

$j_x$  coincide and read

$$\begin{aligned}
 P_{\text{JN}}(\omega) &\equiv - \int_{-\infty}^{\infty} \langle \varrho(x, t_1) \varrho(x, t_2) \rangle_{\beta}^{\text{con}} e^{i\omega t_{12}} dt_{12} = - \int_{-\infty}^{\infty} \langle j_x(x, t_1) j_x(x, t_2) \rangle_{\beta}^{\text{con}} e^{i\omega t_{12}} dt_{12} \\
 &= - \frac{\kappa}{4\beta^2} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \left( \frac{1}{\sinh^2[\pi(t - i\varepsilon)/\beta]} + \frac{1}{\sinh^2[\pi(-t + i\varepsilon)/\beta]} \right) e^{i\omega t} dt \\
 &= - \frac{\kappa}{2\beta^2} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sinh^2[\pi(t - i\varepsilon)/\beta]} dt = \frac{\kappa}{2\pi} \omega \coth(\beta\omega/2) + \frac{\kappa}{2\pi} \omega \quad (5.56)
 \end{aligned}$$

which is independent of  $x$  because of the translation invariance of the CFT. The zero frequency limit of (5.56) gives the Johnson-Nyquist law

$$P_{\text{JN}}(0) = \frac{\kappa}{\pi\beta} \quad (5.57)$$

Comparing (5.53) with (5.56) and also the corresponding zero frequency limits given by (5.54) with (5.57) confirms that the modular evolution has a thermal character with inverse temperature  $\tilde{\beta} = 1$ , which appears in the KMS condition (4.49) satisfied by the modular correlation functions (4.42), (4.43), (4.47) and (4.48). We remark that the noise power (5.54) provides a physical observable [45, 46] where in principle the modular temperature can be measured.

We can also introduce the modular noise power  $P_k(\omega; x, t)$  at frequency  $\omega$  in the point  $(x, t)$  of the spacetime generated by the helicity current (3.22), defined by the r.h.s. (5.51) with  $j_x$  replaced by  $k_x$ . From the explicit expression of  $\langle k_x(\tau_1; x, t) k_x(\tau_2; x, t) \rangle_{\mu}^{\text{con}}$  and (C.14)–(C.15), it is straightforward to obtain the r.h.s. of (5.52) for  $P_k(\omega; x, t)$ ; hence  $P_j(\omega; x, t) = P_k(\omega; x, t)$ . Indeed, the differences due to the diverse relative signs in (3.6) and (3.21) do not lead to a relevant result in this computation because the mixed connected correlators vanish.

Similarly, it is worth investigating the noise generated by the energy current (3.29), namely

$$\mathcal{P}_{\mathcal{J}}(\omega; x, t) \equiv \int_{-\infty}^{\infty} \langle \mathcal{J}_x(\tau_1; x, t) \mathcal{J}_x(\tau_2; x, t) \rangle_{\mu}^{\text{con}} e^{i\omega\tau_{12}} d\tau_{12}. \quad (5.58)$$

From (4.48) and the identity (4.50), one finds (see (F.5) for the evaluation of the integral)

$$\mathcal{P}_{\mathcal{J}}(\omega; x, t) = \frac{\pi^2 c}{8} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \left[ \frac{1}{\sinh^4(\pi\tau - i\varepsilon)} + \frac{1}{\sinh^4(-\pi\tau + i\varepsilon)} \right] e^{i\omega\tau} d\tau \quad (5.59)$$

$$= \frac{\pi^2 c}{4} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\sinh^4(\pi\tau - i\varepsilon)} d\tau = \frac{c}{24\pi} (4\pi^2 + \omega^2) [\omega \coth(\omega/2) + \omega] \quad (5.60)$$

which is independent of  $x$  and  $t$ , and whose zero frequency limit gives

$$\mathcal{P}_{\mathcal{J}}(0; x, t) = \lim_{\omega \rightarrow 0} \mathcal{P}_{\mathcal{J}}(\omega; x, t) = \frac{\pi c}{3}. \quad (5.61)$$

It is worth comparing these results based on the modular evolution with the corresponding ones based on the standard temporal evolution. In a CFT on the line at finite temperature and vanishing chemical potential, we have

$$\langle T_{\pm}(u) T_{\pm}(v) \rangle_{\beta}^{\text{con}} = \frac{c}{8\pi^2} \left( \frac{\pi}{\beta \sinh[\pi(u - v \mp i\varepsilon)/\beta]} \right)^4. \quad (5.62)$$

Since  $T_{tt}(x, t) = T_+(u_+) + T_-(u_-)$ ,  $T_{xt}(x, t) = T_+(u_+) - T_-(u_-)$  and  $\langle T_{\pm}(u) T_{\mp}(v) \rangle_{\beta}^{\text{con}} = 0$ , the noise power  $\mathcal{P}_{\text{JN}}(\omega)$  at frequency  $\omega$  in the point  $x$  of the space of  $T_{tt}$  and  $T_{xt}$  coincide. It reads

$$\begin{aligned} \mathcal{P}_{\text{JN}}(\omega) &\equiv \int_{-\infty}^{\infty} \langle T_{xt}(x, t_1) T_{xt}(x, t_2) \rangle_{\beta}^{\text{con}} e^{i\omega t_{12}} dt_{12} = \int_{-\infty}^{\infty} \langle T_{tt}(x, t_1) T_{tt}(x, t_2) \rangle_{\beta}^{\text{con}} e^{i\omega t_{12}} dt_{12} \\ &= \frac{\pi^2 c}{8\beta^4} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \left( \frac{1}{\sinh^4[\pi(t - i\varepsilon)/\beta]} + \frac{1}{\sinh^4[\pi(-t + i\varepsilon)/\beta]} \right) e^{i\omega t} dt \end{aligned} \quad (5.63)$$

$$= \frac{\pi^2 c}{4\beta^4} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sinh^4[\pi(t - i\varepsilon)/\beta]} dt \quad (5.64)$$

$$= \frac{c}{24\pi\beta^2} [4\pi^2 + (\beta\omega)^2] [\omega \coth(\beta\omega/2) + \omega] \quad (5.65)$$

where (F.5) has been employed and whose zero frequency limit is

$$\mathcal{P}_{\text{JN}}(0) = \frac{\pi c}{3\beta^3}. \quad (5.66)$$

Comparing (5.65) with the modular noise (5.60) provides another consistency check for the fact that the modular evolution has a thermal character with inverse temperature  $\tilde{\beta} = 1$ .

Thus, while  $P_j(0; x, t)$  provides the coefficient  $\kappa$  of the central term in (2.5) through (5.54),  $\mathcal{P}_{\mathcal{J}}(0; x, t)$  gives the central charge  $c$  through (5.61). Notice that in the derivation of (5.60), the expression (4.37) has been employed, which holds for the specific velocity given in (4.7).

The noise  $\mathcal{P}_{\tilde{\mathcal{J}}}(\omega; x, t)$  generated by the momentum current (3.37) is defined by the r.h.s. of (5.58) with  $\mathcal{J}_x$  replaced by  $\tilde{\mathcal{J}}_x$ . By adapting the observations made above to get  $P_j(\omega; x, t) = P_k(\omega; x, t)$  and using (C.17)–(C.18), one finds that  $\mathcal{P}_{\tilde{\mathcal{J}}}(\omega; x, t) = \mathcal{P}_{\mathcal{J}}(\omega; x, t)$ .

The previous analysis shows that the modular noise power generated by the charge and energy currents is uniform in space and time, despite the fact that the translation invariance is broken by the bipartition of the system. This peculiar feature does not hold for the quadratic fluctuations of a generic observable. Indeed, consider for instance the modular noise power relative to the charge density (3.4), namely

$$P_{\varrho}(\omega; x, t) \equiv - \int_{-\infty}^{\infty} \langle \varrho(\tau_1; x, t) \varrho(\tau_2; x, t) \rangle_{\mu}^{\text{con}} e^{i\omega\tau_{12}} d\tau_{12}. \quad (5.67)$$

By using (4.47), (4.50) and (F.3), we find

$$\begin{aligned} P_{\varrho}(\omega; x, t) &= -\frac{\kappa}{4} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \left[ \frac{1}{V(u_+)^2 \sinh^2(\pi\tau - i\varepsilon)} + \frac{1}{V(u_-)^2 \sinh^2(-\pi\tau + i\varepsilon)} \right] e^{i\omega\tau} d\tau \\ &= -\frac{\kappa}{4} \left[ \frac{1}{V^2(u_+)} + \frac{1}{V^2(u_-)} \right] \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\sinh^2(\pi\tau - i\varepsilon)} d\tau \\ &= \frac{\kappa}{4\pi} \left[ \frac{1}{V^2(u_+)} + \frac{1}{V^2(u_-)} \right] [\omega \coth(\omega/2) + \omega] \end{aligned} \quad (5.68)$$

that depends on the frequency and on the position in spacetime. The zero frequency limit of (5.68) gives

$$P_{\varrho}(0; x, t) = \frac{\kappa}{2\pi} \left[ \frac{1}{V(u_+)^2} + \frac{1}{V(u_-)^2} \right] \quad (5.69)$$

which is qualitatively different from (5.54) because of the occurrence of a non trivial dependence on the spacetime position. Notice that, setting  $V(u) = 1$  identically in (5.69) one recovers (5.57) in the special case of  $\beta = 1$ .

We can introduce also the noise  $P_\chi(\omega; x, t)$  generated by  $\chi$  in (3.19) as the Fourier transform in  $\tau_{12}$  of the connected modular two-point function of  $\chi$  at coincident points, as done in (5.67) for  $P_\rho(\omega; x, t)$ . Comparing this computation with the one reported above for  $P_\rho(\omega; x, t)$ , we observe again that the differences due to the different relative sign in (3.4) and (3.19) do not play any role because the mixed connected correlators vanish; hence  $P_\chi(\omega; x, t) = P_\rho(\omega; x, t)$ .

We can perform a non trivial consistency check of the above results by considering the Fourier transform of the anticommutator

$$\mathcal{A}[\mathcal{O}](\omega; x, t) \equiv \frac{1}{2} \int_{-\infty}^{\infty} \langle \{ \mathcal{O}(\tau_1; x, t), \mathcal{O}(\tau_2; x, t) \} \rangle_\mu^{\text{con}} e^{i\omega\tau_{12}} d\tau_{12} \quad (5.70)$$

and the Fourier transform of the commutator

$$\mathcal{C}[\mathcal{O}](\omega; x, t) \equiv \frac{1}{2} \int_{-\infty}^{\infty} \langle [ \mathcal{O}(\tau_1; x, t), \mathcal{O}(\tau_2; x, t) ] \rangle_\mu^{\text{con}} e^{i\omega\tau_{12}} d\tau_{12} \quad (5.71)$$

for the modular correlators of the operators  $\mathcal{O} \in \{j_x, k_x, \mathcal{J}_x, \tilde{\mathcal{J}}_x, \rho, \chi\}$  considered above. Since the Fourier transform  $F(\omega) \equiv \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt$  of a generic function  $f(t)$  satisfies the property  $\frac{1}{2} \int_{-\infty}^{+\infty} [f(t) \pm f(-t)] e^{i\omega t} dt = \frac{1}{2} [F(\omega) \pm F(-\omega)]$ , by employing (5.53), (5.60) and (5.68), for (5.70) and (5.71) we find that the following modular fluctuation-dissipation relation

$$\mathcal{A}[\mathcal{O}](\omega; x, t) = \coth\left(\frac{\omega}{2}\right) \mathcal{C}[\mathcal{O}](\omega; x, t) \quad (5.72)$$

which corresponds to the fluctuation-dissipation relation [49–51] with inverse temperature given by  $\tilde{\beta} = 1$ . In the case of two-dimensional CFT, this result further confirms the thermal nature of the modular evolution with inverse temperature  $\tilde{\beta} = 1$ , in agreement with the KMS condition discussed in section 4.4.

## 6 Finite volume

In this section the analyses of section 4 are extended to a two-dimensional CFT at finite density and finite volume by compactifying each chiral direction on the circle of length  $L$ , hence the resulting spacetime  $\mathbb{M}_\diamond$  has the topology of the torus. In section 6.1 the relevant chiral correlators are discussed. The modular Hamiltonian associated to the bipartition of each chiral direction provided by the interval  $A = [a, b]$  and the corresponding modular correlators are explored in section 6.2 and section 6.3 respectively.

### 6.1 Finite density representation on the circle

The finite density and finite volume representation of a CFT on a circle of length  $L$  can be obtained from its ground state representation in the infinite line, by employing first the conformal transformation  $u \mapsto e^{2\pi i u/L}$  and then the automorphism described in the appendix C.

In this representation, the one-point functions are obtained by first applying the conformal transformation  $u \mapsto e^{2\pi i u/L}$  to the one-point functions in the ground state and on the line, which are given by (4.1) with  $\mu_+ = \mu_- = 0$ , and then employing the automorphism discussed in the appendix C. The result is

$$\langle \phi_{\pm}(u) \rangle_{L, \mu_{\pm}} = 0 \quad \langle j_{\pm}(u) \rangle_{L, \mu_{\pm}} = -\frac{\kappa \mu_{\pm}}{2\pi} \quad \langle \mathcal{T}_{\pm}(u) \rangle_{L, \mu_{\pm}} = \frac{\kappa \mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \quad (6.1)$$

where we used that the Schwarzian derivative (2.35) of the conformal map  $u \mapsto e^{2\pi i u/L}$  is equal to  $2\pi^2/L^2$ . In the infinite volume limit  $L \rightarrow +\infty$ , the one-point functions (6.1) become the ones in (4.1), as expected.

The connected two-point functions in the finite density and finite volume representation can be written through the same procedure, starting from the connected two-point functions in the ground state representation. From (4.2), for the connected two-point expectation values of  $\phi_{\pm}$  we obtain

$$\langle \phi_{\pm}^*(u) \phi_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{e^{\pm i \mu_{\pm}(u-v)}}{2\pi e^{\pm i \pi h_{\pm}}} \left( \frac{\pi}{L \sin[\pi(u-v \mp i\varepsilon)/L]} \right)^{2h_{\pm}} \quad \frac{L\mu_{\pm}}{2\pi} \in \mathbb{Z} \quad (6.2)$$

whose r.h.s. is periodic, as expected. In the appendix D.1, a consistency check of (6.2) is discussed by considering the special case of a free fermion with anti-periodic boundary conditions and obtaining the corresponding two-point function through the Fermi-Dirac distribution.

As for the two-point functions of  $j_{\pm}$  and  $\mathcal{T}_{\pm}$ , from (4.3) and (4.4) we find respectively

$$\langle j_{\pm}(u) j_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{\kappa}{4\pi^2} \left( \frac{\pi}{L \sin[\pi(u-v \mp i\varepsilon)/L]} \right)^2 \quad (6.3)$$

and

$$\langle \mathcal{T}_{\pm}(u) \mathcal{T}_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{c}{8\pi^2} \left( \frac{\pi}{L \sin[\pi(u-v \mp i\varepsilon)/L]} \right)^4. \quad (6.4)$$

We also have that

$$\langle \mathcal{T}_{\pm}(u) j_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}} = -\mu_{\pm} \langle j_{\pm}(u) j_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}}. \quad (6.5)$$

The connected mixed correlators involving fields having different chiralities vanish identically. Notice that the infinite volume limit  $L \rightarrow +\infty$  of the two-point correlators (6.2), (6.3) and (6.4) gives the two-point correlators on line (4.2), (4.3) and (4.4) respectively, as expected.

## 6.2 Modular Hamiltonian and modular conjugation

In the following we consider the portion of Minkowski spacetime described by the light-cone coordinates  $(u_+, u_-)$  when periodic boundary conditions are imposed along both the chiral directions with the same period equal to  $L$ . The resulting spacetime  $\mathbb{M}_{\diamond}$  has the topology of a torus and it is shown in figure 7, where  $u_{\pm} \in (-L/2, L/2)$  and the dashed segments having the same colour are identified. We consider a two-dimensional CFT in the finite density state on this spacetime. Moreover, each chiral direction is partitioned through the

interval  $A = [a, b]$  with length  $\ell \equiv b - a$  and its complement; hence the diamond  $\mathcal{D}_A$  can be introduced. The two panels of figure 7 describe the same setup in two equivalent ways and  $\mathcal{D}_A$  corresponds to the grey region in each panel.

A standard way to compactify a chiral direction exploits the Cayley map, which relates the real line to the unit circle  $\mathbb{S}$  with one point removed (see e.g. [52]) and reads  $u \mapsto z = \frac{1+iv}{1-iv}$  where  $v \in \mathbb{R}$  and  $z \in \mathbb{S} \setminus \{P_0\}$ , being  $P_0$  the point at  $\theta = \pi$  on  $\mathbb{S}$ . By employing the complex number  $z = e^{2\pi i u/L}$  with  $u \sim u + L$  to parameterise  $\mathbb{S}$ , where  $L$  corresponds to the compactification parameter, the Cayley map and its inverse read respectively

$$e^{2\pi i u/L} = \frac{1+iv}{1-iv} \quad v = i \frac{1 - e^{2\pi i u/L}}{1 + e^{2\pi i u/L}} = \tan(\pi u/L) \equiv \mathcal{C}(u). \quad (6.6)$$

Alternatively [53], one first introduces the periodic identification  $u \sim u + L$  on the real line and then uses the exponential map  $u \mapsto e^{2\pi i u/L}$ .

By adapting the general results described in section 2.2, in this CFT setup the modular Hamiltonian of  $A$  and the corresponding full modular Hamiltonian read respectively [9, 10]

$$K_A = \int_A V_L(u_+) \left[ \mathcal{T}_+(u_+) - \frac{\kappa \mu_+^2}{4\pi} + \frac{\pi c}{12L^2} \right] du_+ + \int_A V_L(u_-) \left[ \mathcal{T}_-(u_-) - \frac{\kappa \mu_-^2}{4\pi} + \frac{\pi c}{12L^2} \right] du_- \quad (6.7)$$

and

$$K \equiv K_A \otimes \mathbf{1}_B - \mathbf{1}_A \otimes K_B = \int_{-L/2}^{L/2} V_L(u_+) \left[ \mathcal{T}_+(u_+) - \frac{\kappa \mu_+^2}{4\pi} + \frac{\pi c}{12L^2} \right] du_+ + \int_{-L/2}^{L/2} V_L(u_-) \left[ \mathcal{T}_-(u_-) - \frac{\kappa \mu_-^2}{4\pi} + \frac{\pi c}{12L^2} \right] du_- \quad (6.8)$$

where the velocity  $V_L(u)$  is

$$V_L(u) = 2L \frac{\sin[\pi(b-u)/L] \sin[\pi(u-a)/L]}{\sin[\pi(b-a)/L]} = \frac{1}{w'_L(u)} \quad u \in A \quad (6.9)$$

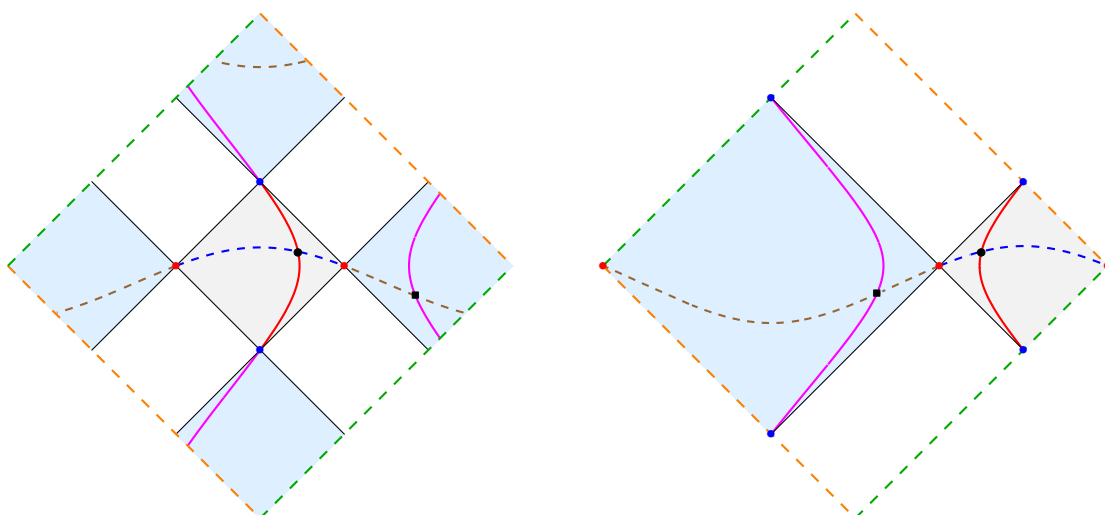
being  $w_L(u)$  defined as follows

$$w_L(u) \equiv \frac{1}{2\pi} \log \left( -\frac{\sin[\pi(u-a)/L]}{\sin[\pi(u-b)/L]} \right). \quad (6.10)$$

The weight function (6.9) can be obtained from (4.7) as follows

$$V_L(v) = \frac{\tilde{V}(e^{2\pi i v/L})}{\partial_v(e^{2\pi i v/L})} = \frac{\hat{V}(\mathcal{C}(v))}{\mathcal{C}'(v)} \quad (6.11)$$

where  $\tilde{V}(v)$  is defined as (4.7) with  $a$  and  $b$  replaced by  $e^{2\pi i a/L}$  and  $e^{2\pi i b/L}$  respectively, while in the last expression the Cayley map (6.6) is employed and  $\hat{V}(v)$  is given by (4.7) with  $a$  and  $b$  replaced by  $\mathcal{C}(a)$  and  $\mathcal{C}(b)$  respectively. The full modular Hamiltonian (6.8) corresponds (2.6) specialised to  $V_+(u) = V_-(u) = V_L(u)$  given by (6.9), which vanishes only at the endpoints of  $A$ . Notice that a vanishing additive constant has been chosen in (6.7).



**Figure 7.** Modular trajectories generated either by the modular Hamiltonian (6.8) (solid lines) or by the modular momentum (dashed lines) for the CFT on the circle. Since periodic boundary conditions are imposed, the green (orange) dashed straight lines must be identified. The solid (dashed) curves having different colours are related through the geometric action of the modular conjugation, constructed from (6.18).

The modular evolution generated by (6.8) can be studied by applying the results discussed in section 2.2 to  $V_+(u) = V_-(u) = V_L(u)$  and  $w_+(u) = w_-(u) = w_L(u)$ , introduced in (6.9) and (6.10) respectively. In this case (2.16) becomes [54]

$$\xi_{\pm}(\tau, u) = \xi_L(\pm\tau, u) \quad \xi_L(\tau, u) \equiv \frac{L}{2\pi i} \log\left(\frac{e^{\pi i(b+a)/L} + e^{2\pi i b/L} e^{2\pi w_L(u)+2\pi\tau}}{e^{\pi i(b-a)/L} + e^{2\pi w_L(u)+2\pi\tau}}\right) \quad (6.12)$$

whose infinite volume limit  $L \rightarrow +\infty$  gives (4.11), as expected. From (6.12) we have

$$e^{2\pi i \xi_L(\tau, u)/L} = \frac{(e^{2\pi i b/L} - e^{2\pi i u/L}) e^{2\pi i a/L} + (e^{2\pi i u/L} - e^{2\pi i a/L}) e^{2\pi i b/L} e^{2\pi\tau}}{(e^{2\pi i b/L} - e^{2\pi i u/L}) + (e^{2\pi i u/L} - e^{2\pi i a/L}) e^{2\pi\tau}} \quad (6.13)$$

$$= e^{\pi i(a+b)/L} \frac{e^{\pi i a/L} \sin[\pi(b-u)/L] + e^{\pi i b/L} \sin[\pi(u-a)/L] e^{2\pi\tau}}{e^{\pi i b/L} \sin[\pi(b-u)/L] + e^{\pi i a/L} \sin[\pi(u-a)/L] e^{2\pi\tau}} \quad (6.14)$$

where the r.h.s. of (6.13) corresponds to the r.h.s. of  $\xi(\tau, u)$  in (4.11) with  $a, b$  and  $u$  replaced by  $e^{2\pi i a/L}, e^{2\pi i b/L}$  and  $e^{2\pi i u/L}$  respectively.

From  $\xi_L(\tau, u)$  in (6.12) we obtain the spacetime coordinates of the modular trajectory in the diamond  $\mathcal{D}_A$  whose initial point at  $\tau = 0$  has light-cone coordinates  $(u_+, u_-)$ , namely

$$x(\tau) = \frac{\xi_L(\tau, u_+) + \xi_L(-\tau, u_-)}{2} \quad t(\tau) = \frac{\xi_L(\tau, u_+) - \xi_L(-\tau, u_-)}{2} \quad (6.15)$$

which correspond to (4.13) with  $\xi(\tau, u)$  replaced by  $\xi_L(\tau, u)$ . In figure 7, the solid red curve is a modular trajectory whose initial point is the black dot.

The modular evolutions of the operators  $\phi_{\pm}, j_{\pm}$  and  $\mathcal{T}_{\pm}$  are obtained by specialising (2.18), (2.23) and (2.34) respectively to (6.12). It is worth writing explicitly the result

for  $\mathcal{T}_\pm$

$$\mathcal{T}_\pm(\tau, u) = [\partial_u \xi_L(\pm\tau, u)]^2 \mathcal{T}_\pm(\xi_L(\pm\tau, u)) + \left( \frac{\kappa \mu_\pm^2}{4\pi} - \frac{\pi c}{12L^2} \right) \left\{ 1 - [\partial_u \xi_L(\pm\tau, u)]^2 \right\} \quad (6.16)$$

where we used that<sup>4</sup>

$$\mathcal{S}_u[\xi_L](\tau, u) = \frac{2\pi^2}{L^2} \left\{ 1 - [\partial_u \xi_L(\tau, u)]^2 \right\}. \quad (6.17)$$

The modular momentum operator can be introduced as (3.40) specialised to  $V_+(u) = V_-(u) = V_L(u)$  given by (6.9), finding (4.14) with  $V(u)$  replaced by  $V_L(u)$ . The coordinates of the corresponding modular trajectories are (4.15) with  $\zeta(\lambda, u)$  replaced by  $\zeta_L(\lambda, u)$ , which is defined by specialising (3.45) to  $w_+(u) = w_-(u) = w_L(u)$  in (6.10). In figure 7, the dashed blue curve in  $\mathcal{D}_A$  is a modular trajectory generated by the momentum operator whose initial point is the black dot.

The modular conjugation  $J$  for the state and the bipartition of the circle that we are considering displays a geometric action in the spacetime characterised by the map  $(x, t) \rightarrow (\tilde{x}(x, t), \tilde{t}(x, t))$  given by (4.22) where  $j(u)$  is replaced by the function  $j_L : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$j_L(u) \equiv \frac{L}{2\pi i} \log \left( \frac{e^{2\pi i b/L} + e^{2\pi i a/L}}{2} + \frac{[(e^{2\pi i b/L} - e^{2\pi i a/L})/2]^2}{e^{2\pi i u/L} - (e^{2\pi i b/L} + e^{2\pi i a/L})/2} \right) \quad (6.18)$$

which is a bijective and idempotent function sending  $A$  onto  $B$  with negative derivative

$$j'_L(u) = - \frac{\sin^2[\pi(b-a)/L]}{|\sin[\pi(b-u)/L] - e^{\pi i(b-a)/L} \sin[\pi(u-a)/L]|^2}. \quad (6.19)$$

Notice that the maps (6.12) and (6.18) commute, namely they satisfy

$$j_L(\xi_L(\tau, u)) = \xi_L(\tau, j_L(u)) \quad (6.20)$$

whose infinite volume limit gives (4.21). In figure 7 the solid and dashed curves in the light blue region are obtained from the corresponding ones in  $\mathcal{D}_A$  through the above mentioned map providing the geometric action of the modular conjugation. These curves also the modular trajectories generated by either the modular Hamiltonian or the modular momentum whose initial point is labelled by the black square, which is the image of the black dot in  $\mathcal{D}_A$  through the geometric action of modular conjugation.

The field transformations of the basic CFT fields are obtained by adapting the observations made in section 4.3 to the finite volume case we are considering. This leads us to conclude that the action of  $J$  on  $\phi_\pm$ ,  $\phi_\pm^*$  and  $j_\pm$  is given by (4.29), (4.30) are (4.31) respectively, with  $j(u)$  replaced by  $j_L(u)$  defined in (6.18). As for  $\mathcal{T}_\pm$ , the non trivial term due to the

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<sup>4</sup>Another interesting result about (6.12) is

$$\mathcal{S}_u[e^{2\pi i \xi_L(\tau, u)/L}](\tau, u) = \mathcal{S}_u[e^{2\pi i u/L}](u) = \frac{2\pi^2}{L^2}.$$

Schwarzian derivative must be taken into account. The result is obtained by setting  $\tau = \pm i/2$  in (6.16) and reads

$$J \mathcal{T}_{\pm}(u) J = j'_L(u)^2 \mathcal{T}_{\pm}(j_L(u)) + \left( \frac{\kappa \mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \right) [1 - j'_L(u)^2]. \quad (6.21)$$

By adapting (4.33) to the finite volume case, this transformation rule combined with the fact that (6.9) and (6.18) satisfy

$$j'_L(u) V_L(u) = V_L(j_L(u)) \quad (6.22)$$

leads to write the full modular Hamiltonian in the form given in (4.35).

At finite volume, we performed a consistency check of these field transformations rules by taking their mean values, employing the fact that  $J$  leaves the state invariant and using (6.1), as done in the end of section 4.3 for the interval in the infinite line. For instance, in the case of  $\mathcal{T}_{\pm}$  we have that  $\langle \mathcal{T}_{\pm}(u) \rangle_{L, \mu_{\pm}} = \langle J \mathcal{T}_{\pm}(u) J \rangle_{L, \mu_{\pm}}$  and we found that the r.h.s. of (6.21) is consistent with the last expression in (6.1).

Following the analysis reported in the final part of section 4.3, we computed the modular evolution of an operator belonging to the complementary region  $B$ , from (4.36) specialised to either  $\phi_{\pm}$  or  $j_{\pm}$  or  $\mathcal{T}_{\pm}$ . In the appendix E, by employing also (6.20) and (6.21), we have found that the expressions (2.18), (2.20), (2.23) and (2.34) with  $\xi_{\pm}(\tau, u)$  given by (6.12) for the modular evolution hold also for  $u \in B$ .

We remark that the compact manifold  $\mathbb{M}_{\diamond}$  considered above does not coincide with the compactification  $\overline{\mathbb{M}} = (\mathbb{S} \times \mathbb{S})/\mathbb{Z}_2$  of the two-dimensional Minkowski spacetime  $\mathbb{M}$  (often called Dirac-Weyl compactification) discussed in [55–59], where  $\mathbb{S}$  is the unit circle. Since  $\overline{\mathbb{M}}$  is not causally orientable, its universal covering  $\widetilde{\mathbb{M}} = \mathbb{S} \times \mathbb{R}$  is employed to define a consistent CFT on the cylinder [55, 57, 59]. However, from the group theoretical point of view, the time  $t_c \in \mathbb{R}$  on  $\widetilde{\mathbb{M}}$  is associated to the conformal Hamiltonian  $\frac{1}{2}(P_0 + K_0)$  rather than to the Hamiltonian  $P_0$  in  $\mathbb{M}$ , where  $K_0$  is the generator of the special conformal transformations [56, 58].

### 6.3 Modular correlators

The modular evolutions of  $\phi_{\pm}$ ,  $j_{\pm}$  and  $\mathcal{T}_{\pm}$  can be written by specialising (2.18), (2.23) and (2.34) to (6.12), as already mentioned above in section 6.2. The result for  $\mathcal{T}_{\pm}$  has been reported explicitly in (6.16). Taking the mean values of the resulting expressions and using (6.1), we find one-point functions that are independent of  $\tau$ , i.e.  $\langle \phi_{\pm}(u) \rangle_{L, \mu_{\pm}} = \langle \phi_{\pm}(\tau, u) \rangle_{L, \mu_{\pm}}$  for the primaries,  $\langle j_{\pm}(u) \rangle_{L, \mu_{\pm}} = \langle j_{\pm}(\tau, u) \rangle_{L, \mu_{\pm}}$  for the currents and  $\langle \mathcal{T}_{\pm}(u) \rangle_{L, \mu_{\pm}} = \langle \mathcal{T}_{\pm}(\tau, u) \rangle_{L, \mu_{\pm}}$  for the operators (2.7) (in the latter case also (6.17) has been used).

As for the modular two-point correlators at finite volume, when  $u \neq v$ ,  $\tau_1 \neq \tau_2$  and for the velocity (6.9) providing (6.12), one finds the following identity

$$\frac{\partial_u \xi_L(\tau_1, u) \partial_v \xi_L(\tau_2, v)}{\left( \frac{L}{\pi} \sin \left[ \frac{\pi}{L} (\xi_L(\tau_1, u) - \xi_L(\tau_2, v)) \right] \right)^2} = \left( \frac{R_L(\tau_{12}; u, v)}{\frac{L}{\pi} \sin \left[ \frac{\pi}{L} (u - v) \right]} \right)^2 \quad (6.23)$$

where

$$\begin{aligned}
 R_L(\tau; u, v) &\equiv \frac{e^{2\pi w_L(u)} - e^{2\pi w_L(v)}}{e^{2\pi w_L(u)+\pi\tau} - e^{2\pi w_L(v)-\pi\tau}} \\
 &= \frac{\sin\left[\frac{\pi}{L}(u-a)\right] \sin\left[\frac{\pi}{L}(v-b)\right] - \sin\left[\frac{\pi}{L}(u-b)\right] \sin\left[\frac{\pi}{L}(v-a)\right]}{\sin\left[\frac{\pi}{L}(u-a)\right] \sin\left[\frac{\pi}{L}(v-b)\right] e^{\pi\tau} - \sin\left[\frac{\pi}{L}(u-b)\right] \sin\left[\frac{\pi}{L}(v-a)\right] e^{-\pi\tau}}
 \end{aligned} \tag{6.24}$$

which satisfies

$$R_L(\tau = 0; u, v) = 1 \quad R_L(-\tau; v, u) = R_L(\tau; u, v) \quad R_L(\tau + i; u, v) = -R_L(-\tau; v, u). \tag{6.25}$$

The infinite volume limit of (6.23) and (6.24) gives (4.37) and (4.38) respectively, as expected. The r.h.s. of (6.23) has been obtained by observing that

$$\sin\left[\frac{\pi}{L}(\xi_L(\tau_1, u) - \xi_L(\tau_2, v))\right] = \frac{p_L(\tau_1, u) p_L(\tau_2, v)}{R_L(\tau_{12}; u, v)} \sin\left[\frac{\pi}{L}(u-v)\right] \tag{6.26}$$

in terms of (6.24) and of

$$p_L(\tau, u) \equiv \frac{2i e^{\pi i(a+b)/L} \sin[\pi(b-a)/L] e^{\pi\tau} e^{\pi i u/L}}{\left[ e^{2\pi i b/L} - e^{2\pi i u/L} + (e^{2\pi i u/L} - e^{2\pi i a/L}) e^{2\pi\tau} \right] e^{\pi i \xi_L(\tau, u)/L}} \tag{6.27}$$

which satisfies  $p_L(\tau = 0, u) = 1$  and is a real function when  $u \in A$ , indeed we notice that its square can be written as

$$\begin{aligned}
 p_L(\tau, u)^2 &= \partial_u \xi_L(\tau, u) \\
 &= \frac{\sin^2\left[\frac{\pi}{L}(b-a)\right]}{\sin^2\left[\frac{\pi}{L}(b-a)\right] + (e^{2\pi\tau} - 1) \sin^2\left[\frac{\pi}{L}(u-a)\right] + (e^{-2\pi\tau} - 1) \sin^2\left[\frac{\pi}{L}(b-u)\right]}
 \end{aligned} \tag{6.28}$$

which is positive for  $u \in A$  and any  $\tau \in \mathbb{R}$ . From (6.28), we have that  $p_L(\tau, u)$  does not vanish for any finite value of  $\tau$ ; hence, since  $p_L(\tau = 0, u) = 1$ , we conclude that  $p_L(\tau, u) > 0$ . The infinite volume limit of (6.26) gives (4.40), as expected.

The modular correlators can be written by adapting the procedure described in section 4.4. Thus, from the expressions in (2.18), (2.20), (2.23) and (2.34), the correlators (6.2), (6.3) and (6.4) and the identity (6.23), for the connected modular correlators of the primary  $\phi_{\pm}$  we get

$$\langle \phi_{\pm}^*(\tau_1, u) \phi_{\pm}(\tau_2, v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{e^{\pm i \mu_{\pm}(u-v)}}{2\pi e^{\pm i \pi h_{\pm}}} W_{L, \pm}(\pm \tau_{12}; u, v)^{2h_{\pm}} \tag{6.30}$$

$$\langle \phi_{\pm}(\tau_1, u) \phi_{\pm}^*(\tau_2, v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{e^{\mp i \mu_{\pm}(u-v)}}{2\pi e^{\pm i \pi h_{\pm}}} W_{L, \pm}(\pm \tau_{12}; u, v)^{2h_{\pm}} \tag{6.31}$$

and for the connected modular correlators of the current  $j_{\pm}$  and the energy-momentum tensor one obtains respectively

$$\langle j_{\pm}(\tau_1, u) j_{\pm}(\tau_2, v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{\kappa}{4\pi^2} W_{L, \pm}(\pm \tau_{12}; u, v)^2 \tag{6.32}$$

$$\langle \mathcal{T}_{\pm}(\tau_1, u) \mathcal{T}_{\pm}(\tau_2, v) \rangle_{L, \mu_{\pm}}^{\text{con}} = \frac{c}{8\pi^2} W_{L, \pm}(\pm \tau_{12}; u, v)^4 \tag{6.33}$$

where  $W_{L,\pm}$  is defined in terms of (6.10) as follows [54]

$$W_{L,\pm}(\tau; u, v) \equiv \frac{e^{2\pi w(u)} - e^{2\pi w(v)}}{\frac{L}{\pi} \sin\left[\frac{\pi}{L}(u-v)\right]} \frac{1}{e^{2\pi w(u)+\pi\tau} - e^{2\pi w(v)-\pi\tau} \mp i\varepsilon} \quad (6.34)$$

which becomes (4.44) in the infinite volume limit  $L \rightarrow +\infty$ .

For (6.34) one finds the following property

$$W_{L,\pm}(\tau + i; u, v) = W_{L,\pm}(\tau - i; u, v) = W_{L,\pm}(-\tau; v, u) \quad (6.35)$$

which implies that the modular correlators (6.30), (6.31), (6.32) and (6.33) satisfy the KMS condition with modular inverse temperature  $\tilde{\beta} = 1$ .

Furthermore, when  $\tau \neq 0$ , the limit  $v \rightarrow u$  of (4.44) is well defined and given by

$$\lim_{v \rightarrow u} W_{L,\pm}(\tau; u, v) = \frac{\pi}{V_L(u) \sinh(\pi\tau \mp i\varepsilon)} \quad (6.36)$$

in terms of (6.9), which becomes (4.50) as  $L \rightarrow +\infty$  and will be employed in section 7.

Finally, by employing the following identity for (6.34) and (6.18)

$$W_{L,\pm}(\tau, u, v)^2 = j'_L(u) j'_L(v) W_{L,\pm}(-\tau, j_L(v), j_L(u))^2 \quad (6.37)$$

we checked that the expressions reported in the r.h.s.'s of (4.29), (4.30), (4.31) and (6.21), with  $j$  replaced by  $j_L$ , are consistent with (4.51) written for finite volume and finite density representation.

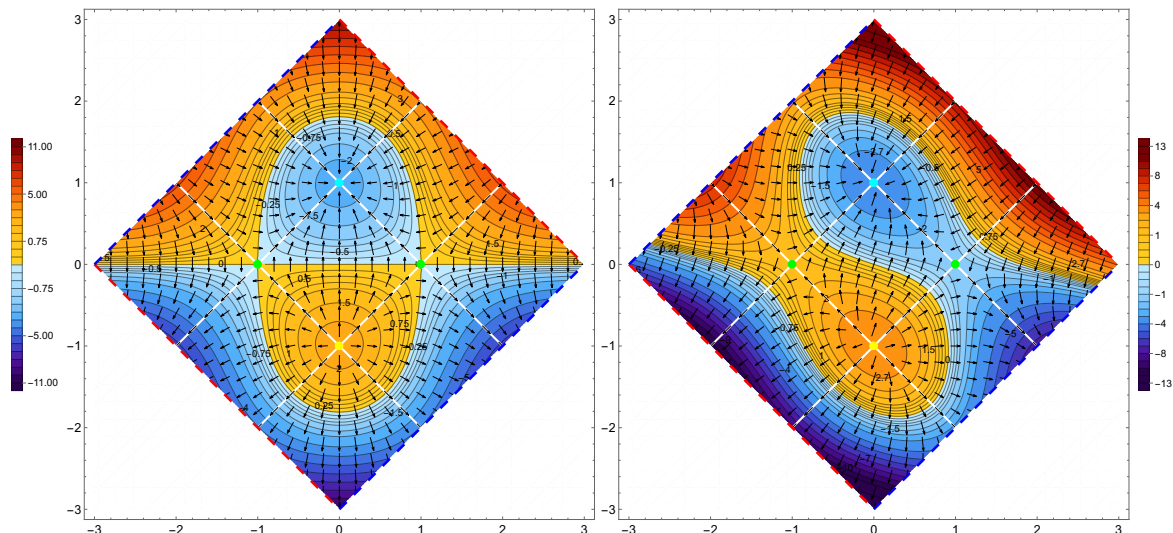
## 7 Modular transport and fluctuations at finite volume

In this section, the analyses discussed in section 5 are extended to the CFT at finite density and finite volume described in section 6.

The mean values of the charge currents  $\langle j_x(\tau; x, t) \rangle_{L,\mu}$  and  $\langle j_t(\tau; x, t) \rangle_{L,\mu}$  are obtained by employing (3.6), (3.9) and (6.1). This gives the r.h.s.'s of (5.1)–(5.2) with the velocity  $V(u)$  replaced by  $V_L(u)$  introduced in (6.9), which provide the components of the vector field  $\mathbf{j}(x, t)$ . Similarly, the mean values of the helicity currents  $\langle k_x(\tau; x, t) \rangle_{L,\mu}$  and  $\langle k_t(\tau; x, t) \rangle_{L,\mu}$ , which are the components of the vector field  $\mathbf{k}(x, t)$ , can be written from (3.22) and (6.1), finding the r.h.s.'s of (5.3)–(5.4) with  $V(u)$  replaced by  $V_L(u)$ .

The smooth planar vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$  in  $\mathbb{M}_\diamond$  are shown in figure 8 and figure 9 for the choice of the parameters described in the caption of figure 8. In all the figures of this section  $\mathcal{D}_A$  and  $\mathbb{M}_\diamond$  have been represented like in the left panel of figure 7. Moreover, the extension of the vector fields to the entire spacetime  $\mathbb{M}_\diamond$  have been displayed, as discussed in section 5.1 for the case of the Minkowski spacetime (see also figure 3).

Both the vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$  have the same critical points (by construction) and all of them have multiplicity 1. In particular, four critical points occur in  $\mathbb{M}_\diamond$ : two nodes (one stable and one unstable) and two saddles, which are denoted through the same notation adopted in the figures of section 5. We recall that the nodes have Poincaré index +1, while the saddles have Poincaré index −1. Thus, the sum of the Poincaré indices of all the isolated critical points in a fundamental region vanishes. This is consistent with the



**Figure 8.** Vector fields for the mean values of the charge currents (7.1) in  $\mathbb{M}_\diamond$ , whose potential is the first expressions in (7.3). The CFT has  $c = 1$ ,  $\kappa = 3$  and either equal chemical potentials  $\mu_+ = \mu_- = 2\pi/L$  (left panel) or different chemical potentials  $\mu_+ = 4\pi/L$  and  $\mu_- = 2\pi/L$  (right panel). Here  $L = 6$  and  $\ell = 2$ .

Poincaré-Hopf theorem mentioned in section 5.1; indeed  $\mathbb{M}_\diamond$  has the topology of the torus, whose Euler characteristic is equal to zero.

The vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$  are curl free and satisfy

$$\langle j_x(\tau; x, t) \rangle_{L, \mu} = -\partial_x \mathbb{W}_{L,j}(x, t) \quad \langle j_t(\tau; x, t) \rangle_{L, \mu} = -\partial_t \mathbb{W}_{L,j}(x, t) \quad (7.1)$$

$$\langle k_x(\tau; x, t) \rangle_{L, \mu} = -\partial_x \mathbb{W}_{L,k}(x, t) \quad \langle k_t(\tau; x, t) \rangle_{L, \mu} = -\partial_t \mathbb{W}_{L,k}(x, t) \quad (7.2)$$

where the potentials read respectively

$$\mathbb{W}_{L,j}(x, t) \equiv \frac{\kappa}{2\pi} [\mu_+ g_L(u_+) - \mu_- g_L(u_-)] \quad \mathbb{W}_{L,k}(x, t) \equiv \frac{\kappa}{2\pi} [\mu_+ g_L(u_+) + \mu_- g_L(u_-)] \quad (7.3)$$

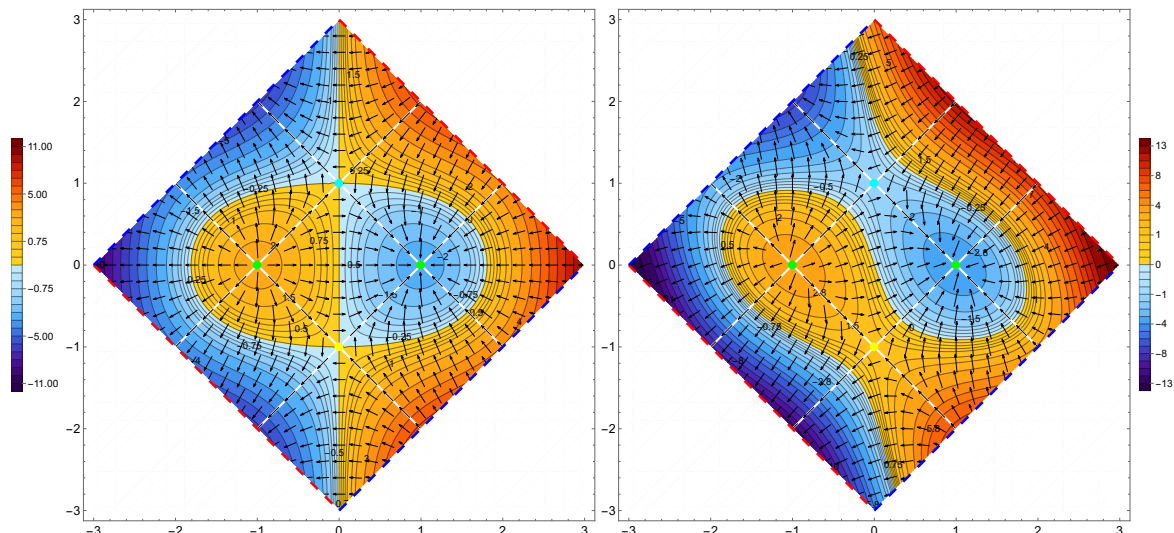
being the function  $g_L(u)$  defined as follows

$$g_L(u) \equiv \frac{L}{\tan[\frac{\pi}{L}(b-a)]} \left( u - \frac{a+b}{2} \right) - \frac{L^2}{2\pi \sin[\frac{\pi}{L}(b-a)]} \sin \left[ \frac{2\pi}{L} \left( u - \frac{a+b}{2} \right) \right]. \quad (7.4)$$

Notice that, although (7.4) is not a periodic function of  $u$  and therefore the potentials in (7.3) are not periodic  $\mathbb{M}_\diamond$  as well, the corresponding vector fields (7.1) and (7.2) are periodic  $\mathbb{M}_\diamond$ . Thus, the potentials in (7.3) are well defined on an open subset of  $\mathbb{M}_\diamond$ ; hence the potentials displayed in figure 8 and figure 9 are not defined in a neighbourhood of boundary made by the union of the dashed straight segments.

Consistency between (7.1)–(7.4) and the mean values of the currents occurs because (7.4) and (6.9) are related as follows

$$-\partial_u g_L(u) = V_L(u). \quad (7.5)$$



**Figure 9.** Vector fields for the mean values of the helicity currents (7.2) in  $\mathbb{M}_\diamond$ , whose potential is the second expressions in (7.3), for either equal (left panel) or different (right panel) chemical potentials, in the same setup of figure 8.

Combining (5.14), (6.11) and (7.5), it is straightforward to find that the functions (5.13) and (7.4) are related as follows

$$\partial_u g_L(u) = \frac{\partial_u \tilde{g}(e^{2\pi i u/L})}{[\partial_u(e^{2\pi i u/L})]^2} \quad (7.6)$$

where  $\tilde{g}(u)$  is defined as (5.13) where  $a$  and  $b$  are replaced by  $e^{2\pi i a/L}$  and  $e^{2\pi i b/L}$  respectively. Moreover, the infinite volume limit  $L \rightarrow +\infty$  of (7.4) gives (5.13); hence the potentials (5.12) obtained for the CFT on the line are the infinite volume limit of (7.3).

The vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$ , which are well defined in  $\mathbb{M}_\diamond$ , have vanishing fluxes through the solid white lines in figure 8 and figure 9; indeed, the absolute value of the ratio of their components is equal to one along these lines. These vanishing fluxes lead us to consider the total charges in the diamond  $\mathcal{D}_A$ . In the finite volume and finite density representation, from (6.1) and (5.16), for the mean values of (3.17) and (3.23) we find respectively

$$\langle Q_A \rangle_{L, \mu} = \langle Q_A \rangle_\mu \quad \langle \tilde{Q}_A \rangle_{L, \mu} = \langle \tilde{Q}_A \rangle_\mu. \quad (7.7)$$

It is worth considering the line integrals of the curl free vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$  along curves anchored to the opposite vertices of  $\mathcal{D}_A$ , as done in section 5.1. The results read

$$\mathcal{L}[\mathbf{j}](\gamma(P_a \rightarrow P_b)) = W_{L, j}|_{P_a} - W_{L, j}|_{P_b} = -\frac{2\pi}{3} \langle \tilde{Q}_A \rangle_\mu M(\pi\ell/L) \quad (7.8)$$

$$\mathcal{L}[\mathbf{j}](\gamma(P_{-\infty} \rightarrow P_{+\infty})) = W_{L, j}|_{P_{-\infty}} - W_{L, j}|_{P_{+\infty}} = -\frac{2\pi}{3} \langle Q_A \rangle_\mu M(\pi\ell/L) \quad (7.9)$$

and

$$\mathcal{L}[\mathbf{k}](\gamma(P_a \rightarrow P_b)) = W_{L, k}|_{P_a} - W_{L, k}|_{P_b} = -\frac{2\pi}{3} \langle Q_A \rangle_\mu M(\pi\ell/L) \quad (7.10)$$

$$\mathcal{L}[\mathbf{k}](\gamma(P_{-\infty} \rightarrow P_{+\infty})) = W_{L, k}|_{P_{-\infty}} - W_{L, k}|_{P_{+\infty}} = -\frac{2\pi}{3} \langle \tilde{Q}_A \rangle_\mu M(\pi\ell/L) \quad (7.11)$$

where

$$M(y) \equiv \frac{3}{y^2} \left(1 - y \cot(y)\right). \quad (7.12)$$

Since  $M(\pi\ell/L) \rightarrow 1$  as  $L/\ell \rightarrow +\infty$ , the line integrals (7.8)–(7.9) and (7.10)–(7.11) become respectively (5.17)–(5.18) and (5.19)–(5.20) in the infinite volume limit. When  $\mu_+ = \mu_-$ , the line integrals in (7.8) and (7.11) vanish (see also the left panel of figure 8 and figure 9 respectively).

The mean values of the energy currents  $\langle \mathcal{J}_x(\tau; x, t) \rangle_{L, \mu}$  and  $\langle \mathcal{J}_t(\tau; x, t) \rangle_{L, \mu}$  for the two-dimensional CFT in  $\mathbb{M}_\diamond$  provide the components of the vector field  $\mathcal{J}(x, t)$ . From the expressions of the operators in (3.29)–(3.30) with  $C_{\mathcal{J}} = -\pi c/6$  (see also in the text below (5.27)), the mean values (6.1) and the velocity  $V_L(u)$  in (6.9) characterising the representation and the bipartition we are considering, for the mean values of these energy currents we find

$$\langle \mathcal{J}_x(\tau; x, t) \rangle_{L, \mu} = - \left[ \left( \frac{\kappa\mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_+)^2 - \left( \frac{\kappa\mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_-)^2 \right] \quad (7.13)$$

and

$$\begin{aligned} \langle \mathcal{J}_t(\tau; x, t) \rangle_{L, \mu} &\equiv -\frac{\kappa}{4\pi} \left\{ \mu_+^2 V_L(u_+)^2 + \mu_-^2 V_L(u_-)^2 \right\} \\ &\quad - \frac{c}{24\pi} \left\{ V_L(u_+)^2 \mathcal{V}[V_L](u_+) + V_L(u_-)^2 \mathcal{V}[V_L](u_-) \right\} - \frac{\pi c}{6} \end{aligned} \quad (7.14)$$

$$= - \left[ \left( \frac{\kappa\mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_+)^2 + \left( \frac{\kappa\mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_-)^2 \right] \quad (7.15)$$

where the non trivial function of the spacetime position occurring in the second line of (7.14) can be simplified by observing that

$$V_L(u)^2 \mathcal{V}[V_L](u) = -2\pi^2 - \frac{2\pi^2}{L^2} V_L(u)^2 \quad (7.16)$$

which becomes constant in the infinite volume limit (see (5.28)).

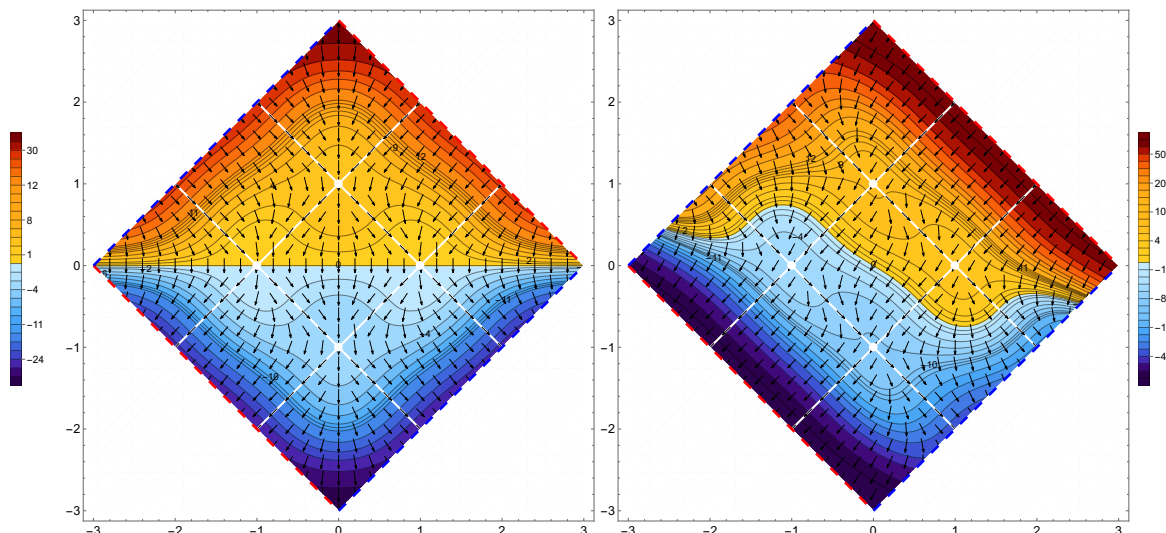
In a similar way, the mean values  $\langle \tilde{\mathcal{J}}_x(\tau; x, t) \rangle_{L, \mu}$  and  $\langle \tilde{\mathcal{J}}_t(\tau; x, t) \rangle_{L, \mu}$  of the momentum currents define the components of the vector field  $\tilde{\mathcal{J}}(x, t)$  in  $\mathbb{M}_\diamond$ . From the expressions of the operators in (3.37)–(3.38) with  $\tilde{C}_{\mathcal{J}} = 0$ , the mean values (6.1) and the velocity  $V_L(u)$  in (6.9), for the mean values of the operators (3.37) and (3.38) with  $\tilde{C}_{\mathcal{J}} = 0$  in the finite density representation in  $\mathbb{M}_\diamond$  we get respectively

$$\langle \tilde{\mathcal{J}}_x(\tau; x, t) \rangle_{L, \mu} = - \left[ \left( \frac{\kappa\mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_+)^2 + \left( \frac{\kappa\mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_-)^2 \right] \quad (7.17)$$

and

$$\begin{aligned} \langle \tilde{\mathcal{J}}_t(\tau; x, t) \rangle_{L, \mu} &\equiv -\frac{\kappa}{4\pi} \left\{ \mu_+^2 V_L(u_+)^2 - \mu_-^2 V_L(u_-)^2 \right\} \\ &\quad - \frac{c}{24\pi} \left\{ V_L(u_+)^2 \mathcal{V}[V_L](u_+) - V_L(u_-)^2 \mathcal{V}[V_L](u_-) \right\} \end{aligned} \quad (7.18)$$

$$= - \left[ \left( \frac{\kappa\mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_+)^2 - \left( \frac{\kappa\mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) V_L(u_-)^2 \right]. \quad (7.19)$$



**Figure 10.** Vector fields for the mean values of the energy density currents (7.20) in  $\mathbb{M}_\diamond$ , whose potential is (7.22), for either equal (left panel) or different (right panel) chemical potentials, in the same setup of figure 8.

The additive constants in (7.13)–(7.14) and (7.17)–(7.18) have been fixed by imposing that the resulting expressions vanish at the vertices of the diamond  $\mathcal{D}_A$ .

In figure 10 and figure 11 we show the vector fields  $\mathcal{J}(x, t)$  and  $\tilde{\mathcal{J}}(x, t)$  in  $\mathbb{M}_\diamond$  for the choice of the parameters described in the caption of figure 8. These vector fields vanish at the same critical point, which correspond to the vertices of  $\mathcal{D}_A$ . All these isolated critical points have multiplicity 2 and Poincaré index 0. Hence, the Poincaré-Hopf theorem can be checked also in these cases; indeed, the sum of the Poincaré indices in  $\mathbb{M}_\diamond$  is zero, consistently with the fact that  $\mathbb{M}_\diamond$  has the topology of the torus.

The vector fields  $\mathcal{J}(x, t)$  and  $\tilde{\mathcal{J}}(x, t)$  are curl free and can be written respectively as

$$\langle \mathcal{J}_x(\tau; x, t) \rangle_{L, \mu} = -\partial_x W_{L, \mathcal{J}}(x, t) \quad \langle \mathcal{J}_t(\tau; x, t) \rangle_{L, \mu} = -\partial_t W_{L, \mathcal{J}}(x, t) \quad (7.20)$$

$$\langle \tilde{\mathcal{J}}_x(\tau; x, t) \rangle_{L, \mu} = -\partial_x W_{L, \tilde{\mathcal{J}}}(x, t) \quad \langle \tilde{\mathcal{J}}_t(\tau; x, t) \rangle_{L, \mu} = -\partial_t W_{L, \tilde{\mathcal{J}}}(x, t) \quad (7.21)$$

where the potentials  $W_{L, \mathcal{J}}$  and  $W_{L, \tilde{\mathcal{J}}}$  read respectively

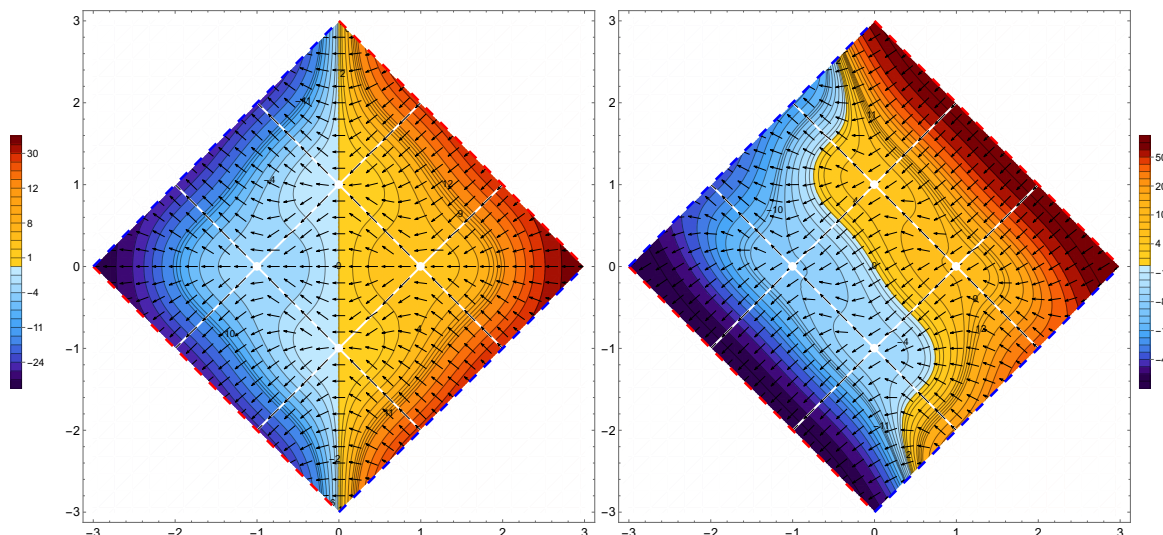
$$W_{L, \mathcal{J}}(x, t) \equiv \left( \frac{\kappa \mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) G_L(u_+) - \left( \frac{\kappa \mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) G_L(u_-) \quad (7.22)$$

$$W_{L, \tilde{\mathcal{J}}}(x, t) \equiv \left( \frac{\kappa \mu_+^2}{4\pi} - \frac{\pi c}{12L^2} \right) G_L(u_+) + \left( \frac{\kappa \mu_-^2}{4\pi} - \frac{\pi c}{12L^2} \right) G_L(u_-) \quad (7.23)$$

in terms of the function  $G_L(u)$  defined as follows

$$G_L(u) \equiv \frac{L^3}{2\pi \sin^2\left[\frac{\pi}{L}(b-a)\right]} \left\{ \sin\left(\frac{2\pi(b-u)}{L}\right) - \sin\left(\frac{2\pi(u-a)}{L}\right) + \frac{1}{4} \sin\left(\frac{4\pi}{L}\left(u - \frac{a+b}{2}\right)\right) + \frac{2\pi}{L} \left[ 1 + \frac{1}{2} \cos\left(\frac{2\pi(b-a)}{L}\right) \right] \left(u - \frac{a+b}{2}\right) \right\} \quad (7.24)$$

which satisfies  $G_L\left(\frac{a+b}{2}\right) = 0$ ; hence (7.22) and (7.23) vanish in the center of the diamond  $\mathcal{D}_A$ .



**Figure 11.** Vector fields for the mean values of the momentum density currents (7.21) in  $\mathbb{M}_\diamond$ , whose potential is (7.23), for either equal (left panel) or different (right panel) chemical potentials, in the same setup of figure 8.

Similarly to (7.4) and (7.1)–(7.2), the function (7.24) is not periodic in  $u$  and therefore the potentials (7.22) and (7.23) are not well defined in the whole spacetime  $\mathbb{M}_\diamond$  but only in a subset of  $\mathbb{M}_\diamond$  where a neighbourhood of boundary made by the union of the dashed straight segments in figure 10 and figure 11 has been subtracted.

Consistency among the expressions reported between (7.13) and (7.24) occurs because (7.24) and  $V_L(u)$  in (6.9) are related as follows

$$\partial_u G_L(u) = V_L(u)^2. \tag{7.25}$$

From (5.40), (6.11) and (7.25), we observe that the functions in (5.39) and (7.24) are related as follows

$$\partial_u G_L(u) = \frac{\partial_u \tilde{G}(e^{2\pi i u/L})}{[\partial_u (e^{2\pi i u/L})]^3} \tag{7.26}$$

where  $\tilde{G}(u)$  is defined as (5.39) where  $a$  and  $b$  are replaced by  $e^{2\pi i a/L}$  and  $e^{2\pi i b/L}$  respectively. In the infinite volume limit  $L \rightarrow +\infty$  of (7.24) gives (5.39); hence the potentials in (7.22) and (7.23) become respectively the first and the second potential in (5.38) in this regime.

The fluxes of the vector fields  $\mathcal{J}(x, t)$  and  $\tilde{\mathcal{J}}(x, t)$  through the straight white lines in figure 10 and figure 11 vanish, as we can show by observing that the absolute value of the ratios of their components is equal to 1 along these lines. As already highlighted in section 5.2 for the corresponding vector fields on the plane, also in this case this analytic result is not properly displayed in the left panels of figure 10 and figure 11 (see around the vertices of  $\mathcal{D}_A$ ) because of a failure in the graphical representation of the vector fields. A similar issue occurs in both the panels of these figures at the vertices of  $\mathcal{D}_A$ , where arrows are displayed, despite the fact that these are critical points of the vector fields. Such failures do not occur for the vector fields  $\mathbf{j}(x, t)$  and  $\mathbf{k}(x, t)$  in figure 8 and figure 9, whose critical points have multiplicity 1.

The line integrals of the curl free vector fields  $\mathcal{J}(x, t)$  and  $\widetilde{\mathcal{J}}(x, t)$  along curves anchored to the opposite vertices of  $\mathcal{D}_A$  read

$$\mathcal{L}[\mathcal{J}](\gamma(P_a \rightarrow P_b)) = \mathbb{W}_{L,\mathcal{J}}|_{P_a} - \mathbb{W}_{L,\mathcal{J}}|_{P_b} = -\frac{4\pi}{5} \widetilde{\mathbb{E}}_{A,L} \mathcal{M}(\pi\ell/L) \quad (7.27)$$

$$\mathcal{L}[\mathcal{J}](\gamma(P_{-\infty} \rightarrow P_{+\infty})) = \mathbb{W}_{L,\mathcal{J}}|_{P_{-\infty}} - \mathbb{W}_{L,\mathcal{J}}|_{P_{+\infty}} = -\frac{4\pi}{5} \mathbb{E}_{A,L} \mathcal{M}(\pi\ell/L) \quad (7.28)$$

and

$$\mathcal{L}[\widetilde{\mathcal{J}}](\gamma(P_a \rightarrow P_b)) = \mathbb{W}_{L,\widetilde{\mathcal{J}}}|_{P_a} - \mathbb{W}_{L,\widetilde{\mathcal{J}}}|_{P_b} = -\frac{4\pi}{5} \mathbb{E}_{A,L} \mathcal{M}(\pi\ell/L) \quad (7.29)$$

$$\mathcal{L}[\widetilde{\mathcal{J}}](\gamma(P_{-\infty} \rightarrow P_{+\infty})) = \mathbb{W}_{L,\widetilde{\mathcal{J}}}|_{P_{-\infty}} - \mathbb{W}_{L,\widetilde{\mathcal{J}}}|_{P_{+\infty}} = -\frac{4\pi}{5} \widetilde{\mathbb{E}}_{A,L} \mathcal{M}(\pi\ell/L) \quad (7.30)$$

where we have introduced

$$\mathbb{E}_{A,L} \equiv \left( \frac{\kappa(\mu_+^2 + \mu_-^2)}{8\pi} - \frac{\pi c}{12L^2} \right) \frac{\pi\ell^3}{3} M(\pi\ell/L) \quad (7.31)$$

$$\widetilde{\mathbb{E}}_{A,L} \equiv \frac{\kappa(\mu_+^2 - \mu_-^2)}{8\pi} \frac{\pi\ell^3}{3} M(\pi\ell/L) \quad (7.32)$$

in terms of the function (7.12) (hence these expressions become the ones defined in (5.45) in the infinite volume limit, as expected) and

$$\mathcal{M}(y) \equiv \frac{5[3 \cot(y) + y(2 - 3 \csc(y)^2)]}{4y[y \cot(y) - 1]}. \quad (7.33)$$

From (6.1), we notice that (7.31) and (7.32) provide the mean values of the total energy (3.32) and of the total momentum (3.39) in  $\mathcal{D}_A$  in the finite density and finite volume representation when  $f_+(u_+) = f_-(u_-) = 0$ . Instead, from (6.7), for the mean values of (3.32) and (3.39) we have  $\langle E_A \rangle_{L,\mu} = \langle \widetilde{E}_A \rangle_{L,\mu} = 0$ .

Since  $\mathcal{M}(\pi\ell/L) \rightarrow 1$  in the infinite volume limit, the line integrals (7.27)–(7.28) and (7.29)–(7.30) become respectively (5.41)–(5.42) and (5.43)–(5.44) in this regime. When  $\mu_+ = \mu_-$ , the integrals in (7.28) and (7.30) vanish, as one can realise also from the left panel of figure 10 and figure 11 respectively.

By using the correlators in section 6.1, also at finite volume we can introduce the modular noise power at frequency  $\omega$  and in the spacetime point  $(x, t) \in \mathbb{M}_\diamond$  generated by the various quantities as done in section 5.3 for the Minkowski spacetime. In  $\mathbb{M}_\diamond$  one introduces  $P_{L,j}(\omega; x, t)$  for the charge current,  $P_{L,k}(\omega; x, t)$  for the helicity current,  $P_{L,\mathcal{J}}(\omega; x, t)$  for the energy current,  $P_{L,\widetilde{\mathcal{J}}}(\omega; x, t)$  for the momentum current,  $P_{L,\varrho}(\omega; x, t)$  for the charge density and  $P_{L,\chi}(\omega; x, t)$  for the density (3.19). Since in the computation of these quantities the coincident points limit must be considered, the only difference with respect to the expressions reported in section 5.3 is due to the fact that  $V(u)$  must be replaced by  $V_L(u)$  in (6.9). Thus,  $P_{L,j}(\omega; x, t) = P_j(\omega; x, t)$ ,  $P_{L,k}(\omega; x, t) = P_k(\omega; x, t)$ ,  $P_{L,\mathcal{J}}(\omega; x, t) = P_{\mathcal{J}}(\omega; x, t)$  and  $P_{L,\widetilde{\mathcal{J}}}(\omega; x, t) = P_{\widetilde{\mathcal{J}}}(\omega; x, t)$ , while in  $P_{L,\varrho}(\omega; x, t)$  and  $P_{L,\chi}(\omega; x, t)$  do not coincide with  $P_\varrho(\omega; x, t)$  and  $P_\chi(\omega; x, t)$  respectively because the velocity explicitly occurs in their expressions (see (5.68)).

These observations lead to introduce  $\mathcal{A}_L[\mathcal{O}](\omega; x, t)$  and  $\mathcal{C}_L[\mathcal{O}](\omega; x, t)$  as the r.h.s. of (5.70) and (5.71) respectively with  $\langle \dots \rangle_\mu^{\text{con}}$  replaced by  $\langle \dots \rangle_{L, \mu}^{\text{con}}$ , finding that they satisfy the following modular fluctuation-dissipation relation

$$\mathcal{A}_L[\mathcal{O}](\omega; x, t) = \coth\left(\frac{\omega}{2}\right) \mathcal{C}_L[\mathcal{O}](\omega; x, t) \tag{7.34}$$

which encodes the fact that the modular evolution has a thermal nature with inverse temperature  $\tilde{\beta} = 1$ , in agreement with the KMS condition for the modular correlators (see section 6.3).

Finally, we emphasise that the heuristic picture for the transport described in the final part of section 5.1 can be adapted to the finite volume case in a straightforward way.

## 8 Conclusions

We investigated the modular quantum transport in a two-dimensional CFT at finite density and zero temperature for the bipartition given by an interval either on the line or on the circle.

The modular flows of the operators that we have considered (primaries, currents and energy-momentum tensor) are generated by modular Hamiltonians which depend also on the chemical potentials (see (2.6), (4.10) and (6.8)) [9, 11]. Their explicit expressions can be written by specialising (2.18), (2.20), (2.23) and (2.34) to the modular evolutions corresponding to (4.11) for the interval on the infinite line and to (6.12) for the interval on the circle, as discussed in section 4.2 and section 6.2 respectively. From these modular flows, we have found modular continuity equations (see (3.15), (3.20), (3.28) and (3.36) specialised to  $V_+(u) = V_-(u)$ , which is equal to (4.7) for the interval on the line and to (6.9) for the interval on the circle) and the corresponding conserved quantities along the modular evolution (see (3.17), (3.23), (3.32), (3.39) and also (3.55) for the modular heat, in the special cases just mentioned), where the dependence on the representation occurs through the expression of the velocities.

In the finite density representations, either on the line or on the circle, the mean values of the modular currents that we have introduced naturally provide two-dimensional curl free vector fields that describe the modular quantum transport in the spacetime (see section 5.1 and section 5.2 for the CFT on the line and in section 7 for the CFT on the circle). This modular quantum transport is different from the one discussed in [60–62], based on the Berry phase.

Finally, we have investigated the modular quantum noise power generated by various currents for a CFT either on the line or on the circle (see section 5.3 and section 7 respectively). A modular Johnson-Nyquist law (see (5.54), which holds also at finite volume) for the modular noise power generated by the charge current and the modular analogue of the fluctuation-dissipation relation are obtained. These results confirm the thermal nature of the modular evolution with inverse temperature  $\tilde{\beta} = 1$ , in agreement with the KMS condition for the modular correlators (see section 4.4 and section 6.3). While the modular noise power generated by the charge or by the helicity current contains the coefficient occurring in the central term in (2.5), the modular noise power generated by the energy or by the momentum current contains the central charge of the CFT. Furthermore, while the modular noise power generated by these currents is independent of the spacetime position (see (5.54) and (5.61)),

the modular noise power generated by the charge density depends on the position in the spacetime (see (5.69)).

We recall that charge and energy (heat) quantum transport experiments represent fundamental tools in contemporary condensed matter physics [2–4]. The physical quantities of interest in these experiments are essentially the conductance and the noise power associated to the charge and energy currents. In this paper we studied the evolution of these quantities along the modular time  $\tau$  instead of the physical time  $t$ , finding (5.32)–(5.34), (5.53)–(5.54) and (5.60)–(5.61). In this respect, recent proposals of experimental setups to explore the modular Hamiltonian, e.g. through cold atoms in optical lattices [63], open the challenging possibility to test experimentally the main features of the charge and energy modular quantum transport that we have described.

The results discussed in this manuscript can be extended in various directions. In two spacetime dimensions, it is worth studying the modular transport properties for CFT at finite temperature [64], for either bi-local or non-local modular Hamiltonians [40, 41, 65–70], including the ones corresponding to systems with boundaries or defects [54, 71, 72], for modular Hamiltonians in inhomogeneous systems [73] and in non-relativistic field theories [11, 74, 75]. It is relevant to explore the modular quantum transport also in higher dimensional quantum field theories [7, 8] and to investigate possible gravitational duals of these results, maybe by using [76–78]. Another direction concerns the connections between our results and the previous analyses in inhomogeneous CFT [26, 79–88]. Finally, it is worth exploring the modular evolution and the corresponding transport properties also in lattice models [89–99].

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## A Correlators in the fundamental representation

In this appendix we summarise the one-point functions and the two-point functions of the chiral fields on the line considered in section 2.1 in the fundamental representation.

A characteristic feature of this representation is the vanishing of the one-point functions

$$\langle \phi_{\pm}(u) \rangle_0 = \langle j_{\pm}(u) \rangle_0 = \langle T_{\pm}(u) \rangle_0 = 0 \tag{A.1}$$

hence the corresponding two-point functions coincide with their connected parts. By using the normalisation adopted in [28, 52], by conformal invariance, for the primaries we have

$$\langle \phi_{\pm}^*(u) \phi_{\pm}(v) \rangle_0 = \langle \phi_{\pm}(u) \phi_{\pm}^*(v) \rangle_0 = \frac{e^{\mp i\pi h_{\pm}}}{2\pi(u-v \mp i\epsilon)^{2h_{\pm}}}. \tag{A.2}$$

The normalisation of the two-point functions of  $j_{\pm}(u)$  and  $T_{\pm}(u)$  follow from (2.5) and (2.1) respectively. This can be shown by considering the following well known distribution

$$\frac{1}{u \pm i\epsilon} = \frac{1}{u} \mp i\pi\delta(u) \tag{A.3}$$

and the ones obtained by taking its first, second and third derivative, that read respectively

$$\frac{1}{(u \pm i\epsilon)^2} = \frac{1}{u^2} \pm i\pi\delta'(u) \quad \frac{1}{(u \pm i\epsilon)^3} = \frac{1}{u^3} \mp \frac{i\pi}{2}\delta''(u) \quad \frac{1}{(u \pm i\epsilon)^4} = \frac{1}{u^4} \pm \frac{i\pi}{6}\delta'''(u). \tag{A.4}$$

By conformal invariance [24], we have that

$$\langle j_{\pm}(x_1 \pm t_1) j_{\pm}(x_2 \pm t_2) \rangle_0 = \frac{C_{j_{\pm}}}{(x_{12} \pm t_{12} \mp i\epsilon)^2} \tag{A.5}$$

where we remind that  $x_{12} \equiv x_1 - x_2$  and  $t_{12} \equiv t_1 - t_2$ . In order to fix  $C_{j_{\pm}}$ , from (A.5) we evaluate the expectation value of the commutator

$$\begin{aligned} \langle [j_{\pm}(x_1 \pm t), j_{\pm}(x_2 \pm t)] \rangle_0 &= C_{j_{\pm}} \left[ \frac{1}{(x_{12} \mp i\epsilon)^2} - \frac{1}{(x_{21} \mp i\epsilon)^2} \right] \\ &= C_{j_{\pm}} \left[ \frac{1}{(x_{12} \mp i\epsilon)^2} - \frac{1}{(x_{12} \pm i\epsilon)^2} \right] = \mp 2\pi i C_{j_{\pm}} \delta'(x_{12}) \end{aligned} \tag{A.6}$$

where the first distribution in (A.4) has been employed. Comparing this result with the expectation value of (2.5), one finds

$$C_{j_{\pm}} = \frac{\kappa}{4\pi^2}. \tag{A.7}$$

Hence, the positivity of (A.5) implies  $\kappa \geq 0$ .

A similar analysis can be performed for  $T_{\pm}(u)$ . In this case the two-point function is

$$\langle T_{\pm}(x_1 \pm t_1) T_{\pm}(x_2 \pm t_2) \rangle_0 = \frac{C_{T_{\pm}}}{(x_{12} \pm t_{12} \mp i\epsilon)^4} \tag{A.8}$$

where the constant  $C_{T_{\pm}}$  has to be fixed. From (A.8), one obtains

$$\begin{aligned} \langle [T_{\pm}(x_1 \pm t), T_{\pm}(x_2 \pm t)] \rangle_0 &= C_{T_{\pm}} \left[ \frac{1}{(x_{12} \mp i\epsilon)^4} - \frac{1}{(x_{21} \mp i\epsilon)^4} \right] \\ &= C_{T_{\pm}} \left[ \frac{1}{(x_{12} \mp i\epsilon)^4} - \frac{1}{(x_{12} \pm i\epsilon)^4} \right] = \mp \frac{i\pi}{3} C_{T_{\pm}} \delta'''(x_{12}) \end{aligned} \tag{A.9}$$

which can be compared with the expectation value of (2.1), finding that consistency leads to

$$C_{T_{\pm}} = \frac{c}{8\pi^2}. \tag{A.10}$$

In this case, positivity of (A.8) implies the well known constraint  $c \geq 0$  for unitary CFT.

## B Currents involving the chiral primaries

In this appendix we apply to the chiral primaries the analyses discussed in section 3.1 and section 3.2 about the continuity equations and the conservation laws involving the electric charge, the helicity, the energy and the momentum.

Multiplying (2.11) by  $V_{\pm}(u_{\pm})^{h_{\pm}-1}$  first and then taking either the sum or the difference of the resulting equations, we obtain

$$\partial_{\tau}\Phi(\tau; x, t) = (\partial_{u_+} + i\mu_+)[V_+(u_+)^{h_+}\phi_+(\tau, u_+)] - (\partial_{u_-} - i\mu_-)[V_-(u_-)^{h_-}\phi_-(\tau, u_-)] \quad (\text{B.1})$$

$$\partial_{\tau}\tilde{\Phi}(\tau; x, t) = (\partial_{u_+} + i\mu_+)[V_+(u_+)^{h_+}\phi_+(\tau, u_+)] + (\partial_{u_-} - i\mu_-)[V_-(u_-)^{h_-}\phi_-(\tau, u_-)] \quad (\text{B.2})$$

where we have introduced

$$\Phi(\tau; x, t) \equiv V_+(u_+)^{h_+-1}\phi_+(\tau, u_+) + V_-(u_-)^{h_- -1}\phi_-(\tau, u_-) \quad (\text{B.3})$$

$$\tilde{\Phi}(\tau; x, t) \equiv V_+(u_+)^{h_+-1}\phi_+(\tau, u_+) - V_-(u_-)^{h_- -1}\phi_-(\tau, u_-). \quad (\text{B.4})$$

By using  $\partial_{u_{\pm}} = \frac{1}{2}(\partial_x \pm \partial_t)$  and renaming (3.52) as  $(\mu_x, \mu_t) \equiv (\mu_e, \mu_h)$ , one finds that (B.1) and (B.2) can be written respectively as the following continuity equations

$$\partial_{\tau}\Phi(\tau; x, t) = (\partial_{\nu} + i\mu_{\nu})\Psi^{\nu}(\tau; x, t) \quad \partial_{\tau}\tilde{\Phi}(\tau; x, t) = (\partial_{\nu} + i\mu_{\nu})\tilde{\Psi}^{\nu}(\tau; x, t) \quad (\text{B.5})$$

where we have introduced

$$\Psi^x(\tau; x, t) \equiv \frac{1}{2}[V_+(u_+)^{h_+}\phi_+(\tau, u_+) - V_-(u_-)^{h_-}\phi_-(\tau, u_-)] \quad (\text{B.6})$$

$$\Psi^t(\tau; x, t) \equiv \frac{1}{2}[V_+(u_+)^{h_+}\phi_+(\tau, u_+) + V_-(u_-)^{h_-}\phi_-(\tau, u_-)] \quad (\text{B.7})$$

and

$$\tilde{\Psi}^x(\tau; x, t) \equiv \Psi^t(\tau; x, t) \quad \tilde{\Psi}^t(\tau; x, t) \equiv \Psi^x(\tau; x, t). \quad (\text{B.8})$$

The differential equations in (B.5) can be expressed more conveniently in the following form

$$\partial_{\tau}[e^{i\mu_{\alpha}x^{\alpha}}\Phi(\tau; x, t)] = \partial_{\nu}[e^{i\mu_{\alpha}x^{\alpha}}\Psi^{\nu}(\tau; x, t)] \quad (\text{B.9})$$

$$\partial_{\tau}[e^{i\mu_{\alpha}x^{\alpha}}\tilde{\Phi}(\tau; x, t)] = \partial_{\nu}[e^{i\mu_{\alpha}x^{\alpha}}\tilde{\Psi}^{\nu}(\tau; x, t)] \quad (\text{B.10})$$

where  $\mu_{\alpha}x^{\alpha} = \mu_x x + \mu_t t = \mu_+ u_+ + \mu_- u_-$ . Considering (B.9), its r.h.s. can be written as

$$\begin{aligned} \partial_{\nu}[e^{i\mu_{\alpha}x^{\alpha}}\Psi^{\nu}(\tau; x, t)] &= \\ &= e^{i\mu_- u_-} \partial_{u_+}[e^{i\mu_+ u_+} V_+(u_+)^{h_+} \phi_+(\tau, u_+)] - e^{i\mu_+ u_+} \partial_{u_-}[e^{i\mu_- u_-} V_-(u_-)^{h_-} \phi_-(\tau, u_-)] \end{aligned} \quad (\text{B.11})$$

hence

$$\begin{aligned} \int_{\mathcal{D}_A} \partial_{\nu}[e^{i\mu_{\alpha}x^{\alpha}}\Psi^{\nu}(\tau; x, t)] dx dt &= C_{a,b}(\mu_-) \left( \int_a^b \partial_{u_+}[e^{i\mu_+ u_+} V_+(u_+)^{h_+} \phi_+(\tau, u_+)] du_+ \right) \\ &+ C_{a,b}(\mu_+) \left( \int_a^b \partial_{u_-}[e^{i\mu_- u_-} V_-(u_-)^{h_-} \phi_-(\tau, u_-)] du_- \right) = 0 \end{aligned} \quad (\text{B.12})$$

where we have introduced

$$C_{a,b}(\mu) \equiv \frac{e^{-i\mu a} - e^{-i\mu b}}{2\mu} \tag{B.13}$$

and (2.15) has been used. This result and the corresponding one obtained by performing a similar analysis for (B.10) imply that

$$F_A \equiv \int_{\mathcal{D}_A} e^{i\mu_\alpha x^\alpha} \Phi(\tau; x, t) dx dt = C_{a,b}(\mu_-) \int_a^b e^{i\mu_+ u_+} V_+(u_+)^{h_+-1} \phi_+(\tau, u_+) du_+ \tag{B.14}$$

$$+ C_{a,b}(\mu_+) \int_a^b e^{i\mu_- u_-} V_-(u_-)^{h_- -1} \phi_-(\tau, u_-) du_-$$

and

$$\tilde{F}_A \equiv \int_{\mathcal{D}_A} e^{i\mu_\alpha x^\alpha} \tilde{\Phi}(\tau; x, t) dx dt = C_{a,b}(\mu_-) \int_a^b e^{i\mu_+ u_+} V_+(u_+)^{h_+-1} \phi_+(\tau, u_+) du_+ \tag{B.15}$$

$$- C_{a,b}(\mu_+) \int_a^b e^{i\mu_- u_-} V_-(u_-)^{h_- -1} \phi_-(\tau, u_-) du_-$$

are conserved, i.e. independent of  $\tau$ .

### C Representations and automorphisms

In this appendix we describe the construction of the finite density representation of a chiral CFT on either the line  $\mathbb{R}$  or on the circle  $\mathbb{S}$  (see section 4.1 and section 6.1 respectively) using a specific automorphism  $\gamma_\mu \equiv \gamma_{\mu_+} \otimes \gamma_{\mu_-}$  and the corresponding fundamental representation.

We begin by considering the line, where the automorphism  $\gamma_\mu \equiv \gamma_{\mu_+} \otimes \gamma_{\mu_-}$  is defined as follows [12–16, 26, 100]

$$\gamma_{\mu_\pm}(\mathcal{O}_\pm) \equiv e^{iD_{\mu_\pm}} \mathcal{O}_\pm e^{-iD_{\mu_\pm}} \tag{C.1}$$

with

$$D_{\mu_\pm} \equiv \pm \mu_\pm \int_{-\infty}^{\infty} u j_\pm(u) du. \tag{C.2}$$

By applying (C.1) to the fields  $\phi_\pm$ ,  $j_\pm$  and  $T_\pm$ , we find respectively

$$\gamma_{\mu_\pm}(\phi_\pm(u)) = e^{\mp i\mu_\pm u} \phi_\pm(u) \tag{C.3}$$

$$\gamma_{\mu_\pm}(j_\pm(u)) = j_\pm(u) - \frac{\kappa \mu_\pm}{2\pi} \tag{C.4}$$

$$\gamma_{\mu_\pm}(T_\pm(u)) = T_\pm(u) + \mu_\pm j_\pm(u) - \frac{\kappa \mu_\pm^2}{4\pi}. \tag{C.5}$$

Thus, for (2.7) we obtain

$$\gamma_{\mu_\pm}(\mathcal{T}_\pm(u)) = \gamma_{\mu_\pm}(T_\pm(u)) - \mu_\pm \gamma_{\mu_\pm}(j_\pm(u)) = T_\pm(u) + \frac{\kappa \mu_\pm^2}{4\pi} \tag{C.6}$$

where in the r.h.s.  $T_{\pm}(u)$  occurs. The inverse of (C.3), (C.4), (C.5) and (C.6) read respectively

$$\gamma_{\mu_{\pm}}^{-1}(\phi_{\pm}(u)) = e^{\pm i\mu_{\pm}u} \phi_{\pm}(u) \tag{C.7}$$

$$\gamma_{\mu_{\pm}}^{-1}(j_{\pm}(u)) = j_{\pm}(u) + \frac{\kappa\mu_{\pm}}{2\pi} \tag{C.8}$$

$$\gamma_{\mu_{\pm}}^{-1}(T_{\pm}(u)) = T_{\pm}(u) - \mu_{\pm}j_{\pm}(u) - \frac{\kappa\mu_{\pm}^2}{4\pi} \tag{C.9}$$

and

$$\gamma_{\mu_{\pm}}^{-1}(\mathcal{T}_{\pm}(u)) = \mathcal{T}_{\pm}(u) - \mu_{\pm}j_{\pm}(u) - \frac{3\kappa\mu_{\pm}^2}{4\pi}. \tag{C.10}$$

We remark that (C.3)–(C.5) preserve the commutation relations (2.1)–(2.5).

The  $n$ -point function of a generic operator  $\mathcal{O}$  in the finite density representation can be constructed through the automorphism (C.1) and the corresponding  $n$ -point function in the fundamental representation as follows

$$\langle \mathcal{O}(u_1) \dots \mathcal{O}(u_n) \rangle_{\mu_{\pm}} = \langle \gamma_{\mu_{\pm}}(\mathcal{O}(u_1)) \dots \gamma_{\mu_{\pm}}(\mathcal{O}(u_n)) \rangle_0 \tag{C.11}$$

The one-point functions at finite density in (4.1) are straightforwardly obtained by combining (C.11) with the fact that for a CFT in its ground state and on the line we have

$$\langle \phi_{\pm}(u) \rangle_0 = \langle j_{\pm}(u) \rangle_0 = \langle \mathcal{T}_{\pm}(u) \rangle_0 = 0 \tag{C.12}$$

The prescription (C.11) tell us that the automorphism  $\gamma_{\mu_{\pm}}$  maps an operator in the finite density representation into the corresponding operator in the fundamental representation; hence  $\gamma_{\mu_{\pm}}^{-1}$  can be employed to construct operators in the finite density representation from the corresponding ones in the fundamental representation. An important example is given by the modular Hamiltonians in (4.6), (4.9) and (4.10), which can be easily obtained by applying  $\gamma_{\mu_{\pm}}^{-1}$  (see (C.9)) to the corresponding modular Hamiltonians in the fundamental representation.

The action of the automorphism  $\gamma_{\mu_{\pm}}$  described in (C.3)–(C.6) provides also the two-point functions at finite density on the line reported in section 4.1. The two-point function for the primaries in (4.2) is obtained by employing (C.3) and (4.1) as follows

$$\langle \phi_{\pm}^*(u) \phi_{\pm}(v) \rangle_{\mu_{\pm}}^{\text{con}} = \langle \gamma_{\mu_{\pm}}(\phi_{\pm}^*(u)) \gamma_{\mu_{\pm}}(\phi_{\pm}(v)) \rangle_0^{\text{con}} = e^{\pm i\mu_{\pm}(u-v)} \langle \phi_{\pm}^*(u) \phi_{\pm}(v) \rangle_0^{\text{con}}. \tag{C.13}$$

This procedure tells us that the correlators involving primaries with different chiralities vanish also at finite density. Similarly, from (C.4) and (4.1) one finds the two-point function for the current given in (4.3) as follows

$$\begin{aligned} \langle j_{\pm}(u) j_{\pm}(v) \rangle_{\mu_{\pm}}^{\text{con}} &= \langle j_{\pm}(u) j_{\pm}(v) \rangle_{\mu_{\pm}} - \langle j_{\pm}(u) \rangle_{\mu_{\pm}} \langle j_{\pm}(v) \rangle_{\mu_{\pm}} \\ &= \langle \gamma_{\mu_{\pm}}(j_{\pm}(u)) \gamma_{\mu_{\pm}}(j_{\pm}(v)) \rangle_0 - \langle \gamma_{\mu_{\pm}}(j_{\pm}(u)) \rangle_0 \langle \gamma_{\mu_{\pm}}(j_{\pm}(v)) \rangle_0 \\ &= \left\langle \left( j_{\pm}(u) - \frac{\kappa\mu_{\pm}}{2\pi} \right) \left( j_{\pm}(v) - \frac{\kappa\mu_{\pm}}{2\pi} \right) \right\rangle_0 - \left( \frac{\kappa\mu_{\pm}}{2\pi} \right)^2 = \langle j_{\pm}(u) j_{\pm}(v) \rangle_0^{\text{con}} \end{aligned} \tag{C.14}$$

and through similar steps also

$$\langle j_{\pm}(u) j_{\mp}(v) \rangle_{\mu}^{\text{con}} = \left\langle \left( j_{\pm}(u) - \frac{\kappa\mu_{\pm}}{2\pi} \right) \left( j_{\mp}(v) - \frac{\kappa\mu_{\mp}}{2\pi} \right) \right\rangle_0 - \frac{\kappa^2\mu_{+}\mu_{-}}{(2\pi)^2} = \langle j_{\pm}(u) j_{\mp}(v) \rangle_0^{\text{con}} = 0 \tag{C.15}$$

which implies (see e.g. [26])

$$\langle j_{\pm}(u) j_{\mp}(v) \rangle_{\mu} = \kappa^2 \frac{\mu + \mu_-}{(2\pi)^2}. \quad (\text{C.16})$$

From (C.6) and (4.1), the two-point function (4.4) is obtained as follows

$$\langle \mathcal{T}_{\pm}(u) \mathcal{T}_{\pm}(v) \rangle_{\mu_{\pm}}^{\text{con}} = \left\langle \left( T_{\pm}(u) + \frac{\kappa \mu_{\pm}^2}{4\pi} \right) \left( T_{\pm}(v) + \frac{\kappa \mu_{\pm}^2}{4\pi} \right) \right\rangle_0 - \left( \frac{\kappa \mu_{\pm}^2}{4\pi} \right)^2 = \langle T_{\pm}(u) T_{\pm}(v) \rangle_0^{\text{con}} \quad (\text{C.17})$$

$$\langle \mathcal{T}_{\pm}(u) \mathcal{T}_{\mp}(v) \rangle_{\mu_{\pm}}^{\text{con}} = \left\langle \left( T_{\pm}(u) + \frac{\kappa \mu_{\pm}^2}{4\pi} \right) \left( T_{\mp}(v) + \frac{\kappa \mu_{\mp}^2}{4\pi} \right) \right\rangle_0 - \frac{\kappa^2 \mu_{\pm}^2 \mu_{\mp}^2}{(4\pi)^2} = \langle T_{\pm}(u) T_{\mp}(v) \rangle_0^{\text{con}} = 0. \quad (\text{C.18})$$

The above considerations can be extended to any chiral field theory on a circle of length  $L$  by fixing the periodicity of the fields. This circle is obtained by imposing the periodicity condition on the line; hence we can restrict to the interval  $[-L/2, L/2]$ , imposing the following boundary conditions

$$\phi_{\pm}(L/2) = (-1)^{2h_{\pm}} \phi_{\pm}(-L/2) \quad j_{\pm}(L/2) = j_{\pm}(-L/2) \quad T_{\pm}(L/2) = T_{\pm}(-L/2). \quad (\text{C.19})$$

In particular,  $\phi_{\pm}$  are periodic for  $h_{\pm} \in \mathbb{Z}$  and anti-periodic for  $h_{\pm} \in \mathbb{Z} + \frac{1}{2}$ . In this case, instead of the automorphism  $D_{\mu_{\pm}}$  in (C.1), one introduces

$$D_{L, \mu_{\pm}} \equiv \pm \mu_{\pm} \int_{-L/2}^{L/2} u j_{\pm}(u) du \quad (\text{C.20})$$

where  $j_{\pm}(u)$  satisfies (C.19). Requiring that  $\gamma_{\mu_{\pm}}$  preserves the periodicity condition (C.19) leads to the following constraint

$$\frac{L \mu_{\pm}}{2\pi} = n_{\pm} \quad n_{\pm} \in \mathbb{Z}. \quad (\text{C.21})$$

Taking into account this condition, the finite density and the zero density correlators on the circle are related as follows

$$\langle \mathcal{O}(u_1) \dots \mathcal{O}(u_n) \rangle_{L, \mu_{\pm}} = \langle \gamma_{\mu_{\pm}}(\mathcal{O}(u_1)) \dots \gamma_{\mu_{\pm}}(\mathcal{O}(u_n)) \rangle_{L, 0}. \quad (\text{C.22})$$

Thus, the one-point functions at finite density (6.1) can be written straightforwardly by combining (C.22) with the fact that for a CFT in its ground state and on the circle we have

$$\langle \phi_{\pm}(u) \rangle_{L, 0} = \langle j_{\pm}(u) \rangle_{L, 0} = 0 \quad \langle \mathcal{T}_{\pm}(u) \rangle_{L, 0} = \langle T_{\pm}(u) \rangle_{L, 0} = -\frac{\pi c}{12L^2}. \quad (\text{C.23})$$

Analogously, the two-point function (6.4) is obtained from (6.1) and (C.6) as follows

$$\begin{aligned} \langle \mathcal{T}_{\pm}(u) \mathcal{T}_{\pm}(v) \rangle_{L, \mu_{\pm}}^{\text{con}} &= \langle \mathcal{T}_{\pm}(u) \mathcal{T}_{\pm}(v) \rangle_{L, \mu_{\pm}} - \langle \mathcal{T}_{\pm}(u) \rangle_{L, \mu_{\pm}} \langle \mathcal{T}_{\pm}(v) \rangle_{L, \mu_{\pm}} \\ &= \left\langle \left( T_{\pm}(u) + \frac{\kappa \mu_{\pm}^2}{4\pi} \right) \left( T_{\pm}(v) + \frac{\kappa \mu_{\pm}^2}{4\pi} \right) \right\rangle_{L, 0} - \left( \frac{\kappa \mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \right)^2 \\ &= \langle T_{\pm}(u) T_{\pm}(v) \rangle_{L, 0} - \left( \frac{\pi c}{12L^2} \right)^2 = \langle T_{\pm}(u) T_{\pm}(v) \rangle_{L, 0}^{\text{con}}. \end{aligned} \quad (\text{C.24})$$

Summarising, the fundamental input are the correlation functions  $\langle \dots \rangle_0$  in the ground state representation on the line. They generate, through the mapping  $u \mapsto e^{2\pi i u/L}$ , the ground state correlation functions  $\langle \dots \rangle_{L,0}$  on the circle. The associated finite density correlators are obtained in turn through the automorphism  $\gamma_{\mu_{\pm}}$  (see (C.11) and (C.22)). The allowed values for the chemical potential are  $\mu_{\pm} \in \mathbb{R}$  on the line and  $L\mu_{\pm}/(2\pi) \in \mathbb{Z}$  on the circle.

## D Consistency checks for the correlators

In this appendix we describe some consistency checks for both the correlation functions (see (4.2) and (6.2)) and the modular correlators (4.42). These checks are based on the special case of free fermions (section D.1), the positivity (section D.2) and the properties of the entanglement spectrum (section D.3). Since both chiralities can be analysed in the same way, we focus on the right movers  $\phi_+(u)$ , by setting  $\phi_+(u) \equiv \phi(u)$ ,  $h_+ \equiv h$  and  $\mu_+ \equiv \mu$  throughout this appendix, to enlighten the notation.

### D.1 Fermionic correlators at finite density

In order to check the first expression in (4.2) in a special case, let us consider the free chiral fermion on the line, whose two-point function at finite density can be written through the Fermi-Dirac distribution as follows

$$C(u, v) = \int_{\mathbb{R}} \frac{dp}{2\pi} \int_{\mathbb{R}} \frac{dk}{2\pi} e^{i(pu-kv)} \frac{2\pi \delta(u-v)}{1 + e^{\beta(p-\mu)}} = \int_{\mathbb{R}} \frac{dp}{2\pi} e^{ip(u-v)} \frac{1}{1 + e^{\beta(p-\mu)}} \quad (\text{D.1})$$

which can be regularised at  $p \rightarrow -\infty$  by introducing the infinitesimal  $\varepsilon > 0$  as follows

$$C_{\varepsilon}(u, v) \equiv \int_{\mathbb{R}} \frac{dp}{2\pi} e^{ip(u-v-i\varepsilon)} \frac{1}{1 + e^{\beta(p-\mu)}} \quad (\text{D.2})$$

the limit of this expression at zero temperature reads

$$\begin{aligned} \lim_{\beta \rightarrow +\infty} C_{\varepsilon}(u, v) &= \int_{\mathbb{R}} \frac{dp}{2\pi} e^{ip(u-v-i\varepsilon)} \theta(\mu - p) \\ &= e^{i\mu(u-v)} \int_{\mathbb{R}} \frac{dq}{2\pi} e^{-iq(u-v-i\varepsilon)} \theta(q) = \frac{e^{i\mu(u-v)}}{2\pi i (u-v-i\varepsilon)} \end{aligned} \quad (\text{D.3})$$

which corresponds to the first expression in (4.2) for the right moving fields when  $h_+ = 1/2$ .

It is worth studying the analogue computation on the circle, in order to check (6.2) in a special case. Considering the free fermion on a circle of length  $L$  satisfying anti-periodic boundary conditions we can employ the complete set of orthonormal functions  $\{\frac{1}{\sqrt{L}} e^{i2\pi n u/L}; n \in \tilde{\mathbb{Z}}\}$ , where  $\tilde{\mathbb{Z}}$  denotes the set made by the half-integers. The periodic boundary conditions are more subtle in the context of entanglement [101]. The two-point function of this field can be expressed through the Fermi-Dirac distribution as follows

$$C_L(u, v) = \frac{1}{L} \sum_{m \in \tilde{\mathbb{Z}}} \sum_{n \in \tilde{\mathbb{Z}}} e^{2\pi i(mu-nv)/L} \frac{\delta_{m,n}}{1 + e^{\beta(2\pi n/L-\mu)}} = \frac{1}{L} \sum_{n \in \tilde{\mathbb{Z}}} e^{2\pi i n(u-v)/L} \frac{1}{1 + e^{\beta(2\pi n/L-\mu)}} \quad (\text{D.4})$$

which can be regularised as  $n \rightarrow -\infty$  through the infinitesimal  $\varepsilon > 0$  as above, namely

$$C_{L,\varepsilon}(u, v) = \frac{1}{L} \sum_{n \in \tilde{\mathbb{Z}}} e^{2\pi i n(u-v-i\varepsilon)/L} \frac{1}{1 + e^{\beta(2\pi n/L-\mu)}} \quad (\text{D.5})$$

which is convergent for both  $n \rightarrow -\infty$  and  $n \rightarrow +\infty$ , because of  $\varepsilon > 0$  and  $\beta > 0$  respectively.

The zero temperature limit of (D.5) selects the values of  $n \in \tilde{\mathbb{Z}}$  in the series such that  $2\pi n/L - \mu \leq 0$ , namely  $n \leq L\mu/(2\pi)$ . Since  $L\mu/(2\pi) \in \mathbb{Z}$  (see appendix C), the values of  $n$  providing a non-vanishing contribution to (D.5) in the zero temperature limit are given by  $n = L\mu/(2\pi) + \tilde{n}$  with  $\tilde{n} \in \tilde{\mathbb{Z}}_-$ , being  $\tilde{\mathbb{Z}}_-$  made by the half-integer and negative numbers (in particular, we have that  $n < L\mu/(2\pi)$ ). Thus, the zero temperature limit of (D.5) when  $L\mu/(2\pi) \in \mathbb{Z}$  gives

$$\begin{aligned} \lim_{\beta \rightarrow +\infty} C_{L,\varepsilon}(u, v) &= \frac{1}{L} \sum_{\tilde{n} \in \tilde{\mathbb{Z}}_-} e^{2\pi i(L\mu/(2\pi) + \tilde{n})(u-v-i\varepsilon)/L} \\ &= \frac{e^{i\mu(u-v)}}{L} \sum_{\tilde{n} \in \tilde{\mathbb{Z}}_-} e^{2\pi i \tilde{n}(u-v-i\varepsilon)/L} = \frac{e^{i\mu(u-v)}}{L} \frac{1}{2i \sin[\pi(u-v-i\varepsilon)/L]} \end{aligned} \tag{D.6}$$

which corresponds to (6.2) for the right moving fields and specialised to  $h_+ = 1/2$ .

## D.2 Positivity

Assuming that we are dealing with a unitary CFT, the positivity of the scalar product  $(\cdot, \cdot)$  in the state space implies that

$$(\mathcal{O}(f) \Omega_\mu, \mathcal{O}(f) \Omega_\mu) \geq 0 \tag{D.7}$$

where the state  $\Omega_\mu$  characterises the finite density representation introduced in section 4.1 and

$$\mathcal{O}(f) \equiv \int_{-\infty}^{\infty} f(u) \mathcal{O}(u) du \tag{D.8}$$

is a chiral field smeared with a generic complex test function  $f$ . Choosing  $\mathcal{O}(u) = \phi(u)$ , the inequality (D.7) implies

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f(u)} \langle \phi^*(u) \phi(v) \rangle_\mu f(v) \geq 0 \tag{D.9}$$

for any test function  $f$ . Plugging in (D.9) the explicit form (4.2) of the correlation function one gets

$$\frac{e^{-i\pi h}}{2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \frac{\overline{f_\mu(u)} f_\mu(v)}{(u-v-i\varepsilon)^{2h}} \geq 0 \tag{D.10}$$

where  $f_\mu(u) \equiv e^{-i\mu u} f(u)$  and the overline denotes complex conjugation.

The inequality (D.10) implies a condition on the dimension  $h$ , which is easily obtained in momentum space. In fact, performing the Fourier transform (see for instance [102]) (D.10) takes the form

$$\frac{1}{\Gamma(2h)} \int_{-\infty}^{\infty} \frac{dp}{2\pi} |\hat{f}_\mu(p)|^2 p_+^{2h-1} = \frac{1}{\Gamma(2h)} \int_0^{\infty} \frac{dp}{2\pi} |\hat{f}_\mu(p)|^2 p^{2h-1} \geq 0 \tag{D.11}$$

where the distribution  $p_+^\sigma \equiv \theta(p) p^\sigma$  has been introduced. The bound (D.11) is satisfied provided that the integral converges. This is the case for large  $p$ , where the integrand is

dominated by the exponential decay of  $|\widehat{f}_\mu(p)|$ . The convergence at  $p = 0$  implies that  $h > 0$ . Summarising, for both chiralities the dimensions of the primary fields in unitary CFT satisfy

$$h_\pm > 0. \quad (\text{D.12})$$

Let us discuss now the impact of positivity on the modular correlator (4.42). Given the interval  $A \equiv [a, b]$ , we consider the chiral field (D.8) localised in  $A$  assuming for the support of the test functions  $\text{supp}(f) \subset A$ . Let  $S$  be the conjugate linear operator occurring in the Tomita-Takesaki theorem (see eq. (V.2.1) of [1]) which acts as follows

$$S \mathcal{O}(f) \Omega_\mu = \mathcal{O}^*(f) \Omega_\mu \quad \text{supp}(f) \subset A. \quad (\text{D.13})$$

The unique polar decomposition of  $S$  reads

$$S = J \Delta^{1/2} \quad (\text{D.14})$$

defining the (antiunitary) modular conjugation  $J$  studied in section 4.3 and the self-adjoint positive (in general unbounded) modular operator  $\Delta$ , which satisfy

$$J = J^* = J^{-1} \quad \Delta \Omega_\mu = \Omega_\mu \quad J \Omega_\mu = \Omega_\mu. \quad (\text{D.15})$$

The modular operator  $\Delta$  is expressed in terms of the full modular Hamiltonian  $K$  in (4.10) by  $\Delta \equiv e^{-K}$ .

Following [38, 39, 103] (see also (4.29)–(4.32)), we introduce the reflected operator

$$\mathcal{O}^{\text{ref}}(f) \equiv J \mathcal{O}(f) J^* \quad (\text{D.16})$$

By employing (D.14) and (D.15), one finds

$$\begin{aligned} (\Omega_\mu, \mathcal{O}^{\text{ref}}(f) \mathcal{O}(f) \Omega_\mu) &= (J^2 \Omega_\mu, J \mathcal{O}(f) J^* \mathcal{O}(f) \Omega_\mu) \\ &= (\mathcal{O}(f) J^* \mathcal{O}(f) \Omega_\mu, J \Omega_\mu) = (\mathcal{O}(f) J^* \mathcal{O}(f) \Omega_\mu, \Omega_\mu) \\ &= (\Omega_\mu, \mathcal{O}^*(f) J \mathcal{O}^*(f) \Omega_\mu) = (\Omega_\mu, \mathcal{O}^*(f) J S \mathcal{O}(f) \Omega_\mu) \\ &= (\Omega_\mu, \mathcal{O}^*(f) J^2 \Delta^{1/2} \mathcal{O}(f) \Omega_\mu) = (\Omega_\mu, \mathcal{O}^*(f) \Delta^{1/2} \mathcal{O}(f) \Omega_\mu) \\ &= (\mathcal{O}(f) \Omega_\mu, \Delta^{1/2} \mathcal{O}(f) \Omega_\mu) \geq 0 \end{aligned} \quad (\text{D.17})$$

which has been called modular reflection positivity [38]. Thus, the modular reflection positivity can be derived from the properties of  $J$ , the positivity of the scalar product  $(\cdot, \cdot)$  and the positivity of the modular operator  $\Delta$ .

For the field  $\mathcal{O}(u) = \phi(u)$  in (D.8), the inequality (D.17) takes the form

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f(u)} (\phi(u) \Omega_\mu, e^{-K/2} \phi(v) \Omega_\mu) f(v) \geq 0. \quad (\text{D.18})$$

On the other hand, by using  $e^{-i\tau K} \Omega_\mu = \Omega_\mu$ , the modular evolution (2.8) and the modular correlator (4.42) for the field  $\phi(u)$ , one gets

$$\begin{aligned} \frac{e^{\pm i\mu(u-v)}}{2\pi e^{i\pi h}} W_+(\tau_{12}; u, v)^{2h} &= (\phi(\tau_1, u) \Omega_\mu, \phi(\tau_2, v) \Omega_\mu) \\ &= (e^{i\tau_1 K} \phi(u) e^{-i\tau_1 K} \Omega_\mu, e^{i\tau_2 K} \phi(v) e^{-i\tau_2 K} \Omega_\mu) \\ &= (\phi(u) \Omega_\mu, e^{-i\tau_{12} K} \phi(v) \Omega_\mu). \end{aligned} \quad (\text{D.19})$$

Combining (D.19) specialised to  $\tau_{12} = -i/2$  with (D.18), one finds

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_{\mu}(u)} \frac{W_{+}(-i/2; u, v)^{2h}}{2\pi e^{i\pi h}} f_{\mu}(v) \geq 0 \quad (\text{D.20})$$

where  $f_{\mu}$  has been defined in the text below (D.10) and we have that  $\text{supp}(f_{\mu}) = \text{supp}(f) \subset A$ . The inequality (D.20) provides a non-trivial consistency condition for the explicit expression of  $W_{+}(\tau; u, v)$  in (4.44). Indeed, when  $A$  has finite length we have that (4.44) for  $\varepsilon = 0$  and  $\tau_{12} = -i/2$  can be written as

$$W_{+}(-i/2; u, v)^{2h} = \left( \frac{i}{\frac{b-a}{2} [1 - s(u) s(v)]} \right)^{2h} \quad s(u) \equiv \frac{u - \frac{a+b}{2}}{\frac{b-a}{2}} \in (-1, 1). \quad (\text{D.21})$$

Plugging this expression into (D.20) and employing the following Taylor series

$$\frac{1}{(1-y)^h} = \sum_{n=0}^{\infty} \alpha_n(h) y^n \quad \alpha_n(h) \equiv \frac{1}{n!} \prod_{k=0}^{n-1} (h+k) > 0 \quad (\text{D.22})$$

where  $h > 0$  and  $|y| < 1$ , for an interval of finite length we find that the inequality (D.20) becomes equivalent to

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_{\mu}(u)} \left[ \sum_{n=0}^{\infty} \alpha_n(2h) s(u)^n s(v)^n \right] f_{\mu}(v) = \sum_{n=0}^{\infty} \alpha_n(2h) \left| \int_{-\infty}^{\infty} du f_{\mu}(u) s(u)^n \right|^2 \geq 0 \quad (\text{D.23})$$

which is verified because  $\alpha_n(2h) > 0$ .

In the special case of the Rindler wedge, i.e. in the limiting regime given by  $b \rightarrow +\infty$ , the inequality (D.20) becomes equivalent to

$$\begin{aligned} & \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_{\mu}(u)} \left( \frac{1}{u-a+v-a} \right)^h f_{\mu}(v) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \left( \frac{1}{u-a+v-a} \right)^h \left[ \overline{f_{\mu}(u)} f_{\mu}(v) + \text{c.c.} \right] \geq \frac{1}{M_f^h} \left[ \int_{-\infty}^{\infty} du f_{\mu}(u) \right]^2 \geq 0 \end{aligned} \quad (\text{D.24})$$

where c.c. denotes the complex conjugate,  $M_f \equiv \max\{u-a+v-a > 0; u, v \in \text{supp}(f)\}$  and the crucial inequality has been obtained by assuming that  $f_{\mu}$  are real and positive functions with compact support properly included in  $A = [a, +\infty)$ .

The above analysis provides a non-trivial consistency check between the expression (4.44) of  $W_{+}(\tau; u, v)$  and the modular reflection positivity condition (D.17).

### D.3 Entanglement spectrum

As further consistency check of the modular correlator in (4.42)–(4.44), we show that

$$(\mathcal{O}(f) \Omega_{\mu}, \mathcal{O}(f) \Omega_{\mu}) \geq (\mathcal{O}(f) \Omega_{\mu}, \Delta^{1/2} \mathcal{O}(f) \Omega_{\mu}) \geq 0 \quad (\text{D.25})$$

where the first inequality comes from the fact that the spectrum of  $\Delta$  (i.e. the entanglement spectrum) is a subset of  $(0, 1)$ , while the last one corresponds to the positivity of the modular operator  $\Delta$  (see (D.17)). For non-coincident  $u, v \in [a, b]$ , from (4.38) we introduce

$$r(u, v) \equiv -i R(-i/2; u, v) = \frac{(b-a)(u-v)}{2 \left[ \frac{a+b}{2}(u+v) - (uv+ab) \right]} \in [-1, 1] \quad (\text{D.26})$$

which satisfies (see also (4.45) and (D.21))

$$\frac{W_+(-i/2; u, v)^{2h}}{e^{i\pi h}} = \frac{r(u, v)}{u - v}. \quad (\text{D.27})$$

From (D.19) specialised to  $\tau_{12} = -i/2$ , (D.20) and (D.27), we obtain

$$\begin{aligned} (\phi(u) \Omega_\mu, \Delta^{1/2} \phi(v) \Omega_\mu) &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_\mu(u)} \frac{W_+(-i/2; u, v)^{2h}}{2\pi e^{i\pi h}} f_\mu(v) \\ &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_\mu(u)} \frac{r(u, v)^{2h}}{2\pi (u - v)^{2h}} f_\mu(v) \\ &\leq \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \overline{f_\mu(u)} \frac{1}{2\pi (u - v)^{2h}} f_\mu(v) \end{aligned} \quad (\text{D.28})$$

where the inequality originates from the fact that  $r(u, v)^2 \in [0, 1]$  for  $u, v \in [a, b]$  and it has been obtained by assuming that  $f_\mu$  are real and positive functions. Finally, from (4.2), one realises that the inequality in (D.28) provides the first inequality in (D.25).

## E Modular evolution in the complementary region

The modular evolution of an operator localised in the region  $B$  complementary to the interval  $A$  can be studied by combining its modular evolution in  $A$  and the modular conjugation, as obtained in (4.36). In this appendix we derive the explicit expressions of these modular evolutions for the fields  $\phi_\pm$ ,  $j_\pm$  and  $\mathcal{T}_\pm$ .

In order to specify (4.36) to the primary fields  $\phi_\pm$ , by employing (2.18), (2.20) and (4.29), let us consider

$$\begin{aligned} e^{iK\tau} (J \phi_\pm(u) J) e^{-iK\tau} &= e^{\mp i\mu_\pm(j(u)-u)} j'(u)^{h_\pm} e^{iK\tau} \phi_\pm^*(j(u)) e^{-iK\tau} \\ &= e^{\mp i\mu_\pm(j(u)-u)} j'(u)^{h_\pm} \left[ e^{\mp i\mu_\pm(\xi_\pm(\tau, j(u))-j(u))} \left( \partial_v \xi_\pm(\tau, v) \Big|_{v=j(u)} \right)^{h_\pm} \phi_\pm^*(\xi_\pm(\tau, j(u))) \right] \\ &= e^{\mp i\mu_\pm(\xi_\pm(\tau, j(u))-u)} [\partial_u \xi_\pm(\tau, j(u))]^{h_\pm} \phi_\pm^*(\xi_\pm(\tau, j(u))). \end{aligned} \quad (\text{E.1})$$

Then, the r.h.s. of (4.36) is obtained by applying  $J(\cdot)J$  to (E.1); hence, by using (4.30), for the primaries we find

$$\phi_\pm(\tau, u) = e^{\pm i\mu_\pm(\tilde{\xi}_\pm(\tau, u)-u)} (\partial_u \tilde{\xi}_\pm(\tau, u))^{h_\pm} \phi_\pm(\tilde{\xi}_\pm(\tau, u)) \quad (\text{E.2})$$

with  $\tilde{\xi}_\pm(\tau, u)$  being defined as follows

$$\tilde{\xi}_\pm(\tau, u) \equiv j(\xi_\pm(\tau, j(u))) = \xi_\pm(\tau, u) \quad (\text{E.3})$$

where (4.21) has been used in the last step. This result tells us that, for the modular evolution, the expression (2.18) combined with (4.11) holds also for  $u \notin A$ . The above analysis can be adapted to the finite volume case by replacing  $j(u)$  with  $j_L(u)$  and  $\xi(\tau, u)$  with  $\xi_L(\tau, u)$  (see (6.18) and (6.12) respectively), finding the same conclusion in terms of (6.12).

As for the chiral currents  $j_{\pm}$ , from the r.h.s. of (4.36), (2.23) and (4.31) we have that

$$\begin{aligned} e^{iK\tau}(J j_{\pm}(u) J) e^{-iK\tau} &= j'(u) e^{iK\tau} j_{\pm}(j(u)) e^{-iK\tau} - \frac{\kappa\mu_{\pm}}{2\pi} [1 - j'(u)] \\ &= j'(u) \left[ \partial_v \xi_{\pm}(\tau, v)|_{v=j(u)} j_{\pm}(\xi_{\pm}(\tau, j(u))) - \frac{\kappa\mu_{\pm}}{2\pi} \left(1 - \partial_v \xi_{\pm}(\tau, v)|_{v=j(u)}\right) \right] - \frac{\kappa\mu_{\pm}}{2\pi} [1 - j'(u)] \\ &= \partial_u \xi_{\pm}(\tau, j(u)) j_{\pm}(\xi_{\pm}(\tau, j(u))) - \frac{\kappa\mu_{\pm}}{2\pi} [1 - \partial_u \xi_{\pm}(\tau, j(u))]. \end{aligned} \quad (\text{E.4})$$

By applying  $J(\cdot)J$  to (E.4), one gets the r.h.s. of (4.36) specified to this case. Hence, by employing again (4.31), we obtain

$$\begin{aligned} j_{\pm}(\tau, u) &= \partial_u \xi_{\pm}(\tau, j(u)) \left[ j'(v)|_{v=\xi_{\pm}(\tau, j(u))} j_{\pm}(j(\xi_{\pm}(\tau, j(u)))) - \frac{\kappa\mu_{\pm}}{2\pi} \left(1 - j'(v)|_{v=\xi_{\pm}(\tau, j(u))}\right) \right] \\ &\quad - \frac{\kappa\mu_{\pm}}{2\pi} [1 - \partial_u \xi_{\pm}(\tau, j(u))] \\ &= \partial_u \tilde{\xi}_{\pm}(\tau, u) j_{\pm}(\tilde{\xi}_{\pm}(\tau, u)) - \frac{\kappa\mu_{\pm}}{2\pi} [1 - \partial_u \tilde{\xi}_{\pm}(\tau, u)] \end{aligned} \quad (\text{E.5})$$

in terms of (E.3). Thus, as for the modular evolution of the chiral currents, the expression (2.23) combined with (4.11) holds also for  $u \notin A$ . This conclusion is found also in the finite volume case, once  $\xi(\tau, u)$  is replaced by  $\xi_L(\tau, u)$  given in (6.12), as one obtains by repeating the above analysis replacing also  $j(u)$  with  $j_L(u)$  given in (6.18).

In the finite volume case, for (2.7), from the r.h.s. of (4.36), (6.21) and (2.34) we arrive to

$$\begin{aligned} e^{iK\tau}(J \mathcal{T}_{\pm}(u) J) e^{-iK\tau} &= j'_L(u)^2 e^{iK\tau} \mathcal{T}_{\pm}(j_L(u)) e^{-iK\tau} + \left( \frac{\kappa\mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \right) [1 - j'_L(u)^2] \\ &= j'_L(u)^2 \left\{ \left( \partial_v \xi_{L,\pm}(\tau, v)|_{v=j_L(u)} \right)^2 \mathcal{T}_{\pm}(\xi_{\pm}(\tau, j_L(u))) + \frac{\kappa\mu_{\pm}^2}{4\pi} \left[ 1 - \left( \partial_v \xi_{\pm}(\tau, v)|_{v=j_L(u)} \right)^2 \right] \right. \\ &\quad \left. - \frac{c}{24\pi} \mathcal{S}_v[\xi_{\pm}](\tau, v)|_{v=j_L(u)} \right\} + \left( \frac{\kappa\mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \right) [1 - j'_L(u)^2] \\ &= (\partial_u \xi_{\pm}(\tau, j_L(u)))^2 \mathcal{T}_{\pm}(\xi_{\pm}(\tau, j_L(u))) + \frac{\kappa\mu_{\pm}^2}{4\pi} [1 - (\partial_u \xi_{\pm}(\tau, j_L(u)))^2] \\ &\quad - \frac{c}{24\pi} j'_L(u)^2 \mathcal{S}_v[\xi_{\pm}](\tau, v)|_{v=j_L(u)} - \frac{\pi c}{12L^2} [1 - j'_L(u)^2] \\ &= (\partial_u \xi_{\pm}(\tau, j_L(u)))^2 \mathcal{T}_{\pm}(\xi_{\pm}(\tau, j_L(u))) + \frac{\kappa\mu_{\pm}^2}{4\pi} [1 - (\partial_u \xi_{\pm}(\tau, j_L(u)))^2] - \frac{c}{24\pi} \mathcal{S}_u[\xi_{\pm} \circ j_L](u) \end{aligned} \quad (\text{E.6})$$

where the following identity for the Schwarzian derivative

$$g'(u)^2 \mathcal{S}_v[f](v)|_{v=g(u)} = \mathcal{S}_u[f \circ g](u) - \mathcal{S}_u[g](u) \quad (\text{E.7})$$

and (6.17) have been employed. By applying  $J(\cdot)J$  to (E.6) and then using (6.21), we get

$$\mathcal{T}_{\pm}(\tau, u) = (\partial_u \tilde{\xi}_{\pm}(\tau, j_L(u)))^2 \mathcal{T}_{\pm}(j_L(u)) + \left( \frac{\kappa\mu_{\pm}^2}{4\pi} - \frac{\pi c}{12L^2} \right) [1 - (\partial_u \tilde{\xi}_{\pm}(\tau, j_L(u)))^2] \quad (\text{E.8})$$

with  $\tilde{\xi}_{\pm}(\tau, u)$  being defined in terms of (6.12) as follows

$$\tilde{\xi}_{\pm}(\tau, u) \equiv j_L(\xi_{\pm}(\tau, j_L(u))) = \xi_{\pm}(\tau, u) \quad (\text{E.9})$$

where (6.20) has been employed in the last step. Thus, for the modular evolution of the chiral operators (2.7) we can use (6.16) combined with (4.11) also for  $u \notin A$ .

## F Integrals for the quantum noise

In this appendix we discuss the explicit computation of the integrals occurring in section 5.3. Let us consider first the following identity

$$\frac{1}{\sinh^2(\pi\tau \pm i\epsilon)} = -\frac{1}{\pi} \partial_\tau \coth(\pi\tau \pm i\epsilon) \tag{F.1}$$

where

$$\coth(\pi\tau \pm i\epsilon) = \coth(\pi\tau) \mp i\delta(\tau) \tag{F.2}$$

where in the r.h.s. the principal value regularization is assumed for the distribution  $\coth(\pi\tau)$ .

The first integral to consider is the one occurring in (5.56), namely

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sinh^2[\pi(t - i\epsilon)/\beta]} dt &= -\frac{\beta}{\pi} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} e^{i\beta\omega\tau} \partial_\tau \coth(\pi\tau - i\epsilon) d\tau \tag{F.3} \\ &= i \frac{\beta^2\omega}{\pi} \int_{-\infty}^{+\infty} [\coth(\pi\tau) + i\delta(\tau)] e^{i\beta\omega\tau} d\tau = -\frac{\beta^2}{\pi} [\omega \coth(\beta\omega/2) + \omega] \end{aligned}$$

where we used (F.1), (F.2) and  $\int_{-\infty}^{+\infty} e^{i\omega\tau} \coth(\pi\tau) d\tau = i \coth(\omega/2)$ . This computation can be adapted to investigate also the other integral we need, which occurs in (5.64). Indeed, by considering the following identity

$$\frac{1}{\sinh^4(\pi\tau \pm i\epsilon)} = \frac{2}{3\pi} \partial_\tau \coth(\pi\tau \pm i\epsilon) - \frac{1}{6\pi^3} \partial_\tau^3 \coth(\pi\tau \pm i\epsilon) \tag{F.4}$$

we find that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\sinh^4[\pi(t - i\epsilon)/\beta]} dt & \tag{F.5} \\ &= \frac{\beta}{\pi} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} \left[ \frac{2}{3} \partial_\tau \coth(\pi\tau - i\epsilon) - \frac{1}{6\pi^2} \partial_\tau^3 \coth(\pi\tau - i\epsilon) \right] e^{i\beta\omega\tau} d\tau \\ &= -i \frac{\beta^2\omega}{\pi} \left[ \frac{2}{3} + \frac{(\beta\omega)^2}{6\pi^2} \right] \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} e^{i\beta\omega\tau} \coth(\pi\tau - i\epsilon) d\tau \\ &= \frac{\beta^2}{6\pi^3} [4\pi^2 + (\beta\omega)^2] [\omega \coth(\beta\omega/2) + \omega] \end{aligned}$$

where (F.4) and (F.2) have been employed.

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