theories [30] with chiral blocks in Virasoro algebra.

There's an interesting duality, where one can 'engineer'¹ some gauge theory starting from *local* toric Calabi-Yau three-fold: non-compactness is necessary in order to decouple gravitational interactions that in large volume limit are suppressed due to the CY volume appearing in Planck mass coefficient of Einstein-Hilbert action after dimensional reduction; interesting physics comes from local singularity structure, namely there will be some interacting cycles in the CY, which typically intersect with ADE pairing and support BPS branes.

One can then apply vertex techniques to show that topological string on the given CY gives precisely Nekrasov's result for the engineered gauge group: this stems from the fact that both theories are computing some Witten index taking contributions from BPS states, which are essentially the same for the two sides.

¹Geometry of the compactification manifold determines the effective gauge theory in the field theory limit.

This circle of ideas is less understood for the case when unoriented and open contributions are present, namely when one has orientifold planes and D-branes wrapping them in the string construction, and is the topic of the Thesis.

1.2 Background material

We briefly review some topics that will be useful when reading the rest of the Thesis. Our aim is to introduce basic notions and give some references.

1.2.1 Topological strings

We review closed oriented topological strings [17]. One possibility is to adopt a CFT viewpoint: start with $\mathcal{N} = 2$ super conformal (worldsheet) algebra, and try to form BRST cohomology of the zero mode of one of the two supercharges, say G_+ ; this is not possible because G_+ has spin $3/2$, and to get a scalar zero mode we need to start with spin 1. To fix this, perform so-called topological twist [102].

Since we are interested in CY three-folds \mathbb{X} , we consider unitary $\mathcal{N} = (2, 2)$ SCFTs, where the numbers in parenthesis denote the numbers of left- and right-moving (the latter with a tilde) supercharges, and the relation between dimension and central charges is $d = c/3 = \tilde{c}/3 = 3$. There are four possible twists (but only two inequivalent choices) dubbed A/B models [103]:

BRST	name	depends on	
(G_+, \tilde{G}_+)	A-model	Kähler moduli on \mathbb{X}	(1.2.1)
(G_-, \tilde{G}_-)	\bar{A} -model (complex conj. correlators)		
(G_+, \tilde{G}_-)	B-model	complex moduli on \mathbb{X}	
(G_-, \tilde{G}_+)	\bar{B} -model		

Using supersymmetric localization² one can show that correlation functions between physical operators in the B-model are purely classical, localized to integrals over the target manifolds. In contrast in the A-model (on which we now focus) correlation functions get quantum corrections, as these correlation functions are localized to integrals over moduli space of holomorphic maps from a Riemann surface to the target space, $f : \Sigma_g \rightarrow \mathbb{X}$; this leads to worldsheet instantons, weighted by $e^{-\int_{\Sigma_g} f^* \omega} = e^{-\int_{\beta} \omega}$, where $\beta = f_*[\Sigma_g] \in H_2(\mathbb{X}; \mathbb{Z})$, and $\omega = \sum_{a=1}^{h^{1,1}(\mathbb{X})} t^a \omega_a$ is made out of the Kähler form of \mathbb{X} , complexified with the B -field.

Construction of topological string correlation functions is done by coupling twisted $\mathcal{N} = 2$ theory to topological gravity: the correlator for $g > 1$ is

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} \int_{\Sigma_g} (G_-)_{zz}(\mu_i)^{\frac{z}{\bar{z}}} \right\rangle^2, \quad (1.2.2)$$

²For supersymmetric localization and topological twist, a good reference is the book [50]. The review [88] covers the field theory viewpoint.

where μ are Beltrami differentials and G_- is the supercharge that we did *not* use as BRST cohomology generator.

Remark 1 Gromov-Witten theory studies precisely holomorphic maps $f : \Sigma_g \rightarrow \mathbb{X}$. $\mathcal{F}_g(t)$ can be expanded³ as $\sum_{\beta} N_{\beta}^g e^{-\int_{\beta} \omega}$, where $N_{\beta}^g := I_{g,0,\beta} \in \mathbb{Q}$,

$$I_{g,n,\beta}(\phi_1, \dots, \phi_n) := \int_{\overline{\mathcal{M}}_{g,n}(\mathbb{X}, \beta)} \pi_1^*(\phi_1 \otimes \dots \otimes \phi_n), \quad (1.2.3)$$

$\pi_1 : \overline{\mathcal{M}}_{g,n}(\mathbb{X}, \beta) \rightarrow \mathbb{X}^n$, and $\phi_i \in H^{\bullet}(\mathbb{X})$. Correlation functions in the A-model define quantum cohomology rings on target manifolds \mathbb{X} .

Remark 2 \mathcal{F}_g vanish $\forall g \neq 1$ unless $d = 3$ for $\mathbb{X} = CY_d$ (this is precisely the case relevant for the superstring on CY_3) since to get non-zero correlators we need

$$\underbrace{d(g-1)}_{\text{U(1) charge from topological twist}} - \underbrace{3(g-1)}_{G_- \text{ insertions}} = 0 \quad (1.2.4)$$

The prepotential for the three-point correlation function on the sphere can be written in terms of classical intersection numbers and genus-zero Gromov-Witten invariants,

$$\mathcal{F}_0 = \frac{1}{3!} c_{abc} t^a t^b t^c + \sum_{0 \neq \beta} N_{\beta}^0 e^{-\int_{\beta} \omega} \quad (1.2.5)$$

and it is not a monodromy invariant quantity, so the sub-leading classical terms are always given only up to monodromy transformations; namely, \mathcal{F}_0 is ambiguous up to some polynomial $P_2(t) = -\frac{\chi(\mathbb{X})}{2} \zeta(3) - \frac{\pi^2}{6} \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \omega + A_{ij} t^i t^j$. At genus one we have

$$\mathcal{F}_1 = -\frac{1}{24} t^a \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \omega_a + \sum_{0 \neq \beta} N_{\beta}^1 e^{-\int_{\beta} \omega}, \quad (1.2.6)$$

and for higher genera (B are Bernoulli numbers, which one gets integrating some Chern class λ of the Hodge bundle over the moduli space [29])

$$\mathcal{F}_{g \geq 2} = (-1)^{g+1} \frac{\chi(\mathbb{X})}{2} \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!} + \sum_{0 \neq \beta} N_{\beta}^g e^{-\int_{\beta} \omega} \quad (1.2.7)$$

Although several techniques are available,⁴ the quantities N_{β}^g are in general hard to compute; for local toric CY three-folds \mathbb{X} , one can work at large volume $\text{Re} \omega$ and use so-called topological vertex [3, 60, 76, 77]: set $q = e^{ig_s}$ (g_s is the string coupling and $i = \sqrt{-1}$) and $n_{\beta}^{jL} = \sum_{j_R} (-1)^{2j_R} (2j_R + 1) N_{\beta}^{jL, j_R}$, where the numbers $N_{\beta}^{jL, j_R} \in \mathbb{Z}$ are

³The (subtle) dependence of \mathcal{F}_g , which is a section of a line bundle over the moduli space of CY manifolds, on \bar{t} is related to holomorphic anomaly.

⁴For example, localization in the A-model, holomorphic anomaly in the B-model, constraints from modular invariance, and asymptotic properties of generating function series.

related to refined counting of BPS states (more below); if we define the topological string partition function as $Z := \exp \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g$, then [61]

$$Z = \frac{\exp \left(\frac{1}{3!} c_{abc} t^a t^b t^c - \frac{1}{24} t^a \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \omega_a \right)}{\exp \left(-\frac{\zeta(3)}{g_s^2} + \sum_{g=2}^{\infty} g_s^{2g-2} (-1)^g \int_{\mathcal{M}_g} \lambda_{g-1}^3 \right)^{-\frac{\chi(\mathbb{X})}{2}}} \quad (1.2.8)$$

$$\times \prod_{\beta, j_L} \prod_{k_L = -j_L}^{+j_L} \prod_{m=1}^{\infty} \left(1 - q^{2k_L+m} e^{-\int_{\beta} \omega} \right)^{m(-1)^{2j_L} n_{\beta}^{j_L}}$$

Gopakumar-Vafa Topological string can be thought of as a localized version of physical string, i.e. it receives contribution only from special path integral configurations, which can be identified with special configurations of the physical string. At the same time, there exist some BPS observables of physical string for which physical string computation localizes on these same special configurations. So let's adopt target space viewpoint and study super-strings on $M_4 \times \mathbb{X}$: the internal properties of \mathbb{X} lead to physical consequences for observers living in Minkowski space M_4 , the main example being the relation between $\mathcal{N} = 2$ gauge theory in dimension $d = 4, 5$ and the A-model prepotential.⁵

Topological string A-model on \mathbb{X} computes F-terms for type IIA on $M_4 \times \mathbb{X}$ with a constant self-dual graviphoton background $F_+ = g_s$ [9]

$$\int d^4x \int d^4\theta \mathcal{F}_g(X^I) (\mathcal{W}^2)^g = \int d^4x \mathcal{F}_g(t_i) F_+^{2g-2} R_+^2 \quad (1.2.9)$$

where X^I is vector multiplet and $\mathcal{W} = F_+ + \theta^2 R_+ + \dots$, R being the Riemann tensor.

Basing on the duality with heterotic string [8], Gopakumar and Vafa argued that the supersymmetric computation in the $F_+^{2g-2} R_+^2$ background is equivalent to a Schwinger loop computation with just F_+^{2g-2} . If we lift type IIA to M-theory, then compactify on \mathbb{S}^1 and integrate out massive D2-branes and their bound states with D0-branes, the count of such BPS states in $SU(2)_L$ representation⁶ $I_g = \left[\left(\frac{1}{2} \right) \oplus 2(0) \right]^{\otimes g}$ of Lorentz group $SO(4) = SU(2)_L \times SU(2)_R$ gives an *integral*⁷ expansion

$$\log Z = \sum_{g, \beta} \sum_{m=1}^{\infty} G V_{\beta}^g \frac{1}{m} \left(2i \sinh \frac{m g_s}{2} \right)^{2g-2} e^{-m \int_{\beta} \omega}. \quad (1.2.10)$$

1.2.2 Supersymmetric gauge theories

Consider a 4d $\mathcal{N} = 2$ gauge theory on \mathbb{R}^4 with gauge group $G = SU(N)$. Its exact quantum dynamics is obtained by the perturbative one-loop contribution and the contribution from the infinite set of BPS instantons. These corrections can be obtained from

⁵If we compactify type IIA on CY_3 we get 8 supercharges, i.e. $\mathcal{N} = 2$ in 4d. IIA theory on \mathbb{X} has $h^{1,1}(\mathbb{X})$ vector multiplets, $h^{2,1}(\mathbb{X})$ hyper-multiplets and 1 gravity multiplet: let's focus on the vector.

⁶The relation with the previously defined invariants is $\sum_g G V_{\beta}^g I_g = \sum_{j_L} n_{\beta}^{j_L} (j_L)$.

⁷Namely, the numbers $G V_{\beta}^g$ are integers.

a 5d theory with 8 supercharges, compactified on \mathbb{S}^1 , as a one-loop contribution from the set of 5d one-particle BPS states: the BPS particles are W-bosons, 4d instantons (viewed as solitons in the 5d theory) and bound states thereof. The connection with 5d gauge theory allows us to make contact with the GV formulation of topological string, namely if we regard the 5d partition function as a Witten index in the SUSY quantum mechanics with target the ADHM moduli space, this same index is computed by the lift to M-theory of the topological string in the GV construction.

The instanton partition function [83, 85] is computed by using equivariant localization. Instead of working on the moduli space of framed instantons, namely anti self dual connections on \mathbb{S}^4 with fixed second Chern class that are pure gauge at ∞ , modulo gauge transformations, one uses its Gieseker partial compactification and desingularization, which is obtained from the ADHM construction by performing the hyperKähler quotient at a nonzero level of the moment map: it is given in terms of framed rank N coherent torsion-free sheaves on $\mathbb{C}\mathbb{P}^2$, where the framing is given by a choice of trivialization on the line at infinity. Localization is done w.r.t. the Cartan subalgebra of the gauge group G (with parameters usually denoted a_i for $i = 1, \dots, \text{rk } G$) and the lift to moduli space of $U(1)^2$ subgroup of Lorentz group (with parameters denoted ϵ_1, ϵ_2): these localize the integral completely, restricting it to a sum over fixed points of the equivariant action, which can be expressed in terms of Young diagrams.

1.2.3 Geometric engineering

We focus on $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions: their low-energy effective prepotential is holomorphic function of the Coulomb branch moduli a_i and masses of matter fields m_i , and receives contributions from space-time instantons. This class of theories can be engineered using type IIA superstring compactified on local Calabi-Yau 3-fold \mathbb{X} .

To obtain interesting non-abelian gauge groups, we include non-perturbative effects. One way [48, 65] is to consider D2-branes wrapped on compact 2-cycles f_i : they are electrically charged under the RR abelian gauge fields and have masses proportional to the volume t_{f_i} of the wrapped 2-cycles, so we need $t_{f_i} \rightarrow 0$ to have massless charged fields in the vector multiplet.

Regard \mathbb{X} as dimensional reduction of a K3 compactification near an ADE singularity. The gauge group originates in six dimensions from the compact homology of the singularity, with the intersection numbers of f_i equal to the Cartan matrix, while further dimensional reduction on $b = \mathbb{C}\mathbb{P}^1$ leads to $\mathcal{N} = 2$ gauge theory in four dimensions. The bare gauge coupling and Kähler class of b are related by

$$t_b \sim 1/g_{\text{YM}}^2, \quad (1.2.11)$$

which follows from dimensional reducing a form from six to four dimensions. In order to decouple gravity and stringy effects it is sufficient to send the coupling constant to zero, since it pushes the string scale to infinity: this means that we take the limit

$$t_b \rightarrow \infty. \quad (1.2.12)$$

In order to satisfy the Calabi-Yau condition, the compactification space cannot be a direct product, but rather must be fibration

$$\begin{array}{ccc} f & \rightarrow & \mathbb{X} \\ & & \downarrow \\ & & b \end{array} \quad (1.2.13)$$

where the fiber geometry f (with ADE singularity) determines the gauge group while the base geometry b the effective 4d gauge coupling. The fibration structure also allows to include matter via local enhancement of the fiber singularity.

The limits $t_b \rightarrow \infty$ and $t_{f_i} \rightarrow 0$ are not independent. Consider the example of pure SU(2): to obtain the two charged gauge bosons it is sufficient to fiber a \mathbb{CP}_f^1 over the base \mathbb{CP}_b^1 .⁸ In the weak coupling regime the running of the gauge coupling is given by

$$\frac{1}{g_{\text{YM}}^2} \sim \log \frac{m_W}{\Lambda}, \quad (1.2.14)$$

where m_W denotes the mass of the W-bosons and Λ the quantum scale. With the above identifications, we have to take the limit in a way such that $t_b \sim \log t_f$ holds. This example can also be used to understand eq. (1.2.12) from another perspective: from wrapping D2-branes over b we get W-bosons as well; these states are the light degrees of freedom of a different $\text{SU}(2)_b$ theory, which however appears at strong coupling of the six dimensional $\text{SU}(2)_f$ theory from \mathbb{CP}_f^1 , since its Coulomb parameter is given by t_b in eq. (1.2.11). In general this theory will decouple in the $t_b \rightarrow \infty$ limit, which we should take in going from string theory to the field theory limit with gauge group $\text{SU}(2)_f$.

There's an equivalent description [45, 63, 104] in IIA string theory: at the origin of Coulomb branch, an $\mathcal{N} = 2$ gauge theory with gauge group $U(N)$ can be realized on a stack of D4-branes stretched between parallel NS5-branes, as in fig. 1.1. This produces an effective 4 dimensional world-volume for the D4 branes and $\mathcal{N} = 2$ supersymmetry is preserved in 4 dimensions. The D4-branes sit at the origin of the $x_4 + ix_5$ coordinate, and the distance between NS5-branes in the x_6 direction (complexified to $x_6 + ix_{10}$ in M-theory lift), measured at various positions in $x_4 + ix_5$, is the running gauge coupling. At a generic point on the Coulomb branch, the D4-branes move apart, with $x_4 + ix_5$ positions corresponding to the eigenvalues of the adjoint scalar. The end points of the D4 branes on the NS5 branes are singular. However an approximate description is that the D4 branes exert a force on the NS5 branes causing them to bend. The resulting theory is then engineered in a toric geometry [73], where the skeleton of toric space is identified with the brane configuration.

⁸The different ways this can be done are labeled by an integer and the corresponding geometries correspond to Hirzebruch surfaces. The Calabi-Yau 3-fold is the total space of the anti-canonical bundle over this complex surface. All Hirzebruch surfaces give rise to pure SU(2) in four dimensions, with different Chern-Simons level if seen from the 5d perspective.

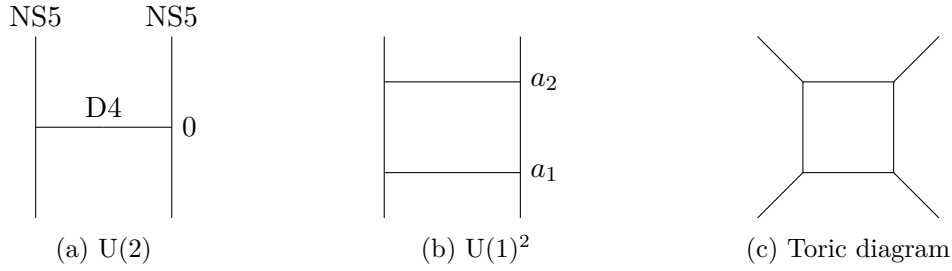


Figure 1.1: Equivalence of various geometric engineering setups for $N = 2$. In the first two pictures, direction $x_4 + ix_5$ is vertical, x_6 horizontal and x_7 comes out of the plane.

The useful property of type IIA string construction is that space-time instanton corrections are mapped to world-sheet instanton corrections: the Euclidean string world-sheet is wrapped around the 2-cycles of the geometry with worldsheet instanton action $S \sim d_b t_b + d_f t_f$, where d_b and d_f refer to wrapping numbers. This means that we do not need to consider the full type IIA string theory to investigate the gauge theory. Rather, the topological sector is sufficient, i.e. the topological string amplitudes that capture world-sheet instanton corrections.

Differences in Coulomb eigenvalues become fiber Kähler parameters, here $e^{-t_f} \sim e^{-\beta(a_1 - a_2)}$ with β the M-theory circle radius, and the scale Λ becomes a base Kähler parameter, $(\beta\Lambda)^4 \sim e^{-t_b}$.

Since the geometric engineering limit involves $t_{f_i} \rightarrow 0$, the compactification geometry is singular, and we are not expanding the string amplitudes around the large volume point in moduli space. Hence, if we compute the topological string amplitudes using the topological vertex, which is valid at large volume, these amplitudes have to be analytically continued before we can take the limit. If we do so, we can indeed check in many examples, e.g. toric $U(N)$ geometries, that Nekrasov partition function matches with the calculation from topological vertex, both in the perturbative sector, and in the instanton sector [28, 57, 58].

1.3 The project

The Ph.D. project is related to the unoriented A-model topological string that J. Walcher has worked on [101]. The basic idea is that one would like to include in the perturbative sum over genera also surfaces with boundaries and crosscaps, and count (appropriately defined) holomorphic maps from these surfaces to a Calabi-Yau three-fold target.

At the computational level, one can either employ localization techniques in the A-model setup (essentially a modified version of results by Kontsevich) or apply mirror symmetry and integrate the B-model holomorphic anomaly equations: one is forced by mathematical consistency (related to the appearance of otherwise ill-defined quantities, and to the existence of real codimension one boundaries in moduli space; these issues have also been studied by Georgieva and Zinger [33, 34]) to combine the rational num-

bers so obtained, and interestingly it's possible to do that in a consistent way and get integer multiplicities, which suggest a possible BPS interpretation, in the spirit of what Gopakumar-Vafa, and Ooguri-Vafa did for closed oriented and open oriented strings respectively.

To define the model properly, one must impose some “topological tadpole cancellation conditions” that correspond to having one boundary, i.e. one D-brane, as seen in the covering. The unoriented topological A-model has a physical theory corresponding to type IIA with an O4-plane, wrapped on a slag 3-cycle of the CY. The “tadpole condition” requires to introduce one D4-brane on the same 3-cycle, and the question is what is special about having one (rather than more, or less) D4.

Our earlier result [89] shows that the configuration of one O4 and one D4 has an interesting lift to M-theory, specifically a purely geometrical background corresponding to an orbifold by the antiholomorphic involution on the CY times a half-shift on the M-theory circle. This allows for a physical definition of the unoriented Gopakumar-Vafa invariants that Walcher (together with Krefl as well) computes by localization and holomorphic anomaly matching, and it explains the cancellation mechanism from a physical viewpoint, namely by canceling contributions from curves that are equal except for trading a crosscap for a disk by means of a relative sign produced by the C_3 form in M-theory surrounding a crosscap, and suggesting a way to glue along real codimension one boundaries in moduli space.

One can also study the above system from the gauge theory viewpoint, and ask whether there's some relation between the real topological string and some modified version of Nekrasov instanton partition function. This is expected on general grounds (duality between topological strings on CY3's and $\mathcal{N} = 2$ gauge theories in 4d/5d) basing on geometric engineering arguments and decoupling of gravity.

What we discovered [47] is that one can start from $\mathcal{N} = 2$ gauge theory in 4d and perform a shift in one of the ϵ -background equivariant parameters, corresponding to the directions transverse to the O4-plane in spacetime. This matches almost exactly (see the subtlety below) the real topological string computations for local toric CY3's, which can be done with the help of a modified version of the topological vertex taking into account the quotient by toric involutions [70], and it suggests a possible gauge theory interpretation for the real topological string.

A natural question one could ask, from the physical superstring viewpoint, is whether the real topological amplitude computes any effective term/correction for superstring in self-dual graviphoton background, similarly to what has been done for the closed oriented case [9]. The answer seems to be positive, and we are able to understand the physical meaning of open and unoriented topological amplitudes: one can check that the relevant RR operators survive orientifold projection, and then the real topological string computation is mapped via a covering trick to a computation on a closed oriented Riemann surface, but restricted to left-movers only. One can also study the heterotic dual of this, and try to recover the \sinh^{-1} power that was previously found in the 2d

Schwinger computation.

1.4 Organization

The rest of the Thesis is organized as follows.

Chapter 2 deals with the M-theory interpretation of the real topological string. In section 2.2 we review the Gopakumar-Vafa reformulation of the closed oriented topological A-model. In section 2.3 we review the properties of the real topological string. In section 2.4 we present the physical IIA theory corresponding to this topological model and construct its M-theory lift (section 2.4.1), compute the partition function in terms of M-theory BPS invariants (section 2.4.2), and describe the M-theory explanation of the tadpole cancellation conditions (section 2.4.3). In section 2.5 we describe related systems, by the inclusion of additional brane pairs (section 2.5.1), or by using other orientifold plane structures (section 2.5.2); in this respect, we clarify certain misidentifications of the M-theory lifts in the earlier literature on unoriented topological models. Section 2.A reviews the basics of the real topological string, while section 2.B discusses the physical couplings computed by the real topological string.

Chapter 3 is about the connection with gauge theory. In section 3.2 we review the M-theory/gauge theory correspondence in the oriented case: in section 3.2.1 we review the topological vertex computation of topological string partition functions on local CY three-folds, in section 3.2.2 we describe the computation of the gauge theoretic Nekrasov partition function via localization. In section 3.3 we review the computation of real topological string amplitudes: section 3.3.1 introduces some general considerations of unoriented theories, and section 3.3.2 describes the real topological string computation using the real topological vertex. Explicit examples are worked out in section 3.4, like the conifold ($U(1)$ gauge theory) in section 3.4.1, where we correct some typos in the previously known result, and the pure $SU(N)$ theories in section 3.4.2, where we also discuss their behavior under flop transitions. In section 3.5 we describe a twisted Nekrasov partition function, whose structure is motivated by the action of the orientifold, and compare it with the real topological string partition function. Section 3.A reviews aspects of the real topological string and the topological vertex formulation, section 3.B presents some new enumerative checks of the BPS integrality, and section 3.C gathers some useful identities.

Chapter 4 deals with the superstring correlation functions that compute real topological string amplitudes: in section 4.2.1 we explain that indeed some operators survive orientifold projection, in section 4.2.2 we discuss the supergravity perspective, and in section 4.2.3 we present the appropriate orientifold invariant vertices that appear in the correlators, which are computed in full detail in section 4.2.4. Section 4.A presents the same method applied to the standard closed oriented case, while section 4.B lists some useful facts about theta functions.

Finally, we offer our conclusions in chapter 5.

Chapter 2

M-theory interpretation

2.1 Introduction

Topological string theory is a fertile arena of interplay between physics and mathematics. A prominent example is the physics-motivated reformulation of the topological A-model on a threefold \mathbb{X}_6 in terms of integer multiplicities of BPS states in the 5d compactification of M-theory on \mathbb{X}_6 [38, 39], and the corresponding mathematical reformulation of the (in general fractional) Gromov-Witten invariants in terms of the *integer* Gopakumar-Vafa invariants (see also [55]).

A natural generalization is to consider A-models with different worldsheet topologies. In particular, there is a similar story for the open topological A-model, in which worldsheets are allowed to have boundaries mapped to a lagrangian 3-cycle in \mathbb{X}_6 , and which via lift to M-theory admits an open BPS invariant expansion [86]. There has also been substantial work to define unoriented topological A-models, for instance in terms of the so-called real topological strings [70–72, 101]. The latter was proposed to require a specific open string sector for consistency, and conjectured to admit a BPS-like expansion ansatz, although no physical derivation in terms of M-theory was provided. Conversely, although some unoriented topological models have been proposed directly from the M-theory picture [1, 4, 20, 21, 90], they do not correspond to this real topological string.

We fill this gap, construct the physical theory corresponding to the real topological string, and show that its M-theory lift reproduces the topological string partition function in terms of certain BPS invariants, which we define and show to be the equivariant subsector of the corresponding closed oriented Gopakumar-Vafa invariants. Along the way, the M-theory picture sheds new light into certain peculiar properties of the topological model, like the so-called tadpole cancellation condition, which requires combining open and unoriented worldsheets in order to produce well-defined amplitudes and integer invariants. Although [101] focused on the quintic and other simple examples (see also [70–72]), we keep the discussion general, using these examples only for illustration at concrete points.

2.2 Review of Gopakumar-Vafa expansion

We start with a brief review of the Gopakumar-Vafa interpretation of the closed oriented topological string in terms of BPS states in M-theory [38, 39].

The 4d compactification of type IIA on a CY threefold \mathbb{X}_6 provides a physical realization of the topological A-model on \mathbb{X}_6 , whose genus g partition function $F_g(t_i)$, which depends on the Kähler moduli t_i , computes the F-term

$$\int d^4x \int d^4\theta F_g(t_i) (\mathcal{W}^2)^g \rightarrow \int d^4x F_g(t_i) F_+^{2g-2} R_+^2 \quad (2.2.1)$$

(where the second expression applies for $g > 1$ only). Here we have used the $\mathcal{N} = 2$ Weyl multiplet, schematically $\mathcal{W} = F_+ + \theta^2 R_+ + \dots$, with F_+ , R_+ being the self-dual components of the graviphoton and curvature 2-form, respectively. These contributions are summed up if we turn on a self-dual graviphoton background in the four non-compact dimensions

$$F_+ = \frac{\lambda}{2} dx^1 \wedge dx^2 + \frac{\lambda}{2} dx^3 \wedge dx^4. \quad (2.2.2)$$

The sum is given by the total A-model partition function, with coupling λ

$$\mathcal{F}(t_i) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t_i). \quad (2.2.3)$$

There is an alternative way to compute this same quantity, by considering the lift of the IIA configuration to M-theory, as follows. We start with the 5d compactification of M-theory on \mathbb{X}_6 . There is a set of massive half BPS particle states, given by either the dimensional reduction of 11d graviton multiplets, or by M2-branes wrapped on holomorphic 2-cycles. These states are characterized by their quantum numbers under the 5d little group $SU(2)_L \times SU(2)_R$. Note that at the classical level, each such particle can have a classical moduli space, but at the quantum level there is only a discrete set of ground states, which provide the BPS particle states we are interested in. For instance, an 11d particle (such as the 11d graviton) has a classical moduli space given by \mathbb{X}_6 itself, but quantization leads to wave functions given by the cohomology of \mathbb{X}_6 , resulting in a net BPS multiplicity given by $\chi(\mathbb{X}_6)$.

In order to relate to type IIA, we compactify on an S^1 . Corrections to the R^2 term will arise from one-loop diagrams in which the above BPS particles run, in the presence of the graviphoton field, which couples to their $SU(2)_L$ quantum numbers. In type IIA language this corresponds to integrating out massive D0- and D2-brane states (and their bound states). In the Schwinger proper time formalism we have

$$\begin{aligned} \mathcal{F} &= \int_{\epsilon}^{\infty} \frac{ds}{s} \text{tr}_{\mathcal{H}} \left[(-1)^F e^{-s(\Delta + m^2 + J \cdot F_+)} \right] \\ &= \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{k \in \mathbb{Z}} \frac{1}{4 \sinh^2(\frac{s\lambda}{2})} \text{tr}_{\mathcal{H}} \left[(-1)^F e^{-2s\lambda J_3^L - sZ - 2\pi i s k} \right]. \end{aligned} \quad (2.2.4)$$

Here the \sinh^2 factor arises from the 4d kinematics, we have included a sum over KK momenta along the \mathbb{S}^1 , the trace is over the Hilbert space \mathcal{H} of 5d one-particle BPS states, with central charge Z , and $F = 2J_3^L + 2J_3^R$.

The Hilbert space \mathcal{H} of 5d one-particle BPS states from an M2-brane on a genus g holomorphic curve Σ_g (in general not the same as the genus of the worldsheet in the type IIA interpretation) in the homology class β is obtained by quantization of zero modes on its worldline. Quantization of the universal Goldstinos contributes to the state transforming as a (half) hypermultiplet, with $SU(2)_L$ representation $I_1 = \left(\frac{1}{2}\right) \oplus 2(0)$. There are in general additional zero modes, characterized in terms of the cohomology groups

$$\mathcal{H} = H^\bullet(\mathcal{M}_{g,\beta}) \otimes H^\bullet(\mathbb{T}^{2g}). \quad (2.2.5)$$

The first factor corresponds to zero modes from the deformation moduli space $\mathcal{M}_{g,\beta}$ of Σ_g in \mathbb{X}_6 , whose quantization determines the $SU(2)_R$ representation. The latter is decoupled from the self-dual graviphoton background, so it only contributes as some extra overall multiplicity in the above trace.¹

The second factor corresponds to zero modes arising from flat connections on the type IIA D2-brane worldvolume gauge field on Σ_g . The \mathbb{T}^{2g} should be regarded as the Jacobian of Σ_g , $\text{Jac } \Sigma_g = \mathbb{T}^{2g}$. Quantization of these zero modes determines further contributions to the $SU(2)_L$ representation of the state as dictated by the $SU(2)$ Lefschetz decomposition of cohomology of \mathbb{T}^{2g} , i.e. with creation, annihilation and number operators

$$J_+ = k\wedge, \quad J_- = k\lrcorner, \quad J_3 = (\text{deg } -n)/2. \quad (2.2.6)$$

Here k is the Kähler form of the torus, \lrcorner denotes contraction, the bidegree deg is $p + q$ for a (p, q) -form and n is complex dimension.

The $SU(2)$ representation is of the form $I_g = I_1^{\otimes g}$, where $I_1 = \left(\frac{1}{2}\right) \oplus 2(0)$. For instance, for $g = 1$ we have a ground state 1 and operators dz and $d\bar{z}$, so that the cohomology of \mathbb{T}^2 splits as

$$\begin{pmatrix} 1 \\ k \end{pmatrix} dz \pm d\bar{z} \quad (2.2.7)$$

where $k \sim dz \wedge d\bar{z}$. These form the representation I_1 . The argument generalizes straightforwardly to higher genera.

The contribution from a state in the $SU(2)_L$ representation I_g to the trace is given by $(-4)^g \sinh^{2g} \frac{s\lambda}{2}$, so we get

$$\mathcal{F} = - \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{g,\beta} \sum_{k \in \mathbb{Z}} \text{GV}_{g,\beta} \left(2i \sinh \frac{s\lambda}{2} \right)^{2g-2} e^{-s\beta \cdot t} e^{-2\pi i s k}, \quad (2.2.8)$$

where we write $Z = \beta \cdot t$. Also, $\text{GV}_{g,\beta}$ are integers describing the multiplicity of BPS states arising from M2-branes on a genus g curve in the class $\beta \in H_2(\mathbb{X}_6; \mathbb{Z})$, with the

¹This is no longer true if one considers refined topological strings as in [2, 4, 24, 42, 60], corresponding to a non self-dual background field of the form $F = \epsilon_1 dx^1 \wedge dx^2 - \epsilon_2 dx^3 \wedge dx^4$.

understanding that $\beta = 0$ corresponds to 11d graviton states. This multiplicity includes that arising from the $SU(2)_R$ representations, in the following sense. Describing the set of BPS states in terms of their $SU(2)_L \times SU(2)_R$ representations

$$\left[\left(\frac{1}{2}, 0 \right) \oplus 2(0, 0) \right] \otimes \sum_{j_L, j_R} N_{j_L, j_R}^\beta(j_L, j_R) \quad (2.2.9)$$

we have

$$\sum_{j_L, j_R} (-1)^{2j_R} (2j_R + 1) N_{j_L, j_R}^\beta(j_L) = \sum_g \text{GV}_{g, \beta} I_g. \quad (2.2.10)$$

Going back to eq. (2.2.8), we use Poisson resummation

$$\sum_{k \in \mathbb{Z}} e^{-2\pi i s k} = \sum_{m \in \mathbb{Z}} \delta(s - m) \quad (2.2.11)$$

to get

$$\mathcal{F} = - \sum_{g, \beta} \sum_{m=1}^{\infty} \text{GV}_{g, \beta} \frac{1}{m} \left(2i \sinh \frac{m\lambda}{2} \right)^{2g-2} e^{-m\beta \cdot t}. \quad (2.2.12)$$

This is known as the GV or BPS expansion of the closed oriented topological string amplitude. There are similar expansions for open oriented topological string amplitudes (see section 2.5.1 for more details). The situation for unoriented topological string amplitudes is the main topic of this chapter.

2.3 Walcher's real topological string

A prominent example of unoriented topological string is Walcher's real topological string introduced in [101] (see also [70–72]). It includes both open and closed unoriented topological strings, subject to a mysterious ‘tadpole cancellation condition’ requiring the open string sector to be described by a single D-brane on top of the fixed locus of the orientifold action. We now review its basic features, the proposed tadpole cancellation condition, and the conjectured expansion in terms of integer multiplicities. For other details, see section 2.A. For simplicity, we take the case of a single Kähler modulus, although the generalization is straightforward. Also, examples in the literature [70–72, 101] have considered cases with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$, for instance the quintic, or local \mathbb{CP}^2 . Since our description in M-theory is more general, and this condition will only play a role in section 2.4.3, we keep the description general here as well.

The A-model target space is a Calabi-Yau threefold \mathbb{X}_6 , equipped with an antiholomorphic involution σ , whose pointwise fixed set is a lagrangian 3-cycle denoted by L . The model is defined by considering maps of (possibly non-orientable) surfaces (possibly with boundaries) into \mathbb{X}_6 , with boundaries lying in L (so that the maps are topologically classified by the relative homology $d = f_*([\Sigma]) \in H_2(\mathbb{X}_6, L; \mathbb{Z})$), and with compatible orientifold actions on the target and worldsheet, as follows. We construct the non-orientable

surface Σ as the quotient of the parent oriented Riemann surface $\widehat{\Sigma}$ by an antiholomorphic involution Ω of the worldsheet,² and demand equivariance of the holomorphic map f as in fig. 2.1.

$$\begin{array}{ccc}
 \widehat{\Sigma} & \xrightarrow{\Omega} & \widehat{\Sigma} \\
 f \downarrow & \circlearrowleft & \downarrow f \\
 \mathbb{X}_6 & \xrightarrow{\sigma} & \mathbb{X}_6
 \end{array}$$

Figure 2.1: Equivariance means the diagram is commutative (we are improperly calling f both the equivariant map and its lift $\Sigma \rightarrow X/\sigma$.)

In the relation $\Sigma = \widehat{\Sigma}/\Omega$, the particular case in which Σ is itself closed oriented and $\widehat{\Sigma}$ has two connected components is not included.

The topological classification of possibly non-orientable surfaces Σ with boundaries, described as symmetric Riemann surfaces, written $(\widehat{\Sigma}, \Omega)$, generalizes the closed oriented case, through the following classic result:

Theorem 2.3.1. *Let $h = h(\widehat{\Sigma}, \Omega) \geq 0$ be the number of components of the fixed point set $\widehat{\Sigma}_\Omega$ of Ω in $\widehat{\Sigma}$ (i.e. the number of boundaries of Σ), and introduce the index of orientability $k = k(\widehat{\Sigma}, \Omega)$, given by $(2 - \# \text{ components of } \widehat{\Sigma} \setminus \widehat{\Sigma}_\Omega)$. Then the topological invariants h and k together with the genus \hat{g} of $\widehat{\Sigma}$ determine the topological type of $(\widehat{\Sigma}, \Omega)$ uniquely. For fixed genus \hat{g} , these invariants satisfy*

- (i) $k = 0$ or $k = 1$ (corresponding to oriented surfaces, or otherwise)
- (ii) if $k = 0$, then $0 < h \leq \hat{g} + 1$ and $h \equiv \hat{g} + 1 \pmod{2}$
- (iii) if $k = 1$ then $h \leq \hat{g}$.

Let us define the (negative of the) Euler characteristic of $\widehat{\Sigma}/\Omega$ by $\chi = \hat{g} - 1$. It is useful to separate the worldsheets into three classes, corresponding to having 0, 1 or 2 crosscaps (recall that two crosscaps are equivalent to a Klein handle, namely two holes glued together with an orientation reversal, which in the presence of a third crosscap can be turned into an ordinary handle). This leads to a split of the topological amplitudes into classes, namely: closed oriented surfaces (with amplitude denoted by $\mathcal{F}^{(g_\chi)}$, with $g_\chi = \frac{1}{2}\chi + 1$ the number of handles), oriented surfaces with h boundaries (with amplitude $\mathcal{F}^{(g,h)}$), non-orientable surfaces with an odd number of crosscaps (with amplitude $\mathcal{R}^{(g,h)}$) and non-orientable surfaces with an even number of crosscaps ($\mathcal{K}^{(g,h)}$). The Euler characteristic is given by $\chi = 2g - 2 + h + c$, with $c = 0, 1$.³

²This is equivalent to considering a dianalytic structure on the surface, which generalizes the notion of complex structure by allowing for antiholomorphic transition functions.

³Note that we define g such that the negative Euler characteristic is $2g + h - 2$, $2g + h - 1$, and $2g + h - 2$ in the \mathcal{F} , \mathcal{R} and \mathcal{K} cases respectively, i.e. it also accounts for Klein handles.

The basic tool used to compute these amplitudes is equivariant localization on the moduli space \mathcal{M} of stable maps, following ideas going back to [69] (see also [26], and [22, 36] for more recent developments on the formal side). Localization is with respect to a torus action which is compatible with the involution, and leads to a formulation in terms of the diagram techniques of [69].

Ref. [101] finds that, in the example of the quintic or local $\mathbb{C}\mathbb{P}^2$, in order to apply this machinery to unoriented and/or open worldsheets, some constraints, dubbed *tadpole cancellation* conditions, have to be imposed: as we discuss in section 2.A, this is a cancellation between contributions from worldsheets with an unpaired crosscap and worldsheets with boundaries, with one boundary ending on L ‘with even degree’ (specifically, wrapping the generator of $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$ an even number of times, hence begin topologically trivial). This results in a condition

$$d \equiv h \equiv \chi \pmod{2}, \tag{2.3.1}$$

where $d \in H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$ is the relevant homology class. It implies that \mathcal{R} -type amplitudes do not contribute, $\mathcal{R}^{(g,h)} \equiv 0$.

Mathematically, this condition applies to real codimension one boundary strata in moduli space, in which a given worldsheet piece near L develops a node which can be smoothed to yield either a disk or a crosscap. The combined count of these homologically trivial disks and crosscaps leads to cancellation of potentially ill-defined pieces, and produces an invariant count.

Strong evidence for this consistency condition comes from the fact that the invariant numbers thus computed turn out to be all integers. This motivated the proposal of an ansatz reminiscent of a BPS expansion, as a sum over holomorphic *embeddings* (rather than maps) equivariant with respect to worldsheet parity $\Omega : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$.

If we write the total topological amplitude as

$$\mathcal{G}^{(\chi)} = \frac{1}{2} \left[\mathcal{F}^{(g_\chi)} + \sum \mathcal{F}^{(g,h)} + \sum \mathcal{K}^{(g,h)} \right], \tag{2.3.2}$$

the conjecture is

$$\sum_{\chi} i^{\chi} \lambda^{\chi} \left(\mathcal{G}^{(\chi)} - \frac{1}{2} \mathcal{F}^{(g_\chi)} \right) = \sum_{\substack{\chi \equiv d \pmod{2} \\ k \text{ odd}}} \widetilde{\text{GV}}_{\hat{g},d} \frac{1}{k} \left(2 \sinh \frac{\lambda k}{2} \right)^{\chi} q^{kd/2}. \tag{2.3.3}$$

In more physical terms, the tadpole cancellation condition means that the background contains a single D-brane wrapped on L , as counted in the covering space. The interpretation in terms of a physical type IIA construction and its lift to M-theory will be discussed in the next section.

2.4 M-theory lift and BPS expansion

2.4.1 Tadpole cancellation, the O4/D4 system and M-theory

It is natural to look for a physical realization of the real topological string in terms of type IIA on the threefold \mathbb{X}_6 , quotiented by worldsheet parity times an involution acting

antiholomorphically on \mathbb{X}_6 . In general, we consider involutions with a fixed point set along the lagrangian 3-cycle L , which therefore supports an orientifold plane. The total dimension of the orientifold plane depends on the orientifold action in the 4d spacetime, and can correspond to an O6-plane or an O4-plane. The choice of 4d action is not specified in the topological string, but can be guessed as follows.

We expect that the topological tadpole cancellation condition has some translation in the physical theory, as a special property occurring when precisely one D-brane (as counted in the covering space) is placed on top of the orientifold plane. Since the charge of a negatively charged O4-plane is -1 (in units of D4-brane charge in the covering), the configuration with a single D4-brane stuck on top of it is special, because it cancels the RR charge locally (on the other hand, the charge of an O6-plane is -4 , and no similarly special property occurs for a single stuck D6-brane).

The presence of a single D4-brane stuck on the O4-plane is not a consistency requirement of the type IIA theory configuration,⁴ but rather a condition that we will show leads to a particularly simple M-theory lift, and a simple extension of the Gopakumar-Vafa BPS expansion of topological amplitudes. This nicely dovetails the role played by tadpole cancellation in the real topological string to achieve the appearance of integer invariants.

The M-theory lift of O4-planes with and without D4-branes has been discussed in [37, 49]. In particular, a negatively charged O4-plane with no stuck D4-brane, spanning the directions 01234 in 10d Minkowski space M_{10} , lifts to M-theory on a \mathbb{Z}_2 orbifold $M_5 \times (\mathbb{R}^5/\mathbb{Z}_2) \times \mathbb{S}^1$ with generator $(x^5, \dots, x^9) \rightarrow (-x^5, \dots, -x^9)$, and which also flips the M-theory 3-form, $C_3 \rightarrow -C_3$. The latter action is required to be a symmetry of the M-theory Chern-Simons term, and matches the effect of the type IIA orientifold action on the NSNS 2-form B_2 . Hence, we will classify the M-theory action as ‘orientifold’ as well.

The construction generalizes to compactification on \mathbb{X}_6 , with the orientifold acting holomorphically on \mathbb{X}_6 . It produces M-theory on the quotient $(M_4 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$, with the \mathbb{Z}_2 acting as the antiholomorphic involution σ on \mathbb{X}_6 and as $x^2, x^3 \rightarrow -x^2, -x^3$ on 4d Minkowski space. This system, and its generalization with additional D4-brane pairs (M5-branes in M-theory), is discussed in section 2.5.2. Here we simply note that the explicit breaking of the $SU(2)_L$ symmetry already in the 5d theory makes necessary to make certain assumptions on the structure of BPS multiplets in the theory, obscuring the derivation of the BPS expansion of the amplitude.

The M-theory lift of a negatively charged O4-plane with a stuck D4-brane is however much simpler, and in particular does not suffer from these difficulties. Because of the already mentioned local cancellation of the RR charge, the M-theory lift is a completely smooth space described by a freely acting quotient $M_5 \times (\mathbb{R}^5 \times \mathbb{S}^1)/\mathbb{Z}_2$, with generator acting as $(x^5, \dots, x^9) \rightarrow (-x^5, \dots, -x^9)$, as a half-period shift on \mathbb{S}^1 ($y \rightarrow y + \pi$

⁴Notice that any configuration with additional pairs of D4-branes is continuously connected to it. See section 2.5.1 for further discussion. Also, topological A-models related to systems of O4-planes with no stuck D4-brane, and their M-theory interpretation, have appeared in [4], see section 2.5.2 for further discussion.

for periodicity $y \simeq y + 2\pi$), and flipping the 3-form C_3 (hence defining an M-theory orientifold).

Concerning the latter, it is important to point out that the negative charge of the O4-plane implies that there is a half-unit NSNS B_2 background on an \mathbb{RP}^2 surrounding the O4-plane; consequently, there is a non-trivial half-unit of 3-form background on the corresponding M-theory lift $(\mathbb{CP}^1 \times \mathbb{S}^1)/\mathbb{Z}_2$. This will play an important role in the M-theory interpretation of the disk/crosscap tadpole cancellation, see section 2.4.3.

The construction generalizes to compactification on \mathbb{X}_6 , with the orientifold acting holomorphically on \mathbb{X}_6 . It produces M-theory on the quotient $(M_4 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 acting as

$$\begin{aligned} \mathbb{X}_6 : \quad x &\mapsto \sigma(x), & \mathbb{S}^1 : \quad y &\mapsto y + \pi, \\ \text{Minkowski} : \quad x^0, x^1 &\mapsto x^0, x^1, & x^2, x^3 &\mapsto -x^2, -x^3. \end{aligned} \tag{2.4.1}$$

The geometry is a (Möbius) fiber bundle with base \mathbb{S}^1 , fiber $M_4 \times \mathbb{X}_6$, and structure group \mathbb{Z}_2 .

As before, it is straightforward to add extra D4-brane pairs away from (or on top of) the O4-plane, since they lift to extra M5-brane pairs in M-theory, see section 2.5.1.

2.4.2 M-theory BPS expansion of the real topological string

The M-theory configuration allows for a simple Gopakumar-Vafa picture of amplitudes, which should reproduce the real topological string amplitudes. Since the quotient is acting on the M-theory \mathbb{S}^1 as a half-shift, its effect is not visible locally on the \mathbb{S}^1 . This means that the relevant 5d picture is exactly the same as for the closed oriented setup, c.f. section 2.2, so the relevant BPS states are counted by the standard Gopakumar-Vafa invariants. When compactifying on \mathbb{S}^1 and quotienting by \mathbb{Z}_2 , these states run in the loop as usual, with the only (but crucial) difference that they split according to their parity under the M-theory orientifold action. In the Möbius bundle picture, even components of the original $\mathcal{N} = 2$ multiplets will run on \mathbb{S}^1 with integer KK momentum, whereas odd components run with half-integer KK momentum.

The split is also in agreement with the reduction of supersymmetry by the orientifold, which only preserves 4 supercharges. Note also that the orientifold is not 4d Poincaré invariant, as Lorentz group is broken as

$$\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R \rightarrow \text{U}(1)_L \times \text{U}(1)_R. \tag{2.4.2}$$

The preserved supersymmetry is not 4d $\mathcal{N} = 1$ SUSY, and in particular it admits BPS particles.

General structure

The states in the Hilbert space \mathcal{H} are groundstates in the SUSY quantum mechanics on the moduli space of wrapped M2-branes. In the orientifold model, these BPS states of the 5d theory fall into two broad classes.

Non-invariant states and the closed oriented contribution

Consider a BPS state $|A\rangle$ associated to an M2 on a curve Σ_g not mapped to itself under the involution σ ; there is an image multiplet $|A'\rangle$ associated to the image curve⁵ Σ'_g . We can now form orientifold even and odd combinations $|A\rangle \pm |A'\rangle$, which run on the \mathbb{S}^1 with integer or half-integer KK momentum, respectively. Since each such pair has $Z_A = Z_{A'}$ and identical multiplet $SU(2)_L \times SU(2)_R$ structure and multiplicities (inherited from the parent theory), we get the following structure:

$$\begin{aligned}
& \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{k \in \mathbb{Z}} \frac{1}{4 \sinh^2(\frac{s\lambda}{2})} e^{-sZ} \operatorname{tr} \left\{ \left[(-1)^F e^{-2s\lambda J_3^L - 2\pi i s k} \right] \right. \\
& \quad \left. + \left[(-1)^F e^{-2s\lambda J_3^L - 2\pi i s(k + \frac{1}{2})} \right] \right\} \\
&= \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{1}{4 \sinh^2(\frac{s\lambda}{2})} e^{-sZ} \operatorname{tr} \left[(-1)^{2J_3^L + 2J_3^R} e^{-2s\lambda J_3^L} \right] \sum_{k \in \mathbb{Z}} \left(e^{-2\pi i s k} + e^{-2\pi i s(k + \frac{1}{2})} \right) \\
&= -2 \int_{\epsilon}^{\infty} \frac{ds}{s} n_{\Sigma_g} \left(2i \sinh \frac{s\lambda}{2} \right)^{2g-2} e^{-sZ} \sum_m \delta(s - 2m) \\
&= -2 \sum_{\substack{\text{even } p > 0}} n_{\Sigma_g} \frac{1}{p} \left(2i \sinh \frac{p\lambda}{2} \right)^{2g-2} e^{-pZ}.
\end{aligned} \tag{2.4.3}$$

The \sinh^{-2} factor corresponds to 4d kinematics, since the orientifold imposes no restriction on momentum in the directions transverse to the fixed locus. We have also denoted n_{Σ_g} the possible multiplicity arising from $SU(2)_R$ quantum numbers. Clearly, because the states are precisely those in the parent $\mathcal{N} = 2$ theory, summing over multiplets reproduces the Gopakumar-Vafa expansion eq. (2.2.12) of the closed oriented contribution to the topological string partition function: this is because even wrappings p on the orientifold \mathbb{S}^1 correspond to both even and odd wrappings on the closed-oriented \mathbb{S}^1 , due to the reduction of the \mathbb{S}^1 by half. This closed oriented contribution must be duly subtracted from the total amplitude, in order to extract the genuine contribution associated to equivariant curves, reproducing the open and unoriented piece of the topological amplitude; this nicely reproduces the subtraction of the closed oriented contribution in the left hand side of eq. (2.3.3).

The conclusion is that contributions with even wrapping belong to the sector of non-invariant states, which heuristically describe disconnected curves in the cover and reproduce the closed oriented topological string.

Invariant states and the open and unoriented contributions

The second kind of BPS states correspond to M2-branes wrapped on curves $\widehat{\Sigma}_{\hat{g}}$ in the cover, mapped to themselves under σ . The overall parity of one such state is determined by the parities of the states in the corresponding $SU(2)_L$ and $SU(2)_R$ representations.

⁵Actually, the BPS state wavefunctions may be partially supported in the locus in moduli space where the curve and its image combine into an irreducible equivariant curve. Hence, part of the contribution of non-invariant states in this section may spill off to contributions of invariant states discussed later.

We introduce the spaces $\mathcal{H}_{\pm}^{\hat{g}}$ describing the even/odd pieces of the $SU(2)_L$ representation $I_{\hat{g}}$ for fixed \hat{g} . In what follows, we drop the \hat{g} label to avoid cluttering notation. We similarly split the equivariant BPS invariant $\text{GV}'_{\hat{g},\beta}$ (i.e. after removing the pairs of states considered in the previous discussion) into even/odd contributions as

$$\text{GV}'_{\hat{g},\beta} = \text{GV}'_{\hat{g},\beta}^{'+} + \text{GV}'_{\hat{g},\beta}{'-}. \quad (2.4.4)$$

Recalling that states with even/odd overall parity have integer/half-integer KK momenta, we have a structure

$$\begin{aligned} & \sum_{\beta,\hat{g}} e^{-sZ} \left[\left(\text{GV}'_{\hat{g},\beta}^{'+} \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{GV}'_{\hat{g},\beta}{'-} \text{tr}_{\mathcal{H}_-} \mathcal{O} \right) \sum_{k \in \mathbb{Z}} e^{-2\pi i s k} \right. \\ & \quad \left. + \left(\text{GV}'_{\hat{g},\beta}^{'+} \text{tr}_{\mathcal{H}_-} \mathcal{O} + \text{GV}'_{\hat{g},\beta}{'-} \text{tr}_{\mathcal{H}_+} \mathcal{O} \right) \sum_{k \in \mathbb{Z}} e^{-2\pi i s (k + \frac{1}{2})} \right] \\ &= \sum_{\beta,\hat{g}} e^{-sZ} \sum_{m \in \mathbb{Z}} \delta(s - m) \left\{ \left[\text{GV}'_{\hat{g},\beta}^{'+} \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{GV}'_{\hat{g},\beta}{'-} \text{tr}_{\mathcal{H}_-} \mathcal{O} \right] \right. \\ & \quad \left. + (-1)^m \left[\text{GV}'_{\hat{g},\beta}^{'+} \text{tr}_{\mathcal{H}_-} \mathcal{O} + \text{GV}'_{\hat{g},\beta}{'-} \text{tr}_{\mathcal{H}_+} \mathcal{O} \right] \right\} \\ &= \sum_{\beta,\hat{g}} e^{-sZ} \sum_{m \in 2\mathbb{Z}} \delta(s - m) \left(\text{GV}'_{\hat{g},\beta}^{'+} + \text{GV}'_{\hat{g},\beta}{'-} \right) \left\{ \text{tr}_{\mathcal{H}_+} \mathcal{O} + \text{tr}_{\mathcal{H}_-} \mathcal{O} \right\} \\ & \quad + \sum_{\beta,\hat{g}} e^{-sZ} \sum_{m \in 2\mathbb{Z}+1} \delta(s - m) \left(\text{GV}'_{\hat{g},\beta}^{'+} - \text{GV}'_{\hat{g},\beta}{'-} \right) \left\{ \text{tr}_{\mathcal{H}_+} \mathcal{O} - \text{tr}_{\mathcal{H}_-} \mathcal{O} \right\}. \end{aligned} \quad (2.4.5)$$

In the next to last line, the traces clearly add up to the total trace over the parent $\mathcal{N} = 2$ multiplet and the GV'^{\pm} add up to the parent BPS invariants, c.f. eq. (2.4.4). Noticing also that it corresponds to even wrapping contributions $m \in 2\mathbb{Z}$, we realize that this corresponds to a contribution to the closed oriented topological string partition function, c.f. footnote 5. As discussed, it should not be included in the computation leading to the open and unoriented contributions.

The complete expression for the latter is

$$\begin{aligned} & \sum_{\substack{\beta,\hat{g} \\ m \in 2\mathbb{Z}+1}} \widehat{\text{GV}}_{\hat{g},\beta} \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\delta(s - m)}{2 \sinh(\frac{s\lambda}{2})} e^{-sZ} \left\{ \text{tr}_{\mathcal{H}_+} \left[(-1)^{2J_3^L} e^{-s\lambda J_3^L} \right] - \text{tr}_{\mathcal{H}_-} \left[(-1)^{2J_3^L} e^{-s\lambda J_3^L} \right] \right\} \\ &= \sum_{\substack{\beta,\hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g},\beta} \frac{1}{m} \frac{1}{2 \sinh(\frac{m\lambda}{2})} e^{-mZ} \left\{ \text{tr}_{\mathcal{H}_+} \left[(-1)^{2J_3^L} e^{-m\lambda J_3^L} \right] - \text{tr}_{\mathcal{H}_-} \left[(-1)^{2J_3^L} e^{-m\lambda J_3^L} \right] \right\}, \end{aligned} \quad (2.4.6)$$

where we have introduced the integers, which we call real BPS invariants,

$$\widehat{\text{GV}}_{\hat{g},\beta} := \text{GV}'_{\hat{g},\beta}^{'+} - \text{GV}'_{\hat{g},\beta}{'-}. \quad (2.4.7)$$

These integer numbers $\widehat{\text{GV}}$'s are those playing the role $\widetilde{\text{GV}}$'s in eq. (2.3.3). Note however that their correct physical interpretation differs from that in [101], where they were rather identified as our $\text{GV}'_{\hat{g},\beta}$. Note also that the correct invariants eq. (2.4.7) are equal mod 2 to the parent $\text{GV}_{\hat{g},\beta}$, proposed in [101], just like the $\text{GV}'_{\hat{g},\beta}$.

In eq. (2.4.6) we have taken into account that these states, being invariant under the orientifold, propagate only in the 2d fixed subspace of the 4d spacetime, resulting in a single power of $(2\sinh)$ in the denominator. This also explains the factor of 2 in the graviphoton coupling relative to eq. (2.2.4). In the next section we fill the gap of showing the promised equality of the even and odd multiplicities, and compute the trace difference in the last expression.

Jacobian and computation of $SU(2)_L$ traces

We must now evaluate the trace over the even/odd components of the Hilbert space of a parent $\mathcal{N} = 2$ BPS multiplet. This is determined by the parity of the corresponding zero modes on the particle worldline. As reviewed in section 2.2, the traces are non-trivial only over the cohomology of the Jacobian of $\widehat{\Sigma}_{\hat{g}}$ which determines the $SU(2)_L$ representation. We now focus on its parity under the orientifold.

Consider for example the case of I_1 , c.f. eq. (2.2.7). We introduce the formal split of the trace into traces over \mathcal{H}_{\pm}

$$t_1 = t_1^+ \ominus t_1^-, \quad (2.4.8)$$

where \pm denotes orientifold behavior and \ominus denotes a formal combination operation, which satisfies $\ominus^2 = 1$ (it corresponds to the $(-1)^m$ factor once the wrapping number m has been introduced, c.f. eq. (2.4.5)). Since the orientifold action is an antiholomorphic involution on the worldsheet, it acts as $dz \leftrightarrow d\bar{z}$, so eq. (2.2.7) splits as

$$I_1 = \left(\frac{1}{2}\right) \oplus 2(0) = \begin{pmatrix} + \\ - \end{pmatrix} \oplus (+) \oplus (-), \quad (2.4.9)$$

which gives a trace

$$t_1 = \underbrace{(1 - e^{s/2})}_{t_1^+} \ominus \underbrace{(1 - e^{-s/2})}_{t_1^-}, \quad (2.4.10)$$

where, to avoid notational clutter, we have reabsorbed λ into s .

Since the creation and annihilation operators associated to different 1-forms commute, the argument generalizes easily to higher genus, and the trace over a representation $I_{\hat{g}}$ has the structure

$$t_{\hat{g}} = (t_1^+ \ominus t_1^-)^{\hat{g}} = t_{\hat{g}}^+ \ominus t_{\hat{g}}^-, \quad (2.4.11)$$

where $t_{\hat{g}}^+$ and $t_{\hat{g}}^-$ contain even and odd powers of t_1^- , respectively. For instance, for I_2 we have to trace over

$$\begin{pmatrix} + \\ - \\ + \end{pmatrix} \oplus 2 \begin{pmatrix} + \\ - \end{pmatrix} \oplus 2 \begin{pmatrix} - \\ + \end{pmatrix} \oplus 2(+) \oplus 3(-) \quad (2.4.12)$$

and obtain

$$t_2 = \underbrace{(t_1^+)^2 + (t_1^-)^2}_{2+e^{-s}+e^s-2e^{-s/2}-2e^{s/2}} \ominus \underbrace{2t_1^+t_1^-}_{4-2e^{-s/2}-2e^{s/2}} = (t_1^+ \ominus t_1^-)^2 = t_2^+ \ominus t_2^-. \quad (2.4.13)$$

We are now ready to compute the final expression for the BPS expansion

The BPS expansion

Recalling eq. (2.3.3), the genuine open and unoriented contribution reduces to the odd wrapping number case eq. (2.4.6). Interestingly, the trace difference can be written (restoring the λ)

$$t_{\hat{g}}^+ - t_{\hat{g}}^- = \left(-2 \sinh \frac{s\lambda}{2} \right)^{\hat{g}}. \quad (2.4.14)$$

This is clear from eqs. (2.4.10) and (2.4.13) for I_1, I_2 respectively, and holds in general.

The final result for the BPS amplitude, which corresponds to the BPS expansion of the open and unoriented partition function, is

$$\sum_{\substack{\beta, \hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g}, \beta} \frac{1}{m} \left[2 \sinh \left(\frac{m\lambda}{2} \right) \right]^{\hat{g}-1} e^{-mZ}. \quad (2.4.15)$$

This has the precise structure to reproduce the conjecture in [101] as in eq. (2.3.3), with the invariants defined by eq. (2.4.7). In particular we emphasize the nice matching of exponents of the sinh factors (achieved since for the covering $\hat{g} - 1 = \chi$) and of the exponential $e^{-Z} = q^{d/2}$ for a one-modulus \mathbb{X}_6 (the factor of $1/2$ coming from the volume reduction due to the \mathbb{Z}_2 quotient.)

The only additional ingredient present in eq. (2.3.3) is the restriction on the degree, which is related to the conjectured tadpole cancellation condition, and which also admits a natural interpretation from the M-theory picture, as we show in the next section. We simply advance that this restriction applies to examples with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$. Our formula above is the general BPS expansion of the real topological string on a general CY threefold.

We anticipate that, once the tadpole cancellation discussed below is enforced, our derivation of eq. (2.4.15) provides the M-theory interpretation for the integer quantities $\widehat{\text{GV}}$ appearing in eq. (2.3.3) as conjectured in [101]. Therefore the real topological string is computing (weighted) BPS multiplicities of equivariant M2-brane states in M-theory, with the weight given by an orientifold parity sign, c.f. eq. (2.4.7). It would be interesting to perform a computation of the numbers appearing in eq. (2.4.15) along the lines of [51, 66].

2.4.3 Tadpole cancellation

In this section we discuss the M-theory description of the tadpole cancellation condition, in examples of the kind considered in the literature, i.e. with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}_6, L; \mathbb{Z}) = \mathbb{Z}$ (like the quintic or local \mathbb{CP}^2), for which one trades the class β for the degree $d \in \mathbb{Z}$. The argument involves several steps.

First step: Restriction to even degree

Consider the relative homology exact sequence

$$\begin{array}{ccccc} H_2(\mathbb{X}_6; \mathbb{Z}) & \xrightarrow{2\cdot} & H_2(\mathbb{X}_6, L; \mathbb{Z}) & \rightarrow & H_1(L; \mathbb{Z}). \\ \parallel & & \parallel & & \parallel \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 \end{array} \quad (2.4.16)$$

Since (the embedded image of) a crosscap doesn't intersect the lagrangian L , its class must be in the kernel of the second map, i.e. the image of the first. Thus, every crosscap contributes an even factor to the degree d . For boundaries, the same argument implies that boundaries wrapped on an odd multiple of the non-trivial generator of $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ contribute to odd degree, while those wrapped on an even multiple of the \mathbb{Z}_2 1-cycle contribute to even degree. This restricts the possible cancellations of crosscaps to even degree boundaries.

Second step: Relative signs from background form fields

We now show that there is a relative minus sign between crosscaps and disks associated to the same (necessarily even degree) homology class. As mentioned in section 2.4.1, the M-theory lift contains a background 3-form C_3 along the 3-cycles $(\mathbb{C}\mathbb{P}^1 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 acting antiholomorphically over $\mathbb{C}\mathbb{P}^1$; this corresponds to a half-unit of NSNS 2-form flux on any crosscap $\mathbb{R}\mathbb{P}^2$ surrounding the O4-plane in the type IIA picture. In M-theory, the reduction of C_3 along the $\mathbb{C}\mathbb{P}^1$ produces a 5d gauge boson, under which any M2-brane is charged with charge c , where c is the number of crosscaps in the embedded curve in \mathbb{X}_6/σ . The 3-form background corresponds to a non-trivial \mathbb{Z}_2 Wilson line turned on along the M-theory \mathbb{S}^1 , and produces an additional contribution to the central charge term Z , which (besides the KK term) reads

$$Z = dt + \frac{i}{2}c. \quad (2.4.17)$$

Once we exponentiate, and perform Poisson resummation eq. (2.2.11), this gives a contribution $(-1)^{m \times c}$, with m the wrapping number, which is odd for the genuine equivariant contributions. This extra sign does not change the contributions of curves with even number of crosscaps. Note that the above also agrees with the fact that the (positive) number of crosscaps is only defined mod 2.

In contrast, boundaries do not receive such contribution,⁶ and therefore there is a relative sign between contributions from curves which fall in the same homology class, but differ in trading a crosscap for a boundary.

⁶Note that odd degree boundaries can receive an extra sign due to a possible \mathbb{Z}_2 Wilson line on the D4-brane. This however does not affect the even degree boundaries, which are those canceling against crosscaps. Hence it does not have any effect in the explanation of the tadpole cancellation condition.

Third step: Bijection between crosscaps and boundaries

To complete the argument for the tadpole cancellation condition, one needs to show that there is a one-to-one correspondence between curves which agree except for a replacement of one crosscap by one boundary. The replacement can be regarded as a local operation on the curve, so the correspondence is a bijection between disk and crosscap contributions.

More precisely we want to show that for every homologically trivial disk which develops a node on top of L we can find a crosscap, and viceversa. This is mathematically a nontrivial statement, for which we weren't able to find an explicit construction going beyond the local model of eq. (2.A.3). Moreover, in higher genera this problem has not been tackled by mathematicians yet. Nonetheless, following [97], we propose a model for gluing boundaries on the moduli space in the genus zero case, which points towards the desired bijection. The original argument applies to holomorphic maps, relevant to Gromov-Witten invariants; we expect similar results for holomorphic embeddings, relevant for Gopakumar-Vafa invariants.

The main point is that integrals over moduli spaces of Riemann surfaces make sense and are independent of the choice of complex structure if the moduli space has a virtually orientable fundamental cycle without real codimension one boundaries (RCOB). If $L \neq \emptyset$, in order to achieve this, one has to consider together contributions coming roughly speaking from open and unoriented worldsheets, as proposed by [101].

We are interested in elements of RCOB in which a piece of the curve degenerates as two spheres touching at a point q :

$$(f, \Sigma = \Sigma_1 \cup_q \Sigma_2), \quad (2.4.18)$$

where f is the holomorphic map, $\Sigma_i = \mathbb{CP}^1$, the involution exchanges the Σ_i and $f(q) \in L$. For real $\epsilon \neq 0$ one can glue Σ into a family of smooth curves, described locally as⁷

$$\Sigma_\epsilon = \left\{ (z, w) \in \mathbb{C}^2 : zw = \epsilon \right\}. \quad (2.4.19)$$

For $\epsilon \in \mathbb{R}$, Σ_ϵ inherits complex conjugation from eq. (2.4.20), and the fixed point set is \mathbb{S}^1 if $\epsilon > 0$, empty if $\epsilon < 0$. They correspond to an equivariant curve with two different involutions, which in terms of homogeneous coordinates on \mathbb{CP}^1 , can be described as

$$(u : v) \mapsto (\bar{u} : \pm \bar{v}), \quad (2.4.20)$$

leading to either a boundary or a crosscap. The RCOB corresponding to sphere bubbling in the '+' case is the same as the RCOB for the '-' case. By attaching them along their common boundary, we obtain a moduli space whose only RCOB corresponds to disk bubbling. The resulting combined moduli space admits a Kuranishi structure and produces well-defined integrals.

⁷In this equivariant covering picture, we only consider singularities of type (1) as in Definition 3.4 of [74], since the covering does not have boundaries.

Final step

Using the above arguments, we can now derive eq. (2.3.1), as follows. First, the tadpole cancellation removes contributions where the number of crosscaps c is odd, so taking $\chi = 2g - 2 + h + c$ we have $\chi \equiv h \pmod{2}$, where h denotes the number of boundaries. Second, the value of $d = \sum d_i \pmod{2}$ can only get contributions from boundaries and crosscaps (since contributions of pieces of the curve away from L cancel mod 2 from the doubling due to the orientifold image); moreover contributions from crosscaps and even degree boundaries cancel. Hence, the only contributions arise from boundaries with odd terms d_i , so clearly $d \equiv h \pmod{2}$. We hence recover eq. (2.3.1).

2.5 Extensions and relations to other approaches

2.5.1 Adding extra D4-brane pairs

The discussion in the previous sections admits simple generalizations, for instance the addition of N extra D4-brane pairs in the type IIA picture.⁸ The two branes in each pair are related by the orientifold projection, but can otherwise be placed at any location, and wrapping general lagrangian 3-cycles in \mathbb{X}_6 . For simplicity, we consider them to wrap the O4-plane lagrangian 3-cycle L , and locate them on top of the O4-plane in the spacetime dimensions as well.

In the M-theory lift, we have the same quotient acting as a half shift on the M-theory \mathbb{S}^1 (times the antiholomorphic involution of \mathbb{X}_6 and the spacetime action $x^2, x^3 \rightarrow -x^2, -x^3$), now with the extra D4-brane pairs corresponding to extra M5-brane pairs, related by the M-theory orientifold symmetry [49]. Notice that since the orientifold generator is freely acting, there is no singularity, and therefore no problem in understanding the physics associated to these M5-branes. In the 5d theory, there is no orientifold, and the introduction of the M5-branes simply introduces sectors of BPS states corresponding to open M2-branes wrapped on holomorphic 2-chains with boundary on the M5-brane lagrangians. Their multiplicity is precisely given by the open oriented Gopakumar-Vafa invariants [86]. These particles run in the M-theory \mathbb{S}^1 and must be split according to their parity under the orientifold action, which determine the appropriate KK momentum quantization. By the same arguments as in section 2.4.2, the contributions which have even wrapping upon Poisson resummation actually belong to the open oriented topological amplitude, and should be discarded. To extract the genuine open unoriented amplitude, we must focus on surfaces mapped to themselves under σ , and restrict to odd wrapping number. In analogy with section 2.4.2 and [86], the amplitude can be written

$$\sum_{\text{odd } m \geq 1} \sum_{\beta, r, \mathcal{R}} \frac{N_{\beta, r, \mathcal{R}}^+ - N_{\beta, r, \mathcal{R}}^-}{2m \sinh\left(\frac{m\lambda}{2}\right)} e^{-m\beta \cdot t - mr\lambda} \text{tr}_{\mathcal{R}} \prod_{i=1}^{b_1(L)} V_i^m. \quad (2.5.1)$$

⁸The addition of one extra unpaired D4-brane would modify drastically the M-theory lift of the configuration, as discussed in the next section.

The sum in m runs only over positive odd integers. The $N_{\beta, \mathcal{R}, r}^{\pm}$ are the multiplicities of (even or odd) states from M2-branes on surfaces in the class β , with spin r under the rotational $U(1)$ in the 01 dimensions,⁹ and in the representation \mathcal{R} of the background brane $SO(2N)$ symmetry. The V_i denote the lagrangian moduli describing the $SO(2N)$ Wilson lines (complexified with deformation moduli), possibly turned on along the non-trivial 1-cycles of L . Since the presence of the M5-branes breaks the $SU(2)_L$ structure, it is not possible to perform a partial sum over such multiplets explicitly.

A clear expectation from the type IIA perspective is that the total amplitude (namely adding the original contribution in the absence of D4-brane pairs) could be rewritten to display an $SO(2N + 1)$ symmetry, combining the stuck and paired D4-branes. This is possible thanks to the close analogy of the above expression with eq. (2.4.15), once we expand the contribution $\sinh^{\hat{g}}$ to break down the $SU(2)_L$ multiplet structure. By suitably subtracting contributions $N^+ - N^-$ to the multiplicities \widehat{GV} , one can expect to isolate the $SO(2N + 1)$ symmetric contribution.

For instance, take the case of only one matrix V (e.g. $H_1(L; \mathbb{Z}) = \mathbb{Z}$ or \mathbb{Z}_2). Combining eq. (2.5.1) with eq. (2.4.15), we have

$$\sum_{\beta, r, m} \frac{e^{-mZ}}{2m \sinh(\frac{m\lambda}{2})} e^{-mr\lambda} \left[\widehat{GV}_{r, \beta} + \left(N_{\beta, r, \mathcal{R}}^+ - N_{\beta, r, \mathcal{R}}^- \right) \text{tr}_{\mathcal{R}} V^m \right], \quad (2.5.2)$$

where we have introduced $\widehat{GV}_{r, \beta}$ as the combination of real BPS invariants $\widehat{GV}_{\hat{g}, \beta}$ describing the multiplicity of M2-brane states with 2d $U(1)$ spin r . For a given representation \mathcal{R} , the requirement that the expression in square brackets combines into traces of $SO(2N + 1)$ seems to imply non-trivial relations between the open and real BPS invariants for different representations of $SO(2N)$. It would be interesting to study these relations further.

2.5.2 Relation to other approaches

In this section we describe the relation of our system with other unoriented A-model topological strings and their physical realization in M-theory.

M-theory lift of the four O4-planes

As discussed in [37, 49] there are four kinds of O4-planes in type IIA string theory, with different lifts to M-theory. We describe them in orientifolds of type IIA on $\mathbb{X}_6 \times M_4$, with the geometric part of the orientifold acting as an antiholomorphic involution on \mathbb{X}_6 and $x^2, x^3 \rightarrow -x^2, -x^3$ on the 4d spacetime.

- An $O4^-$ -plane (carrying -1 units of D4-brane charge, as counted in the covering space) with no D4-branes on top. Its lift to M-theory is a geometric orientifold $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6) / \mathbb{Z}_2 \times \mathbb{S}^1$. Inclusion of additional D4-brane pairs (in the covering space) corresponds to including additional M5-brane pairs in the M-theory lift.

⁹Actually, it is the charge under the $U(1)_L \subset SU(2)_L$, which describes the coupling to the self-dual graviphoton background.

- An $O4^0$ -plane, which can be regarded as an $O4^-$ with one stuck D4-brane. We recall that its M-theory lift, exploited in this chapter, is $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$, with the \mathbb{Z}_2 including a half-period shift of the \mathbb{S}^1 . Additional D4-brane pairs correspond to additional M5-brane pairs, as studied in the previous section.
- An $O4^+$ -plane (carrying +1 units of D4-brane charge). Its lift to M-theory is a geometric orientifold $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$ with two stuck M5-branes on top. The M5-branes are stuck because they do not form an orientifold pair, due to a different worldvolume Wilson line [37].
- An $\widetilde{O4}^+$ -plane, which can be regarded as an $O4^+$ with an extra RR background field. Its M-theory lift is our $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6 \times \mathbb{S}^1)/\mathbb{Z}_2$ geometry, with one stuck M5-brane fixed by the \mathbb{Z}_2 action.

The $O4^-$ vs. the $O4^0$ case

Several references, e.g. [1, 4, 20, 21, 90], consider unoriented A-models with no open string sector, corresponding to an M-theory lift $(M_4 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$. This corresponds to the case of the $O4^-$ -plane (for their relation with the $O4^+$ case, see later). The key difference with our setup is that in general there are \mathbb{Z}_2 fixed points, which correspond to $L \times \mathbb{R}^2 \times \mathbb{S}^1$. The physics near these singularities cannot be addressed with present technology. However, since crosscap embeddings do not intersect L , it is possible to meaningfully propose an M-theory Gopakumar-Vafa interpretation of the unoriented topological string amplitude. This can also be extended to open string sectors, as long as the D4-branes (or M5-branes in the M-theory lift) are introduced in pairs and kept away from the singular locus. As we discuss later on, this limits the possibility of reproducing the right physics for the $O4^+$.

A second difference from our setup is that the orientifold action in M-theory is felt even locally on the \mathbb{S}^1 , i.e. already at the level of the 5d theory, and breaks the $SU(2)_L$ symmetry. Therefore the structure of multiplets need not correspond to full $SU(2)_L$ multiplets, although this is explicitly assumed in most of these references. Although supported by the appropriate integrality properties derived from the analysis, these extra assumptions obscure the physical derivation of the BPS integrality structures.

We emphasize again that these properties differ in our system, which corresponds to the lift of the $O4^0$ -plane. The $SU(2)_L$ multiplet structure is directly inherited from the parent theory, and is therefore manifestly present, without extra assumptions.

The $O4^+$ case

The case of the $O4^+$ -plane has been discussed in the literature as a minor modification of the $O4^-$ case. Indeed, from the viewpoint of the type IIA theory (equiv. of the topological A-model), both systems are related by a weight $(-1)^c$ for any worldsheet amplitude with c crosscaps (corresponding to a change in the NSNS 2-form background around the orientifold plane). This motivates an immediate BPS invariant expansion of the unoriented A-model amplitude corresponding to the $O4^+$, see e.g. [1, 4, 90].

On the other hand, the actual M-theory lift of the $O4^+$ -plane corresponds not to the geometry $M_2 \times (\mathbb{R}^2 \times \mathbb{X}_6)/\mathbb{Z}_2 \times \mathbb{S}^1$ with a different choice of 3-form background, but rather to the same geometry as the $O4^-$ -plane, with the addition of two stuck M5-branes at the \mathbb{Z}_2 fixed point. In this lift, interestingly, the M-theory picture contains both unoriented and open M2-brane curves, which should combine together to reproduce a purely unoriented Gromov-Witten worldsheet expansion; the latter moreover admits a BPS expansion in terms of purely closed M2-brane curves, up to some sign flips. It is non-trivial to verify how these pictures fit together, in particular given the difficulties in dealing with M2-branes ending on M5-branes stuck at the \mathbb{Z}_2 fixed point in the M-theory geometry. The details of this connection are therefore still open.

The $\widetilde{O4}^+$ case

Finally, the case of the $\widetilde{O4}^+$ -plane has not been considered in the literature. Actually, it is closely related to the lift of the $O4^0$, with the addition of one M5-brane. It is therefore very similar to the systems in the previous section, and the corresponding amplitude is essentially given by eq. (2.5.1), for $2N + 1$ M5-branes (allowing for the addition of N brane pairs).

In this case there is also an interesting interplay with the type IIA picture, although in the opposite direction as compared with the $O4^0$ case. Namely, the M-theory lift contains one more brane than the corresponding type IIA picture. This implies that in the BPS expansion both closed and open M2-brane states have to combine together to reproduce the crosscap worldsheet diagram in type IIA. It would be interesting to carry out this comparison further, although this may be difficult due to the presence of a non-trivial RR background in the type IIA orientifold, which may render the worldsheet analysis difficult.

2.6 Conclusions and open issues

We have discussed the BPS integer expansion of the real topological string in [101], using the M-theory lift of the $O4$ -plane with one stuck D4-brane. Since the geometry is a \mathbb{Z}_2 quotient acting freely in the M-theory \mathbb{S}^1 , the 5d setup enjoys an enhancement to 8 supercharges and is identical to that in the closed oriented Gopakumar-Vafa system. The subtleties due to the orientifold quotient arise as a compactification effect modifying the KK momentum of the BPS states on the \mathbb{S}^1 according to their parity under the orientifold action. This allows for a clean derivation of the BPS integer expansion, without the extra assumptions that pop up in other unoriented A-models.

Although we recover the BPS expansion conjectured in [101], our derivation shows the correct identification of the BPS invariants not as the equivariant sector of the parent Gopakumar-Vafa invariants, but rather a weighted version thereof.

The M-theory picture provides a complementary viewpoint on the sign choices implied by the tadpole cancellations in models where the fixed lagrangian 3-cycle L has

$H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ [101]. More in general, the BPS integer expansion we propose is valid for other situations, providing a general definition of the real topological string.

The careful M-theory lift of other O4-planes suggests non-trivial relations between their BPS invariant expansions, for instance the addition of an open M2-brane sector (associated to two stuck M5-branes) to the lift of the $O4^-$ -plane should reproduce a sign flip in odd crosscap contributions. This seems to imply non-trivial relations among the unoriented and open BPS invariants in M-theory orientifolds with fixed points. We hope to return to these and other questions in the future.

2.A Review of Walcher's real topological string

We now give a short review of A-model localization as in [101]. Take for concreteness Fermat quintic, given by

$$\left\{ \sum_{i=1}^5 x_i^5 = 0 \right\} \subset \mathbb{CP}^4 \quad (2.A.1)$$

and involution

$$\sigma : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (\bar{x}_2 : \bar{x}_1 : \bar{x}_4 : \bar{x}_3 : \bar{x}_5) \quad (2.A.2)$$

which gives a fixed point lagrangian locus L with \mathbb{RP}^3 topology, hence $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$.

2.A.1 Tadpole cancellation in the topological string

Requiring a function f to be equivariant implies that fixed points of Ω are mapped to L , but one has to further specify their homology class in $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$. When that class is trivial, then under deformation of the map it can happen that the boundary is collapsed to a point on L .¹⁰

The local model for this phenomenon is a Veronese-like embedding $\mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ defined by a map $(u : v) \mapsto (x : y : z)$ depending on a target space parameter a , concretely

$$x = au^2, \quad y = av^2, \quad z = uv. \quad (2.A.3)$$

The image can be described as the conic $xy - a^2z^2 = 0$, and it is invariant under σ if $a^2 \in \mathbb{R}$. The singular conic $a = 0$ admits two different equivariant smoothings, determined by the nature of a :

$$\begin{array}{lll} a \in \mathbb{R} & (u : v) \sim (\bar{v} : \bar{u}) & \text{disk} \\ a \in i\mathbb{R} & (u : v) \sim (\bar{v} : -\bar{u}) & \text{crosscap.} \end{array} \quad (2.A.4)$$

The proposal of [101] to account for this process is to count disks with collapsible boundaries and crosscaps together. Specifically, there is a one-to-one correspondence between even degree maps leading to boundaries and maps leading to crosscaps (which must

¹⁰This is a real codimension one stratum in the moduli space, as discussed in section 2.3.

be of even degree, in order to be compatible with the antiholomorphic involution, as already manifest in the above local example). The tadpole cancellation condition amounts to proposing the combination of these paired diagrams, such that certain cancellations occur. For instance, the amplitude $\mathcal{R}^{(g,h)}$ (odd number of crosscaps) vanish (due to the cancellation of the unpaired crosscap with an odd degree boundary). Similarly, for the remaining contributions, in terms of χ and d , the tadpole cancellation imposes the restriction

$$d \equiv h \equiv \chi \pmod{2}. \quad (2.A.5)$$

The first equality follows from the requirement that odd degree contributions only come from boundaries (homologically trivial, i.e. even degree, ones cancel against crosscaps), while the second from the requirement that there be no unpaired crosscaps.

2.A.2 Rules of computation

One can then *postulate* the existence of a well-defined fundamental class allowing to integrate over the moduli space $\mathcal{M} = \overline{\mathcal{M}}_{\Sigma}(\mathbb{C}\mathbb{P}^4, d)$ of maps, defined as the top Chern class of an appropriate bundle \mathcal{E}_d over \mathcal{M} , whose fiber is given by $H^0(\Sigma, f^*\mathcal{O}(5))$:

$$\widetilde{\text{GW}}_d^{\Sigma} = \int_{\mathcal{M}} \mathbf{e}(\mathcal{E}_d). \quad (2.A.6)$$

For the integral to make sense, dimensions are constrained as

$$5d + 1 = \dim \mathcal{E}_d \stackrel{!}{=} \text{vdim } \overline{\mathcal{M}}_{g,n}(\mathbb{C}\mathbb{P}^D, \beta) = c_1 \cdot \beta + (3 - D)(g - 1) + n, \quad (2.A.7)$$

where $\beta \in H_2(\mathbb{C}\mathbb{P}^D; \mathbb{Z})$ is to be identified with d in this particular case, and n denotes the number of punctures.

The next step is to apply Atiyah-Bott localization to the subtorus $\mathbb{T}^2 \subset \mathbb{T}^5$ compatible with σ . As explained in [69], the fixed loci of the torus action are given by nodal curves, in which any node or any component of non-zero genus is collapsed to one of the fixed points in target space, and any non-contracted rational component is mapped on one of the coordinate lines with a standard map of given degree d_i :

$$f(w_1 : w_2) = (0 : \dots : 0 : w_1^{d_1} : 0 : \dots : 0 : w_2^{d_2} : 0 : \dots : 0). \quad (2.A.8)$$

The components of the fixed locus can be represented by a decorated graph Γ and one has well-defined rules for associating a graph to a class of stable maps.

For the case of real maps, one has to be extra careful and require the decoration to be compatible with the action of Ω and σ . For example, consider a fixed edge: if we think of $z = w_1/w_2$, then in eq. (2.A.8) $z \mapsto 1/\bar{z}$ is compatible (i.e. f is equivariant) with any degree, while $z \mapsto -1/\bar{z}$ requires even degree, because our involution acts on the target space as $(x_1 : x_2 : \dots) \mapsto (\bar{x}_2 : \bar{x}_1 : \dots)$ i.e. $w = x_1/x_2 \mapsto 1/\bar{w}$.

A careful analysis in [72, 87, 100, 101] allows to conclude that the localization formula takes the form

$$\widetilde{\text{GW}}_d^{\Sigma} = (-1)^{p(\Sigma)} \sum_{\Gamma} \frac{1}{\text{Aut } \Gamma} \int_{\mathcal{M}_{\Gamma}} \frac{\mathbf{e}(\mathcal{E}_d)}{\mathbf{e}(\mathcal{N}_{\Gamma})}, \quad (2.A.9)$$

where the following prescriptions are used:

- (i) the $(-1)^{p(\Sigma)}$ factor in front of the localization formula is put by hand in order to fix the relative orientation between different components of moduli space;
- (ii) for any fixed edge of even degree, the homologically trivial disk and crosscap contributions are summed, with a relative sign such that they cancel. This is the above mentioned *tadpole cancellation*.

Finally, [101] proposes an integer BPS interpretation of the obtained rational numbers $\widetilde{\text{GW}}$: by combining them with the above prescribed signs at fixed χ , one gets integer numbers $\widetilde{\text{GV}}$, which are conjectured to reproduce a BPS expansion for the open-unoriented topological amplitudes.

2.B Physical couplings

An interesting question one can ask is what kind of coupling the topological string is computing in the IIA physical theory. This has a clear answer for the closed oriented sector [9,17], while some proposals have been made for open oriented [86] and unoriented sectors [90]. Here we make a proposal for the analogous expression in our unoriented model.

We have 4 supercharges in 1 + 1 dimensions, and we'd like to find a good splitting of the $\mathcal{N} = 2$ Weyl tensor

$$\mathcal{W}_{\mu\nu}^{ij} = T_{\mu\nu}^{ij} + R_{\mu\nu\rho\lambda}\theta^i\sigma^{\rho\lambda}\theta^j + \dots, \quad (2.B.1)$$

where one requires the graviphoton field strength T to acquire a self-dual background, and hence can write

$$\mathcal{W}_{\alpha\beta} = \frac{1}{2}\varepsilon_{ij}\mathcal{W}_{\mu\nu}^{ij}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}}. \quad (2.B.2)$$

A natural guess is that the amplitude $\mathcal{G}^{(\chi)}$ as in eq. (2.3.2) computes

$$\int d^4x \int d^4\theta \delta^2(\theta)\delta^2(x) \left(\mathcal{G}^{(\chi)}(t) - \frac{1}{2}\mathcal{F}^{(g_\chi)}(t) \right) (\mathcal{W} \cdot v)^{\hat{g}}, \quad (2.B.3)$$

where $\mathcal{W} \cdot v = \mathcal{W}_{\alpha\beta}\sigma_{\alpha\beta}^{\mu\nu}v_{\mu\nu}$. This has a contribution $RT^{\hat{g}-1}$ (in the covering picture $\hat{g}-1 = \chi$) which in principle can generate (via SUSY) the \sinh^{-1} power in the Schwinger computation, taking into account the fact that the orientifold halves the number of fermion zero modes on the Riemann surface.

It would be interesting to discuss the appearance of this contribution from different topologies at fixed χ .

Chapter 3

Towards a gauge theory interpretation

3.1 Introduction

A most interesting connection between gauge theory and string theory is the relation between non-perturbative instanton corrections in 4d/5d gauge theories with 8 supercharges [83, 85], and the topological string partition functions on non-compact toric Calabi-Yau (CY) three-folds \mathbb{X} (in their formulation in terms of BPS invariants [38, 39, 86]). Basically, the BPS M2-branes on compact 2-cycles of \mathbb{X} are instantons of the gauge theory arising from M-theory on \mathbb{X} .

We are interested in extending this correspondence to systems with orientifold projections. A natural starting point is the real topological string, introduced in [101] and studied in compact examples in [71] and in non-compact CY three-folds in [70, 72]. This real topological string is physically related to type IIA on a CY three-fold \mathbb{X} quotiented by an orientifold, given by an antiholomorphic involution σ on \mathbb{X} and a flip on two of the 4d spacetime coordinates (say x^2, x^3); it thus introduces an O4-plane, spanning the lagrangian 3-cycle given by the fixed point set,¹ and the two fixed 4d spacetime dimensions x^0, x^1 . In addition, the topological tadpole cancellation [101] requires the introduction of a single stuck D4-brane on top of the (negatively charged) O4-plane, producing local cancellation of RR charge.

The M-theory lift of this type IIA configuration [89] corresponds to a freely acting quotient of M-theory on $\mathbb{X} \times \mathbb{S}^1$, in which the action σ on \mathbb{X} (and the flip of two 4d coordinates) is accompanied by a half-period shift along the \mathbb{S}^1 . This M-theory lift provides a reinterpretation of the real topological string partition function in terms of real BPS invariants, which are essentially given by a combination of the parent Gopakumar-Vafa (GV) BPS invariants, weighted by the ± 1 eigenvalue of the corresponding state under the orientifold action.

¹We focus on cases with non-trivial fixed point sets; freely acting orientifolds could be studied using similar ideas.

This orientifolding can be applied in the context of type IIA/M-theory on non-compact toric CY three-folds \mathbb{X} which realizes 5d gauge theories. More specifically, one should consider the 5d gauge theory compactified to 4d, with an orientifold acting non-trivially as a shift on the \mathbb{S}^1 . Since the orientifold plane is real codimension 2 in the 4d Minkowski dimensions, the system describes the gauge theory in the presence of a surface defect. Certain surface operators have been studied in [31,41,43,44,62], also in the context of M-theory/gauge theory correspondence, describing them by the introduction of D2-branes/M2-branes in the brane setup or D4-branes in the geometric engineering [27, 86, 96]. An important difference with our discussion is that the holonomy of our surface operators is an outer automorphism of the original gauge group. In our case, the properties of the gauge theory in this orientifold background are implicitly defined by the real topological vertex, even though they should admit an eventual intrinsic gauge-theoretical description. Hence, one can regard our real topological string results as a first step in the study of a novel kind of gauge theory surface operators.

Our strategy is as follows:

- For concreteness, we focus on the geometry which realizes a pure $SU(N)$ gauge theory, and other similarly explicit examples. We construct the real topological string on non-compact toric CY three-folds \mathbb{X} by using the real topological vertex formalism [70].
- In the M-theory interpretation, the real topological string amplitudes correspond to a one-loop diagram of a set of 5d BPS particles from wrapped M2-branes, suitably twisted by the orientifold action as they propagate on the \mathbb{S}^1 . An important point is that the effect of the orientifold action arises only after the compactification to 4d on \mathbb{S}^1 , so the 5d picture is identical to the parent theory. Therefore, the correspondence between M-theory and 5d $\mathcal{N} = 1$ gauge theory is untouched.
- The corresponding statement on the gauge theory side is that the 4d partition function of the gauge theory in the presence of the orientifold surface defect must be given by the compactification of the original 5d gauge theory on \mathbb{S}^1 , but with modified periodic/anti-periodic boundary conditions for fields which are even/odd under the orientifold action. This can in principle be implemented as the computation of the Witten index with an extra twist operator in the trace. This kind of operator has not appeared in the literature. We use the comparison of the oriented and real topological vertex partition functions to better understand the nature and action of this operator on the gauge theory.

Even though we do not achieve a completely successful gauge theoretical definition of the orientifold operation, we obtain a fairly precise picture of this action in some concrete situations. Moreover, our discussion of the topological vertex amplitudes reveals new properties in the unoriented case.

3.2 Review of the oriented case

In this section we review the correspondence between BPS M2-brane invariants of M-theory on a toric CY three-fold singularity \mathbb{X} and the supersymmetric gauge theory Nekrasov partition function [28, 48, 57, 58, 95, 106]. We will take advantage to introduce useful tools and notations to be used in the discussion of the orientifolded case.

3.2.1 The topological string

BPS expansion of the topological string

We start with a brief review of closed oriented topological string interpreted in terms of BPS states in M-theory [38, 39]. Consider type IIA on a CY three-fold \mathbb{X} , which provides a physical realization of the topological A-model on \mathbb{X} . The genus g topological string amplitude $F_g(t_i)$, which depends on the complexified Kähler moduli $t_i = a_i + iv_i$ with a_i coming from the B-field and v_i being the volume, computes the F-term [9]

$$\int d^4x \int d^4\theta F_g(t_i) (\mathcal{W}^2)^g \rightarrow \int d^4x F_g(t_i) F_+^{2g-2} R_+^2, \quad (3.2.1)$$

where the second expression applies for $g > 1$ only, and the $\mathcal{N} = 2$ Weyl multiplet is schematically $\mathcal{W} = F_+ + \theta^2 R_+ + \dots$, in terms of the self-dual graviphoton and curvature. If we turn on a constant self-dual graviphoton background in the four non-compact dimensions

$$F_+ = \frac{\epsilon}{2} dx^1 \wedge dx^2 + \frac{\epsilon}{2} dx^3 \wedge dx^4, \quad (3.2.2)$$

the sum may be regarded as the total A-model free-energy, with coupling ϵ

$$\mathcal{F}(t_i) = \sum_{g=0}^{\infty} \epsilon^{2g-2} F_g(t_i). \quad (3.2.3)$$

This same quantity can be directly computed as a one-loop diagram of 5d BPS states in M-theory compactified on $\mathbb{X} \times \mathbb{S}^1$, corresponding to 11d graviton multiplets and to M2-branes wrapped on holomorphic 2-cycles. These states couple to the graviphoton background via their quantum numbers under the $SU(2)_L$ in the 5d little group $SU(2)_L \times SU(2)_R$. Denoting by $GV_{g,\beta}$ the multiplicity of BPS states corresponding to M2-branes on a genus g curve in the homology class β , we have an expression²

$$\mathcal{F} = \sum_{\beta \in H_2(\mathbb{X}; \mathbb{Z})} \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} GV_{g,\beta} \frac{1}{m} \left(2i \sinh \frac{m\epsilon}{2} \right)^{2g-2} e^{im\beta \cdot t}, \quad (3.2.4)$$

with $|e^{im\beta \cdot t}| < 1$ so that the BPS state counting is well-defined. Namely, the computation is carried out in the large volume point in the Kähler moduli space. From the M-theory point of view, the real part of t may be provided by the Wilson line along \mathbb{S}^1 , originated from the three-form C_3 . Finally, the topological string partition function is defined as

$$Z_{\text{top}} = \exp \mathcal{F}. \quad (3.2.5)$$

²A subtlety regarding the reality condition of the ϵ -background has been discussed in [25].

Topological vertex formalism

From now on we specialize to a particular class of M-theory background geometries, directly related to supersymmetric gauge theories. M-theory on a CY three-fold singularity, in the decoupling limit, implements a geometric engineering realization of 5d gauge theory with 8 supercharges, with gauge group and matter content determined by the singularity structure [54, 78]. Upon compactification on \mathbb{S}^1 , it reproduces type IIA geometric engineering [64, 65]. This setup is therefore well-suited for the matching of gauge theory results in terms of topological string amplitudes in the GV interpretation.

We will use the best known tool at large volume point in moduli space to compute topological string partition functions on local toric CY three-folds, namely the topological vertex formalism [3, 10, 56, 60]. This can be even used to define a refined version of the topological string amplitude [2, 4, 7, 10, 24, 42, 60], associated to the theory in a non self-dual graviphoton background

$$F = \frac{1}{2}\epsilon_1 dx^1 \wedge dx^2 - \frac{1}{2}\epsilon_2 dx^3 \wedge dx^4. \quad (3.2.6)$$

The unrefined topological string amplitude is recovered for $\epsilon_1 = -\epsilon_2$. We will describe the refined topological vertex computation, but will eventually restrict to the unrefined case, since only this is known in the unoriented case. Happily, this suffices to illustrate our main points.

The basic idea is to regard the web diagram of the resolved singularity (the dual of the toric fan) as a Feynman diagram, with rules to produce the topological string partition function. Roughly, one sums over edges that correspond to Young diagrams R_i , with propagators $(-e^{it_i})^{|R_i|}$ depending on Kähler parameters t_i , and vertex functions expressed in terms of skew-Schur functions. The formalism is derived from open-closed string duality and 3d Chern-Simons theory [3, 5, 56].

Let us begin by introducing some useful definitions. A Young diagram R is defined by the numbers of boxes $R(i)$ in the i^{th} column, ordered as $R(1) \geq R(2) \geq \dots \geq R(d) \geq R(d+1) = 0$, see fig. 3.1. We denote by $|R| = \sum_{i=1}^d R(i)$ the total number of boxes, and by \emptyset the empty diagram. For a box $s = (i, j) \in R$, we also define

$$a_R(i, j) := R^t(j) - i, \quad l_R(i, j) := R(i) - j, \quad (3.2.7)$$

where R^t denotes transpose, see fig. 3.1.

We recall some definitions useful to work with refined topological vertex³ by following the conventions used in [46]. An edge is labeled by a Young diagram ν , and has an associated propagator $(-Q_\nu)^{|\nu|}$, where Q is the exponential of the complexified Kähler parameter of the corresponding 2-cycle. Edges join at vertices, which have an associated vertex function

$$C_{\lambda\mu\nu}(t, q) = t^{-\frac{\|\mu^t\|^2}{2}} q^{\frac{\|\mu\|^2 + \|\nu\|^2}{2}} \tilde{Z}_\nu(t, q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta| + |\lambda| - |\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t} q^{-\rho}), \quad (3.2.8)$$

³In subindex-packed formulas, we sometimes adopt Greek letters to label Young diagrams.

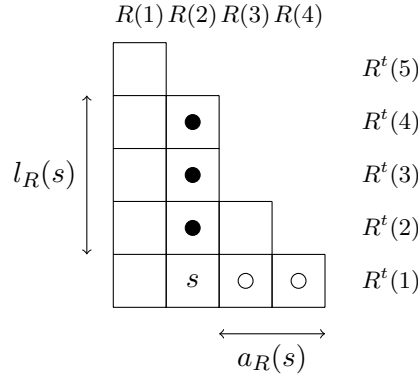


Figure 3.1: Notation for arm and leg lengths of the box $s = (2, 1) \in R$.

where $q = e^{-i\epsilon_2}$ and $t = e^{i\epsilon_1}$, $s_R(q^{-\rho}t^{-\nu})$ means s_R evaluated at $x_i = q^{i-\frac{1}{2}}t^{-\nu(i)}$, and $s_{\mu/\eta}(x)$ are skew-Schur functions: if $s_\nu(x)$ is Schur function and $s_\nu(x)s_\rho(x) = \sum_\mu c_{\nu\rho}^\mu s_\mu(x)$, then $s_{\mu/\nu}(x) := \sum_\rho c_{\nu\rho}^\mu s_\rho(x)$ [75] (in particular, $s_{\emptyset/R} = \delta_{\emptyset,R}$, $s_{R/\emptyset} = s_R$, and $s_{\mu/\nu}(Qq) = Q^{|\mu|-|\nu|}s_{\mu/\nu}(q)$.) Sub-indices are ordered according to fig. 3.2a, and we defined

$$\tilde{Z}_\nu(t, q) = \prod_{s \in \nu} (1 - q^{l_\nu(s)} t^{a_\nu(s)+1})^{-1}. \quad (3.2.9)$$

We also define

$$f_\nu(t, q) = (-1)^{|\nu|} t^{\frac{\|\nu^t\|^2}{2}} q^{-\frac{\|\nu\|^2}{2}}, \quad \tilde{f}_\nu(t, q) = (-1)^{|\nu|} t^{\frac{\|\nu^t\|^2}{2}} q^{-\frac{\|\nu\|^2}{2}} \left(\frac{t}{q}\right)^{\frac{|\nu|}{2}}, \quad (3.2.10)$$

so that when we glue two refined topological vertices, we introduce framing factors $\tilde{f}_{\nu^t}(t, q)^n$ or $f_{\nu^t}(q, t)^n$ depending on whether the internal line is the non-preferred direction (fig. 3.2b) or preferred direction (fig. 3.2c) respectively, with $n := \det(u_1, u_2)$. The rule for the unrefined topological vertex is recovered by setting $t = q$.

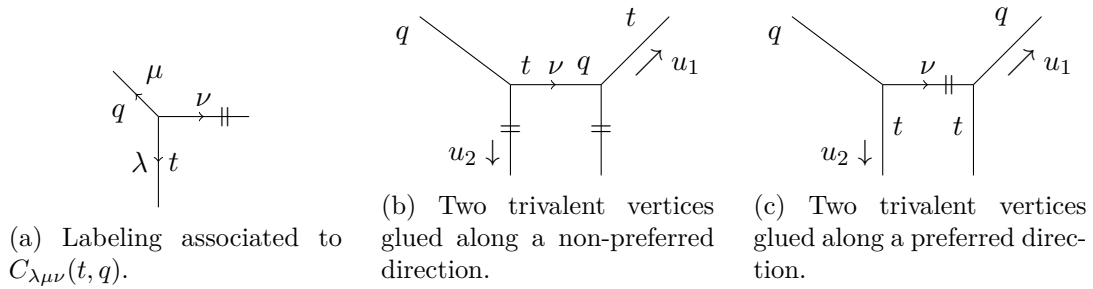


Figure 3.2: Refined vertex conventions. We express a leg in the preferred direction by \parallel .

Examples

SU(N) gauge theory As an illustrative example, consider the toric diagram for an SU(N) gauge theory, given in fig. 3.3, where we use standard notation [3, 57, 58]. Among the different possible ways to get SU(N), we have taken our diagram to be symmetric with respect to a vertical line, for later use in section 3.4 when we impose \mathbb{Z}_2 orientifold involutions. This constrains the slope of external legs in the diagram entering the topological vertex computation (in the gauge theory of the next section, this translates into a choice of 5d Chern-Simons level $K = N - 2$ [92].) We denote by Q_{F_i} the exponential

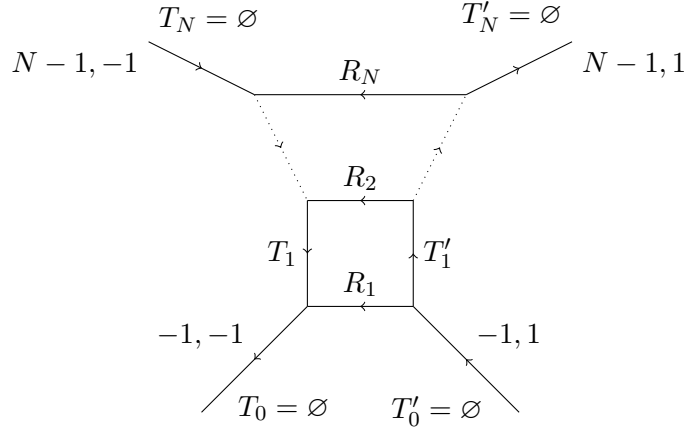


Figure 3.3: Web diagram for a toric singularity engineering pure SU(N) gauge theory with $K = N - 2$.

of the Kähler parameter for the edge T_i (for $i = 1, \dots, N - 1$). The exponentials of the Kähler parameters for the horizontal edges R_i are denoted by Q_{B_i} (for $i = 1, \dots, N$), and they can be expressed in terms of $Q_B := Q_{B_1} = Q_{B_2}$ as

$$Q_{B_i} = Q_B \prod_{m=2}^{i-1} Q_{F_m}^{2(m-1)}. \quad (3.2.11)$$

The refined topological string partition function is written

$$\begin{aligned} Z_{\text{ref top}}^{\text{SU}(N)} = & \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N \\ T'_1, \dots, T'_{N-1}}} \prod_{i=1}^N C_{T_{i-1} T_i^t R_i^t}(t, q) \tilde{f}_{T_i^t}(t, q) (-Q_{F_i})^{|T_i|} \times \\ & \times f_{R_i}(q, t)^{-2i+3} (-Q_{B_i})^{|R_i|} \times C_{T'_i T'_{i-1} R_i}(q, t) \tilde{f}_{T'_i}(q, t) (-Q_{F_i})^{|T'_i|}, \end{aligned} \quad (3.2.12)$$

where the relevant rules and quantities are defined in section 3.2.1. We focus on the unrefined case (i.e. set $t = q = e^{i\epsilon}$), where this can be explicitly evaluated to give

$$Z_{\text{top}}^{\text{SU}(N)} = \sum_{\mathbf{R}} \left(\prod_{i=1}^N q^{(\|R_i^t\|^2 - \|R_i\|^2)(1-i) + \|R_i^t\|^2} \tilde{Z}_{R_i}^2 Q_{B_i}^{|R_i|} \right) \times \prod_{1 \leq k < l \leq N} \prod_{i,j=1}^{\infty} \left[1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1-R_k(i)-R_l^t(j)} \right]^{-2}. \quad (3.2.13)$$

Using eqs. (3.C.1) and (3.C.4) we can recast the expression as sum over N -tuples of Young diagrams $\mathbf{R} = (R_1, \dots, R_N)$ such that $|\mathbf{R}| := \sum_i |R_i| = k$,

$$Z_{\text{top}}^{\text{SU}(N)} = Z_{\text{top pert}}^{\text{SU}(N)} \sum_{k=0}^{\infty} u^k \sum_{|\mathbf{R}|=k} CS_{N,N-2} \prod_{i,j=1}^N \prod_{s \in R_i} \frac{1}{(2i)^2 \sin^2 \frac{1}{2} E_{ij}(s)}, \quad (3.2.14)$$

where

$$E_{ij}(s) = t_i - t_j - \epsilon_1 l_{R_i}(s) + \epsilon_2 [a_{R_j}(s) + 1], \quad (3.2.15)$$

(we also denote by E the unrefined expression, i.e. the one with $\epsilon_1 = -\epsilon_2 =: \epsilon$). Here we have introduced the perturbative contribution

$$Z_{\text{top pert}}^{\text{SU}(N)} := \prod_{1 \leq k < l \leq N} \prod_{i,j=1}^{\infty} \left[1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1} \right]^{-2} \quad (3.2.16)$$

and the instanton fugacity

$$u := \frac{Q_B}{\left(\prod_{i=1}^{N-1} Q_{F_i}^{N-i} \right)^{\frac{2}{N}}}. \quad (3.2.17)$$

We have also introduced with hindsight the quantities $t_i = a_i + iv_i$, $i = 1, \dots, N$, satisfying $\sum_i t_i = 0$, to rephrase the Kähler parameters as

$$Q_{F_i} = e^{i(t_{i+1} - t_i)}. \quad (3.2.18)$$

$v_{i+1} - v_i$ encodes the length of vertical edge F_i in the web diagram (and eventually the gauge theory Coulomb branch parameters). They should satisfy $|Q_{F_i}| < 1$ or $v_{i+1} > v_i$, so that it is a good expansion parameter. In other words, we are at a large volume point in the Kähler moduli space spanned by v_i . We also have the quantity (eventually corresponding to the contribution from the gauge theory Chern-Simons term)

$$CS_{N,K} := \prod_{i=1}^N \prod_{s \in R_i} e^{iK E_{i\emptyset}(s)}. \quad (3.2.19)$$

Conifold Another illustrative example is the resolved conifold, whose web diagram (again restricting to a case symmetric under a line reflection, for future use) is given in fig. 3.4. The unrefined topological vertex computation gives

$$Z_{\text{top}}^{\text{U}(1)} = \exp \left\{ - \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^m}{(q^{m/2} - q^{-m/2})^2} \right\} = \prod_{n=1}^{\infty} (1 - Qq^n)^n, \quad (3.2.20)$$

where Q is the Kähler parameter.

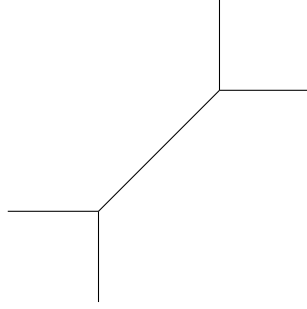


Figure 3.4: Web diagram for the conifold.

3.2.2 Supersymmetric Yang-Mills on ϵ -background

We now consider a 4d $\mathcal{N} = 2$ gauge theory. Its exact quantum dynamics is obtained by the perturbative one-loop contribution and the contribution from the (infinite) set of BPS instantons. These corrections can be obtained from a 5d theory with 8 supercharges, compactified on \mathbb{S}^1 , as a one-loop contribution from the set of 5d one-particle BPS states. These particles are perturbative states of the 5d theory and BPS instanton particles.

The 5d instanton partition function [85] is given by a power series expansion $Z_{\text{inst}} = \sum_{k=0}^{\infty} u^k Z_k^{\text{inst}}$, where the contribution for instanton number $k = c_2$ is

$$Z_k^{\text{inst}}(\{a_i\}; \epsilon_1, \epsilon_2) = \text{tr}_{\mathcal{H}_k} \left[(-1)^F e^{-\beta H} e^{-i\epsilon_1(J_1 + J_{\mathcal{R}})} e^{-i\epsilon_2(J_2 + J_{\mathcal{R}})} e^{-i \sum a_i \Pi_i} \right]. \quad (3.2.21)$$

Here we trace over the 5d Hilbert space \mathcal{H} of one-particle massive BPS states. Also, J_1 and J_2 span the Cartan subalgebra of the $\text{SO}(4)$ little group, $J_{\mathcal{R}}$ the Cartan of the $\text{SU}(2)_{\mathcal{R}}$ R-symmetry, and Π are the Cartan generators for a gauge group G . a_i , $i = 1, \dots, \text{rank } G$ are Wilson lines of G on the \mathbb{S}^1 .

The BPS particles are W-bosons, 4d instantons (viewed as solitons in the 5d theory) and bound states thereof. For $k \neq 0$, the partition function can be regarded as a Witten index in the SUSY quantum mechanics whose vacuum is the ADHM moduli space, with the SUSY algebra

$$\{Q_M^A, Q_N^B\} = P_{\mu}(\Gamma^{\mu} C)_{MN} \epsilon^{AB} + i \frac{4\pi^2 k}{g_{\text{YM}}^2} C_{MN} \epsilon^{AB} + i \text{tr}(v \Pi) C_{MN} \epsilon^{AB}, \quad (3.2.22)$$

where v_i are the 5d Coulomb branch parameters. We will eventually complexify them with the already appeared Wilson lines to complete the complex Coulomb branch parameters, bearing in mind that we will compare the Nekrasov partition function in eq. (3.2.21) with the topological string partition function in eq. (3.2.5).

In the following we focus on $G = \mathrm{U}(N)$ with 5d Chern-Simons level K . The quantity in eq. (3.2.21) can be evaluated by using localization in equivariant K-theory [81, 83, 85] on the instanton moduli space $M(N, k)$, more precisely on its Gieseker partial compactification and desingularization given by framed rank N torsion-free sheaves on $\mathbb{C}\mathbb{P}^2 = \mathbb{R}^4 \cup \ell_\infty$, where the framing is given by a choice of trivialization on the line at infinity ℓ_∞ . The ϵ -background localizes the integral, restricting it to a sum over fixed points of the equivariant action.

The result of computation for pure $G = \mathrm{U}(N)$ gauge theory with Chern-Simons level K is [23, 52, 83, 85, 92, 93]

$$Z_{\mathrm{inst}}^{\mathrm{U}(N)} = \sum_{k=0}^{\infty} u^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{iK(E_{i\emptyset}(s) - \frac{1}{2}(\epsilon_1 + \epsilon_2))}}{\prod_{j=1}^N (2i)^2 \sin \frac{E_{ij}(s)}{2} \sin \frac{E_{ij}(s) - (\epsilon_1 + \epsilon_2)}{2}}. \quad (3.2.23)$$

For comparison with the topological string result, we take $K = N - 2$. We can restrict the result to $\mathrm{SU}(N)$ by constraining the sum of the Coulomb branch moduli to be zero. Note that the $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ instanton partition functions are in general different,⁴ but they agree in our case of zero flavors with Chern-Simons level $K = N - 2$. With this proviso, we can see that eq. (3.2.14) from section 3.2.1 can be written as $Z_{\mathrm{top}}^{\mathrm{SU}(N)} / Z_{\mathrm{top\ pert}}^{\mathrm{SU}(N)} = Z_{\mathrm{inst}}^{\mathrm{U}(N)}$ evaluated for $K = N - 2$, $\sum t_i = 0$, and in the unrefined limit $\epsilon_1 = -\epsilon_2 =: \epsilon$, by identifying the complexified Kähler parameter with the complexified Coulomb branch moduli. Hence we have an exact match up to the perturbative part.

Another interesting example is $\mathrm{U}(1)$ gauge theory. Although it does not support semi-classical gauge instantons, one can consider BPS states corresponding to small instantons. A mathematically more rigorous way to define them is to consider $\mathrm{U}(1)$ instantons on non-commutative \mathbb{R}^4 , or equivalently rank 1 torsion-free sheaves on $\mathbb{C}\mathbb{P}^2$ with fixed framing on the line at infinity [84]. The gauge theory result is [80, 91]

$$\sum_R \frac{Q^{|\mathbf{R}|}}{\prod_{s \in R} (1 - q_1^{-l_R(s)} q_2^{1+a_R(s)}) (1 - q_1^{1+l_R(s)} q_2^{-a_R(s)})} = \exp \left\{ \sum_{r=1}^{\infty} \frac{1}{r} \frac{Q^r}{(1 - q_1^r)(1 - q_2^r)} \right\}, \quad (3.2.24)$$

where $q_1 = e^{i\epsilon_1}$, $q_2 = e^{i\epsilon_2}$. The exponent agrees precisely with the topological vertex result eq. (3.2.20) in the unrefined case $q_1 q_2 = 1$, by setting $q_1 = q$.

⁴The difference between the $\mathrm{U}(N)$ Nekrasov partition functions and the $\mathrm{SU}(N)$ Nekrasov partition functions has been discussed in [12, 14, 15, 46, 52]. It turns out that the web diagram nicely encodes factors that account for the difference.

3.3 Orientifolds and the real topological vertex

In this section we review properties of the unoriented theories we are going to focus on. We first introduce their description in string theory and M-theory, and subsequently review the computation of their partition function using the real topological string theory in the real topological vertex formalism.

3.3.1 Generalities

There are many ways to obtain an unoriented theory from a parent oriented string theory configuration, which in our present setup result in different gauge theory configurations. We will focus on a particular choice, which has the cleanest connection with the parent oriented theory, in a sense that we now explain.

Consider the type IIA version of our systems, namely type IIA on a non-compact toric CY threefold \mathbb{X} singularity. We introduce an orientifold quotient, acting as an antiholomorphic involution σ on \mathbb{X} and as a sign flip in an \mathbb{R}^2 (parametrized by x^2, x^3) of 4d Minkowski space. For concreteness, we consider σ to have a fixed locus L , which on general grounds is a lagrangian 3-cycle of \mathbb{X} (one can build orientifolds with similar M-theory lift even if σ is freely acting). In other words, we have an O4-plane wrapped on L and spanning x^0, x^1 ; we choose the O4-plane to carry negative RR charge (see later for other choices). We complete the configuration by introducing one single D4-brane (as counted in the covering space) wrapped on L and spanning x^0, x^1 , namely stuck on the O4⁻-plane.⁵

In general, if $H_1(L; \mathbb{Z})$ is non-trivial, it is possible to turn on \mathbb{Z}_2 -valued Wilson lines for the D4-branes worldvolume $O(1) \equiv \mathbb{Z}_2$ gauge group. This results in a different sign weight for the corresponding disk amplitudes, as discussed in the explicit example later on.

This setup is the physical realization of the real topological string introduced in [101] (see also [70–72]). In the topological setup, the addition of the D4-brane corresponds to a topological tadpole cancellation condition; in the physical setup, it corresponds to local cancellation of the RR charge, and leads to a remarkably simple M-theory lift, which allows for direct connection with the 5d picture of the oriented case, as follows.

This IIA configuration lifts to M-theory as a compactification on $\mathbb{X} \times \mathbb{S}^1$, with a \mathbb{Z}_2 quotient⁶ acting as σ on \mathbb{X} , as $(x^2, x^3) \rightarrow (-x^2, -x^3)$ on 4d Minkowski space-time, and as a half-shift along the \mathbb{S}^1 . Because the \mathbb{Z}_2 is freely acting on the \mathbb{S}^1 , the configuration can be regarded as an \mathbb{S}^1 compactification of the 5d theory corresponding to M-theory on \mathbb{X} (with the \mathbb{S}^1 boundary conditions for the different fields given by their eigenvalue under the orientifold action). Since the 5d picture is essentially as in the oriented case, these configurations have a direct relation with the oriented Gopakumar-Vafa description of the topological string. Specifically, the real topological string amplitude is given by a

⁵Additional pairs of mirror D4-branes may be added; in the M-theory lift, they correspond to the inclusion of explicit M5-branes, so that open M2-brane states enter the computations.

⁶The M-theory 3-form C_3 is intrinsically odd under this \mathbb{Z}_2 , so we refer to it as ‘orientifold action’ in M-theory as well.

one-loop diagram of 5d BPS states running on \mathbb{S}^1 , with integer (resp. half integer) KK momentum for states even (resp. odd) under the orientifold action [89].

For completeness, we quote the M-theory lifts corresponding to other choices of O4-plane and D4-brane configurations [37, 49]:

- An orientifold introducing an $O4^-$ -plane with no stuck D4-brane lifts to M-theory on $\mathbb{X} \times \mathbb{S}^1$ with a \mathbb{Z}_2 acting as σ on \mathbb{X} , flipping x^2, x^3 in 4d space, and leaving the \mathbb{S}^1 invariant. This M-theory configuration has orbifold fixed points and therefore is not directly related to the 5d picture of the oriented theory.
- An orientifold introducing an $O4^+$ -plane lifts to M-theory on $\mathbb{X} \times \mathbb{S}^1$ with a \mathbb{Z}_2 acting as σ on \mathbb{X} , flipping x^2, x^3 in 4d space, and leaving the \mathbb{S}^1 invariant, with 2 M5-branes stuck at the orbifold locus. Again this M-theory configuration has orbifold fixed points.
- Finally, there is an exotic orientifold, denoted $\widetilde{O4^+}$ -plane, which lifts to M-theory as our above freely acting orbifold (acting with a half-shift on \mathbb{S}^1), with one extra stuck M5-brane. This M-theory configuration contains a sector of closed membranes exactly as in the $O4^-+D4$ case, and in addition an open membrane sector which has no direct relation to the 5d oriented theory (but is described by Ooguri-Vafa invariants [86]).

Hence, as anticipated, we focus on the $O4^-+D4$, whose M-theory lift is the simplest and closest to the parent 5d oriented theory.

To finally determine the orientifold actions, we must specify the antiholomorphic involution σ acting on \mathbb{X} . In general, a toric CY three-fold associated to an $SU(N)$ gauge theory admits two such \mathbb{Z}_2 actions,⁷ illustrated in fig. 3.5 for $SU(4)$. They mainly differ in the effect of the orientifold action on the Coulomb branch moduli of the 5d gauge theory. Namely, the blue quotient in fig. 3.5 reduces the number of independent moduli, whereas the red one preserves this number. Equivalently, the two quotients either reduce or preserve the rank of the gauge group at the orientifold fixed locus. Since the Coulomb branch parameters play an important role in the parent gauge theory localization computation, we will focus on rank-preserving quotients to keep the discussion close to the parent theories. We leave the discussion of rank-reducing involutions for future projects.

3.3.2 The real topological string

The real topological string is a natural generalization of the topological string in section 3.2.1. It provides a topological version of the IIA orientifolds in the previous section. Namely, the real topological string computes holomorphic maps from surfaces with boundaries and crosscaps into a target \mathbb{X} modded out by the orientifold involution σ . Realizing the unoriented world-sheet surface as a quotient of a Riemann surface by an

⁷Certain cases, like the conifold, may admit additional symmetries.

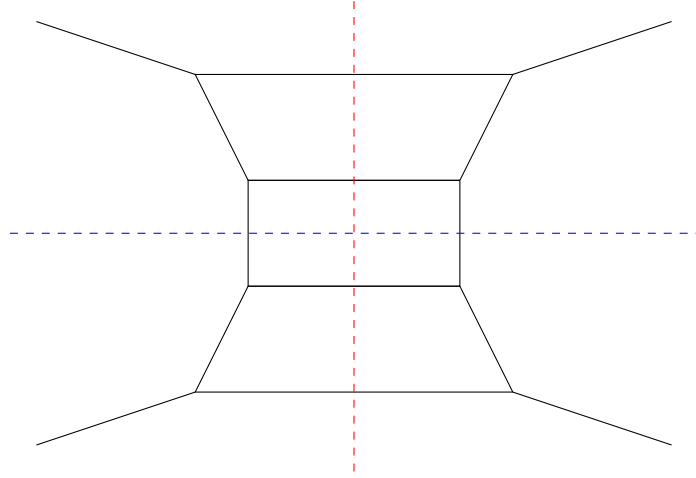


Figure 3.5: SU(4) with two involutions.

antiholomorphic involution,⁸ $\Sigma = \Sigma_g/\Omega$, we must consider equivariant maps f as in fig. 3.6.

$$\begin{array}{ccc}
 \Sigma_g & \xrightarrow{\Omega} & \Sigma_g \\
 f \downarrow & \circlearrowleft & \downarrow f \\
 \mathbb{X} & \xrightarrow{\sigma} & \mathbb{X}
 \end{array}$$

Figure 3.6: Commutative diagram for equivariant map.

The model includes crosscaps, and boundaries (with a single-valued Chan-Paton index to achieve the topological tadpole cancellation) ending on the lagrangian L . Hence, we must consider the relative homology class $f_*([\Sigma_g]) \in H_2(\mathbb{X}, L; \mathbb{Z})$.

The total topological amplitude at fixed Euler characteristic χ may be written as

$$\mathcal{G}^{(\chi)} = \frac{1}{2} \left[\mathcal{F}^{(g_\chi)} + \sum \mathcal{F}^{(g,h)} + \sum \mathcal{R}^{(g,h)} + \sum \mathcal{K}^{(g,h)} \right], \quad (3.3.1)$$

where the different terms account for closed oriented surfaces, oriented surfaces with boundaries, surfaces with one crosscap, and surfaces with two crosscaps. Different consistency conditions, needed to cancel otherwise ill-defined contributions from the enumerative geometry viewpoint,⁹ guarantee integrality of the BPS expansion for

$$\mathcal{F}_{\text{real}} = \sum_{\chi} i^{\chi} \epsilon^{\chi} \left(\mathcal{G}^{(\chi)} - \frac{1}{2} \mathcal{F}^{(g_\chi)} \right). \quad (3.3.2)$$

⁸The case Σ itself is a Riemann surface requires to start from a disconnected Σ_g , and is better treated separately.

⁹For example, for configurations in which $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$, one needs to cancel homologically trivial disks against crosscaps.

This can be taken as the definition of the real topological string.

This integrality of BPS invariants, as well as a physical explanation of the tadpole cancellation and other consistency conditions of the real topological string, may be derived from the M-theory viewpoint [89]. The real topological string amplitude is obtained as a sum over 5d BPS M2-brane states of the oriented theory, running in the compactification \mathbb{S}^1 with boundary conditions determined by the eigenvalue under the orientifold operator. For a short review, see section 3.A.1. Denoting by $\widehat{\text{GV}}_{\hat{g},\beta}$ this weighted BPS multiplicity of M2-branes wrapped on a genus \hat{g} surface (as counted in the quotient) in the homology class β , the equivalent to eq. (3.2.4) is

$$\mathcal{F}_{\text{real}} = \sum_{\substack{\beta, \hat{g} \\ \text{odd } m \geq 1}} \widehat{\text{GV}}_{\hat{g},\beta} \frac{1}{m} \left[2 \sinh \left(\frac{m\epsilon}{2} \right) \right]^{\hat{g}-1} e^{im\beta \cdot t}. \quad (3.3.3)$$

To compute real topological string partition function on local Calabi-Yau, we will use the real topological vertex [70], which is a generalization of the standard topological vertex to take into account involutions of the toric diagram. The formalism is still only available in the unrefined case, on which we focus herefrom.

We apply the formalism to involutions of the kind shown in fig. 3.7, as described in more detail in section 3.A.2. For these involutions there are no legs fixed point-wise in the diagram, and this simplifies the computation of the topological vertex. Due to the symmetry of the diagram in the parent theory, one can use symmetry properties of the vertex functions to cast each summand in the sum over Young diagrams as a square [70]. Then the real topological vertex amplitude is given by the sum of the square roots of the summands. To define these in a consistent way, we follow the choice of sign in [70], see eq. (3.A.6). In all our examples this sign is trivial, since $|R| \pm c(R)$ is even for every R , and in the cases we consider also $n+1 = -2i+4$ is even as well, as can be seen from eq. (3.2.12).

Explicit examples will be described in section 3.4.

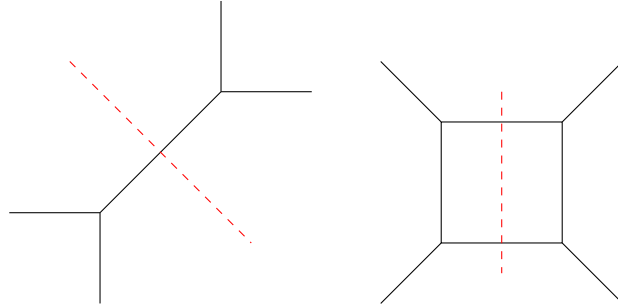


Figure 3.7: Web diagram for our orientifolds of the conifold and $\text{SU}(2)$ theories.

3.4 Explicit examples

In this section, we explicitly compute the real topological string partition functions of the resolved conifold and also the $SU(N)$ geometry using the real topological vertex formalism [70]. We first review the calculation of the real topological string partition function of the resolved conifold [70], correcting some typos. Then, we move on to the computation of the real topological string partition function for the $SU(N)$ geometry, which shows an intriguing new feature.

3.4.1 The real conifold

Let us apply the above recipe to the orientifold of the resolved conifold. This is particularly simple because there are no Coulomb branch moduli, and the only parameters are the instanton fugacity and those defining the ϵ -background.

The topological string side can be computed using the real topological vertex formalism. The result¹⁰ reads

$$Z_{\text{real top}}^{\text{U}(1)} = \exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^m}{(q^{m/2} - q^{-m/2})^2} \pm \sum_{m \text{ odd}} \frac{1}{m} \frac{Q^{m/2}}{q^{m/2} - q^{-m/2}} \right\}. \quad (3.4.1)$$

Our choice of orientifold plane charge corresponds to the negative sign.¹¹

The first term in the exponent corresponds to the closed topological string contribution, while the second reproduces the open and unoriented topological string contributions.

3.4.2 Orientifold of pure $SU(N)$ geometry, and its flops

In this section we study the unoriented version of the $SU(N)$ systems of section 3.2.1, compute their real topological vertex amplitudes following the rules in [70], and describe their behavior under flops of the geometry.

Real topological vertex computation

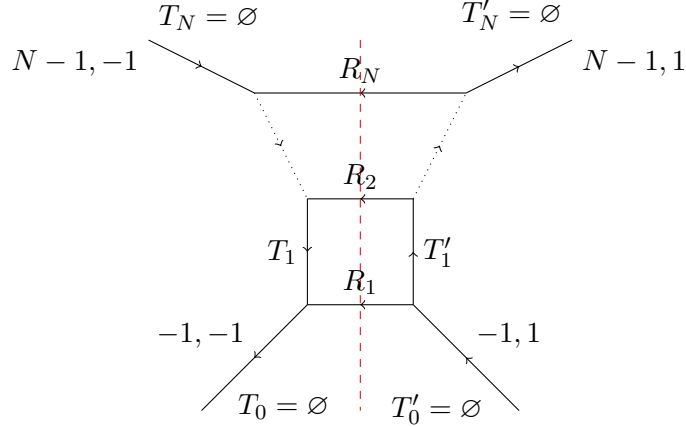
The web diagram is given in fig. 3.8, which describes a \mathbb{Z}_2 involution of fig. 3.3 (which was chosen symmetric in hindsight).

We recall some expressions already introduced in section 3.2.1 for the oriented case. We define the perturbative contribution as

$$Z_{\text{real top pert}}^{\text{SU}(N)} := \prod_{1 \leq k < l \leq N} \prod_{i, j=1}^{\infty} \left(1 - \left(\prod_{m=k}^{l-1} Q_{F_m} \right) q^{i+j-1} \right)^{-1}. \quad (3.4.2)$$

¹⁰This expression corrects some typos in [70].

¹¹The choice of positive sign can be recovered by turning on a non-trivial \mathbb{Z}_2 -valued Wilson line on the D4-brane stuck at the O4-plane in the type IIA picture, since the fixed locus $L = \mathbb{R}^2 \times \mathbb{S}^1$ has one non-trivial circle.


 Figure 3.8: $SU(N)$ with $K = N - 2$ and involution.

We also recall the Chern-Simons level eq. (3.2.19)

$$CS_{N,K} := \prod_{i=1}^N \prod_{s \in R_i} e^{iK E_{i\emptyset}(s)}, \quad (3.4.3)$$

where E_{ij} is defined in eq. (3.2.15) and \emptyset denotes the empty diagram. Finally, we introduce the rescaled instanton fugacity

$$\tilde{u} := \frac{Q_B^{\frac{1}{2}}}{\left(\prod_{i=1}^{N-1} Q_{F_i}^{N-i}\right)^{\frac{1}{N}}}, \quad (3.4.4)$$

which is the square root of the instanton fugacity eq. (3.2.17) in the oriented computation.

Expressing the (complexified) Kähler parameters in terms of the (complexified) edge positions t_i , with $\sum_i t_i = 0$, and edges ordered such that $\text{Im } t_{i+1} > \text{Im } t_i$ in a certain large volume region in the Kähler moduli space, we take as in eq. (3.2.18)

$$Q_{F_i} = e^{i(t_{i+1} - t_i)}. \quad (3.4.5)$$

Notice that in this case the fugacity can be written as $\tilde{u} = Q_B^{\frac{1}{2}} e^{it_1}$.

The computation is as follows: we start from eq. (3.2.12), go to the unrefined limit, and apply real topological vertex rules [70], cf. section 3.A.2. Since our involution does not fix any leg point-wise, we only need to reconstruct the square within summands using permutation properties of the topological vertex $C_{RR'R''}(q)$ (see eq. (3.A.4)), and the fact that $T_i = T'_i$ due to the involution. We get

$$Z_{\text{top}}^{\text{SU}(N)} \Big|_{T_i = T'_i} = \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N}} \prod_{i=1}^N C_{T_i T_{i-1}^t R_i}^2(q) Q_{F_i}^{2|T_i|} Q_{B_i}^{|R_i|} (-1)^{|R_i|(4-2i)} q^{\|T_i\|^2 - \|T_i^t\|^2 + (2-i)(\|R_i^t\|^2 - \|R_i\|^2)}. \quad (3.4.6)$$

We then take the square root, and notice that the sign eq. (3.A.6) for the propagator is always +1,

$$Z_{\text{real top}}^{\text{SU}(N)} = \sum_{\substack{T_1, \dots, T_{N-1} \\ R_1, \dots, R_N}} \prod_{i=1}^N C_{T_i T_{i-1} R_i} (q) Q_{F_i}^{|T_i|} Q_{B_i}^{|R_i|/2} q^{\frac{1}{4}(\|T_i\|^2 - \|T_i^t\|^2) + \frac{1}{4}(\|T_{i-1}\|^2 - \|T_{i-1}^t\|^2) + \frac{2-i}{2}(\|R_i^t\|^2 - \|R_i\|^2)}. \quad (3.4.7)$$

By using combinatorial identities for Young diagrams and skew-Schur functions, described in section 3.C (in particular eqs. (3.C.1) and (3.C.5)), we arrive at the final result

$$Z_{\text{real top}}^{\text{SU}(N)} = Z_{\text{real top pert}}^{\text{SU}(N)} \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} (-1)^{\sum_i (i-1)|R_i|} CS_{N, \frac{N-2}{2}} \prod_{i,j=1}^N \prod_{s \in R_i} \frac{1}{2i \sin \frac{1}{2} E_{ij}(s)}. \quad (3.4.8)$$

Behavior under flops

It is worthwhile to briefly step back and emphasize an important point. In the above computation there is an explicit choice of ordering of edges in the web diagram, which defines a particular large volume limit. Moving in the Kähler moduli space across a wall of a flop transition¹² can reorder the edges, so we need to redefine the expansion parameters eq. (3.2.18). Consider the simplest setup in which the ordering of all edges is reversed, such that $v_i > v_{i+1}$ for all i , and we take

$$Q_{F_i} = e^{i(t_{N-i} - t_{N-i+1})}, \quad (3.4.9)$$

with $|Q_{F_i}| < 1$. In this case, we are at a different large volume point in the enlarged Kähler moduli space compared to the case when we defined t_i by eq. (3.4.5). From the viewpoint of the five-dimensional pure $\text{SU}(N)$ gauge theory, it corresponds to moving to a different Weyl chamber in the Coulomb branch moduli space by a Weyl transformation $t_i \rightarrow t_{N-i+1}$ for all i . If we write the result by using $E_{ij}(s)$ and $CS_{N,K}$ defined in eqs. (3.2.15) and (3.2.19), this gives the same exact result *except* for sign pattern $(-1)^{\sum_i (N-i)|R_i|}$. The computation of this result is similar to eq. (3.4.8) except that we redefined dummy variables $R_i^{\text{new}} := (R_{N-i+1}^{\text{old}})^t$, compared to the geometry before the flop transition. When we regard eq. (3.4.8) as a function of t_i , we have computed

$$Z_{\text{real top}}(t_{N-i+1}, K) =: \tilde{Z}_{\text{real top}}(t_i, K), \quad (3.4.10)$$

which is not equal to $Z_{\text{real top}}(t_i, K)$ as a function of t_i in the unoriented case.¹³ This is different from the oriented case where we have $Z_{\text{top}}(t_i, K) = Z_{\text{top}}(t_{N-i+1}, K)$, which

¹²The flop transition considered here shrinks and introduces a family of rational curves. This is different from the usual flop which shrinks and introduces an isolated rational curve. The behavior of the (non-real) topological string partition function under the usual flop has been studied in [59, 68, 94].

¹³In the case $N = 3$ they are equal to each other accidentally.

should be true since Weyl transformations are part of the gauge transformations. Therefore, this is a feature special to the real topological string partition function of pure $SU(N)$ geometry. From the viewpoint of five-dimensional pure $SU(N)$ gauge theory, the pure $SU(N)$ gauge theory is invariant under the Weyl transformation of $SU(N)$ and this is reflected into the invariance of the partition function under the Weyl transformation in the oriented case. In the unoriented case, however, the non-invariance of the partition function under the transformation implies that the presence of the orientifold or the corresponding defect in field theory breaks the symmetry that existed in the oriented case.

It is similarly easy to consider intermediate cases of partial reorderings. The simplest is to take $N = 3$, and move from $v_1 < v_2 < v_3$ to $v_2 < v_3 < v_1$. In this case, the new expansion parameters are $Q_{F_1} = e^{i(t_3-t_2)}$ and $Q_{F_2} = e^{i(t_1-t_3)}$. The result is basically the same, but with sign $(-1)^{2|R_1|+|R_3|}$. Here we redefined dummy variables as $R_1^{\text{new}} := (R_3^{\text{old}})^t$, $R_2^{\text{new}} := (R_1^{\text{old}})^t$, $R_3^{\text{new}} := (R_2^{\text{old}})^t$.

In other words, starting with the result in a given chamber, moving across a wall of a flop transition exchanging two edges with diagrams R, S produces a change in the amplitude (expressed in the new Kähler parameters) given by a sign $(-1)^{|R|+|S|}$.

This is the explicit manifestation of the fact that the topological string amplitude regarded as a function of t_i in this unoriented theory is not universal throughout the moduli space, but it has a non-trivial behavior.¹⁴

Behavior under other transformations

Let us consider another transformation which is a reflection with respect to a horizontal axis for the pure $SU(N)$ geometry of fig. 3.8. The operation in the original pure $SU(N)$ geometry corresponds to charge conjugation, which is given by a transformation of the Coulomb branch moduli $t_i \rightarrow -t_{N-i+1}$ for $i = 1, \dots, N$ and a flip of the sign of CS level.¹⁵ The transformation can be effectively implemented by defining the Kähler parameters as

$$Q_{F_i} = e^{i(t_i-t_{i+1})}, \quad (3.4.11)$$

with $|Q_{F_i}| < 1$ and the same labeling for R_i for all i as in fig. 3.8. Compared to eq. (3.4.5), we flip the sign of t_i for all i . Hence, we are now assuming $v_i > v_{i+1}$ for all i and hence we effectively consider the pure $SU(N)$ geometry upside-down. If we write the result by using $E_{ij}(s)$ and $CS_{N,K}$ defined in eqs. (3.2.15) and (3.2.19), this gives the same result except for different CS level $-\frac{N-2}{2}$ and sign pattern $(-1)^{\sum_i (N-i)|R_i|}$. In this case, we did not redefine the dummy variables R_i . The change in the sign of CS level is consistent with the fact that the definition eq. (3.4.11) is related to charge conjugation

¹⁴However, when we regard the topological string amplitude as a function of a good expansion parameter which is always Q_{F_i} with $|Q_{F_i}| < 1$, then they are essentially the same function. Namely, the real GV invariants are the same at the two different points in the enlarged Kähler moduli space.

¹⁵When we regard the pure $SU(N)$ geometry as a 5-brane web diagram, the CS level can be read off from the asymptotic behavior of the external legs [13, 14, 67]. In particular, when we turn the 5-brane web upside-down, the sign of the CS level of the gauge theory also flips.

of the original pure $SU(N)$ gauge theory. When we regard eq. (3.4.8) as a function of t_i , we have computed

$$Z_{\text{real top}}(-t_i, -K) =: \tilde{Z}_{\text{real top}}(t_i, K), \quad (3.4.12)$$

which is again not equal to $Z_{\text{real top}}(t_i, K)$ as a function of t_i in the unoriented case. This is different from the oriented case where we have $Z_{\text{top}}(t_i, K) = Z_{\text{top}}(-t_i, -K)$.¹⁶ This is another example of a transformation where the presence of the orientifold defect breaks the invariance of the partition function under a transformation that existed in the original theory without the defect.

3.5 Discussion: towards gauge theory interpretation

As explained in the introduction, the real topological string on the local CY threefold should be related to the partition function of the corresponding gauge theory in the presence of a surface defect. Given the M-theory lift of the orientifold in terms of a freely-acting shift on the S^1 , this should correspond to a partition function of the theory on S^1 , with modified boundary conditions, or equivalently with an extra twist in the Witten index computation,

$$Z_k^{\text{real inst}}(\{a_i\}; \epsilon_1, \epsilon_2) = \text{tr}_{\mathcal{H}_k} \left[(-1)^F e^{-\beta H} e^{-i\epsilon_1(J_1 + J_{\mathcal{R}})} e^{-i\epsilon_2(J_2 + J_{\mathcal{R}})} e^{-i \sum a_i \Pi_i} \mathcal{O}_{\Omega} \right], \quad (3.5.1)$$

where \mathcal{O}_{Ω} is an operator implementing the orientifold action in the corresponding Hilbert space sector.

In this section we exploit the intuitions from the topological vertex computations to describe aspects of this twist in explicit examples.

3.5.1 Invariant states and the conifold example

We start the discussion with the conifold. This is particularly simple, because there is only one BPS (half-)hypermultiplet, whose internal structure is invariant under the orientifold, namely it is an M2-brane wrapped on a 2-cycle mapped to itself under the orientifold. Then the orientifold action is just action on the Lorentz quantum numbers. From the viewpoint of gauge theory, there is a 5d $U(1)$ gauge theory, whose BPS states are instantons. In this simple system it is possible to motivate the structure of the twisted Nekrasov partition function eq. (3.5.1), i.e. of the operator \mathcal{O}_{Ω} . As discussed in section 3.2.1, the ADHM moduli space is just $(\mathbb{C}^2)^k / S_k$ where k is the instanton number and S_k is the symmetric group of order k ; these moduli are intuitively the positions of the k instantons in \mathbb{R}^4 . Therefore the action of \mathcal{O}_{Ω} on this moduli space is simply the geometric action imposed by the orientifold. Since the orientifold action flips the space-time coordinates $(x^2, x^3) \rightarrow (-x^2, -x^3)$, we are motivated to take \mathcal{O}_{Ω} as given by a shift

$$\epsilon_2 \rightarrow \epsilon_2 + \pi \quad (3.5.2)$$

¹⁶We have checked this for $N = 2, 3, 4$ up to 5-instanton order.

in the original parent gauge theory expression eq. (3.2.21). Furthermore, we assume that the operator induces the redefinition by a factor of 2 of certain quantities between the parent theory and the twisted theory. In practice, it requires that the twisted theory result should be expressed in terms of the redefined weight

$$Q \rightarrow Q^{\frac{1}{2}}. \quad (3.5.3)$$

We then consider the twisted Nekrasov partition function of the U(1) instanton. First, we consider the refined amplitude for the original theory eq. (3.2.24), and perform the shift $\epsilon_2 \rightarrow \epsilon_2 + \pi$. Taking the unrefined limit, we obtain

$$\exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{Q^{2m}}{(2i \sin \epsilon m)^2} - \sum_{k \text{ odd}} \frac{1}{k} \frac{Q^k}{2i \sin \epsilon k} \right\}. \quad (3.5.4)$$

We now redefine $\epsilon = \tilde{\epsilon}/2$ and $Q = \tilde{Q}^{\frac{1}{2}}$, and get

$$Z_{\text{real inst}}^{\text{U}(1)} = \exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\tilde{Q}^m}{(2i \sin \frac{1}{2} \tilde{\epsilon} m)^2} - \sum_{k \text{ odd}} \frac{1}{k} \frac{\tilde{Q}^{k/2}}{2i \sin \frac{1}{2} \tilde{\epsilon} k} \right\}, \quad (3.5.5)$$

which agrees with eq. (3.4.1) for the negative overall sign for the unoriented contribution. One may choose a redefinition $Q = -\tilde{Q}^{\frac{1}{2}}$, which agrees with eq. (3.4.1) for the positive overall sign for the unoriented contribution. This choice reflects the choice of \mathbb{Z}_2 Wilson line, although its gauge theory interpretation is unclear.

Note that the conifold geometry is special in that the only degree of freedom is one BPS (half-)hypermultiplet, from an M2-brane on a \mathbb{CP}^1 invariant under the orientifold action, which thus motivates a very simple proposal for \mathcal{O}_Ω . This can in general change in more involved geometries, where there are higher spin states, and/or states not invariant under the orientifold. In these cases, the orientifold action should contain additional information beyond its action on 4d space-time quantum numbers.

3.5.2 A twisted Nekrasov partition function for pure SU(N)

We now consider the case of the SU(N) geometry, whose real topological string amplitude was described in section 3.4.2, and discuss its interpretation in terms of a twisted Nekrasov partition function.

We start with the following observation. Consider the parent theory expression eq. (3.2.23) for CS level $K = N - 2$ as starting point. Since part of the orientifold action includes a space-time rotation which motivates the ϵ_2 -shift, let us carry it out just like in the conifold case and check the result.

Let us thus perform the ϵ_2 -shift and take the unrefined limit ($\epsilon_2 \rightarrow -\epsilon + \pi$, $\epsilon_1 \rightarrow \epsilon$), and rescale $\epsilon \rightarrow \frac{\epsilon}{2}$, and $t_i \rightarrow \frac{t_i}{2}$ with $u \rightarrow \tilde{u} (= u^{\frac{1}{2}})$. We obtain the result

$$Z_{\text{real inst}}^{\text{U}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} (-1)^{k+A+\Gamma} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}, \quad (3.5.6)$$

where $A = \frac{1}{2}N (\sum_i ||R_i^t||^2 + |R_i|)$, and $\Gamma = \sum_{ij} \sum_{s \in R_i} a_{R_j}(s)$. One can prove that $A + \Gamma = \sum_{ij} \sum_{(m,n) \in R_i} R_j^t(n) \equiv k \pmod{2}$, so the final result is

$$Z_{\text{real inst}}^{\text{U}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}. \quad (3.5.7)$$

This expression evaluated for $\sum t_i = 0$ is remarkably close to the real topological string computation eq. (3.4.8), up to i -dependent sign factors. Specifically eq. (3.4.8) can be recast as

$$Z_{\text{real top}}^{\text{SU}(N)} / Z_{\text{real top pert}}^{\text{SU}(N)} = \sum_{k=0}^{\infty} \tilde{u}^k \sum_{|\mathbf{R}|=k} \prod_{i=1}^N \prod_{s \in R_i} \frac{(-1)^{i-1} e^{i \frac{N-2}{2} E_{i\emptyset}(s)}}{\prod_{j=1}^N (2i) \sin \frac{1}{2} E_{ij}(s)}. \quad (3.5.8)$$

As emphasized, our viewpoint is that the real topological string computation defines the rules to describe the properties of the orientifold surface operator in the $\text{SU}(N)$ gauge theory. Let us now discuss the effect of the additional signs from the perspective of the gauge theory, to gain insight into the additional ingredients in \mathcal{O}_Ω beyond the ϵ_2 -shift. The orientifold is acting with different (alternating) signs on the different Young diagram degrees of freedom associated to the edges in the web diagram. This might imply an action with different signs on the states charged under the corresponding Cartans (alternating when ordered as determined by the 5d real Coulomb branch parameters). It would be interesting to gain a more direct gauge theory insight into the definition of this orientifold action.

Before concluding, we would like to mention an important point. We have used the ϵ_2 -shift exactly as in the conifold case in section 3.5.1, and obtained a result very close to the real topological vertex computation. However, one should keep in mind that the BPS states of the $\text{SU}(N)$ theory have a much richer structure. Therefore, the additional signs are of crucial importance to reproduce the correct results for the complete orientifold action on the theory.

For instance, if we isolate the unoriented contribution from eq. (3.5.8), the extra signs are crucial to produce certain non-zero real BPS multiplicities. This can be checked explicitly e.g. for $\text{SU}(2)$ using the results from section 3.B. For instance, consider the real BPS multiplicities n_{d_1, d_2}^g for M2-branes wrapped with degrees d_1, d_2 on the homology classes B and F , respectively. Already at $g = 0$ we have $n_{1,0}^0 = -2$ but $n_{0,1}^0 = 0$. The extra signs are crucial to produce a non-zero result for the unoriented contribution of the vector multiplet from the M2-brane on B (which using eq. (3.5.7) would give zero contribution). Similar considerations can be drawn for many others of the enumerative results in section 3.B.

3.6 Conclusions

In this chapter we have explored the extension of the correspondence between topological strings on toric CY three-folds and 4d/5d supersymmetric gauge theories with 8

supercharges to systems with orientifolds with real codimension 2 fixed locus. On the topological string side, we have focused on quotients which produce the real topological string of [101], because of its remarkably simple physical realization in M-theory. We have analyzed the properties of the systems, and emphasized their behavior under flops of the geometry.

The real topological string amplitudes define the properties of a new kind of surface defect in the corresponding gauge theory. We have rephrased the amplitudes in a form adapted to a gauge theory interpretation, by means of a newly defined twisted Nekrasov partition function, and we have taken the first steps towards providing an intrinsically gauge-theoretic interpretation of the twisting operator.

It would be interesting to complete the gauge theory interpretation of the twist operator. The partition function obtained by the simple ϵ_2 shift does not distinguish invariant states from non-invariant states. Therefore, M2-branes wrapping F and M2-branes wrapping B essentially give the same contribution to the partition function eq. (3.5.7). However, in general, they would give a different contribution in the real topological string amplitude since the former correspond to non-invariant states and the latter correspond to invariant states. Hence, another implementation may be related to some operation that distinguishes the invariant states from the non-invariant states. There may be also a possibility to shift the Coulomb branch moduli like the ordinary orbifolded instantons [62].

A way to complete the gauge theory interpretation may be to give a more specific description of the orientifold in the ADHM quantum mechanics. In [62], instantons with a surface defect were identified with orbifolded instantons via a chain of dualities of string theory. It would be interesting to extend their reasoning to our case and determine an effect of the orientifold defect in the ADHM quantum mechanics. Once we identify the effect in the ADHM quantum mechanics, then we may proceed in the standard localization technique with it.

There are several other interesting directions worth exploring:

- It would be interesting to exploit the M-theory picture to develop a refined real topological vertex formalism, and to compare it with the gauge theory computations.
- As explained, there are different kinds of O4-planes in the physical type IIA picture, which correspond to different M-theory lifts, and different unoriented topological strings (albeit, with intricate relations).
- It would also be interesting to consider the addition of extra D4-branes, either on top of the O4-plane or possibly on other lagrangian 3-cycles, to describe the unoriented version of the relation of open topological strings and vortex counting on surface defects [19, 27].

3.A Real topological string

3.A.1 M-theory interpretation of the real topological string

BPS state counting As already observed, we consider tadpole canceling configurations, such that the M-theory lift of the O4/D4 system is smooth, i.e. there are no fixed points. This guarantees that locally, before moving around the M-circle, the physics looks like in the oriented case. The $SU(2)_L \times SU(2)_R$ group is broken by the orientifold to its Cartan generators, which are enough to assign multiplicities to the 5d BPS states.

There will be two kinds of states, as follows. First, those not invariant under the orientifold, will have their orientifold image curve somewhere in the covering \mathbb{X} , and will contribute to closed oriented amplitudes (thus, we can neglect them). Second, those corresponding to curves mapped to themselves by the involution; their overall \mathbb{Z}_2 parity is determined by their $SU(2)_L$ and $SU(2)_R$ multiplet structure: as in the original GV case, the R part parametrizes our ignorance about cohomology of D-brane moduli space, while the L part is cohomology of Jacobian, and we know how to break it explicitly, due to our simple orientifold action. Let us split the second class of states, for fixed genus \hat{g} and homology class β , according to their overall parity

$$GV'_{\hat{g},\beta} = GV'^+_{\hat{g},\beta} + GV'^-_{\hat{g},\beta}. \quad (3.A.1)$$

Once we move on the circle, these invariant states acquire integer or half-integer momentum, according to their overall parity; this is taken into account in the 2d Schwinger computation, that finally yields, after removing even wrapping states that correspond again to closed oriented sector, the numbers

$$\widehat{GV}_{\hat{g},\beta} := GV'^+_{\hat{g},\beta} - GV'^-_{\hat{g},\beta} \quad (3.A.2)$$

that appear in eq. (3.3.3). The detailed computation is described in [89].

Tadpole cancellation The requirement that physical tadpoles are canceled has an interesting implication for real topological amplitudes, for geometries with $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{X}, L; \mathbb{Z}) = \mathbb{Z}$; these include well-studied examples like the real quintic or real local $\mathbb{C}\mathbb{P}^2$. We discuss this for completeness, even though the geometries in the main text do not have this torsion homology on L .

Denote the degree of a map by $d \in H_2(\mathbb{X}, L; \mathbb{Z})$. By looking at the appropriate exact sequence in homology, one can see that crosscaps contribute an even factor to d , while boundaries may contribute even or odd factor. Moreover, by looking at the M-theory background form C_3 , one can see that it contributes to the central charge a factor of $i/2$ for every crosscap, i.e. $\mathbb{R}\mathbb{P}^2$, that surrounds the O4-plane. This translates into a minus sign once Poisson resummation is performed, more precisely a $(-1)^{mc}$ sign, where c is the crosscap number and m wrapping number around the circle (recall the states we are interested in have odd m .)

Finally, since boundaries do not receive such contributions and one can show, with a heuristic argument regarding real codimension one boundaries in the moduli space of

stable maps, that there is a bijection between curves that agree except for a replacement of a crosscap with a (necessarily even-degree) boundary, we conclude that these two classes of curves cancel against each other. This implies that the only contributions may arise from odd-degree-boundary curves, and it is written as a restriction $\chi = d \bmod 2$ on the summation in eq. (3.3.3), as it has been proposed in [101] based on the fact that these two contributions to the topological amplitude are not mathematically well-defined separately, and the above mentioned prescription produces integer BPS multiplicities.

3.A.2 Real topological vertex

The real topological vertex [70] is a technique that allows to compute the all genus topological string partition functions, in the presence of toric orientifolds, namely a symmetry of the toric diagram with respect to which we quotient. This corresponds to an involution σ of \mathbb{X} , and it introduces boundaries and crosscaps in the topological string theory. We restrict to unrefined quantities, since at the moment real topological vertex technology is only available for that setup.

The recipe morally amounts to taking a square root of the topological vertex amplitude of the corresponding oriented parent theory, as follows. First, we observe that contributions from legs and vertices that are not fixed by the involution can be dealt with using standard vertex rules, and they automatically give rise to a perfect square once paired with their image. We then only need to explain how to deal with a fixed edge connecting two vertices. Their contribution to the partition function is given by a factor

$$\sum_{R_i} C_{R_j R_k R_i} (-e^{-t_i})^{|R_i|} (-1)^{n|R_i|} q^{\frac{1}{2}n(\|R_i\|^2 - \|R_i^t\|^2)} C_{R'_j R'_k R_i^t}, \quad (3.A.3)$$

where $C := C(q, q)$ was introduced in eq. (3.2.8), and notation corresponds to fig. 3.9. Here $n := \det(v_{j'}, v_j)$, where v_m represents an outgoing vector associated to leg m .

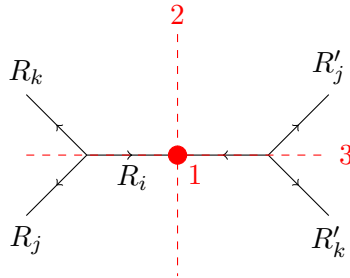


Figure 3.9: Three involutions for a generic internal leg; notice that each involution requires a specific symmetry to be present.

There are three cases: the involution can act as a point reflection at the center of the line (1), a reflection at the line perpendicular to the compact leg (2), or a reflection along the compact leg (3). The case interesting for us is (2), namely a leg that is not point-wise fixed by the involution, and where the representations in one vertex are mapped

to representations in the other. In this case, no restriction is imposed on the internal representation R_i , while leg j is mapped to leg k' and similarly $k \rightarrow j'$. This imposes $R_j = R_k^t$ and $R_k = R_j^t$, where the transposition is implemented since the involution introduces an orientation-reversal of the plane in fig. 3.9. By exploiting the symmetry of function C

$$C_{AB^tC} = q^{\frac{\|A\|^2 - \|A^t\|^2 + \|B^t\|^2 - \|B\|^2 + \|C\|^2 - \|C^t\|^2}{2}} C_{BA^tC^t}, \quad (3.A.4)$$

we can rewrite eq. (3.A.3) as a perfect square, and take the square root:

$$\sum_{R_i} C_{R_j R_k R_i} e^{-\frac{1}{2} t_i |R_i|} (-1)^{\frac{1}{2}(n+1)s(R_i)} q^{\frac{1}{4}(n-1)(\|R_i\|^2 - \|R_i^t\|^2) + \frac{1}{4}(\|R_k^t\|^2 - \|R_k\|^2) + \frac{1}{4}(\|R_j\|^2 - \|R_j^t\|^2)}. \quad (3.A.5)$$

We introduced a sign in eq. (3.A.5)

$$(-1)^{\frac{1}{2}(n+1)s(R_i)}, \quad (3.A.6)$$

determined by $s(R) = |R| \pm c(R)$, where $c(R)$ is defined via $|R| - c(R) = 2 \sum_i R(2i)$. Finally, there is a global prescription for the choice of $c(R_i)$ vs. $c(R_i^t)$.

3.B Enumerative checks

We compute the real GV invariants of the $SU(N)$ geometry with the involution considered in section 3.4.2. We describe some numerical checks that the topological vertex amplitudes indeed produce integer BPS multiplicities, corresponding to the proposed BPS state counting for the real topological string [89, 101]. The enumerative checks support the new result of the real topological string partition function for $SU(N)$ geometry in section 3.4.2.

After removing the purely closed oriented contribution, we perform an expansion

$$Z^{\text{unor}} = \frac{Z^{\text{real}}}{\sqrt{Z^{\text{or}}}} = 1 + Z_{1\text{-inst}}^{\text{real}} u^{\frac{1}{2}} + \left(Z_{2\text{-inst}}^{\text{real}} - \frac{1}{2} Z_{1\text{-inst}} \right) u + \mathcal{O}\left(u^{\frac{3}{2}}\right). \quad (3.B.1)$$

Note that the perturbative contribution does not contribute to the unoriented string part in the current choice of involution. From eq. (3.B.1), we can compute the real GV invariants: they are the numbers n appearing if we rewrite it using eq. (3.3.3),

$$Z^{\text{unor}} = \exp \sum_{d_1, d_2, g; \text{ odd } k} \frac{n_{d_1, d_2}^g}{k} \left(2i \sin \frac{k\epsilon}{2} \right)^{g-1} Q_B^{\frac{kd_1}{2}} Q_F^{\frac{kd_2}{2}}, \quad (3.B.2)$$

where we focus on the $SU(2)$ computation; from eq. (3.4.8) with $N = 2$ we obtain

$$Z_{1\text{-inst}}^{\text{real}} = \frac{1}{2 \sin \frac{\epsilon}{2} \sin t} \quad (3.B.3)$$

and

$$Z_{u^1}^{\text{unor}} = -\frac{3}{8 \sin^2 \frac{\epsilon}{2} (\cos 2t - \cos \epsilon)} - \frac{1}{16 \sin^2 t \sin^2 \frac{\epsilon}{2}}. \quad (3.B.4)$$

For illustration, we obtain real GV numbers up to $d_2 \leq 6$ and $g \leq 6$. We have $n_{1,d_2}^0 = -2$, for $d_2 = 0, 2, 4, 6$, and the others are zero. For n_{2,d_2}^g , we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	3	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	12	0	3	0	0	0
5	0	0	0	0	0	0	0
6	0	30	0	18	0	0	0

(3.B.5)

For $d_1 = 3$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	-6	0	-4	0	0	0	0
3	0	0	0	0	0	0	0
4	-28	0	-58	0	-28	0	-4
5	0	0	0	0	0	0	0
6	-82	0	-324	0	-362	0	-184

(3.B.6)

For $d_1 = 4$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	12	0	5	0	0	0
3	0	0	0	0	0	0	0
4	0	153	0	268	0	177	0
5	0	0	0	0	0	0	0
6	0	900	0	3107	0	4670	0

(3.B.7)

For $d_1 = 5$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	-12	0	-20	0	-6	0	0
3	0	0	0	0	0	0	0
4	-156	0	-744	0	-1212	0	-962
5	0	0	0	0	0	0	0
6	-990	0	-8518	0	-27704	0	-49814

(3.B.8)

For $d_1 = 6$ we have

$d_2 \setminus g$	0	1	2	3	4	5	6	(3.B.9)
0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	
2	0	30	0	30	0	7	0	
3	0	0	0	0	0	0	0	
4	0	900	0	3293	0	5378	0	
5	0	0	0	0	0	0	0	
6	0	10255	0	70128	0	232826	0	

Bound on genus One possible check we can perform is obtain the maximal g for given d [66]. To do this, let us take diagonal combinations $n_d^g := \sum_{d_1+d_2=d} n_{d_1, d_2}^g$:

$d \setminus g$	0	1	2	3	(3.B.10)
1	-2	0	0	0	
2	0	0	0	0	
3	-2	0	0	0	
4	0	3	0	0	
5	-8	0	-4	0	
6	0	24	0	8	

They satisfy tadpole cancellation $d = \chi = g - 1 \pmod{2}$. For fixed $d = d_1 + d_2$, a smooth curve in $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ has genus $g = (d_1 - 1)(d_2 - 1)$, which is the top genus. For $d = 1$, we have a non-zero contribution from $d_1 = 1$ and $d_2 = 0$ for the involution we are considering. The top genus contribution appears from $g = 0$. The Gopakumar-Vafa invariant is then related to the Euler characteristic¹⁷ of the moduli space [66]

$$n_1^0 = n_{1,0}^0 = -e(\mathbb{C}\mathbb{P}^1) = -2, \quad (3.B.11)$$

which is consistent with eq. (3.B.10).

The dimension of the moduli space of a curve inside $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ may be understood as follows. The Gopakumar-Vafa invariants n_{d_1, d_2}^g are related to M2-branes wrapping a two-cycle

$$d_1 [\mathbb{C}\mathbb{P}_B^1] + d_2 [\mathbb{C}\mathbb{P}_F^1], \quad (3.B.12)$$

where $[\mathbb{C}\mathbb{P}^1]$ represents a divisor class of $\mathbb{C}\mathbb{P}^1$. The degrees are also related to the degree of a polynomial that represents the curve by

$$\sum_{a_1, a_2, b_1, b_2} \alpha_{a_1, a_2, b_1, b_2} X_0^{a_1} X_1^{a_2} \tilde{X}_0^{b_1} \tilde{X}_1^{b_2} = 0, \quad a_1 + a_2 = d_2, \quad b_1 + b_2 = d_1, \quad (3.B.13)$$

where (X_0, X_1) are homogeneous coordinates of $\mathbb{C}\mathbb{P}_B^1$ and $(\tilde{X}_0, \tilde{X}_1)$ are homogeneous coordinates of $\mathbb{C}\mathbb{P}_F^1$. We are now considering the case with $d_1 = 1$, $d_2 = 0$, which is

$$\alpha_{0,0,1,0} \tilde{X}_0 + \alpha_{0,0,0,1} \tilde{X}_1 = 0. \quad (3.B.14)$$

¹⁷Recall that $e(\mathbb{C}\mathbb{P}^m) = m + 1$ and $e(\mathbb{R}\mathbb{P}^m) = 1, 0$ for m even/odd respectively.

$\alpha_{0,0,1,0}$ and $\alpha_{0,0,0,1}$ still take value in \mathbb{C} and eq. (3.B.14) is still a complex equation. Therefore, the moduli space parametrized by $\alpha_{0,0,1,0}$ and $\alpha_{0,0,0,1}$ is \mathbb{CP}^1 . On the other hand, the deformation space of the curve class $d_2 \left[\mathbb{CP}_F^1 \right]$ gives a real projective space due to the involution acting on \mathbb{CP}_B^1 . In general, the moduli space may be given by $\mathbb{CP}^{d_1} \times \mathbb{RP}^{d_2}$.

We can also consider the case $d = 2$. Then the top genus comes from $g = 0$. But this contribution will be absent since this does not satisfy the tadpole condition $d = g - 1 \pmod{2}$. This is also consistent with eq. (3.B.10).

The degree 3 case is also the same, namely

$$n_3^0 = n_{1,2}^0 = -e\left(\mathbb{CP}^1\right) e\left(\mathbb{RP}^2\right) = -2. \quad (3.B.15)$$

One may do similarly for $d = 4$. The maximal genus is 1 and hence 3 in eq. (3.B.10) will be related to the Euler characteristic of the moduli space: we find

$$n_4^1 = n_{2,2}^1 = (-1)^2 e\left(\mathbb{CP}^2\right) e\left(\mathbb{RP}^2\right) = 3. \quad (3.B.16)$$

For $d = 5$ we find

$$n_5^2 = n_{3,2}^2 = (-1)^3 e\left(\mathbb{CP}^3\right) e\left(\mathbb{RP}^2\right) = -4, \quad (3.B.17)$$

which is again consistent with the obtained result. For $d = 6$, the top genus is $g = 4$. But this does not satisfy the tadpole condition and hence $n_6^4 = 0$. That is also consistent with the result.

3.C Definitions and useful identities

In this appendix we list some useful identities for quantities appearing in the topological vertex amplitudes. For non-trivial ones, we cite a reference where a proof can be found.

One can prove [11, eqs. (2.9) and (2.16)] the identity

$$\prod_{i,j=1}^{\infty} \left(1 - Qq^{i+j-1}\right) \prod_{s \in R_2} \left(1 - Qq^{-a_{R_1^t}(s)-1-l_{R_2}(s)}\right) \prod_{s \in R_1^t} \left(1 - Qq^{a_{R_2}(s)+1+l_{R_1^t}(s)}\right) = \quad (3.C.1)$$

From definition $\|R\|^2 := \sum_i R(i)^2$, some simple combinatorial identities follow

$$\sum_i i\nu(i) = \frac{\|\nu^t\|^2}{2} + \frac{|\nu|}{2}, \quad \sum_{(i,j) \in \nu} \tilde{\nu}^t(j) = \sum_{(i,j) \in \tilde{\nu}} \nu^t(j), \quad \sum_{s \in \nu} l_\nu(s) = \frac{\|\nu\|^2}{2} - \frac{|\nu|}{2}, \quad (3.C.2)$$

which imply

$$\sum_{(m,n) \in R_i} a_{R_j}(m, n) = \frac{1}{2} \left(\|R_j^t\|^2 + |R_j| - \|R_i^t\|^2 - |R_i| \right) + \sum_{(m,n) \in R_j} a_{R_i}(m, n). \quad (3.C.3)$$

Using the above we get

$$\begin{aligned} & \prod_{s \in R_i} \left(1 - e^{iE_{ij}(s)}\right)^{-1} \prod_{s \in R_j} \left(1 - e^{-iE_{ji}(s)}\right)^{-1} = \\ & (-1)^{|R_i|} e^{i\frac{t_{ji}}{2}(|R_i|+|R_j|)} e^{i\frac{\xi}{4}[\|R_j^t\|^2 - \|R_j\|^2 - \|R_i^t\|^2 + \|R_i\|^2]} \prod_{s \in R_i} \frac{1}{2i \sin \frac{E_{ij}(s)}{2}} \prod_{s \in R_j} \frac{1}{2i \sin \frac{E_{ji}(s)}{2}}. \end{aligned} \quad (3.C.4)$$

Finally, the important identity for skew-Schur functions [75, page 93]

$$\sum_{\mu} s_{\mu/\eta}(x) s_{\mu/\xi}(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \sum_{\mu} s_{\xi/\mu}(x) s_{\eta/\mu}(y) \quad (3.C.5)$$

gives upon iteration [106, Lemma 3.1]

$$\sum_{\substack{\nu^1, \dots, \nu^N \\ \eta^1, \dots, \eta^{N-1}}} \prod_{k=1}^N s_{\nu^k/\eta^{k-1}}(x^k) Q_k^{|\nu^k|} s_{\nu^k/\eta^k}(y^k) = \prod_{\substack{1 \leq k < l \leq N+1 \\ i, j \geq 1}} \left(1 - Q_k Q_{k+1} \cdots Q_{l-1} x_i^k y_j^{l-1}\right)^{-1}, \quad (3.C.6)$$

where $\eta^0 = \emptyset = \eta^N$ and $x^k = (x_1^k, x_2^k, \dots)$.

Chapter 4

Real topological string amplitudes

4.1 Introduction

The system we consider, dubbed real topological string [70–72, 89, 101], may be approached either from type IIA or type I perspective. From type IIA viewpoint, we start from a Calabi-Yau three-fold \mathbb{X} , such that it admits an anti-holomorphic involution σ with non-empty fixed point set L , which is therefore lagrangian and supports an orientifold plane. If we are interested in the topological subsector, the spacetime directions spanned by the orientifold are not constrained a priori, but can be guessed in the following way: if we take the O-plane to span L and two spacetime directions, namely consider also a spacetime involution $x^2, x^3 \rightarrow -x^2, -x^3$, then the O4-plane we get is such that we can cancel its charge by wrapping precisely one D4-brane on the same locus, as seen in the covering. Such local tadpole cancellation condition, on the one hand it guarantees integrality of the BPS expansion one gets from combining the various unoriented and open topological amplitudes, on the other hand it is the topological analogue of physical tadpole cancellation. Moreover, it is ultimately related to decoupling of vector- and hyper-multiplet in spacetime. We take as definition of the real topological string, the counting of holomorphic maps from Riemann surfaces to \mathbb{X} that are equivariant w.r.t. σ and worldsheet parity Ω , subject to local tadpole cancellation.¹

For simplicity we shall consider in our calculations an orbifold limit of a Calabi-Yau space with orbifold of the form $(T^2)^3/G$, G being the orbifold group, for example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. However, our results can be generalized to an arbitrary CY three-fold by using CFT arguments [9]. Let (Z^3, Z^4, Z^5) denote the complex coordinates on the three T^2 . We take the D4-O4 system to be along x^0, x^1 directions and wrapping a lagrangian subspace along $r(i) = \text{Re } Z^i$ for $i = 3, 4, 5$. Defining complex coordinates for space-time as $Z^1 = x^0 + ix^2$ and $Z^2 = x^1 + ix^3$ and the left and right moving fermionic partners of Z^i as ψ^i and $\widetilde{\psi}^i$ we conclude that orientifolding (which takes type II to type I-like

¹In some concrete examples, this amounts to a constraint relating the Euler characteristic of the surface to the degree of the map.

theory) is defined by the action of world sheet parity operator Ω (exchanging ψ and $\tilde{\psi}$) combined with \mathbb{Z}_2 involution $\sigma : (Z^i, \psi^i, \tilde{\psi}^i) \mapsto (\overline{Z^i}, \overline{\psi^i}, \overline{\tilde{\psi}^i})$ for $i = 1, \dots, 5$.

This particular model (in its simplest version) can be obtained starting from the standard Type I theory (i.e. IIB quotiented by Ω with the associated 32 D9-branes and O9-plane) compactified on \mathbb{X} .² One can get our model by doing 5 T-dualities, 3 of them inside \mathbb{X} so that one gets the mirror CY and the remaining 2 T-dualities on x^2 and x^3 directions. This results, by the standard rules of T-duality, in the presence of world-sheet parity projection $\Omega \rightarrow \Omega \cdot \sigma$ and it converts original O9 and D9-branes into O4 and D4-branes along the σ -fixed point set, i.e. for the orbifold realization of \mathbb{X} along $(x^0, x^1, \text{Re } Z^3, \text{Re } Z^4, \text{Re } Z^5)$.

4.1.1 Useful facts

Every surface $\Sigma_{g,h,c}$, where (g, h, c) denote respectively the number of handles, boundaries and crosscaps,³ can be realized as a quotient $\Sigma_{g'}/\Omega$, where $\Sigma_{g'}$ is Riemann surface of genus g' , and Ω an orientation reversing involution. The Euler characteristic of $\Sigma_{g,h,c}$ is $\chi = 1 - g' = 2 - 2g - h - c$.

Let $h = h(\Sigma_{g'}, \Omega) \geq 0$ be the number of components of the fixed point set $\Sigma_{g'}^\Omega$ of Ω in $\Sigma_{g'}$ (i.e. the number of boundaries of $\Sigma_{g'}/\Omega$), and the index of orientability $k = k(\Sigma_{g'}, \Omega) = (2 - \# \text{ components of } \Sigma_{g'} \setminus \Sigma_{g'}^\Omega)$. Then the topological invariants h and k together with g' determine the topological type of $\Sigma_{g'}/\Omega$ uniquely. For fixed genus g' , these invariants satisfy

- $k = 0$ or $k = 1$ (corresponding to oriented surfaces, or otherwise)
- if $k = 0$, then $0 < h \leq g' + 1$ and $h \equiv g' + 1 \pmod{2}$
- if $k = 1$ then $h \leq g'$.

Moreover, the total number of topologically distinct surfaces at fixed g' is given by $\lfloor \frac{-3\chi+8}{2} \rfloor$, where for even χ we also included in the count the case where $\Sigma_{g,h,c}$ itself is a closed oriented Riemann surface, which is not of the form $\Sigma_{g'}/\Omega$ for $\Sigma_{g'}$ connected.

It is useful to separate the worldsheets $\Sigma_{g,h,c}$ into three classes, corresponding to having 0, 1 or 2 crosscaps.⁴ This leads to a split of the topological amplitudes into classes, namely: oriented surfaces with $h \geq 0$ boundaries with amplitude $\mathcal{F}_{g,h}$, non-orientable surfaces with an odd number of crosscaps with amplitude $\mathcal{R}_{g,h}$, and non-orientable surfaces with an even number of crosscaps with amplitude $\mathcal{K}_{g,h}$. We will be

²In more complicated constructions one can also have D5-branes and O5-planes wrapping various 2-cycles of \mathbb{X} . However this will not change the closed string sector, which is what we are concerned with in trying to identify the self-dual background.

³A crosscap is topologically \mathbb{RP}^2 .

⁴Two crosscaps are equivalent to a Klein handle, namely two holes glued together with an orientation reversal, which in the presence of a third crosscap can be turned into an ordinary handle.

interested in their sum at fixed χ , namely

$$\mathcal{G}_\chi = \frac{1}{2} \left(\mathcal{F}_{g_\chi} + \sum_{2-2g-h=\chi} \mathcal{F}_{g,h} + \sum_{2-2g-h=\chi} \mathcal{K}_{g,h} + \sum_{1-2g-h=\chi} \mathcal{R}_{g,h} \right), \quad (4.1.1)$$

where we singled out the $h = 0 = c$ case, denoted simply by \mathcal{F}_{g_χ} with $g_\chi = 1 - \chi/2$.

Remark on relative homology In real topological string and Gromov-Witten theory, one considers maps that are equivariant with respect to worldsheet involution Ω and target involution σ . Such maps are classified by the second homology group: although for purely open subsector it makes sense to consider relative homology $H_2(\mathbb{X}, L; \mathbb{Z})$, namely to reduce modulo 2-cycles supported on the lagrangian L , when dealing with the real theory it is not sufficient mathematically to specify the relative homology, rather the full second homology is needed, as one is formally studying maps from the symmetric surface to \mathbb{X} . In some simple cases, like the real quintic or local \mathbb{CP}^2 , the two objects are isomorphic.

Remark on Kähler moduli Type IIA orientifold action has a well-defined action on the moduli space $\mathcal{M}^k \times \mathcal{M}^q$ [40]: the relevant subspace $\widetilde{\mathcal{M}}^k$ of \mathcal{M}^k is a special Kähler submanifold of dimension $h_-^{1,1}$, namely if we denote the Kähler form J the action $\sigma^* J = -J$ induces a decomposition $h^{1,1} = h_+^{1,1} + h_-^{1,1}$. The simplest examples have $h^{1,1} = h_-^{1,1} = 1$.

4.1.2 What we compute

The mathematical foundations of real GW theory have been recently put on a solid ground [33–35], in particular we know that in constructing a nice moduli space of maps one should not restrict to a single topological type of surface, but rather consider them at once, morally as in eq. (4.1.1), so that orientation issues along real codimension one boundaries in moduli space are taken care of, and one can construct a well-defined enumerative problem.

We are interested in the target space interpretation of topological strings. Here, in the context of standard \mathcal{F}_g in the oriented type II theory, a study on the heterotic dual [8] gave the Schwinger formula describing the singularity structure that explicitly proved the $c = 1$ conjecture and was generalized to all BPS states by Gopakumar and Vafa [38, 39]. Recently their work has been revisited and clarified in the paper [25], which we point to for references and details. A related aspect of target space physics connection to topological string amplitudes is the well-known fact [9] that topological strings compute certain corrections to super string amplitudes. In order to understand how this result can be extended to the real setup, which is our goal, it is useful to briefly recall some facts about the standard closed oriented case.

The physical string amplitude that computes \mathcal{F}_g in type II theories was identified [9] with the F-term coupling $\mathcal{F}_g(\mathcal{W}^2)^g$ in the low energy effective action, where \mathcal{W} is the

chiral Weyl super-field. Expanding in component fields one gets a term proportional to $\mathcal{F}_g R^2 (T^2)^{g-1}$, with R and T being the anti-self-dual Riemann tensor and graviphoton field strength. In the resulting amplitude there were two terms: one where the fermionic bilinear terms of the graviton vertex operator contribute and the other where the bosonic part of the graviton vertex contributes. For the fermionic term, by choosing a particular gauge for the positions of picture changing operators (PCO), it was possible to carry out the spin structure sum and the result was shown to produce the topological partition function. The bosonic term was shown to vanish in a different gauge choice. While the final answer is perhaps true, the choice of a different gauge for the two terms is unsatisfactory. In section 4.A we give an alternative derivation of the result [9] by computing the term $(D_a \mathcal{F}_g)(F_a.R.T)(T^2)^{g-1}$, where the subscript a denotes a vector multiplet and F_a its anti-self-dual field strength. This is obtained from the super-field $\mathcal{F}_g(\mathcal{W}^2)^g$ by extracting two θ from \mathcal{F}_g . The resulting amplitude involves only the fermionic bilinear term of the graviton vertex and with a suitable gauge choice one can perform the spin structure sum and show that the result gives holomorphic derivative of topological partition function along the a direction.

In the real case, we expect a contribution of the form $RT^{g'-1}$ (in the covering picture $1 - g' = \chi$), which can generate (via SUSY) the \sinh^{-1} power in the Schwinger computation, taking into account the fact that orientifold halves the number of fermionic zero modes on the Riemann surface. In the following, to avoid the problem of having two different terms contributing to the amplitude that require different gauge fixings to simplify, we will again consider an amplitude involving a certain number of anti-self-dual graviphotons and one matter anti-self-dual field strength, namely we will compute the term involving one holomorphic covariant derivative of the real topological amplitude that appears in the expansion of the orientifold invariant part of Weyl super-gravity tensor.

4.2 Type II with orientifold

4.2.1 Projection

To get the Ramond-Ramond operators that survive orientifold projection, the easiest way is to start from original type I compactified on \mathbb{X} , which for concreteness we take to be $(T^2)^3/(\mathbb{Z}_2)^2$. In type I the only R-R field is the 3-form field strength and its vertex operator in $(-1/2, -1/2)$ ghost picture is

$$T_{MNP} = \tilde{S}^t . C . \Gamma_{MNP} S \quad (4.2.1)$$

where S and \tilde{S} are left and right moving 10d spin fields, C is the charge conjugation matrix and M, N, P are 10d indices. Orbifold projection $(T^2)^3/(\mathbb{Z}_2)^2$ implies that

$$(MNP) = (\mu\nu\rho), (\mu ij), (ijk) \quad (4.2.2)$$

where indices i, j, k in (ijk) are coming one index from each T^2 and in (μij) i and j come from the same T^2 (this ensures orbifold group invariance). We can do the 5 T-dualities on these R-R operators: they are given by the corresponding 5 parity actions (i.e. action of σ) on S (but not on right moving \tilde{S} — this is because T-duality can be thought of as parity transformation only on left movers):

$$S \rightarrow \Gamma_{23i(a)i(b)i(c)} S, \quad (4.2.3)$$

where $i(a)i(b)i(c)$ indicate the three Im Z directions. So if one starts from Type I R-R field

$$T_{i(a)i(b)i(c)} \rightarrow T_{23} \quad (4.2.4)$$

$$T_{r(a)r(b)r(c)} \rightarrow T_{23r(a)r(b)r(c)i(a)i(b)i(c)} = T_{01} \quad (4.2.5)$$

where $r(a)r(b)r(c)$ denote the three Re Z directions and in the last equality one has used the fact that because of GSO projection S is chiral w.r.t. 10d tangent space Lorentz group:

$$\Gamma_{0123r(a)r(b)r(c)i(a)i(b)i(c)} S = S \quad (4.2.6)$$

So in the simplest version there are the vertex operators T_{01} and T_{23} , precisely the operators that we'll use in our computation.

In more complicated constructions, already the original Type I theory compactified on \mathbb{X} may come with O5 and D5-branes, e.g. in the orbifold we are considering there could be D5-branes along $(0, 1, 2, 3, r(a), i(a))$ ($a = 1, 2, 3$) i.e. D5-branes wrapped on one of the T^2 . These could be thought of as turning on some instantons on the remaining two T^2 , modded by orbifold group, in the D9-branes. Upon 5 T-dualities, for example for $a = 1$, this will become D4-brane along

$$(0, 1, r(1), i(2), i(3)), \quad (4.2.7)$$

although we don't expect there is any effect of these more complicated constructions on the closed string sector.

4.2.2 Supergravity

With the orientifold projection, supersymmetry is reduced to $(2, 2)$ in 2 dimensions, namely 4 supercharges, and we want to understand how to decompose $\mathcal{N} = 2$ chiral Weyl tensor [16]

$$\mathcal{W}_{\mu\nu}^{ij} = T_{\mu\nu}^{ij} - R_{\mu\nu\rho\lambda} \theta^i \sigma^{\rho\lambda} \theta^j + \dots \quad (4.2.8)$$

which is anti-symmetric in $SU(2)$ indices $i, j \in \{1, 2\}$ and anti-self-dual in Lorentz indices μ, ν (namely, we will restrict both the graviphoton field strength T and the Riemann tensor R , which are antisymmetric in μ, ν , to a constant anti-self-dual background), so that we may integrate some of its parts in 2d $(2, 2)$ super-space:

$$\int d^4x \int d^4\theta \delta^2(\theta) \delta^2(x) \left(\mathcal{G}_x - \frac{1}{2} \mathcal{F}_{g_x} \right) (\mathcal{W}_{\parallel})^{g'}, \quad (4.2.9)$$

where we removed from \mathcal{G} in eq. (4.1.1) the purely closed oriented piece. Let's denote such combination as $\mathcal{H}_{g'}$, where $\chi = 1 - g'$.

We now discuss the term $\mathcal{W}_{||}$. Since orientifold action (in Weyl indices) takes the form $\theta_\alpha^i \rightarrow 2(\sigma^2)^i_j (\sigma^{23})_\alpha^\beta \theta_\beta^j$, we can form orientifold even and odd combinations

$$\theta^\pm = (\theta_1^1 \mp i\theta_2^2), \quad \tilde{\theta}^\pm = (\theta_1^2 \pm i\theta_2^1) \quad (4.2.10)$$

where θ^+ and $\tilde{\theta}^+$ are the super-space coordinates that are invariant under the orientifold. For the Riemann sector, we write $\theta^1 \sigma^{\mu\nu} \theta^2$ as

$$\begin{aligned} \theta^1 \sigma^{23} \theta^2 &= -\frac{1}{4}(\theta^+ \tilde{\theta}^+ + \theta^- \tilde{\theta}^-) \\ \theta^1 \sigma^{02} \theta^2 &= -\frac{1}{4}(\theta^+ \tilde{\theta}^- + \theta^- \tilde{\theta}^+) \\ \theta^1 \sigma^{03} \theta^2 &= -\frac{1}{4}(\theta^+ \theta^- + \tilde{\theta}^+ \tilde{\theta}^-) \end{aligned} \quad (4.2.11)$$

This means that for example \mathcal{W}_{01} decomposes as

$$\mathcal{W}_{01}^{12} = T_{01}^{12} + \frac{1}{4}R_{0101}(\theta^+ \tilde{\theta}^+ + \theta^- \tilde{\theta}^-) + \text{odd part} \quad (4.2.12)$$

The relevant (2, 2) super-field is therefore

$$\mathcal{W}_{||} = T_{01} + R_{0101} \theta^+ \tilde{\theta}^+ + \dots \quad (4.2.13)$$

where T_{01} etc. are short hand notation for the anti-self-dual combination $T_{01} + T_{23}$. From the original vector moduli super-fields Φ^a where a labels different vector multiplets, one gets (2, 2) super-fields

$$\Phi^a = \phi^a + F_{01}^a \theta^+ \tilde{\theta}^+ + \dots \quad (4.2.14)$$

The effective action term that we are interested in is

$$\int d\theta^+ d\tilde{\theta}^+ \mathcal{H}_{g'}(\Phi) \mathcal{W}_{||}^{g'} = \mathcal{H}_{g'}(\phi) R_{0101} T_{01}^{g'-1} + D_a \mathcal{H}_{g'}(\phi) F_{01}^a T_{01}^{g'} + \dots \quad (4.2.15)$$

where D_a is the holomorphic covariant derivative w.r.t. ϕ^a . We will focus on the second term above, as its computation is not affected by the issue mentioned in section 4.1.2.

4.2.3 Vertex operators

Recalling that this model is obtained by applying five T-dualities on Type I, left and right moving sectors have opposite GSO projection. We bosonize $(\psi^i, \tilde{\psi}^i) = (e^{i\phi_i}, e^{i\tilde{\phi}_i})$, so that orientifolding exchanges $\phi_i \leftrightarrow -\tilde{\phi}_i$; in terms of bosonized fields the 4-dimensional chiral spin fields with helicities labeled by 1 and 2 are

$$S_1 = e^{\frac{i}{2}(\phi_1 + \phi_2)}, \quad S_2 = e^{-\frac{i}{2}(\phi_1 + \phi_2)}, \quad \tilde{S}_1 = e^{\frac{i}{2}(\tilde{\phi}_1 + \tilde{\phi}_2)}, \quad \tilde{S}_2 = e^{-\frac{i}{2}(\tilde{\phi}_1 + \tilde{\phi}_2)} \quad (4.2.16)$$

The internal spin fields are given by

$$\Sigma = e^{\frac{i}{2}H}, \quad \bar{\Sigma} = e^{-\frac{i}{2}H}, \quad \tilde{\Sigma} = e^{\frac{i}{2}\tilde{H}}, \quad \tilde{\bar{\Sigma}} = e^{-\frac{i}{2}\tilde{H}} \quad (4.2.17)$$

where $H = \phi_3 + \phi_4 + \phi_5$ and $\tilde{H} = \tilde{\phi}_3 + \tilde{\phi}_4 + \tilde{\phi}_5$. Finally, φ and $\tilde{\varphi}$ appear in the bosonization of left and right moving super-ghosts.

The $(-1/2, -1/2)$ ghost picture for graviphoton vertex

$$V_T^{(-1/2)}(p, \epsilon) = p_\nu \epsilon_\mu : e^{-\frac{1}{2}(\varphi + \tilde{\varphi})} \left[S^\alpha (\sigma^{\mu\nu})_\alpha^\beta \tilde{S}_\beta \Sigma(z, \bar{z}) + S_{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \tilde{S}^{\dot{\beta}} \bar{\Sigma}(z, \bar{z}) \right] e^{ip \cdot X} : \quad (4.2.18)$$

becomes for T_{01} to the lowest order in momentum

$$V_T = e^{-\frac{\varphi}{2}} e^{-\frac{\tilde{\varphi}}{2}} (S_1 \Sigma \tilde{S}_1 \tilde{\bar{\Sigma}} - S_2 \Sigma \tilde{S}_2 \tilde{\bar{\Sigma}}) \quad (4.2.19)$$

The graviphoton vertex eq. (4.2.19) is invariant under both the orbifold group as well as under the orientifold projection, since orientifold action preserves the $T_{01}(S_2 \tilde{S}_2 - S_1 \tilde{S}_1)$ part of graviphoton vertex.

We take, for concreteness, the vector multiplet Φ^a to be the one corresponding to the Kähler modulus of the first torus (i.e. in Z^3 direction). In the $(-1, -1)$ picture this vertex operator is just the (chiral, anti-chiral) primary $\psi^3 \bar{\psi}^3$. The gauge field strength for this vector multiplet can be obtained by applying two supersymmetry transformations, whose charges are obtained by integrating currents $Q_\alpha = \oint dz j_\alpha$,

$$j_\alpha = e^{-\frac{\varphi}{2}} S_\alpha \Sigma, \quad j_{\dot{\alpha}} = e^{-\frac{\varphi}{2}} S_{\dot{\alpha}} \bar{\Sigma}, \quad \tilde{j}_\alpha = e^{-\frac{\tilde{\varphi}}{2}} \tilde{S}_\alpha \tilde{\Sigma}, \quad \tilde{j}_{\dot{\alpha}} = e^{-\frac{\tilde{\varphi}}{2}} \tilde{S}_{\dot{\alpha}} \tilde{\bar{\Sigma}} \quad (4.2.20)$$

and we recall that the orientifold only preserves 4 supercharges

$$Q = Q_1 + \tilde{Q}_2, \quad Q' = Q_2 + \tilde{Q}_1, \quad \dot{Q} = Q_1 - \tilde{Q}_2, \quad \dot{Q}' = Q_2 - \tilde{Q}_1 \quad (4.2.21)$$

The result for an orbifold group and orientifold action invariant such vertex is

$$V_{F^a} = e^{-\frac{\varphi}{2}} e^{-\frac{\tilde{\varphi}}{2}} (S_1 e^{\frac{i}{2}(\phi_3 - \phi_4 - \phi_5)} \tilde{S}_1 e^{\frac{i}{2}(-\tilde{\phi}_3 + \tilde{\phi}_4 + \tilde{\phi}_5)}) - \{(S_1, \tilde{S}_1) \rightarrow (S_2, \tilde{S}_2)\} \quad (4.2.22)$$

4.2.4 Computation

We are interested in computing the amplitude involving g' V_T and one V_{F^a} on a surface $\Sigma_{g,h,c}$, where we recall that (g, h, c) denote respectively the number of handles, boundaries and crosscaps and g' is the genus of the double cover of $\Sigma_{g,h,c}$, namely $g' = 2g + h + c - 1$. By using the image method this amplitude can be computed on the compact oriented Riemann surface $\Sigma_{g'}$. Denoting the image of a point $p \in \Sigma_{g,h,c}$ as $\bar{p} \in \Sigma_{g'}$ and using the fact that the right moving part of the vertex at p is mapped to a

left moving part dictated by the orientifold action at \bar{p} ,⁵ we get

$$\begin{aligned} V_T(p)|_{p \in \Sigma_{g,h,c}} &= e^{-\frac{\varphi}{2}}(p)e^{-\frac{\varphi}{2}}(\bar{p})(S_1(p)\Sigma(p)S_2(\bar{p})\Sigma(\bar{p}) - S_2(p)\Sigma(p)S_1(\bar{p})\Sigma(\bar{p}))|_{p \in \Sigma_{g,h,c}} \\ &= e^{-\frac{\varphi}{2}}(p)e^{-\frac{\varphi}{2}}(\bar{p})S_1(p)\Sigma(p)S_2(\bar{p})\Sigma(\bar{p})|_{p \in \Sigma_{g'}} \end{aligned} \quad (4.2.23)$$

Similarly

$$\begin{aligned} V_{F^a}|_{p \in \Sigma_{g,h,c}} &= e^{-\frac{\varphi}{2}}(p)e^{-\frac{\varphi}{2}}(\bar{p})(S_1(p)\hat{\Sigma}(p)S_2(\bar{p})\hat{\Sigma}(\bar{p}) - S_2(p)\hat{\Sigma}(p)S_1(\bar{p})\hat{\Sigma}(\bar{p}))|_{p \in \Sigma_{g,h,c}} \\ &= e^{-\frac{\varphi}{2}}(p)e^{-\frac{\varphi}{2}}(\bar{p})S_1(p)\hat{\Sigma}(p)S_2(\bar{p})\hat{\Sigma}(\bar{p})|_{p \in \Sigma_{g'}} \end{aligned} \quad (4.2.24)$$

where $\hat{\Sigma} = e^{\frac{i}{2}(\phi_3 - \phi_4 - \phi_5)}$. In other words both for V_T and V_{F^a} the region of integration extends to double cover $\Sigma_{g'}$. This is actually due to the fact that both these operators are orientifold invariant. Note that the graviphoton as well as matter field strength vertices in $(-1/2, -1/2)$ picture already come with one momentum giving altogether $(g' + 1)$ momenta. Therefore in the remaining part of the vertices we can set zero momenta. Finally we can write the amplitude of interest on the double cover as

$$\begin{aligned} A &= \int_{\Sigma_{g,h,c}} d^2z \prod_{i=1}^{g'} d^2x_i \langle V_{F^a}(z) \prod_{i=1}^{g'} V_T(x_i) \rangle_{\Sigma_{g,h,c}} \\ &= \int_{\Sigma_{g'}} d^2z \prod_{i=1}^{g'} d^2x_i \langle e^{-\frac{\varphi}{2}}(z)e^{-\frac{\varphi}{2}}(\bar{z})S_1(z)\hat{\Sigma}(z)S_2(\bar{z})\hat{\Sigma}(\bar{z}) \\ &\quad \times \prod_{i=1}^{g'} e^{-\frac{\varphi}{2}}(x_i)e^{-\frac{\varphi}{2}}(\bar{x}_i)S_1(x_i)\Sigma(x_i)S_2(\bar{x}_i)\Sigma(\bar{x}_i) \prod_{a=1}^{3g'-1} \text{PCO}(u_a) \prod_{a=1}^{3g'-3} \int \mu_{ab} \rangle \end{aligned} \quad (4.2.25)$$

where the number $(3g' - 1)$ of PCOs follows from the fact that the total picture (super-ghost charge) on a genus g' surface must be $2g' - 2$ and the operators that are inserted give a total super ghost charge $-(g' + 1)$. As each Σ operator carries internal $U(1)$ charge $3/2$ and $\hat{\Sigma}$ carries a charge $-1/2$, the total internal $U(1)$ charge carried by the vertices is $3/2(2g') - 1/2(2) = 3g' - 1$. This means that to balance the total $U(1)$ charge, each of the PCO must contribute an internal charge (-1) . Thus the relevant part of each PCO is $e^{\varphi}G_-$, where G_- is the super-current of the $\mathcal{N} = 2$ super-conformal field theory describing the Calabi-Yau space (the subscript $-$ refers to the $U(1)$ charge). In the orbifold example we are considering, fermion charge for each plane must be conserved, which implies that of the $(3g' - 1)$ PCO, $(g' + 1)$ contribute $e^{\varphi}\bar{\psi}^3\partial Z^3$ and $(g' - 1)$ each contribute $e^{\varphi}\bar{\psi}^4\partial Z^4$ and $e^{\varphi}\bar{\psi}^5\partial Z^5$ respectively. We shall denote the positions of these three groups of PCOs by $u_a^{(1)}$ for $a = 1, \dots, g' + 1$ and $u_a^{(2)}$ and $u_a^{(3)}$ for $a = 1, \dots, g' - 1$. Of

⁵Note that the image across a boundary is governed by the Neumann or Dirichlet boundary conditions, with Dirichlet directions being accompanied by \mathbb{Z}_2 involution. The resulting \mathbb{Z}_2 involution is the same that appears with orientifolding action. This is so because the D4 and O4 planes are parallel. Had we considered a situation where they were not parallel or a system containing say D2-branes with O4-planes, these two involutions would have been different and we would have to go to quadruple covers for world-sheets containing both boundaries and crosscaps.

course we will need to sum over all the partitions with appropriate anti-symmetrization. Finally μ_a for $a = 1, \dots, 3g' - 3$ are the Beltrami differentials and b are anti-commuting spin 2 ghost fields. Note that b provide the $3g' - 3$ quadratic differentials h_a that are dual to the Beltrami differentials μ_a .

We use chiral bosonization formulae for anti-commuting (b, c) system with conformal dimensions $(\lambda, 1 - \lambda)$ with $\lambda > 1$ and $g' > 1$ [98, 99], namely

$$\langle \prod_{i=1}^{Q(g'-1)} b(x_i) \rangle = \mathcal{Z}_1^{-\frac{1}{2}} \theta(\sum_i x_i - Q\Delta) \prod_{i<j} E(x_i, x_j) \prod_i \sigma^Q(x_i) \quad (4.2.26)$$

where $Q = 2\lambda - 1$ and by Riemann-Roch theorem the number of zero modes of λ -differential b is $Q(g' - 1)$ and the $(1 - \lambda)$ -differential c has no zero mode for $g' > 1$. The above expression is also valid for the case of $\lambda = 1$ when the (b, c) system is twisted. Some relevant facts about θ functions are summarized in section 4.B. We can write the above correlation function as

$$\begin{aligned} A = K \sum_s \frac{\theta_s(\frac{1}{2}(\sum_i(x_i - \bar{x}_i) + z - \bar{z}))^2 \theta_{s,g_1}(\frac{1}{2}(\sum_i(x_i + \bar{x}_i) + z + \bar{z}) - \sum_a u_a^{(1)})}{\theta_s(\frac{1}{2}(\sum_i(x_i + \bar{x}_i) + z + \bar{z}) - \sum_a u_a + 2\Delta)} \\ \times \prod_{k=2}^3 \theta_{s,g_k}(\frac{1}{2}(\sum_i(x_i + \bar{x}_i) - z - \bar{z}) - \sum_a u_a^{(k)}) \end{aligned} \quad (4.2.27)$$

where the sums appearing above are in the appropriate ranges, for example for x_i , $i = 1, \dots, g'$, for $u_a^{(1)}$, $a = 1, \dots, g' + 1$, for $u_a^{(2)}$ and $u_a^{(3)}$, $a = 1, \dots, g' - 1$ and finally $a = 1, \dots, 3g' - 1$ for u_a . The twists g_1, g_2, g_3 are the orbifold twists along the three tori along $2g'$ cycles (in other words g_i are points in Jacobi variety). Since the orbifold group $G \subset SU(3)$ in order to preserve supersymmetry, we have the relation $g_1 + g_2 + g_3 = 0$. Finally K is the spin-structure independent part of the correlation function and can be expressed in terms of prime forms and certain nowhere vanishing holomorphic sections σ that are quasi-periodic and transform as $\frac{g'}{2}$ differential under local coordinate transformations. The prime form $E(x, y)$ has the important property that it vanishes only at $x = y$ in $\Sigma_{g'}$, transforms as holomorphic $-\frac{1}{2}$ differentials in arguments x and y and is quasi-periodic along various cycles. In fact K can be determined by just the leading singularity structures coming from OPE and the total conformal weights at each point. The position dependent part of K is given by

$$\begin{aligned} K = \frac{\prod_{i<j} E(x_i, x_j) E(\bar{x}_i, \bar{x}_j) \prod_{k=2}^3 \prod_{a=1}^{g'-1} E(z, u_a^{(k)}) E(\bar{z}, u_a^{(k)}) \sigma(z) \sigma(\bar{z}) \prod_i \sigma(x_i) \sigma(\bar{x}_i)}{\prod_i E(x_i, \bar{z}) E(\bar{x}_i, z) \prod_{k<l} \prod_{a,b} E(u_a^{(k)}, u_b^{(l)}) \prod_a \sigma(u_a)^2} \\ \times \mathcal{Z}_1^{-2} \langle \prod \partial Z^3(u^{(1)}) \partial Z^4(u^{(2)}) \partial Z^5(u^{(3)}) \prod_k \int \mu_k b \rangle \end{aligned} \quad (4.2.28)$$

where \mathcal{Z}_1 is the chiral non-zero determinant of the Laplacian acting on a scalar and $\langle \dots \rangle$ indicates correlation function in the space of all the bosonic fields (Z^1, \dots, Z^5) and the

(b, c) ghost system. We can now put two of the PCO say at $u_{3g'-1}$ and $u_{3g'-2}$ at points z and \bar{z} respectively. The expression for K above shows that if these two PCO positions appear in the partitioning as $u^{(2)}$ or $u^{(3)}$ the result vanishes due to the appearance of $\prod_{k=2}^3 \prod_{a=1}^{g'-1} E(z, u_a^{(k)}) E(\bar{z}, u_a^{(k)})$ in the numerator. What this means is that the only non-vanishing contribution can come when these two PCOs are in the partitioning $u^{(1)}$. There are now $g' - 1$ remaining $u^{(1)}$, the same number as the ones for $u^{(2)}$ and $u^{(3)}$. The amplitude now becomes much simpler:

$$A = K \sum_s \frac{\theta_s(\frac{1}{2}(\sum_i(x_i - \bar{x}_i) + z - \bar{z}))^2 \prod_{k=1}^3 \theta_{s, g_k}(\frac{1}{2}(\sum_i(x_i + \bar{x}_i) - z - \bar{z}) - \sum_a u_a^{(k)})}{\theta_s(\frac{1}{2}(\sum_i(x_i + \bar{x}_i) - z - \bar{z}) - \sum_a u_a + 2\Delta)} \quad (4.2.29)$$

where the position dependent part of K is

$$K = \frac{\prod_{i < j} E(x_i, x_j) E(\bar{x}_i, \bar{x}_j)}{\prod_i (E(x_i, \bar{z}) E(\bar{x}_i, z))} \frac{1}{\prod_{1 \leq i < j \leq 3} E(u^{(i)}, u^{(j)})} \frac{\prod_i \sigma(x_i) \sigma(\bar{x}_i)}{\sigma(z) \sigma(\bar{z}) \prod_a \sigma(u_a)^2} \times \mathcal{Z}_1^{-2} \langle \partial Z^3(z) \partial Z^3(\bar{z}) \prod_{n=1}^3 \partial Z^{n+2}(u^{(n)}) \rangle \langle \prod_a \int \mu_a b \rangle \quad (4.2.30)$$

We can now choose the following gauge condition so that super-ghost theta function appearing in the denominator cancels with one of the space-time theta functions:

$$\sum_{a=1}^{3g'-3} u_a = \sum_{i=1}^{g'} \bar{x}_i - z + 2\Delta \quad (4.2.31)$$

After performing the spin-structure sum using eq. (4.B.4), the result is

$$A = K \theta(\sum_i x_i - \bar{z} - \Delta) \prod_{k=1}^3 \theta_{g_k}(\sum_a u_a^{(k)} - \Delta) \quad (4.2.32)$$

We now multiply this expression by identity

$$1 = \frac{\theta(\sum_i \bar{x}_i - z - \Delta)}{\theta(\sum_a u_a - 3\Delta)} \quad (4.2.33)$$

which follows from the gauge condition, and make use of chiral bosonization formulae

$$\begin{aligned} \mathcal{Z}_1^{-\frac{1}{2}} \theta_{g_k}(\sum_{a=1}^{g'-1} u_a^{(k)} - \Delta) \prod_{a < b} E(u_a^{(k)}, u_b^{(k)}) \prod_a \sigma(u_a^{(k)}) &= \langle \prod_a \bar{\psi}^k(u_a^{(k)}) \rangle \\ \mathcal{Z}_1^{-\frac{1}{2}} \theta(\sum_{i=1}^{g'} x_i - \bar{z} - \Delta) \frac{\prod_{i < j} E(x_i, x_j)}{\prod_i E(x_i, \bar{z})} \frac{\prod_i \sigma(x_i)}{\sigma(\bar{z})} &= \langle \prod_i \bar{\psi}(x_i) \psi(\bar{z}) \rangle \\ &= \mathcal{Z}_1 \det(\omega_i(x_j)) \\ \mathcal{Z}_1^{-\frac{1}{2}} \theta(\sum_{a=1}^{3g'-3} u_a - 3\Delta) \prod_{a < b} E(u_a, u_b) \prod_a \sigma(u_a)^3 &= \langle \prod_a b(u_a) \rangle \end{aligned} \quad (4.2.34)$$

where $(\overline{\psi^k}, \psi^k)$ are anti-commuting $(1, 0)$ system twisted by g_k , ω_i are the g' abelian differentials and (b, c) anti-commuting spin- $(2, -1)$ system. Note that by Riemann-Roch theorem b has $(3g' - 3)$ zero modes (quadratic differentials) and therefore the last correlation function just soaks these zero modes. It is interesting to note that after the spin structure sum, we have obtained the correlation function in topologically twisted internal theory where $\overline{\psi^k}$ and ψ^k are of dimension $(1, 0)$ for $k = 3, 4, 5$. Taking into account various ∂Z^k in K and summing over all partitions we obtain

$$A = \int_{\Sigma_{g'}} d^2 z \prod_i d^2 x_i (\det \omega(x_i)) (\det \omega(\overline{x}_i)) \mathcal{Z}_1^2 \times \frac{\langle \partial Z^3(z) \partial Z^3(\overline{z}) \prod_{a=1}^{3g'-3} G_-(u_a) \prod_{a=1}^{3g'-3} \int(\mu_a b) \rangle}{\langle \prod_{a=1}^{3g'-3} b(u_a) \rangle} \quad (4.2.35)$$

where $G_- = \sum_{k=3}^5 \overline{\psi^k} \partial Z^k$ is the twisted super current of dimension 2 and $G_+ = \sum_{k=3}^5 \psi^k \overline{\partial Z^k}$ is dimension 1 topological BRST current. G_- in A provide only zero modes both for $\overline{\psi^k}$ and ∂Z^k and hence must be holomorphic quadratic differentials. Therefore $\frac{\prod_{a=1}^{3g'-3} G_-(u_a)}{\langle \prod_{a=1}^{3g'-3} b(u_a) \rangle}$ is independent of u_a . Finally, using the fact that the anti-analytic \mathbb{Z}_2 involution Ω maps abelian differentials as $\omega_i(\overline{x}) = \sum_j \Gamma_{ij}^\Omega \overline{\omega}_j(x)$, where $\Gamma := \Gamma^\Omega$ is a matrix satisfying $\Gamma^2 = I$, we find that

$$\prod_i d^2 x_i (\det \omega_j(x_k)) (\det \omega_j(\overline{x}_k)) = \prod_i d^2 x_i (\det \omega_j(x_k)) (\det(\Gamma \overline{\omega})_j(x_k)) \sim \det \Gamma \det \text{Im } \tau \quad (4.2.36)$$

The amplitude then becomes

$$A = \int_{\mathcal{M}_{g,h,c}} \det \Gamma \det \text{Im } \tau \mathcal{Z}_1^2 \langle \int d^2 z \partial Z^3(z) \partial Z^3(\overline{z}) \prod_a \left(\int \mu_a G_- \right) \rangle = D_{t_3} \int_{\mathcal{M}_{g,h,c}} \det \Gamma (\det \text{Im } \tau) \mathcal{Z}_1^2 \langle \prod_a \left(\int \mu_a G_- \right) \rangle \quad (4.2.37)$$

where we have used the fact that $\int_{\Sigma_{g'}} d^2 z \partial Z^3(z) \partial Z^3(\overline{z}) = 2 \int_{\Sigma_{g,h,c}} d^2 z \partial Z^3(z) \overline{\partial Z^3}(z)$ is the marginal operator corresponding to the complexified Kähler modulus t_3 of the torus along Z^3 direction and hence gives a holomorphic covariant derivative with respect to t_3 . The integral is over $\mathcal{M}_{g,h,c}$ since among the Beltrami on $\Sigma_{g'}$ we should only include those that are invariant under the involution.⁶ Furthermore $\langle \dots \rangle$ denotes correlation function in the space of bosonic fields (Z^1, Z^2) and the topologically twisted $\mathcal{N} = 2$ super-conformal theory describing the Calabi-Yau manifold.

Finally $\mathcal{Z}_1^2 \det \text{Im } \tau$ cancels with the partition function of the spacetime bosonic fields (Z^1, Z^2) . This can be seen using the results of [18], as follows:

⁶On $\Sigma_{g'}$ there are a total of $6g' - 6$ real moduli, but the fact that we are actually restricting to $\Sigma_{g,h,c}$ reduces these to $3g' - 3$ real moduli.

- (i) Consider first diagrams that have no crosscaps. Let $\Sigma_{g'}$ be the double cover of $\Sigma_{g,h}$ and let Ω be the map that takes a point in $\Sigma_{g,h}$ to its image in $\Sigma_{g'}$. Then for a Neumann (N) direction X_N the scalar determinant (denoted by $\det \Delta^+$) is over functions that are even under Ω while the determinant for Dirichlet (D) directions X_D (denoted by $\det \Delta^-$) is over odd functions under Ω . These are given in [18, eqs. (4.1-4.3)]. Note that correction factor denoted by R appears with opposite powers in the two cases. This means that if one has equal number of N and D directions (in our case number of N and D is 2 each), then the correction factor cancels and one just gets the square root of the closed string determinants for 4 scalars. Closed string result for this is $1/(\det \text{Im } \tau)^{4/2}$. The square-root of this gives $1/(\det \text{Im } \tau)$. So for this to work it is crucial that D-branes have equal number of N and D \mathbb{R}^4 directions.
- (ii) Now consider diagrams with just crosscaps (i.e. no boundaries). The formula given in [18, eq. (4.25)] is just for even functions, but this is because in [18] orientifolding is simply world sheet parity operator. In our case however it is combined with a \mathbb{Z}_2 reflection of the two D-directions of \mathbb{R}^4 . This means that if $p \in \Sigma_{g'}$ is mapped to \bar{p} under Ω , then $X_N(p) = X_N(\bar{p})$ while $X_D(p) = -X_D(\bar{p})$. So once again for the two cases one has $\det \Delta^+$ and $\det \Delta^-$ respectively and using [18, eq. (4.3)] again correction factors cancel.
- (iii) The reason why in [18] the authors needed to look at quadruple cover for the surfaces that have both crosscaps and boundaries, is as follows. One goes to the quadruple cover as explained in the third paragraph of page 287: one first goes to an unoriented boundary-less double cover B of $\Sigma_{g,h,c} = B/\Omega_1$, but B still has the crosscaps inherited from $\Sigma_{g,h,c}$. One now goes to oriented double cover Q of $B = Q/\Omega_2$ so that the original $\Sigma_{g,h,c} = Q/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where the two \mathbb{Z}_2 are generated by Ω_1 and Ω_2 . They need to work with Q (as is seen in the table eq. (4.24)) because for them Ω_1 and Ω_2 act differently on X_D : $X_N(p) = X_N(\Omega_1(p)) = X_N(\Omega_2(p))$ but $X_D(p) = -X_D(\Omega_1(p)) = X_D(\Omega_2(p))$.

However for us, since orientifold action comes with \mathbb{Z}_2 reflection on X_D : $X_D(p) = -X_D(\Omega_1(p)) = -X_D(\Omega_2(p))$; therefore one can just work with the complex double described in third paragraph of page 287 and X_N and X_D will be even and odd functions under Ω and once again the correction factors cancels.

So it is crucial not only that there are two N and two D-directions in \mathbb{R}^4 but also that orientifold action comes with \mathbb{Z}_2 reflection on X_D , otherwise the prefactors would not have canceled by integrating the positions of the vertices. This is precisely what happens in our case, as we have one D4-brane (extended along two spacetime directions) stuck on top of the O4-plane.

Thus the amplitude eq. (4.2.37) for a fixed genus of the covering space g' reduces to

$$A_{g'} = D_{t_3} \sum \int_{\mathcal{M}_{g,h,c}} \det \Gamma_{g,h,c} \langle \prod_a \int \mu_a G_- \rangle_{\text{twisted internal theory}} \quad (4.2.38)$$

where the sum is over all (g, h, c) such that $g' = 2g + h + c - 1$ and $\Gamma_{g,h,c}$ is the corresponding involution in $\Sigma_{g'}$.

4.A Old type IIA computation

The quantity⁷ we are looking for is $\mathcal{F}_g(\mathcal{W}^2)^g$. From $(\mathcal{W}^2)^{g-1}$ we take the lowest components to get $(T_+T_-)^{g-1}$. From the remaining \mathcal{W}^2 we take $(R.T)_{\mu\nu}\theta_1\sigma^{\mu\nu}\theta_2$ and finally take the remaining two θ from \mathcal{F}_g , which gives $(\partial_a\mathcal{F}_g)F_{\mu\nu}^a\theta_1\sigma^{\mu\nu}\theta_2$ where T are self-dual graviphoton field strengths, R the self-dual Riemann tensor and $F_{\mu\nu}^a$ is the self-dual field strength in a chiral vector multiplet V^a labeled by the index a .⁸ Recall that \mathcal{F}_g is a function of vector multiplets V^a and so $(\partial_a\mathcal{F}_g) = \frac{\partial\mathcal{F}_g(\chi)}{\partial\chi^a}$ where χ^a are the lowest components (i.e. moduli of Calabi-Yau) of the vector super-fields V^a . Thus we have $2g - 1$ graviphotons, one R and one F^a . All the field strength vertices are in $(-\frac{1}{2})$ picture (we are focusing on the left moving sector — discussion for the right moving part is identical) so total number of PCO on genus g surface is $(2g - 2 + \frac{1}{2}(2g - 1) + \frac{1}{2}) = 3g - 2$. To be explicit let us work with orbifold CY. The internal part of the vertex for T carries charge $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in the three internal planes and for F^a we take it to be $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. This means that the total internal charge of the vertices is $(g - 1, g - 1, g)$ therefore all the $(3g - 2)$ PCOs can only contribute the internal parts of the super-currents. Note that SUSY transformation of F^a vertex gives the vertex operator for χ^a in (-1) picture with charge $(0, 0, 1)$, which is the vertex of the untwisted modulus relating to change in the complex structure (or complexified volume) in IIB (or IIA) of the third 2-torus. The space-time part of the spin field is as follows: $(g - 1)$ of T (at points $x_i, i = 1, \dots, g - 1$) come with S_1 , g come with S_2 (at points $y_m, m = 1, \dots, g$) and F^a at point w comes with S_2 . So altogether there are $(g - 1) S_1$ and $(g + 1) S_2$. This means that the vertex for the Riemann tensor (at z) must be R_{0+0+} i.e. $\psi_1\psi_2$ to balance the space-time charges. In other words the bosonic part of R vertex does *not* contribute.

With this assignment $(g - 1)$ of the PCO must contribute $\bar{\psi}_3\partial X^3$ (say at $u_i^1, i = 1, \dots, g - 1$), $(g - 1)$ of the PCO must contribute $\bar{\psi}_4\partial X^4$ (say at $u_i^2, i = 1, \dots, g - 1$) and g of the PCO must contribute $\bar{\psi}_5\partial X^5$ (say at $u_i^3, i = 1, \dots, g$) where scripts 3, 4, 5 on X and ψ refer to the complex coordinates and their fermionic partners of the three tori respectively.

The correlation function, apart from the prime forms (given by the OPEs) and σ 's that take care of the dimensions and monodromies, is

$$\frac{\theta_s(z + \frac{1}{2}(x - y - w))^2 \theta_{g_3,s}(\frac{1}{2}(x + y + w) - u^3) \prod_{i=1}^2 \theta_{g_i,s}(\frac{1}{2}(x + y - w) - u^i)}{\theta_s(\frac{1}{2}(x + y + w) - u + 2\Delta)} \quad (4.A.1)$$

where summation of x, y, u^1, u^2, u^3 is implied and u denotes the sum over all the $(3g - 2)$

⁷In this section, we assume familiarity with the original computation [9]; therefore we omit to carefully explain some notations and details, and concentrate on the aspects that are new.

⁸Note that this is chiral vector multiplet and not anti-chiral. In the latter case one would be probing holomorphic anomaly and that would be a completely different calculation.

positions of PCO. Now we can choose the gauge

$$u = y + w - z + 2\Delta \quad (4.A.2)$$

then the denominator cancels with one of the space-time θ functions. After the spin structure sum one finds:

$$\theta(z + x - w - \Delta)\theta_{g_1}(\Delta - u^1)\theta_{g_2}(\Delta - u^2)\theta_{g_3}(\Delta + w - u^3) \quad (4.A.3)$$

Finally we multiply the above by (by using gauge condition)

$$1 = \frac{\theta(y - z - \Delta)}{\theta(u - w - 3\Delta)} \quad (4.A.4)$$

Now together with appropriate prime forms and σ 's that are already there in the original correlation function

$$\theta(z + x - w - \Delta)\theta(y - z - \Delta) = (\det \omega(z, x))(\det \omega(y)) \quad (4.A.5)$$

Furthermore by using bosonization and together with appropriate prime forms and σ 's that are already there in the original correlation function

$$\begin{aligned} \theta_{g_1}(\Delta - u^1) &= \langle \prod \bar{\psi}_3(u^1) \rangle, & \theta_{g_2}(\Delta - u^2) &= \langle \prod \bar{\psi}_4(u^2) \rangle, \\ \theta_{g_3}(\Delta + w - u^3) &= \langle \psi_5(w) \prod \bar{\psi}_5(u^3) \rangle \end{aligned} \quad (4.A.6)$$

where these are correlation functions in the twisted theory (i.e. $\bar{\psi}_{3,4,5}$ are dimension one and $\psi_{3,4,5}$ are dimension zero). Combining also $\partial X_3(u^1)$, $\partial X_4(u^2)$ and $\partial X_5(u^3)$ and taking all partitioning of u into the three groups and anti-symmetrization, the above becomes

$$\langle \psi_5(w) \prod G_-(u) \rangle \quad (4.A.7)$$

where G_- is the $\mathcal{N} = 2$ world sheet super-current with $U(1)$ charge (-1) : this is again in the twisted theory, i.e. G_- has dimension 2 and G_+ with dimension 1 is the topological BRST current. Note that the above correlator has first order poles as w approaches any of the u and has first order zeros when any of the u goes to any other u . Finally

$$\frac{1}{\theta(u - w - 3\Delta)} \rightarrow \frac{1}{\langle c(w) \prod b(u) \rangle} \quad (4.A.8)$$

where \rightarrow means after taking into account various prime forms and σ 's, b and c are the standard (b, c) ghost system of dimension $(2, -1)$.

Now we can take one of the u 's (say u_{3g-2}) to approach w :

$$\frac{\langle \psi_5(w) \prod_{i=1}^{3g-2} G_-(u_i) \rangle}{\langle c(w) \prod_{i=1}^{3g-2} b(u_i) \rangle} = \partial X^5(w) \frac{\langle \prod_{i=1}^{3g-3} G_-(u_i) \rangle}{\langle \prod_{i=1}^{3g-3} b(u_i) \rangle} \quad (4.A.9)$$

where we have used the OPE

$$\psi_5(w)G_-(u_{3g-2}) = \partial X^5(w) \frac{1}{w - u_{3g-2}}, \quad c(w)b(u_{3g-2}) = \frac{1}{w - u_{3g-2}} \quad (4.A.10)$$

and the fact that $\partial X^5(w)$ just gives the zero modes (note that G contains only ∂X^5 and not $\partial \bar{X}^5$). Now $\frac{\langle \prod_{i=1}^{3g-3} G_-(u_i) \rangle}{\langle \prod_{i=1}^{3g-3} b(u_i) \rangle}$ is independent of u_i as both numerator and denominator are proportional to $\det h(u)$ where $h(u)$ are the $(3g-3)$ quadratic differentials. So

$$\frac{\langle \prod_{i=1}^{3g-3} G_-(u_i) \rangle}{\langle \prod_{i=1}^{3g-3} b(u_i) \rangle} \langle \prod_{i=1}^{3g-3} (\mu_i b) \rangle = \langle \prod_{i=1}^{3g-3} (\mu_i G_-) \rangle \quad (4.A.11)$$

where μ_i are the Beltrami differentials and $(\mu_i G_-) = \int \mu_i G_-$. Combining also the right moving part and integrating (z, x, y) using eq. (4.A.5) one finds $(\det \text{Im } \tau)^2$, which cancels with the contribution from the space-time X zero mode integrations. The final result is in IIB

$$\int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} (\mu_i G_-) \prod_{i=1}^{3g-3} (\mu_i \widetilde{G_-}) \int_w \partial X^5 \bar{\partial} X^5(w) \rangle = \partial_a \mathcal{F}_g^B \quad (4.A.12)$$

where the derivative is w.r.t. complex structure moduli of the CY, and in IIA

$$\int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} (\mu_i G_-) \prod_{i=1}^{3g-3} (\mu_i \widetilde{G_+}) \int_w \partial X^5 \bar{\partial} X^5(w) \rangle = \partial_a \mathcal{F}_g^A \quad (4.A.13)$$

where derivative is w.r.t. complexified Kähler moduli of CY. In both the cases the derivatives is with respect to the holomorphic vector moduli as is to be expected.

All of the above can be done for an arbitrary CY (i.e. not necessarily orbifold) [9]. So the conclusion is that at least for the holomorphic derivatives of \mathcal{F}_g we have a clear physical string amplitude.

4.B Theta functions

Generalized θ -function for genus g Riemann surface Σ_g is defined on \mathbb{C}^g as

$$\theta(v|\tau) = \sum_{n \in \mathbb{Z}^g} \exp(i\pi n \cdot \tau \cdot n + 2i\pi n \cdot v) \quad (4.B.1)$$

where the positions that enter the arguments of theta functions are defined on the Jacobi variety of Σ_g , for example $v = \frac{1}{2}(x - y) + z$ means $v_\mu = \int_{P_0}^z \omega_\mu + \frac{1}{2} \int_y^x \omega_\mu \in \mathbb{C}^g$, with $x, y, z \in \Sigma_g$ and P_0 some base point. Here ω_μ for $\mu = 1, \dots, g$ are the abelian differentials and τ is the period matrix of Σ_g ; sometimes we drop τ and just write $\theta(v)$.

The generalization with spin structure $s = (a, b) \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^{2g}$ is given by

$$\theta_s(v|\tau) = e^{i\pi a \cdot \tau a + 2i\pi a \cdot (v+b)} \theta(v + \tau a + b|\tau), \quad (4.B.2)$$

while the twisted one is $\theta_{s,g}(v|\tau) = \theta_{(a,b+g)}(v|\tau)$. Since $\theta_s(-x) = (-1)^{4a \cdot b} \theta_s(x)$, we distinguish accordingly between even and odd spin-structures.

Riemann vanishing theorem states that for all $z \in \mathbb{C}^g$ the function $f(P) = \theta(z + \int_{P_0}^P \omega)$ either vanishes identically for all $P \in \Sigma_g$, or it has exactly g zeros Q_i on Σ_g ; moreover,

in the latter case, there exists a vector $\Delta \in \mathbb{C}^g$, called the Riemann class, depending only on P_0 , such that the points Q_i satisfy $z + \sum_i \int_{P_0}^{Q_i} \omega \equiv \Delta$, modulo elements in the period lattice. Note that Δ depends on the choice of P_0 in such a way that e.g.

$$\theta \left(\frac{1}{2} \left(\sum_{i=1}^g (x_i + \bar{x}_i) + z + \bar{z} \right) - \sum_{a=1}^{3g-3} u_a + 2\Delta \right) \quad (4.B.3)$$

is independent of P_0 .

A useful identity [79, II§6 eq. (R_{ch}) p. 214] due to Riemann is

$$2^{-g} \sum_s \theta_s(x) \theta_s(y) \theta_s(u) \theta_s(v) = \theta \left(\frac{x+y+u+v}{2} \right) \theta \left(\frac{x+y-u-v}{2} \right) \theta \left(\frac{x-y+u-v}{2} \right) \theta \left(\frac{x-y-u+v}{2} \right) \quad (4.B.4)$$

Chapter 5

Conclusions and outlook

5.1 Final remarks

In this Thesis, we discussed the BPS integer expansion of the real topological string [101], using the M-theory lift of the O4-plane with one stuck D4-brane. Since the geometry is a \mathbb{Z}_2 quotient acting freely in the M-theory \mathbb{S}^1 , the 5d setup enjoys an enhancement to 8 supercharges and is identical to that in the closed oriented Gopakumar-Vafa system. The subtleties due to the orientifold quotient arise as a compactification effect modifying the KK momentum of the BPS states on the \mathbb{S}^1 according to their parity under the orientifold action. This allows for a clean derivation of the BPS integer expansion, without the extra assumptions that pop up in other unoriented A-models.

Although we recover the conjectured BPS expansion [101], our derivation shows the correct identification of the BPS invariants not as the equivariant sector of the parent Gopakumar-Vafa invariants, but rather a weighted version thereof.

The M-theory picture provides a complementary viewpoint on the sign choices implied by the tadpole cancellations in models where the fixed lagrangian 3-cycle L has $H_1(L; \mathbb{Z}) = \mathbb{Z}_2$ [101]. More generally, the BPS integer expansion we propose is valid for other situations, providing a general definition of the real topological string.

The M-theory lift of other O4-planes suggests non-trivial relations between their BPS invariant expansions, for instance the addition of an open M2-brane sector (associated to two stuck M5-branes) to the lift of the $O4^-$ -plane should reproduce a sign flip in odd crosscap contributions. This seems to imply non-trivial relations among the unoriented and open BPS invariants in M-theory orientifolds with fixed points.

Note that when one tries to generalize the above constructions, due to infrared effects [25] it might not be correct to perform the Schwinger-type computations for cases where the Lagrangian has $b_1 > 0$, it is not topologically $\mathbb{R}^2 \times \mathbb{S}^1$ or $\mathbb{R}\mathbb{P}^3$, or $N > 1$ D4-branes wrap it.

We also explored the extension of the correspondence between topological strings on toric CY three-folds and 4d/5d supersymmetric gauge theories with 8 supercharges to systems with orientifolds with real codimension 2 fixed locus. On the topological string

side, we focused on quotients which produce the real topological string [101], because of its remarkably simple physical realization in M-theory. We analyzed the properties of the systems, and emphasized their behavior under flops of the geometry.

The real topological string amplitudes define the properties of a new kind of surface defect in the corresponding gauge theory. We have rephrased the amplitudes in a form adapted to a gauge theory interpretation, by means of a newly defined twisted Nekrasov partition function, and we have taken the first steps towards providing an intrinsically gauge-theoretic interpretation of the twisting operator.

The ϵ_2 -shift that we performed on Nekrasov instanton partition function, although it passes many consistency checks and produces integer BPS multiplicities, does not completely reproduce the results from real topological vertex, due to different $(-1)^{|R|}$ signs in particular sub-sectors of the theory. One might therefore try to add extra ingredients, e.g. shift the Coulomb branch moduli as for ordinary orbifolded instantons [62], where instantons with a surface defect were identified with orbifolded instantons via a chain of dualities. It would be interesting to extend their reasoning to our case and more generally determine the effect of the orientifold defect in the ADHM quantum mechanics. Once we identify the effect in the ADHM quantum mechanics, then we may use localization techniques.

Finally, we computed using the doubling trick the orientifold-invariant superstring correlators that produce a holomorphic derivative of the real topological amplitude. We also used the same method to clarify previous computations in the standard closed oriented setup.

Having obtained Walcher's topological string in terms of physical type I amplitude, a natural question, which we are currently investigating, is what its heterotic dual would be. This is particularly useful to study the singularity structure of the topological string when some massive state becomes massless as one moves in the moduli space of compactification. In the type I or type II side such states are necessarily some D-brane states wrapped on a vanishing cycle and hence non-perturbative, but on the heterotic side one can realize the would-be massless states perturbatively. In fact in the context of standard \mathcal{F}_g in the oriented type II theory, a study on the heterotic dual [8] gave the Schwinger formula describing the singularity structure that explicitly proved the $c = 1$ conjecture and was generalized to all BPS states by Gopakumar and Vafa [38, 39].

5.2 Possible developments

Some interesting questions that one may investigate further are as follows.

- The relation between O4-planes in type IIA and in topological string: as is well known, there are 4 different O4/D4 systems, whose classification is based on the O-plane charge and the presence of RR background fields; one could study the M-theory lift of the other O4/D4 systems, which are believed to yield different topological string constructions and thus different integrality properties for the

enumerative invariants. In many of these cases the main difficulty lies in the fact that the M-theory lift is not smooth, so one should find a way to study the physics near these singularities and extract relevant information on BPS states.

- Adding N pairs of M5-branes away from the (M-theory lift of the) orientifold should lead to a simple extension of the model we considered, where one introduces extra open string sectors, and expects to see an enhancement of the background brane $SO(2N)$ symmetry to $SO(2N + 1)$.
- Study more general tadpole canceling configurations, as was already initiated by Krefl and Walcher [71], for example when the lagrangian has multiple connected components; one expects that topological tadpole cancellation corresponds to the physical tadpole cancellation, while on the topological string side it guarantees the decoupling of vector and hyper multiplets. It would be interesting to study the enumerative geometry of this configurations, and its implications for BPS state counting.
- One could ask how robust these type of invariants are against complex structure deformations: differently from the standard oriented case, where one finds the invariants are independent of cs deformations, in this setup it is not clear, as for example the notion of lagrangian subspace is changing.¹
- There is a subtlety in the choice of signs, related to how one can take the square root in the real vertex prescription, that needs further investigation; this subtlety is related to the orientifold action on the Coulomb branch parameters of the gauge theory. One could try to clarify this by deriving a localization formula in the orientifold background, by slightly changing the SO/Sp ADHM construction [53], and check whether the real vertex prescriptions are exactly matched.
- Moreover, one could consider a different type of involution of the toric diagram, where the number of Coulomb branch parameters gets reduced, and try to find the gauge theory dual.
- In the framework of BPS state counting in presence of defects, the above result suggests a relation between unoriented GV invariants and codimension two surface defects [27] in 4d gauge theory, possibly clarifying the interpretation of the O4-plane as a defect, in a similar fashion as open BPS invariants and surface defects are related [27, 32]. The peculiarity here is that the holonomy is given by an outer automorphism of the original gauge group, so one has to consider a different kind of defect than the ones studied by Gukov and collaborators [27].

¹Of course they are invariant for deformations that preserve the anti-analytic involution.

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