Rend. Istit. Mat. Univ. Trieste
Volume 50 (2018), 149–159
DOI: 10.13137/2464-8728/22434

Tannakian categories, fundamental groups and Higgs bundles

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Abstract. *After recalling the basic notions concerning profinite and proalgebraic group completions and Tannakian categories, we review how the latter can be used to define generalizations of the notion of fundamental group of a space, such as the Nori and Langer fundamental groups, and the algebraic fundamental group introduced by Simpson. Then we discuss how one can define a Tannakian category whose objects are Higgs bundles on a complex projective variety that are "numerically flat" in a suitable sense, and show how the Higgs fundamental group is related to a conjecture about semistable Higgs bundles.*

Keywords: Fundamental groups, Tannakian categories, Higgs bundles, curve semistability.
MS Classification 2010: 14F05, 14F35, 14H60, 14J60, 18D35.

1. Introduction

Tannakian categories are abelian tensor categories that satisfy some additional properties and are equipped with a functor to the category of vector spaces. They all turn out to be equivalent to categories of representations of proalgebraic affine group schemes, so that there is natural duality between Tannakian categories and such group schemes. This "Tannaka duality" has been used to devise generalizations of the notion of fundamental group, with the purpose of better capturing the geometry of such geometric structures as schemes and algebraic varieties. A classical example is the Nori fundamental group [21, 22], and more recently, the S-fundamental group introduced by Langer [16, 17]. The latter is the Tannaka dual of the category of numerically flat vector bundles, i.e., vector bundles that are numerically effective together with their duals (this group was introduced in the case of curves also in [5]). C. Simpson considered the category of semi-harmonic bundles on a smooth projective variety over $\mathbb{C}$, i.e., semistable Higgs vector bundles with vanishing rational Chern classes [24, 25]. The resulting fundamental group scheme is a proalgebraic completion of the topological fundamental group. Since flat (Higgs) bundles are essentially finite, numerically flat, and semi-harmonic, and the topological fundamental group represents the category of flat bundles, there is a natural morphism from the usual fundamental group to each of these groups.

Notions of numerical effectiveness and numerical flatness for Higgs bundles were introduced in [6, 7] , motivated by the remark that the universal quotient bundles over the Grassmann bundles $\mathrm{Gr}_k(E)$ of a numerically effective vector bundle are numerically effective. Given a Higgs vector bundle $\mathfrak{E} = (E, \phi)$, we consider closed subschemes $\mathfrak{Gr}_k(\mathfrak{E}) \subset \mathrm{Gr}_k(E)$ that parameterize locally free Higgs quotients on $\mathfrak{E}$. Then $\mathfrak{E}$ is said to be H-numerically effective if the universal Higgs quotients on $\mathfrak{Gr}_k(\mathfrak{E})$ are H-numerically effective, according to a definition which is recursive on the rank. Finally, a Higgs bundle is said to be H-numerically flat if $\mathfrak{E}$ and its dual Higgs bundle $\mathfrak{E}^*$ are H-numerically effective. H-numerically flat Higgs bundles make up again a neutral Tannakian category; the corresponding group scheme is denoted $\pi_1^H(X,x)$ [4].

Numerically flat vector bundles equipped with the zero Higgs field are H-numerically flat, hence there is a faitfhfully flat morphism $\pi_1^H(X,x) \to \pi_1^S(X,x)$. The relation of $\pi_1^H(X,x)$ with Simpson's proalgebraic fundamental group $\pi_1^{\mathrm{alg}}(X,x)$ is more subtle: semi-harmonic bundles are H-numerically flat, so that there is faitfhfully flat morphism $\pi_1^H(X,x) \to \pi_1^{\mathrm{alg}}(X,x)$. The fact that the groups may be isomorphic is related with a conjecture about the so-called *curve semistable Higgs bundles* — i.e., Higgs bundles that are semistable after pullback to any smooth projective curve [7, 11, 18] (Conjecture 4.7 in the text). This conjecture states that if a Higgs bundle (E, ϕ) on a projective variety is semistable after pullback to any projective curve, then its rational characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2$$

vanishes (here $r = \mathrm{rk}\, E$).

2. Completions

Generalized fundamental groups are defined in terms of, or are related to, completions of discrete groups. In this section we briefly review the definitions of *profinite* and *proalgebraic* completion of a discrete group.

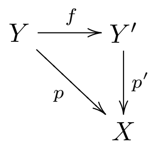
DEFINITION 2.1. *A profinite group is a topological group which is the inverse limit of an inverse system of discrete finite groups. The profinite completion $\hat{G}$ of a group G is the inverse limit of the system formed by the quotients groups G/N of G, where N are normal subgroups of G of finite index, ordered by inclusion.*

For instance, the profinite completion of $\mathbb{Z}$ is

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}(p)\,,$$

where p runs over the prime numbers, and $\mathbb{Z}(p)$ is the ring of p-adic integers [19].

An interesting geometric example of a profinite completion is *Grothendieck's fundamental group* [14]. The idea for its introduction may be regarded as a generalization of the usual fundamental group, recalling that for X a topological space, $\pi_1(X)$ is the group of deck transformations of the universal covering of X. To get a suitable replacement for schemes, one substitutes covering spaces with étale covers. So, if X a connected and locally noetherian scheme over a field $\Bbbk$, let x be a geometric point in X, i.e., a morphism $\operatorname{Spec}\bar{\Bbbk} \to X$, where $\bar{\Bbbk}$ is a separable closure of $\Bbbk$. Let I be the set of pairs (p, y), where $p\colon Y \to X$ is a finite étale cover, and $y \in Y$ is a geometric point such that $p(y) = x$, partially ordered by the relation $(p, y) \geq (p', y')$ if there is a commutative diagram



with with $y' = f(y)$. Then one sets

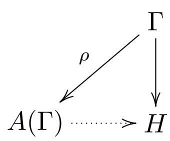
$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{i \in I} \operatorname{Aut}_X(p_i, y_i).$$

If X is a scheme of finite type over $\mathbb{C}$, *the étale fundamental group* $\pi_1^{\text{ét}}(X, x)$ *is a profinite completion of the topological fundamental group* $\pi_1(X, x)$ [14].

In spite of the naturalness of its definition, the étale fundamental group, for a field of positive characteristic, fails to enjoy some quite reasonable properties; for instance, it is not a birational invariant, and is not necessarily zero for rational varieties [21, 22]. Nori's fundamental group solves some of these problems. It is defined in terms of Tannaka duality (see next Section) and involves the notion of proalgebraic completion of a discrete group [3].

A proalgebraic group over $\Bbbk$ is the inverse limite of a system of algebraic groups over $\Bbbk$.

Definition 2.2. *Let* Γ *be a discrete group. A proalgebraic completion of* Γ *over* $\Bbbk$ *is a proalgebraic group* $A(\Gamma)$ *over* $\Bbbk$ *with a homomorphism* $\rho : \Gamma \to A(\Gamma)$ *such that every morphism* $\Gamma \to H$*, where* H *is a proalgebraic group over* $\Bbbk$*, uniquely filters through* $A(\Gamma)$ *via* ρ



A proalgebraic completion for Γ is unique up to unique isomorphism. The image of ρ is Zariski dense in $A(\Gamma)$. A proalgebraic completion can be built via

Tannaka duality, as the group of tensor product preserving automorphisms of the forgetful functor from the category of finite dimensional Γ-modules to the category of finite dimensional $\mathbb{k}$-vector spaces.

3. Tannakian categories

In this section we recall the main notions and establish the basic notation about Tannakian categories. For a detailed introduction the reader may refer to [12].

A category $\mathfrak{C}$ is *additive* if

- the Hom classes are abelian groups and the composition of morphisms is bilinear;
- $\mathfrak{C}$ has finite direct sums and direct products;
- it has a zero object.

An additive category is *abelian* if

- every morphism has both a kernel and a cokernel (the notion of kernel and cokernel are defined in terms of suitable universal properties);
- every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.

An additive category is $\mathbb{k}$-*linear* over a field $\mathbb{k}$ if the Hom groups are $\mathbb{k}$-vector spaces, and the composition of morphisms is $\mathbb{k}$-linear. A *tensor* category is an abelian category with a biproduct satisfying the usual properties of the tensor product (including the existence of a unit object 1 for the tensor product).

A tensor category is *rigid* if

- Hom and $\otimes$ satisfy the natural distributive property over finite families;
- all objects are reflexive, i.e., the natural maps to their double duals are isomorphisms (the dual $A^\vee$ of an object A of $\mathfrak{C}$ is the object $\mathrm{Hom}(A, 1)$).

Definition 3.1. *A* neutral Tannakian category *over a field* $\mathbb{k}$ *is a rigid Abelian* $\mathbb{k}$*-linear tensor category* $\mathfrak{T}$ *together with an exact faithful* $\mathbb{k}$*-linear tensor functor* $\omega\colon \mathfrak{T} \longrightarrow \mathbf{Vect}_{\mathbb{k}}$*, called the* fiber functor.

The archetypical Tannakian category is the category $\mathrm{Rep}(G)$ of representations (on vector spaces over $\mathbb{k}$) of an affine group scheme G over $\mathbb{k}$. The fiber functor is defined as the forgetful functor

$$\omega(\rho, V) = V \qquad \text{if} \qquad \rho\colon G \to \mathrm{Aut}(V).$$

Categories of representations of affine group schemes are much more than just examples: it turns out that every neutral Tannakian category is equivalent to one of them [12].

Theorem 3.2 (Tannaka duality). *For every neutral Tannakian category* $(\mathfrak{T}, \omega)$ *there is a proalgebraic affine group scheme* G *such that* $\mathfrak{T} \simeq \operatorname{Rep}(G)$.

The group G is recovered as the group of automorphisms of the fiber functor that are compatible with the tensor product, $G = \operatorname{Aut}^{\otimes}(\omega)$. If $\mathfrak{T} \simeq \operatorname{Rep}(G)$, one also writes $G = \pi_1(\mathfrak{T})$.

Examples 3.3:

- The category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over $\Bbbk$ with the identity as fiber functor is a neutral Tannakian category. Its corresponding affine group scheme is the trivial group $G = \{e\}$, i.e., $\pi_1(\mathbf{Vect}_{\Bbbk}) = \{e\}$.
- The category of modules over a commutative ring with unit R is an abelian tensor category. It may fail to be rigid as there are R-modules that are not reflexive.
- If $\mathfrak{g}$ is a semisimple Lie algebra over a field $\Bbbk$, the category $\operatorname{Rep}(\mathfrak{g})$ of representations of $\mathfrak{g}$, with the fiber functor given by the forgetful functor that only keeps the vector space structure of $\mathfrak{g}$, is a neutral Tannakian category, and $\pi_1(\operatorname{Rep}(\mathfrak{g})) = G$, where G is the unique connected simply connected Lie group whose Lie algebra is $\mathfrak{g}$.
- If X is a smooth projective variety over $\mathbb{C}$, the category of vector bundles on X with a flat connection (a.k.a. local systems), with a functor which to a bundle E associates its fiber at $x \in X$, is Tannakian, and is equivalent to the category $\operatorname{Rep}(\pi_1(X, x))$ of representations of the topological fundamental group of X. The dual group via Tannaka duality, i.e. the group $\pi_1(\operatorname{Rep}(\pi_1(X, x)))$, is the proalgebraic completion of $\pi_1(X, x)$.

4. Tannakian categories and fundamental groups

The basic idea for using Tannaka duality to define fundamental groups is to single out a class of geometric objects on a scheme X that make up a neutral Tannakian category, and take the associated group scheme. We briefly review two examples, Nori's and Langer's fundamental groups. Next we shall introduce the Higgs fundamental group and discuss its relation with Simpson's proalgebraic fundamental group; this will be related to a conjecture about semistable Higgs bundles on projective varieties.

Nori's fundamental group

The first example of such a fundamental group was provided by Nori [21, 22]. A vector bundle E over a scheme X is *essentialy finite* if there exists a principal bundle $\pi \colon P \to X$, with a finite structure group, such that $\pi^* E$ is trivial. Essentially finite vector bundles make up a neutral Tannakian category, where the fiber functor maps E to the fiber over a fixed point $x \in X$ (some assumptions on the scheme X have to be made). The affine group scheme representing this Tannakian category is the *Nori fundamental group scheme* $\pi_1^N(X,x)$. It turns out that there is a faithfully flat (i.e., flat and surjective) morphism

$$\pi_1^N(X,x) \twoheadrightarrow \pi_1^{\text{ét}}(X,x)$$

which is an isomorphism when $\operatorname{char}\Bbbk = 0$.

A related notion, that of *F-fundamental group,* was introduced in [2], and some properties of it were studied in [1]. Another generalization was proposed in [23].

Langer's fundamental group

Let X be a smooth projective variety over an algebraically closed field. We can define intersections between divisors D and curves C in X by letting

$$C \cdot D = \deg f^*\mathcal{O}_X(D)$$

where $f\colon \tilde{C} \to C$ is a normalization of C. In the same way, we can define the intersection product between a line bundle and a curve. Then we have the usual notion of numerical effectiveness.

DEFINITION 4.1. *L is numerically effective (nef) if $L \cdot C \geq 0$ for all irreducible curves C in X. A vector bundle E on X is numerically effective if its hyperplane line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivization $\mathbb{P}(E)$ is. E is numerically flat if both E and its dual bundle $E^\vee$ are nef.*

As proved by Langer [16, 17], numerically flat vector bundles make up a neutral Tannakian category, so that one can define a "fundamental group" $\pi_1^S(X,x)$ as its dual (this group was introduced in the case of curves also in [5]). Essentially finite vector bundles are numerically flat, so that there is a morphism

$$\pi_1^S(X,x) \twoheadrightarrow \pi_1^N(X,x)$$

which is again faithfully flat, and is an isomorphism when $\operatorname{char}\Bbbk = 0$. Some properties of this fundamental group, e.g. its birational invariance, were proved in [15].

Higgs fundamental group

We follow this pattern to introduce a fundamental group which "feels" the behavior of Higgs bundles on a projective variety. To do that we restrict to varieties over the complex numbers, and start by considering ordinary bundles.

So, let X be a smooth projective variety over $\mathbb{C}$, and E a vector bundle on X of rank r. We shall consider the characteristic class (the discriminant of E)

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 \in H^4(X, \mathbb{R})\,.$$

Moreover, after equipping X with an ample line bundle L, and denoting by H it first Chern class (a polarization on X), we define the degree of E as

$$\deg E = c_1(E) \cdot H^{n-1}$$

where $n = \dim X$. If X is a smooth irreducible projective curve, it has a canonical polarization, given by the class of a closed point of X. Whenever X is such a curve, one implicitly assumes this choice of a polarization.

Definition 4.2. *E is semistable (with respect to the chosen polarization) if for every coherent subsheaf $F \subset E$, with $0 < \operatorname{rk} F < r$, one has*

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{r}\,.$$

E is curve semistable if for all morphisms $f\colon C \to X$, where C is a smooth projective irreducible curve, the pullback bundle $f^*(E)$ is semistable.

The following theorem was proved in a slightly weaker form by Nakayama [20] and strengthened into its present form by Hernández Ruipérez and the author [9].

Theorem 4.3. *The following conditions are equivalent:*

- *E is curve semistable;*
- *E is semistable with respect to a polarization, and $\Delta(E) = 0$.*

The following corollary is not hard to prove [9].

Corollary 4.4. *E is numerically flat if and only if it is curve semistable and $c_1(E) = 0$.*

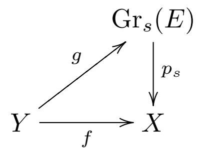
It is quite natural to ask if a result such as Theorem 4.3 also works for Higgs bundles. A Higgs sheaf is a pair (E, ϕ) where E is a coherent sheaf and

$$\phi\colon E \to E \otimes \Omega^1_X, \qquad \phi \wedge \phi = 0.$$

A Higgs bundle is a locally free Higgs sheaf. A notion of semistability is given as for ordinary vector bundles, but the inequality is required to hold only for ϕ-invariant subsheaves. There is a notion of nefness/numerical flatness for Higgs bundles [9, 7], which we briefly review here. If E is a vector bundle of rank r on X, and $s < r$ is a positive integer, we can consider the Grassmann bundle $\mathrm{Gr}_s(E)$ on X. Denote by $p_s : \mathrm{Gr}_s(E) \longrightarrow X$ the natural projection. There is a universal short exact sequence

$$0 \longrightarrow S_{r-s,E} \xrightarrow{\psi} p_s^* E \xrightarrow{\eta} Q_{s,E} \longrightarrow 0 \tag{1}$$

of vector bundles on $\mathrm{Gr}_s(E)$, with $S_{r-s,E}$ the universal subbundle of rank $r-s$ and $Q_{s,E}$ the universal quotient of rank s [13]. The Grassmannian $\mathrm{Gr}_s(E)$ parameterizes locally free rank s quotients of E, in the sense that if $f\colon Y \to X$ is a morphism, and G is a quotient bundle of $f^*(E)$, there is a morphism $g\colon Y \to \mathrm{Gr}_s(E)$ such that $G \simeq g^* Q_{s,E}$, and the diagram



commutes [13].

Given a Higgs bundle $\mathfrak{E} = (E, \phi)$, we define closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parameterizing rank s locally free Higgs quotients, i.e., locally free quotients of E whose corresponding kernels are ϕ-invariant. The Grassmannian of locally free rank s Higgs quotients of $\mathfrak{E}$, denoted $\mathfrak{Gr}_s(\mathfrak{E})$, is the closed subscheme of $\mathrm{Gr}_s(E)$ defined by the vanishing of the composition of morphisms

$$(\eta \otimes \mathrm{Id}) \circ p_s^*(\phi) \circ \psi \colon S_{r-s,E} \longrightarrow Q_{s,E} \otimes p_s^* \Omega_X^1. \tag{2}$$

Let $\rho_s := p_s|_{\mathfrak{Gr}_s(\mathfrak{E})} : \mathfrak{Gr}_s(\mathfrak{E}) \longrightarrow X$ be the induced projection. The restriction of (1) to $\mathfrak{Gr}_s(\mathfrak{E})$ yields a universal exact sequence

$$0 \longrightarrow \mathfrak{S}_{r-s,\mathfrak{E}} \xrightarrow{\psi} \rho_s^* \mathfrak{E} \xrightarrow{\eta} \mathfrak{Q}_{s,\mathfrak{E}} \longrightarrow 0, \tag{3}$$

where $\mathfrak{Q}_{s,\mathfrak{E}} := Q_s|_{\mathfrak{Gr}_s(\mathfrak{E})}$ is endowed with the quotient Higgs field induced by the Higgs field $\rho_s^*\phi$. A morphism of $\Bbbk$-varieties $f : T \to X$ factors through $\mathfrak{Gr}_s(\mathfrak{E})$ if and only if the pullback $f^*(E)$ admits a Higgs quotient of rank s. The pullback of the above universal sequence on $\mathfrak{Gr}_s(E)$ gives a quotient of $f^*(E)$.

Definition 4.5. *A Higgs bundle $\mathfrak{E}$ of rank one is said to be Higgs-numerically effective (H-nef for short) if it is numerically effective in the usual sense. If* $\mathrm{rk}\,\mathfrak{E} \geq 2$*, we inductively define H-nefness by requiring that*

1. *all Higgs bundles $\mathfrak{Q}_{s,\mathfrak{E}}$ are Higgs-nef, and*

2. the determinant line bundle $\det(E)$ *is nef.*

If both $\mathfrak{E}$ *and* $\mathfrak{E}^*$ *are Higgs-numerically effective,* $\mathfrak{E}$ *is said to be Higgs-numerically flat (H-nflat).*

Definition 4.5 immediately implies that the first Chern class of an H-numerically flat Higgs bundle is numerically equivalent to zero.

It was proved in [4] that numerically flat Higgs bundles make up a neutral Tannakian category. Therefore, after fixing a point $x \in X$, we can define the *Higgs fundamental group* $\pi_1^H(X,x)$ as the group which is Tannaka dual to that category. A numerically flat vector bundle, equipped with the zero Higgs field, is a numerically flat Higgs vector bundle, so that there is a morphism

$$\pi_1^H(X,x) \twoheadrightarrow \pi_1^S(X,x)$$

which is again faithfully flat.

The nature of this fundamental group is related to the validity of Theorem 4.3 for Higgs bundles. The following theorem was proved in [7].

THEOREM 4.6. *If* $\mathfrak{E} = (E,\phi)$ *is semistable, and* $\Delta(E) = 0$*, then* $\mathfrak{E}$ *is curve semistable.*

The question whether the opposite result holds true is an open problem.

CONJECTURE 4.7. *If the Higgs bundle* $\mathfrak{E}$ *is curve semistable, then* $\Delta(E) = 0$.

Conjecture 4.7 is known to hold for certain classes of varieties (varieties whose tangent bundle is numerically effective [11] and K3 surfaces [10], and varieties obtained from these two classes by some simple geometric constructions [11]).

The category of semistable Higgs bundles on X having vanishing Chern classes (semi-harmonic Higgs bundles) is Tannakian (the definition of this category does not require the specification of a polarization since such bundles are semistable with respect to all polarizations). Its Tannaka dual is isomorphic to the proalgebraic completion of the topological fundamental group $\pi_1^{\mathrm{alg}}(X,x)$ [24]. Since such semi-harmonic Higgs bundles are Higgs numerically effective, there is a morphism (again, a faithfully flat morphism)

$$\pi_1^H(X,x) \to \pi_1^{\mathrm{alg}}(X,x). \tag{4}$$

THEOREM 4.8. *The morphism* (4) *is an isomorphism if and only the Conjecture 4.7 holds.*

Proof. If the morphism (4) is an isomorphism, the categories of numerically flat Higgs bundles and semi-harmonic bundles are equivalent. Then a numerically flat Higgs bundle has vanishing Chern classes, which implies the conjecture.

Vice versa, if the conjecture holds, and $\mathfrak{E} = (E, \phi)$ is a numerically flat Higgs bundle, then $\mathfrak{E}$ is curve semistable, and since the Conjecture is assumed to hold, $\Delta(E) = 0$; moreover, $\mathfrak{E}$ is semistable and $c_1(E) = 0$ [7], so that by Theorem 2 in [24], all Chern classes of E vanish, and $\mathfrak{E}$ is semi-harmonic. Thus the two above mentioned categories are isomorphic, and (4) is an isomorphism. □

Acknowledgements

The original results presented in this paper were obtained in collaboration with Indranil Biswas, Beatriz Graña Otero, Sudarshan Gurjar, Daniel Hernández Ruipérez, Valeriano Lanza and Alessio Lo Giudice. I thank them all for their collaboration, in particular B.G.O. with whom I wrote the related paper [8]. I also thank V. Lanza and T. Pantev for very useful discussions on Tannaka duality.

The text of this paper reflects the talk I gave on January 29th, 2018, at the "INdAM day" held at the University of Trieste to celebrate the 50th anniversary of the "Rendiconti dell'Istituto di Matematica dell'Università di Trieste" journal. I am grateful to the organizers for their invitation.

This research was partly supported by GNSAGA–INdAM and PRIN "Geometria delle varietà algebriche."

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Received October 28, 2018
Accepted November 7, 2018