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Various Aspects of Holographic Renormalization

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To my parents

Abstract

In this thesis we explore various aspects of holographic renormalization. The thesis comprises the work done by the candidate during the doctorate programme at SISSA and ICTP under the supervision of A. Tanzini. This consists in the following works.

- In [1], reproduced in chapter 2 we consider holographic renormalization in an exotic spacetime such as an asymptotically conical manifold, showing that it has a close relation with variational principle. The variational problem of gravity theories is directly related to black hole thermodynamics. For asymptotically locally AdS backgrounds it is known that holographic renormalization results in a variational principle in terms of equivalence classes of boundary data under the local asymptotic symmetries of the theory, which automatically leads to finite conserved charges satisfying the first law of thermodynamics. We show that this connection holds well beyond asymptotically AdS black holes. In particular, we formulate the variational problem for $\mathcal{N} = 2$ STU supergravity in four dimensions with boundary conditions corresponding to those obeyed by the so-called 'subtracted geometries'. We show that such boundary conditions can be imposed covariantly in terms of a set of asymptotic second class constraints, and we derive the appropriate boundary terms that render the variational problem well posed in two different duality frames of the STU model. This allows us to define finite conserved charges associated with any asymptotic Killing vector and to demonstrate that these charges satisfy the Smarr formula and the first law of thermodynamics. Moreover, by uplifting the theory to five dimensions and then reducing on a 2-sphere, we provide a precise map between the thermodynamic observables of the subtracted geometries and those of the BTZ black hole. Surface terms play a crucial role in this identification.

- In [2], reproduced in chapter 3 we present a systematic approach to supersymmetric holographic renormalization for a generic 5D $\mathcal{N} = 2$ gauged supergravity theory with matter multiplets, including its fermionic sector, with all gauge fields consistently set to zero. We determine the complete set of supersymmetric local boundary counterterms, including the finite counterterms that parameterize the choice of supersymmetric renormalization scheme. This allows us to derive holographically the superconformal Ward identities of a 4D superconformal field theory on a generic background, including the Weyl and super-Weyl anomalies. Moreover, we show that these anomalies satisfy the

Wess-Zumino consistency condition. The super-Weyl anomaly implies that the fermionic operators of the dual field theory, such as the supercurrent, do not transform as tensors under rigid supersymmetry on backgrounds that admit a conformal Killing spinor, and their anticommutator with the conserved supercharge contains anomalous terms. This property is explicitly checked for a toy model. Finally, using the anomalous transformation of the supercurrent, we obtain the anomaly-corrected supersymmetry algebra on curved backgrounds admitting a conformal Killing spinor.

Contents

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Chapter 1

Introduction

Holographic renormalization [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] is a procedure to consistently remove divergences from infinite quantities of gravity theory in non-compact spaces such as anti-de Sitter (AdS) manifold. It is named so because in the context of the AdS/CFT [14, 15, 16] conjecture (or in a broad sense the *holographic principle* [17, 18], introduced by G. 't Hooft and L. Susskind) holographic renormalization corresponds to renormalization of the gauge field theory residing on the boundary of the non-compact manifold. In the first part of this thesis (chapter 2) we study its connection to variational principle[1] in asymptotically conical spacetime, which allows us to get information of black hole thermodynamics in those spaces. In the second part (chapter 3), we apply holographic renormalization for the full (bosonic as well as fermionic) sector of gauged supergravity and study its implications on supersymmetric field theory.

Mathematical aspects of holographic renormalization were already emphasized since the beginning of AdS/CFT. In [19], it was pointed out that the volume of the conformally compact spacetime like $\mathbb{H}^n$ can be made finite by using holographic renormalization developed in [3]. In [20] it was presented a close relationship between holographic renormalization and having a well-defined variational problem in *asymptotically locally AdS spaces* (AlAdS) [21, 19, 22], whose definition is given in section 1.1. Moreover, it turns out that additional boundary terms required in order for the variational problem to be well-posed automatically make the physical quantities (such as thermodynamic observables in black-hole background) finite. It is natural to extend the above story to other spaces like asymptotically flat or de Sitter (dS) spaces, which are quite interesting in the area of cosmology since our universe was once approximately a dS space and is now almost flat. In chapter 2, we, therefore, address the variational problem for asymptotically conical spaces, which can be thought as a mid-step to this goal. We should note however that it is still an open question if the gravity theory in non-AlAdS spaces has a holographic dual gauge theory or not.

Since Witten's seminal work [23], many works have shown that supersymmetric (SUSY) gauge theories on compact spaces possess a deep geometrical interpretation.

[1] As pointed out in [13], holographic renormalization is in fact relevant for a well-posed variational problem independently of holography.

For instance, $\mathcal{N} = 2$ theories on 4D compact manifolds can be used to compute Donaldson invariants, since they describe moduli space of unframed instantons. On the other hand, non-perturbative results in supersymmetric gauge theories can be useful for testing the AdS/CFT conjecture. Conversely, if one believes in AdS/CFT, results in supergravity theory can give a hint on getting non-perturbative quantities in supersymmetric gauge theories. In order to obtain the supersymmetric results of supergravity theory, we study both of its bosonic and fermionic sectors in chapter 3.

In this thesis we perform holographic renormalization in the radial Hamiltonian formulation of the gravitational theory [5, 10, 12, 24, 25]. The advantage of this approach is twofold. First, it allows one to obtain the covariant boundary terms (that are needed for a well-posed variational problem) without relying on a specific solution of the gravity theory. General covariance of these boundary terms is a necessary and sufficient condition to maintain diffeomorphism invariance of the gravity theory. Second, it allows one to derive *holographic Ward identities*, which reflect the global symmetries of the holographic dual gauge field theory. In general, Ward identities are a mathematical description of the global symmetries of the quantum field theory.

Symmetries play an important role in physics, and thus obtaining supersymmetric Ward identities is a key step in understanding the supersymmetric field theories. By applying the prescription introduced above to the full sector of the supergravity we obtain the supersymmetric Ward identities for generic supersymmetric backgrounds, i.e. ones that allow at least one (conformal) Killing spinor [26, 27]. In chapter 3 we present this holographic derivation of the supersymmetric Ward identities for 4D $\mathcal{N} = 1$ superconformal field theory (SCFT) and show the following theorem, see (3.5.28) and (3.5.30).

Theorem 1.1. *The supersymmetric Ward identity for 4D $\mathcal{N} = 1$ superconformal field theory on supersymmetric backgrounds is anomalous, namely*

$$\{Q^s[\eta_+], S^i\} = -\Gamma^j \eta_+ T^i_j + c\, \mathbb{D}_k \Big[\big(\frac{4}{3} R \Gamma^{ki} - 2 R_j{}^k \Gamma^{ji} + 2 R_j{}^i \Gamma^{jk} \big) \eta_- \Big] + \cdots ,$$

with

$$\langle \{Q^s[\eta_+], S^i\} \rangle = 0,$$

where i, j, k denote the spacetime indexes, S^i and T_{ij} are the supercurrent operator and stress-energy tensor of the theory respectively, $\mathbb{D}_i$ represent the covariant derivative, and Γ^i are 4×4 matrices following the Clifford algebra $\{\Gamma^i, \Gamma^j\} = 2g^{ij}$ with $\Gamma^{ij} \equiv \frac{1}{2}[\Gamma^i, \Gamma^j]$. η_+ is a conformal Killing spinor, which implies the existence of a spinor η_- satisfying $\mathbb{D}_i \eta_+ = \Gamma_i \eta_-$. $Q^s[\eta_+]$ is the conserved supercharge associated with the conformal Killing spinor η_+, and c is the central charge of the 4D $\mathcal{N} = 1$ SCFT with $a = c$. The ellipsis indicate the possible entities that depend on the operators entering the theory.

The second term in the above Ward identity is the universal anomalous term which does not appear in the classical supersymmetric Ward identity.

Theorem 1.1 displays that the action of the preserved supercharge on the supercurrent operator is anomalous. From theorem 1.1 we reach the following one that the supersymmetry algebra of 4D $\mathcal{N} = 1$ SCFT is also anomalous, see (3.5.56).

Theorem 1.2. *Let η_+ be a conformal Killing spinor and $\eta_- = \frac{1}{4}\Gamma^i \mathbb{D}_i \eta_+$. When $\mathcal{K}^i \equiv i\overline{\eta}_+ \Gamma^i \eta_+$ becomes a Killing vector, we have the supersymmetry algebra*

$$\boxed{\begin{aligned}
&\{Q^s[\eta_+], Q^s[\overline{\eta}_+]\} = -\frac{i}{2}\mathcal{Q}[\mathcal{K}] + \alpha \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \sqrt{g}\, \overline{\eta}_+ \times \\
&\qquad\qquad\qquad\qquad \times \mathbb{D}_k \Big[\big(\frac{4}{3} R \Gamma^{ki} - 2R_j{}^k \Gamma^{ji} + 2R_j{}^i \Gamma^{jk}\big)\eta_- \Big] + \cdots, \\
&\{\mathcal{Q}[\xi], Q^s[\eta_+]\} = -Q^s[\mathcal{L}_\xi \eta_+], \\
&\{\mathcal{Q}[\xi], Q^s[\overline{\eta}_+]\} = -Q^s[\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi],
\end{aligned}}$$

where ξ denotes any Killing vector of the theory. $\mathcal{Q}[\xi]$ refers to the conserved charge associated with the Killing vector ξ, and $\mathcal{L}_\xi$ indicates the Lie-derivative with respect to ξ. The ellipsis again represent the possible entities that depend on the operators entering the theory.

We emphasize that the anomalous term presented in theorem 1.1 and 1.2 does not vanish for generic supersymmetric backgrounds. We note that a similar conclusion was made also in [28] where they turned on the $U(1)_R$ gauge field of 4D $\mathcal{N} = 1$ SCFT, which is turned off in chapter 3 of this thesis.

1.1 Background material

In this section we provide a brief introduction of holographic renormalization in Al-AdS spaces and supersymmetric field theories in curved space. We refer the interested reader to [29, 30] for a detailed review of AdS/CFT and holographic renormalization, and to [31, 32] for a recent review of supersymmmetric field theories in curved space, respectively.

1.1.1 AdS/CFT

Let us begin with introducing the AdS/CFT correspondence.

Conjecture 1.1 (Maldacena)**.** *AdS/CFT is a conjecture that a superstring theory or M-theory in $AdS_{d+1} \times X$ where X is a compact manifold is equivalent to a conformally invariant theory on the boundary of AdS_{d+1}.*

Its mathematical description is that the generating functional of the CFT is identified with the partition function of the gravity theory, namely

$$\left\langle \exp\left(\int_{S^d} \mathcal{O}\Phi_0 \right) \right\rangle = \exp\left(-S_{\text{on-shell}}[\Phi]\right), \quad \text{with} \quad \Phi|_{\text{boundary}} = \Phi_0, \tag{1.1}$$

in the classical supergravity limit. Note that S^d is the boundary of global Euclidean AdS space AdS_{d+1}. Here Φ_0 denotes the source for every local and single-trace operator $\mathcal{O}$ that is supposed to have a dual field Φ in the 'bulk' gravity theory. The bulk field Φ follows the Dirichlet boundary condition $\Phi|_{\text{boundary}} = \Phi_0$, and $S_{\text{on-shell}}$ means the action that is evaluated on solutions of the classical supergravity.

Several examples of the AdS/CFT correspondence were already presented in [14]; the simplest one is the equivalence between Type IIB string theory on $AdS_5 \times S^5$ and 4 dimensional $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its (super)conformal phase. Here AdS_5 and S^5 have the same radius L, and the 5-form F_5^+ has integer flux $N = \int_{S^5} F_5^+$. The following equivalence relation is established according to AdS/CFT,

$$g_s = g_{YM}^2, \quad L^4 = 4\pi g_s N(\alpha')^2, \tag{1.2}$$

where g_s is the string coupling and α' is the square of the Plank length.

A simple check of the equivalence of these two theories is to compare their symmetries. The continuous global symmetry group of $\mathcal{N} = 4$ SYM with $SU(N)$ in its conformal phase turns out to be the superconformal group $SU(2, 2|4)$. Its maximal bosonic subgroup is $SU(2, 2) \times SU(4)_R \sim SO(2, 4) \times SO(6)_R$ and it has 32 fermionic generators. On the string theory side, $SO(2, 4) \times SO(6)$ is the isometry group of the $AdS_5 \times S^5$ background. Type IIB string theory contains 32 supercharges, all of which the $AdS_5 \times S^5$ background does not break.

We finish introducing AdS/CFT here, emphasizing that as general dualities, the AdS/CFT correspondence is a weak/strong duality, which can be seen as follows. In the large N limit ($N \to \infty$) while keeping the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed, perturbation theory of Yang-Mills gauge theory corresponds to $\lambda \ll 1$, while in string theory the α' expansion of a physical quantity

$$Q = a_1 \alpha R + a_2(\alpha' R)^2 + a_3(\alpha' R)^3 + \cdots , \tag{1.3}$$

where R is the Ricci scalar of AdS_5, is replaced by

$$Q \sim a_1 \lambda^{-\frac{1}{2}} + a_2 \lambda^{-1} + a_3 \lambda^{-\frac{3}{2}} + \cdots , \tag{1.4}$$

since $R \sim 1/L^2 = \lambda^{-\frac{1}{2}}/\alpha'$.

1.1.2 Holographic renormalization

To introduce holographic renormalization clearly we restrict ourselves in the case where space is AlAdS whose definition is given shortly. We emphasize that the main logic here can be extended to another kind of non-compact spaces as done in chapter 2. We again recommend [28] for a detailed discussion on this.

Definition 1.1 (Penrose). *A Riemannian metric g on the interior $\mathcal{M}$ of a compact manifold with boundary $\overline{\mathcal{M}}$ is said to be conformally compact if $\bar{g} \equiv z^2 g$ extends continuously (or with some degree of smoothness) as a metric to $\overline{\mathcal{M}}$, where z is a defining function for $\partial\mathcal{M}$, i.e. $z > 0$ on $\mathcal{M}$ and $z = 0$, $dz \neq 0$ on $\partial\mathcal{M}$.*

It is immediate to see that AdS space is conformally compact. The definition 1.1 allows one to define an AlAdS metric, see [11] for a detailed discussion.

Definition 1.2. *An asymptotically locally AdS (AlAdS) metric is a conformally compact Einstein metric.*

The asymptotic structure of AlAdS spaces was studied by Fefferman and Graham in [21]. Near the boundary, one can always choose a coordinate system such that the AlAdS metric takes the form

$$\mathrm{d}s^2 = \frac{1}{z^2}(\mathrm{d}z^2 + \mathfrak{g}_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j), \quad i,j = 1,2,\cdots,d, \tag{1.5}$$

$$\mathfrak{g}_{ij}(x,z) = \mathfrak{g}_{(0)ij} + z\mathfrak{g}_{(1)ij} + \cdots + z^d\mathfrak{g}_{(d)ij} + \mathfrak{h}_{(d)ij}z^d\log z^2 + \cdots, \tag{1.6}$$

which is so-called Fefferman-Graham (FG) expansion. Note that the logarithmic term appears only when d is even number. In this coordinate system the boundary lies at $z = 0$, but from now on we use the coordinate $r = \exp(-z)$ instead of z so that the boundary lies at $r = +\infty$.

Let us evaluate the partition function of the classical supergravity with negative cosmological constant in the simplest case where all fields but the metric are turned off. The effective action then becomes the Einstein-Hilbert (EH) action. Including the Gibbons-Hawking (GH) term [33],[2] the action is given in the Euclidean signature by

$$S_{EH} = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{g}\,(R[g]-\Lambda) - \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{\gamma}\,K, \tag{1.7}$$

where $\kappa^2 = 8\pi G_{d+1}$, γ is the induced metric on the boundary manifold $\partial\mathcal{M}$ and K is the extrinsic curvature. The cosmological constant Λ can be normalized to $\Lambda = -d(d-1)/\ell^2$. Note that the GH term does not affect the equation of motion of the EH action.

The EH action possesses a maximally symmetric solution, i.e. AdS metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + e^{2r/\ell}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i,j = 1,2,\cdots,d, \tag{1.8}$$

written in modified Poincaré coordinates. Evaluating the action S_{EH+GH} on this AdS metric, one can readily see that it is divergent, since $K = d/\ell$, $R[g] = \frac{2d}{d-2}\Lambda$ and the volume of the AdS space is infinite.

The on-shell action of the gravity theory is equivalent to the partition function of the boundary quantum field theory, according to AdS/CFT. Now it is obvious that to justify this equivalence we need to make the on-shell action finite by renormalization, i.e. holographic renormalization.

Notice that as we have seen, the divergence of the on-shell action is due to the infinite volume of AdS space. Therefore, one can see that holographic renormalization also gives a prescription for obtaining the renormalized volume of AdS space.

The AdS metric is not the only solution of the EH action. It many other solutions. Moreover, the real supergravity action and its classical solutions are much more complicated, and thus it is necessary to develop a generic holographic renormalization

[2] The GH term is needed to well-pose a variational problem for Einstein-Hilbert gravity in compact spaces.

prescription. To this end, we make use of the radial Hamiltonian formulation of the gravity theory, where the bulk metric is written as

$$\mathrm{d}s^2 = (N^2 + N^i N_i)\mathrm{d}r^2 + 2N_i \mathrm{d}r\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1.9}$$

where the radial coordinate r plays a role of the Hamiltonian time of ADM formalism [25]. N and N_i are called the lapse function and the shift vector respectively, and γ_{ij} is the induced metric on the hypersurface Σ_r of constant radial coordinate r. For the metric (1.9), the action (1.7) becomes

$$S_{EH+GH} = \int \mathrm{d}r\ L, \tag{1.10}$$

where L is a 'radial Lagrangian',

$$L = -\frac{1}{2\kappa^2}\int_{\Sigma_r} \mathrm{d}^d x\sqrt{\gamma}N\left(R[\gamma] + K^2 - K_{ij}K^{ij}\right), \tag{1.11}$$

and

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i). \tag{1.12}$$

The 'radial Hamiltonian' is given by the Legendre transform of the Lagrangian (1.11), namely

$$H = \int_{\Sigma_r} \mathrm{d}^d x\ \pi^{ij}\dot{\gamma}_{ij} - L = \int_{\Sigma_r} \mathrm{d}^d x\ (N\mathcal{H} + N_i\mathcal{H}^i), \tag{1.13}$$

where

$$\pi^{ij} \equiv \frac{\delta L}{\delta\dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}) \tag{1.14}$$

are the canonical momenta conjugate to the induced metric γ_{ij} and

$$\mathcal{H} \equiv 2\kappa^2\frac{1}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}(\gamma^{ij}\pi_{ij})^2\right), \tag{1.15a}$$

$$\mathcal{H}^i \equiv -2D_j\pi^{ij}. \tag{1.15b}$$

Since N and N^i enter the Hamiltonian (1.13) as Lagrange multipliers, their equations of motion give the constraints

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0, \tag{1.16}$$

which in fact reflects diffeomorphism invariance of the gravity theory [34].

The radial Hamiltonian formulation above naturally leads us to Hamilton-Jacobi (HJ) formalism to solve the constraints (1.15). That is, by expressing the canonical momenta in terms of gradients of Hamilton's principal function $\mathcal{S}[\gamma]$,

$$\pi^{ij} = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \tag{1.17}$$

and inserting into the constraints (1.15), we obtain the first-order partial differential equations for the principal function $\mathcal{S}$. While the constraint $\mathcal{H}^i = 0$ turns out to be satisfied when $\mathcal{S}[\gamma]$ respects the diffeomorphism invariance on the radial slice Σ_r (that is, takes the covariant form), the constraint $\mathcal{H} = 0$ becomes

$$\frac{2\kappa^2}{\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}} + \frac{\sqrt{\gamma}}{2\kappa^2}(R[\gamma] - \Lambda) = 0. \tag{1.18}$$

The general solution of the HJ equation (1.18) contains *integration constants* that correspond generically to *normalizable* modes. Upon solving the HJ equation (1.18), we get the solution for the metric by using the first-order flow equation

$$\dot{\gamma}_{ij} = \frac{4\kappa^2}{\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}. \tag{1.19}$$

The integration constants that appear when solving the flow equations correspond generically to the *non-normalizable* modes.

Now the main idea of holographic renormalization follows. First, setting the radial 'cut-off' r, we regulate the on-shell action by

$$S_{\mathrm{reg}}[\gamma(r,x)] = \int^r \mathrm{d}r'\, L|_{\mathrm{on-shell}}, \tag{1.20}$$

which naturally satisfies the HJ equation (1.18). As discussed before, the regulated on-shell action (1.20) goes to infinity as $r \to \infty$. Second, this divergence is eliminated consistently by adding the counterterms $\mathcal{S}_{\mathrm{ct}}$, which are obtained from the divergent part of a generic asymptotic solution of the HJ equation (1.18). Finally, the renormalized on-shell action is determined by

$$S_{\mathrm{ren}} := \lim_{r\to\infty}(S_{\mathrm{reg}} + S_{\mathrm{ct}}), \tag{1.21}$$

which is finite by construction.

Let us see how to obtain the solution of (1.18) asymptotically. We define an operator

$$\delta_\gamma = \int 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}, \tag{1.22}$$

and formally write the solution as

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \cdots, \tag{1.23}$$

where

$$\delta_\gamma\mathcal{S}_{(2k)} = \lambda_k\mathcal{S}_{(2k)}, \quad k = 0, 1, \cdots. \tag{1.24}$$

In fact, it is easy to see that $\lambda_k = d - 2k$. The operator δ_γ basically counts twice the number of the γs. Moreover, the eigenvalue of δ_γ can be d at most, provided that it acts on the functional spaces in the covariant form. The eigenfunctional $\mathcal{S}_{(0)}$ with the eigenvalue d should take the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int_{\Sigma_r}\mathrm{d}^d x\sqrt{\gamma}\,U, \tag{1.25}$$

with a certain scalar function U. In this case, it is immediate to see that $U = -(d-1)/\ell$. And thus we get

$$\mathcal{S}_{(0)} = -\frac{1}{\kappa^2}\int_{\Sigma_r} \mathrm{d}^d x\sqrt{\gamma}\,\frac{d-1}{\ell}. \tag{1.26}$$

Now the HJ equation (1.18) turns out to be recursive relation for $\mathcal{S}_{(2)}$, $\mathcal{S}_{(4)}$ and so on. Since it overlaps in many parts with chapter 3, we will not discuss this further, but just provide

$$\mathcal{S}_{(2)} = -\frac{1}{4\kappa^2}\int_{\Sigma_r} \mathrm{d}^d x\sqrt{\gamma}R[\gamma], \tag{1.27}$$

when $d = 4$. The key point is that this recursive procedure stops at order $n = [d/2]$ in AlAdS backgrounds (even when matter fields are added to the model). This is because higher order terms are finite as $r \to \infty$ and arbitrary integration constants (correspond to normalizable modes) enter in the solution. It follows that the counterterms are

$$\mathcal{S}_{\mathrm{ct}} = -\sum_{n=0}^{[d/2]} \mathcal{S}_{(2n)}. \tag{1.28}$$

An important feature of the counterterms (1.28) is that this allows us to obtain the *renormalized canonical momenta* that correspond to the renormalized one-point functions of holographic dual field theory. Namely,

$$\hat{\pi}^{ij} := \lim_{r\to\infty} e^{(d+2)r}\gamma^{-\frac{1}{2}}\left(\pi^{ij} + \frac{\delta\mathcal{S}_{\mathrm{ct}}}{\delta\gamma_{ij}}\right) = \langle T^{ij}\rangle_{\mathrm{ren}}, \tag{1.29}$$

where $\langle T^{ij}\rangle_{\mathrm{ren}}$ is the renormalized one-point function stress-energy tensor of the dual field theory. $e^{(d+2)r}$ is there to match the scaling dimension. Of course, this correspondence between renormalized canonical momenta and one-point functions extends straightforwardly to other operators in the theory.

Another important feature of the counterterms is that these automatically satisfy the constraint $\mathcal{H}^i = 0$. It follows that we have

$$D_i\hat{\pi}^{ij} = D_i\langle T^{ij}\rangle_{\mathrm{ren}} = 0, \tag{1.30}$$

which is in fact the Ward identity for diffeomorphism invariance of the dual field theory. We call it *holographic Ward identity*. The constraint $\mathcal{H} = 0$ also gives a Ward identity (actually the conformal Ward identity), but we will discuss in detail in chapter 3. What we want to say is that each constraint of the bulk theory leads to holographic Ward identity that reflects the corresponding global symmetry of the dual field theory.

1.1.3 Supersymmetry on curved manifold

Since chapter 3 is quite related to supersymmetric field theories in curved space, in this subsection we briefly introduce how to construct supersymmetric field theories in curved manifold and observe the Ward identities.

Let us begin with introducing supersymmetry.

Definition 1.3. *Poincaré group is the group of Minkowski spacetime isometries.*

The generators of the Poincaré group satisfy the following commutation relations.

$$[P_\mu, P_\nu] = 0, \tag{1.31a}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -\imath\eta_{\mu\rho}M_{\nu\sigma} - \imath\eta_{\nu\sigma}M_{\mu\rho} + \imath\eta_{\mu\sigma}M_{\nu\rho} + \imath\eta_{\nu\rho}M_{\mu\sigma}, \tag{1.31b}$$
$$[M_{\mu\nu}, P_\rho] = -\imath\eta_{\rho\mu}P_\nu + \imath\eta_{\rho\nu}P_\mu \tag{1.31c}$$

In group notation, the Poincaré group is $\mathbb{R}^{3,1} \rtimes O(3,1)$. This group is quite interesting, since it is symmetry subgroup of many physical theories defined on $\mathbb{R}^{3,1}$. Moreover, in theoretical physics there is a famous no-go theorem (Coleman-Mandula theorem), stating that space-time and internal symmetries cannot be combined in any but a trivial way. Note that charges of the internal symmetries should be spacetime-scalars.

Later on, it turned out that Coleman-Mandula theorem has some loopholes. One of the loopholes is that the symmetry group can contain additional generators that are not scalars but rather spinors. By extending Poincaré group so as to include the spinor generators, we reach the *super-Poincaré group*, the generators of which satisfy the following $\mathbb{Z}_2$-graded commutation relations.

$$[P_\mu, Q_\alpha] = 0, \qquad [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \tag{1.32a}$$
$$[M_{\mu\nu}, Q_\alpha] = \imath(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = \imath(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}, \tag{1.32b}$$
$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu. \tag{1.32c}$$

Quantum field theories defined on $\mathbb{R}^{3,1}$ is said to be *supersymmetric* when super-Poincaré group is its symmetry subgroup. As mentioned at the beginning of this chapter, many phenomena interesting in geometry and theoretical physic can be observed when placing the supersymmetric field theories on compact (curved) manifold. Of course, in order to observe such interesting phenomena, the first thing to do is to place the supersymmetric field theories on curved manifold. It is, however, non-trivial, since if we do it naively supersymmetries would be broken in general. We should emphasize that it is not always possible to construct supersymmetric field theories on curved manifold because the supersymmetry imposes severe constraints on the background manifold.

A popular way for construction of supersymmetric field theories on curved manifold was presented by Seiberg and Festuccia [35, 36]. Their logic is as follows. First, couple the supersymmetric field theory to supergravity. Then, in order to decouple the supergravity multiplet which includes the metric $g_{\mu\nu}$, gravitino $\psi_{\mu\alpha}$ and auxiliary fields typically, take a *rigid limit* where Newton's constant $G_N \to 0$, so that the metric is sent to a fixed background metric, and the auxiliary fields are also sent to fixed backgrounds. That is, we only demand that the supergravity multiplet is non-dynamical bosonic and supersymmetric.

In general, this is achieved by

$$\Psi_{\mu\alpha} = 0, \quad \delta_{\text{susy}}\Psi_{\mu\alpha} = 0, \tag{1.33}$$

which leads to generalized Killing spinor equations. As mentioned previously, existence of solutions for the above Killing spinor equations impose the constraints on the background manifold.

In most cases, *new-minimal* supergravity formalism [37, 38] is used in coupling the 4-dimensional supersymmetric field theories with $U(1)_R$ symmetry to the supergravity, see e.g. [35] for other kinds of coupling. In new-minimal supergravity, the gravity multiplet consists of the metric, gravitino $\Psi_{\mu\alpha}$, $U(1)_R$-gauge field A_μ and a two-form field $B_{\mu\nu}$, which couples to the $\mathcal{R}$-multiplet

$$\mathcal{R}_\mu = (T_{\mu\nu}, S_{\mu\alpha}, j_\mu^{(R)}, C_{\mu\nu}), \tag{1.34}$$

where $T_{\mu\nu}$ is the stress-energy tensor, $S_{\mu\alpha}$ is the supercurrent, $j_\mu^{(R)}$ is the R-current and $C_{\mu\nu}$ is a closed two-form current. The linearized couplings of $\mathcal{R}$-multiplet to the supergravity multiplet in new-minimal supergravity are given by

$$\triangle\mathcal{L} = -\frac{1}{2}\triangle g^{\mu\nu}T_{\mu\nu} + A_\mu^{(R)} j^{(R)\mu} + B^{\mu\nu}C_{\mu\nu}. \tag{1.35}$$

In new-minimal supergravity, the supersymmetry variation of the gravitino takes the form of

$$\delta\Psi_{\mu\alpha} = -2(\nabla_\mu - \imath A_\mu^{(R)})\zeta_\alpha - \imath V^\nu \sigma_{\mu\alpha\dot{\alpha}}\bar{\sigma}_\nu^{\dot{\alpha}\beta}\zeta_\beta, \tag{1.36a}$$

$$\delta\bar{\Psi}_\mu^{\dot{\alpha}} = -2(\nabla_\mu + \imath A_\mu^{(R)})\bar{\zeta}^{\dot{\alpha}} + \imath V^\nu \sigma_\mu^{\dot{\alpha}\alpha}\bar{\sigma}_{\nu\dot{\alpha}\beta}\bar{\zeta}^\beta, \tag{1.36b}$$

in Lorentzian signature. In Euclidean signature, we change the bar notations to the tilde notations in order to emphasize that the left-handed and right-handed spinors are no more related by complex conjugation but rather independent. We remark that the Killing spinor equations (1.36a) and (1.36b) can be also derived by imposing consistency of the supersymmetry algebra on the curved manifold, see [39] for details. The supersymmetric configuration for the bosonic background fields are then obtained by setting the above variations to zero. It turns out that the above Killing spinor equations have solution in some special cases. For instance, the first equation has solution only if the manifold admits an integrable complex structure and its metric is a compatible Hermitian metric, see [36, 26].

Each Killing spinor (i.e. the solution of either the equation (1.36a) or (1.36b)) correspond to a conserved supercharge. We denote the corresponding charges by Q and $\tilde{Q}$. From the local supergravity transformation rules, we obtain how the conserved supercharge acts on currents in the $\mathcal{R}$-multiplet, namely

$$\{Q, j_\mu^{(R)}\} = -\imath\zeta S_\mu, \tag{1.37a}$$

$$\{Q, S_{\alpha\mu}\} = 0, \tag{1.37b}$$

$$\{Q, \tilde{S}_\mu^{\dot{\alpha}}\} = 2\imath(\tilde{\sigma}^\nu\zeta)^{\dot{\alpha}}(T_{\mu\nu} + \frac{\imath}{4}\varepsilon_{\mu\nu\rho\lambda}C^{\rho\lambda} - \frac{\imath}{4}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho j^{(R)\lambda} - \frac{\imath}{2}\partial_\nu j_\mu^{(R)}), \tag{1.37c}$$

$$\{Q, T_{\mu\nu}\} = \frac{1}{2}\zeta\sigma_{\mu\rho}\partial^\rho S_\nu + \frac{1}{2}\zeta\sigma_{\nu\rho}\partial^\rho S_\mu, \tag{1.37d}$$

$$\{Q, C_{\mu\nu}\} = \frac{\imath}{2}\zeta\sigma_\nu\tilde{\sigma}_\rho\partial_\mu S^\rho - \frac{\imath}{2}\zeta\sigma_\mu\tilde{\sigma}_\rho\partial_\nu S^\rho, \tag{1.37e}$$

at the linearized level.

These transformation rules of the currents are followed by the algebra of the charges, since the conserved charge is obtained by integrating time-component of the corresponding current (in Minkowski signature) over the Cauchy surface. For instance, when both ζ and $\tilde{\zeta}$ exist, one can show that the rigid supersymmetry algebra on a field Φ with $U(1)_R$ charge r is given by

$$\{\delta_Q, \delta_{\tilde{Q}}\}\Phi = 2\imath\Big(\mathcal{L}_K - \imath r K^\mu (A_\mu^{(R)} + \frac{3}{2}V_\mu)\Big)\Phi, \quad \delta_Q^2\Phi = \delta_{\tilde{Q}}^2\Phi = 0, \tag{1.38}$$

where $K^\mu \equiv \zeta\sigma^\mu\tilde{\zeta}$ is a Killing vector.

Let us finish this subsection, presenting a simple supersymmetry field theory on curved manifolds. It is a theory of a free chiral multiplet $\Phi = (\phi, \psi_\alpha, F)$ of R-charge r and its conjugate anti-chiral multiplet $\tilde{\Phi} = (\tilde{\phi}, \tilde{\psi}_{\dot{\alpha}}, \tilde{F})$, the Lagrangian of which is given by [35]

$$\mathcal{L} = D^\mu\tilde{\phi}\partial_\mu\phi - \imath\tilde{\psi}\tilde{\sigma}^\mu D_\mu\psi - \tilde{F}F + V^\mu(\imath\tilde{\phi}\overleftrightarrow{D}_\mu\phi + \tilde{\psi}\tilde{\sigma}_\mu\psi) - r(\frac{1}{4}R - 3V^\mu V_\mu)\tilde{\phi}\phi, \tag{1.39}$$

where $D_\mu = \partial_\mu - \imath r A_\mu^{(R)}$. The Lagrangian is invariant (up to a total derivative) under the *modified* supersymmetry transformations,

$$\delta\phi = \sqrt{2}\zeta\psi, \tag{1.40a}$$
$$\delta\psi = \sqrt{2}\zeta F + \imath\sqrt{2}\sigma^\mu\tilde{\zeta}(\partial_\mu - \imath r A_\mu^{(R)})\phi, \tag{1.40b}$$
$$\delta F = \sqrt{2}\tilde{\zeta}\tilde{\sigma}^\mu\Big(\nabla_\mu - \imath(r-1)A_\mu^{(R)} - \frac{\imath}{2}V_\mu\Big)\psi, \tag{1.40c}$$

and similarly for the fields in the anti-chiral multiplet $\tilde{\Phi}$.

1.2 Organization

The thesis is organized as follows.

In chapter 2 we begin with a review of the STU model and the relevant truncations in two distinct duality frames in section 2.2, paying particular attention to the surface terms that arise from the dualization procedure. In section 2.3 we reparameterize the subtracted geometries in a way that simplifies the separation of the parameters into boundary conditions and dynamical modes that are allowed to vary independently in the variational problem. Moreover, by analyzing the asymptotic symmetries we identify the equivalence classes of boundary conditions in terms of which the variational problem must be formulated. Section 2.4 contains the main technical results of chapter 2. After arguing that the subtraction procedure, i.e. excising the asymptotically flat region in order to zoom into the conical asymptotics of the subtracted geometries, can be implemented in terms of covariant second class constraints on the phase space of the STU model, we derive the covariant boundary terms required in order to formulate the variational problem in terms of equivalence classes of boundary conditions under the asymptotic symmetries. The same boundary terms ensure that the on-shell action is free of long-distance divergences and allows us to construct finite conserved charges associated with any asymptotic Killing vector. In section 2.5 we evaluate explicitly these conserved charges for the subtracted geometries and demonstrate that they satisfy the Smarr formula and the first law of black hole thermodynamics. Section 2.6 discusses the uplift of the STU model to five dimensions and the Kaluza-Klein reduction of the resulting theory on the internal S^2 to three dimensions, which relates the subtracted geometries to the BTZ black hole. By keeping track of all surface terms arising in this sequence of uplifts and reductions, we provide a precise map between the thermodynamics of the subtracted geometries and that of the BTZ black hole. Some technical details are presented in two appendices.

In chapter 3 we consider the supersymmetric holographic renormalization for a generic 5 dimensional $\mathcal{N} = 2$ SUGRA. In section 3.2 we review the generic $\mathcal{N} = 2$ 5D gauged SUGRA action and SUSY variation of the fields. In section 3.3, we first present the radial Hamiltonian and other first class constraints. We then systematically carry out the procedure of holographic renormalization and obtain the flow equations. In section 3.4 we determine the divergent counterterms and the possible finite counterterms. In particular, the complete set of counterterms is obtained explicitly for a toy model. By means of these counterterms, in section 3.5 we obtain the holographic Ward identities and anomalies and show that the anomalies satisfy the Wess-Zumino consistency condition. We then define constraint functions on the phase space of local sources and operators using the Ward identities, and we show that the symmetry transformation of the sources and operators are simply described in terms of the Poisson bracket with the corresponding constraint functions. Finally, the anomaly-corrected supersymmetry algebra on supersymmetric backgrounds. In section 3.6 we show that consistency with SUSY requires that scalars and their SUSY-partner fields should satisfy the same boundary condition. In appendix 3.A, we describe our notations and present some useful identities. In appendix 3.B we carry out some preliminary steps necessary in order to obtain the radial Hamiltonian, including the ADM decomposition, the strong Fefferman-Graham (FG) gauge, and the generalized Penrose-Brown-Henneaux (gPBH) transformations. In appendix 3.C, we present the ADM decomposition of the radial Lagrangian part by part and in appendix 3.D we prove that the gPBH transformations of the operators can be obtained from the holographic Ward identities. In appendix 3.E we derive the anomaly-corrected SUSY algebra in an alternative way.

We end with some concluding remarks in chapter 4.

Chapter 2

Asymptotically conical backgrounds

2.1 Introduction

Although asymptotically flat or (anti) de Sitter (A)dS backgrounds have been studied extensively in (super)gravity and string theory, solutions that are asymptotically supported by matter fields have attracted attention relatively recently. Such backgrounds range from flux vacua in string theory to holographic backgrounds dual to supersymmetric quantum field theories (QFTs) [40, 41] and non-relativistic systems [42, 43, 44, 45], to name a few. Understanding the macroscopic properties of black holes with such exotic asymptotics is not only essential in order to address questions of stability and uniqueness, but also a first step towards their microscopic description.

Thermodynamic quantities such as the black hole entropy or temperature are not sensitive to the asymptotic structure of spacetime, since they are intrinsically connected with the horizon, but observables like conserved charges and the free energy depend heavily on the spacetime asymptotics. This is particularly important for backgrounds that are asymptotically supported by matter fields because the conserved pre-symplectic current that gives rise to conserved charges receives contributions from the matter fields [46, 47]. As a result, the usual methods for computing the conserved charges, such as Komar integrals, often do not work. Moreover, the large distance divergences that plague the free energy and the conserved charges cannot be remedied by techniques such as background subtraction, since it is not always easy, or even possible, to find a suitable background with the same asymptotics. The main motivation behind the work in this chapter is addressing these difficulties using a general and systematic approach that does not rely on the specific details of the theory or its asymptotic solutions, even though we will demonstrate the general methodology using a concrete example.

The backgrounds we are going to consider were originally obtained from generic multi-charge asymptotically flat black holes in four [48, 49, 50, 51, 52] and five dimensions [53] through a procedure dubbed 'subtraction' [54, 55]. The subtraction procedure consists in excising the asymptotic flat region away from the black hole by modifying the warp factor of the solution, in such a way that the scalar wave equation acquires a manifest $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ conformal symmetry. This leaves the near-horizon region intact, but the resulting background is asymptotically conical [56]. Moreover, it is not necessarily a solution of the original equations of motion.

It was later realized that the subtracted geometries are solutions [55, 56] of the STU model in four dimensions, an $\mathcal{N}=2$ supergravity theory coupled to three vector multiplets [57]. The STU model can be obtained from a T^2 reduction of minimal supergravity coupled to a tensor multiplet in six dimensions. In particular, the bosonic action is obtained from the reduction of 6-dimensional bosonic string theory

$$2\kappa_6^2\mathcal{L}_6 = R\star 1 - \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{-\sqrt{2}\phi}\star F_{(3)}\wedge F_{(3)}, \tag{2.1.1}$$

where $F_{(3)}=dB_{(2)}$, and then dualizing the 4-dimensional 2-form to an axion. The resulting 4-dimensional theory has an $O(2,2)\simeq SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ global symmetry, which is enhanced to $SL(2,\mathbb{R})^3$ on-shell, when electric-magnetic S-duality transformations are included [50].

In [56] it was shown that subtracted geometries correspond to a scaling limit of the general non-extremal 4-charge rotating asymptotically flat black hole solutions of the STU model [49, 50], with all four $U(1)$ gauge fields electrically sourced. In [58], starting with the same non-extremal asymptotically flat black holes, but in a frame where only one gauge field is electrically sourced while the remaining three are magnetically sourced, it was shown that the subtracted geometries can also be obtained by Harrison transformations, a solution generating technique exploiting the hidden $SO(4,4)$ symmetry of the STU model upon reduction on a Killing vector [50]. General interpolating solutions between asymptotically flat black holes in four and five dimensions and their subtracted geometry counterparts were subsequently constructed in [59] by extending these techniques.

When uplifted to five dimensions the subtracted geometries become a BTZ$\times S^2$ background, with the 2-sphere fibered over the BTZ black hole [55, 60], which makes manifest the origin of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ symmetry of the wave equation. Using this connection with the BTZ black hole, [61] showed that the parameters that need to be tuned in order to interpolate between the asymptotically flat black holes and the subtracted geometries correspond to the couplings of irrelevant scalar operators in the two-dimensional conformal field theory (CFT) at the boundary of the asymptotically AdS$_3$ factor of the five-dimensional geometry.

The thermodynamics of asymptotically conical black holes were first studied in [62]. In the present work we emphasize the importance of the variational problem in black hole thermodynamics. Using lessons from asymptotically AdS backgrounds [20], we show that a well posed variational problem automatically ensures that all thermodynamic observables are finite and satisfy the first law of thermodynamics. This relegates the problem of seeking the correct definition of conserved charges in backgrounds with new exotic asymptotics to that of properly formulating the variational principle, which in non-compact spaces can be achieved through the following algorithmic procedure:

i) Firstly, the integration constants parameterizing solutions of the equations of motion must be separated into 'normalizable' and 'non-normalizable' modes. A complete set of modes parameterizes the symplectic space of asymptotic solutions. Normalizable modes are free to vary in the variational problem, while non-normalizable modes should be kept fixed.

ii) Secondly, the non-normalizable modes are not determined uniquely, but only up to transformations induced by the local symmetries of the bulk theory, such as bulk diffeomorphisms and gauge transformations. Hence, what should be kept fixed in the variational problem is in fact the equivalence class of non-normalizable modes under such transformations [20].

iii) Formulating the variational problem in terms of equivalence classes of non-normalizable modes requires the addition of a covariant boundary term, S_{ct}, to the bulk action, which can be determined by solving asymptotically the radial Hamilton-Jacobi equation [13]. Since radial translations are part of the local bulk symmetries, formulating the variational problem in terms of equivalence classes ensures that the total action is independent of the radial coordinate, and hence free of long-distance divergences.

iv) Finally, besides determining the boundary term S_{ct}, the first-class constraints of the radial Hamiltonian formulation of the bulk dynamics also lead to conserved charges associated with Killing vectors. The canonical transformation generated by the boundary term S_{ct} 'renormalizes' the phase space variables such that these charges are independent of the radial cutoff, and hence finite. These charges automatically satisfy the first law of thermodynamics, with all normalizable modes treated as free parameters and the non-normalizable modes allowed to vary only within the equivalence class under local bulk symmetries.

Although this algorithm originates in the AdS/CFT correspondence and holographic renormalization [3, 4, 5, 6, 7, 8, 9, 10, 11, 12], it is in principle applicable to any gravity theory, including the subtracted geometries we consider here. However, in this case we find that there are two additional complications, both of which have been encountered before in a holographic context. The first complication arises from the fact that subtracted geometries are obtained as solutions of the STU model provided certain conditions are imposed on the non-normalizable modes. For example, it was shown in [61] that certain modes (interpreted as couplings of irrelevant scalar operators in the dual CFT_2) need to be turned off in the asymptotically flat solutions in order to obtain the subtracted geometries. We show that all conditions among the non-normalizable modes required to obtain the subtracted geometries can be expressed as covariant second class constraints on the phase space of the theory. This is directly analogous to the way Lifshitz asymptotics were imposed in [63]. The presence of asymptotic second class constraints in these backgrounds is crucial for being able to solve the radial Hamilton-Jacobi equation and to obtain the necessary boundary term S_{ct}.

The second complication concerns specifically the duality frame in which the STU model was presented in e.g. [56, 62]. In this particular frame, one of the

$U(1)$ gauge fields supporting the subtracted geometries asymptotically dominates the stress tensor, which is reminiscent of fields in asymptotically AdS space that are holographically dual to an irrelevant operator. The variational problem for such fields is known to involve additional subtleties [64], which we also encounter in this specific duality frame of the STU model. We address these subtleties by first formulating the variational problem in a different duality frame and then dualizing to the frame where these complications arise. Remarkably, the form of the boundary term that we obtain through this procedure is exactly of the same form as the boundary term for fields dual to irrelevant operators in asymptotically AdS backgrounds.

It should be emphasized that our analysis of the variational problem and the derivation of the necessary boundary terms does not assume or imply any holographic duality for asymptotically conical black holes in four dimensions. Nevertheless, subtracted geometries possess a hidden (spontaneously broken) $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry which can be traced to the fact that they uplift to an S^2 fibered over a three-dimensional BTZ black hole in five dimensions [55, 56]. The most obvious candidate for a holographic dual, therefore, would be a two-dimensional CFT at the boundary of the asymptotically AdS_3 factor of the 5D uplift [61]. However, if a holographic dual to asymptotically conical backgrounds in four dimensions exists, it is likely that its Hilbert space overlaps with that of the two-dimensional CFT only partially. In particular, we show that the variational problems in four and five dimensions are not fully compatible in the sense that not all asymptotically conical backgrounds uplift to asymptotically $\mathrm{AdS}_3 \times S^2$ solutions in five dimensions, and conversely, not all asymptotically $\mathrm{AdS}_3 \times S^2$ backgrounds reduce to solutions of the STU model. This is because turning on generic sources on the boundary of AdS_3 leads to Kaluza-Klein modes in four dimensions that are not captured by the STU model, while certain modes that are free in the four-dimensional variational problem must be frozen or quantized in order for the solutions to be uplifted to 5D. Although we do not pursue a holographic understanding of the subtracted geometries in the present work, elucidating the relation between the four and five-dimensional variational problems allows us to find a precise map between the thermodynamics of asymptotically conical black holes in four dimensions and that of the BTZ black hole.

2.2 The STU model and duality frames

In this section we review the bosonic sector of the 2-charge truncation of the STU model that is relevant for describing the subtracted geometries. We will do so in the duality frame discussed in [56], where both charges are electric, as well as in the one used in [58], where there is one electric and one magnetic charge. We will refer to these frames as 'electric' and 'magnetic' respectively. As it will become clear from the subsequent analysis, in order to compare the thermodynamics in the two frames, it is necessary to keep track of surface terms introduced by the duality transformations.

2.2.1 Magnetic frame

The bosonic Lagrangian of the STU model in the duality frame used in [58] is given by

$$\begin{aligned}2\kappa_4^2\mathcal{L}_4 =& R\star 1 - \frac{1}{2}\star d\eta_a\wedge d\eta_a - \frac{1}{2}e^{2\eta_a}\star d\chi^a\wedge d\chi^a \\ &- \frac{1}{2}e^{-\eta_0}\star F^0\wedge F^0 - \frac{1}{2}e^{2\eta_a-\eta_0}\star(F^a+\chi^aF^0)\wedge(F^a+\chi^aF^0) \\ &+ \frac{1}{2}C_{abc}\chi^aF^b\wedge F^c + \frac{1}{2}C_{abc}\chi^a\chi^bF^0\wedge F^c + \frac{1}{6}C_{abc}\chi^a\chi^b\chi^cF^0\wedge F^0, \end{aligned} \quad (2.2.1)$$

where η_a $(a=1,2,3)$ are dilaton fields and $\eta_0=\sum_{a=1}^3\eta_a$. The symbol C_{abc} is pairwise symmetric with $C_{123}=1$ and zero otherwise. The Kaluza-Klein ansatz for obtaining this action from the 6-dimensional action (2.1.1) is given explicitly in [58]. This frame possesses an explicit triality symmetry, exchanging the three gauge fields A^a, the three dilatons η^a and the three axions χ^a. In this frame, the subtracted geometries source all three gauge fields A^a magnetically, while A^0 is electrically sourced. Moreover, holographic renormalization turns out to be much more straightforward in this frame compared with the electric frame.

In order to describe the subtracted geometries it suffices to consider a truncation of (2.2.1), corresponding to setting $\eta_1=\eta_2=\eta_3\equiv\eta$, $\chi_1=\chi_2=\chi_3\equiv\chi$, and $A^1=A^2=A^3\equiv A$. The resulting action can be written in the σ-model form

$$\begin{aligned}S_4 = \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4\mathbf{x}\sqrt{-g}\Big(&R[g] - \frac{1}{2}\mathcal{G}_{IJ}\partial_\mu\varphi^I\partial^\mu\varphi^J - \mathcal{Z}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} \\ &- \mathcal{R}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\Big) + S_{\mathrm{GH}},\end{aligned} \quad (2.2.2)$$

where

$$S_{\mathrm{GH}} = \frac{1}{2\kappa_4^2}\int_{\partial\mathcal{M}} \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\,2K, \quad (2.2.3)$$

is the standard Gibbons-Hawking [33] term and we have defined the doublets

$$\varphi^I = \begin{pmatrix}\eta\\ \chi\end{pmatrix}, \qquad A^\Lambda = \begin{pmatrix}A^0\\ A\end{pmatrix}, \qquad I=1,2, \quad \Lambda=1,2, \quad (2.2.4)$$

as well as the 2×2 matrices

$$\mathcal{G}_{IJ} = \begin{pmatrix}3 & 0\\ 0 & 3e^{2\eta}\end{pmatrix}, \ \mathcal{Z}_{\Lambda\Sigma} = \frac{1}{4}\begin{pmatrix}e^{-3\eta}+3e^{-\eta}\chi^2 & 3e^{-\eta}\chi\\ 3e^{-\eta}\chi & 3e^{-\eta}\end{pmatrix}, \ \mathcal{R}_{\Lambda\Sigma} = \frac{1}{4}\begin{pmatrix}\chi^3 & \frac{3}{2}\chi^2\\ \frac{3}{2}\chi^2 & 3\chi\end{pmatrix}. \quad (2.2.5)$$

As usual, $\epsilon_{\mu\nu\rho\sigma}=\sqrt{-g}\;\varepsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric Levi-Civita tensor, where $\varepsilon_{\mu\nu\rho\sigma}=\pm 1$ is the Levi-Civita symbol. Throughout this chapter we choose the orientation in $\mathcal{M}$ so that $\varepsilon_{rt\theta\phi}=1$. We note in passing that the Lagrangian (2.2.2) is invariant under the global symmetry transformation

$$e^\eta\to\mu^2e^\eta, \quad \chi\to\mu^{-2}\chi, \quad A^0\to\mu^3A^0, \quad A\to\mu A, \quad ds^2\to ds^2, \quad (2.2.6)$$

where μ is an arbitrary non-zero constant parameter.

2.2.2 Electric frame

The STU model in the duality frame in which the subtracted geometries are presented in [56] can be obtained from (2.2.2) by dualizing the gauge field A.[1] Following [52], we dualize A by introducing a Lagrange multiplier, $\widetilde{A}$, imposing the Bianchi identity $dF = 0$, and consider the action

$$\widetilde{S}_4 = S_4 + \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} 3\widetilde{A}\wedge dF = S_4 + \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} 3\widetilde{F}\wedge F - \frac{3}{2\kappa_4^2}\int_{\partial\mathcal{M}} \widetilde{A}\wedge F + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F. \tag{2.2.7}$$

The factor of 3 is a convention, corresponding to a choice of normalization for $\widetilde{A}$, chosen such that the resulting electric frame model agrees with the one in [56]. The term added to S_4 vanishes on-shell and so the on-shell values of $\widetilde{S}_4$ and S_4 coincide. The total derivative term that leads to surface contributions from the boundary, $\partial\mathcal{M}$, and the outer horizon, $\mathcal{H}_+$, is crucial for comparing the physics in the electric and magnetic frames. As we will discuss later on, this surface term is also the reason behind the subtleties of holographic renormalization in the electric frame.

Integrating out F in (2.2.7) we obtain

$$F_{\mu\nu} = -(4\chi^2 + e^{-2\eta})^{-1}\left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, e^{-\eta}(\widetilde{F} - \chi^2 F^0)^{\rho\sigma} + 2\chi\widetilde{F}_{\mu\nu} + \chi(2\chi^2 + e^{-2\eta})F^0_{\mu\nu}\right). \tag{2.2.8}$$

Inserting this expression for F in (2.2.7) leads to the electric frame action

$$\begin{aligned}
\widetilde{S}_4 = \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} \Big(& R\star\mathbf{1} - \frac{3}{2}\star d\eta\wedge d\eta - \frac{3}{2}e^{2\eta}\star d\chi\wedge d\chi - \frac{1}{2}e^{-3\eta}\star F^0\wedge F^0 \\
& -\frac{3}{2}\frac{e^{-\eta}}{(4\chi^2 + e^{-2\eta})}\star(\widetilde{F} - \chi^2 F^0)\wedge(\widetilde{F} - \chi^2 F^0) \\
& -\frac{\chi}{(4\chi^2 + e^{-2\eta})}\left[3\widetilde{F}\wedge\widetilde{F} + 3(2\chi^2 + e^{-2\eta})\widetilde{F}\wedge F^0 - \chi^2(\chi^2 + e^{-2\eta})F^0\wedge F^0\right]\Big) \\
& - \frac{3}{2\kappa_4^2}\int_{\partial\mathcal{M}} \widetilde{A}\wedge F + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F + S_{\rm GH}.
\end{aligned} \tag{2.2.9}$$

As in the magnetic frame, it is convenient to write the bulk part of the action in σ-model form as

$$\begin{aligned}
\widetilde{S}_4 = \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4\mathbf{x}\sqrt{-g}\,&\left(R[g] - \frac{1}{2}\mathcal{G}_{IJ}\partial_\mu\varphi^I\partial^\mu\varphi^J - \widetilde{\mathcal{Z}}_{\Lambda\Sigma}\widetilde{F}^\Lambda_{\mu\nu}\widetilde{F}^{\Sigma\mu\nu} - \widetilde{\mathcal{R}}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}\widetilde{F}^\Lambda_{\mu\nu}\widetilde{F}^\Sigma_{\rho\sigma}\right) \\
& - \frac{3}{2\kappa_4^2}\int_{\partial\mathcal{M}} \widetilde{A}\wedge F + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F + S_{\rm GH},
\end{aligned} \tag{2.2.10}$$

[1] Notice that the duality frame in eq. (1) of [56] is not the one in which the solutions are given in that paper. As mentioned above eq. (3), two of the gauge fields in (1) are dualized in the solutions discussed. The corresponding action, which was not given explicitly in [56], can be obtained from the magnetic frame action (2.2.2) here by first implementing the field redefinitions $A \to -A$ and $\chi \to -\chi$ and then dualizing A as we describe here. The resulting action differs by a few signs from our electric frame action (2.2.10).

where we have defined

$$\widetilde{A}^\Lambda = \begin{pmatrix} A^0 \\ \widetilde{A} \end{pmatrix}, \qquad \widetilde{\mathcal{Z}}_{\Lambda\Sigma} = \frac{1}{4}\begin{pmatrix} e^{-3\eta} + \frac{3e^{-\eta}\chi^4}{4\chi^2+e^{-2\eta}} & -\frac{3e^{-\eta}\chi^2}{4\chi^2+e^{-2\eta}} \\ -\frac{3e^{-\eta}\chi^2}{4\chi^2+e^{-2\eta}} & \frac{3e^{-\eta}}{4\chi^2+e^{-2\eta}} \end{pmatrix},$$
$$\widetilde{\mathcal{R}}_{\Lambda\Sigma} = \frac{\chi}{4(4\chi^2+e^{-2\eta})}\begin{pmatrix} \chi^2(\chi^2+e^{-2\eta}) & -\frac{3}{2}(2\chi^2+e^{-2\eta}) \\ -\frac{3}{2}(2\chi^2+e^{-2\eta}) & -3 \end{pmatrix}. \tag{2.2.11}$$

As in the magnetic frame, the action (2.2.10) is invariant under the global symmetry transformation

$$e^\eta \to \mu^2 e^\eta, \quad \chi \to \mu^{-2}\chi, \quad A^0 \to \mu^3 A^0, \quad \widetilde{A} \to \mu^{-1}\widetilde{A}, \quad ds^2 \to ds^2. \tag{2.2.12}$$

2.3 Asymptotically conical backgrounds

The general rotating subtracted geometry backgrounds are solutions of the equations of motion following from the action (2.2.2) or (2.2.10) and take the form [56, 58][2]

$$\begin{aligned}
ds^2 &= \frac{\sqrt{\Delta}}{X}d\bar{r}^2 - \frac{G}{\sqrt{\Delta}}(d\bar{t}+\mathcal{A})^2 + \sqrt{\Delta}\left(d\theta^2 + \frac{X}{G}\sin^2\theta d\bar{\phi}^2\right),\\
e^\eta &= \frac{(2m)^2}{\sqrt{\Delta}}, \qquad \chi = \frac{a\,(\Pi_c-\Pi_s)}{2m}\cos\theta,\\
A^0 &= \frac{(2m)^4 a\,(\Pi_c-\Pi_s)}{\Delta}\sin^2\theta d\bar{\phi} + \frac{(2ma)^2\cos^2\theta\,(\Pi_c-\Pi_s)^2 + (2m)^4\Pi_c\Pi_s}{(\Pi_c^2-\Pi_s^2)\,\Delta}d\bar{t},\\
A &= \frac{2m\cos\theta}{\Delta}\left(\left[\Delta - (2ma)^2(\Pi_c-\Pi_s)^2\sin^2\theta\right]d\bar{\phi} - 2ma\,(2m\Pi_s + \bar{r}(\Pi_c-\Pi_s))\,d\bar{t}\right),\\
\widetilde{A} &= -\frac{1}{2m}\left(\bar{r} - m - \frac{(2ma)^2(\Pi_c-\Pi_s)}{(2m)^3(\Pi_c+\Pi_s)}\right)d\bar{t} + \frac{(2ma)^2(\Pi_c-\Pi_s)[2m\Pi_s+\bar{r}(\Pi_c-\Pi_s)]\cos^2\theta}{2m\Delta}d\bar{t}\\
&\qquad + a(\Pi_c-\Pi_s)\sin^2\theta\left(1 + \frac{(2ma)^2(\Pi_c-\Pi_s)^2\cos^2\theta}{\Delta}\right)d\bar{\phi},
\end{aligned} \tag{2.3.1}$$

where

$$\begin{aligned}
X &= \bar{r}^2 - 2m\bar{r} + a^2, \qquad G = X - a^2\sin^2\theta, \qquad \mathcal{A} = \frac{2ma}{G}\left((\Pi_c-\Pi_s)\bar{r} + 2m\Pi_s\right)\sin^2\theta d\bar{\phi},\\
\Delta &= (2m)^3(\Pi_c^2-\Pi_s^2)\bar{r} + (2m)^4\Pi_s^2 - (2ma)^2(\Pi_c-\Pi_s)^2\cos^2\theta,
\end{aligned} \tag{2.3.2}$$

and Π_c, Π_s, a and m are parameters of the solution.

In order to study the thermodynamics of these backgrounds it is necessary to identify which parameters are fixed by the boundary conditions in the variational

[2] In order to compare this background with the expressions given in eqs. (24) and (25) of [56], one should take into account the field redefinition $A \to -A$, $\chi \to -\chi$, before the dualization of A, as mentioned in footnote 1, and add a constant pure gauge term. Moreover, there is a typo in eq. (25) of [56]: the term $2m\Pi_s^2\cos^2\theta d\bar{t}$ should be replaced by $2m\Pi_s(\Pi_s-\Pi_c)\cos^2\theta d\bar{t}$.

problem. A full analysis of the variational problem for the actions (2.2.2) or (2.2.10) requires knowledge of the general asymptotic solutions and is beyond the scope of the present chapter. However, we can consider the variational problem within the class of stationary solutions (2.3.1). To this end, it is convenient to reparameterize these backgrounds by means of a suitable coordinate transformation, accompanied by a relabeling of the free parameters. In particular, we introduce the new coordinates

$$\begin{aligned}
&\ell^4 r = (2m)^3(\Pi_c^2-\Pi_s^2)\bar{r} + (2m)^4\Pi_s^2 - (2ma)^2(\Pi_c-\Pi_s)^2,\\
&\frac{k}{\ell^3}t = \frac{1}{(2m)^3(\Pi_c^2-\Pi_s^2)}\bar{t}, \qquad \phi = \bar{\phi} - \frac{2ma(\Pi_c-\Pi_s)}{(2m)^3(\Pi_c^2-\Pi_s^2)}\bar{t},
\end{aligned} \tag{2.3.3}$$

where ℓ and k are additional non-zero parameters, whose role will become clear shortly. Moreover, we define the new parameters

$$\begin{aligned}
&\ell^4 r_\pm = (2m)^3 m(\Pi_c^2+\Pi_s^2) - (2ma)^2(\Pi_c-\Pi_s)^2 \pm \sqrt{m^2-a^2}(2m)^3(\Pi_c^2-\Pi_s^2),\\
&\ell^3\omega = 2ma(\Pi_c-\Pi_s), \qquad B = 2m,
\end{aligned} \tag{2.3.4}$$

which can be inverted in order to express the old parameters in terms of the new ones, namely

$$\begin{aligned}
\Pi_{c,s} &= \frac{\ell^2}{B^2}\left(\frac{1}{2}\left(\sqrt{r_+}+\sqrt{r_-}\right) \pm \sqrt{\ell^2\omega^2 + \frac{1}{4}\left(\sqrt{r_+}-\sqrt{r_-}\right)^2}\right),\\
a &= \frac{B\ell\omega}{2\sqrt{\ell^2\omega^2+\frac{1}{4}\left(\sqrt{r_+}-\sqrt{r_-}\right)^2}}, \qquad m = B/2.
\end{aligned} \tag{2.3.5}$$

Rewriting the background (2.3.1) in terms of the new coordinates and parameters we obtain[3]

$$\begin{aligned}
e^\eta &= \frac{B^2/\ell^2}{\sqrt{r+\ell^2\omega^2\sin^2\theta}}, \qquad \chi = \frac{\ell^3\omega}{B^2}\cos\theta,\\
A^0 &= \frac{B^3/\ell^3}{r+\ell^2\omega^2\sin^2\theta}\left(\sqrt{r_+r_-}\,kdt + \ell^2\omega\sin^2\theta d\phi\right),\\
A &= \frac{B\cos\theta}{r+\ell^2\omega^2\sin^2\theta}\left(-\omega\sqrt{r_+r_-}\,kdt + rd\phi\right),\\
\widetilde{A} &= -\frac{\ell}{B}\left(r-\frac{1}{2}(r_++r_-)\right)kdt + \frac{\omega\ell^3}{B}\cos^2\theta\left(\frac{\omega\sqrt{r_+r_-}\,kdt - rd\phi}{r+\omega^2\ell^2\sin^2\theta}\right) + \frac{\omega\ell^3}{B}d\phi,\\
ds^2 &= \sqrt{r+\ell^2\omega^2\sin^2\theta}\left(\frac{\ell^2dr^2}{(r-r_-)(r-r_+)} - \frac{(r-r_-)(r-r_+)}{r}k^2dt^2 + \ell^2d\theta^2\right)\\
&\quad + \frac{\ell^2 r\sin^2\theta}{\sqrt{r+\ell^2\omega^2\sin^2\theta}}\left(d\phi - \frac{\omega\sqrt{r_+r_-}}{r}kdt\right)^2.
\end{aligned} \tag{2.3.6}$$

[3] Since these solutions carry non-zero magnetic charge, the gauge potential A must be defined in the north ($\theta < \pi/2$) and south hemispheres respectively as [65, 66], $A_{\text{north}} = A - Bd\phi$ and $A_{\text{south}} = A + Bd\phi$, where A is the expression given in (2.3.6).

Several comments are in order here. Firstly, the two parameters $r_\pm$ are the locations of the outer and inner horizons respectively, and clearly correspond to normalizable perturbations. A straightforward calculation shows that ω is also a normalizable mode. We will explicitly confirm this later on by showing that the long-distance divergences of the on-shell action are independent of ω. Setting the normalizable parameters to zero we arrive at the background

$$e^\eta = \frac{B^2/\ell^2}{\sqrt{r}}, \qquad \chi = 0, \qquad A^0 = 0, \qquad A = B\cos\theta d\phi, \qquad \widetilde{A} = -\frac{\ell}{B} rkdt,$$
$$ds^2 = \sqrt{r}\left(\ell^2 \frac{dr^2}{r^2} - rk^2 dt^2 + \ell^2 d\theta^2 + \ell^2 \sin^2\theta d\phi^2\right), \tag{2.3.7}$$

which we shall consider as the vacuum solution. The fact that the background (2.3.7) is singular does not pose any difficulty since it should only be viewed as an asymptotic solution that helps us to properly formulate the variational problem. Changing the radial coordinate to $\varrho = \ell r^{1/4}$, the vacuum metric becomes

$$ds^2 = 4^2 d\varrho^2 - \left(\frac{\varrho}{\ell}\right)^6 k^2 dt^2 + \varrho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.3.8}$$

which is a special case of the conical metrics discussed in [56]. Different conical geometries are supported by different matter fields. Although we focus on the specific conical backgrounds obtained as solutions of the STU model here, we expect that our analysis, modified accordingly for the different matter sectors, applies to general asymptotically conical backgrounds.

The asymptotic structure of (stationary) conical backgrounds is parameterized by the three non-zero constants B, ℓ and k. In the most restricted version of the variational problem, these three parameters should be kept fixed. However, there is a 2-parameter family of deformations of these boundary data still leading to a well posed variational problem, as we now explain. The first deformation corresponds to the transformation of the boundary data induced by reparameterizations of the radial coordinate. Namely, under the bulk diffeomorphism

$$r \to \lambda^{-4} r, \quad \lambda > 0, \tag{2.3.9}$$

the boundary parameters transform as

$$k \to \lambda^3 k, \quad \ell \to \lambda\ell, \quad B \to B. \tag{2.3.10}$$

This transformation is a direct analogue of the so called Penrose-Brown-Henneaux (PBH) diffeomorphisms in asymptotically AdS backgrounds [67], which induce a Weyl transformation on the boundary sources. The PBH diffeomorphisms imply that the bulk fields do not induce boundary fields, but only a *conformal structure*, that is boundary fields up to Weyl transformations [11]. This dictates that the variational problem must be formulated in terms of conformal classes rather than representatives of the conformal class [20]. In the case of subtracted geometries, variations of the boundary parameters of the form

$$\delta_1 k = 3\epsilon_1 k, \quad \delta_1\ell = \epsilon_1\ell, \quad \delta_1 B = 0, \tag{2.3.11}$$

correspond to motion within the equivalence class (anisotropic conformal class) defined by the transformation (2.3.10), and therefore lead to a well posed variational problem.

A second deformation of the boundary data that leads to a well posed variational problem is

$$\delta_2 k = 0, \quad \delta_2 \ell = \epsilon_2 \ell, \quad \delta_2 B = \epsilon_2 B. \tag{2.3.12}$$

To understand this transformation, one must realize that the parameters B and ℓ do not correspond to independent modes, but rather only the ratio B/ℓ, which can be identified with the source of the dilaton η. In particular, keeping B/ℓ fixed ensures that the variational problem is the same in all frames of the form

$$ds_\alpha^2 = e^{\alpha\eta} ds^2, \tag{2.3.13}$$

for some α, which will be important for the uplift of the conical backgrounds to five dimensions. The significance of the parameter B is twofold. It corresponds to the background magnetic field in the magnetic frame and variations of B are equivalent to the global symmetry transformation (2.2.6) or (2.2.12) of the bulk Lagrangian. Moreover, as we will discuss in the next section, it enters in the covariant asymptotic second class constraints imposing conical boundary conditions. The transformation (2.3.12) is a variation of B combined with a bulk diffeomorphism in order to keep the modes k and B/ℓ fixed. The relevant bulk diffeomorphism is a rescaling of the radial coordinate of the form (2.3.9), accompanied by a rescaling $t \to \lambda^3 t$ of the time coordinate.

2.4 Boundary counterterms and conserved charges

The first law of black hole thermodynamics is directly related to the variational problem and the boundary conditions imposed on the solutions of the equations of motion. As we briefly reviewed in the previous section, in non-compact spaces, where the geodesic distance to the boundary is infinite, the bulk fields induce only an equivalence class of boundary fields, which implies that the variational problem must be formulated in terms of equivalence classes of boundary data, with different elements of the equivalence class related by radial reparameterizations. In order to formulate the variational problem in terms of equivalence classes of boundary data one must add a specific boundary term, $S_{\text{ct}} = -\mathcal{S}_o$, to the bulk action, where $\mathcal{S}_o$ is a certain asymptotic solution of the radial Hamilton-Jacobi equation, which we discuss in appendix 2.A. For asymptotically locally AdS spacetimes, this boundary term is identical to the boundary counterterms derived by the method of holographic renormalization [3, 4, 5, 6, 7, 8, 9, 10, 11, 12], which are designed to render the on-shell action free of large-distance divergences. In particular, demanding that the variational problem be formulated in terms of equivalence classes (conformal classes in the case of AdS) of boundary data cures all pathologies related to the long-distance divergences of asymptotically AdS spacetimes, leading to a finite on-shell action and conserved charges that obey the first law and the Smarr formula of black hole thermodynamics [20]. This observation, however, goes beyond asymptotically AdS backgrounds. Provided a suitable asymptotic solution $\mathcal{S}_o$ of the radial Hamilton-Jacobi equation can be found, one can perform a canonical transformation of the form

$$(\phi^\alpha,\ \pi_\beta) \to \left(\phi^\alpha,\ \Pi_\beta = \pi_\beta - \frac{\delta \mathcal{S}_o}{\delta \phi^\beta}\right), \tag{2.4.1}$$

such that the product $\phi^\alpha \Pi_\alpha$ depends only on the equivalence class of boundary data. This in turn implies that formulating the variational problem in terms of the symplectic variables $(\phi^\alpha,\ \Pi_\beta)$ ensures that it be well posed [13].

This analysis of the variational problem presumes that the induced fields ϕ^α on a slice of constant radial coordinate are independent variables, or equivalently, the boundary data induced from the bulk fields are unconstrained. However, this may not be the case. Imposing conditions on the boundary data leads to different asymptotic structures and accordingly different boundary conditions. A typical example is the case of asymptotically Lifshitz backgrounds [44, 45] (see [68] for a recent review), where non-relativistic boundary conditions are imposed on a fully diffeomorphic bulk theory [63]. The conditions imposed on the boundary data correspond to asymptotic second class constraints of the form

$$\mathcal{C}\left(\phi^\alpha\right) \approx 0, \tag{2.4.2}$$

in the radial Hamiltonian formulation of the bulk dynamics. As a result, the asymptotic solution $\mathcal{S}_o$ of the Hamilton-Jacobi equation that should be added as a boundary term may not be unique anymore, since it can be written in different ways, all related to each other by means of the constraints (2.4.2). It should be emphasized that the potential ambiguity in the boundary counterterms arising due to the presence of asymptotic second class constraints is not related to the ambiguity that is commonly referred to as 'scheme dependence' in the context of the AdS/CFT correspondence [9]. The latter is an ambiguity in the *finite* part of the solution $\mathcal{S}_o$, and it exists independently of the presence of second class constraints. On the contrary, the potential ambiguity resulting from the presence of second class constraints may affect both the divergent and finite parts of $\mathcal{S}_o$. As we will see below, in order to obtain asymptotically conical backgrounds from the STU model one must impose certain asymptotic second class constraints, which play a crucial role in the understanding of the variational problem. A subset of these second class constraints corresponds to turning off the modes that, if non-zero, would lead to an asymptotically Minkowski background. As such, the asymptotic second class constraints constitute a covariant way of turning off the couplings of the irrelevant scalar operators identified in [61], or implementing the original subtraction procedure.

After covariantizing the definition of asymptotically conical backgrounds in the STU model by introducing a set of covariant second class constraints, we will determine the boundary terms required in order to render the variational problem well posed, both in the magnetic and electric frames. This will allow us to define finite conserved charges associated with asymptotic Killing vectors, which will be used in section 2.5 in order to prove the first law of thermodynamics for asymptotically conical black holes. We will first consider the magnetic frame because the electric frame presents additional subtleties, which can be easily addressed once the variational problem in the magnetic frame is understood. Since the boundary term we must determine in order to render the variational problem well posed is a solution of the radial Hamilton-Jacobi equation, the analysis in this section relies heavily on the radial Hamiltonian formulation of the bulk dynamics discussed in detail in appendix 2.A. In particular, we will work in the coordinate system (2.A.1) and gauge-fix the Lagrange multipliers as

$$N=\left(r+\ell^2\omega^2\sin^2\theta\right)^{1/4},\quad N_i=0,\quad a^\Lambda=\widetilde{a}^\Lambda=0. \tag{2.4.3}$$

2.4.1 Magnetic frame

Even though we have not determined the most general asymptotic solutions of the equations of motion compatible with conical boundary conditions in the present work, we do need a covariant definition of asymptotically conical backgrounds in order to determine the appropriate boundary term that renders the variational problem well posed. It turns out that the stationary solutions (2.3.6) are sufficiently general in order to provide a minimal set of covariant second class constraints, which can be deduced from the asymptotic form (2.3.7) of conical backgrounds. In the magnetic frame they take the form

$$F_{ij}F^{ij}\approx\frac{2}{B^2}e^{2\eta},\quad R_{ij}[\gamma]\approx e^{-\eta}F_{ik}F_j{}^k,\quad 2R_{ij}[\gamma]R^{ij}[\gamma]\approx R[\gamma]^2, \tag{2.4.4}$$

where the $\approx$ symbol indicates that these constraints should be imposed only asymptotically, i.e. they should be understood as conditions on non-normalizable modes only. Qualitatively, these covariant and gauge-invariant second class constraints play exactly the same role as the second class constraints imposing Lifshitz asymptotics [63].

The fact that we have been able to determine the constraints (2.4.4) in covariant form ensures that the boundary term we will compute below renders the variational problem well posed for general asymptotically conical backgrounds – not merely the stationary solutions (2.3.6). Moreover, this boundary term can be used together with the first order equations (2.A.9) to obtain the general asymptotic form of conical backgrounds, but we leave this analysis for future work.

Boundary counterterms

The general procedure for determining the solution $\mathcal{S}_o$ of the Hamilton-Jacobi equation, and hence the boundary counterterms, is the following. Given the leading asymptotic form of the background, the first order equations (2.A.9) are integrated asymptotically in order to obtain the leading asymptotic form of $\mathcal{S}_o$. Inserting this leading solution in the Hamilton-Jacobi equation, one sets up a recursive procedure that systematically determines all subleading corrections that contribute to the long-distance divergences. Luckily, for asymptotically conical backgrounds in four dimensions, integrating the first order equations (2.A.9) using the leading asymptotic form of the background determines all divergent terms, and so there is no need for solving the Hamilton-Jacobi recursively.

In order to integrate the first order equations (2.A.9) we observe that the radial coordinate u in (2.A.1) is related to the coordinate r in (2.3.6) as

$$du = \frac{\ell dr}{\sqrt{(r-r_+)(r-r_-)}} \sim \ell dr/r, \qquad \partial_u = \ell^{-1}\sqrt{(r-r_+)(r-r_-)} \sim \ell^{-1} r\partial_r. \tag{2.4.5}$$

Using these relations, together with the asymptotic form (2.3.7) of the conical backgrounds, we seek to express the radial derivatives of the induced fields as covariant functions of the induced fields. In particular, focusing on the three first order equations that are relevant for our computation, it is not difficult to see that to leading order asymptotically one can write

$$\frac{1}{N}\dot{\gamma}_{ij} \sim \frac{e^{\eta/2}}{B}\left(\frac{3}{2}\gamma_{ij} - B^2 e^{-2\eta} F_{ik}F_j{}^k\right), \quad \frac{1}{N}\dot{\eta} \sim -\frac{e^{\eta/2}}{2B}, \quad \frac{1}{N}\dot{A}^0_i \sim \frac{B}{2}e^{3\eta}D_j\left(e^{-7\eta/2}F^{0j}{}_i\right). \tag{2.4.6}$$

Notice that the first two expressions are not unique since they can be written in alternative ways using the constraints (2.4.4). Taking into account these expressions, as well as the freedom resulting from the constraints, we conclude that the leading asymptotic form of the solution of the Hamilton-Jacobi equation takes the form

$$\mathcal{S} = \frac{1}{\kappa_4^2}\int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\,\frac{1}{B}e^{\eta/2}\left(a_1 + a_2 B^2 e^{-\eta}R[\gamma] + a_3 B^2 e^{-2\eta}F_{ij}F^{ij} + a_4 B^2 e^{-4\eta}F^0_{ij}F^{0ij} + \cdots\right), \tag{2.4.7}$$

where a_1, a_2, a_3 and a_4 are unspecified constants and the ellipses stand for subleading terms. The functional derivatives of this asymptotic solution take the form

$$\begin{aligned}
\frac{\delta\mathcal{S}}{\delta\gamma_{ij}} &= \frac{\sqrt{-\gamma}}{\kappa_4^2}\frac{1}{B}e^{\eta/2}\Bigg(\frac{1}{2}\gamma^{ij}\left(a_1 + a_2 B^2 e^{-\eta}R[\gamma] + a_3 B^2 e^{-2\eta}F_{kl}F^{kl} + a_4 B^2 e^{-4\eta}F^0_{kl}F^{0kl} + \cdots\right)\\
&\qquad + a_2 B^2 e^{-\eta}\left(-R^{ij} + \frac{1}{4}\partial^i\eta\partial^j\eta - \frac{1}{2}D^iD^j\eta - \frac{1}{4}\gamma^{ij}\partial_k\eta\partial^k\eta + \frac{1}{2}\gamma^{ij}\Box_\gamma\eta\right)\\
&\qquad - 2a_3 B^2 e^{-2\eta}F^{ik}F^j{}_k - 2a_4 B^2 e^{-4\eta}F^{0ik}F^{0j}{}_k + \cdots\Bigg)\\
&\sim \frac{\sqrt{-\gamma}}{\kappa_4^2}\frac{1}{B}e^{\eta/2}\left(\frac{1}{2}(a_1 + 2a_2 + 2a_3)\gamma^{ij} - (a_2 + 2a_3)B^2 e^{-2\eta}F^{ik}F^j{}_k + \cdots\right),
\end{aligned} \tag{2.4.8a}$$

$$\frac{\delta\mathcal{S}}{\delta\eta} = \frac{\sqrt{-\gamma}}{\kappa_4^2}\frac{1}{B}e^{\eta/2}\frac{1}{2}\left(a_1 - a_2 B^2 e^{-\eta}R[\gamma] - 3a_3 B^2 e^{-2\eta}F_{ij}F^{ij} - 7a_4 B^2 e^{-4\eta}F^0_{ij}F^{0ij} + \cdots\right),$$

$$\sim \frac{\sqrt{-\gamma}}{\kappa_4^2}\frac{1}{B}e^{\eta/2}\frac{1}{2}\left(a_1 - 2a_2 - 6a_3\right) + \cdots, \tag{2.4.8b}$$

$$\frac{\delta\mathcal{S}}{\delta A^0_i} = -\frac{\sqrt{-\gamma}}{\kappa_4^2}4Ba_4 D_j\left(e^{-7\eta/2}F^{0ji}\right) + \cdots, \tag{2.4.8c}$$

where the symbol $\sim$ indicates that we have used the constraints (2.4.4) and only kept the leading terms. Inserting these in the first order equations (2.A.9) and comparing with (2.4.6) leads to the set of algebraic equations

$$a_1 - 2a_2 - 6a_3 = \frac{3}{2}, \quad a_2 + 2a_3 = -\frac{1}{4}, \quad a_4 = \frac{1}{16}, \tag{2.4.9}$$

which admit the one-parameter family of solutions

$$a_1 = 1 - \alpha/4, \quad a_2 = (\alpha - 1)/4, \quad a_3 = -\alpha/8, \quad a_4 = 1/16, \tag{2.4.10}$$

where α is unconstrained. One can readily check that (2.4.7), with these values for a_1, a_2, a_3 and a_4, satisfies the Hamilton-Jacobi equations asymptotically for any value of the parameter α.

As we shall see momentarily, for any α, this asymptotic solution suffices to remove all long-distance divergences of the on-shell action and renders the variational problem well posed on the space of equivalence classes of boundary data. We have therefore determined that a complete set of boundary counterterms for the variational problem in the magnetic frame is

$$S_{\rm ct} = -\frac{1}{\kappa_4^2}\int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\,\frac{B}{4}e^{\eta/2}\left(\frac{4-\alpha}{B^2} + (\alpha-1)e^{-\eta}R[\gamma] - \frac{\alpha}{2}e^{-2\eta}F_{ij}F^{ij} + \frac{1}{4}e^{-4\eta}F^0_{ij}F^{0ij}\right). \tag{2.4.11}$$

As we mentioned at the beginning of this section, the freedom to choose the value of the parameter α does not correspond to a choice of scheme. Instead, it is a direct consequence of the presence of the second class constraints (2.4.4). The scheme dependence corresponds to the freedom to include additional finite local terms, which do not affect the divergent part of the solution. Later on we will consider situations where additional conditions on the variational problem require a specific value for α, or particular finite counterterms.

The variational problem

Given the counterterms $S_{\rm ct}$ and following standard terminology in the context of the AdS/CFT duality, we define the 'renormalized' on-shell action in the magnetic frame as the sum of the on-shell action (2.2.2) and the counterterms (2.4.11), with the regulating surface Σ_u removed. Namely,

$$S_{\rm ren} = \lim_{r\to\infty}\left(S_4 + S_{\rm ct}\right). \tag{2.4.12}$$

The boundary counterterms ensure that this limit exists and its value is computed in appendix 2.B.

A generic variation of the renormalized on-shell action takes the form

$$\delta S_{\rm ren} = \lim_{r\to\infty}\int \mathrm{d}^3\mathbf{x}\left(\Pi^{ij}\delta\gamma_{ij} + \Pi^i_\Lambda\delta A^\Lambda_i + \Pi_I\delta\varphi^I\right), \tag{2.4.13}$$

where the renormalized canonical momenta are given by

$$\Pi^{ij} = \pi^{ij} + \frac{\delta S_{\rm ct}}{\delta \gamma_{ij}}, \quad \Pi^i_\Lambda = \pi^i_\Lambda + \frac{\delta S_{\rm ct}}{\delta A^\Lambda_i}, \quad \Pi_I = \pi_I + \frac{\delta S_{\rm ct}}{\delta \varphi^I}. \tag{2.4.14}$$

Inserting the asymptotic form of the backgrounds (2.3.6) into the definitions (2.A.4) of the canonical momenta and in the functional derivatives (2.4.8) we obtain

$$\Pi^t{}_t \sim -\frac{k\ell}{2\kappa_4^2}\left(\frac{1}{4}(r_+ + r_-) + \frac{\alpha-2}{8}\ell^2\omega^2(1+3\cos 2\theta)\right)\sin\theta, \quad \Pi^\phi{}_t \sim -\frac{k^2\ell\omega}{2\kappa_4^2}\sqrt{r_+ r_-}\sin\theta, \tag{2.4.15a}$$

$$\Pi^\theta{}_\theta \sim \frac{k\ell^3\omega^2}{16\kappa_4^2}((2-5\alpha)\cos 2\theta + 2 - 3\alpha)\sin\theta, \quad \Pi^\phi{}_\phi \sim -\frac{k\ell^3\omega^2}{16\kappa_4^2}((5\alpha-4)\cos 2\theta + 3\alpha)\sin\theta, \tag{2.4.15b}$$

$$\Pi^{0t} \sim -\frac{1}{2\kappa_4^2}\frac{\ell^4}{B^3}\sin\theta\left(\sqrt{r_+ r_-} + 3\omega^2\ell^2\cos^2\theta\right), \quad \Pi^{0\phi} \sim -\frac{1}{2\kappa_4^2}\frac{k\omega\ell^4}{2B^3}(r_+ + r_-)\sin\theta, \tag{2.4.15c}$$

$$\Pi^t \sim -\frac{1}{2\kappa_4^2}\frac{3\omega\ell^3}{B}\sin 2\theta, \quad \Pi^\phi \sim -\frac{1}{2\kappa_4^2}\frac{2\alpha k\omega^2\ell^3}{B}\sin 2\theta, \tag{2.4.15d}$$

$$\Pi_\eta \sim -\frac{1}{2\kappa_4^2}\frac{k\ell}{8}\sin\theta\left(6(r_+ + r_-) + \ell^2\omega^2((13\alpha-18)\cos 2\theta + 7\alpha - 6)\right), \tag{2.4.15e}$$

with all other components vanishing identically.

Finally, we can use these expressions to evaluate the variation (2.4.13) of the renormalized action in terms of boundary data. To this end we need to perform the integration over θ and remember that the magnetic potential A is not globally defined, as we pointed out in footnote 3. In particular, taking $A_{\rm north} \sim B(\cos\theta - 1)d\phi$ and $A_{\rm south} \sim B(\cos\theta + 1)d\phi$ we get

$$\delta S_{\rm ren} = -\frac{1}{2\kappa_4^2}\int {\rm d}t{\rm d}\phi\;(r_+ + r_-)k\ell\delta\log\left(kB^3/\ell^3\right), \tag{2.4.16}$$

independently of the value of the parameter α. Note that the combination kB^3/ℓ^3 of boundary data is the unique invariant under both the equivalence class transformation (2.3.11) and the transformation (2.3.12). We have therefore demonstrated that by adding the counterterms (2.4.11) to the bulk action, the variational problem is formulated in terms of equivalence classes of boundary data under the transformations (2.3.11) and (2.3.12). This is an explicit demonstration of the general result that formulating the variational problem in terms of equivalence classes of boundary data under radial reparameterizations is achieved via the same canonical transformation that renders the on-shell action finite. As we will now demonstrate, the same boundary terms ensure the finiteness of the conserved charges, as well as the validity of the first law of thermodynamics.

Conserved charges

Let us now consider conserved charges associated with local conserved currents. This includes electric charges, as well as conserved quantities related to asymptotic Killing vectors. Magnetic charges do not fall in this category, but they can be described in this language in the electric frame, as we shall see later on.

In the radial Hamiltonian formulation of the bulk dynamics, the presence of local conserved currents is a direct consequence of the first class constraints $\mathcal{F}^\Lambda = 0$ and $\mathcal{H}^i = 0$ in (2.A.6).[4] As in the case of asymptotically AdS backgrounds, these constraints lead respectively to conserved electric charges and charges associated with asymptotic Killing vectors.[5] In particular, the gauge constraints $\mathcal{F}^\Lambda = 0$ in (2.A.6) take the form

$$D_i\pi^i = 0, \quad D_i\pi^{0i} = 0, \tag{2.4.17}$$

where π^i and π^{0i} are respectively the canonical momenta conjugate to the gauge fields A_i and A_i^0. Since the boundary counterterms (2.4.11) are gauge invariant, it follows from (2.4.14) that these conservation laws hold for the corresponding renormalized momenta as well, namely

$$D_i\Pi^i = 0, \quad D_i\Pi^{0i} = 0. \tag{2.4.18}$$

This implies that the quantities

$$Q_4^{(e)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\Pi^t, \qquad Q_4^{0(e)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\Pi^{0t}, \tag{2.4.19}$$

where C denotes a Cauchy surface that extends to the boundary $\partial\mathcal{M}$, are both conserved and finite and correspond to the electric charges associated with these gauge fields.

Similarly, the momentum constraint $\mathcal{H}^i = 0$ in (2.A.6), which can be written in explicit form as

$$\begin{aligned}
&-2D_j\pi_i^j + \pi_\eta\partial_i\eta + \pi_\chi\partial_i\chi + F_{ij}^0\pi^{0j} + F_{ij}\pi^j \\
&+ \frac{1}{2\kappa_4^2}\sqrt{-\gamma}\,\epsilon^{jkl}\left(\chi^3F_{ij}^0F_{kl}^0 + \frac{3}{2}\chi^2F_{ij}^0F_{kl} + 3\chi F_{ij}F_{kl} + \frac{3}{2}\chi^2F_{ij}F_{kl}^0\right) = 0,
\end{aligned} \tag{2.4.20}$$

leads to finite conserved charges associated with asymptotic Killing vectors. Note that the terms in the second line are independent of the canonical momenta and originate in the parity odd terms in the STU model Lagrangian.[6] However, for asymptotically conical backgrounds of the form (2.3.6) these terms are asymptotically subleading, the most dominant term being

$$\sqrt{-\gamma}\epsilon^{jkl}\chi F_{ij}F_{kl} = \mathcal{O}(r^{-1}), \tag{2.4.21}$$

[4]These constraints can be derived alternatively by applying the general variation (2.4.13) of the renormalized action to $U(1)$ gauge transformations and transverse diffeomorphisms, assuming the invariance of the renormalized action under such transformations. This method will be used in order to derive the conserved charges in the electric frame.

[5]In asymptotically locally AdS spaces, the Hamiltonian constraint $\mathcal{H} = 0$ can be used in order to construct conserved charges associated with *conformal* Killing vectors of the boundary data [20]. For asymptotically conical backgrounds, the Hamiltonian constraint leads to conserved charges associated with asymptotic transverse diffeomorphisms, ξ^i, that preserve the boundary data up to the equivalence class transformations (2.3.11).

[6]In the AdS/CFT context these terms are interpreted as a gravitational anomaly in the dual QFT.

and so the momentum constraint asymptotically reduces to

$$-2D_j\pi^j_i + \pi_\eta\partial_i\eta + \pi_\chi\partial_i\chi + F^0_{ij}\pi^{0j} + F_{ij}\pi^j \approx 0. \tag{2.4.22}$$

Since the counterterms (2.4.11) are invariant with respect to diffeomorphisms along the surfaces of constant radial coordinate, it follows from (2.4.14) that this constraint holds for the renormalized momenta as well,

$$-2D_j\Pi^j_i + \Pi_\eta\partial_i\eta + \Pi_\chi\partial_i\chi + F^0_{ij}\Pi^{0j} + F_{ij}\Pi^j \approx 0. \tag{2.4.23}$$

Given an asymptotic Killing vector ζ^i satisfying the asymptotic conditions

$$\mathcal{L}_\zeta\gamma_{ij} = D_i\zeta_j + D_j\zeta_i \approx 0, \quad \mathcal{L}_\zeta A^\Lambda_i = \zeta^j\partial_j A^\Lambda_i + A^\Lambda_j\partial_i\zeta^j \approx 0, \quad \mathcal{L}_\zeta\varphi^I = \zeta^i\partial_i\varphi^I \approx 0, \tag{2.4.24}$$

the conservation identity (2.4.23) implies that the quantity

$$\mathcal{Q}[\zeta] = \int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\left(2\Pi^t_j + \Pi^{0t}A^0_j + \Pi^t A_j\right)\zeta^j, \tag{2.4.25}$$

is both finite and conserved, i.e. it is independent of the choice of Cauchy surface C. However, there are a few subtleties in evaluating these charges. Firstly, Gauss' theorem used to prove conservation for the charges (2.4.25) assumes differentiability of the integrand across the equator at the boundary. If the gauge potentials are magnetically sourced, as is the case for A_i in the magnetic frame, then the gauge should be chosen such that A_i is continuous across the equator. In particular, contrary to the variational problem we discussed earlier, the gauge that should be used to evaluate these charges is the one given in (2.3.6), and not the one discussed in footnote 3.

Secondly, the charges (2.4.25) are not generically invariant under the $U(1)$ gauge transformations $A^\Lambda_i \to A^\Lambda_i + \partial_i\alpha^\Lambda$. These gauge transformations though must preserve both the radial gauge (2.4.3) and the asymptotic Killing conditions (2.4.24). Preserving the radial gauge implies that the gauge parameter must depend only on the transverse coordinates, i.e. $\alpha^\Lambda(x)$ (see e.g. [20]), while respecting the Killing symmetry leads to the condition

$$\zeta^i\partial_i\alpha^\Lambda = \text{constant}. \tag{2.4.26}$$

Under such gauge transformations the charges (2.4.25) are shifted by the corresponding electric charges (2.4.19). As will become clear in section 2.5, this compensates a related shift in the electric potential such that the Smarr formula and the first law are gauge invariant. Nevertheless, gauge invariant charges, as well as electric potentials, can be defined if and only if $A^\Lambda_j\zeta^j\big|_{\partial\mathcal{M}} =$ constant. However, this is not true in general.

Finally, another potential ambiguity in the value of the charges (2.4.25) arises from the ambiguity in the choice of boundary counterterms used to define the renormalized momenta. In the case of asymptotically conical backgrounds in the magnetic frame, this ambiguity consists in both the value of the parameter α in (2.4.11), as well as the possibility of adding extra finite and covariant terms. From the explicit expressions (2.4.15) we see that α does lead to an ambiguity in the renormalized momenta. However, as we will see in section 2.5, it does not affect the value of physical observables.

The fact that the value of the charges (2.4.25) is ambiguous in the precise sense we just discussed does not affect the thermodynamic relations among the charges and the first law, which are unambiguous. In fact, the ambiguity in the definition (2.4.25) of the conserved charges allows us to match them to alternative definitions [69, 20].

2.4.2 Electric frame

We will now repeat the above analysis for the variational problem in the electric frame, emphasizing the differences relative to the magnetic frame. Besides the fact that the electric frame is most commonly used in the literature on subtracted geometries, it is also necessary in order to evaluate the magnetic potential, also known as the 'magnetization'. Moreover, the variational problem for asymptotically conical backgrounds in the electric frame presents some new subtleties, from which interesting lessons can be drawn.

Boundary counterterms

By construction, the electric frame action $\widetilde{S}_4$ given in (2.2.10) has the same on-shell value as the magnetic frame action S_4 in (2.2.2). Therefore, the boundary counterterms (2.4.11) that were derived for S_4 must also render the variational problem for $\widetilde{S}_4$ well posed and remove its long-distance divergences. Adding the boundary counterterms (2.4.11) to $\widetilde{S}_4$ we get

$$\widetilde{S}_4 + S_{\rm ct} = \widetilde{S}'_4 + S_{\rm ct} - \frac{3}{2\kappa_4^2}\int_{\partial\mathcal{M}} \widetilde{A}\wedge F + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F, \tag{2.4.27}$$

where $\widetilde{S}'_4$ denotes the σ-model part of (2.2.10) (plus the Gibbons-Hawking term), to which the Hamiltonian analysis of appendix 2.A can be applied.

As for the bulk part of the action in (2.2.10), we need to replace F_{ij} in the boundary terms with the electric gauge field $\widetilde{A}_i$ using (2.2.8), which for the transverse components reduces to the canonical momentum for $\widetilde{A}_i$ in (2.A.4), namely

$$\widetilde{\pi}^i_\Lambda = \frac{\delta L}{\delta \dot{\widetilde{A}}^\Lambda{}_i} = -\frac{2}{\kappa_4^2}\sqrt{-\gamma}\left(N^{-1}\widetilde{\mathcal{Z}}_{\Lambda\Sigma}\gamma^{ij}\dot{\widetilde{A}}^\Sigma{}_j + \widetilde{\mathcal{R}}_{\Lambda\Sigma}\epsilon^{ijk}\widetilde{F}^\Sigma_{jk}\right). \tag{2.4.28}$$

Evaluating this expression leads to the identity

$$F_{ij} = \frac{2\kappa_4^2}{3}\varepsilon_{ijk}\widetilde{\pi}^k = \frac{2\kappa_4^2}{3}\epsilon_{ijk}\widehat{\widetilde{\pi}}^k, \tag{2.4.29}$$

where we have defined $\widehat{\widetilde{\pi}}^i = \widetilde{\pi}^i/\sqrt{-\gamma}$. Hence,

$$\widetilde{S}_4 + S_{\rm ct} = \widetilde{S}'_4 + S_{\rm ct} - \int_{\partial\mathcal{M}} \mathrm{d}^3\mathbf{x}\, \widetilde{\pi}^i\widetilde{A}_i + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F, \tag{2.4.30}$$

where the counterterms are now expressed as

$$S_{\rm ct}=-\frac{B}{4\kappa_4^2}\int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\,e^{\eta/2}\left(\frac{4-\alpha}{B^2}+(\alpha-1)e^{-\eta}R[\gamma]+\frac{1}{4}e^{-4\eta}F^0_{ij}F^{0ij}+\frac{4\alpha\kappa_4^4}{9}e^{-2\eta}\widehat{\widetilde{\pi}}^i\widehat{\widetilde{\pi}}_i\right). \tag{2.4.31}$$

The renormalized action in the electric frame therefore takes the form

$$\widetilde{S}_{\rm ren}=\lim_{r\to\infty}\left(\widetilde{S}'_4+S_{\rm ct}-\int_{\partial\mathcal{M}}\mathrm{d}^3\mathbf{x}\,\widetilde{\pi}^i\widetilde{A}_i\right), \tag{2.4.32}$$

with $S_{\rm ct}$ given by (2.4.31). Moreover, the asymptotic second class constraints (2.4.4) become

$$\widehat{\widetilde{\pi}}^k\widehat{\widetilde{\pi}}_k\approx-\left(\frac{3e^{\eta}}{2\kappa_4^2B}\right)^2,\quad R_{ij}[\gamma]\approx-\left(\frac{2\kappa_4^2}{3}\right)^2e^{-\eta}\left(\gamma_{ij}\widehat{\widetilde{\pi}}^k\widehat{\widetilde{\pi}}_k-\widehat{\widetilde{\pi}}_i\widehat{\widetilde{\pi}}_j\right),\quad 2R_{ij}[\gamma]R^{ij}[\gamma]\approx R[\gamma]^2. \tag{2.4.33}$$

Note that the surface term on the horizon in (2.4.27) is not part of the action defining the theory in the electric frame, which is why we have not included it in the definition of the renormalized action (2.4.32). The theory is specified by the bulk Lagrangian and the boundary terms on $\partial\mathcal{M}$, which dictate the variational problem and the boundary conditions. The horizon is a dynamical surface – not a boundary. This surface term, however, will be essential in section 2.5 for comparing the free energies in the electric and magnetic frames.

Given that the counterterms $S_{\rm ct}$ render the on-shell action in the magnetic frame finite, the limit (2.4.32) is guaranteed to exist: its value differs from the on-shell value of the renormalized action (2.4.12) by the surface term on the horizon given in (2.4.27). However, as we will show shortly, it turns out that the variational problem for the renormalized action (2.4.32) is only well posed provided α takes a specific non-zero value. This value is determined by the term implementing the Legendre transformation in (2.4.32), which has a fixed coefficient. Therefore, even though any value of α leads to a well posed variational problem in the magnetic frame, a specific value of α is required for the variational problem in the electric frame.

Another consequence of the Legendre transform in (2.4.32) is that it changes the boundary conditions from Dirichlet, where $\widetilde{A}_i$ is kept fixed on the boundary (up to equivalence class transformations), to Neumann, where $\widetilde{\pi}^i$ is kept fixed. This in turn forces the counterterms to be a function of the canonical momentum, i.e. $S_{\rm ct}[\gamma,A^0,\widehat{\widetilde{\pi}},\eta,\chi]$. An analogous situation arises in asymptotically AdS backgrounds with fields that are dual to irrelevant operators [64]. An example that shares many qualitative features with the potential $\widetilde{A}_i$ here is a gauge field in AdS_2, coupled to appropriate matter [70]. From the form of the conical backgrounds (2.3.6) we see that $\widetilde{A}_i$ asymptotically dominates the stress tensor as $r\to\infty$ since

$$T_{tt}\sim e^{\eta}g^{rr}(\widetilde{F}_{rt})^2\sim r, \tag{2.4.34}$$

and hence, in this sense, the gauge potential $\widetilde{A}_i$ is analogous to bulk fields dual to irrelevant operators in asymptotically AdS spaces. This property is what makes the variational problem and the boundary counterterms in the electric frame more subtle, which is why we found it easier to formulate the variational problem in the magnetic frame first and then translate the result to the electric frame.

The variational problem

A generic variation of the renormalized action (2.4.32) takes the form

$$\delta\widetilde{S}_{\rm ren}=\lim_{r\to\infty}\int \mathrm{d}^3\mathbf{x}\left(\widetilde{\Pi}^{ij}\delta\gamma_{ij}+\widetilde{\Pi}^{0i}\delta A_i^0-\sqrt{-\gamma}\;\widetilde{A}_i^{\rm ren}\delta\widehat{\widetilde{\pi}}^i+\widetilde{\Pi}_I\delta\varphi^I\right), \tag{2.4.35}$$

where

$$\widetilde{\Pi}^{ij}=\widetilde{\pi}^{ij}-\frac{1}{2}\gamma^{ij}\widetilde{\pi}^k\widetilde{A}_k+\left.\frac{\delta S_{\rm ct}}{\delta\gamma_{ij}}\right|_{\widehat{\widetilde{\pi}}},\quad \widetilde{\Pi}^{0i}=\widetilde{\pi}^{0i}+\frac{\delta S_{\rm ct}}{\delta A_i^0},\quad \widetilde{\Pi}_I=\widetilde{\pi}_I+\frac{\delta S_{\rm ct}}{\delta\varphi^I}, \tag{2.4.36}$$

and

$$\widetilde{A}_i^{\rm ren}=\widetilde{A}_i-\frac{1}{\sqrt{-\gamma}}\left.\frac{\delta S_{\rm ct}}{\delta\widehat{\widetilde{\pi}}^i}\right|_{\gamma}, \tag{2.4.37}$$

are the renormalized canonical variables in the electric frame. It should be emphasized that the functional derivative with respect to γ_{ij} in $\widetilde{\Pi}^{ij}$ is computed keeping $\widehat{\widetilde{\pi}}^i$ fixed instead of $\widetilde{\pi}^i$. The term implementing the Legendre transform in (2.4.32), therefore, gives $-\frac{1}{2}\gamma^{ij}\widetilde{\pi}^k\widetilde{A}_k$, while

$$\begin{aligned}\left.\frac{\delta S_{\rm ct}}{\delta\gamma_{ij}}\right|_{\widehat{\widetilde{\pi}}}=&-\frac{\sqrt{-\gamma}}{\kappa_4^2}\frac{e^{\eta/2}}{B}\Bigg(\frac{1}{2}\gamma^{ij}\left(\frac{1}{4}+\frac{1}{2}B^2e^{-\eta}R[\gamma]+\frac{1}{16}B^2e^{-4\eta}F^0_{ij}F^{0ij}\right)\\&-\frac{1}{8}B^2e^{-4\eta}F^{0ik}F^{0j}{}_k+\frac{1}{2}B^2e^{-\eta}\Big(-R^{ij}+\frac{1}{4}\partial^i\eta\partial^j\eta-\frac{1}{2}D^iD^j\eta\\&-\frac{1}{4}\gamma^{ij}\partial_k\eta\partial^k\eta+\frac{1}{2}\gamma^{ij}\Box_\gamma\eta\Big)\Bigg)-\frac{\kappa_4^2}{\sqrt{-\gamma}}\frac{B}{3}e^{-3\eta/2}\left(\frac{\gamma^{ij}}{2}\widetilde{\pi}^k\widetilde{\pi}_k+\widetilde{\pi}^i\widetilde{\pi}^j\right).\end{aligned} \tag{2.4.38}$$

Moreover, note that (2.A.4) implies that the canonical momenta $\widetilde{\pi}^{ij}$, $\widetilde{\pi}^{0i}$, and $\widetilde{\pi}_I$, remain the same as their magnetic frame counterparts.

What is novel in (2.4.35) from the point of view of holographic renormalization is that the variable that gets renormalized is the induced field $\widetilde{A}_i$, according to (2.4.37), instead of its conjugate momentum. However, as we mentioned earlier, only a specific value of the parameter α correctly renormalizes $\widetilde{A}_i$. In particular, from (2.4.31) we get

$$\widetilde{A}_i^{\rm ren}=\widetilde{A}_i+\frac{2\alpha B\kappa_4^2}{9}\frac{e^{-3\eta/2}}{\sqrt{-\gamma}}\widetilde{\pi}_i. \tag{2.4.39}$$

On the other hand, from (2.4.28) and the asymptotic form of $\widetilde{A}_i$ in (2.3.7) we deduce that asymptotically

$$\widetilde{\pi}^i\sim-\frac{2}{\kappa_4^2}\sqrt{-\gamma}\cdot\frac{3}{4B}e^{3\eta/2}\gamma^{it}\widetilde{A}_t. \tag{2.4.40}$$

It follows that $\widetilde{A}_i^{\mathrm{ren}}$ has a finite limit as $r \to \infty$ provided $\alpha = 3$.

Setting $\alpha = 3$ in (2.4.31) and evaluating the renormalized variables on the conical backgrounds (2.3.6) we obtain

$$\widetilde{\Pi}^t{}_t \sim \frac{k\ell}{2\kappa_4^2}\sin\theta\left(-\frac{1}{4}(r_+ + r_-) + \frac{1}{8}\ell^2\omega^2(11 + 9\cos 2\theta)\right),$$
$$\widetilde{\Pi}^\phi{}_t \sim -\frac{1}{2\kappa_4^2}\frac{k\omega\ell}{2}\sqrt{r_+ r_-}\sin\theta, \tag{2.4.41a}$$
$$\widetilde{\Pi}^\theta{}_\theta \sim -\frac{1}{2\kappa_4^2}\frac{k\ell^3\omega^2}{16}(\sin 3\theta - 11\sin\theta), \quad \widetilde{\Pi}^\phi{}_\phi \sim \frac{1}{2\kappa_4^2}\frac{k\ell^3\omega^2}{16}(\sin 3\theta + 5\sin\theta), \tag{2.4.41b}$$
$$\widetilde{\Pi}^{0t} \sim -\frac{1}{2\kappa_4^2}\frac{\ell^4}{B^3}\sin\theta\left(\sqrt{r_+ r_-} + 3\omega^2\ell^2\cos^2\theta\right), \quad \widetilde{\Pi}^{0\phi} \sim -\frac{1}{2\kappa_4^2}\frac{k\omega\ell^4}{2B^3}(r_+ + r_-)\sin\theta, \tag{2.4.41c}$$
$$\widetilde{A}_t^{\mathrm{ren}} \sim \frac{2\omega^2\ell^3 k}{B}\cos^2\theta, \quad \widetilde{A}_\phi^{\mathrm{ren}} \sim \frac{\omega\ell^3\sin^2\theta}{B}, \tag{2.4.41d}$$
$$\widetilde{\Pi}_\eta \sim -\frac{1}{2\kappa_4^2}\frac{3k\ell}{8}\sin\theta\left(2(r_+ + r_-) + \ell^2\omega^2(5 + 7\cos 2\theta)\right). \tag{2.4.41e}$$

Finally, inserting these expressions in (2.4.35) gives

$$\delta\widetilde{S}_{\mathrm{ren}} = -\frac{1}{2\kappa_4^2}\int \mathrm{d}t\mathrm{d}\phi\ (r_+ + r_-)k\ell\delta\log\left(kB^3/\ell^3\right), \tag{2.4.42}$$

in agreement with the magnetic frame result (2.4.16). In particular, as in the magnetic frame, the variational problem is well posed in terms of equivalence classes of boundary data under the transformation (2.3.11).

Conserved charges

The last aspect of the electric frame we need to discuss before we can move on to study the thermodynamics of conical backgrounds is how to define the conserved charges. Focusing again on charges obtained from local conserved currents, the electric charges follow from the conservation laws

$$D_i\widetilde{\pi}^i = 0, \qquad D_i\widetilde{\Pi}^{0i} = 0. \tag{2.4.43}$$

From (2.4.29) we see that the first of these expressions is simply the Bianchi identity $dF = 0$ and so the corresponding charge is the magnetic charge in the magnetic frame, $Q_4^{(m)}$, while $\widetilde{\Pi}^{0i}$ coincides with the renormalized momentum Π^{0i} in the magnetic frame. Hence,

$$\widetilde{Q}_4^{(e)} = -\int_{\partial\mathcal{M}\cap C}\mathrm{d}^2\mathbf{x}\,\widetilde{\pi}^t = Q_4^{(m)}, \qquad \widetilde{Q}_4^{0(e)} = -\int_{\partial\mathcal{M}\cap C}\mathrm{d}^2\mathbf{x}\,\widetilde{\Pi}^{0t} = Q_4^{0(e)}. \tag{2.4.44}$$

Slightly more subtle are conserved charges associated with asymptotic Killing vectors. The easiest way to derive the conservation laws in the electric frame is by considering the variation of the renormalized action under an infinitesimal diffeomorphism, ξ^i, along the surfaces of constant radial coordinate. Inserting the transformations

$$\begin{aligned}
&\delta_\xi \gamma_{ij} = \mathcal{L}_\xi \gamma_{ij} = D_i \xi_j + D_j \xi_i, \quad \delta_\xi \varphi^I = \mathcal{L}_\xi \varphi^I = \xi^i \partial_i \varphi^I, \\
&\delta_\xi A_i^0 = \mathcal{L}_\xi A_i^0 = \xi^j \partial_j A_i^0 + A_j^0 \partial_i \xi^j, \quad \delta_\xi \widehat{\widetilde{\pi}}^i = \mathcal{L}_\xi \widehat{\widetilde{\pi}}^i = \xi^j D_j \widehat{\widetilde{\pi}}^i - \widehat{\widetilde{\pi}}^j D_j \xi^i,
\end{aligned} \tag{2.4.45}$$

under such a diffeomorphism in the general variation (2.4.35) of the renormalized action gives

$$\begin{aligned}
\delta \widetilde{S}_{\rm ren} &= \lim_{r\to\infty} \int \mathrm{d}^3\mathbf{x} \left(2\widetilde{\Pi}^{ij} D_i \xi_j + \xi^i F^0_{ij} \Pi^{0j} - \sqrt{-\gamma} \widetilde{A}_i^{\rm ren} \delta \widehat{\widetilde{\pi}}^i + \Pi_I \xi^i \partial_i \varphi^I \right) \\
&= \lim_{r\to\infty} \int \mathrm{d}^3\mathbf{x}\, \xi^i \left(-2D_j \widetilde{\Pi}^j_i + F^0_{ij} \Pi^{0j} - D_i \left(\widetilde{A}^{\rm ren}_j \widetilde{\pi}^j \right) + \widetilde{\mathcal{F}}_{ij} \widetilde{\pi}^j + \Pi_I \partial_i \varphi^I \right),
\end{aligned} \tag{2.4.46}$$

from which we arrive at the conservation identity

$$-2D_j \left(\widetilde{\Pi}^j_i + \frac{1}{2} \delta^j_i \widetilde{\pi}^k \widetilde{A}^{\rm ren}_k \right) + F^0_{ij} \Pi^{0j} + \widetilde{\mathcal{F}}_{ij} \widetilde{\pi}^j + \Pi_I \partial_i \varphi^I \approx 0. \tag{2.4.47}$$

An asymptotic Killing vector, ζ^i, in the electric frame satisfies the same conditions (2.4.24) as in the magnetic frame, except that the asymptotic form of the background is now specified in terms of $\widehat{\widetilde{\pi}}^i$ instead of A_i and so the condition $\mathcal{L}_\zeta A_i = \zeta^j \partial_j A_i + A_j \partial_i \zeta^j \approx 0$ in the magnetic frame should be replaced with

$$\mathcal{L}_\zeta \widehat{\widetilde{\pi}}^i = \zeta^j D_j \widehat{\widetilde{\pi}}^i - \widehat{\widetilde{\pi}}^j D_j \zeta^i \approx 0. \tag{2.4.48}$$

With this crucial modification in the definition of an asymptotic Killing vector in the electric frame, the conservation law (2.4.47) leads to the conserved charges

$$\widetilde{\mathcal{Q}}[\zeta] = \int_{\partial \mathcal{M} \cap C} \mathrm{d}^2\mathbf{x} \left(2\widetilde{\Pi}^t_j + \Pi^{0t} A^0_j + \widetilde{\pi}^t \widetilde{A}^{\rm ren}_j \right) \zeta^j, \tag{2.4.49}$$

which are again manifestly finite. The value of these charges is subject to the same ambiguities as the charges (2.4.25), but as we shall see in the next section, the gauge choice we made in the specification (2.3.6) of the conical backgrounds in the two frames ensures that the charges (2.4.49) and (2.4.25) coincide.

2.5 Black hole thermodynamics

In the previous section we derived specific boundary terms that should be added to the STU model action in both the magnetic and electric frames such that the variational problem for asymptotically conical backgrounds of the form (2.3.6) is well posed. Moreover, we showed that the same boundary terms ensure that the on-shell action is free of long-distance divergences and allow us to construct finite conserved charges. In this section we evaluate explicitly these conserved charges and other relevant thermodynamic observables for conical backgrounds and we demonstrate that both the Smarr formula and the first law of thermodynamics hold. Along the way we compare our results with those obtained in [62], and comment on some differences.

2.5.1 Renormalized thermodynamic observables

Let us start by evaluating in turn all relevant thermodynamic variables that we will need in order to prove the first law and the Smarr formula. We will use a subscript '4' to denote the variables computed in this section to distinguish them from their counterparts in 5 and 3 dimensions, which we will discuss in section 2.6.

Entropy

The entropy is given by the standard Bekenstein-Hawking area law and its value for the conical black holes (2.3.6) is[7]

$$S_4 = \frac{\pi \ell^2}{G_4}\sqrt{r_+} \ . \tag{2.5.1}$$

Temperature

The Hawking temperature can be obtained by requiring that the Euclidean section of the black hole solution is smooth at the horizon, which determines

$$T_4 = \frac{k(r_+ - r_-)}{4\pi\ell\sqrt{r_+}} \ . \tag{2.5.2}$$

Angular velocity

We define the physical (diffeomorphism invariant) angular velocity as the difference between the angular velocity at the outer horizon and at infinity, namely

$$\Omega_4 = \Omega_H - \Omega_\infty = \frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{\partial\mathcal{M}} - \frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{\mathcal{H}_+} = \omega k\sqrt{\frac{r_-}{r_+}} \ . \tag{2.5.3}$$

In the coordinate system (2.3.6) there is no contribution to the angular velocity from infinity, but there is in the original coordinate system (2.3.1). The rotation at infinity was not taken into account in [62], which is why our result does not fully agree with the one obtained there.

[7] We hope that using the same symbol for the entropy and the action will not cause any confusion, since it should be clear from the context which quantity we refer to.

Electric charges

In the magnetic frame there is only one non-zero electric charge given by (2.4.19), whose value is

$$Q_4^{0(e)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\Pi^{0t} = \frac{\ell^4}{4G_4 B^3}\left(\sqrt{r_+ r_-} + \omega^2\ell^2\right). \tag{2.5.4}$$

In the electric frame both electric charges defined in (2.4.44) are non-zero:

$$\widetilde{Q}_4^{(e)} = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\,\widetilde{\pi}^t = \frac{3B}{4G_4}\,, \qquad \widetilde{Q}_4^{0(e)} = Q_4^{0(e)}. \tag{2.5.5}$$

Magnetic charge

The only non-zero magnetic charge is present in the magnetic frame and it is equal to one of the electric charges in the electric frame:

$$Q_4^{(m)} = -\frac{3}{2\kappa_4^2}\int_{\partial\mathcal{M}\cap C} F = \widetilde{Q}_4^{(e)}. \tag{2.5.6}$$

Electric potential

We define the electric potential as

$$\Phi_4^{0(e)} = A_i^0\mathcal{K}^i\Big|_{\mathcal{H}_+} = k\left(\frac{B}{\ell}\right)^3\sqrt{\frac{r_-}{r_+}}\,, \tag{2.5.7}$$

where $\mathcal{K} = \partial_t + \Omega_H\partial_\phi$ is the null generator of the outer horizon. Note that $A_i^0\mathcal{K}^i$ is constant over the horizon [20] and so leads to a well defined electric potential. However, as we remarked in the previous section, the electric potential is not gauge invariant. Under gauge transformations it is shifted by a constant (see (2.4.26)) which compensates the corresponding shift of the charges (2.4.25) in the Smarr formula and the first law.

Magnetic potential

Similarly, the magnetic potential is defined in terms of the gauge field $\widetilde{A}_i$ in the electric frame as

$$\Phi_4^{(m)} = \widetilde{A}_i\mathcal{K}^i\Big|_{\mathcal{H}_+} = \frac{\ell k}{2B}\left((r_- - r_+) + 2\omega^2\ell^2\sqrt{\frac{r_-}{r_+}}\right). \tag{2.5.8}$$

Mass

The mass is the conserved charge associated with the Killing vector[8] $\zeta = -\partial_t - \Omega_\infty \partial_\phi$. Since $\Omega_\infty = 0$ in the coordinate system (2.3.6), (2.4.25) gives[9]

$$M_4 = -\int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\, \left(2\Pi^t{}_t + \Pi^t_0 A^0_t + \Pi^t A_t\right) = \frac{\ell k}{8G_4}(r_+ + r_-)\ . \tag{2.5.9}$$

The same result is obtained in the electric frame using (2.4.49).

Angular momentum

The angular momentum is defined as the conserved charge corresponding to the Killing vector $\zeta = \partial_\phi$, which gives

$$J_4 = \int_{\partial\mathcal{M}\cap C} \mathrm{d}^2\mathbf{x}\, (2\Pi^t{}_\phi + \Pi^t_0 A^0_\phi + \Pi^t A_\phi) = -\frac{\omega\ell^3}{2G_4}\ . \tag{2.5.10}$$

The same result is obtained in the electric frame.

Free energy

Finally, the full Gibbs free energy, $\widetilde{\mathcal{G}}_4$, is related to the renormalized Euclidean on-shell action in the electric frame, where all charges are electric. Namely,

$$\widetilde{I}_4 = \widetilde{S}^{\mathrm{E}}_{\mathrm{ren}} = -\widetilde{S}_{\mathrm{ren}} = \beta_4 \widetilde{\mathcal{G}}_4, \tag{2.5.11}$$

with $\beta_4 = 1/T_4$ and $\widetilde{S}_{\mathrm{ren}}$ defined in (2.4.32). The Euclidean on-shell action in the magnetic frame similarly defines another thermodynamic potential, $\mathcal{G}_4$, through

$$I_4 = S^{\mathrm{E}}_{\mathrm{ren}} = -S_{\mathrm{ren}} = \beta_4 \mathcal{G}_4, \tag{2.5.12}$$

where S_{ren} was given in (2.4.12). Evaluating this we obtain (see appendix 2.B)

$$I_4 = \frac{\beta_4 \ell k}{8G_4}\left((r_- - r_+) + 2\omega^2\ell^2\sqrt{\frac{r_-}{r_+}}\right)\ . \tag{2.5.13}$$

[8]The overall minus sign relative to the Killing vector used in [20] can be traced to the fact that the free energy is defined as the Lorentzian on-shell action in section 5 of that paper, while in section 6 it is defined as the Euclidean on-shell action. We adopt the latter definition here.

[9]In [62] the mass for static subtracted geometry black holes was evaluated from the regulated Komar integral and the Hawking-Horowitz prescription and shown to be equivalent. Both the Smarr formula and the first law of thermodynamics were shown to hold in the static case. In the rotating case, the chosen coordinate system of the subtracted metric in [62] has non-zero angular velocity at spatial infinity which was erroneously not included in the thermodynamics analysis of the rotating subtracted geometry. Furthermore, the evaluation of the regulated Komar integral in the rotating subtracted geometry would have to be performed; this would lead to an additional contribution to the regulated Komar mass due to rotation, and in turn ensure the validity of the Smarr formula and the first law of thermodynamics.

Moreover, (2.4.27) implies that the on-shell action is given by

$$\widetilde{I}_4 = I_4 + \frac{3}{2\kappa_4^2}\int_{\mathcal{H}_+} \widetilde{A}\wedge F = I_4 - \beta_4 \Phi_4^{(m)} Q_4^{(m)}. \tag{2.5.14}$$

An interesting observation is that the value of the renormalized action in the magnetic frame, as well as the value of all other thermodynamic variables, is independent of the parameter α in the boundary counterterms (2.4.11). This property is necessary in order for the thermodynamic variables in the electric and magnetic frames to agree, and in order to match with those of the 5D uplifted black holes that we will discuss in section 2.6. Recall that the terms multiplying α are designed so that their leading asymptotic contribution to the Hamilton-Jacobi solution (2.4.7), as well as to the derivatives (2.4.8), vanishes by means of the asymptotic constraints (2.4.4). This is the reason why any value of α leads to boundary counterterms that remove the long-distance divergences. However, the parameter α does appear in the renormalized momenta, as is clear from (2.4.15), and in the unintegrated value of the renormalized action. Nevertheless, α does not enter in any physical observable. This observation results from the explicit computation of the thermodynamic variables, but we have not been able to find a general argument that ensures this so far.

2.5.2 Thermodynamic relations and the first law

We can now show that the thermodynamic variables we just computed satisfy the expected thermodynamic relations, including the first law of black hole mechanics.

Quantum statistical relation

It is straightforward to verify that the total Gibbs free energy $\widetilde{\mathcal{G}}_4$ satisfies the quantum statistical relation [33]

$$\widetilde{\mathcal{G}}_4 = M_4 - T_4 S_4 - \Omega_4 J_4 - \Phi^{0(e)} Q^{0(e)} - \Phi_4^{(m)} Q_4^{(m)}. \tag{2.5.15}$$

Similarly, the thermodynamic potential $\mathcal{G}_4$, which was obtained from the on-shell action in the magnetic frame, satisfies

$$\mathcal{G}_4 = M_4 - T_4 S_4 - \Omega_4 J_4 - \Phi^{0(e)} Q^{0(e)}. \tag{2.5.16}$$

Note that the shift of the mass and angular momentum under a gauge transformation (2.4.26) is compensated by that of the electric potentials so that these relations are gauge invariant.

First law

In order to demonstrate the validity of the first law we must recall the transformations (2.3.11) and (2.3.12) of the non-normalizable boundary data that allow for a well posed variational problem. In particular, variations of B, k and ℓ that are a combination of the two transformations (2.3.11) and (2.3.12) are equivalent to generic transformations keeping kB^3/ℓ^3 fixed. Considering such transformations, as well as arbitrary variations of the normalizable parameters $r_\pm$ and ω, we obtain

$$\mathrm{d}M_4 - T_4\mathrm{d}S_4 - \Omega_4\mathrm{d}J_4 - \Phi_4^{0(e)}\mathrm{d}Q_4^{0(e)} - \Phi_4^{(m)}\mathrm{d}Q_4^{(m)} = 0. \tag{2.5.17}$$

Smarr formula

Finally, one can explicitly check that the Smarr formula

$$M_4 = 2S_4T_4 + 2\Omega_4 J_4 + Q_4^{0(e)}\Phi_4^{0(e)} + Q_4^{(m)}\Phi_4^{(m)}, \tag{2.5.18}$$

also holds. This identity can be derived by applying the first law to the one-parameter family of transformations

$$\delta M_4 = \epsilon M_4, \quad \delta S_4 = 2\epsilon S_4, \quad \delta J_4 = 2\epsilon J_4, \quad \delta Q_4^{0(e)} = \epsilon Q_4^{0(e)}, \quad \delta Q_4^{(m)} = \epsilon Q_4^{(m)}, \tag{2.5.19}$$

which corresponds to the parameter variations

$$\delta\ell = \epsilon\ell, \quad \delta B = \epsilon B, \quad \delta\omega = -\epsilon\omega, \tag{2.5.20}$$

while keeping all other parameters of the solutions fixed. This transformation keeps kB^3/ℓ^3 fixed and, therefore, it is a special case of the allowed transformations for the variational problem and the first law. The weight of ω under this transformation follows from dimensional analysis.

2.6 5D uplift and relation to the BTZ black hole

The STU model (2.2.1) can be obtained by a circle reduction from a five-dimensional theory [58]. Kaluza-Klein reducing the resulting theory on an S^2 gives rise to Einstein-Hilbert gravity in three dimensions, coupled to several matter fields [50, 58, 61]. Through this sequence of uplifts and Kaluza-Klein reductions, the conical backgrounds (2.3.6) can be related to the BTZ black hole in three dimensions [61, 55, 62].

In this section we revisit the uplift of the truncated STU model (2.2.2) to five dimensions, as well as the reduction of the resulting 5D theory to three dimensions, keeping track of all surface terms on the boundary and on the horizon. As we will demonstrate, these terms are essential in order to connect the thermodynamics of the 4D black holes with that of the BTZ black hole. Moreover, we find that some continuous parameters of the 4D solutions must be quantized in order for the uplift to 5D to be possible, which explains the mismatch between the number of thermodynamic variables in four and three dimensions.

2.6.1 4D action from circle reduction

A consistent truncation of the 5D uplift of the STU model is given by the action [58, 61]

$$S_5 = \frac{1}{2\kappa_5^2}\int_{\widehat{\mathcal{M}}} \mathrm{d}^5\mathbf{x}\left(R[\widehat{g}]\star 1 - \frac{3}{2}\star\widehat{F}\wedge\widehat{F} + \widehat{F}\wedge\widehat{F}\wedge\widehat{A}\right) + \frac{1}{2\kappa_5^2}\int_{\partial\widehat{\mathcal{M}}} \mathrm{d}^4\mathbf{x}\,\sqrt{-\widehat{\gamma}}\,2K[\widehat{\gamma}], \tag{2.6.1}$$

where hats signify 5D quantities. If z is a compact dimension of length R_z, then the Kaluza-Klein ansatz

$$d\widehat{s}^2 = e^{\eta}ds^2 + e^{-2\eta}(dz + A^0)^2, \qquad \widehat{A} = \chi(dz + A^0) + A, \tag{2.6.2}$$

gives [71]

$$\sqrt{-\widehat{g}}\;R[\widehat{g}] = \sqrt{-g}\left(R[g] - \frac{3}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{4}e^{-3\eta}F^0_{\mu\nu}F^{0\mu\nu} - \Box_g\eta\right), \tag{2.6.3a}$$

$$\sqrt{-\widehat{g}}\;\frac{1}{4}\widehat{F}^2 = \sqrt{-g}\left(\frac{1}{4}e^{-\eta}(F+\chi F^0)_{\mu\nu}(F+\chi F^0)^{\mu\nu} + \frac{1}{2}e^{2\eta}\partial_\mu\chi\partial^\mu\chi\right), \tag{2.6.3b}$$

$$\widehat{F}\wedge\widehat{F}\wedge\widehat{A} = dz\wedge\left(3\chi F\wedge F + 3\chi^2 F\wedge F^0 + \chi^3 F^0\wedge F^0 - d\left(\chi^2 A\wedge F^0 + 2\chi A\wedge F\right)\right). \tag{2.6.3c}$$

In order to reduce the Gibbons-Hawking term we need the canonical decomposition (2.A.1) of the 5D metric, which takes the form

$$d\widehat{s}^2 = \widehat{N}^2du^2 + \widehat{\gamma}_{\hat{i}\hat{j}}dx^{\hat{i}}dx^{\hat{j}} = e^{\eta}N^2du^2 + e^{\eta}\gamma_{ij}dx^idx^j + e^{-2\eta}(dz + A^0_idx^i)^2, \tag{2.6.4}$$

where $\hat{i} = (z,i)$. In matrix form, therefore, the induced metric, $\widehat{\gamma}_{\hat{i}\hat{j}}$, on the four-dimensional radial slices $\widehat{\Sigma}_u$ is related to the induced fields on the three-dimensional radial slices Σ_u via

$$\widehat{\gamma}_{\hat{i}\hat{j}} = \begin{pmatrix} e^{-2\eta} & e^{-2\eta}A^0_i \\ e^{-2\eta}A^0_i & e^{\eta}\gamma_{ij} + e^{-2\eta}A^0_iA^0_j \end{pmatrix}, \qquad \widehat{\gamma}^{\hat{i}\hat{j}} = \begin{pmatrix} e^{2\eta} + e^{-\eta}A^0_kA^{0k} & -e^{-\eta}A^{0i} \\ -e^{-\eta}A^{0i} & e^{-\eta}\gamma^{ij} \end{pmatrix}. \tag{2.6.5}$$

From these expressions it is straightforward to compute $\det\widehat{\gamma} = e^{\eta}\det\gamma$. Moreover, the extrinsic curvature of $\widehat{\gamma}_{\hat{i}\hat{j}}$ is given by

$$K[\widehat{\gamma}]_{\hat{i}\hat{j}} = \frac{1}{2\widehat{N}}\dot{\widehat{\gamma}}_{\hat{i}\hat{j}}, \tag{2.6.6}$$

and can be expressed in terms of four-dimensional variables as

$$K[\widehat{\gamma}]_{zz} = -\frac{1}{N}e^{-5\eta/2}\dot{\eta}, \tag{2.6.7a}$$

$$K[\widehat{\gamma}]_{zi} = -\frac{1}{N}e^{-5\eta/2}\left(\dot{\eta}A^0_i - \frac{1}{2}\dot{A}^0_i\right), \tag{2.6.7b}$$

$$K[\widehat{\gamma}]_{ij} = \frac{1}{N}e^{-\eta/2}\left(\frac{1}{2}e^{\eta}\dot{\eta}\gamma_{ij} + e^{-2\eta}A^0_{(i}\dot{A}^0_{j)} - e^{-2\eta}\dot{\eta}A^0_iA^0_j + \frac{1}{2}e^{\eta}\dot{\gamma}_{ij}\right). \tag{2.6.7c}$$

In particular, the trace of the extrinsic curvature is given by

$$K[\widehat{\gamma}] = \widehat{\gamma}^{\hat{i}\hat{j}} K[\widehat{\gamma}]_{\hat{i}\hat{j}} = \frac{1}{2N} e^{-\eta/2}\dot{\eta} + e^{-\eta/2} K[\gamma], \tag{2.6.8}$$

which allows us to reduce the 5D Gibbons-Hawking term to 4D.

Combining the reduction formulae for the bulk and Gibbons-Hawking terms leads to the four-dimensional action

$$S_5 = S_4 - \frac{1}{2\kappa_4^2}\int_{\partial\mathcal{M}} \left(\chi^2 A\wedge F^0 + 2\chi A\wedge F\right) + \frac{1}{2\kappa_4^2}\int_{\mathcal{H}_+} \left(\chi^2 A\wedge F^0 + 2\chi A\wedge F\right), \tag{2.6.9}$$

where S_4 is the magnetic frame action (2.2.2), and the 5D and 4D gravitational constants are related via $\kappa_5^2 = R_z\kappa_4^2$. Hence, even though the 4D magnetic frame action can be obtained by a circle reduction from the 5D action (2.6.1), there are additional surface terms that are necessary for connecting the physics in 4 and 5 dimensions. In particular, the surface term on the boundary vanishes on-shell when evaluated on the conical backgrounds (2.3.6), but it is required in order to properly relate the 5D and 4D variational problems. Moreover, the surface term on the horizon is necessary to relate the free energies.

However, we also need to uplift the boundary counterterms (2.4.11) so that the five-dimensional on-shell action is free of long-distance divergences and the variational problem is well posed. Since

$$\sqrt{-\widehat{\gamma}} = \sqrt{-\gamma}\, e^{\eta/2}, \tag{2.6.10a}$$

$$\sqrt{-\widehat{\gamma}}\, R[\widehat{\gamma}] = \sqrt{-\gamma}\, e^{-\eta/2}\left(R[\gamma] - \frac{1}{8}\partial_i\eta\partial^i\eta - \frac{1}{4}e^{-3\eta}F^0_{ij}F^{0ij}\right) + \text{total derivative}, \tag{2.6.10b}$$

$$\sqrt{-\widehat{\gamma}}\, \widehat{F}_{ij}\widehat{F}^{ij} = \sqrt{-\gamma}\, e^{-\eta/2}\left(e^{-\eta}F_{ij}F^{ij} - 2e^{2\eta}\partial_i\chi\partial^i\chi\right), \tag{2.6.10c}$$

it follows that the boundary counterterms for the 4D action can be uplifted to five dimensions provided they are a linear combination of the expressions on the RHS of these identities. Moreover, the same counterterms must coincide with (2.4.11) up to finite local counterterms and at least for some specific value of the parameter α, or else the variational problem in four dimensions would not be well defined. The only way to reconcile these conditions is by setting $\alpha = 0$ in (2.4.11) and adding the finite local counterterm $\sqrt{-\gamma}e^{-\eta/2}(\partial\eta)^2$ with the appropriate coefficient.[10] The resulting boundary counterterms are

$$S'_{\text{ct}} = -\frac{1}{\kappa_4^2}\int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}\,\frac{1}{B}e^{\eta/2}\left(1 - \frac{1}{4}B^2e^{-\eta}R[\gamma] + \frac{1}{16}B^2e^{-4\eta}F^0_{ij}F^{0ij} + \frac{1}{32}B^2e^{-\eta}\partial_i\eta\partial^i\eta\right), \tag{2.6.11}$$

[10]Note that the value of the parameter α required for the uplift to 5D ($\alpha = 0$) is different from that required in the electric frame ($\alpha = 3$). This reflects the fact that the variational problems in the two cases are somewhat different, with the uplift to 5D only being possible provided B is kept fixed and ω is quantized in units of $1/2Bk$, as we will see below.

whose uplift is

$$S'_{\rm ct} = -\frac{1}{\kappa_5^2}\int \mathrm{d}^4\mathbf{x}\sqrt{-\widehat{\gamma}}\,\frac{1}{B}\left(1-\frac{1}{4}B^2R[\widehat{\gamma}]\right). \tag{2.6.12}$$

2.6.2 Uplifting conical backgrounds to 5D

Uplifting the conical black hole solutions (2.3.6) using the Kaluza-Klein ansatz (2.6.2) results in the 5D background [60]

$$\begin{aligned} d\widehat{s}^2 &= \frac{4B^2\rho^2 d\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)} - \frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{4B^2\rho^2}dt^2 + \rho^2\left(d\phi_3 - \frac{\rho_+\rho_-}{2B\rho^2}dt\right)^2 \\ &\quad + B^2\left(d\theta^2 + \sin^2\theta\,(d\phi + 2Bk\omega d\phi_3)^2\right), \qquad (2.6.13\text{a}) \\ \widehat{A} &= B\cos\theta\,(d\phi + 2Bk\omega d\phi_3), \qquad (2.6.13\text{b}) \end{aligned}$$

where the new coordinates ρ and ϕ_3 are defined through the relations

$$z = 2Bk\left(\frac{B}{\ell}\right)^3\phi_3, \qquad r = \frac{1}{(2Bk)^2}\left(\frac{B}{\ell}\right)^{-2}\rho^2. \tag{2.6.14}$$

In this coordinate system the 5D metric (2.6.13a) is immediately recognizable as a 2-sphere of radius B, fibered over a three-dimensional BTZ black hole [72] with AdS$_3$ radius $L = 2B$. Since the BTZ angular coordinate ϕ_3 must have periodicity 2π, the length R_z of the 5D circle is determined through (2.6.14) to be

$$R_z = 4\pi Bk\left(\frac{B}{\ell}\right)^3. \tag{2.6.15}$$

Given that the gravitational constants in four and five dimensions are related by $\kappa_5^2 = R_z\kappa_4^2$, this implies that the variational problem in five dimensions must be formulated keeping B fixed, in addition to kB^3/ℓ^3, which must be kept fixed even in four dimensions. Moreover, the internal S^2 has a conical singularity at the north and south poles unless $2Bk\omega$ is an integer. This implies that the conical backgrounds (2.3.6) can be uplifted to five dimensions if and only if ω is quantized in units of $1/(2Bk)$. With this condition, the internal part of the metric (2.6.13a) becomes the standard metric on S^2 with azimuthal coordinate $\phi' = \phi + n\phi_3$, where $n \in \mathbb{Z}$.

2.6.3 S^2 reduction and BTZ thermodynamics

The 5D action (2.6.1) can be Kaluza-Klein reduced on the internal S^2 using the reduction ansatz [61]

$$d\widehat{s}^2 = ds_3^2 + B^2 d\Omega_2^2, \qquad \widehat{A} = B\cos\theta\,(d\phi + 2Bk\omega d\phi_3). \tag{2.6.16}$$

The resulting theory in three dimensions is Einstein-Hilbert gravity

$$S_5 = S_3 = \frac{1}{2\kappa_3^2}\left(\int_{\mathcal{M}_3}\mathrm{d}^3\mathbf{x}\sqrt{-g_3}(R_3 - 2\Lambda_3) + \int_{\partial\mathcal{M}_3}\mathrm{d}^2\mathbf{x}\sqrt{-\gamma_2}\,2K_2\right), \tag{2.6.17}$$

with cosmological constant $\Lambda_3 = -1/(2B)^2$ and gravitational constant given by

$$\kappa_3^2 = \frac{\kappa_5^2}{(2B)^2\pi} = \frac{\kappa_4^2}{B} k \left(\frac{B}{\ell}\right)^3 . \tag{2.6.18}$$

Moreover, from (2.6.16) follows that

$$R[\widehat{\gamma}] = \frac{2}{B^2} + R[\gamma_2], \tag{2.6.19}$$

and so the boundary counterterms (2.6.12) for the five-dimensional theory reduce in three dimensions to the boundary terms

$$S'_{\rm ct} = -\frac{1}{\kappa_3^2} \int \mathrm{d}^2\mathbf{x}\sqrt{-\gamma_2}\, \left(\frac{1}{2B} - \frac{B}{4} R[\gamma_2]\right) . \tag{2.6.20}$$

The first term is the standard volume divergence of an AdS$_3$ space with radius $L = 2B$. The second term is proportional to the Euler density of the induced metric γ_2 and corresponds to a particular renormalization scheme. It shifts the on-shell action by a finite multiple of the Euler characteristic of the AdS$_3$ boundary. However, three-dimensional solutions with non-trivial z dependence, such as those obtained by turning on a generic metric source γ_2 on the AdS$_3$ boundary, excite Kaluza-Klein fields in the circle reduction to 4D and, therefore, are not captured by the STU model. 4D solutions of the STU model uplift to 5D solutions that are oxidized along the z coordinate, and consequently reduce to 3D solutions that can only have a non-trivial profile along an AdS$_2$ inside the AdS$_3$. For such solutions $R[\gamma_2]$ vanishes identically, which explains why the boundary counterterms (2.6.20) we obtained from the STU model do not include the logarithmic counterterm $-\frac{B}{2}R[\gamma_2]\log\epsilon^2$ corresponding to the conformal anomaly of the dual CFT$_2$ [3].

Combining (2.6.9) and (2.6.17), the renormalized action in three dimensions can be related to that of the STU model in the magnetic frame, namely

$$S_3 + S'_{\rm ct} = S_4 + S'_{\rm ct} + \frac{1}{2\kappa_4^2}\int_{\mathcal{H}_+} \left(\chi^2 A\wedge F^0 + 2\chi A\wedge F\right) , \tag{2.6.21}$$

where we have used the fact that the surface term on the boundary in (2.6.9) vanishes identically for the conical solutions (2.3.6). However, the contribution on the horizon is non-zero, which implies that the value of the Gibbs free energies in three and four dimensions do not coincide. More specifically, the complete set of relations between the BTZ thermodynamic variables [72]

$$\begin{aligned} &T_3 = \frac{\rho_+^2 - \rho_-^2}{2\pi L^2 \rho_+}, \quad S_3 = \frac{4\pi^2\rho_+}{\kappa_3^2}, \quad M_3 = \frac{\pi}{\kappa_3^2 L^2}\left(\rho_+^2 + \rho_-^2\right) , \\ &\Omega_3 = \frac{\rho_-}{L\rho_+}, \quad J_3 = \frac{2\pi\rho_+\rho_-}{\kappa_3^2 L}, \quad I_3 = \frac{\pi\beta_3}{\kappa_3^2 L^2}(\rho_-^2 - \rho_+^2), \end{aligned} \tag{2.6.22}$$

and the 4D ones computed in section 2.5 is

$$\begin{aligned}
&T_4 = T_3, \qquad S_4 = S_3, \qquad M_4 = M_3, \qquad I_4 + \frac{1}{2}\beta_4\Omega_4 J_4 = I_3,\\
&\Omega_4 = (2Bk\omega)\Omega_3 = n\Omega_3, \quad n \in \mathbb{Z}, \qquad J_4 = -(2Bk\omega)\frac{\pi L}{\kappa_3^2} = -\frac{n\pi L}{\kappa_3^2},\\
&\Phi_4^{0(e)} = Lk\left(\frac{B}{\ell}\right)^3\Omega_3, \qquad \Phi_4^{0(e)}Q_4^{0(e)} + \frac{1}{2}\Omega_4 J_4 = \Omega_3 J_3,\\
&\Phi_4^{(m)}Q_4^{(m)} + \frac{3}{2}\Omega_4 J_4 = -\frac{3}{2}T_3 S_3, \qquad Q_4^{(m)} = \frac{6\pi}{\kappa_3^2}k\left(\frac{B}{\ell}\right)^3.
\end{aligned} \tag{2.6.23}$$

Clearly, besides the mass, entropy and temperature, the relation between the 3D and 4D variables is non-trivial. In particular, the 3D thermodynamics ensemble corresponds to a subspace of the 4D ensemble, since the 4D angular momentum and magnetic charge are fixed constants in the 5D and 3D thermodynamics, which also renders the corresponding potentials Ω_4 and $\Phi_4^{(m)}$ redundant. This is a direct consequence of the fact that the magnetic field B must be kept fixed in the 5D and 3D variational problems, while the rotation parameter ω must be quantized in units of $1/(2Bk)$.

We end this section with the observation that inserting the relations (2.6.23) into the 4D quantum statistical relation (2.5.16) and the first law (2.5.17) we obtain the corresponding 3D thermodynamic identities, namely

$$I_3 = \beta_3(M_3 - T_3S_3 - \Omega_3 J_3), \tag{2.6.24}$$

and

$$\mathrm{d}M_3 = T_3\mathrm{d}S_3 + \Omega_3\mathrm{d}J_3. \tag{2.6.25}$$

The fact that J_4 and $Q_4^{(m)}$ must be kept fixed in the 3D variational problem is crucial for deriving the first law in three dimensions from its 4D counterpart. Moreover, the Smarr formula (2.5.18) gives

$$M_3 = \frac{1}{2}T_3S_3 + \Omega_3 J_3, \tag{2.6.26}$$

which can be verified explicitly from the expressions (2.6.22). This identity follows from the scaling transformation $\delta M_3 = 2\epsilon M_3$, $\delta J_3 = 2\epsilon J_3$, $\delta S_3 = \epsilon S_3$, corresponding to rescaling the BTZ parameters according to $\rho_\pm \to (1+\epsilon)\rho_\pm$.

2.A Radial Hamiltonian formalism

In this appendix we present in some detail the radial Hamiltonian formulation of the reduced STU σ-model (2.2.2). This analysis can be done abstractly, without reference to the explicit form of the σ-model functions $\mathcal{G}^{IJ}$, $\mathcal{Z}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$, and it therefore applies to the electric Lagrangian (2.2.10) as well, provided A_L, $\mathcal{Z}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are replaced with their electric frame analogues in (2.2.11).

The first step towards a Hamiltonian formalism is picking a suitable radial coordinate u such that constant-u slices, which we will denote by Σ_u, are diffeomorphic to the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. Moreover, it is convenient to choose u to be proportional to the geodesic distance between any fixed point in $\mathcal{M}$ and a point in Σ_u, such that[11] $\Sigma_u \to \partial\mathcal{M}$ as $u \to \infty$. Given the radial coordinate u, we then proceed with an ADM-like decomposition of the metric and gauge fields [25]

$$\begin{aligned} ds^2 &= (N^2 + N_i N^i)du^2 + 2N_i du dx^i + \gamma_{ij}dx^i dx^j, \\ A^L &= a^\Lambda du + A_i^\Lambda dx^i, \end{aligned} \tag{2.A.1}$$

where $\{x^i\} = \{t, \theta, \phi\}$. This is merely a field redefinition, trading the fully covariant fields $g_{\mu\nu}$ and A_μ^L for the induced fields N, N_i, γ_{ij}, a^Λ and A_i^Λ on Σ_u. Inserting this decomposition in the σ-model action (2.2.2) and adding the Gibbons-Hawking term (2.2.3) leads to the radial Lagrangian

$$\begin{aligned} L = \frac{1}{2\kappa_4^2}\int \mathrm{d}^3\mathbf{x} N\sqrt{-\gamma}\Big\{ &R[\gamma] + K^2 - K_{ij}K^{ij} - \frac{1}{2N^2}\mathcal{G}_{IJ}(\varphi)\left(\dot\varphi^I - N^i\partial_i\varphi^I\right)\left(\dot\varphi^J - N^j\partial_j\varphi^J\right) \\ &-\frac{2}{N^2}\mathcal{Z}_{\Lambda\Sigma}(\varphi)\gamma^{ij}\left(\dot A_i^\Lambda - \partial_i a^\Lambda - N^k F_{ki}^\Lambda\right)\left(\dot A_j^\Sigma - \partial_j a^\Sigma - N^l F_{lj}^\Sigma\right) \\ &-4\mathcal{R}_{\Lambda\Sigma}(\varphi)\epsilon^{ijk}\left(\dot A_i^\Lambda - \partial_i a^\Lambda\right)F_{jk}^\Sigma - \frac{1}{2}\mathcal{G}_{IJ}(\varphi)\partial_i\varphi^I\partial^i\varphi^J - \mathcal{Z}_{\Lambda\Sigma}(\varphi)F_{ij}^\Lambda F^{\Sigma ij}\Big\}, \end{aligned} \tag{2.A.2}$$

where

$$K_{ij} = \frac{1}{2N}\left(\dot\gamma_{ij} - D_i N_j - D_j N_i\right), \tag{2.A.3}$$

is the extrinsic curvature of the radial slices Σ_u, D_i denotes a covariant derivative with respect to the induced metric γ_{ij} on Σ_u, while a dot $\dot{}$ stands for a derivative with respect to the Hamiltonian 'time' u.

The canonical momenta conjugate to the induced fields on Σ_u following from the Lagrangian (2.A.2) are

$$\pi^{ij} = \frac{\delta L}{\delta\dot\gamma_{ij}} = \frac{1}{2\kappa_4^2}\sqrt{-\gamma}\left(K\gamma^{ij} - K^{ij}\right), \tag{2.A.4a}$$

$$\pi_I = \frac{\delta L}{\delta\dot\varphi^I} = -\frac{1}{2\kappa_4^2}N^{-1}\sqrt{-\gamma}\,\mathcal{G}_{IJ}\left(\dot\varphi^J - N^i\partial_i\varphi^J\right), \tag{2.A.4b}$$

$$\pi_\Lambda^i = \frac{\delta L}{\delta\dot A_i^\Lambda} = -\frac{2}{\kappa_4^2}N^{-1}\sqrt{-\gamma}\mathcal{Z}_{\Lambda\Sigma}\left(\gamma^{ij}\left(\dot A_j^\Sigma - \partial_j a^\Sigma\right) - N_j F^{\Sigma ji}\right) - \frac{2}{\kappa_4^2}\sqrt{-\gamma}\,\mathcal{R}_{\Lambda\Sigma}\epsilon^{ijk}F_{jk}^\Sigma. \tag{2.A.4c}$$

Notice that the momenta conjugate to N, N_i, and a^Λ vanish identically, since the Lagrangian (2.A.2) does not contain any radial derivatives of these fields. It follows that the fields N, N_i, and a^Λ are Lagrange multipliers, implementing three first class

[11]We assume $\mathcal{M}$ to be a non-compact space with infinite volume such that the geodesic distance between any point in the interior of $\mathcal{M}$ and a point in $\partial\mathcal{M}$ is infinite.

constraints, which we will derive momentarily. The canonical momenta (2.A.4) allow us to perform the Legendre transform of the Lagrangian (2.A.2) to obtain the radial Hamiltonian

$$H = \int \mathrm{d}^3\mathbf{x} \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_I\dot{\varphi}^I + \pi^i_\Lambda \dot{A}^\Lambda_i\right) - L = \int \mathrm{d}^3\mathbf{x} \left(N\mathcal{H} + N_i\mathcal{H}^i + a^\Lambda \mathcal{F}_\Lambda\right), \quad (2.A.5)$$

where

$$\begin{aligned}
\mathcal{H} =& -\frac{\kappa_4^2}{\sqrt{-\gamma}}\left(2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} + \mathcal{G}^{IJ}(\varphi)\pi_I\pi_J\right. \\
&\left. + \frac{1}{4}\mathcal{Z}^{\Lambda\Sigma}(\varphi)\left(\pi_{\Lambda i} + \frac{2}{\kappa_4^2}\sqrt{-\gamma}\mathcal{R}_{\Lambda M}(\varphi)\epsilon_i{}^{kl}F^M_{kl}\right)\left(\pi^i_\Sigma + \frac{2}{\kappa_4^2}\sqrt{-\gamma}\mathcal{R}_{\Sigma N}(\varphi)\epsilon^{ipq}F^N_{pq}\right)\right) \\
& + \frac{\sqrt{-\gamma}}{2\kappa_4^2}\left(-R[\gamma] + \frac{1}{2}\mathcal{G}_{IJ}(\varphi)\partial_i\varphi^I\partial^i\varphi^J + \mathcal{Z}_{\Lambda\Sigma}(\varphi)F^\Lambda_{ij}F^{\Sigma ij}\right), \qquad (2.A.6a) \\
\mathcal{H}^i =& -2D_j\pi^{ij} + \pi_I\partial^i\varphi^I + F^{\Lambda ij}\left(\pi_{\Lambda j} + \frac{2}{\kappa_4^2}\sqrt{-\gamma}\mathcal{R}_{\Lambda\Sigma}(\varphi)\epsilon_j{}^{kl}F^\Sigma_{kl}\right), \qquad (2.A.6b) \\
\mathcal{F}_\Lambda =& -D_i\pi^i_\Lambda. \qquad (2.A.6c)
\end{aligned}$$

Since the canonical momenta conjugate to the fields N, N_i, and a^Λ vanish identically, the corresponding Hamilton equations lead to the first class constraints

$$\mathcal{H} = \mathcal{H}^i = \mathcal{F}_\Lambda = 0, \quad (2.A.7)$$

which reflect respectively diffeomorphism invariance under radial reparameterizations, diffeomorphisms along the radial slices Σ_u and a $U(1)$ gauge invariance for every gauge field A^Λ_i.

Hamilton-Jacobi formalism

The first class constraints (2.A.7) are particularly useful in the Hamilton-Jacobi formulation of the dynamics, where the canonical momenta are expressed as gradients of Hamilton's principal function $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ as

$$\pi^{ij} = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \quad \pi^i_\Lambda = \frac{\delta\mathcal{S}}{\delta A^\Lambda_i}, \quad \pi_I = \frac{\delta\mathcal{S}}{\delta\varphi^I}. \quad (2.A.8)$$

Since the momenta conjugate to N, N_i, and a^Λ vanish identically, the functional $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ does not depend on these Lagrange multipliers. Inserting the expressions (2.A.8) for the canonical momenta in the first class constraints (2.A.7) leads to a set of functional partial differential equations for $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$. These are the Hamilton-Jacobi equations for the Lagrangian (2.A.2).

Given a solution $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ of the Hamilton-Jacobi equations, the radial evolution of the induced fields γ_{ij}, a^Λ and A^Λ_i is determined through the first order equations obtained by identifying the expressions (2.A.4) and (2.A.8) for the canonical momenta. Namely, gauge-fixing the Lagrange multipliers $N_i = a^\Lambda = 0$, but keeping N arbitrary, the resulting first order equations are

$$\frac{1}{N}\dot{\gamma}_{ij} = -\frac{4\kappa_4^2}{\sqrt{-\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}, \quad (2.A.9a)$$

$$\frac{1}{N}\dot{\varphi}^I = -\frac{2\kappa_4^2}{\sqrt{-\gamma}}\mathcal{G}^{IJ}(\varphi)\frac{\delta\mathcal{S}}{\delta\varphi^J}, \tag{2.A.9b}$$

$$\frac{1}{N}\dot{A}_i^\Lambda = -\frac{\kappa_4^2}{2\sqrt{-\gamma}}\mathcal{Z}^{\Lambda\Sigma}(\varphi)\gamma_{ij}\frac{\delta\mathcal{S}}{\delta A_j^\Sigma} - \mathcal{Z}^{\Lambda\Sigma}(\varphi)\mathcal{R}_{\Sigma P}(\varphi)\epsilon_i{}^{jk}F_{jk}^P. \tag{2.A.9c}$$

The complete solution of the equations of motion can be obtained by solving the Hamilton-Jacobi equations, together with the first order equations (2.A.9), without actually solving the second order equations of motion. Even though this may not seem an easier avenue to solve the system, it is a very efficient approach for obtaining asymptotic solutions of the equations of motion, which is all that is required in order to determine the boundary terms that render the variational problem well posed [13].

These boundary terms, commonly referred to as ‘boundary counterterms’, can in fact be read off a suitable asymptotic solution $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ of the Hamilton-Jacobi equations [13]. This is related to the fact that Hamilton's principal function generically coincides with the on-shell action,[12] up to terms that remain finite as $\Sigma_u \to \partial\mathcal{M}$. In particular, the divergent part of $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ coincides with that of the on-shell action. Adding, therefore, the boundary counterterms $S_{\rm ct} = -\mathcal{S}$ to the action, where $\mathcal{S}[\gamma, A^\Lambda, \varphi^I]$ is a suitable asymptotic solution of the Hamilton-Jacobi equations, not only renders the variational problem well posed, but also automatically ensures that the on-shell action remains finite as $\Sigma_u \to \partial\mathcal{M}$ [20, 13]. For asymptotically AdS backgrounds, the fact that the divergences of the on-shell action can be canceled by a solution of the radial Hamilton-Jacobi equation was first observed in [5].

2.B Evaluation of the 4D renormalized on-shell action

The easiest way to evaluate the renormalized on-shell action of the reduced STU model in the magnetic frame is to utilize the relation (2.6.21), namely

$$S_{\rm ren} = \lim_{r\to\infty}(S_4 + S'_{\rm ct}) = \lim_{r\to\infty}(S_3 + S'_{\rm ct}) - \frac{1}{2\kappa_4^2}\int_{\mathcal{H}_+}\left(\chi^2 A\wedge F^0 + 2\chi A\wedge F\right), \tag{2.B.1}$$

which relates $S_{\rm ren}$ to the renormalized on-shell action in three dimensions, plus a surface contribution from the outer horizon. The renormalized on-shell action in three dimensions is (see (2.6.22))

$$\lim_{r\to\infty}(S_3 + S'_{\rm ct}) = \frac{\pi\beta_3}{\kappa_3^2 L^2}\left(\rho_+^2 - \rho_-^2\right) = \frac{\beta_4 k\ell}{8G_4}(r_+ - r_-), \tag{2.B.2}$$

[12]This holds provided the Hamilton's principal function in question corresponds, through the first order equations (2.A.9), to asymptotic solutions satisfying the same boundary conditions as the solutions on which the action is evaluated.

where (2.6.14) and (2.6.18) have been used in the second step. Moreover, the parity-odd term on the horizon gives

$$\frac{1}{2\kappa_4^2}\int_{\mathcal{H}_+}\left(\chi^2 A\wedge F^0+2\chi A\wedge F\right)=\frac{k\ell^3}{2\kappa_4^2}\int \mathrm{d}^3\mathbf{x}\,\partial_\theta\left(\frac{\omega^2\sqrt{r_+r_-}\cos^3\theta}{r_++\omega^2\ell^2\sin^2\theta}\right)=-\frac{\beta_4}{4G_4}k\ell^3\omega^2\sqrt{\frac{r_-}{r_+}}. \tag{2.B.3}$$

Combining these two results we obtain

$$S_{\text{ren}}=\frac{k\ell}{8G_4}\left(r_+-r_--2\ell^2\omega^2\sqrt{\frac{r_-}{r_+}}\right). \tag{2.B.4}$$

A few comments are in order here. Firstly, although in the gauge in which the backgrounds (2.3.6) are given the parity-odd terms on the boundary in (2.6.9) give a zero contribution, this is not the case for a generic choice of gauge for the potential A. In general both contributions from the boundary and the horizon must be considered, and their difference is clearly gauge invariant.

A second comment concerns the potential dependence of the renormalized on-shell action on the parameter α. Here we have evaluated the renormalized on-shell action through the relation (2.6.21), which holds only for $\alpha=0$. However, evaluating the counterterms (2.4.11) for generic α we obtain

$$S_{\text{ct}}=-\frac{\ell}{\kappa_4^2}\int_{\mathcal{M}_{r_0}}\mathrm{d}^3\mathbf{x}\,\sin\theta\left(\frac{1}{2}r_0-\frac{1}{4}(r_++r_-)-\frac{\alpha}{8}\omega^2\ell^2(1+3\cos 2\theta)+\mathcal{O}\left(r_0^{-1}\right)\right), \tag{2.B.5}$$

where r_0 is the radial cut-off. It is obvious that the α-dependent term drops out after integration over θ, which implies that for all values of α we get the same result (2.B.4). Therefore, the renormalized on-shell action is independent of the choice of α.

The same conclusion holds for the finite counterterm $\sqrt{-\gamma}e^{-\eta/2}(\partial\eta)^2$ that was added in (2.6.11) in order to uplift the counterterms to five dimensions. Namely, this term does not contribute to the on-shell action since

$$\int \mathrm{d}^3\mathbf{x}\sqrt{-\gamma}e^{\eta/2}\left(e^{-\eta}\partial_i\eta\partial^i\eta\right)=\int \mathrm{d}^3\mathbf{x}\,\frac{k\omega^2\ell^3}{B}\left(\sin^3\theta-2\cos^2\theta\sin\theta\right)=0. \tag{2.B.6}$$

Hence, evaluating S_{ren} with S'_{ct} in (2.6.11) or with S_{ct} in (2.4.11) gives the same result (2.B.4).

Chapter 3

Supersymmetric holographic renormalization

3.1 Introduction

Supersymmetric (SUSY) field theories in curved backgrounds [35, 36, 26] (see also [32] for a recent review) have received much attention in recent years, since they provide a playground where physically interesting, non-perturbative, results can often be obtained through localization techniques [73, 74].

Formulating consistent SUSY field theories in curved space usually consists of two steps [35]; the first one is to find the classical supergravity theory (SUGRA) by coupling a flat-space supersymmetric (SUSY) field theory to the gravity multiplet, and the second one is to take a rigid limit of SUGRA such that the gravity multiplet becomes non-dynamical, but maintains a non-trivial background value. Consistency requires that there exists at least one SUSY transformation of the SUGRA under which this background gravity multiplet should be invariant, namely

$$\delta_\eta e^a_{(0)i} = 0, \quad \delta_\eta \Psi_{(0)+i} = 0, \quad \cdots , \tag{3.1.1}$$

where $e^a_{(0)i}$ refers to the vielbein and $\Psi_{(0)+i}$ is the gravitino field and η refers to the spinor parameter of the preserved SUSY. We refer to appendix 3.A and 3.B for notations and conventions. The requirement that the variation of the bosonic fields vanish is trivially satisfied on bosonic backgrounds.

One then derives the SUSY transformation of the local operators and the SUSY algebra in curved space from the corresponding ones of SUGRA. However, they are *classical* in the sense that the SUSY transformation laws and algebra derived in this way do not reflect any quantum effects.

To clarify this point, let us schematically discuss these quantum effects for a theory with an $\mathcal{N} = 1$ 4D superconformal field theory (SCFT) as a UV fixed point. For this aim, we derive the Ward identities which contain UV data of quantum field theories. These Ward identities can be obtained in a local renormalization group language [75] without relying on a classical Lagrangian description, see e.g. section 2.3 in [30] for a recent review. In $\mathcal{N} = 1$ SCFT, we have two local fermionic transformations, supersymmetry and super-Weyl, respectively

$$\delta_{\epsilon_+} e^a_{(0)i} = -\frac{1}{2}\overline{\Psi}_{(0)+i}\Gamma^a\epsilon_+, \quad \delta_{\epsilon_+}\Psi_{(0)+i} = \mathbb{D}_i\epsilon_+ + \cdots, \quad \cdots \tag{3.1.2a}$$

$$\delta_{\epsilon_-} e^a_{(0)i} = 0, \quad \delta_{\epsilon_-}\Psi_{(0)+i} = -\widehat{\Gamma}_i\epsilon_- + \cdots, \quad \cdots \tag{3.1.2b}$$

where the ellipses indicate possible contributions from other fields in the gravity multiplet and higer-order terms in fermions. Requiring the generating functional of connected correlation functions, $W[g_{(0)ij}, \Psi_{(0)+i}, \cdots]$, to be invariant under these local transformations up to a possible anomaly, we obtain two local operator equations, namely

$$\frac{1}{2}\mathcal{T}^i_a\overline{\Psi}_{(0)+i}\Gamma^a - \overline{\mathcal{S}}^i\overleftarrow{\mathbb{D}}_i + \cdots = \overline{\mathcal{A}}_s, \tag{3.1.3a}$$

$$-\overline{\mathcal{S}}^i\widehat{\Gamma}_{(0)i} + \cdots = \overline{\mathcal{A}}_{\mathrm{sW}}, \tag{3.1.3b}$$

where $\mathcal{T}^i_a$ and $\mathcal{S}^i$ refer to the energy-momentum tensor and supercurrent operator, respectively. Note that the Ward identities hold for generic backgrounds, even those where the fermionic sources are turned on. Combining these two Ward identities with the parameters η_+ and η_-, which satisfy conformal Killing spinor (CKS) condition

$$\delta_\eta\Psi_{(0)+i} \equiv \delta_{\eta_+}\Psi_{(0)+i} + \delta_{\eta_-}\Psi_{(0)+i} = \mathbb{D}_i\eta_+ - \widehat{\Gamma}_i\eta_- = 0, \tag{3.1.4}$$

to the lowest order in fermions, we obtain the SUSY-η Ward identity

$$-\frac{1}{2}\mathcal{T}^i_a\overline{\Psi}_{(0)+i}\Gamma^a\eta_+ + D_i(\overline{\mathcal{S}}^i\eta_+) + \cdots = -(\overline{\mathcal{A}}_s\eta_+ + \overline{\mathcal{A}}_{\mathrm{sW}}\eta_-) \equiv \mathcal{A}_\eta, \tag{3.1.5}$$

where the fermionic sources are still turned on, because the CKS equation (3.1.4) to the lowest order in fermions does not require the background to be bosonic. One can see from the operator equation (3.1.5) that the SUSY-η anomaly $\mathcal{A}_\eta$ should depend on the fermionic background sources, such as the gravitino field Ψ_{+i}. Therefore, one may not notice the existence of $\mathcal{A}_\eta$ on a bosonic background.

Ward identities such as (3.1.5) turn out to be rather useful.[1] For instance, they determine the variation of quantum operators under the corresponding symmetry transformations, see e.g. (2.3.7) in [76]. It then follows from (3.1.5) that on (bosonic) supersymmetric backgrounds the supercurrent operator $\mathcal{S}^i$ transforms under the SUSY-η transformation as

$$\delta_\eta\mathcal{S}^i\Big|_{\mathrm{susy-backgrounds}} = \Big(-\frac{1}{2}\mathcal{T}^i_a\Gamma^a\eta_+ - \frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\mathcal{A}_\eta + \cdots\Big)_{\mathrm{susy-backgrounds}}. \tag{3.1.6}$$

We emphasize that the anomalous term $\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\mathcal{A}_\eta$ does not appear in the 'classical' SUSY variation of the supercurrent operator $\mathcal{S}^i$, and it is non-zero in generic curved backgrounds admitting a conformal Killing spinor. Moreover, by integrating (3.1.6)

[1] One should keep in mind that the conservation law which allows to construct the conserved supercharge with non-covariantly-constant rigid parameter η_+ is $D_i(\overline{\mathcal{S}}^i\eta_+) = 0$, not $\overline{\mathcal{S}}^i\overleftarrow{\mathbb{D}}_i\eta_+ = 0$.

over a Cauchy surface, one can obtain the commutator of two supercharges (see e.g. (2.6.14) and (2.6.15) in [76]) and find that it is also corrected by the anomalous term.

The upshot is that once the Ward identities (3.1.3) are found, one can see immediately all these quantum corrections. The main obstacle in obtaining (3.1.3) is to find out the anomalies $\mathcal{A}_s$ and $\mathcal{A}_{\rm sW}$. Fortunately, we have a nice tool for computing the anomalies, namely the AdS/CFT correspondence [14, 16, 15]. The holographic computation of the quantum anomalies, such as the computation of the Weyl anomaly in [3], results in specific values for the anomaly coefficients. For instance, one gets $a = c$ Weyl anomaly from a holographic calculation of two-derivative supergravity in AdS$_5$. To obtain the whole class of anomalies one should consider a higher-derivative action. We emphasize that since the anomalies belonging to the same multiplet are related by SUSY transformations, the super-Weyl anomaly $\mathcal{A}_{\rm sW}$ obtained by a holographic computation also has specific values for the anomaly coefficients.

Henceforth, in order to obtain the Ward identities of 4D $\mathcal{N} = 1$ SCFT[2] by AdS/CFT, we consider a generic $\mathcal{N} = 2$ 5D gauged SUGRA, including its fermionic sector, in asymptotically locally AdS (AlAdS) spaces, particular examples of which were studied in [77, 78, 79, 80, 81, 82].[3] More specifically, the SUGRA theory we consider is specified by a scalar superpotential $\mathcal{W}$ and its field content consists of a vielbein, two gravitini, as well as an equal number of spin-1/2 and scalar fields with negative mass-squared in order for the space to be asymptotically AdS. All gauge fields are consistently set to zero for simplicity. We study this theory up to quadratic order in the fermions. Having a stable AlAdS solution requires that $\mathcal{W}$ has an isolated local extremum. We also demand that $\mathcal{W}$ is a analytic function around that point.

As indicated in [83, 80], the $\mathcal{N} = 2$ 5D gauged SUGRA can have a scalar superpotential $\mathcal{W}$ in several cases. A typical case is when there are only vector multiplets and a $U(1)_R$ (subgroup of $SU(2)_R$ R-symmetry group) is gauged [84]. When there are also hypermultiplets, the gauged SUGRA can have a scalar superpotential under a certain constraint related to the 'very special geometry' on the scalar manifold of the vector multiplets, which we do not discuss here in detail.

As in field theory, renormalization is required also in the bulk holographic computation. Although it has been studied since the early period of the AdS/CFT correspondence, most works on holographic renormalization (HR) [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] have focused on the bosonic sector. [85, 86, 87, 88, 89, 90, 81, 82] obtained some boundary counterterms for the fermionic sector, but typically these were limited to either lower dimensional spacetime (mainly 3 or 4 dimensions) or to homogeneous solutions which do not depend on the transverse directions. We note that in a context different from this chapter, 4D $\mathcal{N} = 1$ SUGRA including the fermionic sector was treated in [91] by a somehow *ad hoc* approach.

It turns out that the $\mathcal{N} = 1$ superconformal symmetry is broken by anomalies. From the bulk point of view, these anomalies are due to the fact that some of the first

[2] By comparing symmetries, one can immediately see that the holographic dual supergravity of 4D $\mathcal{N} = 1$ SCFT should be 5D $\mathcal{N} = 2$ SUGRA.

[3] Even though the solution considered in [82] is not AlAdS, the general form of the action given there is the same with the one here.

class constraints are non-linear functions of the canonical momenta, implying that the corresponding symmetries are broken by the radial cut-off. From the dual field theory point of view, of course, the global anomalies are a quantum effect. We obtain not only the SUSY-completion of the trace-anomaly, but also the holographic super-Weyl anomaly,[4] which are rather interesting by themselves, since they can provide another tool for testing the AdS/CFT correspondence.[5] As discussed before, we find that due to the anomaly, certain operators do not transform as tensors under super-Weyl transformation and the variation of operators gets an anomalous contribution, see (3.5.22). Hence, the Q-transformation of the operators also becomes anomalous, since it is obtained by putting together supersymmetry and super-Weyl transformations. Here Q refers to the preserved supercharge. This is rather remarkable, since it implies that the 'classical' SUSY variation cannot become a total derivative in the path integral of SUSY field theories in curved space, unless the anomaly effects disappear. In this regard, it is shown in [28] that the 'new' non-covariant finite counterterms suggested in [95, 96] should be discarded since they were introduced in order to match with field theory without taking into account the anomaly-effect. From the anomalous transformation of the supercurrent operator, we find that the supersymmetry algebra in curved space is corrected by anomalous terms, see (3.5.56).

We finally note that the boundary conditions consistent with SUSY should be specified before the main computation of HR. In this work we always impose Dirichlet boundary conditions for the metric and the gravitino. As we will see, consistency with SUSY requires that either Dirichlet or Neumann boundary conditions should be imposed for scalars and their SUSY-partner spin 1/2 fields, together at the same time.

3.2 $\mathcal{N} = 2$ gauged SUGRA action in 5D

The action of gauged (on-shell) ($D = d + 1 = 5$) SUGRA admitting a scalar superpotential, with all gauge fields consistently truncated, is given by [82]

$$S = S_b + S_f, \tag{3.2.1}$$

where

$$\begin{aligned} S_b &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\,\left(R[g] - \mathcal{G}_{IJ}(\varphi)\partial_\mu\varphi^I\partial^\mu\varphi^J - \mathcal{V}(\varphi)\right), \qquad (3.2.2)\\ S_f &= -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left\{\left(\overline{\Psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \overline{\Psi}_\mu\overleftarrow{\nabla}_\nu\Gamma^{\mu\nu\rho}\Psi_\rho - \mathcal{W}\overline{\Psi}_\mu\Gamma^{\mu\nu}\Psi_\nu\right)\right.\end{aligned}$$

[4] Notice that the existence of a super-Weyl anomaly is natural, due to the existence of a Weyl anomaly that is related to the super-Weyl anomaly by a SUSY transformation.

[5] As we will see in the main text, our result for the super-Weyl anomaly is different from [92], which was obtained through a field theory calculation using Feynman diagrams. In [93], they tried to obtain the holographic super-Weyl anomaly , but their work is incomplete since contribution from the Ricci curvature is missed. In any case, we show that our result satisfies the Wess-Zumino (WZ) consistency conditions. One can check that the result of [92] does not satisfy the consistency conditions. See [94] for a review of WZ consistency conditions.

$$
\begin{aligned}
&+\left(i\mathcal{G}_{IJ}\overline{\zeta}^I\Gamma^\mu\left(\partial\!\!\!/\varphi^J-\mathcal{G}^{JK}\partial_K\mathcal{W}\right)\Psi_\mu - i\mathcal{G}_{IJ}\overline{\Psi}_\mu(\partial\!\!\!/\varphi^I+\mathcal{G}^{IK}\partial_K\mathcal{W})\Gamma^\mu\zeta^J\right)\\
&+\left(\mathcal{G}_{IJ}\overline{\zeta}^I\left(\delta^J_K\overrightarrow{\nabla\!\!\!\!/}+\Gamma^J_{KL}[\mathcal{G}]\partial\!\!\!/\varphi^L\right)\zeta^K-\mathcal{G}_{IJ}\left[\overline{\zeta}^I\overleftarrow{\nabla\!\!\!\!/}\zeta^J+\overline{\zeta}^K(\partial\!\!\!/\varphi^L)\Gamma^J_{KL}\zeta^I\right]\right)\\
&+2\mathcal{M}_{IJ}(\varphi)\overline{\zeta}^I\zeta^J+\text{quartic terms}\Big\},
\end{aligned}
\tag{3.2.3}
$$

and the scalar potential and the mass matrix $\mathcal{M}_{IJ}$ are expressed in terms of the superpotential as

$$
\mathcal{V}(\varphi) = \mathcal{G}^{IJ}\partial_I\mathcal{W}(\varphi)\partial_J\mathcal{W}(\varphi) - \frac{d}{d-1}\mathcal{W}(\varphi)^2, \tag{3.2.4}
$$

$$
\mathcal{M}_{IJ}(\varphi) = \partial_I\partial_J\mathcal{W} - \Gamma^K_{IJ}[\mathcal{G}]\partial_K\mathcal{W} - \frac{1}{2}\mathcal{G}_{IJ}\mathcal{W}. \tag{3.2.5}
$$

Here κ^2 is related to the gravitational constant by $\kappa^2 = 8\pi G_{(d+1)}$. Note that near the conformal boundary of AlAdS spaces (with radius 1), which we are interested in, the scalar potential and the superpotential take respectively the form

$$
\mathcal{V}(\varphi) = -d(d-1) + \mathcal{O}\left(\varphi^2\right), \quad \mathcal{W}(\varphi) = -(d-1) + \mathcal{O}\left(\varphi^2\right). \tag{3.2.6}
$$

The action (3.2.1) is, up to boundary terms, invariant under the supersymmetry transformation[6]

$$
\delta_\epsilon\varphi^I = \frac{i}{2}\bar{\epsilon}\zeta^I + \text{h.c.} = \frac{i}{2}\left(\bar{\epsilon}\zeta^I - \overline{\zeta}^I\epsilon\right), \tag{3.2.7a}
$$

$$
\delta_\epsilon E^\alpha_\mu = \frac{1}{2}\bar{\epsilon}\Gamma^\alpha\Psi_\mu + \text{h.c.} = \frac{1}{2}\left(\bar{\epsilon}\Gamma^\alpha\Psi_\mu - \overline{\Psi}_\mu\Gamma^\alpha\epsilon\right), \tag{3.2.7b}
$$

where h.c. refers to hermitian conjugation, and

$$
\delta_\epsilon\zeta^I = -\frac{i}{2}\left(\partial\!\!\!/\varphi^I - \mathcal{G}^{IJ}\partial_J\mathcal{W}\right)\epsilon, \tag{3.2.8a}
$$

$$
\delta_\epsilon\Psi_\mu = \left(\nabla_\mu + \frac{1}{2(d-1)}\mathcal{W}\Gamma_\mu\right)\epsilon. \tag{3.2.8b}
$$

for any value of d.

Two comments are in order about the action (3.2.1). Firstly, all the fermions here, including the supersymmetry transformation parameter ϵ, are Dirac fermions. In fact, in $\mathcal{N}=2$ five-dimensional SUGRA, the gravitino field is expressed in terms of a symplectic Majorana spinor [97], which can also be described in terms of a Dirac fermion [79]. Other fermions in the theory can also be expressed in the same way. Secondly, we would like to be as general as possible and thus, we keep d generic in most of the following computations.

[6] In [82] the transformation rule of the gravitino field is given by $\delta_\epsilon\Psi_\mu = (\nabla_\mu + \frac{1}{6}\mathcal{W}\Gamma_\mu)\epsilon$, which is obtained by setting $D=5$ explicitly in (3.2.8).

3.3 Radial Hamiltonian dynamics

According to the holographic dictionary [15] the on-shell action of the supergravity theory is the generating functional of the dual field theory. Therefore, the first step of the holographic computation is usually to consider the on-shell action on the bulk side. As is well-known, this on-shell action always suffers from long-distance divergences, which corresponds to the UV divergences of the dual field theory. Therefore, we need to renormalize the on-shell action of the supergravity theory, through holographic renormalization [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The Hamiltonian formulation of holographic renormalization [5, 10, 12, 24] is arguably the most efficient, and as we will see, it helps make the analysis of the fermions tractable. The Hamiltonian constraint, one of the first class constraints obtained from the radial Hamiltonian, gives the Hamilton-Jacobi (HJ) equation by which we can obtain all the infinite counterterms for generic sources and curved background. Holographic renormalization essentially consists in determining all divergent terms in the on-shell action for generic background and sources in covariant form and subtracting them. Depending on the problem under consideration one can add some extra finite counterterms which actually correspond to the choice of renormalization scheme in the boundary field theory.

In this section we obtain the radial Hamiltonian, from which we extract the first class constraints. Afterwards, we present a general algorithm for obtaining the full counterterms from the HJ equation. We then obtain the flow equations which are needed to form a complete set of equations of motion.

3.3.1 Radial Hamiltonian

The Gibbons-Hawking term (2.2.3) [33]

$$\frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^dx\sqrt{-\gamma}\, K, \tag{3.3.1}$$

where K is the extrinsic curvature on the boundary $\partial\mathcal{M}$, was introduced to have a well-defined variational problem for the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\, R. \tag{3.3.2}$$

As indicated in [81, 86, 85, 89, 87], for the same reason some additional boundary terms are needed when the theory involves fermionic fields. For the action (3.2.1) these fermionic boundary terms turn out to be (for details, see appendix 3.C.1 and 3.C.2)

$$\pm\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^dx\sqrt{-\gamma}\, \overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j, \tag{3.3.3a}$$

$$\pm\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^dx\sqrt{-\gamma}\, \mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J, \tag{3.3.3b}$$

where the signs in front of the terms bilinear in fermionic fields fixes which radiality (see (3.B.10)) of the fermion should be used as a generalized coordinate. Note, however, that the sign depends on the mass of the fermions and the choice of boundary conditions [87]. Since the mass of the gravitino Ψ_μ is $(d-1)/2 > 0$, the sign of (3.3.3a) should be positive (see also appendix 3.B.3 and 3.B.4). The sign of the mass of ζ^I changes according to the model, and thus we cannot choose the sign of (3.3.3b) *a priori*.

For the time being, however, let us pick the + sign. As we will discuss in section 3.6, picking the − sign corresponds to imposing Neumann boundary conditions on the spin-1/2 field ζ^I. We emphasize that this choice of sign will not affect our claim later about the determination of the scalar fields' leading asymptotics. The whole action including the terms (2.2.3) and (3.3.3) is then given by

$$S_{\rm full} = S + \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^d\mathrm{x}\sqrt{-\gamma}\ \left(2K + \overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j + \mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J\right). \tag{3.3.4}$$

The full action $S_{\rm full}$ can be written as $S_{\rm full} = \int dr\ L$, where the radial Lagrangian L is

$$\begin{aligned}
L = \frac{1}{2\kappa^2}\int_{\Sigma_r} d^dx\ N\sqrt{-\gamma}\Bigg\{ & R[\gamma] - \mathcal{G}_{IJ}\partial_i\varphi^I\partial^i\varphi^J - \mathcal{V}(\varphi) + (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{ij}K_{kl} \\
& - \frac{\mathcal{G}_{IJ}}{N^2}(\dot\varphi^I - N^i\partial_i\varphi^I)(\dot\varphi^J - N^j\partial_j\varphi^J) + \frac{2}{N}\left(\dot{\overline{\Psi}}_{+i}\widehat{\Gamma}^{ij}\Psi_{-j} + \overline{\Psi}_{-i}\widehat{\Gamma}^{ij}\dot\Psi_{+j}\right) \\
& + \frac{1}{N}\dot e^i_a e^j_b\left(\overline{\Psi}_i\Gamma^{ab}\Psi_j + \overline{\Psi}_j\Gamma^{ba}\Psi_i\right) + \left(K + \frac{1}{N}D_kN^k\right)\overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j + \frac{1}{4N}e_{ak}\dot e_b{}^k\overline{\Psi}_i\Gamma\{\widehat{\Gamma}^{ij},\Gamma^{ab}\}\Psi_j \\
& + \frac{1}{2N}K_{kl}\left[\left(\overline{\Psi}_r - N^i\overline{\Psi}_i\right)[\widehat{\Gamma}^{kj},\widehat{\Gamma}^l]\Psi_j - \overline{\Psi}_j[\widehat{\Gamma}^{kj},\widehat{\Gamma}^l]\left(\Psi_r - N^i\Psi_i\right)\right] \\
& + \frac{1}{4N}\overline{\Psi}_i\left(2\partial_kN[\widehat{\Gamma}^{ij},\widehat{\Gamma}^k] - (D_kN_l)\Gamma\{\widehat{\Gamma}^{ij},\widehat{\Gamma}^{kl}\}\right)\Psi_j \\
& - \frac{N^i}{N}\left(\overline{\Psi}_j\Gamma\widehat{\Gamma}^{jk}\mathbb{D}_i\Psi_k - \overline{\Psi}_j\overleftarrow{\mathbb{D}}_i\Gamma\widehat{\Gamma}^{jk}\Psi_k\right) - \overline{\Psi}_i\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_k + \overline{\Psi}_i\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ijk}\Psi_k \\
& - \frac{1}{N}\overline{\Psi}_k\overleftarrow{\mathbb{D}}_j\Gamma\widehat{\Gamma}^{jk}\left(\Psi_r - N^i\Psi_i\right) - \frac{1}{N}\left(\overline{\Psi}_r - N^i\overline{\Psi}_i\right)\Gamma\widehat{\Gamma}^{jk}\mathbb{D}_j\Psi_k \\
& + \frac{1}{N}\overline{\Psi}_k\Gamma\widehat{\Gamma}^{jk}\left(\mathbb{D}_j\Psi_r - N^i\mathbb{D}_j\Psi_i\right) + \frac{1}{N}\left(\overline{\Psi}_r\overleftarrow{\mathbb{D}}_j - N^i\overline{\Psi}_i\overleftarrow{\mathbb{D}}_j\right)\Gamma\widehat{\Gamma}^{jk}\Psi_k \\
& + \frac{1}{N}\mathcal{W}\left[\left(\overline{\Psi}_r - N^i\overline{\Psi}_i\right)\Gamma\widehat{\Gamma}^j\Psi_j + \overline{\Psi}_j\widehat{\Gamma}^j\Gamma\left(\Psi_r - N^i\Psi_i\right)\right] + \mathcal{W}\overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j \\
& + \frac{2}{N}\mathcal{G}_{IJ}\left(\overline{\zeta}^I_+\dot\zeta^J_- + \dot{\overline{\zeta}}^I_-\zeta^J_+\right) + \left(K + \frac{1}{N}D_kN^k\right)\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J - \frac{1}{2N}\mathcal{G}_{IJ}e_{ai}\dot e^i_b\overline{\zeta}^I\Gamma^{ab}\Gamma\zeta^J \\
& + \frac{1}{N}\left(\dot\varphi^K - N^i\partial_i\varphi^K + N^i\partial_i\varphi^K\right)\partial_K\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J - \mathcal{G}_{IJ}\left(\overline{\zeta}^I\widehat{\Gamma}^i\mathbb{D}_i\zeta^J - \overline{\zeta}^I\overleftarrow{\mathbb{D}}_i\widehat{\Gamma}^i\zeta^J\right) \\
& - \frac{1}{N}\mathcal{G}_{IJ}\left[-\frac{1}{2}D_iN_j\left(\overline{\zeta}^I\widehat{\Gamma}^{ij}\Gamma\zeta^J\right) - N^i\overline{\zeta}^I\Gamma\mathbb{D}_i\zeta^J + N^i(\overline{\zeta}^I\overleftarrow{\mathbb{D}}_i)\Gamma\zeta^J\right] \\
& - \frac{i}{N}\mathcal{G}_{IJ}\Bigg[\frac{1}{N}\left(\dot\varphi^J - N^j\partial_j\varphi^J\right)\left[\overline{\zeta}^I\left(\Psi_r - N^i\Psi_i + N\widehat{\Gamma}^i\Gamma\Psi_i\right) - \left(\overline{\Psi}_r - N^i\overline{\Psi}_i + N\overline{\Psi}_i\Gamma\widehat{\Gamma}^i\right)\zeta^I\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \partial_i \varphi^J \left[\overline{\zeta}^I \Gamma \widehat{\Gamma}^i \left(\Psi_r - N^j \Psi_j \right) - \left(\overline{\Psi}_r - N^j \overline{\Psi}_j \right) \widehat{\Gamma}^i \Gamma \zeta^I \right] + N \partial_i \varphi^J \left(\overline{\zeta}^I \widehat{\Gamma}^j \widehat{\Gamma}^i \Psi_j - \overline{\Psi}_j \widehat{\Gamma}^i \widehat{\Gamma}^j \zeta^I \right) \Bigg] \\
&+ \frac{i}{N} \partial_I \mathcal{W} \left[\overline{\zeta}^I \Gamma \left(\Psi_r - N^i \Psi_i \right) + \left(\overline{\Psi}_r - N^i \overline{\Psi}_i \right) \Gamma \zeta^I + N \left(\overline{\Psi}_i \widehat{\Gamma}^i \zeta^I + \overline{\zeta}^I \widehat{\Gamma}^i \Psi_i \right) \right] \\
&- \frac{1}{N} \partial_K \mathcal{G}_{IJ} \left[\left(\dot{\varphi}^J - N^i \partial_I \varphi^J \right) \left(\overline{\zeta}^I \Gamma \zeta^K - \overline{\zeta}^K \Gamma \zeta^I \right) + N \partial_i \varphi^J \left(\overline{\zeta}^I \widehat{\Gamma}^i \zeta^K - \overline{\zeta}^K \widehat{\Gamma}^i \zeta^I \right) \right] \\
&- 2 \mathcal{M}_{IJ} \overline{\zeta}^I \zeta^J \Bigg\}.
\end{aligned} \tag{3.3.5}
$$

Given the radial Lagrangian L we can derive the canonical momenta

$$
\begin{aligned}
\pi_a{}^i = \frac{\delta L}{\delta \dot{e}^a_i} = \left(\delta^i_j e_{ak} + \delta^i_k e_{aj} \right) \frac{\sqrt{-\gamma}}{2\kappa^2} \Bigg[& \left(\gamma^{jk} \gamma^{lm} - \gamma^{jl} \gamma^{km} \right) K_{lm} + \frac{1}{2} \gamma^{jk} \left(\mathcal{G}_{IJ} \overline{\zeta}^I \zeta^J + \overline{\Psi}_p \widehat{\Gamma}^{pq} \Psi_q \right) \\
& - \frac{1}{4N} \left(\overline{\Psi}_p [\widehat{\Gamma}^{jp}, \widehat{\Gamma}^k] \left(\Psi_r - N^l \Psi_l \right) - \left(\overline{\Psi}_r - N^l \overline{\Psi}_l \right) [\widehat{\Gamma}^{jp}, \widehat{\Gamma}^k] \Psi_p \right) \Bigg] \\
& - \frac{\sqrt{-\gamma}}{2\kappa^2} \left[e^{bi} \left(\frac{1}{4} \overline{\Psi}_j \Gamma \{ \widehat{\Gamma}^{jk}, \Gamma_{ab} \} \Psi_k - \frac{1}{2} \mathcal{G}_{IJ} \overline{\zeta}^I \Gamma_{ab} \Gamma \zeta^J \right) + e_{aj} \left(\overline{\Psi}^j \widehat{\Gamma}^{ik} \Psi_k + \overline{\Psi}_k \widehat{\Gamma}^{ki} \Psi^j \right) \right],
\end{aligned} \tag{3.3.6a}
$$

$$
\begin{aligned}
\pi^\varphi_I = \frac{\delta L}{\delta \dot{\varphi}^I} = \frac{\sqrt{-\gamma}}{2N\kappa^2} \Big[& - 2 \mathcal{G}_{IJ} \left(\dot{\varphi}^J - N^i \partial_i \varphi^J \right) + N \partial_I \mathcal{G}_{JK} \overline{\zeta}^J \zeta^K - N \partial_K \mathcal{G}_{IJ} \left(\overline{\zeta}^J \Gamma \zeta^K - \overline{\zeta}^K \Gamma \zeta^J \right) \\
& - i \mathcal{G}_{IJ} \left(\overline{\zeta}^J \left(\Psi_r - N^i \Psi_i + N \widehat{\Gamma}^i \Gamma \Psi_i \right) - \left(\overline{\Psi}_r - N^i \overline{\Psi}_i + N \overline{\Psi}_i \Gamma \widehat{\Gamma}^i \right) \zeta^J \right) \Big],
\end{aligned} \tag{3.3.6b}
$$

$$
\pi^\zeta_I = L \frac{\overleftarrow{\delta}}{\delta \dot{\zeta}^I_-} = \frac{\sqrt{-\gamma}}{\kappa^2} \mathcal{G}_{IJ} \overline{\zeta}^J_+, \tag{3.3.6c}
$$

$$
\pi^{\overline{\zeta}}_I = \frac{\overrightarrow{\delta}}{\delta \dot{\overline{\zeta}}^I_-} L = \frac{\sqrt{-\gamma}}{\kappa^2} \mathcal{G}_{IJ} \zeta^J_+, \tag{3.3.6d}
$$

$$
\pi^i_\Psi = L \frac{\overleftarrow{\delta}}{\delta \dot{\Psi}_{+i}} = \frac{\sqrt{-\gamma}}{\kappa^2} \overline{\Psi}_{-j} \widehat{\Gamma}^{ji}, \tag{3.3.6e}
$$

$$
\pi^i_{\overline{\Psi}} = \frac{\overrightarrow{\delta}}{\delta \dot{\overline{\Psi}}_{+i}} L = \frac{\sqrt{-\gamma}}{\kappa^2} \widehat{\Gamma}^{ij} \Psi_{-j}. \tag{3.3.6f}
$$

One should keep in mind that $\pi^i_{\overline{\Psi}}$ and π^i_Ψ have negative radiality, and π^ζ_I and π^ζ_I have positive radiality.

From $K_{ij} = K_{ji}$, we obtain the constraint

$$
\begin{aligned}
0 = \mathcal{J}_{ab} \equiv & \frac{\kappa^2}{\sqrt{-\gamma}} (e^i_a \pi_{bi} - e^i_b \pi_{ai}) - \frac{1}{4} \overline{\Psi}_j \Gamma \{ \widehat{\Gamma}^{jk}, \Gamma_{ab} \} \Psi_k + \frac{1}{2} \mathcal{G}_{IJ} \overline{\zeta}^I \Gamma_{ab} \Gamma \zeta^J \\
& - \frac{1}{2} e^i_a e^j_b (\overline{\Psi}_i \widehat{\Gamma}_{jk} \Psi^k + \overline{\Psi}^k \widehat{\Gamma}_{kj} \Psi_i - \overline{\Psi}_j \widehat{\Gamma}_{ik} \Psi^k - \overline{\Psi}^k \widehat{\Gamma}_{ki} \Psi_j),
\end{aligned} \tag{3.3.7}
$$

which, as we will see, corresponds to the local Lorentz generator of the frame bundle on the slice Σ_r [89].

Inverting the canonical momenta[7] and implementing the Legendre transform we obtain the radial Hamiltonian

$$
\begin{aligned}
H = & \int d^d x \left(\dot{e}^a_i \pi_a^{\ i} + \dot{\varphi}^I \pi^\varphi_I + \pi^\zeta_I \dot{\zeta}^I_- + \dot{\overline{\zeta}}{}^I_- \pi^{\overline{\zeta}}_I + \pi^i_\Psi \dot{\Psi}_{+i} + \dot{\overline{\Psi}}_{+i} \pi^i_{\overline{\Psi}} \right) - L \\
= & \int d^d x \left[N\mathcal{H} + N_i \mathcal{H}^i + \left(\overline{\Psi}_r - N^i \overline{\Psi}_i \right) \mathcal{F} + \overline{\mathcal{F}} \left(\Psi_r - N^i \Psi_i \right) \right],
\end{aligned}
\tag{3.3.8}
$$

where

$$
\begin{aligned}
\mathcal{H} = & \frac{\kappa^2}{2\sqrt{-\gamma}} \Bigg[\left(\frac{1}{d-1} e^a_i e^b_j - e^a_j e^b_i \right) \pi^i_a \pi^j_b - \mathcal{G}^{IJ} \pi^\varphi_I \pi^\varphi_J + \mathcal{G}^{IJ} \left(\pi^\zeta_I \not{\mathbb{D}} \pi^{\overline{\zeta}}_J - \pi^\zeta_I \overleftarrow{\not{\mathbb{D}}} \pi^{\overline{\zeta}}_J \right) \\
& - \frac{1}{2(d-1)} \left(e^{aj} \pi^i_a + e^{ai} \pi^j_a \right) \Big[(d-1)(\overline{\Psi}_{+i} \pi_{\overline{\Psi} j} + \pi_{\Psi j} \Psi_{+i}) + \pi^p_\Psi \left(\widehat{\Gamma}_{pi} - (d-2)\gamma_{pi} \right) \widehat{\Gamma}_j^{\ k} \Psi_{+k} \\
& + \overline{\Psi}_{+k} \widehat{\Gamma}^k{}_j \left(\widehat{\Gamma}_{ip} - (d-2)\gamma_{ip} \right) \pi^p_{\overline{\Psi}} \Big] + \frac{1}{d-1} e^a_i \pi^i_a \left(-\overline{\zeta}{}^I_- \pi^{\overline{\zeta}}_I - \pi^\zeta_I \zeta^I_- + \overline{\Psi}_{+j} \pi^j_{\overline{\Psi}} + \pi^j_\Psi \Psi_{+j} \right) \\
& + 2\mathcal{G}^{IJ} \Gamma^L_{JK}[\mathcal{G}] \pi^\varphi_I \left(\overline{\zeta}{}^K_- \pi^{\overline{\zeta}}_L + \pi^\zeta_L \zeta^K_- \right) + i\pi^\varphi_I \Big[\frac{1}{d-1} \left(\overline{\zeta}{}^I_- \widehat{\Gamma}_i \pi^i_{\overline{\Psi}} + \pi^i_\Psi \widehat{\Gamma}_i \zeta^I_- \right) \\
& - \mathcal{G}^{IJ} \left(\pi^\zeta_I \widehat{\Gamma}^i \Psi_{+i} + \overline{\Psi}_{+i} \widehat{\Gamma}^i \pi^{\overline{\zeta}}_J \right) \Big] - \pi^k_\Psi \left[\left(\frac{1}{d-1} \widehat{\Gamma}_k \widehat{\Gamma}_j - \gamma_{kj} \right) \not{\mathbb{D}} - \overleftarrow{\not{\mathbb{D}}} \left(\frac{1}{d-1} \widehat{\Gamma}_k \widehat{\Gamma}_j - \gamma_{kj} \right) \right] \pi^j_{\overline{\Psi}} \\
& + \frac{i}{d-1} \left(\pi^\zeta_I \not{\partial} \varphi^I \widehat{\Gamma}_i \pi^i_{\overline{\Psi}} - \pi^i_\Psi \widehat{\Gamma}_i \not{\partial} \varphi^I \pi^{\overline{\zeta}}_I \right) - 2i\partial_i \varphi^I \left(\pi^\zeta_I \pi^i_{\overline{\Psi}} - \pi^i_\Psi \pi^{\overline{\zeta}}_I \right) \\
& + \mathcal{G}^{IM} \mathcal{G}^{KN} \partial_i \varphi^J \left(\partial_K \mathcal{G}_{IJ} - \partial_I \mathcal{G}_{KJ} \right) \pi^\zeta_M \widehat{\Gamma}^i \pi^{\overline{\zeta}}_N \Bigg] \\
& - \frac{1}{2} \mathcal{W} \left(\overline{\Psi}_{+i} \pi^i_{\overline{\Psi}} + \pi^i_\Psi \Psi_{+i} \right) + \mathcal{M}_{IJ} \left(\mathcal{G}^{IK} \pi^\zeta_K \zeta^J_- + \mathcal{G}^{JK} \overline{\zeta}{}^I_- \pi^{\overline{\zeta}}_K \right) \\
& - \frac{i}{2} \partial_I \mathcal{W} \left[\mathcal{G}^{IJ} \left(\overline{\Psi}_{+i} \widehat{\Gamma}^i \pi^{\overline{\zeta}}_J + \pi^\zeta_J \widehat{\Gamma}^i \Psi_{+i} \right) + \frac{1}{d-1} \left(\pi^i_\Psi \widehat{\Gamma}_i \zeta^I_- + \overline{\zeta}{}^I_- \widehat{\Gamma}_i \pi^i_{\overline{\Psi}} \right) \right] \\
& + \frac{\sqrt{-\gamma}}{2\kappa^2} \Bigg[- R[\gamma] + \mathcal{G}_{IJ} \partial_i \varphi^I \partial^i \varphi^J + \mathcal{V}(\varphi) + \mathcal{G}_{IJ} \overline{\zeta}{}^I_- \left(\not{\mathbb{D}} - \overleftarrow{\not{\mathbb{D}}} \right) \zeta^J_- + \overline{\Psi}_{+i} \widehat{\Gamma}^{ijk} \left(\mathbb{D}_j - \overleftarrow{\mathbb{D}}_j \right) \Psi_{+k} \\
& + D_k \left(\overline{\Psi}_{+i} \left(\gamma^{jk} \widehat{\Gamma}^i - \gamma^{ik} \widehat{\Gamma}^j \right) \Psi_{+j} \right) + i\mathcal{G}_{IJ} \partial_i \varphi^J \left(\overline{\zeta}{}^I_- \widehat{\Gamma}^j \widehat{\Gamma}^i \Psi_{+j} - \overline{\Psi}_{+j} \widehat{\Gamma}^i \widehat{\Gamma}^j \zeta^I_- \right) \\
& + \partial_K \mathcal{G}_{IJ} \partial_i \varphi^J \left(\overline{\zeta}{}^I_- \widehat{\Gamma}^i \zeta^K_- - \overline{\zeta}{}^K_- \widehat{\Gamma}^i \zeta^I_- \right) \Bigg],
\end{aligned}
\tag{3.3.9}
$$

$$
\begin{aligned}
\mathcal{H}^i = & - e^{ai} D_j \pi_a^{\ j} + (\partial^i \varphi^I) \pi^\varphi_I + (\overline{\zeta}{}^I_- \overleftarrow{\mathbb{D}}{}^i) \pi^{\overline{\zeta}}_I + \pi^\zeta_I (\mathbb{D}^i \zeta^I_-) + \pi^j_\Psi \left(\mathbb{D}^i \Psi_{+j} \right) + \left(\overline{\Psi}_{+j} \overleftarrow{\mathbb{D}}{}^i \right) \pi^j_{\overline{\Psi}} \\
& - D_j (\pi^j_\Psi \Psi^i_+ + \overline{\Psi}{}^i_+ \pi^j_{\overline{\Psi}}),
\end{aligned}
\tag{3.3.10}
$$

[7]For instance, the inverse of the canonical momentum $\pi^i_{\overline{\Psi}}$ is $\Psi_{-i} = \frac{\kappa^2}{\sqrt{-\gamma}} \frac{1}{d-1} [\widehat{\Gamma}_{ij} - (d-2)\gamma_{ij}] \pi^j_{\overline{\Psi}}$.

$$\mathcal{F} = \frac{2\kappa^2}{\sqrt{-\gamma}}\left\{\frac{1}{4(d-1)}\widehat{\Gamma}_i\pi^i_{\Psi}e^a_j\pi^j_a - \frac{1}{8}\Gamma^a\gamma_{ik}\pi^k_{\Psi}\pi^i_a - \frac{1}{8}e^a_l\widehat{\Gamma}_i\pi^l_{\Psi}\pi^i_a + \frac{i}{4}\mathcal{G}^{IJ}\pi^{\varphi}_I\pi^{\overline{\zeta}}_J\right\}$$
$$+ \frac{1}{4}\Gamma^a\Psi_{+i}\pi^i_a + \frac{1}{4}\widehat{\Gamma}_i\Psi_{+j}e^{aj}\pi^i_a + \frac{i}{2}\pi^{\varphi}_I\zeta^I_- - \mathbb{D}_i\pi^i_{\Psi} - \frac{1}{2(d-1)}\mathcal{W}\widehat{\Gamma}_i\pi^i_{\Psi} - \frac{i}{2}\partial_i\varphi^I\widehat{\Gamma}^i\pi^{\overline{\zeta}}_I$$
$$- \frac{i}{2}\mathcal{G}^{IJ}\partial_I\mathcal{W}\pi^{\overline{\zeta}}_J + \frac{\sqrt{-\gamma}}{2\kappa^2}\left(2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} + \mathcal{W}\widehat{\Gamma}^i\Psi_{+i} + i\mathcal{G}_{IJ}\partial_i\varphi^J\widehat{\Gamma}^i\zeta^I_- + i(\partial_I\mathcal{W})\zeta^I_-\right). \tag{3.3.11}$$

We note that in the above computations we used the local Lorentz constraint (3.3.7).

By radiality we split $\mathcal{F}$ into two parts

$$\mathcal{F}_+ \equiv \Gamma_+\mathcal{F} = \frac{\kappa^2}{2\sqrt{-\gamma}}\left[\pi^j_a e^{ak}\left(\frac{1}{d-1}\gamma_{jk}\widehat{\Gamma}_i - \frac{1}{2}\gamma_{ij}\widehat{\Gamma}_k - \frac{1}{2}\gamma_{ik}\widehat{\Gamma}_j\right)\pi^i_{\Psi} + i\mathcal{G}^{IJ}\pi^{\varphi}_I\pi^{\overline{\zeta}}_J\right]$$
$$- \frac{1}{2(d-1)}\mathcal{W}\widehat{\Gamma}_i\pi^i_{\Psi} - \frac{i}{2}\mathcal{G}^{IJ}\partial_I\mathcal{W}\pi^{\overline{\zeta}}_J + \frac{\sqrt{-\gamma}}{2\kappa^2}\left(2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} + i\mathcal{G}_{IJ}\partial_i\varphi^J\widehat{\Gamma}^i\zeta^I_-\right), \tag{3.3.12}$$

and

$$\mathcal{F}_- \equiv \Gamma_-\mathcal{F} = \frac{1}{4}\left(\widehat{\Gamma}_i\Psi_{+j} + \widehat{\Gamma}_j\Psi_{+i}\right)e^{aj}\pi^i_a + \frac{i}{2}\pi^{\varphi}_I\zeta^I_-$$
$$- \mathbb{D}_i\pi^i_{\Psi} - \frac{i}{2}\partial_i\varphi^I\widehat{\Gamma}^i\pi^{\overline{\zeta}}_I + \frac{\sqrt{-\gamma}}{2\kappa^2}\left(\mathcal{W}\widehat{\Gamma}^i\Psi_{+i} + i\partial_I\mathcal{W}\zeta^I_-\right). \tag{3.3.13}$$

The canonical momenta for N, N_i and Ψ_r vanish identically, and it then follows from Hamilton's equations that

$$\mathcal{H} = \mathcal{H}^i = \mathcal{F}_- = \mathcal{F}_+ = 0. \tag{3.3.14}$$

These first class constraints reflect respectively radial reparameterization invariance and diffeomorphism, supersymmetry and super-Weyl invariance along the radial slice Σ_r, which can be seen by comparing with (3.B.38).

Inserting (3.3.6) in (3.3.7), we obtain

$$0 = e^i_a\pi_{bi} - e^i_b\pi_{ai} + \frac{1}{2}\overline{\zeta}^I_-\Gamma_{ab}\pi^{\overline{\zeta}}_I - \frac{1}{2}\pi^{\zeta}_I\Gamma_{ab}\zeta^I_- - \frac{1}{2}\pi^i_{\Psi}\Gamma_{ab}\Psi_{+i} + \frac{1}{2}\overline{\Psi}_{+i}\Gamma_{ab}\pi^i_{\overline{\Psi}}, \tag{3.3.15}$$

which reflects the local frame rotation symmetry of the theory according to (3.B.38). We emphasize that at the bosonic level this local Lorentz constraint reduces to

$$e^i_a\pi_{bi} = e^i_b\pi_{ai}, \tag{3.3.16}$$

which implies that we can define a symmetric canonical momentum for the metric through the relations

$$\frac{\delta}{\delta\dot{\gamma}_{ij}}L^B \equiv \pi^{ij} = \frac{1}{2}e^{aj}\frac{\delta}{\delta\dot{e}^a_i}L^B. \tag{3.3.17}$$

Here L^B denotes bosonic part of the radial Lagrangian (3.3.5).

We emphasize that the linearity of the constraints $\mathcal{H}^i = \mathcal{F}_- = 0$ and the local Lorentz constraint reflect the fact that their corresponding symmetries are not broken by the radial cut-off. Meanwhile, the constraints $\mathcal{H} = 0$ and $\mathcal{F}_+ = 0$ are quadratic in the momenta, implying that in fact the cut-off breaks these symmetries, though they are non-linearly realized in the bulk.

3.3.2 Hamilton-Jacobi equations and the holographic renormalization

The HJ equations are obtained by inserting the expressions

$$\pi_a^{\ i} = \frac{\delta}{\delta e_i^a}\mathbb{S}, \quad \pi_I^\varphi = \frac{\delta}{\delta \varphi^I}\mathbb{S}, \quad \pi_I^\zeta = \mathbb{S}\frac{\overleftarrow{\delta}}{\delta \zeta_-^I}, \quad \pi_I^{\bar{\zeta}} = \frac{\overrightarrow{\delta}}{\delta \bar{\zeta}_-^I}\mathbb{S}, \quad \pi_\Psi^i = \mathbb{S}\frac{\overleftarrow{\delta}}{\delta \Psi_{+i}}, \quad \pi_{\bar{\Psi}}^i = \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}_{+i}}\mathbb{S} \tag{3.3.18}$$

for the canonical momenta in the first class constraints (3.3.14). Here $\mathbb{S}[e, \varphi, \zeta_-, \Psi_+]$ is Hamilton's principal functional.

Hamilton's principal functional $\mathbb{S}$ is particularly important since it can be identified with the on-shell action evaluated with a radial cut-off Σ_r. Holographically renormalizing the on-shell action only requires solving these HJ equations for $\mathbb{S}$ up to the finite terms, without relying on the specific solution of the equations of motion. Since this asymptotic solution of the HJ equations is obtained in covariant form for generic sources, we can identify the divergent terms with the sought after boundary counterterms, which cancel the divergences of the on-shell action as well as of all correlation functions.

As pointed out in [98], the constraint $\mathcal{H}^i = 0$ and the local Lorentz constraint (3.3.15) which reflects the bulk diffeomorphism invariance along the transverse direction is automatically satisfied as long as we look for a local and covariant solution. Hence, the equations we have to solve are the constraints $\mathcal{H} = \mathcal{F}_- = \mathcal{F}_+ = 0$.

Let us briefly review the algorithm of solving the HJ equation in AlAdS geometry. In general, the Hamiltonian constraint is solved asymptotically by using the formal expansion of $\mathbb{S}$ with respect to the dilatation operator δ_D [12] (see section 5.2 of [30] for a recent review)

$$\delta_D = \int d^d x \ \sum_\Phi (\Delta_\Phi - d)\frac{\delta}{\delta \Phi}, \tag{3.3.19}$$

where Φ refers to every field in the theory and Δ_Φ denotes the scaling dimension of the operator dual to Φ. The solution takes form of

$$\mathbb{S} = \int_{\Sigma_r} d^d x \sqrt{-\gamma}\ \mathcal{L} = \int_{\Sigma_r} d^d x \sqrt{-\gamma}\ \Big(\mathcal{L}_{[0]} + \mathcal{L}_{[1]} + \cdots + \widetilde{\mathcal{L}}_{[d]} \log e^{-2r} + \mathcal{L}_{[d]} + \cdots\Big), \tag{3.3.20}$$

where

$$\delta_D \mathcal{L}_{[n]} = -n\mathcal{L}_{[n]}, \ 0 \le n < d, \qquad \delta_D \widetilde{\mathcal{L}}_{[d]} = -d\widetilde{\mathcal{L}}_{[d]}. \tag{3.3.21}$$

Since the dilatation operator δ_D asymptotically coincides with the radial derivative

$$\partial_r = \int_{\Sigma_r} d^d x \ \sum_\Phi \dot{\Phi}\frac{\delta}{\delta \Phi}, \tag{3.3.22}$$

in AlAdS, one can see that $\mathcal{L}_{[n]}$ for $n < d$ and $\widetilde{\mathcal{L}}_{[d]}$ are asymptotically divergent, and can therefore be identified with the boundary counterterms, namely

$$\mathbb{S}_{\text{ct}} = -\int_{\Sigma_r} d^dx\sqrt{-\gamma}\,\Big(\mathcal{L}_{[0]} + \mathcal{L}_{[1]} + \cdots + \widetilde{\mathcal{L}}_{[d]}\log e^{-2r}\Big). \tag{3.3.23}$$

By construction, this is the full set of all possible divergent terms.

This general argument of finding $\mathbb{S}_{\text{ct}}$ is not suitable in our case, since the operator δ_D requires knowledge of all scaling dimensions in the theory from the onset. Since we do not want to specify the scaling dimension of the scalars φ^I and of the fermions ζ^I in advance, we will instead seek a solution for $\mathbb{S}$ in an expansion in eigenfunctions of the alternative operator

$$\delta_e = \int d^dx\,\left(e_i^a\frac{\delta}{\delta e_i^a} + \frac{1}{2}\overline{\Psi}_{+i}\frac{\delta}{\overline{\Psi}_{+i}} + \frac{1}{2}\frac{\overleftarrow{\delta}}{\delta\Psi_{+i}}\Psi_{+i}\right), \tag{3.3.24}$$

rather than δ_D [24, 98], since we know that the scaling dimension of the operators dual to e_i^a and Ψ_{+i} in AlAdS are $d+1$ and $d+1/2$ respectively, see appendix 3.B.3. Note that δ_e basically counts powers of the vielbein and the gravitino. The formal expansion of Hamilton's principal function $\mathbb{S}[e,\varphi,\zeta_-,\Psi_+]$ with respect to δ_e is thus

$$\mathbb{S} = \mathbb{S}_{(0)} + \mathbb{S}_{(1)} + \mathbb{S}_{(2)} + \cdots, \quad \mathbb{S}_{(k)} \equiv \int d^dx\,\mathbb{L}_{(k)}, \tag{3.3.25}$$

where $\delta_e\mathbb{S}_{(k)} = (d-k)\mathbb{S}_{(k)}$. This implies that

$$\pi^i_{(k)a}e_i^a + \frac{1}{2}\pi^i_{(k)\Psi}\Psi_{+i} + \frac{1}{2}\overline{\Psi}_{+i}\pi^i_{(k)\overline{\Psi}} = (d-k)\mathbb{L}_{(k)} + \partial_i v^i_{(k)}, \tag{3.3.26}$$

for certain $v^i_{(k)}$. However, the Lagrangian $\mathbb{L}_{(k)}$ is defined up to a total derivative, and thus we can put [24]

$$\pi^i_{(k)a}e_i^a + \frac{1}{2}\pi^i_{(k)\Psi}\Psi_{+i} + \frac{1}{2}\overline{\Psi}_{+i}\pi^i_{(k)\overline{\Psi}} := (d-k)\mathbb{L}_{(k)}. \tag{3.3.27}$$

As we will see later, this identification of $\mathbb{L}_{(k)}$ greatly simplifies the HJ equation and makes it almost algebraic.

By using (3.3.27) we can solve the HJ equation recursively, but this procedure stops at $\mathbb{S}_{(d)}$, which has δ_e weight zero. The reason why higher-order terms, which are finite in $r\to\infty$ limit, cannot be determined in this recursive procedure is that they are related to the arbitrary integration constants which form a complete integral together with the integration constants from the flow equations, see [24] for explanation in more detail.

Assuming that the all scalar and spin-1/2 operators are not irrelevant, we find that any term with negative δ_e weight should have negative dilatation weight, see (3.3.19). This implies that all the divergent terms appear up to $\mathbb{S}_{(d)}$ so that we can identify the counterterms as

$$S_{\text{ct}} = -\sum_{k=0}^{2d}\mathbb{S}_{(k/2)}, \tag{3.3.28}$$

Note that the logarithmically divergent terms are distributed in almost all of the $\mathbb{S}_{(k)}$s with $0 \leq k \leq d$. Since our radial slice is four-dimensional, these terms appear with the pole $1/(d-4)$. Converting this pole by (dimensional regularization) [12, 24]

$$\frac{1}{d-4} \to -\frac{1}{2}\log e^{-2r}, \tag{3.3.29}$$

and summing up all of them, we obtain the logarithmically divergent terms $\widetilde{\mathcal{L}}_{[d]}$. We emphasize that the two algorithms we described in fact give the same result for $S_{\rm ct}$.

Once the local counterterms $S_{\rm ct}$ are obtained, we renormalize the on-shell action by

$$\widehat{S}_{ren} = \lim_{r\to+\infty}(S_{\rm full} + S_{\rm ct}) = \lim_{r\to+\infty}\int_{\Sigma_r} d^dx\ \mathcal{L}_{[d]}. \tag{3.3.30}$$

The canonical momenta are automatically renormalized by $\mathbb{S}_{\rm ct}$, namely

$$\widehat{\pi}^{\Phi} \equiv \pi^{\Phi} + \frac{\delta}{\delta\Phi}S_{\rm ct},\ \text{for every field } \Phi, \tag{3.3.31}$$

and the variation of the renormalized on-shell action under any variation of fields is given by the chain rule

$$\delta\widehat{S}_{ren} = \lim_{r\to+\infty}\int d^dx\left(\widehat{\pi}^i_a\delta e^a_i + \widehat{\pi}^{\varphi}_I\delta\varphi^I + \delta\overline{\zeta}^I_-\widehat{\pi}^{\zeta}_I + \widehat{\pi}^{\zeta}_I\delta\zeta^I_- + \delta\overline{\Psi}_{+i}\widehat{\pi}^i_{\Psi} + \widehat{\pi}^i_{\Psi}\delta\Psi_{+i}\right). \tag{3.3.32}$$

3.3.3 Flow equations and leading asymptotics

The flow equations are obtained by substituting (3.3.18) into Hamilton's equations

$$\dot{e}^a_i = \frac{\delta H}{\delta\pi^i_a},\quad \dot{\pi}^i_a = -\frac{\delta H}{\delta e^a_i}, \tag{3.3.33a}$$

$$\dot{\varphi}^I = \frac{\delta H}{\delta\pi^{\varphi}_I},\quad \dot{\pi}^{\varphi}_I = -\frac{\delta H}{\delta\varphi^I}, \tag{3.3.33b}$$

$$\dot{\zeta}^I_- = \frac{\delta}{\delta\pi^{\zeta}_I}H,\quad \dot{\pi}^{\zeta}_I = -H\frac{\delta}{\delta\zeta_-},\quad \dot{\overline{\zeta}}^I_- = H\frac{\delta}{\delta\pi^{\overline{\zeta}}_I},\quad \dot{\pi}^{\overline{\zeta}}_I = -\frac{\delta}{\delta\overline{\zeta}^I_-}H, \tag{3.3.33c}$$

$$\dot{\Psi}_{+i} = \frac{\delta}{\delta\pi^i_{\Psi}}H,\quad \dot{\pi}^i_{\Psi} = -H\frac{\delta}{\delta\Psi_{+i}},\quad \dot{\overline{\Psi}}_{+i} = H\frac{\delta}{\delta\pi^i_{\overline{\Psi}}},\quad \dot{\pi}^i_{\overline{\Psi}} = -\frac{\delta}{\delta\overline{\Psi}_{+i}}H. \tag{3.3.33d}$$

The resulting flow equations are

$$\begin{aligned}\dot{e}^a_i = \frac{\kappa^2}{2\sqrt{-\gamma}}\Bigg\{&2\left(\frac{1}{d-1}e^a_ie^b_j - e^a_je^b_i\right)\pi^j_b - \frac{1}{2(d-1)}e^{aj}\Big[(d-1)(\overline{\Psi}_{+i}\pi_{\overline{\Psi}j} + \pi_{\Psi j}\Psi_{+i})\\ &- \pi^p_{\Psi}[\widehat{\Gamma}_{pi} - (d-2)\gamma_{pi}]\widehat{\Gamma}_j{}^k\Psi_{+k} + \overline{\Psi}_{+k}\widehat{\Gamma}^k{}_j[\widehat{\Gamma}_{ip} - (d-2)\gamma_{ip}]\pi^p_{\overline{\Psi}} + (i\leftrightarrow j)\Big]\\ &+ \frac{1}{d-1}e^a_i\left(-\overline{\zeta}^I_-\pi^{\overline{\zeta}}_I - \pi^{\zeta}_I\zeta^I_- + \overline{\Psi}_{+j}\pi^j_{\overline{\Psi}} + \pi^j_{\Psi}\Psi_{+j}\right)\Bigg\},\end{aligned} \tag{3.3.34}$$

$$
\begin{aligned}
\dot{\varphi}^I = \frac{\kappa^2}{\sqrt{-\gamma}} \mathcal{G}^{IJ} \Big[& - \pi^{\varphi}_J + \Gamma^K_{JL}[\mathcal{G}] \left(\pi^{\zeta}_K \zeta^L_- + \overline{\zeta}^L_- \pi^{\overline{\zeta}}_K \right) - \frac{i}{2} \left(\pi^{\zeta}_J \widehat{\Gamma}^i \Psi_{+i} + \overline{\Psi}_{+i} \widehat{\Gamma}^i \pi^{\overline{\zeta}}_J \right) \Big] \\
& + \frac{\kappa^2}{\sqrt{-\gamma}} \frac{i}{2(d-1)} \left(\overline{\zeta}^I_- \widehat{\Gamma}_i \pi^i_{\overline{\Psi}} + \pi^i_{\Psi} \widehat{\Gamma}_i \zeta^I_- \right),
\end{aligned} \tag{3.3.35}
$$

$$
\begin{aligned}
\dot{\Psi}_{+i} = \frac{\kappa^2}{2\sqrt{-\gamma}} \Big[& - \frac{1}{2} \left(\delta^k_i e^{aj} + \gamma^{jk} e^a_i \right) \pi^k_a \Psi_{+j} + \frac{1}{d-1} e^a_j \pi^j_a \Psi_{+i} + i \pi^{\varphi}_I \widehat{\Gamma}_i \zeta^I_- \\
& - \frac{1}{2(d-1)} \left(e^{aj} \pi^l_a + e^{al} \pi^j_a \right) \left(\widehat{\Gamma}_{il} - (d-2) \gamma_{il} \right) \widehat{\Gamma}_j{}^k \Psi_{+k} - \frac{i}{d-1} \widehat{\Gamma}_i \partial\!\!\!/ \varphi^I \pi^{\overline{\zeta}}_J \\
& - \frac{2}{d-1} \left(\widehat{\Gamma}_{ijk} - (d-2) \gamma_{ij} \widehat{\Gamma}_k \right) \mathbb{D}^k \pi^j_{\overline{\Psi}} + 2i \partial_i \varphi^I \pi^{\overline{\zeta}}_I \Big] - \frac{1}{2} \mathcal{W} \Psi_{+i} \\
& - \frac{i}{2(d-1)} \partial_I \mathcal{W} \widehat{\Gamma}_i \zeta^I_-,
\end{aligned} \tag{3.3.36}
$$

and

$$
\begin{aligned}
\dot{\zeta}^I_- = \frac{\kappa^2}{2\sqrt{-\gamma}} \Big[& 2 \mathcal{G}^{IJ} \mathbb{D}\!\!\!/ \pi^{\overline{\zeta}}_J + \partial_i \mathcal{G}^{IJ} \widehat{\Gamma}^i \pi^{\overline{\zeta}}_J - \frac{1}{d-1} e^a_i \pi^i_a \zeta^I_- + 2 \mathcal{G}^{LJ} \Gamma^I_{JK}[\mathcal{G}] \pi^{\varphi}_L \zeta^K_- \\
& - i \mathcal{G}^{IJ} \pi^J_{\varphi} \widehat{\Gamma}^i \Psi_{+i} - 2i \partial_i \varphi^I \pi^i_{\overline{\Psi}} + \mathcal{G}^{IM} \mathcal{G}^{KN} \partial_i \varphi^J (\partial_K \mathcal{G}_{JM} - \partial_M \mathcal{G}_{KJ}) \widehat{\Gamma}^i \pi^{\overline{\zeta}}_N \Big] \\
& + \mathcal{M}_{JK} \mathcal{G}^{IK} \zeta^J_- - \frac{i}{2} \partial^I \mathcal{W} \widehat{\Gamma}^i \Psi_{+i}.
\end{aligned} \tag{3.3.37}
$$

Here for simplicity we choose the gauge (3.B.13), which reduces the radial Hamiltonian H to $H = \int d^dx\ \mathcal{H}$. We emphasize that the flow equations (3.3.34), (3.3.35), (3.3.36) and (3.3.37), together with the HJ equations, form a complete set of equations of motion of the theory.[8]

3.4 Solution of the Hamilton-Jacobi equation

To solve the HJ equation efficiently we divide Hamilton's principal function into several parts according to the structure of the various terms. Namely, we first split $\mathbb{S}$ into two sectors: $\mathbb{S}^B$, the purely bosonic part, and $\mathbb{S}^F$, which is quadratic in fermions. The terms in $\mathbb{S}^F$ are further split into three parts: $\mathbb{S}^{\zeta\zeta}$ which contains quadratic terms in ζ^I_-s, $\mathbb{S}^{\Psi\Psi}$ containing quadratic terms in Ψ_{+i} and $\mathbb{S}^{\zeta\Psi}$, containing bilinears in ζ^I_- and Ψ_{+i}. In total,

$$\mathbb{S} = \mathbb{S}^B + \mathbb{S}^{\zeta\zeta} + \mathbb{S}^{\Psi\Psi} + \mathbb{S}^{\zeta\Psi}. \tag{3.4.1}$$

[8] One can use the flow equations (3.3.36) and (3.3.37) to determine the asymptotic behavior of Ψ_{+i} and ζ^I_-, as is done in appendix 3.B.3, instead of using the Euler-Lagrange equations (3.B.19) and (3.B.20).

Due to radiality and the Lorentz structure of the fermionic sources, the asymptotic expansion of $\mathbb{S}^B$, $\mathbb{S}^{\zeta\Psi}$, $\mathbb{S}^{\zeta\zeta}$ and $\mathbb{S}^{\Psi\Psi}$ should be

$$\mathbb{S}^B = \mathbb{S}^B_{(0)} + \mathbb{S}^B_{(2)} + \mathbb{S}^B_{(4)} + \cdots , \tag{3.4.2a}$$

$$\mathbb{S}^{\zeta\Psi} = \mathbb{S}^{\zeta\Psi}_{(3/2)} + \mathbb{S}^{\zeta\Psi}_{(7/2)} + \cdots , \tag{3.4.2b}$$

$$\mathbb{S}^{\zeta\zeta} = \mathbb{S}^{\zeta\zeta}_{(1)} + \mathbb{S}^{\zeta\zeta}_{(3)} + \mathbb{S}^{\zeta\zeta}_{(5)} + \cdots , \tag{3.4.2c}$$

$$\mathbb{S}^{\Psi\Psi} = \mathbb{S}^{\Psi\Psi}_{(2)} + \mathbb{S}^{\Psi\Psi}_{(4)} + \cdots . \tag{3.4.2d}$$

How to solve the HJ equation for the bosonic sector has been discussed in the literature [12, 24, 99], though it is difficult to solve the HJ equation for a completely general model.[9] The key feature is that after finding the solution of the HJ equation to leading order, we only need to solve a (almost algebraic) first-order differential equation for the higher orders, thanks to the relation (3.3.27). Nevertheless, these first-order differential equations are not easy to solve at the first attempt.

Here we have another set of first-order differential equations, namely $\mathcal{F}_- = \mathcal{F}_+ = 0$. These are relatively simpler than the Hamiltonian constraint $\mathcal{H} = 0$, so one can try to solve these constraints first. Not surprisingly, it works well, in particular for the fermionic sector, and the solution is totally consistent with the other constraints, as we will see soon.

3.4.1 Bosonic sector

Let us first consider the bosonic sector. The corresponding Hamiltonian constraint $\mathcal{H} = 0$ is

$$\frac{\kappa^2}{2\sqrt{-\gamma}}\left[4\left(\frac{1}{d-1}\gamma_{ij}\gamma_{kl} - \gamma_{ik}\gamma_{jl}\right)\frac{\delta\mathbb{S}^B}{\delta\gamma_{ij}}\frac{\delta\mathbb{S}^B}{\delta\gamma_{kl}} - \mathcal{G}^{IJ}\frac{\delta\mathbb{S}^B}{\delta\varphi^I}\frac{\delta\mathbb{S}^B}{\delta\varphi^J}\right] + \frac{\sqrt{-\gamma}}{2\kappa^2}\left(-R[\gamma] + \mathcal{G}_{IJ}\partial_i\varphi^I\partial^i\varphi^J + \mathcal{V}(\varphi)\right) = 0. \tag{3.4.3}$$

One can readily see that the HJ equation for $\mathbb{S}_{(0)}$ is

$$\frac{\kappa^2}{2\sqrt{-\gamma}}\left[4\left(\frac{1}{d-1}\gamma_{ij}\gamma_{kl} - \gamma_{ik}\gamma_{jl}\right)\frac{\delta\mathbb{S}_{(0)}}{\delta\gamma_{ij}}\frac{\delta\mathbb{S}_{(0)}}{\delta\gamma_{kl}} - \mathcal{G}^{IJ}\frac{\delta\mathbb{S}_{(0)}}{\delta\varphi^I}\frac{\delta\mathbb{S}_{(0)}}{\delta\varphi^J}\right] + \frac{\sqrt{-\gamma}}{2\kappa^2}\mathcal{V}(\varphi) = 0. \tag{3.4.4}$$

The leading term of $\mathbb{S}$, $\mathbb{S}_{(0)}$, should not contain any derivatives and must be purely bosonic so that its ansatz becomes

$$\mathbb{S}_{(0)} = -\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\,U(\varphi). \tag{3.4.5}$$

Substituting this ansatz into the constraint $\mathcal{F}_- = 0$, we obtain

$$\frac{1}{4}\left(\widehat{\Gamma}_i\Psi_{+j} + \widehat{\Gamma}_j\Psi_{+i}\right)e^{aj}\frac{\delta\mathbb{S}_{(0)}}{\delta e^a_i} + \frac{\sqrt{-\gamma}}{2\kappa^2}\mathcal{W}\widehat{\Gamma}_i\Psi_{+i} = 0, \tag{3.4.6}$$

[9] One might try to solve the HJ equation for the general scalar-gravity model by using the argument in [24].

and find the unique solution for $U(\varphi)$ given by $U = \mathcal{W}(\varphi)$, or

$$\mathbb{S}_{(0)} = -\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\,\mathcal{W}. \tag{3.4.7}$$

As promised, we obtain (3.4.7) regardless of the sign of (3.3.3b). It follows that the leading asymptotics of the scalar field φ^I is also determined, independently of the sign chosen in (3.3.3b), as we see in (3.4.8c). From (3.4.7) we can now determine the leading asymptotics of the fields by using the above flow equations, namely

$$e_i^a(r,x) \sim e^r e^a_{(0)i}(x), \tag{3.4.8a}$$

$$\Psi_{+i}(r,x) \sim e^{r/2}\Psi_{(0)+i}(x), \tag{3.4.8b}$$

$$\dot{\varphi}^I \sim \mathcal{G}^{IJ}\partial_J\mathcal{W}, \quad \text{or } \varphi^I \sim e^{-\mu^I r}\varphi^I_{(0)}, \tag{3.4.8c}$$

$$\dot{\zeta}^I_- \sim -\frac{1}{2}\zeta^I_- + (\mathcal{G}^{IK}\partial_J\partial_K\mathcal{W})\zeta^J_-, \quad \text{or } \zeta^I_- \sim e^{-(\mu^I+\frac{1}{2})r}\zeta^I_{-(0)}, \tag{3.4.8d}$$

where μ^I stands for the radial weight of φ^I when the scalars are properly diagonalized.

Now let us go to the next order of the bosonic sector. The HJ equation for $\mathbb{S}^B_{(2)}$ is then

$$-\frac{2}{d-1}\mathcal{W}\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}\mathbb{S}^B_{(2)} + \mathcal{G}^{IJ}\partial_I\mathcal{W}\frac{\delta}{\delta\varphi^J}\mathbb{S}^B_{(2)} + \frac{\sqrt{-\gamma}}{2\kappa^2}\left(-R[\gamma] + \mathcal{G}_{IJ}\partial_i\varphi^I\partial^i\varphi^J\right) = 0. \tag{3.4.9}$$

The most general ansatz for $\mathbb{S}^B_{(2)}$ is as follows:

$$\mathbb{S}^B_{(2)} = \frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\left(\Xi(\varphi)R + A_{IJ}(\varphi)\partial_i\varphi^I\partial^i\varphi^J\right). \tag{3.4.10}$$

Then,

$$\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}\mathbb{S}^B_{(2)} = \frac{\sqrt{-\gamma}}{\kappa^2}\frac{d-2}{2}\left(\Xi R + A_{IJ}\partial_i\varphi^I\partial^i\varphi^J\right) - \frac{\sqrt{-\gamma}}{\kappa^2}(d-1)\Box\Xi, \tag{3.4.11}$$

$$\frac{\delta}{\delta\varphi^J}\mathbb{S}^B_{(2)} = \frac{\sqrt{-\gamma}}{\kappa^2}\left(R\partial_J\Xi + \partial_J A_{IK}\partial_i\varphi^I\partial^i\varphi^K - 2D_i\left(A_{JK}\partial^i\varphi^K\right)\right), \tag{3.4.12}$$

where we used the relation

$$\gamma^{ij}\delta R_{ij} = D^iD^j\delta\gamma_{ij} - \gamma^{ij}\Box\left(\delta\gamma_{ij}\right). \tag{3.4.13}$$

One can notice from (3.4.11) that

$$\mathbb{L}^B_{(2)} = \frac{\sqrt{-\gamma}}{\kappa^2}\left(\Xi R + A_{IJ}\partial_i\varphi^I\partial^i\varphi^J - \frac{2(d-1)}{d-2}\Box\Xi\right). \tag{3.4.14}$$

Therefore, (3.4.9) becomes

$$0 = R\left(-\frac{d-2}{d-1}\mathcal{W}\Xi^{[1]} + \mathcal{G}^{IJ}\partial_I\mathcal{W}\partial_J\Xi - \frac{1}{2}\right) + \partial_i\varphi^I\partial^i\varphi^J\Big(- \frac{d-2}{d-1}\mathcal{W}A_{IJ} + 2\mathcal{W}\partial_I\partial_J\Xi$$

$$+\mathcal{G}^{KL}\partial_L\mathcal{W}\partial_K A_{IJ}-2\mathcal{G}^{KL}\partial_K\mathcal{W}\partial_I A_{LJ}+\frac{1}{2}\mathcal{G}_{IJ}\Big)+2\Box\varphi^I\left(\mathcal{W}\partial_I\Xi-\mathcal{G}^{JK}\partial_J\mathcal{W}A_{IK}\right), \tag{3.4.15}$$

and we obtain the equations for Ξ and A_{IJ}

$$0=-\frac{d-2}{d-1}\Xi+V^I\partial_I\Xi-\frac{1}{2\mathcal{W}}, \tag{3.4.16a}$$

$$0=-\frac{d-2}{d-1}A_{IJ}+V^K\partial_K A_{IJ}+\partial_I V^K A_{JK}+\partial_J V^K A_{IK}+\frac{1}{2\mathcal{W}}\mathcal{G}_{IJ}, \tag{3.4.16b}$$

$$0=\partial_I\Xi-V^J A_{IJ}, \tag{3.4.16c}$$

where

$$V^I\equiv\frac{1}{\mathcal{W}}\mathcal{G}^{IJ}\partial_J\mathcal{W}. \tag{3.4.17}$$

Note that A_{IJ} should satisfy the condition

$$\partial_I(V^K A_{JK})=\partial_J(V^K A_{IK}). \tag{3.4.18}$$

We emphasize that we do not discuss the existence of a solution for A_{IJ} and Ξ here. Nevertheless, equations (3.4.16) are useful for determining $\mathbb{S}^{\zeta\zeta}_{(1)}$, $\mathbb{S}^{\Psi\Psi}_{(2)}$ and $\mathbb{S}^{\zeta\Psi}_{(3/2)}$.

$\mathbb{S}^{B}_{(2n)}$ $(n\geq 2)$ is obtained by the following recursive equation

$$0=-\frac{2}{d-1}\mathcal{W}\gamma^{ij}\pi^B_{(2n)ij}+\mathcal{W}V^I\pi^B_{(2n)I}+\frac{\kappa^2}{2\sqrt{-\gamma}}\sum_{m=1}^{n-1}\left[4\left(\frac{1}{d-1}\gamma_{ij}\gamma_{kl}-\gamma_{ik}\gamma_{jl}\right)\pi^{ij}_{B(2m)}\pi^{kl}_{B(2n-2m)}-\mathcal{G}^{IJ}\pi_I^{B(2m)}\pi_J^{B(2n-2m)}\right]. \tag{3.4.19}$$

In particular, when $d=4$ the inhomogeneous terms on the RHS become

$$2\frac{\kappa^2}{\sqrt{-\gamma}}\left(\frac{1}{d-1}\gamma_{ij}\gamma_{kl}-\gamma_{ik}\gamma_{jl}\right)\pi^{ij}_{(2)}\pi^{kl}_{(2)}=\frac{\sqrt{-\gamma}}{\kappa^2}\Xi^2\left(\frac{d}{2(d-1)}R^2-2R_{kl}R^{kl}\right), \tag{3.4.20}$$

where

$$\Xi=\frac{1}{2(d-2)}+\mathcal{O}(\varphi^2), \tag{3.4.21}$$

is the solution of (3.4.16a), while other inhomogeneous terms are asymptotically suppressed.

3.4.2 Fermionic sector

After substituting the leading order solution (3.4.7) into the Hamiltonian constraint (3.3.9), we get the following first-order differential equation for $\breve{\mathbb{S}}\equiv\mathbb{S}-\mathbb{S}_{(0)}$

$$0=\mathcal{W}\left(-\frac{1}{d-1}e^a_i\breve{\pi}^i_a+V^I\breve{\pi}^\varphi_I\right)-\frac{1}{2(d-1)}\mathcal{W}\left(\overline{\Psi}_{+i}\pi^i_\Psi+\pi^i_\Psi\Psi_{+i}\right)+\mathcal{W}\left(\frac{1}{2(d-1)}\delta^J_I+\partial_I V^J\right)\times$$

$$
\begin{aligned}
&\times \left(\overline{\zeta}^I_- \pi^\zeta_J + \pi^\zeta_J \zeta^I_-\right) + \frac{\kappa^2}{2\sqrt{-\gamma}} \Bigg\{ \left(\frac{1}{d-1} e^a_i e^b_j - e^a_j e^b_i\right) \breve{\pi}^i_a \breve{\pi}^j_b - \mathcal{G}^{IJ} \breve{\pi}^\varphi_I \breve{\pi}^\varphi_J + \mathcal{G}^{IJ} \left(\pi^\zeta_I \not{\mathbb{D}} \pi^\zeta_J - \pi^\zeta_I \overleftarrow{\not{\mathbb{D}}} \pi^\zeta_J\right) \\
&- 2\breve{\pi}^{ij} \Big[(\overline{\Psi}_{+i} \pi_{\Psi j} + \pi_{\Psi j} \Psi_{+i}) + \frac{1}{d-1} \pi^p_\Psi \left(\widehat{\Gamma}_{pi} - (d-2)\gamma_{pi}\right) \widehat{\Gamma}_j{}^k \Psi_{+k} \\
&+ \frac{1}{d-1} \overline{\Psi}_{+k} \widehat{\Gamma}^k{}_j \left(\widehat{\Gamma}_{ip} - (d-2)\gamma_{ip}\right) \pi^p_\Psi - i \frac{\partial_I \mathcal{W}}{\mathcal{W}} \gamma_{jk} \left(\overline{\zeta}^I_- \widehat{\Gamma}_i \pi^k_\Psi + \pi^k_\Psi \widehat{\Gamma}_i \zeta^I_-\right) \Big] \\
&+ \frac{2}{d-1} \gamma_{ij} \breve{\pi}^{ij} \left(-\overline{\zeta}^I_- \pi^\zeta_I - \pi^\zeta_I \zeta^I_- + \overline{\Psi}_{+k} \pi^k_\Psi + \pi^k_\Psi \Psi_{+k} - i \frac{\partial_I \mathcal{W}}{\mathcal{W}} \overline{\zeta}^I_- \widehat{\Gamma}_k \pi^k_\Psi - i \frac{\partial_I \mathcal{W}}{\mathcal{W}} \pi^k_\Psi \widehat{\Gamma}_k \zeta^I_-\right) \\
&+ \left[\mathcal{G}^{IJ} \mathcal{G}^{LM} (\partial_J \mathcal{G}_{MK} - \partial_M \mathcal{G}_{JK}) - \mathcal{W} \partial_K \left(\frac{\mathcal{G}^{IL}}{\mathcal{W}}\right)\right] \breve{\pi}^\varphi_I \left(\overline{\zeta}^K_- \pi^\zeta_L + \pi^\zeta_L \zeta^K_-\right) + i \breve{\pi}^\varphi_I \Big[\frac{1}{d-1} \left(\overline{\zeta}^I_- \widehat{\Gamma}_i \pi^i_\Psi + \pi \right. \\
&\left. - \mathcal{G}^{IJ} \left(\pi^\zeta_J \widehat{\Gamma}^i \Psi_{+i} + \overline{\Psi}_{+i} \widehat{\Gamma}^i \pi^\zeta_J\right) \right] - \pi^k_\Psi \left[\left(\frac{1}{d-1} \widehat{\Gamma}_k \widehat{\Gamma}_j - \gamma_{kj}\right) \not{\mathbb{D}} - \overleftarrow{\not{\mathbb{D}}} \left(\frac{1}{d-1} \widehat{\Gamma}_k \widehat{\Gamma}_j - \gamma_{kj}\right)\right] \pi^j_\Psi \\
&+ \frac{i}{d-1} \left(\pi^\zeta_I \not{\partial} \varphi^I \widehat{\Gamma}_i \pi^i_\Psi - \pi^i_\Psi \widehat{\Gamma}_i \not{\partial} \varphi^I \pi^\zeta_I\right) - 2i \partial_i \varphi^I \left(\pi^\zeta_I \pi^i_\Psi - \pi^i_\Psi \pi^\zeta_I\right) \\
&+ \mathcal{G}^{IM} \mathcal{G}^{KN} \partial_i \varphi^J \left(\partial_K \mathcal{G}_{IJ} - \partial_I \mathcal{G}_{KJ}\right) \pi^\zeta_M \widehat{\Gamma}^i \pi^\zeta_N \Bigg\} \\
&+ \frac{\sqrt{-\gamma}}{2\kappa^2} \Bigg[- R[\gamma] + \mathcal{G}_{IJ} \partial_i \varphi^I \partial^i \varphi^J + \mathcal{G}_{IJ} \overline{\zeta}^I_- \left(\not{\mathbb{D}} - \overleftarrow{\not{\mathbb{D}}}\right) \zeta^J_- + \overline{\Psi}_{+i} \widehat{\Gamma}^{ijk} \left(\mathbb{D}_j - \overleftarrow{\mathbb{D}}_j\right) \Psi_{+k} \\
&+ D_k \left(\overline{\Psi}_{+i} \left(\gamma^{jk} \widehat{\Gamma}^i - \gamma^{ik} \widehat{\Gamma}^j\right) \Psi_{+j}\right) + i \mathcal{G}_{IJ} \partial_i \varphi^J \left(\overline{\zeta}^I_- \widehat{\Gamma}^j \widehat{\Gamma}^i \Psi_{+j} - \overline{\Psi}_{+j} \widehat{\Gamma}^i \widehat{\Gamma}^j \zeta^I_-\right) \\
&+ \partial_K \mathcal{G}_{IJ} \partial_i \varphi^J \left(\overline{\zeta}^I_- \widehat{\Gamma}^i \zeta^K_- - \overline{\zeta}^K_- \widehat{\Gamma}^i \zeta^I_-\right) - 2i V_I \left(\overline{\zeta}^I_- \widehat{\Gamma}^{ij} \mathbb{D}_i \Psi_{+j} + \overline{\Psi}_{+j} \overleftarrow{\mathbb{D}}_i \widehat{\Gamma}^{ij} \zeta^I_-\right) \\
&+ V_I \mathcal{G}_{JK} \partial_i \varphi^J \left(\overline{\zeta}^I_- \widehat{\Gamma}^i \zeta^K_- - \overline{\zeta}^K_- \hat{\Gamma}^i \zeta^I_-\right) \Bigg],
\end{aligned} \tag{3.4.22}
$$

where

$$
\breve{\pi}^i_a \equiv \frac{\delta \breve{\mathbb{S}}}{\delta e^a_i}, \quad \breve{\pi}^\varphi_I \equiv \frac{\breve{\mathbb{S}}}{\delta \varphi^I}. \tag{3.4.23}
$$

From this one could write a recursive equation for every $\mathbb{S}_{(k)}$. However, it looks too complicated, and thus we first write down the equations for $\mathbb{S}^{\zeta\zeta}_{(1)}$, $\mathbb{S}^{\zeta\Psi}_{(3/2)}$ and $\mathbb{S}^{\Psi\Psi}_{(2)}$, namely

$$
\begin{aligned}
0 = & -\mathcal{W} \mathbb{L}^{\zeta\zeta}_{(1)} + \mathcal{W} \Bigg\{ V^I \partial_I + \frac{1}{2(d-1)} \left(\overline{\zeta}^I_- \frac{\delta}{\delta \overline{\zeta}^I_-} + \frac{\overleftarrow{\delta}}{\delta \zeta^I_-} \zeta^I_-\right) + \partial_I V^J \left(\overline{\zeta}^I_- \frac{\delta}{\delta \overline{\zeta}^J_-} + \frac{\overleftarrow{\delta}}{\delta \zeta^J_-} \zeta^I_-\right) \Bigg\} \mathbb{S}^{\zeta\zeta}_{(1)} \\
& + \frac{\sqrt{-\gamma}}{2\kappa^2} \Big[\mathcal{G}_{IJ} \left(\overline{\zeta}^I_- \not{\mathbb{D}} \zeta^J_- - \overline{\zeta}^I_- \overleftarrow{\not{\mathbb{D}}} \zeta^J_-\right) + \left(V_I \mathcal{G}_{JK} + \partial_K \mathcal{G}_{IJ}\right) \partial_i \varphi^J \left(\overline{\zeta}^I_- \widehat{\Gamma}^i \zeta^K_- - \overline{\zeta}^K_- \widehat{\Gamma}^i \zeta^I_-\right) \Big].
\end{aligned} \tag{3.4.24a}
$$

$$
0 = -\frac{d - \frac{3}{2}}{d-1} \mathcal{W} \mathbb{L}^{\zeta\Psi}_{(3/2)} + \mathcal{W} \Bigg[V^I \partial_I + \frac{1}{2(d-1)} \left(\overline{\zeta}^I_- \frac{\delta}{\delta \overline{\zeta}^I_-} + \frac{\overleftarrow{\delta}}{\delta \zeta^I_-} \zeta^I_-\right) + \partial_L V^K \left(\overline{\zeta}^L_- \frac{\delta}{\delta \overline{\zeta}^K_-} + \frac{\overleftarrow{\delta}}{\delta \zeta^K_-} \zeta^L_-\right) \Bigg] \mathbb{S}^{\zeta\Psi}_{(3/
$$

$$+\frac{\sqrt{-\gamma}}{2\kappa^2}i\Big[-2V_I\left(\overline{\zeta}^I_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}+\overline{\Psi}_{+j}\overleftarrow{\mathbb{D}}_i\widehat{\Gamma}^{ij}\zeta^I_-\right)+\mathcal{G}_{IJ}\left(\overline{\zeta}^I_-\widehat{\Gamma}^j\partial\!\!\!/\varphi^J\Psi_{+j}-\overline{\Psi}_{+j}\partial\!\!\!/\varphi^J\widehat{\Gamma}^j\zeta^I_-\right)\Big], \tag{3.4.24b}$$

$$0=-\frac{d-2}{d-1}\mathcal{W}\mathbb{L}^{\Psi\Psi}_{(2)}+\mathcal{G}^{IJ}\partial_I\mathcal{W}\frac{\delta}{\delta\varphi^J}\mathbb{S}^{\Psi\Psi}_{(2)}$$
$$+\frac{\sqrt{-\gamma}}{2\kappa^2}\Big[\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}(\mathbb{D}_j-\overleftarrow{\mathbb{D}}_j)\Psi_{+k}+D_k\left(\overline{\Psi}_{+i}(\gamma^{jk}\widehat{\Gamma}^i-\gamma^{ik}\widehat{\Gamma}^j)\Psi_{+j}\right)\Big], \tag{3.4.24c}$$

where we used (3.3.27).

While (3.4.24a) and (3.4.24b) are not so easy to treat at first sight, the solution of (3.4.24c) is obvious once we take into account (3.4.16a), namely

$$\mathbb{L}^{\Psi\Psi}_{(2)}=-\frac{\sqrt{-\gamma}}{\kappa^2}\Xi\left[\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}(\mathbb{D}_j-\overleftarrow{\mathbb{D}}_j)\Psi_{+k}+D_k\left(\overline{\Psi}_{+i}(\gamma^{jk}\widehat{\Gamma}^i-\gamma^{ik}\widehat{\Gamma}^j)\Psi_{+j}\right)\right]. \tag{3.4.25}$$

Instead of solving (3.4.24a) and (3.4.24b) directly, we now try to solve the $\mathcal{F}_+$ constraint (3.3.13), which requires much less effort. They are respectively at the order 1 and 3/2

$$i\mathcal{G}^{IJ}\partial_I\mathcal{W}\frac{\delta}{\delta\overline{\zeta}^J_-}\mathbb{S}^{\zeta\zeta}_{(1)}+\frac{1}{d-1}\mathcal{W}\widehat{\Gamma}_i\frac{\delta}{\delta\overline{\Psi}_{+i}}\mathbb{S}^{\zeta\Psi}_{(3/2)}=\frac{\sqrt{-\gamma}}{2\kappa^2}i\mathcal{G}_{IJ}\partial_i\varphi^J\widehat{\Gamma}^i\zeta^I_-, \tag{3.4.26a}$$

$$\frac{1}{d-1}\mathcal{W}\widehat{\Gamma}_i\frac{\delta}{\delta\overline{\Psi}_{+i}}\mathbb{S}^{\Psi\Psi}_{(2)}+i\mathcal{G}^{IJ}\partial_I\mathcal{W}\frac{\delta}{\delta\overline{\zeta}^J_-}\mathbb{S}^{\zeta\Psi}_{(3/2)}=\frac{\sqrt{-\gamma}}{\kappa^2}\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}. \tag{3.4.26b}$$

The solution (3.4.25) allows us to solve (3.4.26b) immediately and we obtain

$$\frac{\delta}{\delta\overline{\zeta}^I_-}\mathbb{S}^{\zeta\Psi}_{(3/2)}=i\frac{\sqrt{-\gamma}}{\kappa^2}\left(-2\partial_I\Xi\overline{\zeta}^I_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}+A_{IJ}\overline{\zeta}^I_-\widehat{\Gamma}^i\partial\!\!\!/\varphi^J\Psi_{+i}\right). \tag{3.4.27}$$

One can readily see that

$$\mathbb{S}^{\zeta\Psi}_{(3/2)}=\frac{i}{\kappa^2}\int d^dx\sqrt{-\gamma}\Big[2\partial_I\Xi\left(\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\zeta^I_--\overline{\zeta}^I_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}\right)+$$
$$+A_{IJ}\left(\overline{\zeta}^I_-\widehat{\Gamma}^i\partial\!\!\!/\varphi^J\Psi_{+i}-\overline{\Psi}_{+i}\partial\!\!\!/\varphi^I\widehat{\Gamma}^i\zeta^J_-\right)\Big]. \tag{3.4.28}$$

In the same way, we find from (3.4.26a) that

$$\mathbb{S}^{\zeta\zeta}_{(1)}=\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\ \left(A_{IJ}\overline{\zeta}^I_-(\mathbb{D}\!\!\!/-\overleftarrow{\mathbb{D}\!\!\!/})\zeta^J_-+(\partial_JA_{Ik}-\partial_IA_{JK})\overline{\zeta}^I_-\partial\!\!\!/\varphi^K\zeta^J_-\right). \tag{3.4.29}$$

Moreover, we can confirm that the solutions (3.4.29) and (3.4.28) satisfy the Hamiltonian constraints (3.4.24a) and (3.4.24b) respectively. That is not the whole story, however, and one has to convince themselves that the $\mathcal{F}_-=0$ constraint also holds for these solutions. From (3.3.13), we obtain

$$0=\mathbb{D}_i\frac{\delta}{\delta\overline{\Psi}_{+i}}\mathbb{S}^{\Psi\Psi}_{(2k)}+\frac{i}{2}\partial_i\varphi^I\widehat{\Gamma}^i\frac{\delta}{\delta\overline{\zeta}^I_-}\mathbb{S}^{\zeta\Psi}_{(2k-1/2)}-\widehat{\Gamma}_i\Psi_{+j}\frac{\delta}{\delta\gamma_{ij}}\mathbb{S}^B_{(2k)}, \tag{3.4.30a}$$

$$0 = \mathbb{D}_i \frac{\delta}{\delta \overline{\Psi}_{+i}} \mathbb{S}^{\zeta\Psi}_{(2k-1/2)} + \frac{i}{2} \partial_i \varphi^I \widehat{\Gamma}^i \frac{\delta}{\delta \overline{\zeta}^I_-} \mathbb{S}^{\zeta\zeta}_{(2k-1)} - \frac{i}{2} \zeta^I_- \frac{\delta}{\delta \varphi^I} \mathbb{S}^B_{(2k)}, \tag{3.4.30b}$$

where k is an arbitrary positive integer. It is not so difficult to check that the solutions we obtained satisfy the constraints (3.4.30b) and (3.4.30a) for $k = 1$, implying that the combination

$$\mathbb{S}^B_{(2)} + \mathbb{S}^{\Psi\Psi}_{(2)} + \mathbb{S}^{\zeta\zeta}_{(1)} + \mathbb{S}^{\zeta\Psi}_{(3/2)}, \tag{3.4.31}$$

is (ϵ_+) supersymmetric.

We have seen how to obtain Hamilton's principal function in the fermionic sector from its bosonic supersymmetric partner, but at the lowest order. It was relatively easy because we could give the most general ansatz for $\mathbb{S}^B_{(2)}$ which has a small number of terms. To go further we should first determine $\mathbb{S}^B_{(4)}, \mathbb{S}^B_{(6)}, \cdots$ and obtain their SUSY partners by using the above trick. The ansatz for $\mathbb{S}^B_{(2n)}$ ($n \geq 2$), however, has lots of terms and is complicated, hence finding its SUSY partner is too tedious.

Although we stop finding the general solution of the HJ equations in the fermionic sector here, we remark that the solution we have found is almost sufficient for providing the divergent counterterms in the low dimensions, say, $d = 4$. This is because in the generic case that there are no scalar fields dual to marginal operators, $\mathbb{S}^{\zeta\zeta}_{(3)}$ and $\mathbb{S}^{\zeta\Psi}_{(7/2)}$ are asymptotically suppressed in 4 dimensions. As a result, what remains in the case $d = 4$ is only to determine $\mathbb{S}^{\Psi\Psi}_{(4)}$, corresponding to the logarithmically divergent terms, which are directly related to the holographic Weyl anomaly [3].

We should emphasize that from the general analysis here the divergent counterterms (except for $\mathbb{S}_{(0)}$) always satisfy the constraint $\mathcal{F}_- = 0$ and so does the renormalized on-shell action $\widehat{S}_{ren}$.

We finish this subsection by presenting the recursive relation obtained from (3.3.12), namely

$$\begin{aligned} 0 = & -\frac{1}{d-1} \mathcal{W} \widehat{\Gamma}_i \pi^i_{\overline{\Psi}(n-1/2)} - i \mathcal{G}^{IJ} \partial_I \mathcal{W} \pi^{\overline{\zeta}}_{J(n-1)} + \frac{\kappa^2}{\sqrt{-\gamma}} \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor - 1} \Big[\frac{i}{2} \mathcal{G}^{IJ} \pi^{\varphi}_{I(2m)} \pi^{\overline{\zeta}}_{J(n-2m-1)} \\ & + \pi^{jk}_{(2m)} \left(\frac{1}{d-1} \gamma_{jk} \widehat{\Gamma}_i - \gamma_{ij} \widehat{\Gamma}_k \right) \pi^i_{\overline{\Psi}(n-2m-1/2)} \Big], \end{aligned} \tag{3.4.32}$$

where (integer or half-integer) $n \geq 4$. This will be useful for determining the super-Weyl anomaly in section 3.5.1.

3.4.3 Logarithmically divergent terms in 4D

As mentioned in subsection 3.3.2, every $\mathbb{S}_{(k)}$ in the asymptotic expansion (3.3.25) with respect to the operator δ_e contains poles related to logarithmically divergent terms. Let us denote such terms by $\widetilde{\mathbb{S}}_{(k)}$. Whereas $\widetilde{\mathbb{S}}^B_{(4)}$ and $\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}$ are purely gravitational (meaning that they are related only to the metric and the gravitino field) and universal, $\widetilde{\mathbb{S}}^{\zeta\zeta}_{(1)}$, $\widetilde{\mathbb{S}}^{\zeta\Psi}_{(3/2)}$, $\widetilde{\mathbb{S}}^B_{(2)}$ and $\widetilde{\mathbb{S}}^{\Psi\Psi}_{(2)}$ are model-dependent. We first discuss the former and then study the latter for a simple model.

$\widetilde{\mathbb{S}}^B_{(4)}$ is easily obtained from (3.4.19) and (3.4.20), namely

$$\begin{aligned}\widetilde{\mathbb{S}}^B_{(4)} &\equiv \int d^dx\sqrt{-\gamma}\ \widetilde{\mathcal{L}}^B_{(4)}\log e^{-2r}\\ &= \frac{1}{4\kappa^2(d-2)^2}\int d^dx\sqrt{-\gamma}\left(\frac{d}{4(d-1)}R^2 - R_{ij}R^{ij}\right)\log e^{-2r}, \qquad (3.4.33)\end{aligned}$$

which is already well-known. Meanwhile, $\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}$ is determined by the inhomogeneous terms of the Hamiltonian constraint (3.4.22) at order 4, namely[10]

$$\begin{aligned}\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)} &\equiv \int d^dx\sqrt{-\gamma}\ \widetilde{\mathcal{L}}^{\Psi\Psi}_{(4)}\log e^{-2r}\\
&= \int d^dx\frac{\kappa^2}{4\sqrt{-\gamma}}\Bigg\{2\left(\frac{1}{d-1}\gamma_{ij}\gamma_{kl}-\gamma_{ik}\gamma_{jl}\right)\widetilde{\pi}^{ij}_{(2)}e^{ak}\pi^{l\Psi}_{a(2)} - \widetilde{\pi}^{ij}_{(2)}(\overline{\Psi}_{+k}\widehat{\Gamma}_j\widehat{\Gamma}^k\pi^{(2)}_{\overline{\Psi}i}+\pi^{(2)}_{\Psi i}\widehat{\Gamma}^k\widehat{\Gamma}_j\Psi_{+k})\\
&\quad - \frac{1}{d-1}\gamma_{ij}\widetilde{\pi}^{ij}_{(2)}\left(\overline{\Psi}^k_+\widehat{\Gamma}_{kl}\pi^l_{(2)\overline{\Psi}}+\pi^{(2)l}_{\Psi}\widehat{\Gamma}_{lk}\Psi^k_+\right)+\frac{1}{2(d-1)}(\pi^{(2)k}_{\Psi}\widehat{\Gamma}_k\not{\mathbb{D}}\widehat{\Gamma}_j\pi^{(2)j}_{\overline{\Psi}}-\pi^{(2)k}_{\Psi}\widehat{\Gamma}_k\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}_j\pi^{(2)j}_{\overline{\Psi}})\\
&\quad + \frac{1}{2}(\pi^{(2)i}_{\Psi}\not{\mathbb{D}}\pi^{(2)}_{\overline{\Psi}i}-\pi^{(2)i}_{\Psi}\overleftarrow{\not{\mathbb{D}}}\pi^{(2)}_{\overline{\Psi}i})\Bigg\}\log e^{-2r}\\
&= \frac{1}{8(d-2)^2\kappa^2}\int d^dx\sqrt{-\gamma}\Bigg\{(d-3)R(\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k}-\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ijk}\Psi_{+k})\\
&\quad + \frac{d}{d-1}RD_j\left[\overline{\Psi}_{+i}(\gamma^{ij}\widehat{\Gamma}^k-\gamma^{jk}\widehat{\Gamma}^i)\Psi_{+k}\right]-(d-4)R(\overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jk}\mathbb{D}_j\Psi_{+k}-\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\widehat{\Gamma}^k\Psi_{+k})\\
&\quad + \frac{(d-2)^2}{d-1}R\Big[\overline{\Psi}^k_+\widehat{\Gamma}^j\mathbb{D}_k\Psi_{+j}-\overline{\Psi}^i_+\not{\mathbb{D}}\Psi_{+i}-\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}^k\widehat{\Gamma}^i\Psi_{+k}+\overline{\Psi}^k_+\overleftarrow{\not{\mathbb{D}}}\Psi_{+k}\Big]\\
&\quad + 2R_{kl}\Big[\overline{\Psi}_{+i}[(\gamma^{ip}\widehat{\Gamma}^k-\gamma^{ik}\widehat{\Gamma}^p)\mathbb{D}^l-\overleftarrow{\mathbb{D}}^l(\gamma^{ip}\widehat{\Gamma}^k-\gamma^{pk}\widehat{\Gamma}^i)]\Psi_{+p}-\overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jl}\mathbb{D}_j\Psi^k_+ + \overline{\Psi}^k_+\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{lj}\widehat{\Gamma}^i\Psi_{+i}\\
&\quad - D_j[\overline{\Psi}^l_+\widehat{\Gamma}^{kji}\Psi_{+i}-\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\Psi^l_+-\overline{\Psi}_{+i}(\gamma^{jk}\gamma^{pl}\widehat{\Gamma}^i-\gamma^{jk}\gamma^{il}\widehat{\Gamma}^p+\gamma^{jp}\gamma^{il}\widehat{\Gamma}^k-\gamma^{pl}\gamma^{ij}\widehat{\Gamma}^k)\Psi_{+p}]\Big]\\
&\quad - \frac{2(d-2)^2}{d-1}(\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\not{\mathbb{D}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l}-\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l})\\
&\quad - 2(\overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\not{\mathbb{D}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k}-\overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k})\Bigg\}\log e^{-2r}. \qquad (3.4.34)\end{aligned}$$

Although nontrivial, one can show that $\widetilde{\mathbb{S}}^B_{(4)}+\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}$ satisfies the constraints $\mathcal{H}=\mathcal{F}_-=\mathcal{F}_+=0$ (i.e. conformal, supersymmetry and super-Weyl invariance), namely

$$0 = \left(e^a_i\frac{\delta}{\delta e^a_i}+\frac{1}{2}\overline{\Psi}_{+i}\frac{\delta}{\delta\overline{\Psi}_{+i}}+\frac{\overleftarrow{\delta}}{\delta\Psi_{+i}}\Psi_{+i}\right)\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}, \qquad (3.4.35a)$$

$$0 = \widehat{\Gamma}_i\Psi_{+j}\frac{\delta}{\delta\gamma_{ij}}\widetilde{\mathbb{S}}^B_{(4)}+\mathbb{D}_i\frac{\delta}{\delta\overline{\Psi}_{+i}}\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}, \qquad (3.4.35b)$$

$$0 = \widehat{\Gamma}_i\frac{\delta}{\delta\overline{\Psi}_{+i}}\mathbb{S}^{\Psi\Psi}_{(4)}. \qquad (3.4.35c)$$

[10]When the boundary metric is flat, (3.4.34) matches with the result in [81].

3.4.4 Generic finite counterterms in 4D and summary

Up to now we obtained the generic part of the divergent counterterms. $\mathcal{S}_{\rm ct}$ can involve additional finite terms which satisfy the first class constraints (3.3.14), though.[11] The possible bosonic finite counterterms are the Euler density and the Weyl invariant in 4D, namely,

$$E_{(4)} = \frac{1}{64}\left(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2\right), \quad I_{(4)} = -\frac{1}{64}\left(R^{ijkl}R_{ijkl} - 2R^{ij}R_{ij} + \frac{1}{3}R^2\right). \tag{3.4.36}$$

The integral of the Euler density $E_{(4)}$ by itself satisfies all the first class constraints, since it is a topological quantity, any local variation of which vanishes. Therefore, we find that the possible supersymmetric finite counterterms are a linear combination of

$$\begin{aligned}
X_I = {}& 64I_{(4)} + (d-3)R(\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ijk}\Psi_{+k}) + \frac{d}{d-1}RD_j\left[\overline{\Psi}_{+i}(\gamma^{ij}\widehat{\Gamma}^k - \gamma^{jk}\widehat{\Gamma}^i)\Psi_{+k}\right] \\
& - (d-4)R(\overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\widehat{\Gamma}^k\Psi_{+k}) \\
& + \frac{(d-2)^2}{d-1}R\Big[\overline{\Psi}^k_+\widehat{\Gamma}^j\mathbb{D}_k\Psi_{+j} - \overline{\Psi}^i_+\not{\mathbb{D}}\Psi_{+i} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}^k\widehat{\Gamma}^i\Psi_{+k} + \overline{\Psi}^k_+\overleftarrow{\not{\mathbb{D}}}\Psi_{+k}\Big] \\
& + 2R_{kl}\Big[\overline{\Psi}_{+i}[(\gamma^{ip}\widehat{\Gamma}^k - \gamma^{ik}\widehat{\Gamma}^p)\mathbb{D}^l - \overleftarrow{\mathbb{D}}^l(\gamma^{ip}\widehat{\Gamma}^k - \gamma^{pk}\widehat{\Gamma}^i)]\Psi_{+p} - \overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jl}\mathbb{D}_j\Psi^k_+ + \overline{\Psi}^k_+\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{lj}\widehat{\Gamma}^i\Psi_{+i} \\
& - D_j[\overline{\Psi}^l_+\widehat{\Gamma}^{kji}\Psi_{+i} - \overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\Psi^l_+ - \overline{\Psi}_{+i}(\gamma^{jk}\gamma^{pl}\widehat{\Gamma}^i - \gamma^{jk}\gamma^{il}\widehat{\Gamma}^p + \gamma^{jp}\gamma^{il}\widehat{\Gamma}^k - \gamma^{pl}\gamma^{ij}\widehat{\Gamma}^k)\Psi_{+p}]\Big] \\
& - \frac{2(d-2)^2}{d-1}(\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\not{\mathbb{D}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l}) \\
& - 2(\overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\not{\mathbb{D}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k}),
\end{aligned} \tag{3.4.37}$$

and

$$X_E = E_{(4)}, \quad X_P = \mathcal{P} = \frac{1}{64}\epsilon^{ijkl}R_{ijpq}R_{kl}{}^{pq}, \tag{3.4.38}$$

where $\mathcal{P}$ is the Pontryagin density. Notice that the integral of $\mathcal{P}$ is a topological quantity and thus can be a finite counterterm as in the case of the Euler density, as long as there is no other symmetry which prevents its appearance.

In summary, collecting all of these finite counterterms and the previous divergent

[11] Otherwise, these finite terms would generate trivial cocycle terms, which do not have any physical implication.

ones we obtain

$$\boxed{\begin{aligned} S_{\rm ct} = &-\left(\mathbb{S}_{(0)}+\mathbb{S}_{(1)}+\mathbb{S}_{(2)}\right)-\left(\widetilde{\mathbb{S}}^B_{(4)}+\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}\right)+\cdots \\ =&\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\Big\{\mathcal{W}-\Xi R-A_{IJ}\partial_i\varphi^I\partial^i\varphi^J-A_{IJ}\overline{\zeta}^I_-(\not{\mathbb{D}}-\overleftarrow{\not{\mathbb{D}}})\zeta^J_- \\ &\qquad -(\partial_J A_{IK}-\partial_I A_{JK})\overline{\zeta}^I_-\not{\partial}\varphi^K\zeta^J_- -2i\partial_I\Xi(\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\zeta^I_- -\overline{\zeta}_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}) \\ &\qquad -iA_{IJ}(\overline{\zeta}^I_-\widehat{\Gamma}^i\not{\partial}\varphi^J\Psi_{+i}-\overline{\Psi}_{+i}\not{\partial}\varphi^I\widehat{\Gamma}^i\zeta^J_-)+\Xi\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}(\mathbb{D}_j-\overleftarrow{\mathbb{D}}_j)\Psi_{+k} \\ &\qquad +\overline{\Psi}_{+i}(\partial^i\Xi\widehat{\Gamma}^j-\partial^j\Xi\widehat{\Gamma}^i)\Psi_{+j}\Big\} \\ &-\widetilde{\mathbb{S}}^B_{(4)}-\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}+\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\,(\alpha_I X_I+\alpha_E X_E+\alpha_P X_P)+\cdots\,, \end{aligned}}$$

$$(3.4.39)$$

where $\widetilde{\mathbb{S}}^B_{(4)}$ and $\widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}$ are given in (3.4.33) and (3.4.34) and α_I, α_E and α_P are arbitrary constants. Here the ellipses stand for model-dependent terms, which we discuss in section 3.4.5 for a simple toy model.

3.4.5 Application to a toy model

For completeness, we present an application of our general procedure to a simple toy model.

In the toy model there is only one scalar field, which corresponds to an operator with the scaling dimension $\Delta = d-1$ with $d=4$. In principle, there are two possibilities for the coefficient of φ^2-term of the bulk superpotential $\mathcal{W}$; $-\frac{1}{2}$ corresponds to deformation, and $-\frac{d-1}{2}$ corresponds to RG-flow due to giving a VEV [99]. However, as we have seen, SUSY requires that the counterterm should be the same as the bulk superpotential $\mathcal{W}$, and thus we just need to consider the $-\frac{1}{2}$ case. It follows that

$$\mathcal{W}=-(d-1)-\frac{1}{2}\varphi^2+k_3\varphi^3+k_4\varphi^4+\mathcal{O}(\varphi^5), \tag{3.4.40}$$

where k_3 and k_4 are arbitrary constants, and therefore the solution of (3.4.16a), (3.4.16b) and (3.4.16c) becomes

$$\Xi=\frac{1}{2(d-2)}-\frac{1}{d-4}\cdot\frac{1}{4(d-1)}\varphi^2+\cdots\,,\quad A_{IJ}=-\frac{1}{d-4}\cdot\frac{1}{2}+\cdots\,. \tag{3.4.41}$$

The divergent counterterms that we need, other than those in (3.4.39), are only the logarithmically divergent terms. Following the argument in section 3.3.2 again we can determine them from the poles (when $d=4$) in Ξ and A_{IJ} and are responsible for additional logarithmically divergent terms.

We thus obtain

$$\widetilde{\mathbb{S}}^B_{(2)}=\frac{1}{4\kappa^2}\int d^dx\sqrt{-\gamma}\left(\frac{1}{2(d-1)}\varphi^2R+\partial_i\varphi\partial^i\varphi\right)\log e^{-2r}, \tag{3.4.42a}$$

$$\widetilde{\mathbb{S}}^{\zeta\zeta}_{(1)} = \frac{1}{4\kappa^2}\int d^dx\sqrt{-\gamma}\,(\overline{\zeta}_-\not{\mathbb{D}}\zeta_- + \text{h.c.})\log e^{-2r}, \tag{3.4.42b}$$

$$\widetilde{\mathbb{S}}^{\zeta\Psi}_{(3/2)} = \frac{i}{4\kappa^2}\int d^dx\sqrt{-\gamma}\left(\overline{\zeta}_-\widehat{\Gamma}^i\not{\partial}\varphi\Psi_{+i} - \frac{2}{(d-1)}\varphi\overline{\zeta}_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} + \text{h.c.}\right)\log e^{-2r}, \tag{3.4.42c}$$

$$\widetilde{\mathbb{S}}^{\Psi\Psi}_{(2)} = -\frac{1}{4\kappa^2}\int d^dx\sqrt{-\gamma}\,\frac{1}{d-1}\left(\frac{1}{2}\varphi^2\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} + \varphi\overline{\Psi}_{+i}\partial^i\varphi\widehat{\Gamma}^j\Psi_{+j} + \text{h.c.}\right)\log e^{-2r}. \tag{3.4.42d}$$

One can easily check that $\widetilde{\mathbb{S}}^B_{(2)} + \widetilde{\mathbb{S}}^{\zeta\zeta}_{(1)} + \widetilde{\mathbb{S}}^{\zeta\Psi}_{(3/2)} + \widetilde{\mathbb{S}}^{\Psi\Psi}_{(2)}$ again satisfies the constraints $\mathcal{H} = \mathcal{F}_- = \mathcal{F}_+ = 0$.

Besides X_I and X_E, the possible finite counterterms (conformal and ϵ_+ supersymmetric) are

$$\begin{aligned} X_0 = {} & \frac{1}{2(d-1)}\varphi^2 R + \partial_i\varphi\partial^i\varphi + \overline{\zeta}_-\not{\mathbb{D}}\zeta_- + i\overline{\zeta}_-\widehat{\Gamma}^i\not{\partial}\varphi\Psi_{+i} \\ & - \frac{2i}{d-1}\varphi\overline{\zeta}_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} - \frac{1}{2(d-1)}\varphi^2\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} - \frac{1}{d-1}\varphi\overline{\Psi}_{+i}\partial^i\varphi\widehat{\Gamma}^j\Psi_{+j} + \text{h.c.}, \end{aligned} \tag{3.4.43}$$

and the finite term $k_4\varphi^4$ in $\mathcal{W}$ should be in the counterterms without any ambiguity, due to the $\mathcal{F}_-$ constraint.

In total, the divergent counterterms for the toy model are

$$\mathbb{S}^{div}_{\text{ct}} = -\left(\mathbb{S}_{(0)} + \mathbb{S}_{(1)} + \mathbb{S}_{(3/2)} + \mathbb{S}_{(2)}\right) - \int d^dx\sqrt{-\gamma}\,\widetilde{\mathcal{L}}_{[4]}\log e^{-2r}, \tag{3.4.44}$$

where the logarithmically divergent counterterms are

$$\int d^dx\sqrt{-\gamma}\,\widetilde{\mathcal{L}}_{[4]}\log e^{-2r} = \widetilde{\mathbb{S}}^{\zeta\zeta}_{(1)} + \widetilde{\mathbb{S}}^{\zeta\Psi}_{(3/2)} + \widetilde{\mathbb{S}}^B_{(2)} + \widetilde{\mathbb{S}}^{\Psi\Psi}_{(2)} + \widetilde{\mathbb{S}}^B_{(4)} + \widetilde{\mathbb{S}}^{\Psi\Psi}_{(4)}. \tag{3.4.45}$$

Adding possible finite ones, the whole counterterms are

$$\boxed{\begin{aligned} \mathbb{S}_{\text{ct}} = {} & \frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\Big[-(d-1) - \frac{1}{2}\varphi^2 + k_3\varphi^3 + k_4\varphi^4 - \frac{1}{2(d-2)}R \\ & + \frac{1}{2(d-2)}\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}(\mathbb{D}_j - \overleftarrow{\mathbb{D}}_j)\Psi_{+k}\Big] - \int d^dx\sqrt{-\gamma}\,\widetilde{\mathcal{L}}_{[4]}\log e^{-2r} \\ & + \frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\,(\alpha_I X_I + \alpha_E X_E + \alpha_P X_P + \alpha_0 X_0), \end{aligned}} \tag{3.4.46}$$

where α_E, α_I, α_P and α_0 are arbitrary constants and determine the renormalization scheme.

3.5 Holographic dictionary and Ward identities

Now that all the counterterms are determined, we can relate by the holographic dictionary [15] the renormalized canonical momenta to the renormalized local operators of the boundary field theory, namely

$$\mathcal{T}^i_a = -\lim_{r\to\infty} e^{(d+1)r}\frac{1}{\sqrt{-\gamma}}\left(\pi^i_a + \frac{\delta S_{\rm ct}}{\delta e^a_i}\right) := -\frac{1}{|\boldsymbol{e}_{(0)}|}\Pi^i_a, \tag{3.5.1a}$$

$$\mathcal{O}^\varphi_I = \lim_{r\to\infty} e^{(d+\mu^I)r}\frac{1}{\sqrt{-\gamma}}\left(\pi^\varphi_I + \frac{\delta S_{\rm ct}}{\delta \varphi^I}\right) := \frac{1}{|\boldsymbol{e}_{(0)}|}\Pi^\varphi_I, \tag{3.5.1b}$$

$$\mathcal{O}^{\overline{\zeta}}_I = \lim_{r\to\infty} e^{(d+\mu^I+\frac{1}{2})r}\frac{1}{\sqrt{-\gamma}}\left(\pi^{\overline{\zeta}}_I + \frac{\delta S_{\rm ct}}{\delta \overline{\zeta}^I}\right) := \frac{1}{|\boldsymbol{e}_{(0)}|}\Pi^{\overline{\zeta}}_I, \tag{3.5.1c}$$

$$\mathcal{S}^i = \lim_{r\to\infty} e^{(d+\frac{1}{2})r}\frac{1}{\sqrt{-\gamma}}\left(\pi^i_{\overline{\Psi}} + \frac{\delta S_{\rm ct}}{\delta \overline{\Psi}_{+i}}\right) := \frac{1}{|\boldsymbol{e}_{(0)}|}\Pi^i_{\overline{\Psi}}, \tag{3.5.1d}$$

where $\mathcal{T}^i_a$ is the energy-momentum tensor,[12] $\mathcal{S}^i$ is the supercurrent[13] and $\boldsymbol{e}_{(0)} = \det(e^a_{(0)i})$. We note that since these local renormalized operators are obtained in the presence of arbitrary sources we can obtain higher-point functions simply by taking functional derivatives of them with respect to the sources.

3.5.1 Ward identities and anomalies

One can find from the computation of section 3.4 and 3.4.5 that $\mathbb{S}_{\rm ct}$ satisfies the first class constraints $\mathcal{H}^i = \mathcal{F}_- = 0$ and the local Lorentz constraint (3.3.7), and so does the renormalized on-shell action $\widehat{S}_{ren}$. This is also related to the fact that these constraints are linear functional derivative equations.

Since $\mathcal{H}$ and $\mathcal{F}_+$ are not linear constraints, one should expect that the counterterms do not satisfy the constraints $\mathcal{H} = 0$ and $\mathcal{F}_+ = 0$ in general and thus generate non-trivial cocycle terms, which appear in the constraints for the renormalized on-shell action. Also, the poles appearing in solving the constraints contribute to the corresponding anomaly. In total, after removing all divergent counterterms, the first class constraints (3.3.12), (3.3.13), (3.3.9), (3.3.10) and (3.3.15) are reduced into

$$0 = -\frac{1}{2}\Gamma^a\Psi_{(0)+i}\mathcal{T}^i_a + \frac{i}{2}\zeta^I_{(0)-}\mathcal{O}^\varphi_I - \frac{i}{2}\partial\!\!\!/\varphi^I_{(0)}\mathcal{O}^{\overline{\zeta}}_I - \mathbb{D}_i\mathcal{S}^i, \tag{3.5.2a}$$

$$\mathcal{A}_{\rm sW} = -i\mathcal{G}^{IJ}\partial_I\mathcal{W}\mathcal{O}^{\overline{\zeta}}_J + \widehat{\Gamma}_i\mathcal{S}^i, \tag{3.5.2b}$$

$$\begin{aligned}\mathcal{A}_{\rm W} = e^a_{(0)i}\mathcal{T}^i_a - \mathcal{G}^{IJ}\partial_I\mathcal{W}\mathcal{O}^\varphi_J - \frac{1}{2}\left(\overline{\Psi}_{(0)+i}S^i + \text{h.c.}\right) + \\ + \left(\frac{1}{2}\delta^J_I - \partial_I\partial^J\mathcal{W}\right)\left(\overline{\zeta}^I_{(0)-}\mathcal{O}^{\overline{\zeta}}_J + \text{h.c.}\right),\end{aligned} \tag{3.5.2c}$$

$$0 = e^{ai}_{(0)}D_j\mathcal{T}^j_a + \partial^i\varphi^I_{(0)}\mathcal{O}^\varphi_I + \left(\overline{\zeta}^I_{(0)-}\overleftarrow{\mathbb{D}}^i\mathcal{O}^{\overline{\zeta}}_I + \text{h.c.}\right) + \left(\overline{\Psi}_{(0)+j}\overleftarrow{\mathbb{D}}^i\mathcal{S}^j + \text{h.c.}\right)$$

[12] The definition of the energy-momentum tensor is modified when the vielbein is used instead of the metric, see e.g. (2.198) in [100].

[13] The spinor index of the supercurrent $\mathcal{S}^i$ is implicit.

$$- D_j \left(\overline{\Psi}^i_{(0)+} \mathcal{S}^j + \text{h.c.} \right), \tag{3.5.2d}$$

$$0 = e_{(0)ai} \mathcal{T}^i_b - e_{(0)bi} \mathcal{T}^i_a + \frac{1}{2} \left(\overline{\zeta}^I_{(0)-} \Gamma_{ab} \mathcal{O}^{\overline{\zeta}}_I + \overline{\Psi}_{(0)+i} \Gamma_{ab} \mathcal{S}^i + \text{h.c.} \right), \tag{3.5.2e}$$

where $\mathcal{A}_{\text{sW}}$ and $\mathcal{A}_{\text{W}}$ are the super-Weyl and Weyl anomaly densities respectively. In (3.5.2) we keep only up to quadratic order and zero order in φ^I in the Taylor expansion of $\mathcal{W}$ and $\mathcal{G}_{IJ}$ respectively.

We identify the constraints (3.5.2) with the Ward identities, which relate the local sources and their dual operators of the field theory. These Ward identities play a key role in the following discussion and reflect the remaining local symmetries of the bulk SUGRA after fixing the strong FG gauge (3.B.14), on which we did HR for the bulk theory in section 3.4. The remaining local symmetry transformations of SUGRA are called generalized Penrose-Brown-Henneaux (gPBH) transformations, whose action on the sources is carefully treated in appendix 3.B.4. The resulting expressions are (3.B.38). Before discussing the gPBH action on the renormalized canonical momenta, let us first determine the anomalies explicitly in the case of $d = 4$.

Weyl anomaly

Although there are many ways to find the Weyl anomaly, a direct way is to read it from the HJ equation. One can see that in (3.4.22) at order 4 the first linear terms are indeed the RHS of the trace Ward identity (3.5.2c) and the rest of the terms give part of the trace anomaly. The terms with the pole $1/(d-4)$ which appeared in the HJ equations for $\mathbb{S}_{(1)}, \cdots, \mathbb{S}_{(4)}$ are also inherited into (3.4.22) for $\mathbb{S}_{[4]}$. These non-homogeneous terms are already identified with the logarithmically divergent terms and thus we only need to multiply them by 2 to obtain the trace anomaly [30]. For the metric and gravitino parts the trace anomaly density is[14]

$$\begin{aligned}
\mathcal{A}^{(G)}_{\text{W}}[e, \Psi_+] = \frac{1}{4(d-2)^2\kappa^2} \Bigg\{ & \frac{d}{2(d-1)} R^2 - 2R_{ij}R^{ij} \\
& + (d-3)R(\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ijk}\Psi_{+k}) \\
& + \frac{d}{d-1} R D_j \left[\overline{\Psi}_{+i}(\gamma^{ij}\widehat{\Gamma}^k - \gamma^{jk}\widehat{\Gamma}^i)\Psi_{+k} \right] \\
& - (d-4)R(\overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\widehat{\Gamma}^k\Psi_{+k}) \\
& + \frac{(d-2)^2}{d-1} R \Big[\overline{\Psi}^k_+\widehat{\Gamma}^j\mathbb{D}_k\Psi_{+j} - \overline{\Psi}^i_+\not{\mathbb{D}}\Psi_{+i} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}^k\widehat{\Gamma}^i\Psi_{+k} + \overline{\Psi}^k_+\overleftarrow{\not{\mathbb{D}}}\Psi_{+k} \Big] \\
& + 2R_{kl} \Big[\overline{\Psi}_{+i}[(\gamma^{ip}\widehat{\Gamma}^k - \gamma^{ik}\widehat{\Gamma}^p)\mathbb{D}^l - \overleftarrow{\mathbb{D}}^l(\gamma^{ip}\widehat{\Gamma}^k - \gamma^{pk}\widehat{\Gamma}^i)]\Psi_{+p} \\
& - \overline{\Psi}_{+i}\widehat{\Gamma}^i\widehat{\Gamma}^{jl}\mathbb{D}_j\Psi^k_+ + \overline{\Psi}^k_+\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{lj}\widehat{\Gamma}^i\Psi_{+i} - D_j[\overline{\Psi}^l_+\widehat{\Gamma}^{kji}\Psi_{+i} - \overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\Psi^l_+
\end{aligned}$$

[14]The SUSY completion of the Weyl anomaly in a 4 dimensional supersymmetric theory was obtained in [101, 102] by using the superspace formalism. To get the fermionic sector explicitly, however, one has yet to expand it further around the bosonic coordinates.

$$
\begin{aligned}
&- \overline{\Psi}_{+i}(\gamma^{jk}\gamma^{pl}\widehat{\Gamma}^i - \gamma^{jk}\gamma^{il}\widehat{\Gamma}^p + \gamma^{jp}\gamma^{il}\widehat{\Gamma}^k - \gamma^{pl}\gamma^{ij}\widehat{\Gamma}^k)\Psi_{+p}]\Big] \\
&- \frac{2(d-2)^2}{d-1}(\overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\not{\mathbb{D}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l} - \overline{\Psi}_{+i}\overleftarrow{\mathbb{D}}_j\widehat{\Gamma}^{ij}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}^{kl}\mathbb{D}_k\Psi_{+l}) \\
&- 2(\overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\not{\mathbb{D}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k} - \overline{\Psi}_{+p}\overleftarrow{\mathbb{D}}_q\widehat{\Gamma}^{pqi}\overleftarrow{\not{\mathbb{D}}}\widehat{\Gamma}_i{}^{jk}\mathbb{D}_j\Psi_{+k})\Bigg\}. \qquad (3.5.3)
\end{aligned}
$$

The holographic computation of the supersymmetric Weyl anomaly in 4D is quite remarkable; even though its bosonic part has already been known for a long time, it seems really tough to obtain its SUSY partner terms by means of giving an ansatz and finding out the coefficients, whereas holography enables us to compute them directly.

We comment that although the bosonic sector of $\mathcal{A}_{\mathrm{W}}^{G}$ is the sum of the a anomaly density $E_{(4)}$ and c anomaly $I_{(4)}$, the fermionic sector is in fact the SUSY partner of the c anomaly density up to a total derivative. This is because the integral of $E_{(4)}$ is supersymmetric by itself, as mentioned before.

For the toy model of section 3.4.5, we have an additional contribution to the Weyl anomaly density, which is

$$
\begin{aligned}
\mathcal{A}_W^{(\mathrm{model})}[\Phi] = \frac{1}{2\kappa^2}\Bigg(&\frac{1}{2(d-1)}\varphi^2 R + \partial_i\varphi\partial^i\varphi + \overline{\zeta}_-\not{\mathbb{D}}\zeta_- + i\overline{\zeta}_-\widehat{\Gamma}^i\not{\partial}\varphi\Psi_{+i} \\
&- \frac{2i}{d-1}\varphi\overline{\zeta}_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} - \frac{1}{2(d-1)}\varphi^2\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} \\
&- \frac{1}{d-1}\varphi\overline{\Psi}_{+i}\partial^i\varphi\widehat{\Gamma}^j\Psi_{+j} + \mathrm{h.c.}\Bigg). \qquad (3.5.4)
\end{aligned}
$$

The total Weyl anomaly density is thus given by[15]

$$
\boxed{
\begin{aligned}
\mathcal{A}_W[\Phi] &= \mathcal{A}_W^{(G)}[\Phi] + \mathcal{A}_W^{(\mathrm{model})}[\Phi] \\
&= \mathcal{A}_W^{(G)} + \frac{1}{2\kappa^2}\Bigg(\frac{1}{2(d-1)}\varphi^2 R + \partial_i\varphi\partial^i\varphi + \overline{\zeta}_-\not{\mathbb{D}}\zeta_- + i\overline{\zeta}_-\widehat{\Gamma}^i\not{\partial}\varphi\Psi_{+i} \\
&\quad - \frac{2i}{d-1}\varphi\overline{\zeta}_-\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} - \frac{1}{2(d-1)}\varphi^2\overline{\Psi}_{+i}\widehat{\Gamma}^{ijk}\mathbb{D}_j\Psi_{+k} \\
&\quad - \frac{1}{d-1}\varphi\overline{\Psi}_{+i}\partial^i\varphi\widehat{\Gamma}^j\Psi_{+j} + \mathrm{h.c.}\Bigg).
\end{aligned}
} \qquad (3.5.5)
$$

[15]It seems that the bosonic sector of the conformal anomaly density $\mathcal{A}$ here is different from the one given in [30] (see (162) there), because of the φ^4 term in $\widetilde{\mathcal{L}}_{(4)}$. However, one can easily check that it actually vanishes, taking into account (3.2.4). This is because in our model the superpotential $\mathcal{W}$ is analytic in φ by construction, while a non-zero φ^4 term in $\widetilde{\mathcal{L}}_{(4)}$ requires that the solution $\mathcal{W}$ of (3.2.4) contains $\log\varphi$ terms.

Super-Weyl anomaly

Here we compute the super-Weyl anomaly for the toy model. As pointed out in section 3.4.2, (3.4.26b) holds up to the finite order. For the toy model, it means that the RHS of (3.4.26b) is not canceled out and an additional finite term

$$+\frac{\sqrt{-\gamma}}{\kappa^2}\frac{\varphi^2}{2(d-1)}\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}, \tag{3.5.6}$$

comes out from the LHS of (3.4.26b). As in the case of the Weyl anomaly, we thus get from (3.4.32)

$$\begin{aligned}
&-i\mathcal{G}^{IJ}\partial_J\mathcal{W}\pi^{\bar{\zeta}}_{(7/2)I}-\frac{1}{d-1}\mathcal{W}\widehat{\Gamma}_i\pi^i_{(4)\overline{\Psi}}\\
=&-\frac{\kappa^2}{\sqrt{-\gamma}}\pi^{jk}_{(2)}\left(\frac{1}{d-1}\gamma_{jk}\widehat{\Gamma}_i-\gamma_{ij}\widehat{\Gamma}_k\right)\pi^i_{(2)\overline{\Psi}}-\frac{\sqrt{-\gamma}}{2\kappa^2}i\partial_i\varphi\widehat{\Gamma}^i\zeta_-+\frac{\sqrt{-\gamma}}{\kappa^2}\frac{\varphi^2}{2(d-1)}\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}\\
=&\frac{\sqrt{-\gamma}}{\kappa^2}\left[\frac{1}{4(d-2)^2}\left(\frac{d}{d-1}R\widehat{\Gamma}^{kl}-2R_i{}^k\widehat{\Gamma}^{il}+2R_i{}^l\widehat{\Gamma}^{ik}\right)\mathbb{D}_k\Psi_{+l}-\right.\\
&\left.\qquad-\frac{i}{2}\partial_i\varphi\widehat{\Gamma}^i\zeta_-+\frac{1}{2(d-1)}\varphi^2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}\right],
\end{aligned} \tag{3.5.7}$$

or

$$\boxed{\begin{aligned}\mathcal{A}_{\text{sW}}[\Phi]=\frac{1}{\kappa^2}&\left[\frac{1}{4(d-2)^2}\left(\frac{d}{d-1}R\widehat{\Gamma}^{kl}-2R_i{}^k\widehat{\Gamma}^{il}+2R_i{}^l\widehat{\Gamma}^{ik}\right)\mathbb{D}_k\Psi_{+l}-\right.\\
&\left.\qquad-\frac{i}{2}\partial_i\varphi\widehat{\Gamma}^i\zeta_-+\frac{1}{2(d-1)}\varphi^2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}\right].\end{aligned}} \tag{3.5.8}$$

Notice that the terms in the first bracket

$$\mathcal{A}^{(G)}_{\text{sW}}[e,\Psi_+]=\frac{1}{\kappa^2}\frac{1}{4(d-2)^2}\left(\frac{d}{d-1}R\widehat{\Gamma}^{kl}-2R_i{}^k\widehat{\Gamma}^{il}+2R_i{}^l\widehat{\Gamma}^{ik}\right)\mathbb{D}_k\Psi_{+l}, \tag{3.5.9}$$

are universal, in the sense that they do not depend on the model.

Wess-Zumino consistency condition

From the relation (3.4.35a) and corresponding equation for the toy model we find that the Weyl anomaly (3.5.3) and (3.5.5) satisfies the Wess-Zumino (WZ) consistency condition, which can be seen as follows. Defining the Weyl transformation operator δ_σ by

$$\delta_\sigma\equiv\int_{\partial\mathcal{M}}d^dx\sum_{\Phi_{(0)}}\delta_\sigma\Phi_{(0)}\frac{\delta}{\delta\Phi_{(0)}}, \tag{3.5.10}$$

where $\Phi_{(0)}$ refers to the source for every field Φ, the WZ consistency condition becomes $[\delta_{\sigma_1}, \delta_{\sigma_2}]S_{ren} = 0$. This is equivalent to demanding that $\delta_{\sigma_1} \int d^d x \ \mathcal{A}_{\mathrm{W}}\sigma_2$ be symmetric in σ_1 and σ_2, which can be seen from (3.4.35a) since

$$\sum_{\Phi_{(0)}} \delta_{\sigma_1}\Phi_{(0)} \frac{\delta}{\delta\Phi_{(0)}} \int d^d y \ \mathcal{A}_{\mathrm{W}}\sigma_2 = \sigma_1\partial^i(T\partial_i\sigma_2), \tag{3.5.11}$$

for a certain scalar function T. We note that the SUSY and super-Weyl invariance of the Weyl anomaly follows from (3.4.35b) and (3.4.35c), which can be thought as the WZ consistency checks.

In order to see that the super-Weyl anomaly (3.5.8) satisfies the WZ consistency condition, first we need to find the algebra of relevant symmetries. From (3.B.38), one can readily see that[16]

$$[\delta_{\epsilon_+}, \delta_{\bar{\epsilon}'_-}]e^a_i = (\delta_\sigma + \delta_\lambda)e^a_i, \quad [\delta_{\epsilon_+}, \delta_{\bar{\epsilon}'_-}]\varphi^I = (\delta_\sigma + \delta_\lambda)\varphi^I, \tag{3.5.12}$$

with the parameters $\sigma = \frac{1}{2}\bar{\epsilon}'_-\epsilon_+, \quad \lambda = \frac{1}{2}\bar{\epsilon}'_-\Gamma^{ab}\epsilon_+$. Notice that in our analysis it is impossible to see the above commutator for the fermionic sources, since our consideration is limited to quadratic order in fermions. However, (3.5.12) provides the WZ consistency condition for the super-Weyl anomaly, namely

$$\begin{aligned}\left(\delta_{\epsilon_+}\int d^d x|\boldsymbol{e}_{(0)}|\bar{\epsilon}'_-\mathcal{A}_{\mathrm{sW}}[\Phi_{(0)}]\right)\Big|_{\text{bosonic}} &= \Big([\delta_{\epsilon_+}, \delta_{\bar{\epsilon}'_-}]S_{ren}\Big)\Big|_{\text{bosonic}} = \\ &= -\int d^d x|\boldsymbol{e}_{(0)}|\sigma\mathcal{A}^{(B)}_W[\Phi_{(0)}],\end{aligned} \tag{3.5.13}$$

since $\delta_\lambda S_{ren} = 0$. Here $\mathcal{A}^{(B)}_W$ refers to the bosonic sector of the Weyl anomaly. In the following we show (3.5.13) in detail, namely

$$\begin{aligned}
&\delta_{\epsilon_+}\int d^d x\sqrt{-\gamma}\,\bar{\epsilon}'_-\mathcal{A}_{\mathrm{sW}} = \\
&= \frac{1}{\kappa^2}\int d^d x\sqrt{-\gamma}\bar{\epsilon}'_-\Big[\frac{1}{4(d-2)^2}\left(\frac{d}{d-1}R\widehat{\Gamma}^{kl} - 2R_i{}^k\widehat{\Gamma}^{il} + 2R_i{}^l\widehat{\Gamma}^{ik}\right)\mathbb{D}_k\mathbb{D}_l\epsilon_+ \\
&\qquad - \frac{1}{4}\partial_i\varphi\widehat{\Gamma}^i\widehat{\Gamma}^j\partial_j\varphi\epsilon_+ + \frac{1}{2(d-1)}\varphi^2\widehat{\Gamma}^{ij}\mathbb{D}_i\mathbb{D}_j\epsilon_+\Big] \\
&= \frac{1}{\kappa^2}\int d^d x\sqrt{-\gamma}\,\bar{\epsilon}'_-\Big[\frac{1}{32(d-2)^2}\left(\frac{d}{d-1}R\widehat{\Gamma}^{kl} - 2R_i{}^k\widehat{\Gamma}^{il} + 2R_i{}^l\widehat{\Gamma}^{ik}\right)R_{mnkl}\widehat{\Gamma}^{mn} \\
&\qquad - \frac{1}{4}\partial_i\varphi\partial^i\varphi + \frac{1}{16(d-1)}\varphi^2\widehat{\Gamma}^{ij}\widehat{\Gamma}^{kl}R_{ijkl}\Big]\epsilon_+ \\
&= \frac{1}{\kappa^2}\int d^d x\sqrt{-\gamma}\,\bar{\epsilon}'_-\Big[\frac{1}{32(d-2)^2}\left(-\frac{2d}{d-1}R^2 + 8R_{ij}R^{ij}\right) - \frac{1}{4}\partial_i\varphi\partial^i\varphi - \frac{1}{8(d-1)}\varphi^2 R\Big]\epsilon_+ \\
&= -\int d^d x\sqrt{-\gamma}\,\sigma\mathcal{A}^{(B)}_W,
\end{aligned} \tag{3.5.14}$$

[16]Here the subscript o is omitted again, which was used to denote the leading asymptotics of the variation parameters in appendix 3.B.4.

where again $\sigma = \frac{1}{2}\overline{\epsilon}'_{-}\epsilon_{+}$. In the above computation we omitted the subscript $_{(0)}$ for simplicity. In the same spirit, one can find another WZ consistency condition for the super-Weyl anomaly from

$$[\delta_{\epsilon_-}, \delta_{\epsilon'_-}]e^a_i = [\delta_{\epsilon_-}, \delta_{\epsilon'_-}]\varphi^I = 0. \tag{3.5.15}$$

We therefore have

$$\Big([\delta_{\epsilon_-}, \delta_{\epsilon'_-}]S_{ren}\Big)\Big|_{\text{bosonic}} = 0, \tag{3.5.16}$$

which can be shown in the same way.

3.5.2 SUSY transformation of operators

Now that the Ward identities are completely determined, we can use (3.5.2) to derive the gPBH transformation of the renormalized canonical momenta, without using the FG expansions of the induced fields [34, 13, 103]. In order to describe the gPBH transformation of the induced fields and their renormalized canonical momenta in an integrated way, we introduce the concept of a generalized Poisson bracket, which is defined by (see e.g. (6.30) in [34])

$$\begin{aligned}
\{A[\Phi_{(0)}, \Pi^\Phi], B[\Phi_{(0)}, \Pi^\Phi]\} &\equiv \int_{\partial\mathcal{M}} d^d x \sum_{\Phi_{(0)}} \left(\frac{\delta A}{\delta \Phi_{(0)}}\frac{\delta B}{\delta \Pi^\Phi} - \frac{\delta B}{\delta \Phi_{(0)}}\frac{\delta A}{\delta \Pi^\Phi}\right) \\
&= \int_{\partial\mathcal{M}} d^d x \Bigg(\frac{\delta A}{\delta e^a_{(0)i}}\frac{\delta B}{\delta \Pi^i_a} - \frac{\delta B}{\delta e^a_{(0)i}}\frac{\delta A}{\delta \Pi^i_a} + \frac{\delta A}{\delta \varphi^I_{(0)}}\frac{\delta B}{\delta \Pi^\varphi_I} - \frac{\delta B}{\delta \varphi^I_{(0)}}\frac{\delta A}{\delta \Pi^\varphi_I} \\
&\quad + A\frac{\overleftarrow{\delta}}{\delta \Psi_{(0)+i}}\frac{\overrightarrow{\delta}}{\delta \Pi^i_\Psi}B - B\frac{\overleftarrow{\delta}}{\delta \Psi_{(0)+i}}\frac{\overrightarrow{\delta}}{\delta \Pi^i_\Psi}A + A\frac{\overleftarrow{\delta}}{\delta \zeta^I_{(0)-}}\frac{\overrightarrow{\delta}}{\delta \Pi^\zeta_I}B - B\frac{\overleftarrow{\delta}}{\delta \zeta^I_{(0)-}}\frac{\overrightarrow{\delta}}{\delta \Pi^\zeta_I}A \\
&\quad + B\frac{\overleftarrow{\delta}}{\delta \Pi^i_{\overline{\Psi}}}\frac{\overrightarrow{\delta}}{\delta \overline{\Psi}_{(0)+i}}A - A\frac{\overleftarrow{\delta}}{\delta \Pi^i_{\overline{\Psi}}}\frac{\overrightarrow{\delta}}{\delta \overline{\Psi}_{(0)+i}}B + B\frac{\overleftarrow{\delta}}{\delta \Pi^{\overline{\zeta}}_I}\frac{\overrightarrow{\delta}}{\delta \overline{\zeta}^I_{(0)-}}A - A\frac{\overleftarrow{\delta}}{\delta \Pi^{\overline{\zeta}}_I}\frac{\overrightarrow{\delta}}{\delta \overline{\zeta}^I_{(0)-}}B \Bigg),
\end{aligned} \tag{3.5.17}$$

where $A[\Phi_{(0)}, \Pi^\Phi]$ and $B[\Phi_{(0)}, \Pi^\Phi]$ are arbitrary functions on the phase space ($\Phi_{(0)}$, Π^Φ). The Ward identities (3.5.2) then allow us to define a constraint function on the phase space

$$\begin{aligned}
\mathcal{C}[\xi, \sigma, \epsilon_\pm, \lambda] \equiv \int_{\partial\mathcal{M}} d^d x \Bigg\{ &\xi_i \Big(e^{ai}_{(0)} D_j \Pi^j_a - (\partial^i \varphi^I_{(0)})\Pi^\varphi_I - (\overline{\zeta}^I_{(0)-}\overleftarrow{\mathbb{D}}^i)\Pi^{\overline{\zeta}}_I - \Pi^\zeta_I(\mathbb{D}^i \zeta^I_{(0)-}) \\
&- \Pi^j_\Psi \left(\mathbb{D}^i \Psi_{(0)+j}\right) - \left(\overline{\Psi}_{(0)+j}\overleftarrow{\mathbb{D}}^i\right)\Pi^j_{\overline{\Psi}} + D_j(\Pi^j_\Psi \Psi^i_{(0)+} + \overline{\Psi}^i_{(0)+}\Pi^j_{\overline{\Psi}})\Big) \\
&+ \sigma\Big[- e^a_{(0)i}\Pi^i_a - \mathcal{G}^{IJ}\partial_I \mathcal{W}\Pi^\varphi_J - \frac{1}{2}(\overline{\Psi}_{(0)+i}\Pi^i_{\overline{\Psi}} + \text{h.c.}) \\
&+ \big(\frac{1}{2}\delta^J_I - \partial_I \partial^J \mathcal{W}\big)\big(\overline{\zeta}^I_{(0)-}\Pi^{\overline{\zeta}}_J + \text{h.c.}\big) - |\boldsymbol{e}_{(0)}|\mathcal{A}_{\rm W}[\Phi_{(0)}]\Big] \\
&+ \overline{\epsilon}_+\Big(- \frac{1}{2}\Gamma^a \Psi_{(0)+i}\Pi^i_a - \frac{i}{2}\zeta^I_{(0)-}\Pi^\varphi_I + \frac{i}{2}\partial\!\!\!/ \varphi^I_{(0)}\Pi^{\overline{\zeta}}_I + \mathbb{D}_i \Pi^i_{\overline{\Psi}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{2}\Pi^i_a\overline{\Psi}_{(0)+i}\Gamma^a+\frac{i}{2}\Pi^{\varphi}_I\overline{\zeta}^I_{(0)-}+\frac{i}{2}\Pi^{\zeta}_I\partial\!\!\!/\varphi^I_{(0)}+\Pi^i_{\Psi}\overleftarrow{\mathbb{D}}_i\Big)\epsilon_+\\
&+\overline{\epsilon}_-\Big(i\mathcal{G}^{IJ}\partial_I\mathcal{W}\Pi^{\overline{\zeta}}_J-\widehat{\Gamma}_i\Pi^i_{\overline{\Psi}}+|\boldsymbol{e}_{(0)}|\mathcal{A}_{\rm sW}[\Phi_{(0)}]\Big)\\
&+\Big(\Pi^i_{\Psi}\widehat{\Gamma}_i-i\mathcal{G}^{IJ}\partial_I\mathcal{W}\Pi^{\zeta}_J+|\boldsymbol{e}_{(0)}|\overline{\mathcal{A}}_{\rm sW}[\Phi_{(0)}]\Big)\epsilon_-\\
&-\lambda^{ab}\Big[e_{(0)[ai}\Pi^i_{b]}+\frac{1}{4}\big(\overline{\zeta}^I_{(0)-}\Gamma_{ab}\Pi^{\overline{\zeta}}_I+\overline{\Psi}_{(0)+i}\Gamma_{ab}\Pi^i_{\overline{\Psi}}+\text{h.c.}\big)\Big]\Bigg\},
\end{aligned}
\tag{3.5.18}
$$

which generates the gPBH transformation (3.B.4) through the Poisson bracket[17]

$$\delta_{\sigma,\epsilon_\pm,\lambda}\Phi_{(0)}=\{\mathcal{C}[\sigma,\epsilon_\pm,\lambda],\Phi_{(0)}\},\quad \delta_{\sigma,\epsilon_\pm,\lambda}\Pi^{\Phi}=\{\mathcal{C}[\sigma,\epsilon_\pm,\lambda],\Pi^{\Phi}\},\tag{3.5.19a}$$

$$\delta^{(\rm cgct)}_{\xi}\Phi_{(0)}=\{\mathcal{C}[\xi],\Phi_{(0)}\},\quad \delta^{(\rm cgct)}_{\xi}\Pi^{\Phi}=\{\mathcal{C}[\xi],\Pi^{\Phi}\}.\tag{3.5.19b}$$

Here $\delta^{(\rm cgct)}_{\xi}$ refers to the covariant general coordinate transformation (see e.g. section 11.3 of [97]), under which variation of the fields is given by

$$\delta^{(\rm cgct)}_{\xi}e^a_{(0)i}=D_i\xi^a,\quad \delta^{(\rm cgct)}_{\xi}\varphi^I_{(0)}=\xi^a\partial_a\varphi^I_{(0)}\equiv\xi^i\partial_i\varphi^I_{(0)},\tag{3.5.20a}$$

$$\delta^{(\rm cgct)}_{\xi}\Psi_{(0)+i}=\xi^j\mathbb{D}_j\Psi_{(0)+i}+(D_i\xi^j)\Psi_{(0)+j},\quad \delta^{(\rm cgct)}_{\xi}\zeta^I_{(0)-}=\xi^a\mathbb{D}_a\zeta^I_{(0)-}\equiv\xi^i\mathbb{D}_i\zeta^I_{(0)-},\tag{3.5.20b}$$

where $\xi^a\equiv\xi^i e^a_{(0)i}$. Meanwhile, δ_ξ given in (3.B.4) is the general coordinate transformation and it is related to $\delta^{(\rm cgct)}_{\xi}$ by

$$\delta^{(\rm cgct)}_{\xi}=\delta_\xi-\delta_{\lambda_{ab}=\omega_{jab}\xi^j}.\tag{3.5.21}$$

The reason why diffeomorphisms and local Lorentz transformations appear in a mixed way is that the constraint function and the Poisson bracket can only give a covariant quantity but δ_ξ in (3.B.38) is not covariant by itself. Moreover, SUSY transformations require the sources to be covariant and thus we are forced to see the covariant general coordinate transformation rather than the general coordinate transformation.

The useful variations of renormalized canonical momenta extracted from (3.5.19) are

$$\delta_{\epsilon_+}\Pi^i_{\overline{\Psi}}=\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\mathcal{C}[\epsilon_+]=\frac{1}{2}\Pi^i_a\Gamma^a\epsilon_+\tag{3.5.22a}$$

$$
\begin{aligned}
\delta_{\epsilon_-}\Pi^i_{\overline{\Psi}}&=\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\mathcal{C}[\epsilon_-]=\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\int d^dx|\boldsymbol{e}_{(0)}|\,\overline{\mathcal{A}}_{\rm sW}[\Phi_{(0)}]\epsilon_-\\
&=-\frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{1}{8}\mathbb{D}_k\left(\Big[\frac{2}{3}R_{(0)}\widehat{\Gamma}^{ik}_{(0)}-R_{(0)j}{}^k\widehat{\Gamma}^{ij}_{(0)}+R_{(0)j}{}^i\widehat{\Gamma}^{kj}_{(0)}\Big]\epsilon_-\right)-\\
&\qquad\underline{-\frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{1}{6}\widehat{\Gamma}^{ij}\mathbb{D}_j(\varphi^2_{(0)}\epsilon_-)},
\end{aligned}
\tag{3.5.22b}
$$

[17]It is obvious that the gPBH transformation of the sources can be obtained through this Poisson bracket. In appendix 3.D we show that the same holds for the canonical momenta.

$$\delta_{\epsilon_+}\Pi_I^{\overline{\zeta}} = \frac{\delta}{\delta\overline{\zeta}^I_{(0)-}}\mathcal{C}[\epsilon_+] = \frac{i}{2}\Pi_I^{\varphi}\epsilon_+, \tag{3.5.22c}$$

$$\delta_{\epsilon_-}\Pi_I^{\overline{\zeta}} = \frac{\delta}{\delta\overline{\zeta}^I_{(0)-}}\mathcal{C}[\epsilon_-] = \frac{\delta}{\delta\overline{\zeta}^I_{(0)-}}\int d^dx|\boldsymbol{e}_{(0)}|\,\overline{\mathcal{A}}_{\rm sW}[\Phi_{(0)}]\epsilon_- = \underline{-\frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{i}{2}\partial_i\varphi_{(0)}\widehat{\Gamma}^i_{(0)}\epsilon_-}, \tag{3.5.22d}$$

$$\delta_{\epsilon_+}\Pi_I^{\varphi} = \frac{\delta}{\delta\varphi^I_{(0)}}\mathcal{C}[\epsilon_+] = -\frac{i}{2}\partial_i\left(\Pi_I^{\zeta}\widehat{\Gamma}^i\epsilon_+\right), \tag{3.5.22e}$$

$$\begin{aligned}\delta_{\epsilon_-}\Pi_I^{\varphi} &= \frac{\delta}{\delta\varphi^I_{(0)}}\mathcal{C}[\epsilon_-] = -i\partial_I(\mathcal{G}^{JK}\partial_K\mathcal{W})\Pi_J^{\zeta}\epsilon_- + \frac{\delta}{\delta\varphi^I_{(0)}}\int d^dx|\boldsymbol{e}_{(0)}|\,\overline{\mathcal{A}}_{\rm sW}[\Phi_{(0)}]\epsilon_- \\ &= \underline{i\Pi^{\zeta}\epsilon_- + \frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{1}{3}\varphi_{(0)}\overline{\Psi}_{(0)+j}\overleftarrow{\mathbb{D}}_i\widehat{\Gamma}^{ji}_{(0)}\epsilon_-},\end{aligned} \tag{3.5.22f}$$

where $R_{(0)}$, $R_{(0)i}{}^j$ and $\widehat{\Gamma}^i_{(0)}$ denote the Ricci scalar, Ricci tensor, the Gamma matrix and the determinant of the metric for the vielbein $e^a_{(0)i}$. Here the underlined terms are computed specifically for the toy model. Notice that due to the super-Weyl anomaly, the ϵ_- variation of the renormalized canonical momenta contains bosonic anomalous terms, which have a similar origin as the Schwarzian derivative appearing in the conformal transformation of the energy-momentum tensor of 2D CFT.

3.5.3 BPS relations

A bulk (bosonic) BPS configuration, which is a bosonic solution of the classical SUGRA action as well as is invariant under bulk SUSY transformation with a certain parameter, corresponds to a supersymmetric vacuum state of the dual field theory. Since the vacuum expectation value (vev) of many observables is computed in SUSY field theories, it is necessary to pay special attention to the bulk BPS solutions. The existence of a bulk BPS configuration implies that there exists a boundary SUSY parameter, under the gPBH transformation with which the fermionic sources are invariant, namely[18]

$$\delta_\eta\Psi_{(0)+i} \equiv \delta_{\eta_+}\Psi_{(0)+i} + \delta_{\eta_-}\Psi_{(0)+i} = \mathbb{D}_i\eta_+ - \widehat{\Gamma}_{(0)i}\eta_- = 0, \tag{3.5.23a}$$

$$\delta_\eta\zeta^I_{(0)-} = -\frac{i}{2}\widehat{\Gamma}^i_{(0)}\partial_i\varphi^I_{(0)}\eta_+ + i\mathcal{G}^{IJ}\partial_J\mathcal{W}\eta_- = 0, \tag{3.5.23b}$$

where the first equation is usually referred to as the conformal Killing spinor (CKS) condition. Actually, the rigid supersymmetry of the boundary field theory is found by solving (3.5.23) [35, 26, 104].[19]

[18] Here we do not discuss the integrability condition of (3.5.23). For a discussion of the geometry of (3.5.23a), which is also known as the twistor equation, see e.g. section 3.1 in [26].

[19] More precisely, most of the rigid $\mathcal{N}=1$ SUSY field theories on curved backgrounds require a $U(1)$ R-symmetry gauge field to be turned on. In this case, which is discussed in [28], the covariant derivative $\mathbb{D}_i$ in (3.5.23a) becomes $\mathbb{D}_i + igA_i$, where g is the R-charge of the corresponding field.

Now we show that the η-variation of any renormalized canonical momentum vanishes on a BPS solution, i.e.

$$\boxed{\delta_\eta \Pi^\Phi\Big|_{\text{BPS}} \equiv \delta_{\eta_+}\Pi^\Phi\Big|_{\text{BPS}} + \delta_{\eta_-}\Pi^\Phi\Big|_{\text{BPS}} = 0, \quad \text{for any source } \Phi_{(0)},} \tag{3.5.24}$$

where for the fermionic operators we have from (3.5.22)

$$\delta_\eta \Pi^i_{\overline{\Psi}} = \frac{1}{2}\Pi^i_a \Gamma^a \eta_+ + \frac{\delta}{\overline{\Psi}_{(0)+i}} \int_{\Sigma_r} d^d x |\boldsymbol{e}_{(0)}|\, \overline{\mathcal{A}}_{\text{sW}}[\Phi_{(0)}]\eta_-, \tag{3.5.25a}$$

$$\delta_\eta \Pi^{\overline{\zeta}}_I = \frac{i}{2}\Pi^\varphi_I \eta_+ + \frac{\delta}{\delta\overline{\zeta}^I_{(0)-}} \int_{\Sigma_r} d^d x |\boldsymbol{e}_{(0)}|\, \overline{\mathcal{A}}_{\text{sW}}[\Phi_{(0)}]\eta_-. \tag{3.5.25b}$$

This is the holographic version of the fact that the vev of any Q-exact operator vanishes on SUSY vacua. We only need to consider the variation of the fermionic canonical momenta, since the η-variation of the bosonic canonical momenta trivially vanishes on a bosonic solution. One can in principle see (3.5.24) by expanding the bulk BPS equations. But since we have the SUSY and super-Weyl Ward identities, the form of which is the same for all SCFTs, we take advantage of the Ward identities (3.5.2a) for η_+ and (3.5.2b) for η_-.

Taking into account the CKS condition (3.5.23), we obtain from the Ward identities that

$$\begin{aligned}
0 = \int_{\partial\mathcal{M}} d^d x \bigg[&\Big(-\frac{1}{2}\overline{\Psi}_{(0)+i}\Gamma^a \Pi^i_a - \frac{i}{2}\overline{\zeta}^I_{(0)-}\Pi^\varphi_I - \frac{i}{2}\Pi^\zeta_I \partial\!\!\!/ \varphi^I_{(0)} - \Pi^i_\Psi \overleftarrow{\mathbb{D}}_i \Big)\eta_+ \\
&+ \Big(i\mathcal{G}^{IJ}\partial_I \mathcal{W}\Pi^\zeta_J - \Pi^i_\Psi \widehat{\Gamma}_{(0)i} - |\boldsymbol{e}_{(0)}|\overline{\mathcal{A}}_{\text{sW}}[\Phi_{(0)}]\Big)\eta_- \bigg] \\
= \int_{\partial\mathcal{M}} d^d x \Big(&-\frac{1}{2}\overline{\Psi}_{(0)+i}\Gamma^a \Pi^i_a \eta_+ - \frac{i}{2}\Pi^\varphi_I \overline{\zeta}^I_{(0)-}\eta_+ - |\boldsymbol{e}_{(0)}|\overline{\mathcal{A}}_{\text{sW}}[\Phi_{(0)}]\eta_- \Big).
\end{aligned} \tag{3.5.26}$$

We emphasize that because the Ward identities are valid for any background, (3.5.26) holds at least to linear order in fermions for any value of $\overline{\Psi}_{(0)+i}$ and $\overline{\zeta}^I_{(0)-}$ as long as the bosonic sources admit a CKS. There might be a correction at order of $O\Big((\Psi_{(0)+})^2, (\zeta_{(0)-})^2\Big)$, though. Note that non-trivial dependence of bosonic momenta Π^i_a and Π^φ_I on the fermionic sources occurs from the quadratic order in fermions, i.e.

$$\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\Pi^i_a\bigg|_{\Psi_{(0)+i}=\zeta^I_{(0)-}=\cdots=0} = 0, \tag{3.5.27}$$

and so on. Therefore, by taking the functional derivative of (3.5.26) with respect to the fermionic sources and evaluating on a (bosonic) supersymmetric background, we obtain the (bosonic) identities

$$\frac{1}{2}\Pi^i_a \Gamma^a \eta_+ + \frac{\delta}{\overline{\Psi}_{(0)+i}} \int_{\Sigma_r} d^d x |\boldsymbol{e}_{(0)}|\, \overline{\mathcal{A}}_{\text{sW}}[\Phi_{(0)}]\eta_- = 0, \tag{3.5.28a}$$

$$-\frac{i}{2}\Pi^{\varphi}_{I}\eta_{+} - \frac{\delta}{\delta\overline{\zeta}^{I}_{(0)-}}\int_{\Sigma_r} d^dx|\boldsymbol{e}_{(0)}|\;\overline{\mathcal{A}}_{\rm sW}[\Phi_{(0)}]\eta_{-} = 0, \tag{3.5.28b}$$

where we used (3.3.17). Therefore, we find that on BPS backgrounds

$$\delta_\eta \Pi^{i}_{\overline{\Psi}} = 0, \quad \delta_\eta \Pi^{\overline{\zeta}}_{I} = 0, \tag{3.5.29}$$

which confirms our claim.

Note that from the field theory point of view (3.5.24) is quite natural, since supersymmetric vacua are annihilated by the preserved supercharge Q.

In order to convince ourselves, let us check (3.5.24) for the toy model. First, let us recall that in the toy model, $d = 4$ and scaling dimension of φ is 3. Then, (3.5.28)s become

$$\begin{aligned} 0 =& -\Gamma^a\eta_+\Pi^i_a + \frac{1}{\kappa^2}\frac{1}{4(d-2)^2}\mathbb{D}_k\Big[\big(\frac{d}{d-1}R_{(0)}\widehat{\Gamma}^{ki}_{(0)} - 2R_{(0)j}{}^{k}\widehat{\Gamma}^{ji}_{(0)} + 2R_{(0)j}{}^{i}\widehat{\Gamma}^{jk}_{(0)}\big)\eta_-\Big] \\ &+ \frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{1}{2(d-1)}\varphi^2_{(0)}\widehat{\Gamma}^{ij}_{(0)}\mathbb{D}_j\eta_-, \end{aligned} \tag{3.5.30}$$

$$0 = -\frac{i}{2}\eta_+\Pi^{\varphi} + \frac{|\boldsymbol{e}_{(0)}|}{\kappa^2}\frac{i}{2}\widehat{\Gamma}^{i}_{(0)}\eta_-\partial_i\varphi_{(0)}. \tag{3.5.31}$$

By combining (3.5.31) with the conformal Killing spinor equation for the toy model

$$\mathbb{D}_i\eta_+ = \widehat{\Gamma}_{(0)i}\eta_-, \tag{3.5.32a}$$

$$\frac{1}{2}\widehat{\Gamma}^{i}_{(0)}\partial_i\varphi_{(0)}\;\eta_+ + \varphi_{(0)}\eta_- = 0, \tag{3.5.32b}$$

we get

$$-\varphi_{(0)}\Pi^{\varphi} + \frac{|\boldsymbol{e}_{(0)}|}{2\kappa^2}\partial_i\varphi_{(0)}\partial^i\varphi_{(0)} = 0. \tag{3.5.33}$$

This formula can be verified in the toy model by using the bulk BPS equation.

From the bulk BPS equation for ζ with the bulk SUSY parameter $\widehat{\epsilon}$

$$\delta_{\widehat{\epsilon}}\zeta = \big(\partial\!\!\!/\varphi - \mathcal{W}'\big)\,\widehat{\epsilon} = 0, \quad \mathcal{W}' \equiv \frac{d}{d\varphi}\mathcal{W}(\varphi), \tag{3.5.34}$$

one can obtain

$$\dot{\varphi} = -\sqrt{(\mathcal{W}')^2 + \partial_i\varphi\partial^i\varphi}, \tag{3.5.35}$$

where we fix the sign from leading asymptotics of φ. It then follows from the definition of π^{φ} that

$$\pi^{\varphi} = -\frac{\sqrt{-\gamma}}{\kappa^2}\dot{\varphi} = \frac{\sqrt{-\gamma}}{\kappa^2}\sqrt{(-\mathcal{W}')^2 + \partial_i\varphi\partial^i\varphi}. \tag{3.5.36}$$

On the other hand, the full bosonic counterterms are given by

$$S_{\rm ct} = \frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\Big[\mathcal{W} - \frac{1}{4}R - \frac{1}{2}\log e^{-2r}\Big(\frac{1}{6}\varphi^2 R + \partial_i\varphi\partial^i\varphi + \cdots\Big)\Big], \tag{3.5.37}$$

where the ellipses denote the terms which do not depend on φ. The counterterms for the canonical momenta $\pi^{\varphi}_{\rm ct}$ are then given by

$$\pi^{\varphi}_{\rm ct} = \frac{\delta}{\delta\varphi} S_{\rm ct} = \frac{\sqrt{-\gamma}}{\kappa^2}\Big[-(-\mathcal{W}') - \frac{1}{2}\log e^{-2r}\Big(\frac{1}{3}\varphi R - 2\Box\varphi\Big)\Big]. \tag{3.5.38}$$

Furthermore, from the conformal Killing spinor condition (3.5.23), we obtain

$$0 = \Big(\Box_{(0)}\varphi_{(0)} - \frac{1}{6}\varphi_{(0)}R_{(0)}\Big)\eta_+, \tag{3.5.39}$$

which implies that the logarithmically divergent terms in (3.5.38) actually do not contribute to the counterterms. Eventually, the renormalized canonical momentum Π^{φ} becomes

$$\Pi^{\varphi} = \frac{1}{\kappa^2}\lim_{r\to+\infty} e^{-3r}\sqrt{-\gamma}\frac{\partial_i\varphi\partial^i\varphi}{\sqrt{(-\mathcal{W}')^2 + \partial_i\varphi\partial^i\varphi} + (-\mathcal{W}')} = \frac{|\boldsymbol{e}_{(0)}|}{2\kappa^2}\frac{\partial_i\varphi_{(0)}\partial^i\varphi_{(0)}}{\varphi_{(0)}}, \tag{3.5.40}$$

which confirms the result (3.5.33) as well as the anomalous SUSY variation of the renormalized canonical momenta (3.5.22).

3.5.4 Conserved charges and supersymmetry algebra

We recall that given a Killing vector ξ^i which satisfies the Killing condition[20]

$$\mathcal{L}_\xi g_{(0)ij} = D_{(0)i}\xi_j + D_{(0)j}\xi_i = 0, \tag{3.5.41a}$$

$$\mathcal{L}_\xi \varphi^I_{(0)} = \xi^i\partial_i\varphi^I_{(0)} = 0, \tag{3.5.41b}$$

$$\mathcal{L}_\xi \zeta^I_{(0)-} = \xi^i\mathbb{D}_{(0)i}\zeta^I_{(0)-} + \frac{1}{4}D_{(0)i}\xi_j\widehat{\Gamma}^{ij}_{(0)}\zeta^I_{(0)-} = 0, \tag{3.5.41c}$$

$$\mathcal{L}_\xi \Psi_{(0)+j} = \xi^i\mathbb{D}_{(0)i}\Psi_{(0)+j} + (D_{(0)j}\xi_i)\Psi^i_{(0)+} + \frac{1}{4}D_{(0)k}\xi_l\widehat{\Gamma}^{kl}_{(0)}\Psi_{(0)+j} = 0, \tag{3.5.41d}$$

we obtain a conservation law by combining (3.5.2d) with (3.5.2e), namely

$$D_i\left[e^a_j\xi^j\Pi^i_a + \xi^j(\Pi^i_\Psi\Psi_{+j} + \overline{\Psi}_{+j}\Pi^i_{\overline{\Psi}})\right] = 0. \tag{3.5.42}$$

Note that we use the Kosmann's definition for the spinorial Lie derivative (see e.g. [105] and (A.11) of [36][21]) and the Lie derivative is related to gPBH transformations by

$$\mathcal{L}_\xi = \delta^{(\rm cgct)}_\xi + \delta_{\lambda_{ab}=-e^i_ae^j_bD_{[i}\xi_{j]}}. \tag{3.5.43}$$

We emphasize that (3.5.42) holds for any background that admits a Killing vector. The conservation law (3.5.42) allows us to define a conserved charge associated with ξ^i, namely [20, 1]

$$\boxed{\mathcal{Q}[\xi] \equiv \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i\ \left(e^a_j\Pi^i_a + \Pi^i_\Psi\Psi_{+j} + \overline{\Psi}_{+j}\Pi^i_{\overline{\Psi}}\right)\xi^j,} \tag{3.5.44}$$

[20] $g_{(0)ij} \equiv e^a_{(0)i}e_{(0)aj}$ is the induced metric on the boundary $\partial\mathcal{M}$.

[21] In the literature, including [105], the spinoral Lie derivative is defined by $\mathcal{L}_\xi\zeta = \xi^i\mathbb{D}_i\zeta - \frac{1}{4}D_i\xi_j\widehat{\Gamma}^{ij}\zeta$. The sign of the last term is minus, since the Gamma matrices there satisfy a Grassman algebra in Euclidean signature, while here we use the Minkowskian signature.

which is independent of the choice of Cauchy surface $\mathcal{C}$. Note that the conserved charge $Q[\xi]$ is related to the constraint function by

$$\mathcal{Q}[\xi] = \mathcal{C}[\xi, \lambda_{ab} = -e^i_a e^j_b D_{[i}\xi_{j]}]. \tag{3.5.45}$$

We also have the conservation laws

$$D_i(\Pi^i_\Psi \eta_+) = D_i(\overline{\eta}_+ \Pi^i_{\overline{\Psi}}) = 0, \tag{3.5.46}$$

which follow from the SUSY and super-Weyl Ward identities (3.5.2a) and (3.5.2b) for the CKS parameters η_+ and $\overline{\eta}_+$. Note that the conservation laws (3.5.46) hold only on bosonic backgrounds. This allows us to define the conserved supercharges

$$Q^s[\eta_+] \equiv \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; \Pi^i_\Psi \eta_+, \quad Q^s[\overline{\eta}_+] \equiv \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; \overline{\eta}_+ \Pi^i_{\overline{\Psi}}. \tag{3.5.47}$$

On a bosonic background we can identify these conserved charges with the constraint functions, namely

$$Q^s[\eta_+] = \mathcal{C}[\eta_+, \eta_-], \quad Q^s[\overline{\eta}_+] = \mathcal{C}[\overline{\eta}_+, \overline{\eta}_-]. \tag{3.5.48}$$

It then follows from (3.5.22) that on a bosonic background we have

$$\begin{aligned} \{Q^s[\eta_+], Q^s[\overline{\eta}_+]\}\Big|_{\text{Bosonic}} &= \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; \overline{\eta}_+ \{\mathcal{C}[\eta_+, \eta_-], \Pi^i_\Psi\}\Big|_{\text{Bosonic}} = \\ = \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; &\left[\frac{1}{2}\Pi^i_a \overline{\eta}_+ \Gamma^a \eta_+ + \overline{\eta}_+ \Big(\frac{\delta}{\delta\overline{\Psi}_{(0)+i}} \int_{\partial\mathcal{M}} d^d x |\boldsymbol{e}_{(0)}| \overline{\mathcal{A}}_{\text{sW}} \eta_-\Big)\right]_{\text{Bosonic}}. \end{aligned} \tag{3.5.49}$$

In the case where the conformal Killing vector[22]

$$\mathcal{K}^i \equiv i\overline{\eta}_+ \widehat{\Gamma}^i \eta_+, \tag{3.5.50}$$

becomes a Killing vector, we can see that on a bosonic background the above commutator becomes

$$\{Q^s[\eta_+], Q^s[\overline{\eta}_+]\} = -\frac{i}{2}\mathcal{Q}[\mathcal{K}] + \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; \overline{\eta}_+ \Big(\frac{\delta}{\delta\overline{\Psi}_{(0)+i}} \int_{\partial\mathcal{M}} d^d x |\boldsymbol{e}_{(0)}| \overline{\mathcal{A}}_{\text{sW}} \eta_-\Big). \tag{3.5.51}$$

Not surprisingly, the super-Weyl anomaly corrects the supersymmetry algebra, too.

We can obtain other commutators such as $\{\mathcal{Q}[\xi], Q^s[\eta_+]\}$. It is possible because $\mathcal{Q}[\xi]$ for the Killing vector ξ^i is conserved for any background so that

$$\begin{aligned} \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \; \{\mathcal{Q}[\xi], \Pi^i_\Psi\}\eta_+ &= \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_k \; \{\mathcal{C}[\xi, \lambda_{ab} = -e^i_a e^j_b D_{[i}\xi_{j]}], \Pi^k_\Psi\}\eta_+ = \\ = \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \Big[&- \Pi^i_\Psi \mathcal{L}_\xi \eta_+ + D_j[(\xi^j \Pi^i_\Psi - \xi^i \Pi^j_\Psi)\eta_+] + \xi^i D_j(\Pi^j_\Psi \eta_+)\Big], \end{aligned}$$

[22]One can easily check that $\mathcal{K}^i$ satisfies the conformal Killing condition, by using (3.5.23).

where the second term vanishes by using Stokes' theorem. The third term is also zero on a bosonic background, due to the conservation law. Therefore, we have

$$\{\mathcal{Q}[\xi], Q^s[\eta_+]\} = -\int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i\ \Pi^i_\Psi \mathcal{L}_\xi \eta_+ = -Q^s[\mathcal{L}_\xi \eta_+], \tag{3.5.52}$$

and in the same way

$$\{\mathcal{Q}[\xi], Q^s[\overline{\eta}_+]\} = -\int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i\ (\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi)\Pi^i_{\overline{\Psi}} = -Q^s[\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi], \tag{3.5.53}$$

since $\mathcal{L}_\xi \eta_+$ and $\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi$ become conformal Killing spinors [105], i.e.

$$\mathbb{D}_i(\mathcal{L}_\xi \eta_+) = \frac{1}{d}\widehat{\Gamma}_i \widehat{\Gamma}^j \mathbb{D}_j(\mathcal{L}_\xi \eta_+), \quad (\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi)\overleftarrow{\mathbb{D}}_i = \frac{1}{d}(\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi)\overleftarrow{\mathbb{D}}_j \widehat{\Gamma}^j \widehat{\Gamma}_i. \tag{3.5.54}$$

We note that (3.5.52) and (3.5.53) can be obtained in the other way, namely by computing

$$\{Q^s[\eta_+], e^a_j \Pi^i_a + \Pi^i_\Psi \Psi_{+j} + \overline{\Psi}_{+j}\Pi^i_{\overline{\Psi}}\}, \quad \{Q^s[\overline{\eta}_+], e^a_j \Pi^i_a + \Pi^i_\Psi \Psi_{+j} + \overline{\Psi}_{+j}\Pi^i_{\overline{\Psi}}\}. \tag{3.5.55}$$

In summary, the supersymmetry algebra on a curved (bosonic) background is

$$\boxed{\begin{aligned}
&\{Q^s[\eta_+], Q^s[\overline{\eta}_+]\} = -\frac{i}{2}\mathcal{Q}[\mathcal{K}] + \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i\ \overline{\eta}_+\Big(\frac{\delta}{\delta\overline{\Psi}_{(0)+i}}\int_{\partial\mathcal{M}} d^dx |\boldsymbol{e}_{(0)}| \overline{\mathcal{A}}_{\mathrm{sW}}\eta_-\Big),\\
&\{\mathcal{Q}[\xi], Q^s[\eta_+]\} = -Q^s[\mathcal{L}_\xi \eta_+],\\
&\{\mathcal{Q}[\xi], Q^s[\overline{\eta}_+]\} = -Q^s[\overline{\eta}_+ \overleftarrow{\mathcal{L}}_\xi].
\end{aligned}} \tag{3.5.56}$$

(3.5.56) closely resembles the SUSY algebra presented in the literature (see e.g. [35, 27, 106]), except for the super-Weyl anomaly-effect term.

We comment that (3.5.56) can be obtained without using the Poisson bracket, but in an equivalent and rather simple way. Recall that a symmetry of the field theory leads to a conservation of the corresponding (anomalous) Noether current J^i (with the anomaly $\mathcal{A}_J$)

$$D_i J^i = \mathcal{A}_J, \tag{3.5.57}$$

from which we derive the variation of any operator $\mathcal{O}$ under the symmetry transformation (see e.g. (2.3.7) in [76]), namely

$$\delta\mathcal{O}(x) + \int_{\partial\mathcal{M}} d^dy\ [D_i J^i(y) - \mathcal{A}_J(y)]\mathcal{O}(x) = 0, \tag{3.5.58}$$

where the second term can be computed by differentiating the relevant Ward identities with the source dual to operator $\mathcal{O}(x)$. Now one can readily see that the commutator of charges becomes

$$\{Q_1, Q_2\} = \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i\ (\delta_1 J^i_2) = -\int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \left(\int_{\partial\mathcal{M}} d^dy\ [D_j J^j_1(y) - \mathcal{A}_J(y)] J^i_2\right), \tag{3.5.59}$$

and this prescription gives the same result with (3.5.56). See e.g. appendix 3.E for derivation of $\{\mathcal{Q}[\xi], Q^s[\eta_+]\}$.

Now that we know from the last section that the LHS of (3.5.51) vanishes on BPS backgrounds, we can conclude that the conserved charge associated with $\mathcal{K}^i$ on BPS backgrounds is totally fixed to be a functional derivative of the fermionic anomaly, namely

$$\boxed{\mathcal{Q}[\mathcal{K}]\Big|_{\text{BPS}} = -2i \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_i \, \overline{\eta}_+ \left\{ \frac{\delta}{\delta\overline{\Psi}_{(0)+i}} \int_{\partial\mathcal{M}} d^d x |\boldsymbol{e}_{(0)}| \overline{\mathcal{A}}_{\text{sW}} \eta_- \right\}.} \tag{3.5.60}$$

Depending on the theory, $\mathcal{K}^i$ can be a combination of other Killing vectors such as ∂_t and ∂_ψ, where ψ refers to an angular coordinate. If this is the case, (3.5.60) can be regarded as a relation of the conserved charges on the supersymmetric background, but accompanied with an anomalous contribution. A similar relation is found in [28], which explains the discrepancy of the BPS condition (see e.g. (C.16) of [96])

$$\langle H \rangle + \langle J \rangle + \gamma \langle Q \rangle = 0, \tag{3.5.61}$$

for pure AdS_5 is precisely due to the anomalous contribution coming from the fermionic anomalies.

3.6 Neumann boundary conditions

Most of the computations so far are for the plus sign choice of (3.3.3b) at the beginning of section 3.3. This plus sign is actually equivalent to imposing Dirichlet boundary conditions on the spin 1/2 field ζ. Independently from this choice, we could determine the leading asymptotics of the scalar field, as emphasized before. This allows us to use the result of appendix 3.B.3 and 3.B.4 to conclude that the minus sign choice can be supersymmetric only when mass of its scalar SUSY-partner field belongs to the window [107, 108, 109]

$$-\left(\frac{d}{2}\right)^2 \leq m^2 \leq -\left(\frac{d}{2}\right)^2 + 1. \tag{3.6.1}$$

In this window (3.3.3b) is already finite, implying that the canonical momentum of ζ_- is not renormalized. Since ζ_+ by itself becomes the renormalized canonical momentum, the change of the sign from plus to minus is in fact a Legendre transformation of the renormalized on-shell action $\widehat{S}_{ren}$, which is equivalent to imposing Neumann boundary conditions on ζ_- [110]. We have seen that $\widehat{S}_{ren}$ in the case of the plus sign choice is (ϵ_+) supersymmetric (Dirichlet boundary conditions for scalar the field were implicitly imposed). Therefore, in order to preserve SUSY, one can expect that the boundary conditions for the scalar field should also be changed from Dirichlet to Neumann by a Legendre transformation.

To see this, one has to prove that the total Legendre transformation action

$$S_L = -\int_{\Sigma_r} (\widehat{\pi}^\zeta \zeta_- + \overline{\zeta}_- \widehat{\pi}^{\overline{\zeta}} + \varphi \widehat{\pi}^\varphi), \quad \widehat{\pi}^{\overline{\zeta}} = \frac{\sqrt{-\gamma}}{\kappa^2} \zeta_+, \tag{3.6.2}$$

is invariant under an ϵ_+ transformation. Note that the variation of $\Pi_I^{\bar{\zeta}}$ gives directly how gPBH transformations act on ζ_+. We again consider only one scalar field, and it is straightforward to extend the result here to the case for several scalar fields. From (3.5.22), one can find that the action of ϵ_+ on S_L gives

$$\delta_{\epsilon_+} S_L \sim -\int_{\Sigma_r} \Big(\frac{i}{2}\widehat{\pi}^{\varphi}\overline{\zeta}_-\epsilon_+ - \frac{i}{2}\partial_i\varphi\overline{\epsilon}_+\widehat{\Gamma}^i\widehat{\pi}^{\overline{\zeta}} - \frac{i}{2}\widehat{\pi}^{\varphi}\overline{\zeta}_-\epsilon_+ - \frac{i}{2}\varphi\partial_i(\overline{\epsilon}_+\widehat{\Gamma}^i\widehat{\pi}^{\varphi}) + \text{h.c.}\Big) = 0.$$

This confirms that the total action $S + S_L$ for the Neumann boundary condition is still invariant under an ϵ_+ transformation.

When it comes to the ϵ_- variation of S_L, one finds that all the momentum-related terms are canceled, as before. The anomalous terms in the ϵ_- variation of the renormalized canonical momenta, however, are not canceled but contribute to the ϵ_- anomaly of $S + S_L$, together with $\mathcal{A}_{\text{sW}}$. Namely, we obtain for the toy model that

$$\delta_{\overline{\epsilon}_-}(S + S_L) \sim \int_{\Sigma_r} d^dx\sqrt{-\gamma}\,\overline{\epsilon}_- \left(\mathcal{A}_{\text{sW}}^{(G)} - \frac{1}{6\kappa^2}\varphi^2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j} \right) \equiv \int_{\Sigma_r} d^dx\sqrt{-\gamma}\,\overline{\epsilon}_-\mathcal{A}_{\text{sW}}^N, \tag{3.6.3}$$

where the super-Weyl anomaly for Neumann boundary conditions is

$$\mathcal{A}_{\text{sW}}^N = \mathcal{A}_{\text{sW}}^{(G)} - \frac{1}{6\kappa^2}\varphi^2\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_{+j}. \tag{3.6.4}$$

3.A Notation, conventions for Gamma matrices and useful identities

Throughout this chapter Greek indexes μ, ν and $\alpha, \beta, \cdots$ refer to the coordinate and flat directions in the bulk respectively, and the Latin indexes $i, j, m, n, p, q, \cdots$ and $a, b, \cdots$ refer to the coordinate and flat directions on the radial slice respectively. The flat indices which correspond to radial-like and time-like directions are special, so we denote them by $\bar{r}$ and $\bar{t}$ respectively. The capital Latin letters $A, B, \cdots$ indicate the coordinate directions on the scalar and hyperino manifold. ∇_μ, D_i and $\mathbb{D}_i$ refer to the covariant derivative in the bulk and the covariant derivatives of the bosonic and fermionic fields on the radial slice respectively.

We use the hermitian representation of the Lorentzian Gamma matrices, following the convention in [97]. Γ^α and Γ^a indicate the Gamma matrices along the flat directions in the bulk and the boundary, while Γ^μ and $\widehat{\Gamma}^i$ refer to the Gamma matrices along the coordinate directions in the bulk and the boundary. The relations between these Gamma matrices are provided in appendix 3.C. Both in the bulk and on the boundary the hermitian conjugation of the Gamma matrix is given by

$$\Gamma^{\mu\dagger} = \Gamma^{\bar{t}}\Gamma^\mu\Gamma^{\bar{t}}, \quad \widehat{\Gamma}^{i\dagger} = \Gamma^{\bar{t}}\widehat{\Gamma}^i\Gamma^{\bar{t}}. \tag{3.A.1}$$

The following formulas, which hold in any D dimensional spacetime (see e.g. section 3 in [97]), are frequently used in this chapter.

$$\Gamma^{\mu\nu\rho} = \frac{1}{2}\{\Gamma^\mu, \Gamma^{\nu\rho}\}, \tag{3.A.2a}$$

$$\Gamma^{\mu\nu\rho\sigma} = \frac{1}{2}[\Gamma^{\mu}, \Gamma^{\nu\rho\sigma}], \tag{3.A.2b}$$
$$\Gamma^{\mu\nu\rho}\Gamma_{\sigma\tau} = \Gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\Gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\Gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}, \tag{3.A.2c}$$
$$\Gamma^{\mu\nu\rho\sigma}\Gamma_{\tau\lambda} = \Gamma^{\mu\nu\rho\sigma}{}_{\tau\lambda} + 8\Gamma^{[\mu\nu\rho}{}_{[\lambda}\delta^{\sigma]}{}_{\tau]} + 12\Gamma^{[\mu\nu}\delta^{\rho}{}_{[\lambda}\delta^{\sigma]}{}_{\tau]}, \tag{3.A.2d}$$
$$[\Gamma_{\mu\nu}, \Gamma_{\rho\sigma}] = 2(g_{\nu\rho}\Gamma_{\mu\sigma} - g_{\mu\rho}\Gamma_{\nu\sigma} - g_{\nu\sigma}\Gamma_{\mu\rho} + g_{\mu\sigma}\Gamma_{\nu\rho}), \tag{3.A.2e}$$
$$\Gamma^{\mu\nu\rho}\Gamma_{\rho} = (D-2)\Gamma^{\mu\nu}, \tag{3.A.2f}$$
$$\Gamma^{\mu\nu\rho}\Gamma_{\rho\sigma} = (D-3)\Gamma^{\mu\nu}{}_{\sigma} + 2(D-2)\Gamma^{[\mu}\delta^{\nu]}{}_{\sigma}, \tag{3.A.2g}$$
$$\Gamma^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\zeta = -\frac{1}{4}R\zeta, \tag{3.A.2h}$$
$$\Gamma^{\mu\nu\rho}\nabla_{\nu}\nabla_{\rho}\zeta = -\frac{1}{4}(R\Gamma^{\mu} - 2R_{\nu}{}^{\mu}\Gamma^{\nu})\zeta, \tag{3.A.2i}$$

where δ refers to the Kronecker delta.

There are left and right acting functional derivatives with respect to fermionic variable ψ, namely

$$\frac{\overrightarrow{\delta}}{\delta\overline{\psi}}, \quad \frac{\overleftarrow{\delta}}{\delta\psi}, \tag{3.A.3}$$

and in most cases the rightarrow symbol $\rightarrow$ is omitted. Here $\overline{\psi}$ denotes the Dirac adjoint of the spinor ψ, namely

$$\overline{\psi} \equiv \psi^{\dagger}(i\Gamma^{\bar{t}}). \tag{3.A.4}$$

The affine connection $\Gamma^{\mu}_{\nu\rho}$ is related to the spin connection by (see e.g. (7.100) in [97])

$$\Gamma^{\rho}_{\mu\nu} = E^{\rho}_{\alpha}(\partial_{\mu}E^{\alpha}_{\nu} + \omega_{\mu}{}^{\alpha}{}_{\beta}E^{\beta}_{\nu}). \tag{3.A.5}$$

In this work we consider the supergravity theory in the second order formalism. This means that our theory is torsionless and thus the spin connection can bre expressed in terms of the vielbein as

$$\omega_{\mu\alpha\beta} = E_{\nu\alpha}\partial_{\mu}E^{\nu}_{\beta} + \Gamma^{\rho}_{\mu\nu}E_{\rho\alpha}E^{\nu}_{\beta}. \tag{3.A.6}$$

The variation of the torsionless spin connection is

$$\delta\omega_{\mu\alpha\beta} = E^{\nu}_{[\alpha}D_{\mu}\delta E_{\beta]\nu} - E^{\nu}_{[\alpha}D_{\nu}\delta E_{\beta]\mu} + e^{\rho}_{\alpha}E^{\nu}_{\beta}E_{\gamma\mu}D_{[\nu}\delta E^{\gamma}_{\rho]}. \tag{3.A.7}$$

which is useful for many of our computations. The covariant derivatives of the fermionic fields are given by

$$\nabla_{\mu}\Psi_{\nu} = \partial_{\mu}\Psi_{\nu} + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\Psi_{\nu} - \Gamma^{\rho}_{\mu\nu}\Psi_{\rho}, \tag{3.A.8}$$
$$\nabla_{\mu}\zeta^{I} = \partial_{\mu}\zeta^{I} + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\zeta^{I}. \tag{3.A.9}$$

3.B ADM decomposition and generalized PBH transformation

A preliminary step of the Hamiltonian analysis of the gravitational theory is to decompose the variables of theory including the metric (or the vielbeins) into a radial-like (or time-like) direction and the other transverse directions (a.k.a. ADM decomposition [25]). Coupling gravity to spinor fields requires vielbeins to appear in the action explicitly and thus the ADM decomposition of the vielbeins instead of the metric should be done.

The ADM decomposition brings us a natural choice of gauge for the variables of the theory, which is referred to as the Fefferman-Graham (FG) gauge. In the FG gauge, the Hamiltonian analysis becomes much simpler.

3.B.1 ADM decomposition and the strong Fefferman-Graham gauge

We begin with picking a suitable radial coordinate r and doing the ADM decomposition of the metric to run the Hamiltonian formalism. Since the vielbein explicitly appears in the action through the covariant derivative of the spinor fields we need to decompose the vielbein itself rather than the metric.

Choosing the radial coordinate r, we describe the bulk space as a foliation of the constant r-slices, which we denote by Σ_r. Let E^α be the vielbeins of the bulk and we decompose them as

$$E^\alpha = \left(Nn^\alpha + N^j e_j^\alpha\right) dr + e_j^\alpha dx^j, \tag{3.B.1}$$

such that

$$g_{\mu\nu} = E_\mu^\alpha E_\nu^\beta \eta_{\alpha\beta}, \quad \gamma_{ij} = e_i^\alpha e_j^\beta \eta_{\alpha\beta}, \quad n_\alpha e_i^\alpha = 0, \quad \eta_{\alpha\beta} n^\alpha n^\beta = 1, \tag{3.B.2}$$

where α, β are bulk tangent space indices and $\eta = \mathrm{diag}\,(1, -1, 1, \ldots, 1)$ (where $\eta_{\underline{tt}} = -1$). Note that N and N^α are known as *lapse* and *shift* functions respectively. One can check that

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = (N^2 + N^i N_i) dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j, \tag{3.B.3}$$

which usually appears in textbooks. The inverse vielbeins are then given by

$$E_\alpha^r = \frac{1}{N} n_\alpha, \quad E_\alpha^i = e_\alpha^i - \frac{N^i}{N} n_\alpha. \tag{3.B.4}$$

It follows that

$$\Gamma^r = \Gamma^\alpha E_\alpha^r = \frac{1}{N} n_\alpha \Gamma^\alpha \equiv \frac{1}{N}\Gamma. \tag{3.B.5}$$

The extrinsic curvature on the radial slice Σ_r is defined as

$$K_{ij} \equiv \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i) \tag{3.B.6}$$

and $K \equiv \gamma^{ij} K_{ij}$. Moreover,

$$\Gamma^i = \Gamma^\alpha E_\alpha{}^i = \widehat{\Gamma}^i - \frac{N^i}{N}\Gamma, \tag{3.B.7}$$

where $\widehat{\Gamma}^i \equiv \Gamma^\alpha e^i_\alpha$. These vielbeins satisfy the relation

$$e^i_\alpha e^\beta_i + n_\alpha n^\beta = \delta^\beta_\alpha. \tag{3.B.8}$$

One can also see that the $\widehat{\Gamma}^i$s satisfy the Clifford algebra on the slice and Γ anticommutes with all $\widehat{\Gamma}^i$s, i.e.

$$\{\widehat{\Gamma}^i, \widehat{\Gamma}^j\} = 2\gamma^{ij}, \quad \{\widehat{\Gamma}^i, \Gamma\} = 0. \tag{3.B.9}$$

It follows that the matrix Γ can be used to define the 'radiality' (see e.g. [91]) on the slice, so that a generic spinor ψ on the slice can be split into two by radiality [23],

$$\psi_\pm \equiv \Gamma_\pm \psi, \tag{3.B.10}$$

where $\Gamma_\pm \equiv \frac{1}{2}(1 \pm \Gamma)$.

We recall that splitting spinor fields by their radiality is inevitable because different radiality leads to different asymptotic behavior [86, 85] as well as the constraints that relate the fermionic fields and their conjugate momenta should be solved in a Lorentz invariant way [89]. Remind that the fermionic fields that follow the first-derivative principle, differently from the bosonic fields that follow the second-derivative one, are related to their conjugate momenta by definition. Taking the Dirac Lagrangian as an example, we find that

$$\mathcal{L}_{\text{Dirac}} = -\overline{\Psi}\Gamma^\mu \mathcal{D}_\mu \Psi - m\overline{\Psi}\Psi \implies \Pi_\Psi \equiv \mathcal{L}_{\text{Dirac}} \frac{\overleftarrow{\delta}}{\delta\dot{\Psi}} = -\overline{\Psi}\Gamma^r. \tag{3.B.11}$$

In order to simplify the calculations that follow it is convenient to pick a particular vielbein frame so that

$$n_\alpha = (1, 0), \quad e^i_{\bar{r}} = 0, \quad e^{\bar{r}}_i = 0, \tag{3.B.12}$$

and e^a_i becomes the vielbein on the slice Σ_r. We will call the gauge (3.B.12) combined with the traditional Fefferman-Graham (FG) gauge

$$N = 1, \quad N^i = 0, \quad \Psi_r = 0, \tag{3.B.13}$$

as the *strong* FG gauge. Namely, the strong FG gauge refers to

$$E^{\bar{r}}_r = 1, \quad E^a_r = 0, \quad E^{\bar{r}}_i = 0, \quad \Psi_r = 0. \tag{3.B.14}$$

[23] When $d = D - 1$ is even number, radiality can be regarded as chirality.

3.B.2 Decomposition of the covariant derivatives

We obtain (see also (88) and (89) in [89])

$$\omega_{r\alpha\beta} = n_{[\alpha}\dot{n}_{\beta]} + e_{i[\alpha}\dot{e}_{\beta]}{}^{i} + 2n_{[\alpha}e_{\beta]}{}^{i}\left(\partial_i N - N^j K_{ji}\right) - D_i N_j e_{[\alpha}{}^{i} e_{\beta]}{}^{j}, \tag{3.B.15}$$

$$\omega_{i\alpha\beta} = n_\alpha \partial_i n_\beta + e_{j\alpha}\partial_i e_\beta{}^j + \Gamma^k_{ij}[\gamma] e_{k\alpha} e_\beta{}^j + 2K_i^j e_{j[\alpha} n_{\beta]}, \tag{3.B.16}$$

where we have used the Christoffel symbols

$$\begin{aligned}
\Gamma^r_{rr} &= N^{-1}\left(\dot{N} + N^i\partial_i N - N^i N^j K_{ij}\right),\\
\Gamma^r_{ri} &= N^{-1}\left(\partial_i N - N^j K_{ij}\right),\\
\Gamma^r_{ij} &= -N^{-1}K_{ij},\\
\Gamma^i_{rr} &= -N^{-1}N^i\dot{N} - ND^iN - N^{-1}N^iN^j\partial_j N + \dot{N}^i + N^j D_j N^i +\\
&\qquad\qquad\qquad\qquad +2NN^jK^i_j + N^{-1}N^iN^kN^lK_{kl},\\
\Gamma^i_{rj} &= -N^{-1}N^i\partial_j N + D_j N^i + N^{-1}N^iN^kK_{kj} + NK^i_j,\\
\Gamma^k_{ij} &= \Gamma^k_{ij}[\gamma] + N^{-1}N^kK_{ij}.
\end{aligned}$$

Denoting the spin connection on the radial cut-off as $\widehat{\omega}_{iab}$, we get

$$\widehat{\omega}_{iab} = e_{ja}\partial_i e_b^j + \Gamma^k_{ij}[\gamma]e_{ka}e_b^j = \omega_{iab}, \tag{3.B.17a}$$

$$\omega_{i\alpha\beta}\Gamma^{\alpha\beta} = \widehat{\omega}_{iab}\Gamma^{ab} + 2K_{ji}e_\alpha^j n_\beta\Gamma^{\alpha\beta} = \widehat{\omega}_{iab}\Gamma^{ab} + 2K_{ji}\widehat{\Gamma}^j\Gamma, \tag{3.B.17b}$$

$$\omega_{r\alpha\beta}\Gamma^{\alpha\beta} = e_{ia}\dot{e}_b^i\Gamma^{ab} + 2\Gamma\widehat{\Gamma}^i\left(\partial_i N - N^jK_{ji}\right) - \widehat{\Gamma}^{ij}D_iN_j, \tag{3.B.17c}$$

$$\nabla_i\Psi_j = \mathbb{D}_i\Psi_j + \frac{1}{2}K_{li}\widehat{\Gamma}^l\Gamma\Psi_j + \frac{1}{N}K_{ij}(\Psi_r - N^k\Psi_k), \tag{3.B.17d}$$

$$\nabla_i\Psi_r = \mathbb{D}_i\Psi_r + \frac{1}{2}K_{ji}\widehat{\Gamma}^j\Gamma\Psi_r - \Gamma^j_{ir}\Psi_j - \Gamma^r_{ir}\Psi_r, \tag{3.B.17e}$$

$$\nabla_r\Psi_i = \dot{\Psi}_i + \frac{1}{4}\left[e_{ai}\dot{e}_b^i\Gamma^{ab} + 2\Gamma\widehat{\Gamma}^j\left(\partial_j N - N^lK_{lj}\right) - \widehat{\Gamma}^{jl}D_jN_l\right]\Psi_i - \Gamma^j_{ir}\Psi_j - \Gamma^r_{ir}\Psi_r, \tag{3.B.17f}$$

$$\nabla_i\zeta = \mathbb{D}_i\zeta + \frac{1}{2}K_{ji}\widehat{\Gamma}^j\Gamma\zeta, \tag{3.B.17g}$$

$$\nabla_r\zeta = \dot{\zeta} + \frac{1}{4}\left[e_{ai}\dot{e}_b^i\Gamma^{ab} + 2\Gamma\widehat{\Gamma}^j\left(\partial_j N - N^lK_{lj}\right) - \widehat{\Gamma}^{jl}D_jN_l\right]\zeta, \tag{3.B.17h}$$

where

$$\mathbb{D}_i\Psi_j = \partial_i\Psi_j + \frac{1}{4}\widehat{\omega}_{iab}\Gamma^{ab}\Psi_j - \Gamma^k_{ij}[\gamma]\Psi_k, \tag{3.B.18a}$$

$$\mathbb{D}_i\Psi_r = \partial_i\Psi_r + \frac{1}{4}\widehat{\omega}_{iab}\Gamma^{ab}\Psi_r, \tag{3.B.18b}$$

$$\mathbb{D}_i\zeta = \partial_i\zeta + \frac{1}{4}\omega_{iab}\Gamma^{ab}\zeta, \tag{3.B.18c}$$

are the covariant derivatives of the spinors on the slice Σ_r. Note that in the final computations we used the gauge (3.B.12).

3.B.3 Equations of motion and leading asymptotics of fermionic fields

In order to discuss with the transformation law of the induced fields, we first study the leading asymptotic behavior of the fields, which can be understood from equations of motion. For Ψ_μ and ζ^I they are respectively,

$$\Gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \mathcal{W}\Gamma^{\mu\nu}\Psi_\nu - \frac{i}{2}\mathcal{G}_{IJ}\left(\partial\!\!\!/\varphi^I + \mathcal{G}^{IK}\partial_K\mathcal{W}\right)\Gamma^\mu\zeta^J = 0, \tag{3.B.19}$$

and

$$\mathcal{G}_{IJ}\left(\delta^J_K\nabla\!\!\!\!/ + \Gamma^J_{KL}[\mathcal{G}]\partial\!\!\!/\varphi^L\right)\zeta^K + \mathcal{M}_{IJ}(\varphi)\zeta^J + \frac{i}{2}\mathcal{G}_{IJ}\Gamma^\mu\left(\partial\!\!\!/\varphi^J - \mathcal{G}^{JK}\partial_K\mathcal{W}\right)\Psi_\mu = 0. \tag{3.B.20}$$

Extracting the relevant terms, we obtain in the gauge (3.B.13)

$$0 \sim -\widehat{\Gamma}^{ij}\left(\dot{\Psi}_{+j} - \frac{1}{2}\Psi_{+j}\right) + \widehat{\Gamma}^{ij}\left(\dot{\Psi}_{-j} + \frac{2d-3}{2}\Psi_{-j}\right) + \widehat{\Gamma}^{ijk}\mathbb{D}_j(\Psi_{+k} + \Psi_{-k}), \tag{3.B.21}$$

$$\begin{aligned}0 \sim \dot{\zeta}_+ + \left(\frac{d}{2} + M_\zeta\right)\zeta_+ - \dot{\zeta}_- - \left(\frac{d}{2} - M_\zeta\right)\zeta_- + \widehat{\Gamma}^i\mathbb{D}_i\zeta_+ - \widehat{\Gamma}^i\mathbb{D}_i\zeta_- + \frac{i}{2}(\dot{\varphi} + \mu\varphi)\widehat{\Gamma}^i\Psi_{+i}\\ + \frac{i}{2}\widehat{\Gamma}^i\widehat{\Gamma}^j\partial_j\varphi\Psi_{+i},\end{aligned} \tag{3.B.22}$$

where we assume that there is only one scalar φ and one spin-1/2 field ζ for simplicity, and M_ζ which is the mass of ζ and μ are respectively

$$\mu = -\partial_\varphi\partial_\varphi\mathcal{W}\Big|_{\varphi=0}, \quad M_\zeta = \mathcal{M}_{\varphi\varphi}\Big|_{\varphi=0}, \tag{3.B.23}$$

under the assumption that the scalar manifold metric is canonically normalized. μ and M_ζ are related by

$$M_\zeta = -\mu + \frac{d-1}{2}. \tag{3.B.24}$$

When $d > 2$, the leading asymptotics of Ψ_{+i} and Ψ_{-i} are

$$\Psi_{+i}(r,x) \sim e^{\frac{r}{2}}\Psi_{(0)+i}(x), \tag{3.B.25}$$

$$\Psi_{-i}(r,x) \sim -\frac{1}{2}e^{-\frac{1}{2}r}\left(\frac{d-2}{d-1}\widehat{\Gamma}^{(0)}{}_i\widehat{\Gamma}^{(0)kl} - \widehat{\Gamma}^{(0)}{}_i{}^{kl}\right)\mathbb{D}^{(0)}_k\Psi_{(0)+l}(x), \tag{3.B.26}$$

where we used $e^a_i(r,x) \sim e^r e^a_{(0)i}(x)$ in AlAdS geometry, and $\Gamma^{(0)i}$ and $\mathbb{D}^{(0)}$ refer to the Gamma matrices and the covariant derivative with respect to $e^a_{(0)i}$.

We need to be more careful, regarding ζ. First, we note that since we would like to turn on an arbitrary source for the scalar field, the leading asymptotics of φ should always be $\varphi(r,x) \sim e^{-\mu r}\varphi_{(0)}(x)$ as can be seen from (3.4.8c). Therefore, the final two terms in (3.B.22) can be discarded from the argument. Now there are 3 cases to consider:

1. $M_\zeta > 1/2$ (or $\mu < \frac{d}{2} - 1$)

   The leading asymptotics of ζ_- and ζ_+ are respectively

$$\zeta_-(r,x) \sim e^{-(\mu+\frac{1}{2})r}\zeta_{(0)-}(x), \tag{3.B.27}$$

$$\zeta_+(r,x) \sim -\frac{1}{\mu+\frac{3}{2}}\left(e^{-(\mu+\frac{3}{2})r}\widehat{\Gamma}^{(0)i}\mathbb{D}_i^{(0)}\zeta_{(0)-}(x) - \frac{i}{2}\widehat{\Gamma}^{(0)i}\widehat{\Gamma}^{(0)j}\partial_j\varphi_{(0)}(x)\Psi_{(0)+i}(x)\right). \tag{3.B.28}$$

2. $M_\zeta < -1/2$ (or $\mu > \frac{d}{2}$)

   Here the behavior of ζ_- and ζ_+ is opposite to the first case, namely

$$\zeta_+(r,x) \sim e^{-(d-\mu-\frac{1}{2})r}\zeta_{(0)+}(x), \tag{3.B.29}$$

$$\zeta_-(r,x) \sim \frac{1}{d-\mu+\frac{1}{2}}e^{-(d-\mu+\frac{1}{2})r}\widehat{\Gamma}^{(0)i}\mathbb{D}_i^{(0)}\zeta_{(0)+}(x). \tag{3.B.30}$$

3. $1/2 \geq M_\zeta \geq -1/2$ (or $\frac{d}{2} \geq \mu \geq \frac{d}{2} - 1$)

   This case actually coincides with the double quantization window [107, 108, 109] of the scalar field. The leading asymptotics are

$$\zeta_-(r,x) \sim e^{-(\mu+\frac{1}{2})r}\zeta_{(0)-}(x), \tag{3.B.31}$$

$$\zeta_+(r,x) \sim e^{-(d-\mu-\frac{1}{2})r}\zeta_{(0)+}(x). \tag{3.B.32}$$

3.B.4 Generalized PBH transformations

Let us find the most general bulk symmetry transformations that preserve the strong FG gauge (3.B.14), which we refer to as the generalized Penrose-Brown-Henneaux (gPBH) transformations [111, 112, 67]. We can immediately see that the local symmetries of the bulk SUGRA action (3.2.1) are diffeomorphisms, local Lorentz and supersymmetry transformations. Their infinitesimal action on the bulk fields takes the form

$$\delta_{\xi,\lambda,\epsilon}E_\mu^\alpha = \xi^\nu\partial_\nu E_\mu^\alpha + (\partial_\mu\xi^\nu)E_\nu^\alpha - \lambda^\alpha{}_\beta E_\mu^\beta + \frac{1}{2}(\bar{\epsilon}\Gamma^\alpha\Psi_\mu - \overline{\Psi}_\mu\Gamma^\alpha\epsilon), \tag{3.B.33a}$$

$$\delta_{\xi,\lambda,\epsilon}\Psi_\mu = \xi^\nu\partial_\nu\Psi_\mu + (\partial_\mu\xi^\nu)\Psi_\nu - \frac{1}{4}\lambda^{\alpha\beta}\Gamma_{\alpha\beta}\Psi_\mu + (\nabla_\mu + \frac{1}{2(d-1)}\mathcal{W}\Gamma_\mu)\epsilon, \tag{3.B.33b}$$

$$\delta_{\xi,\lambda,\epsilon}\varphi^I = \xi^\mu\partial_\mu\varphi^I + \frac{i}{2}(\bar{\epsilon}\zeta^I - \overline{\zeta}^I\epsilon), \tag{3.B.33c}$$

$$\delta_{\xi,\lambda,\epsilon}\zeta^I = \xi^\mu\partial_\mu\zeta^I - \frac{1}{4}\lambda^{\alpha\beta}\Gamma_{\alpha\beta}\zeta^I - \frac{i}{2}(\partial\!\!\!/\varphi^I - \mathcal{G}^{IJ}\partial_J\mathcal{W})\epsilon, \tag{3.B.33d}$$

with parameters ξ^μ, $\lambda^{\alpha\beta}$ ($\lambda^{\alpha\beta} = -\lambda^{\beta\alpha}$) and ϵ respectively. The condition that imposes the strong FG gauge is then

$$0 = \dot{\xi}^r, \tag{3.B.34a}$$

$$0 = \dot{\xi}^i e_i^a - \lambda^a{}_{\bar{r}}, \tag{3.B.34b}$$

$$0 = \partial_i \xi^r - \lambda^{\bar{r}}{}_a e_i^a + \frac{1}{2}(\bar{\epsilon}_- \Psi_{+i} + \overline{\Psi}_{+i}\epsilon_- - \bar{\epsilon}_+ \Psi_{-i} - \overline{\Psi}_{-i}\epsilon_+), \tag{3.B.34c}$$

$$0 = \dot{\epsilon}_+ + \dot{\epsilon}_- + \dot{\xi}^i(\Psi_{+i} + \Psi_{-i}) + \frac{1}{4}e_{ai}\dot{e}_b^i\Gamma^{ab}(\epsilon_+ + \epsilon_-) + \frac{1}{2(d-1)}\mathcal{W}(\epsilon_+ - \epsilon_-), \tag{3.B.34d}$$

and its solution is

$$\xi^r = \sigma(x), \tag{3.B.35a}$$

$$\xi^i(r,x) = \xi_o^i(x) - \int^r dr'\ \gamma^{ij}(r',x)\Big[\partial_j\sigma + \frac{1}{2}(\bar{\epsilon}_- \Psi_{+j} + \overline{\Psi}_{+j}\epsilon_- - \bar{\epsilon}_+ \Psi_{-j} - \overline{\Psi}_{-j}\epsilon_+)\Big], \tag{3.B.35b}$$

$$\lambda^{\bar{r}a} = e^{ai}\Big[\partial_i\sigma + \frac{1}{2}(\bar{\epsilon}_- \Psi_{+i} + \overline{\Psi}_{+i}\epsilon_- - \bar{\epsilon}_+ \Psi_{-i} - \overline{\Psi}_{-i}\epsilon_+)\Big], \tag{3.B.35c}$$

$$\lambda^a{}_b = \lambda_o{}^a{}_b(x) + \cdots, \tag{3.B.35d}$$

$$\epsilon_+(r,x) = \exp\left[\frac{r}{2} + \int^r dr'\Big(- \frac{\mathcal{W} + (d-1)}{2(d-1)} + \gamma^{ij}(r',x)\partial_j\sigma - \frac{1}{4}e_{ai}\dot{e}_b^i\Gamma^{ab} + O(\Psi^2)\Big)\right]\epsilon_{o+}(x), \tag{3.B.35e}$$

$$\epsilon_-(r,x) = \exp\left[- \frac{r}{2} + \int^r dr'\Big(\frac{\mathcal{W} + (d-1)}{2(d-1)} + \gamma^{ij}(r',x)\partial_j\sigma - \frac{1}{4}e_{ai}\dot{e}_b^i\Gamma^{ab} + O(\Psi^2)\Big)\right]\epsilon_{o-}(x), \tag{3.B.35f}$$

where $\sigma(x)$, $\xi_o^i(x)$, $\lambda_o{}^a{}_b(x)$ and $\epsilon_{o\pm}(x)$ are ‘integration constants’ which depend only on the transverse coordinates. Taking into account the leading behavior of the vielbeins and the gravitino one can see that the integral terms are subleading in (3.B.35). It follows that the leading asymptotics of the generalized PBH transformations are parameterized by the arbitrary independent transverse functions

$$\sigma(x), \quad \xi_o^i(x), \quad \lambda_o{}^a{}_b(x), \quad \epsilon_{o\pm}(x), \tag{3.B.36}$$

which in fact correspond to the local conformal, diffeomorphism, Lorentz, SUSY, and super-Weyl transformations of the induced fields on the radial slice Σ_r respectively, as we will see soon.

Extracting the leading terms in (3.B.33) and taking into account the asymptotic behavior of the induced fields, we obtain how the sources transform, namely (from now on and also in the main text we do not write the subscript o)

$$\delta_{\xi,\lambda,\epsilon}e_i^a \sim \xi^j\partial_j e_i^a + \partial_i\xi^j e_j^a + e_i^a\sigma - \lambda^a{}_b e_i^b + \frac{1}{2}\left(\bar{\epsilon}_+\Gamma^a\Psi_{+i} + \text{h.c.}\right), \tag{3.B.37a}$$

$$\delta_{\xi,\lambda,\epsilon}\Psi_{+i} \sim \frac{1}{2}\Psi_{+i}\sigma + \xi^j\partial_j\Psi_{+i} + (\partial_i\xi^j)\Psi_{+j} + \mathbb{D}_i\epsilon_+ - \widehat{\Gamma}_i\epsilon_- - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\Psi_{+i}, \tag{3.B.37b}$$

$$\delta_{\xi,\lambda,\epsilon}\varphi^I \sim -\mathcal{G}^{IJ}\partial_J\mathcal{W}\sigma + \xi^i\partial_i\varphi^I + \frac{i}{2}\left(\bar{\epsilon}_+\zeta_-^I + \text{h.c.}\right) + \frac{i}{2}\left(\bar{\epsilon}_-\zeta_+^I + \text{h.c.}\right), \tag{3.B.37c}$$

where we do not write down the variation of Ψ_{-i} since unlike Ψ_{+i} its leading term (3.B.26) does not transform as a source so that it cannot be used as a generalized coordinate [87].

As for ζ^I, we need a careful discussion, since its leading behavior changes according to its mass. In the first case where $M_\zeta \geq 1/2$, ζ_+^I cannot be treated as a source, like the case of gravitino Ψ_{-i}. We also find that in the second case where $M_\zeta^I \leq -1/2$ (3.B.37c) is not consistent with the leading behavior of $\varphi \sim e^{-\mu r}$ due to the term $\frac{i}{2}(\bar{\epsilon}_-\zeta_+^I + \text{h.c.}) \sim e^{-(d-\mu^I)r}$, which implies that ζ_+^I cannot be turned on as a source, in order for the theory to be supersymmetric. In the final case where $1/2 > M_\zeta > -1/2$, both ζ_+^I and ζ_-^I can be used as sources. The transformation law in this case is discussed in section 3.6. In summary, what we obtain is

$$\delta_{\xi,\lambda,\epsilon} e_i^a \sim \xi^j \partial_j e_i^a + \partial_i \xi^j e_j^a + e_i^a \sigma - \lambda^a{}_b e_i^b + \frac{1}{2}\left(\bar{\epsilon}_+ \Gamma^a \Psi_{+i} + \text{h.c.}\right), \tag{3.B.38a}$$

$$\delta_{\xi,\lambda,\epsilon} \Psi_{+i} \sim \frac{1}{2}\Psi_{+i}\sigma + \xi^j \partial_j \Psi_{+i} + (\partial_i \xi^j)\Psi_{+j} + \mathbb{D}_i \epsilon_+ - \widehat{\Gamma}_i \epsilon_- - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\Psi_{+i}, \tag{3.B.38b}$$

$$\delta_{\xi,\lambda,\epsilon} \varphi^I \sim \mathcal{G}^{IJ}\partial_J \mathcal{W}\sigma + \xi^i \partial_i \varphi^I + \frac{i}{2}\left(\bar{\epsilon}_+ \zeta_-^I + \text{h.c.}\right), \tag{3.B.38c}$$

$$\begin{aligned} \delta_{\xi,\lambda,\epsilon} \zeta_-^I \sim -\left(\frac{d}{2}\delta_K^I - \mathcal{G}^{IJ}\mathcal{M}_{JK}\right)\zeta_-^K \sigma + \xi^i \partial_i \zeta_-^I + i\mathcal{G}^{IJ}\partial_J \mathcal{W}\epsilon_- - \\ - \frac{i}{2}\widehat{\Gamma}^i \partial_i \varphi^I \epsilon_+ - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\zeta_-^I, \end{aligned} \tag{3.B.38d}$$

where we inverted the mass of ζ_-^I into the (scalar) σ-manifold language.

3.C Decomposition of the action and the fermion boundary terms

In this appendix we decompose the terms in the fermionic sector of the action (3.2.1).

3.C.1 Decomposition of the kinetic action of the hyperino field

The kinetic term for ζ^I in the action (3.2.1) is decomposed as

$$\begin{aligned} &\mathcal{G}_{IJ}\left(\bar{\zeta}^I \Gamma^\mu \nabla_\mu \zeta^J - (\nabla_\mu \bar{\zeta}^I)\Gamma^\mu \zeta^J\right) \\ &= \mathcal{G}_{IJ}\bar{\zeta}^I\left(\Gamma^r \nabla_r \zeta^J + \Gamma^i \nabla_i \zeta^J\right) - \mathcal{G}_{IJ}\bar{\zeta}^I \overleftarrow{\nabla}_r \Gamma^r \zeta^J \zeta^J - \mathcal{G}_{IJ}\bar{\zeta}^I \overleftarrow{\nabla}_i \Gamma^i \zeta^J \\ &= \mathcal{G}_{IJ}\bar{\zeta}^I\left[\frac{1}{N}\Gamma\dot{\zeta}^J + \frac{1}{4N}\Gamma\left(e_{ai}\dot{e}_b^i \Gamma^{ab} + 2\Gamma\widehat{\Gamma}^i\left(\partial_i N - N^j K_{ij}\right) - \widehat{\Gamma}^{ij} D_i N_j\right)\zeta^J \right. \\ &\qquad \left. + \left(\widehat{\Gamma}^i - \frac{N^i}{N}\Gamma\right)\left(\mathbb{D}_i \zeta^J + \frac{1}{2}K_{ij}\widehat{\Gamma}^j \Gamma \zeta^J\right)\right] \end{aligned}$$

$$
\begin{aligned}
&- \mathcal{G}_{IJ}\left[\dot{\overline{\zeta}}^I - \frac{1}{4}\overline{\zeta}^I\left[e_{ai}\dot{e}^i_b\Gamma^{ab} + 2\Gamma\widehat{\Gamma}^i\left(\partial_i N - N^j K_{ij}\right) - \widehat{\Gamma}^{ij}D_i N_j\right]\right]\frac{1}{N}\Gamma\zeta^J \\
&- \mathcal{G}_{IJ}\left(\overline{\zeta}^I\overleftarrow{\mathbb{D}}_i - \frac{1}{2}K_{ij}\overline{\zeta}^I\widehat{\Gamma}^j\Gamma\right)\left(\widehat{\Gamma}^i - \frac{N^i}{N}\Gamma\right)\zeta^J \\
= &\frac{1}{N}\mathcal{G}_{IJ}\left(\overline{\zeta}^I_-\dot{\zeta}^J_+ - \overline{\zeta}^I_+\dot{\zeta}^J_- - \dot{\overline{\zeta}}^I_-\zeta^J_+ + \dot{\overline{\zeta}}^I_+\zeta^J_-\right) + \frac{1}{2N}\mathcal{G}_{IJ}e_{ai}\dot{e}^i_b\overline{\zeta}^I\Gamma\Gamma^{ab}\zeta^J \\
&- \frac{1}{2N}D_iN_j\mathcal{G}_{IJ}\overline{\zeta}^I\Gamma\widehat{\Gamma}^{ij}\zeta^J + \mathcal{G}_{IJ}\left(\overline{\zeta}^I\not{\mathbb{D}}\zeta^J - \overline{\zeta}^I\overleftarrow{\not{\mathbb{D}}}\zeta^J\right) \\
&- \frac{N^i}{N}\mathcal{G}_{IJ}\left(\overline{\zeta}^I\Gamma\mathbb{D}_i\zeta^J - \overline{\zeta}^I\overleftarrow{\mathbb{D}}_i\Gamma\zeta^J\right),
\end{aligned}
\tag{3.C.1}
$$

where the terms in the first bracket can be recast into

$$
\begin{aligned}
&\mathcal{G}_{IJ}\left(\overline{\zeta}^I_-\dot{\zeta}^J_+ - \overline{\zeta}^I_+\dot{\zeta}^J_- - \dot{\overline{\zeta}}^I_-\zeta^J_+ + \dot{\overline{\zeta}}^I_+\zeta^J_-\right) = \mathcal{G}_{IJ}\partial_r\left(\overline{\zeta}^I_-\zeta^J_+ + \overline{\zeta}^I_+\zeta^J_-\right) - 2\mathcal{G}_{IJ}\overline{\zeta}^I_+\dot{\zeta}^J_- - 2\mathcal{G}_{IJ}\dot{\overline{\zeta}}^I_-\zeta^J_+ \\
= &\frac{1}{\sqrt{-\gamma}}\partial_r\left(\mathcal{G}_{IJ}\sqrt{-\gamma}\overline{\zeta}^I\zeta^J\right) - \left(NK + D_kN^k\right)\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J \\
&- \left(\dot{\varphi}^K - N^i\partial_i\varphi^K + N^i\partial_i\varphi^K\right)\partial_K\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J - 2\mathcal{G}_{IJ}\overline{\zeta}^I_+\dot{\zeta}^J_- - 2\mathcal{G}_{IJ}\dot{\overline{\zeta}}^I_-\zeta^J_+.
\end{aligned}
\tag{3.C.2}
$$

Finally, the hyperino kinetic terms are decomposed into

$$
\begin{aligned}
&\mathcal{G}_{IJ}\left(\overline{\zeta}^I\Gamma^\mu\nabla_\mu\zeta^J - (\nabla_\mu\overline{\zeta}^I)\Gamma^\mu\zeta^J\right) \\
= &\frac{1}{N\sqrt{-\gamma}}\partial_r\left(\sqrt{-\gamma}\,\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J\right) - \frac{2}{N}\mathcal{G}_{IJ}\left(\overline{\zeta}^I_+\dot{\zeta}^J_- + \dot{\overline{\zeta}}^I_-\zeta^J_+\right) - \left(K + \frac{1}{N}D_kN^k\right)\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J \\
&+ \frac{1}{2N}\mathcal{G}_{IJ}e_{ai}\dot{e}^i_b\overline{\zeta}^I\Gamma^{ab}\Gamma\zeta^J - \frac{1}{N}\left(\dot{\varphi}^K - N^i\partial_i\varphi^K + N^i\partial_i\varphi^K\right)\partial_K\mathcal{G}_{IJ}\overline{\zeta}^I\zeta^J \\
&+ \mathcal{G}_{IJ}\left(\overline{\zeta}^I\widehat{\Gamma}^i\mathbb{D}_i\zeta^J - \overline{\zeta}^I\overleftarrow{\mathbb{D}}_i\widehat{\Gamma}^i\zeta^J\right) \\
&+ \frac{1}{N}\mathcal{G}_{IJ}\left[-\frac{1}{2}D_iN_j\left(\overline{\zeta}^I\widehat{\Gamma}^{ij}\Gamma\zeta^J\right) - N^i\overline{\zeta}^I\Gamma\mathbb{D}_i\zeta^J + N^i(\overline{\zeta}^I\overleftarrow{\mathbb{D}}_i)\Gamma\zeta^J\right].
\end{aligned}
\tag{3.C.3}
$$

3.C.2 Gravitino part

Repeating the same computation for the kinetic terms for gravitino as before, we obtain

$$
\begin{aligned}
&\left(\overline{\Psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \overline{\Psi}_\mu\overleftarrow{\nabla}_\nu\Gamma^{\mu\nu\rho}\Psi_\rho\right) + \frac{1}{(D-2)}\overline{\Psi}_\mu\Gamma^{\mu\nu\rho}\left(\mathcal{W}\Gamma_\nu\right)\Psi_\rho \\
= &\frac{1}{N\sqrt{-\gamma}}\partial_r\left(\sqrt{-\gamma}\overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j\right) - \frac{2}{N}\left(\dot{\overline{\Psi}}_{+i}\widehat{\Gamma}^{ij}\Psi_{-j} + \overline{\Psi}_{-i}\widehat{\Gamma}^{ij}\dot{\Psi}_{+j}\right) \\
&- \left(K + \frac{1}{N}D_kN^k\right)\overline{\Psi}_i\widehat{\Gamma}^{ij}\Psi_j - \frac{1}{4N}e_{ak}\dot{e}^k_b\overline{\Psi}_i\Gamma\{\widehat{\Gamma}^{ij},\Gamma^{ab}\}\Psi_j \\
&+ \frac{1}{2N}K_{lk}\overline{\Psi}_i\Big(N[\widehat{\Gamma}^{ikj},\widehat{\Gamma}^l]\Gamma + N^i[\widehat{\Gamma}^{kj},\widehat{\Gamma}^l] - N^j[\widehat{\Gamma}^{ki},\widehat{\Gamma}^l]\Big)\Psi_j
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2N}K_{ki}\left(\overline{\Psi}_j[\widehat{\Gamma}^{ij},\widehat{\Gamma}^k]\Psi_r-\overline{\Psi}_r[\widehat{\Gamma}^{ij},\widehat{\Gamma}^k]\Psi_j\right)\\
&+\frac{1}{N}\left(\overline{\Psi}_j\overleftarrow{\mathbb{D}}_i\Gamma\widehat{\Gamma}^{ij}\Psi_r+\overline{\Psi}_r\Gamma\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_j-\overline{\Psi}_j\Gamma\widehat{\Gamma}^{ij}\mathbb{D}_i\Psi_r-\overline{\Psi}_r\overleftarrow{\mathbb{D}}_i\Gamma\widehat{\Gamma}^{ij}\Psi_j\right)\\
&-\frac{1}{N}\mathcal{W}\left(\overline{\Psi}_r\Gamma\widehat{\Gamma}^i\Psi_i+\overline{\Psi}_i\widehat{\Gamma}^i\Gamma\Psi_r\right)-\frac{1}{4N}\overline{\Psi}_i\left(2\partial_kN[\widehat{\Gamma}^{ij},\widehat{\Gamma}^k]-(D_kN_l)\Gamma\{\widehat{\Gamma}^{ij},\widehat{\Gamma}^{kl}\}\right)\Psi_j\\
&+\frac{1}{N}\overline{\Psi}_j\left(N\widehat{\Gamma}^{jik}-N^j\Gamma\widehat{\Gamma}^{ik}-N^i\Gamma\widehat{\Gamma}^{kj}-N^k\Gamma\widehat{\Gamma}^{ji}\right)\mathbb{D}_i\Psi_k\\
&+\frac{1}{N}\overline{\Psi}_k\overleftarrow{\mathbb{D}}_i\left(N\widehat{\Gamma}^{jik}-N^j\Gamma\widehat{\Gamma}^{ik}-N^i\Gamma\widehat{\Gamma}^{kj}-N^k\Gamma\widehat{\Gamma}^{ji}\right)\Psi_j\\
&-\frac{1}{N}\mathcal{W}\overline{\Psi}_i\left(N\widehat{\Gamma}^{ij}-N^i\Gamma\widehat{\Gamma}^j+N^j\Gamma\widehat{\Gamma}^i\right)\Psi_j.
\end{aligned}
\tag{3.C.4}
$$

3.C.3 Decomposition of the other terms

For the other terms, we get

$$
\begin{aligned}
&i\mathcal{G}_{IJ}\overline{\zeta}^I\Gamma^\mu\left(\partial\!\!\!/\varphi^J-\mathcal{G}^{JK}\partial_K\mathcal{W}\right)\Psi_\mu-i\mathcal{G}_{IJ}\overline{\Psi}_\mu(\partial\!\!\!/\varphi^I+\mathcal{G}^{IK}\partial_K\mathcal{W})\Gamma^\mu\zeta^J\\
&=\frac{i}{N}\mathcal{G}_{IJ}\Bigg\{\frac{1}{N}\left(\dot\varphi^J-N^j\partial_j\varphi^J\right)\left[\overline{\zeta}^I\left(\Psi_r-N^i\Psi_i+N\widehat{\Gamma}^i\Gamma\Psi_i\right)-\left(\overline{\Psi}_r-N^i\overline{\Psi}_i+N\overline{\Psi}_i\Gamma\widehat{\Gamma}^i\right)\zeta^I\right]\\
&\qquad+\partial_i\varphi^J\left[\overline{\zeta}^I\Gamma\widehat{\Gamma}^i\left(\Psi_r-N^j\Psi_j\right)-\left(\overline{\Psi}_r-N^j\overline{\Psi}_j\right)\widehat{\Gamma}^i\Gamma\zeta^I\right]\\
&\qquad\qquad+N\partial_i\varphi^I\left(\overline{\zeta}^I\widehat{\Gamma}^j\widehat{\Gamma}^i\Psi_j-\overline{\Psi}_j\widehat{\Gamma}^i\widehat{\Gamma}^j\zeta^I\right)\Bigg\}\\
&\quad-\frac{i}{N}\partial_I\mathcal{W}\left[\overline{\zeta}^I\Gamma\left(\Psi_r-N^i\Psi_i\right)+\left(\overline{\Psi}_r-N^i\overline{\Psi}_i\right)\Gamma\zeta^I+N\left(\overline{\Psi}_i\widehat{\Gamma}^i\zeta^I+\zeta^I\widehat{\Gamma}^i\Psi_i\right)\right],
\end{aligned}
\tag{3.C.5}
$$

and

$$
\begin{aligned}
&\mathcal{G}_{IJ}\left[\overline{\zeta}^I\left(\Gamma^J_{\ KL}\partial\!\!\!/\varphi^L\right)\zeta^K-\overline{\zeta}^K\left(\Gamma^J_{\ KL}\partial\!\!\!/\varphi^L\right)\zeta^I\right]\\
&=\frac{1}{N}\partial_K\mathcal{G}_{IJ}\left[\left(\dot\varphi^J-N^i\partial_I\varphi^J\right)\left(\overline{\zeta}^I\Gamma\zeta^K-\overline{\zeta}^K\Gamma\zeta^I\right)+N\partial_i\varphi^J\left(\overline{\zeta}^I\widehat{\Gamma}^i\zeta^K-\overline{\zeta}^K\widehat{\Gamma}^i\zeta^I\right)\right].
\end{aligned}
\tag{3.C.6}
$$

3.D Variation of the canonical momenta

In this appendix, we review how to obtain the commutation relation (3.5.19) between symmetry generators and canonical momenta, see e.g. (7.3.19) of [113]. By chain rule,

$$
\delta\widehat{S}_{ren}=\int d^dx\ \sum_\Phi\Pi^\Phi\delta\Phi,
\tag{3.D.1}
$$

and let us define a symmetry transformation of $\widehat{S}_{ren}$ by

$$
\delta_\xi=\int d^dx\ \sum_\Phi\delta_\xi\Phi(x)\frac{\delta}{\delta\Phi(x)}.
\tag{3.D.2}
$$

Let us also assume that this symmetry has an anomaly, i.e.

$$\delta_\xi \widehat{S}_{ren} = \int d^d x \sum_\Phi \Pi^\Phi \delta_\xi \Phi = \int d^d x |\boldsymbol{e}_{(0)}| \, \xi \mathcal{A}_\xi. \tag{3.D.3}$$

Then, the definition of the constraint function $\mathcal{C}[\xi]$ (3.5.18) can be written as

$$\mathcal{C}[\xi] = -\int d^d x \left(\sum_\Phi \Pi^\Phi \delta_\xi \Phi - |\boldsymbol{e}_{(0)}| \, \xi \mathcal{A}_\xi \right). \tag{3.D.4}$$

Now we derive how the ξ-symmetry acts on Π^Φ. It is

$$\begin{aligned}
\delta_\xi \Pi^\Phi(x) = \delta_\xi \frac{\delta}{\delta\Phi(x)} \widehat{S}_{ren} &= \left[\delta_\xi, \frac{\delta}{\delta\Phi(x)} \right] \widehat{S}_{ren} + \frac{\delta}{\delta\Phi(x)} \delta_\xi \widehat{S}_{ren} \\
&= -\int d^d y \sum_{\Phi'} \left(\frac{\delta}{\delta\Phi(y)} \delta_\xi \Phi'(x) \right) \Pi^{\Phi'}(x) + \frac{\delta}{\Phi(x)} \int d^d y |\boldsymbol{e}_{(0)}| \, \xi \mathcal{A}_\xi \\
&= -\frac{\delta}{\delta\Phi(x)} \int d^d y \sum_{\Phi'} \left(\Pi^{\Phi'}(y) \delta_\xi \Phi'(y) - |\boldsymbol{e}_{(0)}| \xi \mathcal{A}_\xi \right) \\
&= \{ \mathcal{C}[\xi], \Pi^\Phi \},
\end{aligned} \tag{3.D.5}$$

which confirms (3.5.19).

3.E Derivation of the SUSY algebra without using Poisson bracket

In this appendix we compute the anticommutator $\{\mathcal{Q}[\xi], Q^s[\eta_+]\}$. By differentiating the diffeomorphism Ward identity (3.5.2d) in the integral form with respect to $\Psi_{+k}(y)$, we get

$$\begin{aligned}
0 = \int_{\partial\mathcal{M}} d^d x \, \xi_i \Big[& e^{ai}_{(0)} D_j \Pi_a^j - (\partial^i \varphi^I_{(0)}) \Pi_I^\varphi - (\overline{\zeta}^I_{(0)-} \overleftarrow{\mathbb{D}}^i) \Pi_I^{\overline{\zeta}} - \Pi_I^\zeta (\mathbb{D}^i \zeta^I_{(0)-}) \\
& - \Pi^j_\Psi \left(\mathbb{D}^i \Psi_{(0)+j} \right) - \left(\overline{\Psi}_{(0)+j} \overleftarrow{\mathbb{D}}^i \right) \Pi^j_{\overline{\Psi}} + D_j (\Pi^j_\Psi \Psi^i_{(0)+} + \overline{\Psi}^i_{(0)+} \Pi^j_{\overline{\Psi}}) \Big]_x \Pi^k_\Psi(y) \\
& + \left(\xi^i \Pi^k_\Psi \right) \overleftarrow{\mathbb{D}}_i(y) - D_j \xi^k \Pi^j_\Psi(y).
\end{aligned} \tag{3.E.1}$$

From the local Lorentz Ward identity (3.5.2e), we obtain

$$0 = \int_{\partial\mathcal{M}} d^d x \, \lambda^{ab} \Big[e_{(0)[ai} \Pi^i_{b]} + \frac{1}{4} \big(\overline{\zeta}^I_{(0)-} \Gamma_{ab} \Pi_I^{\overline{\zeta}} + \overline{\Psi}_{(0)+i} \Gamma_{ab} \Pi^i_{\overline{\Psi}} + \text{h.c.} \big) \Big]_x \Pi^k_\Psi(y) - \frac{1}{4} \lambda^{ab} \Pi^k_\Psi \Gamma_{ab}(y). \tag{3.E.2}$$

Summing these two expressions for the parameter $\lambda_{ab} = e^i_a e^j_b D_{[i} \xi_{j]}$, we obtain

$$0 = \int_{\partial\mathcal{M}} d^d x \, D_j [\xi^i (e^{ai} \Pi^j_a + \Pi^j_\Psi \Psi_{(0)+i} + \overline{\Psi}_{(0)+i} \Pi^j_{\overline{\Psi}})]_x \Pi^k_\Psi(y) +$$

$$+\left(\xi^i\Pi_\Psi^k\right)\overleftarrow{\mathbb{D}}_i(y)-D_j\xi^k\Pi_\Psi^j(y)-\frac{1}{4}D_i\xi_j\Pi_\Psi^k\widehat{\Gamma}^{ij}(y). \qquad (3.E.3)$$

It follows from (3.5.59) that

$$\begin{aligned}
\{\mathcal{Q}[\xi],Q^s[\eta_+]\} &= -\int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_k(y)\int_{\partial\mathcal{M}} d^dxD_j[\xi^i(e^{ai}\Pi_a^j+\Pi_\Psi^j\Psi_{(0)+i}+\overline{\Psi}_{(0)+i}\Pi_{\overline{\Psi}}^j)]_x(\Pi_\Psi^k\eta_+)_y \\
&= \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_k\left[\left(\xi^i\Pi_\Psi^k\right)\overleftarrow{\mathbb{D}}_i-D_j\xi^k\Pi_\Psi^j-\frac{1}{4}D_i\xi_j\Pi_\Psi^k\widehat{\Gamma}^{ij}\right]\eta_+ \\
&= \int_{\partial\mathcal{M}\cap\mathcal{C}} d\sigma_k\left[D_i(\xi^i\Pi_\Psi^k\eta_+-\xi^k\Pi_\Psi^i\eta_+)+\xi^kD_j(\Pi_\Psi^j\eta_+)-\Pi_\Psi^k\mathcal{L}_\xi\eta_+\right] \\
&= -Q^s[\mathcal{L}_\xi\eta_+], \qquad (3.E.4)
\end{aligned}$$

where the first term in the third line is zero by Stokes' theorem and the second term vanishes due to the conservation law. One can confirm that the other commutators in (3.5.56) can be obtained in the same way.

Chapter 4

Conclusion and outlook

The main message we would like to get across in chapter 2 is that a well defined thermodynamics, including finite conserved charges and thermodynamic identities, is an immediate consequence of a well posed variational problem, formulated in terms of equivalence classes of boundary data under the asymptotic local symmetries of the theory. This has been known for some time in the case of asymptotically AdS black holes, but we argue that it applies to more general asymptotics, including cases where matter fields are required to support the background.

We demonstrated this claim by carefully analyzing the variational problem for asymptotically conical backgrounds of the STU model in four dimensions and deriving the thermodynamics of subtracted geometry black holes. Moreover, by uplifting these solutions to five dimensions, we provided a precise map between all thermodynamic variables of subtracted geometries and those of the BTZ black hole. Crucial to this matching was the fact that some free parameters of the four-dimensional black holes must be fixed or quantized in order for the solutions to be uplifted to five dimensions.

Although our analysis in chapter 2 does not assume or imply a holographic duality for asymptotically conical backgrounds, we would like to view it as the first step in this direction. Our comparison of the variational problems in four and five dimensions indicates that not all asymptotically conical solutions of the STU model in four dimensions correspond to asymptotically $\mathrm{AdS}_3 \times S^2$ solutions in five dimensions and vice versa. This suggests that the Hilbert space of a putative holographic dual to subtracted geometries can at most have a partial overlap with that of the two-dimensional CFT at the boundary of AdS_3. The next steps in order to construct a genuine dual to asymptotically conical backgrounds, as well as to understand the connection with the two-dimensional CFT, would be a systematic analysis of the most general asymptotically conical solutions of the STU model (i.e. not merely stationary), and the identification of the symmetry algebra acting on the modes as a result of the asymptotic local symmetries. We plan to address both these problems in future work.

In chapter 3 we have considered a generic $\mathcal{N} = 2$ 5D supergravity theory with its fermionic sector in the context of holographic renormalization, through which we have obtained a complete set of supersymmetric counterterms. We have also found that scalars and their superpartners should satisfy the same boundary conditions in order for the theory to be consistent with SUSY.

The Ward identities (3.5.2) and the anomalies lead to rather remarkable consequences. By means of them, we showed that the SUSY transformation of local operators and the SUSY algebra of a theory which has $\mathcal{N}=1$ 4D SCFT in curved space as a UV fixed point become anomalous at the quantum level, see (3.5.25) and (3.5.56). We comment that once the R-symmetry gauge field is turned on, the R-charge and the related terms appear on RHS of the first line (3.5.56), see [28]. Note that the anomalous terms are non-vanishing in general on curved backgrounds, even where all anomalies vanish.

By all means, one has to verify existence of the supersymmetric anomaly in the field theory, independently of holography. One possible test that could be carried out is to check the BPS relation (3.5.61) for a simple 4D $\mathcal{N}=1$ supersymmetric field theory in curved space.

We emphasize that our whole analysis here crucially relies on the existence of a scalar superpotential $\mathcal{W}$, in terms of which the Lagrangian is expressed. If the theory does not possess any superpotential, one could introduce a local and *approximate* superpotential which is sufficient for reproducing all divergent terms of the scalar potential, as done in [114]. Now one can see that the approximate superpotential should meet more restrictive criteria for the supersymmetric holographic renormalization. To make this point clear, let us discuss the approximate superpotential suggested in [114], see (5.15) there. One can find from the BPS equations (3.20) and (3.25) and the algebraic equation (3.26) in [114] that the BPS solution's flow to leading order is

$$\frac{d\psi}{dr} \sim -\psi, \tag{4.1a}$$

$$\frac{d\varphi}{dr} \sim -\left(2\varphi + \sqrt{\frac{2}{3}}\psi^2\right), \tag{4.1b}$$

$$\frac{d\chi}{dr} \sim -2\chi\left(1 + \frac{\psi^2}{\sqrt{6}\varphi}\right), \tag{4.1c}$$

where the RHS of the last equation is a *non-analytic* function of φ around $\varphi=0$. Hence it is impossible to find a *local and approximate* superpotential consistent with the BPS flow equations, which means that we need a more generic $\mathcal{N}=2$ gauged SUGRA model to study [114]. Notice that this inconsistency of the approximate superpotential with the BPS flow equations implies that the superpotential suggested in [114] is *not approximate* for the fermionic sector of SUGRA.

As long as there exists a superpotential (or at least an approximate one for the whole sector of SUGRA), many of our results here can be extended straightforwardly to other dimensions. A direct application of the analysis of this chapter to other dimensions is to obtain the 2D super-Virasoro algebra with a central extension. Let us explain this here schematically. The super-Weyl anomaly in 2D SCFT can be easily found by using the trick of section 3.5.1, namely that the SUSY variation of the super-Weyl anomaly is equal to the Weyl anomaly. Since the Weyl anomaly is

$e^a_i\mathcal{T}^i_a = \frac{c}{24\pi}R$, we see immediately that the super-Weyl anomaly in 2D is $\Gamma_i\mathcal{S}^i \sim i\frac{c}{24\pi}\Gamma^{ij}\mathbb{D}_i\Psi_j$ up to a constant coefficient, depending on the convention. It follows that the anomalous variation of the super-current operator is

$$\delta_\eta\mathcal{S}^i = -\frac{i}{4}\Gamma^a\eta\mathcal{T}^i_a - \frac{ic}{48\pi}\widehat{\Gamma}^{ij}\widehat{\Gamma}^k\mathbb{D}_j\mathbb{D}_k\eta, \tag{4.2}$$

where η is the 2D CKS, satisfying the condition

$$\mathbb{D}_i\eta = \frac{1}{2}\widehat{\Gamma}_i\widehat{\Gamma}^j\mathbb{D}_j\eta, \quad \text{or} \quad \widehat{\Gamma}^j\widehat{\Gamma}^i\mathbb{D}_j\eta = 0. \tag{4.3}$$

Note that the anomalous term in (4.2) vanishes only when the 2D Ricci scalar $R = 0$ and η is a spinor, all second derivatives of which vanish. Since (4.3) admits an infinite number of solutions, as 2D conformal Killing vector equation, one gets infinite number of conserved super-charges G_r, which are added to the Virasoro algebra to form the super-Virasoro algebra. Now one can see that the central extension in (see e.g. (10.2.11b) in [115])

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \tag{4.4}$$

of the super-Virasoro algebra in 2D flat background is derived from the anomalous term of (4.2).

One should keep in mind, however, that since the representation of the spinor fields strongly depends on the dimension of spacetime it might not be easy to put the SUGRA action into the form of (3.2.1) in other (especially odd) dimensions.

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