Internationl School for Advanced Studies (SISSA/Trieste)

Pure spinor formalism and gauge/string duality

by

Houman Safaai

Supervisor

Giulio Bonelli

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the
Sector of Elementary Particles
Scuola Internazionale Superiore di Studi Avanzati

September 2009

“ And that inverted Bowl we call The Sky, Whereunder crawling coop’t we live and die, Lift not your hands to It for help−for It. As impotently moves as you or I.”

Omar Khayym (11th century), Persian astronomer and poet, Rubiyt of Omar Khayym

Abstract

Sector of Elementary Particles
Scuola Internazionale Superiore di Studi Avanzati
Doctor of Philosophy
by Houman Safaai

A worldsheet interpretation of AdS/CFT duality from the point of view of a topological open/closed string duality using the power of pure spinor formalism will be studied. We will show that the pure spinor superstring on some maximally supersymmetric backgrounds which admit a particular $\mathbb{Z}_4$ automorphism can be recasted as a topological A-model action on a fermionic coset plus a BRST exact term. This topological model will be interpreted as the superstring theory at zero radius. Using this decomposition we will prove the exactness of the σ-model on these backgrounds. We then show that corresponding to this topological model, there exist a gauged linear sigma model which makes it possible to sketch the superstring theory in the small radius limit as the dual limit of the perturbative gauge theory. Studying the branch geometry of this gauged sigma model in different phases gives information about how the gauge/string duality is realized at small radius from a similar point of view of the topological open/closed conifold duality studied by Gopakumar, Ooguri and Vafa. Moreover, we will discuss possible D-brane boundary conditions in this model. Using this D-branes, we will make an exact check in the $\mathcal{N} = 4$ SYM/$AdS_5 \times S^5$ duality. We will show that the exact computation of the expectation value of the circular Wilson loops in the gauge theory side can be obtained from the amplitudes of some particular D-branes as the dual of the Wilson loops in the superstring side. The next step will be to construct a BV action for G/G principal chiral model with $G \in PSU(2,2|4)$, we will show that after applying different gauge fixings of the model, we will get either a topological A-model theory or a topological thery whose supersymmetric charge is equal to the pure spinor BRST charge. Using this model one can explore the cohomology of the pure spinor action from the topological BV model. Then we show that there exist a particular consistent deformation of the G/G action equal to the pure spinor superstring action. In this way we generate the superstring action on a non-zero radius AdS background as a perturbation around a topological model corresponding to zero radius limit of the superstring theory. Using this picture we will give an argument based on the worldsheet interpretation of open/closed duality to give a worldsheet explanation for AdS/CFT duality. A better understanding of this picture might give a prove of Maldacena's conjecture. At the end we will show that using the topological A-model, one can also give a prescription for computing multiloop amplitudes in the superstring theory on $AdS_5 \times S^5$ background.

Acknowledgements

Tracing back my interest into physics reminds me the old days of the early-school time where I, together with very ambitious friends, was always dreaming about physics and deeply indulgated by the beauty of science. I would like to thank all those friends and teachers and very specially my parrents for all their support, encouragement and love they gave me during my life to help me pursuing my dreams. Anything I get in my scientific carrier is the crop of the seeds they planted years ago by their immaculate training and never-ending sacrifices.

I would like to express my deep thanks to my supervisor in SISSA, Dr. Giulio Bonelli, for all the suuport, time and scientific knowledge he gave me. I enjoyed his way of thinking and working in which I learned many things from him. We spent many hours of our lifes together discussing about physics but I should admit that the part out of the work and the fact that I could count on him as a near and supportive frined was invaluable for me too.

I benefited a lot in the collaborations and discussions with Dr. Pietro Antonio Grassi to whom I want to express my thanks for all the things I learned from him and all his support.

It is a pleasure for me also to thank my external referee, Prof. Dimitry Sorokin, for his valuable comments and suggestions on the thesis. I would like to thank also my ex-supervisor in Iran, Prof. H. Arfaei, for many things he learned me and for his guidance in the early stages of my scintific carrier. Also I am thankful to all my professors in SISSA.

I enjoyed sharing many deep and happy moments of my life with many good friends in Trieste during these four years. Their presence made the life here more familiar and melted the ices of the lonliness. I am thankful to all of them specially to my very early and endless friends Andrea Brini, Davide Forcella and Diego Gallego who has been very near during all these years.

Last but of course not least, I would like to thank deeply Ladan, who has been beside me as the nearest one during the last stages of my PhD, for all the invaluable friendship, support, extreme happiness, energy and love she gave me.

Trieste, September 2009

Contents

To my parents, Parvaneh and Behrouz
And to the one I love.

Chapter 1

Introduction

1.1 General gauge/string duality

One of the most attractive subjects of theoretical physics giving deeper understanding of the string theory is the large N duality between gauge theories and string theories. There are several examples of this duality in physics and mathematics. Most of these dualities are motivated and studied from the target space point of view to relate the string theory on the target space to the dual gauge theory. However, the original idea of large N duality which originated from 't Hooft was based on the worldsheet perspective in his paper on 1974 [1]. According to 't Hooft, we start with a $U(N)$ gauge theory with the following action

$$S = \frac{1}{g_{YM}^2} \int \mathcal{L}(A) \tag{1.1.1}$$

where A is the gauge field and g_{YM} is the coupling constant of the YM theory with Lagrangian $\mathcal{L}(A)$. The gauge field takes value in the adjoint of the complex gauge group $U(N)$ and can be represented with two fundamental and antifundamental indices as $A_{i\mu}^{j}$ where $i, j = 1$ to N and μ is the space-time index. In this notation the propagator and all the interactions can be drawn as *ribbon graph* Feynman diagrams as is shown in figure (1.1)[1]. Here an upper index is denoted with an incoming arrow and a lower index by an outgoing arrow. The vertices's consist of Kronecker delta functions connecting upper and lower indices in which we do not need the explicit relation here. As usual, all the amplitude and Green functions can be computed by considering all planar and non-planar diagrams with their appropriate weight factor. An important observation is that any loop, which exists whenever an index line closes, contributes a factor of N

[1] Here we just considered the propagator and vertices's of the pure gauge fields for simplicity and do not consider other matter degrees of freedom of the theory.

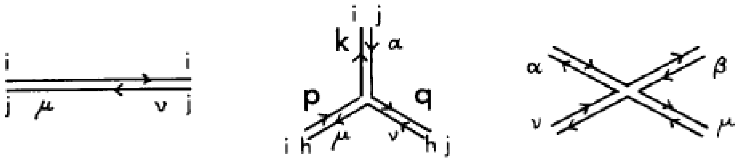

FIGURE 1.1: Feynman diagrams as ribbon graphs.

to a particular scattering amplitude because of the multiplicity of that index for the Kronecker delta

$$\sum_{i=1}^{N} \delta_i^i = N \tag{1.1.2}$$

As it was shown in [1], we can classify the diagrams according to their powers of coupling constant g_{YM} and their powers of N. The next step is to realize this Feynman diagrams as big connected surfaces. It was shown in [1] that for large N we can draw the diagrams on closed surfaces as in figure (1.2).

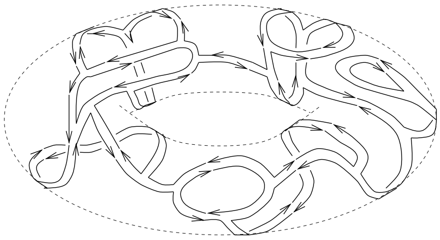

FIGURE 1.2: A ribbon diagram drawn on a surface.

Then the next step is to fill each loop with a disk as follows

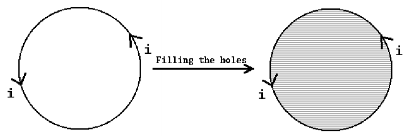

FIGURE 1.3: Filling each loop with a disk produce a Riemann surface.

This generates a surface which has g handles and h holes as is shown in figure (1.4) where the number of holes is equal to the number of the loops of the corresponding ribbon diagram.

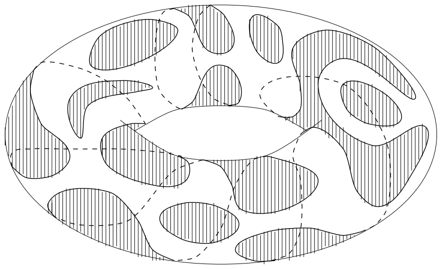

FIGURE 1.4: A ribbon diagram makes a surface with holes.

Each Feynman diagram with h faces or loops, V vertices's and E propagators corresponds a factor

$$r = (g_{YM}^2)^{-V+E} N^h \tag{1.1.3}$$

to the amplitude. This is because each loop brings a factor N and each vertex contributes a factor g_{YM}^2 and each propagator a facto of g_{YM}^{-2} in the result.

This contribution can be also specified referring to the topology of the surface in which the diagram is drawn. This can be seen after rewriting (1.1.3) as

$$r = (g_{YM}^2)^{-V+E} N^h = (g_{YM}^2)^{-V+E-h} (g_{YM}^2 N)^h \tag{1.1.4}$$

According to Euler formula for a surface with h holes, E edges and V vertices's we have

$$h - E + V = 2 - 2g \tag{1.1.5}$$

where g is the genus of the surface. So (1.1.4) can be rewritten as

$$r = (g_{YM}^2)^{2g-2} t^h \tag{1.1.6}$$

where $t = g_{YM}^2 N$ is named as the 't Hooft parameter.

The full amplitude is obtained after summing over all the ribbon graphs with the weight factor (1.1.6)

$$F_{YM} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} (g_{YM}^2)^{2g-2} t^h F_{g,h} \tag{1.1.7}$$

where $F_{g,h}$ is a function of the other parameters exsiting in the theory. We can rewrite it as a sum over the genera

$$F_{YM} = \sum_{g=0}^{\infty} (g_{YM}^2)^{2g-2} F_g(t) \tag{1.1.8}$$

where

$$F_g(t) = \sum_{h=1}^{\infty} t^h F_{g,h} \tag{1.1.9}$$

In the large N limit, the graphs corresponding to low-genus dominate in the computations and taking $t = g_{YM}^2 N$ fixed, the expansion in $\frac{1}{N}$ becomes similar to the expansion in g_{YM}^2.

On the other hand, we can write the free energy of the closed string theory as a sum over the amplitudes correponding to connected closed worldsheets with genus g contributing a factor g_s^{2g-2} where g_s is the string coupling constant. The free energy can be written as the following expansion

$$\mathcal{F}_s = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(t) \tag{1.1.10}$$

Here t is a geometric modulus of the target space. Comparing (1.1.8) and (1.1.10), we can see that the two theories can be interpreted as they are computing the same thing stating the 't Hooft conjecture.

't Hooft conjecture: There is a closed string theory whose $g-$loop amplitude is given by $\mathcal{F}_g(t) = F_{YM}(t)$ where the target space modulus is identified with the 't Hooft coupling $t = g_{YM}^2 N$ and the string coupling constant is related to the YM coupling as $g_s = g_{YM}^2$. The two theories are the dual descriptions of the same physical theory.

This duality can be seen from a higher perspective as a duality between open and closed string theories by considering the simple fact that the $U(N)$ gauge theories can be realized as the Chan-Paton degrees of freedom of open strings ending on a stack of N D-branes.

A ribbon graphs has natural interpretation as describing open string worldsheet ending on D-branes. The left figure in (1.5) is an open string describing a worldsheet in which its end points carrying the indices i and j of the gauge group are on the D-brane making the ribbon graph. On the other picture, the open surface with genus g and h holes is the worldsheet of open string whose amplitude is equivalent to the scattering amplitude of the corresponding ribbon graph in figure (1.4). Each open worldsheet amplitude which has genus g and h holes like figure (1.5) is weighted with a factor $g_s^{2g-2}(Ng_s)^h$ where the factor $(Ng_s)^h$ comes from the Chan-Paton factors for each hole which is the intersection of the worldsheet with D-branes. In fact, one can give a conjecture for the duality between open and closed string theories in the sense that there is a closed string theory

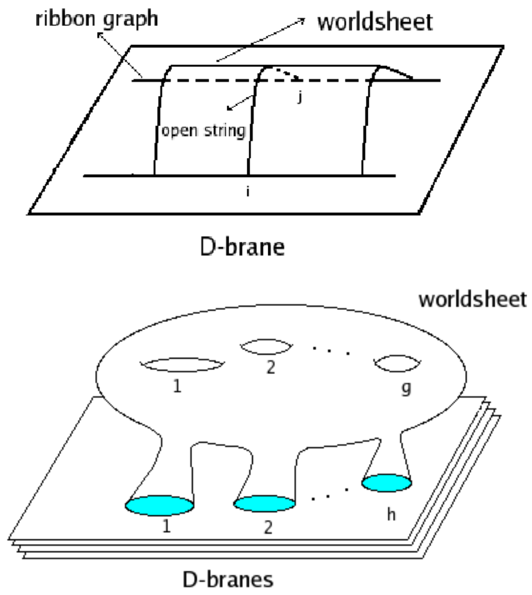

FIGURE 1.5: Ribbon graphs as the intersection of D-brane and worldsheet.

whose free energy $\mathcal{F}_g$ on a closed worldsheet with genus g is related to the amplitude of the open string theory with fixed genus g as follows

$$\mathcal{F}_g = \sum_{h=1}^{\infty} (Ng_s)^h F_{g,h} \tag{1.1.11}$$

where $F_{g,h}$ is the amplitude of open strings propagating on a worldsheet with genus g and h holes like the one in figure (1.5). If we take the low energy limit, we get the original 't Hooft conjecture since the gauge theory can be realized as the low energy limit of the open string theory in the presence of D-branes.

1.2 AdS/CFT correspondence

By now, several examples of large N dualities have been discovered including the famous Maldacena's AdS/CFT duality and the topological string dualities which we will briefly discuss. We are going to relate them at least for some specific cases and we will show that we can use the techniques and advantages of the topological string duality in the more elaborated AdS/CFT correspondence.

In [2], Maldacena considered a system of N D-branes which according to different ways of interpretation, give rise to different theories. If one takes the near horizon limit in which the string coupling g_s and N is fixed, while the length of strings goes to zero $l_s \to 0$, the open and closed strings decouple and we get two decoupled descriptions of the same physics. The first description is in terms of the closed strings which is a superstring theory on a $AdS \times X$ background with N units of flux on X and the second description is in terms of open strings whose low energy limit is a superconformal gauge theory on the boundary of the AdS space. In this sense it can be seen as a holographic duality. In both sides of the duality the superstring and superconformal gauge theory have equal supergroup as their super-isometry or super-conformal group.

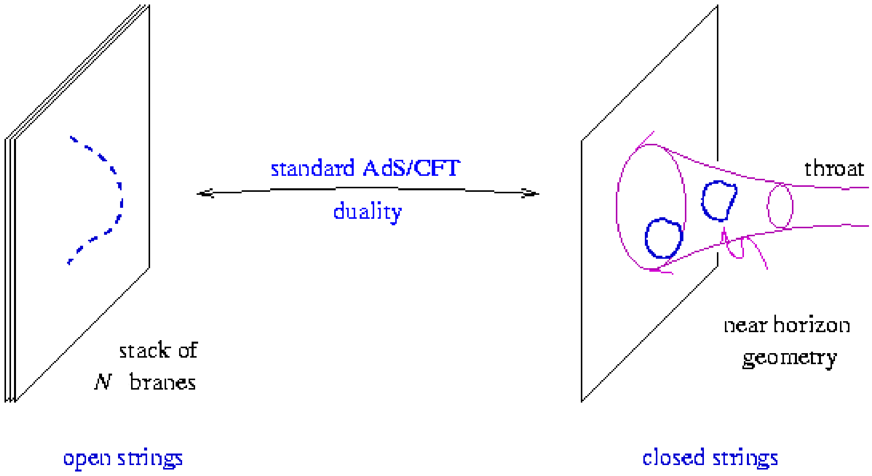

FIGURE 1.6: Standard AdS/CFT duality.

The main three examples of AdS/CFT duality which was addressed by Maldacena is listed in table (1.1). We should note that there is also a long list of non-conformal SYM dualities which we are not referring them here.

Brane	**AdS theory**	**dual SCFT**	**Superisometry**
M2-branes	M-theory, $AdS_4 \times S^7$	some 3d SCFT	$OSp(8\|4)$
D3-branes	IIB superstring, $AdS_5 \times S^5$	4d $\mathcal{N} = 4$ $SU(N)$ SYM	$PSU(2,2\|4)$
M5-branes	M-theory, $AdS_7 \times S^4$	some 6d SCFT	$OSp(6,2\|4)$

TABLE 1.1: Three main Maldacena AdS/CFT correspondences.

The most by far studied example of the AdS/CFT duality is the type IIB supersting on $AdS_5 \times S^5$ with N units of RR flux

$$\int_{S^5} F_5 = N \tag{1.2.1}$$

on the S^5 with the $\mathcal{N} = 4$ super Yang-Mills theory with $SU(N)$ gauge group in four dimensions which appears to be a conformal theory. The gauge theory amplitudes can be expanded in powers of $1/N$ for large N at fixed 't Hooft parameter t defined as $t = g_{YM}^2 N$. This 't Hooft expansion corresponds to the string loop expansion in the superstring side after the following identifications

$$R^2 \leftrightarrow \sqrt{t} \quad , \quad g_s \leftrightarrow \frac{t}{N} \tag{1.2.2}$$

where we take α' to be one. The parameter R is the radius of the AdS_5 and S^5 which should be the same in order to maintain worldsheet conformal invariance and g_s is the string coupling constant determined by the value of the dilaton field. The source of the curvature lies in the nonzero value of the self-dual 5-form flux belonging to the SUGRA multiplet.

To match conformal supersymmetry in $4d$ with AdS supersymmetry in $5d$ the symmetry supergroups in both cases happen to coincide, as they should. There are $8\mathcal{N}$ real SUSY generators and the bosonic part consists of the conformal AdS group $Spin(4,2) \cong SO(4,2)$ times an internal group $SU(\mathcal{N})_T \times U(1)_A$. For the case $\mathcal{N} = 4$, we have 32 real SUSY generators and an internal group $SU(4)_T \times U(1)_A$. Now, $SU(4) \cong Spin(6)$ and $Spin(6)$ is the isometry group of S^5 with spinorial fields. The bosonic spatial isometry group of $AdS_5 \times S^5$ is $SO(4,2) \times SU(4)$ which together with fermionic degrees of freedom completes the supergroup $PSU(2,2|4)$.

In $\mathcal{N} = (2,0)$ 10D IIB superstring theory, we have 32 real SUSY generators. However, the bosonic spatial isometries with 55 generators in the flat case is now substituted by $SO(4,2) \times SU(4)$ with 30 generators. $\mathcal{N} = (2,0)$ also has a $U(1)_R$ symmetry and this is identified with $U(1)_A$. The $AdS_5 \times S^5$ superstring action was shown to be given as a supercoset sigma model on $PSU(2,2|4)/SO(5) \times SO(4,1)$ [3] whose isometry group is exactly the superconformal group of the dual SCFT.

The other important AdS/CFT duality which attracts many attentions during the last two years is the type IIA/ABJM duality which is the first duality of table (1.1). This duality got progressed because new achievments in understanding of the M2-brane systems. Bagger, Lambert and Gustavsson [4], followed a suggestion by Schwarz [5] to use Chern-Simons theory, constructed a three-dimensional superconformal gauge theory with $\mathcal{N} = 8$ maximal supersymmetry based on the so called three-algebra structute. Because of its special consistency condition, their construction works just only for the gauge group $SO(4)$ and so it does not provide the dual of M-theory on $AdS_4 \times S^7$ unless for a very special case of $N = 2$.

The correct dual however was obtained by Aharony, Bergman, Jafferis and Maldacena (ABJM) in [6]. They considered M-theory on the orbifold $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of flux which gives 3/4 supersymmetry for $k > 2$. The dual gauge theory which they proposed was a supersymmetric $\mathcal{N} = 6$ superconformal Chern-Simons theory which has two Chern-Simons terms corresponding to a $U(N)_k \times U(N)_{-k}$ gauge group where k and $-k$ are the levels of the Chern-Simons terms. Because of a nontrivial property of the quantum theory which was explained in [6], the supersymmetry increases to $\mathcal{N} = 8$ for $k = 1, 2$. The ABJM theory also includes bifundamental scalar and spinor fields. The 't

Hooft expansion of the gauge theory is defined as follows

$$t = \frac{N}{k} \tag{1.2.3}$$

The difference with the previous case is that here the 't Hooft parameter appears to be a rational number.

The orbifold $S^7/\mathbb{Z}_k$ can be described as a circle bundle over a $\mathbb{CP}^3$ basis upon reduction to string theory

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^7 \\ & & \downarrow \\ & & \mathbb{CP}^3 \end{array} \tag{1.2.4}$$

The circle has radius R/k, where R is the radius of S^7. In the limit $k^5 \gg N$ there is a weakly coupled type IIA on $AdS_4 \times \mathbb{CP}^3$ with a string coupling given by

$$g_s = \left(\frac{N}{k^5}\right)^{1/4} \tag{1.2.5}$$

and upon the following identification with the dual superconformal gauge theory

$$R^2 \leftrightarrow \sqrt{t} \quad , \quad g_s \leftrightarrow t^{\frac{5}{4}}/N \tag{1.2.6}$$

The difference which arise here with the maximally supersymmetric $AdS_5 \times S^5$ case is that in this case the construction of the action for $AdS_4 \times \mathbb{CP}^3$ is more complicated as the background preserves only 24 out of the 32 supersymmetries of the type IIA supergravity. A supercoset space $OSp(6|4)/U(3) \times SO(1,3)$ with 24 fermionic coordinates has been used in [7, 8, 9, 10] to construct the superstring sigma model in $AdS_4 \times \mathbb{CP}^3$ but it was shown in [11] that this supercoset is a subspace of the complete superspace and it describes just a subsector of the full type IIA superstring theory in $AdS_4 \times \mathbb{CP}^3$. The complete type IIA superspace with 32 fermionic coordinates in $AdS_4 \times \mathbb{CP}^3$ background is not a supercoset space and is more complicated as it was stated in [11].

The AdS/CFT correspondence gives a dictionary between the objects on the two sides of the duality. For example we can relate n-point correlation functions in the gauge theory to the corresponding quantities in the string theory side [12]. If one considers two-point functions, the duality relates the energy E_a of a state $|a\rangle$ to the conformal dimension Δ_a of the operator $\mathcal{O}_a$ which is defined as

$$\langle \mathcal{O}_a \mathcal{O}_b \rangle \approx \frac{\Delta_{ab}}{|x-y|^{2\delta_a}} \tag{1.2.7}$$

specifically the duality says

$$\Delta_a(t, 1/N) = E_a(R^2, g_s) \tag{1.2.8}$$

The studies on AdS/CFT have been pointed mainly to test the duality by finding, comparing and matching similar objects on both sides of the duality. In this sense it is an state/operator duality. Being a weak/strong duality, almost all the studies have been done in the weak string theory side which corresponds to supergravity theory, because from the target-space point of view in which most of the studies have been directed, there is no possibility to explore the string theory in the strongly coupled regime.

As we saw before, the 't Hooft parameter is proportional to the radius of the AdS space and if one wants to obtain the 't Hooft expansion in the gauge theory side which corresponds to $t \to 0$ as it is a perturbative expansion, one should study the string theory in the corresponding limit which is the highly curved $R \to 0$ limit. Since the string theory sigma model coupling constant is proportional to the inverse of R^2, this corresponds to the strongly coupled regime of the sigma model and can not be discussed with usual techniques. One has to find a good description of the theory which makes it possible to explore superstring theory in the small radius limit. With a "good" description we mean the one in which the worldsheet quantum field theory is well-defined and can be used to study the physics near $R = 0$ as the dual of the perturbative gauge theory side of which we have a good perturbative description. This seems puzzling from the duality point of view but as we will see later one can resolve this puzzle by defining a well-defined description of the string theory in this limit. We can state this puzzle more clearly as follows.

A puzzle in large N dualities: Usually in large N dualities, the 't Hooft parameter t is identified with a geometric modulus of the string theory target space and the limit $t \to 0$ leads to vanishing cycles and breakdown of the closed string perturbation theory by leading to the divergence of some amplitudes. But, in the dual side, this limit corresponds to a reliable perturbative regime. This seems puzzling which should be overcomed by giving a better description to the string theory side which does not breakdown in this limit.

1.3 Topological A-model conifold open/closed duality

Actually there is an example in which this good description was found and the duality was proved. This is the case of topological gauge/string dualities which we will explain here. The first type of topological gauge/string dualities was discovered by Gopakumar and Vafa [13] which states:

Topological gauge/string duality: Chern-Simons gauge theory in three dimensions which can be obtained from the low energy limit of open string theory ending on A-branes in a deformed CY_3 is dual to a topological closed string theory with A-twist in resolved CY_3.

where the A-twisted topological string theory will be defined and discussed later and the A-branes are consistent boundary conditions in the Calabi-Yau manifold which does not spoil the supersymmetric structure of the topological A-model theory.

The correspondence is between open strings ending on D-branes on the deformed CY_3 and closed strings on the resolved CY_3 where deforming and resolving are two ways of removing the singularity of the Calabi-Yau manifold which we take it to be the conifold.

There is also the mirror Version of this duality which was discovered by Dijkgraaf and Vafa [14]. The topological theory in question is the B-model and the roles of the deformed and resolved CY_3 are exchanged.

In order to have a more precise statement of the duality, consider the Chern-Simons theory on S^3 with gauge group $U(N)$

$$S = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{1.3.1}$$

where k is the level of the Chern-Simons action and the gauge coupling constant is given in terms of k and N as $g_{YM}^2 = \frac{i}{k+N}$. The statement is that the large N expansion of this Chern-Simons theory on S^3 produces the closed A-model topological string theory defined on the resolved conifold with string coupling constant $g_s = \frac{i}{k+N}$.

The singular conifold is topologically a cone over $S^2 \times S^3$ defined by the equation

$$\sum_{i=1}^{4} z_i^2 = 0 \tag{1.3.2}$$

in $\mathbb{C}^4$. We can remove the cone singularity in two ways as is shown in figure (1.7). Either by blowing an S^3 in the singularity or by replacing the singularity with a S^2, the first geometry is a deformed conifold which can be seen as the contangent bundle T^*S^3 and the second one is the resolved superconfold which can be described as a sum of line bundles over the base which is the S^2 as $\mathcal{O}(-1)+\mathcal{O}(-1)\to\mathbb{P}^1$. The more detailed description of the geometries will given later when we will discuss the super generalization of the conifold duality.

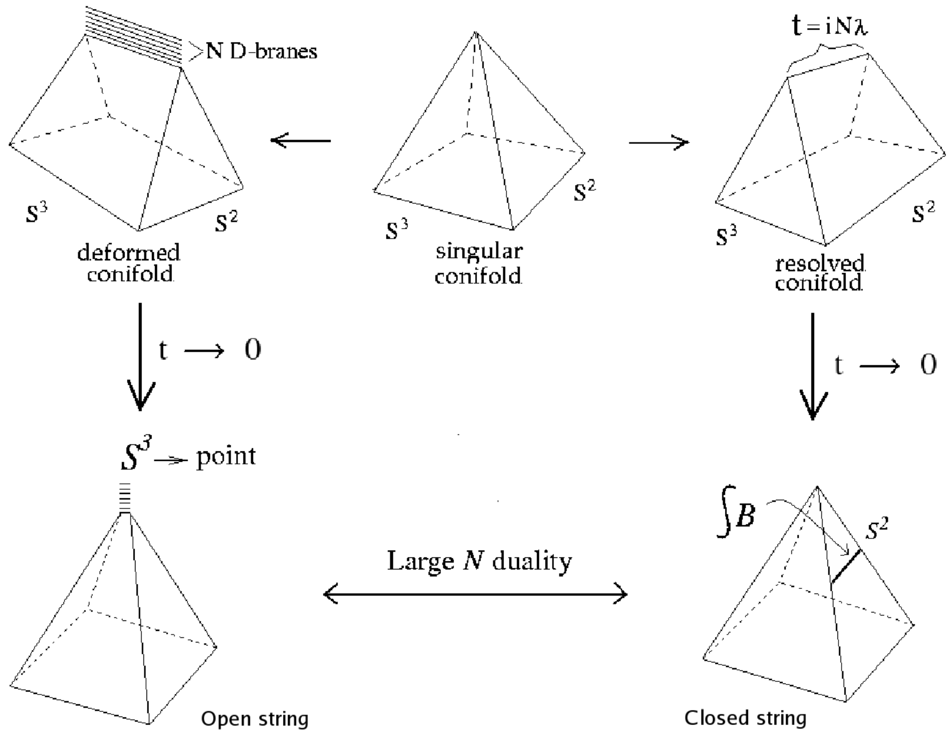

FIGURE 1.7: Large N duality as a geometric transition between topological open string on the defomred conifold and closed string theory on resolved conifld.

The base of the deformed conifold is S^3 with radius t. It can be checked that it is a Lagrangian submanifold and so it is a proper place to wrap the branes in the topological A-model theory. Putting N D-branes on the base S^3, it was shown by Witten [15], using open string field theory, that the theory reduces to the Chern-Simons theory on S^3 with $SU(N)$ gauge group. On the other hand, in the closed string side the complexified Kähler class is given by $t = iN/(k+N)$. This parameter corresponds to the size of the S^2 on the base. Since t is pure imaginary, this size is corresponding actually to a non-vanishing NS-NS two-form field whose integral over the two-cycle is equal to $|t|$

$$\int_{S^2} B = \frac{N}{k+N} \tag{1.3.3}$$

The duality is between the closed string theory with this flux and the open string theory with the open strings wrapping the base. The duality converts the resolved and deformed geometries and switches the branes into the flux. Since the theory is topological A-model the action have the general form[2]

$$S = it\int_{\Sigma}\{Q,V\} + t\int_{\Sigma} x^*(K) \tag{1.3.4}$$

[2]Here the integration is over the worldsheet Σ and the explicit form of V and the pullback of the Kähler form K, expressed as $x^*(K)$, can be be found for example in [16].

where Q is the conserved supersymmetry charge of the $N = 2$ topological A-model theory and t is the Kähler modulus of the manifold which is identified with the 't Hooft parameter in the gauge/string duality. It can be shown that the topological string amplitudes are independent of the Kähler parameter t and so we can in principle send it to zero as is shown in figure (1.7). We can send the size of the S^3 as the Kähler parameter to zero in the open string side. On the other hand, in the closed string side, in this $t \to 0$ limit the string amplitudes diverges and string theory perturbation breaks down. This is in accordance with the puzzle we addressed before for general large N dualities.

Having a singularity in a theory usually means lacking some degrees of freedom whose dynamics in the singular limit describes the theory. The idea proposed by Ooguri and Vafa in [17] was to introduce a new sector in the theory containing a gauge field. The

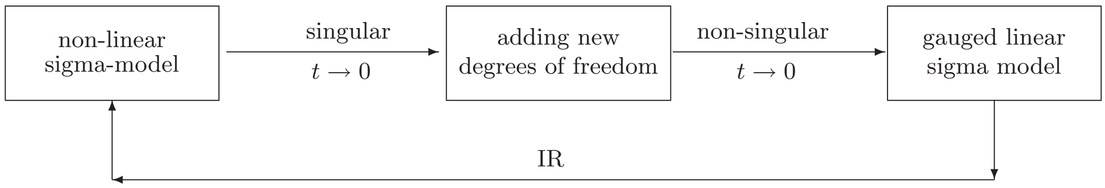

FIGURE 1.8: Adding a new sector to non-linear sigma model (1.3.4) gives a linear sigma model which is no longer singular in the limit $t \to 0$. The gauged linear sigma model gives the non-linear action after the flow to infra-red.

basic idea is to start with the closed string side and use a 'good' description of the string theory being able to explore the physics in the $t \to 0$ limit. This description was proposed by Witten [18] to be the gauged linear sigma model which flows to the original non-linear theory in the infra-red where the new degrees of freedom are integrated out.

The gauged linear sigma model in the case of the conifold duality contains one vector multiplet V, and four chiral multiplet A_1, A_2, B_1 and B_2 which are charged differently under the gauge group. The gauged linear sigma model action can be written as follows

$$S = \int d^4\kappa \left(\sum_{i=1}^{2} \bar{A}_i e^{2V} A_i + \sum_{i=1}^{2} \bar{B}_i e^{-2V} B_i + \frac{1}{e^2}\Sigma^2 \right) + t \int d^2\tilde{\kappa}\ \mathrm{Tr}\Sigma \tag{1.3.5}$$

where Σ is the gauge field strength which is a twisted chiral superfield $\Sigma = \bar{D}DV$ and e is the gauge coupling constant, the κ's are the fermionic coordinates of the $N = 2$ supersymmetry on the worldsheet. We see that the Kähler parameter t corresponding to the 't Hooft coupling is appears here as a Fayet-Illiopoulos parameter.

It was shown by Witten in [15] that the vacuum of theory admits two branches corresponding to different regimes of the t parameter. Here we consider a $U(1)$ gauge field in which the chiral superfields A_i and B_i have charge $+1$ and -1 respectively.

In order to find the vacuum we should find the zero points of the scalar potential which is given by

$$V = 2|\sigma|^2 \left(|a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2\right) + \frac{e^2}{2} \left(|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 - t\right)^2 \quad (1.3.6)$$

where σ, a_i and b_i are the first components of the Σ, A_i and B_i superfields respectively. Both e and σ have the dimension of mass on the worldsheet and in the infrared they can be sent to infinity. We can study the classical solution $V = 0$ in two regimes separately

- $t \neq 0$:

  In this case the condition $V = 0$ is satisfied when $\sigma = 0$ and with the following constraint on the other fields

  $$|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = t \quad (1.3.7)$$

  modulo the gauge transformations

  $$a_i \to a_i e^{i\theta} \quad , \quad b_i \to b_i e^{-i\theta} \quad (1.3.8)$$

  The space defined by this scalar fields as its coordinates can be shown to be the resolved conifold. The S^2 in the base of the resolved conifold is placed at $b_i = 0$ with radius $\sqrt{t}$ with coordinates a_i and b_i's as the fiber coordinates over the base S^2. Expanding the gauged linear sigma model (4.1.5) in components one can see that in this regime the gauge field get mass and in the infrared it can be integrated out and the gauge symmetry gets broken. This phase of the theory is the Higgs phase denoted by H. The physics in this phase is totally described with the chiral superfields A_i and B_i which can be mapped to the fields of the corresponding topological A-model non-linear sigma-model action.

- $t \to 0$:

  In this limit, it can be seen that the zero point of the potential can be obtained in two ways and the theory develops two phases:

  – Higgs phase:

    This phase is similar to the previous case and is obtained by putting $\sigma = 0$ for $a_i \neq 0$ and $b_i \neq 0$ subject to (1.3.7). This solution defines the resolved conifold as its target space and the gauge symmetry is broken because the gauge field becomes massive. The physics is described with the matter fields A_i and B_i which as we saw are related to the A-model non-linear sigma model fields.

- Coulomb phase:

  On top of the Higgs phase, here we have another solution which obtained for $\sigma \neq 0$ and putting $a_i = b_i = 0$. This is actually the new phase which appears in the singular point of the theory and describes the physics in this limit. One can check that the mass of the scalar fields are proportional to $|\sigma|^2$ and so in this phase they become massive and can be integrated out. The physics is given only by the dynamics of the gauge field. The emergence of the dynamics of this new degree of freedom removes the singularity of the theory.

There is an important issue here that one can check that there is no energy gap between these two phases and they can actually co-exist.

From the worldsheet point of view, the gauged linear sigma model tells us that the singularity at $t \to 0$ in the non-linear sigma-model action is due to the emergence of this new Coulomb phase which does not have interpretation in terms of the geometry of the conifold and physical content of the non-linear sigma model action. It was used to prove the duality from a worldsheet perspective [17].

In order to proof the gauge/string duality as in the original idea of 't Hooft, we start writing the closed topological string theory and try to obtain the open string amplitudes as an expansion in the 't Hooft parameter for small t as we described before. A particular g-loop topological closed string amplitude can be written as

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} \int_X d^2z \eta_i(z) G_L^-(z) \int_X d^2z \bar{\eta}_i(z) G_R^-(z) \right\rangle \tag{1.3.9}$$

where $\mathcal{M}_g$ is the moduli space of the genus-g Riemann surfaces, G_L^- and G_R^- are $\mathcal{N} = 2$ supersymmetry charges of the topological A-model and η_i and $\bar{\eta}_i$ are the Beltrami differentials of the Riemann surface X. In the corresponding path integral over the field configuration, we have to do integration over the fields A_i and B_i corresponding to the physical degrees of freedom of topological string theory on the resolved conifold, and also over the new degree of freedom σ. Cosinsidering a particular field configuration on a closed string worldsheet in the $t \to 0$ limit, as we saw before the field configuration separates into two parts. Either we have the fields in the Higgs phase or the Coulomb phase according to the value of the σ, we can put a cutoff parameter σ_0 to identify this phases on the worldsheet as follows

$$\begin{aligned} \text{C domain} &= \{z \in X : |\sigma(z)| > \sigma_0\} \\ \text{H domain} &= \{z \in X : |\sigma(z)| < \sigma_0\} \\ \text{Interface } \gamma &= \{z \in X : |\sigma(z)| = \sigma_0\} \end{aligned} \quad (1.3.10)$$

If we consider a genus-g Riemann surface as the worldsheet, it separates into h connected pieces in the H-domain with genus g_i, $i = 1$ to h and c pieces in the C-domain with genus g_j , $j = 1$ to c in which

$$\sum_{i=1}^{h} g_i + \sum_{j=1}^{c} g_j = g \quad (1.3.11)$$

all the pieces are connected to other pieces which is in a different branch with a circle which we named as γ. We can present a particular closed worldsheet in figure (1.9).

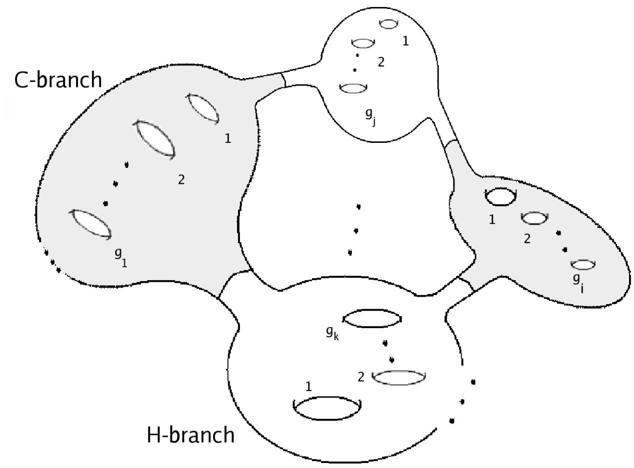

FIGURE 1.9: The worldsheet splits into connected pieces, some in Coulomb branch and some in the Higgs branch.

As we explained before, in the H-branch, the field configuration in the infrared is suppressed with the string theory fields A_i and B_i which can be mapped to the non-linear topological sigma-model fields. But in the C-branch, these physical fields get mass and we can integrate them out and they do not appear in the functional integration on these branch. In C-branch, the dynamical field is the gauge field and from the string theory field content which is (A_i, B_i) these branches are like empty Riemann surfaces connected to the worldsheet with Dirichlet boundary conditions $a_i = b_i = 0$. It is like that the worldsheet is connected to D-branes, as we expect for an open string worldsheet. It is a consequence of adding the new degree of freedom which generates holes on a closed string worldsheet. So, starting from a closed string worldsheet without holes we obtain open string worldsheet with many holes and the next step is to generate the partition function of the open string theory from this mechanism.

The functional path integral also decompose into the C and H-branches and we can do the functional integration over the fields in these two separate branches. The contribution from the C-branch is just comming from the dynamics of the σ field. It was observed by Ooguri and Vafa [17] that the functional integration over the C-branches satisfies the following properties

1. The only configuration of the C-branch contributing in the path integral is the disk. The contribution of the other topologies with at least one hole is zero. This is consistent from the way we realized the 't Hooft expansion by filling any hole with a disk and we did not have any other non-trivial topology. Starting from a close Riemann surface as the worldsheet for the closed string theory, the worldsheet partitions into open Riemann surfaces with h holes which can be interpreted as the worldsheet of the open string theory side. The number of the holes is equal to the number of the C-branches on the worldsheet and their contribution will be shown to be a constant later.

   The partition of a close Riemann surface into open Riemann surfaces with h holes can be visualized as in figure (1.10).

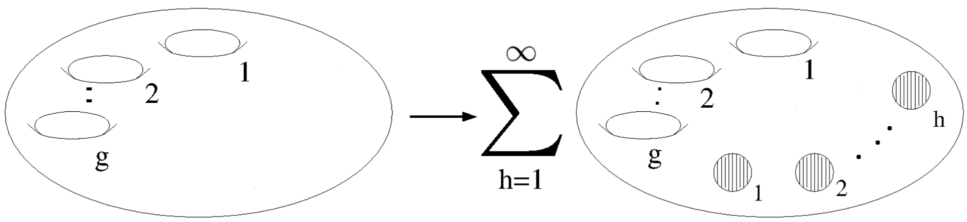

FIGURE 1.10: The partition of a close Riemann surface into open Riemann surfaces with holes.

   The closed string amplitude for a worldsheet with genus g also partitions into open string amplitudes with worldsheets with genus g and any number h of holes $\mathcal{F}_{g,\,h}$ as follows

$$\mathcal{F}_g = \sum_{h=1}^{\infty} \left(\mathcal{F}^{(C)}\right)^h \mathcal{F}_{g,\,h}^{(H)} \tag{1.3.12}$$

   where $\mathcal{F}^{(C)}$ is the contribution of the C-branch computed as a path integral just over the gauge field in the Coulomb phase and $\mathcal{F}_{g,\,h}^{(H)}$ is the amplitude corresponding to H-branch on a Riemann surface with genus g and h holes.

2. Using A-model topological theory it can be shown that the contribution of any C-branch can be computed exactly as a contour integral over the complex σ-plane

around the boundary of the hole. Because of the topological localization, it was shown in [17] that the boundary value σ_0 does not have angular dependence and so it is constant along the boundary as the Dirichlet boundary conditions. The contribution from the C-branch was computed in [17] using the fact that in the C-branch the theory becomes like a Landau-Ginzburg theory with superpotential $W = t\Sigma$ as follows

$$\mathcal{F}^{(C)} = t = g_{YM}^2 N \tag{1.3.13}$$

So, using (1.3.12) we see that we generate the 't Hooft expansion as follows

$$\mathcal{F}_g = \sum_{h=1}^{\infty} t^h \mathcal{F}_{g,\, h} \tag{1.3.14}$$

This is the direct consequence of the generation of the open string worldsheets from the mechanism derived by gauged linear sigma-model for small 't Hooft parameter.

Using the fact that the amplitude in the H-branch is independent of the Kähler parameter t, it can be shown that the $\mathcal{F}_{g,\, h}$ is the amplitude of the open strings with D-branes wrapping on the base S^3 of the deformed conifold T^*S^3. It comes as a worldsheet proof of the topological open/closed duality on conifold.

1.4 Towards a worldsheet proof of AdS/CFT duality

The proof of the topological open/closed duality was done using the power of the gauged linear sigma-model which is useful when the 't Hooft parameter is small. It would be very interesting to find similar description for the case of superstring on AdS spaces which might make it possible to give a similar proof for AdS/CFT duality in the regime which is not accessible with the usual target space methods. AS we will see, this can be done by using pure spinor superstring theory on some maximally supersymmetric string theory on AdS backgrounds and a topological decomposition of their action. The similarity between Chern-Simon and $\mathcal{N} = 4$ $d = 4$ SYM comes actually not surprising because, using pure spinor formalism, the $d = 10$ SYM action which can be reduced to the four-dimensional one upon dimensional reduction can be written in the Chern-Simons form as follows [19]

$$S = \int \left\langle VQV + \frac{2}{3}V^3 \right\rangle \tag{1.4.1}$$

where Q is the pure spinor BRST operator and V is the super-Yang-Mills vertex operator. One would expect to find a similar worldsheet derivation for large N duality like in the case of CS/conifold duality.

The first step towards having this description was done by Berkovits. Using pure spinor formalism for superstring on $AdS_5 \times S^5$, he found that there is a particular limit in which the sigma-model reduces to a topological A-model constructed from fermionic $\mathcal{N} = 2$ superfields. In an $AdS_5 \times S^5$ background, one can write the sigma model action in terms of a supercoset as follows[3]

$$S = \frac{1}{R^2}\int d^2 z \left[\frac{1}{2}\eta_{ab} J^a \bar{J}^b + \eta_{\alpha\hat{\beta}}\left(J^{\hat{\beta}}\bar{J}^{\alpha} - \frac{1}{4}\bar{J}^{\hat{\beta}}J^{\alpha}\right) + \text{pure spinor contribution}\right] \quad (1.4.2)$$

where J^a with $a = 0$ to 9 are bosonic and $(J^\alpha, J^{\hat{\alpha}})$ with $\alpha, \hat{\alpha} = 1$ to 16 are the fermionic left-invariant currents construct from $J = g^{-1}dg$ where g belongs to the supercoset $g \in \frac{PSU(2,2|4)}{SO(5)\times SO(4,1)}$ which was used by Matsaev and Tseytlin [3] to construct the worldsheet Green-Schwarz superstring action. The η_{ab} and $\eta_{\alpha\hat{\beta}}$ are the bilinear metrics of the supercoset. The pure spinor part of the superstring was introduced by Berkovits [20] in order to give a covariant quantization of the superstring theory. Unlike Green-Schwarz formalism in which the $\kappa-$symmetry gauge fixing is poorly understood even in a flat background unless in some particular gauges like the light-cone gauge, the pure spinor formalism as we will describe it better in the next chapter of the thesis give a quantization scheme, using a BRST charge which is constructed after introducing the ghost degrees of freedom $(\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}})$ and their conjugate momenta $(w_\alpha, \hat{w}_{\hat{\alpha}})$ subject to *pure spinor constraints*

$$\lambda\gamma^a\lambda = 0 \quad , \quad \hat{\lambda}\gamma^a\hat{\lambda} = 0 \quad (1.4.3)$$

These constraints leaves eleven complex degrees of freedom in ten dimensions. On top of many advantages of the pure spinor formalism we are interested to a particular application which is the construction of a topological theory based on superstring on $AdS_5 \times S^5$ [21] or as it was shown in [10] on any supercoset background which admits a particular $\mathbb{Z}_4$ automorphism.

Superstring on $AdS_5 \times S^5$ has a limit which is the $d = 10$ flat space limit in which the radius of the $AdS_5 \times S^5$ goes to infinity or in a covariant way one can do the limit by

[3] We put always $\alpha' = 1$.

rescaling the metric $g_{\alpha\hat{\beta}} = \eta_{\alpha\hat{\beta}}$ and $g_{[ab][cd]} = \eta_{[ab][cd]}$ [4]as

$$g_{\alpha\hat{\beta}} = R\eta_{\alpha\hat{\beta}} \quad , \quad g_{[ab][cd]} = R^2\eta_{[ab][cd]} \tag{1.4.4}$$

also the structure constants of the $PSU(2,2|4)$ supergroup should be rescaled consistently. In the $AdS_5 \times S^5$ background, the torsions $T_{\alpha a}{}^{\hat{\beta}}$ and $T_{\alpha\beta}{}^{b}$ are proportional to the corresponding structure constants and they satisfy $T_{\alpha a}{}^{\hat{\beta}}\eta_{\beta\hat{\beta}} = T_{\alpha\beta}{}^{b}\eta_{ab}$ but in the flat background $T_{\alpha a}{}^{\hat{\beta}} = 0$. The rescaling of the structure constants and the metric used by Berkovits implies the following rescaling of the torsion

$$\frac{T_{\alpha\beta}{}^{b}\eta_{ab}}{T_{\alpha a}{}^{\hat{\beta}}\eta_{\beta\hat{\beta}}} = R \tag{1.4.5}$$

where in the $R \to \infty$ limit implies $T_{\alpha a}{}^{\hat{\beta}} = 0$ as it should be for the flat background. Also the left invariant one-forms of the supergroup simplify after this rescaling as follows

$$J^a = \partial x^a + \theta\gamma^a\partial\theta + \hat{\theta}\gamma^a\partial\hat{\theta} \quad , \quad J^\alpha = \partial\theta^\alpha \quad , \quad J^{\hat{\alpha}} = \partial\hat{\theta}^{\hat{\alpha}} \quad , \quad J^{[ab]} = 0 \tag{1.4.6}$$

which after putting back into the pure spinor action (1.4.2) gives the pure spinor superstring action for flat background[5]

$$S = \frac{1}{R^2}\int\left[\frac{1}{2}\eta_{ab}\partial x^a\bar{\partial}x^b - p_\alpha\bar{\partial}\theta^\alpha - \hat{p}_{\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} + w_\alpha\bar{\partial}\lambda^\alpha + \hat{w}_{\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}}\right] \tag{1.4.7}$$

On top of this limit, Berkovits showed [21] that there is another rescaling sending $R \to 0$ which giving a topological theory corresponding to highly curved $AdS_5 \times S^5$ superstring. To go to this limit we have to apply another rescaling for the metric and the structure constants as follows

$$g_{ab} = R^{-1}\eta_{ab} \quad , \quad g_{[ab][cd]} = R^{-1}\eta_{[ab][cd]} \tag{1.4.8}$$

When $R \to 0$, the structure constants $f^A_{\alpha\beta} \to 0$. This implies that the 32 fermionic isometries become Abelian. In this limit the supercoset splits into its bosonic and fermionic parts. The Bosonic part composed of the coset $\frac{SU(2,2)}{SO(4,1)}$ which parametrizes AdS_5 and the coset $\frac{SU(4)}{SO(5)}$ which parametrizes the sphere S^5, The fermionic part of the supercoset is parametrized with two matrices θ^{α^+} and $\hat{\theta}^{\alpha^-}$ where $\theta^{\alpha^+} = \theta^\alpha + i\hat{\theta}^{\hat{\alpha}}$ and

[4]The generators of the $PSU(2,2|4)$ are represented by $(T_{[ab]}, T_a, T_\alpha, T_{\hat{\alpha}})$ where $T_{[ab]}$ are the generators of the $SO(5) \times SO(4,1)$ group and T_α, $T_{\hat{\alpha}}$ are the fermionic generators and T_a are the translation generators for $a, b = 0$ to 9 and $\alpha, \hat{\alpha} = 1$ to 16.

[5]Note that eventhough it looks like the flat superstring action, it is different from the flat superstring theory quantum mechanically since it does not exist any continous deformation of the $PSU(2,2|4)$ supergroup into the super Poincaré group since they have different number of generators.

$\hat{\theta}^{\alpha^-} = \theta^{\alpha} - i\hat{\theta}^{\hat{\alpha}}$ are the upper-right and lower-left blocks of the supergroup elements as we will discuss it better later.

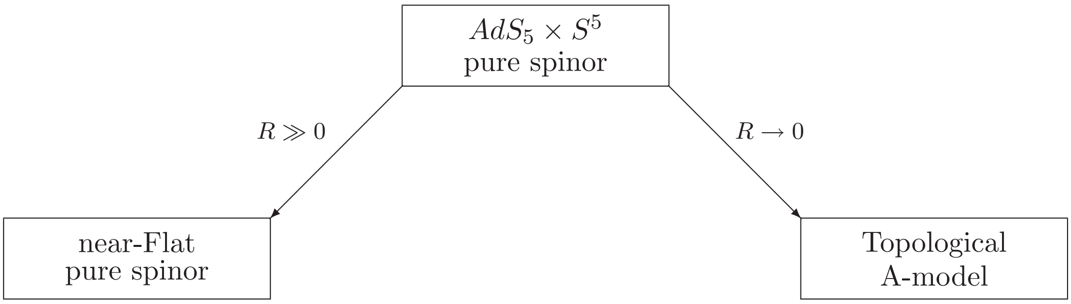

FIGURE 1.11: Taking a particular rescaling, Berkovits showed that the zero radius limit of the pure spinor superstring theory on $AdS_5 \times S^5$ gives a topological A-model.

This rescaling (1.4.8) implies ${T_{\alpha\beta}}^a = 0$ but since the usual construction of supergravity backgrounds assumes that ${T_{\alpha\beta}}^a = \gamma^a_{\alpha\beta}$ so this limit does not correspond to a standard supergravity background as we expected since we are exploring the strongly coupled regime of superstring theory.

It was shown by Berkovits that after taking the limit the superstring action can be written as a $\mathcal{N} = 2$ worldsheet action in terms of fermionic superfields Θ^{α^+} and $\hat{\Theta}^{\alpha^-}$ as follows

$$\begin{aligned} S &= \frac{1}{R^2}\int d^2z d^4\kappa \left[\hat{\Theta}\Theta - \frac{1}{2}\hat{\Theta}\Theta\hat{\Theta}\Theta + \frac{1}{3}\hat{\Theta}\Theta\hat{\Theta}\Theta\hat{\Theta}\Theta + \cdots\right] \\ &= \int d^2z d^4\kappa \mathrm{Tr}\left[\log\left(1 + \hat{\Theta}\Theta\right)\right] \end{aligned} \tag{1.4.9}$$

where κ's are the $\mathcal{N} = 2$ worldsheet supersymmetry coordinates and fermionic chiral superfields Θ^{α^+} and $\hat{\Theta}^{\alpha^-}$ can be expanded as follows

$$\begin{aligned} \Theta^{\alpha^+} &= \theta^{\alpha^+} + \kappa_+ Z^{\alpha^+} + \kappa_- \hat{Y}^{\alpha^+} + \kappa_+\kappa_- f^{\alpha^+} \\ \hat{\Theta}^{\alpha^-} &= \hat{\theta}^{\alpha^-} + \bar{\kappa}_+ \hat{Z}^{\alpha^-} + \bar{\kappa}_- Y^{\alpha^-} + \bar{\kappa}_+\bar{\kappa}_- \hat{f}^{\alpha^-} \end{aligned} \tag{1.4.10}$$

where θ and $\hat{\theta}$ are the fermionic degrees of freedom of the superspace and Z and Y fields are bosonic twisted variables constructed from the bosonic degrees of freedom of the superspace and the pure spinors and their conjugate momenta. The fields f^{α^+} and $\hat{f}^{\alpha^-}$ are auxiliary fields. The fact that there are 11 complex independent pure spinor degrees of freedom is a very crucial fact which make it possible to construct such unconstrained twisted-like variables as follows

$$\begin{aligned} Z^{\alpha^+} &= f^{\alpha^+}_{m\beta^+} H^m \lambda^{\beta^+} \quad , \quad \hat{Z}^{\alpha^-} = f^{\alpha^-}_{m\beta^-} H^m \hat{\lambda}^{\beta^-} \\ \hat{Y}^{\alpha^+} &= f^{\alpha^+}_{m\beta^+} H^m w^{\beta^+} \quad , \quad Y^{\alpha^-} = f^{\alpha^-}_{m\beta^-} H^m \hat{w}^{\beta^-} \end{aligned} \tag{1.4.11}$$

where $H^m = (H^a, H^{a'})$ are the bosonic cosets and $f^{\alpha^+}_{m\beta^+}$ and $f^{\alpha^-}_{m\beta^-}$ are structure constants of the supergroup.

The matching of the bosonic and fermionic degrees of freedom appears as follows

$$\begin{cases} 10\ \mathbb{R} & x^m \\ 11\ \mathbb{C} & \lambda^\alpha, \hat{\lambda}^{\hat{\alpha}} \\ 16\ \mathbb{C} & \theta^\alpha, \hat{\theta}^{\hat{\alpha}} \end{cases} \longleftrightarrow \begin{cases} 16\ \mathbb{C} & Z^{\alpha^+},\ \hat{Z}^{\alpha^-} \\ 16\ \mathbb{C} & \theta^{\alpha^+}, \hat{\theta}^{\alpha^-} \end{cases} \tag{1.4.12}$$

where $m = 1$ to 9 and $\alpha, \hat{\alpha} = 1$ to 16.

In fact the chiral fermionic superfield Θ belongs to the fermionic supercoset

$$\Theta \in \frac{PSU(2,2|4)}{SU(4) \times SU(2,2)} \tag{1.4.13}$$

for the case of $AdS_5 \times S^5$.

The action (1.4.9) is an A-model topological action which after expanding its Kähler potential and integrating over the auxiliary fields we get the following action

$$S = \int d^2z \left[\eta_{\alpha\hat{\beta}} J^\alpha \bar{J}^{\hat{\beta}} - \eta_{\alpha^-\alpha^+} Y^{\alpha^-} \bar{\nabla} Z^{\alpha^+} + \eta_{\alpha^-\alpha^+} \hat{Y}^{\alpha^+} \nabla \hat{Z}^{\alpha^-} - [Y, Z]_m [\hat{Y}, \hat{Z}]^m \right] \tag{1.4.14}$$

which as we will see later in this thesis and in [22] it has the right structure of an A-twisted topological action.

Actually it was shown in [23] for $AdS_5 \times S^5$ and in [10] for a general maximally supersymmetric supercoset background which admits a particular $\mathbb{Z}_4$ automorphism that this A-model topological action is related to the pure spinor action with a BRST trivial term and one can split the pure string action as follows

$$S_{\text{pure spinor}} = S_{\text{A-model}} + Q\,\Omega \tag{1.4.15}$$

where Q is the pure spinor BRST charge. Note that the topological charge of the A-model action is different from Q and so this decomposition gives the ability to explore the BPS sector of the superstring theory using the A-model action.

A symmetry argument were used in [23] and [10] to find this decomposition. Being an element of the supercoset $PSU(2,2|4)/SU(4) \times SU(2,2)$, the A-model action preserves all the $PSU(2,2|4)$ supergroup but it has a 'bonus' $U(1)$ symmetry which does not exist in $AdS_5 \times S^5$ background. It appears that after expanding the A-model action (1.4.14) in terms of the pure spinors and left-invariant Cartan one-forms, the fermionic currents appear as the following kinetic term in the $\mathbb{Z}_4$ grading language

$$\frac{1}{2}J_1\bar{J}_3 - \frac{1}{2}\bar{J}_1 J_3 \tag{1.4.16}$$

It is clear that there is a $U(1)$ symmetry which switches the fermionic currents $J_1 \leftrightarrow J_3$ and under this $U(1)$ the currents $J_1 + iJ_3$ and $J_1 - iJ_3$ have opposite charges.

However, this symmetry does not exist in the $AdS_5 \times S^5$ action because the corresponding fermionic currents appear as follows

$$\frac{3}{4}J_1\bar{J}_3 - \frac{1}{4}\bar{J}_1 J_3 \tag{1.4.17}$$

In order to relate the pure spinor action and the A-model action, we add a BRST-trivial term $S_{bonus} = Q\,X$ to the pure spinor action which forces the pure spinor action to preserve this $U(1)$ symmetry

$$S'_{\text{pure spinor}} \longrightarrow S_{\text{pure spinor}} + S_{bonus} \tag{1.4.18}$$

Then using BRST transformation rules of the pure spinor formalism, one can check that $S'_{\text{pure spinor}} - S_{\text{A-model}}$ is a BRST trivial term which proves (1.4.15).

Using this topological decomposition, one can use similar techniques in the AdS/CFT duality as in the topological CS/conifold duality which we discussed before to explore AdS/CFT duality from a worldsheet point of view. The first tool which we need is a gauged linear sigma model for the topological $AdS_5 \times S^5$ superstring theory. Based on an observation in [24, 25] it was shown in [23, 10] that there is a gauged linear sigma model corresponding to the nonlinear topological A-model defined on the Grassmannian $\frac{PSU(2,2|4)}{SU(4)\times SU(2,2)}$ by gauging the theory under a $U(4)$ gauge group after introducing the following vector, chiral and antichiral superfields in the $N = (2,2)$ superspace

$$V_S^R(z,\bar{z},\kappa^+,\kappa^-,\bar{\kappa}^+,\bar{\kappa}^-) \quad , \quad \Phi_\Sigma^R(z,\bar{z},\kappa^+,\kappa^-) \quad , \quad \bar{\Phi}_R^\Sigma(z,\bar{z},\bar{\kappa}^+,\bar{\kappa}^-) \tag{1.4.19}$$

where the indices R, S correspond to the gauge indices of the $U(4)$ group and the index $\Sigma = (I, A)$ is a global $SU(4) \times SU(2,2)$ index where A is a $SU(2,2)$ and I is a $SU(4)$ index. The difference here with the CS/conifold duality is that the chiral superfields can be eihter fermionic or bosonic according to their global index as follows

$$\begin{cases} \Phi_I^R & \longrightarrow \quad \text{bosonic} \\ \Phi_A^R & \longrightarrow \quad \text{fermionic} \end{cases} \tag{1.4.20}$$

They are related to the non-linear sigma model fields Θ as follows

$$\Theta^A_J = \Phi^A_R(\Phi^J_R)^{-1} \quad , \quad \hat{\Theta}^J_A = \bar{\Phi}^R_A(\bar{\Phi}^R_J)^{-1} \tag{1.4.21}$$

where we used the matrix notation of Θ superfields.

The gauged linear sigma model can be written as follows

$$S = \int d^2z \int d^4\kappa \, [\bar{\Phi}^S_\Sigma(e^V)^R_S \Phi^\Sigma_R - tV^R_R] \tag{1.4.22}$$

which after integrating out the gauge field V it produces the non-linear action (1.4.9) in the infra-red.

Having the gauged linear sigma-model one can study study the theory in the $t \to 0$ limit in which we have not access from the non-linear sigma model. The Vacua of the gauged linear sigma model of $AdS_5 \times S^5$ and also for more general backgrounds including $\frac{OSp(6|4)}{SO(6)\times Sp(4)}$ was studied in [23] and [10] and it was shown that in $t \to 0$ on top of the Higgs branch corresponding to the phase in which the gauge field is integrated out, we have the emergence of a Coulomb branch in which the gauge field is dynamical and produces holes on the worldsheet. In principle, one can use the same technique of Ooguri and Vafa for the case of CS/conifold [17] to prove the AdS/CFT duality.

The branch geometry in the case of $\frac{PSU(2,2|4)}{SO(5)\times SO(4,1)}$ and $\frac{OSp(6|4)}{SO(6)\times Sp(4)}$ supercosets were studied in [23, 10] and it was shown that it produces the following geometries as the target space of the topological model

$$\begin{cases} \frac{PSU(2,2|4)}{SO(5)\times SO(4,1)} \implies \left(\mathbb{CP}^{(3|4)}\right)^4 //S_4 \\ \\ \frac{OSp(6|4)}{SO(6)\times Sp(4)} \implies \left(\mathbb{S}^{(5|4)}\right)^3 //S_3 \times \mathbb{Z}_2 \end{cases} \tag{1.4.23}$$

where the double slash is used to stress the fact that it is not the S_4 or $S_3 \times \mathbb{Z}_2$ orbifold but it is the set of maximal S_4 or $S_3 \times \mathbb{Z}_2$ orbits, meaning that the orbits whose elements are not fixed by any non-trivial subgroup of S_4 or $S_3 \times \mathbb{Z}_2$.

Also it was shown that the open sector of this topological A-model corresponds to the free $\mathcal{N} = 4$ SYM theory. It would be very interesting to try to accomplish the same analysis of the worldsheet CS/conifold duality for AdS/CFT using this construction. For example, one has to see if it is possible to produce the 't Hooft expansion from the worldsheet picture and the emergence of the holes on the closed string worldsheet by checking if any hole produce a factor of t in path integral. A possible prescription will be given later together some other applications of the topological construction which will be addressed some of them here in this thesis.

Conformal exactness of the background: One can check easily by computing the Ricci scalar of the topological A-model action (1.4.9) as the one-loop conformal anomaly that the A-model does not have conformal anomaly at one loop.

$$R = \log\det(\partial\bar{\partial}(1+\Theta\hat{\Theta})) = 0 \tag{1.4.24}$$

Being an $N = 2$ supersymmetric theory, the conformal anomaly and ghost anomaly belong to the same superfield and the vanishing of the conformal anomaly at one loop ensures its vanishing at any loop. So the A-model action is an exact conformal theory. Since the A-model and the pure spinor action are related through a BRST trivial term, It ensures also the exactness of any superstring background which admits such a topological decomposition.

An exact check of AdS/CFT duality: One can use the A-model to do some exact check in the $\mathcal{N} = 4/AdS_5 \times S^5$ correspondence. As an example we will explore an exact check between some D-brane amplitudes and the exact result for the expectation value of circular Wilson loops in gauge theory side which their exact value was computed to be given by Gaussian matrix model. In order to check the duality we use the topological A-model to compute the correponding dual objects of the circular Wilson loop in the superstring side. We first consider the A-model for closed strings on $AdS_5 \times S^5$ and its gauged linear σ- model in the limit of small Fayet-Illiopoulos which corresponds to the large curvature regime. In this limit as we saw the model reduces to the invariant quotient $\left(\hat{\mathbb{CP}}^{(3|4)}\right)^4 //S_4$. Its maximal orbit under the cyclic permutation is isomorphic to a single copy of the superprojective space $\hat{\mathbb{CP}}^{(3|4)}$. We can consider then a mirror of such a geometry in the form of a deformed fermionic conifold, dubbed superconifold [26]. This is actually the cotangent bundle over $S^{(1|2)}$ and we get the closed B-model with N-units of flux along the $S^{(1|2)}$. We can follow then the theory in a dual formulation after a geometric transition analogous to the Dijkgraaf-Vafa one [14, 27]. In the superconifold case one calculates the minimal resolution as the resolved superconifold over $\hat{\mathbb{CP}}^{(0|1)} = \left\{\mathbb{C}^{(1|1)} \setminus (0,0)\right\}/\mathbb{C}^*$. This will be discussed in detail in this thesis. Here the dual theory is that of N D-branes wrapping the base manifold and therefore the theory is described by the dimensional reduction of the holomorphic $U(N)$ Chern-Simons theory to the branes [15]. This results to be the hermitian $N \times N$ Gaussian matrix model similar to the purely bosonic case [14].

In order to generate gauge invariant observables in the topological string, let us return to the gauged linear σ-model of $AdS_5 \times S^5$ and look for the A-branes there. These are wrapped around special Lagrangian's of the supercoset and their geometry is dictated

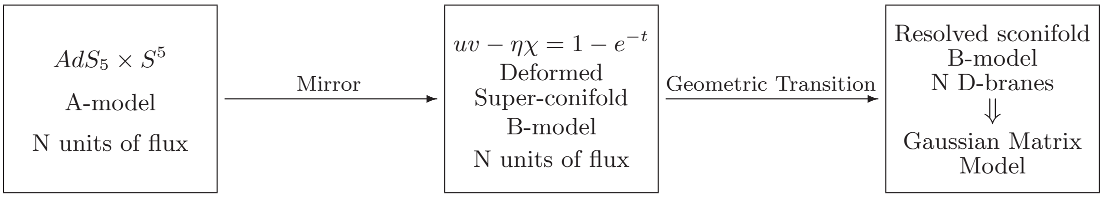

FIGURE 1.12: The duality chain: the mirror symmetry maps to the B-model on the deformed superconifold and the geometric transition to the resolved one corresponding to the Gaussian matrix model.

by the possible supersymmetric boundary conditions. On top of the AdS_4 branes considered in [23], there are also other possibilities among which we choose that of the real supercoset $OSp(4^*|4)/SO^*(4) \times USp(4)$. As such, the choice of Dirichlet boundary conditions for open strings on such a submanifold breaks the original $U(2,2|4)$ isometry to $Osp(4^*|4)$. Notice that this is the same symmetry breaking which corresponds to placing 1/2-BPS circular Wilson loops in Minkowski space as in [28]. These D-branes can be shown to correspond to $D5$-branes wrapping $AdS_2 \times S^4$ geometries [29]. As such, these states realize the Wilson loops in an alternative way – suitable for the large curvature regime – compared to the string world-sheet with boundary condition along the loop on the AdS_5 boundary. Analogue constructions were actually elaborated in [30] (and references therein) from the point of view of the effective Dirac-Born-Infeld theory, while it is obtained here directly for the microscopic theory.

We have then to follow these D-branes along the duality map described above (see Figure 1.13). Actually the Lagrangian cycle is mapped to a transverse non-compact holomorphic cycle in the superconifold geometry. Therefore, the computation of the corresponding topological string amplitude gets mapped to the computation in the Gaussian matrix model of the corresponding observables. The relevant observables are obtained by integrating over the open strings with mixed boundary conditions similar to [31].

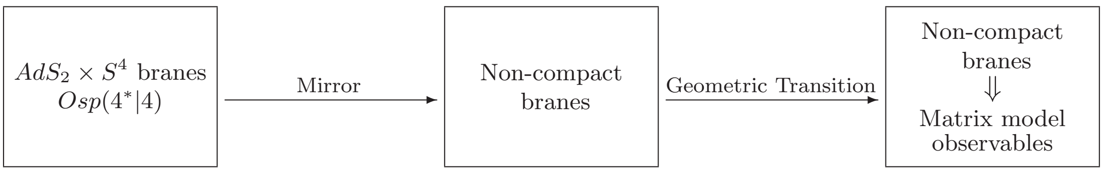

FIGURE 1.13: The duality chain for the $AdS_2 \times S^4$-branes. Following them we obtain Gaussian matrix model amplitudes.

This construction therefore leads to express the topological string amplitudes for the A-model on the fermionic quotient with $AdS_2 \times S^4$-branes boundary conditions as correlators of Wilson loops in the Gaussian matrix model. As such, these amplitudes should obey the holomorphic anomaly equations of BCOV [32]. It has been actually proved that it is indeed the case in [33]. This not only applies to the construction in [14], but more in general also to the ones given in [34]. This consistency check strongly supports the validity of our derivation. This problem will be addressed more in detail in chapter (4).

Multiloop amplitudes computation: Having a topological A-model theory as the theory which is exploring the BPS sector of the superstring theory, there is the possibility to use the well-defined multiloop amplitude computations of the topological theory in order to give a multiloop amplitude prescription for the pure spinor superstring on $AdS_5 \times S^5$ since we can construct a general n-point topological amplitude for genus $g > 1$ as follows

$$A^g_{i_1,\ldots i_n} = \int_{\Sigma_g} d^2 z \int_{\mathcal{M}_g} \left\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \prod_{k=1}^{3g-3} (b, \mu^k)(\bar{b}, \bar{\mu}^k) \right\rangle \tag{1.4.25}$$

. where $\mathcal{O}_i$'s are observables and b and μ are the b-field and the Beltrami differential respectively. We have to integrate over the moduli space of Riemann surfaces with genus g, which is denoted as $\mathcal{M}_g$.

Here in this work using the topological A-model of the $AdS_5 \times S^5$ we try to give a prescription for worldsheets with $g = 0$, $g = 1$ and $g > 1$ by introducing appropriate picture changing operators to soak up the sigma-model fields. The main difference which arise from the usual topological A-model sigma models is that here we have a fermionic target space and the zero modes of the fields are different from the bosonic case.

This is one of the important possible applications of this construction in which we will try to address a little bit in chapter (7). A more clear construction is very appealing since a multiloop amplitude computation of the superstring on Ramond-Ramond backgrounds is not known.

1.5 G/G principal chiral model and its deformation

Another step towards a better understanding of the zero radius limit of pure spinor superstring on $AdS_5 \times S^5$ was started by Berkovits in [35] by showing that the topological A-model corresponding to this limit can be obtained from a gauge-fixed version of the G/G principal chiral model with $G = PSU(2, 2|4)$ with the following action

$$S_{G/G} = \mathrm{Str} \int d^2 z \, \eta_{AB} (J - A)^A (\bar{J} - \bar{A})^B \tag{1.5.1}$$

where J and $\bar{J}$ are the left and right components of the one-form $J = g^{-1}dg$ with respect to the worldsheet derivatives ∂ and $\bar{\partial}$ and g is a group element of the $PSU(2,2|4)$ supergroup and $(A, \bar{A})$ are $PSU(2,2|4)$ gauge groups on the worldsheet.

Using the same G/G topological action it was proposed in [36] that there exist another gauge fixing in which we get a topological theory which its supersymmetric charge is equal to the pure spinor BRST charge of the $AdS_5 \times S^5$ superstring

$$Q_{\text{topological}} = Q_{\text{pure spinor BRST}} \tag{1.5.2}$$

Here in this thesis, based on a work in progress [22], the Batalin-Vilkovisky version of the G/G principal chiral model will be constructed as follows

$$S_{\text{BV}} = S_{G/G} + \int d^2z \Big[A^*_A (dC + [A, C])^A + \bar{A}^*_A (\bar{d}C + [\bar{A}, C]) + g^*_A C^A - \frac{1}{2} C^*_A \, [C, C]^A \Big] \tag{1.5.3}$$

where C^A are the ghosts[6] and the fields with the star are the antifields corresponding to the fields. These are new ingredients of the BV formalism which make it possible to have a covariant description of the gauge-fixing based on the BRST quantization as we will explain more clearly later.

Having the BV action of the G/G model, we can define different gauge fixing fermions Ψ which are projecting the field space into a Lagrangian submanifold Σ in the field-antifield space as the gauge fixing orbit by putting the following constraint on a particular antifield Φ^*,

$$\Sigma : \qquad \Phi^* = \frac{\partial \Psi}{\partial \Phi} \tag{1.5.4}$$

Here we take two different gauge fermions ψ_1 and ψ_2 which their resulting gauge fixing produce two different topological theories, one is the A-model topological action which we got from the decomposition of $AdS_5 \times S^5$ pure spinor action with a topological supercharge which is different from the pure spinor BRST charge. On the other hand, following the second gauge fixing we will show that it will produce a topological action with a topological charge equal to the BRST charge of the pure spinor superstring theory on $AdS_5 \times S^5$ as the gauge fixed G/G model. The fact that the topological A-model and the second topological model are the gauge fixed version of the same BV theory means that they are describing the same physics. Since the second theory is exploring the cohomology of the $Q_{\text{pure spinor}}$ this means that all the physical states of the superstring action might be encoded in the cohomology of the A-model action too. One can in principle study the cohomology of the superstring using the topological A-model after

[6] The index $A = \{[ab], a.\alpha, \hat{\alpha}\}$ is a $PSU(2,2|4)$ index.

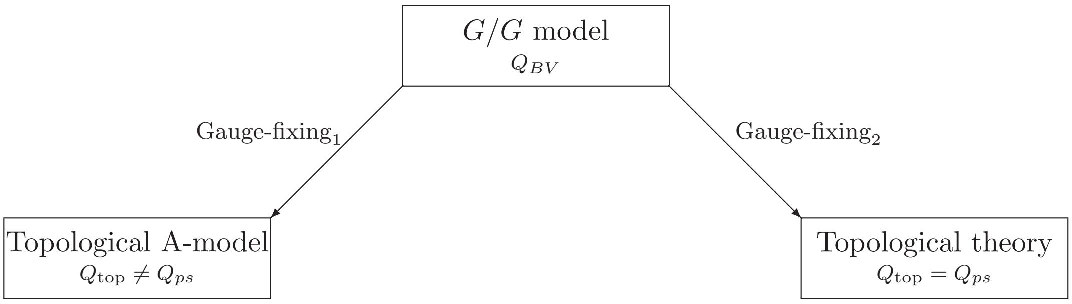

FIGURE 1.14: Upon different gauge-fixings, the G/G principal chiral model gives either an A-model topological action or another topological action with the same supersymmetry charge as the BRST charge of pure spinor superstring.

passing through the bridge sketched in figure (1.14) from the A-model to the other topological theory whose supersymmetric charge is equal to the pure spinor BRST charge. In principle we can use the cohomological techniques of the BV formalism including the homological perturbation theory to analyze the cohomology of the superstring theory in this topological language. The BRST charge splits according to a grading which is named as the antighost number as follows

$$Q = \delta + d + \text{"}more\ \ terms\text{"} \tag{1.5.5}$$

where δ is the *Koszul-Tata* differential with antighost number -1 and d is the differential with antighost number 0 which their form we will give later. It happens that for G/G model the expansion stops after the second term. One can use this decomposition of the BRST charge to simplify the computations of the cohomology using homological perturbation theory.

Another important issue is to study whether there exist possible consistent deformations for the G/G principal chiral model. For a general BV action we can expand around a particular classical solution of the master equation as follows

$$S = S^{(0)} + rS^{(1)} + r^2 S^{(2)} + \cdots \tag{1.5.6}$$

where r is the perturbative parameter.

In order to find the first order deformation of the BV action we have to study the relative cohomology group $H(\delta|d)$ at ghost number zero as we will see later. A particular deformation of the BV action is obtained after solving a set of descent equations. We will see later that for the case of the BV action of the G/G principal chiral model with $G \in PSU(2,2|4)$, there is a particular deformation which after the gauge fixing upon the second gauge fixing in figure (1.14), it produces exactly the $AdS_5 \times S^5$ pure spinor superstring action and the action of the topological theory deforms as follows

$$S_{\text{topological}} \longrightarrow S_{\text{topological}} + R^2 S_{\text{pure spinor}} \tag{1.5.7}$$

where R is the radius of the AdS space. This deformation corresponds to turning on the radius modulus of the AdS by inserting an integrated vertex operator which is equal to the superstring action. In this way the topological theory which is the zero radius limit of superstring extended with a perturbative term to the small radius limit of the superstring theory on $AdS_5 \times S^5$. This small radius superstring action with the BRST charge of the pure spinor superstring theory is dual to the perturbative limit of the gauge theory side which we know it produces the 't Hooft expansion. We will discuss the possible modification of the large N gauge/string duality from the original 't Hooft idea which we explained before by adding these vertex operators to the worldsheet. Getting a more clear understanding of this picture might help to give an exact worldsheet proof of the AdS/CFT duality.

1.6 Résumé of the thesis

In this thesis we will try to build a new way of studying and understanding of the most important examples of the gauge/string duality which is AdS/CFT correspondence. In particular we will focus more on the particular example of $\mathcal{N} = 4$ SYM/$AdS_5 \times S^5$ duality and try to find a way of extracting more information about this duality from a worldsheet perspective. We will see that using pure spinor formalism we can trade the AdS/CFT duality into a duality which is similar to a topological open/closed duality. One can then use the better known understanding of the topological string theory to get deeper information about the superstring on the backgrounds which admit this construction like $AdS_5 \times S^5$ and also to study or even prove the AdS/CFT duality using the techniques of the topological open/closed duality.

In chapter (2) we will review the pure spinor formalism of string theory. First we discuss the Green-Schwarz formalism of superstrings and then we introduce the pure spinor formalism for flat and curved backgrounds. In particular we will explore the action for the supercoset backgrounds which admit a $\mathbb{Z}_4$ automorphism like $AdS_5 \times S^5$ we will see that we can write the action in a simple form in terms of the left-invariant Cartan one-forms of the super-isometry group of the background.

In chapter (3) we will show how the pure spinor action on the maximally supersymmetric backgrounds which admit a particular $\mathbb{Z}_4$ automorphism decomposes into a topological A-model action plus a BRST trivial term. We use a simple symmetry argument which was explained briefly in the introduction to find the map from the A-model action to the superstring action. At the end it will be shown how this topological decomposition can be used to proof the conformal exactness of these backgrounds.

In chapter (4), we start using the topological A-model as the theory which explores the BPS sector of the superstring action and as the zero radius limit of the superstring action towards studying the gauge/string duality. First we will show that we can write a gauged linear sigma model based on the non-linear topological A-model action. This makes it possible to study the physics in the limit $R \to 0$ of the superstring side corresponding to the perturbative regime in the gauge theory. We will study the vacuo of the gauged linear sigma model and explore its branch geometry. We will see that the emergence of the Coulomb branch will produce holes with Dirichlet boundary conditions on the closed string worldsheet which makes it possible to give a worldsheet interpretation to the AdS/CFT duality as a closed/open duality. We then explore the open string sector of the theory both for $AdS_5 \times S^5$ and the $\tilde{AdS}_4 \times \mathbb{CP}^3$ supercosets[7]. At the end we will do an exact check on AdS/CFT duality by showing that an exact result in the gauge theory side, namely the expectation value of some circular Wilson loops, can be computed exactly from the scattering amplitudes of some D-branes in the superstring action using the topological A-model.

Chapter (5) is an introductory to the Batalin-Vilkovisky quantization of gauge theories. After introducing the Faddeev-Popov and BRST quantization procedures, we will give a detailed introduction to BV antifield formalism. At the end we will see how one can find a consistent deformation of a particular BV action which does not spoil the gauge structure of the theory.

In chapter (6), we will write a BV action for the G/G principal chiral model with $G \in PSU(2,2|4)$. Then we will study two different gauge fixings of G/G and will see that one of them produces the topological A-model action and the other one gives a topological action whose supersymmetry charge is equal to the pure spinor BRST charge. We will then explore the possible deformation of this G/G BV action and we will see there exist a particular deformation which in the second gauge fixing produces the pure spinor superstring action on $AdS_5 \times S^5$. In this way we will produce the pure spinor superstring as a deformation over a topological theory with the perturbation parameter which is equal to the radius of the AdS space. We will give an argument to sketch the AdS/CFT duality from the worldsheet point of view similar to the analysis of Ooguri and Vafa on topological conifold duality.

[7]From now on, whenever we refer to $\tilde{AdS}_4 \times \mathbb{CP}^3$ we mean the supercoset $\frac{OSp(6|4)}{SO(1,3)\times Sp(4)}$ as a subspace of the full superspace of the $AdS_4 \times \mathbb{CP}^3$. We specify the difference with the tilde which we put on AdS_4

In chapter (7), we will show that using A-model topological action we can give a possible prescription to compute multiloop amplitudes in the pure spinor superstring on $AdS_5 \times S^5$.

In the last chapter we will briefly discuss the possible open questions and problems on this construction and also the possible extensions and applications of the following work will be reviewed.

The contents of chapter (3) is based on [10], chapter (4) and part of chapter (2) are based on the paper [37] and chapters (6) and (7) are based on [22].

Chapter 2

Pure spinor formalism of superstring theory

Before 2001 there were mainly two standard formalisms to describe the superstring theory, the Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS) formalisms. Although the RNS formalism has a manifest N=1 worldsheet supersymmetry, it lacks manifest target-space supersymmetry makes a lot of problems for some applications. For example, to compute amplitudes involving more than four external fermions, it is almost impossible to compute in a Lorentz-covariant manner because of the complexity of picture-changing operators and the bosonization procedure [38]. The other important problem of this formalism is that there is not a well defined description for superstring theory in the presence of Ramond-Ramond fluxes using this formalism. This is usually the case for most of the backgrounds which we are discussing in the gauge/string duality like superstring $AdS_5 \times S^5$ with N units of flux as the dual of $\mathcal{N} = 4$ $SU(N)$ SYM in four dimensions.

On the other hand, in the GS formalism the target-space supersymmetry is manifest, but we do not know how to realize the worldsheet supersymmetry. This prevents to have a quantization of the theory except in light-cone gauge. Although we can use the light-cone gauge to determine the physical spectrum of the theory, it is difficult to use it to compute scattering amplitudes because of the lack of manifest Lorentz covariance and the need to introduce interaction-point operators and contact terms. For these reasons, only four-point tree and one-loop amplitudes have been explicitly computed using the GS formalism [39]. Furthermore, the necessity of choosing light-cone gauge means that quantization is only possible in those backgrounds which allow a light-cone gauge choice.

As will be discussed in this chapter, a new formalism for the superstring was proposed by Berkovits [20] which combines the advantages of the RNS and GS formalisms.

This new formalism was inspired, among other things, by the so called superembedding description of superparticles, superstrings and superbranes. The superembedding approach was first proposed in [40, 41]. In these works, on examples of superparticles in $D = 3, 4, 6$ and 10 dimensions, it was shown that the kappa-symmetry is a somewhat weird realization of the conventional N-extended worldsheet supersymmetry $N = 8$ in $D = 10$ dimensions. As a result corresponding irreducible set of Lorentz-covariant first-class fermionic constraints was obtained by projecting the fermionic covariant momenta D_α along commuting spinor variables λ^α which are superpartners of θ. Berkovits was the first who noticed these results and generalized them to the heterotic superstring in [42]. It was shown later in [43], that the pure spinor condition is part of a so called superembedding condition which is the key fundamental condition of the superembedding formalism [44]. Later, Berkovits proposed the pure spinor formalism as a covariant approach of quantizing superstring [20]. The superembedding origin of the pure spinor formalism for the heterotic string was demonstrated in [45].

In pure spinor formalism, the worldsheet action is quadratic in a flat background so quantization is as easy as in the RNS formalism. Since we have D=10 super-Poincaré covariance, we can compute covariant tree-amplitudes and also we can quantize Ramond-Ramnod backgrounds. It will also be shown how this approach can be used to quantize the superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux in particular.

We will see in the next chapters how the pure spinor formalism enables us to give a topological realization of the superstring theory on some RR background which makes it possible to give a worldsheet approach to study gauge/string duality and Maldacena's conjecture. This power originates from the new degrees of freedom which is used in this formalism, namely the pure spinors.

2.1 Green-Schwartz formalism of superstrings

The classical type II Green Schwarz (GS) superstring [46] describes the embedding of a string worldsheet into a target type II superspace with coordinates

$$x^M \equiv (x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}) \tag{2.1.1}$$

The bosonic coordinates x^m locally parametrize the ten-dimensional spacetime manifold, while the fermionic coordinates θ^α and $\hat{\theta}^{\hat{\alpha}}$ have dimension of Majorana Weyl spinors which is 16 real for the $10-$dimensional critical superstring theory. In the flat case, where one can identify the manifold with its tangent space, the θ's are the spinors. In the context of a curved supermanifold that we will treat later on, this will not be the case a priori.

The difference between type IIA and IIB arises from the presence or absence of the left-right chiral symmetry meaning to require either $\hat{\theta}^{\hat{\alpha}} \equiv \hat{\theta}_\alpha$ for type IIA or $\hat{\theta}^{\hat{\alpha}} \equiv \hat{\theta}^\alpha$for type IIB.

In Green-Schwarz formalism we can manifestly observe the $N = 2$ supersymmetry. The target space supersymmetry in flat space-time can be written in this way

$$\begin{aligned} \delta\theta^\alpha &= \epsilon^\alpha \quad , \quad \delta\hat{\theta}^{\hat{\alpha}} = \hat{\epsilon}^{\hat{\alpha}} \\ \delta x^m &= \epsilon\gamma^m\theta + \hat{\epsilon}\gamma^m\hat{\theta} \end{aligned} \tag{2.1.2}$$

where γ's are the ten-dimensional gamma matrices. In order to write a supersymmetric sigma-model on the superspace we can write the superspace vielbein one-forms of the target space

$$E^A \equiv dx^M E_M^A = \left(dx^m + d\theta\gamma^m\theta + d\hat{\theta}\gamma^m\hat{\theta}, d\theta, d\hat{\theta}\right) \tag{2.1.3}$$

Their pullback on the worldsheet can be written as follows

$$\Pi_z^A \equiv \partial x^M E_M^A \quad , \quad \Pi_{\bar{z}}^A \equiv \bar{\partial} x^M E_M^A \tag{2.1.4}$$

whose bosonic components are known as *supersymmetric momentum* can be written as follows

$$\Pi_z^m \equiv \partial x^m + \partial\theta\gamma^m\theta + \partial\hat{\theta}\gamma^m\hat{\theta} \quad , \quad \Pi_{\bar{z}}^m \equiv \bar{\partial} x^m + \bar{\partial}\theta\gamma^m\theta + \bar{\partial}\hat{\theta}\gamma^m\hat{\theta} \tag{2.1.5}$$

The Green-Schwarz superstring action can be constructed from the square of this supersymmetric momentum as its kinetic term plus a Wess-Zumino term which is quadratic in the derivatives of the fields and is necessary to maintain the conformal invariance of the worldsheet theory. It also establishes a fermionic gauge symmetry named as $\kappa-$*symmetry.*

The GS action gets the following form in conformal gauge

$$S_{GS} = \frac{1}{2}\int_{\Sigma} d^2 z \eta_{mn} \Pi_z^m \Pi_{\bar{z}}^n + \int_M \Omega_3 \tag{2.1.6}$$

where Σ is the worldsheet and M is a 3-manifold which $\partial M = \Sigma$. The second term is the WZ term which to construct it we can write the general $N = 2$ supersymmetry and $SO(9,1)$ invariant closed three-form in flat space which is

$$\Omega_3 = f_{MNP} E^M \wedge E^N \wedge E^P \tag{2.1.7}$$

with some constant f_{MNP} and one-forms E^M which were defined in (2.1.3).

This three-form is closed and Lorentz invariant only for the following choice as it was shown in [46]

$$\Omega_3 = E^m \wedge d\hat{\theta}\gamma_m \wedge d\theta \tag{2.1.8}$$

It appears that not only Ω_3 is closed ($d\Omega_3 = 0$) but also it is exact, namely

$$\Omega_3 = d\Omega_2 \tag{2.1.9}$$

where

$$\Omega_2 = -\frac{1}{2}\Pi_z^m \left(\theta\gamma_m\bar{\partial}\theta - \hat{\theta}\gamma_m\bar{\partial}\hat{\theta}\right) + \frac{1}{2}(\theta\gamma^m\partial\theta)(\hat{\theta}\gamma_m\bar{\partial}\hat{\theta}) - (z \leftrightarrow \bar{z}) \tag{2.1.10}$$

This should be integrated over the worldsheet to generate the WZ term.

The GS action (2.1.6) is covariant and spacetime supersymmetric. It is the difference with respect to RNS formalism in which we have manifest worldsheet supersymmetry since we build the model using worldsheet fermions but then we lost the covariance and the spacetime supersymmetry. However the problem of GS formalism is related to its quantization which does not let to find a covariant way of quantization with the standard BRST quantization. The reason for this is related to the existence of second class constraints which we will discuss here.

Let $p_{z\alpha}$ be the conjugate momentum of θ^α which can be written in terms of other phase space variables as $p_\alpha = \delta\mathcal{L}/\delta\partial\theta^\alpha$. The Dirac constraints corresponding to this relation is given by the field d_α with the following explicit form

$$d_\alpha \equiv p_{z\alpha} - (\gamma_m\theta)_\alpha \left(\partial x^m - \frac{1}{2}\theta\gamma^m\partial\theta - \frac{1}{2}\hat{\theta}\gamma^m\partial\hat{\theta}\right) \tag{2.1.11}$$

One can use the canonical commutation relations to find the Poisson bracket

$$\{d_\alpha, d_\beta\} = 2\gamma^m_{\alpha\beta}\Pi_m \tag{2.1.12}$$

Because of the Virasoro constraint

$$T = -\frac{1}{2}\Pi^m\Pi_m = 0 \tag{2.1.13}$$

the non-closure of the Poisson algebra (2.1.12), because of (2.1.13), implies that eight of d_α constraints are first-class and the other eight are second-class constraints. The eight first class constraints correspond to the fermionic $\kappa-$symmetry. Since the anti-commutator of the second class constraints d_α is proportional to an operators Π instead of a number, we can not use the standard Dirac quantization since it needs inserting some operators. Except in a special frame in which the right hand side of (2.1.12) becomes a constant like in the light-cone gauge, we can not easily quantize the covariant GS superstring action.

In order to get ride of this problem, Siegel [47] made the open algebra of (2.1.12) to be a closed algebra by adding the generators which arise via the Poisson bracket. This leads to the following centrally extended but closed algebra

$$\begin{aligned}
\{d_{z\alpha}, \Pi_m\} &= 2(\gamma_m)_{\alpha\beta}\partial\theta^\beta & (2.1.14)\\
\{\Pi_{zm}, \Pi_{zn}\} &= \eta_{mn} & (2.1.15)\\
\left\{d_{z\alpha}, \partial\theta^\beta\right\} &= \delta^\beta_\alpha & (2.1.16)
\end{aligned}$$

The important observation is that this closed algebra can be obtained from a free-field action

$$\begin{aligned}
S_{free} &= \int d^2z[\frac{1}{2}\partial x^m\eta_{mn}\bar{\partial}x^n + \bar{\partial}\theta^\alpha p_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}}\hat{p}_{\bar{z}\hat{\alpha}}] & (2.1.17)\\
&= \int d^2z[\underbrace{\frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \mathcal{L}_{WZ}}_{\mathcal{L}_{GS}} + \bar{\partial}\theta^\alpha d_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}}\hat{d}_{\bar{z}\hat{\alpha}}]
\end{aligned}$$

which coincide with the GS action (2.1.6) for $d_\alpha = \hat{d}_{\hat{\alpha}} = 0$. This reformulation does not remove the mixed first-second class constraints of d_α but it gives a simple free-field action which make it trivial to compute the following OPE's

$$
\begin{aligned}
x^m(y)x^n(z) &\rightarrow -2\eta^{mn}\log|y-z| \quad , \quad p_\alpha(y)d^\beta(z) \rightarrow \delta^\beta_\alpha(y-z)^{-1} \\
d_\alpha(y)d_\beta(z) &\rightarrow -\frac{1}{(y-z)}\gamma^m_{\alpha\beta}\Pi_m(z) \quad , \quad d_\alpha(y)\Pi_m(z) \rightarrow \frac{1}{(y-z)}\gamma^m_{\alpha\beta}\partial\theta^\beta(z)
\end{aligned} \tag{2.1.18}
$$

Using these OPE's we can compute the conformal anomaly of the model and also the ghost degree needed to cancel this anomaly. The pair $(p_\alpha, \theta^\alpha)$ contributes a factor of -32 to the conformal anomaly, there is another 10 coming from ten x^m's. The total conformal anomaly appears to be -22 which should be canceled with an appropriate ghost sector which we will discuss in the next section the one suggested by Berkovits [48]. Furthermore, looking into the Lorentz currents of the Siegel approach which are $M_{mn} = p\gamma_{mn}\theta$, we can compute their OPE and it appears that they produce a double pole with a numerator which is +4. In order that this matches with the RNS formalism which is 1, the ghost sector should have Lorentz currents which produce a double pole with a factor of -3 in the numerator. We will see that the pure spinors of Berkovits are in fact the appropriate ghost sector which satisfies the above requirements.

One can also write the generalization of the flat space-time GS action to a curved background as follows

$$
S = \frac{1}{4\pi\alpha'}\int d^2z\,(G_{MN} + B_{MN})\,\partial x^M \partial x^N \tag{2.1.19}
$$

where G_{MN} and B_{MN} corresponds to the background superfields.

Since we are interested to study superstring theory on maximally supersymmetric backgrounds like $AdS_5 \times S^5$ and $\tilde{AdS_4} \times \mathbb{CP}^3$[1] and because it was shown that the theory on these backgrounds can be formulated as sigma models on supercosets [3, 7, 8, 9, 10], here we give the prescription to write the superstring theory on such supercoset backgrounds.

2.1.1 Structure of $AdS_5 \times S^5$ and $\tilde{AdS_4} \times \mathbb{CP}^3$ supercosets

Two examples we are considering here are superstring theory on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ which as we said before the superstring on these backgrounds was shown to be written completely in terms of the following supercosets

$$
AdS_5 \times S^5 \longrightarrow \frac{PSU(2,2|4)}{SO(5)\times SO(4,1)} \tag{2.1.20}
$$

$$
\tilde{AdS_4} \times \mathbb{CP}^3 \longrightarrow \frac{Osp(6|4)}{SO(6)\times Sp(4)} \tag{2.1.21}
$$

[1] By $\tilde{AdS_4} \times \mathbb{CP}^3$ we mean a subsector of the $AdS_4 \times \mathbb{CP}^3$ superstring which captured by the supercoset $\frac{Osp(6|4)}{SO(6)\times Sp(4)}$ sigma model. Even though it is not the full superspace of the superstring on $AdS_4 \times \mathbb{CP}^3$, it is a particular subsector of it which deserves studying.

which we investigate their Lie algebra separately here.

The algebra of the $psu(2,2|4)$ supergroup is the algebra of 8×8 matrices with bosonic diagonal blocks and fermionic off-diagonal blocks as follows

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \tag{2.1.22}$$

which A, B, X and Y are 4×4 matrices which satisfy $\mathrm{tr}A = \mathrm{tr}B = 0$. The operation of transposition for supermatrices is defined as follows

$$M^t = \begin{pmatrix} A^t & Y^t \\ -X^t & B^t \end{pmatrix} \tag{2.1.23}$$

that is compatible with the *supertrace* which is defined as $\mathrm{Str}M = \mathrm{tr}A - \mathrm{tr}B$ and satisfies

$$Str(MN) = Str(NM) \tag{2.1.24}$$

The super antihermiticity $M^\dagger = -M$ for the $psu(2,2|4)$ implies [49, 50]

$$M^\dagger \equiv \Sigma M \Sigma^{-1} = \begin{pmatrix} \sigma A^\dagger \sigma & -i\sigma Y^\dagger \\ -iX^\dagger \sigma & B^\dagger \end{pmatrix} = -M \tag{2.1.25}$$

where Σ is a block diagonal matrix defined as

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & i\mathbb{1} \end{pmatrix} \tag{2.1.26}$$

satisfying $\Sigma^2 = \mathbb{1}$ and $\Sigma^\dagger = \Sigma$. The condition (2.1.25) implies

$$A = -\sigma A^\dagger \sigma \quad , \quad B = -B^\dagger \quad , \quad X = i\sigma Y^\dagger \tag{2.1.27}$$

Choosing

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.1.28}$$

implies that $A \in su(2,2)$ and $B \in su(4)$.

For the case of $Osp(6|4)$ supergroup the bosonic blocks A and B are 4×4 and 6×6 matrices and the fermionic blocks X and Y are 4×6 and 6×4 matrices respectively. The supermatrices instead of anti-hermiticity condition (2.1.25) should satisfy the following orthosymplecticity condition [50]

$$\begin{aligned} \Omega M \Omega^{-1} &= -M^t \\ H M H^{-1} &= -M^{-1} \end{aligned} \tag{2.1.29}$$

where

$$\Omega = \begin{pmatrix} \Omega_{(4\times4)} & 0 \\ 0 & \Omega_{(6\times6)} \end{pmatrix} \quad , \quad H = \begin{pmatrix} H_{(4\times4)} & 0 \\ 0 & H_{(6\times6)} \end{pmatrix} \tag{2.1.30}$$

where the block matrices satisfy

$$\Omega^2_{(4)} = -1\!\!1_{(4)} \quad , \quad \Omega^t_{(4)} = -\Omega_{(4)} \quad , \quad \Omega^t_{(6)} = \Omega_{(6)} \tag{2.1.31}$$

Using (2.1.30), (2.1.29) implies the following conditions on the blocks of M

$$A^t \Omega_{(4)} + \Omega_{(4)} A = 0 \quad , \quad B^t H_{(6)} + H_{(6)} B = 0 \tag{2.1.32}$$

then (2.1.31) means that $A \in sp(4)$ and $B \in o(6)$.

It appears that this supergroups admit a particular $\mathbb{Z}_4$ automorphism which will become very important in our construction of the action. This $\mathbb{Z}_4$ automorphism is generated by the following conjugation

$$M \to \Omega(M) \equiv \Omega^{-1} M \Omega \tag{2.1.33}$$

where Ω is a matrix with eigenvalues equal to i^k for $k = 0$ to 3 and $\Omega^4(M) = M$. For the case of $psu(2,2|4)$ it can be realized by the following matrix presentation [51]

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \to \Omega(M) \equiv \begin{pmatrix} JA^tJ & -JY^tJ \\ JX^tJ & JB^tJ \end{pmatrix} \tag{2.1.34}$$

where

$$J = \begin{pmatrix} 0 & -1\!\!1_{2\times2} \\ 1\!\!1_{2\times2} & 0 \end{pmatrix} \tag{2.1.35}$$

This is a Lie algebra automorphism which is compatible with the antihermiticity condition of $psu(2,2|4)$.

The action of this $\mathbb{Z}_4$ automorphism decomposes the Lie algebra $\mathcal{G}$ of the supergroups $PSU(2,2|4)$ and $Osp(6|4)$ as follows

$$\mathcal{G} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \tag{2.1.36}$$

where each subspace $\mathcal{H}_k$ is the eigenspace of the operator Ω with eigenvalue i^k. The subspaces $\mathcal{H}_0$ and $\mathcal{H}_2$ are bosonic while $\mathcal{H}_1$ and $\mathcal{H}_3$ are fermionic which are related through hermitian conjugation for the $PSU(2,2|4)$ group.

Since $\mathbb{Z}_4$ is an automorphism of the Lie algebra, the decomposition (2.1.36) satisfies

$$[\mathcal{H}_m, \mathcal{H}_n] \subset \mathcal{H}_{m+n} \text{ (mod 4)} \tag{2.1.37}$$

Also the bilinear form is $\mathbb{Z}_4$ invariant an so we have

$$\langle \mathcal{H}_m, \mathcal{H}_n \rangle = 0 \quad \text{unless} \quad n + m = 0 \text{ (mode 4)} \tag{2.1.38}$$

the more illustrative realization of this $\mathbb{Z}_4$ will be given later separately for the generators of supercosets we will study.

2.1.2 Sigma model action for supercosets with $\mathbb{Z}_4$ automorphism

Consider a supercoset G/H which admits the $\mathbb{Z}_4$ discussed before with $\mathcal{G}$ as the Lie algebra of G and $H \in \mathcal{H}_0$. As in the flat space case, the sigma model for such backgrounds is constructed from two terms, the kinetic term S_{kin} and the Wess-Zumino term S_{WZ} as

$$S = \int_\Sigma d^2z L_{kin} + \int_M d^3z \mathcal{W} \tag{2.1.39}$$

where $\partial M = \Sigma$ and $d\mathcal{W} = 0$.

In order to satisfy G invariance, both L_{kin} and $\mathcal{W}$ should be constructed in terms of the left-invariant Cartan one-forms $J = g^{-1}dg$ valued in the Lie algebra $\mathcal{G}$ for $g \in G$ and can be expanded in the supergroup basis $J = J^A T_A$ where T_A are the generators of the supergroup G. This comes from the fact that under the action of an arbitrary element of the isometry supergroup G, the Vielbein transforms as tangent vectors of the stability group H

$$J(y)g = J(y')h \tag{2.1.40}$$

for $g \in G$ and $h \in H$. So any invariant of the stability group H constructed in terms of J will be automatically invariant under the full group G.

To write the kinetic term for the supercoset, we can gauge the kinetic term of the G sigma model with a gauge field A which takes value in $\mathcal{H}_0$ as follows

$$S_{kin} = R^2 \int d^2z Str(J - A)^2 \tag{2.1.41}$$

which is the most general quadratic action can be written in terms of the left-invariant one-forms and the gauge field A to be invariant under G.

Let us decompose J in two pieces $J_0 \in \mathcal{H}_0$ and $J_G \in (\mathcal{G} \backslash \mathcal{H}_0)$. We can take the following metric on $\mathcal{G}$ as

$$\langle A, B \rangle = Str(AB) \tag{2.1.42}$$

where together with (2.1.38) implies

$$Str(J_G J_0) = 0 \tag{2.1.43}$$

This means that under the gauge transformation $g(z) \to g(z)h(z)$ the currents J_G and J_0 transform as

$$J_G \to h^{-1} J_G h \quad , \quad J_0 \to h^{-1} J_0 h + h^{-1} dh \tag{2.1.44}$$

where the term $h^{-1}dh$ can be canceled with a gauge transformation of the gauge field $A \to h^{-1}Ah + h^{-1}dh$ and so the action is invariant under this gauge transformation.

Because of (2.1.43) the action decomposes into two terms, one with the Lagrangian J_G^2 and the other as $(J_0 - A)^2$. Integrating out the gauge field will cancel the second term and we end with the following action for the supercoset

$$S_{kin}^{G/H} = R^2 \int d^2z Str(J_G^2) \tag{2.1.45}$$

The action (2.1.45) is not conformal in general and its conformal anomaly is proportional to the Ricci tensor. So in order to make conformal theory as a string theory one should add a Wess-Zumino term which compensates this conformal anomaly.

The general structure of the Wess-Zumino term for a supergroup is constructed from a closed 3-form $\mathcal{W}$ which can be written in terms of the left-invariant one-forms of the supergroup as follows

$$\mathcal{W} = Str(J \wedge [J \wedge J]) = f_{ABC} J^A \wedge J^B \wedge J^C \tag{2.1.46}$$

where $f_{ABC} = g_{AD} f^D_{BC}$ are the structure constants of the supergroup and g_{AB} is a G invariant bilinear form on the supergroup manifold.

In order to construct an exact three-form one can use the Maurer-Cartan equations of the supergroup

$$dJ^A = -\frac{1}{2} f^A_{BC} J^B \wedge J^C \tag{2.1.47}$$

The possibility of constructing the exact three-form which arises from a two-form is very specially depend on the fact that the supergroup has the $\mathbb{Z}_4$ automorphism. This action has been written in [3, 52] for the case of $AdS_5 \times S^5$ but it can be generalized to any maximally supersymmetric supergroup manifold which admits the $\mathbb{Z}_4$ automorphism. The reasoning is simply related to the fact that if we denote the projections of the Cartan one-form J on the subspaces $\mathcal{H}_k$ as

$$J_k = J|_{\mathcal{H}_k} \tag{2.1.48}$$

and because of (2.1.37), using the fact that $\mathcal{W}$ should be invariant under $H \in \mathcal{H}_0$, then it comes out that the only three-form $\mathcal{W}$ which can be composed of the currents $J_G \in \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ and stay in $\mathcal{H}_0$ is the following one

$$\mathcal{W} = Str(\alpha J_1 \wedge J_1 \wedge J_2 + \beta J_3 \wedge J_3 \wedge J_2) \tag{2.1.49}$$

because any combination of three form $J_m \wedge J_n \wedge J_p \in \mathcal{H}_{m+n+p \text{ (mod) } 4}$ and one can see that $(1,1,2)$ and $(3,3,2)$ is the only way of partitioning a number into 0 (mod) 4 out of the numbers $\{1,2,3\}$.

Using the Maurer-Cartan equations and the fact that $\mathcal{W}$ should be closed

$$d\mathcal{W} = 0 \tag{2.1.50}$$

forces the coefficients α and β to satisfy $\alpha = -\beta$. Then we can check easily that $\mathcal{W}$ is also a d-exact three-form

$$\mathcal{W} = d\, Str(J_1 \wedge J_3) \equiv d\, \mathcal{W}_{(2)} \tag{2.1.51}$$

The sigma model action (2.1.39) can be written as follows

$$S = \frac{R^2}{2} \int d^2 z\, Str(J_2 \bar{J}_2 + (1+k) J_1 \bar{J}_3 + (1-k) \bar{J}_1 J_3) \tag{2.1.52}$$

where k is the overall factor of the Wess-Zumino term and is determined after considering the conformal invariance of the background. Using background field method, the one-loop conformal anomaly of the model was computed in [51] to be zero for $k = \pm 1/2$ where the difference between and plus signs is just to change the role of the J_1 and J_3 as covariant holomorphic or antiholomorphic currents. So we can take $k = 1/2$ and write the sigma-model action for these backgrounds as follows

$$S = R^2 \int d^2z\, Str(\frac{1}{2}J_2\bar{J}_2 + \frac{3}{4}J_1\bar{J}_3 + \frac{1}{4}\bar{J}_1 J_3) \tag{2.1.53}$$

2.2 Pure spinor action for flat background

As we have seen the main problem of the GS formalism were related to the fact that we could not find a covariant way of quantizing the theory because of the presence of second class constraints. Berkovits [48] implemented the constraints cohomologically with a BRST operator disregarding its non-closure. He proposed the following left and right-moving BRST operators which are constructed simply from the structure $Q =$ (ghost $\times$ constraint) as follows

$$Q = \oint \lambda^\alpha d_\alpha \quad , \quad \hat{Q} = \oint \hat{\lambda}^{\hat{\alpha}} d_{\hat{\alpha}} \tag{2.2.1}$$

Also the non-closure of the second class constraints d_α implies a lack of the nilpotency of the BRST operator, we can maintain the nilpotency

$$Q^2 = \frac{1}{2}\{Q, Q\} = -\frac{1}{2}\oint dz(\lambda\gamma^m\lambda)\Pi_m \tag{2.2.2}$$

by putting the following constraints on the bosonic ghosts

$$\lambda^\alpha \gamma^m_{\alpha\beta}\lambda^\beta = 0 \quad , \quad \hat{\lambda}^{\hat{\alpha}}\gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\hat{\beta}} = 0 \tag{2.2.3}$$

These are named as *pure spinors* which are bosonic ghost degrees of freedom. They are ten independent constraints and so the pure spinor has eleven complex degrees of freedom which exactly compensate the -22 conformal anomaly of the GS action. We can also introduce the conjugate momenta w_α and $\hat{w}_{\hat{\alpha}}$ corresponding to these ghosts and add the ghost sector to the action (2.1.17) to get the following worldsheet action for the flat background

$$S = \int d^2z[\frac{1}{2}\partial x^m \eta_{mn}\bar{\partial}x^n + \bar{\partial}\theta^\alpha p_\alpha + \partial\hat{\theta}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}} - w_\alpha\bar{\partial}\lambda^\alpha - w_{\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}}] \tag{2.2.4}$$

The pure spinor constraints (2.2.3) prevents a direct computation of the OPE's between λ and w but we can solve the pure spinor constraint by Wick-rotating the Lorentz group $SO(10)$ into a $SU(5) \times U(1)$ subgroup [48]. The sixteen complex components of the λ^α splits into $(\lambda^+, \lambda_{ab}, \lambda^a)$ for $a, b = 1$ to 5 which transform as $(1_{\frac{5}{2}}, \bar{10}_{\frac{1}{2}}, 5_{-\frac{3}{2}})$ of the $SU(5) \times U(1)$ group. We can solve the pure spinor constraints (2.2.3) with eleven complex degree of freedom γ and u_{ab} transforming as 1_5 and $\bar{10}_{-2}$ respectively as follows

$$\lambda^+ = \gamma \quad , \quad \lambda_{ab} = \gamma u_{ab} \quad , \quad \lambda^a = -\frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de} \tag{2.2.5}$$

which satisfies the pure spinor constraints.

Using the solution (2.2.5) we can compute the OPE's of the theory. Although the OPE's of the unconstrained variables are not manifestly Lorentz-covariant, all the OPE's and other computations involving the pure spinors λ can be written in a manifestly Lorentz-covariant way.

Because of the pure spinor constraints, the conjugate momenta of the pure spinors contain the gauge transformation

$$\delta w_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha \quad , \quad \delta \hat{w}_{\hat{\alpha}} = \hat{\Lambda}^m (\gamma_m \hat{\lambda})_{\hat{\alpha}} \tag{2.2.6}$$

Because of this gauge symmetry, five out of the sixteen components of w_α can be gauged away and since we want to preserve the Lorentz invariance, the momenta w_α and $\hat{w}_{\hat{\alpha}}$ can only appear in the gauge-invariant combinations which are the Lorentz current and the ghost current defined as follows

$$\begin{aligned} N_{mn} &= \frac{1}{2} w_\alpha (\gamma_{mn})_\beta{}^\alpha \lambda^\beta \quad , \quad J = w_\alpha \lambda^\alpha \\ \hat{N}_{mn} &= \frac{1}{2} \hat{w}_{\hat{\alpha}} (\gamma_{mn})_{\hat{\beta}}{}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \quad , \quad \hat{J} = \hat{w}_{\hat{\alpha}} \hat{\lambda}^{\hat{\alpha}} \end{aligned} \tag{2.2.7}$$

Using the solution (2.2.5), one can find the following Lorentz-covariant OPE's

$$\begin{aligned} N_{mn}(y)\lambda^\alpha(z) &\to \frac{(\gamma_{mn}\lambda)^\alpha}{2(y-z)} \quad , \quad J(y)\lambda^\alpha(z) \to \frac{\lambda^\alpha}{(y-z)} \\ N^{kl}(y)N^{mn}(z) &\to \frac{\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}}{(y-z)} - \frac{3\eta^{n[k}\eta^{l]m}}{(y-z)^2} \\ J(y)N_{mn}(z) &\to \text{regular} \quad , \quad J(y)J(z) \to \frac{-4}{(y-z)^2} \\ N_{mn}(y)T(z) &\to \frac{N_{mn}}{(y-z)^2} \quad , \quad J(y)T(z) \to \frac{J(z)}{(y-z)^2} - \frac{-8}{(y-z)^3} \end{aligned} \tag{2.2.8}$$

where

$$T = -\frac{1}{2}\partial x^m \bar{\partial} x_m - p_\alpha \partial\theta^\alpha + w_\alpha \lambda^\alpha \tag{2.2.9}$$

is the left-moving stress-energy tensor of the pure spinor flat superstring theory. We can see from these OPE's (2.2.8) that levels for the Lorentz and ghost currents are -3 and -4 respectively and the ghost anomaly is -8. From the first OPE it is obvious that the pure spinor λ transforms as a spinor under the action of the Lorentz current. We can also see that the stress-energy tensor (2.2.9) has vanishing central charge because the (10-32) contribution of $(x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}})$ cancels with +22 from the eleven $(\lambda^\alpha, w_\alpha)$ ghost variables.

In order to define physical states of the theory we naturally use the BRST operator

(2.2.1) and the physical vertex operators would be the ghost-number one elements of the cohomology of Q. Note that we assign ghost number one to λ and zero to all the rest. In this way easily we can show that the most general unintegrated ghost number 1 and (1,1) operators are the following ones for open and closed superstring theory

$$V_{open} = \lambda^\alpha A_\alpha(x,\theta) \tag{2.2.10}$$
$$V_{closed} = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x,\theta,\hat{\theta}) \tag{2.2.11}$$

where $A_\alpha(x,\theta)$ and $A_{\alpha\hat{\alpha}}(x,\theta,\hat{\theta})$ are spinor and bispinor superfields depending only on the worldsheet zero modes of the x^m, θ^α and $\hat{\theta}^{\hat{\theta}}$.

These operators are in the cohomology of the BRST operator satisfying $QV_{open} = QV_{closed} = 0$ as their equation of motion and also they transform as $\delta V = Q\Lambda$ under gauge symmetry. This implies for the open vertex operator using the fact that $\lambda^\alpha\lambda^\beta \propto (\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$, the following equation

$$QV_{open} = \oint dz \lambda^\alpha d_\alpha \lambda^\beta A_\beta = \lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0 \tag{2.2.12}$$

where the OPE

$$d_\alpha(y) f(x(z),\theta(z)) \to \frac{D_\alpha f}{(y-z)} \tag{2.2.13}$$

is used in which the covariant derivative is defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta\gamma^m_{\alpha\beta}\partial_m \tag{2.2.14}$$

The gauge variation reads as $\delta A_\alpha = D_\alpha\Lambda$ for the spinor superfield.

The relation $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$ implies

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m \tag{2.2.15}$$

for some vector superfield A_m which transforms under the gauge symmetry as $\delta A_m = \partial_m\Lambda$.

Using (2.2.15), and the gauge invariance of the spinor and vector superfields, it was shown [53] that there is the following solution for A_α and A_m

$$A_\alpha(x,\theta) = e^{ikx}(\frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)_\alpha + \cdots) \tag{2.2.16}$$
$$A_m(x,\theta) = e^{ikx}(a_m + (\xi\gamma^m\theta) + \cdots) \tag{2.2.17}$$

where $k^2 = k^m a_m = k^m(\gamma_m\xi)_m = 0$. These, as the ghost-number one elements of the BRST operators, were shown to coincide exactly with the super-Maxwell multiplet and so this BRST operator produces correctly the massless spectrum of the open superstring.

One can write the integrated massless vertex operators as follows

$$V_{open} = \int dz[\partial\theta^\alpha A_\alpha + \Pi^m A_m + "more"] \tag{2.2.18}$$

where the "*more*" terms are needed to get BRST invariance.

Similarly we can investigate the massless closed string vertex operator by imposing $QV_{closed} = \bar{Q}V_{closed} = 0$ which implies the following equations

$$\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_{\beta\hat{\gamma}} = 0 \quad , \quad \gamma^{\hat{\alpha}\hat{\beta}}_{mnpqr} D_{\hat{\alpha}} A_{\gamma\hat{\beta}} = 0 \tag{2.2.19}$$

whose solutions can be written as follows

$$A_{n\hat{\gamma}} = -\frac{1}{8} D_\alpha \gamma^{\alpha\beta}_n A_{\beta\hat{\gamma}} \tag{2.2.20}$$

$$A_{\gamma n} = -\frac{1}{8} \hat{D}_{\hat{\alpha}} \gamma^{\hat{\alpha}\hat{\beta}}_n A_{\gamma\hat{\beta}} \tag{2.2.21}$$

$$A_{mn} = \frac{1}{64} D_\alpha \hat{D}_{\hat{\gamma}} \gamma^{\alpha\beta}_m \gamma^{\hat{\gamma}\hat{\delta}}_n A_{\beta\hat{\delta}} \tag{2.2.22}$$

where the covariant derivatives are defined as follows

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta \gamma^m_{\alpha\beta}\partial_m \quad , \quad D_{\hat{\alpha}} = \frac{\partial}{\partial\hat{\theta}^{\hat{\alpha}}} + \frac{1}{2}\hat{\theta}^{\hat{\beta}} \gamma^m_{\hat{\alpha}\hat{\beta}}\partial_m \tag{2.2.23}$$

which are the $N = 2$ $D = 10$ supersymmetric derivatives. These solutions are the linearized $N = 2$ supergravity equations of motion which is written in terms of superfield $A_{\alpha\hat{\beta}}$ and the linearized supergravity connections in terms of $A_{\alpha\hat{\beta}}$ [54].

In order to construct the sigma model for the type II superstring, it is useful to construct the integrated vertex operator as follows

$$V_{closed} = \int d^2z[\partial\theta^\alpha \bar{\partial}\hat{\theta}^{\hat{\beta}} A_{\alpha\hat{\beta}} + \partial\theta^\alpha \bar{\Pi}^m A_{\alpha m} + \Pi^m \bar{\partial}\hat{\theta}^{\hat{\alpha}} A_{m\hat{\alpha}} + \Pi^m \bar{\Pi}^n A_{mn} + "more"] \tag{2.2.24}$$

this is similar to the Green-Schwarz type II superstring vertex operator except the "*more*" term which is necessary for this to be BRST invariant. The superstring closed vertex operator (2.2.24) can be seen as the square of the open vertex operator (2.2.18) for the left and right movers because the theory is holomorphic-antiholomorphic for flat background. This holomorphicity does not exist for more general backgrounds like

$AdS_5 \times S^5$ and the closed supergravity vertex operator can not be seen as the product of the open string left and right vertex operators [36].

2.3 Pure spinor formalism for curved backgrounds

The pure spinor formalism for a general curved background which is constructed either by adding the closed superstring vertex operator (2.2.24) to the flat background and then covariantizing with respect to the $N = 2$ $D = 10$ super-reparametrization invariance or by writing the most general action constructed from the worldsheet variables and retain its worldsheet conformal invariance as follows

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\int d^2z[\frac{1}{2}(G_{MN}(Z)+B_{MN}(Z))\partial Z^M\bar{\partial}Z^N \\ &+ P^{\alpha\hat{\beta}}(Z)d_\alpha\hat{d}_{\hat{\beta}} + E^\alpha_M(Z)d_\alpha\bar{\partial}Z^M + E^{\alpha}_M(Z)\hat{d}_{\hat{\alpha}}\partial Z^M \\ &+ +\Omega_{M\alpha}{}^\beta(Z)\lambda^\alpha w_\beta\bar{\partial}Z^M + \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}(Z)\hat{\lambda}^{\hat{\alpha}}\hat{w}_{\hat{\alpha}}\partial Z^M \\ &+ C^{\beta\hat{\gamma}}_\alpha(Z)\lambda^\alpha w_\beta\hat{d}_{\hat{\gamma}} + \hat{C}^{\hat{\beta}\gamma}_{\hat{\alpha}}(Z)\hat{\lambda}^{\hat{\alpha}}\hat{w}_{\hat{\beta}}d_\gamma + S^{\beta\hat{\delta}}_{\alpha\hat{\gamma}}(Z)\lambda^\alpha w_\beta\hat{\lambda}^{\hat{\gamma}}\hat{w}_{\hat{\delta}} \\ &+ \frac{1}{2}\alpha'\Phi(Z)r] + S_\lambda + S_{\hat{\lambda}} \end{aligned} \tag{2.3.1}$$

where S_λ and $S_{\hat{\lambda}}$ are the action for pure spinors, r is the worldsheet curvature, and $(G_{MN} = \eta_{cd}E^c_M E^d_N,\ B_{MN},\ E^\alpha_M,\ E^{\hat{\alpha}}_M,\ \Omega_{M\alpha}{}^\beta,\ \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}},\ P^{\alpha\hat{\beta}},\ C^{\beta\hat{\gamma}}_\alpha,\ \hat{C}^{\hat{\beta}\gamma}_{\hat{\alpha}},\ S^{\beta\hat{\delta}}_{\alpha\hat{\gamma}}, \Phi)$ are the background superfields. Putting their value which comes from their supergravity equations of motion, will give the pure spinor action for that curved background. The superfields E^A_M, B_{MN} and Φ are the supervielbein, two-form potential and dilaton superfields, $P^{\alpha\hat{\beta}}$ is the superfield whose lowest components are the Type II Ramond-Ramond field strengths and the fields $C^{\beta\hat{\gamma}}_\alpha$ and $\hat{C}^{\hat{\beta}\gamma}_{\hat{\alpha}}$ are related to $N = 2$ $D = 10$ dilatino and gravitino field strengths. It was shown in [54] that one can get all the supergravity constraints from the type II pure spinor superstring integrability conditions.

We saw in the previous section how to construct Green-Schwarz action on the backgrounds which are constructed on supercosets like $AdS_5 \times S^5$ and $\tilde{AdS}_4 \times \mathbb{CP}^3$ whose Lie algebra admits a particular $\mathbb{Z}_4$ automorphism. Here we use that construction and write the pure spinor action for those backgrounds, but we have to add the corresponding ghost sector to the action in order to make it BRST invariant.

The worldsheet action here has also the pure spinor ghost sector on top of the matter sector of the Green-Schwarz sigma model action. The matter fields are written in terms of the left-invariant Cartan one-forms $J = g^{-1}dg$ where $g : \Sigma \to G$ is the map from the worldsheet to the superisometry group. The currents J and $\bar{J}$ decomposes exactly like (2.1.48) into graded components.

The pure spinor ghosts and their conjugate momenta can also be expanded into the generators of the supergroup and also according to the $\mathbb{Z}_4$ grading [55, 56] as follows

$$\lambda = \lambda^\alpha T_\alpha \quad , \quad w = w_\alpha \eta^{\alpha\hat{\alpha}} T_{\hat{\alpha}} \quad , \quad \hat{\lambda} = \hat{\lambda}^{\hat{\alpha}} T_{\hat{\alpha}} \quad , \quad \hat{w} = \hat{w}_{\hat{\alpha}} \eta^{\alpha\hat{\alpha}} T_\alpha \tag{2.3.2}$$

where T_α and $T_{\hat{\alpha}}$ are the generators of G taking their value in the Lie algebras $\mathcal{H}_1$ and $\mathcal{H}_3$ respectively and $\eta^{\alpha\hat{\alpha}}$ is the inverse of the Cartan metric.

The Lorentz currents are defined to be

$$N = -\{\lambda, w\} \quad , \quad \hat{N} = -\{\hat{\lambda}, \hat{w}\} \tag{2.3.3}$$

which take value in $\mathcal{H}_0$.

The sigma model, being a G/H supercoset sigma model is invariant under the global transformation $\delta g = \Sigma g$ where Σ is a constant element of G. The left invariant currents J and $\bar{J}$ are invariant by their definition. The sigma model is also invariant under the following gauge transformation

$$\delta_g J = \partial\Omega + [J, \Omega] \quad , \quad \delta_g \lambda = [\lambda, \Omega] \quad , \quad \delta_g w = [w, \Omega] \tag{2.3.4}$$

where $\Omega \in \mathcal{H}_0$ which is an element of the Lorentz group.

one can check that the most general action invariant under the local symmetry (2.3.32) and the global symmetry G has the following form [55]

$$S = R^2 \int d^2z Str(\alpha J_2 \bar{J}_2 + \beta J_1 \bar{J}_3 + \gamma J_3 \bar{J}_1 + w\bar{\partial}\lambda + \hat{w}\partial\hat{\lambda} + N\bar{J}_0 + \hat{N} J_0 + aN\hat{N}) \tag{2.3.5}$$

As we saw before, the pure spinor theory is invariant under a BRST symmetry which is generated with the following BRST operator written in the $\mathbb{Z}_4$ grading

$$Q = \oint dz\, Str(\lambda J_3) + \oint d\bar{z}\, Str(\hat{\lambda}\bar{J}_1) \tag{2.3.6}$$

This generates the following BRST transformations. Note that Q takes value in $\mathcal{H}_0$ and so does not change the $\mathbb{Z}_4$ equivalence class. The BRST transformations can be written in the following way

$$\begin{aligned}
\delta_b\lambda = 0 \quad , \quad \delta_b\hat{\lambda} = 0 \quad &, \quad \delta_b w = -J_3 \quad , \quad \delta_b\hat{w} = -\bar{J}_1 \\
\delta_b J_0 = [J_3, \lambda] + [J_1, \hat{\lambda}] \quad &, \quad \delta_b J_1 = \partial\lambda + [J_0, \lambda] + [J_2, \hat{\lambda}] \\
\delta_b J_2 = [J_1, \lambda] + [J_3, \hat{\lambda}] \quad &, \quad \delta_b J_1 = \partial\hat{\lambda} + [J_2, \lambda] + [J_0, \hat{\lambda}] \\
\delta_b N = \{J_3, \lambda\} \quad &, \quad \delta_b\hat{N} = \{\bar{J}_1, \hat{\lambda}\}
\end{aligned} \tag{2.3.7}$$

Requiring (2.3.5) to be invariant under the transformations (2.3.7) and using the fact that $\{N, \lambda\} = \{\hat{N}, \hat{\lambda}\} = 0$ because of the pure spinor constraints, one can solve for the coefficients α, β, γ and a as follows [55]

$$\alpha = \frac{1}{2} \quad , \quad \beta = \frac{1}{4} \quad , \quad \gamma = \frac{3}{4} \quad , \quad a = -1 \tag{2.3.8}$$

which produces the following action

$$\begin{aligned} S &= R^2 \int d^2z Str(\frac{1}{2}J_2\bar{J}_2 + \frac{1}{4}J_1\bar{J}_3 + \frac{3}{4}J_3\bar{J}_1 + w\bar{\partial}\lambda + \hat{w}\partial\hat{\lambda} + N\bar{J}_0 + \hat{N}J_0 - N\hat{N}) \\ &= S_{GS} + S_{ghost} \end{aligned} \tag{2.3.9}$$

where the Green-Schwarz action S_{GS} was given in (2.1.53) and using the definition of the currents N and $\hat{N}$, the ghost action can be rewritten in this way

$$S_{ghost} = R^2 \int d^2z Str(w\bar{\nabla}\lambda + \hat{w}\nabla\hat{\lambda} - N\hat{N}) \tag{2.3.10}$$

where the covariant derivative is defined as follows

$$\nabla X = \partial X + [J_0, X] \quad , \quad \bar{\nabla} X = \bar{\partial} X + [\bar{J}_0, X] \tag{2.3.11}$$

Using this general construction, in the next two subsections, we will explore more in detail the form of the pure spinor action for the $AdS_5 \times S^5$ and $Ad\tilde{S}_4 \times \mathbb{CP}^3$ backgrounds.

2.3.1 Pure spinor action for superstring on $AdS^5 \times S^5$

To write the explicit form of the action for the $AdS_5 \times S^5$ background we should first study the structure of the $\frac{G}{H} = \frac{PSU(2,2|4)}{SO(5)\times SO(4,1)}$ supercoset which was shown to produce the correct sigma model for this background [3]. The action is written in terms of the left-invariant currents $J^A = (g^{-1}dg)^A$ where g takes value in the supercoset and $A = ([ab], a, \alpha, \hat{\alpha})$ is the index of the generators of the supergroup $PSU(2,2|4)$.

The 30 bosonic and 32 fermionic generators of the supergroup can be represented as $(T_{[ab]}, T_a, T_\alpha, T_{\hat{\alpha}})$ where $a = 0 \cdots 9$ is an index of the bosonic $SO(5) \times SO(4,1)$ Lorentz and $\frac{SO(6)}{SO(5)} \times \frac{SO(4,2)}{SO(4,1)}$ translation generators which are denoted by $T_{[ab]}$ and T_a respectively and $\alpha, \hat{\alpha} = 1 \cdots 16$ are the indices for the fermionic generators T_α and $T_{\hat{\alpha}}$. The bosonic generators T_a are the coset representatives for the bosonic manifolds AdS_5 for $a = 0 \cdots 4$ and S^5 for $a = 5 \cdots 9$ respectively.

The non-zero structure constants of the supergroup $PSU(2,2|4)$ for the following superalgebra

$$\begin{array}{lll}\{T_\alpha, T_\beta\} = f^a_{\alpha\beta}T_a, & \{T_{\hat\alpha}, T_{\hat\beta}\} = f^a_{\hat\alpha\hat\beta}T_a, & \{T_\alpha, T_{\hat\beta}\} = f^{[ab]}_{\alpha\hat\beta}T_{[ab]} \\ [T_a, T_b] = f^{[cd]}_{ab}T_{[cd]}, & [T_a, T_\alpha] = f^{\hat\beta}_{a\alpha}T_{\hat\beta}, & [T_a, T_{\hat\alpha}] = f^{\beta}_{a\hat\alpha}T_\beta \\ \left[T_{[ab]}, T_{[cd]}\right] = f^{[ef]}_{[ab][cd]}T_{[ef]}, & \left[T_{[ab]}, T_\alpha\right] = f^{\beta}_{[ab]\alpha}T_\beta, & \left[T_{[ab]}, T_{\hat\alpha}\right] = f^{\hat\beta}_{a\hat\alpha}T_{\hat\beta}\end{array} \tag{2.3.12}$$

can be written in terms of the ten-dimensional γ matrices γ_a which are the 16×16 off-diagonal blocks of the Weyl representation of the 32×32 10-dimensional Γ matrices as follows

$$\begin{array}{rcl} f^a_{\alpha\beta} = \gamma^a_{\alpha\beta} & , & f^a_{\hat\alpha\hat\beta} = \gamma^a_{\hat\alpha\hat\beta} \\ f^{[mn]}_{\alpha\hat\beta} = \frac{1}{2}(\gamma^{mn})_\alpha^{\gamma}\eta_{\gamma\hat\beta} & , & f^{[m'n']}_{\alpha\hat\beta} = -\frac{1}{2}(\gamma^{m'n'})_\alpha^{\gamma}\eta_{\gamma\hat\beta} \\ f^{\hat\beta}_{\alpha a} = (\gamma_a)_{\alpha\beta}\eta^{\beta\hat\beta} & , & f^{\beta}_{\hat\alpha a} = -(\gamma_a)_{\hat\alpha\hat\beta}\eta^{\beta\hat\beta} \end{array} \tag{2.3.13}$$
$$f^{[ef]}_{[ab][cd]} = \frac{1}{2}(\eta_{cd}\delta^{[g}_d\delta^{h]}_f - \eta_{cf}\delta^{[g}_d\delta^{h]}_e + \eta_{df}\delta^{[g}_c\delta^{h]}_e - \eta_{de}\delta^{[g}_c\delta^{h]}_f)$$
$$f^f_{[cd]e} = \eta_{e[c}\delta^f_{d]} \quad , \quad f^\beta_{[cd]\alpha} = \frac{1}{2}(\gamma_c d)^\beta_\alpha \quad , \quad f^{\hat\beta}_{[cd]\hat\alpha} = \frac{1}{2}(\gamma_c d)^{\hat\beta}_{\hat\alpha}$$

where $m,n = 0\cdots4$ and $m',n' = 5\cdots9$. The non-zero components of the metric $\eta_{AB} = \langle T_A, T_B\rangle = Str(T_AT_B)$ are given by

$$\begin{array}{rcl} \eta_{ab} = \eta_{ba} & , & \eta_{\alpha\hat\beta} = -\eta_{\hat\beta\alpha} = (\gamma_{01234})_{\alpha\hat\beta} \\ \eta_{[mn][pq]} = \frac{1}{2}\eta_{m[p}\eta_{q]n} & , & \eta_{[m'n'][p'q']} = -\frac{1}{2}\eta_{m'[p'}\eta_{q']n'} \end{array} \tag{2.3.14}$$

where η_{ab} is the Euclidean and Minkowski metric for $a,b = 0,\cdots,4$ and $a,b = 5,\cdots,9$ corresponding to coordinates of S^5 and AdS_5 respectively. The matrix γ_{01234} is defined as the product of $\gamma-$matrices $\gamma_{01234} = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$. Any group index can be raised or lowered with the metric η_{AB} or its inverse η^{AB} satisfying $\eta_{AB}\eta^{BC} = \delta^C_A$.

As we see from the superalgebra (2.3.12), the supergroup $PSU(2,2|4)$ admits the previously mentioned $\mathbb{Z}_4$ automorphism which can be realized easily. The generators can be classified according to the $\mathbb{Z}_4$ grading as follows

$$T_{[ab]} \in \mathcal{H}_0 \quad , \quad T_\alpha \in \mathcal{H}_1 \quad , \quad T_a \in \mathcal{H}_2 \quad , \quad T_{\hat\alpha} \in \mathcal{H}_3 \tag{2.3.15}$$

Using (2.3.12) and (2.3.14), one can see that they satisfy

$$\begin{array}{rcl} [\mathcal{H}_m, \mathcal{H}_n\} & = & \mathcal{H}_{m+n \text{ (mod) } 4} \\ \langle\mathcal{H}_m, \mathcal{H}_n\rangle & = & 0 \text{ unless } m+n = 0 \text{ (mod) } 4 \end{array} \tag{2.3.16}$$

To write the superstring sigma-model action we can either use the general type II Green-Schwarz action of (2.3.1) and generalize it to pure spinor formalism or to use the fact that $AdS_5 \times S^5$ is a maximally supercoset background which admits a $\mathbb{Z}_4$ in which we wrote their sigma model action in the previous section. We will review both derivations both in brief here.

Let us start with the following Green-Schwarz action for a curved background

$$S_{GS} = \frac{R^2}{2} \int d^2z (G_{MN} + B_{MN}) \partial Z^M \bar{\partial} Z^N \tag{2.3.17}$$

where $Z^M = (x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$ are the coordinates of the target space with $m = 0 \cdots 9$ and $\mu, \hat{\mu} = 1 \cdots 16$. The background metric and two form B can be written in terms of the Vielbeins as follows

$$G_{MN} = \eta_{ab} E^a_M E^b_N \quad , \quad B_{MN} = E^A_M E^B_N B_{AB} \tag{2.3.18}$$

in which $A = (a, \alpha, \hat{\alpha})$ are tangent superspace variables and η_{AB} is the bilinear metric of the $PSU(2,2|4)$ supergroup and $M = (m, \mu, \hat{\mu})$ are coordinate variables. The Vielbeins can be written in terms of the left-invariant one-forms $J = g^{-1}dg$ as

$$J^A = E^A_M \partial Z^M \quad , \quad J^{[ab]} = \omega^{[ab]}_M \partial Z^M \tag{2.3.19}$$

where $\omega^{[ab]}$ is the spin connection and the one-form J was expanded in the $PSU(2,2|4)$ basis

$$J = J^{[ab]} T_{[ab]} + J^a T_a + J^\alpha T_\alpha + J^{\hat{\alpha}} T_{\hat{\alpha}} \tag{2.3.20}$$

The non-zero components of the B_{AB} were computed from the supergravity equations to be given as follows [3, 51]

$$B_{\alpha\hat{\beta}} = B_{\hat{\beta}\alpha} = \frac{1}{2} (\gamma_{01234})_{\alpha\hat{\beta}} \equiv \frac{1}{2} \eta_{\alpha\hat{\alpha}} \tag{2.3.21}$$

Putting (2.3.18) and (2.3.21) into the action (2.3.17) one finds the Green-Schwarz action for the $AdS_5 \times S^5$ as follows

$$S_{GS} = \frac{R^2}{2} \int d^2z (J_2 \bar{J}_2 + \frac{1}{2} J_1 \bar{J}_3 + \frac{1}{2} J_3 \bar{J}_1) \tag{2.3.22}$$

using the $\mathbb{Z}_4$ grading (2.3.15).

In order to generalize this GS action to pure spinors, we can add the canonical momenta $(d_\alpha, \hat{d}_{\hat{\alpha}})$ for $(\theta^\alpha, \hat{\theta}^{\hat{\theta}})$ fermionic variables and also the pure spinor ghosts and their momenta $(\lambda^\alpha, w_\alpha)$ and $(\hat{\lambda}^{\hat{\alpha}}, \hat{w}_{\hat{\alpha}})$ which can be expanded according to the $\mathbb{Z}_4$ grading [55, 56] with the following action [36]

$$S = S_{GS} + R^2 \int d^2 z (-d_\alpha \bar{J}^\alpha + \hat{d}_{\hat{\alpha}} J^{\hat{\alpha}} + d_\alpha \hat{d}_{\hat{\alpha}} F^{\alpha\hat{\alpha}} - w_\alpha \bar{\nabla} \lambda^\lambda + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} + R_{abcd} N^{[ab]} \hat{N}^{[cd]}) \tag{2.3.23}$$

where $F^{\alpha\hat{\alpha}}$ is the bispinor Ramond-Ramond background fields strength which for $AdS_5 \times S^5$ background is given by

$$F^{\alpha\hat{\alpha}} = (\gamma_{01234})^{\alpha\hat{\alpha}} = \eta^{\alpha\hat{\alpha}} \tag{2.3.24}$$

and R_{abcd} is the curvature of the supercoset which for a particular supercoset G/H is given by [57, 51]

$$R_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} + (\tilde{B} \leftrightarrow \tilde{C}) = \frac{1}{4} \eta_{\tilde{E}\tilde{F}} f^{\tilde{E}}_{\tilde{A}\tilde{B}} f^{\tilde{F}}_{\tilde{C}\tilde{D}} + \eta_{IJ} f^{I}_{\tilde{A}\tilde{B}} f^{J}_{\tilde{C}\tilde{D}} + (\tilde{B} \leftrightarrow \tilde{C}) \tag{2.3.25}$$

where the covariant derivative is defined with respect the Lorentz currents

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + f^{\alpha}_{[ab]\beta} \bar{J}^{[ab]} \lambda^\beta \quad , \quad \nabla \hat{\lambda}^{\hat{\alpha}} = \partial \hat{\lambda}^{\hat{\alpha}} + f^{\hat{\alpha}}_{[ab]\hat{\beta}} J^{[ab]} \hat{\lambda}^{\hat{\beta}} \tag{2.3.26}$$

The $(\tilde{A}, \tilde{B}, \cdots)$ are indices for the generators of $\mathcal{H}$ algebra which here is $SO(5) \times SO(4,1)$ with generators $T_{[ab]}$ and $(I, J, \cdots)$ are indices for $\mathcal{G} \backslash \mathcal{H}$ generators which are $(T_a, T_\alpha, T_{\hat{\alpha}})$ for $AdS_5 \times S^5$.

Computing (2.3.25) explicitly for the symmetric supergroup $PSU(2,2|4)$ gives

$$R_{abcd} = \pm \eta_{a[c} \eta_{d]b} \equiv \eta_{[ab][cd]} \tag{2.3.27}$$

where the $+$ sign is for the S^5 directions $a, b = 0$ to 4 and the $-$ sign is for the AdS_5 directions $a, b = 5$ to 9.

Putting these back into the action (2.3.23) and considering the fact that the momenta d_α and $\hat{d}_{\hat{\alpha}}$ fields are auxiliary because of the non-vanishing Ramond-Ramond flux, we can integrate them out and get the following action

$$\begin{aligned} S &= S_{GS} + R^2 \int d^2 z \left(\frac{1}{2} \eta_{\alpha\hat{\beta}} (J^\alpha \bar{J}^{\hat{\beta}} - J^{\hat{\beta}} \bar{J}^\alpha) - w_\alpha \bar{\nabla} \lambda^\lambda + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} - \eta_{[ab][cd]} N^{[ab]} \hat{N}^{[cd]} \right) \\ &= S_{GS} + S_{GF} + S_{ghost} \end{aligned} \tag{2.3.28}$$

where S_{GF} is the $\kappa-$gauge fixing part of the action defined as follows

$$S_{GF} = \frac{R^2}{2} \int d^2 z \eta_{\alpha\hat{\beta}} (J^\alpha \bar{J}^{\hat{\beta}} - J^{\hat{\beta}} \bar{J}^\alpha) \tag{2.3.29}$$

The gauge-fixing action breaks the fermionic κ-symmetry and adds kinetic terms of the fermions and the coupling to the Ramond-Ramond flux and S_{ghost} is the pure spinor ghost part of the action.

The action (2.3.28) can be written in $\mathbb{Z}_4$ grading as follows

$$S = R^2 \int d^2z(\frac{1}{2}J_2\bar{J}_2 + \frac{1}{2}J_1\bar{J}_3 - \frac{1}{4}J_3\bar{J}_1) + S_{ghost} \tag{2.3.30}$$

As we saw before, the pure spinor theory is invariant under a BRST symmetry which is generated with the following BRST operator written in the $\mathbb{Z}_4$ language

$$\begin{aligned} Q &= \oint dz\lambda^\alpha d_\alpha + \oint d\bar{z}\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}} = \oint dz\eta_{\alpha\hat{\alpha}}\lambda^\alpha J^{\hat{\alpha}} + \oint d\bar{z}\hat{\lambda}^{\hat{\alpha}}\bar{J}^\alpha \\ Q &= \oint dz\, Str(\lambda J_3) + \oint d\bar{z}\, Str(\hat{\lambda}\bar{J}_1) \end{aligned} \tag{2.3.31}$$

which generates the BRST transformations (2.3.7). Note that the operator Q is $\mathcal{H}_0$ invariant and so does not change the $\mathbb{Z}_4$ grading of any field it acts on. The sigma model, as a supercoset sigma model on the supercoset $\frac{PSU(2,2|4)}{SO(5)\times SO(4,1)}$ is invariant under the global transformation $\delta g = \Sigma g$ where Σ is a constant element of $PSU(2,2|4)$ supergroup. The Cartan one-forms J and $\bar{J}$ are invariant by their definition. The sigma model is also invariant under the following gauge transformation

$$\delta_g J = \partial\Omega + [J,\Omega] \quad , \quad \delta_g\lambda = [\lambda,\Omega] \quad , \quad \delta_g w = [w,\Omega] \tag{2.3.32}$$

where $\Omega \in \mathcal{H}_0 = SO(5)\times SO(4,1)$ which is the Lorentz group.

Actually, as it was shown in [36] and [56], the action (2.3.30) is the unique action which has the global and local symmetries which were mentioned and is invariant under BRST charge (2.3.31).

2.3.2 Pure spinor action for superstring on $\tilde{AdS}_4 \times \mathbb{CP}^3$

Here in this section we use similar techniques to construct the pure spinor superstring sigma model action on the $\tilde{AdS}_4 \times \mathbb{CP}^3$ as a subspace of $AdS_4 \times \mathbb{CP}^3$ background which got many attentions after it was discovered that they are related in somehow to the string theory compactification of the M2-brane backgrounds. As we will see this background has many similarities with the $AdS_5 \times S^5$ background, one of which is that its sigma model can also can be represented with a supercoset which admits a $\mathbb{Z}_4$ automorphism. We will use this grading in order to construct the sigma model action. Here in this section we will construct the pure spinor formalism on this background and we will study and solve the corresponding pure spinor constraints according to [7, 8, 9, 10].

By the superstring on $\tilde{AdS}_4\times\mathbb{CP}^3$ we mean the superstring on the supercoset $\frac{Osp(6|4)}{U(3)\times SO(1,3)}$ which as it was shown in [7, 8, 9, 10] it is a subspace of the full superspace of the superstring on $AdS_4\times\mathbb{CP}^3$ background.

Its Maurer-Cartan left invariant 1-forms can be expanded into the generators of $Osp(6|4)$ as follows

$$J=J^a\gamma_a+J_{IJ}T^{IJ}+J^{IJ}T_{IJ}+H^{ab}\gamma_{ab}+H_I{}^JT_J^I+J_I{}^{\alpha}Q_{\alpha}^I+J_I{}^{\dot\alpha}Q_{\dot\alpha}^I+J^{I\alpha}Q_{I\alpha}+J^{I\dot\alpha}Q_{I\dot\alpha}\quad(2.3.33)$$

where (T_{IJ},T^{IJ},T_J^I) are the generators of $SO(6)$, $T_{[AB]}$ with $A,B=1\ldots6$ decomposes according to irreducible representations of $U(3)$ as it will be explained later, and T_J^I are the generators of $U(3)$. Then, J_{IJ} and J^{IJ} are the Maurer-Cartan forms associated to the generators of the coset $\frac{SU(4)}{U(3)}$ and $H_I{}^J$ are the corresponding spin connections of the coset. Similarly, (γ_a,γ_{ab}) with $a,b=1\ldots4$ are the generators of the anti de Sitter group $SO(2,3)$ which as is shown in [9] they all turn out to be given by real symplectic matrices and γ_{ab} are the generators of the Lorentz group $SO(1,3)$. The matrices $Q_I^{\alpha},Q_I^{\dot\alpha},Q_{\alpha}^I$ and $Q_{\dot\alpha}^I$ are the 24 fermionic generators where we split the symplectic indices $x=1\ldots4$ into $SO(1,3)$ spinorial indices $\alpha,\dot\alpha=1,2$. The Maurer-Cartan 1-forms of the symplectic group $Sp(4,\mathbb{R})$ are related to the Maurer-Cartan of $SO(2,3)$ with the relation $J^{xy}=J^a\gamma_a^{xy}+H^{ab}\gamma_{ab}^{xy}$. The fermionic 1-forms J_A^x are real and transform in the fundamental 4-dimensional representation of $\mathfrak{sp}(4,\mathbb{R})$ and in the fundamental 6-dimensional representation of $\mathfrak{so}(6)$ with the symplectic invariant antisymmetric metric $\epsilon_{xy}=i\sigma_1\otimes\mathbb{1}$.

Notice that η^{ab} is the invariant metric on AdS_4 and $g_{I\bar{J}}$ is the $U(3)$ invariant metric on $\mathbb{P}^3$ and we denote by $k_{I\bar{J}}$ as the Kähler form on $\mathbb{P}^3$. The index I can be raised and lowered with the inverse metric $g^{\bar{I}J}$ as $J^{\bar{I}\bar{J}}=g^{\bar{I}K}g^{\bar{J}L}J_{KL}$ which is independent of J^{IJ}, similarly we can make $J_{\bar{I}\bar{J}}$ out of J_{IJ}.

The $\mathfrak{osp}(6|4)$ algebra $\mathcal{H}$ admits a $\mathbb{Z}_4$ grading with decomposition $\mathcal{H}=\sum_{i=0}^3\mathcal{H}_i$ as follows[2]

$$\begin{aligned}\mathcal{H}_0&=\left\{H_{\alpha\beta},H_{\dot\alpha\dot\beta},H_I{}^J\right\},&\mathcal{H}_1=\left\{J^{\alpha I},J^{\dot\alpha\bar{I}}\right\},\\ \mathcal{H}_2&=\left\{J_{\alpha\dot\alpha},J_{IJ},J^{IJ}\right\},&\mathcal{H}_3=\left\{J_I{}^{\alpha},J_{\bar{I}}{}^{\dot\alpha}\right\}.\end{aligned}\quad(2.3.34)$$

satisfying

$$[\mathcal{H}_m,\mathcal{H}_n]\subset\mathcal{H}_{m+n\ (\mathrm{mod}\ 4)}\quad(2.3.35)$$

[2]Sometimes the notation $\hat{J}$ will be used to denote the currents of the subset $\mathcal{H}_3$.

We can check that the bilinear metric is also $\mathbb{Z}_4$ invariant. Recall that the invariant supermetric for $Osp(6|4)$ is given by

$$\begin{aligned} \mathrm{Str}(T_{AB}T_{CD}) &= \delta_{AC}\delta_{DB} - \delta_{AD}\delta_{CB}\,, \\ \mathrm{Str}(T_{xy}\,T_{zt}) &= \epsilon_{xz}\epsilon_{ty} + \epsilon_{xt}\epsilon_{zy}\,, \\ \mathrm{Str}(T_x\,T_y) &= \epsilon_{xy}\,, \\ \mathrm{Str}(Q^x_A Q^y_B) &= \delta_{AB}\epsilon^{xy}\,. \end{aligned} \tag{2.3.36}$$

where T_{AB} and T_{xy} are the generators of the bosonic subgroups $SO(6)$ and $Sp(4,\mathbb{R})$, and Q^x_A are the fermionic generators of the supergroup. It is convenient to adopt a complex basis for the generators of $SO(6)$. We can define $T_{AB} = U^{IJ}_{AB}T_{IJ} + U^{I}_{J,AB}T_I{}^J + U_{IJ,AB}T^{IJ}$ where $U^{IJ}_{AB}, U^{I}_{J,AB}, U_{IJ,AB}$ are the Clebsh-Gordon matrices mapping from $\underline{15}$ of $SO(6)$ to the representations $\underline{3}(-1)$, $\underline{8}(0)$, $\underline{3}^*(+1)$ of $U(3)$, respectively. In the same way, we decompose the fermionic generators into Q^x_I and Q^{xI} of $\underline{3}(-1)$ and $\underline{3}^*(1)$, respectively.

The metric of the supergroup can be written explicitly as follows

$$\begin{aligned} \mathrm{Str}(T_{IJ}T^{KL}) &= \delta_I{}^K\delta_J{}^L - \delta_J{}^K\delta_I{}^L\,, \\ \mathrm{Str}(T_I{}^J T_K{}^L) &= \delta_I{}^L\delta_K{}^J\,, \\ \mathrm{Str}(Q^x_I Q^{yJ}) &= \delta^J_I\epsilon^{xy}\,. \end{aligned} \tag{2.3.37}$$

while the other traces vanish. This means that the bilinear metric is $\mathbb{Z}_4$ invariant, satisfying

$$\langle \mathcal{H}_m, \mathcal{H}_n\rangle = \mathrm{Str}(\mathcal{H}_m\mathcal{H}_n) = 0, \quad \text{unless } m+n = 0 \bmod 4 \tag{2.3.38}$$

Using this $\mathbb{Z}_4$ automorphism similar to the $AdS_5\times S^5$ sigma model action, it was shown that the pure spinor sigma model action can be decomposed in the following way

$$S = S_{GS} + S_{GF} + S_{\text{ghost}} \tag{2.3.39}$$

where S_{GS} is the Green-Schwarz action was shown in [7, 8, 9, 10] to exhibit the usual quadratic form after using the fact that it is possible to write the Wess-Zumino term as a total derivative in this background which produces the following term as an integral over a two-form on the worldsheet

$$S_{GS} = R^2\int d^2z\,\mathrm{Str}\left[\frac{1}{2}J_2\bar{J}_2 + \frac{1}{4}\left(J_1\bar{J}_3 - J_3\bar{J}_1\right)\right] \tag{2.3.40}$$

Here $J_i = J|_{\mathcal{H}_i}$ are the projections of the MC left invariant currents into different subclasses according to $\mathbb{Z}_4$ automorphism as it was defined in (2.3.34). The action can be written in terms of the left-invariant currents of the coset in the following form

$$S_{GS} = R^2 \int d^2z[\epsilon_{xy} J^x \bar{J}^y + \frac{1}{2} J_{IJ} \bar{J}^{IJ} + \frac{1}{4}(J_{\alpha I} \bar{J}^{\alpha I} + J_{\dot{\alpha}\bar{I}} \bar{J}^{\dot{\alpha}\bar{I}} - J_{\alpha\bar{I}} \bar{J}^{\alpha\bar{I}} - J_{\dot{\alpha}I} \bar{J}^{\dot{\alpha}I})] \quad (2.3.41)$$

To this, one has to add a term which breaks $\kappa-$symmetry and adds kinetic terms for the target-space fermions and the coupling to the RR flux. This gauge fixing action S_{GF} was shown to be given by [9]

$$S_{GF} = R^2 \int d^2z \left(J_{\alpha\bar{I}} \bar{J}^{\alpha\bar{I}} + J_{\dot{\alpha}I} \bar{J}^{\dot{\alpha}I} \right) , \quad (2.3.42)$$

which gives the following action

$$\begin{aligned} S &= S_{GS} + S_{GF} \quad (2.3.43) \\ &= R^2 \int d^2z[\epsilon_{xy} J^x \bar{J}^y + \frac{1}{2} J_{IJ} \bar{J}^{IJ} + \frac{1}{4}(J_{\alpha I} \bar{J}^{\alpha I} + J_{\dot{\alpha}\bar{I}} \bar{J}^{\dot{\alpha}\bar{I}}) + \frac{3}{4}(J_{\alpha\bar{I}} \bar{J}^{\alpha\bar{I}} + J_{\dot{\alpha}I} \bar{J}^{\dot{\alpha}I})] \end{aligned}$$

In order to write the pure spinor ghost part of the action, we introduce the pure spinors $(\lambda_I^\alpha, \lambda_{\bar{I}}^{\dot{\alpha}})$, $(\hat{\lambda}^{\alpha I}, \hat{\lambda}^{\dot{\alpha}\bar{I}})$ and their conjugate momenta $(w_\alpha^I, w_{\dot{\alpha}}^{\bar{I}})$, $(\hat{w}_{\alpha I}, \hat{w}_{\dot{\alpha}\bar{I}})$, belonging to the $\mathcal{H}_1$ and $\mathcal{H}_3$ respectively. The pure spinor constraints can be written as follows

$$\begin{cases} \lambda_I^\alpha \lambda^{\dot{\alpha} I} = 0 \\ \lambda_I^\alpha \epsilon_{\alpha\beta} \lambda_J^\beta = 0 \\ \lambda^{\dot{\alpha} I} \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^{\dot{\beta} J} = 0 \end{cases} , \qquad \begin{cases} \hat{\lambda}^{\alpha I} \hat{\lambda}_I^{\dot{\alpha}} = 0 \\ \hat{\lambda}^{I\alpha} \epsilon_{\alpha\beta} \hat{\lambda}^{\beta J} = 0 \\ \hat{\lambda}_I^{\dot{\alpha}} \epsilon_{\dot{\alpha}\dot{\beta}} \hat{\lambda}_J^{\dot{\beta}} = 0 \end{cases} \quad (2.3.44)$$

To solve this constraint, we can use the following ansatz

$$\begin{aligned} \lambda_I^\alpha &= \lambda^\alpha u_I, \qquad \lambda^{\dot{\alpha} I} = \lambda^{\dot{\alpha}} v^I , \quad (2.3.45) \\ \hat{\lambda}^{\alpha I} &= \hat{\lambda}^\alpha \hat{u}^I, \qquad \hat{\lambda}_I^{\dot{\alpha}} = \hat{\lambda}^{\dot{\alpha}} \hat{v}_I , \end{aligned}$$

subject to the following gauge transformations

$$\begin{aligned} &\lambda^\alpha \to \frac{1}{\rho} \lambda^\alpha, \quad \lambda^{\dot{\alpha}} \to \frac{1}{\sigma} \lambda^{\dot{\alpha}}, \quad u_I \to \rho u_I, \quad v^I \to \sigma v^I , \quad (2.3.46) \\ &\hat{\lambda}^\alpha \to \frac{1}{\hat{\rho}} \hat{\lambda}^\alpha, \quad \hat{\lambda}^{\dot{\alpha}} \to \frac{1}{\hat{\sigma}} \hat{\lambda}^{\dot{\alpha}}, \quad \hat{u}^I \to \hat{\rho} \hat{u}^I, \quad \hat{v}_I \to \hat{\sigma} \hat{v}_I , \end{aligned}$$

where $\rho, \sigma, \hat{\rho}, \hat{\sigma} \in \mathbb{C}^*$.

Inserting these factorization into (2.3.44), we arrive to the following constraints

$$u_I v^I = 0, \qquad \hat{v}_I \hat{u}^I = 0 . \quad (2.3.47)$$

So, the counting of the degrees of freedom gives $2 \times (2+3-1) - 1 = 7$ complex for λ and the same for $\hat{\lambda}$. The geometry of the pure spinor space can be easily described. Using the gauge symmetries ρ and σ we can fix the norm of u_I and v^I as such $u_I \bar{u}^I = 1$ and $v^I \bar{v}_I = 1$. Then, together the constraint $u_I v^I = 0$, the matrix $(u_I, \bar{v}_I, \epsilon_{IJK} \bar{u}^J v^K)$ is an $SU(3)$ matrix. In addition, using the remaining phases of the gauge symmetries ρ and σ, we see that the variables u_I and v^I parametrize the space $SU(3)/U(1) \times U(1)$ which is the space of the harmonic variables of the $N = 3$ harmonic superspace (It is also known as the flag manifold $F(1,2,3)$).

Another way to solve the constraints (2.3.44) is decomposing the pure spinor into $\lambda^\alpha_I = (\lambda^\alpha_a, \lambda^\alpha)$ and $\lambda^{\dot\alpha I} = (\lambda^{\dot\alpha a}, \lambda^{\dot\alpha})$ where $a = 1,2$. It is easy to show that the pure spinor constrains become $\lambda^\alpha_a \lambda^{\dot\alpha a} + \lambda^\alpha \lambda^{\dot\alpha} = 0$, $\det(\lambda^\alpha_a) = 0$, $\det(\lambda^{\dot\alpha}_a) = 0$, $\lambda^\alpha_a \epsilon_{\alpha\beta} \lambda^\beta = 0$ and $\lambda^{\dot\alpha}_a \epsilon_{\dot\alpha\dot\beta} \lambda^{\dot\beta} = 0$. The first set of constraints implies that we can solve 3 parameters in terms of the rest and we get a consistency condition $\det(\lambda^\alpha_a)\det(\lambda^{\dot\alpha}_a) = 0$. This is solved by imposing the second and the third conditions. The latter also imply the existence of a solution for the forth and for the fifth constraints. Again the counting of the parameters gives 7 complex numbers.

The pure spinor constraints are first class constraints and they commute with the Hamiltonian, therefore they generate the gauge symmetries on the antighost fields w's. In particular if we denote by $\eta_{\alpha\dot\alpha}, \eta^{IJ}, \eta_{IJ}$ and by $\kappa_{\alpha\dot\alpha}, \kappa^{IJ}, \kappa_{IJ}$ the infinitesimal parameters of the gauge symmetries we have that

$$
\begin{aligned}
\delta w^I_\alpha &= \eta_{\alpha\dot\alpha} \lambda^{\dot\alpha I} + 2\eta^{IJ} \epsilon_{\alpha\beta}\, \lambda^\beta_J\,, \qquad \delta w_{\dot\alpha I} = \eta_{\alpha\dot\alpha} \lambda^\alpha_I + 2\eta_{IJ} \epsilon_{\dot\alpha\dot\beta}\, \lambda^{\beta J}\,, \\
\delta \hat{w}_{\alpha I} &= \kappa_{\alpha\dot\alpha} \hat{\lambda}^{\dot\alpha}_I + 2\kappa^{IJ} \epsilon_{\alpha\beta}\, \hat{\lambda}^{\beta J}\,, \quad \delta \hat{w}^I_{\dot\alpha} = \kappa_{\alpha\dot\alpha} \hat{\lambda}^{\alpha I} + 2\eta^{IJ} \epsilon_{\dot\alpha\dot\beta}\, \hat{\lambda}^\beta_I\,.
\end{aligned}
\tag{2.3.48}
$$

One can introduce pure spinor Lorentz generators $(N = -\{w, \lambda\}, \hat{N} = -\{\hat{w}, \hat{\lambda}\}) \in \mathcal{H}_0$, bringing the couplings between the pure spinor fields and matter fields, as follows

$$
\begin{aligned}
N_{\alpha\beta} &= w^I_{(\alpha} \lambda_{\beta)I}, \qquad \hat{N}_{\alpha\beta} = w_{I(\alpha} \lambda^I_{\beta)}\,, \\
N_{\dot\alpha\dot\beta} &= w_{(\dot\alpha I} \lambda^I_{\dot\beta)}, \qquad \hat{N}_{\dot\alpha\dot\beta} = \hat{w}^I_{(\dot\alpha} \hat{\lambda}_{\dot\beta)I}\,, \\
N_I{}^J &= w^I_\alpha \lambda^\alpha_I + w_{I\dot\alpha} \lambda^{I\dot\alpha}\,, \\
\hat{N}_I{}^J &= \hat{w}^\alpha_I \hat{\lambda}^I_\alpha + \hat{w}^{I\dot\alpha} \hat{\lambda}_{I\dot\alpha}\,.
\end{aligned}
\tag{2.3.49}
$$

They are gauge invariant under the transformations (5.1.5). Finally, we can write the pure spinor ghost piece of the action

$$
\begin{aligned}
S_{\text{ghost}} &= R^2 \int d^2 z \Big(w^I_\alpha \bar\nabla \lambda^\alpha_I + w_{\dot\alpha I} \bar\nabla \lambda^{\dot\alpha I} + \hat{w}_{\alpha I} \nabla \hat{\lambda}^{I\alpha} + \hat{w}^I_{\dot\alpha} \nabla \hat{\lambda}^{\dot\alpha}_I \\
&- \eta^{(\alpha\beta)(\gamma\delta)} N_{\alpha\beta} \hat{N}_{\alpha\delta} - \eta^{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)} N_{\dot\alpha\dot\beta} \hat{N}_{\dot\gamma\dot\delta} - \eta^I{}_J{}^K{}_L N_I{}^J \hat{N}_K{}^L \Big)\,,
\end{aligned}
\tag{2.3.50}
$$

where the bilinear metrics η are given from (2.3.36) and (2.3.37) as

$$\eta^{(\alpha\beta)(\gamma\delta)} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}, \qquad \eta^{I}{}_{J}{}^{K}{}_{L} = \delta_L{}^I\delta_J{}^K\,. \tag{2.3.51}$$

Putting everything together we get the pure spinor action for $\tilde{AdS}_4 \times \mathbb{CP}^3$ background as follows

$$\begin{aligned} S &= R^2\int d^2z\Big[\epsilon_{xy}J^x\bar{J}^y + \frac{1}{2}J_{IJ}\bar{J}^{IJ} \\ &+ \frac{1}{4}\left(J_{\alpha I}\bar{J}^{\alpha I} + J_{\dot\alpha \bar I}\bar{J}^{\dot\alpha\bar I}\right) + \frac{3}{4}\left(J_{\alpha\bar I}\bar{J}^{\alpha\bar I} + J_{\dot\alpha I}\bar{J}^{\dot\alpha I}\right) \\ &+ w^I_\alpha\bar\nabla\lambda^\alpha_I + w_{\dot\alpha I}\bar\nabla\lambda^{\dot\alpha I} + \hat w_{\alpha I}\nabla\hat\lambda^{I\alpha} + \hat w^I_{\dot\alpha}\nabla\hat\lambda^{\dot\alpha}_I \\ &- \eta^{(\alpha\beta)(\gamma\delta)}N_{\alpha\beta}\hat N_{\gamma\delta} - \eta^{(\dot\alpha\dot\beta)(\dot\gamma\dot\delta)}N_{\dot\alpha\dot\beta}\hat N_{\dot\gamma\dot\delta} - \eta^{I}{}_{J}{}^{K}{}_{L}N_I{}^J\hat N_K{}^L\Big] \end{aligned} \tag{2.3.52}$$

The theory admits a BRST transformation with the following BRST charge

$$\begin{aligned} Q+\bar Q &= \oint\left\langle dz\lambda J_3 + d\bar z\hat\lambda\bar J_1\right\rangle \\ &= \oint dz\left(\lambda_{I\alpha}\hat J^{\alpha I} + \lambda^{\dot\alpha I}\hat J_{\dot\alpha I}\right) + \oint d\bar z\left(\hat\lambda^{\alpha I}\bar J_{\alpha I} + \hat\lambda^{\dot\alpha}_I\bar J^I_{\dot\alpha}\right)\,. \end{aligned} \tag{2.3.53}$$

The BRST transformation (2.3.7) for a general supercoset background admitting the $\mathbb{Z}_4$ automorphism can be written in the following form for the $\tilde{AdS}_4 \times \mathbb{CP}^3$ background

$$\begin{aligned} \delta_B J_{\alpha\beta} &= -2\lambda_{(\alpha I}\hat J^I_{\beta)} - 2J_{(\alpha I}\hat\lambda^I_{\beta)}\,, \qquad \delta_B J_{\dot\alpha\dot\beta} = -2\lambda^I_{(\dot\alpha}\hat J_{\dot\beta)I} - 2J^I_{(\dot\alpha}\hat\lambda_{\beta)I}\,, \\ \delta_B\hat J^{\alpha I} &= (\nabla\hat\lambda)^{\alpha I} + J^{IJ}\lambda^\alpha_J + J^\alpha{}_{\dot\alpha}\lambda^{\dot\alpha I}, \quad \delta_B\hat J^{\dot\alpha}_I = (\nabla\hat\lambda)^{\dot\alpha}_I + J_{IJ}\lambda^{\dot\alpha J} + J_\alpha{}^{\dot\alpha}\lambda^\alpha_I \\ \delta_B J^\alpha_I &= (\nabla\lambda)^\alpha_I + J_{IJ}\hat\lambda^{\alpha J} + J^\alpha{}_{\dot\alpha}\hat\lambda^{\dot\alpha}_I, \quad \delta_B J^{\dot\alpha I} = (\nabla\lambda)^{\dot\alpha I} + J^{IJ}\hat\lambda^{\dot\alpha}_J + J^\alpha{}_{\dot\alpha}\hat\lambda^{\dot\alpha I}\,, \\ \delta_B J_{\alpha\dot\beta} &= \lambda_{\alpha I}J^I_{\dot\beta} + J_{\alpha I}\lambda^I_{\dot\beta} + \hat J_{\dot\beta I}\hat\lambda^I_\alpha + \hat\lambda_{\dot\beta I}\hat J^I_\alpha\,, \\ \delta_B J_{IJ} &= 2\,\epsilon^{\alpha\beta}\lambda_{\alpha[I}J_{J]\beta} + 2\,\epsilon^{\dot\alpha\dot\beta}\hat J_{\dot\alpha[I}\hat\lambda_{J]\dot\beta}\,, \\ \delta_B J^{IJ} &= 2\,\epsilon^{\alpha\beta}\lambda^{[I}_\alpha J^{J]}_\beta + 2\,\epsilon^{\dot\alpha\dot\beta}\hat J^{[I}_{\dot\alpha}\hat\lambda^{J]}_{\dot\beta}\,, \\ \delta_B\omega^I_\alpha &= -\hat J^I_\alpha\,, \qquad \delta_B\omega_{\dot\alpha I} = -\hat J_{\dot\alpha I}\,, \\ \delta_B\hat\omega_{\alpha I} &= -J_{\alpha I}\,, \qquad \delta_B\hat\omega^I_{\dot\alpha} = -J^I_{\dot\alpha}\,, \end{aligned} \tag{2.3.54}$$

the variations of $N_{\alpha\beta}, N_{\dot\alpha\dot\beta}, \hat N_{\alpha\beta}, \hat N_{\dot\alpha\dot\beta}$ can be easily derived by their definitions (2.3.49). Using this notation, we can assign a further quantum number by assigning 0 to $J_{\alpha\dot\alpha}$, +1 to J^{IJ}, −1 to J_{IJ}, −1/2 to $J_{\alpha I}, \hat J_{\dot\alpha,I}$ and +1/2 to $\hat J_{\alpha I}, J_{\dot\alpha,I}$. This is the center of $U(1)$ inside of $U(3)$. Notice that the symmetry is a $\mathbb{Z}_5$ symmetry. The action, the BRST transformations and the pure spinor conditions respect such a symmetry. It would be nice to see if this symmetry corresponds to some geometric symmetry in the background or can be used to simplify the superstring formulation.

Chapter 3

Topological decomposition of pure spinor superstring action

In this chapter we show that the same way Berkovits and Vafa [23] obtained the embedding of the A-model action in the pure spinor superstring on $AdS_5 \times S^5$ background, we can obtain the existence of such an embedding and decomposition for any superscoset background admitting a $\mathbb{Z}_4$ automorphism, as is the case also for the $\tilde{AdS}_4 \times \mathbb{CP}^3$ supercoset.

3.1 Topological A-model on a Grassmannian

Let's consider the following Kähler potential defining a topological theory on a fermionic Grassmannian coset G/H [58]

$$K(\Theta, \bar{\Theta}) = \frac{1}{2} \ln \det \left(\bar{\xi}(\bar{\Theta}) \xi(\Theta) \right) \tag{3.1.1}$$

The $\xi(\Theta) \in G/H$ is a representative of the fermionic coset G/H where for any $h \in H$ and $g \in G$ satisfies

$$g\xi(\Theta) = \xi(\Theta')h(\Theta, g) \tag{3.1.2}$$

For $G/H = \frac{PU(2,2|4)}{SU(4) \times SU(2,2)}$ coset and for $G/H = \frac{Ops(6|4)}{SO(6) \times Sp(4)}$, we can present the coset representative in the following form

$$G/H = \frac{Ops(6|4)}{SO(6)\times Sp(4)} \qquad \xi = \begin{pmatrix} 1\!\!1_{4\times 4} & \Theta \\ \bar\Theta & 1\!\!1_{6\times 6}\end{pmatrix}, \; \bar\xi = \begin{pmatrix} 1\!\!1_{4\times 4} & \Theta \\ -\bar\Theta & 1\!\!1_{6\times 6}\end{pmatrix} \tag{3.1.3}$$

$$G/H = \frac{PU(2,2|4)}{SU(4)\times SU(2,2)} \qquad \xi = \begin{pmatrix} 1\!\!1_{4\times 4} & \Theta \\ \bar\Theta & 1\!\!1_{4\times 4}\end{pmatrix}, \; \bar\xi = \begin{pmatrix} 1\!\!1_{4\times 4} & \Theta \\ -\bar\Theta & 1\!\!1_{4\times 4}\end{pmatrix} \tag{3.1.4}$$

where here Θ and $\bar\Theta$ are fermionic matrices.

Using the convention $i\bar\Theta = \Theta^\dagger$, the Kähler potential (3.1.1) can be written as

$$\begin{aligned} K(\Theta,\bar\Theta) &= \frac{1}{2} ln \det\left[\begin{pmatrix} 1\!\!1 & \Theta \\ \bar\Theta & 1\!\!1\end{pmatrix}\begin{pmatrix} 1\!\!1 & \Theta \\ -\bar\Theta & 1\!\!1\end{pmatrix}\right] \\ &= \frac{1}{2} ln \det\left[\begin{pmatrix} 1\!\!1 - \Theta\bar\Theta & 0 \\ 0 & 1\!\!1 + \bar\Theta\Theta\end{pmatrix}\right] \\ &= \frac{1}{2} ln \left[\det(1\!\!1 - \Theta\bar\Theta)\times\det(1\!\!1 + \bar\Theta\Theta)\right] \\ &= \mathrm{T}r\, ln(1\!\!1 + \bar\Theta\Theta) \end{aligned} \tag{3.1.5}$$

which in the last line we used the fact that

$$\mathrm{T}r(\Theta\bar\Theta)^n = -\mathrm{T}r(\bar\Theta\Theta)^n, \quad \text{for } n > 0\,, \tag{3.1.6}$$

This Kähler potential gives the following action which after writing the explicit form of the superfields Θ and $\bar\Theta$ it will produce the same topological A-model action as it was found by Berkovits from a particular limit of the superstring and was proposed to correspond to the zero radius limit of the superstring as we explained in the introduction.

The action of the A-model topological theory can be written as follows

$$S = \int d^2z d^4\kappa \mathrm{Tr}\left[\log\left(1 + \eta_{\alpha^+\alpha^-}\bar\Theta^{\alpha^-}\Theta^{\alpha^+}\right)\right] \tag{3.1.7}$$

where κ's are the $\mathcal{N} = 2$ worldsheet supersymmetry coordinates and fermionic chiral superfields Θ^{α^+} and $\hat\Theta^{\alpha^-}$ can be expanded as follows

$$\begin{aligned} \Theta^{\alpha^+} &= \theta^{\alpha^+} + \kappa_+ Z^{\alpha^+} + \kappa_- \bar Y^{\alpha^+} + \kappa_+\kappa_- f^{\alpha^+} \\ \bar\Theta^{\alpha^-} &= \bar\theta^{\alpha^-} + \bar\kappa_+ \bar Z^{\alpha^-} + \bar\kappa_- Y^{\alpha^-} + \bar\kappa_+\bar\kappa_- \bar f^{\alpha^-} \end{aligned} \tag{3.1.8}$$

where θ and $\bar\theta$ are the fermionic degrees of freedom of the superspace and Z and Y fields are bosonic twisted variables constructed from the bosonic degrees of freedom of the superspace and the pure spinors and their conjugate momenta. The fields f^{α^+} and $\bar{f}^{\alpha^-}$ are auxiliary fields. The fact that there are 11 complex independent pure spinor degrees of freedom is a very crucial fact which make it possible to construct such unconstrained twisted-like variables as follows

$$\begin{aligned} Z^{\alpha^+} &= f^{\alpha^+}_{m\beta^+} H^m \lambda^{\beta^+} \quad , \quad \bar{Z}^{\alpha^-} = f^{\alpha^-}_{m\beta^-} H^m \hat{\lambda}^{\beta^-} \\ \bar{Y}^{\alpha^+} &= f^{\alpha^+}_{m\beta^+} H^m w^{\beta^+} \quad , \quad Y^{\alpha^-} = f^{\alpha^-}_{m\beta^-} H^m \hat{w}^{\beta^-} \end{aligned} \tag{3.1.9}$$

where $H^m = (H^a, H^{a'})$ are the bosonic cosets corresponding to the geometry of the background[1] and $f^{\alpha^+}_{m\beta^+}$ and $f^{\alpha^-}_{m\beta^-}$ are structure constants of the supergroup.

The action (3.1.7) is an A-model topological action which after expanding its Kähler potential and integrating over the auxiliary fields we get the following action

$$S = \int d^2z \left[\eta_{\alpha\hat{\beta}} J^\alpha \bar{J}^{\hat{\beta}} - \eta_{\alpha^-\alpha^+} Y^{\alpha^-} \bar{\nabla} Z^{\alpha^+} + \eta_{\alpha^-\alpha^+} \bar{Y}^{\alpha^+} \nabla \bar{Z}^{\alpha^-} - [Y, Z]_m [\bar{Y}, \bar{Z}]^m \right] \tag{3.1.10}$$

For the case of $AdS_5 \times S^5$ the fermionic chiral superfield Θ belongs to the fermionic supercoset

$$\Theta \in \frac{PSU(2,2|4)}{SU(4) \times SU(2,2)} \tag{3.1.11}$$

which can be expanded in terms of the bosonic fields Z and Y as follows

$$\begin{aligned} \Theta^A_J(\kappa_+, \kappa_-) &= \theta^A_J + \kappa_+ Z^A_J + \kappa_- \hat{Y}^A_J + \kappa_+\kappa_- f^A_J \\ \bar{\Theta}^J_A(\bar{\kappa}_+, \bar{\kappa}_-) &= \hat{\theta}^J_A + \bar{\kappa}_+ \bar{Z}^J_A + \bar{\kappa}_- Y^J_A + \bar{\kappa}_+\bar{\kappa}_- \hat{f}^J_A \end{aligned} \tag{3.1.12}$$

where $A = 1$ to 4 and $J = 1$ to 4 label fundamental representation of $SU(2,2)$ and $SU(4)$ respectively. The θ^A_J and $\bar{\theta}^J_A$ are the fermionic coordinates of the $AdS_5 \times S^5$ superspace. The fields f^A_J and $\bar{f}^J_A$ are auxiliary fields and the other fields are twistor-like variables encoding the bosonic and the pure spinor and their conjugate momenta degrees of freedom expanded as follows

$$\begin{aligned} Z^A_J &= H^A_{A'}(x)(\tilde{H}^{-1}(\tilde{x}))^{J'}_J \lambda^{A'}_{J'} \quad , \quad \bar{Z}^J_A = (H^{-1}(x))^{A'}_A \tilde{H}^J_{J'}(\tilde{x}) \hat{\lambda}^{J'}_{A'} \\ \bar{Y}^A_J &= H^A_{A'}(x)(\tilde{H}^{-1}(\tilde{x}))^{J'}_J \hat{w}^{A'}_{J'} \quad , \quad Y^A_J = (H^{-1}(x))^{A'}_A \tilde{H}^J_{J'}(\tilde{x}) w^{J'}_{A'} \end{aligned} \tag{3.1.13}$$

where $A' = 1$ to 4 and $J' = 1$ to 4 are $SO(4,1)$ and $SO(5)$ spinor index respectively. The $H^A_{A'}$ is a coset representative of $\frac{SU(2,2)}{SO(4,1)}$ corresponding to AdS_5 with coordinates $\tilde{x}^{\tilde{m}}$

[1] The bosonic cosets are either $H^a = \frac{SO(6)}{SO(5)}$ and $H^{a'} = \frac{SO(4,2)}{SO(4,1)}$ or $H^a = \frac{SO(6)}{U(3)}$ and $H^{a'} = \frac{Sp(4)}{SO(3,1)}$ for $AdS_5 \times S^5$ and $\tilde{AdS}_4 \times \mathbb{CP}^3$ backgrounds respectively.

and $H^J_{J'}$ is a coset representative of the coset $\frac{SU(4)}{SO(5)}$ corresponding to S^5 with coordinates x^m.

The matching of the bosonic and fermionic degrees of freedom can be represented as follows

$$\begin{cases} 10\ \mathbb{R} & x^m, \tilde{x}^{\tilde{m}} \\ 11\ \mathbb{C} & \lambda^A_J, \hat{\lambda}^J_A \\ 16\ \mathbb{C} & \theta^A_J, \bar{\theta}^J_A \end{cases} \Longleftrightarrow \begin{cases} 16\ \mathbb{C} & Z^A_J,\ \bar{Z}^J_A \\ 16\ \mathbb{C} & \theta^A_J, \bar{\theta}^J_A \end{cases} \tag{3.1.14}$$

where $m = 0$ to 4, $\tilde{m} = 5$ to 9 and $\alpha, \hat{\alpha} = 1$ to 16. The bosonic degrees of freedom $(x^m, \tilde{x}^{\tilde{m}})$ corresponding to the coordinates of AdS_5 and S^5 together with 11 complex pure spinors $(\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}})$ are encoded in the sixteen complex degrees of freedom of $(Z^A_J, \bar{Z}^J_A)$.

The worldsheet variables for this Kähler N=2 sigma-model on $\tilde{AdS_4} \times \mathbb{CP}^3$ are fermionic superfields Θ^x_A and $\bar{\Theta}^A_x$ belong to the following Grassmannian coset

$$\Theta = \frac{Osp(6|4)}{SO(6) \times Sp(4)} \tag{3.1.15}$$

where $A = 1, \ldots, 6$ and $x = 1, \ldots, 4$ label fundamental representations of $SO(6)$ and $Sp(4)$ respectively.

These $N = 2$ chiral and anti-chiral superfields can be expanded in terms of the fields of the pure spinor superstring theory as follows

$$\begin{aligned} \Theta^x_A(\kappa_+, \kappa_-) &= \theta^x_A + \kappa_+ Z^x_A + \kappa_- \bar{Y}^x_A + \kappa_+ \kappa_- f^x_A\,, \\ \bar{\Theta}^A_x(\bar{\kappa}_+, \bar{\kappa}_-) &= \bar{\theta}^A_x + \bar{\kappa}_+ \bar{Z}^A_x + \bar{\kappa}_- Y^A_x + \bar{\kappa}_+ \bar{\kappa}_- \bar{f}^A_x\,, \end{aligned} \tag{3.1.16}$$

where $(\kappa_+, \bar{\kappa}_+)$ are left-moving and $(\kappa_-, \bar{\kappa}_-)$ are right-moving Grassmannian parameters of the worldsheet N=2 supersymmetry. The matching of the bosonic and fermionic degrees of freedom can be represented as follows

$$\begin{cases} 10\ \mathbb{R} & x_A, x_P \\ 7\ \mathbb{C} & \lambda^x_A, \bar{\lambda}^A_x \\ 24\ \mathbb{R} & \theta^x_A, \bar{\theta}^A_x \end{cases} \Longleftrightarrow \begin{cases} 24\ \mathbb{R} & Z^A_J,\ \bar{Z}^J_A \\ 24\ \mathbb{R} & \theta^x_A, \bar{\theta}^A_x \end{cases} \tag{3.1.17}$$

The 24 lowest components θ^x_A and $\bar{\theta}^A_x$ are 24 fermionic coordinates of the $\frac{Osp(6|4)}{U(3)\times SO(1,3)}$ supercoset. The 24 bosonic variables Z^x_A and $\bar{Z}^A_x$ which are twistor-like variables combining the 10 spacetime coordinates of AdS_4 and $\mathbb{CP}^3$ with pure spinors $(\lambda^x_A, \bar{\lambda}^A_x)$ which the number of their degrees of freedom was calculated in [9, 10] to be 14.

The twistor-like variables can be expressed explicitly as follows

$$Z^x_A = H^x_{x'}(x_A)(\tilde{H}^{-1}(x_P))^{A'}_A \, \lambda^{x'}_{A'} \quad , \quad \bar{Z}^A_x = (H^{-1}(x_A))^{x'}_x \, \tilde{H}^A_{A'}(x_P) \, \bar{\lambda}^{A'}_{x'} \tag{3.1.18}$$

Here $H^x_{x'}(x_A)$ is a coset representative for the AdS_4 coset $\frac{Sp(4)}{SO(1,3)}$ and $\tilde{H}^A_{A'}(x_P)$ is a coset representative for the $\mathbb{CP}^3$ coset $\frac{SO(6)}{U(3)}$. Similarly, the conjugate twistor-like variables Y^A_J and $\bar{Y}^J_A$ are constructed from the conjugate momenta to the pure spinors and f^x_A and $\bar{f}^A_x$ are auxiliary fields.

3.1.1 From Kähler action to topological A-model action

Here we will see how we can get the A-model action from the Kähler action defined on the Grassmannian discussed before. To do so we have to expand the Kähler potential and integrate over the fermionic coordinated of the worldsheet supersymmetry.

The Kähler action (3.1.7) for the case of $AdS_5 \times S^5$ can be written as follows

$$S = \int d^2z \int d^4\kappa \mathrm{Tr}\left[\log\left(1 + \bar{\Theta}\Theta\right)\right] \tag{3.1.19}$$

We can expand chiral Θ and antichiral $\bar{\Theta}$ superfields in terms of their components as follows [18] as follows

$$\Theta^{\alpha^+}(z,\bar{z},\kappa_+,\kappa_-) = \theta^{\alpha^+}(y) + \kappa_+ Z^{\alpha^+}(y) + \kappa_- \bar{Y}^{\alpha^+}(y) + \kappa_+\kappa_- f^{\alpha^+}(y) \tag{3.1.20}$$

$$\bar{\Theta}^{\alpha^-}(z,\bar{z},\bar{\kappa}_+,\bar{\kappa}_-) = \bar{\theta}^{\alpha^-}(\bar{y}) + \bar{\kappa}_+ \bar{Z}^{\alpha^-}(\bar{y}) + \bar{\kappa}_- Y^{\alpha^-}(\bar{y}) + \bar{\kappa}_+\bar{\kappa}_- \bar{f}^{\alpha^-}(\bar{y}) \tag{3.1.21}$$

where their components depend on coordinates $(y, \bar{y})$, instead of the usual worldsheet coordinates $(z, \bar{z}, \kappa, \bar{\kappa})$, defined as follows

$$y = \bar{z} + i\kappa\sigma\bar{\kappa} \qquad , \qquad \bar{y} = z - i\kappa\bar{\sigma}\bar{\kappa} \tag{3.1.22}$$

where 2×2 matrices σ and $\bar{\sigma}$ are defined as

$$\sigma = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \bar{\sigma} = \sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.1.23}$$

and σ_1 and σ_2 are the following Pauli matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.1.24}$$

This implies

$$y = \bar{z} + \kappa_- \bar{\kappa}_+ \qquad , \qquad \bar{y} = z - \kappa_+ \bar{\kappa}_- \tag{3.1.25}$$

Using this new set of coordinate makes it possible to expand around z and $\bar{z}$ and integrate over the fermionic worldsheet supersymmetric coordinates. Defining

$$\Psi^{\alpha^+} \equiv \Theta^{\alpha^+}(z, \bar{z}, \kappa_+, \kappa_-) - \theta^{\alpha^+}(y) \ , \ \bar{\Psi}^{\alpha^-} \equiv \bar{\Theta}^{\alpha^-}(z, \bar{z}, \bar{\kappa}_+, \bar{\kappa}_-) - \bar{\theta}^{\alpha^-}(\bar{y}) \tag{3.1.26}$$

which are linear in κ and $\bar{\kappa}$, implies that

$$\Psi^{\alpha^+}\Psi^{\beta^+}\Psi^{\gamma^+} = 0 \quad , \quad \bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}\bar{\Psi}^{\gamma^-} = 0 \tag{3.1.27}$$

So at most two Ψ and two $\bar{\Psi}$ can appear in any expansion around z and $\bar{z}$.

We can expand these superfields as a function of y and $\bar{y}$ around z and $\bar{z}$ as follows

$$\begin{aligned}
\Psi^{\alpha^+}(z, \bar{z}, \kappa, \bar{\kappa}) &= \kappa_+ Z^{\alpha^+}(z, \bar{z}) + \kappa_- \bar{Y}^{\alpha^+}(z, \bar{z}) + \kappa_+\kappa_- f^{\alpha^+}(z, \bar{z}) \qquad (3.1.28)\\
&+ \kappa_+\kappa_-\bar{\kappa}_+ \bar{\partial} Z^{\alpha^+}(z, \bar{z}) + \kappa_+\bar{\kappa}_- \bar{\partial}\theta^{\alpha^+}(z, \bar{z})\\
\bar{\Psi}^{\alpha^-}(z, \bar{z}, \kappa, \bar{\kappa}) &= \bar{\kappa}_+ \bar{Z}^{\alpha^-}(z, \bar{z}) + \bar{\kappa}_- Y^{\alpha^-}(z, \bar{z}) + \bar{\kappa}_+\bar{\kappa}_- \bar{f}^{\alpha^-}(z, \bar{z}) \qquad (3.1.29)\\
&+ \bar{\kappa}_-\kappa_+\bar{\kappa}_+ \partial \bar{Z}^{\alpha^-}(z, \bar{z}) + \kappa_-\bar{\kappa}_+ \partial\bar{\theta}^{\alpha^-}(z, \bar{z})
\end{aligned}$$

Using (3.1.27), a general Kähler potential $K(\Theta, \bar{\Theta})$ can be Taylor expanded as follows

$$\begin{aligned}
K(\bar{\Theta}, \Theta) = {} & K(\bar{\theta}, \theta) + K_{\alpha^+}\Psi^{\alpha^+} + K_{\alpha^-}\bar{\Psi}^{\alpha^-} + \frac{1}{2}K_{\alpha^-,\beta^-}\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-} + \frac{1}{2}K_{\alpha^+,\beta^+}\Psi^{\alpha^+}\Psi^{\beta^+}\\
& + K_{\alpha^+,\beta^-}\Psi^{\alpha^+}\bar{\Psi}^{\beta^-} + \frac{1}{2}K_{\alpha^+,\beta^+,\alpha^-}\Psi^{\alpha^+}\Psi^{\beta^+}\bar{\Psi}^{\alpha^-} + \frac{1}{2}K_{\alpha^-,\beta^-,\alpha^+}\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}\Psi^{\alpha^+}\\
& + \frac{1}{4}K_{\alpha^+,\beta^+,\alpha^-,\beta^-}\Psi^{\alpha^+}\Psi^{\beta^+}\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}
\end{aligned} \tag{3.1.30}$$

where we used the following definitions

$$K_{\alpha^+} = \frac{\partial}{\partial\theta^{\alpha^+}}K(\bar{\theta}, \theta) \ , \ K_{\alpha^-} = \frac{\partial}{\partial\bar{\theta}^{\alpha^-}}K(\bar{\theta}, \theta) \ , \ K_{\alpha^+,\alpha^-} = \frac{\partial}{\partial\theta^{\alpha^+}\partial\bar{\theta}^{\alpha^-}}K(\bar{\theta}, \theta) \tag{3.1.31}$$

and also similar other definitions as derivatives of the Kähler potential for $K_{\alpha^+\beta^+}$, $K_{\alpha^-\beta^-}$ and $K_{\alpha^+\beta^-\alpha^+}$ and so on.

In order to integrate over the fermionic coordinates κ and $\bar{\kappa}$, we have to expand the Kähler potential (3.1.19) using (3.1.28), (3.1.29) and (3.1.30) and keep only the terms which has a factor $\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-$. The $\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-$ components of (3.1.30) can be obtained using the following identities

$$
\begin{aligned}
\Psi^{\alpha^+}\Psi^{\beta^+}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& 0 \\
\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& 0 \\
\Psi^{\alpha^+}\bar{\Psi}^{\beta^-}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& Y^{\beta^-}\bar{\partial}Z^{\alpha^+} + \bar{Y}^{\alpha^+}\partial\bar{Z}^{\beta^-} + f^{\alpha^+}\bar{f}^{\beta^-} + \bar{\partial}\theta^{\alpha^+}\partial\bar{\theta}^{\beta^-} \\
\Psi^{\alpha^+}\Psi^{\beta^+}\bar{\Psi}^{\alpha^-}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& Z^{\alpha^+}\bar{Y}^{\beta^+}\bar{f}^{\alpha^-} + \bar{Y}^{\alpha^+}\bar{\partial}\theta^{\beta^+}\bar{Z}^{\alpha^-} \\
\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}\Psi^{\alpha^+}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& \bar{Z}^{\alpha^-}Y^{\beta^-}f^{\alpha^+} + Y^{\alpha^-}\partial\bar{\theta}^{\beta^-}Z^{\alpha^+} \\
\Psi^{\alpha^+}\Psi^{\beta^+}\bar{\Psi}^{\alpha^-}\bar{\Psi}^{\beta^-}|_{\kappa_+\kappa_-\bar{\kappa}_+\bar{\kappa}_-} =& Z^{\alpha^+}\bar{Y}^{\beta^+}\bar{Z}^{\alpha^-}Y^{\beta^-}
\end{aligned}
\tag{3.1.32}
$$

Putting (3.1.32) into the Kähler potential (3.1.30) we get the following sigma-model action

$$
\begin{aligned}
S =& \int d^2z\Big[K_{\alpha^+,\beta^-}\bar{\partial}\theta^{\alpha^+}\partial\bar{\theta}^{\beta^-} + \frac{1}{4}K_{\alpha^+,\beta^+,\alpha^-,\beta^-}Z^{\alpha^+}\bar{Y}^{\beta^+}\bar{Z}^{\alpha^-}Y^{\beta^-} \\
+& K_{\alpha^+,\beta^-}(Y^{\beta^-}\bar{\partial}Z^{\alpha^+} + \bar{Y}^{\alpha^+}\partial\bar{Z}^{\beta^-} + f^{\alpha^+}\bar{f}^{\beta^-}) \\
+& \frac{1}{2}K_{\alpha^-,\beta^-,\alpha^+}(\bar{Z}^{\alpha^-}Y^{\beta^-}f^{\alpha^+} + Y^{\alpha^-}\partial\bar{\theta}^{\beta^-}Z^{\alpha^+}) \\
+& \frac{1}{2}K_{\alpha^+,\beta^+,\alpha^-}(Z^{\alpha^+}\bar{Y}^{\beta^+}\bar{f}^{\alpha^-} + \bar{Y}^{\alpha^+}\bar{\partial}\theta^{\beta^+}\bar{Z}^{\alpha^-})\Big]
\end{aligned}
\tag{3.1.33}
$$

The equations of motion for auxiliary fields f^{α^+} and $\bar{f}^{\alpha^-}$ can be written as follows

$$
f^{\alpha^+} = -\frac{1}{2}K^{\alpha^+,\alpha^-}K_{\delta^+,\beta^+,\alpha^-}Z^{\delta^+}\bar{Y}^{\beta^+} \quad , \quad \bar{f}^{\alpha^-} = \frac{1}{2}K^{\alpha^+,\alpha^-}K_{\delta^-,\beta^-,\alpha^+}\bar{Z}^{\delta^-}Y^{\beta^-} \tag{3.1.34}
$$

where K^{α^+,α^-} is the inverse of K_{α^+,α^-}.

Putting (3.1.34) in (3.1.33) we get the following action

$$
\begin{aligned}
S =& \int d^2z K_{\alpha^+,\beta^-}\Big[\bar{\partial}\theta^{\alpha^+}\partial\bar{\theta}^{\beta^-} + \bar{Y}^{\alpha^+}\nabla\bar{Z}^{\beta^-} + Y^{\beta^-}\bar{\nabla}Z^{\alpha^+} \\
+& \frac{1}{4}K^{\alpha^+,\beta^-}K_{\delta^+,\beta^+,\alpha^-,\delta^-}Z^{\delta^+}\bar{Y}^{\beta^+}\bar{Z}^{\alpha^-}Y^{\delta^-}\Big]
\end{aligned}
\tag{3.1.35}
$$

where the covariant derivatives are defined as follows

$$
\nabla = \partial + \frac{1}{2}\eta^{\delta^+\beta^-}K^{\alpha^+,\delta^-}K_{\delta^+,\beta^+,\delta^-}\bar{\partial}\theta^{\beta^+} \tag{3.1.36}
$$

$$
\bar{\nabla} = \bar{\partial} + \frac{1}{2}\eta^{\alpha^+\alpha^-}K^{\delta^+,\beta^-}K_{\alpha^-,\delta^-,\delta^+}\partial\bar{\theta}^{\delta^-} \tag{3.1.37}
$$

For a general Kähler potential K, the second derivative

$$
K_{\alpha^+,\alpha^-} = \frac{\partial}{\partial\theta^{\alpha^+}\partial\bar{\theta}^{\alpha^-}}K(\bar{\theta},\theta) \tag{3.1.38}
$$

is the Kähler metric of the manifold whose coordinates are θ and $\bar{\theta}$. The metric is invariant under the Kähler transformation

$$K(\theta,\bar{\theta}) \to K(\theta,\bar{\theta}) + g(\theta) + \bar{g}(\bar{\theta}) \tag{3.1.39}$$

The coefficient of the quartic term $Z\bar{Z}Y\bar{Y}$ in the action is actually the curvature of the corresponding Kähler manifold with metric (3.1.38).

The explicit expression for derivatives of the Kähler potential can be obtained after using $K = \mathrm{Tr}\log(1+\bar{\theta}\theta)$, for example

$$K_{\alpha^+,\alpha^-} = [(1+\bar{\theta}\theta)^{-1}]_{\alpha^+} \otimes [(1+\theta\bar{\theta})^{-1}]_{\alpha^-} \tag{3.1.40}$$

Also we can write the curvature term.

The next step is to relate the A-model action (3.1.35) with the one we wrote before as a sigma model on the Grassmannian $\frac{PSU(2,2|4)}{SU(2,2)\times SU(4)}$. We have to relate the Kähler metric and curvature to the supercoset geometry.

The left-invariant one-forms $J = g^{-1}dg$ for $g \in PSU(2,2|4)$ can be expanded into the generators of the $PSU(2,2|4)$ supergroup as follows

$$J = J^M T_M + J^{\alpha^+} T_{\alpha^+} + J^{\alpha^-} T_{\alpha^-} \tag{3.1.41}$$

where $(T_M, T_{\alpha^+}, T_{\alpha^-})$ are generators of $PSU(2,2|4)$ supergroup. The bosonic generators T_M for $M = ([ab], a)$ with $a = 0$ to 9 are the diagonal block generators corresponding to $SU(4)\times SU(2,2)$ group and the fermionic generators T_{α^+} and T_{α^-} are related to the usual T_α and T_{α^-} generators as follows

$$T_{\alpha^+} \equiv T_\alpha + iT_{\hat{\alpha}} \quad , \quad T_{\alpha^-} \equiv T_\alpha - iT_{\hat{\alpha}} \tag{3.1.42}$$

In $(4+4)\times(4+4)$ matrix representation of $PSU(2,2|4)$ a supergroup element can be represented as follows

$$G = \begin{pmatrix} A_{4\times 4} & X_{4\times 4} \\ Y_{4\times 4} & B_{4\times 4} \end{pmatrix} \tag{3.1.43}$$

the generators $T_{\alpha+}$, $T_{\alpha-}$ and T_M correspond to the upper-right, lower-left and the block-diagonal matrices respectively. Hence, we have the following algebra for $PSU(2,2|4)$

$$\begin{aligned} [T_M, T_{\alpha^+}] &= f^{\beta^+}_{M\alpha^+} T_{\beta^+} \quad , \quad [T_M, T_{\alpha^-}] = f^{\beta^-}_{M\alpha^-} T_{\beta^-} \\ \{T_{\alpha^+}, T_{\beta^-}\} &= f^{M}_{\alpha^+\beta^-} T_M \quad , \quad [T_M, T_N] = f^{P}_{MN} T_P \end{aligned} \tag{3.1.44}$$

The non-zero components of the metric are the symmetric and antisymmetric tensors η_{MN} and $\eta_{\alpha^+\beta^-}$ respectively.

An element of the Grassmannian $\frac{PSU(2,2|4)}{SU(2,2)\times SU(4)}$ instead can be represented as follows

$$g(\theta) = \exp\begin{pmatrix} 0 & \zeta \\ \bar{\zeta} & 0 \end{pmatrix} \tag{3.1.45}$$

where each block is a 4×4 matrix. This can be written as follows

$$g(\theta) = \begin{pmatrix} (1+\theta\bar{\theta})^{\frac{1}{2}} & \theta \\ \bar{\theta} & (1+\bar{\theta}\theta)^{\frac{1}{2}} \end{pmatrix} \tag{3.1.46}$$

where

$$\theta = \zeta \frac{\sinh\sqrt{\bar{\zeta}\zeta}}{\bar{\zeta}\zeta} \quad , \quad \bar{\theta} = \bar{\zeta}\frac{\sinh\sqrt{\bar{\zeta}\zeta}}{\bar{\zeta}\zeta} \tag{3.1.47}$$

and θ and $\bar{\theta}$ are the fermionic coordinates of the target superspace. The inverse of the group element is obtained as follows

$$g^{-1}(\theta) = g(-\theta) = \begin{pmatrix} (1+\theta\bar{\theta})^{\frac{1}{2}} & -\theta \\ -\bar{\theta} & (1+\bar{\theta}\theta)^{\frac{1}{2}} \end{pmatrix} \tag{3.1.48}$$

Using (3.1.46) and (3.1.48), we can write the left-invariant one-forms as follows

$$J^M = f^M_{\alpha^+\alpha^-}\bar{\theta}^{\alpha^-}\bar{\partial}\theta^{\alpha^+} \tag{3.1.49}$$

$$\bar{J}^M = -f^M_{\alpha^+\alpha^-}\theta^{\alpha^+}\partial\bar{\theta}^{\alpha^-} \tag{3.1.50}$$

$$J^{\alpha^+} = f^{\alpha^+\alpha^-}_M[(1+\theta\bar{\theta})^{\frac{1}{2}}]^M\partial\bar{\theta}^{\alpha^-} \tag{3.1.51}$$

$$\bar{J}^{\alpha^-} = f^{\alpha^-\alpha^+}_M[(1+\bar{\theta}\theta)^{\frac{1}{2}}]^M\bar{\partial}\theta^{\alpha^+} \tag{3.1.52}$$

One can check that the kinetic term of (3.1.35) can be written in terms of Maurer-Cartan one-forms as follows

$$K_{\alpha^+,\alpha^-}\bar{\partial}\theta^{\alpha^+}\partial\bar{\theta}^{\beta^-} = \eta_{\alpha^+\alpha^-}J^{\alpha^+}\bar{J}^{\alpha^-} \tag{3.1.53}$$

Also we can check that

$$J^M = \frac{1}{2}\eta^{\alpha^+\alpha^-}K^{\delta^+,\beta^-}K_{\alpha^-,\delta^-,\delta^+}\partial\bar{\theta}^{\delta^-} \tag{3.1.54}$$

$$\bar{J}^M = \frac{1}{2}\eta^{\delta^+\beta^-}K^{\alpha^+,\delta^-}K_{\delta^+,\beta^+,\delta^-}\bar{\partial}\theta^{\beta^+} \tag{3.1.55}$$

putting all these in the action we can reproduce the following A-model action

$$S = \int d^2z\left[\eta_{\alpha\hat{\beta}}J^\alpha\bar{J}^{\hat{\beta}} - \eta_{\alpha^-\alpha^+}Y^{\alpha^-}\bar{\nabla}Z^{\alpha^+} + \eta_{\alpha^-\alpha^+}\hat{Y}^{\alpha^+}\nabla\hat{Z}^{\alpha^-} - [Y,Z]_m[\hat{Y},\hat{Z}]^m\right] \tag{3.1.56}$$

where the covariant derivatives are defined as follows

$$\nabla = \partial + J \tag{3.1.57}$$

$$\bar{\nabla} = \bar{\partial} + \bar{J} \tag{3.1.58}$$

This is the A-model action which is capturing the zero radius limit of $AdS_5 \times S^5$ superstring.

3.2 Pure spinor superstring on supercosets with $\mathbb{Z}_4$ automorphism and the "bonus" symmetry

Consider a supercoset G/H which admits a $\mathbb{Z}_4$ automorphism under which its generators can be decomposed into invariant subspaces $\mathcal{H}_i, i = 0 \cdots 3$. The matter fields of the sigma model can be written in terms of the left-invariant currents $J = g^{-1}\partial g$, $\bar{J} = g^{-1}\bar{\partial} g$, where $g \in G$. The left-invariant currents are decomposed according to the invariant subspaces of the $\mathbb{Z}_4$ into $J = J_0 + J_1 + J_2 + J_3$ as follows

$$\begin{array}{cccc} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\ J^{[AB]} & J^{\alpha} & J^{M} & J^{\hat{\alpha}} \end{array} \tag{3.2.1}$$

where the left-invariant current $J = g^{-1}\partial g$ is expanded by the generators of the superalgebra as

$$J = \sum_{i=0}^{3} J_i = J^{[AB]}T_{[AB]} + J^m T_m + J^{\alpha} T_{\alpha} + J^{\hat{\alpha}} T_{\hat{\alpha}}\,, \tag{3.2.2}$$

here, $J^{[AB]} \in H$ are the spin connections of the supercoset and J^m and $(J^{\alpha}, J^{\hat{\alpha}})$ are the bosonic and fermionic components of the supervielbein respectively. The generators of the supercoset are $(T_{[AB]}, T_m, T_{\alpha}, T_{\hat{\alpha}})$ which are the Lorentz generators, translations and fermionic generators respectively with the following non-zero structure constants

$$f_{mn}{}^{[AB]},\quad f_{[AB][CD]}{}^{[EF]},\quad f_{\alpha\hat{\beta}}{}^{[AB]},\quad f_{\alpha\beta}{}^{m}\,, \tag{3.2.3}$$

The sigma model is invariant under the global transformations $\delta g = \Sigma g$, $\Sigma \in \mathcal{G}$ and under the BRST transformations, using the fact that $\langle AB \rangle \neq 0$ only for $A \in \mathcal{H}_i$ and $B \in \mathcal{H}_{4-i}$. It can be written in the following form

$$S = R^2 \int d^2 z \left\langle \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 + \frac{3}{4} J_3 \bar{J}_1 + w\bar{\partial}\lambda + \hat{w}\partial\hat{\lambda} + N\bar{J}_0 + \hat{N} J_0 - N\hat{N} \right\rangle\,, \tag{3.2.4}$$

for any supercoset admitting a $\mathbb{Z}_4$ automorphism including $AdS_5 \times S^5$ and $\tilde{AdS}_4 \times \mathbb{CP}^3$ examples as discussed also before (see also [59, 55] for non-critical examples based on different sets of pure spinor variables).

On top of the global bosonic isometry group G_b of the supergroup G, the A-model action has a 'bonus' chiral symmetry exchanging left and right movers which appears in the sigma model as a symmetry between left and right moving fermions J^{α} and $J^{\hat{\alpha}}$. Apparently (3.2.4) does not have such a symmetry because of the different coefficients of $J_1\bar{J}_3$ and $J_3\bar{J}_1$ terms. To promote the symmetry of (3.2.4), one can add an additional term to the action including a $-\frac{1}{2}J_3\bar{J}_1$ to cancel the asymmetry of the fermionic currents together with its appropriate companion in order that the whole term stays a BRST-closed term

$$\begin{aligned} S_{trivial} =& S_m + S_g \\ =& \frac{R^2}{2}\int d^2z\left(C_{mn}J^m\bar{J}^n - \langle J_3\bar{J}_1\rangle + \langle\omega\bar{\nabla}\lambda + \hat{\omega}\nabla\hat{\lambda} - N\hat{N}\rangle\right) \\ =& \frac{R^2}{2}\int d^2z\left(C_{mn}J^m\bar{J}^n + \eta_{\alpha\hat{\beta}}J^{\hat{\beta}}\bar{J}^\alpha + \omega_\alpha\bar{\nabla}\lambda^\alpha + \hat{\omega}_{\hat{\alpha}}\nabla\hat{\lambda}^{\hat{\alpha}} - \eta_{[AB][CD]}N^{[AB]}\hat{N}^{[CD]}\right) \end{aligned} \tag{3.2.5}$$

where $S_g = \frac{R^2}{2}\int d^2z(\omega\bar{\nabla}\lambda + \hat{\omega}\nabla\hat{\lambda} - N\hat{N})$ is exactly the ghost part of the original action (3.2.4) and $\eta_{XY} = \langle T_X, T_Y\rangle = Str(T_XT_Y)$. The requirement of BRST invariance of the $S_{trivial}$ will determine the unknown coefficients C_{mn}.

Using the classical equations of motion

$$\nabla\hat{\lambda} - [N,\hat{\lambda}] = 0, \quad \bar{\nabla}\lambda - [\hat{N},\lambda] = 0\,, \tag{3.2.6}$$

and the identities $[N,\lambda] = [\hat{N},\hat{\lambda}] = 0$ coming from the pure spinor constraints, it can be shown that under the BRST transformations, S_g and S_m vary as follows

$$\begin{aligned} \delta_B(S_g) =& \frac{R^2}{2}\int d^2z\langle -J_3\bar{\partial}\lambda - \bar{J}_1\partial\hat{\lambda} - J_3[\bar{J}_0,\lambda] - \bar{J}_1[J_0,\hat{\lambda}]\rangle \\ =& \frac{R^2}{2}\int d^2z\;\eta_{\alpha\hat{\beta}}(-J^{\hat{\beta}}\bar{\nabla}\lambda^\alpha + \bar{J}^\alpha\nabla\hat{\lambda}^{\hat{\beta}}) \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} \delta_B(S_m) =& \frac{R^2}{2}\int d^2z\left[C_{mn}\left(J^\alpha\lambda^\beta f_{\alpha\beta}{}^m + J^{\hat{\alpha}}\hat{\lambda}^{\hat{\beta}}f_{\hat{\alpha}\hat{\beta}}{}^m\right)\bar{J}^n + C_{mn}J^m\left(\bar{J}^\alpha\lambda^\beta f_{\alpha\beta}{}^n + \bar{J}^{\hat{\alpha}}\hat{\lambda}^{\hat{\beta}}f_{\hat{\alpha}\hat{\beta}}{}^n\right)\right. \\ & \left. -\,\eta_{\alpha\hat{\beta}}\left(\nabla\hat{\lambda}^{\hat{\beta}} + J^m\lambda^\beta f_{m\beta}{}^{\hat{\beta}}\right)\bar{J}^\alpha + \eta_{\alpha\hat{\beta}}J^{\hat{\beta}}\left(\bar{\nabla}\lambda^\alpha + \bar{J}^n\hat{\lambda}^{\hat{\alpha}}f_{n\hat{\alpha}}{}^\alpha\right)\right] \end{aligned} \tag{3.2.8}$$

which gives

$$\begin{aligned} \frac{1}{R^2}\delta_B(S_{trivial}) &= \frac{1}{2}C_{mn}J^m\bar{J}^\alpha\lambda^\beta f_{\alpha\beta}{}^n + \frac{1}{2}\eta_{\alpha\hat{\beta}}J^m\bar{J}^\alpha\lambda^\beta f_{m\beta}{}^{\hat{\beta}}\,, \\ &+ \frac{1}{2}C_{mn}\bar{J}^nJ^{\hat{\beta}}\hat{\lambda}^{\hat{\alpha}}f_{\hat{\beta}\hat{\alpha}}{}^m + \frac{1}{2}\eta_{\alpha\hat{\beta}}\bar{J}^nJ^{\hat{\beta}}\hat{\lambda}^{\hat{\alpha}}f_{n\hat{\alpha}}{}^\alpha \\ &+ \frac{1}{2}C_{mn}\bar{J}^nJ^\alpha\lambda^\beta f_{\alpha\beta}{}^m + \frac{1}{2}C_{mn}J^m\bar{J}^{\hat{\alpha}}\hat{\lambda}^{\hat{\beta}}f_{\hat{\alpha}\hat{\beta}}{}^n \\ &= 0\,, \end{aligned} \tag{3.2.9}$$

which admits the following solution for $\delta_B(S_{trivial}) = 0$ after using the Jacobi identities for the structural constants

$$C_{mn} = \frac{1}{2}\frac{\eta_{\alpha\hat{\beta}}(\hat{\lambda}^{\hat{\alpha}}f_{n\hat{\alpha}}{}^\alpha)(\lambda^\beta f_{m\beta}{}^{\hat{\beta}})}{\eta_{\alpha\hat{\beta}}\lambda^\alpha\hat{\lambda}^{\hat{\beta}}}\,. \tag{3.2.10}$$

The first and the second lines of (3.2.9) vanish because of the identity $\eta_{\beta\hat{\alpha}} = Str\,(T_\beta T_{\hat{\alpha}}) = f_{\alpha\beta}^{\ n} f_{n\hat{\alpha}}^{\ \alpha}$ and the terms in the last line vanish because of the following Jacobi identity,

$$f_{\alpha\gamma}^{\ m} f_{m\beta}^{\ \hat{\beta}} + f_{\alpha\beta}^{\ m} f_{m\gamma}^{\ \hat{\beta}} = f_{\beta\gamma}^{\ m} f_{m\alpha}^{\ \hat{\beta}}\,, \tag{3.2.11}$$

which implies

$$\lambda^\beta \lambda^\gamma \left(f_{\alpha\gamma}^{\ m} f_{m\beta}^{\ \hat{\beta}} + f_{\alpha\beta}^{\ m} f_{m\gamma}^{\ \hat{\beta}} \right) = 0\,. \tag{3.2.12}$$

So $S_{trivial}$ of (3.2.5) with C_{mn} given in (3.2.10) is BRST-closed. We should also show that it is really a BRST-trivial term satisfying $S_{trivial} = Q\bar{Q}X$, up to the equations of motion. In order to do that, we introduce the antifields w^*_α and $\hat{w}^*_{\hat{\alpha}}$ which after adding the term

$$R^2 \int d^2 z \eta^{\alpha\hat{\beta}} {w^*}_\alpha \hat{w}^*_{\hat{\beta}}\,, \tag{3.2.13}$$

the full action stay invariant under the new BRST transformations,

$$\begin{aligned}
Q' w_\alpha &= -\eta_{\alpha\hat{\alpha}} J^{\hat{\alpha}}, & \bar{Q}' w_\alpha &= w^*_\alpha\,, \\
Q' \hat{w}_{\hat{\alpha}} &= \hat{w}^*_{\hat{\alpha}}, & \bar{Q}' \hat{w}_{\hat{\alpha}} &= -\eta_{\hat{\alpha}\alpha} \bar{J}^\alpha\,, \\
Q' w^*_\alpha &= \eta_{\alpha\hat{\alpha}} (\nabla \hat{\lambda}^{\hat{\alpha}} - [N, \hat{\lambda}]^{\hat{\alpha}}), & \bar{Q}' w^*_\alpha &= 0\,, \\
Q' \hat{w}^*_{\hat{\alpha}} &= 0, & \bar{Q}' \hat{w}^*_{\hat{\alpha}} &= \eta_{\hat{\alpha}\alpha} (\bar{\nabla} \lambda^\alpha - [\hat{N}, \lambda]^\alpha)\,, \\
Q' N &= [J_3, \lambda], & \bar{Q}' N &= [w^*, \lambda]\,, \\
Q' \hat{N} &= [\bar{J}_1, \hat{\lambda}], & \bar{Q}' \hat{N} &= [\hat{w}^*, \hat{\lambda}]\,,
\end{aligned} \tag{3.2.14}$$

These BRST transformations are nilpotent off-shell.

Now, considering the following identities we can make a BRST-close term which its BRST variation produces $S_{trivial}$.

$$\begin{aligned}
Q'\bar{Q}' \left(C_{mn} J^m \bar{J}^n \right) &= C_{mn} \left\{ Q'\bar{Q}'(J^m) \bar{J}^n + Q'(J^m) \bar{Q}'(\bar{J}^n) \right\} \\
&+ C_{mn} \left\{ \bar{Q}'(J^m) Q'(\bar{J}^n) + J^m Q'\bar{Q}'(\bar{J}^n) \right\} \\
&= C_{mn} \left\{ \nabla \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} f_{\hat{\alpha}\hat{\beta}}{}^m \bar{J}^n + J^m \bar{\nabla} \lambda^\alpha \lambda^\beta f_{\alpha\beta}{}^n \right\} \\
&+ C_{mn} \left\{ J^p \bar{J}^n \lambda^\alpha \hat{\lambda}^{\hat{\beta}} f_{p\alpha}{}^{\hat{\alpha}} f_{\hat{\alpha}\hat{\beta}}{}^m + J^m \bar{J}^p \lambda^\alpha \hat{\lambda}^{\hat{\beta}} f_{p\alpha}{}^{\hat{\alpha}} f_{\hat{\alpha}\hat{\beta}}{}^n \right\} \\
&+ C_{mn} \left\{ J^\alpha \bar{J}^{\hat{\alpha}} \lambda^\beta \hat{\lambda}^{\hat{\beta}} f_{\alpha\beta}{}^m f_{\hat{\alpha}\hat{\beta}}{}^n + J^{\hat{\alpha}} \bar{J}^\alpha \hat{\lambda}^{\hat{\beta}} \lambda^\beta f_{\hat{\alpha}\hat{\beta}}{}^m f_{\alpha\beta}{}^n \right\} \\
&= 2 C_{mn} J^m \bar{J}^n (\eta \lambda \hat{\lambda})\,,
\end{aligned} \tag{3.2.15}$$

also we have

$$\begin{aligned} Q'\bar{Q}'\left(N\hat{N}\right) &= Q'\bar{Q}'(N)\hat{N} + Q'(N)\bar{Q}'(\hat{N}) + \bar{Q}'(N)Q'(\hat{N}) + NQ'\bar{Q}'(\hat{N}) \\ &= [(\nabla\hat{\lambda} - [N,\hat{\lambda}]),\lambda]\hat{N} + [J_3,\lambda][\bar{J}_1,\hat{\lambda}] \\ &+ [w^*,\lambda][\hat{w}^*,\hat{\lambda}] + N[(\bar{\nabla}\lambda - [\hat{N},\lambda]),\hat{\lambda}] \end{aligned} \tag{3.2.16}$$

and,

$$\begin{aligned} Q'\bar{Q}'\left((\omega\lambda)(\hat{\omega}\hat{\lambda})\right) &= Q'\bar{Q}'(\omega\lambda)(\hat{\omega}\hat{\lambda}) + Q'(\omega\lambda)\bar{Q}'(\hat{\omega}\hat{\lambda}) \\ &+ \bar{Q}'(\omega\lambda)Q'(\hat{\omega}\hat{\lambda}) + (\omega\lambda)Q'\bar{Q}'(\hat{\omega}\hat{\lambda}) \\ &= \frac{1}{2}[(\nabla\hat{\lambda} - [N,\hat{\lambda}]),\lambda](\hat{w}\hat{\lambda}) + \frac{1}{4}[J_3,\lambda][\bar{J}_1,\hat{\lambda}] + \frac{1}{4}[w^*,\lambda][\hat{w}^*,\hat{\lambda}] \\ &+ \frac{1}{2}(w\lambda)[(\bar{\nabla}\lambda - [\hat{N},\lambda]),\hat{\lambda}] \end{aligned} \tag{3.2.17}$$

to get these identities, we used the equation of motions, (3.2.9), (3.2.11) and (3.2.12) together with the following Jacobi identity

$$f_{M\underline{\alpha}}{}^{\underline{\beta}} f_{N\underline{\beta}}{}^{\underline{\gamma}} - f_{N\underline{\alpha}}{}^{\underline{\beta}} f_{M\underline{\beta}}{}^{\underline{\gamma}} = f_{MN}{}^{P} f_{P\underline{\alpha}}{}^{\underline{\beta}} \tag{3.2.18}$$

where $M, N, \cdots = \{m, [mn]\}$ and $\underline{\alpha}, \underline{\beta}, \cdots = \{\alpha, \hat{\alpha}\}$. From (3.2.15), (3.2.16) and (3.2.17) one can see that there exists a linear combination of them such that $S_{trivial} = Q\bar{Q}X$ up to the anti-ghost term, that is up to the momenta equations of motion

$$X = \frac{1}{2}\int d^2z \frac{1}{\eta_{\alpha\hat{\alpha}}\lambda^\alpha\hat{\lambda}^{\hat{\alpha}}}\left[\frac{1}{4}C_{mn}J^m\bar{J}^n + \frac{1}{4}(\omega\lambda)(\hat{\omega}\hat{\lambda}) - \frac{1}{8}N\hat{N}\right] \tag{3.2.19}$$

The sigma model action after adding $S_{trivial}$ to the pure spinor action becomes

$$\begin{aligned} S_b &= \frac{R^2}{2}\int d^2z \Big[\left(\frac{1}{2}\frac{\eta_{\alpha\hat{\beta}}(\hat{\lambda}^{\hat{\alpha}} f_{n\hat{\alpha}}{}^{\alpha})(\lambda^\beta f_{m\beta}{}^{\hat{\beta}})}{\eta_{\alpha\hat{\beta}}\lambda^\alpha\hat{\lambda}^{\hat{\beta}}} + \eta_{mn}\right) J^m\bar{J}^n \\ &+ \frac{1}{2}\langle J_3\bar{J}_1 - J_1\bar{J}_3 + \omega\bar{\nabla}\lambda + \hat{\omega}\nabla\hat{\lambda} - N\hat{N}\rangle\Big] \end{aligned} \tag{3.2.20}$$

The analysis follows the considerations in the literature, but it is derived in a very general way.

3.3 Mapping pure spinor superstring action to topological A-model action

In order to relate S_b and the A-model action, we should relate the supercoset element $g(x,\theta,\bar\theta)\in\frac{G}{H}$ with the Grassmannian coset element $G(\theta,\bar\theta)\in\frac{G}{G_b}$.

We can define the following bosonic twisted variables out of the bosonic coset elements $H(x)\in\frac{G_b}{H}$ and the pure spinors as follows

$$\begin{aligned} Z^\alpha &= [H,\lambda] = H^{[AB]}(x)\lambda^\beta f_{[AB]\beta}{}^\alpha \\ \bar Z^{\hat\alpha} &= [H^{-1},\hat\lambda] = (H^{-1})^{[AB]}(x)\hat\lambda^{\hat\beta} f_{[AB]\hat\beta}{}^{\hat\alpha} \\ Y^{\hat\alpha} &= [H^{-1},w] = (H^{-1})^{[AB]}(x)\eta^{\beta\hat\beta}w_\beta f_{[AB]\hat\beta}{}^{\hat\alpha} \\ \bar Y^\alpha &= [H,\hat w] = H^{[AB]}(x)\eta^{\beta\hat\beta}\hat w_{\hat\beta} f_{[AB]\beta}{}^\alpha \end{aligned} \tag{3.3.1}$$

The supercoset element g can be parametrized as follows

$$g(x,\theta,\bar\theta) = G(\theta,\bar\theta)H(x) \tag{3.3.2}$$

where $G(\theta,\bar\theta)=e^{\theta^\alpha T_\alpha+\bar\theta^{\hat\alpha}T_{\hat\alpha}}$ and $H(x)=e^{x^m T_m}$ in which $(T_m,T_\alpha,T_{\hat\alpha})$ are the generators of the supercoset G/H.

According to (3.3.2), we can also decompose the left-invariant currents $J=g^{-1}\partial g$. The pure spinor action can be written into H and G components, corresponding to the purely bosonic part and purely fermionic part of the supercoset as follows

$$J = H^{-1}\partial H + H^{-1}(G^{-1}\partial G)H \tag{3.3.3}$$

Its components $J=J^m T_m + J^{[AB]}T_{[AB]} + J^\alpha T_\alpha + J^{\hat\alpha}T_{\hat\alpha}$ can be written as

$$J^M = (H^{-1}\partial H)^M + (H^{-1})^M(G^{-1}\partial G)^P H^Q f_{NP}{}^R f_{RQ}{}^M \tag{3.3.4}$$

$$J^{\underline{\alpha}} = (H^{-1})^M(G^{-1}\partial G)^{\underline{\beta}} H^N f_{M\underline{\beta}}{}^{\underline{\gamma}} f_{\underline{\gamma}N}{}^{\underline{\alpha}} \tag{3.3.5}$$

where $M,N,\cdots=\{m,[AB]\}$ and $\underline{\alpha},\underline{\beta},\cdots=\{\alpha,\hat\alpha\}$.

The A-model action can be written in terms of the fermionic superfields $(\Theta^\alpha,\bar\Theta^{\hat\alpha})$ which was defined before as $S=\int \mathrm{Tr}ln[1+\bar\Theta\Theta]$. Here we assume that for the Grassmannian supercoset G/G_b, there exist a gauging in which the supercoset elements G can be written in the following form

$$G^m = 1\!\!1,\quad G^{[AB]} = 1\!\!1,\quad G^\alpha = \theta^\alpha,\quad G^{\hat\alpha} = \bar\theta^{\hat\alpha} \tag{3.3.6}$$

Finally, the A-model action, after integration over the auxiliary fields can be written in this form

$$\begin{aligned} S_A &= t\int d^2z\Big[(G^{-1}\partial G)(G^{-1}\bar\partial G) + Y\bar\nabla Z + \bar Y\nabla\bar Z - (YZ)(\bar Z\bar Y)\Big] \\ &= t\int d^2z[\eta_{\alpha\hat\alpha}(G^{-1}\partial G)^{\alpha}(G^{-1}\bar\partial G)^{\hat\alpha} + \eta_{MN}(G^{-1}\partial G)^M(G^{-1}\bar\partial G)^N \\ &+ \eta_{\alpha\hat\alpha}Y^{\hat\alpha}(\bar\nabla Z)^{\alpha} + \eta_{\alpha\hat\alpha}\bar Y^{\alpha}(\nabla\bar Z)^{\hat\alpha} \\ &- \eta_{mn}f_{\alpha\hat\alpha}{}^{m}f_{\beta\hat\beta}{}^{n}\Big[(Y^{\hat\alpha}Z^{\alpha})(\bar Z^{\hat\beta}\bar Y^{\beta}) + (Z^{\alpha}Y^{\hat\alpha})(\bar Y^{\beta}\bar Z^{\hat\beta})\Big]] \end{aligned} \tag{3.3.7}$$

where,

$$\begin{aligned} (\bar\nabla Z)^{\alpha} &= \bar\partial Z + [G^{-1}\bar\partial G, Z] \\ &= \bar\partial Z^{\alpha} + (G^{-1}\bar\partial G)^{[AB]}Z^{\beta}f_{[AB]\beta}{}^{\alpha} \\ (\nabla\bar Z)^{\hat\alpha} &= \partial\bar Z + [G^{-1}\partial G, \bar Z] \\ &= \partial\bar Z^{\hat\alpha} + (G^{-1}\partial G)^{[AB]}Z^{\hat\beta}f_{[AB]\hat\beta}{}^{\hat\alpha} \end{aligned} \tag{3.3.8}$$

To relate the pure spinor action (3.2.20) and the A-model action (6.2.17), we use the explicit form of the twisted variables (3.3.1).

Using (3.3.1) and Jacobi identity (3.2.18), one can write

$$\begin{aligned} Y\bar\partial Z &= [H^{-1}, w]\bar\partial\left([H,\lambda]\right) \\ &= [H^{-1}, w]\left([\bar\partial H,\lambda] + [H,\bar\partial\lambda]\right) \\ &= w\bar\partial\lambda + [H^{-1}\bar\partial H, w\lambda] \\ &= w\bar\partial\lambda + [H^{-1}\bar\partial H, w\lambda] + [H^{-1}(G^{-1}\bar\partial G)H, w\lambda] - [H^{-1}(G^{-1}\bar\partial G)H, w\lambda] \\ &= w\bar\partial\lambda + [\bar J, w\lambda] - [(G^{-1}\bar\partial G), YZ] \end{aligned} \tag{3.3.9}$$

which after using (3.3.8), we get

$$\begin{aligned} Y\bar\nabla Z &= w\bar\partial\lambda + [\bar J, w\lambda] \\ &= w_{\alpha}\bar\partial\lambda^{\alpha} + \bar J^{[AB]}w_{\alpha}\lambda^{\beta}f_{[AB]\beta}{}^{\alpha} + \eta_{mn}\eta^{\alpha\beta}\bar J^{m}w_{\alpha}\lambda^{\gamma}f_{\gamma\beta}{}^{n} \\ &= w_{\alpha}\bar\nabla\lambda^{\alpha} + \eta_{mn}\eta^{\alpha\beta}\bar J^{m}w_{\alpha}\lambda^{\gamma}f_{\gamma\beta}{}^{n} \end{aligned} \tag{3.3.10}$$

similarly, one can see that

$$\begin{aligned} \bar Y\nabla\bar Z &= \hat w\partial\hat\lambda + [J, \hat w\hat\lambda] \\ &= \hat w_{\hat\alpha}\partial\hat\lambda^{\hat\alpha} + J^{[AB]}\hat w_{\hat\alpha}\hat\lambda^{\hat\beta}f_{[AB]\hat\beta}{}^{\hat\alpha} + \eta_{mn}\eta^{\hat\alpha\hat\beta}J^{m}\hat w_{\hat\alpha}\hat\lambda^{\hat\gamma}f_{\hat\gamma\hat\beta}{}^{n} \\ &= \hat w_{\hat\alpha}\nabla\hat\lambda^{\hat\alpha} + \eta_{mn}\eta^{\hat\alpha\hat\beta}J^{m}\hat w_{\hat\alpha}\hat\lambda^{\hat\gamma}f_{\hat\gamma\hat\beta}{}^{n} \end{aligned} \tag{3.3.11}$$

The last term simplifies as follows

$$
\begin{aligned}
(YZ)(\bar{Z}\bar{Y}) &= \left([H^{-1},w][H,\lambda]\right)\left([H^{-1},\hat{\lambda}][H,\hat{w}]\right) \\
&= (w\lambda)(\hat{w}\hat{\lambda}) \\
&= \eta^{[AB][CD]}\left(f_{\alpha[AB]}{}^{\beta}w_{\beta}\lambda^{\alpha}\right)\left(\hat{w}_{\hat{\beta}}\hat{\lambda}^{\hat{\alpha}}f_{\hat{\alpha}[CD]}{}^{\hat{\beta}}\right) \\
&- \eta_{mn}\left(\eta^{\alpha\gamma}f_{\alpha\beta}{}^{m}w_{\gamma}\lambda^{\beta}\right)\left(\eta^{\hat{\alpha}\hat{\gamma}}f_{\hat{\alpha}\hat{\beta}}{}^{n}\hat{w}_{\hat{\gamma}}\hat{\lambda}^{\hat{\beta}}\right)
\end{aligned} \tag{3.3.12}
$$

Putting everything together, we obtain the A-model action in terms of the pure spinor fields as follows

$$
\begin{aligned}
S_A &= t\int d^2z\Big[\frac{1}{2}\eta_{\alpha\hat{\beta}}(J^{\hat{\beta}}\bar{J}^{\alpha}-J^{\alpha}\bar{J}^{\hat{\beta}})+w\bar{\nabla}\lambda+\hat{w}\nabla\hat{\lambda}-N\hat{N} \\
&\qquad +\eta^{\alpha\beta}\bar{J}^{m}w_{\alpha}\lambda^{\gamma}f_{m\gamma}{}^{\alpha}+\eta^{\hat{\alpha}\hat{\beta}}J^{m}\hat{w}_{\hat{\alpha}}\hat{\lambda}^{\hat{\gamma}}f_{m\hat{\gamma}}{}^{\hat{\alpha}} \\
&- \eta_{mn}\left(\eta^{\alpha\gamma}f_{\alpha\beta}{}^{m}w_{\gamma}\lambda^{\beta}\right)\left(\eta^{\hat{\alpha}\hat{\gamma}}f_{\hat{\alpha}\hat{\beta}}{}^{n}\hat{w}_{\hat{\gamma}}\hat{\lambda}^{\hat{\beta}}\right)\Big]
\end{aligned} \tag{3.3.13}
$$

The equations of motion for w and $\hat{w}$ comes from the variation of the action under the transformations $\delta w_{\alpha}=f_{\alpha\beta}{}^{m}\lambda^{\beta}\Lambda_m$ and $\delta\hat{w}_{\hat{\alpha}}=f_{\hat{\alpha}\hat{\beta}}{}^{m}\hat{\lambda}^{\hat{\beta}}\tilde{\Lambda}_m$ can be written as follows

$$
(f_{m\alpha}{}^{\hat{\delta}}\lambda^{\alpha})\left(\bar{J}^{m}-\eta^{\hat{\beta}\hat{\gamma}}f_{\hat{\beta}\hat{\alpha}}{}^{m}\hat{w}_{\hat{\gamma}}\hat{\lambda}^{\hat{\alpha}}\right) = 0 \tag{3.3.14}
$$

$$
(f_{m\hat{\alpha}}{}^{\delta}\hat{\lambda}^{\hat{\alpha}})\left(J^{m}-\eta^{\beta\gamma}f_{\beta\alpha}{}^{m}w_{\gamma}\lambda^{\alpha}\right) = 0 \tag{3.3.15}
$$

After inserting these equations (3.3.13), the second line of (3.3.13) produces the following kinetic term for the bosonic Maurer-Cartan currents,

$$
t\int d^2z\Big[\frac{1}{2}\frac{\eta_{\alpha\hat{\beta}}(\hat{\lambda}^{\hat{\alpha}}f_{n\hat{\alpha}}{}^{\alpha})(\lambda^{\beta}f_{m\beta}{}^{\hat{\beta}})}{\eta_{\alpha\hat{\beta}}\lambda^{\alpha}\hat{\lambda}^{\hat{\beta}}}+\eta_{mn}\Big]J^{n}\bar{J}^{m} \tag{3.3.16}
$$

Then the action (3.3.13), becomes

$$
\begin{aligned}
S_A &= t\int d^2z\Big[\left(\frac{1}{2}\frac{\eta_{\alpha\hat{\beta}}(\hat{\lambda}^{\hat{\alpha}}f_{n\hat{\alpha}}{}^{\alpha})(\lambda^{\beta}f_{m\beta}{}^{\hat{\beta}})}{\eta_{\alpha\hat{\beta}}\lambda^{\alpha}\hat{\lambda}^{\hat{\beta}}}+\eta_{mn}\right)J^{n}\bar{J}^{m}+\frac{1}{2}\eta_{\alpha\hat{\beta}}(J^{\hat{\beta}}\bar{J}^{\alpha}-J^{\alpha}\bar{J}^{\hat{\beta}}) \\
&\qquad +w\bar{\nabla}\lambda+\hat{w}\nabla\hat{\lambda}-N\hat{N}\Big]
\end{aligned} \tag{3.3.17}
$$

which coincides with the action (3.2.20) after identifying $t=\frac{1}{2}R^2$.

We conclude that the pure spinor superstring action on supercoset backgrounds which admit a particular $\mathbb{Z}_4$ automorphism can be decomposed into a topological A-model action and a BRST trivial term as follows

$$
S_{\text{pure spinor}} = S_{A-model}+Q\bar{Q}X \tag{3.3.18}
$$

Since the supersymmetric charge of the topological A-model theory is different from the BRST charge of the pure spinor formalism, the cohomologies of the two theories are not coincide[2] but the topological theory captures at least the BPS sector of the superstring theory. In principle, one can study the BPS sector of the superstring theory using the topological A-model theory. The A-model topological theory was conjectured to describe the superstring theory at zero AdS radius which is the dual of the free gauge theory. We will see that we can extend this picture by adding some vertex operators to the topological theory which turn on the radius modulus in the string theory side, corresponding to turning on a nonzero coupling constant in the gauge theory side. This picture will be used to study AdS/CFT duality from a worldsheet point of view.

[2] Actually as we will see later the two theories can be seen as different gauge fixings of the same theory, so the cohomologies seems to be related non trivially.

3.4 On conformal exactness of superstring backgrounds

Using the A-model action defined on the fermionic Grassmannian, we can compute the one loop conformal anomaly of the A-model as its Ricci scalar which is given as follows

$$R = \log\det(\partial_\Theta \partial_{\bar{\Theta}} K) \tag{3.4.1}$$

where

$$K(\Theta, \bar{\Theta}) = \mathrm{tr}\log(1 + \bar{\Theta}\Theta) \tag{3.4.2}$$

Writing the superfields with their matrix indices as Θ^{rj} and $\bar{\Theta}_{jr}$ where (r, s) and (i, j) are different kind of indices corresponding to upper and lower diagonal blocks of the supergroup element. Then we can write

$$\begin{aligned} \partial_{\Theta^{ks}} \partial_{\bar{\Theta}_{rj}} K &= \partial_{\Theta^{ks}} \left[\Theta^{rl} [(1 + \bar{\Theta}\Theta)^{-1}]_l^j \right] \\ &= \delta_s^r [(1 + \bar{\Theta}\Theta)^{-1}]_k^j - \Theta^{rl} [(1 + \bar{\Theta}\Theta)^{-1}]_l^m \bar{\Theta}_{ms} [(1 + \bar{\Theta}\Theta)^{-1}]_k^j \end{aligned} \tag{3.4.3}$$

but we have

$$\delta_s^r - \Theta^{rl} [(1 + \bar{\Theta}\Theta)^{-1}]_l^m \bar{\Theta}_{ms} = [(1 + \Theta\bar{\Theta})^{-1}]_s^r \tag{3.4.4}$$

which implies

$$\partial_{\Theta^{ks}} \partial_{\bar{\Theta}_{rj}} K = [(1 + \Theta\bar{\Theta})^{-1}]_s^r [(1 + \bar{\Theta}\Theta)^{-1}]_k^j \tag{3.4.5}$$

This a tensor product of two matrices with different kind of indices. We can compute the Ricci scalar using the fact that

$$\mathrm{tr}\log(M) = \log\det(M) \tag{3.4.6}$$

So we have

$$\begin{aligned} R &= \log\det(\partial_\Theta \partial_{\bar{\Theta}} K) \\ &= \log\det[(1 + \Theta\bar{\Theta})^{-1}] + \log\det[(1 + \bar{\Theta}\Theta)^{-1}] \\ &= -\mathrm{tr}\log(1 + \Theta\bar{\Theta}) - \mathrm{tr}\log(1 + \bar{\Theta}\Theta) \end{aligned} \tag{3.4.7}$$

We can write the Taylor expansion for the logarithm as follows

$$\log(1 + T) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} T^n \tag{3.4.8}$$

Then (3.4.7) can be written as follows

$$R = -\mathrm{tr}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left((\Theta\bar{\Theta})^n + (\bar{\Theta}\Theta)^n\right)\right] = 0 \tag{3.4.9}$$

where in the last step we used the fact that

$$\mathrm{T}r(\Theta\bar{\Theta})^n = -\mathrm{T}r(\bar{\Theta}\Theta)^n, \ \text{ for } n > 0\,, \tag{3.4.10}$$

since Θ and $\bar{\Theta}$ are fermionic.

So the A-model is conformal at one-loop. Being a $N = 2$ supersymmetric model in two dimensions, ensures the cancellation of the all-loop conformal anomaly since the conformal anomaly and the ghost anomaly belong to the same multiplet. The pure spinor superstring action and the A-model action are related through a BRST-exact term then the $N = 2$ supersymmetry non-renormalization theorem implies its all-loop conformal invariance in the A-model term, this conformal exactness in the A-model also implies the conformal exactness in the superstring theory.

Chapter 4

An exact check of AdS/CFT duality using the topological A-model

4.1 Gauged linear sigma-model for the superstring action

AS we saw in the introduction, to prove the open/closed duality for the $d = 3$ Chern-Simons/resolved conifold duality, we used the fact that sigma model could be written as a gauged linear sigma model in which the Kähler modulus t of the A-model becomes the Fayet-Illiopoulos parameter. This is useful when we want to study the limit $t \to 0$ limit in which the nonlinear A-model becomes unable to explore the physics. On the other hand the gauged linear sigma model for the resolved conifold can develop both a Coulomb phase and a Higgs phase, and the Coulomb phase was interpreted as D-brane holes which corresponds to loops in the Feynman diagrams of the Chern-Simons gauge theory.

In this section, we suggest that a similar technique which might be useful to give a worldsheet derivation of the Maldacena's conjecture as a duality between $\mathcal{N} = 4$ d=4 super-Yang-Mils and the $AdS_5 \times S^5$ sigma model. We will write the A-model action of $AdS_5 \times S^5$ studied in the previous chapter as a gauged linear sigma model with a $U(4)$ worldsheet gauge field. Then, we will argue that in the limit where $t \to 0$, a Coulomb phase develops which can be interpreted as D-brane holes. Furthermore, it will be argued that these D-brane holes are associated with gauge-invariant $\mathcal{N} = 4$ d=4 super-Yang-Mills operators. In particular an exact check in the AdS/CFT will be discussed using this construction.

4.1.1 Gauged linear sigma-model of $AdS_5 \times S^5$ background

The worldsheet variables are fermionic superfields Θ_J^A and $\bar{\Theta}_A^J$ where $A = 1$ to 4 and $J = 1$ to 4 label fundamental representations of $SU(2,2)$ and $SU(4)$ respectively. These $N = 2$ chiral superfields can be expanded in components as

$$\begin{aligned} \Theta_J^A(\kappa_+, \kappa_-) &= \theta_J^A + \kappa_+ Z_J^A + \kappa_- \bar{Y}_J^A + \kappa_+ \kappa_- f_J^A \\ \bar{\Theta}_A^J(\bar{\kappa}_+, \bar{\kappa}_-) &= \bar{\theta}_A^J + \bar{\kappa}_+ \bar{Z}_A^J + \bar{\kappa}_- Y_A^J + \bar{\kappa}_+ \bar{\kappa}_- \bar{f}_A^J \end{aligned} \tag{4.1.1}$$

where $(\kappa_+, \bar{\kappa}_+)$ are left-moving and $(\kappa_-, \bar{\kappa}_-)$ are right-moving Grassmannian parameters.

The 32 lowest components θ_J^A and $\bar{\theta}_A^J$ are related to the 32 fermionic coordinates of the $\frac{PSU(2,2|4)}{SU(2,2)\times U(4)}$ supercoset which parametrizes the $AdS_5 \times S^5$ superspace. The 32 bosonic variables Z_J^A and $\bar{Z}_A^J$ are twistor-like variables combining the 10 spacetime coordinates of AdS_5 and S^5 with 11 pure spinors $(\lambda_J^A, \bar{\lambda}_A^J)$ of the pure spinor formalism. They can be expressed explicitly as follows

$$\begin{aligned} Z_J^A &= H_{A'}^A(x)(\tilde{H}^{-1}(\tilde{x}))_J^{J'} \lambda_{J'}^{A'} \\ \bar{Z}_A^J &= (H^{-1}(x))_A^{A'} \tilde{H}_{J'}^J(\tilde{x}) \bar{\lambda}_{A'}^{J'} \end{aligned} \tag{4.1.2}$$

where the pure spinors are written in $SO(4,1) \times SO(5)$ notation. Here $H_{A'}^A$ is a coset representative for the AdS_5 coset $\frac{SU(2,2)}{SO(4,1)}$ where $A' = 1$ to 4 is an $SO(4,1)$ spinor index and $\tilde{H}_{J'}^J(\tilde{x})$ is a coset representative for the S^5 coset $\frac{SU(4)}{SO(5)}$ where $J' = 1$ to 4 is an $SO(5)$ spinor index. Similarly, the conjugate twistor-like variables Y_J^A and $\bar{Y}_A^J$ are constructed from the conjugate momenta to the pure spinors and f_J^A and $\bar{f}_A^J$ are auxiliary fields.

As discussed before, the $U(2,2|4)$ invariant action for the topological A-model can be written in the $N = (2,2)$ superfield notation as follows

$$S = t \int d^2 z \int d^4 \kappa Tr \left[\log(\delta_K^J + \bar{\Theta}_A^J \Theta_K^A) \right] \tag{4.1.3}$$

where t is a constant parameter proportional to the σ-model coupling $R_{AdS_5}^2/\alpha'$.

This A-model is based on a Grassmannian coset $\frac{U(2,2|4)}{U(2,2)\times U(4)}$ and as it was shown before, a nonlinear σ-model action based on a Grassmannian can be obtained as the Higgs phase of an appropriate gauged linear σ-model.

This is obtained by introducing a $U(4)$ worldsheet gauge field V_S^R, together with an appropriate set of matter fields transforming in the fundamental representation of the gauge group

$$\Phi_R^\Sigma(z, \bar{z}, \kappa^+, \kappa^-), \quad \bar{\Phi}_\Sigma^R(z, \bar{z}, \bar{\kappa}^+, \bar{\kappa}^-) \tag{4.1.4}$$

where $R, S = 1$ to 4 are local gauge $U(4)$ indices, and $\Sigma = (A, J)$ is referred to the global A and J indices for $U(2,2)$ and $U(4)$ respectively. Note that Φ^A_R is a fermionic superfield whereas Φ^J_R is a bosonic superfield. The gauged linear sigma model can be written in $U(2,2|4)$, $N = (2,2)$ and gauge invariant notation as

$$S = \int d^2z \int d^4\kappa[\bar{\Phi}^S_\Sigma \, (e^V)^R_S \Phi^\Sigma_R - tTrV] \tag{4.1.5}$$

where t enters as the Fayet-Illiopoulos parameter. When t is nonzero, one can show using the equations of motion that the action (4.1.5) is equivalent to the $A-$model action (4.1.3) with the following parametrization for the chiral and antichiral superfields Θ^A_J and $\bar{\Theta}^J_A$ as follows

$$\Theta^A_J \equiv \Phi^A_R(\Phi^J_R)^{-1}, \quad \bar{\Theta}^J_A \equiv \bar{\Phi}^R_A(\bar{\Phi}^R_J)^{-1} \tag{4.1.6}$$

As it will be shown later, in the small t regime, the above gauged linear σ-model is equivalent by applying an observation at the end of [25], to the geometric quotient $\left(\hat{\mathbb{CP}}^{(3|4)}\right)^4 //S_4$. we will concentrate on the twisted sector corresponding to the cyclic permutation. This is equivalent to a single copy of the twistorial space $\hat{\mathbb{CP}}^{(3|4)}$.

4.1.2 Gauged linear sigma-model of $A\tilde{d}S_4 \times \mathbb{CP}^3$ background

The nonlinear A-model action of this background was studied in the previous chapter. Similar to $AdS_5 \times S^5$ background, we can write a gauged linear sigma model corresponding to this background. The two-dimensional N=(2,2) linear gauged sigma model can be described by a set of matter fields which are chiral and antichiral superfields Φ^Σ_R and $\bar{\Phi}^R_\Sigma$ gauged under the real worldsheet superfield V^R_S taking value in the $SO(6)$ gauge group where $R, S, ... = 1, \ldots, 6$ are gauge field indices and $\Sigma = (x, A)$ is a global $Osp(6|4)$ index. We can take Φ^x_R to be fermionic while Φ^A_R are bosonic superfields.

The gauged linear sigma model action can be written in a $Osp(6|4)$ invariant way as

$$S = \int d^2z \int d^4\kappa \left[\bar{\Phi}^S_\Sigma(e^V)^R_S\Phi^\Sigma_R + t\,\mathrm{Tr}V + \frac{1}{e^2}\Sigma^2\right] \tag{4.1.7}$$

where $\Sigma = \bar{D}DV$ is the field strength of the gauge field V and is a twisted chiral superfield.

As it is clear from the matter content of the theory, it contains 24 fermions and 36 bosons and so the theory actually has conformal anomaly if we ask the bosons and fermions to be gauged in the same representation of the gauge group as we did. But still the theory has a conformal IR fixed point corresponding to the large volume and gauge coupling limit which after integrating out the auxiliary equations of motion for the gauge field we obtain the non-linear sigma model (when $e \to \infty$)

$$S = t \int d^2z \int d^4\kappa \mathrm{Tr} \left[\bar{\Phi}^R_\Sigma \Phi^\Sigma_S\right] \tag{4.1.8}$$

which can be rewritten in terms of the meson fields Θ^x_A and $\bar{\Theta}^A_x$ defined as

$$\Theta^x_A \equiv \Phi^x_R (\Phi^{-1})^R_A, \qquad \bar{\Theta}^A_x \equiv (\bar{\Phi}^{-1})^A_R \bar{\Phi}^R_x \tag{4.1.9}$$

which gives exactly the A-model sigma model which was obtained from the pure spinor string for $\tilde{AdS}_4 \times \mathbb{CP}^3$ as

$$S = t \int d^2z \int d^4\kappa \mathrm{Tr} \ln \left[1 + \bar{\Theta}\Theta\right] \tag{4.1.10}$$

The FI parameter corresponds to the Kähler parameter of the supercoset Grassmannian target space $\frac{Osp(6|4)}{SO(6)\times Sp(4)}$.

4.2 Vacua of the gauged linear sigma-model

The small radius limit of the gauged linear sigma-model is convenient to study the perturbative regime of the gauge theory since the introduction of the Coulomb branch. The presence of the gauge group which is an additional degree of freedom in the gauged linear sigma model with respect to non-linear sigma model, resolves the singularity of the non-linear sigma-model in the small radius limit. To study different phases of the theory, we should solve the D-term equations of the gauged linear sigma-model. It is enough to focus on the fields which have conformal weight zero because they are the only fields which can get non-zero expectation value. We analyze the gauged linear σ-model following the standard techniques of [18] and [60].

The gauge superfield V_S^R in Wess-Zumino gauge can be expanded as follows

$$V_S^R = \sigma_S^R \kappa_+ \bar{\kappa}_- + \bar{\sigma}_S^R \kappa_+ \bar{\kappa}_+ + \ldots + \kappa_+ \kappa_- \bar{\kappa}_+ \bar{\kappa}_- D_S^R \tag{4.2.1}$$

Similarly we can expand the fermionic and bosonic superfields as follows

$$\Phi_R^\Sigma = \phi_R^\Sigma + \kappa_+ \psi_R^\Sigma + \ldots, \qquad \bar{\Phi}_\Sigma^R = \bar{\phi}_\Sigma^R + \bar{\kappa}_- \bar{\psi}_\Sigma^R + \ldots \tag{4.2.2}$$

where we just kept the components which have zero conformal weight after the A-twist because they are the only fields which can attain nonzero expectation value and so can appear in the D-term equations. Here the index Σ refers to both x and A indices. Note that $(\phi_R^A, \psi_R^x, \bar{\phi}_A^R, \bar{\psi}_x^R)$ are bosonic and $(\phi_R^x, \psi_R^A, \bar{\phi}_x^R, \bar{\psi}_A^R)$ are fermionic fields.

Using the vector superfield and the usual superderivatives $D_\pm$ and $\bar{D}_\pm$, one can define the covariant superderivatives as follows

$$\mathcal{D}_\pm = e^{-V} D_\pm e^{+V}, \quad \bar{\mathcal{D}}_\pm = e^{+V} \bar{D}_\pm e^{-V} \tag{4.2.3}$$

Then the field strength Σ which is a twisted chiral superfield is constructed as follows

$$\begin{aligned} \Sigma &= \{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \\ &= \sigma + \ldots + \kappa_+ \kappa_- \bar{\kappa}_+ \bar{\kappa}_- (D^m D_m \sigma + [\sigma, [\sigma, \bar{\sigma}]] + i[\partial^m v_m, \sigma]) \end{aligned} \tag{4.2.4}$$

which produces the following gauge field kinetic term in the Lagrangian

$$\begin{aligned} L_{gauge} &= -\frac{1}{e^2} \int d^4\kappa \mathrm{Tr} \bar{\Sigma} \Sigma \\ &= \frac{1}{e^2} \mathrm{Tr} \left(-D_i \bar{\sigma} D^i \sigma - \frac{1}{2} [\sigma, \bar{\sigma}]^2 + \ldots \right) \end{aligned} \tag{4.2.5}$$

Also we have the following FI term $L_{D,\theta}$

$$\begin{aligned} L_{D,\theta} &= it\int d\kappa_+ d\bar{\kappa}_- \mathrm{Tr}\Sigma\Big|_{\kappa_-=\bar{\kappa}_+=0} - i\bar{t}\int d\kappa_- d\bar{\kappa}_+ \mathrm{Tr}\bar{\Sigma}\Big|_{\kappa_+=\bar{\kappa}_-=0} \\ &= \mathrm{Tr}\left(-rD + \frac{\theta}{2\pi}v_{01}\right) \end{aligned} \tag{4.2.6}$$

Now we can consider the matter part of the gauged linear sigma model consisting of the kinetic terms for the fermionic and bosonic superfields which carries the kinetic and interaction terms for the bosonic and fermionic fields as follows

$$\begin{aligned} L^b_{kin} &= \int d^4\kappa \bar{\Phi}^R_A e^V \Phi^A_R \\ &= -(\bar{D}_j\bar{\phi}^R_A)(D^j\phi^A_R) + \bar{F}^R_A F^A_R - \bar{\phi}^S_A\{\sigma,\bar{\sigma}\}^R_S\phi^A_R + \bar{\phi}^S_A D^R_S\phi^A_R + \ldots \end{aligned} \tag{4.2.7}$$

Similarly we can write the kinetic term for the fermionic chiral superfields,

$$\begin{aligned} L^f_{kin} &= \int d^4\kappa \bar{\Phi}^R_x e^V \Phi^x_R \\ &= -(\bar{D}_j\bar{\phi}^R_x)(D^j\phi^x_R) + \bar{F}^R_x F^x_R - \bar{\phi}^S_x\{\sigma,\bar{\sigma}\}^R_S\phi^x_R + \bar{\phi}^S_x D^R_S\phi^x_R + \ldots \end{aligned} \tag{4.2.8}$$

We can see that $\{\sigma,\bar{\sigma}\}$ appears as the mass for the matter fields and so whenever σ gets VEV, the matter fields become massive and can be integrated out in the effective theory as it happens in the Coulomb phase.

The potential of the theory can be written as,

$$\begin{aligned} L_V &= \frac{1}{2e^2}\mathrm{Tr}D^2 - r\mathrm{Tr}D + \bar{\phi}^S_x D^R_S\phi^x_R + \bar{\phi}^S_A D^R_S\phi^A_R \\ &- \frac{1}{2e^2}\mathrm{Tr}[\sigma,\bar{\sigma}]^2 - \bar{\phi}^S_x\{\sigma,\bar{\sigma}\}^R_S\phi^x_R - \bar{\phi}^S_A\{\sigma,\bar{\sigma}\}^R_S\phi^A_R \end{aligned} \tag{4.2.9}$$

After eliminating the D-field by using the following D-term equation

$$D^S_R = \bar{\phi}^S_x\phi^x_R + \bar{\phi}^S_A\phi^A_R - r\delta^S_R \tag{4.2.10}$$

one obtains the potential

$$\begin{aligned} V &= \frac{e^2}{2}\left[\bar{\phi}^S_x\phi^x_R + \bar{\phi}^S_A\phi^A_R - r\delta^S_R\right]\left[\bar{\phi}^R_x\phi^x_S + \bar{\phi}^R_A\phi^A_S - r\delta^R_S\right] \\ &+ \frac{1}{2e^2}\mathrm{Tr}[\sigma,\bar{\sigma}]^2 + \bar{\phi}^S_x\{\sigma,\bar{\sigma}\}^R_S\phi^x_R + \bar{\phi}^S_A\{\sigma,\bar{\sigma}\}^R_S\phi^A_R \end{aligned} \tag{4.2.11}$$

The space of the classical vacua is given by putting the potential to zero up to gauge transformations. We can study the vacua in two regimes, when $r > 0$ and not small, the constraint $V = 0$ implies that $\sigma = 0$ which implies the following condition as the classical vacua for the matter fields

The space of the classical vacua is given by putting the potential to zero up to gauge transformations. We can study the vacua in two regimes, when $r > 0$ and not small, the constraint $V = 0$ implies that $\sigma = 0$ which implies the following condition as the classical vacua for the matter fields

$$D_R^S = \bar{\phi}_x^S \phi_R^x + \bar{\phi}_A^S \phi_R^A - r\delta_R^S = 0 \tag{4.2.12}$$

For the $\frac{OSp(6|4)}{SO(6)\times Sp(4)}$ it means actually that the vectors (ϕ_R^x, ψ_R^A) for any $R = 1, ..., 4$ are orthonormal. Any such vector, after diagonalization, is subject to the constraint

$$\sum_{A=1}^{6} \bar{\phi}_A \phi^A + \sum_{x=1}^{4} \bar{\phi}_x \phi^x = r \tag{4.2.13}$$

which defines a supersphere $\mathbb{S}^{(5|4)}$.[1] The space of classical vacua is the gauge invariant subspace of the product of such vectors [25] giving the orbit space

$$(\mathbb{S}^{(5|4)})^3 // S_3 \times \mathbb{Z}_2 \tag{4.2.14}$$

obtained by dividing the action of $S_3 \times \mathbb{Z}_2$ on the three copies, where $\mathbb{Z}_2$ is the simultaneous reflection. This phase corresponds to the Higgs phase of the theory because the gauge symmetry completely breaks.

For the gauged linear sigma-model of $AdS_5 \times S^5$ one can rewrite the D-term equation as follows

$$\sum_{A=1}^{4} \bar{\phi}_A \phi^A + \sum_{J=1}^{4} \bar{\phi}_J \phi^J = r \tag{4.2.15}$$

where J and A are $SU(4)$ and $SU(2,2)$ indices respectively. This D-term equation defines a projective space $\mathbb{CP}^{(3|4)}$. The space of classical vacua is the gauge invariant subspace of the product of such vectors [10] divided by the S_4 permutations over the four copies. The Higgs phase is given by the following superspace

$$\left(\mathbb{CP}^{(3|4)}\right)^4 // S_4 \tag{4.2.16}$$

If one looks into $r \to 0$ limit, on top of the above Higgs phase, one can have another possibility as it is explained in [17] and [23]. In this phase, the σ_R^S is unconstrained but the matter variables are constrained to satisfy

$$\mathcal{O}_R^S = \bar{\phi}_x^S \phi_R^x + \bar{\phi}_A^S \phi_R^A = 0 \tag{4.2.17}$$

[1] The conditions for a supermanifold of being a super-Ricci flat are discussed in [61].

The mass term for the fermions and bosons are written as

$$\bar{\phi}_x^S\{\sigma,\bar{\sigma}\}_S^R\phi_R^x + \bar{\phi}_A^S\{\sigma,\bar{\sigma}\}_S^R\phi_R^A \tag{4.2.18}$$

And so whenever the σ gets expectation value the matter fields become massive and one can integrate them out from the theory. One can easily compute the 1-loop correction to the condition (4.2.17) which should be proportional to r by doing the path integral with a cut-off μ,

$$\begin{aligned} \langle\mathcal{O}\rangle_{\text{1-loop}} &= -\sum_{A=1}^{6}\int d^2p\frac{1}{p^2+\{\sigma,\bar{\sigma}\}} + \sum_{x=1}^{4}\int d^2p\frac{1}{p^2+\{\sigma,\bar{\sigma}\}} \\ &= -\frac{1}{2\pi}\log\left(\frac{\{\sigma,\bar{\sigma}\}}{2\mu^2}\right) = r \end{aligned} \tag{4.2.19}$$

which has a solution as

$$\{\sigma,\bar{\sigma}\} = 2\mu^2\exp\left(-2\pi r\right) \tag{4.2.20}$$

After integrating over all the matter fields, the classical vacua $V=0$ is given by condition $\mathrm{T}r[\sigma,\bar{\sigma}]^2=0$ which together with (4.2.20) gives the following solution,

$$\sigma = \sigma_0\mu\exp\left(-2\pi r\right) \tag{4.2.21}$$

where here σ_0 is an orthogonal 6×6 or 4×4 constant matrix for $OSp(6|4)$ and $PSU(2,2|4)$ supergroups respectively. This means that σ can be diagonalized and for each diagonal component of the σ in the small radius regime, one gets a copy of the $\mathbb{S}^{(5|4)}$ or $\mathbb{CP}^{(3|4)}$ in each case as it was seen before.

4.3 Open sector and D-branes

Let us now pass to the discussion of observables and D-branes which are made after putting consistent boundary conditions on the fields in our theory. In order to discuss open strings and D-branes we have to see how to put the boundary conditions.

4.3.1 Open sector of topological $AdS_5 \times S^5$

We take the boundary conditions for open strings in the coset σ-model as follows [2]

$$(\bar{\Theta}^t)^A_J = \epsilon^A_B \Theta^{*B}_K \delta^K_J \tag{4.3.1}$$

where [3] δ and ϵ are four by four constant matrices such that $\epsilon = a\epsilon^{-1}$ and $\delta = b\delta^{-1}$ with a and b complex numbers such that $ab = -1$.

In order to preserve the correct 1/2 supersymmetry, we chose

$$\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.3.2}$$

This breaks the $U(2,2|4)$ isometry to $OSp(4^*|4)$.

Notice that this remnant symmetry is exactly the same symmetry preserved by 1/2 BPS circular Wilson loops in $\mathcal{N} = 4$ SYM of Drukker and Gross [28].

These A-branes wrap the Lagrangian submanifolds of the target space, as

$$\frac{OSp(4^*|4)}{SO^*(4) \times USp(4)} \longrightarrow \frac{U(2,2|4)}{U(2,2) \times U(4)} \tag{4.3.3}$$

which is the fixed locus under the anti-involution

$$\bar{\Theta} \rightarrow \delta^t \Theta^\dagger \epsilon^t \quad \textit{and} \quad \Theta \rightarrow \epsilon^{*-1} \bar{\Theta}^\dagger \delta^{*-1} \tag{4.3.4}$$

which is explicitly a symmetry of the σ-model action since $\delta^{-1} = \delta^\dagger = -\delta$ and $\epsilon^{-1} = \epsilon^\dagger = \epsilon$ in our case. Recall that $SO^*(4) = SU(1,1) \times SU(2)$ and $USp(4) = SO(5)$ (see [62]).

[2] Note that these boundary condition are different from the ones which was used in [23] as $(\bar{\Theta}^t)^A_J = \epsilon^A_B \Theta^B_K \delta^K_J$. It can be shown that these two type of boundary conditions are producing different types of D-branes.

[3] We work in the conventions $\Theta^\dagger = i\bar{\Theta}$, $\bar{\Theta}^\dagger = i\Theta$ and $(\psi\zeta)^\dagger = -\zeta^\dagger\psi^\dagger$ for fermionic ψ and ζ.

In the gauged linear σ- model the boundary conditions (4.4.81) become

$$(\Phi^\dagger)^R_J \delta^{t}{}^J_I = \kappa^\dagger{}^R_S \bar{\Phi}^S_I \quad \textit{and} \quad (\Phi^\dagger)^R_A \epsilon^{t}{}^A_B = \kappa^\dagger{}^R_S \bar{\Phi}^S_B \tag{4.3.5}$$

which is the fixed point of the transformation

$$\Phi^I_R \rightarrow (\delta^\dagger)^I_J (\bar{\Phi}^\dagger)^J_S \kappa^S_R \quad \textit{and} \quad \Phi^A_R \rightarrow (\epsilon^\dagger)^A_B (\bar{\Phi}^\dagger)^B_S \kappa^S_R \tag{4.3.6}$$

while $(e^V) \rightarrow \kappa e^V \kappa^\dagger$ and κ is, because of the reality condition on the fields, a constant element in $O(4)$. This breaks the gauge symmetry to ones preserving κ, namely $\Lambda \in U(4)$ such that $\Lambda^t \kappa \Lambda = \kappa$.

4.3.2 Open sector of the topological $\tilde{AdS_4} \times \mathbb{CP}^3$

For the supercoset $Osp(6|4)/SO(6)\times Sp(4)$, we reduce it as follows: the bosonic subcoset: $SO(6)\times Sp(4)$ is reduced to $U(3)\times Sp(2)$ and the fermionic part is halved. This achieved by using the boundary conditions

$$\Theta^{\alpha I} = \delta^\alpha_{\dot\alpha} \, \mathcal{J}^I_J \, \bar{\Theta}^{\dot\alpha J} \, , \; \bar{\Theta}^{\dot\alpha}_I = \delta^{\dot\alpha}_\alpha \, \mathcal{J}^J_I \, \Theta^\alpha_J \, , \tag{4.3.7}$$

where $\mathcal{J}^I_j$ is the complex structure on $\mathbb{P}^3$. The tensor $\delta^{\dot\alpha}_\alpha$ reduce the subgroup $Sp(4)$ to $Sp(2)$. We recall that using the symplectic matrices Λ of $Sp(4,\mathbb{R})$ as the 4×4 matrices satisfying $\Lambda^T \epsilon \Lambda = \epsilon$ where $\epsilon = \mathrm{i}\,\sigma_2 \otimes 1\!\!1$, we can see immediately the two subgroups $Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})$. In the above equation, we have selected the diagonal subgroup $Sp(2,\mathbb{R})$. The above equations are invariant under $Sp(2,\mathbb{R}) \times U(3)$. Notice that we have identified on the boundary of the Riemann surface the fermionic variables of the subset $\mathcal{H}_1 = \{\Theta^{\alpha I}, \Theta^{\dot\alpha}_I\}$ with those of the other subset $\mathcal{H}_3 = \{\bar{\Theta}^\alpha_I, \bar{\Theta}^{\dot\alpha I}\}$. This simply reduces the 24 fermions to 12 ones. The new set of states can be represented in terms of the supercoset

$$\frac{SU(3|1,1)}{U(3)\times SU(1,1)} \tag{4.3.8}$$

(where we have used the isomorphism $Sp(2,\mathbb{R}) \simeq SL(2,\mathbb{R}) \simeq SU(1,1)$). The 6 fermions are in the $(3,2)$ or in the $(\bar{3},2)$ representation of the bosonic subgroup.

In addition, we have to recall $SL(2,\mathbb{R}) \simeq AdS_3$, which can be seen by parameterizing a group element of $SL(2,\mathbb{R})$ as follows

$$g = \begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix} \tag{4.3.9}$$

with the condition $\det g = X_{-1}^2 - X_1^2 + X_0^2 - X_2^2 = 1$ which shows that the $SL(2,\mathbb{R})$ group manifold is a 3-dimensional hyperboloid. The metric on AdS_3 is given by $ds^2 = -dX_{-1}^2 + dX_1^2 - dX_0^2 + dX_2^2$, which is the invariant metric on the group manifold. Then, we have that these boundary conditions imply a boundary theory of the type $N = 6$ super-YM/Chern-Simons on AdS_3 space.

There is another possibility which is given by the following boundary conditions

$$\Theta^{\alpha I} = \delta^{\alpha}_{\dot{\alpha}}\, \delta^{I}_{J}\, \bar{\Theta}^{\dot{\alpha} J}\,,\ \bar{\Theta}^{\dot{\alpha}}_{I} = \delta^{\dot{\alpha}}_{\alpha}\, \delta^{J}_{I}\, \Theta^{\alpha}_{J}\,, \tag{4.3.10}$$

In this case the supergroup $Osp(6|4)$ is broken to $Osp(6|2) \times SO(2)$. Notice that using the delta δ_I^J in place of $\mathcal{J}_I^J$ we do not break the $SO(6)$. In addition, the subgroup $Sp(4)$ is broken to $Sp(2) \times SO(2)$. Now, using the isomorphism $SU(4) \simeq SO(6)$, we can see the coset $SO(6) \times SO(2)/SU(3) \times U(1) \simeq \mathbb{S}^7/\mathbb{Z}_p$ where p defines how the $U(1)$ is embedded in the groups of the numerator. This observation would help us to lift the D-branes solution to KK monopoles of M-theory. The fermions are halved by the boundary conditions. So, the boundary open topological model can be described as the Grassmannian

$$\frac{Osp(6|2) \times SO(2)}{U(4) \times Sp(2)}\,. \tag{4.3.11}$$

This solution deserves more attention.

4.4 An exact check in $AdS_5 \times S^5/\mathcal{N} = 4, d = 4$ SYM duality

As we have seen, the $AdS_5 \times S^5$ string admit a formulation in the pure spinor framework [20, 56]. In particular we have shown that to calculate 1/2-BPS string amplitudes, one can use a topologically A-twisted version of the $\mathcal{N} = (2, 2)$ σ-model on the fermionic coset $U(2, 2|4)/U(2, 2) \times U(4)$ [21, 23, 37, 10]. This non-linear sigma-model can be obtained by an auxiliary gauged linear one which has been proposed as the correct framework to describe the string theory in the large curvature regime.

Here in this section we collect a set of arguments which lead to reproduce the known perturbative gauge theory results alluded above by making use of this topological decomposition proposal. Our line of reasoning goes as it was explained in the introduction chapter. Here we will do the steps more in detail and will show that there are some particular Wilson loops in the gauge theory side which can be computed as the amplitude of some D-branes in the string theory side.

4.4.1 Mirror symmetry, superconifold and matrix model

AS we showed before, the BPS sector of the superstring on $AdS_5 \times S^5$ can be studied by a topological A-model theory defined on four products of the superprojective space $\mathbb{CP}^{(3|4)}$. Here we consider a particular sector of the superstring theory which is captured by just one of these four copies. We start from the closed topological A-model theory on the super Calabi-Yau $\mathbb{CP}^{(3|4)}$, passing through a duality map which was explained in the introduction chapter, first we obtained its closed topological B-model theory by using mirror symmetry, then we use the geometric transition to go to the open B-model theory and at then end we will show that this open topological theory can be reduced to a Gaussian matrix model. We will show that there are some particular D-branes in this topological theory whose amplitudes computed as observables of the Gaussian matrix model produce exactly the result of their dual objects in the gauge theory side which are the circular Wilson loops whose expectation value was computed exactly and was shown to be given by a Gaussian matrix model. This will serve as an exact check on AdS/CFT duality.

4.4.1.1 Mirror symmetry

The first step is to use mirror symmetry to relate the A-model which we get from the superstring action to a B-model theory. This has been already calculated in [63] and further elaborated in [26] for the case at hand. The mirror symmetry is an equivalence between two topological $N = 2$ string theories which are defined on on different

Calabi-Yau manifolds as the proper target spaces for topological string theories. We will discuss in more detail about the structure of topological string theories mainly a class of the namely the A-model later but here we take as granted that we have two types of topological string theories named as A-model and B-model according to the way we do the topological twist which is explored in [64]. As we see from figure(4.1), mirror symmetry is a symmetry between A-model theories and B-model theories and also the proper boundary conditions of these two theories can be related by mirror symmetry too which are Lagrangian submanifolds for the A-model and holomorphic cycles for the B-model [64].

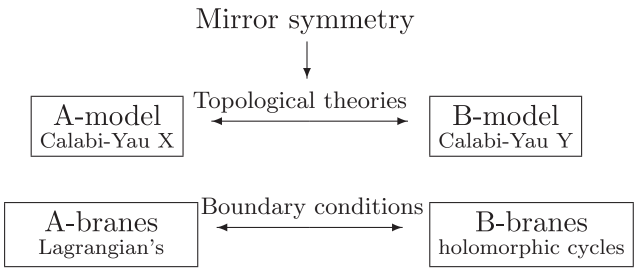

FIGURE 4.1: Mirror symmetry as a symmetry between topological theories and their boundary conditions.

The idea behind mirror symmetry is very similar to the T-duality, it is an equivalence between two ways of writing an effective action for a *mother theory*. Starting from the mother theory which is a worldsheet supersymmetric sigma-model, we can write an effective theory in two different ways which since they are all effective description of the same theory, they are equivalent theories in this sense.

In order to realize this T-duality let us consider the following mother Lagrangian as the starting point

$$\mathcal{L}_M = \int d^4\kappa \left(e^{2QV+B} - \frac{1}{2}(Y+\bar{Y})B \right) - \frac{1}{2}\int d^2\tilde{\kappa}\, t\Sigma \tag{4.4.1}$$

where $(\kappa^-, \kappa^+, \bar{\kappa}^-, \bar{\kappa}^+)$ are the $N=2$ superspace coordinates and V is a $N=2$ vector superfield which in the Wess-Zumino gauge can be expanded as follows

$$\begin{aligned} V &= \kappa^-\bar{\kappa}^-(v_0 - v_1) + \kappa^+\bar{\kappa}^+(v_0 + v_1) - \kappa^-\bar{\kappa}^+\sigma - \kappa^+\bar{\kappa}^-\bar{\sigma} \\ &+ i\sqrt{2}\kappa^-\kappa^+(\bar{\kappa}^-\bar{\lambda}_- + \bar{\kappa}^+\bar{\lambda}_+) + i\sqrt{2}\bar{\kappa}^+\bar{\kappa}^-(\theta^-\lambda_- + \theta^+\lambda_+) + 2\kappa^-\kappa^+\bar{\kappa}^+\bar{\kappa}^- D \end{aligned} \tag{4.4.2}$$

where D is the D-term which is an auxiliary field.

In (4.4.1), the field B is a real superfield and Y is a twisted chiral superfield and Σ is the twisted chiral field strength superfield. For the twisted chiral superfield Y and real chiral superfield B we have

$$\bar{D}_+ Y = D_- Y = 0, \quad , \quad \bar{D}_+ B = \bar{D}_- B = 0 \tag{4.4.3}$$

where $D_\pm$ and $\bar{D}_\pm$ are the $N = 2$ supersymmetry derivatives

$$D_\pm = \frac{\partial}{\partial \kappa^\pm} - i\bar{\kappa}^\pm \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right) \quad , \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\kappa}^\pm} + i\kappa^\pm \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right) \tag{4.4.4}$$

The superfields B and Y are expanded into their components as follows

$$B = b + \sqrt{2}\kappa^+ \psi_+ + \sqrt{2}\kappa^- \psi_- + 2\kappa^+ \kappa^- F + \cdots \tag{4.4.5}$$

$$Y = y + \sqrt{2}\kappa^+ \bar{\chi}_+ + \sqrt{2}\kappa^- \chi_- + 2\kappa^+ \bar{\kappa}_- G + \cdots \tag{4.4.6}$$

where F and G are auxiliary fields and " $\cdots$ " involves only the derivatives of the component fields.

We can now write an effective theory for the Lagrangian (4.4.1) in two ways, which we investigate them here separately following the lines of [63] as we can see schematically in figure(4.2).

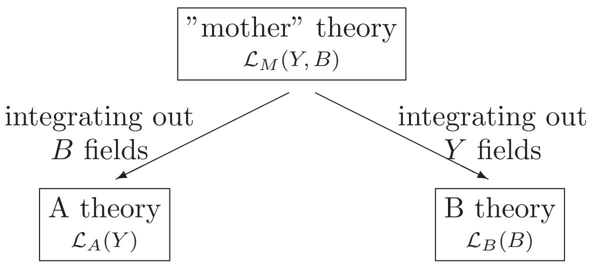

FIGURE 4.2: Mirror symmetry as different descriptions of the same mother theory.

First description: First we can integrate over Y which gives the following constraints on the real chiral superfield B as its equations of motion

$$\bar{D}_+ D_- B = D_+ \bar{D}_- B = 0 \tag{4.4.7}$$

These has the following solution

$$B = \Psi + \bar{\Psi} \tag{4.4.8}$$

for a chiral superfield Ψ.

Inserting (4.4.8) into the Lagrangian (4.4.1) we get the following effective Lagrangian

$$\mathcal{L} = \int d^4\kappa \, e^{2QV + \Psi + \bar{\Psi}} - \frac{1}{2} \int d^2\tilde{\kappa} \, t\Sigma \tag{4.4.9}$$

which after using the redefinition

$$\Phi = e^{\Psi} \tag{4.4.10}$$

it becomes

$$\mathcal{L}_A = \int d^4\kappa \Phi e^{2QV} \bar{\Phi} - \frac{1}{2} \int d^2\tilde{\kappa}\, t\Sigma \tag{4.4.11}$$

This is the gauged linear sigma model action with Fayet-Illiopoulos parameter t and as we will see is defined on a particular super Calabi-Yau. This is the action which we got from the A-model action of the pure spinor superstring theory and will be used to find the dual of the certain gauge theory observables. The target space of the theory is determined by looking into the D-term equations and the vacua of the theory which will was explored before.

The proper boundary conditions, as D-branes, in this A-model action which are preserving the $N = 2$ supersymmetry structure were shown to be given by the Lagrangian submanifolds of the Calabi-Yau which are half-dimensional subspace with vanishing symplectic form.

Second description: To get the second description of the mother action (4.4.1) which is the mirror dual of the linear gauged sigma model we integrate over the real chiral superfield B using the following equations of motion

$$B = -2QV + \log\left(\frac{Y + \bar{Y}}{2}\right) \tag{4.4.12}$$

inserting this into the mother action (4.4.1) gives the following action

$$\mathcal{L}_B = -\frac{1}{2} \int d^4\kappa (Y + \bar{Y}) \log\left(Y + \bar{Y}\right) + \frac{1}{2} \int d^2\tilde{\kappa} \Sigma (Y - t) \tag{4.4.13}$$

which is the dual theory of the linear gauged sigma model (4.4.11). As we can see, the chiral superfield of (4.4.11) is playing the role of the neutral chiral superfield Y which couples to the field strength Σ.

We can see from (4.4.8) and (4.4.12) that the superfield Φ of the linear gauged sigma model is related to the Y field with the following relation

$$Re\ Y = 2\bar{\Phi} e^{2QV} \Phi \tag{4.4.14}$$

This is the map between the dual fields.

Hori and Vafa showed that the superpotential of the action (4.4.13) is subject to instantonic non-perturbative corrections which is given by e^{-Y} and the exact superpotential becomes

$$\tilde{W} = \Sigma\,(Y - t) + e^{-Y} \tag{4.4.15}$$

for a theory with n chiral superfield, we get the following dual superpotential

$$\tilde{W} = \sum_{i=1}^{n} (Y_i - t)\Sigma + e^{-Y_i} \tag{4.4.16}$$

integrating out Σ gives

$$\sum_{i=1}^{n} Y_i = t \tag{4.4.17}$$

which is dual of the D-term equation for the A-model action. Putting this back in the superpotential gives the following superpotential

$$\tilde{W} = \sum_{i=1}^{n} e^{-Y_i} \tag{4.4.18}$$

The theory with the superpotential (4.4.15) defines a Landau-Ginzburg theory which is the mirror dual of the linear gauged sigma model we obtained as the first description.

Up to now we just considered that the chiral superfields are bosonic meaning that their first components are bosonic fields, which means that the target space which is specified from the D-term equation of the action (4.4.11) is a bosonic manifold. But as the linear gauged sigma model which we got from the A-model action of the pure spinor formalism is a super Calabi-Yau, we should generalize these results to this case. Starting with a linear gauge sigma model with A bosonic chiral superfields Φ and S fermionic chiral superfields Ψ, the dual fields in the B-model side would be the bosonic fields X^i and Y^j in which

$$\begin{aligned} Re\ Y^i &= |\Phi^i|^2 \quad , \qquad i = 1 \text{ to } A \\ Re\ X^j &= |\Psi^j|^2 \quad , \qquad j = 1 \text{ to } S \end{aligned} \tag{4.4.19}$$

It was shown in [63] that in this case, on top of these bosonic fields we need to add also some pairs of fermionic variables (η, χ) to the dual Landau-Ginzburg superpotential in order to preserve the superdimension which is the difference of the number of the bosonic and fermionic coordinates. These fermionic fields contribute as follows to the superpotential

$$\tilde{W} = \sum_{i=1}^{A} e^{-Y^i} + \sum_{j=1}^{S} e^{-X^j}(1 + \eta^j \chi^j) \tag{4.4.20}$$

which gives the Landau-Ginzburg dual of the linear gauged sigma model for the case of supermanifolds, where we have also fermionic coordinates.

We have observed that these two theories, the A-model linear gauged sigma model and the B-model Landau-Ginzburg with superpotential (4.4.15), are the mirror duals. We will use this mirror dual to do computations in the B-model side for the topological A-model of the $AdS_5 \times S^5$ superstring theory.

In order to relate the geometries on which the two theories are defined, as we will see later, we have start from the path-integral of one of them and after integrating over some family of fields, we will get some delta functions in the path integral which define constraints over the coordinates of the dual theory. These constraints are translated as the geometry of the mirror theory.

Here we start with the linear gauge sigma model and use the mirror symmetry to investigate the mirror of the linear gauged sigma model of the $AdS_5 \times S^5$ following [37].

As we saw before, the Coulomb branch of the linear gauged sigma model of $AdS_5 \times S^5$ is equal to four copies of the super projective Calabi Yau $\mathbb{CP}^{(3|4)}$'s but since we are going to consider some observables which are coupled just with a particular sector of the theory which can be explained just by just one of these four $\mathbb{CP}^{(3|4)}$'s we consider the linear gauged sigma model on the $\mathbb{CP}^{(3|4)}$ and will follow the duality map which was explained before before.

Let us consider the A-model on the $\hat{\mathbb{CP}}^{(3|4)}$ with bosonic and fermionic coordinates ϕ^I and ϕ^A which are the first components of the chiral superfields Φ in (4.4.10). Since all the fields have charge one under the remnant $U(1)$ gauge group, the D-term equation can be written, in terms of the first components of the superfields, as

$$\sum_{I=1}^{4} |\phi^I|^2 + \sum_{A=1}^{4} |\phi^A|^2 = r \tag{4.4.21}$$

subject to the following symmetry

$$\phi^I \rightarrow e^{ia}\phi^I \quad , \quad \phi^A \rightarrow e^{-ia}\phi^A \tag{4.4.22}$$

The fields ϕ^I and ϕ^A are corresponding to the coordinates of the super Calabi-Yau $\hat{\mathbb{CP}}^{(3|4)}$. The twistor space $\hat{\mathbb{CP}}^{(m|n)}$ is defined as

$$\hat{\mathbb{CP}}^{(m|n)} = \left\{ \mathbb{C}^{(m+1|n)} \setminus \{(0,0)\} \right\} / \mathbb{C}^* \tag{4.4.23}$$

where $\{(0,0)\}$ is the origin in $\mathbb{C}^{(m+1|n)}$. This space describes the vacua of the gauged linear sigma-model with $m+1$ bosonic chiral multiplets Φ^M and n fermionic ones Φ^N all of them with unit charge under the Abelian $U(1)$ gauge symmetry. Its defining equation is $\sum_M |\phi^M|^2 + \sum_N |\phi^N|^2 = r$ modulo the $U(1)$ action $\phi_\Sigma \to e^{i\alpha}\phi_\Sigma$. We can trade the D-term equation for a complexification of the group action and obtain the symplectic quotient $\hat{\mathbb{CP}}^{(m|n)}$ as defined above. In the mathematical literature, one defines the superprojective space $\mathbb{CP}^{(m|n)} = \left\{ \mathbb{C}^{(m+1|n)} \setminus \left\{ \mathbb{C}^{(0|n)} \right\} \right\} / \mathbb{C}^*$, where $\mathbb{C}^{(0|n)}$ is sitting at the origin $\Phi^M = 0$ of the commuting variables. This is a supermanifold contained in $\hat{\mathbb{CP}}^{(m|n)}$. It is clear that the choice of the sublocus containing the origin one has to remove, makes the difference between the two spaces. The gauged linear σ-model chooses the sublocus closed under the action of the global $U(m+1|n)$ symmetry of the D-term equations, namely the origin of the whole space. For more formal issues related to supergeometries and all that, see for example [65] and references therein.

The first step to get the mirror dual is to define the dual fields which appear in the mirror theory as follows

$$\begin{aligned} Re\, Y^I &= |\phi^I|^2 \\ Re\, X^A &= -|\phi^A|^2 \end{aligned} \tag{4.4.24}$$

The superpotential for the mirror Landau-Ginzburg description results to be

$$\tilde{W} = \sum_{I=1}^{4} e^{-Y^I} + \sum_{A=1}^{4} e^{-X^A}(1 + \eta^A \chi^A) \tag{4.4.25}$$

where the fermionic fields η and χ were added to the bosonic field X to match the central charge of the original σ-model and to ensure the exact matching of the effective superpotentials. To find the mirror we can follow the lines of [26] by starting with the path integral for the mirror Landau-Ginzburg model as follows

$$\int \prod_{I=1}^{4} dY^I \prod_{A=1}^{4} dX^A d\eta^A d\chi^A \delta\left(\sum_{I=1}^{4} Y^I - \sum_{A=1}^{4} X^A - t \right) e^{\left(\sum_{I=1}^{4} e^{-Y^I} + \sum_{A=1}^{4} e^{-X^A}(1+\eta^A \chi^A) \right)} \tag{4.4.26}$$

where the delta function is specifies the dual of the D-term equation of the linear gauged sigma model. The delta function can be solved by integrating over one of the fields for example X^1, in terms of the other fields.

The path integral then becomes

$$\int \prod_{I=1}^{4} dY^I \prod_{A=1}^{4} d\eta^A d\chi^A \prod_{A=2}^{4} dX^A \qquad (4.4.27)$$
$$\exp\left(\sum_{I=1}^{4} e^{-Y^I} + e^t \prod_{I=1}^{4} e^{-Y^I} \prod_{A=2}^{4} e^{-X^A}(1+\eta^1\chi^1) + \sum_{B=2}^{4} e^{-X^B}(1+\eta^B\chi^B) \right)$$

The next step is to integrate over the fermionic fields η^A and χ^A, one by one, except η^4 and χ^4 to get the following path integral

$$\int \prod_{I=1}^{4} dY^I e^{-Y^I} \prod_{A=2}^{4} dX^A e^{X^4} d\eta^4 d\chi^4 \qquad (4.4.28)$$
$$\exp\left(\sum_{I=1}^{4} e^{-Y^I} + e^t \prod_{I=1}^{4} e^{-Y^A} \prod_{A=2}^{3} e^{-X^A} + \sum_{A=2}^{3} e^{-X^A} + e^{-X^4}(1+\eta^4\chi^4) \right)$$

After the following field redefinition

$$x^A = e^{-X^A} \quad , \quad y^1 = e^{-Y^1} \quad , \quad y^J = e^{X^J - Y^J} \text{ for } J = 2,3,4 \qquad (4.4.29)$$

we get the following path integral

$$\int \prod_{I=1}^{4} dy^I \prod_{A=2}^{4} dx^A \quad \frac{dx^4}{x^4} d\eta^4 d\chi^4 \qquad (4.4.30)$$
$$\exp\left(y^1 + \sum_{J=2}^{4} y^J x^J + e^t \prod_{I=1}^{4} y^I + \sum_{A=2}^{4} x^A + x^4(1+\eta^4\chi^4) \right)$$

The factor $\frac{1}{x^4}$ in the path integral can be rewritten after introducing the auxiliary bosonic variables u and v as follows

$$\frac{1}{x^4} = \int du dv e^{uvx^4} \qquad (4.4.31)$$

then we can write the path integral in terms of the new variables as

$$\int \prod_{I=1}^{4} dy^I \prod_{A=2}^{4} dx^A d\eta^4 d\chi^4 du dv \qquad (4.4.32)$$
$$\exp\left(y^1 \left(1 + e^t \prod_{I=2}^{4} y^I \right) + \sum_{J=2}^{4} x^J(y^J+1) + x^4(1+\eta^4\chi^4 + uv + y^4) \right)$$

Integrating over y^1, x^2, x^3 and x^4 gives

$$\int \prod_{J=2}^{4} dy^J du dv \delta(1+\eta^4\chi^4+uv+y^4) \prod_{I=2}^{3} \delta(y^I+1)\delta(1+e^t \prod_{I=2}^{4} y^I) \qquad (4.4.33)$$

The delta functions impose the following constraints

$$\begin{aligned} y^4 &= -\frac{e^{-t}}{y^2 y^3} \\ y^2 &= y^3 = -1 \\ 0 &= 1 + \eta^4 \chi^4 + uv + y^4 \end{aligned} \tag{4.4.34}$$

These constraints can be solved together to obtain

$$1 + \eta^4 \chi^4 + uv = e^{-t} \tag{4.4.35}$$

For small t this gives

$$uv - \eta\chi = t \tag{4.4.36}$$

where $\eta \equiv \chi^4$ and $\chi \equiv \eta^4$.

Therefore, we see that the geometry which is defined by (4.4.36) and is named as *superconifold*[4] is the dual geometry of the $\hat{\mathbb{CP}}^{3|4}$ and so as far as the calculation of 1/2 BPS invariant observables in Type IIB String theory on $AdS_5 \times S^5$ concerns, one can use the mirror geometry formulation for the A-model, which is the B-model on the superconifold (4.4.36)in the regime $t \sim 0$. The geometry in such a regime gets singular. In these situations the string theory target space gets represented by a blown up geometry via the conifold transition, like in the cases which were analyzed in [17] and [14]. One can actually extend the geometric transition to this Grassmann odd version of the conifold.

We study the aspects of the singular superconifold by starting with the singular superconifold corresponding to $t = 0$ point

$$uv - \eta\chi = 0 \tag{4.4.37}$$

we can use another parametrization of the coordinates which can show better the geometry. Using $u = u_1 + iu_2$ and $v = u_1 + iu_2$, the conifold equation (4.4.37) can be rewritten as

$$u_1^2 + u_2^2 - \eta\chi = 0 \tag{4.4.38}$$

[4]It is the generalization of the conifold $\sum_i x_i = 0$ to supermanifolds in which we have also fermionic coordinates.

for complex bosonic variables u_1 and u_2. Now writing (4.4.38) in the real and complex components

$$\begin{aligned} u_1 = v_1 + iw_1 \quad &, \quad u_2 = v_2 + iw_2 \\ \eta = \eta_1 + i\nu_1 \quad &, \quad \chi = \eta^1 + i\nu^1 \end{aligned} \tag{4.4.39}$$

we get the following equations from the real and imaginary parts of (4.4.38)

$$\sum_{i=1}^{2}(v_i^2 - w_i^2) + \sum_{\alpha=1}^{2}(\nu_\alpha\nu^\alpha - \eta_\alpha\eta^\alpha) = 0 \tag{4.4.40}$$

$$\sum_{i=1}^{2} v_i w_i + \sum_{\alpha=1}^{2}\eta_\alpha\nu^\alpha = 0 \tag{4.4.41}$$

Using (4.4.40) and (4.4.41) we can see the supergeometry can be viewed as $T^*S^{(1|2)}$ where coordinates (w_i, ν_α) are parameterizing the fiber and (v_i, η_α) are parameterizing the base $S^{(1|2)}$ which is defined by $\sum_{i=1}^{2} v_i^2 + \sum_{\alpha=1}^{2}\eta^\alpha\eta_\alpha = 0$ on the base which has zero radius here. As we see in figure(4.3), the singular superconifold can be seen as a cone over $S^{(1|2)} \times \mathbb{P}^{(0|1)}$. We can get ride of the singularity by blowing up into this

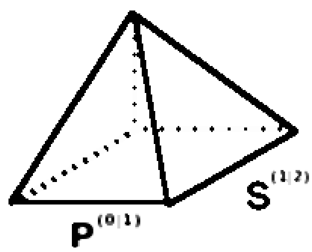

FIGURE 4.3: The singular superconifold.

supersphere to make it having a non zero radius. This means the $t = r$ parameter in (4.4.36) becomes nonzero too and this modifies (4.4.40) and (4.4.41) as follows

$$\sum_{i=1}^{2}(v_i^2 - w_i^2) + \sum_{\alpha=1}^{2}(\nu_\alpha\nu^\alpha - \eta_\alpha\eta^\alpha) = t \tag{4.4.42}$$

$$\sum_{i=1}^{2} v_i w_i + \sum_{\alpha=1}^{2}\eta_\alpha\nu^\alpha = 0 \tag{4.4.43}$$

This supergeometry which has a $S^{(0|1)}$ projective space at the singular point is named as the *deformed superconifold.*

One can observe that the base $S^{(0|1)}$ is a Lagrangian submanifold and so it is a proper boundary condition for the topological A-model by looking into the symplectic form of

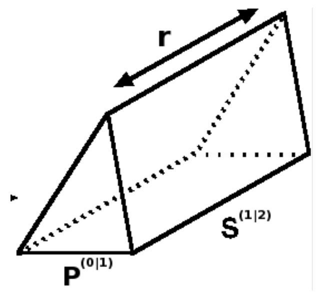

FIGURE 4.4: The deformed superconifold.

the super Calabi-Yau which is given by [37, 26]

$$\omega = \sum_{i=1}^{2} dv_i dw_i + \sum_{\alpha=1}^{2} d\eta_\alpha d\nu^\alpha \tag{4.4.44}$$

which is zero on $S^{(1|2)}$. Also we can check that the imaginary part of the holomorphic measure

$$\Omega = \frac{du d\chi d\eta}{v} \tag{4.4.45}$$

We can easily see that ω is zero on $S^{(0|1)}$ and so this submanifold is actually a special Lagrangian submanifold.

Another way to cure the singularity of (4.4.37) is to define the so called *resolved superconifold* which is defined by blowing a $\mathbb{CP}^{(0|1)}$ into the singularity as we can see in figure(4.4.1.1).

The resolved superconifold can be parametrized with the following relations

$$\begin{pmatrix} u & \eta \\ \chi & v \end{pmatrix} \begin{pmatrix} z \\ \zeta \end{pmatrix} = 0 \tag{4.4.46}$$

where $(z, \zeta) \in \left\{ \mathbb{C}^{(1|1)} \setminus (0,0) \right\} / \mathbb{C}^* = \hat{\mathbb{CP}}^{(0|1)}$. Away from the singularity it gets mapped to the singular cone $uv - \eta\chi = 0$, the singularity being replaced by $\hat{\mathbb{CP}}^{(0|1)}$ very much like in the bosonic case. This space is covered by two patches which we now describe. If $z \neq 0$, then we can fix our coordinates [5] at any given $z_0 \neq 0$ as (z_0, ζ) which is a $\mathbb{C}^{(0|1)}$ patch, while if $\zeta \neq 0$, then we can fix our coordinates at any given $\zeta_0 \neq 0$ as (z, ζ_0) which is a $\mathbb{C}^{(1|0)}$ patch. Clearly, on the intersection, the two patches are related by $z\zeta = z_0\zeta_0$. The last condition is the choice of representative upon the $\mathbb{C}^*$ equivalent points exactly as in the usual $\mathbb{CP}^1$.

[5] Notice that also in the usual bosonic geometric analog, one usually specifies the reference points to $z_0 = 1$, but this is not compulsory at all.

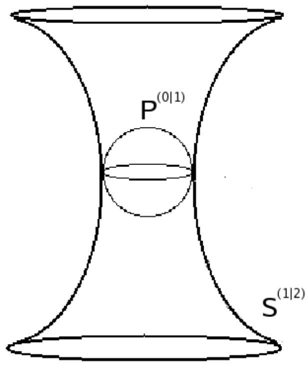

FIGURE 4.5: The resolved superconifold.

So, following mirror symmetry, it was shown that the closed topological A-model theory on $\mathbb{CP}^{3|4}$ is dual to the closed topological B-model on the deformed conifold. The next step is to use geometric transition to relate the closed topological B-model on the deformed superconifold to an open topological B-model on a particular background which will be conjectured to be the resolved superconifold.

4.4.1.2 Geometric transition

As we have seen, we got the mirror dual of the closed string topological A-model on $\mathbb{CP}^{(3|4)}$ to be the closed topological B-model on the deformed superconifold. One can trade this closed string theory to an open topological string theory by doing the so called *geometric transition* an example of which was discussed in the case of the Ooguri-Vafa duality. In that case it was shown that the closed topological A-model theory on the resolved conifold is equivalent to an open topological A-model theory on the deformed conifold as it was shown in the introduction.

Let us now apply the construction of the open string dual theory after geometric transition , by following [14], for the generalization to the superconifold. This is obtained by realizing the fermionic resolved conifold geometry as a complex structure deformation of the local super-$K3$ geometry, namely $\mathcal{O}(-2) \oplus \mathcal{O}(0)$ over $\hat{\mathbb{CP}}^{(0|1)}$. The gluing conditions among the northern and southern hemispheres which are bosonic and fermionic respectively are

$$\begin{aligned} \zeta' z &= \zeta_0 z_0 \\ \zeta' \psi' &= z\psi + z_0 \phi \\ \zeta_0 \phi' &= z_0 \phi \end{aligned} \tag{4.4.47}$$

where ψ' and ϕ' are fermionic while ψ and ϕ are bosonic variables. The complex structure deformation is induced by the non-diagonal patching term in the second line. Let us call

X this superCalabi-Yau space. The invariant three-form Ω on X can be defined in this parametrization as follows

$$\Omega = z_0 d\phi\, d\psi\, dz = \zeta_0\, d\phi' d\psi'\, d\zeta' \tag{4.4.48}$$

in the two coordinate patches.

Similarly to the purely bosonic case, the geometry obtained by imposing the gluing rules can be projected via the blow-down map

$$\begin{aligned} \eta &= \zeta_0 \psi \\ \chi &= z_0 \psi' \\ u &= z\psi \\ x &= z_0 \phi \end{aligned} \tag{4.4.49}$$

which defines the following blown-down geometry

$$\begin{aligned} \eta\chi &= \zeta_0 \psi z_0 \psi' \\ &= \zeta' z \psi \psi' \\ &= z\psi(z\psi + z_0\phi) \\ &= u(u+x) \end{aligned} \tag{4.4.50}$$

which is the singular superconifold (4.4.37) with $v = u + x$.

So, starting from the closed topological B-model superstring theory on deformed superconifold, we conjecture to get open topological B-model superstring theory on resolved superconifold with the D-branes residing on the base of the resolved superconifold.

4.4.1.3 From open topological B-model superstring to holomorphic Chern-Simons

After the geometric transition, we get the open topological B-model action on the deformed superconifold. In [15] it is explained how the Fock space of a particular topological open string theory can be explained in terms of a functional $\mathcal{A}$ which composed of the Bose and Fermi zero modes, with other modes in their Fock vacuum. In order to look at this construction let us remind first the general structure of a topological B-model action.

The topological B-model sigma model governs maps from Riemann surface Σ as the worldsheet to a target space X which would have a complex structure and in our case is the resolved superconifold. The B-model is well-defined on target spaces with vanishing first Chern class $c_1(X) = 0$ namely the Calabi-Yau's. The fields of the B-model are simply maps $\Phi : \Sigma \to X$ which can be described with functions $\phi^I(x^\alpha)$ as local coordinates of the target space X. In the case we are discussing, the target space is a supermanifold, so the coordinates ϕ can be either bosonic or fermionic fields.

Writing an $N = (2, 2)$ supersymmetric theory, we have to consider also the fermions which accompany the bosonic fields of the sigma model. These fermionic fields in the B-model include sections $\theta^{\bar{i}}$, $\eta^{\bar{i}}$ of the pullback $\Phi^*(T^{0,1}X)$ to the worldsheet and the other way $\theta_i = g_{i\bar{j}}\theta^{\bar{j}}$ which $g_{i\bar{j}}$ is the Ricci flat metric of the Calabi-Yau target space X considering its complex structure. In order to complete the supersymmetric multiplet we should add also a one-form fermionic field ρ^i taking value in $\Phi^*(T^{1,0}X)$ which they transform as follows under supersymmetry [15]

$$\begin{aligned} \delta\phi^i = 0 \quad &, \quad \delta\phi^{\bar{i}} = i\alpha\eta^{\bar{i}} \\ \delta\eta^i = \delta\theta_i = 0 \quad &, \quad \delta\rho^i = -\alpha d\phi^i \end{aligned} \tag{4.4.51}$$

We can define a BRST operator Q acting on any field as $\delta\Lambda = -i\alpha\{Q, \Lambda\}$, then the Lagrangian can be obtained as $\mathcal{L} = i\{Q, V\}$ for a suitable V, one gets the action for the topological B-model sigma model as follows

$$\mathcal{L} = t\int_\Sigma \Big(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{i}}(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_z)g_{i\bar{i}} + i\theta_i(D_{\bar{z}}\rho^i_z - D_z\rho^i_{\bar{z}}) + R_{i\bar{i}j\bar{j}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{i}}\theta^{\bar{k}} \Big) \tag{4.4.52}$$

where $R_{i\bar{i}j\bar{j}}$ is the curvature of the target space. Note that since the Lagrangian is written as $\mathcal{L} = i\{Q, V\}$, the t dependence and the metric dependence is of the form $\{Q, \cdots\}$ which means that it does not change the BRST cohomology of the theory and so the theory does not depend on the coupling t and the metric of the target space.

We can look into the Hamiltonian version of (4.4.52) for open strings. As it was shown in [15], the Hilbert space $\mathcal{H}$ consists of some functionals $\mathcal{A}$ which depend on the maps Φ from the worldsheet to the target space which the dependence comes from the representation of the canonical anticommutation relations. This functional depends on the zero modes of the fields ϕ^I, $\eta^{\bar{i}}$ as follows

$$\mathcal{A}(\phi^I, \eta^{\bar{i}}) = c(\phi^I) + \eta^{\bar{i}}A_{\bar{i}}(\phi^I) + \eta^{\bar{i}}\eta^{\bar{j}}B_{\bar{i}\bar{j}}(\phi^I) + \cdots \tag{4.4.53}$$

which because of the commutations relations $[Q, \phi^{\bar{i}}] = -\eta^{\bar{i}}$ and $\{Q, \eta^{\bar{i}}\} = 0$, we can interpret $\eta^{\bar{i}}$ as $d\phi^{\bar{i}}$ and see (4.4.53) as a sum over $(0, q)$ forms over X.

In order to give a space time interpretation to the B-model topological sigma model

(4.4.52), Witten proposed a string field theory based on the functional $\mathcal{A}$ which is interpreted as a ghost number one element in the associative algebra $\mathcal{B}$ with a multiplication law which is denoted as $\star$ and a derivative Q of degree one satisfying $Q^2 = 0$. There is a functional $\int : \mathcal{B} \to \mathbb{C}$ which is non vanishing just for ghost number -3 operators and obeying the following relation

$$\int a \star b = (-1)^{\deg a \deg b} \int b \star a \tag{4.4.54}$$

and also we have $\int Qb = 0$ for any $b \in \mathcal{B}$. Using the fact that $\mathcal{A}$ is a ghost number one operator, Witten wrote the following Lagrangian using the properties of the $\star$ and $\int$ [15]

$$\mathcal{L} = \frac{1}{2g_s} \int \left(\mathcal{A} \star Q\mathcal{A} + \frac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A} \right) \tag{4.4.55}$$

This is invariant under the following gauge transformation

$$\delta \mathcal{A} = Q\alpha - \alpha \star \mathcal{A} + \mathcal{A} \star \alpha \tag{4.4.56}$$

Actually we can associate Chan-Paton factors to the string fields $\mathcal{A}$ by considering to take value in the space of $N \times N$ hermitian matrices and this way the theory defined by (4.4.55) as the string field theory is describing open string theory with the boundary conditions which is given from the Chan-Paton factors of the string fields $\mathcal{A}$. In this way the associative algebra $\mathcal{B}$ is the space of the open string states and the operations $\star$ and $\int$ are related to the string theory with the gluing of the open strings which we don't discuss it here. The operator Q of the open string field theory also is going to be interpreted as the BRST charge of the string theory.

It was shown by Witten [15] that the string field theory action (4.4.55) describing open strings ending on space-filling D-branes, meaning that no specific place for the boundary condition is chosen, has a simpler realization in terms of a specific Chern-Simons theory named as *holomorphic Chern-Simons* theory.

The boundary condition should preserve the supersymmetric structure of the sigma model, for B-model this means that we should have

$$\partial_\perp \Phi = 0 \tag{4.4.57}$$

where $\partial_\perp$ is normal derivative on $\partial\Sigma$. This condition means that $\theta|_{\partial\Sigma} = 0$ and also it implies the vanishing of pullback to $\partial\Sigma$ of $\star\eta$ where $\star$ is the Hodge star product.

The string field should have ghost number one which is raised with the BRST operator Q. Giving ghost number zero to the target space maps ϕ, one can see from (4.4.51) and so from (4.4.53) that we should keep only the linear term in the field η

$$\mathcal{A} = \eta^{\bar{i}} A_{\bar{i}}(\phi^I) \tag{4.4.58}$$

It is a one-form which takes value in the endomorphisms of some holomorphic vector bundle E. Since the string field just depends on the bosonic and fermionic zero modes, the star product $\star$ becomes the wedge product of forms in $\Omega^{(0,p)}(\mathrm{End}(E))$ and the integration operator $\int$ becomes ordinary integration over the forms on the target space X wedged by the holomorphic form Ω which completes the functional form in order to define a non-zero integral. We can write the following dictionary from the string field theory to a field theory defined in terms of the one-form connection A on the target space X

$$\begin{array}{lcl} \mathcal{A} \to A & , & Q \to \bar{\partial} \\ \star \to \wedge & , & \int \to \int \Omega \wedge \end{array} \tag{4.4.59}$$

The string field action (4.4.55) after this identification becomes

$$S = \frac{1}{2g_s} \int_X \Omega \wedge Tr \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right) \tag{4.4.60}$$

where Ω is the invariant holomorphic three-form of the resolved superconifold we discussed before.

We have shown that the closed topological A-model string theory is equivalent to the holomorphic Chern-Simons (4.4.60) on the resolved superconifold. This was obtained with the assumption that the open strings are free and we have space-filling D-branes, but as we know, in our case the D-branes which are the counterparts of the fluxes in the closed string side, are reside just on the base of the superconifold and not in all the target space. So in the next section we will modify the result by taking into account this consideration.

4.4.1.4 Dimensional reduction and The Gaussian Matrix model

Here we consider the case in which the branes which as the end points of the open strings wrap only the holomorphic two-cycles $\mathbb{CP}^{(0|1)}$ as the base of the resolved superconifold we discussed before. We have to dimensionally reduce the action (4.4.60) on the worldvolume of these branes in the Calabi-Yau

$$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{CP}^{(0|1)} \tag{4.4.61}$$

in the geometry defined in (4.4.47). To do this dimensional reduction we follow the passage of [14, 31] which was reviewed in [27]. For some comments on the Chern-Simons theory on supermanifolds, see also [66].

We discussed before the geometry of the Calabi-Yau (4.4.61) which is described by the gluing (4.4.47). Since we want to consider the branes wrapping $\mathbb{CP}^{(0|1)}$, this means that the gauge field A which is describing the field theory on the worldvolume of the D-branes, splits into a gauge potential on the worldvolume of the branes and a Higgs field which describes the motion along the noncompact, transverse direction of the fiber. These Higgs fields Φ_0 and Φ_1 are actually sections of the corresponding normal bundles $\mathcal{O}(0)$ and $\mathcal{O}(-2)$ respectively. One can decompose the gauge field as follows

$$A = a(z, \phi, \psi) + \Phi_0(z)d\phi + \Phi_1(z)d\psi \tag{4.4.62}$$

where a is a one-form residing on the base with coordinates z and (ϕ, ψ) are the coordinates of the fibers as it was explained in (4.4.47). Assuming that we have N D-branes, all the fields take value in the adjoint representation of $U(N)$. Putting (4.4.62) into (4.4.60) one gets the following action

$$S = \frac{1}{2g_s}\left[\int_{\hat{\mathbb{CP}}^{(0|1)}} Tr(\Phi_1 \bar{D}\Phi_0) + \oint Tr\, W(\Phi_0)\right] \tag{4.4.63}$$

where $\bar{D} = \bar{\partial} + [a, \cdot]$ is the covariant derivative and $W(x) = \frac{1}{2}x^2$ is the complex structure deformation which as we are working on a supermanifold which has fermionic coordinates, the only possibility is a quadratic function. As we will see this quadratic complex structure deformation gives rise to produce the hermitian *Gaussian matrix model.*

The gauge connection appear in the action as a Lagrange multiplier giving rise to the constraint

$$[\Phi_0, \Phi_1] = 0 \tag{4.4.64}$$

This means that we can diagonalize Φ_0 and Φ_1 simultaneously. Also we can integrate out the other field Φ_1 since it appears linearly in the action (4.4.63) which gives rise to the following equation of motion for Φ_0

$$\bar{\partial}\Phi_0 = 0 \tag{4.4.65}$$

Since we are on $\mathbb{CP}^{(0|1)}$, $\bar{\partial}$-operator has just one constant zero mode and the solution of (4.4.65) is a constant diagonal matrix. Notice the fact that here, although the base geometry is half fermionic and half bosonic, this does not influence the endpoint result, because as ϕ and ψ change statistics while patching, their propagating contributions continue to cancel against the ghost determinants. The important fact is that the $\bar{\partial}$-operator on scalars still has a single (constant) zero mode.

On the other hand, the equation of motion for Φ_0 implies

$$\bar{\partial}\Phi_1 = W'(\Phi_0)\omega \tag{4.4.66}$$

where ω is a $(1,1)$ form which can be taken to have unit volume on the $\mathbb{CP}^{(0|1)}$. Note that the integral of $\bar{\partial}\Phi_1$ over $\mathbb{CP}^{(0|1)}$ should be zero for non-singular Φ_1 which leads to the following relations after integrating (4.4.66)

$$\Phi_1 = W'(\Phi_0) = 0 \tag{4.4.67}$$

This means that the classical vacua are localized on the critical points of $W(\Phi_0)$ which is a quadratic function in our case.

Putting all these together and remembering the fact that $W(x) = \frac{1}{2}x^2$ for the superconifold, we get the Gaussian hermitian $N \times N$ matrix model with the following measure factor

$$\mu = d\Phi \; e^{-\frac{1}{2g_s}Tr\Phi^2} \tag{4.4.68}$$

where Φ is a $N \times N$ matrix. This corresponds to the Drukker-Gross one if $g_s = g_{YM}^2$ as predicted by gauge string duality. In the next section we will do explicit computations on both side of the duality by using these Matrix models and we will see that there are some particular observables producing exact result of the circular Wilson loops in the gauge theory side.

4.4.2 Circular Wilson loop and its dual in topological model

Using the construction we did in the previous section, we saw that the closed topological A-model string theory on the superprojective space $\mathbb{CP}^{(3|4)}$ is equivalent to a Gaussian hermitian matrix model. And, as we saw before this is a particular sector of the $AdS_5 \times S^5$ superstring and can be used in order to study the dual sector in the $\mathcal{N} = 4$ $D = 4$ SYM theory. The particular observables residing in the dual of this sector are proposed to be the circular Wilson loops which from their symmetry we will argue that their duals in the string theory side can be captured by particular observables in the topological A-model on $\mathbb{CP}^{(3|4)}$ or its equivalent matrix model. We start by studying briefly the half-BPS circular Wilson loops which was studied in [28] and confirmed in [67] to produce exactly a Gaussian matrix model. Then we go to the string theory side, using our topological construction, we will show that the exact result of the gauge theory side can be calculated in the string theory side. It is an exact check of the Maldacena conjecture.

4.4.2.1 Circular Wilson loops in $\mathcal{N}=4$ $d=4$ SYM

In gauge theory, a *Wilson loop* is a gauge-invariant observable obtained from the holonomy of the gauge connection around a given loop. In the classical theory, the collection of all Wilson loops contains sufficient information to reconstruct the gauge connection, up to gauge transformations [68].

If we consider a $SU(N)$ gauge theory, the Wilson loop can be defined as the path-ordered exponential of the gauge field as follows

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left(i\oint A_\mu dx^\mu\right) \tag{4.4.69}$$

where the trace is defined in the fundamental representation. We can define this Wilson loop for any closed path in the target space and they define a class of observables in the gauge theory forming a complete basis of gauge invariant operators for pure Yang-Mills theory. The Wilson loop is actually the phase of a quark in the fundamental representation of the gauge group.

In $\mathcal{N}=4$ SYM we have the gauge field A_μ, six scalars ϕ_i for $i=1$ to 6 and four Weyl fermions λ_a, $a=1$ to 4 in the adjoint representation of the $SU(N)$ gauge group. This theory does not have any quark in the fundamental representation and we have to use the W-bosons to probe the theory and make the Wilson loops. To do this we consider a non-zero expectation value for the six scalars and parametrize the vacuum expectation values with a point θ^i on the unit five-sphere which is defined as $\theta^2=1$. The phase factor associated to the trajectory of the W-boson in the path we defined gives the following Wilson loop operator

$$W = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp i\oint\left(A_\mu \dot{x}^\mu + i\Phi_i|\dot{x}(s)|\theta(s)\right) \tag{4.4.70}$$

where s parametrizes the point on the five-sphere. This special loop is taken to be locally supersymmetric.

We can write the expectation value of the Wilson loop (4.4.70) around some contour $\mathcal{C}$ order by order in perturbation theory as follows

$$\langle W_{\mathcal{C}}\rangle = \sum_{n=0}^{\infty} A_n\lambda^n \tag{4.4.71}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. The first terms are computed as follows

$$\begin{aligned} A_0 &= 1 \qquad (4.4.72)\\ A_1 &= \frac{1}{2}\oint ds_1\oint ds_2 \frac{1}{N}\mathrm{Tr}\left(-\dot{x}_1^\mu \dot{x}_2^\nu \langle A_\mu(x_1)A_\nu(x_2)\rangle + |\dot{x}_1|\theta_1^i|\dot{x}_2|\theta_2^j\langle\Phi_i(x_1)\Phi_j(x_2)\rangle\right)\end{aligned}$$

We are working in $\mathbb{R}^4$ and all the propagators are translationally invariant because of the symmetry of the background. Also, since $\mathcal{N} = 4$ SYM in four space time dimension is conformal invariant, the Wilson loop is also conformal invariant and in this way we can relate the expectation values of the Wilson loops of the contours which are related through a conformal transformation.

The contour which we are interested in is the circle in which we want to do explicit computations. Since the circle and the line are related through a large conformal transformation as we see in figure (4.4.2.1), so we use a conformal anomaly statement to compute the value for the circle from the one of the straight line.

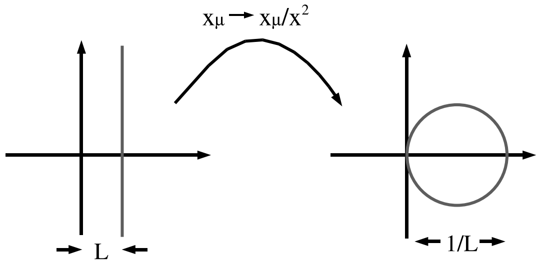

FIGURE 4.6: Line and circle are related through an inversion on the plane.

We start by computing the expectation value for the line. The line can be parametrized on $\mathbb{R}^4$ as follows

$$x_\mu = (\tau, 0, 0, 0) \tag{4.4.73}$$

One can check that it preserves 16 out of 32 supersymmetries and so it is a 1/2-BPS object. Using parametrization (4.4.73), it was shown [28] that the sum of the gluon and scalar propagetors vanishes and so in the perturbative series (4.4.71) the only non zero term is $A_0 = 1$ and so we have the exact result

$$\langle W_{\text{line}} \rangle_{\mathbb{R}^4} = 1 \tag{4.4.74}$$

One reason for this simplicity is related to the fact that it is a BPS object which ensures that there are no contributions to any order of λ in (4.4.71).

Instead, the circle can be parametrized as follows

$$x_\mu = R(\sin\tau, \cos\tau, 0, 0) \tag{4.4.75}$$

It preserves a subgroup $SU(1,1) \times SU(2) \times SO(5)$ of the global symmetry, half of the supersymmetries and so it is also a 1/2-BPS object but we can not apply the same reasoning we used for the straight line because the difference of the gluon and scalar propagators is not zero here but it is a constant. In order to compute the value of $\langle W_{\text{circle}}\rangle_{\mathbb{R}^4}$ we use a simple conformal anomaly statement which relates the line and circle thorough the large conformal transformation $x_\mu \to x_\mu/x^2$ and the fact that four-dimensional $\mathcal{N}=4$ SYM is a superconformal theory. Note that even if the theory is a superconformal theory, the inversion is not a symmetry of $\mathbb{R}^4$ but a symmetry of the S^4. We have that

$$\langle W_{\text{line}}\rangle_{\mathbb{R}^4} \neq \langle W_{\text{line}}\rangle_{S^4} \tag{4.4.76}$$

This comes from the fact that under the inversion the two end points of the line are mapped to a single point on S^4 and we are missing a point as it is seen in figure (4.4.2.1)

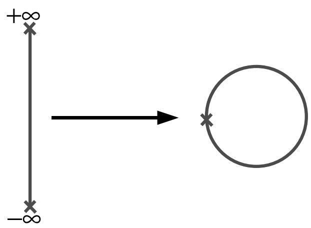

FIGURE 4.7: The two end points of the line are mapped to a single point under the inversion.

Noting that $S^4 = \mathbb{R}^4 \cup \{\infty\}$, the difference between the Wilson loop of the line on $\mathbb{R}^4$ and S^4 just comes from the point at infinity

$$\langle W_{\text{line}}\rangle_{S^4} = \langle W_{\text{line}}\rangle_{\mathbb{R}^4} + \langle W_{\text{line}}\rangle_{\infty} \tag{4.4.77}$$

where $\langle W_{\text{line}}\rangle_{\infty}$ is the contribution of the point at infinity.

On the other hand, a circle is mapped to a circle under inversion, also the line and the circle are conformally the same on S^4, so we have

$$\langle W_{\text{circle}}\rangle_{\mathbb{R}^4} = \langle W_{\text{circle}}\rangle_{S^4} = \langle W_{\text{line}}\rangle_{S^4} \tag{4.4.78}$$

Comparing (4.4.77) and (4.4.78) and using $\langle W_{\text{line}}\rangle_{\mathbb{R}^4} = 1$ we get

$$\langle W_{\text{circle}}\rangle_{\mathbb{R}^4} = \langle W_{\text{line}}\rangle_{\infty} \tag{4.4.79}$$

Which means that the result for the circle is equivalent to the result for a single point at infinity which is like a 0-dimensional field theory, namely a matrix model. This was shown in [28] and [67] that is computed through the following hermitian Gaussian matrix model

$$\langle W_{\text{circle}}\rangle_{\mathbb{R}^4} = \left\langle \frac{1}{N}\text{Tr }\exp(M)\right\rangle = \frac{1}{Z}\int \mathcal{D}M\frac{1}{N}\text{Tr }\exp(M)\exp\left(-\frac{2}{g_{YM}^2}\text{Tr}M^2\right) \tag{4.4.80}$$

with a set of observables expanded in the basis of the matrix model.

We will see in the next section that this result can be produced exactly with a set of D-branes which we proposed to be the dual of this circular Wilson loops in the $AdS_5\times S^5$ superstring side.

4.4.2.2 Dual of the circular Wilson loops in the superstring

Here we want to find the dual observables of the circular Wilson loops and to do the computations in the superstring side. The dual objects which was proposed as the dual of the Wilson loops in the superstring theory side were the D-branes which the Wilson is realized as the loop which made by the D-brane on the boundary of the AdS_5 [30, 69].

In section 4.3.1 we took the following boundary conditions for strings as the defining equations of D-branes

$$(\bar{\Theta}^t)^A_J = \epsilon^A_B\Theta^{*B}_K\delta^K_J \tag{4.4.81}$$

We observed that for a particular choice of the matrices ϵ and δ as in (4.3.2), the D-brane breaks the symmetry of the $AdS_5\times S^5$ coset to the following supercoset

$$\frac{OSp(4^*|4)}{SO^*(4)\times USp(4)} \tag{4.4.82}$$

which can be shown to be correspond to D-branes wrapping $AdS_2\times S^4$ geometries inside $AdS_5\times S^5$ [29]. As such, this states realize the circular Wilson loops in an alternative way because they are preserving the same amount of suppersymmetry and global isometry.

It was also shown that the boundary conditions (4.4.81) translated into the following boundary conditions in the gauged linear sigma model

$$(\Phi^\dagger)^R_J\delta^{tJ}_{\ I} = \kappa^{\dagger R}_{\ S}\bar{\Phi}^S_I \quad \text{and} \quad (\Phi^\dagger)^R_A\epsilon^{tA}_{\ B} = \kappa^{\dagger R}_{\ S}\bar{\Phi}^S_B \tag{4.4.83}$$

where the matrix κ specifies to which twisted sector of the coulomb branch vacuum $(\hat{\mathbb{CP}}^{(3|4)})^4//S_4$ the D-branes couple. For the particular choice of κ

$$\kappa = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.4.84}$$

the gauge symmetry is maximally breaks and the D-branes are just couple with one copy of $\hat{\mathbb{CP}}^{(3|4)}$ because it commute with the permutation S_4.

For this particular observables defined from the choice of (4.4.84) and the choice of δ and ϵ as in (4.3.2) the boundary conditions can be explained as the boundary conditions in the projective space $\hat{\mathbb{CP}}^{(3|4)}$. So we just need to study the D-branes in the topological A-model on $\hat{\mathbb{CP}}^{(3|4)}$ which is defined with the gauged linear sigma model with bosonic and fermionic fields ϕ_I and ψ_A with $I, A = 1$ to 4 with the following boundary conditions coming from (4.4.83)

$$\begin{cases} \phi_I = \bar{\phi}_I \\ \psi_a = \bar{\psi}'_a \\ \psi'_a = -\bar{\psi}_a \end{cases} \quad \text{where } I = 1\cdots 4,\ a = 1,2 \text{ and } \psi'_1 \equiv \psi_3,\ \psi'_2 \equiv \psi_4 \tag{4.4.85}$$

these boundary conditions correspond to a Lagrangian submanifold because the Kähler form

$$\omega = d\phi_I d\bar{\phi}_I + d\psi_a d\bar{\psi}'_a + d\psi_{a+2} d\bar{\psi}'_{a+2} \tag{4.4.86}$$

vanishes on this subspace because (4.4.85) sends $\omega \to -\omega$ whose fixed locus identifies the Lagrangian cycle.

This Lagrangian cycle can be traced back in the mirror geometry as in [70]. Therefore, applying to the mirror dual at hand, the Lagrangian submanifold in $\hat{\mathbb{CP}}^{(3|4)}$ gets mapped to the non compact holomorphic cycle defined as follows

$$\eta = 0\ , \quad uv - \eta\chi = t' \tag{4.4.87}$$

in the superconifold mirror picture. In the singular limit these turn out to be $\mathbb{C}^{(1|1)}$ non compact branes. Their fate after geometric transition is to stay non compact, so these are along a fibration on the base $\hat{\mathbb{CP}}^{(0|1)}$ via a complex curve in the fiber direction which has to compensate the superdimension counting.

Therefore, if in the A-model we add M $D5$-branes, these correspond after the duality to M B-branes along the above non-compact cycles. Now, the open string at hand therefore, on top of the sector of N D-branes along the base, also has the open strings connecting them with the dual image of the M D-branes. Correspondingly, the reduced gauge field in the holomorphic Chern-Simons theory becomes

$$\mathcal{A} = \begin{pmatrix} A & Y \\ \tilde{Y} & X \end{pmatrix} \quad \Longrightarrow \begin{cases} A & \to \text{open strings ending on compact branes} \\ Y, \tilde{Y} & \to \text{open strings ending on both type of branes} \\ X & \to \text{open strings ending on non-compact branes} \end{cases} \tag{4.4.88}$$

where the gauge field components Y and $\tilde{Y}^t$ are the $M \times N$ components with mixed boundary conditions and A and X are $N \times N$ and $M \times M$ components for open strings ending on just one type of branes with the following holomorphic Chern-Simons action

$$S_{hCS}(\mathcal{A}) = \frac{1}{g_s} \int_X \Omega \wedge \mathrm{Tr}_{M+N} \left(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \qquad (4.4.89)$$

Being the transverse branes non-compact, the relative gauge field has been kept frozen because its contribution to the holomorphic CS action

$$\frac{1}{g_s} \int_X \Omega \wedge \mathrm{Tr}_M \left[X \wedge \bar{\partial} C + \frac{2}{3} X \wedge X \wedge X + \tilde{Y} \wedge X \wedge Y \right] \qquad (4.4.90)$$

is zero upon the reduction to the base $\hat{\mathbb{CP}}^{(0|1)}$ because neither X nor Y has a $\bar{\partial} dz$ term to complete the holomorphic three-form Ω and so the integral becomes identically zero and we can just neglect the open strings in the non-compact $M \times M$ sector since they decouple from the rest.

Therefore the action gets reduced as

$$S_{hCS}(\mathcal{A}) = S_{hCS}(A) + \frac{1}{g_s} \int_X \Omega \wedge Y \bar{D}_A \tilde{Y} \qquad (4.4.91)$$

where $\bar{D}_A$ is the covariant $\bar{\partial}$ operator.

Dimensionally reducing to the base and integrating the reduced $(Y, \tilde{Y})$ sector one generates the corresponding observable in the matrix model. In formulas, we have therefore

$$\int dF e^{-\frac{1}{2g_s} TrF^2} \mathcal{O}_M(F) \qquad (4.4.92)$$

By expanding the observable in characters as

$$\mathcal{O}_M(F) = \sum_{i, \{n_i\}} \mathcal{O}_M \left(i, \{n_i\}\right) \prod_i Tre^{n_i F} \qquad (4.4.93)$$

one obtains the expansion of the $D5$-brane amplitudes in terms of 1/2 BPS circular Wilson lines (see Section 4 in [28]). The explicit dictionary needs a much deeper elaboration on the specific form of the observables which will follow from the analysis of the reduced theory on the base of the resolved superconifold. The prototype of such an analysis for the usual conifold is in [17], although to be adapted to our case.

Here, we proposed a dual picture for the calculation of 1/2 BPS open string amplitudes on $AdS_5 \times S^5$ with boundary conditions (4.4.81) in the large curvature regime. These has been shown to reduce to observables in the hermitian Gaussian matrix model. Identifying

$g_s = g_{YM}^2$, we can interpret those topological string amplitudes as 1/2 BPS circular Wilson loops.

There are two consistency checks of this result which are independent on the duality chain we formulated. The first is a symmetry argument, which we already recalled, that is the fact that $AdS_2 \times S^4$-branes break exactly the same 1/2 superconformal symmetry as the 1/2 BPS circular Wilson loops do.

The second has to do with the ability of the matrix model to reproduce topological strings amplitudes. Actually, in order for a candidate set of amplitudes to be compatible with the topological gauge symmetry, these have to satisfy the consistency conditions of BCOV [32], namely the holomorphic anomaly equations. This is a strict constraint on any dual picture one might find for topological string amplitudes. The fact that this proposed matrix model passes such a non trivial test is due to the analysis performed in [33] where this was shown much more in general for the matrix models. Actually, the $D5$-branes amplitudes then gets reduced to matrix integrals at finite N. The coinciding genus expansion is consistent for the corresponding non local observable insertions which we get in the form

$$Tre^{nF} = \oint \frac{dx}{2\pi i} e^{nx} \mathrm{Tr} \frac{1}{F - x} \tag{4.4.94}$$

which is the natural form of the open string generated observables. It would be interesting to further elucidate the properties of the specific realization via the Gaussian hermitian matrix model also in direct comparison with the analysis in [71].

Chapter 5

The Antifield Lagrangian quantization of gauge theories

5.1 Basics of gauge theories

Gauge theories are one of the most important ingredients of theoretical physics which can be thought as a theory in which whose dynamical variables are specified with respect to a reference frame in which one has the freedom to choose it arbitrary at any instant of time. The physics is determined with the variables which are independent on the choice of the local reference frame. This freedom to choose the local frame is the *gauge symmetry* of the theory.

Gauge theories may be quantized by specialization of methods which are applicable to any quantum field theory. However, because of the subtleties imposed by the gauge constraints there are many technical problems to be solved which do not arise in other field theories. At the same time, the richer structure of gauge theories allow simplification of some computations: for example Ward identities connect different renormalization constants. Here with quantization we mean the path integral quantization. A classical mechanical theory is given by an action with the permissible configurations being the ones which are extremal with respect to functional variations of the action. A quantum-mechanical description of the classical system can also be constructed from the action of the system by means of the path integral formulation. One should start from a physical system with degrees of freedom labeled by ϕ^i, which for simplicity here we take them to be bosons. On this configuration space of fields, an action $S[\phi]$ can be defined as a functional over field configurations which governs the dynamics of the theory. One can define the partition function as the path integral over all the possible field configurations as follows

$$Z[\phi] = \int [d\phi] e^{\frac{i}{\hbar}S[\phi]} \tag{5.1.1}$$

where each configuration contributes with a phase which is determined with the action for that field configuration.

The *classical solution* of the theory is determined as usual from the *stationary surface* which is defined from the equations of motion or *Euler-Lagrange equations* given by

$$y(\phi^i) = \frac{\delta S}{\delta \phi^i} \tag{5.1.2}$$

we assume that we are studying theories in which they have at least one classical solution. The *quantum solution* is obtained after adding the quantum fluctuations around the classical solution.

The next step is to enter some *symmetry* in the theory. Suppose there exist a set of operators $R^i_\alpha[\phi]$ with the following relation

$$y^i(\phi) R^i_\alpha[\phi] \epsilon^\alpha = 0 \tag{5.1.3}$$

for any value of the parameter ϵ. For the case in which the parameter ϵ^α depends on the target space coordinates, we have a *gauge theory* and the symmetry is *local*. The R^i_α are the gauge generators.

A consequence of (5.1.3) is that there are some zero modes for the Hessian, namely solutions for the following equation

$$\frac{\overrightarrow{\delta}}{\delta \phi^j} \frac{\overleftarrow{\delta} S[\phi^l_0]}{\delta \phi^i} \cdot R^i_\alpha[\phi^l_0] = 0 \tag{5.1.4}$$

where the zero modes $R^i_\alpha[\phi^l_0]$ impose the following infinitesimal transformation on the stationary surface

$$y_i(\phi^l_0 + R^l_\alpha[\phi^k_0]\epsilon^\alpha) = 0. \tag{5.1.5}$$

This maps a classical solution to another one.

For the case of the global symmetry where ϵ's are independent of space-time, the stationary surface becomes a finite dimensional space but in the case of the local symmetry we have an infinite dimensional space as the stationary surface since we can take a set of parameters ϵ^α for any point in space-time. This is the very basic difference of the global and local symmetries which shows itself already in the level of classical solutions. The global symmetries are relating a set of classical solutions but the existence of a local symmetry means that not all field equations are independent and so not all the field degrees of freedom are fixed by the equations of motion. There is an arbitrariness in the field space which relates to the gauge degree of freedom. To get ride of this arbitrariness one should impose some constraints in the field space which are named *gauge fixing conditions*.

The next step after studying the classical solutions of a gauge theory is to quantize it. Here, if simply exponentiate the action like (5.1.1) and sum over the field configurations we will end with problems which are originating from the arbitrariness arises from the gauge symmetry.

In order to quantify this problem more in detail, consider quantum fluctuations around the classical solution which can be decomposed in two components

$$\phi^i = \phi_0^i + \delta_{||}\phi^i + \delta_{\perp}\phi_i \tag{5.1.6}$$

here $\delta_{||}$ and $\delta_{\perp}$ are the variations parallel and orthogonal to the stationary surface respectively. One can see from (5.1.5) that $\delta_{||}\phi^i = R^i_\alpha \epsilon^\alpha$. The measure of the path integral also splits as follows

$$[d\phi] \to [d\epsilon][d\delta_{\perp}\phi] \tag{5.1.7}$$

The action is expanded around the classical solution as follows

$$S[\phi^i] = S[\phi_0^i] + \delta_{\perp}\phi^j \frac{\overleftarrow{\delta}}{\delta\phi^i}\frac{\overrightarrow{\delta}}{\delta\phi^j} S[\phi_0^i]\delta_{\perp}\phi^i. \tag{5.1.8}$$

As we can see, the integrand of the path integral is independent of the gauge transformation parameters ϵ^α, so they can be factorized. This might lead to divergences which should be cured with the methods which will be explained.

Another important property of the gauge systems is that the gauge transformations usually form a *gauge algebra*. To study this consider the field ϕ^i has Grassmann parity ϵ^i_G. Consider gauge generators R^i_α which satisfy

$$\begin{aligned} y_i R^i_\alpha &= 0 \\ \epsilon^R_G &= \epsilon^i_G + \epsilon^\alpha_G \end{aligned} \tag{5.1.9}$$

where as before, $y = \frac{\overleftarrow{\delta} S}{\partial \phi}$ is the stationary surface equation. Suppose that the gauge generators form a complete set, namely they satisfy

$$y_i X^i_{\bar{\beta}}(\phi) = 0 \;\; \Rightarrow \;\; X^i_{\bar{\beta}}(\phi) = R^i_\alpha \epsilon^\alpha_{\bar{\beta}}(\phi) + y_j M^{ij}_{\bar{\beta}}(\phi), \tag{5.1.10}$$

for all possible $X^i_{\bar{\beta}}$. Where $\bar{\beta}$ is an arbitrary set of indices. And where $M^{ij}_{\bar{\beta}}$ is graded antisymmetric $M^{ij}_{\bar{\beta}} = (-1)^{\epsilon^i_G \epsilon^j_G + 1} M^{ji}_{\bar{\beta}}$.

Using (5.1.10), we can find a relation which should be satisfied by R^i_α to maintain consistent gauge transformations

$$\frac{\overleftarrow{\delta} R^i_\alpha}{\delta \phi^j} R^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta} \frac{\overleftarrow{\delta} R^i_\beta}{\delta \phi^j} R^j_\alpha = R^i_\gamma T^\gamma_{\alpha\beta} (-1)^{\epsilon_\alpha} - y_j E^{ji}_{\alpha\beta} (-1)^{\epsilon_i} (-1)^{\epsilon_\alpha} \tag{5.1.11}$$

for some tensors $T^\gamma_{\alpha\beta}$ and $E^{ji}_{\alpha\beta}$ which classify the gauge algebra. In the case when $E^{ji}_{\alpha\beta} = 0$ the algebra is closed and the $T^\gamma_{\alpha\beta}$ becomes the structure constants of the Lie algebra. In [72] it was shown that always there exist a set of generators for the gauge algebra in which it becomes closed.

Another important property of a gauge algebra is its reducibility. If operators $Z^\alpha_\beta[\phi_0]$ exist such that

$$R^i_\alpha[\phi_0] Z^\alpha_\beta[\phi_0] = 0 \tag{5.1.12}$$

on the stationary surface, this means that not all the zero modes (5.1.4) of the Hessian are independent and so the gauge algebra is reducible. If there is no such Z^α_β then the gauge algebra is irreducible and all the zero modes are independent. This is the case in some particular gauge theories we will explore in the next sections.

5.2 Faddeev-Popov quantization procedure

On of the quantization methods was developed by L.D. Faddeev and V.N. Popov [73] based on the path integral formulation of gauge theories. As we saw before, in the general path integral for a gauge theory, in order not to get infinities, one should integrate over whole configuration space, but over the space of *gauge orbits*. Gauge orbits are subspaces of the configuration space which can be connected by gauge transformations and therefore they have the same action. In order to get ride of the infinity each gauge orbit should contribute once in the path integral. The procedure which selects just one configuration on each gauge orbit is through introducing *gauge fixing conditions* which selects just one particular configuration from each gauge orbit contributing in the path integral.

To determine explicitly a gauge orbit we note that each gauge orbit is parametrized with a set of parameters θ_α where its index is a gauge index running over all the gauge symmetries parametrized by gauge generators R^i_α and so they have the same dimension. Different gauge orbits are connected with infinitesimal gauge transformations. A particular gauge orbit is defined as follows

$$\frac{\delta\phi^i(\theta)}{\delta\theta^\beta} = R^i_\alpha[\phi^l(\theta)]\lambda^\alpha_\beta(\theta) \tag{5.2.1}$$

for some unspecified function $\lambda^\alpha_\beta(\theta)$ which relates to the fact that we can choose different gauge generators R^i_α. As it comes from its definition, the action is equal for all the configurations on a particular gauge orbit

$$\frac{\delta S[\phi^i(\theta)]}{\delta\theta^\alpha} = y_i R^i_\alpha \lambda^\alpha_\beta = 0 \tag{5.2.2}$$

where the functions λ^α_β satisfy

$$\frac{\delta\lambda^\alpha_\beta}{\delta\theta^\gamma} - \frac{\delta\lambda^\alpha_\gamma}{\delta\theta^\beta} + t^\alpha_{\mu\nu}\lambda^\mu_\beta\lambda^\nu_\gamma = 0 \tag{5.2.3}$$

which comes from the integrability of (5.2.1), namely the analogue of the Maurer-Cartan equations

$$\frac{\delta^2\phi^i(\theta)}{\delta\theta^\gamma\delta\theta^\beta} - \frac{\delta^2\phi^i(\theta)}{\delta\theta^\beta\delta\theta^\gamma} = 0 \tag{5.2.4}$$

different choices of λ^α_β give rise to different ways of defining the coordinates of the gauge orbits.

After defining the gauge orbits, the measure of the path integral splits naturally into two pieces, over the space of gauge orbits and over the configurations of a particular gauge orbit

$$[d\phi] = [d\Phi_0^i].\prod_l [d\theta_l^\alpha].\det\lambda \tag{5.2.5}$$

where θ_l^α are the coordinates on the l-th orbit and $[d\theta_l^\alpha]$ is the measure for integrating over the configurations which lie on a particular gauge orbit. The $\det\lambda$ just makes the integration over the gauge orbits to be coordinate invariant. The first part of the measure $[d\Phi_0^i]$ is the measure over the space of different orbits.

In order to render infinities which is made by integration over the gauge orbits, since they encode the same field configuration, we can make use the fact that the action is constant under the change of the gauge orbit coordinate θ^α. To do this, we choose just one configuration on each orbit by using $\delta-$functions and rewrite the measure over the gauge orbits $[d\theta_l^\alpha]$ as follows

$$[d\Phi_0^i]\prod_l [d\theta_l]\det\lambda.\frac{\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right)}{\det\lambda} = [d\phi].\prod_l \frac{\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right)}{\det\lambda} \tag{5.2.6}$$

where Θ_l^α are the base coordinates for the configurations which is selected over $l-$th gauge orbit. The more practical way to select one configuration over each gauge orbit is by choosing a set of gauge fixing functions $F^\alpha(\phi^i)$ which are as many as the coordinates of the gauge orbit. Then we put delta functions $\delta(F^i\alpha(\phi^i) - f^\alpha(\phi))$ in the path integral for some function f^α. This can be related to a $\delta-$function in terms of the coordinates as follows

$$\delta\left(F^\alpha(\phi(\theta_l)) - f^\alpha\right) = \frac{1}{\det M}\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right) \tag{5.2.7}$$

where here Θ_l^α is defined as the solution of $F^\alpha(\phi(\theta_l)) - f^\alpha = 0$ and the matrices M_β^α are defined as

$$M_\beta^\alpha = \frac{\delta F^\alpha(\phi(\theta))}{\delta\theta^\beta} \tag{5.2.8}$$

After using (5.2.1) this can be rewritten as follows

$$M_\beta^\alpha = \frac{\delta F^\alpha(\phi(\theta))}{\delta\phi^i} R_\gamma^i \lambda_\beta^\gamma \tag{5.2.9}$$

Then one can write the measure of the path integral as follows

$$[d\phi]\frac{1}{\det\lambda}.\det M.\delta(F^\alpha - f^\alpha) \tag{5.2.10}$$

ant the path integral becomes

$$Z = \int [d\phi] \frac{1}{\det \lambda} . \det M . \delta(F^\alpha - f^\alpha) . e^{\frac{i}{\hbar} S} \tag{5.2.11}$$

In order to make the determinant computable, we can enlarge the field space by adding one pair of fields (b_α, c^α) of reverse Grassmann parity for every gauge generator R^i_α and use the following identity

$$\det M = \int [db][dc] \exp \left[\frac{i}{\hbar} b_\alpha M^\alpha_\beta c^\beta \right] = \int [db][dc] e^{\frac{i}{\hbar} S_{ghost}} \tag{5.2.12}$$

This new sector is the *ghost* sector in which c^α and b_α are the ghosts and antighosts respectively. The action for the ghost sector S_{ghost} can be defined after using $\lambda^\gamma_\beta c^\beta$ instead of c^α as the ghost, then the action becomes

$$S_{ghost} = b_\alpha \frac{\delta F^\alpha}{\delta \phi^i} R^i_\beta c^\beta \tag{5.2.13}$$

where now the Jacobian of the transformation $\lambda^\gamma_\beta c^\beta \to c^\alpha$ cancels the $\det \lambda$ factor in the path integral (5.2.11).

Finally, in order to write the partition function in a way that is independent of the choice of the functions f^α, one can integrate over f^α with a suitable factor $W[f]$ such that

$$\int [df] W[f] = 1 \tag{5.2.14}$$

Taking a Gaussian gauge fixing factor $W[f] = N e^{\frac{i}{2\hbar} f^2}$ we get the following partition function

$$\begin{aligned} Z &= \int [df] \; Z \; W[f] \\ &= \int [d\phi][db][dc] \; e^{\frac{i}{\hbar} S_{complete}} \end{aligned} \tag{5.2.15}$$

where the complete action splits into three terms, the original action, the ghost term and the gauge fixing term

$$\begin{aligned} S_{complete} &= S_0 + S_{ghost} + S_{gf} \\ &= S + b_\alpha \frac{\delta F^\alpha}{\delta \phi^i} R^i_\beta c^\beta + \frac{1}{2} F^2 \end{aligned} \tag{5.2.16}$$

This was the basic ingredients of the first gauge quantization procedure. As we can see the gauge fixed action (5.2.16) has no longer the gauge invariance anymore. However, the gauge invariance is expected to be present also in the quantum theory and it is the subject of the next section which studies the traded form of the gauge invariance into a global symmetry which exist during all the quantization procedure and is called BRST symmetry. Also we can obtain the gauge fixed action using this symmetry.

5.3 BRST quantization of gauge theories

Another quantization method which we are going to discuss in this section is the BRST quantization which was developed by C. Becchi, A. Rouet, R. Stora and I.V. Tyutin in [74]. It is based on the observation that the local gauge symmetry can be replaced with a global symmetry which is present during all the quantization procedure and even for the gauge fixed action. This is the basic procedure will be used to construct the antifield formalism in the next section.

First, we start by introducing the BRST symmetry and then we see the construction of the gauge fixed action of the quantum theory. It appears that the gauge fixed action $S_{complete}$ is indeed invariant under the BRST symmetry.

We start from a classical action $S_0[\phi^i]$ with a gauge symmetry which specified by the generators $R^i_\alpha[\phi]$. In order to construct the BRST symmetry first we have to use the BRST operator s which is a fermionic, linear differential operator acting from the right as follows

$$s(X.Y) = X.sY + (-1)^{\epsilon_Y} sX.Y \tag{5.3.1}$$

with the following action on the Fields

$$s\phi^i = R^i_\alpha[\phi]c^\alpha \tag{5.3.2}$$

where we introduced a ghost degree of freedom c^α for every gauge generator with opposite statistics. For function(al)s that depend on the classical fields, the BRST invariance (5.3.2) is equivalent to gauge invariance, an example of which is the classical action which is invariant under the BRST transformation as $sS_0 = y_i R^i_\alpha c^\alpha = 0$.

The BRST operator should be nilpotent *nilpotent* which is related to the closure of the corresponding gauge algebra, namely

$$s^2 = 0 \tag{5.3.3}$$

which is guaranteed by choosing the BRST transformation for the ghost fields as follows

$$sc^\gamma = T^\gamma_{\alpha\beta}[\phi]c^\beta c^\alpha \tag{5.3.4}$$

We can always enlarge the set of fields by pairs like D^a and d^a named as *trivial system* for an arbitrary set of indices a with the following BRST transformation rules

$$sD^a = d^a \quad , \quad sd^a = 0 \tag{5.3.5}$$

Since the classical action we start with does not depend on D^a fields, shifting these fields is a local symmetry of the theory and the d^a fields are the associated ghost fields corresponding to this gauge symmetry. The form of their BRST transformations (5.3.5) ensures its nilpotency. Adding this trivial sector to theory will be shown to have useful applications. An example of which is the Faddeev-Popov antighosts which are introduced exactly in this way

$$sb_\alpha = \lambda_\alpha \quad , \quad s\lambda_\alpha = 0 \tag{5.3.6}$$

We can assign a *grading* to each field which is called the *ghost number* with the following assignments

$$\begin{aligned} \mathrm{gh}(\phi^i) = 0 \quad &, \quad \mathrm{gh}(c^\alpha) = 1 \\ \mathrm{gh}(b_\alpha) = -1 \quad &, \quad \mathrm{gh}(\lambda_\alpha) = 0 \end{aligned} \tag{5.3.7}$$

The BRST operator itself carries a ghost number $\mathrm{gh}(s) = 1$.

The next step after introducing the BRST operator is to use this operator to construct the gauge fixed action. The claim is that there exist a functional Ψ which is named as *gauge fermion* such that

$$S_{complete} = S_0 + s\Psi \tag{5.3.8}$$

The gauge fermion has ghost number -1 and odd Grassmann parity. Since ϕ^i and c_α have ghost number 0 and 1 respectively, we can not make a suitable gauge fermion out of them. We should introduce a trivial pair $(b_\alpha, \lambda^\alpha)$ with $\mathrm{gh}(b_\alpha) = -1$. It can be proved that as long as Ψ leads to path integrals that are well-defined, means that they do not have gauge invariance, the path integral is independent of the form of the gauge fermion. This is an important consequence of the BRST quantization which admits systematically to apply different gauge fixings of the theory which are useful for their specific purpose. This will be used to find different gauge fixings of the G/G principal chiral model in the next chapter. An example of gauge fermion is the one which produces the Faddeev-Popov gauge fixed action by taking Ψ to be $\Psi = b_\alpha \left(F^\alpha - \lambda^\alpha a\right)$ which gives the following action

$$S_{complete} = S_0 + s\Phi = S_0 + b_\alpha \frac{\overleftarrow{\delta} F^\alpha(\phi)}{\delta \phi^i} R^i_\beta c^\beta + F^\alpha \lambda_\alpha - a\lambda^2(-1)^{\epsilon_\alpha} \tag{5.3.9}$$

This is equivalent to Faddeev-Popov gauge fixed action (5.2.16) after integrating over λ.

Since the gauge fixed action is written as (5.3.8), and the classical action S_0 is invariant under the BRST transformations by construction, it is invariant under the BRST transformation too because the BRST charge is nilpotent. The original gauge symmetry now translates into the BRST symmetry which is manifest also for the gauge fixed action.

One can show easily that for an action which is invariant under the BRST transformation we have

$$\langle sX(\phi)\rangle = \int [d\phi].sX.e^{\frac{i}{\hbar}S[\phi]} = 0 \tag{5.3.10}$$

for any $X(\phi)$. These are the *Ward identities* which can be proved easily by using BRST technique. Using these Ward identities it is possible to show that the gauge invariance is replaced with the BRST symmetry also for the quantum theory. And the gauge invariant observables can be characterized by conditions

$$s\,\Omega[\phi] = 0 \tag{5.3.11}$$

which means they have to be BRST closed.

Using Ward identity it is possible to show that two functions which differ with a BRST exact term, i.e. a BRST variation of something, like $X_2 = X_1 + s\Omega$ have the same expectation value $\langle X_1\rangle = \langle X_2\rangle$. Since the BRST operator is nilpotent we can conclude the last two sentences in the following theorem

Theorem 5.1. *The gauge invariant operators of the gauge theory are given with the non-trivial cohomology classes of the BRST operator s at ghost number zero.*

Namely if we define the cohomology group of the BRST differential as

$$H(s) = \frac{ker\ s}{Im\ s} \tag{5.3.12}$$

where it can be splited in different sectors according to the ghost number grading as $H(s) = \sum_g H^{(g)}(s)$, then we have

$$H^{(0)}(s) = \{\text{classical gauge invariant observables}\} \tag{5.3.13}$$

where $H^{(0)}(s)$ consists of the functions A with $\mathrm{gh}(A) = 0$ such that

$$sA \equiv [A, S] = 0 \tag{5.3.14}$$

where S is the BRST generator.

In the next sections we will explore more in detail the cohomology group of the BRST operator and its elements for different ghost numbers.

5.4 The BV antifield formalism

The antifield formalism is another way of quantization of gauge theories which is mainly based on the BRST Lagrangian quantization which we discussed in the previous section. It was developed by Batalin and Vilkovisky [75, 72]. In this approach the BRST symmetry of the theory is enlarged in a way that Schwinger-Dyson equations which are the shift symmetries of the theory become Ward identities of the theory and one can have a direct Lagrangian formalism of the theory without the need to pass to the Hamiltonian formalism.Using Lagrangian formalism we have all the adapted covariance in hand. This makes it easier to study and quantize the gauge theories.

Here again we start from a classical action $S_0[\phi]$ depending on the fields ϕ which admits a gauge symmetry which we assume to be irreducible and closed. We can construct the nilpotent BRST operator in a way we spoke before acting on the extended set of fields $\phi^A = \left\{\phi^i, c^\alpha, D^a, d^a\right\}$ including also the ghosts and the trivial pairs.

Another ingredient of the antifield formalism is an antibracket $(\cdot, \cdot)$ which replaces the Poisson bracket of the Hamiltonian formalism which acts as a canonical transformation $sA = [A, s]$. Here the BRST transformation of the BV formalism is a canonical transformation in the antibracket which acts as follows

$$sA = (A, S) \tag{5.4.1}$$

where S is the generator of the BRST symmetry.

After introducing new degrees of freedom named as *antifields* for each field and ghost, we can see that using this antibracket there is a symmetry between the fields and ghosts on one hand and the antifields on the other side. This symmetry appears as a conjugate relation between fields and antifields as follows

$$(\phi^i, \phi^*_j) = \delta^i_j \quad , \quad (c^\alpha, c^*_\beta) = \delta^\alpha_\beta \tag{5.4.2}$$

where the $\phi^*_A = \{\phi^*_i, c^*_\alpha\}$ is the antifield sector which we add in order to realize this symmetry. The antifields are related to the variation of the action with respect to a particular differential which we will speak later. They have also opposite Grassmann parity with their corresponding field.

We can assign the following ghost number to the fields, ghosts and antifields as follows

$$\begin{array}{cccc} \phi^i & c^\alpha & \phi^*_i & c^*_\alpha \\ \hline 0 & 1 & \text{-1} & \text{-2} \end{array} \quad \longrightarrow \text{ ghost number} \tag{5.4.3}$$

The definition of antibracket can be extended to any arbitrary functional X, Y of the fields, ghost and the antifields as

$$(X,Y) = \frac{\delta_R X}{\delta z^A}\omega^{AB}\frac{\delta_L Y}{\delta z^B} = \frac{\delta_R X}{\delta \phi^A}\frac{\partial_L Y}{\delta \phi^*_A} - \frac{\delta_R X}{\delta \phi^*_A}\frac{\delta_L Y}{\delta \phi^A} \tag{5.4.4}$$

where $z^A = \left\{\phi^A, \phi^*_A\right\}$ and $\omega^{AB}(z)$ is the inverse of the symplectic form of the space of fields and antifields $\omega = \omega_{AB} dz^A dz^B$ which can be written explicitly in this way

$$\omega = \delta\phi^A \wedge \delta\phi^*_A \tag{5.4.5}$$

and the variations are with respect to the BRST charge defined before.

Because of the parity and ghost number of the antibracket which can be seen from (5.4.2) and (5.4.8), the generator of the BRST transformation S should have the following Grassmann parity and ghost number

$$\epsilon_G(S) = 0 \quad , \quad \text{gh}(S) = 0 \tag{5.4.6}$$

The nilpotency of BRST operator s translated into the following condition

$$(S,S) = 0 \tag{5.4.7}$$

this is named as the *master equation.* The problem of finding the BRST symmetry as a canonical transformation in the antibracket becomes equivalent to the problem of finding the solution of S of (5.4.7) with the consideration that it should produce the classical action after putting the antifields to zero.

To solve the master equation is a crucial step of getting the gauge fixed action of the theory. In order to do so, we should introduce another grading which as we will see it has a natural geometric interpretation. This new grading is named as the *antighost number* with the following assignment

$$\begin{array}{cccc} \phi^i & c^\alpha & \phi^*_i & c^*_\alpha \\ \hline 0 & 0 & 1 & 2 \end{array} \quad \longrightarrow \text{ antighost number} \tag{5.4.8}$$

which then one can define the pure ghost number as $pure\ gh = gh + antigh$.

we can decompose everything including the BRST operator s and BRST generator S according the antighost number as

$$s = \sum_{n\geq -1} s^{(n)} \quad , \quad S = \sum_{n\geq 0} S^{(n)} \tag{5.4.9}$$

with

$$\text{antigh}(s^{(n)}) = n \quad , \quad \text{antigh}(S^{(n)}) = n \tag{5.4.10}$$

The fact that we started from antighost number -1 for s is related to the important theorem that we will discuss later when we will explore the cohomology of the theory. The minimum antighost number for S is one more than the one of s because of (5.4.2) since it implies $\text{antigh}((A, S)) = \text{antigh}(S) + \text{antigh}(A) - 1 = \text{antigh}(sA) = \text{antigh}(s) + \text{antigh}(A)$ which gives

$$\text{antigh}(S) = \text{antigh}(s) + 1 \tag{5.4.11}$$

Here we assume that the BRST operator can be expanded explicitly as follows

$$s = \delta + d + \text{"}more\text{"} \tag{5.4.12}$$

where δ is the *Koszul-Tata* differential with antighost number -1 and d is a differential with antighost number zero which anticommutes with δ and is longitudinal exterior derivative along the gauge orbits on the stationary surface. This splitting has many influences in deriving the cohomology of BV action using the homological perturbation theory.

We can use this form of BRST charge according to antighost number to solve the master equation using the fact that any field-antifield pair transform under the Koszul-Tata differentiate as follows

$$\delta\phi^i = 0 \quad , \quad \delta\phi^*_i = \frac{\delta S_0}{\delta\phi^i} \tag{5.4.13}$$

correspondingly we have the following transformations coming from (5.4.1)

$$\begin{aligned} (\phi^i, S) &= d\phi^i + \text{"}more\text{"} \quad , \quad (\phi^*_i, S) = \delta\phi^*_i + \text{"}more\text{"} \\ (c^\alpha, S) &= dc^\alpha + \text{"}more\text{"} \quad , \quad (c^*_\alpha, S) = \delta c^*_i + \text{"}more\text{"} \end{aligned} \tag{5.4.14}$$

Since the differential d is the exterior derivative along the gauge orbits on the stationary surface, it just measures how the $p-$forms change as one moves along a particular gauge orbit. It contains no information about the transverse directions and is such that $H^0(d) = \{\text{gauge invariant functions}\}$. It generates the following BRST transformations coming from (5.3.2) and (5.3.4)

$$dF = \frac{\delta F}{\delta\phi^i} R^i_\alpha c^\alpha \quad , \quad dc^\alpha = \frac{1}{2} T^\alpha_{\beta\gamma} c^\beta c^\gamma \tag{5.4.15}$$

The differential d is actually the BRST differential which we used in the previous section. Actually one of the effects of adding the antifields to the theory is to generalize the BRST operator with terms which they are absent in the usual interpretation of the BRST symmetry as the global extension of the local gauge symmetry.

The nilpotency of the BRST operator s translates into the following conditions

$$\delta^2 = 0 \quad , \quad d\delta + \delta d = 0 \tag{5.4.16}$$

taking the following gradings

$$\text{pure gh}(\delta) = 0 \quad , \quad \text{pure gh}(d) = 1 \tag{5.4.17}$$

$$\text{antigh}(\delta) = -1 \quad , \quad \text{antigh}(d) = 0 \tag{5.4.18}$$

Now we should solve the master equation (5.4.7) by using (5.4.14) using the expansion (5.4.9), we get for each antifield degree of the expansion of the action the following equations coming from the master equation

$$S^{(0)} = S_0 \tag{5.4.19}$$

$$S^{(1)} = \phi_i^* R_\alpha^i c^\alpha \tag{5.4.20}$$

$$S^{(2)} = \frac{1}{2} T_{\beta\gamma}^\alpha c_\alpha^* c^\beta c^\gamma \tag{5.4.21}$$

The next terms in S are then determined recursively by equations of this form

$$2\delta S^{(n)} + H^{(n-1)} = 0 \tag{5.4.22}$$

where the local functional $H^{(n-1)}$ is the antighost number $n-1$ component of the $(R^{(n-1)}, R^{(n-1)})$ where $R^{(n-1)} = \sum_{k\leq n-1} S^{(k)}$. So as we saw, a solution for master equation exists. Furthermore, in order to find it as local functional, the expansion of S should stop at some antighost number *i.e.* $\exists N,\ S^{(n)} = 0$ for $n \geq N$ because the number of derivatives may increase with n. Therefore to have a local functional we should not have infinite number of $S^{(n)}$'s.

Finally we can write the first terms of S as the solution of master equation as an expansion over antifield degree as follows

$$S = S_0[\phi] + \phi_i^* R_\alpha^i c^\alpha + \frac{1}{2} T_{\alpha\beta}^\gamma c_\gamma^* c^\alpha c^\beta + \text{"}more\text{"} \tag{5.4.23}$$

where "*more*" means possible terms with higher antighost number. This solution starts from the classical action and continues as a polynomial in antifields.

Always we have considered the gauge algebra to be irreducible and closed. For the cases in which the gauge algebra is reducible or open, we have to add more terms in the solution of the master equation to take into account the irreducibility.

The solution (5.4.23) of the master equation is not unique since because of (5.4.22) we can always add a δ-exact term to $S^{(n)}$. Furthermore, we can always add to a given solution further variables that are cohomologically trivial which don not modify the cohomology. This trivial pairs construct the *nonminimal solution* of the master equation.

As we said before, the trivial pair $(\mathcal{D}, d)$ fulfill the following transformations

$$s\mathcal{D}^a = d^a \quad , \quad sd^a = 0 \tag{5.4.24}$$

with the ghost numbers which are related as follows

$$\text{gh}(\mathcal{D}^a) = \text{gh}(d^a) - 1 \tag{5.4.25}$$

These trivial pairs does not appear in the cohomology of the BRST operator, actually the condition $sF = 0$ eliminates $\mathcal{D}^a$ from the cohomology because of the nilpotency of s, and since $\mathcal{D}^a$ is not BRST closed. The further step in the cohomology eliminates d which is BRST exact and its presence cancels in the $Ker(s)$ and $Im(s)$ and so does not appear in the cohomology $H(s) = \frac{Ker(s)}{Im(s)}$.

We can add another trivial pair which let us to write a canonical action for the non-minimal sector after introducing their antifields $\mathcal{D}^*_s$ and d^*_s which are conjugate to $\mathcal{D}^a$ and d^a satisfying the following antibrackets

$$(\mathcal{D}^a, \mathcal{D}^*_b) = \delta^a_b \quad , \quad (d^a, d^*_b) = \delta^a_b \tag{5.4.26}$$

which implies also $\text{gh}(\mathcal{D}^*) = -\text{gh}(\mathcal{D}) - 1$ and $gh(d^*) = -gh(d) - 1$.

A possible non-minimal term in the action which produces (5.4.24) through the antibracket of $(\mathcal{D}, S) = d$ and $(d, S) = 0$ is

$$S_{nm} = \mathcal{D}^*_a d^a \tag{5.4.27}$$

Using the antibrackets (5.4.26) it generates the following BRST transformations for their antifields

$$sd^* = \mathcal{D}^* \quad , \quad s\mathcal{D}^* = 0 \tag{5.4.28}$$

Finally the full action as the solution of the master equation is obtained after adding the non-minimal sector

$$S = S_0[\phi] + \phi_i^* R_\alpha^i c^\alpha + \frac{1}{2} T_{\alpha\beta}^\gamma c_\gamma^* c^\alpha c^\beta + "more" + \mathcal{D}_a^* d^a \tag{5.4.29}$$

The number of required extra trivial pairs depends on the gauge fixing conditions which is desired as we will discuss later.

A very important property of S is that it is invariant under a set of gauge transformations. Let name collectively all the fields, ghosts and non-minimal fields as ϕ^A and their antifields as ϕ_A^* and the set of all fields and antifields as $z^a = (\phi^A; \phi_A^*)$. If we differentiate the master equation (5.4.4) with respect to z^a we get

$$\frac{\delta_R S}{\delta z^a} \mathcal{R}_c^a = 0 \tag{5.4.30}$$

where $\mathcal{R}_c^a = \omega^{ab} \frac{\delta_L \delta_R S}{\delta z^b \delta z^c}$. This implies that the action S is invariant under the following gauge transformation

$$\delta z^a = \mathcal{R}_c^a \epsilon^c \tag{5.4.31}$$

The fact that the theory admits such a gauge symmetry means that the path integral

$$Z = \int [D\phi][D\phi^*] e^{\frac{i}{\hbar} S(\phi, \phi^*)} \tag{5.4.32}$$

where the integration is over all the filed and antifield space and is divergent and ill-defined before gauge fixing. In order to get ride of this problem one can put some constraints on the antifields in which in the space of field-antifields get projected to a submanifold on which the divergent part of the measure factorizes and the rest become a well defined path integral. This is a consequence of the gauge fixing (5.4.31) as it appears as a symmetry between fields and antifields.

We can take the constraint to be a gradient in order to use Stokes theorem later

$$\phi_A^* = \frac{\delta \psi}{\delta \phi^A} \tag{5.4.33}$$

which after using the Stokes theorem a general path integral over a general polynomial of antifields $V(\phi, \phi^*) = V_0(\phi) + V^(1)\phi^* + V^{(2)} \phi^{*2} + \cdots$ becomes

$$\int [D\phi][D\phi^*] V \left(\phi, \phi^* = \frac{\delta \psi}{\delta \phi} \right) = \int [D\phi] V \tag{5.4.34}$$

the function $\psi(\phi)$ must have ghost number -1 and Grassmann parity 1 in order to preserve the grading properties of the antifields and fields in (5.4.33), it is named as *gauge fixing fermion* since it project the field-antifield space to the gauge fixed subspace. Another important thing which should be checked is the fact that the path integral does not depend on the choice of the gauge fermion and it happens actually when the Laplacian of the action vanishes since in order to get (5.4.34) we used the Stokes theorem. So we should have

$$\Delta S = 0 \tag{5.4.35}$$

where the Laplacian is defined as

$$\Delta = \frac{\delta_R}{\delta\phi^A}\frac{\delta_R}{\delta\phi^*_A}(-)^{\epsilon^A_G - 1} \tag{5.4.36}$$

here ϵ^A_G is the Grassmann parity of the field ϕ^A. This operator has the following properties

$$\Delta^2 = 0 \quad , \quad \epsilon_G(\Delta) = 1 \tag{5.4.37}$$

One can check that the surface which is defined in the field-antifield space with the equations

$$O_A(z) \equiv \phi^*_A - \frac{\delta\psi}{\delta\phi^A} \tag{5.4.38}$$

satisfies the following condition

$$(O_A, O_B) = 0 \tag{5.4.39}$$

This means that the symplectic form (5.4.5) vanishes on this surface and since they have half of the dimension of the field-antifield space because of their defining equation (5.4.38), they are *Lagrangian submanifolds* of the field-antifield space.

In order to gauge fix the theory, we should define a Lagrangian submanifold on which the path integral becomes well defied and its divergent part gets factorized.

The gauge fixed action can be written explicitly as the projection of the action to the Lagrangian submanifold as follows

$$S|_\psi = S_0 + \frac{\delta\psi}{\delta\phi^i}s\phi^i + \frac{1}{2}T^\gamma_{\alpha\beta}\frac{\delta\psi}{\delta c^\gamma}c^\alpha c^\beta + \text{"}more\text{"}|_\psi \tag{5.4.40}$$

Also the new BRST charge of the theory can be obtained as the projection of the original BRST operator to the Lagrangian submanifold

$$s_\psi\phi^A = (s\phi^A)\left(\phi, \phi^*_A = \frac{\delta\psi}{\delta\phi^A}\right) \equiv \frac{\delta_L S}{\delta\phi^*_A}\left(\phi, \phi^*_A = \frac{\delta\psi}{\delta\phi^A}\right) \tag{5.4.41}$$

under which the gauge fixed action is invariant

$$s_{\psi} S|_{\psi} = 0 \tag{5.4.42}$$

It is nilpotent only on-shell

$$s|_{\psi}^{2} \phi^{A} = \text{equations of motion} \tag{5.4.43}$$

In chapter (6) we will explore an example of the BV action, for the G/G principal chiral model, and its different gauge fixings to produce the topological actions which are related to pure spinor formalism and the A-model of the $AdS_5 \times S^5$ will be explored.

5.5 Consistent deformation of the BV action

The BV formalism gives the power to find consistent interactions among fields which does not spoil the original gauge invariance of the theory in the sense of the deformation theory.

Indeed, having a consistent interaction, the solution $S^{(0)}$ of the master equation can be deformed as an expansion of the interaction parameter r as follows

$$S = S^{(0)} + rS^{(1)} + r^2 S^{(2)} + \cdots \tag{5.5.1}$$

The master equation for a consistent deformation should also be satisfied which implies

$$(S, S) = 0 \tag{5.5.2}$$

This splits according to different orders of the deformation parameter r as

$$\begin{aligned} (S^{(0)}, S^{(0)}) &= 0 && (5.5.3)\\ (S^{(0)}, S^{(1)}) &= 0 && (5.5.4)\\ 2(S^{(0)}, S^{(2)}) + (S^{(1)}, S^{(1)}) &= 0 && (5.5.5)\\ &\vdots \end{aligned}$$

The first equation is satisfied because we are deforming around a solution of the master equation. The second equation (5.5.4) implies that $S^{(1)}$ is a cocycle for the BRST operator $s^{(0)} = (\cdot, S^{(0)})$.[1] It can be shown [76] that if $S^{(1)}$ is a coboundary, it corresponds to a trivial deformation which can be absorbed into $S^{(0)}$. This means that the nontrivial deformations are elements of the cohomological space $H^0(s^{(0)})$ which is isomorphic to the space of the observables of the original undeformed theory. From (5.5.5) we see that the second order deformation $S^{(2)}$ exists if and only if the cocycle $(S^{(1)}, S^{(1)})$ is trivial in $H^1(s^{(0)})$. If not, there is no $S^{(2)}$ and the deformation gets obstructed at order r^2.

It is the cohomological groups $H^0(s^{(0)})$ and $H^1(s^{(0)})$ which are giving the information of the first-order deformations and the obstruction of continuing the deformation to higher orders.

Here we are interested to find local deformations, deformations which can be expressed as local functionals. Every term $S^{(k)}$ can be written as an integral over a n-form $S^{(k)} = \int \mathcal{L}^{(k)}$ which depends on the fields, antifields and ghosts and a finite number of their derivatives. A vanishing local functional $A = \int a = 0$ implies $a = df$ where d is the

[1] It is the BRST operator we have always used before.

space-time exterior derivative and $\oint f = 0$. We can define an antibracket $\{a, b\}$ for these n-forms as

$$(A, B) = \int \{a, b\} \tag{5.5.6}$$

which is defined up to $d-$exact terms.

One can rewrite the descent equations (5.5.3-5.5.5) as follows

$$\begin{aligned} s^{(0)}\mathcal{L}^{(1)} &= dj^{(1)} && (5.5.7) \\ s^{(0)}\mathcal{L}^{(2)} + \{\mathcal{L}^{(1)}, \mathcal{L}^{(1)}\} &= dj^{(2)} && (5.5.8) \\ &\vdots \end{aligned}$$

which implies again that the non-trivial local deformations are elements of the $H^0(s^{(0)}|d)$ which is the cohomology of $s^{(0)}$ modulo the exterior derivative d. The Second equation (5.5.8) implies that $S^{(2)}$ exists if and only if the cocycle $(S^{(1)}, S^{(1)})$ is trivial in $H^1(s^{(0)}|d)$.

For a general theory, the BRST charge s contains all the fields and ghosts and solving the descent equations (5.5.7-5.5.8) is not an easy task. In order to make it possible, one can use the homological perturbation theory which relates the cohomology of s to the cohomology of δ and γ. This relation goes through the following theorem [77]

Theorem 5.2. *The cohomology group $H^k(s|d)$ is given by*

$$H^k(s|d) \simeq \begin{cases} H_{-k}(\delta|d) & k < 0 \\ H^k(\gamma|d, H_0(\delta)) & k \geq 0 \end{cases} \tag{5.5.9}$$

where $\simeq$ means up to trivial terms and $H^k(\gamma|d, H_0(\delta))$ is the cohomology of $H^k(\gamma|d)$ in $H_0(\delta)$.

Accordingly, in order to find the deformation space and to find $H^0(s|d)$ and $H^1(s|d)$, it is enough to study their antifield independent components.

So, the next step is to find the cohomological group $H(\delta)$ of the G/G principal chiral model as it was studied in [78] in order to find the possible deformations one can find without spoiling the structure of the theory. This we will do in the next chapter after introducing its BV action and we will see that the pure spinor superstring action arises as a possible consistent interaction which one can add to the topological action.

Chapter 6

Towards a worldsheet description of AdS/CFT duality

It was proposed by Berkovits in [35] and [36] that starting from a G/G principal chiral model based on the supergroup $PSU(2,2|4)$ there are at least two different ways of gauge fixing it which gives two topological models, the first is the A-model topological action we described before, and the other is a topological theory whose supersymmetric charge is equivalent to the pure spinor BRST charge

$$Q_{SUSY} = Q_{BRST} \tag{6.0.1}$$

Here in this section based on an unpublished work [22], it will be explained that constructing the BV action for the G/G principal chiral model, in fact we can systematically gauge fix it and get two different topological theories. This topological theories are conjectured to correspond to the zero radius limit of the pure spinor superstring on $AdS_5 \times S^5$ and so they are dual to the free $\mathcal{N} = 4$ SYM. The connection of the A-model and the other topological model with the same supersymmetry charge as the BRST charge of the pure spinor hints that they have actually the same cohomology and the topological model can explore all the physics of the pure spinor superstring and not just its BPS sector. Next, we show that we can consistently deform the G/G model by turning on the radius modulus and we see that the form of the deformation becomes the pure spinor action itself as the vertex for this deformation. Using this picture one might in principle apply the same analysis of Ooguri and Vafa for the case of conifold and give a worldsheet proof for the Maldacena's conjecture as we will comment later.

6.1 BV action for the G/G principal chiral model

The G/G principal chiral model action can be written in this way

$$S_0 = \mathrm{Str}\int d^2z\eta_{AB}(J-A)^A(\bar{J}-\bar{A})^B \tag{6.1.1}$$

where J and $\bar{J}$ are the left and right components of the one-form $J = g^{-1}dg$ with respect to the worldsheet ∂ and $\bar{\partial}$ derivatives constructed from group elements of $PSU(2,2|4)$ supergroup and $(A, \bar{A})$ are $PSU(2,2|4)$ gauge groups on the worldsheet. They can be expanded in the generators of the supergroup as follows

$$J = J^A T_A \quad , \quad \bar{J} = \bar{J}^A T_A \tag{6.1.2}$$

where A here is a general $PSU(2,2|4)$ supergroup index.

An element $g \in PSU(2,2|4)$ can be represented in terms of the local coordinates h^A on the supergroup manifold as

$$g = e^{h^A T_A} \tag{6.1.3}$$

The Cartan one-forms can be written in terms of the local coordinates too

$$J^A = \omega^A_B(h)dh^B \tag{6.1.4}$$

The matrix $\omega^A_B(h)$ is invertible because the Cartan forms J^A form a basis for the super algebra. One can find their inverse $\omega^A_B(h)$ satisfying the following relation

$$\omega^A_B(h)\Omega^B_C(h) = \delta^A_C \tag{6.1.5}$$

The action is invariant under the following local symmetry transformations,

$$\delta A = d\epsilon + [A, \epsilon] \tag{6.1.6}$$

Which can be promoted to a BRST transformation after introducing the ghost fields C taking value in the Lie algebra $psu(2,2|4)$.

The nilpotent BRST transformation acts on the fields and the ghosts as follows,

$$\begin{aligned} sA &= dC + [A, C]\,, \\ s\bar{A} &= \bar{d}C + [\bar{A}, C] \\ sJ &= dC + [J, C] \\ s\bar{J} &= \bar{d}C + [\bar{J}, C] \\ sC &= -\frac{1}{2}[C, C] \end{aligned} \tag{6.1.7}$$

To see the structure of these transformations we can use the $\mathbb{Z}_4$ automorphism grading in which any field or ghost is decomposed in four classes (F_0, F_1, F_2, F_3) belong to $\mathbb{Z}_4$ equivalence classes $(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$. Because of the supergroup structure in which the generators $(T_0 \in \mathcal{H}_0, T_2 \in \mathcal{H}_2)$ are bosonic and $(T_1 \in \mathcal{H}_1, T_3 \in \mathcal{H}_3)$ are fermionic, different components of the fields and ghosts acquire the following Grassmann parities[1]

$$\begin{aligned} \epsilon_G(A_0, A_2, J_0, J_2, C_1, C_3) &= 0 \\ \epsilon_G(A_1, A_3, J_1, J_3, C_0, C_2) &= 1 \end{aligned} \tag{6.1.8}$$

So the BRST transformations (6.1.7) can be rewritten in $\mathbb{Z}_4$ decomposition as follows

$$\begin{aligned} sA_0 &= dC_0 + [A_0, C_0] + [A_1, C_3] + [A_3, C_1] + [A_2, C_2]\,, \\ sA_1 &= dC_1 + [A_1, C_0] + [A_0, C_1] + [A_2, C_3] + [A_3, C_2]\,, \\ sA_2 &= dC_2 + [A_2, C_0] + [A_0, C_2] + [A_1, C_1] + [A_3, C_3]\,, \\ sA_3 &= dC_3 + [A_1, C_2] + [A_2, C_1] + [A_0, C_3] + [A_3, C_0]\,, \\ sC_0 &= -\frac{1}{2}[C_3, C_1]\,, \\ sC_1 &= -\frac{1}{2}[C_1, C_0] - \frac{1}{2}[C_3, C_2]\,, \\ sC_2 &= -\frac{1}{2}[C_2, C_0]\,, \\ sC_3 &= -\frac{1}{2}[C_3, C_0] + \frac{1}{2}[C_2, C_1] \end{aligned} \tag{6.1.9}$$

which is nilpotent off-shell. The BRST transformation of the left-invariant currents J and $\bar{J}$ are similar to the one of A and $\bar{A}$ replacing J with A.

The minimal BV action $S_{min}(\Phi, \Phi^*)$ can be written as an expansion in powers of anti-fields around the classical solution as we described in the previous chapter of this thesis,

$$S_{min} = S_0 + \Phi^*_{(i)} s\Phi^{(i)} \tag{6.1.10}$$

[1] The Grassmann parity ϵ_G is zero for a boson and one for a fermion.

where $\Phi^{(i)} = (g, A, C)$ is the minimal set of fields and we introduced a set of their anti-fields $\Phi^*_{(i)} = (g^*, A^*, C^*)$ for them. The antifield of the group element g belongs to T^*G where G is the supergroup $\mathcal{G} = psu(2,2|4)$. It and can be expanded as follows

$$g^* = g^*_A T^A g^{-1} \tag{6.1.11}$$

They satisfy the following antibracket actions,

$$\begin{aligned} \left(\Phi^*_A, \Phi^B\right) &= \delta^B_A, \\ (g^*_A, g^*_B) &= -f^C_{AB} g^*_C, \\ (g, g^*_A) &= -g T_A \end{aligned} \tag{6.1.12}$$

The minimal BV action can be written explicit for G/G as follows

$$S_{\min} = S_{G/G} + \int d^2 z \Big[A^*_A (dC + [A, C])^A + \bar{A}^*_A (\bar{d}C + [\bar{A}, C])^A + g^*_A C^A - \frac{1}{2} C^*_A \, [C, C]^A \Big] \tag{6.1.13}$$

where A is a $PSU(2,2|4)$ index.

As we discussed before, we are allowed to add some cohomologically trivial pairs into the action as the non-minimal sector. We take the non-minimal sector as follows

$$\begin{aligned} s\mathcal{D}_A = d^*_A \quad &, \quad s d^*_A = 0, \\ s d^A = \mathcal{D}^{*A} \quad &, \quad s\mathcal{D}^{*A} = 0, \\ (\mathcal{D}_A, \mathcal{D}^{*B}) = \delta^B_A \quad &, \quad (d^A, d^*_B) = \delta^A_B, \\ \text{gh}(\mathcal{D}^*) = -\text{gh}(\mathcal{D}) - 1 \quad &, \quad \text{gh}(d^*) = -\text{gh}(d) - 1 \end{aligned} \tag{6.1.14}$$

with the following Grassmann parity in the $\mathbb{Z}_4$ grading

$$\begin{aligned} \epsilon_G(\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}^*_0, \mathcal{D}^*_2, d_0, d_2, d^*_1, d^*_3) &= 0 \\ \epsilon_G(\mathcal{D}_0, \mathcal{D}_2, \mathcal{D}^*_1, \mathcal{D}^*_3, d_1, d_3, d^*_0, d^*_2) &= 1 \end{aligned} \tag{6.1.15}$$

The action which its variation produces these BRST transformations is the following non-minimal action

$$S_{non-min} = \int d^2 z \; \mathcal{D}^{*A} d_A \tag{6.1.16}$$

The full action is obtained after adding the non-minimal action (6.1.16) to the minimal action (6.1.13)

$$S = S_{min} + S_{non-min} \tag{6.1.17}$$

The action (6.1.17) satisfies the master equation

$$(S,S)=0 \tag{6.1.18}$$

This can be written explicitly as follows

$$\begin{aligned}(S,S) &= \frac{\delta^r S}{\delta A^A}\frac{\delta^l S}{\delta A^*_A}-\frac{\delta^r S}{\delta A^*_A}\frac{\delta^l S}{\delta A^A} \\ &+ \frac{\delta^r S}{\delta c^A}\frac{\delta^l S}{\delta c^*_A}-\frac{\delta^r S}{\delta c^*_A}\frac{\delta^l S}{\delta c^A} \\ &+ \frac{\delta^r S}{\delta g}g\frac{\delta^l S}{\delta g^*_A}T^A-\frac{\delta^r S}{\delta g^*_A}T^A g\frac{\delta^l S}{\delta g}\end{aligned} \tag{6.1.19}$$

where each of line of which is vanishing identically.

In order that path integral be independent of the gauge fixing, one can check that the action (6.1.13) also satisfies the following quantum master equation,

$$\frac{1}{2}(S,S)=i\hbar\Delta S \tag{6.1.20}$$

where Δ is defined as follows

$$\Delta S=(-1)^{\epsilon_I}\frac{\delta_l}{\delta\Phi^*_I}\frac{\delta_l}{\delta\Phi^I}S \tag{6.1.21}$$

and $\epsilon_I=(0,1)$ is the Grassmann parity of the field Φ^I. Both sides of the identity (6.1.20) is vanishing for the action (6.1.13).

Since the gauge transformation closes off-shell, we can split the the BRST charge according to antighost degree starting from -1 for the Koszul-Tate differential δ and the longitudinal differential γ as[2]

$$s=\delta+\gamma,\quad \text{antigh}(\delta)=-1,\quad \text{antigh}(\gamma)=0 \tag{6.1.22}$$

No extra terms of higher antighost number is needed in the G/G BV action and the full BRST charge is given as the sum of Koszul-Tate differential δ and the longitudinal derivative γ. The first application of this decomposition comes from the the fact that the cohomology of s coincides with the cohomology of the longitudinal derivative γ according to homological perturbation theory and so one can study the cohomology of γ which might be easier instead of studying the cohomology of the full BRST charge.

[2]Before we used d as the longitudinal derivative but in this section d is used as the spacetime exterior differential and for the longitudinal derivative we use γ.

The BRST transformations (6.1.7) and their generalization to the antifield sector also decompose into transformations under δ and γ separately using the fact that

$$\delta\Phi = 0 \quad , \quad \delta\Phi^* = \frac{\delta S_0}{\delta\Phi} \tag{6.1.23}$$

which gives the following transformations for γ and δ

$$\begin{aligned}
&\gamma A^A = dC^A + f^A_{BC} A^B C^C, && \delta A^A = 0 \\
&\gamma g = gT_A C^A, && \delta g = 0 \\
&\gamma C^A = -\tfrac{1}{2} f^A_{BC} C^B C^C, && \delta C^A = 0 \\
&\gamma A^*_A = f^B_{AC} A^*_B C^C, && \delta A^*_A = \tfrac{\delta S_0}{\delta A^A} \\
&\gamma g^*_A = f^B_{AC} g^*_B C^C, && \delta g^*_A = \tfrac{\delta S_0}{\delta h^B} \Omega^B_A(h) \\
&\gamma C^*_A = f^B_{AC} C^*_B C^C, && \delta C^*_A = -dA^*_A - f^B_{AC} A^*_B A^C + g^*_A
\end{aligned} \tag{6.1.24}$$

we will use the explicit form of the BRST transformations to compute the cohomological groups $H(s|d)$ of the BRST differential s modulo the spacetime exterior differential d, in the space of local forms. As we explained in the previous chapter, these groups characterize the counterterms in ghost degree zero, while in ghost degree one, they classify the anomalies.

The longitudinal derivative γ is nilpotent off-shell. Therefore, we can analyze first the γ-cohomology, $H(\gamma)$, and the γ- cohomology modulo the exterior derivative d, $H(\gamma|d)$, in the space of all fields and antifields.

In the next chapter we will first explore the gauge fixing of this BV action for G/G principal chiral model to gauge fix it.

6.2 Gauge-fixing the G/G principal chiral model

In order to fix the gauge we do the usual procedure of the BV formalism explained before to project the field-antifield sector to a Lagrangian submanifold whose symplectic form is vanishing. To do so, one introduce the gauge fermion ψ and the projection to the Lagrangian submanifold is defined by putting the following constraint for the antifields

$$\Phi^* = \frac{\delta\psi}{\delta\Phi} \tag{6.2.1}$$

as it was explained in section (5.4).

Here we apply two different gauge fixings and we will see that corresponding to each gauge fixing fermion, we will get either the topological A-model or a topological theory with the BRST charge of the pure spinor action as its supersymmetric charge.

6.2.1 Gauge fixing to topological A-model

To define the first gauge fermion we use the set of $\{T_M, T_{\alpha^+}, T_{\alpha^-}\}$ generators of the $PSU(2,2|4)$ supergroup in which $T_M = \{T_{SU(4)}, T_{SU(2,2)}\}$ are the bosonic generators and T_{α^+} and T_{α^-} are the fermionic generators which are related to the usual $\{T_\alpha, T_{\hat{\alpha}}\}$ generators as follows

$$T_{\alpha^+} \equiv T_\alpha + iT_{\hat{\alpha}} \quad , \quad T_{\alpha^-} \equiv T_\alpha - iT_{\hat{\alpha}} \tag{6.2.2}$$

They satisfy the following anticommutation relations

$$\{T_{\alpha^+}, T_{\beta^+}\} = 0 \qquad , \qquad \{T_{\alpha^-}, T_{\beta^-}\} = 0 \tag{6.2.3}$$

These are the generators we already used in the introduction chapter to define the topological A-model action.

In $(4+4)\times(4+4)$ matrix representation of $g \in PSU(2,2|4)$ supergroup as

$$g = \begin{pmatrix} A_{4\times 4} & X_{4\times 4} \\ Y_{4\times 4} & B_{4\times 4} \end{pmatrix} \tag{6.2.4}$$

the generators $T_{\alpha+}$, $T_{\alpha-}$ and T_M correspond to the upper-right, lower-left and the block-diagonal matrices respectively. Hence, we have the following algebra for $PSU(2,2|4)$

$$\begin{aligned} [T_M, T_{\alpha^+}] &= f_{M\alpha^+}^{\beta^+} T_{\beta^+} \qquad , \qquad [T_M, T_{\alpha^-}] = f_{M\alpha^-}^{\beta^-} T_{\beta^-} \\ \{T_{\alpha^+}, T_{\beta^-}\} &= f_{\alpha^+\beta^-}^{M} T_M \qquad , \qquad [T_M, T_N] = f_{MN}^{P} T_P \end{aligned} \tag{6.2.5}$$

The non-zero components of the metric are the symmetric and antisymmetric tensors η_{MN} and $\eta_{\alpha^+\beta^-}$ respectively.

Any Lie algebra valued object can be expanded in terms of these generators and we have the following contractions

$$X_\alpha Y^\alpha + X_{\hat{\alpha}} Y^{\hat{\alpha}} = \frac{1}{2}\left(X_{\alpha^+} Y^{\alpha^+} - X_{\alpha^-} Y^{\alpha^-}\right) \tag{6.2.6}$$

To do the first gauge fixing, we choose the following gauge fermion,

$$\psi_1 \quad = \quad \int d^2 z \left[\mathcal{D}_{\alpha^+} A^{\alpha^+} + \mathcal{D}_{\alpha^-} \bar{A}^{\alpha^-} + C^M d_M\right] \tag{6.2.7}$$

where

$$A^{\alpha^+} \equiv A^\alpha + iA^{\hat{\alpha}} \quad , \quad \bar{A}^{\alpha^-} \equiv \bar{A}^\alpha - i\bar{A}^{\hat{\alpha}} \tag{6.2.8}$$

The antifields are fixed on the Lagrangian submanifold with the following constraint

$$\Phi^* = \frac{\delta \psi_1}{\delta \Phi} \tag{6.2.9}$$

which gives the following conditions on the antifields

$$\begin{array}{ll}
A^*_M = 0, & \bar{A}^*_M = 0, \\
A^*_{\alpha^+} = \mathcal{D}_{\alpha^+}, & \bar{A}^*_{\alpha^+} = 0, \\
A^*_{\alpha^-} = 0, & \bar{A}^*_{\alpha^-} = \mathcal{D}_{\alpha^-}, \\
g^* = 0, & \mathcal{D}^{*M} = 0, \\
\mathcal{D}^{*\alpha^+} = A^{\alpha^+}, & \mathcal{D}^{*\alpha^-} = \bar{A}^{\alpha^-}, \\
d^{*\alpha^+} = 0, & d^{*\alpha^-} = 0 \\
d^{*M} = C^M, & C^*_M = d_M, \\
C^*_{\alpha^+} = 0, & C^*_{\alpha^-} = 0.
\end{array} \tag{6.2.10}$$

One can easily check that these conditions fixing half of the degrees of freedom and the symplectic form

$$\omega = \sum_{A,\Phi} \delta\Phi^*_A \wedge \delta\Phi^A \tag{6.2.11}$$

is vanishing on this subspace where Φ is a collective notation for all the fields and ghosts and antighosts and Φ^* is its antifield. The index A is a gauge index. The vanishing of the symplectic form (6.2.11) for a half-dimensional subspace means that the gauge fixing (6.2.7) defines a Lagrangian submanifold.

The equations of motion for d_{α^+} and d_{α^-} coming from the non-minimal action implies

$$\mathcal{D}^{*\alpha^+} = 0 \quad , \quad \mathcal{D}^{*\alpha^-} = 0 \tag{6.2.12}$$

which together with (6.2.7) fix the fermionic part of the gauge group as follows

$$A^{\alpha^+} = 0 \quad , \quad \bar{A}^{\alpha^-} = 0 \tag{6.2.13}$$

Putting (6.2.7) and (6.2.13) back in the BV action (6.1.13) we obtain the following action

$$\begin{aligned} S &= \int d^2z [\eta_{MN}(J-A)^M(\bar{J}-\bar{A})^N + \frac{1}{2}\eta_{\alpha^+\alpha^-}J^{\alpha^+}\bar{J}^{\alpha^-} - \frac{1}{2}\eta_{\alpha^+\alpha^-}(J^{\alpha^-}-A^{\alpha^-})(\bar{J}^{\alpha^+}-\bar{A}^{\alpha^+}) \\ &+ \frac{1}{2}\mathcal{D}_{\alpha^+}\left(\partial C^{\alpha^+} + f^{\alpha^+}_{M\beta^+}A^M C^{\beta^+}\right) - \frac{1}{2}\mathcal{D}_{\alpha^-}\left(\bar{\partial}C^{\alpha^-} + f^{\alpha^-}_{M\beta^-}\bar{A}^M C^{\beta^-}\right) \\ &- f^M_{\alpha^+\alpha^-}d_M C^{\alpha^+}C^{\alpha^-} - f^M_{PQ}d_M C^P C^Q] \end{aligned} \tag{6.2.14}$$

This is a gauge fixed action but still has some auxiliary degrees of freedom which one can safely integrate them out using their equations of motion. The equations of motion for A^M, $\bar{A}^M$, A^{α^-}, $\bar{A}^{\alpha^+}$ reads as follows

$$\begin{aligned} A^M = J^M - \frac{1}{2}f^{M\alpha^-}_{\beta^-}\mathcal{D}_{\alpha^-}C^{\beta^-} \quad &, \quad \bar{A}^M = \bar{J}^M + \frac{1}{2}f^{M\alpha^+}_{\beta^+}\mathcal{D}_{\alpha^+}C^{\beta^+} \\ A^{\alpha^-} = J^{\alpha^-} \quad &, \quad \bar{A}^{\alpha^+} = \bar{J}^{\alpha^+} \end{aligned} \tag{6.2.15}$$

Also we can write the following equation of motion for d_M

$$f^M_{\alpha^+\alpha^-}C^{\alpha^+}C^{\alpha^-} = -f^M_{PQ}\, C^P C^Q \tag{6.2.16}$$

This implies that one can solve for the ghost C^P in terms of the ghosts C^{α^+} and C^{α^-} and since they do not appear anywhere else in the action, they will disappear from the action. As we will see later, this constraint changes the correct measure of the bosonic ghosts C^{α^+} and C^{α^-} in the path integral.

Putting (6.2.15) and (6.2.16) back into (6.2.14) we obtain the following action

$$\begin{aligned} S &= \int d^2z \Big[\eta_{\alpha^+\alpha^-}(J^{\alpha^+}\bar{J}^{\alpha^-} - J^{\alpha^-}\bar{J}^{\alpha^+}) \\ &+ \mathcal{D}_{\alpha^+}\nabla C^{\alpha^+} - \mathcal{D}_{\alpha^-}\bar{\nabla}C^{\alpha^-} + \left[\mathcal{D}_+, C^+\right]_M \left[\mathcal{D}_-, C^-\right]^M\Big] \end{aligned} \tag{6.2.17}$$

where

$$\left[\mathcal{D}_+, C^+\right]_M = f^{\alpha^+}_{M\beta^+}\mathcal{D}_{\alpha^+}C^{\beta^+} \quad , \quad \left[\mathcal{D}_-, C^-\right]_M = f^{M\alpha^-}_{\beta-}\mathcal{D}_{\alpha^-}C^{\beta^-} \tag{6.2.18}$$

The covariant derivatives are defined as follows

$$\nabla C^{\alpha^+} = \partial C^{\alpha^+} + f^{\alpha^+}_{M\beta^+} J^M C^{\beta^+} \quad , \quad \bar\nabla C^{\alpha^-} = \bar\partial C^{\alpha^-} + f^{\alpha^-}_{M\beta^-} \bar J^M C^{\beta^-} \tag{6.2.19}$$

The action (6.2.17) is the A-model action which we obtained as the zero radius limit of the pure spinor superstring on $AdS_5 \times S^5$ after the following identifications

$$\begin{array}{ccc} Y_{\alpha^-} \to \mathcal{D}_{\alpha^-} & , & \bar Y_{\alpha^+} \to \mathcal{D}_{\alpha^+} \\ Z^{\alpha^-} \to C^{\alpha^-} & , & \bar Z^{\alpha^+} \to C^{\alpha^+} \end{array} \tag{6.2.20}$$

The BRST transformation can be obtained from the following variation of the action projected on the Lagrangian submanifold

$$Q\Phi = \frac{\delta S}{\delta \Phi^*}|_{\Phi^* = \frac{\delta S}{\delta \Phi}} \tag{6.2.21}$$

which is the gauge fixed BRST charge of the BV action (6.1.13) after putting (6.2.10) into (6.1.24) we get

$$\begin{array}{ccc} QJ^{\alpha^+} = \nabla C^{\alpha^+} & , & QJ^{\alpha^-} = \bar\nabla C^{\alpha^-} \\ QC^{\alpha^+} = 0 & , & QC^{\alpha^+} = 0 \end{array} \tag{6.2.22}$$

In order to compute the BRST variation of $\mathcal{D}_{\alpha^+}$ and $\mathcal{D}_{\alpha^-}$ we should use the fact that $\mathcal{D}_{\alpha^+} = A^*_{\alpha^+}$ and $\mathcal{D}_{\alpha^-} = \bar A^*_{\alpha^-}$ from (6.2.10) and after using

$$QA^*_{\alpha^+} = f^M_{\alpha^+\beta^-} A^*_M C^{\beta^-} + f^{\beta^+}_{\alpha^+ M} A^*_{\beta^+} C^M + \eta_{\alpha^+\alpha^-} \bar J^{\alpha^-} = f^{\beta^+}_{\alpha^+ M} \mathcal{D}_{\beta^+} C^M + \eta_{\alpha^+\alpha^-} \bar J^{\alpha^-} \tag{6.2.23}$$

$$Q\bar A^*_{\alpha^-} = f^M_{\alpha^-\beta^+} \bar A^*_M C^{\beta^+} + f^{\beta^-}_{\alpha^- M} \bar A^*_{\beta^-} C^M - \eta_{\alpha^+\alpha^-} J^{\alpha^+} = f^{\beta^-}_{\alpha^- M} \mathcal{D}_{\beta^-} C^M - \eta_{\alpha^+\alpha^-} J^{\alpha^+} \tag{6.2.24}$$

from (6.1.24) and considering the fact that $A^*_M = 0$, $\mathcal{D}_{\alpha^+} = A^*_{\alpha^+}$ and $\mathcal{D}_{\alpha^-} = \bar A^*_{\alpha^-}$ on the Lagrangian submanifold, finally we get

$$Q\mathcal{D}_{\alpha^+} = f^{\beta^+}_{\alpha^+ M} \mathcal{D}_{\beta^+} C^M + \eta_{\alpha^+\alpha^-} \bar J^{\alpha^-} \quad , \quad Q\mathcal{D}_{\alpha^-} = f^{\beta^-}_{\alpha^- M} \mathcal{D}_{\beta^-} C^M - \eta_{\alpha^+\alpha^-} J^{\alpha^+} \tag{6.2.25}$$

and the BRST charge can be written as follows

$$Q = \int dz \eta_{\alpha^+\alpha^-} C^{\alpha^+} \bar J^{\alpha^-} + \int d\bar z \eta_{\alpha^-\alpha^+} C^{\alpha^-} J^{\alpha^+} \tag{6.2.26}$$

which is the BRST charge of the A-model of the pure spinor superstring on $AdS_5 \times S^5$.

6.2.2 Gauge fixing to a topological model with $Q_{\textbf{top}} = Q_{\textbf{pure spinor}}$

In this section, we will see that there is another way of fixing the gauge symmetry of the G/G principal chiral model which gives a BRST trivial action in which its BRST charge coincides with the BRST charge of the pure spinor formalism on $AdS_5 \times S^5$. The pure spinors and their constraints also comes as the gauge fixing constraints imposed on the BV ghosts C^A.

Here we use the $SO(5) \times SO(4,1)$ invariant representation of $PSU(2,2|4)$ supergroup with generators $\left(T_{[ab]}, T_a.T_\alpha, T_{\hat{\alpha}}\right)$ which are Lorentz, translation and fermionic generators respectively.

We use the non-minimal fields $(\mathcal{D}_A, \bar{\mathcal{D}}_A, d_A, \bar{d}_A)$ together with their antifields which transform under the BRST transformation as follows

$$
\begin{aligned}
Q\mathcal{D}_A = d_A \quad &, \quad Qd_A = 0 \\
Qd^{*A} = \mathcal{D}^{*A} \quad &, \quad Q\mathcal{D}^{*A} = 0 \\
Q\bar{\mathcal{D}}_A = \bar{d}_A \quad &, \quad Q\bar{d}_A = 0 \\
Q\bar{d}^{*A} = \bar{\mathcal{D}}^{*A} \quad &, \quad Q\bar{\mathcal{D}}^{*A} = 0
\end{aligned}
\tag{6.2.27}
$$

where $A = ([ab], a, \alpha, \hat{\alpha})$ is a $PSU(2,2|4)$ index. The non-minimal fields have the following Grassmann parities

$$
\begin{aligned}
\epsilon_G(\mathcal{D}_\alpha, \mathcal{D}_{\hat{\alpha}}, \mathcal{D}^{*[ab]}, \mathcal{D}^{*a}, d_{[ab]}, d_a, d^{*\alpha}, d^{*\hat{\alpha}}) = 0 \\
\epsilon_G(\mathcal{D}_{[ab]}, \mathcal{D}_a, \mathcal{D}^{*\alpha}, \mathcal{D}^{*\hat{\alpha}}, d_\alpha, d_{\hat{\alpha}}, d^{*[ab]}, d^{*a}) = 1
\end{aligned}
\tag{6.2.28}
$$

The dynamics of the non-minimal fields is defined by the following action

$$
S_{\text{non-minimal}} = \int d^2z[\mathcal{D}^{*A} d_A + \bar{\mathcal{D}}^{*A} \bar{d}_A] \tag{6.2.29}
$$

Using BRST transformation (6.2.27), it can be shown that this non-minimal action is a BRST trivial term

$$
S_{\text{non-minimal}} = \int d^2z Q(\Omega) \tag{6.2.30}
$$

with

$$
\Omega = d^{*A} d_A + \bar{d}^{*A} \bar{d}_A \tag{6.2.31}
$$

Being a trivial term making out of the cohomologically trivial pairs, the non-minimal action (6.2.29) does not change the cohomology of the original theory.

The next step is to gauge fix the theory by taking the following gauge fermion defining a Lagrangian submanifold

$$\begin{aligned}\psi_2 &= \int d^2z[\mathcal{D}_I \mathcal{N}_a^I(C)A^a + \mathcal{D}_{I+5}\bar{\mathcal{N}}_a^I(C)\bar{A}^a + \bar{\mathcal{D}}_I C^I + \bar{\mathcal{D}}_{I+5}C^{I+5} + \bar{\mathcal{D}}_{[ab]}C^{[ab]} \\ &+ \mathcal{D}_{\hat{\alpha}}A^{\hat{\alpha}} + \mathcal{D}_\alpha \bar{A}^\alpha]\end{aligned} \tag{6.2.32}$$

where $I = 0$ to 4 and $a = 0$ to 9.

This gauge fermion putts the following constraints on the antifields of the theory

$$\begin{array}{ll} A^*_{[ab]} = 0, & \bar{A}^*_{[ab]} = 0, \\ A^*_a = \mathcal{D}_I \mathcal{N}_a^I(C), & \bar{A}^*_a = \mathcal{D}_{I+5}\bar{\mathcal{N}}_a^I(C), \\ A^*_\alpha = 0, & \bar{A}^*_\alpha = \mathcal{D}_\alpha, \\ A^*_{\hat{\alpha}} = \bar{\mathcal{D}}_{\hat{\alpha}}, & \bar{A}^*_{\hat{\alpha}} = 0, \\ \mathcal{D}^{*I} = \mathcal{N}_a^I(C)A^a, & \mathcal{D}^{*I+5} = \bar{\mathcal{N}}_a^I(C)\bar{A}^a, \\ \mathcal{D}^{*\alpha} = \bar{A}^\alpha, & \bar{\mathcal{D}}^{*\alpha} = 0, \\ \mathcal{D}^{*\hat{\alpha}} = A^{\hat{\alpha}}, & \bar{\mathcal{D}}^{*\hat{\alpha}} = 0, \\ \mathcal{D}^{*[ab]} = 0, & \bar{\mathcal{D}}^{*[ab]} = C^{[ab]}, \\ \bar{\mathcal{D}}^{*a} = C^a, & \bar{d}^*_a = 0, \\ \bar{d}^*_\alpha = 0, & \bar{d}^*_{\hat{\alpha}} = 0 \\ d^*_\alpha = 0, & d^*_{\hat{\alpha}} = 0 \\ C^*_I = \bar{\mathcal{D}}_I, & C^*_{I+5} = \bar{\mathcal{D}}_{I+5}, \\ C^*_\alpha = \mathcal{D}_I \frac{\partial \mathcal{N}_a^I(C)}{\partial C^\alpha} A^a, & C^*_{\hat{\alpha}} = \mathcal{D}_{I+5}\frac{\partial \bar{\mathcal{N}}_a^I(C)}{\partial C^{\hat{\alpha}}}\bar{A}^a, \\ C^*_{[ab]} = \bar{\mathcal{D}}^{[ab]}, & g^* = 0 \end{array} \tag{6.2.33}$$

which defines a half-dimensional subspace in the field-antifield space. In order to have a Lagrangian submanifold, the symplectic form (6.2.11) should be vanishing which implies the following completeness and orthonormality conditions on matrices $\mathcal{N}$ and $\bar{\mathcal{N}}$

$$\eta^{ab}\mathcal{N}_a^I \mathcal{N}_b^I = 0 \quad , \quad \eta^{ab}\bar{\mathcal{N}}_a^I \bar{\mathcal{N}}_b^I = 0 \tag{6.2.34}$$

In order to find a solution of (6.2.34) we can use the null vectors $(\gamma_a)_{\alpha\beta}C^\alpha C^\beta$ and $(\gamma_a)_{\hat{\alpha}\hat{\beta}}C^{\hat{\alpha}}C^{\hat{\beta}}$ satisfying

$$\eta^{ab}\left[(\gamma_a)_{\alpha\beta}C^\alpha C^\beta\right]\left[(\gamma_b)_{\delta\gamma}C^\delta C^\gamma\right] = 0\ ,\ \eta^{ab}\left[(\gamma_a)_{\hat{\alpha}\hat{\beta}}C^{\hat{\alpha}}C^{\hat{\beta}}\right]\left[(\gamma_b)_{\hat{\delta}\hat{\gamma}}C^{\hat{\delta}}C^{\hat{\gamma}}\right] = 0 \tag{6.2.35}$$

They decompose under $SO(5)\times SO(4,1)$ into

$$\begin{array}{lll} \Phi_I = (\gamma_I)_{\alpha\beta}C^\alpha C^\beta & , & \Phi_{\tilde{I}} = (\gamma_{\tilde{I}})_{\alpha\beta}C^\alpha C^\beta \\ \hat{\Phi}_I = (\gamma_I)_{\hat{\alpha}\hat{\beta}}C^{\hat{\alpha}}C^{\hat{\beta}} & , & \hat{\Phi}_{\tilde{I}} = (\gamma_{\tilde{I}})_{\hat{\alpha}\hat{\beta}}C^{\hat{\alpha}}C^{\hat{\beta}} \end{array} \tag{6.2.36}$$

where $I = 0$ to 4 and $\tilde{I} = 5$ to 9.

There is another way of presenting the constraints (6.2.34) for matrices $\mathcal{N}_a^I$ and $\bar{\mathcal{N}}_a^I$ as follows

$$(\gamma_a)_{\alpha\beta} C^\alpha C^\beta = (\gamma_I)_{\delta\gamma} C^\delta C^\gamma \mathcal{N}_a^I \quad , \quad (\gamma_a)_{\hat{\alpha}\hat{\beta}} C^{\hat{\alpha}} C^{\hat{\beta}} = (\gamma_I)_{\hat{\delta}\hat{\gamma}} C^{\hat{\delta}} C^{\hat{\gamma}} \bar{\mathcal{N}}_a^I \tag{6.2.37}$$

This is equivalent to (6.2.34) written in a ghost-dependent way. These matrices are the same as the ones used in [79] to construct extended pure spinor formalism and also in [36].

From the equations of motion for the non-minimal antighosts which implies

$$\mathcal{D}^{*A} = 0 \quad , \quad \bar{\mathcal{D}}^{*A} = 0 \tag{6.2.38}$$

Using the gauge fixing constraints (6.2.33), we get the following conditions on the gauge fields and the fermionic ghost components

$$\begin{aligned} A^{\hat{\alpha}} &= 0 \quad &,& \quad \bar{A}^{\alpha} = 0 \\ \mathcal{N}_a^I A^a &= 0 \quad &,& \quad \bar{\mathcal{N}}_a^I \bar{A}^a = 0 \\ C^a &= 0 \quad &,& \quad C^{[ab]} = 0 \end{aligned} \tag{6.2.39}$$

The gauge fixing (6.2.33) kills half of the gauge degrees of freedom of A^a and half of the $\bar{A}^a$ because of the rank of $\mathcal{N}$ and $\bar{\mathcal{N}}$.

Putting the gauge fixing (6.2.33) back in the action (6.1.17) we get the following gauge-fixed action

$$\begin{aligned} S &= \int d^2z [\eta_{[ab][cd]}(J-A)^{[ab]}(\bar{J}-\bar{A})^{[cd]} + \eta_{ab}(J-A)^a(\bar{J}-\bar{A})^b \\ &+ \eta_{\alpha\hat{\alpha}}(J-A)^\alpha(\bar{J}-\bar{A})^{\hat{\alpha}} - \eta_{\alpha\hat{\alpha}} J^{\hat{\alpha}} \bar{J}^\alpha \\ &+ \mathcal{D}_I \mathcal{N}_a^I(\partial C^a + f^a_{\alpha\beta} A^\alpha C^\beta) + \mathcal{D}_{I+5} \bar{\mathcal{N}}_a^I(\bar{\partial} C^a + f^a_{\hat{\alpha}\hat{\beta}} \bar{A}^{\hat{\alpha}} C^{\hat{\beta}}) \\ &- \mathcal{D}_\alpha(\bar{\nabla}_A C^\alpha + f^\alpha_{a\hat{\beta}} \bar{A}^a C^{\hat{\beta}}) + \mathcal{D}_{\hat{\alpha}}(\nabla_A C^{\hat{\alpha}} + f^{\hat{\alpha}}_{a\beta} A^a C^\beta) \\ &- \frac{1}{2} f^I_{\alpha\beta} \bar{\mathcal{D}}_I C^\alpha C^\beta - \frac{1}{2} f^I_{\hat{\alpha}\hat{\beta}} \bar{\mathcal{D}}_{I+5} C^{\hat{\alpha}} C^{\hat{\beta}} - \frac{1}{2} f^{[ab]}_{\alpha\hat{\alpha}} \bar{\mathcal{D}}_{[ab]} C^\alpha C^{\hat{\alpha}} \\ &+ \mathcal{N}_a^I A^a d_I + \bar{\mathcal{N}}_a^I \bar{A}^a d_{I+5}] \end{aligned} \tag{6.2.40}$$

where the covariant derivatives are defined with respect to the gauge connections as

$$\nabla_A C^\alpha = \partial C^\alpha + f^\alpha_{[ab]\beta} A^{[ab]} C^\beta \quad , \quad \bar{\nabla} C^{\hat{\alpha}} = \bar{\partial} C^{\hat{\alpha}} + f^{\hat{\alpha}}_{[ab]\hat{\beta}} \bar{A}^{[ab]} C^{\hat{\beta}} \tag{6.2.41}$$

The third and fourth lines of (6.2.40) are the $A^*(\partial C+[A,C])$ terms, the last two lines are the non-zero $C^*[C,C]$ terms and the non-minimal sector of the BV action (6.1.13) respectively after imposing constraints (6.2.33).

The equations of motion for the anthighosts $\bar{\mathcal{D}}_{[ab]}$, $\bar{\mathcal{D}}_I$ and $\bar{\mathcal{D}}_{I+5}$ coming from the action (6.2.40) can be written as follows

$$\begin{aligned} f_{\alpha\hat{\alpha}}^{[ab]}C^\alpha C^{\hat{\alpha}} &= 0, \\ f_{\alpha\beta}^{I}C^\alpha C^\beta &= 0, \\ f_{\hat{\alpha}\hat{\beta}}^{I}C^{\hat{\alpha}} C^{\hat{\beta}} &= 0. \end{aligned} \tag{6.2.42}$$

The last two are the pure spinor constraint for the ghosts C^α and $C^{\hat{\alpha}}$ since because of (6.2.37) they imply

$$(\gamma_a)_{\alpha\beta}C^\alpha C^\beta = 0 \quad , \quad (\gamma_a)_{\hat{\alpha}\hat{\beta}}C^{\hat{\alpha}}C^{\hat{\beta}} = 0 \tag{6.2.43}$$

which are the usual ten-dimensional pure spinor constraints for the pure spinors C^α and $C^{\hat{\alpha}}$.

The first equation of (6.2.42) implies that the pure spinors C^α and $\eta_{\alpha\hat{\alpha}}C^{\hat{\alpha}}$ has to be interpreted as complex conjugate and so this becomes a trivial equation whenever they satisfy the pure spinor constraint. This is consistent with the observation in [36] that the term $\eta\lambda\hat{\lambda}$ is in the cohomology of the pure spinors.

Because of (6.2.37) and the pure spinor constraints (6.2.42) we get the following identities

$$f_{\alpha\beta}^{a}\mathcal{N}_a^I C^\alpha = 0 \quad , \quad f_{\hat{\alpha}\hat{\beta}}^{a}\bar{\mathcal{N}}_a^I C^{\hat{\alpha}} = 0 \tag{6.2.44}$$

Putting all these into (6.2.40), the action simplifies as follows

$$\begin{aligned} S &= \int d^2z[\eta_{[ab][cd]}(J-A)^{[ab]}(\bar{J}-\bar{A})^{[cd]} + \eta_{ab}(J-A)^a(\bar{J}-\bar{A})^b \\ &- \eta_{\alpha\hat{\alpha}}J^{\hat{\alpha}}\bar{J}^\alpha + \eta_{\alpha\hat{\alpha}}(J-A)^\alpha(\bar{J}-\bar{A})^{\hat{\alpha}} \\ &- \mathcal{D}_\alpha(\bar{\nabla}_A C^\alpha + f_{a\hat{\beta}}^{\alpha}\bar{A}^a C^{\hat{\beta}}) + \mathcal{D}_{\hat{\alpha}}(\nabla_A C^{\hat{\alpha}} + f_{a\beta}^{\hat{\alpha}}A^a C^\beta) \\ &+ \mathcal{N}_a^I A^a d_I + \bar{\mathcal{N}}_a^I \bar{A}^a d_{I+5}] \end{aligned} \tag{6.2.45}$$

The action (6.2.45) implies the following equations of motion for the auxiliary fields $A^{[ab]}$, $\bar{A}^{[ab]}$, A^a, $\bar{A}^a$, $A^{\hat{\alpha}}$ and $\bar{A}^{\alpha}$

$$\begin{aligned} A^{[ab]} &= J^{[ab]} + N^{[ab]} \quad , \quad \bar{A}^{[ab]} = \bar{J}^{[ab]} + \hat{N}^{[ab]} \\ A^a &= J^a + f^{a\alpha}_{\hat{\beta}} \mathcal{D}_\alpha C^{\hat{\beta}} - \eta^{ab} \bar{\mathcal{N}}^I_b d_{I+5} \quad , \quad \bar{A}^a = \bar{J}^a - f^{a\hat{\alpha}}_{\beta} \mathcal{D}_{\hat{\alpha}} C^{\beta} - \eta^{ab} \mathcal{N}^I_b d_I \\ A^\alpha &= J^\alpha \quad , \quad \bar{A}^{\hat{\alpha}} = \bar{J}^{\hat{\alpha}} \end{aligned} \tag{6.2.46}$$

where we defined

$$N^{[ab]} \equiv f^{[ab]\alpha}_{\beta} \mathcal{D}_\alpha C^\beta \quad , \quad \hat{N}^{[ab]} \equiv -f^{[ab]\hat{\alpha}}_{\hat{\beta}} \mathcal{D}_{\hat{\alpha}} C^{\hat{\beta}} \tag{6.2.47}$$

as the usual pure spinor Lorentz currents.

Putting back the equations of motion (6.2.46) into the action (6.2.45) we get

$$\begin{aligned} S &= \int d^2z [\eta_{\alpha\hat{\alpha}} \bar{J}^\alpha J^{\hat{\alpha}} + \eta_{[ab][cd]} N^{[ab]} \hat{N}^{[cd]} \qquad (6.2.48) \\ &- \eta_{ab} (f^{a\hat{\alpha}}_{\beta} \mathcal{D}_\alpha C^{\hat{\beta}} - \eta^{ac} \bar{\mathcal{N}}^I_c d_{I+5})(f^{b\hat{\alpha}}_{\beta} \mathcal{D}_{\hat{\alpha}} C^\beta + \eta^{bd} \mathcal{N}^I_d d_I) \\ &- \mathcal{D}_\alpha (\bar{\nabla} C^\alpha + f^{\alpha}_{[ab]\beta} \hat{N}^{[ab]} C^\beta + f^{\alpha}_{a\hat{\beta}} (\bar{J}^a - f^{a\hat{\alpha}}_{\beta} \mathcal{D}_{\hat{\alpha}} C^\beta - \eta^{ab} \mathcal{N}^I_b d_I) C^{\hat{\beta}}) \\ &+ \mathcal{D}_{\hat{\alpha}} (\nabla C^{\hat{\alpha}} + f^{\hat{\alpha}}_{[ab]\hat{\beta}} N^{[ab]} C^{\hat{\beta}} + f^{\hat{\alpha}}_{a\beta} (J^a + f^{a\alpha}_{\hat{\beta}} \mathcal{D}_\alpha C^{\hat{\beta}} - \eta^{ab} \bar{\mathcal{N}}^I_b d_{I+5}) C^\beta) \\ &+ \mathcal{N}^I_a (J^a + f^{a\alpha}_{\hat{\beta}} \mathcal{D}_\alpha C^{\hat{\beta}} - \eta^{ab} \bar{\mathcal{N}}^J_b d_{J+5}) d_I + \bar{\mathcal{N}}^I_a (\bar{J}^a - f^{a\hat{\alpha}}_{\beta} \mathcal{D}_{\hat{\alpha}} C^\beta - \eta^{ab} \mathcal{N}^J_b d_J) d_{I+5}] \end{aligned}$$

where now the covariant derivatives are defined with respect to the left-invariant currents as follows

$$\bar{\nabla} C^\alpha = \bar{\partial} C^\alpha + f^{\alpha}_{[ab]\beta} \bar{J}^{[ab]} C^\beta \quad , \quad \nabla C^{\hat{\alpha}} = \partial C^{\hat{\alpha}} + f^{\hat{\alpha}}_{[ab]\hat{\beta}} J^{[ab]} C^{\hat{\beta}} \tag{6.2.49}$$

We can also integrate out the auxiliary fields d_I and d_{I+5} using their equations of motion

$$d_I = (R^{-1})_{IJ} \bar{\mathcal{N}}^J_a (\bar{J}^a - f^{a\hat{\alpha}}_{\beta} \mathcal{D}_{\hat{\alpha}} C^\beta) \quad , \quad d_{I+5} = (R^{-1})_{IJ} \mathcal{N}^J_a (J^a + f^{a\alpha}_{\hat{\beta}} \mathcal{D}_\alpha C^{\hat{\beta}}) \tag{6.2.50}$$

where

$$R^{IJ} \equiv \eta^{ab} \mathcal{N}^I_a \bar{\mathcal{N}}^J_b \tag{6.2.51}$$

Finally inserting back (6.2.50) in the action (6.2.48) and after using the following identity

$$\mathcal{N}^I_a (R^{-1})_{IJ} \bar{\mathcal{N}}^J_b = \eta_{ab} \tag{6.2.52}$$

we get the following action

$$S = \int d^2z [J^a \mathcal{N}^I_a R^{-1}_{IJ} \bar{\mathcal{N}}^I_b \bar{J}^b + \eta_{\alpha\hat{\alpha}} \bar{J}^\alpha J^{\hat{\alpha}} - \mathcal{D}_\alpha \bar{\nabla} C^\alpha + \mathcal{D}_{\hat{\alpha}} \nabla C^{\hat{\alpha}} - \eta_{[ab][cd]} N^{[ab]} \hat{N}^{[cd]}] \tag{6.2.53}$$

It is the topological action proposed in [36] as coming from a gauge fixing of the G/G principal chiral model which was shown explicitly before.

The BRST transformation is the "gauge fixed BRST charge" defined as follows

$$Q\Phi = (s\Phi)\left(\Phi, \Phi^* = \frac{\delta\psi}{\delta\Phi}\right) \tag{6.2.54}$$

The BRST transformation of the left-invariant currents and the ghosts C can be obtained directly from (6.1.24) after using gauge fixing constraints (6.2.39) and (6.2.42),

$$\begin{array}{rcl} QJ^a = f^a_{\beta\alpha}J^\beta C^\alpha + f^a_{\hat\beta\hat\alpha}J^{\hat\beta}C^{\hat\alpha} & , & QJ^{[ab]} = f^{[ab]}_{\alpha\hat\beta}J^\alpha C^{\hat\beta} + f^{[ab]}_{\hat\alpha\beta}J^{\hat\alpha}C^\beta \\ Q\bar J^\alpha = \bar\nabla C^\alpha + f^\alpha_{a\hat\alpha}\bar J^a C^{\hat\alpha} & , & QJ^{\hat\alpha} = \nabla C^{\hat\alpha} + f^{\hat\alpha}_{a\alpha}J^a C^\alpha \\ QC^\alpha = 0 & , & QC^{\hat\alpha} = 0 \end{array} \tag{6.2.55}$$

To compute the BRST transformation of the antighosts $\mathcal{D}$ we should use from (6.2.33) the fact that $\mathcal{D}_\alpha = -\bar A^*_\alpha$ and $\mathcal{D}_{\hat\alpha} = A^*_{\hat\alpha}$ and so

$$\begin{aligned} Q\mathcal{D}_\alpha &= -Q\bar A^*_\alpha = -[\bar A^*, C]_\alpha - \frac{\delta S_0}{\delta \bar A^\alpha} = -f^a_{\alpha\beta}\bar A^*_a C^\beta + \eta_{\alpha\hat\alpha}J^{\hat\alpha} = -f^a_{\alpha\beta}\mathcal{D}^*_{I+5}\bar{\mathcal{N}}^I_a C^\beta + \eta_{\alpha\hat\alpha}J^{\hat\alpha} \\ &= \eta_{\alpha\hat\alpha}J^{\hat\alpha} \end{aligned} \tag{6.2.56}$$

$$\begin{aligned} Q\mathcal{D}_{\hat\alpha} &= QA^*_{\hat\alpha} = [A^*, C]_{\hat\alpha} + \frac{\delta S_0}{\delta A^{\hat\alpha}} = f^a_{\hat\alpha\hat\beta}A^*_a C^{\hat\beta} - \eta_{\hat\alpha\alpha}\bar J^\alpha = f^a_{\hat\alpha\hat\beta}\mathcal{D}^*_I\mathcal{N}^I_a C^{\hat\beta} - \eta_{\hat\alpha\alpha}\bar J^\alpha \\ &= \eta_{\alpha\hat\alpha}\bar J^\alpha \end{aligned} \tag{6.2.57}$$

where we used the gauge fixing constraints $A^{\hat\alpha} = 0$ and $\bar A^\alpha = 0$ and (6.2.33) and the fact that $\mathcal{D}^*_I = \mathcal{D}^*_{I+5} = 0$ on-shell.

The BRST operator can be written as

$$Q = \int dz \eta_{\alpha\hat\alpha}C^\alpha J^{\hat\alpha} + \int d\bar z \eta_{\alpha\hat\alpha}C^{\hat\alpha}\bar J^\alpha \tag{6.2.58}$$

It is exactly the $AdS_5 \times S^5$ pure spinor BRST charge after the following identifications,

$$\begin{array}{rcl} C^\alpha \to \lambda^\alpha & , & C^{\hat\alpha} \to \hat\lambda^{\hat\alpha} \\ \mathcal{D}_\alpha \to w_\alpha & , & \bar{\mathcal{D}}_{\hat\alpha} \to \hat w_{\hat\alpha} \end{array} \tag{6.2.59}$$

The action (6.2.53) is invariant under this "gauge-fixed BRST charge".

The action (6.2.53) is a topological action because as it was shown in [36], it can be written as a trivial term

$$S = \int d^2z\, Q(\Omega) \tag{6.2.60}$$

for some Ω which its exact form was given in [36] and we do not need to write it here.

So, starting from the BV action of the topological G/G principal chiral model, and after doing a proper gauge fixing we get a topological action with the BRST charge exactly equal to the superstring action. As it is conjectured in [36], this topological action describes the zero-radius limit of the $AdS_5 \times S^5$ superstring which according to the Maldacena conjecture is dual to the free $\mathcal{N} = 4$ on $d = 4$ super-Yang-Mills theory.

The fact that this topological action and the A-model action are both obtained from different gauge fixings of the same theory implies that they are describing the same physics. Since the physical states explored by the second topological theory is defined from the cohomology of its topological charge which is equal to pure spinor BRST charge, so it is natural to say that this theory and the topological A-model theory are both exploring the full cohomology of the pure spinor superstring on $AdS_5 \times S^5$ in this limit and not just its BPS sector. It seems puzzling and more understanding in this direction deserves more consideration since it might help to give a better understanding of a 'physical' theory using a topological one.

6.3 Consistent deformation of the G/G model

As we saw in section (5.5), the BV formalism gives the power to find consistent interactions among fields keeping the original gauge invariance in the sense of the deformation theory.

Indeed, we showed that having a consistent interaction, the solution $S^{(0)}$ of the master equation can be deformed as an expansion of the interaction parameter r as follows

$$S = S^{(0)} + rS^{(1)} + r^2 S^{(2)} + \cdots \tag{6.3.1}$$

We observed that the local deformations are determined after studying the relative cohomological groups $H^0(\gamma|d)$ and $H^1(\gamma|d)$. In fact, $H^0(\gamma|d)$ determines the first order deformation and $H^1(\gamma|d)$ determines whether the deformation continues or it is obstructed after the first order term. So, we have to study these cohomological groups for the G/G principal chiral model. Note that, since the topological theories which are obtained after the gauge fixing of the G/G model correspond to the zero radius limit of the superstring on $AdS_5 \times S^5$, the deformation corresponds to the vertex operator for the radius modulus and the perturbative parameter is proportional to the radius of the $AdS_5 \times S^5$.

The local deformation of a BV action has been studied mainly for Yang-Mills theories in [76, 80, 77, 81, 82, 83]. It was shown as an example that for the case of Abelian Chern-Simons theory the consistent interaction which one can add to the theory without spoiling the BV gauge invariant structure is the non-abelian Chern-Simons terms. For more general non-linear sigma models including the G/G principal chiral model the same problem has been addressed partly in [78] for a bosonic compact group G.

We have to study the cohomology groups $H(\gamma)$ and then its relative cohomology group $H(\gamma|d)$. The first step of calculating the cohomology is to define the field space in which this calculation should be done. This space is named as *jet space* and is simply the space whose coordinates are the fields and their corresponding antifields as well as their subsequent partial derivatives

$$\Phi = \{A^A, g, C^A\}\ ,\ \Phi^* = \{A^{*A}, g^*, C^{*A}\}\ ,\ d\Phi\ ,\ d\Phi^*\ ,\ \cdots \tag{6.3.2}$$

Since the differential γ commutes with the exterior derivative d, so the transformation laws (6.1.24) can be applied globally to all the jet space.

In order to describe the $\gamma-$cohomology it is convenient to find a set of jet space coordinates in which it has more compact form. To do so, it was shown in [78] that after eliminating the trivial pairs the jet space can be coordinatized with the following fields

$$g, C^A, [\chi^A] \tag{6.3.3}$$

where $\chi^A = \{J^A - A^A, \Phi^*\}$ and according to (6.1.24) it transforms under γ linearly as follows

$$\gamma\chi^A = (T_D)^B_A \chi_B C^D \tag{6.3.4}$$

where T_A are the generators of the group G. But actually we can define another set of fields which are invariant under γ transformation as follows

$$\tilde{\chi}^A = U(g)^A_B \chi^B \tag{6.3.5}$$

in which $U(g)$ stands for the matrix representation of group element g. Because of the following transformation [3]

$$\gamma U(g) = -(-1)^{\epsilon_G(\chi)} U(g) C^A T_A \tag{6.3.6}$$

we have $\gamma\tilde{\chi} = 0$.

We can take the following set of fields as a possible jet space coordinates which is also consistent with the definition given in [84] for a good jet coordinate

$$\text{jet space coordinates} = \{g, C^A, [\tilde{\chi}^A]\} \tag{6.3.7}$$

They transform with γ as follows

$$\gamma g = gC \quad , \quad \gamma C = -C^2 \quad , \quad \gamma[\tilde{\chi}] = 0 \tag{6.3.8}$$

where $[\tilde{\chi}]$ means $\tilde{\chi}$ and all its subsequent ordinary derivatives.

It was shown in [78] that the most general solution for the cocycle condition $\gamma m = 0$ is given as a polynomial in the gauge-invariant variables $[\tilde{\chi}]$ times a solution of $\gamma n = 0$ which just depends on g and C. So we have to compute the cohomology defined by the first two equations of (6.3.8) which is done by relating it to the De Rham cohomology of the supergroup manifold. The relation comes from the fact that we can identify γ with the exterior derivative d, also one can identify the ghosts C with the one-forms J as follows

$$\gamma \leftrightarrow d \quad , \quad C \leftrightarrow J \tag{6.3.9}$$

[3] Here and everywhere $\epsilon_G(F)$ denotes the Grassmann parity of the field F.

This way the de Rham cohomology is identified with the cohomology of the longitudinal derivative γ.

The corresponding of the γ transformation $\gamma C = -C^2$ and $\gamma g = gC$ is given by the Maurer-Cartan equations and the definition of the left-invariant Cartan one-forms as follows

$$dJ = -J \wedge J \quad , \quad dg = gJ \tag{6.3.10}$$

There is a point here that since we are studying a supergroup, the one-forms J have fermionic components J^α and $J^{\hat{\alpha}}$, correspondingly we have also bosonic ghosts C^α and $C^{\hat{\alpha}}$ but the identification seems to work in the same way.

Taking the De Rham group to be $H_{DR}(G)$ with the basis $\omega_I(g,J)$ and denoting by $\omega(g,C)$ the function obtained after inserting C in place of J, a general cocycle solving $\gamma m = 0$ has the following form

$$m = \sum_I P^I\left([\tilde{\chi}], dx\right)\omega(g,C) + \gamma n \tag{6.3.11}$$

where P^I is an arbitrary polynomial in $\tilde{\chi}$ and its ordinary derivatives.

In fact the spacetime forms $\omega(g,J)$ are related to the $\gamma-$cocycles $\omega(g,C)$ by the descent equations as it was shown in [81]. To write the descent equations we can expand $\tilde{\omega}_I \equiv \omega_I(g, J+C)$ according to ghost number and form degree [78] as follows

$$\tilde{\omega}_I = \omega_I^{(0,2)} + \omega_I^{(1,1)} + \omega_I^{(2,0)} \tag{6.3.12}$$

The first superscript stands for the form degree and the second one is the ghost number. They are limited to be less than or equal two since we are working on the worldsheet.

The $\tilde{\omega}$ has to be annihilated $\tilde{\gamma} = \gamma + d$ by construction

$$\tilde{\gamma}\tilde{\omega}_I = 0 \tag{6.3.13}$$

This is named as *Russian formula* [78, 85].

After using the expansion (6.3.12) we get the following descent equations for each ghost degree

$$d\omega^{(0,2)} = 0 \tag{6.3.14}$$

$$\gamma\omega^{(0,2)} + d\omega^{(1,1)} = 0 \tag{6.3.15}$$

$$\gamma\omega^{(1,1)} + d\omega^{(2,0)} = 0 \tag{6.3.16}$$

$$\gamma\omega^{(2,0)} = 0 \tag{6.3.17}$$

where its solution gives the cohomology group $H(\gamma)$.

We can see from (6.3.14) that the integral

$$\int_\Sigma \omega_I^{(0,2)} \tag{6.3.18}$$

is gauge-invariant since its integrand is gauge-invariant up to a total derivative term. So we can add this term to the action without spoiling the gauge invariance of the theory. But, on the other hand, since $\omega^{0,2}$ is locally exact, the topological term (6.3.18) does not modify the equations of motion and this is a consistent deformation of the BV action. These terms are called *winding number* terms and they are consistent with the topological observables which we expect to get from our G/G theory since its gauge fixed form gives an A-model topological action as we saw before.

The next step is to find a solution for the descent equations (6.3.14-6.3.17) to find the deformation term (6.3.18) for the G/G principal chiral model. We assume that we can use this construction also for the case of supergroup as in our case for $AdS_5 \times S^5$ supercoset.

6.3.1 Descent equations

The longitudinal differential along the gauge orbits γ can be written using the transformation laws (6.1.24) as follows

$$\begin{aligned}\gamma \;=\; & C^A\frac{\partial}{\partial A^*_A}+(\partial C^A+[A,C]^A)\frac{\partial}{\partial A^A}+C^A\frac{\partial}{\partial g^*_A}+C^A\frac{\partial}{\partial C^*_A}+C^AT_A\frac{\partial}{\partial g} \\ - \;& f^C_{AB}C^AC^B\frac{\partial}{\partial C^C}+d_A\frac{\partial}{\partial \mathcal{D}^{*A}}\end{aligned} \tag{6.3.19}$$

which squares identically to zero.

In order to find a consistent deformation, we start solving the descent equations from the bottom equation (6.3.17) and go up until we solve the upper descent equation (6.3.14) which its integral over the worldsheet gives the deformation for G/G action.

Here we propose the following solution but we should admit that here we are not considering the fact that the supergroup G is a non-compact supergroup, and this should enlarge the general solution of the descent equations.

There exist a ghost number two as a solution for the bottom equation of the descent equations as a function of g and C is follows

$$\omega^{(2,0)} = \eta_{AB} C^A C^B \tag{6.3.20}$$

This scalar function satisfies (6.3.17). We take the invariant bilinear η_{AB} to be the metric of the supergroup $PSU(2,2|4)$ supergroup Since $C^{[ab]}$ and C^a are fermionic and $\eta_{[ab][cd]}$ and η_{ab} are symmetric, the only possibility for $\omega^{(2,0)}$ is the following component

$$\omega^{(2,0)} = \eta_{\alpha\hat{\alpha}} C^\alpha C^{\hat{\alpha}} \tag{6.3.21}$$

Putting this in (6.3.16) we can solve for $\omega^{(1,1)}$ as follows

$$\omega^{(1,1)} = \eta_{\alpha\hat{\alpha}}((J-A)^\alpha C^{\hat{\alpha}} + C^\alpha (J-A)^{\hat{\alpha}}) + \eta_{\alpha\hat{\alpha}}((\bar{J}-\bar{A})^\alpha C^{\hat{\alpha}} + C^\alpha (\bar{J}-\bar{A})^{\hat{\alpha}}) \tag{6.3.22}$$

Using the Maurer-Cartan equation

$$dJ = -J \wedge J \tag{6.3.23}$$

and the fact that the currents J transform as

$$\gamma J = dC + [J, C] \tag{6.3.24}$$

we can see that the (6.3.15) has the following solution for $\omega^{(0,2)}$

$$\begin{aligned} \omega^{(0,2)} &= \eta_{[ab][cd]}(J-A)^{[ab]}(\bar{J}-\bar{A})^{[cd]} + \eta_{ab}(J-A)^a(\bar{J}-\bar{A})^b \\ &+ \frac{1}{2}\eta_{ab}(J-A)^\alpha(\bar{J}-\bar{A})^{\hat{\alpha}} - \frac{1}{2}\eta_{\alpha\hat{\alpha}}(\bar{J}-\bar{A})^\alpha(J-A)^{\hat{\alpha}} \end{aligned} \tag{6.3.25}$$

Off course one can add any $\gamma-$trivial term to $\omega^{(0,2)}$ which doesn't change the cohomology.

A general local deformation of the BV action (6.1.13) is given by (6.3.25),

$$S^{(0)}_{\rm BV} \longrightarrow S^{(0)}_{\rm BV} + R^2 \int d^2z\, \omega^{(0,2)} \tag{6.3.26}$$

which is a ghost number zero integrated vertex operator.

In the second gauge fixing which gives the topological model with $Q_{\rm top.} = Q_{\rm pure\ spinor}$ after we add the insertion (6.3.18) with $\omega^{0,2}$ as (6.3.25) the auxiliary field equations for the fields $A^{[ab]}$ and A^a get contribution also from the deformation term

$$\begin{aligned} A^{[ab]} &= (1+R^2)J^{[ab]} + N^{[ab]} \quad , \quad \bar{A}^{[ab]} = (1+R^2)\bar{J}^{[ab]} + \hat{N}^{[ab]} \\ A^a &= (1+R^2)J^a + f^{a\alpha}_{\hat{\beta}}\mathcal{D}_\alpha C^{\hat{\beta}} - \eta^{ab}\bar{\mathcal{N}}^I_b d_{I+5} \quad , \quad \bar{A}^a = (1+R^2)\bar{J}^a - f^{a\hat{\alpha}}_{\beta}\mathcal{D}_{\hat{\alpha}} C^\beta - \eta^{ab}\mathcal{N}^I_b d_I \\ A^\alpha &= (1+R^2)J^\alpha \quad , \quad \bar{A}^{\hat{\alpha}} = (1+R^2)\bar{J}^{\hat{\alpha}} \end{aligned} \tag{6.3.27}$$

which after putting back in the BV action and keeping just the first order terms in R^2 we get the following deformation action

$$S_{def} = R^2 \int d^2z \Big[\eta_{ab} J^a \bar{J}^b + \eta_{\alpha\hat{\beta}} (\frac{3}{2} J^\alpha \bar{J}^{\bar{\beta}} - \frac{1}{2} \bar{J}^\alpha \bar{J}^{\hat{\beta}}) - \mathcal{D}_\alpha \bar{\partial} C^\alpha + \mathcal{D}_{\hat{\alpha}} \partial C^{\hat{\alpha}} - \eta_{[ab][cd]} N^{[ab]} \hat{N}^{[cd]} \Big] \tag{6.3.28}$$

This is the original pure spinor action for the $AdS_5 \times S^5$ background after applying the identification (6.2.59). So we get the pure spinor action as a deformation over the topological theory. This is the consequence of giving nonzero expectation value to the radius modulus in the topological theory which corresponds to the zero radius limit of the superstring on $AdS_5 \times S^5$ background.

6.4 A worldsheet description of AdS/CFT duality

Using the observation that the superstring action for small radius can be seen as perturbation over a topological theory, one can explore AdS/CFT from a worldsheet point of view similar to the topological open/closed duality an example of which we studied for the case of CS/conifold duality in which the 't Hooft expansion of the gauge theory side was obtained from a topological closed string theory. The difference in the case of AdS duality is that in the superstring theory we have also propagating degrees of freedom and local deformations on both sides of the duality which they should be mapped to each other. As we emphasized in the introductory chapter, in the AdS/CFT correspondence, any state in the gravity side relates to an operator in the gauge theory side. Actually this map should be exactly one-to-one since this is the way we can produce the correct 't Hooft expansion from the gravity side. Here, In our topological G/G construction of the superstring theory we observed that the topological theory as the dual to free gauge theory, can be consistently deformed with a vertex operator which is the result of giving the radius of the AdS geometry a non-zero value. This non-zero value of the radius generated with the vertex operator $\int_\Sigma d^2z\omega^{(0,2)}$ corresponds to an operator in the gauge theory side which makes the SYM theory an interacting theory with a perturbative coupling $g_{YM}^2 = t/N$which is proportional to the radius parameter in the string theory side. One might try to follow the technique used by Ooguri and Vafa in proving the topological conifold duality from a worldsheet point of view for AdS/CFT duality considering the fact that here we have also some local vertex insertions in the closed string side located in the Coulomb branches or holes on the worldsheet. These vertices's correspond to some D-brane operators at the boundary of the hole in accordance with the state/operator correspondence.

So starting from a closed string worldsheet, we end with a partition over open string worldsheets with h holes. After the emergence of the Coulomb branch, on top of the 'holes' which appear as the regions in which the gauge field of the linear gauged sigma-model becomes dynamical, we have also some punctures, as the insertion points of the gravitational vertex operators in the closed string theory. In order to do so, let's consider that $S_{def}^{(A)}$ is the gauge fixed version of the $\int_\Sigma d^2z\omega^{(0,2)}$ according to the first gauge fixing which produces the A-model topological action from G/G, then the A-model action deforms as follows

$$S_{\text{A-model}} \longrightarrow S_{\text{A-model}} + R^2\, S_{def}^{(A)} \tag{6.4.1}$$

this is equivalent to inserting an exponential set of closed string vertex operators in the closed string side. Starting from a closed string theory, the free energy is partitioned into a sum over the worldsheets with genus g and p vertex insertions with a weight factor $f(R)$ for each insertion as follows

$$\mathcal{F} = \sum_{g=0}^{\infty} \sum_{p=0}^{\infty} \frac{f^p}{p!} g_s^{2g-2} \mathcal{F}_{g,p} \tag{6.4.2}$$

where $\mathcal{F}_{g,p}$ is the amplitude corresponding to a worldsheet with genus g and p punctures. After defining $\mathcal{F}_g$ as

$$\mathcal{F}_g = \sum_{p=1}^{\infty} \frac{f^p}{p!} \mathcal{F}_{g,p} \tag{6.4.3}$$

we can rewrite (6.4.2) as a sum over amplitudes for a particular genus $\mathcal{F}_g$ as follows

$$\mathcal{F} = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g \tag{6.4.4}$$

On the other side, in the open string side, we have the following expansion over the open worldsheets with h holes and p punctures with the D-brane boundary states as dual to the vertex insertions in the closed string side, where each contributes a factor $R^2 f(R)$[4]

$$F = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{p=0}^{\infty} \binom{h+p}{h} (g_{YM}^2 N)^h (R^2 f)^p (g_{YM}^2)^{2g-2} F_{g,h,p} \tag{6.4.5}$$

where the number coefficient is determined from the partitioning of p punctures and h 'holes' into $H = h + p$ holes and is given by $\frac{(h+p)!}{p!h!}$. One can rewrite (6.4.6) as a sum over $H = h + p$ as follows

$$F = \sum_{g=0}^{\infty} \sum_{H=1}^{\infty} (g_{YM}^2 N + R^2 f)^H (g_{YM}^2)^{2g-2} F_{g,H} \tag{6.4.6}$$

where we defined

$$F_{g,H} = \sum_{h,p}^{h+p=H} F_{g,h,p} \tag{6.4.7}$$

[4]As it stated in [36], each puncture contributes a factor R^2 but after inserting the vertices's, the corresponding D-brane operator on the boundary of the hole corresponding to the puncture will contribute a $R^2 f(R)$ factor.

Similar to the original 't Hooft idea we explained in the introduction, this can written as a sum over genus as follows

$$F = \sum_{g=0}^{\infty} (g_{YM}^2)^{2g-2} F_g \tag{6.4.8}$$

with the following definition for F_g

$$F_g = \sum_{H=1}^{\infty} (t')^H F_{g,H} \tag{6.4.9}$$

where t' is the shifted 't Hooft parameter

$$t' = g_{YM}^2 N + R^2 f \tag{6.4.10}$$

The shift of the 't Hooft parameter is a consequence of the insertion of the vertex operators in the closed string side and their corresponding D-brane operators in the open string side.

As we see in figure (6.1) here the partition of the closed string worldsheets contains also the puncture insertions on top of the usual holes originated from the emergence of the coulomb branch. Using the explicit form of the vertex operator in the A-model $S_{def}^{(A)}$,

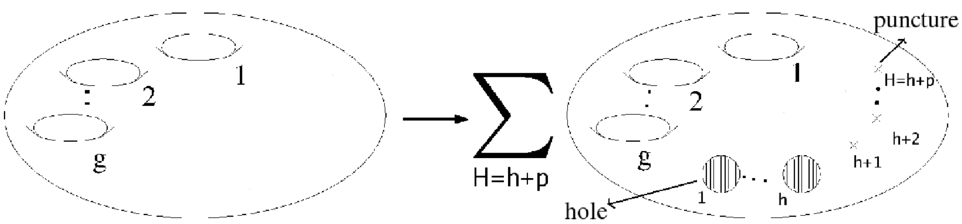

FIGURE 6.1: A close worldsheet partitions into open worldsheets in which some of them have D-brane operators corresponding to the vertex operator in the closed string side.

one should compute the exact value of the function $f(R)$ which is necessary to find the explicit perturbation expansion in the gauge theory side in terms of the radius of the AdS. Note that if one can show that $f(R) = R^2$, this implies that $t' = R^4$ for $N = 0$ and so the relation $t = R^4$ would be valid both for small and large radius. Computing explicit form of the factor would be an important next step towards a perturbative proof of the AdS/CFT duality.

Chapter 7

Amplitudes computation

Here in this chapter we will show that one can use the power of topological A-model action in order to give a multiloop prescription for the amplitude computations of pure spinor superstring on $AdS_5 \times S^5$. We just sketch the first steps towards this computation and a better understanding of the problem seems very appealing and important to us.

7.1 Pure spinor amplitude for $AdS_5 \times S^5$

The next step towards understanding superstring theory on a Ramond-Ramond background is to give a prescription to compute the string amplitudes for a generic multiloop worldsheet. It can also help to give a better understanding to the gauge/string duality since the perturbative Yang-Mills correlation functions should be generated from the topological $AdS_5 \times S^5$ closed string amplitudes as the zero radius version of the superstring on this background.

In order to give a prescription for the amplitude computation on a particular background we have to find the zero mode measure factor of the pure spinor formalism. A prescription for multiloop amplitude computation for the flat background was proposed in [53]. But for the case of $AdS_5 \times S^5$ it was shown in [35] and [36] that a simplification in pure spinor measure factor makes the computation easier than the flat background. For instance in flat background the zero mode measure factor of the pure spinors satisfy the following relation at tree level

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1 \tag{7.1.1}$$

while using the fact that $\eta\lambda\hat{\lambda}$ is in the cohomology of the pure spinors, and λ^α and $\hat{\lambda}^{\hat{\alpha}}$ can be interpreted as complex conjugate, a new simple measure factor for the pure spinors on $AdS_5 \times S^5$ was given in [35] and [36] satisfying

$$\langle (\eta_{\alpha\hat{\alpha}} \lambda^{\alpha} \hat{\lambda}^{\hat{\alpha}})^3 \rangle = 1 \tag{7.1.2}$$

One can regularize the pure spinor measure by restricting zero-modes of λ and $\hat{\lambda}$ to satisfy $\eta\lambda\hat{\lambda} = \Lambda$ for some positive Λ. The dependence on Λ can be absorbed by shifting the coupling constant $e^{\phi} \to \Lambda^{-\frac{3}{2}} e^{\phi}$ which gives the following zero mode integration for tree amplitudes

$$\langle f(x, \theta, \lambda, \hat{\theta}, \hat{\lambda}) \rangle = \int d^{10}x \int d^{16}\theta d^{16}\hat{\theta} \ \mathrm{Sdet}(E_M^A) \int d^{10}\lambda d^{10}\hat{\lambda} f(x, \theta, \lambda, \hat{\theta}, \hat{\lambda}) \tag{7.1.3}$$

For the supergravity vertex operator

$$V = \lambda^{\alpha} \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}} \tag{7.1.4}$$

one can write the three-point supergravity tree amplitudes as follows

$$\mathcal{A} = \int d^{10}x \int d^{16}\theta d^{16}\hat{\theta} \ \mathrm{Sdet}(E_M^A) T^{((\alpha\beta\gamma)(\hat{\alpha}\hat{\beta}\hat{\gamma}))} A^{(1)}_{\alpha\hat{\alpha}}(X) A^{(2)}_{\beta\hat{\beta}}(X) A^{(3)}_{\gamma\hat{\gamma}}(X) \tag{7.1.5}$$

note the integration over $16(\theta\hat{\theta})$'s here instead of over $5(\theta\hat{\theta})$'s for the flat case.

A prescription to compute amplitudes with higher genus was also proposed by Berkovits in [36]. One should insert $(3g-3)$ b and $\bar{b}$ ghosts and N integrated vertex operators into the functional integral. Then we should integrate over the zero modes of $(x, \theta, \hat{\theta}, \lambda, \hat{\lambda})$ and g zero modes of the spin-one fields w_{α} and $\hat{w}_{\hat{\alpha}}$. There is another way of writing and comparing the prescription of Berkovits by doing the computation from the topological A-model action in which because of the topological property of the theory we know how to compute a multiloop amplitude.

7.2 Topological A-model amplitude

A general A-model action with BRST charge $Q_A = Q_- + \bar{Q}_+$ can be written as

$$\begin{aligned} S &= t\int d^2z \Big(g_{i\bar{j}}\partial_\alpha\phi^i\partial^\alpha\phi^{\bar{j}} - g_{i\bar{j}}\rho_z^{\bar{j}}D_{\bar{z}}\chi^i + g_{i\bar{j}}\rho_{\bar{z}}^i D_z\chi^{\bar{j}} + \partial_i\partial_{\bar{k}}g_{j\bar{l}}\rho_{\bar{z}}^i\rho_z^{\bar{k}}\chi^j\chi^{\bar{l}} \\ &- g_{i\bar{j}}F_{\bar{z}}^i F_z^{\bar{j}} - g_{j\bar{j}}\Gamma^j_{ik}\chi^i F_z^{\bar{j}}\rho_{\bar{z}}^k - g_{j\bar{j}}\Gamma^{\bar{j}}_{\bar{i}\bar{k}}\chi^{\bar{i}}F_{\bar{z}}^j\rho_z^{\bar{k}}\Big) \end{aligned} \tag{7.2.1}$$

The Kähler metric can locally be written as $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(x^i, x^{\bar{j}})$ for a Kähler potential $K(x^i, x^{\bar{j}})$ and the covariant derivative is defined from the Levi-Civita connection $\Gamma^i_{jk} = g^{i\bar{j}}\partial_j g_{k\bar{j}}$ for a flat world-sheet as follows

$$\begin{aligned} D_{\bar{z}}\chi^i &= \partial_{\bar{z}}\chi^i + \Gamma^i_{kl}\partial_{\bar{z}}\phi^k\chi^l \\ D_z\chi^{\bar{i}} &= \partial_z\chi^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{l}}\partial_z\phi^{\bar{k}}\chi^{\bar{l}} \end{aligned} \tag{7.2.2}$$

Four supersymmetry generators $Q_\pm$ and $\bar{Q}_\pm$ of the A-model can be generated by the following operator

$$\delta = i\epsilon_+Q_- - i\epsilon_-Q_+ - i\bar{\epsilon}_+\bar{Q}_- + i\bar{\epsilon}_-\bar{Q}_+, \tag{7.2.3}$$

where $(Q^\pm)^\dagger$ and $\delta^\dagger = -\delta$. All the generators are nilpotent and they compromise the $N = 2$ supersymmetry algebra. They have opposite charged under the R symmetry group.

The operator (7.2.3) generates the following off-shell transformations

$$\begin{aligned} \delta\phi^i &= -\epsilon_-\rho_{\bar{z}}^i + \epsilon_+\chi^i \\ \delta\phi^{\bar{i}} &= \bar{\epsilon}_-\chi^{\bar{i}} - \bar{\epsilon}_+\rho_z^{\bar{i}} \\ \delta\chi^i &= -2i\bar{\epsilon}_+\partial_-\phi^i + \epsilon_-F_{\bar{z}}^i \\ \delta\chi^{\bar{i}} &= -2i\epsilon_-\partial_+\phi^{\bar{i}} + \bar{\epsilon}_+F_z^{\bar{i}} \\ \delta\rho_{\bar{z}}^i &= 2i\bar{\epsilon}_-\partial_+\phi^i + \epsilon_+F_{\bar{z}}^i \\ \delta\rho_z^{\bar{i}} &= 2i\epsilon_+\partial_-\phi^{\bar{i}} + \bar{\epsilon}_-F_z^{\bar{i}} \end{aligned} \tag{7.2.4}$$

One can also write the Euler-Lagrange equations as follows

$$\partial_\alpha\phi^i + \epsilon_{\alpha\beta}J^i_j\partial^\beta\phi^j = 0 \tag{7.2.5}$$

where J is the complex structure of the worldsheet Riemann surface. It is the equation for a holomorphic map $\phi : \Sigma \to M$, which is called a worldsheet instanton.

We can rewrite the action (7.2.1) in a more covariant by introducing the following auxiliary fields $\tilde{F}^i_{\bar{z}}$ and $\tilde{F}^{\bar{i}}_z$

$$\begin{aligned} \tilde{F}^i_{\bar{z}} &= F^i_{\bar{z}} - \rho^j_{\bar{z}}\Gamma^i_{jm}\chi^m \\ \tilde{F}^{\bar{i}}_z &= F^{\bar{i}}_z - \rho^{\bar{j}}_z\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\chi^{\bar{m}} \end{aligned} \tag{7.2.6}$$

The action (7.2.1) simplifies as follows

$$S = t\int d^2z\left(g_{i\bar{j}}\partial_\alpha\phi^i\partial^\alpha\phi^{\bar{j}} - g_{i\bar{j}}\rho^{\bar{j}}_z D_{\bar{z}}\chi^i + g_{i\bar{j}}\rho^i_{\bar{z}}D_z\chi^{\bar{j}} + R_{i\bar{k}j\bar{l}}\rho^i_{\bar{z}}\rho^{\bar{k}}_z\chi^j\chi^{\bar{l}} - g_{i\bar{j}}\tilde{F}^i_{\bar{z}}\tilde{F}^{\bar{i}}_z\right) \tag{7.2.7}$$

After integrating over the auxiliary fields F^i_z and $F^{\bar{i}}_{\bar{z}}$ using their equations of motion

$$\begin{aligned} F^i_{\bar{z}} &= \rho^j_{\bar{z}}\Gamma^i_{jm}\chi^m \\ F^{\bar{i}}_z &= \rho^{\bar{j}}_z\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\chi^{\bar{m}} \end{aligned} \tag{7.2.8}$$

we get the following action

$$S = t\int d^2z\left(g_{i\bar{j}}\partial_\alpha\phi^i\partial^\alpha\phi^{\bar{j}} - g_{i\bar{j}}\rho^{\bar{j}}_z D_{\bar{z}}\chi^i + g_{i\bar{j}}\rho^i_{\bar{z}}D_z\chi^{\bar{j}} + R_{i\bar{k}j\bar{l}}\rho^i_{\bar{z}}\rho^{\bar{k}}_z\chi^j\chi^{\bar{l}}\right) \tag{7.2.9}$$

One can check that the action (7.2.7) is Q_A-exact. Using BRST transformations (7.2.4), One can show that

$$S = \left\{Q_A, t\int d^2z g_{i\bar{j}}\left(\rho^{\bar{j}}_z\tilde{F}^i_{\bar{z}} + \rho^j_{\bar{z}}\tilde{F}^{\bar{i}}_z - \rho^i_{\bar{z}}D_z\chi^{\bar{j}} - \rho^{\bar{i}}_z D_{\bar{z}}\chi^j\right)\right\} \tag{7.2.10}$$

therefore the theory is topological since this implies that the energy-momentum tensor is also Q_A-exact.

It appears that the BRST cohomology obtained from $Q_A = Q_- + \bar{Q}_+$ corresponds exactly to the de Rham cohomology classes obtained from $d = \partial + \bar{\partial}$ after identifying $\chi^i \leftrightarrow d\phi^i$ and $\chi^{\bar{i}} \leftrightarrow d\phi^{\bar{i}}$.

For each form

$$A = a_{i_1,\ldots,i_n}(\phi)d\phi^{i_1}\wedge \ldots \wedge d\phi^{i_n} \tag{7.2.11}$$

on the target space, there is a topological operator

$$\mathcal{O}^{(0)}_A = a_{i_1,\ldots,i_n}(\phi)\chi^{i_1}\ldots\chi^{i_n} \tag{7.2.12}$$

of the A-model and the operation of BRST charge Q_A is identified with the exterior derivative

$$\{Q_A, \mathcal{O}_A\} = \mathcal{O}_{dA} \tag{7.2.13}$$

Because of the splitting of the tangent bundle of $M = T^{(1,0)} \oplus T^{(0,1)}$ we can associate to any observable $\mathcal{O}_A$ an element in the Dolbeault cohomology group $H^{(p_k,q_k)}$.

The $U(1)_A$ is anomalous and its anomaly is given by the index of the Dolbault operator which is given from the Hirzbruch-Riemann-Roch theorem as follows

$$q_A = \#(\chi \text{ zero modes}) - \#(\rho \text{ zero modes}) = 2\dim_{\mathbb{C}} M(1-g) \tag{7.2.14}$$

The correlation function of the physical operators is obtained as

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i \right\rangle_g = \int_{\mathcal{M}_g} \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\rho e^{-S} \prod_{i=1}^{n} \mathcal{O}_i \tag{7.2.15}$$

where $\mathcal{M}_g$ is the moduli space of holomorphic maps at genus g.

7.3 Topological A-model of $AdS_5 \times S^5$

The proposed A-model action for superstring theory on $AdS_5 \times S^5$ can be obtained after twisting an action similar to (7.2.7) in off-shell form as follows

$$\begin{aligned} S &= \int d^2z \Big[\eta_{\alpha^+\alpha^-}(J^{\alpha^+}\bar{J}^{\alpha^-} - J^{\alpha^-}\bar{J}^{\alpha^+}) + \mathcal{D}_{\alpha^+}\nabla C^{\alpha^+} - \mathcal{D}_{\alpha^-}\bar{\nabla} C^{\alpha^-} \\ &+ \eta_{MN} f_{\alpha^-}^{\alpha^+ M} f_{\alpha^+}^{\alpha^- N} \mathcal{D}_{\alpha^+} C^{\alpha^-} \mathcal{D}_{\alpha^-} C^{\alpha^+} - \eta_{MN}\tilde{F}^M \bar{\tilde{F}}^N\Big] \end{aligned} \tag{7.3.1}$$

here the difference with the previous sigma-model is that it is based on a fermionic Kähler manifold parametrized by coordinates θ, so the Grassmann parity of all the fields are reversed from the target space point of view. The fields Z and Y are bosonic while the auxiliary fields are fermionic. After integrating over the auxiliary fields $\tilde{F}$ and $\bar{\tilde{F}}$ we will obtain the A-model action (6.2.17) which we got from the gauge fixing of the G/G principal chiral model.

Comparing (7.2.1) and (7.2.7) we find that this action has the correct structure of an A-twisted topological action after the following identifications

$$\begin{aligned} \nabla \leftrightarrow D_z \quad &, \quad \bar{\nabla} \leftrightarrow D_{\bar{z}} \\ J^{\alpha^+} \leftrightarrow D_z\phi^{\bar{i}} \quad &, \quad \bar{J}^{\alpha^-} \leftrightarrow D_{\bar{z}}\phi^{i} \\ \mathcal{D}_{\alpha^+} \leftrightarrow \rho^i_{\bar{z}} \quad &, \quad \mathcal{D}_{\alpha^-} \leftrightarrow \rho^{\bar{i}}_{z} \\ C^{\alpha^+} \leftrightarrow \chi^i \quad &, \quad C^{\alpha^-} \leftrightarrow \chi^{\bar{i}} \end{aligned} \tag{7.3.2}$$

The difference with the original A-model action is in the Grassmann parity of the fields. The parity of all $\mathcal{D}$, C and F fields are fermionic here.

Although $SU(2,2|4)$ symmetry is manifest in this action, the $N = (2,2)$ worldsheet supersymmetry is not manifest but its supersymmetry generators can be constructed as follows

$$\begin{aligned} Q = \int dz \eta_{\alpha^+\alpha^-} C^{\alpha^+} J^{\alpha^-} \quad &, \quad \bar{Q} = \int d\bar{z} \eta_{\alpha^-\alpha^+} C^{\alpha^-} \bar{J}^{\alpha^+} \\ b = \mathcal{D}_{\alpha^+} J^{\alpha^+} \quad &, \quad \bar{b} = \mathcal{D}_{\alpha^-} \bar{J}^{\alpha^-} \end{aligned} \tag{7.3.3}$$

The fact that $\mathcal{D}_{\alpha^+}$ and $\mathcal{D}_{\alpha^-}$ have conformal weight $(1,0)$ and $(0,1)$ is consistent with the fact that it is a A-twisted $N = (2,2)$ supersymmetry algebra which defines a topological A-model.

Now, we want to compute topological amplitudes and to do this we look into the simplest observables in a topological A-model theory which was discussed in the previous section.

They are objects defined in $H^2(M,\mathbb{R})$ which correspond to infinitesimal deformations of the Kähler moduli k. Since our Grassmannian is symmetric, the only observables we can compute are given by the pullback of k on the tangent space which can be represented schematically as

$$\mathcal{O} = X^*(k)\eta_{\alpha^+\alpha^-}C^{\alpha^+}C^{\alpha^-} \tag{7.3.4}$$

which is both local and Q-closed.

In order to compute the amplitudes, we have to give a prescription. The difference here with the usual topological A-model action is in the way you have to soak up the zero modes since now we have bosonic zero modes corresponding to bosonic fields Z and Y in the observables.

A general n-point amplitude for genus $g > 1$ is given by

$$A^g_{i_1,\ldots i_n} = \int_{\Sigma_g} d^2z \int_{\mathcal{M}_g} \left\langle \mathcal{O}_{i_1}\ldots\mathcal{O}_{i_n} \prod_{k=1}^{3g-3} (b,\mu^k)(\bar{b},\bar{\mu}^k) \right\rangle \tag{7.3.5}$$

The Grassmannian $U(2,2|4)/U(2,2)\times U(4)$ has complex dimension sixteen and so the anomaly (7.2.14) is equal to $q_A = 16(g-1)$. This anomaly is generated by the following current

$$\mathcal{J} = \mathcal{D}_{\alpha^+}C^{\alpha^+} + \mathcal{D}_{\alpha^-}C^{\alpha^-} \tag{7.3.6}$$

We will consider different genera separately, starting from the tree level.

Before going to amplitude computation, let us look at the ghost measure factor. As we saw before, the A-model action is obtained as a gauge fixed version of the G/G principal chiral model. The ghost measure factor is also obtained from the gauge fixing of the G/G measure factor which can be written as follows

$$\prod_M dC^M \prod_{\alpha^+} dC^{\alpha^+} \prod_{\alpha^-} dC^{\alpha^-} \prod_{\alpha^+} d\mathcal{D}_{\alpha^+} \prod_{\alpha^-} d\mathcal{D}_{\alpha^-} \tag{7.3.7}$$

where C^M and $C^{\alpha^\pm}$ are the fermionic and bosonic ghosts and $\mathcal{D}_{\alpha^+}$ and $\mathcal{D}_{\alpha^-}$ belong to the non-minimal sector.

As we saw in section (6.2.1), after the first gauge fixing, we get the constraint (6.2.16) between C^M and $C^{\alpha^\pm}$ as follows

$$f^M_{\alpha^+\alpha^-}C^{\alpha^+}C^{\alpha^-} = -f^M_{PQ}\,C^PC^Q \tag{7.3.8}$$

This constraint gives a solution for C^M in terms of $C^{\alpha^\pm}$ and they can be removed from the theory after inserting the following delta function in the path integral

$$\delta((C^M)^2 + C^{\alpha^+}C^{\alpha^-}) \tag{7.3.9}$$

but we have

$$\int dx\delta(f(x)) = \sum_{x_0} \frac{1}{f'(x_0)} \tag{7.3.10}$$

where x_0 are solutions of $f(x) = 0$.

Using (7.3.10) we can see that we get the following measure factor for the ghosts C^{α^+} and C^{α^-} after integrating over the ghosts C^M considering the delta function (7.3.9)

$$\prod_{\alpha^+} \frac{dC^{\alpha^+}}{C^{\alpha^+}} \prod_{\alpha^-} \frac{dC^{\alpha^-}}{C^{\alpha^-}} \prod_{\alpha^+} d\mathcal{D}_{\alpha^+} \prod_{\alpha^-} d\mathcal{D}_{\alpha^-} \tag{7.3.11}$$

The case $g = 0$: In tree-level we have an anomaly equal to $q_A = 16$ from the measure of the path integral. In order to cancel the anomaly we have to insert the following insertion in the path integral

$$\mathcal{O}_1 \ldots \mathcal{O}_{16} = \prod_{\alpha^+} \prod_{\alpha^-} X^*(k) C^{\alpha^+} C^{\alpha^-} \tag{7.3.12}$$

which saturates 16 units of ghost number coming from the measure of the path integral since each such operator has ghost number -1.

Since C^{α^+} and C^{α^+} are bosonic fields, they have one zero mode on a $g = 0$ surface which should be soaked up by inserting the following picture changing operators

$$\Upsilon = \prod_{\alpha^+} \theta^{\alpha^+} \delta(C^{\alpha^+}) \prod_{\alpha^-} \theta^{\alpha^-} \delta(C^{\alpha^-}) \tag{7.3.13}$$

which can be shown easily that is a BRST-invariant operator and does not depend on $(C^{\alpha^+}, C^{\alpha^-})$. The path integral now becomes

$$\langle \prod_{\alpha^+,\alpha^-} (X^*(k) C^{\alpha^+} C^{\alpha^-}) \prod_{\alpha^+,\alpha^-} \theta^{\alpha^+} \delta(C^{\alpha^+}) \theta^{\alpha^-} \delta(C^{\alpha^-}) \rangle \tag{7.3.14}$$

The picture changing operators Υ cancel the zero modes of both C and θ fields.

Note that because of the presence of the factor $\prod_{\alpha^+,\alpha^-}\frac{1}{C^{\alpha^+}}\frac{1}{C^{\alpha^-}}$ in the measure (7.3.11), we get the following integral over the bosonic ghosts

$$\int [dC^{\alpha^+}]\delta(C^{\alpha^+}) \tag{7.3.15}$$

which is non-zero.

The 16 powers of the θ in the picture changing operator also gives a factor one after doing the Berezin integral over the fermionic coordinated θ and $\bar{\theta}$. The conformal weight one fields $\mathcal{D}$ does not have any zero mode on a genus zero worldsheet. and Using this construction we have a well-defined path integral which should be compared with the tree level amplitude of the pure spinor formalism.

The case $g = 1$: In higher genera we have to insert as many (b, μ) insertions as needed. For $g = 1$ we have to insert a factor of $|(b, \mu)|^2$ in the path integral where as we showed before it is from $b = \partial\theta^{\alpha^+}\mathcal{D}_{\alpha^+}$ and $\bar{b} = \partial\theta^{\alpha^-}\mathcal{D}_{\alpha^-}$.

Since $\mathcal{D}$'s are bosonic fields they will bring one zero mode on a genus one worldsheet and also we have one zero mode for scalars C.

To soak up the zero modes of $\mathcal{D}$'s we have to insert the following picture changing operators

$$\Psi = \tilde{F}^M C^{\beta^+}\delta(f^{\beta^+}_{\alpha^- M}\mathcal{D}_{\beta^+})\bar{\tilde{F}}^M C^{\beta^-}\delta(f^{\beta^-}_{\alpha^+ M}\mathcal{D}_{\beta^-}) \tag{7.3.16}$$

to cancel the corresponding zero modes.

The correct path integral with balanced number of zero modes and ghost number anomaly becomes

$$\langle \Psi\Upsilon\, |(b,\mu)|^2 \prod_1^{16} \mathcal{O}_i\rangle \tag{7.3.17}$$

The insertion $X^*(k)C^{\alpha^+}C^{\alpha^-}$ should be added to cancel the conformal anomaly at one-loop.

Finally, the amplitude at one-loop can be written as follows

$$\left\langle \Psi\Upsilon \left|\int \delta\theta^{\alpha^+}\mathcal{D}_{\alpha^+}\mu\right| \left|\int \delta\theta^{\alpha^-}\mathcal{D}_{\alpha^-}\bar{\mu}\right| \prod_{\alpha^+,\alpha^-}(X^*(k)C^{\alpha^+}C^{\alpha^-})\right\rangle \tag{7.3.18}$$

The integral over the fermionic auxiliary fields $\tilde{F}$ and $\bar{\tilde{F}}$ having one zero mode in $g = 1$ is the following non-zero Berezin integral

$$\int [d\tilde{F}]\tilde{F} \tag{7.3.19}$$

The integral over the zero modes of the ghosts C is similar to the one before. The factor $(C)^{16}$ from the observable cancels the factor $\left(\frac{1}{C}\right)^{16}$ from the measure and the delta function in the picture changing operator integrates to one. There are still some C factors in the picture changing operator Ψ which together with the $\mathcal{D}$ factors in the (b, μ) terms give the contractions $\langle C^{\alpha^+}, \mathcal{D}_{\alpha^+}\rangle$ and $\langle C^{\alpha^-}, \mathcal{D}_{\alpha^-}\rangle$. Also the integral over the conformal weight-one fields $\mathcal{D}$ using the delta function in the picture changing operator Ψ becomes as follows

$$\int [d\mathcal{D}]\delta(\mathcal{D}) \tag{7.3.20}$$

which is non-zero.

So the zero modes of the fields go away in the path integral and we end with the contractions of $\langle C^{\alpha^+}, \mathcal{D}_{\alpha^+}\rangle$ and $\langle C^{\alpha^-}, \mathcal{D}_{\alpha^-}\rangle$ as the result of the amplitude computation.

The case $g > 1$: At higher genera, we have to insert $16(g-1)$of (b, μ) operators. Then to cancel the zero modes of the $\mathcal{D}$'s which are inside b's, we have to insert $16(g-1)$ picture changing operators Ψ in the path integral to cancel the g zero modes of the $\mathcal{D}$ fields. Note that we are on a 16$\mathbb{C}$-dimensional Calabi-Yau and so the number of zero modes for a scalar, a one-form and a two form are 16, g and $16(g-1)$ respectively.

Then we have to insert the $X^* C^{\alpha^+} C^{\alpha^-}$ operators. In order to cancel the zero modes of the C's in the measure, we have to insert the picture changing operator Υ.

All in all, the amplitude can be written as follows

$$\left\langle \Upsilon \Psi^{16(g-1)} \left| \int \delta\theta^{\alpha^+} \mathcal{D}_{\alpha^+} \mu \right|^{16(g-1)} \left| \int \delta\theta^{\alpha^-} \mathcal{D}_{\alpha^-} \bar{\mu} \right|^{16(g-1)} \prod_{\alpha^+, \alpha^-} (X^*(k) C^{\alpha^+} C^{\alpha^-}) \right\rangle \tag{7.3.21}$$

The integral over the zero modes of the fields are done as the previous case and we end with the contractions $\langle C^{\alpha^+}, \mathcal{D}_{\alpha^+}\rangle^{16(g-1)}$ and $\langle C^{\alpha^-}, \mathcal{D}_{\alpha^-}\rangle^{16(g-1)}$ and also an integral over the θ fields which is related to the instanton counting of the particular solution of the A-model solution.

Here we just put the first steps toward a topological prescription for multiloop amplitude computations of the pure spinor on $AdS_5 \times S^5$.. There many things to be done, one should first check the tree level amplitude given here with the one of the pure spinors and then to see if higher loop prescription can be related to the pure spinor superstring multiloop amplitudes.

Chapter 8

Open questions and outlook

We have seen that the pure spinor formalism might be useful to give a new perspective on the gauge/string dualities. There are many open questions related to issues we have discussed. For the basic problem of constructing a pure spinor formalism, there are many things to be done since the knowledge of the pure spinor space is limited to very particular backgrounds. The construction of the pure spinor formalism on less supersymmetric backgrounds and also more general backgrounds which can not be expressed fully as a supercoset background like the $AdS_4 \times \mathbb{CP}^3$ is an open question which should be studied later.

There are many open questions regarding the topological decomposition we will present in the thesis. It is interesting to find other backgrounds which admit this decomposition in order to use the topological construction to explore large N gauge/string dualities on these backgrounds.

It is clear that the reduction of the calculation of specific perturbative SYM amplitudes via a topological string model on the twistor space $\hat{\mathbf{CP}}^{(3|4)}$ recalls the duality for MHV amplitudes which started in [86]. The relation with this analysis of what it has been discussed here could led to a better understanding of the features and limits of topological string approach to the string realization of the perturbative gauge theory.

In particular, we focused on a particular twisted sector of the string on the geometric quotient $\left(\hat{\mathbf{CP}}^{(3|4)}\right)^4 //S_4$, while the complete theory has all the other sectors too. The SYM dual interpretation of those sectors has to be understood and found. Also, as we have discussing, there are different possible choices of BPS boundary conditions parametrized by the ϵ and δ matrix parameters which are corresponding to different D-brane configurations. These could be used also to produce lower BPS sectors to be implemented in the gauge/string correspondence as lower BPS Wilson loops [87] which some of them have been described as D-brane configurations. Also one can combine different D-brane configurations to get less supersymmetric objects, an example of which can be obtained by combining the AdS_4 boundary conditions in [23] and [37] which one may generate lower BPS D-branes configurations. Moreover, a precise analysis of the $D5$-branes observables (4.4.93) has to be performed in order to produce a detailed D-branes / circular Wilson loops dictionary. This analysis passes by the complete reduction to the base of the holomorphic Chern-Simons theory on the resolved superconifold. In particular, this passes by the calculation of the determinant of the relevant $\bar{\partial}_A$-operator on supermanifolds.

Another interesting issue to study would also be the clarification of how to add non perturbative contributions in the topological strings to get the instanton corrected version of 1/2 BPS circular Wilson loops [88] [67]. The gauge amplitude contains, on top of the matrix model integral, also the inverse of the gauge group volume and an instanton contribution. The first should be calculated in the complete topological string by the contribution of the pure Coulomb phase, very much like as in [89]. The instanton contribution should be obtained by including D-instantons in the Berkovits-Vafa context.

Let us stress that we conjectured here that the conifold transition extends to supergeometries. As such, one should be able to test it for the A-model too, along the lines of [89, 17, 13]. That is one should be able to recast in such a different case, the amplitudes in the Chern-Simons theory on $S^{(1|2)}$ in terms of the gauged linear σA-model amplitudes on the resolved superconifold. This is another open issue we are letting for future works.

In the G/G principal chiral model construction and its relation to the pure spinor superstring on $AdS_5 \times S^5$, there are many open questions. In particular it would be very interesting to explore more in detail the cohomology of the G/G principal chiral model which it seems to produce all the physical states of the superstring theory. Also it is interesting to see if there are other gauge fixings which they might give simpler interpretation of the superstring on $AdS_5 \times S^5$.

The worldsheet description of the AdS/CFT duality using the topological decomposition of the superstring action is one of the most appealing applications of this construction. One can see if in more explicit way as we will see in this thesis, it is possible to produce the perturbative gauge theory 't Hooft expansion from a particular closed string amplitude computation. This way one might be able to give a worldsheet proof of Maldacena's conjecture.

The next problem is related to the multiloop amplitude computation to give a more clear prescription to compute the superstring amplitudes using the known A-model topological amplitude computations and to compare it with the pure spinor superstring amplitudes on $AdS_5 \times S^5$.

There are many open problems to be studied further in this direction. It seems that using pure spinor formalism give us the power to explore more explicitly the understanding we have of the superstring backgrounds and their correspondence to gauge theories.

Bibliography

[1] G. 't Hooft, *A planar diagram theory for stron interactions*, *Nucl. Phys.* **B72** (1974) 461.

[2] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[3] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in AdS(5) x S(5) background*, *Nucl. Phys.* **B533** (1998) 109–126, [hep-th/9805028].

[4] J. Bagger and N. Lambert, *Comments On Multiple M2-branes*, *JHEP* **02** (2008) 105, [arXiv:0712.3738].

J. Bagger and N. Lambert, *Gauge Symmetry and Supersymmetry of Multiple M2-Branes*, *Phys. Rev.* **D77** (2008) 065008, [arXiv:0711.0955].

A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys.* **B811** (2009) 66–76, [arXiv:0709.1260].

J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D75** (2007) 045020, [hep-th/0611108].

[5] J. H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078, [hep-th/0411077].

[6] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091, [arXiv:0806.1218].

[7] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model*, *JHEP* **09** (2008) 129, [arXiv:0806.4940].

[8] j. Stefanski, B., *Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$*, *Nucl. Phys.* **B808** (2009) 80–87, [arXiv:0806.4948].

[9] P. Fre and P. A. Grassi, *Pure Spinor Formalism for Osp(N—4) backgrounds*, arXiv:0807.0044.

[10] G. Bonelli, P. A. Grassi, and H. Safaai, *Exploring Pure Spinor String Theory on $AdS_4 \times \mathbb{CP}^3$*, *JHEP* **10** (2008) 085, [arXiv:0808.1051].

[11] J. Gomis, D. Sorokin, and L. Wulff, *The complete AdS(4) x CP(3) superspace for the type IIA superstring and D-branes*, *JHEP* **03** (2009) 015, [arXiv:0811.1566].

P. A. Grassi, D. Sorokin, and L. Wulff, *Simplifying superstring and D-brane actions in AdS(4) x CP(3) superbackground*, arXiv:0903.5407.

[12] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[13] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415–1443, [hep-th/9811131].

[14] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories*, *Nucl. Phys.* **B644** (2002) 3–20, [hep-th/0206255].

[15] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637–678, [hep-th/9207094].

[16] A. Klemm, *Topological string theory and integrable structures*, *Fortsch. Phys.* **53** (2005) 720–769.

[17] H. Ooguri and C. Vafa, *Worldsheet Derivation of a Large N Duality*, *Nucl. Phys.* **B641** (2002) 3–34, [hep-th/0205297].

[18] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B403** (1993) 159–222, [hep-th/9301042].

[19] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, *JHEP* **09** (2001) 016, [hep-th/0105050].

[20] N. Berkovits, *Super-Poincare covariant quantization of the superstring*, *JHEP* **04** (2000) 018, [hep-th/0001035].

[21] N. Berkovits, *A New Limit of the $AdS_5 \times S^5$ Sigma Model*, *JHEP* **08** (2007) 011, [hep-th/0703282].

[22] G. Bonelli, P. A. Grassi, and H. Safaai , *In preparaion.*

[23] N. Berkovits and C. Vafa, *Towards a Worldsheet Derivation of the Maldacena Conjecture*, *JHEP* **03** (2008) 031, [arXiv:0711.1799].

[24] U. Lindstrom and M. Rocek, *Scalar Tensor Duality and N=1, N=2 Nonlinear Sigma Models*, *Nucl. Phys.* **B222** (1983) 285–308.

[25] S. Cecotti and C. Vafa, *On classification of N=2 supersymmetric theories*, *Commun. Math. Phys.* **158** (1993) 569–644, [hep-th/9211097].

[26] R. Ricci, *Super Calabi-Yau's and special Lagrangians*, *JHEP* **03** (2007) 048, [hep-th/0511284].

[27] M. Marino, *Les Houches lectures on matrix models and topological strings*, hep-th/0410165.

[28] N. Drukker and D. J. Gross, *An exact prediction of* $N = 4$ *SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896–2914, [hep-th/0010274].

[29] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, *JHEP* **06** (2002) 025, [hep-th/0204054].

[30] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C22** (2001) 379–394, [hep-th/9803001].

N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010, [hep-th/0501109].

J. Gomis and F. Passerini, *Wilson loops as D3-branes*, *JHEP* **01** (2007) 097, [hep-th/0612022].

J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074, [hep-th/0604007].

[31] G. Bonelli, L. Bonora, and A. Ricco, *Conifold geometries, topological strings and multi-matrix models*, *Phys. Rev.* **D72** (2005) 086001, [hep-th/0507224].

[32] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311–428, [hep-th/9309140].

[33] B. Eynard, M. Marino, and N. Orantin, *Holomorphic anomaly and matrix models*, *JHEP* **06** (2007) 058, [hep-th/0702110].

R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B644** (2002) 21–39, [hep-th/0207106].

R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[34] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, *Topological strings and integrable hierarchies*, *Commun. Math. Phys.* **261** (2006) 451–516, [hep-th/0312085].

[35] N. Berkovits, *Perturbative Super-Yang-Mills from the Topological $AdS_5 \times S^5$ Sigma Model*, *JHEP* **09** (2008) 088, [arXiv:0806.1960].

[36] N. Berkovits, *Simplifying and Extending the $AdS_5 \times S^5$ Pure Spinor Formalism*, arXiv:0812.5074.

[37] G. Bonelli and H. Safaai, *On gauge/string correspondence and mirror symmetry*, *JHEP* **06** (2008) 050, [arXiv:0804.2629].

[38] V. A. Kostelecky *et. al.*, *The six fermion amplitude in the superstring*, *Phys. Lett.* **B183** (1987) 299.

[39] M. B. Green and J. H. Schwarz, *Supersymmetrical Dual String Theory*, *Nucl. Phys.* **B181** (1981) 502–530.

[40] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, *Superparticles, Twistors and Siegel Symmetry*, *Mod. Phys. Lett.* **A4** (1989) 901–908.

[41] D. P. Sorokin, V. I. Tkach, D. V. Volkov, and A. A. Zheltukhin, *From the Superparticle Siegel Symmetry to the Spinning Particle Proper Time Supersymmetry*, *Phys. Lett.* **B216** (1989) 302–306.

[42] N. Berkovits, *A covariant action for the heterotic superstring with manifest space-time supersymmetry and world sheet superconformal invariance*, *Phys. Lett.* **B232** (1989) 184.

[43] M. Tonin, *kappa symmetry as world sheet supersymmetry in $D = 10$ heterotic superstring*, *Int. J. Mod. Phys.* **A7** (1992) 6013–6024.

[44] D. P. Sorokin, *Superbranes and superembeddings*, *Phys. Rept.* **329** (2000) 1–101, [hep-th/9906142].

[45] M. Matone, L. Mazzucato, I. Oda, D. Sorokin, and M. Tonin, *The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings*, *Nucl. Phys.* **B639** (2002) 182–202, [hep-th/0206104].

[46] M. B. Green and J. H. Schwarz, *Properties of the Covariant Formulation of Superstring Theories*, *Nucl. Phys.* **B243** (1984) 285.

M. B. Green and J. H. Schwarz, *Covariant Description of Superstrings*, *Phys. Lett.* **B136** (1984) 367–370.

M. Henneaux and L. Mezincescu, *A Sigma Model Interpretation of Green-Schwarz Covariant Superstring Action*, *Phys. Lett.* **B152** (1985) 340.

[47] W. Siegel, *Classical Superstring Mechanics*, *Nucl. Phys.* **B263** (1986) 93.

[48] N. Berkovits, *Super-Poincare covariant quantization of the superstring*, *JHEP* **04** (2000) 018, [hep-th/0001035].

[49] V. G. Kac, *A Sketch of Lie Superalgebra Theory*, *Commun. Math. Phys.* **53** (1977) 31–64.

[50] L. Castellani, R. D'Auria, and P. Fre, *Supergravity and superstrings: A Geometric perspective. Vol. 1: Mathematical foundations*, . Singapore, Singapore: World Scientific (1991) 1-603.

[51] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov, and B. Zwiebach, *Superstring theory on AdS(2) x S(2) as a coset supermanifold*, *Nucl. Phys.* **B567** (2000) 61–86, [hep-th/9907200].

[52] R. Kallosh, J. Rahmfeld, and A. Rajaraman, *Near horizon superspace*, *JHEP* **09** (1998) 002, [hep-th/9805217].

R. Kallosh and J. Rahmfeld, *The GS string action on AdS(5) x S(5)*, *Phys. Lett.* **B443** (1998) 143–146, [hep-th/9808038].

[53] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047, [hep-th/0406055].

[54] N. Berkovits and P. S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys.* **B635** (2002) 75–105, [hep-th/0112160].

[55] I. Adam, A. Dekel, L. Mazzucato, and Y. Oz, *Integrability of type II superstrings on Ramond-Ramond backgrounds in various dimensions*, *JHEP* **06** (2007) 085, [hep-th/0702083].

[56] N. Berkovits, *Quantum consistency of the superstring in AdS(5) x S**5 background*, *JHEP* **03** (2005) 041, [hep-th/0411170].

[57] S. Kobayashi and K. Nomizu, *foundations of differential geometry, Vols. I and II.* 1996. John Wiley & Sons.

[58] K. Itoh, T. Kugo, and H. Kunitomo, *Supersymmetric nonlinear realization for arbitrary Kahlerian coset space G/H*, *Nucl. Phys.* **B263** (1986) 295.

[59] I. Adam, P. A. Grassi, L. Mazzucato, Y. Oz, and S. Yankielowicz, *Non-critical pure spinor superstrings*, *JHEP* **03** (2007) 091, [hep-th/0605118].

[60] S. Seki, K. Sugiyama, and T. Tokunaga, *Superconformal symmetry in linear sigma model on supermanifolds*, *Nucl. Phys.* **B753** (2006) 295–312, [hep-th/0605021].

[61] P. A. Grassi and M. Marescotti, *Flux vacua and supermanifolds*, *JHEP* **01** (2007) 068, [hep-th/0607243].

[62] B. Craps, J. Gomis, D. Mateos, and A. Van Proeyen, *BPS solutions of a D5-brane world volume in a D3-brane background from superalgebras*, *JHEP* **04** (1999) 004, [hep-th/9901060].

[63] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

M. Aganagic and C. Vafa, *Mirror symmetry and supermanifolds*, hep-th/0403192.

[64] E. Witten, *Topological Sigma Models*, *Commun. Math. Phys.* **118** (1988) 411.

[65] R. Catenacci, M. Debernardi, P. Grassi, and D. Matessi, *Balanced Superprojective Varieties*, arXiv:0707.4246.

[66] P. A. Grassi and G. Policastro, *Super-Chern-Simons theory as superstring theory*, hep-th/0412272.

[67] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, arXiv:0712.2824.

[68] R. Giles, *The reconstruction of gauge potentials from Wilson loops*, *Phys. Rev.* **D24** (1981) 2160.

[69] J. M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859–4862, [hep-th/9803002].

L. F. Alday and J. Maldacena, *Null polygonal Wilson loops and minimal surfaces in Anti- de-Sitter space*, arXiv:0904.0663.

[70] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, hep-th/0012041.

[71] S. Yamaguchi and S.-T. Yau, *Topological string partition functions as polynomials*, *JHEP* **07** (2004) 047, [hep-th/0406078].

[72] I. A. Batalin and G. A. Vilkovisky, *Closure of the Gauge Algebra, Generalized Lie Equations and Feynman Rules*, *Nucl. Phys.* **B234** (1984) 106–124.

[73] L. D. Faddeev and V. N. Popov, *Feynman diagrams for the Yang-Mills field*, *Phys. Lett.* **B25** (1967) 29–30.

[74] C. Becchi, A. Rouet, and R. Stora, *The Abelian Higgs-Kibble Model. Unitarity of the S Operator*, *Phys. Lett.* **B52** (1974) 344.

C. Becchi, A. Rouet, and R. Stora, *Renormalization of the Abelian Higgs-Kibble Model*, *Commun. Math. Phys.* **42** (1975) 127–162.

C. Becchi, A. Rouet, and R. Stora, *Renormalization of Gauge Theories*, *Annals Phys.* **98** (1976) 287–321.

I. V. Tyutin, *Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism*, arXiv:0812.0580.

[75] I. A. Batalin and G. A. Vilkovisky, *Gauge Algebra and Quantization*, *Phys. Lett.* **B102** (1981) 27–31.

I. A. Batalin and G. a. Vilkovisky, *Feynman rules for reducible gauge theories*, *Phys. Lett.* **B120** (1983) 166–170.

I. A. Batalin and G. A. Vilkovisky, *Quantization of Gauge Theories with Linearly Dependent Generators*, *Phys. Rev.* **D28** (1983) 2567–2582.

I. A. Batalin and G. A. Vilkovisky, *Existence Theorem for Gauge Algebra*, *J. Math. Phys.* **26** (1985) 172–184.

[76] G. Barnich and M. Henneaux, *Consistent couplings between fields with a gauge freedom and deformations of the master equation*, *Phys. Lett.* **B311** (1993) 123–129, [hep-th/9304057].

[77] G. Barnich, F. Brandt, and M. Henneaux, *Local BRST cohomology in the antifield formalism. 1. General theorems*, *Commun. Math. Phys.* **174** (1995) 57–92, [hep-th/9405109].

[78] M. Henneaux and A. Wilch, *Local BRST cohomology of the gauged principal non-linear sigma model*, *Phys. Rev.* **D58** (1998) 025017, [hep-th/9802118].

[79] Y. Aisaka and Y. Kazama, *A new first class algebra, homological perturbation and extension of pure spinor formalism for superstring*, *JHEP* **02** (2003) 017, [hep-th/0212316].

[80] G. Barnich, M. Henneaux, and R. Tatar, *Consistent interactions between gauge fields and the local BRST cohomology: The Example of Yang-Mills models*, *Int. J. Mod. Phys.* **D3** (1994) 139–144, [hep-th/9307155].

[81] G. Barnich, F. Brandt, and M. Henneaux, *Local BRST cohomology in Einstein Yang-Mills theory*, *Nucl. Phys.* **B455** (1995) 357–408, [hep-th/9505173].

[82] G. Barnich, M. Henneaux, T. Hurth, and K. Skenderis, *Cohomological analysis of gauge-fixed gauge theories*, *Phys. Lett.* **B492** (2000) 376–384, [hep-th/9910201].

[83] G. Barnich, F. Brandt, and M. Henneaux, *Local BRST cohomology in gauge theories*, *Phys. Rept.* **338** (2000) 439–569, [hep-th/0002245].

[84] F. Brandt, *Local BRST Cohomology and Covariance*, *Commun. Math. Phys.* **190** (1997) 459–489, [hep-th/9604025].

[85] J. Manes, R. Stora, and B. Zumino, *Algebraic Study of Chiral Anomalies*, *Commun. Math. Phys.* **102** (1985) 157.

[86] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189–258, [hep-th/0312171].

A. Neitzke and C. Vafa, *$N = 2$ strings and the twistorial Calabi-Yau*, hep-th/0402128.

[87] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, *Supersymmetric Wilson loops on S^3*, *JHEP* **05** (2008) 017, [arXiv:0711.3226].

N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, *On the D3-brane description of some 1/4 BPS Wilson loops*, *JHEP* **04** (2007) 008, [hep-th/0612168].

[88] M. Bianchi, M. B. Green, and S. Kovacs, *Instanton corrections to circular Wilson loops in $N = 4$ supersymmetric Yang-Mills*, *JHEP* **04** (2002) 040, [hep-th/0202003].

[89] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B577** (2000) 419–438, [hep-th/9912123].