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Constructions of stable generalized complex 6-manifolds and their fundamental groups

Doctoral Thesis

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To my beloved parents and brother.

Abstract

Torus surgeries in dimension four (or called C^∞-log transformations) have been widely employed to construct a stable generalized complex 4-manifold with nonempty type change locus. We find a torus surgery in dimension six which can be applied to a stable generalized 6-complex manifold to yield a new stable generalized complex 6-manifold. Each torus surgery has an effect of increasing the number of path-connected components on the type change locus by one as in dimension four. Using this torus surgery, we prove that there exist infinitely many stable generalized complex 6-manifolds with nonempty type change locus that are not homologically equivalent to a product of lower dimensional manifolds. Also, it is shown that any finitely presented group is the fundamental group of a stable generalized complex 6-manifold with nonempty type change locus on which each path-connected component is diffeomorphic to T^4.

Contents

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Chapter 1

Introduction and main results

Generalized complex structures first proposed by Hitchin [35] and further developed by Gualtieri [30, 31] and others [5, 15–17, 28, 45, 47] are a natural extension of complex and symplectic structures. They generalize complex structures by replacing the tangent bundle with the generalized tangent bundle (i.e., the Whitney sum of the tangent bundle and the cotangent bundle) and the Lie bracket with the Courant bracket.

Generalized complex structures have drawn a lot of attention from both mathematicians and physicists due to their intriguing properties. They incorporate complex structures and symplectic structures as their extremal cases and provide a unified language to describe these two structures in the same framework. This inspired physicists to employ them to tackle some physical problems, for instance, mirror symmetry [23, 29, 33, 35].

An important invariant of generalized complex structures is the type [30]. Generalized complex structures can be classified by means of their types. In dimension $2n$, all the complex structures have constant type n, while all the symplectic structures have constant type 0. Interestingly, the type of a generalized complex structure is not necessarily constant throughout the manifold and it can jump along a subset of the manifold called the type change locus. It was firstly demonstrated in [30] that there exist some 4-manifolds admitting the generalized complex structures with type change jumps.

Among the generalized complex structures with type change jumps, the most interesting ones would be stable generalized complex structures [17], in which a section of the corresponding anticanonical bundle has solely nondegenerate zeros. A careful study on stable generalized complex structures was carried out by Cavalcanti and Gualtieri in the paper [17]. They established the symplectic viewpoint of stable generalized complex structures, which paves the way for the use of the symplectic techniques to study stable generalized complex structures. Another interesting result there is that the type change locus inherits a constant type 1 generalized Calabi-Yau structure from the underlying stable generalized complex structure. It builds the bridge between stable generalized complex structures and constant type 1 generalized Calabi-Yau structures. The topology of constant type 1 generalized Calabi-Yau manifolds is very restricted [6], so is the type change locus of stable generalized complex manifolds.

So far, a lot of effort has been devoted to finding stable generalized complex manifolds. In [15, 16], the authors put forward the very first idea of using a cut-and-paste construction, in particular, a $C^\infty-$ logarithmic transformation [26] of multiplicity zero, in order to construct a stable generalized complex 4-manifold with nonempty type change locus. This idea was taken by several mathematicians such as Goto, Hayano, Torres, Yazinski, to find a myriad of examples of stable generalized complex 4-manifolds which are neither symplectic nor complex [28, 45, 47]. On top of $C^\infty-$logarithmic transformations, the other constructions such as blow-up/blow-down [16], boundary Lefschetz fibration [12, 18] were studied to find a stable generalized complex 4-manifold.

The study on the construction of stable generalized complex manifolds has been mainly carried out in dimension four, and very little is known in high dimensions. As a first step to go beyond dimension four, in this thesis, we study the construction of a stable generalized complex 6-manifold. We find a torus surgery (called $C^\infty-$logarithmic transformation in dimension four) which results in a new stable generalized complex 6-manifold. Our result is summarized in the following theorem.

Theorem 1.0.1. *Let $(M, \mathcal{J}_t, H)$ be a stable generalized complex 6-manifold with $t \in \mathbb{N} \sqcup \{0\}$ path-connected components on its type change locus. Assume that there is an embedded 4-torus $T \subset M$ with trivial normal bundle which is $\mathcal{J}_\omega-$symplectic (see Definition 2.3.6). For any $m \in \mathbb{Z}$, the multiplicity m torus surgery on $(M, \mathcal{J}_t, H)$ along T produces a stable generalized complex 6-manifold $(\hat{M}(m), \hat{\mathcal{J}}_{t+1}(m), \hat{H}(m))$ with $(t+1)$ path-connected components* *on the type change locus, each of which admits a constant type 1 generalized Calabi-Yau structure.*

In fact, the examples of stable generalized complex 6-manifolds with nonempty type change locus could be found as a product of the stable generalized complex 4-manifolds with nonempty type change locus and the complex projective line $\mathbb{C}P^1$ (see [30, Example 4.12]). Also, Cavalcanti and Gualtieri [17] proved that $S^5 \times S^1$ has a stable generalized complex structure, even though neither S^5 nor S^1 admits a generalized complex structure . The existence of a 6-manifold that admits a stable generalized complex structure with nonempty type change locus and that is not a product of lower dimensional manifolds is as-yet-unknown. We prove the following result using Theorem 1.0.1 and the result of Wall [48, Thoerem 1].

Theorem 1.0.2. *For any integer $n \geq 2$, there exists a stable generalized complex 6-manifold $(\hat{W}(n), \hat{\mathcal{J}}(n), \hat{H}(n))$ with nonempty type change locus such that*

- $\hat{W}(n) \simeq \hat{W}_0(n) \# (12n+1)(S^3 \times S^3)$
- $\pi_1(\hat{W}(n)) \cong \{1\}$
- *$\hat{W}_0(n)$ is a simply connected closed orientable 6-manifold that has $b_2(\hat{W}_0(n)) = 12n$ and $b_3(\hat{W}_0(n)) = 0$.*

We also study the realization of any finitely presented group by a stable generalized complex 6-manifold. As an immediate consequence of Gompf [25] and Taubes [43], it follows that any finitely presented group G is the fundamental group of a generalized complex manifold of a constant type. And due to the result of Torres [45], any finitely presented group G is the fundamental group of a stable generalized complex $2n-$manifold $(M, \mathcal{J}, H)$ with nonempty type change locus for $n \geq 2$. In particular, for $n = 3$, each path-connected component on the type change locus is diffeomorphic to $T^2 \times \mathbb{C}P^1$ and the Euler characteristic satisfies $\chi(M) \geq 16$. We obtain the following result by using the symplectic sum, which is distinguished from Torres's in [45].

Theorem 1.0.3 (Theorem 4.2.8, Theorem 4.2.9 and Corollary 4.2.11)**.** *Let G be a finitely presented group. For any pair (n, k) of natural numbers, there are stable generalized complex 6-manifolds $(\hat{M}_i(n,k), \hat{\mathcal{J}}_i(n,k), \hat{H}_i(n,k))$ for $i = 1, 2, 3$ such that*

- $\pi_1(\hat{M}_i(n,k)) \cong G$

- $\chi(\hat{M}_1(n,k)) = 0$
- $\chi(\hat{M}_2(n,k)) = -16n$ *and* $\chi(\hat{M}_3(n,k)) = -24n$
- *there are* k *path-connected components on the corresponding type change locus, each of which is diffeomorphic to* T^4.

A weaker result is also obtained (Theorem 4.2.5) by using Theorem 1.0.1.

Organization of the thesis: This thesis is structured as follows. In Chapter 2, we review the fundamental concepts and results in generalized complex structures to the extent necessary to follow the thesis. We give three different but equivalent definitions of a generalized complex structure and define the type of a generalized complex structure. We classify generalized complex structures into stable ones and unstable ones, and provide some corresponding examples. We also discuss topological obstructions to the existence of a generalized almost-complex manifold.

In Chapter 3, we describe three cut-and-paste constructions of stable generalized complex manifolds; the symplectic sum, torus surgery and a combined operation of a $C^\infty-$logarithmic transformation and a Gluck twist [24], which can be used to produce a new stable generalized complex manifold. In particular, we show that a torus surgery of arbitrary multiplicity gives rise to a new stable generalized complex manifold with nonempty type change locus, therefore prove Theorem 1.0.1. We present the way to distinguish homologically closed orientable simply connected 6-manifolds from a product of lower dimensional manifolds. Then we apply torus surgeries of multiplicity zero to a closed symplectic 6-manifold and construct a stable generalized complex 6-manifold which is closed orientable simply connected and is not a product of lower dimensional manifolds. In such a way, we show that there exist infinitely many closed orientable simply connected stable generalized complex 6-manifolds that are not a product of lower dimensional manifolds and prove Theorem 1.0.2.

In Chapter 4, we study the realization of any finitely presented group by a generalized complex manifold. We build on Torres's result [45] to show that any finitely presented group G is the fundamental group of a generalized complex manifold with any type change jump. More importantly, using a multiplicity zero torus surgery and the symplectic sum, we study the realization of any finitely presented group G as the fundamental group of a stable generalized complex 6-manifold with nonempty type change locus which has different topological properties from the one followed from [45]. This leads to a proof of Theorem 1.0.3.

Notations: Throughout this thesis, we will denote the exterior algebra of a vector bundle $\mathcal{E}$ by $\Lambda^\bullet(\mathcal{E})$ and the space of smooth sections of $\mathcal{E}$ by $\Gamma(\mathcal{E})$. We will follow the standard convention that the interior product of a differential form ρ is given by a contraction $i_X\rho = \rho(X, \cdots)$ for a vector field X. And we will denote by $(M, \mathcal{J}, H)$ both a generalized complex structure on a manifold M and a generalized complex manifold itself, unless otherwise specified (the only exception appears in Section 2.1). Also, all the manifolds will be assumed to be smooth and by $M_1 \simeq M_2$ we will mean that two manifolds M_1 and M_2 are diffeomorphic.

Chapter 2

Generalized complex structures

This chapter serves as the introduction to generalized complex structures. Gualtieri's thesis is the main source of the chapter and we follow closely his exposition in [30, 31](cf. [17]). We begin with three different definitions of a generalized complex structure and introduce the type of generalized complex structures. We provide several examples of stable and unstable generalized complex structures. We also discuss topological obstructions to the existence of a generalized almost-complex structure.

2.1 Definitions

Let M be a $2n$-manifold equipped with a closed 3-form $H \in \Gamma(\Lambda^3(TM))$. The *generalized tangent bundle* $\mathbb{T}M$ over M is defined as the Whitney sum of the tangent and cotangent bundles

$$\mathbb{T}M = TM \oplus T^*M. \tag{2.1}$$

A section $v \in \Gamma(\mathbb{T}M)$ is expressed pointwise as $v = X + \xi$ with a vector field $X \in \Gamma(TM)$ and a 1-form $\xi \in \Gamma(T^*M)$. The generalized tangent bundle is automatically endowed with a natural pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)), \tag{2.2}$$

for $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$. We can write (2.2) in matrix form

$$\langle X + \xi, Y + \eta \rangle = \begin{pmatrix} X & \xi \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1}_{2n\times 2n} \\ \mathbf{1}_{2n\times 2n} & 0 \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix}, \tag{2.3}$$

with the indefinite signature $(2n, 2n)$. In addition, we can equip $\mathbb{T}M$ with the *Courant bracket* defined as[1]

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y)) + i_X i_Y H. \quad (2.4)$$

Remark 2.1.1. The Courant bracket exhibits chirality between vector fields and 1-forms. When restricted to the tangent bundle, i.e., $\xi = \eta = 0$, it recovers the Lie bracket for $H = 0$, while it vanishes when restricted to the cotangent bundle.

Now we come to the first definition of a generalized complex structure.

Definition 2.1.2 ([30])**.** A generalized complex structure $(M, \mathcal{J}, H)$ is an endomorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ such that:

- $\mathcal{J}^2 = -1$,
- $\mathcal{J}$ preserves the natural pairing $\langle \cdot, \cdot \rangle$,
- $\mathcal{J}$ is integrable, i.e., $L \subset \mathbb{T}M \otimes \mathbb{C}$, $+i$-eigenbundle of $\mathcal{J}$ is Courant involutive, i.e., $[L, L]_H \subset L$.

Weakening the requirements on $\mathcal{J}$ by dropping out the integrability condition yields the concept of a *generalized almost-complex structure.*

Definition 2.1.3 ([30])**.** A generalized almost-complex structure $(M, \mathcal{J})$ is an endomorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ such that $\mathcal{J}^2 = -1$ and

$$\langle \mathcal{J}u,\ \mathcal{J}v \rangle = \langle u,\ v \rangle \quad (2.5)$$

for any $u, v \in \Gamma(\mathbb{T}M)$.

It turns out that the $+i$-eigenbundle of $\mathcal{J}$ is a *maximally isotropic* subbundle (see the proof of Proposition 2.1.9), which we define below.

Definition 2.1.4 ([30])**.** A subbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$ is called maximally isotropic or Lagrangian if $\langle L, L \rangle = 0$ and $\mathrm{Rank}(L) = \frac{1}{2}\mathrm{Rank}(\mathbb{T}M \otimes \mathbb{C})$.

[1]This is not quite the Courant bracket in view of the original definition [20, 21], but the twisted Courant bracket by a 3-form H. In this thesis, nevertheless, we call the twisted Courant bracket just the Courant bracket.

Example 1 ([30, Example 2.5])**.** Let $E \subset TM \otimes \mathbb{C}$ be a subbundle and $\sigma \in \Gamma(\Lambda^2(E^*))$ a 2-form with E^* being dual to E. Consider a subbundle $L(E,\sigma) \subset \mathbb{T}M \otimes \mathbb{C}$ defined as

$$L(E,\sigma) = \{X + \xi \in \Gamma(E \oplus T^*M \otimes \mathbb{C}) : \xi|_E = i_X\sigma\}. \tag{2.6}$$

Then we observe

$$\begin{aligned} \mathrm{Rank}(L(E,\sigma)) &= \dim(L(E,\sigma)_p) = \dim(E_p) + \dim((T_p^*M \otimes \mathbb{C})/E_p^*) \\ &= \dim(E_p) + \dim(T_p^*M \otimes \mathbb{C}) - \dim(E_p^*) \\ &= \dim(T_p^*M \otimes \mathbb{C}) = \frac{1}{2}\dim(\mathbb{T}_pM \otimes \mathbb{C}) = \frac{1}{2}\mathrm{Rank}(\mathbb{T}M \otimes \mathbb{C}). \end{aligned} \tag{2.7}$$

Here we used the subscript p to denote the fiber at a point $p \in M$. We also have

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)) = \frac{1}{2}(\sigma(X,Y) + \sigma(Y,X)) = 0, \tag{2.8}$$

for any $X+\xi$, $Y+\eta \in \Gamma(L(E,\sigma))$. In (2.8), the second equality follows from (2.6) and the last equality holds due to the skew symmetric property of σ. From (2.7) and (2.8), we see that $L(E,\sigma)$ is indeed maximally isotropic.

Proposition 2.1.5 ([30, Proposition 2.6])**.** *Every maximally isotropic subbundle of* $\mathbb{T}M \otimes \mathbb{C}$ *takes the form* $L(E,\sigma)$*.*

Proof. Let $L \subset \mathbb{T}M \otimes \mathbb{C}$ be a maximally isotropic subbundle. Since L is isotropic, for any $X + \xi$, $Y + \eta \in \Gamma(L)$,

$$\langle X + \xi,\, Y + \eta\rangle = 0, \tag{2.9}$$

which combines with (2.2) to imply

$$\xi(Y) = -\eta(X). \tag{2.10}$$

Based on (2.10), we define a skew symmetric bilinear pairing as

$$P(X + \xi, Y + \eta) = \xi(Y) = -\eta(X). \tag{2.11}$$

Note that (2.11) is uniquely determined by X and Y, regardless of ξ and η, which can be shown as follows. For any $Y + \eta' \in \Gamma(L)$ and $X + \xi' \in \Gamma(L)$, we observe

$$P(X + \xi, Y + \eta') = \xi(Y) = -\eta'(X), \tag{2.12}$$

$$P(X+\xi', Y+\eta) = \xi'(Y) = -\eta(X), \tag{2.13}$$
$$P(X+\xi', Y+\eta') = \xi'(Y) = -\eta'(X). \tag{2.14}$$

Using (2.12) and (2.13) into (2.14), we get

$$P(X+\xi', Y+\eta') = \xi(Y) = -\eta(X). \tag{2.15}$$

Comparing (2.15) with (2.11), we see

$$P(X+\xi, Y+\eta) = P(X+\xi', Y+\eta'). \tag{2.16}$$

Let $E = \pi L$ with the canonical projection $\pi : \mathbb{T}M \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ and define a 2-from $\sigma \in \Gamma(\Lambda^2(E^*))$ by

$$\sigma(X, Y) = P(X+\xi, Y+\eta). \tag{2.17}$$

For any $X \in \Gamma(E)$, we can always find $\xi \in \Gamma(T^*M \otimes \mathbb{C})$ such that $X + \xi \in L$. Then, from (2.17), we get $\sigma(X, Y) = \xi(Y)$ for any $Y \in \Gamma(E)$, and consequently $i_X\sigma = \xi|_E$ as required. Therefore, every maximally isotropic subbundle L is expressed as $L(E, \sigma)$. □

Generalized complex structures can also be completely described by means of the maximally isotropic subbundle of $\mathbb{T}M \otimes \mathbb{C}$ instead of the endomorphism $\mathcal{J}$.

Definition 2.1.6 ([30])**.** A generalized complex structure (M, L, H) is a complex Lagrangian subbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$ such that:

- $L \cap \overline{L} = \{0\}$ with $\overline{L}$ being the complex conjugate of L,
- L is Courant involutive, i.e., $[L, L]_H \subset L$.

Maximally isotropic subbundles are related to differential forms via the Clifford action. Recall that the Clifford action of the generalized tangent bundle $\mathbb{T}M \otimes \mathbb{C}$ on the exterior algebra $\Lambda^\bullet(T^*M) \otimes \mathbb{C}$ is given by

$$(X+\xi) \cdot \rho = i_X\rho + \xi \wedge \rho, \tag{2.18}$$

for $X \in \Gamma(TM \otimes \mathbb{C})$, $\xi \in \Gamma(T^*M \otimes \mathbb{C})$ and $\rho \in \Gamma(\Lambda^\bullet(T^*M) \otimes \mathbb{C})$. The differential form ρ is called a *pure spinor* if it is annihilated by a maximally isotropic subbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$, i.e.,

$$L \cdot \rho = 0. \tag{2.19}$$

Notice that two pure spinors are annihilated by the same maximally isotropic subspace if they are a multiple of each other. Hence, the maximally isotropic subbundle L is uniquely assigned to a complex line bundle $K \subset \Lambda^\bullet(T^*M) \otimes \mathbb{C}$, which is called the *canonical bundle*, and it is locally generated by the pure spinor ρ. This brings us to a third definition of a generalized complex structure.

Definition 2.1.7 ([30])**.** A generalized complex structure (M, K, H) is the canonical bundle $K \subset \Lambda^\bullet(T^*M) \otimes \mathbb{C}$ locally generated by a pure spinor

$$\rho = e^{B+i\omega} \wedge \Omega \ , \tag{2.20}$$

which satisfies

- non-degeneracy, i.e.,
$$\Omega \wedge \bar{\Omega} \wedge \omega^{n-k} \neq 0, \tag{2.21}$$
where $k = deg(\Omega)$,

- integrability, i.e.,
$$d^H\rho = d\rho + H \wedge \rho = v \cdot \rho, \tag{2.22}$$
for some section $v \in \Gamma(\mathbb{T}M \otimes \mathbb{C})$.

Here B, ω are real 2-forms, and $\Omega = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_k$ with linearly independent complex 1-forms $\theta_1, \theta_2, \cdots, \theta_k$.

Definition 2.1.8 ([35])**.** A generalized complex structure is called generalized Calabi-Yau if ρ is nowhere vanishing and $d^H\rho = 0$.

For the remaining part of the section, we show the equivalence of the three definitions of a generalized complex structure that we have discussed.

Proposition 2.1.9 ([30])**.** *Definitions 2.1.2, 2.1.6 and 2.1.7 are equivalent.*

Proof.

- We begin with the proof of the equivalence of Definition 2.1.2 and Definition 2.1.6. First, we show that Definition 2.1.2 implies Definition 2.1.6. Assume that $(M, \mathcal{J}, H)$ is a generalized complex structure. From Definition 2.1.2, we have $\mathcal{J}^2 = -1$. Let $L \subset \mathbb{T}M \otimes \mathbb{C}$ be the $+i$-eigenspace of $\mathcal{J}$, then the complex conjugate $\bar{L}$ of L is the $-i$-eigenspace of $\mathcal{J}$ and
$$L \cap \bar{L} = \{0\}. \tag{2.23}$$

This leads to the split

$$\mathbb{T}M \otimes \mathbb{C} = L \oplus \bar{L}, \tag{2.24}$$

hence,

$$\dim(L) = \frac{1}{2}\dim(\mathbb{T}M \otimes \mathbb{C}). \tag{2.25}$$

Since $\mathcal{J}$ preserves the natural pairing, for any sections $u, v \in \Gamma(L)$,

$$\langle u, v\rangle = \langle \mathcal{J}u, \mathcal{J}v\rangle = \langle iu, iv\rangle = -\langle u, v\rangle \tag{2.26}$$

which implies

$$\langle u, v\rangle = 0. \tag{2.27}$$

Therefore, (2.23), (2.25) and (2.27) along with the third criterion of Definition. 2.1.2 determine the maximally isotropic subspace $L \subset \mathbb{T}M \otimes \mathbb{C}$ that satisfies all the requirements of Definition 2.1.6. Conversely, let us prove that Definition 2.1.6 leads to Definition 2.1.2. Let $L \subset \mathbb{T}M \otimes \mathbb{C}$ be a maximally isotropic subspace used in Definition 2.1.6. Since $L \cap \bar{L} = \{0\}$, we have the decomposition

$$\mathbb{T}M \otimes \mathbb{C} = L \oplus \bar{L}. \tag{2.28}$$

We can define a linear map $\mathcal{J}$ as multiplication by $+i$ on L and by $-i$ on $\bar{L}$, i.e.,

$$\mathcal{J}L = iL, \tag{2.29}$$
$$\mathcal{J}\bar{L} = -i\bar{L}. \tag{2.30}$$

Such a $\mathcal{J}$ is a real linear transformation, because from (2.29) and (2.30) $\overline{\mathcal{J}L} = \mathcal{J}\bar{L}$. Also notice that L and $\mathbb{T}M$ have the same dimension because L is the maximally isotropic subspace. Hence $\mathcal{J}$ can act on $\mathbb{T}M$ as an endomorphism, and $\mathcal{J}^2 = -1$ from (2.29) or (2.30). And for any $u, v \in \Gamma(\mathbb{T}M \otimes \mathbb{C})$,

$$\langle \mathcal{J}u, \mathcal{J}v\rangle = \begin{cases} \langle \pm iu, \pm iv\rangle = -\langle u, v\rangle, & \text{if } u,\, v \in L \text{ or } u,\, v \in \bar{L} \\ \langle \pm iu, \mp iv\rangle = \langle u, v\rangle, & \text{otherwise.} \end{cases} \tag{2.31}$$

Since L and $\bar{L}$ are maximally isotropic, we get

$$\langle u, v\rangle = -\langle u, v\rangle = 0, \tag{2.32}$$

for the first case of (2.31). It follows from (2.31) and (2.32) that $\mathcal{J}$ preserves the natural pairing of any $u, v \in \Gamma(\mathbb{T}M \otimes \mathbb{C})$. Therefore, the endomorphism $\mathcal{J}$ constructed based on Definition 2.1.6 satisfies all the conditions given in Definition 2.1.2. We now can conclude that Definition 2.1.2 is equivalent to 2.1.6.

- We move on to the proof of the equivalence of Definitions 2.1.6 and 2.1.7. We first show that a maximally isotropic subspace uniquely determines the canonical bundle generated by a pure spinor (2.20), and also vice versa. By Proposition 2.1.5, every maximally isotropic subspace is given by $L(E, \sigma)$. Given such an $L(E, \sigma)$, we define a differential form by

$$\rho = e^{-\sigma} \wedge \Omega. \tag{2.33}$$

Here $\Omega = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_k$ in which $\theta_1, \theta_2, \cdots, \theta_k$ are linearly independent complex 1-forms that locally span $(T^*M \otimes \mathbb{C})/E^*$. The differential form (2.33) is indeed the pure spinor, since it is annihilated by $L(E, \sigma)$ as follows:

$$\begin{aligned}(X+\xi)\cdot\left(e^{-\sigma}\wedge\Omega\right) &= i_X\left(e^{-\sigma}\wedge\Omega\right) + \xi\wedge\left(e^{-\sigma}\wedge\Omega\right)\\ &= i_X e^{-\sigma}\wedge\Omega + \xi\wedge(1-\sigma+\frac{1}{2}\sigma\wedge\sigma-\cdots)\wedge\Omega\\ &= i_X(1-\sigma+\frac{1}{2}\sigma\wedge\sigma-\frac{1}{3!}\sigma\wedge\sigma\wedge\sigma+\cdots)\wedge\Omega\\ &\quad + \xi\wedge(1-\sigma+\frac{1}{2}\sigma\wedge\sigma-\cdots)\wedge\Omega\\ &= (-\xi+\xi\wedge\sigma-\frac{1}{2}\xi\wedge\sigma\wedge\sigma+\cdots)\wedge\Omega\\ &\quad + (\xi-\xi\wedge\sigma+\frac{1}{2}\xi\wedge\sigma\wedge\sigma-\cdots)\wedge\Omega = 0,\end{aligned} \tag{2.34}$$

for any $X + \xi \in \Gamma(L(E, \sigma))$. In (2.34), the first equality is the result of the Clifford action (2.18) and the second equality is due to $i_X\Omega = 0$, and the forth equality is obtained using (2.6) and

$$i_X(\underbrace{\sigma \wedge \cdots \wedge \sigma}_{n}) = n\, i_X\sigma. \tag{2.35}$$

We extend σ to $\alpha \in \Gamma(\Lambda^2(T^*M \otimes \mathbb{C}))$ so that $\alpha|_E = -\sigma$. Note that the choice of the extension α does not change the pure spinor (2.33), since only $\alpha|_E = -\sigma$ survives by the wedge with Ω. Hence, we can still write ρ as

$$\rho = e^{\alpha} \wedge \Omega. \tag{2.36}$$

Let B, ω be real and imaginary parts of α, respectively, i.e., $\alpha = B + i\omega$. With this decomposition of α, the pure spinor (2.36) becomes (2.20). As the maximally isotropic subspace $L(E, \sigma)$ annihilates all the differential forms which are a multiple of the pure spinor (2.36), it determines uniquely the canonical bundle $K = \langle e^{B+i\omega} \wedge \Omega \rangle$. Conversely, provided the canonical bundle generated by the pure spinor (2.20), we define $E \subset TM \otimes \mathbb{C}$ by

$$\Gamma(E) = \{X \in \Gamma(TM \otimes \mathbb{C}) : i_X \Omega = 0\}. \tag{2.37}$$

Based on (2.37), we can find a restriction of $B + i\omega$ to E

$$\sigma = -(B + i\omega)|_E. \tag{2.38}$$

Then, (2.37) and (2.38) completely specify $L(E, \sigma)$.
Now we address the equivalence of the two criteria of Definition 2.1.6 and the integrability and non-degeneracy conditions of the pure spinor. To this end, we use the following results, which build bridges between the two definitions.

Proposition 2.1.10 ([19], III.2.4)**.** *Let $L, L' \subset \mathbb{T}M \otimes \mathbb{C}$ be maximally isotropic subspaces. Then, $L \cap L' = \{0\}$ if and only if Mukai pairing of their pure spinors ρ, ρ' is nowhere vanishing $(\rho, \rho') \neq 0$.*

The Mukai pairing of the differential forms [40] is given by

$$(\rho, \rho') = (\mathcal{A}(\rho) \wedge \rho')_{top}. \tag{2.39}$$

Here $\mathcal{A}$ is the antiautomorphism of the exterior algebra $\Lambda^{\bullet}(T^*M \otimes \mathbb{C})$ acting on the differential forms by

$$\mathcal{A}(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k) = \xi_k \wedge \xi_{k-1} \wedge \cdots \wedge \xi_1, \tag{2.40}$$

where $\xi_1, \xi_2, \cdots, \xi_k \in \Gamma(T^*M \otimes \mathbb{C})$

Proposition 2.1.11 ([30], Proposition 3.44)**.** *A maximally isotropic subspace $L \subset \mathbb{T}M \otimes \mathbb{C}$ is Courant involutive if and only if there exists a section $v \in \Gamma(\mathbb{T}M)$ satisfying*

$$d\rho + H \wedge \rho = v \cdot \rho, \tag{2.41}$$

where the pure spinor ρ is a local generator of the canonical bundle $K \subset \Lambda^{\bullet}(T^*M) \otimes \mathbb{C}$ annihilated by L.

Let ρ and $\bar{\rho}$ be pure spinors annihilated by the maximally isotropic subspaces L and $\bar{L}$ given in Definition 2.1.6. The Mukai pairing of $\rho, \bar{\rho}$ can be calculated as follows:

$$\begin{aligned}
(\rho, \bar{\rho}) &= (e^{B+i\omega} \wedge \Omega, e^{B-i\omega} \wedge \bar{\Omega}) \\
&= \left[\mathcal{A}(e^{B+i\omega} \wedge \Omega) \wedge e^{B-i\omega} \wedge \bar{\Omega})\right]_{top} \\
&= \left[\mathcal{A}\left(\sum_{j=0}^{n-k} \frac{1}{j!}(B+i\omega)^j \wedge \Omega\right) \wedge e^{B-i\omega} \wedge \bar{\Omega}\right]_{top} \\
&= \left[\sum_{j=0}^{n-k} \frac{(-1)^{(2j+k)(2j+k-1)/2}}{j!}(B+i\omega)^j \wedge \Omega \wedge e^{B-i\omega} \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\left[\sum_{j=0}^{n-k} \frac{(-1)^{(4j^2+4jk-2j)/2}}{j!}(B+i\omega)^j \wedge \Omega \right. \\
&\qquad\qquad \left. \wedge e^{B-i\omega} \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\left[\sum_{j=0}^{n-k} \frac{(-1)^{-j}}{j!}(B+i\omega)^j \wedge \Omega \wedge e^{B-i\omega} \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\left[e^{-(B+i\omega)} \wedge \Omega \wedge e^{B-i\omega} \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\left[e^{-(B+i\omega)} \wedge e^{B-i\omega} \wedge \Omega \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\left[e^{-2i\omega} \wedge \Omega \wedge \bar{\Omega}\right]_{top} \\
&= (-1)^{(k^2-k)/2}\frac{(-2i\omega)^{n-k}}{(n-k)!} \wedge \Omega \wedge \bar{\Omega} \\
&= (-1)^{(k^2-k)/2+n-k}\frac{\omega^{n-k}}{(n-k)!} \wedge \Omega \wedge \bar{\Omega},
\end{aligned} \tag{2.42}$$

for which $k = deg(\Omega)$. By Proposition 2.1.10 and (2.42), the condition $L \cap \bar{L} = \{0\}$ of Definition 2.1.6 can be rendered into the non-degeneracy condition (2.21) on the pure spinor and also vice versa. The equivalence between Courant involutivity of L and integrability of ρ immediately follows from Proposition 2.1.11. Therefore, Definitions 2.1.6 and 2.1.7 are equivalent each other.

- Finally, Definitions 2.1.2 and 2.1.7 are equivalent as a syllogistic consequence of the equivalence of Definitions 2.1.2 and 2.1.6, and the equivalence of Definitions 2.1.6 and 2.1.7.

□

Remark 2.1.12 (On our notation)**.** We used three different notations for a generalized complex structure in Definitions 2.1.2, 2.1.6 and 2.1.7 to contrast the differences. Hereafter we will adopt the notation $(M, \mathcal{J}, H)$ for a generalized complex structure or a generalized complex manifold in order to avoid confusion.

We close this section by the definition of the equivalent generalized complex structures.

Definition 2.1.13 (Equivalence of generalized complex structures, [17])**.** Two generalized complex structures $(M, \mathcal{J}, H)$ and $(M', \mathcal{J}', H')$ are said to be equivalent if there is a diffeomorphism $\varphi : M \to M'$ and a real 2-form $B \in \Gamma(\Lambda^2(T^*M))$ such that:

$$\varphi^* H' = H + dB, \tag{2.43}$$

$$\mathcal{J}' \circ (\varphi_* e^B) = (\varphi_* e^B) \circ \mathcal{J}. \tag{2.44}$$

Here e^B is an automorphism of $\mathbb{T}M$ given by

$$e^B : X + \xi \to X + \xi + i_X B \tag{2.45}$$

for $X + \xi \in \Gamma(\mathbb{T}M)$.

2.2 Type of a generalized complex structure

The type is a fundamental invariant of generalized complex structures. In the following, we recall its definition and discuss the standard examples.

Definition 2.2.1 ([30])**.** Let K be the canonical bundle of a generalized complex structure $(M, \mathcal{J}, H)$, and $\rho \in \Gamma(K)$ a nonzero local section of K given by (2.20). The *type* of $(M, \mathcal{J}, H)$ at a point $p \in M$, simply denoted by $\text{type}(\mathcal{J})$, is defined by the lowest degree of ρ,

$$\text{type}(\mathcal{J}) = deg(\Omega). \tag{2.46}$$

From (2.46), the possible values of $\text{type}(\mathcal{J})$ are $\{0, 1, 2, \cdots, n\}$ in dimension $2n$. Standard examples of generalized complex structures of constant types are as follows.

Example 2 ([30, Example 4.10])**.** Let (M, ω) be a symplectic $2n-$manifold. The endomorphism of $\mathbb{T}M$ given by

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{2.47}$$

determines a maximally isotropic subbundle $L_\omega \subset \mathbb{T}M \otimes \mathbb{C}$ by

$$L_\omega = \{X - i\omega(X) : X \in \Gamma(TM \otimes \mathbb{C})\}. \tag{2.48}$$

L_ω annihilates the canonical bundle generated by

$$\rho_\omega = e^{i\omega}. \tag{2.49}$$

The differential form (2.49) is a pure spinor with $\Omega = 1$, in view of the standard form (2.20). Indeed, (2.49) is non-degenerate since $\omega^n \neq 0$, and integrable with respect to $H = 0$ due to $d\omega = 0$. Hence, using (2.46) and (2.49),

$$\text{type}(\mathcal{J}_\omega) = 0. \tag{2.50}$$

Example 3 ([30, Example 4.11])**.** Let (M, I) be a complex $2n-$manifold. We can define an endomorphism of $\mathbb{T}M$ as

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \tag{2.51}$$

whose $+i-$subeigenbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$ is

$$L = T^{0,1}M \oplus T^{*1,0}M, \tag{2.52}$$

where $T^{0,1}M$ and $T^{*1,0}M$ are subbundles of $TM \otimes \mathbb{C}$ and $T^*M \otimes \mathbb{C}$ with $-i$ and $+i$ eigenvalues with respect to I and I^*, respectively. The canonical bundle is

$$K = \Lambda^{n,0}T^*M, \tag{2.53}$$

and it is generated by the pure spinor

$$\rho_I = d\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_n, \tag{2.54}$$

with $d\theta_i \in \Gamma(T^{*1,0}M)$ for $i = 1, 2, \cdots, n$. Comparing (2.54) with (2.20), it is straightforward to see

$$\Omega = d\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_n. \tag{2.55}$$

The pure spinor (2.54) satisfies non-degeneracy condition, since $\Omega \wedge \bar{\Omega} \neq 0$. Also the integrability condition (2.22) holds for $H = 0$, because $d\rho_I = 0$. It immediately follows from (2.46) and (2.55)

$$\text{type}(\mathcal{J}_I) = n. \tag{2.56}$$

From Examples 2 and 3, it is clear that generalized complex structures incorporate the symplectic and complex structures as their extremal cases. The following proposition justifies calling generalized complex structures of constant type 0 "symplectic" and generalized complex structures of constant type $\frac{1}{2}\dim_{\mathbb{R}} M$ "complex" (see [30, Examples 4.10 and 4.11] for the proof).

Proposition 2.2.2. *Let $(M, \mathcal{J}, H)$ be a generalized complex $2n-$manifold. If* $\text{type}(\mathcal{J}) = 0$*, there is a 2-form $B \in \Gamma(\Lambda^2(T^*M))$ such that*

$$e^B \circ \mathcal{J} \circ e^{-B} = \mathcal{J}_\omega \tag{2.57}$$

for some symplectic structure (M, ω). If $\text{type}(\mathcal{J}) = n$*, there is a 2-form $B \in \Gamma(\Lambda^2(T^*M))$ such that*

$$e^B \circ \mathcal{J} \circ e^{-B} = \mathcal{J}_I \tag{2.58}$$

for some complex structure (M, I).

Generalized complex structures of constant type 0 are isomorphic ($B-$field equivalent) to a generalized complex structure arising from a symplectic structure. Generalized complex structures of constant type $\frac{1}{2}\dim_{\mathbb{R}} M$ are isomorphic ($B-$field equivalent) to a generalized complex structure arising from a complex structure.

Concerning the type of a product of generalized complex structures, we have the following result.

Proposition 2.2.3 ([30, Example 4.12])**.** *Let $(M_1, \mathcal{J}_1, H_1)$ and $(M_2, \mathcal{J}_2, H_2)$ be generalized complex structures. Then we have the generalized complex structure $(M_1 \times M_2,\ \mathcal{J}_1 \times \mathcal{J}_2,\ p_1^* H_1 + p_2^* H_2)$ with the canonical projections $p_i : M_1 \times M_2 \to M_i$ for $i = 1, 2$, whose type is given by*

$$\text{type}(\mathcal{J}_1 \times \mathcal{J}_2) = \text{type}(\mathcal{J}_1) + \text{type}(\mathcal{J}_2). \tag{2.59}$$

The type of a generalized complex structure need not be constant across the manifold, it can rather vary.

Definition 2.2.4 ([30])**.** The type change locus of a generalized complex manifold $(M, \mathcal{J}, H)$ is a subset of M where $\text{type}(\mathcal{J})$ is not locally constant.

Along the type change locus, the type of a generalized complex structure always jumps, i.e., it increases (see [30, Section 4.8]). The type change jumps obey the following constrain.

Proposition 2.2.5 ([30])**.** *The type change jump of a generalized complex structure respects its parity.*

It directly follows from Proposition 2.2.5 that the type of a generalized complex structure can only be changed by an even number.

2.3 Stable generalized complex structures

Let K be the canonical bundle that determines a generalized complex structure $(M, \mathcal{J}, H)$, and a pure spinor $\rho \in \Gamma(K)$ a local section of K. Denote by K^* the anticanonical bundle locally generated by a section $s \in \Gamma(K^*)$ given by $s(\rho) = \rho_0$ with ρ_0 the degree 0 part of ρ. Then, we observe the type change jump along the zero section $s^{-1}(0)$, outside of which $\operatorname{type}(\mathcal{J}) = 0$.

Definition 2.3.1 ([17, Definition 2.10])**.** A generalized complex structure $(M, \mathcal{J}, H)$ is called *stable* if s is transverse to the zero section of K^*, i.e, s has only non-degenerate zeros.

Example 4 (Symplectic manifold, [17])**.** Let (M, ω) be a symplectic manifold. As we already showed in Example 2, (M, ω) can be equipped with a generalized complex structure $(M, \mathcal{J}_\omega, H)$ (cf. Proposition 2.2.2) determined by the canonical bundle K with the local section $\rho = e^{B+i\omega}$, where a 2-form $B \in \Gamma(\Lambda^2(T^*M))$ is chosen such that $H = -dB$. Then the degree 0 part $\rho_0 = 1$, resulting in the nowhere vanishing section of the anticanonical bundle. Hence, $(M, \mathcal{J}_\omega, H)$ is stable.

Examples of stable generalized complex structures with nonempty type change locus are shown below.

Example 5 ([16, Example 1.6])**.** Let (z, w) be complex coordinates on $\mathbb{C}^2$ and consider a complex differential form

$$\rho = z + dz \wedge dw. \tag{2.60}$$

(2.60) can be written as

$$\rho = dz \wedge dw, \tag{2.61}$$

when $z = 0$, and elsewhere as

$$\rho = z e^{\frac{1}{z} dz \wedge dw}. \tag{2.62}$$

Comparing (2.61) and (2.62) with (2.20), we see that the differential form given by (2.60) is a pure spinor. Furthermore it is non-degenerate, since

$$(\rho, \bar{\rho}) = dz \wedge d\bar{z} \wedge dw \wedge d\bar{w} \neq 0, \tag{2.63}$$

and it is integrable, given that

$$d\rho = -\partial_w \rho. \tag{2.64}$$

Hence, the pure spinor (2.60) defines a generalized complex structure that is of type 2 along the zero locus $\{z = 0\}$ and of constant type 0 along the $z \neq 0$. In addition, this generalized complex structure is stable, since the zero points $\{z = 0\}$ are non-degenerate.

Example 6. Let us consider $\mathbb{C}^2 \times T^{2n-4}$ and the differential form

$$\rho = \rho_1 \wedge \rho_2, \tag{2.65}$$

where ρ_1 is the differential form on $\mathbb{C}^2$ given by (2.60) and ρ_2 is the differential form on T^{2n-4} given by

$$\rho_2 = e^{i\omega_T}, \tag{2.66}$$

for a symplectic form ω_T on T^{2n-4}. We explicitly write (2.65) by combining (2.60) and (2.66) as

$$\rho = (z + dz \wedge dw) \wedge e^{i\omega_T}. \tag{2.67}$$

The differential form (2.67) is a pure spinor that defines a type change stable generalized complex structure on $\mathbb{C}^2 \times T^{2n-4}$ as we now show. First of all, the form (2.67) is a pure spinor in view of (2.20), since it can be expressed as

$$\rho = dz \wedge dw \wedge e^{i\omega_T} \tag{2.68}$$

when $z = 0$, and as

$$\rho = z e^{z^{-1} dz \wedge dw + i\omega_T} \tag{2.69}$$

when $z \neq 0$ [2]. Moreover, (2.67) is non-degenerate, because the Mukai pairing

$$(\rho, \bar{\rho}) = dz \wedge d\bar{z} \wedge dw \wedge d\bar{w} \wedge \omega_T^{n-2} \neq 0. \tag{2.70}$$

Finally, (2.67) is integrable, i.e.,

$$d\rho + H \wedge \rho = (X + \xi) \cdot \rho \ , \tag{2.71}$$

since

$$d\rho = -\partial_w \rho. \tag{2.72}$$

Therefore, the pure spinor (2.67) defines a generalized complex structure on $\mathbb{C}^2 \times T^{2n-4}$ that has type 2 along the $\{z = 0\}$ locus and constant type 0 elsewhere. Also, such a generalized complex structure is stable because $\{z = 0\}$ points are non-degenerate.

[2] Here we used $dz \wedge dw \wedge \omega_T = \omega_T \wedge dz \wedge dw$ thanks to the property of the wedge product between p-form and q-form $\omega_p \wedge \omega_q = (-1)^{pq} \omega_q \wedge \omega_p$.

The local structure of stable generalized complex manifolds is determined by the following result due to Bailey.

Proposition 2.3.2 ([5])**.** *Any stable generalized complex structure* $(M, \mathcal{J}, H)$ *is locally equivalent to the generalized complex structure on* $\mathbb{C}^2 \times \mathbb{R}^{2n-4}$ *whose canonical bundle is generated by the pure spinor*

$$\rho = (z + dz \wedge dw) \wedge e^{i\omega_0}, \tag{2.73}$$

where (z, w) *are complex coordinates on* $\mathbb{C}^2$*, and* ω_0 *is the standard symplectic form on* $\mathbb{R}^{2n-4}$*.*

Proof. This proposition is the corollary of the following result.

Lemma 2.3.3 ([5, Main theorem], [1, Theorem 1.4])**.** *Any generalized complex structure* $(M, \mathcal{J}, H)$ *is locally equivalent to the generalized complex structure on* $\mathbb{C}^k \times \mathbb{R}^{2n-2k}$*, determined by the pure spinor*

$$\rho = e^{\pi} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_k \wedge e^{i\tilde{\omega}_0}. \tag{2.74}$$

Here $\pi \in \Gamma(\Lambda^{2,0}(T\mathbb{C}^k))$ *denotes a holomorphic Poisson structure and* $dz_i \in \Gamma(\Lambda^{1,0}(T^*\mathbb{C}^k))$ *are holomorphic 1-forms, and* $\tilde{\omega}_0$ *is the standard symplectic form on* $\mathbb{R}^{2n-2k}$*.*

First of all, notice that a stable generalized complex structure has constant type 0 besides the type change locus, so the pure spinor (2.74) should consist of the differential forms with even degrees due to Proposition 2.2.5. Hence, in (2.74), k should be even, i.e., $k = 2m$ with $m \in \mathbb{N}$. Since the integrability condition (2.22) for the pure spinor should be satisfied degree by degree, by taking the degree 1 component, we obtain

$$i_X \rho_2 = d\rho_0 + \xi \rho_0, \tag{2.75}$$

for some section $X + \xi \in \Gamma(\mathbb{T}M \otimes \mathbb{C})$. Here ρ_0 and ρ_2 are the degree 0 and degree 2 components of ρ, respectively. Because ρ determines the stable generalized complex structure, we have $\rho_0 = 0$ and $d\rho_0 \neq 0$ on the type change locus. Then (2.75) implies $\rho_2 \neq 0$ on the type change locus. As a result, type($\mathcal{J}$) jumps from 0 to 2, which in turn, implies that the rank of the underlying Poisson structure changes from $2m$ to $2m - 2$ on the type change locus. Applying Weinstein's splitting theorem [49], we can write the Poisson structure π as

$$\pi = f(z_1, z_2)\partial_{z_1} \wedge \partial_{z_2} + \sum_{i=3}^{2m} \sum_{j>i}^{2m} \partial_{z_i} \wedge \partial_{z_j}, \tag{2.76}$$

where $f(z_1, z_2)$ is a holomorphic function vanishing on the type change locus and its zero points are non-degenerate. Using (2.76) into (2.74), we get

$$\rho = e^{f(z_1,z_2)\partial_{z_1}\wedge\partial_{z_2}} dz_1 \wedge dz_2 \wedge e^{\sum_{i=3}^{2m}\sum_{j>i}^{2m}\partial_{z_i}\wedge\partial_{z_j}} dz_3 \wedge dz_4 \wedge \cdots \wedge dz_{2m} \wedge e^{i\tilde{\omega}_0}$$
$$\equiv \rho_{\mathcal{J}_1} \wedge \rho_{\mathcal{J}_2} \wedge \rho_{\mathcal{J}_3}. \tag{2.77}$$

Here $\{\rho_{\mathcal{J}_1}, \rho_{\mathcal{J}_2}, \rho_{\mathcal{J}_3}\}$ are pure spinors defined as

$$\rho_{\mathcal{J}_1} = e^{f(z_1,z_2)\partial_{z_1}\wedge\partial_{z_2}} dz_1 \wedge dz_2, \tag{2.78}$$
$$\rho_{\mathcal{J}_2} = e^{\sum_{i=3}^{2m}\sum_{j>i}^{2m}\partial_{z_i}\wedge\partial_{z_j}} dz_3 \wedge dz_4 \wedge \cdots \wedge dz_{2m}, \tag{2.79}$$

and

$$\rho_{\mathcal{J}_3} = e^{i\tilde{\omega}_0}. \tag{2.80}$$

The pure spinor $\rho_{\mathcal{J}_1}$ defines a stable generalized complex structure on $\mathbb{C}^2$, and it can be written up to a $B-$field transformation [3] as

$$\rho_{\mathcal{J}_1} = z_1 + dz_1 \wedge dz_2, \tag{2.81}$$

due to [16, Theorem 1.13]. Concerning $\rho_{\mathcal{J}_2}$, it provides a generalized complex structure on $\mathbb{C}^{2m-2}$ of symplectic type, since its underlying Poisson structure has constant rank $2m-2$ everywhere on $\mathbb{C}^{2m-2}$. Then by a $B-$transformation, (2.79) can be cast into

$$\rho_{\mathcal{J}_2} = e^{i\tilde{\omega}}, \tag{2.82}$$

with the standard symplectic form $\tilde{\omega}$ on $\mathbb{C}^{2m-2} \simeq \mathbb{R}^{4m-4}$. Putting (2.81), (2.82) and (2.80) all together, we obtain

$$\rho = (z_1 + dz_1 \wedge dz_2) \wedge e^{i\tilde{\omega}} \wedge e^{\tilde{\omega}_0} = (z_1 + dz_1 \wedge dz_2) \wedge e^{i\omega_0}, \tag{2.83}$$

where $\omega_0 = \tilde{\omega} + \tilde{\omega}_0$ is the standard symplectic form on

$$\mathbb{R}^{4m-4} \times \mathbb{R}^{2n-4m} \simeq \mathbb{R}^{2n-4}. \tag{2.84}$$

This completes the proof of the proposition. □

The following result immediately follows from Proposition 2.3.2.

Corollary 2.3.4. *The only possible type change jump of a stable generalized complex structure is* $0 \mapsto 2$.

[3] The $B-$field transformation of a pure spinor ρ is given by $e^B \wedge \rho$.

Regarding products of generalized complex structures, there is the following property.

Proposition 2.3.5 (Product of stable structures)**.** *Let* $(M_1, \mathcal{J}_1, H_1)$ *and* $(M_2, \mathcal{J}_2, H_2)$ *be stable generalized complex structures. Then the product generalized complex structure* $(M_1 \times M_2, \mathcal{J}_1 \times \mathcal{J}_2, p_1^* H_1 + p_2^* H_2)$ *with canonical projections* $p_i : M_1 \times M_2 \to M_i$ *for* $i = 1, 2$ *is stable if and only if at least one of type*$(\mathcal{J}_i)$ *is 0 everywhere.*

Proof. Denote by ρ_i the pure spinor that generates the canonical bundle of the generalized complex structure $(M_i, \mathcal{J}_i, H_i)$ for $i = 1, 2$. The pure spinor that defines the generalized complex structure

$$(M_1 \times M_2,\ \mathcal{J}_1 \times \mathcal{J}_2,\ p_1^* H_1 + p_2^* H_2) \tag{2.85}$$

is given by ([30, Example 4.12])

$$\rho = \rho_1 \wedge \rho_2. \tag{2.86}$$

Since ρ_i satisfies non-degeneracy and integrability conditions, ρ automatically becomes non-degenerate and integrable. By Proposition 2.2.3, we have

$$\mathrm{type}(\mathcal{J}_1 \times \mathcal{J}_2) = \mathrm{type}(\mathcal{J}_1) + \mathrm{type}(\mathcal{J}_2). \tag{2.87}$$

If $(M_1 \times M_2, \mathcal{J}_1 \times \mathcal{J}_2, p_1^* H_1 + p_2^* H_2)$ is stable and there is no type change locus, then

$$\mathrm{type}(\mathcal{J}_1 \times \mathcal{J}_2) = 0 \tag{2.88}$$

everywhere, which holds if and only if

$$\mathrm{type}(\mathcal{J}_1) = \mathrm{type}(\mathcal{J}_2) = 0 \tag{2.89}$$

from (2.87). On the other hand, if $(M_1 \times M_2, \mathcal{J}_1 \times \mathcal{J}_2, p_1^* H_1 + p_2^* H_2)$ is stable with nonempty type change locus, then

$$\mathrm{type}(\mathcal{J}_1 \times \mathcal{J}_2) = 2 \tag{2.90}$$

along the type change locus, which is true if and only if one of the following holds on the type change locus:

$$\mathrm{type}(\mathcal{J}_1) = \mathrm{type}(\mathcal{J}_2) = 1, \tag{2.91}$$

$$\mathrm{type}(\mathcal{J}_1) = 0 \ \text{ and } \ \mathrm{type}(\mathcal{J}_2) = 2, \tag{2.92}$$

$$\mathrm{type}(\mathcal{J}_1) = 2 \ \text{ and } \ \mathrm{type}(\mathcal{J}_2) = 0. \tag{2.93}$$

Since $(M_i, \mathcal{J}_i, H_i)$ is stable, type$(\mathcal{J}_i) \neq 1$ due to Corollary 2.3.4 and (2.91) is ruled out. The case that corresponds to (2.92) complies with type$(\mathcal{J}_1) = 0$ everywhere, while type$(\mathcal{J}_2) = 0$ everywhere for the case (2.93). This completes the proof.

□

Stable generalized complex structures closely resemble symplectic structures, in the sense that apart from the type change locus those are $B-$field equivalent to a symplectic structure. This allows us to introduce a $\mathcal{J}_\omega$-symplectic submanifold, by analogy with a symplectic submanifold. In [28], Goto - Hayano introduced the notion of $\mathcal{J}_\omega-$symplectic 2-torus and we generalize this notion as follows.

Definition 2.3.6 (cf. [28])**.** Let $(M, \mathcal{J}, H)$ be a stable generalized complex manifold. A submanifold $Y \subset M$ is said to be $\mathcal{J}_\omega-$symplectic if the following hold

- there exists a neighborhood $N_Y \subset M$ of Y equipped with a symplectic form ω such that Y is symplectic with respect to ω
- on N_Y, the generalized complex structure $(M, \mathcal{J}, H)$ is $B-$field equivalent to a symplectic type generalized complex structure $(N_Y, \mathcal{J}_\omega, 0)$ with

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{2.94}$$

Remark 2.3.7. Let us assume that the $\mathcal{J}_\omega-$symplectic submanifold Y has codimension two and trivial normal bundle. Let D^2_ϵ be a 2-disk with radius ϵ and equip it with the standard symplectic form ω_D. Then the manifold $D^2_\epsilon \times Y$ admits a symplectic structure given by the product symplectic form $p_1^*\omega_D + p_2^*\omega_Y$, where $p_1 : D^2_\epsilon \times Y \to D^2_\epsilon$, $p_2 : D^2_\epsilon \times Y \to Y$ are canonical projections. Since Y is a symplectic submanifold of N_Y, by Weinstein's neighborhood theorem (see [39]), there exists a symplectic embedding of $(D^2_\epsilon \times Y,\, p_1^*\omega_D + p_2^*\omega_Y)$ in (N_Y, ω) for sufficiently small $\epsilon > 0$.

2.4 A pair of unstable generalized complex structures

In the previous section, we discussed stable generalized complex structures, in which the only allowed type change jump is $0 \mapsto 2$. We begin this section by pointing out that a generalized complex structure of type change jump $0 \mapsto 2$ need not be stable.

Example 7. Let us consider the differential form

$$\rho = z^2 + dz \wedge dw \tag{2.95}$$

on $\mathbb{C}^2$ with complex coordinates (z, w). The form (2.95) is integrable with respect to $H = 0$ given that $d\rho = -2z\partial_w$, and it is non-degenerate since

$$(\rho, \bar{\rho}) = dz \wedge dw \wedge d\bar{z} \wedge d\bar{w} \neq 0. \tag{2.96}$$

We can also write (2.95) as

$$\rho = \begin{cases} dz \wedge dw & \text{for } \ z = 0 \\ z^2 e^{z^{-2} dz \wedge dw} & \text{for } \ z \neq 0, \end{cases} \tag{2.97}$$

which shows that the generalized complex structure determined by the pure spinor (2.95) has type change jump $0 \mapsto 2$ along $\{z = 0\}$. Also note that the degree 0 component of (2.95) is $\rho_0 = z^2$, so $d\rho_0 = 2z = 0$ along the type change locus $\{z = 0\}$. Therefore, the pure spinor (2.95) defines an unstable generalized complex structure.

Most generalized complex structures are unstable, and some of them can be constructed out of the stable ones. We show that a product of the stable generalized complex structures with nonempty type change locus gives rise to an unstable generalized complex structure in the example below.

Example 8. Let us consider $\mathbb{C}^4 \simeq \mathbb{C}^2 \times \mathbb{C}^2$ and denote complex coordinates on each $\mathbb{C}^2$ by (z_1, w_1) and (z_2, w_2), respectively. Take a differential form on $\mathbb{C}^4$ as

$$\rho = \rho_1 \wedge \rho_2 = (z_1 + dz_1 \wedge dw_1) \wedge (z_2 + dz_2 \wedge dw_2). \tag{2.98}$$

We can show that (2.98) is non-degenerate by using Example 5 and integrable since $d\rho = -(\partial_{w_1} + \partial_{w_2})\rho$ for $H = 0$.

We recast (2.98) as

$$\rho = \begin{cases} dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2 & \text{for } \ z_1 = z_2 = 0 \\ dz_1 \wedge dw_1 \wedge e^{z_2^{-1} dz_2 \wedge dw_2} & \text{for } \ z_1 = 0 \ \text{ and } \ z_2 \neq 0 \\ dz_2 \wedge dw_2 \wedge e^{z_1^{-1} dz_1 \wedge dw_1} & \text{for } \ z_2 = 0 \ \text{ and } \ z_1 \neq 0 \\ e^{z_1^{-1} dz_1 \wedge dw_1 + z_2^{-1} dz_2 \wedge dw_2} & \text{for } \ z_1 \neq 0 \ \text{ and } \ z_2 \neq 0. \end{cases} \tag{2.99}$$

from which we see that ρ determines a generalized complex structure with type change jumps: $0 \mapsto 2$, $2 \mapsto 4$ and $0 \mapsto 4$. Denoting the degree zero component of (2.98) by ρ_0, we have

$$d\rho_0 = d(z_1 z_2) = z_1 dz_2 + z_2 dz_1 = 0, \tag{2.100}$$

along the type change locus $\{z_1 = z_2 = 0\}$. Hence, this locus is degenerate and the generalized complex structure determined by (2.98) is unstable.

2.5 Topological obstructions to the existence of almost-complex and generalized almost-complex structures

In this section, we recall topological obstructions to the existence of an almost-complex structure, which turn out to be the same as the ones to the existence of a generalized almost-complex structure.

Definition 2.5.1. An almost-complex structure (M, J) is an endomorphism $J : TM \to TM$ such that $J^2 = -1$.

An almost-complex structure (M, J) implies that the tangent bundle TM is a complex vector bundle. Indeed, given an almost-complex structure (M, J), TM can be turned into a complex vector bundle by setting

$$(a + bi)X = aX + J(bX) \tag{2.101}$$

for any real numbers a, b and $X \in \Gamma(TM)$. We may then naturally assign the Chern class $c_k(M, J) = c_k(TM, J) \in H^{2k}(M; \mathbb{Z})$.

Any product of the Chern classes of total degree $\frac{1}{2}\dim_{\mathbb{R}} M$ can be integrated over M to yield the Chern number, and the number of the independent Chern numbers is equal to the number of partitions of $\frac{1}{2}\dim_{\mathbb{R}} M$.

Definition 2.5.2. An almost-complex structure (M, J) on a $2n-$manifold is the reduction of the structure group from $SO(2n)$ to $U(n)$.

Similarly, a generalized almost-complex structure can be defined in terms of the reduction of the structure group.

Definition 2.5.3 ([30, Proposition 4.6])**.** A generalized almost-complex structure $(M, \mathcal{J})$ on a $2n-$manifold is the reduction of the structure group from $SO(2n, 2n)$ to $U(n, n)$.

Proposition 2.5.4 ([30]). *A manifold M admits a generalized almost-complex structure if and only if it admits an almost-complex structure.*

Proof. Let us first prove that a generalized almost-complex manifold admits an almost-complex structure. Let $(M, \mathcal{J})$ be a generalized almost-complex $2n-$manifold. Then, we have the reduction of the structure group from $SO(2n, 2n)$ to $U(n, n)$ by Definition 2.5.3. Since $U(n, n)$ is homotopy equivalent to $U(n) \times U(n)$, it immediately follows that M admits an almost-complex structure by Definition 2.5.2. To prove the converse, let (M, J) be an almost-complex $2n-$manifold. We can simply construct the generalized almost-complex structure $(M, \mathcal{J})$ by setting $\mathcal{J} = \mathrm{diag}(J, -J^*)$. □

The necessary conditions for the existence of an almost-complex structure are stated in the following result (see [30, Section 4.2] for the proof).

Proposition 2.5.5 ([44]). *Let (M, J) be an almost-complex $2n-$manifold. Then the following hold:*

- *The odd Stiefel-Whitney classes of TM vanish.*
- *There exist classes $c_i \in H^{2i}(M; \mathbb{Z})$ for $i = 1, 2, \cdots, n$ whose mod 2 reductions are the even Stiefel-Whitney classes of TM, and c_n agrees with the Euler class of TM. In addition,*

  $$\sum_{i=0}^{[n/2]} (-1)^i p_i = \sum_{j=0}^{n} c_j \cup \sum_{k=0}^{n} (-1)^k c_k, \tag{2.102}$$

  where p_i denote the Pontrjagin classes of TM.

Sufficient conditions for the existence of an almost-complex structure are only known in several cases. Every orientable 2-manifold admits an almost complex structure [39, Example 4.3]. In dimension four, the following result of Wu provides the necessary and sufficient conditions for the existence of an almost-complex structure.

Lemma 2.5.6 ([50]). *A 4-manifold M admits an almost-complex structure with the first Chern class $h \in H^2(M; \mathbb{Z})$ if and only if*

$$h^2([M]) = 3\sigma(M) + 2\chi(M), \tag{2.103}$$

$$h \equiv w_2 \, (mod\ 2), \tag{2.104}$$

where $\sigma(M)$, $\chi(M)$ indicate the signature and the Euler Characteristic of M, respectively, and w_2 stands for the second Stiefel-Whitney class of TM.

The obstructions to the existence of an almost-complex structure on a $2n-$manifold M lie in the cohomology groups

$$H^{j+1}(M;\pi_j(SO(2n)/U(n)). \tag{2.105}$$

In the case of 6-manifolds, $SO(6)/U(3) = \mathbb{C}P^3$ and the only nontrivial homotopy group for $j \leq 5$ is $\pi_2(SO(6)/U(3)) = \mathbb{Z}$ [48]. The obstruction to the existence of an almost-complex structure is identified with the image of the second Stiefel-Whitney class w_2 under the Bockstein homomorphism $\beta : H^2(M;\mathbb{Z}/2) \to H^3(M;\mathbb{Z})$, which is the third integral Stiefel-Whitney class $W_3(M) = \beta w_2(M)$.

Theorem 2.5.7 ([48, Theorem 9], [41, Proposition 8])**.** *Let M be a closed orientable 6-manifold and suppose there is no 2-torsion in $H^3(M;\mathbb{Z})$. Then the following hold:*

- *There is an almost complex structure (M, J)*
- *There is a 1-1 correspondence between almost complex structures (M, J) and integral lift $W \in H^2(M;\mathbb{Z})$ of $w_2(M)$*
- *The Chern classes of (M, J_W) are given by*

$$\begin{aligned} c_1(M, J_W) &= W, \\ c_2(M, J_W) &= \frac{1}{2}(W^2 - p_1(M)), \\ c_3(M, J_W) &= e(M), \end{aligned} \tag{2.106}$$

where (M, J_W) denotes the almost complex manifold with the integral lift W of $w_2(M)$, and $p_1(M)$ and $e(M)$ indicate the first Pontryagin class and the Euler class of TM, respectively.

The following example points out the contrast of the existence of an almost-complex structure in dimension four and six.

Example 9 (S^4 and $S^4 \times S^2$)**.** Lemma 2.5.6 implies that the 4-sphere S^4 does not admit an almost complex structure. Indeed, $H^2(S^4;\mathbb{Z}) = 0$, so $\sigma(S^4) = 0$ and there is no class $h \in H^2(S^4;\mathbb{Z})$ whose square is $2\chi(S^4) = 4$. On the other hand, an explicit almost-complex structure can be constructed on $S^4 \times S^2$ as in [14] and [39, Example 4.4], since there is an embedding $S^4 \times S^2 \hookrightarrow \mathbb{R}^7 \subset \mathbb{R}^8$.

Chapter 3

Cut-and-paste constructions of stable generalized complex 6-manifolds

This chapter is devoted to the construction of stable generalized complex 6-manifolds. In Section 3.1, we reproduce Gompf's result [25] on the symplectic sum from the viewpoint of generalized complex structures. In Section 3.2, we describe the general procedure of a torus surgery and we find a torus surgery which produces a stable generalized complex 6-manifold with nonempty type change locus. In Section 3.3, we discuss a cut-and-paste construction along $T^2 \times S^2$ as a combination of a $C^\infty-$logarithmic transformation of multiplicity zero [15, 16] and a Gluck twist [24], to produce a stable generalized complex 6-manifold with nonempty type change locus. In Section 3.4, we provide a way to distinguish a closed orientable simply connected 6-manifold from a product of lower dimensional manifolds. In Section 3.5, we show that there exist infinitely many stable generalized complex 6-manifolds which are not a product of lower dimensional manifolds.

3.1 Symplectic sum

Gompf [25] and McCarthy - Wolfson [36] introduced the symplectic sum of two symplectic manifolds to produce a new symplectic manifold. Since stable generalized complex manifolds are everywhere symplectic apart from the type change locus, the symplectic sum can be employed to construct a new stable generalized complex manifold. In the following, we quote their result, in the language of generalized complex structures.

Theorem 3.1.1 ([25, Theorem 1.3], [36, Theorem 1.1])**.** *Let* $(X_i, \mathcal{J}_i, H_i)$ *be stable generalized complex* $2n-$*manifolds with* $t_i \in \mathbb{N} \sqcup \{0\}$ *path-connected components on the corresponding type change locus for* $i = 1, 2$. *Let* (Y, ω_Y) *be a closed symplectic* $(2n-2)$*-manifold. Assume that there are* $\mathcal{J}_\omega-$*symplectic embeddings* $j_i : Y \hookrightarrow X_i$ *with trivial normal bundles. Then the symplectic sum*

$$X = X_1 \#_Y X_2 := (X_1 - j_1(Y)) \cup_\varphi (X_2 - j_2(Y)) \tag{3.1}$$

admits a stable generalized complex structure $(X, \mathcal{J}, H)$ *with* $t_1 + t_2$ *path-connected components on its type change locus. Here the gluing map*

$$\varphi : \nu_1 - j_1(Y) \to \nu_2 - j_2(Y) \tag{3.2}$$

is chosen to be a symplectomorphism with ν_i *being small neighborhoods of* $j_i(Y)$ *such that it preserves the* S^1 *fibration when restricted to the boundary* $\partial(\nu_i - j_i(Y)) \simeq S^1 \times Y$.

Proof. We proceed following Gompf's arguments to find a symplectomorphism used to glue two pieces in (3.1). Let N_i denote the neighborhoods of $j_i(Y)$ and ω_i the symplectic forms on N_i. Since the embeddings $j_i : Y \hookrightarrow X_i$ are $\mathcal{J}_\omega-$symplectic, by Weinstein's neighborhood theorem, there exist symplectic embeddings

$$f_i : (D^2_\epsilon \times Y,\, p_1^*\omega_D + p_2^*\omega_Y) \to (N_i, \omega_i) \tag{3.3}$$

such that $f_i(\{0\} \times Y) = j_i(Y)$ (see Remark 2.3.7). Here D^2_ϵ stands for a 2-disk of radius ϵ and ω_D is the standard symplectic form on it, and p_i denote the canonical projections of $D^2_\epsilon \times Y$ to each factor. Denote the images of f_i in N_i by ν_i so that

$$f_i\left(\left(D^2_\epsilon - \{0\}\right) \times Y\right) = \nu_i - j_i(Y). \tag{3.4}$$

Choose two polar coordinates $(\tilde{r}, \tilde{\theta})$, (r, θ) on D^2_ϵ and two symplectic forms

$$\tilde{\omega}_D = \tilde{r} d\tilde{r} \wedge d\tilde{\theta} \tag{3.5}$$
$$\omega_D = r dr \wedge d\theta \tag{3.6}$$

and consider a self-diffeomorphism

$$g : D^2_\epsilon - \{0\} \to D^2_\epsilon - \{0\} \tag{3.7}$$

defined as

$$(\tilde{r}, \tilde{\theta}) = g(r, \theta) = (\sqrt{\epsilon^2 - r^2}, -\theta). \tag{3.8}$$

We can show that the diffeomorphism (3.8) is indeed a symplectomorphism by the following computation

$$\begin{aligned}
g^*\tilde{\omega}_D &= g^*(\tilde{r}d\tilde{r}\wedge d\tilde{\theta}) = g^*(\tilde{r})g^*(d\tilde{r})\wedge g^*(d\tilde{\theta}) \\
&= \sqrt{\epsilon^2 - r^2}\left[d\tilde{r}(g_*\partial_r)\,dr\wedge d\tilde{\theta}(g_*\partial_\theta)\,d\theta\right] \\
&= \sqrt{\epsilon^2 - r^2}\left[d\tilde{r}\left(\frac{-r\partial_{\tilde{r}}}{\sqrt{\epsilon^2 - r^2}}\right)\,dr\wedge d\tilde{\theta}(-\partial_{\tilde{\theta}})\,d\theta\right] \\
&= rdr\wedge d\theta = \omega_D. \qquad (3.9)
\end{aligned}$$

We immediately can define the self-diffeomorphism of $D^2_\epsilon \times Y$ by

$$\psi = g\times \mathrm{id} : (D^2_\epsilon \times Y,\, p_1^*\omega_D + p_2^*\omega_Y) \to (D^2_\epsilon \times Y,\, p_1^*\tilde{\omega}_D + p_2^*\omega_Y) \qquad (3.10)$$

so that ψ automatically becomes a symplectomorphism and preserves the S^1 fibration on the boundary $\partial\left(D^2_\epsilon \times Y\right) \simeq S^1\times Y$. Then the gluing map

$$\varphi : \nu_1 - j_1(Y) \to \nu_2 - j_2(Y) \qquad (3.11)$$

is defined by the commutative diagram

$$\begin{array}{ccc}
\left(D^2_\epsilon - \{0\}\right)\times Y & \xrightarrow{\psi} & \left(D^2_\epsilon - \{0\}\right)\times Y \\
\big\downarrow f_1 & & \big\downarrow f_2 \\
\nu_1 - j_1(Y) & \xrightarrow{\varphi} & \nu_2 - j_2(Y)
\end{array} \qquad (3.12)$$

so that φ becomes naturally a symplectomorphism and preserves S^1 fibration on the boundary $\partial(\nu_i - j_i(Y)) \simeq S^1 \times Y$.

The following discussions additionally arise from the aspects of generalized complex structures. Let ρ_i be the pure spinors that define the stable generalized complex structures $(X_i, \mathcal{J}_i, H_i)$. On the neighborhoods N_i, they can be written as

$$\rho_i = e^{B_i + i\omega_i}, \qquad (3.13)$$

where B_i are real 2-forms satisfying $H_i = -dB_i$ on N_i. On the overlap

$$(X_1 - j_1(Y)) \cap (X_2 - j_2(Y)) \simeq \left(D^2_\epsilon - \{0\}\right)\times Y, \qquad (3.14)$$

the gluing map (3.11) is a symplectomorphism, so we observe

$$\varphi^*\omega_2 = \omega_1. \qquad (3.15)$$

Let us define a 2-form $\tilde{B}$ on $X_1 - j_1(Y)$ such that

$$\tilde{B} = \begin{cases} \varphi^* B_2 - B_1 & \text{on } \nu_1 - j_1(Y) \\ 0 & \text{on } X_1 - \nu_1 \end{cases} . \tag{3.16}$$

Correspondingly, performing a $B-$field transformation on $(X_1 - j_1(Y), \mathcal{J}_1, H_1)$ by setting B to $\tilde{B}$, we obtain

$$\rho_1' = e^{B_1 + \tilde{B} + i\omega_1} \tag{3.17}$$

$$H_1' = H_1 - d\tilde{B} \tag{3.18}$$

$$\mathcal{J}_1' = e^{-\tilde{B}} \circ \mathcal{J}_1 \circ e^{\tilde{B}} \tag{3.19}$$

so that on the overlap of $X_1 - j_1(Y)$ and $X_2 - j_2(Y)$ we observe

$$\varphi^* \rho_2 = \rho_1' \tag{3.20}$$

$$\varphi^* H_2 = H_1'. \tag{3.21}$$

Thus, by the symplectomorphism φ we can glue the generalized complex manifolds $(X_1, \mathcal{J}_1', H_1')$ and $(X_2, \mathcal{J}_2, H_2)$ in such a way that on the overlap their structures coincide and away from the attaching area everything remains unchanged. As a consequence, we obtain the generalized complex structure $(X, \mathcal{J}, H)$ which coincides with $(X_1, \mathcal{J}_1', H')$ on $X_1 - j_1(Y)$ and with $(X_2, \mathcal{J}_2, H_2)$ on $X_2 - j_2(Y)$. Moreover, the number of path-connected components on the type change locus of $(X, \mathcal{J}, H)$ is $t_1 + t_2$, since the type change loci of $(X_1, \mathcal{J}_1, H_1)$ and $(X_2, \mathcal{J}_2, H_2)$ do not change under the gluing. □

3.2 Torus surgery

Torus surgeries have been employed in many literatures (see e.g. [4, 8, 10, 15, 16, 28, 45, 47]) to produce new examples of the manifolds equipped with a certain geometric structure. In particular, in the context of the generalized complex geometry, its application has been restricted to the dimension four [15, 16, 28, 45, 47]. In this section, we show that torus surgeries allow us to find a new stable generalized complex 6-manifold. For this, we first find a multiplicity zero torus surgery on a stable generalized complex 6-manifold to produce a new stable generalized complex 6-manifold and extend it to a torus surgery of arbitrary multiplicity.

Let T be a $(2n-2)-$torus embedded in a $2n-$manifold M and denote by N_T its tubular neighborhood. We assume that T has trivial normal bundle, so there is a diffeomorphism

$$N_T \simeq D^2 \times T^{2n-2}. \tag{3.22}$$

The torus surgery is composed of two operations: cutting out the tubular neighborhood N_T and pasting the thickened torus $D^2 \times T^{2n-2}$. Keeping this in mind, we in the following describe the technical side of the cut-and-past construction.

Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_{n-1}, \beta_{n-1}$ denote the 1-cycles generating the first homology group $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^{2n-2}$ and μ the meridian of T inside $M - N_T$. Let us consider the push offs $S^1_{\alpha_i}, S^1_{\beta_i} \subset \partial(M - N_T)$ of α_i, β_i for $i = 1, 2, \cdots n-1$, which are homologous to α_i, β_i in N_T. Then the set $\{S^1_{\alpha_1}, S^1_{\beta_1}, \cdots, S^1_{\alpha_{n-1}}, S^1_{\beta_{n-1}}, \mu\}$ spans the homology group

$$H_1(\partial N_T;\mathbb{Z}) \cong H_1(\partial D^2 \times T^{2n-2};\mathbb{Z}) \cong H_1(T^{2n-1};\mathbb{Z}) \cong \mathbb{Z}^{2n-1}. \tag{3.23}$$

Define a surgical curve

$$\gamma := p_1 S^1_{\alpha_1} q_1 S^1_{\beta_1} p_2 S^1_{\alpha_2} q_2 S^1_{\beta_2} \cdots p_{n-1} S^1_{\alpha_{n-1}} q_{n-1} S^1_{\beta_{n-1}}, \tag{3.24}$$

where $p_1, q_1, p_2, q_2, \cdots, p_{n-1}, q_{n-1} \in \mathbb{Z}$. A manifold obtained from M by the torus surgery on T along the curve γ is defined as

$$\hat{M} = (M - N_T) \cup_\phi (D^2 \times T^{2n-2}). \tag{3.25}$$

Here the gluing diffeomorphism

$$\phi : \partial D^2 \times T^{2n-2} \to \partial N_T \tag{3.26}$$

is chosen such that the induced homomorphism

$$SL(2n-1,\mathbb{Z}) \ni \phi_* : H_1(\partial D^2 \times T^{2n-2};\mathbb{Z}) \to H_1(\partial N_T;\mathbb{Z}) \tag{3.27}$$

satisfies

$$\phi_*[\partial D^2] = \sum_{i=1}^{n-1} (p_i[S^1_{\alpha_i}] + q_i[S^1_{\beta_i}]) + r[\mu], \tag{3.28}$$

where $r \in \mathbb{Z}$ is called multiplicity.

Proposition 3.2.1. *The Euler characteristic remains unchanged under torus surgery, i.e.,* $\chi(M) = \chi(\hat{M})$*.*

Proof. Let $A = M - N_T$ and $B = D^2 \times T^{2n-2}$, then we have $\hat{M} = A \cup_\phi B$ and $A \cap B \simeq T^{2n-1}$. Well known properties of the Euler characteristic imply that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \tag{3.29}$$

(see [42, Exercise 4.B.2]) and the Euler characteristic is multiplicative (see [32, Exercise 3.3.13]), it follows that

$$\begin{aligned}
\chi(\hat{M}) &= \chi(A) + \chi(B) - \chi(A \cap B) \\
&= \chi(M - N_T) + \chi(D^2 \times T^{2n-2}) - \chi(T^{2n-1}) \\
&= \chi(M - N_T) + \chi(D^2)\chi(T^{2n-2}) \\
&= \chi(M - N_T).
\end{aligned} \tag{3.30}$$

In the second and the third lines, we used $\chi(T^{2n-2}) = \chi(T^{2n-1}) = 0$. Since $M = (M - N_T) \cup N_T$, we have

$$\chi(M) = \chi(M - N_T) + \chi(N_T) - \chi(\partial N_T) \tag{3.31}$$

which yields

$$\begin{aligned}
\chi(M - N_T) &= \chi(M) - \chi(N_T) + \chi(\partial N_T) \\
&= \chi(M) - \chi(D^2 \times T^{2n-2}) + \chi(T^{2n-1}) \\
&= \chi(M).
\end{aligned} \tag{3.32}$$

Combining (3.30) and (3.32), we get $\chi(\hat{M}) = \chi(M)$. Hence, we complete the proof of Proposition 3.2.1. □

3.2.1 Multiplicity 0 torus surgery

We build on Cavalcanti and Gualtieri's work [15, 16] on the construction of stable generalized complex 4-manifolds and prove the following theorem.

Theorem 3.2.2. *Let $(M, \mathcal{J}_t, H)$ be a stable generalized complex 6-manifold with $t \in \mathbb{N} \sqcup \{0\}$ path-connected components on its type change locus. Assume that there is an embedded 4-torus $T \subset M$ with trivial normal bundle which is $\mathcal{J}_\omega-$symplectic. A multiplicity zero torus surgery on M along T produces a stable generalized complex 6-manifold $(\hat{M}, \hat{\mathcal{J}}_{t+1}, \hat{H})$ with $t+1$ path-connected components on the type change locus, each of which admits a constant type 1 generalized Calabi-Yau structure.*

Proof. Let N_T be a tubular neighborhood of T and ω its symplectic form. We denote by D^2_k a 2-disk with radius k. Since T is $\mathcal{J}_\omega-$symplectic with trivial normal bundle, we find a symplectic embedding of $(D^2_\epsilon \times T^4, \tilde{\omega})$ in (N_T, ω) by Weinstein's neighborhood theorem (see Remark 2.3.7), where $\tilde{\omega}$ is the product symplectic form. Denoting by $(\tilde{r}, \tilde{\theta}_1)$ the polar coordinates on D^2_ϵ and by $(\tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5)$ the standard angular coordinates on T^4, we can write $\tilde{\omega}$ as

$$\tilde{\omega} = \tilde{r} d\tilde{r} \wedge d\tilde{\theta}_1 + d\tilde{\theta}_2 \wedge d\tilde{\theta}_3 + d\tilde{\theta}_4 \wedge d\tilde{\theta}_5 \ . \tag{3.33}$$

Let ρ be a pure spinor corresponding to $(M, \mathcal{J}_t, H)$ and define

$$U_0 = M - \left(D^2_{\epsilon/\sqrt{2}} \times T^4 \right) \tag{3.34}$$

which automatically admits a stable generalized complex structure locally generated by the pure spinor

$$\rho_0 = \rho|_{U_0}. \tag{3.35}$$

Since $(M, \mathcal{J}_t, H)$ is $B-$field equivalent to a symplectic type generalized complex structure on N_T, we observe

$$\rho_0|_{\left(D^2_\epsilon - D^2_{\epsilon/\sqrt{2}}\right) \times T^4} = e^{\tilde{B} + i\tilde{\omega}} \tag{3.36}$$

with a 2-form $\tilde{B}$ satisfying $H = -d\tilde{B}$ on $\left(D^2_\epsilon - D^2_{\epsilon/\sqrt{2}} \right) \times T^4$.

We define

$$U_1 = D^2_1 \times T^4 \ , \tag{3.37}$$

and equip it with a stable generalized complex structure with a type change jump as follows. We first endow $\mathbb{C}^2 \times T^2$ with the stable generalized complex structure determined by the pure spinor (see Example 6)

$$\rho_1 = (z + dz \wedge dw) \wedge e^{i\omega_T} \ , \tag{3.38}$$

where (z, w) are the complex coordinates on $\mathbb{C}^2$ and ω_T is the symplectic form on T^2. Note that the pure spinor (3.38) is invariant under translations in the $w-$direction. Hence, by taking the quotient of $w-$plane by the lattice group $\Lambda(1, i)$, we can build the stable generalized complex structure generated by (3.38) on $D^2_1 \times T^4 \simeq D^2_1 \times T^2 \times T^2$, where the first 2-torus is defined by $T^2 \simeq \frac{\mathbb{C}}{\Lambda(1,i)}$. This stable generalized complex structure has type

2 along the core torus $\{0\} \times T^4 \subset D_1^2 \times T^4$ and type 0 elsewhere. Let us choose coordinates on $D_1^2 \times T^4$ as $((r, \theta_1), \theta_2, \theta_3, \theta_4, \theta_5)$ such that

$$z = re^{i\theta_1} \tag{3.39}$$

$$w = \theta_2 + i\theta_3 \tag{3.40}$$

$$\omega_T = d\theta_4 \wedge d\theta_5. \tag{3.41}$$

We can rewrite (3.38) in terms of the new coordinates $(r, \theta_1, \cdots, \theta_5)$ when $z \neq 0$ as

$$\rho_1 = re^{i\theta_1} e^{B+i(\sigma+\omega_T)} , \tag{3.42}$$

where B, σ are closed 2-forms defined as

$$B = d\log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 , \tag{3.43}$$

$$\sigma = d\log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2 . \tag{3.44}$$

The gluing diffeomorphism

$$\varphi : U_1 \supset (D_1^2 - D^2_{\sqrt{r_0}}) \times T^4 \to (D^2_\epsilon - D^2_{\epsilon/\sqrt{2}}) \times T^4 \subset U_0 \tag{3.45}$$

is given by

$$\begin{aligned}(\tilde{r}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5) &= \varphi\,(r, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \\ &= \left(\sqrt{\log{(r_0^{-2} r^2)}}, \theta_3, \theta_4, \theta_5, \theta_1, \theta_2\right) ,\end{aligned} \tag{3.46}$$

with $r_0 = e^{-\epsilon^2/2}$, by which we can attach U_1 to U_0 and produce a manifold

$$\hat{M} = U_0 \cup_\varphi U_1. \tag{3.47}$$

Remark 3.2.3. We might choose the symplectic form ω_T instead of (3.41) as

$$\omega_T = d\theta_5 \wedge d\theta_4 . \tag{3.48}$$

Accordingly, the gluing diffeomorphism

$$\varphi : U_1 \supset \left(D_1^2 - D^2_{\sqrt{r_0}}\right) \times T^4 \to \left(D^2_\epsilon - D^2_{\epsilon/\sqrt{2}}\right) \times T^4 \subset U_0 \tag{3.49}$$

is determined by

$$\begin{aligned}(\tilde{r}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5) &= \varphi(r, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \\ &= \left(\sqrt{\log{(r_0^{-2} r^2)}}, \theta_3, \theta_5, \theta_4, -\theta_2, \theta_1\right) .\end{aligned} \tag{3.50}$$

Note that the coordinates $\tilde{\theta}_1$ and θ_1 parametrize the meridians of T^4 in U_0 and U_1, respectively, and the diffeomorphisms (3.46) and (3.50) determine a torus surgery of multiplicity zero.

Now we need to show that $\hat{M}$ admits a well-defined stable generalized complex structure. For this, we use the following result.

Proposition 3.2.4. *The diffeomorphism* (3.46) *is indeed a symplectomorphism, i.e.,*

$$\varphi^* \tilde{\omega} = \sigma + \omega_T. \tag{3.51}$$

Proof. We kindly refer the reader who might not be familiar with the computation of pullback of differential forms to Appendix A. We first compute the pushforward of the vector fields $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_5}\}$ by the diffeomorphism (3.46) as

$$\begin{aligned}
\varphi_* \left(\frac{\partial}{\partial r} \right) &= \frac{1}{r \tilde{r}} \frac{\partial}{\partial \tilde{r}} &(3.52)\\
\varphi_* \left(\frac{\partial}{\partial \theta_1} \right) &= \frac{\partial}{\partial \tilde{\theta}_4} &(3.53)\\
\varphi_* \left(\frac{\partial}{\partial \theta_2} \right) &= \frac{\partial}{\partial \tilde{\theta}_5} &(3.54)\\
\varphi_* \left(\frac{\partial}{\partial \theta_3} \right) &= \frac{\partial}{\partial \tilde{\theta}_1} &(3.55)\\
\varphi_* \left(\frac{\partial}{\partial \theta_4} \right) &= \frac{\partial}{\partial \tilde{\theta}_2} &(3.56)\\
\varphi_* \left(\frac{\partial}{\partial \theta_5} \right) &= \frac{\partial}{\partial \tilde{\theta}_3}. &(3.57)
\end{aligned}$$

We use (3.52)-(3.57) into (A.2) and (A.3) to obtain the pullback of 1-forms $\{d\tilde{r}, d\tilde{\theta}_1, \cdots, d\tilde{\theta}_5\}$ as

$$\begin{aligned}
\varphi^*(d\tilde{r}) &= \frac{1}{r \tilde{r}} dr &(3.58)\\
\varphi^*(d\tilde{\theta}_1) &= d\theta_3 &(3.59)\\
\varphi^*(d\tilde{\theta}_2) &= d\theta_4 &(3.60)\\
\varphi^*(d\tilde{\theta}_3) &= d\theta_5 &(3.61)\\
\varphi^*(d\tilde{\theta}_4) &= d\theta_1 &(3.62)\\
\varphi^*(d\tilde{\theta}_5) &= d\theta_2. &(3.63)
\end{aligned}$$

From (3.58)-(3.63) and using Proposition A.0.1, we conclude

$$\varphi^*\tilde{\omega} = d\log r \wedge d\theta_3 + d\theta_4 \wedge d\theta_5 + d\theta_1 \wedge d\theta_2 = \sigma + \omega_T. \tag{3.64}$$

□

Remark 3.2.5. We can repeat the above calculation verbatim for the diffeomorphism (3.50) to show that it is also symplectomorphism. The arguments below do not rely on the explicit form of the symplectomorphism and the choice of ω_T.

In order to equip $\hat{M}$ with a well-defined generalized complex structure, not only do symplectic forms σ and $\tilde{\omega}$ must coincide in the overlap $U_0 \cap U_1$, but also the 2-forms B and $\tilde{B}$. This can be done by redefining the pure spinor ρ_0. Indeed, let $\tilde{B}_0$ be an extension of $\varphi^{-1*}B - \tilde{B}$ to U_0 and redefine ρ_0 as $e^{\tilde{B}_0}\rho_0$ so that

$$e^{\tilde{B}_0}\rho_0|_{U_0\cap U_1} = e^{\varphi^{-1*}B + i\tilde{\omega}}. \tag{3.65}$$

Hence, on $U_0 \cap U_1$ we immediately see that

$$\varphi^*(\varphi^{-1*}B + i\tilde{\omega}) = B + i\sigma \tag{3.66}$$

as desired. Now we define a closed 3-form $\hat{H}$ on $\hat{M}$ as

$$\hat{H}|_{U_0} = H|_{U_0} - d\tilde{B}_0 \tag{3.67}$$

$$\hat{H}|_{U_1} = 0, \tag{3.68}$$

which is well-defined since

$$(H|_{U_0} + d\tilde{B})|_{U_0\cap U_1} = -d\tilde{B} - d(\varphi^{-1*}B - \tilde{B}) = 0. \tag{3.69}$$

Thus, the pure spinor $\{e^{\tilde{B}}\rho_0, \rho_1\}$ determines a stable generalized complex structure $(\hat{M}, \hat{\mathcal{J}}_{t+1}, \hat{H})$. Moreover, given that the type change locus of $(M, \mathcal{J}_t, H)$ has t path-connected components, the number of path-connected components on the type change locus of $(\hat{M}, \hat{\mathcal{J}}_{t+1}, \hat{H})$ becomes $t+1$, since U_1 contains a type change locus $\{0\} \times T^4 \subset U_1$.

The type change locus of a stable generalized complex manifold admits a constant type 1 generalized Calabi-Yau structure inherited from the ambient generalized complex structure [17]. The pure spinor which gives a constant type 1 generalized Calabi-Yau structure on the newly added type change locus $\{0\} \times T^4 \subset U_1$ can be determined by the following result.

Proposition 3.2.6 ([17], Theorem 2.13)**.** *Let ρ be a pure spinor defining a stable generalized complex structure $(M, \mathcal{J}, H)$ with the type change locus D. Then D inherits a constant type 1 generalized Calabi-Yau structure determined by the pure spinor that is the residue of ρ, i.e.,*

$$\rho_D = \mathrm{Res}(\rho/\rho_0), \tag{3.70}$$

with ρ_0 being the degree 0 part of ρ.

Applying Proposition 3.2.6, the generalized Calabi-Yau form on the type change locus $T^4 \subset U_1 \subset \hat{M}$ is determined by using (3.38),(3.40) and (3.70) as

$$\rho_{T^4} = \mathrm{Res}(d \log z \wedge dw \wedge e^{i\omega_T}) = (d\theta_2 + i d\theta_3) \wedge e^{i\omega_T}. \tag{3.71}$$

Therefore, we complete the proof of Theorem 3.2.2. □

3.2.2 Torus surgery of arbitrary multiplicity

In [28], Goto - Hayano generalized the results of Cavalcanti - Gualtieri to torus surgeries of arbitrary multiplicity. We proceed to generalize the work done in Section 3.2.1 as follows.

Theorem 3.2.7. *Let $(M, \mathcal{J}_t, H)$ be a stable generalized complex 6-manifold with $t \in \mathbb{N} \sqcup \{0\}$ path-connected components on its type change locus. Assume that there is an embedded 4-torus $T \subset M$ with trivial normal bundle which is $\mathcal{J}_\omega-$symplectic. For any $m \in \mathbb{Z}$, the multiplicity m torus surgery on $(M, \mathcal{J}_t, H)$ along T produces a stable generalized complex manifold $(\hat{M}(m), \hat{\mathcal{J}}_{t+1}(m), \hat{H}(m))$ with $(t+1)$ path-connected components on the type change locus, each of which admits a constant type 1 generalized Calabi-Yau structure.*

Proof. The proof heavily relies on the Goto - Hayano's work [28]. To generalize a multiplicity zero torus surgery to arbitrary multiplicity, we tweak two things:

- the pure spinor that determines the type change generalized complex structure on U_1
- the symplectomorphism gluing two pieces U_0 and U_1

where U_0 and U_1 are defined as in Section 3.2.1. In the following, we will use the same notations as in Section 3.2.1, unless otherwise specified.

Let us choose complex coordinates (z_1, z_2, z_3) on $U_1 = D^2 \times T^2 \times T^2$ and consider a differential form given by (cf. [28])

$$\rho_1 = \left(z_1 - \frac{m}{2} f(|z_1|) dz_1 \wedge \frac{d\bar{z}_1}{\bar{z}_1} + \frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right) \wedge e^{-dz_3 \wedge d\bar{z}_3/2}, \tag{3.72}$$

where $m, a, p \in \mathbb{Z}$ with $ap \neq 0$[1] and $f : \mathbb{R} \to [0,1]$ is a monotonic increasing function satisfying $f(r) = 0$ for $r < \sqrt{r_0}$ and $f(r) = 1$ for $r \geq \sqrt{r_0}$. The differential form (3.72) is integrable, since

$$d\rho_1 = dz_1 \wedge e^{-dz_3 \wedge d\bar{z}_3/2} = -\frac{1}{a}(\partial_{z_2} - \partial_{\bar{z}_2}) \cdot \rho_1. \tag{3.73}$$

Also the following computation of the Mukai pairing $(\rho_1, \bar{\rho}_1)$ shows that the differential form (3.72) satisfies the non-degeneracy condition:

$$\begin{aligned}
(\rho_1, \bar{\rho}_1) &= (\mathcal{A}(\rho_1) \wedge \bar{\rho}_1)_{top} \\
&= \mathcal{A}\left(\frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right) \\
&\quad \wedge \left(\frac{1}{2}(a-p) d\bar{z}_1 \wedge d\bar{z}_2 - \frac{1}{2}(a+p) d\bar{z}_1 \wedge dz_2 \right) \wedge \left(-\frac{1}{2} d\bar{z}_3 \wedge dz_3 \right) \\
&\quad + \mathcal{A}\left[\left(\frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right) \right. \\
&\quad \left. \wedge \left(-\frac{1}{2} dz_3 \wedge d\bar{z}_3 \right) \right] \wedge \left(\frac{1}{2}(a-p) d\bar{z}_1 \wedge d\bar{z}_2 - \frac{1}{2}(a+p) d\bar{z}_1 \wedge dz_2 \right) \\
&= -\left(\frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right) \\
&\quad \wedge \left(\frac{1}{2}(a-p) d\bar{z}_1 \wedge d\bar{z}_2 - \frac{1}{2}(a+p) d\bar{z}_1 \wedge dz_2 \right) \wedge \left(-\frac{1}{2} d\bar{z}_3 \wedge dz_3 \right) \\
&\quad + \left(\frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right) \wedge \left(-\frac{1}{2} dz_3 \wedge d\bar{z}_3 \right) \\
&\quad \wedge \left(\frac{1}{2}(a-p) d\bar{z}_1 \wedge d\bar{z}_2 - \frac{1}{2}(a+p) d\bar{z}_1 \wedge dz_2 \right) \\
&= -2 \left(\frac{1}{2}(a-p) dz_1 \wedge dz_2 - \frac{1}{2}(a+p) dz_1 \wedge d\bar{z}_2 \right)
\end{aligned}$$

[1] It turns out that it should hold $ap = -1$ to glue U_0 and U_1 by an orientation preserving diffeomorphism(see (3.86)).

$$\wedge \left(\frac{1}{2}(a-p)d\bar{z}_1 \wedge d\bar{z}_2 - \frac{1}{2}(a+p)d\bar{z}_1 \wedge dz_2\right) \wedge \left(-\frac{1}{2}d\bar{z}_3 \wedge dz_3\right)$$
$$= -ap\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 \neq 0. \tag{3.74}$$

When $z_1 = 0$, the differential form (3.72) becomes

$$\rho_1 = \left(\frac{1}{2}(a-p)dz_1 \wedge dz_2 - \frac{1}{2}(a+p)dz_1 \wedge d\bar{z}_2\right) \wedge e^{-dz_3 \wedge d\bar{z}_3/2}, \tag{3.75}$$

and when $z_1 \neq 0$, it can be cast into

$$\rho_1 = z_1 \exp\Bigg(-\frac{m}{2} f(|z_1|) \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} + \frac{1}{2}(a-p)\frac{dz_1}{z_1} \wedge dz_2$$
$$- \frac{1}{2}(a+p)\frac{dz_1}{z_1} \wedge d\bar{z}_2 - \frac{1}{2}dz_3 \wedge d\bar{z}_3\Bigg)$$
$$= z_1 e^{B+i\sigma}. \tag{3.76}$$

In the new coordinates defined by

$$z_1 = re^{i\theta_1} \tag{3.77}$$
$$z_2 = \theta_2 + i\theta_3 \tag{3.78}$$
$$z_3 = \theta_4 + i\theta_5, \tag{3.79}$$

the real 2-forms B, σ are given by

$$B = -p\, d\log r \wedge d\theta_2 - a\, d\theta_1 \wedge d\theta_3, \tag{3.80}$$
$$\sigma = m\, d\log r \wedge d\theta_1 + a\, d\log r \wedge d\theta_3 - p\, d\theta_1 \wedge d\theta_2 + d\theta_4 \wedge d\theta_5. \tag{3.81}$$

From (3.73)-(3.76), we see that the differential form (3.72) is the pure spinor that determines the generalized complex structure on U_1 which has type 2 along $\{z_1 = 0\}$ locus, i.e., $\{0\} \times T^4 \subset U_1$ and has type 0 on $\{z_1 \neq 0\}$.

The gluing diffeomorphism

$$\varphi : U_1 \supset \left(D_1^2 - D_{\sqrt{r_0}}^2\right) \times T^4 \to \left(D_\epsilon^2 - D_{\epsilon/\sqrt{2}}^2\right) \times T^4 \subset U_0 \tag{3.82}$$

is given by

$$(\tilde{r}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5) = \varphi(r, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$$
$$= \left(\sqrt{\log\left(r_0^{-2} r^2\right)}, m\,\theta_1 + a\,\theta_3, \theta_4, \theta_5, -p\,\theta_1, \theta_2\right). \tag{3.83}$$

When restricted to the boundary, the diffeomorphism (3.83) boils down to the self-diffeomorphism $\phi : T^5 \to T^5$ which can be represented by

$$(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\theta}_5) = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)\, A_\phi, \tag{3.84}$$

where

$$A_\phi = \begin{pmatrix} m & 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.85}$$

The diffeomorphism ϕ preserves the orientation if and only if $\det A_\phi = 1$, yielding

$$-\,p\,a = 1 \tag{3.86}$$

which does not impose any constraints on the multiplicity m.

Proposition 3.2.8. *The gluing map φ given by* (3.83) *is a symplectomorphism, i.e., $\varphi^* \tilde{\omega} = \sigma$.*

Proof. We compute the pushforward of the vector fields $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_5}\}$ by the diffeomorphism φ given by (3.83) as

$$\varphi_* \left(\frac{\partial}{\partial r} \right) = \frac{1}{r \tilde{r}} \frac{\partial}{\partial \tilde{r}} \tag{3.87}$$

$$\varphi_* \left(\frac{\partial}{\partial \theta_1} \right) = m \frac{\partial}{\partial \tilde{\theta}_1} - p\, \frac{\partial}{\partial \tilde{\theta}_4} \tag{3.88}$$

$$\varphi_* \left(\frac{\partial}{\partial \theta_2} \right) = \frac{\partial}{\partial \tilde{\theta}_5} \tag{3.89}$$

$$\varphi_* \left(\frac{\partial}{\partial \theta_3} \right) = a\, \frac{\partial}{\partial \tilde{\theta}_1} \tag{3.90}$$

$$\varphi_* \left(\frac{\partial}{\partial \theta_4} \right) = \frac{\partial}{\partial \tilde{\theta}_2} \tag{3.91}$$

$$\varphi_* \left(\frac{\partial}{\partial \theta_5} \right) = a\, \frac{\partial}{\partial \tilde{\theta}_3}. \tag{3.92}$$

Using (3.87)-(3.92) in (A.2) and (A.3), we obtain the pullback of the 1-forms $\{d\tilde{r}, d\tilde{\theta}_1, \cdots, d\tilde{\theta}_5\}$ as

$$\varphi^*(d\tilde{r}) = \frac{1}{r\tilde{r}} dr \tag{3.93}$$

$$\varphi^*(d\tilde{\theta}_1) = m\,d\theta_1 + a\,d\theta_3 \tag{3.94}$$
$$\varphi^*(d\tilde{\theta}_2) = d\theta_4 \tag{3.95}$$
$$\varphi^*(d\tilde{\theta}_3) = d\theta_5 \tag{3.96}$$
$$\varphi^*(d\tilde{\theta}_4) = -p\,d\theta_1 \tag{3.97}$$
$$\varphi^*(d\tilde{\theta}_5) = d\theta_2. \tag{3.98}$$

From (3.33) and (3.93)-(3.98) and using Proposition A.0.1, we conclude

$$\varphi^*\tilde{\omega} = d\log r \wedge (m\,d\theta_1 + a\,d\theta_5) + p\,d\theta_4 \wedge d\theta_1 + d\theta_2 \wedge d\theta_3 = \sigma. \tag{3.99}$$

□

Finding a 2-form $\tilde{B}_0$ on U_0 which agrees with B on $U_0 \cap U_1$ and a well-defined closed 3-form $\hat{H}$ on $\hat{M}$ is exactly same as in Section 3.2.1, so we do not repeat it here. Accordingly, for each multiplicity m torus surgery, we obtain a stable generalized complex manifold $(\hat{M}(m), \hat{\mathcal{J}}_{t+1}(m), \hat{H}(m))$ with $t+1$ path-connected components on its type change locus, each of which admits a constant type 1 generalized Calabi-Yau structure due to Proposition 3.2.6. In particular, the pure spinor which generates a constant type 1 generalized Calabi-Yau structure on the newly produced type change component $\{0\} \times T^4 \subset U_1$ is determined by using (3.72) in (3.70) as

$$\begin{aligned}\rho_{T^4} &= \mathrm{Res}\left[\left(\frac{1}{2}(a-p)\,d\log z_1 \wedge dz_2 - \frac{1}{2}(a+p)\,d\log z_1 \wedge d\bar{z}_2\right) e^{-dz_3\wedge d\bar{z}_3/2}\right] \\ &= (ia\,d\theta_3 - p\,d\theta_2) \wedge e^{id\theta_4 \wedge d\theta_5}.\end{aligned} \tag{3.100}$$

Thus, we prove Theorem 3.2.7. □

We observe from (3.72), (3.80), (3.81), (3.83) and (3.100) that the result obtained in the present section recovers the one of Section 3.2.1 by choosing $\{m = 0, a = 1, p = -1\}$.

3.3 C^∞−logarithmic transformations and Gluck twists

In this section, we consider a cut-and-paste construction of stable generalized complex 6-manifolds with nonempty type change locus that is based in a combination of a C^∞−logarithmic transformation [15, 16] (or called a torus surgery in dimension 4 as already mentioned in Introduction section) and a Gluck twist [24]. Such 6-manifolds are not diffeomorphic to product manifolds.

We first describe Gluck twists (see [2, Section 6.1]). Let us choose coordinates on $S^1 \times S^2$ as $(\theta_0, (z, \theta_S))$ such that $\theta_0 \in S^1$ is standard angular coordinate and $(z, \theta_S) \in S^2$ are cylindrical coordinates. Let

$$\psi_{\theta_0} : S^1 \times S^2 \to S^1 \times S^2 \tag{3.101}$$

be an orientation preserving self-diffeomorphism defined by

$$\psi_{\theta_0} : (\theta_0, z, \theta_S) \mapsto (\theta_0, z, \theta_S + \theta_0) \tag{3.102}$$

which amounts to rotating S^2 about the $z-$axis by an angle $\theta_0 \in S^1$.

Definition 3.3.1. Let S be a 2-sphere embedded in a 4-manifold M with self-intersection zero and denote by $N_S \simeq D^2 \times S$ its tubular neighborhood. A Gluck twist of M is defined as

$$\widetilde{M} = (M - N_S) \cup_{\psi_{\theta_0}} D^2 \times S^2 \tag{3.103}$$

where the gluing map

$$\psi_{\theta_0} : \partial D^2 \times S^2 \simeq S^1 \times S^2 \to S^1 \times S^2 \simeq \partial N_S \tag{3.104}$$

is given by (3.102).

To proceed, recall that a gluing map of a multiplicity zero $C^\infty-$logarithmic transformation is an orientation preserving self-diffeomorphism

$$\psi_T : T^3 \to T^3 \tag{3.105}$$

determined by (see [15, Theorem 3.1])

$$\psi_T : (\theta_0, \theta_1, \theta_2) \mapsto (\theta_2, -\theta_1, \theta_0) \tag{3.106}$$

where $(\theta_0, \theta_1, \theta_2) \in \partial D^2 \times T^2 \simeq T^3$ are standard angular coordinates.

The six dimensional cut-and-paste construction is as follows. Equip the closed 4-manifold

$$Y = T^2 \times S^2 \tag{3.107}$$

with a product symplectic form

$$\omega_Y = d\theta_1 \wedge d\theta_2 + dz \wedge d\theta_S, \tag{3.108}$$

and let

$$\varphi : S^1 \times T^2 \times S^2 \to S^1 \times T^2 \times S^2 \tag{3.109}$$

be a self-diffeomorphism which is given by

$$\varphi : (\theta_0, \theta_1, \theta_2, z, \theta_S) \mapsto (\theta_2, -\theta_1, \theta_0, z, \theta_S + \theta_0). \tag{3.110}$$

We see that the diffeomorphism (3.110) is restricted to the diffeomorphism (3.102) of a Gluck twist on $S^1 \times \{pt\} \times S^2$, while it is restricted to the diffeomorphism (3.106) of a multiplicity zero $C^\infty-$logarithmic transformation on $S^1 \times T^2 \times \{pt\}$. Using the diffeomorphism (3.110), we obtain the following result.

Theorem 3.3.2. *Let (M, ω_M) be a closed symplectic 6-manifold. Assume that there exists a symplectic embedding of (Y, ω_Y) in (M, ω_M) with trivial normal bundle. Let $N_Y \simeq D^2 \times Y$ denote a tubular neighborhood of Y. Then the manifold*

$$\hat{M} = (M - N_Y) \cup_\varphi D^2 \times T^2 \times S^2 \tag{3.111}$$

admits a stable generalized complex structure with a type change locus which is diffeomorphic to $T^2 \times S^2$. Here the gluing diffeomorphism

$$\varphi : \partial N_Y \to \partial D^2 \times T^2 \times S^2 \tag{3.112}$$

is given by (3.110).

Proof. Since (Y, ω_Y) is symplectically embedded in (M, ω_M) with trivial normal bundle, by Weinstein's neighborhood theorem (see [39]) we find a symplectic embedding of $\left(D^2_\epsilon \times Y, \omega_D + \omega_Y\right)$ in (M, ω_M), where ω_D is standard symplectic form on D^2_ϵ and in standard polar coordinates $(r, \theta_0) \in D^2_\epsilon$, it is given by

$$\omega_D = dr \wedge d\theta_0. \tag{3.113}$$

Let

$$U_0 = M - \left(D^2_{\epsilon/\sqrt{2}} \times Y\right) \tag{3.114}$$

which can automatically be equipped with a constant type 0 generalized complex structure generated by a pure spinor

$$\rho_0 = e^{i\omega_M}|_{U_0}. \tag{3.115}$$

We observe

$$\rho_0|_{\left(D^2_\epsilon - D^2_{\epsilon/\sqrt{2}}\right) \times Y} = e^{i(\omega_D + \omega_Y)}. \tag{3.116}$$

Define

$$U_1 = D_1^2 \times T^2 \times S^2 \tag{3.117}$$

and choose complex coordinates $(u, v, \tau) \in D_1 \times T^2 \times S^2$. Consider a differential form [2]

$$\rho_1 = \left(u + du \wedge dv - \frac{2}{(1+|\tau|^2)^2} du \wedge (\bar{\tau} d\tau + \tau d\bar{\tau}) \right) \wedge e^{i\omega_{S^2}} \tag{3.118}$$

where ω_{S^2} is a symplectic form on S^2 given by (see [39, Exercise 4.23])

$$\omega_{S^2} = \frac{2i\, d\tau \wedge d\bar{\tau}}{(1+|\tau|^2)^2}. \tag{3.119}$$

We can rewrite ρ_1 as

$$\rho_1 = \begin{cases} \left(du \wedge dv - \frac{2}{(1+|\tau|^2)^2} du \wedge (\bar{\tau} d\tau + \tau d\bar{\tau}) \right) \wedge e^{i\omega_{S^2}} & \text{if } u = 0 \\ u \exp \left(\frac{du}{u} \wedge dv - \frac{2}{(1+|\tau|^2)^2} \frac{du}{u} \wedge (\bar{\tau} d\tau + \tau d\bar{\tau}) + i\omega_{S^2} \right) & \text{if } u \neq 0 \end{cases} \tag{3.120}$$

which shows that ρ_1 is indeed a pure spinor. Moreover, ρ_1 is integrable since

$$\begin{aligned} d\rho_1 &= \left(du - d \left(\frac{2}{(1+|\tau|^2)^2} du \wedge (\bar{\tau} d\tau + \tau d\bar{\tau}) \right) \right) \wedge e^{i\omega_{S^2}} \\ &= \left(du + \frac{2}{(1+|\tau|^2)^2} du \wedge (d\bar{\tau} \wedge d\tau + d\tau \wedge d\bar{\tau}) \right. \\ &\qquad \left. + \frac{4(\tau d\bar{\tau} + \bar{\tau} d\tau) \wedge du \wedge (\bar{\tau} d\tau + \tau d\bar{\tau})}{(1+|\tau|^2)^3} \right) \wedge e^{i\omega_{S^2}} \\ &= du \wedge e^{i\omega_{S^2}} = -\partial_v \cdot \rho_1. \end{aligned} \tag{3.121}$$

The non-degeneracy of the pure spinor ρ_1 can be shown by the following computation

$$\begin{aligned} (\rho_1, \bar{\rho}_1) &= (\mathcal{A}(\rho_1) \wedge \bar{\rho}_1)_{top} \\ &= \mathcal{A}(du \wedge dv \wedge i\omega_{S^2}) \wedge d\bar{u} \wedge d\bar{v} + \mathcal{A}(du \wedge dv) \wedge d\bar{u} \wedge d\bar{v} \wedge (-i\omega_{S^2}) \\ &= 2i\, du \wedge dv \wedge d\bar{u} \wedge d\bar{v} \wedge \omega_{S^2} \neq 0. \end{aligned} \tag{3.122}$$

Thus, the pure spinor ρ_1 defines a stable generalized complex structure on U_1 with type change locus $\{0\} \times T^2 \times S^2 \subset U_1$.

[2] One might think that the pure spinor could be chosen as the product of the pure spinor (2.60) and $e^{i\omega_{S^2}}$. However, it turns out that under the choice of this pure spinor the gluing map (3.110) can not be a symplectomorphism.

We introduce new coordinates on U_1 defined by

$$u = \tilde{r}e^{i\tilde{\theta}_0} \tag{3.123}$$
$$v = \tilde{\theta}_1 + i\tilde{\theta}_2 \tag{3.124}$$
$$\tau = \sqrt{\frac{1+\tilde{z}}{1-\tilde{z}}}\, e^{i\tilde{\theta}_S}. \tag{3.125}$$

In terms of these coordinates, we can rewrite ρ_1 as

$$\rho_1 = \tilde{r}e^{i\tilde{\theta}_0}e^{B+i\sigma} \tag{3.126}$$

when $\tilde{r} \neq 0$, where the real two forms B, σ are given by

$$\begin{aligned} B &= d\log\tilde{r} \wedge d\tilde{\theta}_1 - d\tilde{\theta}_0 \wedge d\tilde{\theta}_2 + d\tilde{z} \wedge d\log\tilde{r}, \\ \sigma &= d\log\tilde{r} \wedge d\tilde{\theta}_2 + d\tilde{\theta}_0 \wedge d\tilde{\theta}_1 + d\tilde{z} \wedge d\tilde{\theta}_0 + d\tilde{z} \wedge d\tilde{\theta}_S. \end{aligned} \tag{3.127}$$

Now we glue U_1 in U_0 by a diffeomorphism

$$\varphi : U_1 \supset \left(D_1^2 - D^2_{\sqrt{\tilde{r}_0}}\right) \times T^2 \times S^2 \to \left(D^2_\epsilon - D^2_{\epsilon/\sqrt{2}}\right) \times T^2 \times S^2 \subset U_0 \tag{3.128}$$

represented as

$$\begin{aligned} (r, \theta_0, \theta_1, \theta_2, z, \theta_S) &= \varphi(\tilde{r}, \tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{z}, \tilde{\theta}_S) \\ &= \left(\sqrt{\log(\tilde{r}_0^{-2}\tilde{r}^2)}, \tilde{\theta}_2, -\tilde{\theta}_1, \tilde{\theta}_0, \tilde{z}, \tilde{\theta}_S + \tilde{\theta}_0\right), \end{aligned} \tag{3.129}$$

where $\tilde{r}_0 = e^{-\epsilon^2/2}$.

Lemma 3.3.3. *The gluing diffeomorphism φ is a symplectomorphism, i.e., $\varphi^*(\omega_D + \omega_Y) = \sigma$.*

Proof. The pushforward of the vector fields $\left\{\frac{\partial}{\partial\tilde{r}}, \frac{\partial}{\partial\tilde{\theta}_0}, \frac{\partial}{\partial\tilde{\theta}_1}, \frac{\partial}{\partial\tilde{\theta}_2}, \frac{\partial}{\partial\tilde{z}}, \frac{\partial}{\partial\tilde{\theta}_S}\right\}$ are determined by

$$\varphi_*\left(\frac{\partial}{\partial\tilde{r}}\right) = \frac{1}{\tilde{r}r}\frac{\partial}{\partial r} \tag{3.130}$$
$$\varphi_*\left(\frac{\partial}{\partial\tilde{\theta}_0}\right) = \frac{\partial}{\partial\theta_2} + \frac{\partial}{\partial\theta_S} \tag{3.131}$$
$$\varphi_*\left(\frac{\partial}{\partial\tilde{\theta}_1}\right) = -\frac{\partial}{\partial\theta_1} \tag{3.132}$$

$$\varphi_*\left(\frac{\partial}{\partial\tilde{\theta}_2}\right) = \frac{\partial}{\partial\theta_0} \tag{3.133}$$

$$\varphi_*\left(\frac{\partial}{\partial\tilde{z}}\right) = \frac{\partial}{\partial z} \tag{3.134}$$

$$\varphi_*\left(\frac{\partial}{\partial\tilde{\theta}_S}\right) = \frac{\partial}{\partial\theta_S}. \tag{3.135}$$

Hence, the pullback of the 1-forms $\{dr, d\theta_0, d\theta_1, d\theta_2, dz, d\theta_S\}$ are given by

$$\varphi^*(dr) = \frac{1}{r\tilde{r}}d\tilde{r} \tag{3.136}$$

$$\varphi^*(d\theta_0) = d\tilde{\theta}_2 \tag{3.137}$$

$$\varphi^*(d\theta_1) = -d\tilde{\theta}_1 \tag{3.138}$$

$$\varphi^*(d\theta_2) = d\tilde{\theta}_0 \tag{3.139}$$

$$\varphi^*(dz) = d\tilde{z} \tag{3.140}$$

$$\varphi^*(d\theta_S) = d\tilde{\theta}_0 + d\tilde{\theta}_S. \tag{3.141}$$

Using (3.136)-(3.140), it is straightforward to check that

$$\varphi^*(\omega_D + \omega_Y) = \sigma. \tag{3.142}$$

□

As was shown in Section 3.2.1, we can redefine ρ_0 by a $B-$field transformation (see the exposition below Remark 3.2.5), so that on the overlap $U_0 \cap U_1$ two stable generalized complex structures determined by ρ_0 and ρ_1 coincide due to Lemma 3.3.3. As a consequence, the manifold $\hat{M} = U_0 \cup_\varphi U_1$ admits a stable generalized complex structure with the type change locus $\{0\} \times T^2 \times S^2 \subset U_1 \subset \hat{M}$. □

Remark 3.3.4. Though we have discussed the surgery along $T^2 \times S^2$ on a closed symplectic 6-manifold for simplicity, this surgery in principle can be generalized to a stable generalized complex 6-manifold which contains a $\mathcal{J}_\omega-$symplectic embedding of $T^2 \times S^2$ (cf. Section 3.2.1).

The cut-and-paste construction that we discussed in this section produces a stable generalized complex 6-manifold with the type change locus being diffeomorphic to $T^2 \times S^2$, which is not a product generalized complex manifold due to the following result.

Lemma 3.3.5. *Let $M = X^4 \times S^2$ be a closed orientable 6-manifold such that there is an embedded 2-torus $T \subset X^4$ with self-intersection zero. Then the manifold defined by* (3.111) *is a S^2-bundle over X^4 with the second Stiefel-Whitney class $w_2 \neq 0$.*

3.4 Identification of closed orientable simply connected 6-manifolds

After we construct a stable generalized complex 6-manifold by the cut-and-paste constructions discussed in the previous section, we need to identify it topologically in order to show that it is a new example of stable generalized complex 6-manifolds. We are interested in a stable generalized complex manifold $(M, \mathcal{J}, H)$ which is not topologically a product of lower dimensional manifolds, i.e.,

$$M \neq \begin{cases} X^5 \times S^1 \\ N^4 \times \Sigma_g \\ Y_1^3 \times Y_2^3 \end{cases}, \tag{3.143}$$

where all the manifolds are assumed to be closed orientable and the superscript indicates the dimension of the manifold and Σ_g is a genus g surface. The following result can be used to distinguish a closed orientable simply connected 6-manifold from a product of lower dimensional manifolds.

Proposition 3.4.1. *Let M be a closed orientable simply connected 6-manifold with zero Euler characteristic $\chi(M) = 0$. Then M is not homologically equivalent to either $X^5 \times S^1$ or $N^4 \times \Sigma_g$. If M further satisfies $H_3(M; \mathbb{Z}) \ncong \mathbb{Z}^2$, M is not homologically equivalent to $Y_1^3 \times Y_2^3$ either.*

Proof. We proceed by contradiction case by case.

- Suppose that M is homologically equivalent to $X^5 \times S^1$, then its fundamental group is ([34, Proposition 1.12])

  $$\pi_1(X^5 \times S^1) \cong \pi_1(X^5) \times \pi_1(S^1) = \pi_1(X^5) \times \mathbb{Z}. \tag{3.144}$$

  This contradicts the initial hypothesis of $\pi_1(M) \cong \{1\}$. Thus, M is not homologically equivalent to a product manifold $X^5 \times S^1$.

- Assume M is homologically equivalent to a product $N^4 \times \Sigma_g$. Using [34, Proposition 1.12], the fundamental group

  $$\pi_1(M) \cong \pi_1(N^4) \times \pi_1(\Sigma_g), \tag{3.145}$$

which is trivial by hypothesis. Thus, (3.145) implies $\pi_1(N^4) \cong \pi_1(\Sigma_g) \cong \{1\}$, hence Σ_g is homeomorphic to S^2 by the classification theorem of closed surfaces (see [37, Theorem 4.14]). Since the Euler characteristic is multiplicative (see [32, Exercise 3.3.13]), the Euler characteristic of M is

$$\chi(M) = \chi(N^4 \times S^2) = \chi(N^4) \times \chi(S^2) = 2\chi(N^4). \tag{3.146}$$

The Euler characteristic of N^4 is

$$\chi(N^4) = \sum_{i}^{4} (-1)^i b_i(N^4) \tag{3.147}$$

by definition. We claim $\chi(N^4) > 0$; along with (3.146), and the hypothesis $\chi(M) = 0$, this yields the desired contradiction. Since N^4 is connected, then $H_0(N^4; \mathbb{Z}) = \mathbb{Z}$. The abelianization of the fundamental group of N^4 is $H_1(N^4; \mathbb{Z})$. Since $\pi_1(M) = \{1\}$, then $H_1(N^4; \mathbb{Z}) = 0$. The manifold N^4 is closed and oriented, thus $H_4(N^4; \mathbb{Z}) = \mathbb{Z}$ and Poincaré duality can be applied. Since there is no torsion, we conclude

$$0 = H_1(N^4; \mathbb{Z}) = H^1(N^4; \mathbb{Z}) = H_3(N^4; \mathbb{Z}) \tag{3.148}$$

using the universal coefficients theorem and Poincaré duality. Thus (3.147) becomes $\chi(N^4) = 2 + b_2(N^4)$, and $\chi(N^4) > 0$ as it was claimed.

- Finally, M is homologically equivalent to a product of 3-manifolds $Y_1^3 \times Y_2^3$. The hypothesis of $\pi_1(M) \cong \{1\}$ implies

$$\pi_1(Y_1^3) \cong \pi_1(Y_2^3) \cong \{1\}, \tag{3.149}$$

since $\pi_1(M) \cong \pi_1(Y_1^3) \times \pi_1(Y_2^3)$ by [34, Proposition 1.12]. Hence, Y_1 and Y_2 are homotopy equivalent to the 3-sphere S^3 (see [34, Exercise 4.2.15]), so

$$H_k(M; \mathbb{Z}) \cong H_k(S^3 \times S^3; \mathbb{Z}). \tag{3.150}$$

We now compute $H_3(M; \mathbb{Z})$ using the Künneth formula ([13, Theorem 1.6])

$$H_3(M; \mathbb{Z}) = \bigoplus_{i=0}^{3} H_i(S^3; \mathbb{Z}) \otimes H_{3-i}(S^3; \mathbb{Z}). \tag{3.151}$$

Since the only nontrivial homology groups of S^3 are $H_0(S^3; \mathbb{Z}) = \mathbb{Z}$ and $H_3(S^3; \mathbb{Z}) = \mathbb{Z}$, (3.151) boils down to

$$H_3(M; \mathbb{Z}) = H_0(S^3; \mathbb{Z}) \otimes H_3(S^3; \mathbb{Z}) \oplus H_3(S^3; \mathbb{Z}) \otimes H_0(S^3; \mathbb{Z}) \cong \mathbb{Z}^2.$$

(3.152)

This contradicts the hypothesis $H_3(M;\mathbb{Z}) \not\cong \mathbb{Z}^2$. Thus, M is not homologically equivalent to a product of 3-manifolds.

□

Proposition 3.4.1 provides a way to distinguish a closed orientable simply connected 6-manifold from a product of lower dimensional manifolds, but it does not identify the 6-manifold completely. Wall's theorem [48] is used to identify the diffeomorphism type of closed orientable simply connected 6-manifolds, which we quote below.

Theorem 3.4.2. ([48, Theorem 1]) *Let M be a closed orientable simply connected 6-manifold. Then there exists a 6-manifold M_0 with finite $H_3(M_0;\mathbb{Z})$ such that*

$$M \simeq M_0 \# \frac{b_3(M)}{2}(S^3 \times S^3). \tag{3.153}$$

3.5 Examples of stable generalized complex 6-manifolds

In this section, we provide examples of simply connected 6-manifolds equipped with a stable generalized complex structure with nonempty type change locus by using the cut-and-paste construction described in Section 3.2.1. More precisely, we apply two torus surgeries simultaneously to a closed symplectic 6-manifold to construct a stable generalized complex structure with the type change jump on a closed simply connected 6-manifold that is not homologically equivalent to a product of lower dimensional manifolds.

The following result provides a manifold which is used as the building block in our construction of a stable generalized complex 6-manifold.

Proposition 3.5.1 ([9, Theorem 18, Proposition 12])**.** *There exists a closed simply connected symplectic 4-manifold (B, ω_B) which is homeomorphic but not diffeomorphic to $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ such that:*

- *The second homology group $H_2(B;\mathbb{Z})$ is spanned by six 2-tori T_1, T_2, $\tilde{T}_1$, $\tilde{T}_2$, H_1, H_2 and a genus-2 surface F and a genus-3 surface H_3. The intersection form Q_B is a direct sum of two 2-dimensional summands generated by the pairs $T_i, \tilde{T}_i$ with $i = 1, 2$ and one 4-dimensional* *summand generated by* H_1, H_2, H_3, F *such that*

$$Q_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{3.154}$$

- *The tori* T_1 *and* T_2 *can be assumed to be symplectic and*

$$\pi_1(B - (T_1 \sqcup T_2)) \cong \pi_1(B) \cong \{1\}. \tag{3.155}$$

Taking a product of (B, ω_B) with a symplectic 2-torus (T^2, ω_{T^2}), we construct a symplectic 6-manifold (M, ω_M), where $\omega_M = p_1^* \omega_B + p_2^* \omega_{T^2}$ with the canonical projections $p_1 : M \to B$ and $p_2 : M \to T^2$. By Proposition 3.5.1, $T_i \subset B$ with $i = 1, 2$ have self-intersection zero, so their normal bundles are trivial. This allows us to find codimension 2 submanifolds $T_i \times T^2$ in M with trivial normal bundles and tubular neighborhoods $N_{T_i} \simeq D^2 \times T_i \times T^2$. Note that the submanifolds $T_i \times T^2$ are disjoint and symplectic with respect to the symplectic structure ω_M on M, since T_i are symplectic submanifolds of (B, ω_B) by Proposition 3.5.1. By performing simultaneously two torus surgeries on M, we obtain a 6−manifold

$$\hat{M} = (M - (N_{T_1} \sqcup N_{T_2})) \cup_\varphi \left((D^2 \times T_1^4) \sqcup (D^2 \times T_2^4) \right), \tag{3.156}$$

where the gluing map φ consists of two orientation preserving diffeomorphisms

$$\varphi_1 : \partial D^2 \times T_1^4 \to \partial N_{T_1} \tag{3.157}$$

$$\varphi_2 : \partial D^2 \times T_2^4 \to \partial N_{T_2} \tag{3.158}$$

determined by (3.46) and (3.50), respectively. Careful readers may have doubt about two different choices of gluing diffeomorphisms. As it will become clear later, this allows us to kill two generators of the fundamental group of $\{pt\} \times T^2 \subset B \times T^2 = M$.

Choose coordinates $(\tilde{\theta}_{i;1}, \tilde{\theta}_{i;2}, \tilde{\theta}_{i;3}, \tilde{\theta}_4, \tilde{\theta}_5)$ on $\partial N_{T_i} \simeq \partial D^2 \times T_i \times T^2$ and $(\theta_{i;1}, \theta_{i;2}, \theta_{i;3}, \theta_{i;4}, \theta_{i;5})$ on $\partial D^2 \times T_i^4$ such that $\tilde{\theta}_{i;1} \in \partial D^2$, $(\tilde{\theta}_{i;2}, \tilde{\theta}_{i;3}) \in T_i$, $(\tilde{\theta}_4, \tilde{\theta}_5) \in T^2$ and $\theta_{i;1} \in \partial D^2$, $(\theta_{i;2}, \theta_{i;3}, \theta_{i;4}, \theta_{i;5}) \in T_i^4$. The diffeomorphisms φ_i explicitly read

$$\begin{aligned} (\tilde{\theta}_{1;1}, \tilde{\theta}_{1;2}, \tilde{\theta}_{1;3}, \tilde{\theta}_4, \tilde{\theta}_5) &= \varphi_1(\theta_{1;1}, \theta_{1;2}, \theta_{1;3}, \theta_{1;4}, \theta_{1;5}) \\ &= (\theta_{1;3}, \theta_{1;4}, \theta_{1;5}, \theta_{1;1}, \theta_{1;2}), \end{aligned} \tag{3.159}$$

$$\begin{aligned}(\tilde\theta_{2;1},\tilde\theta_{2;2},\tilde\theta_{2;3},\tilde\theta_4,\tilde\theta_5) &= \varphi_2(\theta_{2;1},\theta_{2;2},\theta_{2;3},\theta_{2;4},\theta_{2;5})\\ &= (\theta_{2;3},\theta_{2;5},\theta_{2;4},-\theta_{2;2},\theta_{2;1}). \qquad (3.160)\end{aligned}$$

Then the manifold $\hat{M}$ admits the stable generalized complex structure with two path-connected components $\{0\}\times T_i\times T^2 \subset \hat{M}$ on the type change locus according to Theorem 3.2.2.

In order to identify the manifold $\hat{M}$, we have to know its topological invariants. To this end, we compute the homology groups of M and $\hat{M}$.

Proposition 3.5.2. *The 6-manifolds $M = B\times T^2$ and $\hat{M}$ determined by* (3.156) *have the following homology groups:*

(a)

$$H_0(M;\mathbb{Z}) = H_6(M;\mathbb{Z}) = \mathbb{Z} \qquad (3.161)$$
$$H_1(M;\mathbb{Z}) = H_5(M;\mathbb{Z}) = \mathbb{Z}^2 \qquad (3.162)$$
$$H_2(M;\mathbb{Z}) = H_4(M;\mathbb{Z}) = \mathbb{Z}^9 \qquad (3.163)$$
$$H_3(M;\mathbb{Z}) = \mathbb{Z}^{16}. \qquad (3.164)$$

(b)

$$H_0(\hat{M};\mathbb{Z}) = H_6(\hat{M};\mathbb{Z}) = \mathbb{Z} \qquad (3.165)$$
$$H_1(\hat{M};\mathbb{Z}) = H_5(\hat{M};\mathbb{Z}) = 0 \qquad (3.166)$$
$$H_2(\hat{M};\mathbb{Z}) = H_4(\hat{M};\mathbb{Z}) = \mathbb{Z}^{10} \qquad (3.167)$$
$$H_3(\hat{M};\mathbb{Z}) = \mathbb{Z}^{22}. \qquad (3.168)$$

Proof. We start with the computation of the homology groups of M. Since M is connected, $H_0(M;\mathbb{Z}) = \mathbb{Z}$. Given that M is closed and orientable, $H_6(M;\mathbb{Z}) = \mathbb{Z}$ and Poincaré duality can be applied. By Proposition 3.5.1, B is simply connected, so using [34, Proposition 1.12],

$$\pi_1(M) \cong \pi_1(B)\times\pi_1(T^2) \cong \mathbb{Z}^2, \qquad (3.169)$$

which yields

$$H_1(M;\mathbb{Z}) = \mathbb{Z}^2, \qquad (3.170)$$

since $H_1(M;\mathbb{Z})$ is the abelianization of $\pi_1(M)$. Poincaré duality and the universal coefficients theorem imply

$$H_5(M;\mathbb{Z}) = H^1(M;\mathbb{Z}) = H_1(M;\mathbb{Z}) = \mathbb{Z}^2. \qquad (3.171)$$

The homology groups $H_2(M;\mathbb{Z})$ and $H_3(M;\mathbb{Z})$ are computed by using the Künneth formula (see e.g. [13, Theorem 1.6]) as follows:

$$\begin{aligned}
H_2(M;\mathbb{Z}) &= H_2(B;\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \oplus H_1(B;\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\
&\quad \oplus H_0(B;\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \\
&= \mathbb{Z}^8 \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}^2 \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}^8 \oplus \mathbb{Z} = \mathbb{Z}^9 \qquad (3.172) \\
H_3(M;\mathbb{Z}) &= H_3(B;\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \oplus H_2(B;\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\
&\quad \oplus H_1(B;\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \oplus H_0(B;\mathbb{Z}) \otimes H_3(T^2;\mathbb{Z}) \\
&= 0 \otimes \mathbb{Z} \oplus \mathbb{Z}^8 \otimes \mathbb{Z}^2 \oplus 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes 0 = \mathbb{Z}^{16}. \qquad (3.173)
\end{aligned}$$

In the above computations, we used $H_0(B;\mathbb{Z}) = \mathbb{Z}$, $H_1(B;\mathbb{Z}) = H_3(B;\mathbb{Z}) = 0$ and $H_2(B;\mathbb{Z}) = \mathbb{Z}^8$ due to Proposition 3.5.1. Applying Poincaré duality and the universal coefficients theorem, we obtain

$$H_4(M;\mathbb{Z}) = H^2(M;\mathbb{Z}) = H_2(M;\mathbb{Z}) = \mathbb{Z}^9. \qquad (3.174)$$

Thus, we prove claim (a) of Proposition 3.5.2.

Let us proceed with the computation of the homology groups of $\hat{M}$. It follows from the connectedness of $\hat{M}$ that $H_0(\hat{M};\mathbb{Z}) = \mathbb{Z}$. Moreover, $\hat{M}$ is closed and orientable, so $H_6(\hat{M};\mathbb{Z}) = \mathbb{Z}$ and Poincaré duality can be applied. In order to compute the rest of the homology groups, we use the following result.

Lemma 3.5.3. *$\hat{M}$ is simply connected, i.e., $\pi_1(\hat{M}) \cong \{1\}$.*

Proof. For the notational convenience, we define

$$X = M - (N_{T_1} \sqcup N_{T_2}) \simeq B \times T^2 - D^2 \times (T_1 \sqcup T_2) \times T^2, \qquad (3.175)$$

$$Y = Y_1 \sqcup Y_2 = \left(D^2 \times T_1^4\right) \sqcup \left(D^2 \times T_2^4\right), \qquad (3.176)$$

so that

$$\hat{M} = X \cup_\varphi Y, \qquad (3.177)$$

$$X \cap Y = (X \cap Y_1) \sqcup (X \cap Y_2) \simeq T_1^5 \sqcup T_2^5. \qquad (3.178)$$

Since $X \cap Y$ is disconnected, we can not directly apply Seifert-van Kampen theorem to $\hat{M}$. However, we can facilitate the use of Seifert-van Kampen theorem by attaching Y_1 and Y_2 to X one by one, viewing

$$\hat{M} = X \cup_{\varphi_1} Y_1 \cup_{\varphi_2} Y_2, \qquad (3.179)$$

in which φ_1 and φ_2 are the diffeomorphisms given by (3.159) and (3.160). When we glue Y_1 in X, Seifert-van Kampen theorem yields

$$\pi_1(X \cup_{\varphi_1} Y_1) \cong \frac{\pi_1(X) * \pi_1(Y_1)}{N_1}, \tag{3.180}$$

with the normal subgroup $N_1 \subset \pi_1(X) * \pi_1(Y_1)$. The generators of N_1 are of the form $i_{X_*}(a)\, i_{Y_{1*}}^{-1}(a)$, where

$$i_{X_*} : \pi_1(X \cap Y_1) \to \pi_1(X), \tag{3.181}$$

$$i_{Y_{1*}} : \pi_1(X \cap Y_1) \to \pi_1(Y_1) \tag{3.182}$$

are group homomorphisms induced by the inclusions $i_X : X \cap Y_1 \hookrightarrow X$ and $i_{Y_1} : X \cap Y_1 \hookrightarrow Y_1$. Using [34, Proposition 1.12] and Proposition 3.5.1, we compute

$$\begin{aligned} \pi_1(X) &= \pi_1((B \times T^2 - D^2 \times (T_1 \sqcup T_2) \times T^2)) \\ &\cong \pi_1(B) \times \pi_1(T^2) \cong \{1\} \times \langle \tilde{\gamma}, \tilde{\delta} \,|\, [\tilde{\gamma}, \tilde{\delta}] \rangle \cong \langle \tilde{\gamma}, \tilde{\delta} \,|\, [\tilde{\gamma}, \tilde{\delta}] \rangle \end{aligned} \tag{3.183}$$

where

$$\tilde{\gamma}, \tilde{\delta} \subset \{pt\} \times T^2 \subset X \tag{3.184}$$

are the loops generating $\pi_1(T^2)$ and parameterized by $\tilde{\theta}_4$, $\tilde{\theta}_5$. Since Y_1 is homotopy equivalent to T^4, we have

$$\begin{aligned} \pi_1(Y_1) &\cong \pi_1(T^4) \\ &= \langle \alpha_1, \beta_1, \gamma_1, \delta_1 \,|\, [\alpha_1, \beta_1], [\gamma_1, \beta_1], [\alpha_1, \gamma_1], [\alpha_1, \delta_1], [\beta_1, \gamma_1], [\beta_1, \delta_1] \rangle. \end{aligned} \tag{3.185}$$

Here

$$\alpha_1,\, \beta_1,\, \gamma_1,\, \delta_1 \subset T_1^4 \times \{pt\} \subset Y_1 \tag{3.186}$$

are the loops generating $\pi_1(Y_1)$ and parameterized by $\theta_{1;2}$, $\theta_{1;3}$, $\theta_{1;4}$, $\theta_{1;5}$, respectively. The meridian $\mu_1 = \partial D^2 \times \{pt\} \subset \partial Y_1$ of T_1^4 is null homotopic in Y_1 and parameterized by $\theta_{1;1}$.

Let us denote the push offs of the generators of $\pi_1(T_1)$ by

$$\tilde{\alpha}_1,\, \tilde{\beta}_1 \subset (B - D^2 \times T_1) \times \{pt\} \subset \partial X \tag{3.187}$$

that are parameterized by $\tilde{\theta}_{1;2}$, $\tilde{\theta}_{1;3}$, respectively, and the meridian of $T_1 \times T^2$ in ∂X by $\tilde{\mu}_1$ which is parameterized by $\tilde{\theta}_{1;1}$. These loops are all null homotopic in X. The diffeomorphism φ_1 induces the following identifications of the generators of the fundamental groups:

$$\begin{aligned} \alpha_1 &\mapsto \tilde{\tilde{\delta}} \\ \beta_1 &\mapsto \tilde{\mu}_1 = 1 \\ \gamma_1 &\mapsto \tilde{\alpha}_1 = 1 \\ \delta_1 &\mapsto \tilde{\beta}_1 = 1 \\ 1 = \mu_1 &\mapsto \tilde{\gamma}. \end{aligned} \tag{3.188}$$

Hence, the normal subgroup N_1 is generated by

$$\{\tilde{\tilde{\delta}}\alpha_1^{-1}, \beta_1^{-1}, \gamma_1^{-1}, \delta_1^{-1}, \tilde{\gamma}\}. \tag{3.189}$$

Using (3.183), (3.185) and (3.189) in (3.180), we conclude

$$\pi_1(X \cup_{\varphi_1} Y_1) \cong \langle \tilde{\tilde{\delta}} \rangle. \tag{3.190}$$

We proceed to glue Y_2 in $X \cup_{\varphi_1} Y_1$. Notice that the torus surgery is a local operation, so the previous surgery performed along N_{T_1} does not affect the embedding of N_{T_2} in M, since N_{T_2} has been chosen to be disjoint from N_{T_1}. This allows us to go on with the second torus surgery to attach Y_2. Y_2 is homotopy equivalent to T^4, so its fundamental group is

$$\begin{aligned} \pi_1(Y_2) &\cong \pi_1(T^4) \\ &= \langle \alpha_2, \beta_2, \gamma_2, \delta_2 \,|\, [\alpha_2, \beta_2], [\gamma_2, \beta_2], [\alpha_2, \gamma_2], [\alpha_2, \delta_2], [\beta_2, \gamma_2], [\beta_2, \delta_2] \rangle. \end{aligned} \tag{3.191}$$

Here

$$\alpha_2, \beta_2, \gamma_2, \delta_2 \subset T_2^4 \times \{pt\} \subset Y_2 \tag{3.192}$$

are the loops which generate $\pi_1(Y_2)$ and are parameterized by $\theta_{2;2}$, $\theta_{2;3}$, $\theta_{2;4}$, $\theta_{2;5}$. Denote the meridian of T_2^4 in ∂Y_2 by μ_2 that is parameterized by $\theta_{2;1}$, then μ_2 is null homotopic in Y_1. Let

$$\tilde{\alpha}_2, \tilde{\beta}_2 \subset \partial(X \cup_{\varphi_1} Y_1) \tag{3.193}$$

be the push offs of the generators of $\pi_1(T_2)$ and $\tilde{\mu}_2 \subset \partial(X \cup_{\varphi_1} Y_1)$ the meridian of $T_2 \times T^2$, so that $\tilde{\alpha}_2, \tilde{\beta}_2$ and $\tilde{\mu}_2$ are all null homotopic in $X \cup_{\varphi_1} Y_1$ and parameterized by $\tilde{\theta}_{2;2}$, $\tilde{\theta}_{2;3}$ and $\tilde{\theta}_{2;1}$, respectively. The diffeomorphism (3.160) gives rise to the following identifications of the generators of $\pi_1(Y_2)$ and $\pi_1(X \cup_{\varphi_1} Y_1)$

$$\alpha_2 \mapsto -\tilde{\gamma} = 1$$

$$
\begin{aligned}
\beta_2 &\mapsto \tilde{\mu}_2 = 1 \\
\gamma_2 &\mapsto \tilde{\beta}_2 = 1 \\
\delta_2 &\mapsto \tilde{\alpha}_2 = 1 \\
1 = \mu_2 &\mapsto \tilde{\delta}.
\end{aligned} \tag{3.194}
$$

Accordingly, using Seifert-van Kampen theorem along with (3.190)-(3.194), we conclude

$$
\pi_1(X \cup_{\varphi_1} Y_1 \cup_{\varphi_2} Y_2) \cong \{1\}. \tag{3.195}
$$

Thus we prove Lemma 3.5.3. □

Remark 3.5.4. There is an alternative way to prove Lemma 3.5.3. The main obstacle to the application of Seifert-van Kampen theorem was that $X \cap Y$ is not connected. We can overcome this by using the arguments in [8, 51]. Choose a base point $x_0 \in X$ and a point $p_1 \in \partial Y_1$. Let η_1 be a path connecting $\varphi(p_1)$ with x_0. Define Y_1' as the union of Y and a neighborhood of η_1. Similarly, we define Y_2'. Then $X \cap (Y_1' \cup Y_2')$ is connected so that Seifert-van Kampen theorem can be applied.

Remark 3.5.5. In fact, Lemma 3.5.3 can be generalized as follows. Note that in the the proof of Lemma 3.5.3, we only used that M satisfies the following criteria:

(i) M is a product $B \times T^2$ where B is a closed simply connected 4-manifold

(ii) B contains two disjoint symplectic 2-tori T_1 and T_2 with trivial normal bundles

(iii) the inclusions $j_{T_i} : T_i \hookrightarrow B$ for $i = 1, 2$ induce group homomorphisms

$$
j_{T_i *} : \pi_1(T_i) \to \pi_1(B) \tag{3.196}
$$

which are trivial

(iv) the inclusion $j : (B - (T_1 \sqcup T_2)) \hookrightarrow B$ induces a group homomorphism

$$
j_* : \pi_1(B - (T_1 \sqcup T_2)) \to \pi_1(B) \tag{3.197}
$$

which is an isomorphism.

Also, notice that each torus surgery kills one generator of $\pi_1(T^2)$ and does not have an effect on $\pi_1(B)$. This suggests that Lemma 3.5.3 can immediately be extended to the following result.

Lemma 3.5.6. *Let N be a closed symplectic 4-manifold satisfying the criteria (ii)-(iv) in Remark 3.5.5. Let $\hat{W}$ denote a manifold obtained by performing two torus surgeries of multiplicity zero on a closed symplectic 6-manifold $W = N \times T^2$ along the two embedded symplectic 4-tori. Then $\pi_1(\hat{W}) \cong \pi_1(N)$.*

Getting back to the computation of the homology groups of $\hat{M}$, $H_1(\hat{M};\mathbb{Z})$ is the abelianization of $\pi_1(\hat{M})$, so from Lemma 3.5.3 it immediately follows that

$$H_1(\hat{M};\mathbb{Z}) = 0. \tag{3.198}$$

Poincaré duality and the universal coefficients theorem yield

$$H_5(\hat{M};\mathbb{Z}) = H^1(\hat{M};\mathbb{Z}) = H_1(\hat{M};\mathbb{Z}) = 0, \tag{3.199}$$

given that $H_0(\hat{M};\mathbb{Z})$ is torsion free. In order to determine $H_2(\hat{M};\mathbb{Z})$, let us consider the Mayer-Vietoris sequence (see [34, Section 2.2])

$$\begin{aligned} \cdots \longrightarrow H_2(X\cap Y;\mathbb{Z}) \xrightarrow{j_2} H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) \xrightarrow{i_2} H_2(\hat{M};\mathbb{Z}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H_1(X\cap Y;\mathbb{Z}) \xrightarrow{j_1} H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z}) \xrightarrow{i_1} H_1(\hat{M};\mathbb{Z}) \longrightarrow \cdots . \end{aligned} \tag{3.200}$$

Taking into account $H_1(\hat{M};\mathbb{Z}) = 0$, the above sequence is reduced to

$$\begin{aligned} \cdots \longrightarrow H_2(X\cap Y;\mathbb{Z}) \xrightarrow{j_2} H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) \xrightarrow{i_2} H_2(\hat{M};\mathbb{Z}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H_1(X\cap Y;\mathbb{Z}) \xrightarrow{j_1} H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z}) \xrightarrow{\quad i_1 \quad} 0 . \end{aligned} \tag{3.201}$$

Lemma 3.5.7. *The homomorphism*

$$j_1 : H_1(X\cap Y;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z}) \tag{3.202}$$

is injective, i.e., $\ker j_1 = 0$.

Proof. The exactness of the sequence (3.201) implies

$$\operatorname{im} j_1 = \ker i_1 \cong H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z}). \tag{3.203}$$

Since $\pi_1(X) = \mathbb{Z}^2$ (see below (3.180)), by abelianizing $\pi_1(X)$, we get

$$H_1(X;\mathbb{Z}) = \mathbb{Z}^2. \tag{3.204}$$

Y is homotopy equivalent to $T^4 \sqcup T^4$, hence

$$H_1(Y;\mathbb{Z}) \cong H_1(T^4 \sqcup T^4;\mathbb{Z}) = H_1(T^4;\mathbb{Z}) \oplus H_1(T^4;\mathbb{Z}) = \mathbb{Z}^4 \oplus \mathbb{Z}^4 = \mathbb{Z}^8. \tag{3.205}$$

Applying the first isomorphism theorem (see e.g. [22, Theorem 16]) to the homomorphism j_1, we obtain

$$H_1(X \cap Y;\mathbb{Z})/\ker j_1 \cong \operatorname{im} j_1. \tag{3.206}$$

Since $X \cap Y$ is diffeomorphic to $T^5 \sqcup T^5$, we have

$$H_1(X \cap Y;\mathbb{Z}) \cong H_1(T^5 \sqcup T^5;\mathbb{Z}) = H_1(T^5;\mathbb{Z}) \oplus H_1(T^5;\mathbb{Z}) = \mathbb{Z}^5 \oplus \mathbb{Z}^5 = \mathbb{Z}^{10}. \tag{3.207}$$

We see from (3.203)-(3.207) that

$$\mathbb{Z}^{10}/\ker j_1 = \mathbb{Z}^{10}. \tag{3.208}$$

Along with splitting lemma (see [34, p.147-148]) , (3.208) implies $\ker j_1 = 0$ as required. □

As the sequence (3.201) is exact, Lemma 3.5.7 implies $\operatorname{im} \delta = 0$, so that the sequence (3.201) boils down to

$$\cdots \longrightarrow H_2(X \cap Y;\mathbb{Z}) \xrightarrow{j_2} H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) \xrightarrow{i_2} H_2(\hat{M};\mathbb{Z}) \xrightarrow{\delta} 0\,. \tag{3.209}$$

Remark 3.5.8. Notice that the exact sequence (3.209) holds for a manifold $\hat{M}$ constructed by applying two torus surgeries to a symplectic 6-manifold satisfying the criteria (i)-(iv) given in Remark 3.5.5.

By the first isomorphism theorem along with the exactness of the sequence (3.209), we get

$$\begin{aligned} H_2(\hat{M};\mathbb{Z}) &\cong \operatorname{im} i_2 \cong \left(H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z})\right)/\ker i_2 \\ &= \left(H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z})\right)/\operatorname{im} j_2. \end{aligned} \tag{3.210}$$

We need to compute the homology groups $H_2(X;\mathbb{Z})$, $H_2(Y;\mathbb{Z})$ and $H_2(X\cap Y;\mathbb{Z})$ to determine $H_2(\hat{M};\mathbb{Z})$ by (3.210). Since Y is homotopy equivalent to $T^4 \sqcup T^4$,

$$H_2(Y;\mathbb{Z}) \cong H_2(T^4 \sqcup T^4;\mathbb{Z}) = H_2(T^4;\mathbb{Z}) \oplus H_2(T^4;\mathbb{Z}) = \mathbb{Z}^6 \oplus \mathbb{Z}^6 = \mathbb{Z}^{12}, \tag{3.211}$$

which is spanned by twelve 2-cycles being the product of the 1-cycles generating $H_1(Y;\mathbb{Z})$ (cf. (3.185), (3.191)), i.e.,

$$\{\alpha_i \times \beta_i, \quad \alpha_i \times \gamma_i, \quad \alpha_i \times \delta_i, \quad \beta_i \times \gamma_i, \quad \beta_i \times \delta_i, \quad \gamma_i \times \delta_i\} \tag{3.212}$$

for $i = 1, 2$. On the other hand, $X \cap Y$ is diffeomorphic to $T^5 \sqcup T^5$, hence

$$\begin{aligned} H_2(X\cap Y;\mathbb{Z}) &\cong H_2(T^5 \sqcup T^5;\mathbb{Z}) = H_2(T^5;\mathbb{Z}) \oplus H_2(T^5;\mathbb{Z}) \\ &= \mathbb{Z}^{10} \oplus \mathbb{Z}^{10} = \mathbb{Z}^{20}. \end{aligned} \tag{3.213}$$

Let $\{\mu'_i, \alpha'_i, \beta'_i, \gamma'_i, \delta'_i\}$ be the 1-cycles which span $H_1(X\cap Y;\mathbb{Z})$ for $i = 1, 2$, then the generators of $H_2(X\cap Y;\mathbb{Z})$ can be chosen as

$$\begin{aligned} \{\alpha'_i \times \beta'_i, \quad \alpha'_i \times \gamma'_i, \quad \alpha'_i \times \delta'_i, \quad \beta'_i \times \gamma'_i, \quad \beta'_i \times \delta'_i, \quad \gamma'_i \times \delta'_i \\ \mu'_i \times \alpha'_i, \quad \mu'_i \times \beta'_i, \quad \mu'_i \times \gamma'_i, \quad \mu'_i \times \delta'_i\}. \end{aligned} \tag{3.214}$$

Lemma 3.5.9. *$H_2(X;\mathbb{Z}) = \mathbb{Z}^{11}$.*

Proof. To compute $H_2(X;\mathbb{Z})$, it is convenient to write

$$X = \tilde{X} \times T^2, \tag{3.215}$$

where $\tilde{X} = B - D^2 \times (T_1 \sqcup T_2)$ (see (3.175)). By the relative Künneth formula (see e.g. [34, Theorem 3.18]) and the excision theorem (see e.g. [34, Theorem 2.20]), we have the following isomorphisms (see [10])

$$\begin{aligned} H_{n-2}(T_1 \sqcup T_2;\mathbb{Z}) &\cong H_{n-2}(T_1 \sqcup T_2;\mathbb{Z}) \otimes H_2(D^2, S^1;\mathbb{Z}) \\ &\cong H_n((T_1 \sqcup T_2) \times D^2, (T_1 \sqcup T_2) \times S^1;\mathbb{Z}) \\ &\cong H_n(B, \tilde{X};\mathbb{Z}), \end{aligned} \tag{3.216}$$

which send an $(n-2)-$cycle α in $H_{n-2}(T_1 \sqcup T_2;\mathbb{Z})$ to an $n-$ cycle $\alpha \times (D^2, S^1)$ in $H_n(B, \tilde{X};\mathbb{Z})$. Let us consider the following exact sequence in relative homology (see [34, Section 2.1])

$$\begin{aligned}\cdots \longrightarrow H_3(B;\mathbb{Z}) \longrightarrow H_3(B,\tilde{X};\mathbb{Z}) \longrightarrow H_2(\tilde{X};\mathbb{Z}) \longrightarrow \\ \longrightarrow H_2(B;\mathbb{Z}) \longrightarrow H_2(B,\tilde{X};\mathbb{Z}) \longrightarrow H_1(\tilde{X};\mathbb{Z}) \longrightarrow \cdots\end{aligned} \tag{3.217}$$

By Proposition 3.5.1, $H_3(B;\mathbb{Z}) = 0$ and $H_1(\tilde{X};\mathbb{Z}) = 0$, so the sequence (3.217) is reduced to

$$0 \longrightarrow H_3(B,\tilde{X};\mathbb{Z}) \xrightarrow{f} H_2(\tilde{X};\mathbb{Z}) \xrightarrow{g} H_2(B;\mathbb{Z}) \xrightarrow{h} H_2(B,\tilde{X};\mathbb{Z}) \longrightarrow 0 \ . \tag{3.218}$$

Using the isomorphisms in (3.216) for $n = 3$ and $n = 2$, we obtain

$$\begin{aligned}H_3(B,\tilde{X};\mathbb{Z}) \cong H_1(T_1 \sqcup T_2;\mathbb{Z}) &= H_1(T_1;\mathbb{Z}) \oplus H_1(T_2;\mathbb{Z}) \\ &= \mathbb{Z}^2 \oplus \mathbb{Z}^2 = \mathbb{Z}^4,\end{aligned} \tag{3.219}$$

$$\begin{aligned}H_2(B,\tilde{X};\mathbb{Z}) \cong H_0(T_1 \sqcup T_2;\mathbb{Z}) &= H_0(T_1;\mathbb{Z}) \oplus H_0(T_2;\mathbb{Z}) \\ &= \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2.\end{aligned} \tag{3.220}$$

Remark 3.5.10. Note that the sequence (3.218) and (3.219), (3.220) hold not only for the manifold B but for any simply connected 4-manifold that satisfies the criteria (ii)-(iv) in Remark 3.5.5.

Using (3.219) and (3.220) along with $H_2(B;\mathbb{Z}) = \mathbb{Z}^8$, the sequence (3.218) becomes

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{f} H_2(\tilde{X};\mathbb{Z}) \xrightarrow{g} \mathbb{Z}^8 \xrightarrow{h} \mathbb{Z}^2 \longrightarrow 0 \ . \tag{3.221}$$

The first isomorphism theorem along with the exactness of the sequence (3.221) implies

$$H_2(\tilde{X};\mathbb{Z})/\ker g \cong \operatorname{im} g = \ker h. \tag{3.222}$$

Since f is injective, $\operatorname{im} f \cong \mathbb{Z}^4$, and the exactness of the sequence (3.221) yields

$$\ker g = \operatorname{im} f \cong \mathbb{Z}^4. \tag{3.223}$$

More explicitly, $\operatorname{im} f$ is generated by $\tilde{\alpha}_i \times \tilde{\mu}_i$ and $\tilde{\beta}_i \times \tilde{\mu}_i$ for $i = 1, 2$, where $\tilde{\alpha}_i$, $\tilde{\beta}_i$ span $H_1(T_i; \mathbb{Z})$ and $\tilde{\mu}_i$ are the meridians of T_i in $\tilde{X}$. Since h is surjective, we observe

$$\mathbb{Z}^8 / \ker h \cong \operatorname{im} h \cong \mathbb{Z}^2 \tag{3.224}$$

which implies $\ker h = \mathbb{Z}^6$ from splitting lemma (see [34, p.147-148]) and so

$$\operatorname{im} g = \mathbb{Z}^6. \tag{3.225}$$

Indeed, $\operatorname{im} g$ is spanned by the six generators T_1, T_2, H_1, H_2, H_3, F of $H_2(B; \mathbb{Z})$ which intersect neither T_1 nor T_2. Each of these 2-cycles generates an infinite cyclic group $\mathbb{Z}$ and has obviously nonzero preimage in $H_2(\tilde{X}; \mathbb{Z})$. Combining (3.222), (3.223) and (3.225), we conclude that

$$H_2(\tilde{X}; \mathbb{Z}) = \ker g \oplus \operatorname{im} g = \mathbb{Z}^{10}. \tag{3.226}$$

Here taking into account $\operatorname{im} g$ being free abelian group from (3.225), we used the splitting lemma (see [34, p.147-148]) for the exact sequence

$$0 \longrightarrow \ker g \longrightarrow H_2(\tilde{X}; \mathbb{Z}) \longrightarrow \operatorname{im} g \longrightarrow 0 \ , \tag{3.227}$$

which follows from (3.222). The generators of $H_2(\tilde{X}; \mathbb{Z}))$ are

$$\{\tilde{\alpha}_i \times \tilde{\mu}_i, \quad \tilde{\beta}_i \times \tilde{\mu}_i, \quad T_i, \quad H_1, \quad H_2, \quad H_3, \quad F\} \tag{3.228}$$

with $i = 1, 2$. Applying the Künneth formula (see e.g. [13, Theorem 1.6]) and using (3.226), we obtain

$$\begin{aligned} H_2(X; \mathbb{Z}) &= H_2(\tilde{X} \times T^2; \mathbb{Z}) \\ &= H_0(\tilde{X}; Z) \otimes H_2(T^2; \mathbb{Z}) \oplus H_1(\tilde{X}; \mathbb{Z}) \otimes H_1(T^2; \mathbb{Z}) \\ &\quad \oplus H_2(\tilde{X}; \mathbb{Z}) \otimes H_0(T^2; \mathbb{Z}) \\ &= \mathbb{Z} \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}^2 \oplus \mathbb{Z}^{10} \otimes \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}^{10} = \mathbb{Z}^{11}. \end{aligned} \tag{3.229}$$

Indeed, $H_2(X; \mathbb{Z})$ is spanned by the following eleven generators

$$\{\tilde{\alpha}_i \times \tilde{\mu}_i, \quad \tilde{\beta}_i \times \tilde{\mu}_i, \quad T_i, \quad H_1, \quad H_2, \quad H_3, \quad F, \quad \tilde{\gamma} \times \tilde{\delta}\} \tag{3.230}$$

where $\tilde{\gamma} \times \tilde{\delta}$ is the generator of $H_2(T^2; \mathbb{Z})$. □

Let us get back to the main sequence (3.209) and prove the following result.

Lemma 3.5.11. *The image of the homomorphism*

$$j_2 : H_2(X \cap Y; \mathbb{Z}) \to H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) \tag{3.231}$$

is spanned by thirteen 2-cycles, i.e., $\operatorname{im} j_2 = \mathbb{Z}^{13}$.

Proof. The homomorphism j_2 is a pair of homomorphisms $(j_X, -j_Y)$ for which $j_X : H_2(X \cap Y; \mathbb{Z}) \to H_2(X;\mathbb{Z})$ and $j_Y : H_2(X \cap Y;\mathbb{Z}) \to H_2(Y;\mathbb{Z})$. The homomorphism j_Y is induced by the natural inclusion $X \cap Y \hookrightarrow Y$ and given by

$$\begin{aligned}
\alpha_i' \times \beta_i' &\mapsto \alpha_i \times \beta_i \\
\alpha_i' \times \gamma_i' &\mapsto \alpha_i \times \gamma_i \\
\alpha_i' \times \delta_i' &\mapsto \alpha_i \times \delta_i, \\
\beta_i' \times \gamma_i' &\mapsto \beta_i \times \gamma_i \\
\beta_i' \times \delta_i' &\mapsto \beta_i \times \delta_i \\
\gamma_i' \times \delta_i' &\mapsto \gamma_i \times \delta_i, \\
\mu_i' \times \alpha_i' &\mapsto \mu_i \times \alpha_i = 0 \\
\mu_i' \times \beta_i' &\mapsto \mu_i \times \beta_i = 0 \\
\mu_i' \times \gamma_i' &\mapsto \mu_i \times \gamma_i = 0, \\
\mu_i' \times \delta_i' &\mapsto \mu_i \times \delta_i = 0,
\end{aligned} \tag{3.232}$$

for $i = 1, 2$. On the other hand, j_X is determined via the following commutative diagram as $j_X = \varphi_* \circ j_Y$

$$\begin{array}{ccc}
H_2(X \cap Y;\mathbb{Z}) & \xrightarrow{\quad j_Y \quad} & H_2(Y;\mathbb{Z}) \\
& {}_{j_X}\searrow \quad \swarrow_{\varphi_*} & \\
& H_2(X;\mathbb{Z}) &
\end{array} \tag{3.233}$$

with φ_* induced by the gluing map $\varphi = (\varphi_1, \varphi_2)$ given in (3.159) and (3.160). Then the generators of $H_2(X \cap Y;\mathbb{Z})$ given in (3.214) are mapped into the generators of $H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z})$ as follows (cf. (3.188) and (3.194)):

$$\begin{aligned}
\alpha_1' \times \beta_1' &\mapsto (\tilde{\delta} \times \tilde{\mu}_1, -\alpha_1 \times \beta_1) = (0, -\alpha_1 \times \beta_1) \\
\alpha_1' \times \gamma_1' &\mapsto (\tilde{\delta} \times \tilde{\alpha}_1, -\alpha_1 \times \gamma_1) = (0, -\alpha_1 \times \gamma_1) \\
\alpha_1' \times \delta_1' &\mapsto (\tilde{\delta} \times \tilde{\beta}_1, -\alpha_1 \times \delta_1) = (0, -\alpha_1 \times \delta_1) \\
\beta_1' \times \gamma_1' &\mapsto (\tilde{\mu}_1 \times \tilde{\alpha}_1, -\beta_1 \times \gamma_1)
\end{aligned}$$

$$
\begin{aligned}
&\beta_1' \times \delta_1' \mapsto (\tilde{\mu}_1 \times \tilde{\beta}_1, -\beta_1 \times \delta_1) \\
&\gamma_1' \times \delta_1' \mapsto (\tilde{\alpha}_1 \times \tilde{\beta}_1, -\gamma_1 \times \delta_1) \\
&\mu_1' \times \alpha_1' \mapsto (\tilde{\gamma} \times \tilde{\delta}, -\mu_1 \times \alpha_1) = (\tilde{\gamma} \times \tilde{\delta}, 0) \\
&\mu_1' \times \beta_1' \mapsto (\tilde{\gamma} \times \tilde{\mu}_1, -\mu_1 \times \beta_1) = (0,0) \\
&\mu_1' \times \gamma_1' \mapsto (\tilde{\gamma} \times \tilde{\alpha}_1, -\mu_1 \times \gamma_1) = (0,0) \\
&\mu_1' \times \delta_1' \mapsto (\tilde{\gamma} \times \tilde{\beta}_1, -\mu_1 \times \delta_1) = (0,0) \\
&\alpha_2' \times \beta_2' \mapsto (-\tilde{\gamma} \times \tilde{\mu}_2, -\alpha_2 \times \beta_2) = (0, -\alpha_2 \times \beta_2) \\
&\alpha_2' \times \gamma_2' \mapsto (-\tilde{\gamma} \times \tilde{\beta}_2, -\alpha_2 \times \gamma_2) = (0, -\alpha_2 \times \gamma_2) \\
&\alpha_2' \times \delta_2' \mapsto (-\tilde{\gamma} \times \tilde{\alpha}_2, -\alpha_2 \times \delta_2) = (0, -\alpha_2 \times \delta_2) \\
&\beta_2' \times \gamma_2' \mapsto (\tilde{\mu}_2 \times \tilde{\beta}_2, -\beta_2 \times \gamma_2) \\
&\beta_2' \times \delta_2' \mapsto (\tilde{\mu}_2 \times \tilde{\alpha}_2, -\beta_2 \times \delta_2) \\
&\gamma_2' \times \delta_2' \mapsto (\tilde{\beta}_2 \times \tilde{\alpha}_2, -\gamma_2 \times \delta_2) \\
&\mu_2' \times \alpha_2' \mapsto (-\tilde{\delta} \times \tilde{\gamma}, -\mu_2 \times \alpha_2) = (-\tilde{\delta} \times \tilde{\gamma}, 0) \\
&\mu_2' \times \beta_2' \mapsto (-\tilde{\delta} \times \tilde{\mu}_2, -\mu_2 \times \beta_2) = (0,0) \\
&\mu_2' \times \gamma_2' \mapsto (-\tilde{\gamma} \times \tilde{\beta}_2, -\mu_2 \times \gamma_2) = (0,0) \\
&\mu_2' \times \delta_2' \mapsto (-\tilde{\gamma} \times \tilde{\alpha}_2, -\mu_2 \times \delta_2) = (0,0). \qquad (3.234)
\end{aligned}
$$

From (3.234), we see that $\operatorname{im} j_2$ is spanned by the following independent thirteen 2-cycles:

$$
\begin{aligned}
\operatorname{im} j_2 =\big\{&(0, -\alpha_i \times \beta_i),\ (0, -\alpha_i \times \gamma_i),\ (0, -\alpha_i \times \delta_i),\ (\tilde{\gamma} \times \tilde{\delta}, 0), \\
&(\tilde{\mu}_i \times \tilde{\alpha}_i, -\beta_i \times \gamma_i),\ (\tilde{\mu}_i \times \tilde{\beta}_i, -\beta_i \times \delta_i),\ (\tilde{\alpha}_i \times \tilde{\beta}_i, -\gamma_i \times \delta_i)\big\}
\end{aligned} \qquad (3.235)
$$

for $i = 1, 2$ and each of which generates an infinite cyclic group $\mathbb{Z}$, so we conclude $\operatorname{im} j_2 = \mathbb{Z}^{13}$. □

Remark 3.5.12. The 2-cycles being disjoint from T_i in B (i.e., T_i, H_1, H_2, H_3 and F) are completely isolated from the torus surgery so that they are irrelevant to the computation of $\operatorname{im} j_2$. This tells us that Lemma 3.5.11 holds not only for X but also for any complement of the disjoint union of two symplectic 4-tori with trivial normal bundles in a closed simply connected symplectic 6-manifold that satisfies the criteria (i)-(iv) in Remark 3.5.5.

From (3.211) and Lemma 3.5.9, we have

$$
H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) = \mathbb{Z}^{23} \qquad (3.236)
$$

which are generated by the 2-cycles given in (3.212) and (3.230) among which the thirteen independent 2-cycles generate $\operatorname{im} j_2$ by Lemma 3.5.11. Hence, the remaining ten cycles span $(H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}))/\operatorname{im} j_2$ which are

$$\big\{(H_k,0),\ (F,0),\ (\tilde{\mu}_i \times \tilde{\alpha}_i, \beta_i \times \gamma_i),\ (\tilde{\mu}_i \times \tilde{\beta}_i, \beta_i \times \delta_i),\ (\tilde{\alpha}_i \times \tilde{\beta}_i, \gamma_i \times \delta_i)\big\} \tag{3.237}$$

with $i = 1, 2$ and $k = 1, 2, 3$, each of which generates an infinite cyclic group $\mathbb{Z}$. Thus we get

$$(H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}))/\operatorname{im} j_2 \cong \mathbb{Z}^{10} \tag{3.238}$$

and by (3.210), we conclude

$$H_2(\hat{M};\mathbb{Z}) = \mathbb{Z}^{10}. \tag{3.239}$$

Applying Poincaré duality and the universal coefficients theorem, we get

$$H_4(\hat{M};\mathbb{Z}) = H^2(\hat{M};\mathbb{Z}) = H_2(\hat{M};\mathbb{Z}) = \mathbb{Z}^{10}. \tag{3.240}$$

In order to determine $H_3(\hat{M};\mathbb{Z})$, observe that the Euler characteristic of $\hat{M}$ is given by

$$\chi(\hat{M}) = \sum_{k=0}^{6} (-1)^k\, b_k(\hat{M}) = 22 - b_3(\hat{M}), \tag{3.241}$$

using $b_0(\hat{M}) = b_6(\hat{M}) = 1$, $b_1(\hat{M}) = b_5(\hat{M}) = 0$ and $b_2(\hat{M}) = b_4(\hat{M}) = 10$. The Euler characteristic of M is

$$\chi(M) = \chi(B \times T^2) = \chi(B) \times \chi(T^2) = 0, \tag{3.242}$$

where we used the multiplicativity of the Euler characteristic (see [32, Exercise 3.3.13]) and $\chi(T^2) = 0$. By Proposition 3.2.1

$$\chi(\hat{M}) = \chi(M) = 0 \tag{3.243}$$

so we claim $b_3(\hat{M}) = 22$ from (3.241) and (3.243). Given that $H_2(\hat{M})$ is torsion free, the universal coefficients theorem implies that there is no torsion in $H^3(\hat{M};\mathbb{Z})$, so that

$$H^3(\hat{M};\mathbb{Z}) = \mathbb{Z}^{b_3(\hat{M})} = \mathbb{Z}^{22} \tag{3.244}$$

and by Poincaré duality

$$H_3(\hat{M};\mathbb{Z}) = H^3(\hat{M};\mathbb{Z}) = \mathbb{Z}^{22}. \tag{3.245}$$

Thus we complete the proof of Proposition 3.5.2. $\square$

Since $\hat{M}$ is simply connected by Lemma 3.5.3 and has zero Euler characteristic along with $H_3(\hat{M};\mathbb{Z}) = \mathbb{Z}^{22}$ (see (3.243) and (3.245)), it immediately follows from Proposition 3.4.1 that $\hat{M}$ is not homologically equivalent to a product of lower dimensional manifolds.

We can find more examples of the closed simply connected 6-manifolds equipped with a stable generalized complex structure with nonempty type change locus by doing two torus surgeries on a closed symplectic 6-manifold $W = N \times T^2$. Like the closed simply connected symplectic 4-manifold B defined in Proposition 3.5.1, N is assumed to be a closed simply connected symplectic 4-manifold which satisfies the criteria (ii)-(iv) in Remark 3.5.5. Let T_1, T_2 be two disjoint symplectic 2-tori in N. Denote $\hat{W}$ the manifold obtained by applying two torus surgeries with multiplicity zero simultaneously to W along $T_1 \times T^2$ and $T_2 \times T^2$, which admits a stable generalized complex structure due to Theorem 3.2.2. We show that $\hat{W}$ is not homologically equivalent to a product of lower dimensional manifolds by computing its homology groups. Denoting $b_i(W)$ the $i-$th Betti number of W, we obtain the following result.

Proposition 3.5.13. *The homology groups of $\hat{W}$ are determined by*

$$H_0(\hat{W};\mathbb{Z}) = H_6(\hat{W};\mathbb{Z}) = \mathbb{Z} \tag{3.246}$$

$$H_1(\hat{W};\mathbb{Z}) = H_5(\hat{W};\mathbb{Z}) = 0 \tag{3.247}$$

$$H_2(\hat{W};\mathbb{Z}) = H_4(\hat{W};\mathbb{Z}) = \mathbb{Z}^{b_2(W)+1} \tag{3.248}$$

$$H_3(\hat{W};\mathbb{Z}) = \mathbb{Z}^{b_3(W)+6}. \tag{3.249}$$

Proof. Since $\hat{W}$ is connected, we have $H_0(\hat{W};\mathbb{Z}) = \mathbb{Z}$. Moreover, $\hat{W}$ is closed and orientable, so $H_6(\hat{W};\mathbb{Z}) = \mathbb{Z}$ and we can apply Poincaré duality. We claim

$$\pi_1(\hat{W}) \cong \pi_1(N) \cong \{1\} \tag{3.250}$$

due to Lemma 3.5.6, as W satisfies the criteria (i)-(iv) mentioned in Remark 3.5.5. Since $H_1(\hat{W};\mathbb{Z})$ is the abelianization of $\pi_1(\hat{W})$, we get $H_1(\hat{W};\mathbb{Z}) = 0$. Poincaré duality and the universal coefficients theorem yield

$$H_5(\hat{W};\mathbb{Z}) \cong H^1(\hat{W};\mathbb{Z}) \cong H_1(\hat{W};\mathbb{Z}) = 0. \tag{3.251}$$

To compute $H_2(\hat{W};\mathbb{Z})$, we consider the exact sequence (see Remark 3.5.8)

$$\cdots \longrightarrow H_2(X \cap Y;\mathbb{Z}) \xrightarrow{j_2} H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) \xrightarrow{i_2} H_2(\hat{W};\mathbb{Z}) \xrightarrow{\delta} 0\,. \tag{3.252}$$

Here X, Y are defined as (cf. (3.175)-(3.176))

$$\begin{aligned} X &= W - \left(D^2 \times T_1 \times T^2 \sqcup D^2 \times T_2 \times T^2\right) \\ &= \left(N - D^2 \times (T_1 \sqcup T_2)\right) \times T^2 \qquad (3.253) \\ Y &= D^2 \times T_1^4 \sqcup D^2 \times T_2^4. \qquad (3.254) \end{aligned}$$

Let $\tilde{X} = N - D^2 \times (T_1 \sqcup T_2)$, then the Künneth formula (see e.g. [13, Theorem 1.6]) gives

$$\begin{aligned} H_2(X;\mathbb{Z}) &= H_0(\tilde{X};\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \oplus H_1(\tilde{X};\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\ &\quad \oplus H_2(\tilde{X};\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \\ &= \mathbb{Z} \otimes \mathbb{Z} \oplus 0 \times \mathbb{Z}^2 \oplus H_2(\tilde{X};\mathbb{Z}) \otimes \mathbb{Z} = \mathbb{Z} \oplus H_2(\tilde{X};\mathbb{Z}), \qquad (3.255) \end{aligned}$$

where we used $H_1(\tilde{X};\mathbb{Z}) = 0$ since $\pi_1(\tilde{X}) \cong \pi_1(N) \cong \{1\}$ by assumption. As it was pointed out in Remark 3.5.10, by replacing B with N, $H_2(\tilde{X};\mathbb{Z})$ fits in the exact sequence (see (3.218)-(3.220))

$$0 \longrightarrow H_3(N,\tilde{X};\mathbb{Z}) \xrightarrow{f} H_2(\tilde{X};\mathbb{Z}) \xrightarrow{g} H_2(N;\mathbb{Z}) \xrightarrow{h} H_2(N,\tilde{X};\mathbb{Z}) \longrightarrow 0 \ , \qquad (3.256)$$

and

$$\begin{aligned} H_3(N,\tilde{X};\mathbb{Z}) \cong H_1(T_1 \sqcup T_2;\mathbb{Z}) &= H_1(T_1;\mathbb{Z}) \oplus H_1(T_2;\mathbb{Z}) \\ &= \mathbb{Z}^2 \oplus \mathbb{Z}^2 = \mathbb{Z}^4, \qquad (3.257) \end{aligned}$$

$$\begin{aligned} H_2(N,\tilde{X};\mathbb{Z}) \cong H_0(T_1 \sqcup T_2;\mathbb{Z}) &= H_0(T_1;\mathbb{Z}) \oplus H_0(T_2;\mathbb{Z}) \\ &= \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2. \qquad (3.258) \end{aligned}$$

Then we can write the sequence (3.256) as

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{f} H_2(\tilde{X};\mathbb{Z}) \xrightarrow{g} H_2(N;\mathbb{Z}) \xrightarrow{h} \mathbb{Z}^2 \longrightarrow 0 \ . \qquad (3.259)$$

Since N is a closed and orientable simply connected 4-manifold, its homology groups are torsion free. Indeed, the connectedness of N implies $H_0(N;\mathbb{Z}) = \mathbb{Z}$, and we have $H_4(N;\mathbb{Z}) = \mathbb{Z}$ since N is closed and orientable. Moreover, N is simply connected, so $H_1(N;\mathbb{Z}) = 0$ and

$$H_3(N;\mathbb{Z}) \cong H^1(N;\mathbb{Z}) \cong H_1(N;\mathbb{Z}) = 0 \qquad (3.260)$$

by Poincaré duality and the universal coefficients theorem. $H_1(N;\mathbb{Z})$ is torsion free, so is $H^2(N;\mathbb{Z})$ due to the universal coefficients theorem, which implies $H_2(N;\mathbb{Z})$ is torsion free by Poincaré duality. Let $b_k(N)$ be the $k-$th Betti number of N, then $H_2(N;\mathbb{Z}) = \mathbb{Z}^{b_2(N)}$ and we obtain

$$\operatorname{im} g = \ker h = \mathbb{Z}^{b_2(N)-2}. \tag{3.261}$$

This can be seen from the following exact sequence due to the first isomorphism theorem

$$0 \longrightarrow \ker h \longrightarrow H_2(N;\mathbb{Z}) \longrightarrow \operatorname{im} h \longrightarrow 0 \ . \tag{3.262}$$

This sequence is split

$$H_2(N;\mathbb{Z}) \cong \ker h \oplus \operatorname{im} h, \tag{3.263}$$

since $\operatorname{im} h = \mathbb{Z}^2$ is free abelian (see [34, p.147-148]), which yields the desired result for $\ker h$. Then the exact sequence (3.259) boils down to

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{f} H_2(\tilde{X};\mathbb{Z}) \xrightarrow{g} \mathbb{Z}^{b_2(N)-2} \xrightarrow{h} 0 \ , \tag{3.264}$$

which is split since $\mathbb{Z}^{b_2(N)-2}$ is free abelian (see [34, p.147-148]). This allows us to conclude

$$H_2(\tilde{X};\mathbb{Z}) \cong \mathbb{Z}^4 \oplus \mathbb{Z}^{b_2(N)-2} = \mathbb{Z}^{b_2(N)+2}. \tag{3.265}$$

Plugging (3.265) into (3.255), we obtain

$$H_2(X;\mathbb{Z}) = \mathbb{Z}^{b_2(N)+3} = \mathbb{Z}^{b_2(W)+2}, \tag{3.266}$$

where we used $b_2(W) = b_2(N) + 1$ which follows from the Künneth formula (see e.g. [13, Theorem 1.6])

$$\begin{aligned} H_2(W;\mathbb{Z}) = {} & H_2(N;\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \oplus H_1(N;\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\ & \oplus H_0(N;\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \\ = {} & \mathbb{Z}^{b_2(N)} \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}^2 \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}^{b_2(N)} \oplus \mathbb{Z} = \mathbb{Z}^{b_2(N)+1}. \end{aligned} \tag{3.267}$$

Note that $b_2(W) \geq 5$, since N contains at least two homologically essential tori and their dual spheres, implying that $b_2(N) \geq 4$.

Getting back to the exact sequence (3.252), we know that

$$\ker i_2 = \operatorname{im} j_2 \cong \mathbb{Z}^{13} \tag{3.268}$$

generated by the independent thirteen 2-cycles (3.235) in (see Remark 3.5.12)

$$H_2(X;\mathbb{Z}) \oplus H_2(Y;\mathbb{Z}) = \mathbb{Z}^{b_2(W)+2} \oplus \mathbb{Z}^{12}. \tag{3.269}$$

Thus, the remaining

$$b_2(W) + 2 + 12 - 13 = b_2(W) + 1 \tag{3.270}$$

independent 2-cycles generate (cf. (3.237))

$$H_2(\hat{W};\mathbb{Z}) = \operatorname{im} i_2 \cong \mathbb{Z}^{b_2(W)+1}. \tag{3.271}$$

Poincaré duality and the universal coefficients theorem imply

$$H_4(\hat{W};\mathbb{Z}) \cong H^2(\hat{W};\mathbb{Z}) \cong H_2(\hat{W};\mathbb{Z}) = \mathbb{Z}^{b_2(W)+1}. \tag{3.272}$$

Regarding the computation of $H_3(\hat{W};\mathbb{Z})$, we repeat the computation of $H_3(\hat{M};\mathbb{Z})$ verbatim by replacing $\hat{M}$ with $\hat{W}$ (see (3.241)-(3.245)), which yields

$$H_3(\hat{W};\mathbb{Z}) = \mathbb{Z}^{b_3(W)+6}. \tag{3.273}$$

This completes the proof of Proposition 3.5.13. □

$\hat{W}$ is simply connected and its Euler characteristic is zero, and $H_2(\hat{W};\mathbb{Z}) \neq \mathbb{Z}^2$ taking into account (3.271) along with $b_2(W) \geq 5$. Hence, we conclude that $\hat{W}$ is not homologically equivalent to a product of lower dimensional manifolds due to Proposition 3.4.1.

In what follows we demonstrate that there are countably infinite closed orientable simply connected 6-manifolds that admit a stable generalized complex structure and that are not homologically equivalent to a product of lower dimensional manifolds. Let us consider a closed symplectic 6-manifold

$$W(n) = E(n) \times T^2 \tag{3.274}$$

equipped with the product symplectic structure $\omega_{W(n)} = p_1^*\omega_{E(n)} + p_2^*\omega_{T^2}$ with the canonical projections $p_1 : W(n) \to E(n)$ and $p_2 : W(n) \to T^2$. Here $E(n)$ denotes the elliptic surface described in Appendix B. For $n \geq 2$, one can always find two symplectic 2-tori with self-intersection zero inside $E(n)$ and the complement of the disjoint union of these 2-tori is simply connected (see Lemma B.0.4). Now we simultaneously apply two torus surgeries of multiplicity zero to $W(n)$, resulting in a stable generalized complex 6-manifold $(\hat{W}(n), \hat{\mathcal{J}}(n), \hat{H}(n))$ with two path-connected components of the type change locus by Theorem 3.2.2. Varying the natural number $n \geq 2$, we can construct infinitely many examples of stable generalized complex 6-manifolds from $E(n) \times T^2$. The homology groups of $\hat{W}(n)$ are determined by Proposition 3.5.13, showing that $\hat{W}(n)$ is simply connected and is not homologically equivalent to a product of lower dimensional manifolds (see the exposition below (3.273)).

In particular, using the Künneth formula (see e.g. [13, Theorem 1.6]) we compute (cf. Appendix B)

$$\begin{aligned} H_2(W(n);\mathbb{Z}) &\cong H_2(E(n);\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \oplus H_1(E(n);\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\ &\qquad \oplus H_0(E(n);\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \\ &= \mathbb{Z}^{12n-2} \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}^2 \oplus \mathbb{Z} \otimes \mathbb{Z} \\ &= \mathbb{Z}^{12n-1} \end{aligned} \tag{3.275}$$

$$\begin{aligned} H_3(W(n);\mathbb{Z}) &\cong H_3(E(n);\mathbb{Z}) \otimes H_0(T^2;\mathbb{Z}) \oplus H_2(E(n);\mathbb{Z}) \otimes H_1(T^2;\mathbb{Z}) \\ &\qquad \oplus H_1(E(n);\mathbb{Z}) \otimes H_2(T^2;\mathbb{Z}) \oplus H_0(E(n;\mathbb{Z})) \otimes H_3(T^2;\mathbb{Z}) \\ &= 0 \otimes \mathbb{Z} \oplus \mathbb{Z}^{12n-2} \otimes \mathbb{Z}^2 \oplus 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes 0 \\ &= \mathbb{Z}^{4(6n-1)}. \end{aligned} \tag{3.276}$$

Proposition 3.5.13 along with (3.275) and (3.276) yields

$$H_2(\hat{W}(n);\mathbb{Z}) = \mathbb{Z}^{12n} \tag{3.277}$$

$$H_3(\hat{W}(n);\mathbb{Z}) = \mathbb{Z}^{24n+2} \tag{3.278}$$

which implies $b_3(\hat{W}(n)) = 24n + 2$. Due to Theorem 3.4.2, we observe

$$\hat{W}(n) \simeq \hat{W}_0(n) \# (12n+1)(S^3 \times S^3) \tag{3.279}$$

such that $b_3(\hat{W}_0(n)) = 0$. Since we have (see [34, Exercise 3.3.6])

$$H_2(\hat{W}(n);\mathbb{Z}) = H_2(\hat{W}_0(n);\mathbb{Z}) \oplus \underbrace{H_2(S^3 \times S^3;\mathbb{Z}) \oplus \cdots \oplus H_2(S^3 \times S^3;\mathbb{Z})}_{12n+1}, \tag{3.280}$$

taking into account $H_2(S^3 \times S^3;\mathbb{Z}) = 0$, we see that

$$H_2(\hat{W}_0(n);Z) = H_2(\hat{W}(n);\mathbb{Z}) = \mathbb{Z}^{12n}. \tag{3.281}$$

Hence, $b_2(\hat{W}_0(n)) = 12n$. We summarize the work done in this section in the following result, which is Theorem 1.0.2 in the Introduction.

Theorem 3.5.14. *For any integer $n \geq 2$, there exists a stable generalized complex 6-manifold $(\hat{W}(n), \hat{\mathcal{J}}(n), \hat{H}(n))$ with nonempty type change locus such that*

- $\hat{W}(n) \simeq \hat{W}_0(n) \# (12n+1)(S^3 \times S^3)$
- $\pi_1(\hat{W}(n)) \cong \{1\}$
- $\hat{W}_0(n)$ *is a simply connected closed orientable 6-manifold that has* $b_2(\hat{W}_0(n)) = 12n$ *and* $b_3(\hat{W}_0(n)) = 0$.

Chapter 4

Fundamental groups of generalized complex manifolds

It is well known that any finitely presented group is the fundamental group of a manifold equipped with some geometric structure (see e.g. [7, 25, 38, 43, 45]). In this chapter we discuss the realization of any finitely presented group as the fundamental group of a manifold equipped with a generalized complex structure. In Section 4.1, it is shown that any finitely presented group is the fundamental group of a generalized complex manifold of any constant type in dimension at least eight due to the results of Gompf (Theorem 4.1.2) and Taubes (Theorem 4.1.3) (see Corollary 4.1.4). In Section 4.2, we build on Torres's result [45] to show that any finitely presented group is the fundamental group of a generalized complex manifold with nonempty type change locus and which realizes every possible type change jump. Moreover, we prove that any finitely presented group is the fundamental group of a stable generalized complex 6-manifold with nonempty type change locus whose path-connected components are all diffeomorphic to T^4 (Theorem 4.2.8, Theorem 4.2.9 and Corollary 4.2.11).

4.1 Realization of constant types and finitely presented groups

Definition 4.1.1. For a given integer $k \geq 0$, it is said that constant type k is realized if there exists a generalized complex manifold $(M, \mathcal{J}, H)$ such that type($\mathcal{J}$) $= k$.

Any finitely presented group can be realized as the fundamental group of a symplectic $2n-$manifold for $n \geq 2$ [25] and as the fundamental group of a complex $2n-$manifold for $n \geq 3$ [43]. Using Proposition 2.2.2, we can quote these results as the following theorems.

Theorem 4.1.2 ([25, Theorem 0.1])**.** *Let G be a finitely presented group. For any integer $n \geq 2$, constant type 0 is realized by a generalized complex $2n-$manifold $(M, \mathcal{J}, H)$ whose fundamental group is $\pi_1(M) \cong G$.*

Theorem 4.1.3 ([43, Corollary of Theorem 1.1])**.** *Let G be a finitely presented group. For any integer $n \geq 3$, constant type n is realized by a generalized complex $2n-$manifold $(M, \mathcal{J}, H)$ whose fundamental group is $\pi_1(M) \cong G$.*

Combining Theorem 4.1.2 and Theorem 4.1.3, we observe the following immediate consequence.

Corollary 4.1.4. *Let G be a finitely presented group. For any integers $n \geq 4$ and $n \geq k \geq 0$, constant type k is realized by a generalized complex $2n-$manifold $(M, \mathcal{J}, H)$ whose fundamental group is $\pi_1(M) \cong G$.*

A proof of Corollary 4.1.4 is given by taking products of the generalized complex manifolds of Theorem 4.1.2 or Theorem 4.1.3 and 2-spheres (S^2, ω) or (S^2, I) as in the proof of Theorem 4.2.3 and using Proposition 2.2.3.

4.2 Realization of type change jumps and finitely presented groups

Definition 4.2.1. For any integers $0 \leq i < j$ of the same parity, it is said that the type change jump $i \mapsto j$ is realized if there exists a generalized complex manifold $(M, \mathcal{J}, H)$ which contains the type change locus where type($\mathcal{J}$) jumps from i to j.

Regarding the realization of any type change jump and any finitely presented group, the following result was reported by Torres.

Theorem 4.2.2 ([45, Theorem 16])**.** *Let G be a finitely presented group. For any integer $n \geq 2$, the type change jump $0 \mapsto 2$ is realized by a stable generalized complex $2n$-manifold $(M, \mathcal{J}, H)$ whose fundamental group is $\pi_1(M) \cong G$ and its Euler characteristic satisfies $\chi(M) \geq 2^{n+1}$. In particular, for $n = 2$, M is neither symplectic nor complex.*

Type change jumps other than $0 \mapsto 2$ can not be realized by a stable generalized complex manifold due to Corollary 2.3.4. Such type change jumps are realized by unstable generalized complex manifolds. In this direction, we obtain the following result.

Theorem 4.2.3. *Let G be a finitely presented group. For any integer $n \geq 4$, there exists a family of the generalized complex $2n-$manifolds*

$$\left\{(M_k, \mathcal{J}_k^i, H)|\pi_1(M_k) \cong G, k = 0, 1, \cdots, \left[\frac{n-2}{2}\right], i = 0, 1, \cdots, n-2-2k\right\} \tag{4.1}$$

which realizes all possible type change jumps.

Proof. Theorem 4.2.2 states that the type change jump $0 \mapsto 2$ is realized by a stable generalized complex 4-manifold $(M, \mathcal{J}, H)$ with $\pi_1(M) \cong G$. Since the complex projective line $\mathbb{C}P^1$ is a Kähler manifold, it admits both symplectic and complex structures. So we have two generalized complex structures $(\mathbb{C}P^1, \mathcal{J}_l, 0)$ of constant type$(\mathcal{J}_l) = l$ with $l = 0, 1$ (see Examples 2 and 3). Moreover, the type change jump $0 \mapsto 2$ can be realized by the generalized complex manifold $(\mathbb{C}P^2, \mathcal{J}_{0\mapsto 2}, 0)$ [31, Example 5.6] . For a given integer $[\frac{n-2}{2}] \geq k \geq 0$, we define a $2n-$manifold

$$M_k := M \times \underbrace{\mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2}_{k} \times \underbrace{\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1}_{n-2-2k}. \tag{4.2}$$

We have that $\pi_1(M_k) \cong G$ by [34, Proposition 1.12]. For any integer $n - 2 - 2k \geq i \geq 0$, the generalized complex manifolds $(M_k, \mathcal{J}_k^i, H)$ realize the type change jumps $i \mapsto i + 2 + 2k$ by Proposition 2.2.3, where

$$\mathcal{J}_k^i = \mathcal{J} \times \underbrace{\mathcal{J}_{0\mapsto 2} \times \cdots \times \mathcal{J}_{0\mapsto 2}}_{k} \times \underbrace{\mathcal{J}_1 \times \cdots \times \mathcal{J}_1}_{i} \times \underbrace{\mathcal{J}_0 \times \cdots \times \mathcal{J}_0}_{n-2-2k-i}. \tag{4.3}$$

It is immediate to check that the family (4.1) realizes all possible type change jumps for $n \geq 4$. Therefore we prove Theorem 4.2.3. □

Remark 4.2.4. The generalized complex structures of Theorem 4.2.2 are product structures of the form $(M_0, \mathcal{J}_0^0, H)$, where M_0 is given by (4.2) for $k = 0$ and $\mathcal{J}_0^0$ is given by (4.3) for $i = k = 0$. So each path-connected component on the type change locus of the $2n-$manifolds of Theorem 4.2.2 is diffeomorphic to $T^2 \times \underbrace{\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1}_{n-2}$. The following result shows that in dimension 6, any finitely presented group can be realized as the fundamental group of a stable generalized complex 6-manifold such that each path-connected component on the type change locus is diffeomorphic to T^4. This result can be extended to every dimension $2n \geq 8$ where the type change locus is diffeomorphic to T^{2n-2}.

Theorem 4.2.5. *Let G be a finitely presented group. There exists a stable generalized complex 6-manifold $(\hat{M}, \hat{\mathcal{J}}, \hat{H})$ with nonempty type change locus such that the fundamental group is $\pi_1(\hat{M}) \cong G$ and the Euler characteristic is $\chi(\hat{M}) = 0$. Moreover, each path-connected component on the type change locus is diffeomorphic to T^4.*

Proof. Before going into the proof of Theorem 4.2.5, we first prove the following auxiliary lemma.

Lemma 4.2.6. *Let G be a finitely presented group. There exists a closed symplectic 4-manifold M_G such that*

(i) $\pi_1(M_G) \cong G$

(ii) M_G *contains two disjoint symplectic 2-tori* T_1 *and* T_2 *with* $self-intersection$ *zero*

(iii) the inclusions $j_{T_i} : T_i \hookrightarrow M_G$ *for* $i = 1, 2$ *induce group homomorphisms*

$$j_{T_i*} : \pi_1(T_i) \to \pi_1(M_G) \tag{4.4}$$

which are trivial

(iv) the inclusion $h : (M_G - (T_1 \sqcup T_2)) \hookrightarrow M_G$ *induces a group homomorphism*

$$h_* : \pi_1((M_G - (T_1 \sqcup T_2))) \to \pi_1(M_G) \tag{4.5}$$

which is an isomorphism.

Proof. Due to Gompf's result [25], we can find a closed symplectic 4-manifold M_0 with $\pi_1(M_0) \cong G$ which contains a symplectic 2-torus T_0 with self-intersection zero such that the inclusion $i_{T_0} : T_0 \hookrightarrow M_0$ induces a group homomorphism

$$i_{T_0*} : \pi_1(T_0) \to \pi_1(M_0) \tag{4.6}$$

which is trivial. Since T_0 is codimension two submanifold in M_0, any loop in M_0 can be perturbed to be disjoint from T_0, so the inclusion $i : (M_0 - T_0) \hookrightarrow M_0$ induces a group homomorphism

$$i_* : \pi_1(M_0 - T_0) \to \pi_1(M_0) \cong G \tag{4.7}$$

which is surjective, and its kernel is generated by the meridian $\mu_0 \subset \partial(M_0 - T_0)$ of T_0. By the first isomorphism theorem (see e.g. [22, Theorem 16]), we get

$$\frac{\pi_1(M_0 - T_0)}{\{\mu_0\}} \cong G. \tag{4.8}$$

Let $E(2)$ denote the symplectic sum [25] of two copies of the elliptic surface $E(1)$ (see Appendix B). By Lemma B.0.4, the symplectic 4-manifold $E(2)$ contains three symplectic tori T_i for $i = 1, 2, 3$ with self-intersection zero and the inclusion $j : (E(2) - T_1 \sqcup T_2 \sqcup T_3) \hookrightarrow E(2)$ induces a group isomorphism

$$j_* : \pi_1(E(2) - (T_1 \sqcup T_2 \sqcup T_3)) \to \pi_1(E(2)) \cong \{1\} \tag{4.9}$$

which implies

$$\pi_1(E(2) - T_3) \cong \pi_1(E(2)) \cong \{1\}. \tag{4.10}$$

Now we perform the symplectic sum of M_0 and $E(2)$ along the 2-tori T_0 and T_3, and call it M_G, i.e.,

$$M_G = M_0 \#_{T^2} E(2). \tag{4.11}$$

We claim $\pi_1(M_G) \cong G$. Indeed, Seifert-van Kampen theorem yields

$$\pi_1(M_G) \cong \frac{\pi_1(M_0 - T_0) * \pi_1(E(2) - T_3)}{N_G}. \tag{4.12}$$

Here N_G denotes the normal subgroup of $\pi_1(M_0 - T_0) * \pi_1(E(2) - T_3)$. The generators of N_G take the form $i_{C*}(a) j_{C*}^{-1}(a)$, where

$$i_{C*} : \pi_1(C) \to \pi_1(M_0 - T_0), \tag{4.13}$$

$$j_{C*} : \pi_1(C) \to \pi_1(E(2) - T_3) \tag{4.14}$$

are homomorphisms induced by the inclusions $i_C : C \hookrightarrow (M_0 - T_0)$ and $j_C : C \hookrightarrow (E(2) - T_3)$, with

$$C = (M_0 - T_0) \cap (E(2) - T_3) \simeq T^3. \tag{4.15}$$

Let α_0, $\beta_0 \subset \partial(M_0 - T_0)$ be the push offs of the generators of $\pi_1(T_0)$, which are null homotopic in $(M_0 - T_0)$, since the homomorphism (4.6) is trivial and the kernel of the homomorphism (4.7) is solely generated by the meridian μ_0 (see (4.8)). Let $\mu_3 \subset \partial(E(2) - T_3)$ denote the meridian of T_3 and α_3, $\beta_3 \subset$

$\partial(E(2) - T_3)$ the push offs of the generators of $\pi_1(T_3)$, which are all null homotopic in $(E(2) - T_3)$ due to (4.10). The gluing diffeomorphism

$$\varphi : \partial D^2 \times T_0 \to \partial D^2 \times T_3 \tag{4.16}$$

can be chosen such that the generators of the fundamental groups are identified as follows

$$\begin{aligned} \mu_0 &\mapsto \mu_3 = 1 \\ 1 = \alpha_0 &\mapsto \alpha_3 = 1 \\ 1 = \beta_0 &\mapsto \beta_3 = 1. \end{aligned} \tag{4.17}$$

Accordingly, the normal subgroup N_G is generated by

$$\{\mu_0\mu_3^{-1} = \mu_0,\ \alpha_0\alpha_3^{-1} = 1,\ \beta_0\beta_3^{-1} = 1\}. \tag{4.18}$$

Therefore, from (4.8), (4.10), (4.12) and (4.18), we conclude

$$\pi_1(M_G) \cong G \tag{4.19}$$

as its was claimed. Moreover, the symplectic sum in (4.11) is performed along T_3 in $E(2)$ which is disjoint from T_1, T_2 and their dual spheres. This ensures along with Proposition B.0.3 that M_G contains the symplectic 2-tori T_1 and T_2 which satisfy the conditions $(ii) - (iv)$ of Lemma 4.2.6. Thus we complete the proof of Lemma 4.2.6. □

Let us get back to the proof of Theorem 4.2.5 and define a closed symplectic 6-manifold

$$M = M_G \times T^2. \tag{4.20}$$

Due to Lemma 4.2.6, M contains two disjoint symplectic 4-tori $T_1 \times T^2$ and $T_2 \times T^2$ with trivial normal bundles. We perform simultaneously two torus surgeries of multiplicity zero to M along $T_1 \times T^2$ and $T_2 \times T^2$ to obtain a stable generalized complex 6-manifold $(\hat{M}, \hat{\mathcal{J}}, \hat{H})$ with nonempty type change locus (cf. Theorem 3.2.2). The type change locus has two path-connected components each of which is diffeomorphic to T^4. Moreover, by Lemma 4.2.6 M_G satisfies the criteria (ii)-(iv) in Remark 3.5.5, so it immediately follows from Lemma 3.5.6 (see also [46, Proposition 1]) that

$$\pi_1(\hat{M}) \cong \pi_1(M_G) \cong G. \tag{4.21}$$

As for the Euler characteristic of $\hat{M}$, we first compute the Euler characteristic of M

$$\chi(M) = \chi(M_G \times T^2) = \chi(M_G) \times \chi(T^2) = 0, \tag{4.22}$$

where we used the multiplicativity of the Euler characteristic (see [32, Exercise 3.3.13]) and $\chi(T^2) = 0$. Then we immediately obtain

$$\chi(\hat{M}) = \chi(M) = 0 \tag{4.23}$$

due to Proposition 3.2.1. Thus, we complete the proof of Theorem 4.2.5. □

Remark 4.2.7. Note that the Euler characteristic of the stable generalized complex 6-manifold $(\hat{M}, \hat{\mathcal{J}}, \hat{H})$ given in Theorem 4.2.5 is different from that of $(M, \mathcal{J}, H)$ given in Theorem 4.2.2, so these manifolds are topologically distinguished.

The type change locus of the stable generalized complex 6-manifold of Theorem 4.2.5 has two path-connected components. We can construct a stable generalized complex 6-manifold which has the same properties but arbitrary path-connected components on the type change locus, by using the symplectic sum of stable generalized complex manifolds described in Section 3.1, so we prove the following result.

Theorem 4.2.8. *Let G be a finitely presented group. For any $k \in \mathbb{N}$, there is a stable generalized complex 6-manifold $(\hat{M}(k), \hat{\mathcal{J}}(k), \hat{H}(k))$ such that*

- $\pi_1(\hat{M}(k)) \cong G$
- $\chi(\hat{M}(k)) = 0$
- *there are k path-connected components on the type change locus, each of which is diffeomorphic to T^4.*

Proof. We would like to construct a stable generalized complex 6-manifold $(\hat{M}(k), \hat{\mathcal{J}}(k), \hat{H}(k))$ by performing the symplectic sum of two stable generalized complex 6-manifolds along T^4, i.e.,

$$\hat{M}(k) = M_1(k) \sharp_{T^4} M_2. \tag{4.24}$$

Here $M_1(k)$ and M_2 are chosen as follows. Let $(X(k), \mathcal{J}_X(k), H_X(k))$ be a stable generalized complex 4-manifold with $k \in \mathbb{N}$ path-connected components on the type change locus and the fundamental group $\pi_1(X(k)) \cong G$ constructed by Torres (see [45, Section 3 and Remark 17]). Each path-connected component on the type change locus is diffeomorphic to T^2. Also, $X(k)$ contains a $\mathcal{J}_\omega$−symplectic 2-torus T with self-intersection zero which is disjoint from the type change locus and the inclusion $i : (X(k) - T) \hookrightarrow X(k)$ induces a group homomorphism

$$i_* : \pi_1(X(k) - T) \to \pi_1(X(k)) \cong G \tag{4.25}$$

which is an isomorphism. Define

$$M_1(k) = X(k) \times T^2 \tag{4.26}$$

and equip it with a generalized complex structure $(M_1(k), \mathcal{J}_{M_1}(k), H_1(k))$ which is a product of the generalized complex structure $(X(k), \mathcal{J}_X(k), H_X(k))$ and a constant type 0 generalized complex structure $(T^2, \mathcal{J}_{\omega_{T^2}}, 0)$ induced by a symplectic structure ω_{T^2} on T^2. Then $(M_1(k), \mathcal{J}_{M_1}(k), H_1(k))$ is stable by Proposition 2.3.5 and its type change locus has k path-connected components each of which is diffeomorphic to $T^2 \times T^2 \simeq T^4$. Moreover, it contains a $\mathcal{J}_\omega-$symplectic 4-torus $T \times T^2$ with trivial normal bundle. We compute

$$\begin{aligned}\pi_1(M_1(k) - T \times T^2) &= \pi_1(X(k) \times T^2 - T \times T^2) \cong \pi_1(X(k) \times T^2)\\ &\cong \pi_1(X(k)) \times \pi_1(T^2) \cong G \times \langle \alpha_1, \beta_1 | [\alpha_1, \beta_1] \rangle. \end{aligned} \tag{4.27}$$

Here we used that the homomorphism (4.25) is an isomorphism and [34, Proposition 1.12], and

$$\alpha_1, \beta_1 \subset \{pt\} \times T^2 \subset (X - T) \times T^2 \tag{4.28}$$

are the loops generating $\pi_1(T^2)$. Concerning M_2, we define

$$M_2 = E(1) \times T^2 \tag{4.29}$$

with $E(1)$ the elliptic surface defined in Appendix B and equip it with a constant type 0 generalized complex structure $(M_2, \mathcal{J}_{M_2}, 0)$ that arises from a symplectic structure $p_1^*\omega_{E(1)} + p_2^*\omega_{T^2}$ with the projections $p_1 : M_2 \to E(1)$, $p_2 : M_2 \to T^2$. Since $E(1)$ contains a symplectic 2-torus F with self-intersection zero such that

$$\pi_1(E(1) - F) \cong \pi_1(E(1)) \cong \{1\} \tag{4.30}$$

(see Lemma (B.0.4)), $(M_2, \mathcal{J}_{M_2}, 0)$ contains a $\mathcal{J}_\omega-$symplectic 4-torus $F \times T^2$ with trivial normal bundle and using [34, Proposition 1.12], we compute

$$\begin{aligned}\pi_1(M_2 - F \times T^2) &= \pi_1(E(1) \times T^2 - F \times T^2)\\ &\cong \pi_1(E(1)) \times \pi_1(T^2) \cong \{1\} \times \langle \alpha_2, \beta_2 | [\alpha_2, \beta_2] \rangle\\ &= \langle \alpha_2, \beta_2 | [\alpha_2, \beta_2] \rangle \end{aligned} \tag{4.31}$$

where

$$\alpha_2, \beta_2 \subset \{pt\} \times T^2 \subset (E(1) - F) \times T^2 \tag{4.32}$$

represent the loops generating $\pi_1(T^2)$.

We now perform the symplectic sum (4.24) by identifying $T \times T^2 \subset M_1(k)$ with $F \times T^2 \subset M_2$, which gives rise to a stable generalized complex structure $(\hat{M}(k), \hat{\mathcal{J}}(k), \hat{H}(k))$. The gluing diffeomorphism

$$\varphi : \partial D^2 \times T \times T^2 \to \partial D^2 \times F \times T^2 \tag{4.33}$$

can be chosen such that $\pi_1(\hat{M}(k)) \cong G$. Let

$$\gamma_1, \delta_1 \subset \partial(X(k) - T) \times \{pt\} \subset M_1(k) - T \times T^2 \tag{4.34}$$

denote the push offs of the generators of $\pi_1(T)$ and μ_1 the meridian of $T \times T^2$ in $\partial(M_1(k) - T \times T^2)$, then they are all null homotopic in $M_1(k) - T \times T^2$ by construction. Denote

$$\gamma_2, \delta_2 \subset \partial(E(1) - F) \times \{pt\} \subset M_2 - F \times T^2 \tag{4.35}$$

the push offs of the generators of $\pi_1(F)$ and μ_2 the meridian of $F \times T^2$ in $\partial(M_2 - F \times T^2)$. Then, the loops $\gamma_2, \delta_2, \mu_2$ are all null homotopic in $M_2 - F \times T^2$ (see (4.31)). Now we choose the gluing diffeomorphism φ such that the generators of the fundamental groups are identified as follows:

$$\begin{aligned} 1 = \mu_1 &\mapsto \mu_2 = 1 \\ \alpha_1 &\mapsto \gamma_2 = 1 \\ \beta_1 &\mapsto \delta_2 = 1 \\ 1 = \gamma_1 &\mapsto \alpha_2 \\ 1 = \delta_1 &\mapsto \beta_2 \end{aligned} \tag{4.36}$$

This results in the generators α_1, β_1, α_2, β_2 being killed during the gluing. Thus, taking into account (4.27) and (4.31), Seifert-van Kampen theorem implies

$$\pi_1(\hat{M}(k)) \cong G \tag{4.37}$$

as desired.

Since the type change locus is disjoint from the attaching region, the type change locus of $M_1(k)$ remains unchanged under the symplectic sum. As a consequence, the stable generalized complex manifold $(\hat{M}(k), \hat{\mathcal{J}}(k), \hat{H}(k))$ has nonempty type change locus with k path-connected components each of which is diffeomorphic to T^4. Well known properties of the Euler characteristic (see [42, Exercise 4.B.2] and [32, Exercise 3.3.13]) imply

$$\chi(\hat{M}(k)) = \chi(M_1(k)) + \chi(M_2) - 2\chi(T^4)$$

$$= \chi(X(k) \times T^2) + \chi(E(1) \times T^2)$$
$$= \chi(X(k)) \times \chi(T^2) + \chi(E(1)) \times \chi(T^2) = 0. \quad (4.38)$$

Thus, we prove Theorem 4.2.8. □

The stable generalized complex manifolds of Theorem 4.2.5 and Theorem 4.2.8 have zero Euler characteristic. We finish this section by giving a proof of the following result.

Theorem 4.2.9. *Let G be a finitely presented group. For any pair (n, k) of natural numbers, there exists a stable generalized complex 6-manifold $\big(\hat{Z}(n,k), \hat{\mathcal{J}}(n,k), \hat{H}(n,k)\big)$ such that*

- $\pi_1(\hat{Z}(n,k)) \cong G$
- $\chi(\hat{Z}(n,k)) = -24n$
- *there are k path-connected components on the type change locus, each of which is diffeomorphic to T^4.*

Proof. The proof is pretty much similar to that of Theorem 4.2.8. We would like to construct a stable generalized complex 6-manifold which is the symplectic sum

$$\hat{Z}(n,k) = Z_1(n) \#_{T^2 \times \Sigma_2} Z_2(k). \quad (4.39)$$

Here

$$Z_1(n) = E(n) \times \Sigma_2 \quad (4.40)$$

which admits a stable generalized complex structure $(Z_1(n), \mathcal{J}_{Z_1}(n), H_{Z_1}(n))$ of constant type zero which arises from a symplectic structure on it. Moreover, since a generic fiber $F \subset E(n)$ can be symplectically embedded in $E(n)$ with self-intersection zero, there is a $\mathcal{J}_\omega$−symplectic embedding of $F \times \Sigma_2$ in $(Z_1(n), \mathcal{J}_{Z_1}(n), H_{Z_1}(n))$ with trivial normal bundle. By Lemma B.0.4 and [34, Proposition 1.12], we compute

$$\pi_1(Z_1(n) - F \times \Sigma_2) = \pi_1(E(n) \times \Sigma_2 - F \times \Sigma_2)$$
$$\cong \pi_1(E(n)) \times \pi_1(\Sigma_2) \cong \pi_1(\Sigma_2). \quad (4.41)$$

To construct the stable generalized complex 6-manifold $(Z_2(k), \mathcal{J}_{Z_2}(k), H_{Z_2}(k))$, we use the following result due to Gompf.

Lemma 4.2.10 ([25, Lemma 5.5])**.** *There exists a simply connected symplectic 4-manifold $S_{1,1}$ such that*

- *there are disjoint symplectic embeddings of a 2-torus F_1 and a genus 2 surface F_2 with self-intersection zero*
- *the inclusion $i : (S_{1,1} - (F_1 \sqcup F_2)) \hookrightarrow X$ induces a group homomorphism*

$$i_* : \pi_1(S_{1,1} - (F_1 \sqcup F_2)) \to \pi_1(S_{1,1}) \cong \{1\} \tag{4.42}$$

which is an isomorphism.

For any $k \in \mathbb{N}$, we can find a stable generalized complex 4-manifold $(X(k), \mathcal{J}_X(k), H_X(k))$ with k path-connected components on its type change locus and $\pi_1(X(k)) \cong G$ (see [45, Section 3 and Remark 17]). Let $T \subset X(k)$ be a $\mathcal{J}_\omega$−symplectic 2-torus with self-intersection zero (see exposition below (4.24)). Perform the symplectic sum

$$\tilde{Z}_2(k) = S_{1,1} \#_{T^2} X(k), \tag{4.43}$$

by identifying $F_1 \subset S_{1,1}$ with $T \subset X(k)$. $\tilde{Z}_2(k)$ admits the stable generalized complex structure $(\tilde{Z}_2(k), \mathcal{J}_{\tilde{Z}_2}(k), H_{\tilde{Z}_2}(k))$ with k path-connected components on the type change locus due to Theorem 3.1.1. We have

$$\pi_1(S_{1,1} - F_1) \cong \{1\} \tag{4.44}$$

due to Lemma 4.2.10 and the push offs of the generators of $\pi_1(F_1)$ in $S_{1,1} - F_1$ are automatically null homotopic. As we already discussed in the proof of Theorem 4.2.8,

$$\pi_1(X(k) - T) \cong G \tag{4.45}$$

from (4.25) and the push offs of the generators of $\pi_1(T)$ in $X(k) - T$ are null homotopic. Hence, an application of Seifert-van Kampen theorem yields

$$\pi_1(\tilde{Z}_2(k)) \cong G. \tag{4.46}$$

Moreover, due to Lemma 4.2.10, the genus 2-surface $F_2 \subset S_{1,1}$ is disjoint from F_1, so F_2 is $\mathcal{J}_\omega$−symplectically embedded in $\tilde{Z}_2(k)$ with self-intersection zero and the inclusion $j : (\tilde{Z}_2(k) - F_2) \hookrightarrow \tilde{Z}_2(k)$ induces a group isomorphism

$$j_* : \pi_1(\tilde{Z}_2(k) - F_2) \to \pi_1(\tilde{Z}_2(k)) \cong G. \tag{4.47}$$

Define

$$Z_2(k) = \tilde{Z}_2(k) \times T^2, \tag{4.48}$$

which admits the stable generalized complex structure $(Z_2(k), \mathcal{J}_{Z_2}(k), H_{Z_2}(k))$ by Proposition 2.3.5 that is a product of $(\tilde{Z}_2(k), \mathcal{J}_{\tilde{Z}_2}(k), H_{\tilde{Z}_2}(k))$ and a constant type 0 generalized complex structure $(T^2, \mathcal{J}_{T^2}, 0)$ induced from a symplectic structure on T^2. Moreover, there is a $\mathcal{J}_\omega$−symplectic embedding of

$F_2 \times T^2$ with trivial normal bundle. Using the isomorphism (4.47) and [34, Proposition 1.12], we compute

$$\pi_1(Z_2(k) - F_2 \times T^2) \cong \pi_1(\tilde{Z}_2(k)) \times \pi_1(T^2) \cong G \times \pi_1(T^2) \tag{4.49}$$

Now we perform the symplectic sum (4.39) by the gluing diffeomorphism

$$\varphi : \partial D^2 \times F \times \Sigma_2 \to \partial D^2 \times T^2 \times F_2. \tag{4.50}$$

Let

$$\alpha_1, \beta_1 \subset \partial(Z_1(n) - F \times \Sigma_2) \tag{4.51}$$

be the push offs of the generators of $\pi_1(F)$ and μ_1 the meridian of $F \times \Sigma_2$, which are null homotopic in $Z_1(n) - F \times \Sigma_2$. Let

$$a_1, b_1, c_1, d_1 \subset \partial(Z_1(n) - F \times \Sigma_2) \tag{4.52}$$

be the push offs of the generators of $\pi_1(\Sigma_2)$ and

$$\alpha_2, \beta_2 \subset \partial(Z_2(k) - F_2 \times T^2) \tag{4.53}$$

the push offs of the generators of $\pi_1(T^2)$. Denote by

$$a_2, b_2, c_2, d_2 \subset \partial(Z_2(k) - F_2 \times T^2) \tag{4.54}$$

the push offs of the generators of $\pi_1(F_2)$ and μ_2 the meridian of $F_2 \times T^2$, which are all null homotopic in $Z_2(k) - F_2 \times T^2$. We can choose the gluing diffeomorphism φ such that the generators of the fundamental groups are identified as follows

$$\begin{aligned} 1 = \mu_1 &\mapsto \mu_2 = 1 \\ 1 = \alpha_1 &\mapsto \alpha_2 \\ 1 = \beta_1 &\mapsto \beta_2 \\ a_1 &\mapsto a_2 = 1 \\ b_1 &\mapsto b_2 = 1 \\ c_1 &\mapsto c_2 = 1 \\ d_1 &\mapsto d_2 = 1. \end{aligned} \tag{4.55}$$

As a consequence, the generators $a_1, b_1, c_1, d_1, \alpha_2, \beta_2$ are killed during the gluing. Taking into account (4.41) and (4.49), Seifert-van Kampen theorem implies

$$\pi_1(\hat{Z}(n,k)) \cong G. \tag{4.56}$$

Using the well known properties of the Euler characteristic, we compute (see (4.38))

$$\begin{aligned}\chi(\hat{Z}(n,k)) &= \chi(Z_1(n)) + \chi(Z_2(k)) - 2\chi(T^2 \times \Sigma_2) \\ &= \chi(E(n) \times \Sigma_2) + \chi(\tilde{Z}_2(k) \times T^2) - 2\chi(T^2) \times \chi(\Sigma_2) \\ &= -2\chi(E(n)) + \chi(\tilde{Z}_2(k)) \times \chi(T^2) = -24n. \end{aligned} \tag{4.57}$$

By construction, $(\hat{Z}(n,k), \hat{\mathcal{J}}(n,k), \hat{H}(n,k))$ has the type change locus with k path-connected components each of which is diffeomorphic to T^4. Thus, we prove Theorem 4.2.9. □

Notice that the Euler characteristic of $\hat{Z}(n,k)$ is controlled by $\chi(E(n))$. As it was pointed out in [9], the symplectic 4-manifold B given in Proposition 3.5.1 can be used instead of $E(1)$. Define $P(n)$ as the $n-$fold symplectic sum of B along an embedded 2-torus in B. Using the properties of the Euler characteristic, we can show (cf. [27, Proposition 3.1.11])

$$\chi(P(n)) = 8n. \tag{4.58}$$

Then we immediately obtain the following result.

Corollary 4.2.11. *Let G be a finitely presented group. For any pair (n,k) of natural numbers, there exists a stable generalized complex 6-manifold $\big(\hat{Z}(n,k), \hat{\mathcal{J}}(n,k), \hat{H}(n,k)\big)$ such that*

- $\pi_1(\hat{Z}(n,k)) \cong G$
- $\chi(\hat{Z}(n,k)) = -16n$
- *there are k path-connected components on the type change locus, each of which is diffeomorphic to T^4.*

Appendix A

Pullback of differential forms

We briefly recall the computation of pullback of differential forms. For the basics of differential topology we refer to [42].

Let M, N be $n-$dimensional manifolds and $\phi : M \to N$ be a diffeomorphism. Consider a differential form $\omega_N \in \Gamma(\Lambda^\bullet T^*N)$, then its pullback at a point $p \in M$ is defined as

$$\phi^*\omega_N(u)|_p = \omega_N(\phi_* u)|_{\phi(p)}, \tag{A.1}$$

where $u \in \Gamma(TM)$ is a vector field. For simplicity, let us consider a basic 1-form $\omega_N = dy_i$. Since $\phi^*\omega_N \in \Gamma(\Lambda^\bullet T^*M)$, we have

$$\phi^* dy_i|_p = \sum_{j=1}^{n} \alpha_j dx_j|_p, \tag{A.2}$$

where $\{dx_j|_p\}$ forms a basis of T_p^*M. We can determine α_j by using duality between 1-forms and vector fields, together with (A.1),

$$\alpha_j|_p = \phi^* dy_i \left(\frac{\partial}{\partial x_j} \right) \Bigg|_p = dy_i \left(\phi_* \frac{\partial}{\partial x_j} \right) \Bigg|_p. \tag{A.3}$$

Using (A.2) and (A.3), we obtain the pullback of differential forms with any degrees due to the proposition quoted below (see [42, p. 164] for a proof).

Proposition A.0.1. *For any differential forms ω_1 and ω_2 on N, the following hold*

$$\phi^*(\omega_1 + \omega_2) = \phi^*\omega_1 + \phi^*\omega_2, \tag{A.4}$$

$$\phi^*(\omega_1 \wedge \omega_2) = \phi^*\omega_1 \wedge \phi^*\omega_2. \tag{A.5}$$

Appendix B

Elliptic surfaces: E(n)

In this appendix, we discuss some properties of elliptic surfaces which are frequently used as the building blocks in several sections to produce a stable generalized complex 6-manifold. For details, we kindly refer to [27, Section 3.1].

We begin with the definition of an elliptic surface.

Definition B.0.1. An elliptic surface is a complex surface S which has a holomorphic map $\pi : S \to C$ to a complex curve C such that for a generic point $p \in C$, the inverse image $\pi^{-1}(p)$ is a smooth elliptic curve. And the map π is called an elliptic fibration.

Since an elliptic curve is topologically a 2-torus, a generic fiber of the elliptic fibration is a 2-torus.

A basic example of an elliptic surface is $E(1) := \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, which is obtained by blowing up $\mathbb{C}P^2$ at 9 points. The following properties of $E(1)$ are particularly useful for our purpose.

Lemma B.0.2 ([25, Lemma 3.16])**.** *The elliptic surface $E(1)$ is simply connected and has a homologically essential generic fiber $F \simeq T^2$ of the elliptic fibration $\pi : E(1) \to \mathbb{C}P^1$ such that $[F]^2 = 0$ and F intersects an embedded 2-sphere in $E(1)$ transversely in exactly one point.*

Every simply connected complex surface is Kähler [11], so $E(1)$ admits a symplectic structure as well. Since a generic fiber $F \subset E(1)$ is homologically essential, we can make it symplectic submanifold by perturbing a symplectic form on $E(1)$ by [25, Lemma 1.6]. Moreover, we observe the complement $E(1) - F$ is simply connected from the following result (see [3] for a proof).

Proposition B.0.3. *Let X be a smooth manifold and $Y \subset X$ a codimension 2 submanifold with trivial normal bundle. If there exists an embedded 2-sphere $S^2 \subset X$ which intersects transversely Y in exactly one point, the inclusion $i : (X - Y) \hookrightarrow X$ induces a group homomorphism*

$$i_* : \pi_1(X - Y) \to \pi_1(X), \tag{B.1}$$

which is an isomorphism.

Let $E(n)$ denote the $n-$fold symplectic sum [25] of $E(1)$ along the generic fiber F. The homology groups of $E(n)$ are given by (see [27, Proposition 3.1.11])

$$H_0(E(n); \mathbb{Z}) = H_4(E(n); \mathbb{Z}) = \mathbb{Z} \tag{B.2}$$
$$H_1(E(n); \mathbb{Z}) = H_3(E(n); \mathbb{Z}) = 0 \tag{B.3}$$
$$H_2(E(n); \mathbb{Z}) = \mathbb{Z}^{12n-2}. \tag{B.4}$$

Moreover, $E(n)$ contains $(2n-1)$ homologically essential 2-tori which have self-intersection zero and intersect their own dual 2-spheres in exactly one point [27, p. 73]. These tori can be assumed to be symplectic due to [25, Lemma 1.6] and the complement of their disjoint union is simply connected by Proposition B.0.3. For the convenience, we summarize the observation of this section in the following lemma.

Lemma B.0.4. *The elliptic surface $E(n)$ contains $(2n-1)$ symplectic 2-tori (torus) T_i for $i = 1, 2, \cdots, (2n-1)$ with self intersection zero such that the inclusion $i : (E(n) - (T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{2n-1})) \hookrightarrow E(n)$ induces a group homomorphism*

$$i_* : \pi_1(E(n) - (T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{2n-1})) \to \pi_1(E(n)) \cong \{1\} \tag{B.5}$$

which is an isomorphism.

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