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# HEUN CONNECTION FORMULAE FROM LIOUVILLE CORRELATORS

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# Abstract

We perform a detailed study of a class of irregular correlators in Liouville Conformal Field Theory, of the related Virasoro conformal blocks with irregular singularities and of their connection formulae. Upon considering their semi-classical limit, we provide explicit expressions of the connection matrices for the Heun function and a class of its confluences. Their calculation is reduced to concrete combinatorial formulae from conformal block expansions. Since Heun functions solve wave equations on various black holes backgrounds, we exploit our result to give exact expressions for different observables in black hole physics such as greybody factors, quasinormal modes and Love numbers. In the context of anti de Sitter black holes, we use our connection formulas in order to give novel exact expressions for thermal correlators of the boundary theory.

# Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following published works:

- G. Bonelli, C. Iossa, D. Panea Lichtig, A. Tanzini, *Irregular Liouville Correlators and Connection Formulae for Heun Functions*, Commun. Math. Phys. **397** (2023) no. 2 635-727, [arXiv:2201.004491]

- G. Bonelli, C. Iossa, D. Panea Lichtig, A. Tanzini, *Exact solution of Kerr black hole perturbations via CFT2 and instanton counting: Greybody factor, quasinormal modes, and Love numbers*, Phys. Rev. D **105** (2022), no. 4 044047, [arXiv:2105.04483]

- M. Dodelson, A. Grassi, C. Iossa, D. Panea Lichtig, A. Zhiboedov, *Holographic thermal correlators from supersymmetric instantons*, SciPost Phys. **14** (2023) 116, [arXiv:2206.07720]

I also coauthored these works, which are not part of this thesis:

- F. Benini, C. Iossa, M. Serone, *Conformality Loss, Walking and 4D Complex Conformal Field Theories at Weak Coupling, Phys. Rev. Lett.* **124** (2020) 051602, [arXiv:1908.04325]

- M. Dodelson, C. Iossa, R. Karlsson, A. Zhiboedov, *A thermal product formula,* [arXiv:2304.12339]

- S. Giusto, C. Iossa, R. Russo, *The black hole behind the cut*, [arXiv:2306.15305]

# Contents

# Chapter 1

# Introduction

Laws of physics are often formulated in terms of differential equations. This is the case for most of classical mechanics, quantum mechanics and general relativity. As a result, the development of efficient methodologies for obtaining analytical solutions to these equations becomes a matter of utmost importance. This thesis is dedicated to this very pursuit.

We will be concerned with a very special class of differential equations: second order Fuchsian differential equations. They are homogeneous linear 2nd order ordinary differential equations (ODE) with rational coefficients defined on the Riemann sphere. Being second order ODEs, they can always be recasted in normal form, namely

$$\left(\partial_z^2 + V(z)\right)\psi(z) = 0\,. \tag{1.0.1}$$

Fuchsian ODEs are classified by their singularities. In the form of (1.0.1), this means that they are classified by the singularities of $V(z)$. The usual definition of Fuchsian ODEs requires that all the singularities of $V(z)$ should be regular, i.e. quadratic, but we will also consider confluent limits where regular singularities collide to produce higher order (irregular) ones.

Since we can always fix up to 3 points on a sphere, we expect that the equation (1.0.1) will be qualitatively simpler if it has at most 3 regular singularities. The simplest possible case is when we have just 2 singularities, which we can fix to be at $0, \infty$ without loss of generality. A convenient parametrization of the potential is $V(z) = \frac{\frac{1}{4}-a^2}{z^2}$. This case is indeed very simple and for $a \neq 0$ the solution reads[1]

$$\psi(z) = c_1 z^{\frac{1}{2}-a} + c_2 z^{\frac{1}{2}+a}\,. \tag{1.0.2}$$

The next case is when we have 3 regular singularities, that we can without loss of generality fix to be at $0, 1, \infty$. The solution is given in terms of the hypergeometric functions, which are amongst the most studied special functions in mathematics. Firstly introduced by Gauss [1] [2], they are defined as

$${}_2F_1(a,b,c,z) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\,, \tag{1.0.3}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the rising Pochhammer symbol. The function (1.0.3) is the Frobenius series of the solution of the ODE close to the singularity at $z = 0$. It converges until it reaches the next singularity at $z = 1$. Even though ${}_2F_1$ is a series with a finite radius of convergence, it admits an integral representation which is globally well defined that allow us to analytically continue it everywhere on the Riemann sphere.

When we add a 4th singularity the situation gets qualitatively more complicated. In fact we cannot fix the position of the 4th singularity, and we are left with new modulus. The

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[1] For $a = 0$ logarithms appear in the solution.

[2] See [2] for a passionated review on hypergeometric functions.

power series defining the solutions of the ODE is not known in closed form, and no integral representation comes to help. The mysterious function that solves a 2nd order Fuchsian ODE with 4 regular singularities is dubbed the Heun function.

Heun functions[3] [3, 4, 5, 6] are much less understood than their simpler relative ${}_2F_1$, and not due to a lack of interest: they are ubiquitous in mathematical and theoretical physics [7, 8, 9, 10, 11]. An example that will be relevant for this thesis is the one of general relativity. Perturbations of a black hole background and the consequent emission of gravitational waves are often controlled by Heun functions and their confluences. They efficiently describe the ringdown of black hole mergers, and the quasinormal modes which characterize this phase can be determined exactly by solving a spectral problem where the Heun equation appears as a Schrödinger equation. Similarly scattering off a black hole is controlled by Heun functions and their analytic continuation. Needless to say, the recent experimental verification of gravitational waves [12] renewed the interest in these theoretical problems, and made of the study of analytical solutions of the corresponding ODEs of paramount importance both to deepen our comprehension of physical phenomena and to reveal possible physical fine structure effects.

A less phenomenological context where Heun functions appear is the one of holographic correlators. As we will briefly review later, holography represents a correspondence between a semiclassical gravity theory and a strongly coupled conformal field theory (CFT). In particular, semiclassical gravity in asymptotically anti de Sitter (AdS) spacetimes is believed to be equivalent to a strongly coupled CFT living on the boundary of AdS. This correspondence is often dubbed AdS/CFT, and was firstly introduced in [13]. When a sufficiently large black hole lives in the gravitational theory, the boundary theory will be in thermal equilibrium with the Hawking radiation emitted by the black hole. In particular, black hole perturbations in asymptotically AdS spacetimes govern the dynamics of the dual thermal CFT at strong coupling: above the Hawking-Page transition [14] the two point function of the boundary theory is computed by studying the wave equation on the AdS-Schwarzschild background [15], which is solved by Heun functions. Thermal correlation functions contain a wealth of fascinating physics related to the richness of the black hole geometry. For example, two-point functions encode the transport properties of the system, see e.g. [16, 17], the approach to equilibrium [18], as well as chaotic dynamics via pole-skipping [19, 20]. Moreover their analytic structure is determined in terms of the quasinormal modes of the bulk black hole [21]. Thermal correlators have also been used to formulate a version of the information paradox [22], as well as to look for subtle signatures of the black hole singularity [23, 24, 25, 26].

Fuchsian equations and their confluences appear also when studying perturbations of different geometric backgrounds. An example is the one of fuzzball geometries, which are atypical black holes microstates that admit a supergravity description [27, 28, 29] (for a review see [30]). In this context having analytic control on the solutions of wave equations is crucial in order to capture the features that distinguish black holes from their microstates. While in some simple models perturbations are solved in terms of hypergeometric functions [31], in more realistic models Heun equations make their appearance.

In the scenarios discussed above the relevant physical problem can be outlined as follows. When solving the wave equation satisfied by perturbations of the background of interest, one has to impose boundary conditions. For black holes, the relevant boundary condition is that the perturbation should look like an ingoing wave close to the horizon, which appears as a singularity of the Fuchsian ODE. Since closed form solutions are not available, in order to select the ingoing solution one considers a Frobenius expansion of the solution of the ODE

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[3] For a huge bibliography take a look at https://theheunproject.org .

close to the horizon and selects the series that looks like an ingoing wave:

$$\psi_{in}(r) = (r - r_+)^{c_{in}} \sum_{k \geq 0} c_k (r - r_+)^k \,. \tag{1.0.4}$$

Physical observables are read off at radial infinity, which appears as another singularity of the ODE. The point $r = \infty$ is typically outside the radius of convergence of the ingoing solution, and a nontrivial analytic continuation is needed. One has

$$\psi_{in}(r) = \mathcal{A}\psi_\infty^{(1)}(r) + \mathcal{B}\psi_\infty^{(2)}(r) \,, \tag{1.0.5}$$

where $\psi_\infty^{(1,2)}(r)$ are the two linearly independent series convergent close to $r = \infty$. The coefficients $\mathcal{A}$ and $\mathcal{B}$ select the linear combination that continues the ingoing solution at infinity. They are the so called connection coefficients of the ODE. The relevant mathematical problem is the analytic continuation of the series centered at one singularity up to another singularity. This is called the connection problem in the mathematical literature.

Due to the lack of closed form results regarding Heun functions, the corresponding connection problems are usually studied in the WKB approximation. This method was firstly proposed in [32], and then developed in [33, 34, 35] (see [36] for a review). WKB approximation has been then successfully applied in the context of perturbations of geometric backgrounds (see for example [37, 38, 39]). This is a powerful and rather general method that applies even beyond the Heun function. However WKB approximation works in narrow corners of parameter space, and computes the connection coefficients as asymptotic series.

In the context of pure mathematics, Heun functions solve the classical Poincaré uniformisation problem of a Riemann sphere with four punctures [40, 41]. Moreover Heun equations arise from the linear system whose isomonodromic deformation problems reduce to the classical Painlevé VI equations. This was firstly noted by Fuchs [42, 43] and Garnier [44]. A modern approach to this problem is discussed in [45, 46, 47] and [48].

Crucially for this thesis, there is another context where Fuchsian equations, and in particular Heun equations, make their appearance. In fact recent developments in the study of two-dimensional conformal field theories, their relation with supersymmetric gauge theories and equivariant localization in quantum field theory produced new tools which are very effective to study classical problems in the theory of Fuchsian ODEs. It has been known for some time now that the study of two dimensional Conformal Field Theories [49] and of the representations of its infinite-dimensional symmetry algebra provide exact solutions to partial differential equations in terms of conformal blocks and the appropriate fusion coefficients. The prototypical example is the null-state equation at level 2 for primary operators of Virasoro algebra which reduce, in the large central charge limit, to a Schrödinger-like equation with regular singularities of the form (1.0.1) at the position of the insertion of the operators. In this way one can engineer solutions of second-order linear differential equations of Fuchsian type by making use of the appropriate two dimensional CFT.

While under the operator/state correspondence the vertex operators in the above construction correspond to primary (highest weight) states, one can insert more general irregular vertex operators corresponding to irregular states. The latter generate irregular singularities in the corresponding null-state equation and therefore allow engineering more general potentials with singularities of order higher than two. Schematically, given a multi-vertex operator $\mathcal{O}_V(z_1, \dots, z_N)$ satisfying the OPE

$$T(z)\mathcal{O}_V(z_1, \dots, z_N) \sim V(z; z_i, \partial_{z_i})\mathcal{O}_V(z_1, \dots, z_N) \quad \text{as} \quad z \sim z_i \tag{1.0.6}$$

one finds the corresponding level 2 null-state equation (BPZ equation, from the authors of [49])

$$[b^{-2}\partial_z^2 + V(z; z_i, \partial_{z_i})]\Psi(z) = 0 \quad \Psi(z) = \langle \Phi_{2,1}(z)\mathcal{O}_V(z_1, \dots, z_N)\rangle \,, \tag{1.0.7}$$

satisfied by the correlation function of the multi-vertex and the level 2 degenerate field $\Phi_{2,1}(z)$. In the previous equation, $b$ parametrizes the central charge of the CFT and the scaling dimension of the degenerate field as

$$c = 1 + 6(b^2 + b^{-2} + 2) , \quad \Delta_{2,1} = -\frac{1}{2} - \frac{3}{4} b^2 . \tag{1.0.8}$$

If the multi-vertex contains primary operators only, the OPE (1.0.6) and the potential in (1.0.7) contains at most quadratic poles, while the insertions of irregular vertices generate higher order singularities in $V(z; z_i, \partial_{z_i})$. $V(z; z_i, \partial_{z_i})$ is a function of $z$ and of differential operators with respect to the $z_i$. The dependence on the latter trivializes in the semiclassical limit $c \to \infty$, and one finds a Schrödinger-like equation of the form (1.0.1). This ODE is a representation theoretical object, and will be the same for any CFT whose spectrum (or its analytic continuation) contains null-states. For various reasons that will be clear in the following, throughout the thesis we will specify to Liouville CFT. The mathematical interest of Liouville quantum field theory has been already highlighted by A.M. Polyakov who proposed to interpret it as a quantum extension of the Poincaré uniformisation problem [50]. A consequence of the above interpretation is that one can make use of the classical limit of Liouville theory to obtain new exact solutions of classical uniformisation [51]. This inspired the work of several authors [52, 53, 54, 55] and received a renewed interest after the discovery of AGT correspondence [56, 57, 58, 59, 60, 61, 62]. Perhaps more relevant to this thesis, Liouville CFT has already been used to solve differential equations in the context of Painlevé transcendents [63]. This correspondence between Painlevé and Liouville CFT has been extended to the full Painlevé confluence diagram in [64] and more general contexts in [65, 66, 67, 68]. All these results are related to the $c = 1$ limit of Liouville CFT.

All in all, the solution of (1.0.1) can be derived from the explicit computation of a large $c$ CFT correlator and from its expansions in different intermediate channels. The main advantage over, for example, WKB computations, is that the solutions obtained with this method will inherit the convergence properties of conformal blocks, and be in this sense exact. An important ingredient to accomplish this program is a deep control on the analytic structure of Virasoro conformal blocks. This has been obtained after the seminal AGT paper [69]. As we will briefly review later, according to the AGT correspondence in fact conformal blocks of Virasoro algebra are identified with concrete combinatorial formulae arising from equivariant instanton counting in the context of $\mathcal{N} = 2$ four-dimensional supersymmetric gauge theories [70, 71]. The explicit solution of the instanton counting problem has been decoded in the CFT language in terms of overlap of regular and irregular states in [72, 73, 74, 75].

This program, which started with [76], was firstly applied to black hole physics in [77], and has been used to compute the connection coefficients of Heun functions [78]. They have been verified in a more rigorous context in [79], and applied to a variety of black holes backgrounds [80, 81, 82] and fuzzball geometries [83, 84, 85, 86, 87, 88] [4].

The general round of ideas underlying this thesis is the following: we start from a Fuchsian differential equation that appears in the context of black hole (or some other geometrical background) perturbations and/or holography; then we engineer in Liouville CFT a correlation functions that satisfies in the semiclassical limit the same ODE; finally, we read off the exact solution of the connection problem of the ODE exploiting our knowledge of Liouville CFT, and use the AGT duality to express the result in terms of concrete and convergent combinatorial series. The interplay between the various characters of this game is summarized in figure (1.1).

For concrete purposes it is convenient to solve the connection problems of the Fuchsian ODEs of interest once for all, and then make use of the results when needed. With this in mind, this thesis is structured as follows:

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[4] For a similar approach based on integrability see [89, 90, 91]

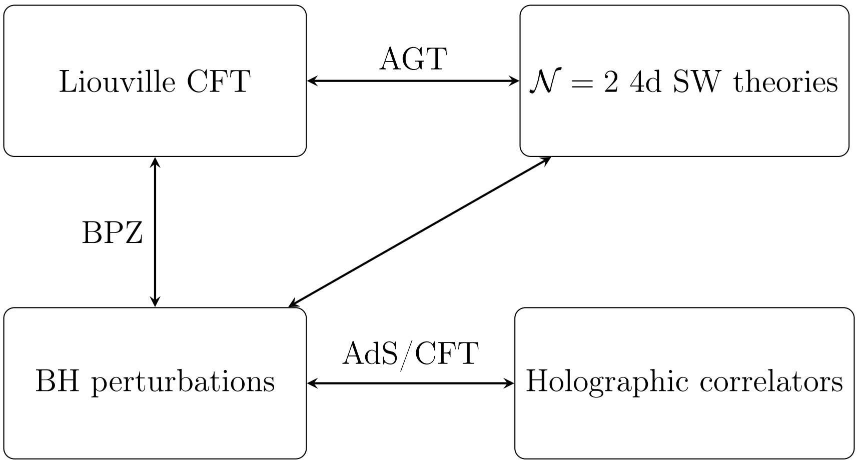

Figure 1.1: Flow diagram of this thesis.

- In **Chapter 2** we will briefly introduce the main characters of the game, that is Liouville CFT, wave equations in black holes backgrounds and their holographic counterpart.
- **Chapter 3** contains the main core of the thesis: here we will exploit our knowledge of Liouville theory to construct new exact solutions of a large class of differential equations. For concreteness, we will focus on Heun equations.
- In **Chapter 4** we apply the results from chapter 3 to perturbations of asymptotically flat black holes.
- In **Chapter 5** we apply the results from chapter 3 to perturbations of asymptotically AdS black holes and holographic correlators at finite temperature.
- In **Chapter 6** we draw our conclusions and list some open problems.

# Chapter 2

# Background

In this chapter we briefly introduce the main characters of the thesis: Liouville CFT, wave equations in black hole backgrounds and holographic correlators at finite temperature. Rather than a comprehensive review, this chapter is intended as a list of preliminary facts that sets the stage (and the notation) for the rest of the thesis.

There is a fourth class of objects that will be mentioned throughout the thesis: $\mathcal{N} = 2$ supersymmetric 4d gauge theories. Partition functions of such theories are related to Liouville CFT by the celebrated AGT duality, and they can be efficiently computed via localizations techniques as convergent combinatorial series. Thanks to the AGT duality such series turn into explicit expressions for 2d conformal blocks. For our purposes these gauge theories serve only as a technical tool to compute Liouville conformal blocks, therefore we will not introduce them in detail.

## 2.1 Liouville CFT

Let us start by recalling some general facts about 2d CFT (for a comprehensive review see e.g. [92]). A 2d CFT is a 2d quantum field theory which is covariant under two copies of the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}\,, \quad m,n \in \mathbb{Z}\,, \tag{2.1.1}$$

and its antiholomorphic counterpart

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}\,, \quad m,n \in \mathbb{Z}\,, \tag{2.1.2}$$

with $[\bar{L}_n, L_m] = 0$. A special class of operators in the theory are the so called primary operators $V_{\Delta,J}$: they transform as tensors under conformal transformations, meaning that when $z \to w(z)$ one has

$$V_{\Delta,J}(z)dz^{\Delta} \to V_{\Delta,J}(w)dw^{\Delta}\,. \tag{2.1.3}$$

Their commutation relations with the $L_n$'s are

$$\begin{aligned} [L_n, V_{\Delta,J}(z)] &= \left(z^{n+1}\partial_z + \Delta(n+1)z^n\right) V_{\Delta,J}(z)\,, \\ [\bar{L}_n, V_{\Delta,J}(z)] &= \left(z^{n+1}\partial_z + \bar{\Delta}(n+1)z^n\right) V_{\Delta,J}(z)\,. \end{aligned} \tag{2.1.4}$$

These operators are labeled by their (anti)holomorphic scaling dimension $(\bar{\Delta})\Delta$ and their spin $J = \Delta - \bar{\Delta}$. Another important operator of the theory is the (anti)holomorphic energy momentum tensor $(\bar{T}(\bar{z}))$ $T(z)$. Its mode expansion reads

$$T(z) = \sum_n L_n z^{-n-2}\,, \quad \bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}\,. \tag{2.1.5}$$

Since

$$L_{-k}=\frac{1}{2\pi i}\oint dz\frac{T(z)}{z^{k-1}}\,, \tag{2.1.6}$$

where the contour is a small circle centered at $z=0$, we can define Virasoro generators acting at a generic spacetime point $w$ as

$$L_{-k}(w)=\frac{1}{2\pi i}\oint dz\frac{T(z)}{(z-w)^{k-1}}\,, \tag{2.1.7}$$

where now the contour is a circle centered at $z=w$. We have $L_n=L_n(0)$.

Primary states are created by primaries acting on the conformal vacuum $|0\rangle$ s.t. $L_{\pm1},L_0|0\rangle=0$:

$$V_{\Delta,J}(0)|0\rangle=|\Delta,J\rangle\,. \tag{2.1.8}$$

Such states diagonalize the $L_0$ action since $L_0|\Delta,J\rangle=\Delta|\Delta,J\rangle$, and are annihilated by all Virasoro descendants $L_n$ with $n>0$ (annihilation operators). They define a lowest weight representation of the Virasoro algebra. The other states in the same representation, the so called conformal descendants of $|\Delta,J\rangle$, are obtained acting with negative Virasoro generators. They are organized according to their $L_0$ weight:

$$\begin{aligned}
&L_0=\Delta+1: \quad L_{-1}|\Delta,J\rangle\,,\\
&L_0=\Delta+2: \quad L_{-2}|\Delta,J\rangle\,,\quad L_{-1}^2|\Delta,J\rangle\,,\\
&L_0=\Delta+3: \quad L_{-3}|\Delta,J\rangle\,,\quad L_{-2}L_{-1}|\Delta,J\rangle\,,\quad L_{-1}^3|\Delta,J\rangle\,,\\
&\dots
\end{aligned} \tag{2.1.9}$$

$L_0$ is the generator of dilatations, and since 2d CFT are typically radially quantized, it morally plays the role of the Hamiltonian of the theory. Unitarity then requires $\Delta,\bar{\Delta}\geq0$. A convenient parametrization is

$$\Delta=\Delta_\alpha=\frac{c-1}{24}-\alpha^2\,,\quad \alpha\in i\mathbb{R}\,. \tag{2.1.10}$$

$\alpha$ is the so called (holomorphic) momentum of the field $V_{\Delta,J}$. The set of all primaries of a given theory, together with their descendants, forms a complete basis of the Hilbert space, and one has

$$\mathrm{Id}=\sum_{\Delta,J}\sum_{Y,Y'}\sum_{\tilde{Y},\tilde{Y}'}L_{-Y}\bar{L}_{-\tilde{Y}}|\Delta,J\rangle Q_\Delta^{-1}(Y,Y')Q_{\Delta-J}^{-1}(\tilde{Y},\tilde{Y}')\langle\Delta,J|L_{Y'}\bar{L}_{\tilde{Y}'}\,, \tag{2.1.11}$$

where Y is a Young diagram $Y=(n_1,n_2,n_3,\dots,n_{max})$, with $n_1\geq n_2\cdots\geq n_{max}$, $L_Y=L_{n_1}L_{n_2}\dots L_{n_{max}}$ and accordingly $L_{-Y'}=L_{-n_1'}L_{-n_2'}\dots L_{-n_{max}'}$. Moreover, the bra $\langle\Delta,J|$ is given by

$$\langle\Delta,J|=\lim_{z\to\infty}z^{2\Delta}\bar{z}^{2\Delta}\langle0|V_{\Delta,J}(z) \tag{2.1.12}$$

and $Q_\Delta(Y,Y')$ is the so called Shapovalov form, that is the matrix with entries

$$Q_\Delta(Y,Y')=\langle\Delta,J|L_YL_{-Y'}|\Delta,J\rangle\,. \tag{2.1.13}$$

Note that $Q_\Delta$ is block diagonal since, as it can easily be proven form the commutations relations (2.1.2), if $Y,Y'$ have different number of boxes $Q_\Delta(Y,Y')=0$.

Conformal symmetry poses great constraints on correlation functions. In particular, since one can always fix (up to) 3 points on a sphere to be at 0, 1 and $\infty$ up to conformal transformations, spacetime dependence in 2 and 3 point functions is trivial. Moreover since

$$\langle\Delta_1,J_1|L_0|\Delta_2,J_2\rangle=\Delta_1\langle\Delta_1,J_1|\Delta_2,J_2\rangle=\Delta_2\langle\Delta_1,J_1|\Delta_2,J_2\rangle\,, \tag{2.1.14}$$

and similarly for the antiholomorphic part, the two point function vanishes unless $\Delta_1 = \Delta_2$ and $J_1 = J_2$. All in all one has

$$\begin{aligned} \langle \Delta_1, J_1 | \Delta_2, J_2 \rangle &= G_{\Delta_1} \delta_{\Delta_1 \Delta_2} \,, \\ \langle \Delta_1, J_2 | V_{\Delta_2, J_2}(1) | \Delta_3, J_3 \rangle &= C_{123} \,. \end{aligned} \tag{2.1.15}$$

$C_\Delta$ and $C_{123}$ are the so called structure constants of the CFT. They are theory dependent constants that are not determined by conformal symmetry. Higher point function are more complicated. Inserting the identity (2.1.11) in a 4 point function one gets

$$\begin{aligned} &\langle \Delta_\infty, J_\infty | V_{\Delta_1, J_1}(1) V_{\Delta_z, J_z}(z) | \Delta_0, J_0 \rangle = \\ &= \sum_\Delta \left| \sum_{Y,Y'} Q_\Delta^{-1}(Y, Y') \langle \Delta_\infty, J_\infty | V_{\Delta_1, J_1}(1) L_{-Y} | \Delta, J \rangle \langle \Delta, J | L_{Y'} V_{\Delta_z, J_z}(z) | \Delta_0, J_0 \rangle \right|^2 \end{aligned} \tag{2.1.16}$$

where the modulus squared sends $\Delta \to \Delta - J = \bar{\Delta}$. One has for example, again from (2.1.2)

$$\begin{aligned} \langle \Delta_1, J_1 | L_n V_{\Delta_2, J_2}(t) | \Delta_3, J_3 \rangle &= \langle \Delta_1, J_1 | [L_n, V_{\Delta_2, J_2}(t)] | \Delta_3, J_3 \rangle = \\ &= C_{123} t^{\Delta_3 - \Delta_2 - \Delta_1} t^n (\Delta_3 + n\Delta_2 - \Delta_1) \,, \end{aligned} \tag{2.1.17}$$

and

$$Q_\Delta(1,1) = 2\Delta \,, \quad Q_\Delta(2,2) = 4\Delta + \frac{c}{2} \,, \quad Q_\Delta(2,(1,1)) = \dots \tag{2.1.18}$$

Every $n$-th descendant contributes with a power $z^n$ to (2.1.16). One has

$$\begin{aligned} &\langle \Delta_\infty, J_\infty | V_{\Delta_1, J_1}(1) V_{\Delta_z, J_z}(z) | \Delta_0, J_0 \rangle = \\ &= \sum_\Delta C_{\infty 1 \Delta} C_{\Delta z 0} G_\Delta \left| z^{\Delta - \Delta_z - \Delta_0} \left( 1 + \frac{(\Delta + \Delta_1 - \Delta_\infty)(\Delta + \Delta_z - \Delta_0)}{2\Delta} z + \mathcal{O}(z^2) \right) \right|^2 . \end{aligned} \tag{2.1.19}$$

The $z$ series inside the modulus squared is a so called conformal block, that we will denote as

$$\mathfrak{F} \left( \begin{matrix} \Delta_1 & & \Delta_z \\ & \Delta & \\ \Delta_\infty & & \Delta_0 \end{matrix} ; z \right) = z^{\Delta - \Delta_z - \Delta_0} \left( 1 + \frac{(\Delta + \Delta_1 - \Delta_\infty)(\Delta + \Delta_z - \Delta_0)}{2\Delta} z + \mathcal{O}(z^2) \right) . \tag{2.1.20}$$

It is given as a power series in $z$ whose coefficients are determined by conformal invariance. It is believed to be a convergent series whose radius of convergence arrives at the position of the next insertion (in this case $z = 1$). In terms of conformal blocks we can rewrite

$$\langle \Delta_\infty, J_\infty | V_{\Delta_1, J_1}(1) V_{\Delta_z, J_z}(z) | \Delta_0, J_0 \rangle = \sum_\Delta C_{\infty 1 \Delta} C_{\Delta z 0} G_\Delta \left| \mathfrak{F} \left( \begin{matrix} \Delta_1 & & \Delta_z \\ & \Delta & \\ \Delta_\infty & & \Delta_0 \end{matrix} ; z \right) \right|^2 . \tag{2.1.21}$$

If we want to expand our series close to the insertion at 1 we simply send $z \to w = 1 - z$. From (2.1.3), and decomposing again in conformal blocks, we find

$$\langle \Delta_\infty, J_\infty | V_{\Delta_1, J_1}(1) V_{\Delta_z, J_z}(z) | \Delta_0, J_0 \rangle = \sum_\Delta C_{\infty 0 \Delta} C_{\Delta z 1} G_\Delta \left| \mathfrak{F} \left( \begin{matrix} \Delta_0 & & \Delta_z \\ & \Delta & \\ \Delta_\infty & & \Delta_1 \end{matrix} ; 1 - z \right) \right|^2 . \tag{2.1.22}$$

In a common domain of convergence the two expressions must agree:

$$\sum_\Delta C_{\infty 1 \Delta} C_{\Delta z 0} G_\Delta \left| \mathfrak{F} \left( \begin{matrix} \Delta_1 & & \Delta_z \\ & \Delta & \\ \Delta_\infty & & \Delta_0 \end{matrix} ; z \right) \right|^2 = \sum_\Delta C_{\infty 0 \Delta} C_{\Delta z 1} G_\Delta \left| \mathfrak{F} \left( \begin{matrix} \Delta_0 & & \Delta_z \\ & \Delta & \\ \Delta_\infty & & \Delta_1 \end{matrix} ; 1 - z \right) \right|^2 . \tag{2.1.23}$$

This statement is the so called crossing symmetry of the correlator. The constraint (2.1.23) poses severe and nontrivial constraints on both the spectrum of the CFT and its structure constants.

Instead of inserting the identity in correlators, it is useful to restate what we just said using a different tool: the operator product expansion (OPE). This is a formal Taylor expansion at the operatorial level. When the insertion points of two operators are close to each other one can expand in the separation as follows:

$$V_{\Delta_1,J_1}(z)V_{\Delta_2,J_2}(0)=(z\bar{z})^{\Delta-\Delta_1-\Delta_2}\sum_{\Delta}C^{\Delta}_{\Delta_1\Delta_2}\sum_{Y,Y'}z^{|Y|}\bar{z}^{|Y'|}c_Y c_{Y'}L_{-Y}\bar{L}_{-Y'}V_{\Delta,J}(0)\,,\qquad(2.1.24)$$

where $|Y|$ is the total size of the Young diagram and the sum over $\Delta$ runs over all the spectrum of the theory. The coefficients $c_Y$ are again completely fixed by Virasoro symmetry, while the constant (OPE coefficient) $C^{\Delta}_{\Delta_1\Delta_2}$ and the spectrum are not. Inserting (2.1.24) in a 4 point function and noting that

$$C^{\Delta}_{\Delta_1\Delta_2}=C_{\Delta_1\Delta_2\Delta}G_{\Delta}\qquad(2.1.25)$$

one finds again (2.1.21).

We now specify our discussion to Liouville CFT. As we mentioned, the only theory dependent objects in a 2d CFT are the spectrum and the structure constant. Liouville CFT is the unique [93, 94] 2d CFT with continuous and diagonal ($\Delta=\bar{\Delta}$, or equivalently $J=0$) spectrum whose structure constants depend meromorphically on the Liouville momentums $\alpha$. Central charge and scaling dimensions are usually parametrized as

$$c=1+6Q^2\,,\quad Q=b+\frac{1}{b}\,,\quad \Delta=\bar{\Delta}=\Delta_\alpha=\frac{Q^2}{4}-\alpha^2\,,\quad b\geq 0\,,\quad \alpha\in i\mathbb{R}\,.\qquad(2.1.26)$$

With the previous assumptions one can explicitly compute the structure constants [95, 96] and finds

$$\begin{aligned}C_{\alpha_1\alpha_2\alpha_3}&=\frac{\Upsilon_b'(0)\Upsilon_b\left(Q+2\alpha_1\right)\Upsilon_b\left(Q+2\alpha_2\right)\Upsilon_b\left(Q+2\alpha_3\right)}{\Upsilon_b\left(\frac{Q}{2}+\alpha_1+\alpha_2+\alpha_3\right)\Upsilon_b\left(\frac{Q}{2}+\alpha_1+\alpha_2-\alpha_3\right)}\times\\&\times\frac{1}{\Upsilon_b\left(\frac{Q}{2}+\alpha_1-\alpha_2+\alpha_3\right)\Upsilon_b\left(\frac{Q}{2}-\alpha_1+\alpha_2+\alpha_3\right)}\,,\\G_\alpha&=\frac{\Upsilon_b\left(Q+2\alpha\right)}{\Upsilon_b\left(2\alpha\right)}\,,\end{aligned}\qquad(2.1.27)$$

where $\Upsilon_b(x)$ is a special function characterized[1] by the functional relation

$$\Upsilon_b(x+b)=\gamma(bx)b^{1-2bx}\Upsilon_b(x)\,,\quad \gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}\,.\qquad(2.1.28)$$

A special role in conformal field theory theory is played by the so called degenerate fields. The spectrum of Liouville theory can be analytically continued to contain zero norm states. Let us discuss the first nontrivial example. Consider the primary operator

$$\Phi_{2,1}(z)\,,\quad \Delta_{2,1}=-\frac{1}{2}-\frac{3}{4}b^2\,,\quad \alpha_{2,1}=-\frac{2b+b^{-1}}{2}\,.\qquad(2.1.29)$$

This operator acting on the vacuum generates the primary state $\Phi_{2,1}(0)|0\rangle=|\Delta_{2,1}\rangle$. Consider the descendant

$$|\chi\rangle=\left(b^{-2}L_{-1}^2+L_{-2}\right)|\Delta_{2,1}\rangle\,.\qquad(2.1.30)$$

---

[1] See [97] for a complete characterization of $\Upsilon_b$.

It can be easily proven that $\langle\chi|\chi\rangle = 0$. This state has zero norm! Moreover we have

$$L_0|\chi\rangle = (\Delta_{2,1}+2)|\chi\rangle\,, \quad L_{n>0}|\chi\rangle = 0\,. \tag{2.1.31}$$

This zero norm descendant is a primary state itself, and in particular is orthogonal to every state in the Hilbert space. This means that all correlation functions involving $|\chi\rangle$ have to vanish identically:

$$\langle V_1(z_1)V_2(z_2)\dots\chi(z)\dots V_n(z_n)\rangle = \langle V_1(z_1)V_2(z_2)\dots\left(b^{-2}L_{-1}^2+L_{-2}\right)\cdot\Phi_{2,1}(z)\dots V_n(z_n)\rangle = 0\,, \tag{2.1.32}$$

where we denoted $L_{-n}(z)\cdot\Phi_{2,1}(z) = L_{-n}(z)\Phi_{2,1}(z)$. Since Virasoro generators act as differential operators on primaries, (2.1.32) turns into a differential equation for the correlator $\langle V_1(z_1)V_2(z_2)\dots\Phi_{2,1}(z)\dots V_n(z_n)\rangle$. Such equation will be of second order with respect to $z$ since $\chi(z)$ is a second level descendant of $\Phi_{2,1}$, and of first order with respect to the various $z_i$, and looks like

$$\left(b^{-2}\partial_z^2 + f(z,z_i)\partial_z + V(z,z_i,\partial_{z_i})\right)\langle V_1(z_1)V_2(z_2)\dots\Phi_{2,1}(z)\dots V_n(z_n)\rangle = 0\,. \tag{2.1.33}$$

This is the so called BPZ equation [49]. Note that the potential $V$ behaves as

$$V(z,z_i,\partial_{z_i}) \simeq \frac{\Delta_i}{(z-z_i)^2}\,, \quad \text{as } z\to z_i\,. \tag{2.1.34}$$

Inserting a primary operators excites a quadratic (i.e. regular) singularities in $V$. It is possible to excite higher (i.e. irregular) singularities by considering the so called irregular states [72, 75, 98, 74]. Such states are obtained by colliding together primary states appropriately rescaling their scaling dimensions. This procedure mimics the confluence of singularities in general theory of differential equations. We will discuss these states in detail in the main text. For the moment the BPZ equation takes the form of a partial differential equation. As we will discuss at length in chapter 3, it will reduce to an ODE of Fuchsian type as $b\to 0$.

Before moving on, let us comment on why we specialized to Liouville CFT. First of all, in Liouville CFT both the spectrum and the structure constants are known nonperturbatively. Since, as sketched in the introduction, we plan to use CFT in order to solve Fuchsian differential equations, it is to say the least convenient to choose a theory where everything is under control. Another important reason is the structure of the Liouville spectrum. Being continuous, it can be analytically continued to contain degenerate and irregular fields, that will play a crucial role in our discussion, and similarly scaling dimensions appearing as parameters in the BPZ equation can be continued to basically any value. This will give us the freedom to solve the Fuchsian ODE in its full generality.

## 2.2 AGT correspondence

As we mentioned above, the AGT correspondence relates Virasoro conformal blocks to instanton partition functions of $\mathcal{N}=2$ supersymmetric 4d gauge theories [69] (see [99] for a review). Such theories are formulated on a background that regulates IR divergences, the Omega background. Regular and irregular insertions in the CFT are AGT dual to matter multiplets in the gauge theory, and degenerate fields correspond to defect operators. Crucially for us this correspondence gives concrete combinatorial formulas to conformal blocks, since their dual instanton partition functions can be computed via localization. In order to limit the technical details in this thesis, we will just state the relevant formulas. Let us go back to

$$\langle\Delta_\infty|V_{\alpha_1}(1)V_{\alpha_z}(z)|\Delta_0\rangle = \int_{\alpha\in i\mathbb{R}} C_{\alpha_\infty\alpha_1\alpha}C_{\alpha\alpha_z\alpha_0}G_\alpha\left|\mathfrak{F}\begin{pmatrix}\alpha_1 & & \alpha_z \\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{pmatrix}; z\right|^2, \tag{2.2.1}$$

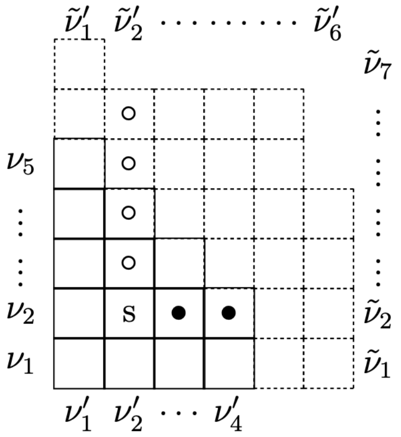

Figure 2.1: Arm length $A_{\tilde{Y}}(s) = 4$ (white circles) and leg length $L_Y(s) = 2$ (black dots) of a box at the site $s = (2, 2)$ for the pair of superimposed diagrams $Y$ (solid lines) and $\tilde{Y}$ (dotted lines).

where with respect to (2.1.21) we highlighted the dependence on Liouville momenta and suppressed the dependence on $J$ since Liouville spectrum always has $J = 0$. Localization in the AGT dual gauge theory gives for the conformal blocks [100, 101]

$$\begin{aligned}\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_z \\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{matrix}; z\right) &= z^{\Delta-\Delta_z-\Delta_0}(1-z)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_z)}\times \\ &\times \sum_{\vec{Y}} z^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_z+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_\infty\right)\,,\end{aligned} \tag{2.2.2}$$

where the sum runs over all pairs of Young diagrams $(Y_1, Y_2)$. We denote the size of the pair $|\vec{Y}| = |Y_1| + |Y_2|$, and [100, 101]

$$\begin{aligned} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\mu\right) &= \prod_{k=1,2}\prod_{(i,j)\in Y_k}\left(\alpha_k+\mu+b^{-1}\left(i-\frac{1}{2}\right)+b\left(j-\frac{1}{2}\right)\right)\,, \\ z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) &= \prod_{k,l=1,2}\prod_{(i,j)\in Y_k} E^{-1}\left(\alpha_k-\alpha_l,Y_k,Y_l,(i,j)\right)\prod_{(i',j')\in Y_l}\left(Q-E\left(\alpha_l-\alpha_k,Y_l,Y_k,(i',j')\right)\right)^{-1}\,, \\ E\left(\alpha,Y_1,Y_2,(i,j)\right) &= \alpha-b^{-1}L_{Y_2}((i,j))+b\left(A_{Y_1}((i,j))+1\right)\,. \end{aligned} \tag{2.2.3}$$

Here $L_Y((i,j)), A_Y((i,j))$ denote respectively the leg-length and the arm-length of the box at the site $(i,j)$ of the diagram $Y$. If we denote a Young diagram as $Y = (\nu'_1 \geq \nu'_2 \geq \dots)$ and its transpose as $Y^T = (\nu_1 \geq \nu_2 \geq \dots)$, then $L_Y$ and $A_Y$ read

$$A_Y(i,j) = \nu'_i - j\,,\ L_Y(i,j) = \nu_j - i\,. \tag{2.2.4}$$

Note that they can be negative if the box $(i,j)$ are the coordinates of a box outside the diagram. Also, the previous formulae has to be evaluated at $\vec{\alpha} = (\alpha_1, \alpha_2) = (\alpha, -\alpha)$.

Equation (2.2.2) is a concrete combinatorial series that can be easily evaluated to high order. It is believed that inherits the convergence properties of the Virasoro conformal blocks (see for example [102]), that is it should be convergent up to the next insertion in the correlation function, in this case at $z = 1$. In appendix C we expand on these combinatorial formulas.

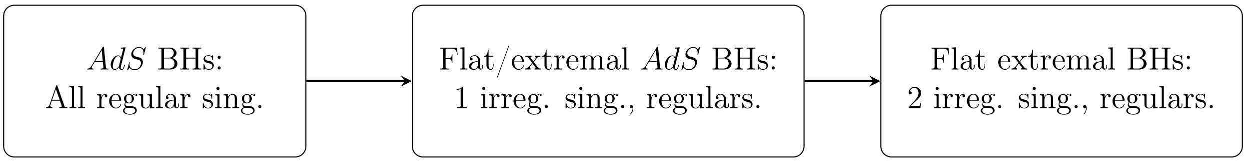

Figure 2.2: Singularities in black holes wave equations.

## 2.3 Black holes perturbations

Black hole perturbations, as well as matter fields propagating in black hole backgrounds, satisfy wave equations in the geometric background excited by the black hole:

$$\Box_{BH}\phi(t,r,\theta_i) = m^2\phi(t,r,\theta_i)\,. \tag{2.3.1}$$

Such wave equations can be obtained by linearizing Einstein equations around a given background. The Laplacian is computed in the black hole metric, which for spherically symmetric $d+1$ dimensional cases takes the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-1}\,. \tag{2.3.2}$$

Real and positive zeros of the function $f(r)$, often dubbed redshift factor, correspond to black hole horizons. The temperature corresponding to a given horizon $r_i$ is $f'(r_i)/(2\pi)$. If $f(r)$ has a double zero at $r = r_i$, the corresponding temperature vanishes and the horizon is said to be extremal.

The wave equation (2.3.1) is in general a 2nd order partial differential equation which depends on the radial coordinate $r$, time $t$ and angles $\theta_i$. In this thesis we will always consider separable backgrounds, that is backgrounds in which the wave equation reduces to a system of ODEs. Under the Ansatz

$$\phi(t,r,\theta_i) = e^{-i\omega t}\psi(r)\prod_i S_i(\theta_i) \tag{2.3.3}$$

equation (2.3.1) reduces to a system of ODEs for $(\psi, S_i)$. For spherically symmetric black holes, $S_i$ are spherical harmonics. The equation for $\psi(r)$ will be a 2nd order Fuchsian ODE with singularities at the zeroes of $f(r)$, plus possibly $r = 0, \infty$. It can be recasted in canonical form as

$$\left(\partial_z^2 + V(z)\right)\psi(z) = 0\,, \tag{2.3.4}$$

where $z = z(r)$. The structure of the singularities of the ODE, that is of $V(z)$, generically is the following:

- simple zeros of $f(r)$ (i.e. non extremal horizons) produce regular (i.e. quadratic) singularities in $V(z)$.
- double zeros of $f(r)$ (i.e. extremal horizons) produce irregular (i.e. higher than quadratic) singularities in $V(z)$.
- $V(z(r = \infty))$ has a regular singularity for asymptotically $AdS$ or $dS$ spacetimes, and an irregular singularity for asymptotically flat spacetimes.
- the singularity at $r = 0$, if present, is regular.

This structure is summarized in figure 2.2. The total number of singularities depends on the spacetime dimensions and the details of the black hole geometry.

Since the wave equation (2.3.1) is a second order ODE, in order to find a solution one needs to specify the relevant boundary conditions. For a black hole to really be black, nothing should come out of the horizon, at least classically. To make sure that this is the case we will require that perturbations look like purely ingoing waves close to the outer horizon $r_+$, that is the largest real solution of $f(r) = 0$. Expanding the equation for $\psi(r)$ for $r \sim r_+$ one will find two linearly independent series expansions centered at the outer horizon, one corresponding to an ingoing wave and the other to an outgoing wave. Choosing the solution corresponding to an ingoing wave we get the Frobenius series

$$\psi(r) = (r - r_+)^{c_{in}} \sum_{k\geq 0} c_k (r - r_+)^k \,, \tag{2.3.5}$$

where $c_{in}$ correspond to the wave being ingoing, and the various $c_k$ can be determined recursively from the ODE. Most of the physical quantities of interest that we will discuss in the thesis however can be read off from the behavior of (2.3.5) at $r \to \infty$. This requires a nontrivial analytic continuation since (2.3.5) is only convergent close to $r_+$. Close to $r = \infty$ the wave equation will be a superposition of the linearly independent behavior well defined in that region:

$$\psi(r) = \mathcal{A}\psi_\infty^{(1)}(r) + \mathcal{B}\psi_\infty^{(2)}(r) \,, \tag{2.3.6}$$

where $\psi_\infty^{(1,2)}(r)$ are Frobenius series that converge as $r \to \infty$ and can be easily determined treating perturbatively the wave equation. The coefficients $\mathcal{A}, \mathcal{B}$ on the other hand are much more complicated to compute. They encode all the details of the analytic continuation of the series (2.3.5) from $r \sim r_+$ to $r \sim \infty$. As mentioned in the introduction, they are called connection coefficients of the ODE. For asymptotically flat spacetimes the behavior of the Frobenius series at infinity is given by

$$\psi_\infty^{(1,2)}(r) = r^{-1\pm 2iM\omega} e^{\pm i\omega r} \left(1 + \mathcal{O}(r^{-1})\right) \,, \tag{2.3.7}$$

where $M$ is the mass of the black hole and $\omega$ the frequency of the perturbation. They correspond respectively to outgoing and ingoing waves at infinity. A particularly relevant physical observable is the one of quasinormal modes (QNM). These modes represent the characteristic frequencies emitted by the black hole after it gets perturbed. Since a black hole is a dissipative system, the quasinormal frequencies have a nonvanishing imaginary part. They satisfy the boundary conditions

$$\begin{aligned} \psi_{QNM}(r) &\simeq (r - r_+)^{c_{in}} \,, \quad r \sim r_+ \,, \\ \psi_{QNM}(r) &\simeq r^{-1+2iM\omega} e^{i\omega r} \,, \quad r \sim \infty \,, \end{aligned} \tag{2.3.8}$$

that is the wave is purely ingoing in the horizon and purely outgoing at infinity. These boundary conditions are only satisfied when $\mathcal{A} = \mathcal{A}(\omega) = 0$. This equation is on the same ground as quantization equations in quantum mechanics, and is solved by a numerable infinity of frequencies $\omega_n$. QNMs are extremely relevant observables: due to the no hair theorem in fact, in general relativity the whole tower of QNMs is determined only in terms of the mass and the spin (and the charge, if present) of the black hole. Matching analytic predictions with observations provides a nontrivial test for general relativity. Other relevant observable for asymptotically flat black holes will be discussed in chapter 4.

Most of the previous discussion holds for spherically symmetric black holes, that is when the metric is entirely determined in terms of the single function $f(r)$. In the following we will also discuss more general (but still separable) cases. Even though the details will be more complicated, all the general ideas will still apply.

## 2.4 Holographic correlators

A special class of observables that one can extract from black hole perturbations is the one of holographic correlators. The $AdS/CFT$ duality states that quantum gravity in asymptotically $AdS$ spaces in $d+1$ spacetime dimensions is equivalent to a CFT living on the $d-$dimensional boundary of $AdS$. The holographic dictionary sets the Newton constant $G_N$ to be proportional to the inverse of the central charge in the boundary CFT, and the string scale $\alpha'$ that governs higher derivative corrections to the gravitational action goes like $\lambda^{-\frac{1}{2}}$, where $\lambda$ is the CFT coupling[2]. In the limit $\alpha' \to 0$ and $G_N \to 0$ one finds that classical gravity in the $AdS$ bulk is dual to the boundary CFT in the large central charge and large coupling limit. In particular the classical gravity action acts as the generating functional of correlators in the boundary CFT at strong coupling. Accordingly, CFT 2 point functions are captured by quadratic perturbations in the bulk. When a black hole sits in the bulk, if its radius is sufficiently large, the boundary CFT will be in thermal equilibrium with the Hawking radiation emitted at the horizon, therefore its correlation functions will be thermal. The prescription to compute thermal two point functions goes as follows [103]. We first impose ingoing boundary at the horizon. Then the solution close to the $AdS$ boundary behaves as (fix $d=4$ for concreteness)

$$\psi(r) = \mathcal{A}(\omega,\ell) r^{\Delta-4}\left(1+\mathcal{O}(r^{-1})\right) + \mathcal{B}(\omega,\ell) r^{-\Delta}\left(1+\mathcal{O}(r^{-1})\right)\,, \quad m^2 = \Delta(\Delta-4)\,, \qquad (2.4.1)$$

where $m^2$ is the squared mass of the perturbation, and $\omega$ and $\ell$ are respectively its frequency and angular momentum. Since $\Delta > 4$, $\psi$ is a superposition of a growing (non normalizable) and decaying (normalizable) mode. It is natural to interpret the non normalizable mode as the source of the perturbation and the normalizable one as the response. Since 2 point functions usually measure the response of the system to perturbation, we have

$$G_R(\omega,\ell) = \mathcal{B}/\mathcal{A}\,, \qquad (2.4.2)$$

$G_R$ in the previous formula is the retarder correlator of the CFT two point function, namely (more in chapter 5 and appendix F)

$$i\theta(t)\langle[\mathcal{O}(t,x),\mathcal{O}(t,x')]\rangle_\beta = \frac{1}{4\pi(\Delta-1)(\Delta-2)}\int_{\mathbb{R}} d\omega e^{-i\omega t}\sum_{\ell=0}^{\infty}(\ell+1)G_R(\omega,\ell)\frac{\sin(\ell+1)\theta}{\sin\theta}\,. \qquad (2.4.3)$$

Again the relevant observable is entirely determined by connection coefficients of the Fuchsian equation.

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[2] Since the boundary CFT is typically a large $N$ QFT, with central charge $c \sim N$, $\lambda$ is the 't Hooft coupling.

# Chapter 3

# Liouville correlators and Heun connection formulae

## 3.1 Introduction

In this chapter we perform a detailed study of irregular correlators in Liouville Conformal Field Theory, of the related Virasoro conformal blocks with irregular singularities and of their connection formulae. Upon considering their semi-classical limit, we provide explicit expressions of the connection matrices for the Heun function and a class of its confluences. These result from the semi-classical limit of Virasoro conformal blocks for the five-point correlation function of four primaries and a degenerate field and a class of its coalescence limits to irregular conformal blocks. While the five-point correlator satisfies a linear PDE, namely the BPZ equation [49], its confluences satisfy a PDE obtained by an appropriate rescaling procedure. As we will discuss in detail in the paper, BPZ equations reduce in the semi-classical limit to ODEs. For the particular five-point correlation function mentioned above, this gets identified with Heun's equation upon a suitable dictionary. Let us also mention that the method we use can be generalised to general Fuchsian equations and their confluences upon considering the relevant conformal blocks.

Following a class of coalescences of the singularities and/or specific parameter scalings, from the configuration of four regular points one naturally obtains a set of confluent irregular blocks satisfying the corresponding confluent BPZ equations. The Heun functions and its confluences are solutions of the resulting semiclassical reduced equations.

In the AGT dual gauge theory context, the confluence procedure is interpreted as the decoupling of massive hypermultiplets [72] or the limit to strongly interacting Argyres-Douglas theories [64, 75] in the $SU(2)$ Seiberg-Witten theory. The semi-classical limit of CFT coincides via AGT correspondence with an asymmetric limit in the $\Omega$-background parameters known as the Nekrasov-Shatashvili (NS) limit [104]. This provides a quantization procedure of the classical integrable systems associated to the Seiberg-Witten theory [105]. From this viewpoint Heun equations can be interpreted as Schrödinger equations for these quantum systems.

This chapter is organised as follows.

- In **section 2**, as a warm-up, we recall the relation between four-point conformal blocks with the insertion of three primary fields and one level 2 degenerate field and hypergeometric functions and we study in detail the confluences to irregular conformal blocks and the related special functions. We obtain the connection formulae for the latter as solutions of the constraints imposed by crossing symmetry.
- In **section 3** we systematically study the five point conformal blocks with the insertion of four primary fields and one level 2 degenerate field. We focus on the explicit computation of the connection formulae as solutions of the constraints imposed by crossing symmetry

for the regular case and a class of its confluences. In each case, we also compute the semi-classical limit.

- In **section 4** we provide a dictionary between semiclassical CFT data and Heun equations in the standard form, we apply the results of the previous section identifying the relevant semiclassical CFT blocks with Heun functions and provide the connection formulae.

Few technical points are relegated to the Appendices.

The accompanying table collects the dictionary between (irregular) conformal blocks, supersymmetric gauge theories and the corresponding Heun functions.

| CFT - CB | | $SU(2)$ Gauge Theory | Heun |
|---|---|---|---|
| $\mathfrak{F}$ | Regular | $N_f = 4$ | HeunG |
| ${}_1\mathfrak{F}$ | Confluent | $N_f = 3$ | HeunC |
| ${}_{\frac{1}{2}}\mathfrak{F}$ | Reduced Confluent | $N_f = 2$ asymmetric | HeunRC |
| ${}_1\mathfrak{D}_1$ | Doubly Confluent | $N_f = 2$ symmetric | HeunDC |
| ${}_1\mathfrak{E}_{\frac{1}{2}}$ | Reduced Doubly Confluent | $N_f = 1$ | HeunRDC |
| ${}_{\frac{1}{2}}\mathfrak{E}_{\frac{1}{2}}$ | Doubly Reduced Doubly Confluent | $N_f = 0$ | HeunDRDC |

$$\begin{tikzcd}
 & & {}_{\frac{1}{2}}\mathfrak{F} \arrow[dr] & & \\
\mathfrak{F} \arrow[r] & {}_1\mathfrak{F} \arrow[ur] \arrow[dr] & & {}_1\mathfrak{E}_{\frac{1}{2}} \arrow[r] & {}_{\frac{1}{2}}\mathfrak{E}_{\frac{1}{2}} \\
 & & {}_1\mathfrak{D}_1 \arrow[ur] & &
\end{tikzcd}$$

Figure 3.1: Confluence diagram of conformal blocks.

## 3.2 Warm-up: 4-point degenerate conformal blocks and classical special functions

We start reviewing standard facts about four-point degenerate conformal blocks on the sphere and their confluence limits. In particular we review their relation to the hypergeometric function and its confluent limits, namely Whittaker and Bessel functions.

The hypergeometric function is the solution to the most general second-order linear ODE with three regular singularities. On the CFT side it arises as the four-point conformal block on the Riemann sphere when one of the insertions is a degenerate vertex operator.

### 3.2.1 Hypergeometric functions

Consider the four-point correlation function on the sphere with one degenerate field insertion $\Phi_{2,1}$ of momentum $\alpha_{2,1} = -\frac{2b+b^{-1}}{2}$ (corresponding to $\Delta_{2,1} = -\frac{1}{2} - \frac{3b^2}{4}$):

$$\langle \Delta_\infty | V_1(1) \Phi_{2,1}(z) | \Delta_0 \rangle \,. \tag{3.2.1}$$

In the following we will drop the subscript $2,1$ and just denote by $\Phi(z)$ this degenerate field. The corresponding BPZ equation takes the form

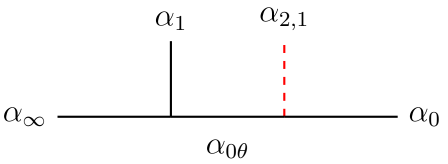
$$\left(b^{-2}\partial_z^2 - \left(\frac{1}{z-1}+\frac{1}{z}\right)\partial_z + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_0}{z^2} + \frac{\Delta_\infty - \Delta_1 - \Delta_{2,1} - \Delta_0}{z(z-1)}\right)\langle\Delta_\infty|V_1(1)\Phi(z)|\Delta_0\rangle = 0\,. \tag{3.2.2}$$

This equation has regular singularities at $0, 1, \infty$. As mentioned above, the corresponding correlator should therefore be expressed in terms of hypergeometric functions. Indeed, the above differential equation by definition is solved by the conformal blocks corresponding to the correlator (3.2.1), which in turn are given in terms of hypergeometric functions. In particular, the conformal block corresponding to the expansion $z \sim 0$ is

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & \\ \alpha_\infty & & \alpha_0\end{matrix}; z\right) = z^{\frac{bQ}{2}+\theta b\alpha_0}(1-z)^{\frac{bQ}{2}+b\alpha_1}{}_2F_1\left(\tfrac{1}{2}+b\left(\theta\alpha_0+\alpha_1-\alpha_\infty\right), \tfrac{1}{2}+b\left(\theta\alpha_0+\alpha_1+\alpha_\infty\right), 1+2b\theta\alpha_0, z\right)\,, \tag{3.2.3}$$

where $\theta = \pm$ and $\alpha_{0\pm} = \alpha_0 \pm \frac{-b}{2}$ are the two fusion channels allowed by the degenerate fusion rules. Similar formulae hold for the expansions around $z \sim 1$ and $\infty$. Conventionally, this conformal block is denoted diagrammatically by

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & \\ \alpha_\infty & & \alpha_0\end{matrix}; z\right) = \tag{3.2.4}$$



.

We now want to expose the interplay between crossing symmetry, DOZZ factors and the connection formulae for the hypergeometric functions. To this end, let us expand the correlator once for $z \sim 0$ and once for $z \sim 1$:

$$\begin{aligned}\langle\Delta_\infty|V_1(1)\Phi_{2,1}(z)|\Delta_0\rangle &= \sum_{\theta=\pm} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C_{\alpha_\infty\alpha_1\alpha_{0\theta}}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & \\ \alpha_\infty & & \alpha_0\end{matrix}; z\right)\right|^2 = \\ &= \sum_{\theta'=\pm} C^{\alpha_{1\theta'}}_{\alpha_{2,1}\alpha_1} C_{\alpha_\infty\alpha_{1\theta'}\alpha_0}\left|\mathfrak{F}\left(\begin{matrix}\alpha_0 & & \alpha_{2,1} \\ & \alpha_{1\theta'} & \\ \alpha_\infty & & \alpha_1\end{matrix}; 1-z\right)\right|^2\,.\end{aligned} \tag{3.2.5}$$

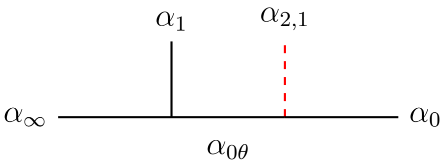
Here $C_{\alpha\beta\gamma}$ are the DOZZ three-point functions, and $C^{\alpha}_{\beta\gamma} = G^{-1}_{\alpha}C_{\alpha\beta\gamma}$ are the OPE coefficients (see Appendix A.1). Equation (3.2.5) is just the statement of crossing symmetry, due to the associativity of the OPE. The two expansions are related by the connection matrix $\mathcal{M}_{\theta\theta'}$ as follows

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & \\ \alpha_\infty & & \alpha_0\end{matrix}; z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_1; b\alpha_\infty)\mathfrak{F}\left(\begin{matrix}\alpha_0 & & \alpha_{2,1} \\ & \alpha_{1\theta'} & \\ \alpha_\infty & & \alpha_1\end{matrix}; 1-z\right)\,. \tag{3.2.6}$$

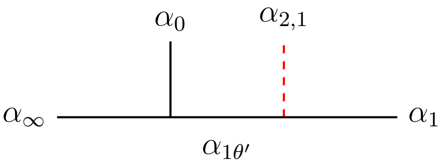
Plugging the latter into (3.2.5) determines $\mathcal{M}_{\theta\theta'}$ to be

$$\mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_1; b\alpha_\infty) = \frac{\Gamma(-2\theta' b\alpha_1)\Gamma(1+2\theta b\alpha_0)}{\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 + b\alpha_\infty\right)\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 - b\alpha_\infty\right)}\,, \tag{3.2.7}$$

which is indeed the connection matrix for hypergeometric functions. Diagrammatically, we can express the connection formula as



$$= \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}$$



. (3.2.8)

### 3.2.2 Whittaker functions

Colliding the singularities at 1 and $\infty$ of the hypergeometric functions we obtain the Whittaker functions, which are related to the confluent hypergeometric functions. They have a regular singularity at 0 and an irregular singularity of rank 1 at $\infty$. To describe the confluence of two regular singularities in CFT we introduce the rank 1 irregular state, denoted by $\langle \mu, \Lambda|$. It lives in a Whittaker module and it is defined by the following properties

$$\begin{aligned}
\langle \mu, \Lambda| L_0 &= \Lambda \partial_\Lambda \langle \mu, \Lambda| \\
\langle \mu, \Lambda| L_{-1} &= \mu\Lambda \langle \mu, \Lambda| \\
\langle \mu, \Lambda| L_{-2} &= -\frac{\Lambda^2}{4} \langle \mu, \Lambda| \\
\langle \mu, \Lambda| L_{-n} &= 0\,, \quad n > 2\,.
\end{aligned} \tag{3.2.9}$$

Note that the action of $L_0$ is not diagonal, and hence $\langle \mu, \Lambda|$ makes no reference to any Verma module. Equivalently, one can describe this state by a confluence limit of primary operators:

$$\langle \mu, \Lambda| \propto \lim_{\eta\to\infty} t^{\Delta_t - \Delta} \langle \Delta | V_t(t) \tag{3.2.10}$$

with[1]

$$\Delta = \frac{Q^2}{4} - \left(\frac{\mu+\eta}{2}\right)^2, \quad \Delta_t = \frac{Q^2}{4} - \left(\frac{\mu-\eta}{2}\right)^2, \quad t = \frac{\eta}{\Lambda}\,. \tag{3.2.11}$$

We fix the normalization of the irregular state by giving its overlap with a primary state, namely

$$\langle \mu, \Lambda | \Delta \rangle = |\Lambda|^{2\Delta} C_{\mu\alpha}\,, \tag{3.2.12}$$

with

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b\left(\frac{Q}{2}+\mu+\alpha\right)\Upsilon_b\left(\frac{Q}{2}+\mu-\alpha\right)}\,. \tag{3.2.13}$$

The $\Lambda$-dependence is fixed by the $L_0$-action, and $C_{\mu\alpha}$ is a normalization function that only depends on $\mu$ and $\alpha$, and is calculated in Appendices A.2, B.1. The notation reflects the fact that $C$ can be interpreted as a collided three-point function [75]. The correlator

$$\langle \mu, \Lambda | \Phi(z) | \Delta \rangle \tag{3.2.14}$$

satisfies the BPZ equation

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} + \frac{\mu\Lambda}{z} - \frac{\Lambda^2}{4}\right)\langle \mu, \Lambda | \Phi(z) | \Delta \rangle = 0\,, \tag{3.2.15}$$

that has a rank 1 irregular singularity at $z = \infty$ and a regular singularity at $z = 0$. Correspondingly, we expect this correlator to be given in terms of confluent hypergeometric functions. Indeed, for $z \sim 0$ one finds by solving the differential equation that the corresponding *confluent* (or *irregular*) conformal block is given by a Whittaker function. In particular, the two solutions are $z^{\frac{b^2}{2}} M_{b\mu,\pm b\alpha}(b\Lambda z)$, where the Whittaker $M$-function has a simple expansion around $z \sim 0$:

$$M_{b\mu,b\alpha}(b\Lambda z) = (b\Lambda z)^{\frac{1}{2}+b\alpha}\left(1 + \mathcal{O}(b\Lambda z)\right)\,. \tag{3.2.16}$$

We can compute the confluent conformal block as

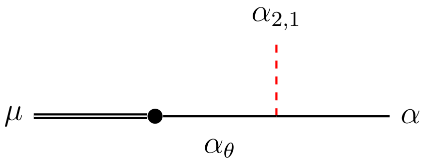
$${}_1\mathfrak{F}\left(\mu\, \alpha_\theta \, {\overset{\alpha_{2,1}}{\alpha}} ; \Lambda z\right) = \Lambda^{\Delta_\theta}(b\Lambda)^{-\frac{1}{2}-\theta b\alpha} z^{\frac{b^2}{2}} M_{b\mu,\theta b\alpha}(b\Lambda z)\,. \tag{3.2.17}$$

---

[1] Note that this procedure mimics the decoupling of a mass in the AGT-dual gauge theory.

by expanding the OPE between $\Phi(z)$ and $|\Delta\rangle$ and projecting on $\langle\mu,\Lambda|$. Comparing this with the expansion of $M$ one obtains the prefactors written above. Here the subscript 1 indicates the presence of a rank 1 irregular singularity at infinity. We represent this block diagramatically by

$${}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,\overset{\alpha_{2,1}}{\alpha};\Lambda z\right) = \qquad . \tag{3.2.18}$$

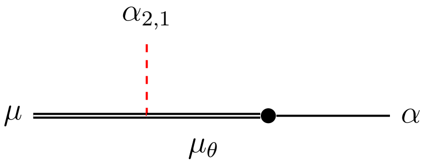


The double line denotes the rank 1 irregular state, and the fat dot the projection onto a primary state. For $z\sim\infty$ we get an intrinsically different kind of confluent conformal block since we are now expanding $z$ near an *irregular singularity* of rank 1, dubbed in [106] confluent conformal block *of 2nd kind*. We denote such a conformal block by the letter $\mathfrak{D}$ and find

$$\begin{aligned}{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_+\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}}e^{-i\pi b\mu}b^{b\mu}(\Lambda z)^{\frac{b^2}{2}}W_{-b\mu,b\alpha}(e^{-i\pi}b\Lambda z)\,,\\ {}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_-\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}}b^{-b\mu}(\Lambda z)^{\frac{b^2}{2}}W_{b\mu,b\alpha}(b\Lambda z)\,,\end{aligned} \tag{3.2.19}$$

where $W$ is the Whittaker function with a simple asymptotic expansion around $z\sim\infty$. This block is obtained by doing the OPE between the irregular state and the degenerate field, which is derived in Appendix B.1, and then projecting on $|\Delta\rangle$. Once again, the prefactors are fixed by comparing with the expansion of $W$. We represent this conformal block diagramatically by

$${}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_\theta\,\alpha;\frac{1}{\Lambda z}\right) = \qquad . \tag{3.2.20}$$



Crossing symmetry now implies

$$\langle\mu,\Lambda|\Phi(z)|\Delta\rangle = \sum_{\theta=\pm}C^{\alpha_\theta}_{\alpha_{2,1},\alpha}C_{\mu\alpha_\theta}\left|{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,\overset{\alpha_{2,1}}{\alpha};\Lambda z\right)\right|^2 = \sum_{\theta'=\pm}B^{\mu_{\theta'}}_{\alpha_{2,1},\mu}C_{\mu_{\theta'}\alpha}\left|{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right)\right|^2 . \tag{3.2.21}$$

Here $B$ is the irregular OPE coefficient arising from the OPE between the irregular state and the degenerate field. We calculate it in Appendices A.2, B.1, and it is given by

$$B^{\mu_\pm}_{\alpha_{2,1},\mu} = e^{i\pi\left(\frac{1}{2}\pm b\mu+\frac{b^2}{4}\right)}\,. \tag{3.2.22}$$

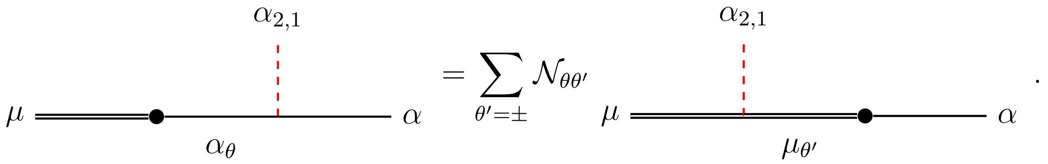
As for the hypergeometric function, we can make an Ansatz for the connection formula for these irregular conformal blocks of the form

$$b^{\theta b\alpha}{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,\overset{\alpha_{2,1}}{\alpha};\Lambda z\right) = \sum_{\theta'=\pm}b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu)\;{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right)\,. \tag{3.2.23}$$

The constraints coming from crossing symmetry (3.2.21) are solved by the irregular connection coefficients

$$\mathcal{N}_{\theta\theta'}(b\alpha,b\mu) = \frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}\,. \tag{3.2.24}$$

These are just the connection coefficients for Whittaker functions. In fact, in Appendix B.1 we argue the other way around, namely we determine the normalization function $C_{\mu\alpha}$ and the irregular OPE coefficient $B^{\mu\pm}_{\alpha_{2,1},\mu}$ by using the known connection coefficients $\mathcal{N}_{\theta\theta'}$ for Whittaker functions. This shows the consistency of our approach. Let us emphasize for latter purposes that the functions $\mathcal{N}_{\theta\theta'}$ solve the constraint (3.2.21), which will appear later in a different context. We represent this connection formula diagrammatically by



(3.2.25)

## 3.2.3 Bessel functions

There is a natural limiting procedure which reduces a rank 1 irregular singularity to a rank 1/2 one. To describe the latter in CFT, let us introduce the rank 1/2 irregular state $\langle \Lambda^2|$ via defining properties

$$\begin{aligned} \langle \Lambda^2|L_0 &= \Lambda^2 \partial_{\Lambda^2} \langle \Lambda^2| \\ \langle \Lambda^2|L_{-1} &= -\frac{\Lambda^2}{4} \langle \Lambda^2| \\ \langle \Lambda^2|L_{-n} &= 0\,, \quad n > 1\,. \end{aligned} \tag{3.2.26}$$

It can be obtained from the rank 1 irregular state via the limit[2]

$$\langle \Lambda^2| = \lim_{\mu\to\infty} \langle \mu, -\frac{\Lambda^2}{4\mu}|\,. \tag{3.2.27}$$

We see that reducing a rank 1 to a rank 1/2 singularity corresponds to further decoupling a mass in the AGT dual gauge theory. We normalize the rank 1/2 state as

$$\langle \Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta} C_\alpha\,, \quad C_\alpha = 2^{-4\Delta} e^{-2\pi i \Delta} \Upsilon_b(Q + 2\alpha)\,. \tag{3.2.28}$$

This normalization function is calculated in Appendices A.3, B.2. Consider the following correlation function involving the rank 1/2 state:

$$\langle \Lambda^2|\Phi(z)|\Delta\rangle\,. \tag{3.2.29}$$

which correspondingly displays a rank 1/2 singularity at infinity. This is reflected in the BPZ equation

$$\left( b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} - \frac{\Lambda^2}{4z} \right) \langle \Lambda^2|\Phi(z)|\Delta\rangle = 0\,. \tag{3.2.30}$$

Solving this differential equation one finds that the corresponding *rank 1/2 irregular* conformal block is given by a modified Bessel function $I_\nu(x)$ as

$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha;\, \Lambda\sqrt{z}\right) = \Gamma(1 + 2\theta b\alpha) \Lambda^{2\Delta_\theta} \left(\frac{b\Lambda}{2}\right)^{-2\theta b\alpha} z^{\frac{bQ}{2}} I_{2\theta b\alpha}(b\Lambda\sqrt{z})\,. \tag{3.2.31}$$

Here the subscript $\frac{1}{2}$ indicates the presence of a rank 1/2 singularity at infinity. This conformal block is obtained by doing the OPE between $\Phi$ and $|\Delta\rangle$ and then projecting the result on $\langle \Lambda^2|$. The prefactors are fixed by comparing this with the following expansion of the Bessel function

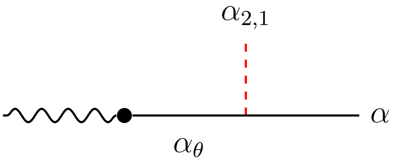
$$I_{2\theta b\alpha}(b\Lambda\sqrt{z}) = \frac{(b\Lambda\sqrt{z}/2)^{2\theta b\alpha}}{\Gamma(1 + 2\theta b\alpha)} \left(1 + \mathcal{O}(b\Lambda\sqrt{z})\right)\,. \tag{3.2.32}$$

---

[2] Note that this limit corresponds to the well known holomorphic decoupling limit of a massive hypermultiplet in the AGT dual gauge theory.

We represent this conformal block diagramatically by

$$
{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \qquad . \tag{3.2.33}
$$

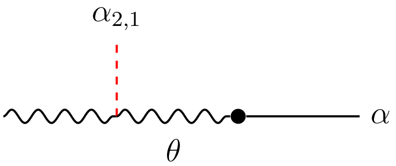


Here the wiggly line denotes the rank 1/2 irregular state, and the fat dot represents the pairing with a primary state. For $z \sim \infty$ we get a different kind of irregular conformal block, since we are now expanding for $z$ near an irregular singularity of rank 1/2. We denote such a conformal block by the letter $\mathfrak{E}$

$$
\begin{aligned}
{}_{\frac{1}{2}}\mathfrak{E}^{(+)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}e^{-\frac{i\pi}{2}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(e^{-i\pi}b\Lambda\sqrt{z})\,,\\
{}_{\frac{1}{2}}\mathfrak{E}^{(-)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(b\Lambda\sqrt{z})\,,
\end{aligned} \tag{3.2.34}
$$

where $K$ is the modified Bessel function of the second kind, which has a nice asymptotic expansion for $z \sim \infty$. This block is obtained from the OPE between the irregular rank 1/2 state and the degenerate field which we derived in Appendix B.2, and then by taking the scalar product with $|\Delta\rangle$. We represent this block diagramatically by

$$
{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) = \qquad . \tag{3.2.35}
$$



Crossing symmetry implies that

$$
\langle\Lambda^2|\Phi(z)|\Delta\rangle = \textstyle\sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha}C_{\alpha_\theta}\left|{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right)\right|^2 = \sum_{\theta'=\pm} B_{\alpha_{2,1}}C_\alpha\left|{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right)\right|^2 . \tag{3.2.36}
$$

Here $B_{\alpha_{2,1}}$ is the irregular OPE coefficient arising from the OPE between the irregular rank 1/2 state and the degenerate field:

$$
B_{\alpha_{2,1}} = 2^{b^2}e^{\frac{i\pi bQ}{2}}\,. \tag{3.2.37}
$$

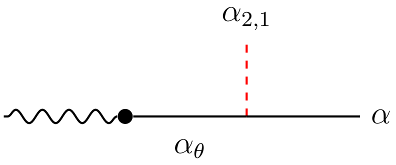
These functions are derived in Appendix B.2. We can now make an Ansatz for the connection formula for these irregular conformal blocks:

$$
b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \sum_{\theta'=\pm} b^{-1/2}\mathcal{Q}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) . \tag{3.2.38}
$$

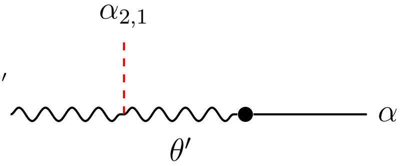
The crossing symmetry condition (3.2.36) gives constraints on the irregular connection coefficients, which are solved by

$$
\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}}\Gamma(1+2\theta b\alpha)e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)}\,. \tag{3.2.39}
$$

These are of course nothing else than the connection coefficients for Bessel functions, including the relevant prefactors. Similar constraints of the form (3.2.36) will reappear later. We represent the connection formula by



$$
= \sum_{\theta'=\pm}\mathcal{Q}_{\theta\theta'} \qquad . \tag{3.2.40}
$$



## 3.3 5-point degenerate conformal blocks, confluences and connection formulae

In this section we consider the relevant CFT correlators obeying the BPZ equations which reduce to Heun equations in the appropriate classical limit. Notice that for more than three vertex insertions BPZ equations on the sphere are richer than the corresponding ODE due to the presence of the corresponding moduli. This implies that a suitable classical limit (NS limit), engineered to decouple the moduli dynamics, is needed to recover the corresponding ODE.

We derive explicit connection formulae for the relevant conformal blocks by making use of crossing symmetry of the CFT correlators. In the classical limit, these generate explicit solutions of the connection problem for the Heun equations.

### 3.3.1 Regular conformal blocks

**General case**

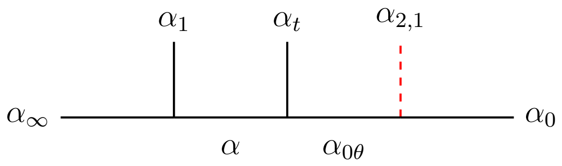
The five-point function with one degenerate insertion in Liouville CFT satisfies the BPZ equation

$$\left(b^{-2}\partial_z^2+\frac{\Delta_1}{(z-1)^2}-\frac{\Delta_1+t\partial_t+\Delta_t+z\partial_z+\Delta_{2,1}+\Delta_0-\Delta_\infty}{z(z-1)}+\frac{\Delta_t}{(z-t)^2}+\frac{t}{z(z-t)}\partial_t-\frac{1}{z}\partial_z+\frac{\Delta_0}{z^2}\right)\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle=0\,. \tag{3.3.1}$$

The five-point function can be expanded in the region $z \ll t \ll 1$ as follows

$$\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle=\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0}C^{\alpha}_{\alpha_t\alpha_{0\theta}}C_{\alpha_\infty\alpha_1\alpha}\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1}\\ \alpha_\infty & \alpha & & \alpha_{0\theta} & \alpha_0\end{matrix};t,\frac{z}{t}\right)\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1}\\ \alpha_\infty & \alpha & & \alpha_{0\theta} & \alpha_0\end{matrix};\bar{t},\frac{\bar{z}}{\bar{t}}\right). \tag{3.3.2}$$

As usual the conformal blocks can be computed via OPEs. The result is naturally an expansion in the variables $t$ and $z/t$. Conformal blocks are usually denoted diagrammatically as

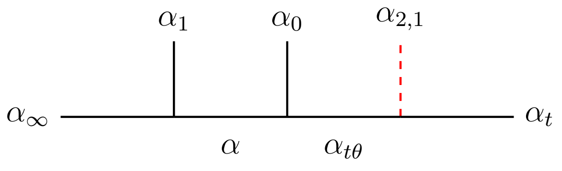


$$=\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1}\\ \alpha_\infty & \alpha & & \alpha_{0\theta} & \alpha_0\end{matrix};t,\frac{z}{t}\right). \tag{3.3.3}$$

An explicit combinatorial formula for this conformal block is given in Appendix D. The same correlator can be expanded for $z \sim t$ and small $t$ after the Möbius transformation $x \to \frac{x-t}{1-t}$, yielding

$$\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle=\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\right|^2\langle\Delta_\infty|V_1(1)V_0\left(\frac{t}{t-1}\right)\Phi\left(\frac{z-t}{1-t}\right)|\Delta_t\rangle=$$
$$=\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t}C^{\alpha}_{\alpha_0\alpha_{t\theta}}C_{\alpha_\infty\alpha_1\alpha}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_0 & & \alpha_{2,1}\\ \alpha_\infty & \alpha & & \alpha_{t\theta} & \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\right|^2. \tag{3.3.4}$$

Diagramatically, this conformal block is



$$=\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_0 & & \alpha_{2,1}\\ \alpha_\infty & \alpha & & \alpha_{t\theta} & \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right). \tag{3.3.5}$$

We notice that the diagrams just represent the order in which the OPEs are performed, neglecting factors such as Jacobians that arise from the Möbius transformations. By crossing symmetry the two expansions should agree, so that

$$\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0}C^{\alpha}_{\alpha_t\alpha_{0\theta}}C_{\alpha_\infty\alpha_1\alpha}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)\right|^2=$$

$$=\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t}C^{\alpha}_{\alpha_0\alpha_{t\theta}}C_{\alpha_\infty\alpha_1\alpha}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\right|^2. \tag{3.3.6}$$

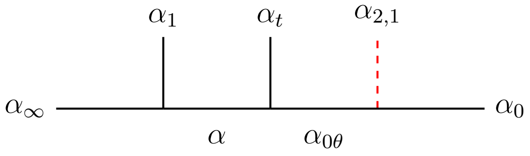
which can be conveniently recast as

$$\begin{aligned}&\int d\alpha\, C_{\alpha_\infty\alpha_1\alpha}\sum_{\theta=\pm}\Bigg(C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0}C^{\alpha}_{\alpha_t\alpha_{0\theta}}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)\right|^2+\\ &-C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t}C^{\alpha}_{\alpha_0\alpha_{t\theta}}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\right|^2\Bigg)=0\,.\end{aligned} \tag{3.3.7}$$

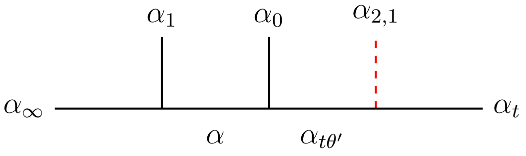
By imposing the vanishing of the integrand we get a constraint analogous to (3.2.5), which analogously to (3.2.6) we solve as[3]

$$\begin{aligned}&\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)=\\ &=\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_t;b\alpha)e^{i\pi(\Delta-\Delta_0-\Delta_{2,1}-\Delta_t)}(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right),\end{aligned} \tag{3.3.8}$$

where $\mathcal{M}_{\theta\theta'}$ are the hypergeometric connection coefficients defined in (3.2.7). Note indeed that in (3.3.8) the functional form of the connection coefficients depends on the local properties of the conformal block in the vicinity of the degenerate vertex insertion as can be seen form the factorized form of (3.3.7). Diagrammatically, the connection formula (3.3.8) reads

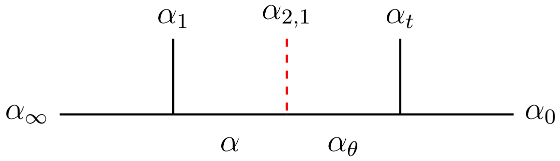


$$=\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}$$



. (3.3.9)

Conformal blocks for small $z$ can also be connected to the expansion for $z\sim 1, z\sim\infty$ passing through the region $t\ll z\ll 1$. The conformal block in that region is



$$=\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_{2,1}}\,\alpha_\theta\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};z,\frac{t}{z}\right). \tag{3.3.10}$$

Then, crossing symmetry relates this block to the expansion for $z\sim 0$ via

$$\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle=\langle\Delta_\infty|V_1(1)\Phi(z)V_t(t)|\Delta_0\rangle\,, \tag{3.3.11}$$

---

[3] The phase appearing in the RHS of equation (3.3.8) is fixed imposing that the overall leading powers of

$$(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\sim e^{-i\pi(\Delta-\Delta_t-\Delta_{2,1}-\Delta_0)}t^{\Delta-\Delta_0-\Delta_{t\theta}}(t-z)^{\Delta_{t\theta}-\Delta_{2,1}-\Delta_{2,1}}(1+\dots)$$

agree with the leading powers of the OPEs of the full correlator, where no explicit phase appears.

therefore, by comparing (3.3.11) with (3.3.3) we get

$$\begin{aligned}\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t\alpha_{0\theta}} C_{\alpha_\infty\alpha_1\alpha}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1} \\ \alpha_\infty & \alpha & & \alpha_{0\theta} & \alpha_0\end{matrix}; t, \frac{z}{t}\right)\right|^2 = \\ = \sum_{\theta=\pm}\int d\alpha\, C^{\alpha}_{\alpha_{2,1}\alpha_\theta} C^{\alpha_\theta}_{\alpha_t\alpha_0} C_{\alpha_\infty\alpha_1\alpha}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} & & \alpha_t \\ \alpha_\infty & \alpha & & \alpha_\theta & \alpha_0\end{matrix}; z, \frac{t}{z}\right)\right|^2 ,\end{aligned} \tag{3.3.12}$$

and following the same argument as for the previous case we find

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1} \\ \alpha_\infty & \alpha & & \alpha_{0\theta} & \alpha_0\end{matrix}; t, \frac{z}{t}\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha; b\alpha_t)\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} & & \alpha_t \\ \alpha_\infty & \alpha & & \alpha_{\theta'} & \alpha_0\end{matrix}; z, \frac{t}{z}\right). \tag{3.3.13}$$

Now we can connect expansions in the intermediate region to expansions for $z \sim \infty$ again invoking crossing symmetry. Performing the transformation $x \to t/x$ on the LHS of (3.3.11) we get

$$\langle\Delta_\infty|V_1(1)\Phi(z)V_t(t)|\Delta_0\rangle = \left|t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\right|^2\langle\Delta_0|V_t(1)V_1(t)\Phi\left(\frac{t}{z}\right)|\Delta_\infty\rangle , \tag{3.3.14}$$

that implies

$$\begin{aligned}\sum_{\theta=\pm}\int d\alpha\, C_{\alpha_t\alpha_0\alpha} C^{\alpha}_{\alpha_{2,1}\alpha_\theta} C^{\alpha_\theta}_{\alpha_\infty\alpha_1}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} & & \alpha_t \\ \alpha_\infty & \alpha_\theta & & \alpha & \alpha_0\end{matrix}; z, \frac{t}{z}\right)\right|^2 = \\ = \sum_{\theta=\pm}\int d\alpha\, C_{\alpha_t\alpha_0\alpha} C^{\alpha_{\infty\theta}}_{\alpha_{2,1}\alpha_\infty} C^{\alpha}_{\alpha_{\infty\theta}\alpha_1}\left|t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ \alpha_0 & \alpha & & \alpha_{\infty\theta'} & \alpha_\infty\end{matrix}; t, \frac{1}{z}\right)\right|^2 ,\end{aligned} \tag{3.3.15}$$

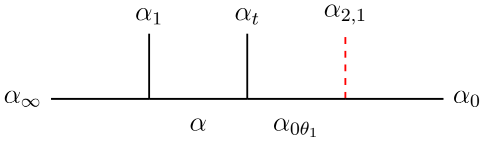
and finally

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_{2,1} & & \alpha_t \\ \alpha_\infty & \alpha_\theta & & \alpha & \alpha_0\end{matrix}; z, \tfrac{t}{z}\right) = \textstyle\sum_{\theta'}\mathcal{M}_{\theta\theta'}(b\alpha, b\alpha_\infty; b\alpha_1)t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ \alpha_0 & \alpha & & \alpha_{\infty\theta'} & \alpha_\infty\end{matrix}; t, \tfrac{1}{z}\right). \tag{3.3.16}$$

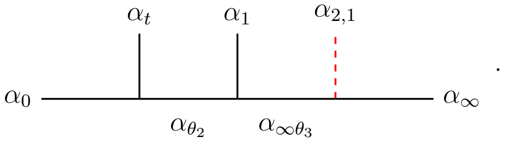
Combining equations (3.3.13) and (3.3.16) we can write

$$\begin{aligned}\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1} \\ \alpha_\infty & \alpha & & \alpha_{0\theta_1} & \alpha_0\end{matrix}; t, \frac{z}{t}\right) = \\ = \sum_{\theta_2\theta_3}\mathcal{M}_{\theta_1\theta_2}(b\alpha_0, b\alpha; b\alpha_t)\mathcal{M}_{(-\theta_2)\theta_3}(b\alpha, b\alpha_\infty; b\alpha_1)t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ \alpha_0 & \alpha_{\theta_2} & & \alpha_{\infty\theta_3} & \alpha_\infty\end{matrix}; t, \frac{1}{z}\right).\end{aligned} \tag{3.3.17}$$

Diagrammatically, this reads



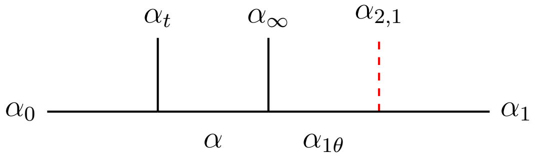
$= \sum_{\theta_2\theta_3}\mathcal{M}_{\theta_1\theta_2}\mathcal{M}_{(-\theta_2)\theta_3}$



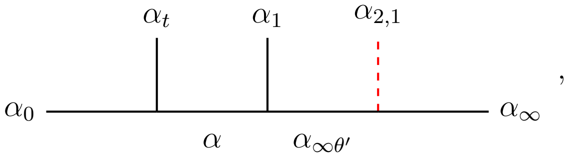
.

(3.3.18)

The diagrams provide a straightforward way to generalize the connection formula to an arbitrary pair of points. Indeed, writing down the diagram it is immediate to guess the correct $\mathcal{M}_{\theta\theta'}$ factors and the conformal blocks that will enter the connection formula. As an example, the connection formula for the expansions for $z \sim 1$ and $z \sim \infty$ with $t \ll 1$ are given by



$= \sum_{\theta'}\mathcal{M}_{\theta\theta'}$



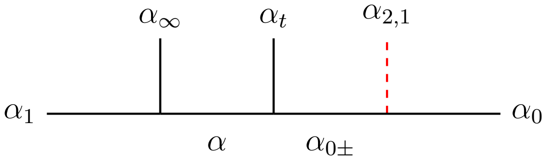
,

(3.3.19)

that is

$$t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_t-\Delta_0}(1-t)^{\Delta_\infty+\Delta_0+\Delta_{2,1}-\Delta_t-\Delta_1}(z-t)^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_0\\ \alpha_t\end{matrix}\,\alpha\,{}^{\alpha_\infty}\,\alpha_{1\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_1\end{matrix};t,\frac{z-1}{z-t}\right)=$$
$$=\sum_{\theta'}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha_\infty;b\alpha)t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha\,{}^{\alpha_1}\,\alpha_{\infty\theta'}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\frac{1}{z}\right). \tag{3.3.20}$$

Note that combining all the previous formulae we manage to analytically continue the expansion in $z\sim 0$ of the conformal block in all the complex plane for $t\ll 1$. It is straightforward to generalize the previous formulae for $t\sim 1, t\sim\infty$. All in all, for any value of $t$ we can connect all the possible expansions in $z$. The analytic continuation in the $t-$plane is more involved and can be done via the fusion kernel. As a concluding remark, note that there is a Möbius tranformation in each region of expansions of the correlator, say $z\ll t\ll 1$ for reference, that only exchanges $\alpha_\infty$ and $\alpha_1$ and that does not change the region of validity of the expansion. This transformation is usually called *braiding*. This gives, up to a Jacobian,



$$=\mathfrak{F}\left(\begin{matrix}\alpha_\infty\\ \alpha_1\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};\frac{t}{t-1},\frac{z}{t}\frac{t-1}{z-1}\right). \tag{3.3.21}$$

Braiding changes the expansion variables in the conformal blocks according to the new positions of the insertions and as such can be used to generate other expansions and the related connection coefficients.

### Semiclassical limit

Let us consider the semiclassical limit of Liouville theory, that is the double scaling limit

$$b\to 0,\ \alpha_i\to\infty,\ b\alpha_i=a_i \text{ finite}. \tag{3.3.22}$$

In this limit the conformal blocks and the corresponding BPZ equation greatly simplify. The divergence exponentiates and the $z$ dependence becomes subleading, namely[4]

$$\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)=t^{\Delta-\Delta_t-\Delta_{0\theta}}z^{\frac{bQ}{2}+\theta b\alpha_0}\exp\left[\frac{1}{b^2}\left(F(t)+b^2W(z/t,t)+\mathcal{O}(b^4)\right)\right]. \tag{3.3.23}$$

Here $F(t)$ is the classical conformal block, related to the conformal block without degenerate insertion via

$$\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};t\right)=t^{\Delta-\Delta_t-\Delta_0}e^{b^{-2}\left(F(t)+\mathcal{O}(b^2)\right)}. \tag{3.3.24}$$

The divergences in the conformal blocks can be cured by dividing by the conformal block without the degenerate insertion. We denote the resulting finite, semiclassical conformal block by the letter $\mathcal{F}$:

$$\mathcal{F}\left(\begin{matrix}a_1\\ a_\infty\end{matrix}\,a\,{}^{a_t}\,a_{0\theta}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};t,\tfrac{z}{t}\right)=\lim_{b\to 0}\frac{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};t\right)}=t^{-\theta a_0}z^{\frac{1}{2}+\theta a_0}e^{-\frac{\theta}{2}\partial_{a_0}F(t)}\left(1+\mathcal{O}(t,z/t)\right). \tag{3.3.25}$$

---

[4] Here and in the following we do not indicate the dependence of $F$ and $W$ on the rescaled momenta.

Note that the conformal block with the degenerate insertion and $z, t \sim 0$ contains a classical conformal block depending on $a_{0\theta} = a_0 - \theta\frac{b^2}{2}$. Dividing by the four-point function without the degenerate insertion, which depends on $a_0$, gives an incremental ratio that in the limit (3.3.22) becomes the derivative $\partial_{a_0} F\left(t\right)$. The BPZ equation (3.3.1) simplifies in the semiclassical limit as well. The $t-$derivative acting on the conformal block gives

$$t\partial_t\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\tfrac{z}{t}\right) = b^{-2}\left(-\tfrac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F(a_i, a, t) + \mathcal{O}(b^2)\right)\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\tfrac{z}{t}\right), \tag{3.3.26}$$

therefore the $t-$derivative becomes a multiplication by a $z$-independent factor at leading order in $b^2$ and the BPZ equation becomes an ODE. Defining

$$u^{(0)} = \lim_{b\to 0} b^2 t\partial_t \log\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};t\right), \tag{3.3.27}$$

where the superscript indicates that the block is expanded for $t \sim 0$, the BPZ equation (3.3.1) in the semiclassical limit reads

$$\left(\partial_z^2 + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} - \frac{\frac{1}{2} - a_1^2 - a_t^2 - a_0^2 + a_\infty^2 + u^{(0)}}{z(z-1)} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} + \frac{u^{(0)}}{z(z-t)} + \frac{\frac{1}{4} - a_0^2}{z^2}\right)\mathcal{F}\left(\begin{matrix}a_1\\ a_\infty\end{matrix}\,a\,{}^{a_t}\,a_{0\theta}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};t,\frac{z}{t}\right) = 0\,. \tag{3.3.28}$$

The solution of the previous ODE for $z \sim t$ is given by the semiclassical block

$$\begin{aligned}(t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_1\\ a_\infty\end{matrix}\,a\,{}^{a_0}\,a_{t\theta}\,\begin{matrix}a_{2,1}\\ a_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right) &= \lim_{b\to 0}(t-1)^{-\Delta_{2,1}}\frac{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_0\\ \alpha_t\end{matrix};\frac{t}{t-1}\right)} = \\ = \lim_{b\to 0}\frac{e^{i\pi(\Delta-\Delta_0-\Delta_{2,1}-\Delta_t)}(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};t\right)}&,\end{aligned} \tag{3.3.29}$$

therefore the connection formula (3.3.8) descends to the semiclassical blocks to be

$$\mathcal{F}\left(\begin{matrix}a_1\\ a_\infty\end{matrix}\,a\,{}^{a_t}\,a_{0\theta}\,\begin{matrix}a_{,2,1}\\ a_0\end{matrix};t,\frac{z}{t}\right) = \sum_{\theta'}\mathcal{M}_{\theta\theta'}(a_0, a_t; a)(t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_1\\ a_\infty\end{matrix}\,a\,{}^{a_0}\,a_{t\theta'}\,\begin{matrix}a_{2,1}\\ a_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right). \tag{3.3.30}$$

Note that the intermediate momentum $a$ can be computed as a function of the parameters appearing in the semiclassical BPZ equation inverting the relation (3.3.27). Similarly, keeping $t \sim 0$ we can analytically continue the solution to the other singularities, that is for $z \sim 1$ and $z \sim \infty$. In particular, we can directly connect $z \sim 0$ and $z \sim \infty$ passing though the intermediate region. The semiclassical block for $z \sim \infty$ reads

$$\begin{aligned}&t^{-\frac{1}{2}}z\mathcal{F}\left(\begin{matrix}a_t\\ a_0\end{matrix}\,a\,{}^{a_1}\,a_{\infty\theta}\,\begin{matrix}a_{2,1}\\ a_\infty\end{matrix};t,\frac{1}{z}\right) = \\ &= \lim_{b\to 0}\frac{t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha\,{}^{\alpha_1}\,\alpha_{\infty\theta'}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\frac{1}{z}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \alpha_0\end{matrix};t\right)} = \lim_{b\to 0}\frac{t^{\Delta_{2,1}}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha\,{}^{\alpha_1}\,\alpha_{\infty\theta'}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\frac{1}{z}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha\,\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix};t\right)}\,.\end{aligned} \tag{3.3.31}$$

The connection formula (3.3.17) from $z \sim 0$ to $z \sim \infty$ involves a conformal block with two shifted momenta, that is

$$\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha_{\theta'}\,{}^{\alpha_1}\,\alpha_{\infty\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\tfrac{1}{z}\right) = t^{\Delta_{\theta'}-\Delta_1-\Delta_{\infty\theta}}\left(\tfrac{t}{z}\right)^{\frac{bQ}{2}+\theta b\alpha_\infty}\exp\left[\tfrac{1}{b^2}F\left(a - \theta'\tfrac{b^2}{2}, t\right) + W\left(a - \theta'\tfrac{b^2}{2}, t\right) + \mathcal{O}(b^2)\right]. \tag{3.3.32}$$

At first order in $b^2$

$$F\left(a-\theta'\frac{b^2}{2},t\right)+b^2W\left(a-\theta'\frac{b^2}{2},t\right)=F(a,t)-\frac{\theta' b^2}{2}\partial_aF(a,t)+b^2W\left(a,t\right)+\mathcal{O}(b^4)\,, \quad (3.3.33)$$

therefore in the semiclassical limit

$$\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ & \alpha_{\theta'} & & \alpha_{\infty\theta} & \\ \alpha_0 & & & & \alpha_\infty\end{matrix};t,\frac{1}{z}\right)\sim t^{-\theta'\alpha}e^{-\frac{\theta'}{2}\partial_aF(t)}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ & \alpha & & \alpha_{\infty\theta} & \\ \alpha_0 & & & & \alpha_\infty\end{matrix};t,\frac{1}{z}\right)\,, \quad \text{as } b\to 0\,. \quad (3.3.34)$$

This is consistent with the fact that we expect only two linearly independent $z$ behaviors. The connection formula (3.3.17) simplifies to

$$\mathcal{F}\left(\begin{matrix}a_1 & & a_t & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0\end{matrix};t,\frac{z}{t}\right)=$$
$$=\sum_{\theta'}\left(\sum_\sigma\mathcal{M}_{\theta\sigma}(a_0,a;a_t)\mathcal{M}_{(-\sigma)\theta'}(a,a_\infty;a_1)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_aF}\right)t^{-\frac{1}{2}}z\mathcal{F}\left(\begin{matrix}a_t & & a_1 & & a_{2,1} \\ & a & & a_{\infty\theta'} & \\ a_0 & & & & a_\infty\end{matrix};t,\frac{1}{z}\right)\,. \quad (3.3.35)$$

Explicitly, the connection coefficients are

$$\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_0,a;a_t)\mathcal{M}_{(-\sigma)\theta'}(a,a_\infty;a_1)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_aF}=$$
$$=\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_0)\Gamma(-2\theta' a_\infty)t^{-\sigma a}e^{-\frac{\sigma}{2}\partial_aF}}{\Gamma\left(\frac{1}{2}+\theta a_0-\sigma a+a_t\right)\Gamma\left(\frac{1}{2}+\theta a_0-\sigma a-a_t\right)\Gamma\left(\frac{1}{2}-\sigma a-\theta' a_\infty+a_1\right)\Gamma\left(\frac{1}{2}-\sigma a-\theta' a_\infty-a_1\right)}\,. \quad (3.3.36)$$

For future reference, the semiclassical block for small $t$ and $z\sim 1$ is given by

$$(t(1-t))^{-\frac{1}{2}}(t-z)\mathcal{F}\left(\begin{matrix}a_0 & & a_\infty & & a_{2,1} \\ & a & & a_{1\theta} & \\ a_t & & & & a_1\end{matrix};t,\tfrac{1-z}{t-z}\right)=\lim_{b\to 0}(t(1-t))^{\Delta_{2,1}}(t-z)^{-2\Delta_{2,1}}\frac{\mathfrak{F}\left(\begin{matrix}\alpha_0 & & \alpha_\infty & & \alpha_{2,1} \\ & \alpha & & \alpha_{1\theta} & \\ \alpha_t & & & & \alpha_1\end{matrix};t,\tfrac{1-z}{t-z}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_0 & & \alpha_\infty \\ & \alpha & \\ \alpha_t & & \alpha_1\end{matrix};t\right)}\,. \quad (3.3.37)$$

Similarly one can obtain the connection coefficients for the other $t-$expansions. As an example, let us schematically consider the case $t\gg 1$. The semiclassical block for $z\sim 0$ reads

$$t^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0\end{matrix};\frac{1}{t},z\right)=\lim_{b\to 0}\frac{t^{-\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 & & \alpha_{2,1} \\ & \alpha & & \alpha_{0\theta} & \\ \alpha_\infty & & & & \alpha_0\end{matrix};\frac{1}{t},z\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 \\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{matrix};\frac{1}{t}\right)}\,. \quad (3.3.38)$$

Still the $t-$derivative decouples, leaving behind

$$u^{(\infty)}=\lim_{b\to 0}b^2t\partial_t\log t^{\Delta-\Delta_t-\Delta_1-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1 \\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{matrix};\frac{1}{t}\right)\,. \quad (3.3.39)$$

Note that the semiclassical BPZ equation formally remains the same, with the substitution[5] of $u^{(0)}$ with $u^{(\infty)}$. Indeed, the intermediate momentum $\alpha$ is now determined in terms of $u^{(\infty)}$. The $z\sim 1$ expansion gives

$$(t-1)^{\frac{1}{2}}e^{i\theta\pi a}\mathcal{F}\left(\begin{matrix}a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0\end{matrix};\tfrac{1}{t-1},1-z\right)=\lim_{b\to 0}(t-1)^{-\Delta_{2,1}}e^{i\theta\pi b\alpha}\frac{\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_0 & & \alpha_{2,1} \\ & \alpha & & \alpha_{1\theta} & \\ \alpha_\infty & & & & \alpha_1\end{matrix};\tfrac{1}{1-t},1-z\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_0 \\ & \alpha & \\ \alpha_\infty & & \alpha_1\end{matrix};\tfrac{1}{1-t}\right)}\,, \quad (3.3.40)$$

---

[5] From the gauge theory viewpoint this amounts to a change of frame from the electric to the monopole one.

and the corresponding connection formula reads

$$t^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} & \\ & a & & a_{0\theta} & & ;\frac{1}{t},z \\ a_\infty & & & & a_0 & \end{matrix}\right) = \sum_{\theta'=\pm1}\mathcal{M}_{\theta\theta'}\left(a_0,a_1;a\right)(t-1)^{\frac{1}{2}}e^{i\theta\pi a}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} & \\ & a & & a_{0\theta'} & & ;\frac{1}{t-1},1-z \\ a_\infty & & & & a_0 & \end{matrix}\right). \tag{3.3.41}$$

All other connection formulae at $t \gg 1$ can be obtained similarly. The same can be done when $t \sim 1$. Note that again the semiclassical BPZ equation looks formally as (3.3.28) upon the substitution[6] of $u^{(0)}$ with

$$u^{(1)} = \lim_{b\to0} b^2 t\partial_t \log \mathfrak{F}\left(\begin{matrix} \alpha_0 & & \alpha_t & \\ & \alpha & & ;1-t \\ \alpha_\infty & & \alpha_1 & \end{matrix}\right). \tag{3.3.42}$$

### 3.3.2 Confluent conformal blocks

**General case**

Consider the correlation function

$$\langle\mu,\Lambda|V_1(1)\Phi(z)|\Delta_0\rangle\,. \tag{3.3.43}$$

It solves the BPZ equation

$$\left(b^{-2}\partial_z^2-\left(\frac{1}{z}+\frac{1}{z-1}\right)\partial_z+\frac{\Lambda\partial_\Lambda-\Delta_{2,1}-\Delta_1-\Delta_0}{z(z-1)}+\frac{\Delta_1}{(z-1)^2}+\frac{\Delta_0}{z^2}+\frac{\mu\Lambda}{z}-\frac{\Lambda^2}{4}\right)\langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle=0\,, \tag{3.3.44}$$

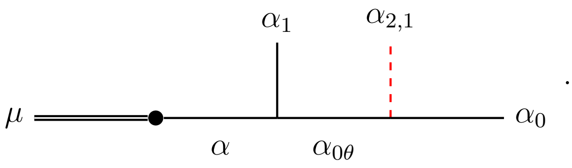
and can be decomposed into *confluent* conformal blocks in different ways. They are all given as collision limits of *regular* conformal blocks.

**Small $\Lambda$ blocks** We focus first on the case where the conformal blocks are given as an expansion in $\Lambda$. The block for $z \sim 0$ is defined as[7]

$${}_1\mathfrak{F}\left(\begin{matrix} & & \alpha_1 & & \alpha_{2,1} & \\ \mu & \alpha & & \alpha_{0\theta} & & ;\Lambda,z \\ & & & & \alpha_0 & \end{matrix}\right) = \Lambda^{\Delta}z^{\frac{bQ}{2}+\theta b\alpha_0}\,{}_1\widetilde{\mathfrak{F}}\left(\begin{matrix} & & \alpha_1 & & \alpha_{2,1} & \\ \mu & \alpha & & \alpha_{0\theta} & & ;\Lambda,z \\ & & & & \alpha_0 & \end{matrix}\right) = \Lambda^{\Delta}z^{\frac{bQ}{2}+\theta b\alpha_0}\lim_{\eta\to\infty}\widetilde{\mathfrak{F}}\left(\begin{matrix} \frac{\eta-\mu}{2} & & \alpha_1 & & \alpha_{2,1} & \\ & \alpha & & \alpha_{0\theta} & & ;\frac{\Lambda}{\eta},z \\ \frac{\eta+\mu}{2} & & & & \alpha_0 & \end{matrix}\right). \tag{3.3.45}$$

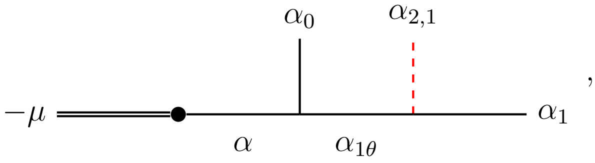
This is nothing but the standard collision limit of $\langle\Delta_\infty|$ and $V_t(t)$ as defined in (3.2.11). The tilde on the conformal block means it has no classical part, i.e. is normalized such that the first term is 1. This conformal block can also be computed directly by doing the OPE of $\Phi(z)$ with $|\Delta_0\rangle$, then the OPE of $V_1(1)$ with the result which we specify to be in the Verma module $\Delta_\alpha$, and then contracting with $\langle\mu,\Lambda|$. In the diagrammatic notation introduced in section 3.2.2, we represent it by

$${}_1\mathfrak{F}\left(\begin{matrix} & & \alpha_1 & & \alpha_{2,1} & \\ \mu & \alpha & & \alpha_{0\theta} & & ;\Lambda,z \\ & & & & \alpha_0 & \end{matrix}\right) = \tag{3.3.46}$$



The double line represents the rank 1 irregular state, and the dot the pairing with a primary state. For $z \sim 1$, the corresponding block can be expressed as

$$e^{\mu\Lambda}{}_1\mathfrak{F}\left(\begin{matrix} & & \alpha_0 & & \alpha_{2,1} & \\ -\mu & \alpha & & \alpha_{1\theta} & & ;\Lambda,1-z \\ & & & & \alpha_1 & \end{matrix}\right) = \tag{3.3.47}$$



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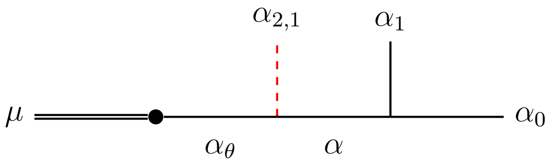
[6] This is the dyon frame.

[7] The argument $\frac{\eta+\mu}{2}$ should appear with a minus sign as in Appendix A.2. Here and in the following we don't write it due to the symmetry of the conformal block. The reader wishing to compare with the Nekrasov partition function should take this sign into account as in Appendix C.

where the exponential factor and the argument $-\mu$ arise from the corresponding Möbius transformation[8]. In the intermediate region, where $z \gg 1$ but $\Lambda z \ll 1$, the corresponding block is

$$z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right) =$$



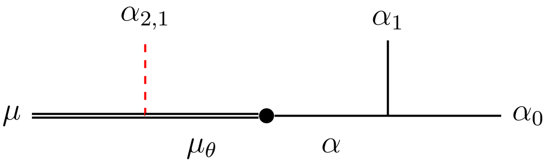
. (3.3.48)

In the deep irregular region where $z \gg 1$ and $\Lambda z \gg 1$, the conformal block is given by a different collision limit, proposed in [106]:

$$\begin{aligned}{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right) &= e^{\theta b\Lambda z/2}\Lambda^{\Delta_{2,1}+\Delta}\,(\Lambda z)^{-\theta b\mu+\frac{b^2}{2}}\times \\ \times \lim_{\eta\to\infty}\left(1-\frac{\eta}{\Lambda z}\right)^{-\frac{bQ}{2}-\theta\frac{b}{2}(\mu-\eta)} &\widetilde{\mathfrak{F}}\left({}^{\alpha_1}_{\alpha_0}\ \alpha\ {}^{\frac{\mu-\eta}{2}}\ \tfrac{\mu+\eta-\theta b}{2}\ {}^{\alpha_{2,1}}_{\frac{\mu+\eta}{2}};\frac{\Lambda}{\eta},\frac{\eta}{\Lambda z}\right).\end{aligned} \tag{3.3.49}$$

Whenever $z$ approaches an irregular singularity of rank 1, we denote the corresponding conformal block by $\mathfrak{D}$. This conformal block can also be computed directly by doing the OPE between $\langle\mu,\Lambda|$ and $\Phi(z)$, then the OPE of the result with $V_1(1)$ and contracting with $|\Delta_0\rangle$. Diagramatically, we write

$${}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right) =$$

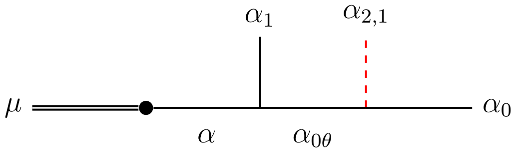


. (3.3.50)

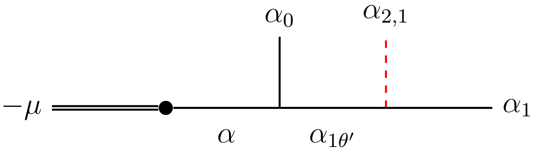
The connection problem between 0 and 1 is solved in the same way as for the regular conformal blocks, since we are never near the irregular singularity. The result is

$${}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_1;b\alpha)e^{\mu\Lambda}\,{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta'}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right). \tag{3.3.51}$$

Diagrammatically:



$$= \textstyle\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}$$



. (3.3.52)

Instead, to solve the connection problem between 1 and $\infty$ one has to do two steps: from 1 to the intermediate region, and then to $\infty$. At each step we decompose the correlator into conformal blocks in the different regions and then use crossing symmetry to determine the connection coefficients. The relevant formulae for the irregular state are reviewed in Appendix B.1. We have

$$\begin{aligned}\langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C_{\mu\alpha}\sum_{\theta=\pm}C^{\alpha_{1\theta}}_{\alpha_{2,1}\alpha_1}C^{\alpha}_{\alpha_{1\theta}\alpha_0}\left|e^{\mu\Lambda}\,{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)\right|^2 = \\ &= \int d\alpha\, C_{\mu\alpha}\sum_{\theta'=\pm}C^{\alpha}_{\alpha_{2,1}\alpha_{\theta'}}C^{\alpha_{\theta'}}_{\alpha_1\alpha_0}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right)\right|^2.\end{aligned} \tag{3.3.53}$$

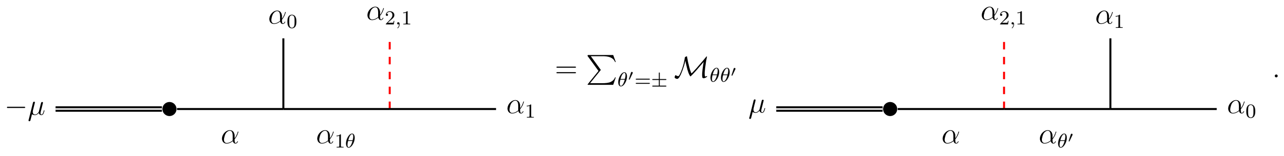
---

[8] Actually, doing the Möbius transformation one gets $-\Lambda$ but since the block depends only on $\mu\Lambda$ and $\Lambda^2$ except for the classical part, one can trade $-\Lambda$ for $-\mu$.

We recognize this condition from the hypergeometric function (3.2.5). Therefore we can readily solve it in terms of the hypergeometric connection coefficients $\mathcal{M}$ and the connection formula between 0 and the intermediate region is then

$$e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\ \alpha\,{}^{\alpha_0}\,\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)=\textstyle\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha;b\alpha_0)z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha\,{}^{\alpha_{2,1}}\,\alpha_{\theta'}\,{}^{\alpha_1}_{\alpha_0};\Lambda z,\tfrac{1}{z}\right).\tag{3.3.54}$$

Diagrammatically:



(3.3.55)

If one decomposes the correlator into conformal blocks in the intermediate region and near $\infty$, one obtains the crossing symmetry condition

$$\begin{aligned}\langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle&=\int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0}\sum_{\theta=\pm}C_{\mu\alpha_\theta}C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\,{}^{\alpha_{2,1}}\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right)\right|^2=\\&=\int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0}\sum_{\theta'=\pm}C_{\mu_{\theta'}\alpha}B^{\mu_{\theta'}}_{\alpha_{2,1}\mu}\left|{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right)\right|^2.\end{aligned}\tag{3.3.56}$$

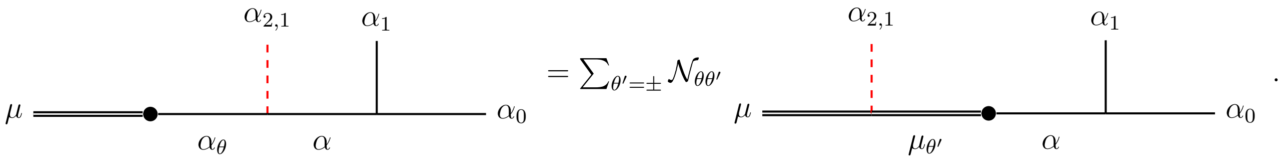
This condition is analogous to the one we found for the Whittaker functions (3.2.21) so that the connection formula between the intermediate region and $\infty$ reads

$$b^{\theta b\alpha}z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\,{}^{\alpha_{2,1}}\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda z,\tfrac{1}{z}\right)=\textstyle\sum_{\theta'=\pm}b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu){}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda,\tfrac{1}{\Lambda z}\right)\tag{3.3.57}$$

with irregular connection coefficients as in (B.1.18):

$$\mathcal{N}_{\theta\theta'}(b\alpha,b\mu)=\frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}.\tag{3.3.58}$$

In diagrams:

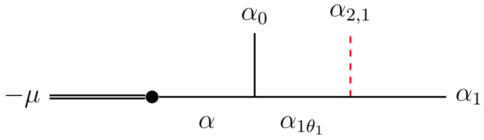


(3.3.59)

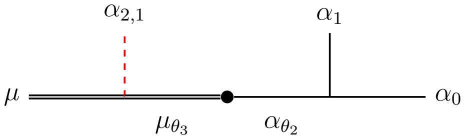
Let us write explicitly the more interesting connection formula between 1 and $\infty$, which is obtained by concatenating the two connection formulae above. Since the $\mathfrak{F}$ block in the intermediate region has different arguments in formula (3.3.54) and (3.3.57), we need to rename some of them. In the end we obtain the following connection formula from 1 directly to $\infty$:

$$\begin{aligned}&e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\ \alpha\,{}^{\alpha_0}\,\alpha_{1\theta_1}\,{}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)=\\&=\sum_{\theta_2,\theta_3=\pm}b^{-\frac{1}{2}+\theta_2 b\alpha_{\theta_2}-\theta_3 b\mu}\mathcal{M}_{\theta_1\theta_2}(b\alpha_1,b\alpha;b\alpha_0)\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2},b\mu){}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta_3}\ \alpha_{\theta_2}\,{}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right).\end{aligned}\tag{3.3.60}$$

Again, in diagrams this is represented by:



$$= \sum_{\theta_2,\theta_3=\pm} \mathcal{M}_{\theta_1\theta_2}\mathcal{N}_{(-\theta_2)\theta_3}$$

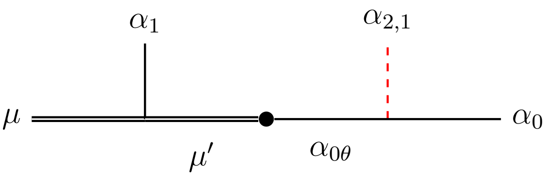


, (3.3.61)

where we have suppressed the arguments of the connection coefficients for brevity.

**Large $\Lambda$ blocks** The conformal blocks considered up to now are expansions in $\Lambda$. One can however play the same game using expansions in $\frac{1}{\Lambda}$. For example, for large $\Lambda$ and for $z \sim 0$, we have

$${}_1\mathfrak{D}\left(\mu \; {}^{\alpha_1} \; \mu' \; \alpha_{0\theta} \; \begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix}; \frac{1}{\Lambda}, \Lambda z\right) = \tag{3.3.62}$$



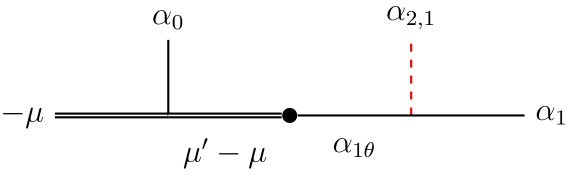
.

One can compute it via OPE as in (B.1.1) or as a collision limit of a regular conformal block as proposed in [106]:

$$\begin{aligned}{}_1\mathfrak{D}\left(\mu \; {}^{\alpha_1} \; \mu' \; \alpha_{0\theta} \; \begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix}; \frac{1}{\Lambda}, \Lambda z\right) =& e^{-(\mu'-\mu)\Lambda}\Lambda^{\Delta_{0\theta}+2\mu'(\mu'-\mu)} z^{\frac{bQ}{2}+\theta b\alpha_0} \times \\ &\times \lim_{\eta\to\infty}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} \widetilde{\mathfrak{F}}\left(\begin{matrix}\alpha_1\\ \frac{\eta+\mu}{2}\end{matrix} \; \frac{\eta-\mu}{2}+\mu' \; {}^{\frac{\eta-\mu}{2}} \; \alpha_{0\theta} \; \begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix}; \frac{\eta}{\Lambda}, \frac{\Lambda z}{\eta}\right).\end{aligned} \tag{3.3.63}$$

Similarly, we have a conformal block for large $\Lambda$ and $z \sim 1$, which as usual we can write in the same form as the one for $z \sim 0$ by doing a Möbius transformation:

$$e^{\mu\Lambda}{}_1\mathfrak{D}\left(-\mu \; {}^{\alpha_0} \; \mu'-\mu \; \alpha_{1\theta} \; \begin{matrix}\alpha_{2,1}\\ \alpha_1\end{matrix}; \frac{1}{\Lambda}, \Lambda(1-z)\right) =$$



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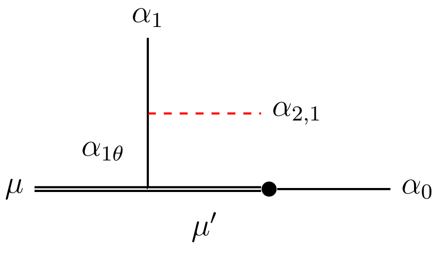


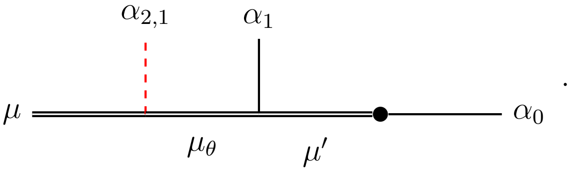
. (3.3.64)

The first line of (3.3.64) is the diagrammatic representation of the conformal block, while the second line is an equality of two a priori seemingly different conformal blocks, which can be explicitly checked order by order . This is consistent with the fact that the corresponding DOZZ factors are equal:

$$B^{\mu'-\mu}_{-\mu\alpha_0}C_{\mu'-\mu,\alpha_{1\theta}} = B^{\mu'}_{\mu\alpha_{1\theta}}C_{\mu',\alpha_0}\,, \tag{3.3.65}$$

as can easily be proven by using their explicit expressions given in Appendix A.2. The most exotic block is the one for large $\Lambda$ and large $z$, which by a slight abuse of notation we still denote by $\mathfrak{D}$:

$$ {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ {}^{\alpha_1}\ \mu'\ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right) = \tag{3.3.66}$$



This block is fully irregular in the sense that to calculate it, we have to perform two irregular OPEs as indicated by the diagram. It is more convenient to calculate it as a collision limit of a regular block:

$$\begin{aligned} &{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ {}^{\alpha_1}\ \mu'\ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right) = e^{\theta b\Lambda z/2}\Lambda^{\Delta_{2,1}}\left(\Lambda z\right)^{-\theta b\mu+\frac{b^2}{2}} e^{-(\mu'-\mu_\theta)\Lambda}\Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu_\theta)}\times \\ &\times \lim_{\eta\to\infty}\left(1-\frac{\eta}{\Lambda z}\right)^{\Delta_{2,1}-(\mu_\theta-\mu)(\eta-\mu_\theta)}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu_\theta)(\eta-\mu')-(\mu'-\mu_\theta)(\mu_\theta-\mu)}\widetilde{\mathfrak{F}}\left(\genfrac{}{}{0pt}{}{\alpha_{2,1}}{\frac{\eta+\mu}{2}}\ {}_{\frac{\eta-\mu}{2}+\mu_\theta}\ {}^{\alpha_1}\ {}_{\frac{\eta-\mu}{2}+\mu'}\ \genfrac{}{}{0pt}{}{\frac{\eta-\mu}{2}}{\alpha_0}; \frac{\eta}{\Lambda}, \frac{\Lambda z}{\eta}\right). \end{aligned} \tag{3.3.67}$$

Having defined all the necessary conformal blocks we now derive their connection formulae. Let us start by connecting $z \sim 1$ with $\infty$. Expanding the correlator in these regions, we get the crossing symmetry condition

$$\begin{aligned} \langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle &= \int \mathrm{d}\mu' \sum_{\theta=\pm} B^{\mu'-\mu}_{-\mu\alpha_0} C_{\mu'-\mu,\alpha_{1\theta}} C^{\alpha_{1\theta}}_{\alpha_1\alpha_{2,1}} \left| e^{\mu\Lambda}\, {}_1\mathfrak{D}\left(-\mu\ {}^{\alpha_0}\ \mu'-\mu\ \alpha_{1\theta}\ \genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\right)\right|^2 = \\ &= \int \mathrm{d}\mu' \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'}_{\mu_{\theta'}\alpha_1} C_{\mu',\alpha_0} \left| {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ {}^{\alpha_1}\ \mu'\ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 . \end{aligned} \tag{3.3.68}$$

Using the following remarkable identity, which can easily be proven using the explicit expression of the structure functions given in Appendix A.2,

$$B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'}_{\mu_{\theta'}\alpha_1} C_{\mu'\alpha_0} = B^{\mu'-\mu}_{-\mu\alpha_0} B^{\mu'-\mu_{\theta'}}_{\mu'-\mu,\alpha_{2,1}} C_{\mu'-\mu_{\theta'},\alpha_1}\,, \tag{3.3.69}$$

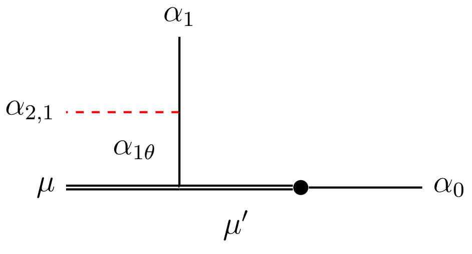
we find that the above crossing symmetry condition (after relabelling the dummy variable $\theta' \to -\theta'$) becomes:

$$\begin{aligned} \langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle &= \int \mathrm{d}\mu' B^{\mu'-\mu}_{-\mu\alpha_0} \sum_{\theta=\pm} C_{\mu'-\mu,\alpha_{1\theta}} C^{\alpha_{1\theta}}_{\alpha_1\alpha_{2,1}} \left| e^{\mu\Lambda}\, {}_1\mathfrak{D}\left(-\mu\ {}^{\alpha_0}\ \mu'-\mu\ \alpha_{1\theta}\ \genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\right)\right|^2 = \\ &= \int \mathrm{d}\mu' B^{\mu'-\mu}_{-\mu\alpha_0} \sum_{\theta'=\pm} B^{\mu'_{\theta'}-\mu}_{\mu'-\mu,\alpha_{2,1}} C_{\mu'_{\theta'}-\mu,\alpha_1} \left| {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{-\theta'}\ {}^{\alpha_1}\ \mu'\ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 . \end{aligned} \tag{3.3.70}$$

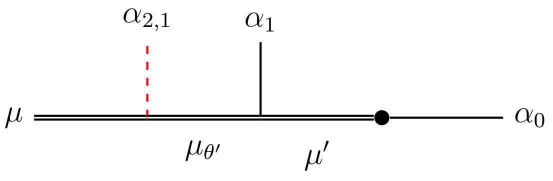
We recognize this constraint from the Whittaker functions (3.2.24), and can readily write the connection formula from 1 to $\infty$:

$$b^{\theta b\alpha_1} e^{\mu\Lambda}{}_1\mathfrak{D}\left(-\mu\ {}^{\alpha_0}\ \mu'-\mu\ \alpha_{1\theta}\ \genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha_1}; \tfrac{1}{\Lambda}, \Lambda(1-z)\right) = \textstyle\sum_{\theta'} b^{-\frac{1}{2}+\theta' b(\mu'-\mu)} \mathcal{N}_{\theta(-\theta')}(b\alpha_1, b\mu'-b\mu)\,{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ {}^{\alpha_1}\ \mu'\ \alpha_0; \tfrac{1}{\Lambda}, \tfrac{1}{z}\right), \tag{3.3.71}$$

where $\mathcal{N}$ are the connection coefficients for the Whittaker functions (3.2.24). Diagrammatically this is clear:



$$= \textstyle\sum_{\theta'=\pm} \mathcal{N}_{\theta(-\theta')}$$



.

(3.3.72)

To connect 0 and $\infty$ we expand the correlator in the relevant regions. By crossing symmetry we have:

$$\begin{aligned}\langle \mu, \Lambda | V_1(1) \Phi(z) | \Delta_0 \rangle &= \int \mathrm{d}\mu' \sum_{\theta=\pm} B^{\mu'}_{\mu\alpha_1} C_{\mu'\alpha_{0\theta}} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} \left| {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_1} \ \mu' \ \alpha_{0\theta} \ \substack{\alpha_{2,1} \\ \alpha_0}; \frac{1}{\Lambda}, \Lambda z \right) \right|^2 = \\ &= \int \mathrm{d}\mu' \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'_{\theta'}}_{\mu_{\theta'}\alpha_1} C_{\mu'_{\theta'},\alpha_0} \left| {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_{2,1}} \ \mu_{\theta'} \ {}^{\alpha_1} \ \mu'_{\theta'} \ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z} \right) \right|^2 ,\end{aligned} \tag{3.3.73}$$

for later convenience we have labelled the intermediate channel in the second line by $\mu'_{\theta'}$ instead of $\mu'$. By using an identity similar to (3.3.69):

$$B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'_{\theta'}}_{\mu_{\theta'}\alpha_1} C_{\mu'_{\theta'},\alpha_0} = B^{\mu'}_{\mu\alpha_1} B^{\mu'_{\theta'}}_{\mu'\alpha_{2,1}} C_{\mu'_{\theta'}\alpha_0} , \tag{3.3.74}$$

the above crossing symmetry equation then becomes:

$$\begin{aligned}\langle \mu, \Lambda | V_1(1) \Phi(z) | \Delta_0 \rangle &= \int \mathrm{d}\mu' B^{\mu'}_{\mu\alpha_1} \sum_{\theta=\pm} C_{\mu'\alpha_{0\theta}} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} \left| {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_1} \ \mu' \ \alpha_{0\theta} \ \substack{\alpha_{2,1} \\ \alpha_0}; \frac{1}{\Lambda}, \Lambda z \right) \right|^2 = \\ &= \int \mathrm{d}\mu' B^{\mu'}_{\mu\alpha_1} \sum_{\theta'=\pm} B^{\mu'_{\theta'}}_{\mu'\alpha_{2,1}} C_{\mu'_{\theta'}\alpha_0} \left| {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_{2,1}} \ \mu_{\theta'} \ {}^{\alpha_1} \ \mu'_{\theta'} \ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z} \right) \right|^2 .\end{aligned} \tag{3.3.75}$$

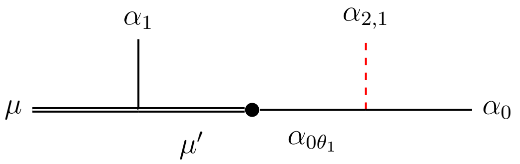
We recognize this constraint from the Whittaker functions (3.2.21) and can readily write the connection formula from 0 to $\infty$:

$$b^{\theta b\alpha_0} {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_1} \ \mu' \ \alpha_{0\theta} \ \substack{\alpha_{2,1} \\ \alpha_0}; \tfrac{1}{\Lambda}, \Lambda z \right) = \textstyle\sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu'} \mathcal{N}_{\theta\theta'}(b\alpha_0, b\mu') {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_{2,1}} \ \mu_{\theta'} \ {}^{\alpha_1} \ \mu'_{\theta'} \ \alpha_0; \tfrac{1}{\Lambda}, \tfrac{1}{z} \right) . \tag{3.3.76}$$

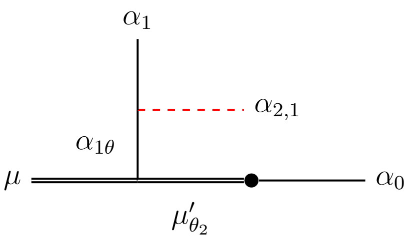
Combining (3.3.76) with the inverse of (3.3.71) we obtain the connection formula from 0 to 1:

$$\begin{aligned}&b^{\theta_1 b\alpha_0} {}_1\mathfrak{D}\left( \mu \ {}^{\alpha_1} \ \mu' \ \alpha_{0\theta_1} \ \substack{\alpha_{2,1} \\ \alpha_0}; \frac{1}{\Lambda}, \Lambda z \right) = \\ &= \sum_{\theta_2,\theta_3=\pm} b^{-\frac{1}{2}-\theta_2 b\mu'} \mathcal{N}_{\theta_1\theta_2}(b\alpha_0, b\mu') b^{\frac{1}{2}-\theta_2 b(\mu'_{\theta_2}-\mu)+\theta_3 b\alpha_1} \mathcal{N}^{-1}_{(-\theta_2)\theta_3}(b\mu'_{\theta_2} - b\mu, b\alpha_1) e^{\mu\Lambda} {}_1\mathfrak{D}\left( -\mu \ {}^{\alpha_0} \ \mu'_{\theta_2} - \mu \ \alpha_{1\theta_3} \ \substack{\alpha_{2,1} \\ \alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z) \right) .\end{aligned} \tag{3.3.77}$$

Diagrammatically:



$$= \textstyle\sum_{\theta_2,\theta_3=\pm} \mathcal{N}_{\theta_1\theta_2} \mathcal{N}^{-1}_{(-\theta_2)\theta_3}$$



. (3.3.78)

One might expect the existence of conformal blocks expanded in an intermediate region, as was the case for small $\Lambda$. Indeed, in the case of large $\Lambda$ one can define a block expanded in the intermediate region $\frac{1}{\Lambda} \ll z \ll 1$. However, by the identity (3.3.74), this block is actually the same as the block (3.3.66) corresponding to $z \sim \infty$, in the sense that the analytic continuation between the two is trivial. Similarly, one can define another intermediate block in the region $\frac{1}{\Lambda} \ll 1 - z \ll 1$ which is also the same as (3.3.66) by virtue of the identity (3.3.69).

**Semiclassical limit**

In the semiclassical limit $b \to 0$ and $\alpha_i, \mu, \Lambda \to \infty$ such that $a_i = b\alpha_i$, $m = b\mu$, $L = b\Lambda$ are finite. We denote the quantities which are finite in the semiclassical limit by latin letters instead of greek ones.

**Small $L$ blocks** The conformal blocks in this limit are expected to exponentiate, and the $z$-dependence becomes subleading: schematically they take the form

$$\mathfrak{F}(\Lambda, z) \sim e^{\frac{1}{b^2}F(L)+W(L,z)+\mathcal{O}(b^2)}\,, \tag{3.3.79}$$

and they diverge in this limit. The classical conformal block $F(L)$ is related to the conformal block $\mathfrak{F}$ without the degenerate field insertion, i.e.

$$ {}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right) = \Lambda^{\Delta} e^{\frac{1}{b^2}\left(F(L)+\mathcal{O}(b^2)\right)}\,. \tag{3.3.80}$$

Normalizing by this block, we obtain finite semiclassical conformal blocks. Consider for concreteness the block corresponding to the expansion for $z \sim 0$. We define the corresponding (finite) semiclassical conformal block by

$$ {}_1\mathcal{F}\left(m\;a\,{}^{a_1}\;a_{0\theta}\,{}^{a_{2,1}}_{a_0};L,z\right) = \lim_{b\to 0}\frac{{}_1\mathfrak{F}\left(\mu\;\alpha\,{}^{\alpha_1}\;\alpha_{0\theta}\,{}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right)}{{}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right)} = e^{-\frac{\theta}{2}\partial_{a_0}F}z^{\frac{1}{2}+\theta a_0}(1+\mathcal{O}(L,z))\,. \tag{3.3.81}$$

The term $\exp -\frac{\theta}{2}\partial_{a_0}F$ on the RHS of the above equation comes from the fact that the leading behaviour of the numerator is $\exp b^{-2}F(a_{0\theta})$ while the denominator behaves as $\exp b^{-2}F(a_0)$. The fact that the $z$-dependence is subleading means that to leading order, the $\Lambda$-derivative in the BPZ equation (3.3.44) becomes $z$-independent, since we have $\Lambda\partial_\Lambda\mathfrak{F}(\Lambda,z) \sim b^{-2}\Lambda\partial_\Lambda F(\Lambda)\mathfrak{F}(\Lambda,z)$. Then the BPZ equation in the semiclassical limit reduces to an ODE. In particular, multiplying (3.3.44) by $b^2$, this semiclassical conformal block now satisfies the equation

$$\left(\partial_z^2 + \frac{u-\frac{1}{2}+a_0^2+a_1^2}{z(z-1)} + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} + \frac{\frac{1}{4}-a_0^2}{z^2} + \frac{mL}{z} - \frac{L^2}{4}\right){}_1\mathcal{F}\left(m\;a\,{}^{a_1}\;a_{0\theta}\,{}^{a_{2,1}}_{a_0};L,z\right) = 0\,. \tag{3.3.82}$$

We have introduced

$$u = \lim_{b\to 0} b^2\Lambda\partial_\Lambda \log {}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right) = \frac{1}{4} - a^2 + \mathcal{O}(L) \tag{3.3.83}$$

Similarly, we define the semiclassical block for $z \sim 1$ to be

$$\begin{aligned} {}_1\mathcal{F}\left(-m\;a\,{}^{a_0}\;a_{1\theta}\,{}^{a_{2,1}}_{a_1};L,1-z\right) &= \lim_{b\to 0}\frac{e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\;\alpha\,{}^{\alpha_0}\;\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)}{{}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right)} = \\ &= \lim_{b\to 0}\frac{{}_1\mathfrak{F}\left(-\mu\;\alpha\,{}^{\alpha_0}\;\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)}{{}_1\mathfrak{F}\left(-\mu\,\alpha\,{}^{\alpha_0}_{\alpha_1};\Lambda\right)} = e^{-\frac{\theta}{2}\partial_{a_1}F}(1-z)^{\frac{1}{2}+\theta a_1}(1+\mathcal{O}(L,1-z))\,, \end{aligned} \tag{3.3.84}$$

and in the deep irregular region:

$$
{}_1\mathcal{D}\left(m \;{}^{a_{2,1}}\; m_\theta \;\; a \;{}^{a_1}_{a_0}; L, \tfrac{1}{Lz}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta m} \frac{{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;\; \alpha \;{}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right)}{{}_1\mathfrak{F}\left(\mu \;\alpha \;{}^{\alpha_1}_{\alpha_0}; \Lambda\right)} = e^{-\frac{\theta}{2}\partial_m F} e^{\theta Lz/2} L^{-\frac{1}{2}-\theta m} z^{-\theta m}(1+\mathcal{O}(L, 1/Lz))\,. \tag{3.3.85}
$$

The explicit power of $b$ is needed to combine with $\Lambda$ to form $L$. All these blocks satisfy the same equation (3.3.82). Note that in the connection formula (3.3.60) we have four different conformal blocks on the right hand side. Since in the semiclassical limit the BPZ equation becomes a second-order ODE, these four different blocks have to reduce to the two linearly independent solutions near the irregular singular point. They are given by

$$
{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;\; \alpha \;{}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right) = e^{\theta b\Lambda z/2}\Lambda^{\Delta_{2,1}+\Delta}\,(\Lambda z)^{-\theta b\mu + \frac{b^2}{2}}\; e^{\frac{1}{b^2}F(a)+W(a)+\mathcal{O}(b^2)}\,, \tag{3.3.86}
$$

where we have suppressed the dependence of $F$ and $W$ on the other parameters. Instead, in (3.3.60) we have

$$
{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;\; \alpha_{\theta'} \;{}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right) = e^{\theta b\Lambda z/2}\Lambda^{\Delta_{2,1}+\Delta_{\theta'}}\,(\Lambda z)^{-\theta b\mu + \frac{b^2}{2}}\; e^{\frac{1}{b^2}F(a_{\theta'})+W(a_{\theta'})+\mathcal{O}(b^2)}\,. \tag{3.3.87}
$$

Since we are taking the limit $b \to 0$, we can safely substitute $W(a_{\theta'}) \to W(a)$. This is not true for $F(a_{\theta'})$ however, since it multiplies a pole in $b^2$. Instead, in the semiclassical limit we have

$$
{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;\; \alpha_{\theta'} \;{}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right) \sim \Lambda^{\theta' a} e^{-\frac{\theta'}{2}\partial_a F(a)}\,{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;\; \alpha \;{}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right), \quad \text{as}\, b \to 0\,, \tag{3.3.88}
$$

as in (3.3.34). Therefore, we can simplify the connection formula from 1 to ∞ (3.3.60) in the semiclassical limit and state it as

$$
{}_1\mathcal{F}\left(-m \;a\;{}^{a_0}\; a_{1\theta} \;{}^{a_{2,1}}_{a_1}; L, 1-z\right) = \textstyle\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1, a; a_0)\mathcal{N}_{(-\sigma)\theta'}(a, m) L^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}\right) {}_1\mathcal{D}\left(m \;{}^{a_{2,1}}\; m_{\theta'} \;\; a \;{}^{a_1}_{a_0}; L, \tfrac{1}{Lz}\right), \tag{3.3.89}
$$

with connection coefficients

$$
\textstyle\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1, a; a_0)\mathcal{N}_{(-\sigma)\theta'}(a, m) L^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} = \sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_1) e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m-\sigma a\right)} L^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma\left(\frac{1}{2}+\theta a_1 - \sigma a + a_0\right)\Gamma\left(\frac{1}{2}+\theta a_1 - \sigma a - a_0\right)\Gamma\left(\frac{1}{2}-\sigma a - \theta' m\right)}\,. \tag{3.3.90}
$$

Note that all the powers of $b$ appearing in (3.3.60) have been absorbed to give finite quantities.[9] The connection formula from 0 to 1 trivially reduces to the semiclassical one:

$$
{}_1\mathcal{F}\left(m \;a\;{}^{a_1}\; a_{0\theta} \;{}^{a_{2,1}}_{a_0}; L, z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(a_0, a_1; a)\,{}_1\mathcal{F}\left(-m \;a\;{}^{a_0}\; a_{1\theta'} \;{}^{a_{2,1}}_{a_1}; L, 1-z\right)\,. \tag{3.3.91}
$$

**Large $L$ blocks** For the conformal blocks valid for large $\Lambda$, the story is analogous. Taking the semiclassical limit, the conformal blocks are expected to exponentiate and the $z$-dependence becomes subleading. Schematically we have

$$
\mathfrak{D}(\Lambda^{-1}, z) \sim e^{\frac{1}{b^2}F_D(L^{-1}) + W_D(L^{-1}, z) + \mathcal{O}(b^2)}\,. \tag{3.3.92}
$$

---

[9] Note also that the Gamma functions in the denominator precisely correspond to the one-loop factors of the three hypermultiplets of the corresponding AGT dual gauge theory.

Here $F_D$ is the classical conformal block for large [10] $\Lambda$ and is related to the conformal block without the degenerate field insertion, i.e.

$$ {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \, \alpha_0; \frac{1}{\Lambda}\right) = e^{-(\mu'-\mu)\Lambda} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)} e^{\frac{1}{b^2}(F_D(L^{-1})+\mathcal{O}(b^2))} \,. \tag{3.3.93} $$

We use this block as a normalization for large $\Lambda$. For $z \sim 0$ we have

$$ {}_1\mathcal{D}\left(m \overset{a_1}{\ } m' \; a_{0\theta} \begin{smallmatrix} a_{2,1} \\ a_0 \end{smallmatrix}; \tfrac{1}{L}, Lz\right) = \lim_{b\to 0} b^{\theta a_0} \frac{ {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \; \alpha_{0\theta} \begin{smallmatrix} \alpha_{2,1} \\ \alpha_0 \end{smallmatrix}; \tfrac{1}{\Lambda}, \Lambda z\right)}{ {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \, \alpha_0; \tfrac{1}{\Lambda}\right)} = L^{\theta a_0} e^{-\frac{\theta}{2}\partial_{a_0} F_D} z^{\frac{1}{2}+\theta a_0} (1 + \mathcal{O}(L^{-1}, Lz)) \,. \tag{3.3.94} $$

This block and all the other large-$L$ blocks defined in the following satisfy the same equation (3.3.82) as the small-$L$ blocks, with the substitution

$$ u \to u_D = \lim_{b\to 0} b^2 \Lambda \partial_\Lambda \log {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \, \alpha_0; \frac{1}{\Lambda}\right) \,. \tag{3.3.95} $$

For $z \sim 1$ we have the block

$$ \begin{aligned} &{}_1\mathcal{D}\left(-m \overset{a_0}{\ } m' - m \; a_{1\theta} \begin{smallmatrix} a_{2,1} \\ a_1 \end{smallmatrix}; \frac{1}{L}, L(1-z)\right) = \lim_{b\to 0} b^{\theta a_1} \frac{e^{\mu\Lambda} {}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu' - \mu \; \alpha_{1\theta} \begin{smallmatrix} \alpha_{2,1} \\ \alpha_1 \end{smallmatrix}; \tfrac{1}{\Lambda}, \Lambda(1-z)\right)}{ {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \, \alpha_0; \tfrac{1}{\Lambda}\right)} = \\ &= \lim_{b\to 0} \frac{ {}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu' - \mu \; \alpha_{1\theta} \begin{smallmatrix} \alpha_{2,1} \\ \alpha_1 \end{smallmatrix}; \tfrac{1}{\Lambda}, \Lambda(1-z)\right)}{ {}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu' - \mu \, \alpha_1; \tfrac{1}{\Lambda}\right)} = L^{\theta a_1} e^{-\frac{\theta}{2}\partial_{a_1} F_D} (1-z)^{\frac{1}{2}+\theta a_1} (1 + \mathcal{O}(L^{-1}, L(1-z))) \,, \end{aligned} \tag{3.3.96} $$

and for $z \sim \infty$:

$$ \begin{aligned} {}_1\mathcal{D}\left(m \overset{a_{2,1}}{\ } m_\theta \overset{a_1}{\ } m' \; a_0; \frac{1}{L}, \frac{1}{z}\right) &= \lim_{b\to 0} b^{-\frac{1}{2}+\theta(m'-m)} \frac{ {}_1\mathfrak{D}\left(\mu \overset{\alpha_{2,1}}{\ } \mu_\theta \overset{\alpha_1}{\ } \mu' \; \alpha_0; \tfrac{1}{\Lambda}, \tfrac{1}{z}\right)}{ {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \, \alpha_0; \tfrac{1}{\Lambda}\right)} = \\ &= e^{\theta L z/2} e^{-\theta L/2} e^{-\frac{\theta}{2}\partial_m F_D} L^{-\frac{1}{2}+\theta(m'-m)} z^{-\theta m} (1 + \mathcal{O}(L^{-1}, z^{-1})) \,. \end{aligned} \tag{3.3.97} $$

In the connection formula from 0 to 1 for large $\Lambda$ (3.3.77), there appear four different conformal blocks on the right hand side. In the semiclassical limit these four reduce to two, by the same argument as for small $\Lambda$. Indeed we have

$$ \begin{aligned} &e^{\mu\Lambda} {}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu'_{\theta_2} - \mu \; \alpha_{1\theta_3} \begin{smallmatrix} \alpha_{2,1} \\ \alpha_1 \end{smallmatrix}; \frac{1}{\Lambda}, \Lambda(1-z)\right) = e^{-(\mu'_{\theta_2}-\mu)\Lambda} \Lambda^{\Delta_{1\theta_3}+2\mu'_{\theta_2}(\mu'_{\theta_2}-\mu)} (1-z)^{\frac{bQ}{2}+\theta b\alpha_1} e^{\frac{1}{b^2}F_D(\mu'_{\theta_2})+W_D(\mu'_{\theta_2})} \\ &\sim e^{\theta_2 L/2} \Lambda^{-\theta_2(2m'-m)} e^{-\frac{\theta_2}{2}\partial_{m'} F_D(m')} e^{\mu\Lambda} {}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu' - \mu \; \alpha_{1\theta_3} \begin{smallmatrix} \alpha_{2,1} \\ \alpha_1 \end{smallmatrix}; \frac{1}{\Lambda}, \Lambda(1-z)\right) , \quad \text{as } b \to 0 \,. \end{aligned} \tag{3.3.98} $$

The connection formula (3.3.77) from 0 to 1 in the semiclassical limit then becomes

$$ \begin{aligned} &{}_1\mathcal{D}\left(m \overset{a_1}{\ } m' \; a_{0\theta} \begin{smallmatrix} a_{2,1} \\ a_0 \end{smallmatrix}; \frac{1}{L}, Lz\right) = \\ &= \sum_{\theta'=\pm} \left( \sum_{\sigma=\pm} \mathcal{N}_{\theta\sigma}(a_0, m') \mathcal{N}^{-1}_{(-\sigma)\theta'}(m'-m, a_1) e^{\frac{\sigma}{2}L} L^{-\sigma(2m'-m)} e^{-\frac{\sigma}{2}\partial_{m'} F_D(m')} \right) {}_1\mathcal{D}\left(-m \overset{a_0}{\ } m' - m \; a_{1\theta'} \begin{smallmatrix} a_{2,1} \\ a_1 \end{smallmatrix}; \frac{1}{L}, L(1-z)\right) , \end{aligned} \tag{3.3.99} $$

---

[10] As the notation suggests, it is nothing else but the dual prepotential of the gauge theory.

where explicitly the connection coefficients read:

$$\begin{aligned}&\sum_{\sigma=\pm}\mathcal{N}_{\theta\sigma}(a_0,m')\mathcal{N}^{-1}_{(-\sigma)\theta'}(m'-m,a_1)e^{\frac{\sigma}{2}L}L^{-\sigma(2m'-m)}e^{-\frac{\sigma}{2}\partial_{m'}F_D(m')}=\\ &=\sum_{\sigma=\pm}\frac{\Gamma(1+2\theta a_0)\Gamma(-2\theta' a_1)e^{\frac{\sigma}{2}L}L^{-\sigma(2m'-m)}e^{-\frac{\sigma}{2}\partial_{m'}F_D(m')}e^{i\pi\left(\frac{1-\sigma}{2}\right)(\theta a_0-\theta' a_1-2m'+m)}}{\Gamma\left(\frac{1}{2}+\theta a_0-\sigma m'\right)\Gamma\left(\frac{1}{2}-\theta' a_1-\sigma(m'-m)\right)}\,.\end{aligned}\tag{3.3.100}$$

Again, all the spurious powers of $b$ and $\Lambda$ have beautifully recombined to give the finite combination $L$.
The connection formula from 1 to $\infty$ (3.3.71) on the other hand becomes

$${}_1\mathcal{D}\left(-m\;{}^{a_0}\;m'-m\;a_{1\theta}\;{}^{a_{2,1}}_{a_1};\tfrac{1}{L},L(1-z)\right)=\textstyle\sum_{\theta'=\pm}\mathcal{N}_{\theta(-\theta')}(a_1,m'-m)\,{}_1\mathcal{D}\left(m\;{}^{a_{2,1}}\;m_{\theta'}\;{}^{a_1}\;m'\;a_0;\tfrac{1}{L},\tfrac{1}{z}\right),\tag{3.3.101}$$

where $\mathcal{N}$ is:

$$\mathcal{N}_{\theta(-\theta')}(a_1,m'-m,)=\frac{\Gamma(1+2\theta a_1)}{\Gamma\left(\frac{1}{2}+\theta a_1+\theta'(m'-m)\right)}e^{i\pi\left(\frac{1+\theta'}{2}\right)\left(\frac{1}{2}-(m'-m)+\theta a_1\right)}\,.\tag{3.3.102}$$

### 3.3.3 Reduced confluent conformal blocks

**General case**

Consider the correlation function

$$\langle\Lambda^2|V_1(1)\Phi(z)|\Delta_0\rangle\,,\tag{3.3.103}$$

which solves the BPZ equation

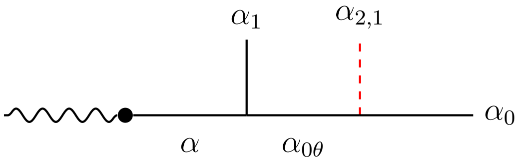
$$\left(b^{-2}\partial_z^2-\left(\frac{1}{z}+\frac{1}{z-1}\right)\partial_z+\frac{\Lambda^2\partial_{\Lambda^2}-\Delta_{2,1}-\Delta_1-\Delta_0}{z(z-1)}+\frac{\Delta_1}{(z-1)^2}+\frac{\Delta_0}{z^2}-\frac{\Lambda^2}{4z}\right)\langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle=0\,.\tag{3.3.104}$$

We can decompose it into irregular conformal blocks in different ways. The blocks corresponding to the expansion of $z$ around a regular singular point can be given as a further decoupling limit of the confluent conformal blocks. For the blocks corresponding to the expansion of $z$ around the irregular singular point of rank 1/2, no closed form expression as (3.3.49) is presently known to us. The block for $z\sim 0$ can be defined as

$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;{}^{\alpha_{2,1}}_{\alpha_0};\Lambda^2,z\right)=\lim_{\eta\to\infty}(4\eta)^{\Delta}\,{}_1\mathfrak{F}\left(-\eta\;\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;{}^{\alpha_{2,1}}_{\alpha_0};\frac{\Lambda^2}{4\eta},z\right)\,.\tag{3.3.105}$$

We multiply by the factor of $(4\eta)^{\Delta}$ to take care of the leading divergence in the limit. In the diagrammatic notation of section 3.2.3, we represent it by

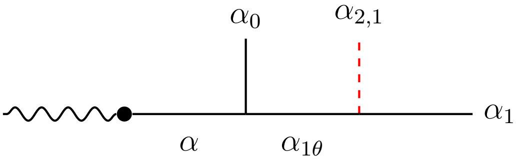
$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;{}^{\alpha_{2,1}}_{\alpha_0};\Lambda^2,z\right)=$$



. (3.3.106)

As indicated by the diagram, all OPEs are regular in this case. The wiggly line represents the rank 1/2 irregular state, and the dot the pairing with a Verma module. The block for $z\sim 1$ is then simply

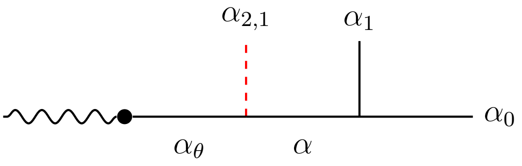
$$e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)=$$



. (3.3.107)

The overall phase compensates the sign in $e^{-i\pi}\Lambda^2$ such that the classical part is still $\Lambda^{2\Delta}$. In the intermediate region where $1 \ll z \ll \frac{1}{\Lambda^2}$ the corresponding block is

$$z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\;{}^{\alpha_{2,1}}\;\alpha\;{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\frac{1}{z}\right) =$$



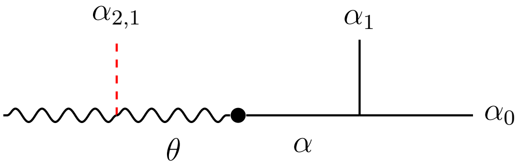
. (3.3.108)

Instead, in the deep irregular region, where $z \gg \frac{1}{\Lambda^2} \gg 1$, a decoupling limit of the form (3.3.49) does not work. Of course one can still calculate this block by solving the BPZ equation iteratively with a series Ansatz, or directly using the Ward identities determining the descendants of the OPE with the irregular state (see Appendix B.1). In any case we will denote the conformal block in this region by

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\;\alpha\;{}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)\sim(\Lambda^2)^{\Delta_{2,1}+\Delta}(\Lambda\sqrt{z})^{\frac{1}{2}+b^2}e^{\theta b\Lambda\sqrt{z}}\left[1+\mathcal{O}\left(\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)\right]. \qquad (3.3.109)$$

The $\sim$ refers to the fact that this expansion is asymptotic. In diagrams we represent this block by

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\;\alpha\;{}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right) =$$



. (3.3.110)

The solution of the connection problems goes in the same way as for the (unreduced) confluent Heun equation (section 3.3.2). In particular the connection problem between 0 and 1 works in the same way as for the general Heun equation. We have

$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;{}^{\alpha_{2,1}}_{\alpha_0};\Lambda^2,z\right)=\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_1;b\alpha)e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right). \qquad (3.3.111)$$

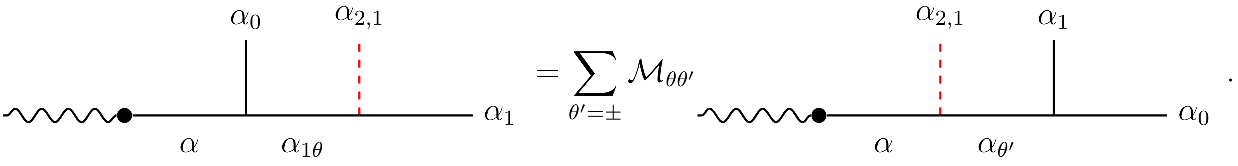
To solve the connection problem between 1 and $\infty$ one has to do two steps: from 1 to the intermediate region, and then to $\infty$. In each step we decompose the correlator into conformal blocks in the different regions and then use crossing symmetry to determine the connection coefficients. The relevant formulae for the rank 1/2 irregular state are reviewed in Appendix B.2. We have

$$\begin{aligned}\langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C_\alpha \sum_{\theta=\pm} C^{\alpha_{1\theta}}_{\alpha_{2,1}\alpha_1}C^{\alpha}_{\alpha_{1\theta}\alpha_0}\left|e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)\right|^2 = \\ &= \int d\alpha\, C_\alpha \sum_{\theta'=\pm} C^{\alpha}_{\alpha_{2,1}\alpha_{\theta'}}C^{\alpha_{\theta'}}_{\alpha_1\alpha_0}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_{2,1}}\;\alpha_{\theta'}\;{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\frac{1}{z}\right)\right|^2 .\end{aligned} \qquad (3.3.112)$$

This is precisely the same condition as for the hypergeometric functions (3.2.5). The connection formula between 1 and the intermediate region is then

$$e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)=\textstyle\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha;b\alpha_0)z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_{2,1}}\;\alpha_{\theta'}\;{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\frac{1}{z}\right). \qquad (3.3.113)$$

Diagrammatically:



(3.3.114)

Now we decompose the correlator into conformal blocks in the intermediate region and near $\infty$, obtaining the crossing symmetry condition

$$\begin{aligned}\langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C^{\alpha}_{\alpha_1,\alpha_0}\sum_{\theta=\pm} C_{\alpha_\theta} C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left| z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\;{}^{\alpha_{2,1}}\;\alpha\;\genfrac{}{}{0pt}{}{\alpha_1}{\alpha_0};\Lambda^2 z,\frac{1}{z}\right)\right|^2 = \\ &= \int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0}\sum_{\theta'=\pm} C_{\alpha} B_{\alpha_{2,1}}\left|{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\;\alpha\;\genfrac{}{}{0pt}{}{\alpha_1}{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)\right|^2 .\end{aligned} \tag{3.3.115}$$

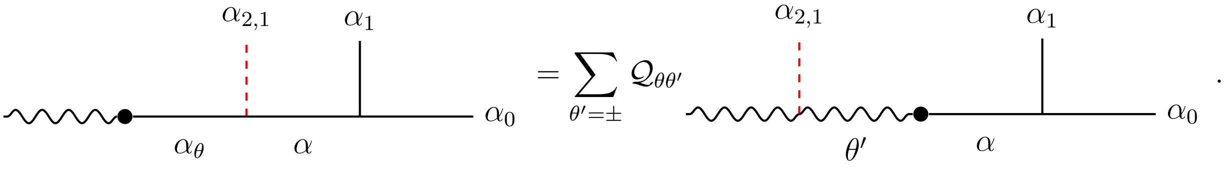
We recognize this condition from the Bessel functions (3.2.36). We then immediately find the connection formula between the intermediate region and $\infty$:

$$b^{2\theta b\alpha} z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\;{}^{\alpha_{2,1}}\;\alpha\;\genfrac{}{}{0pt}{}{\alpha_1}{\alpha_0};\Lambda^2 z,\frac{1}{z}\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}}\mathcal{Q}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\;\alpha\;\genfrac{}{}{0pt}{}{\alpha_1}{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right) \tag{3.3.116}$$

with irregular connection coefficients as in (B.2.15):

$$\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}}\Gamma(1+2\theta b\alpha)e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)} . \tag{3.3.117}$$

In diagrams:

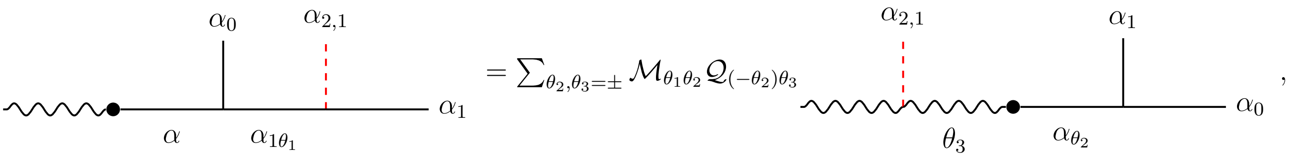


(3.3.118)

Let us write explicitly the more interesting connection formulae between 1 and $\infty$, which is obtained by concatenating the two connection formulae above. Since the $\mathfrak{F}$ block in the intermediate region has different arguments in formula (3.3.113) and (3.3.116), we need to rename some arguments. In the end we obtain the following connection formula from 1 directly to $\infty$:

$$\begin{aligned}&e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta_1}\;\genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right) = \\ &= \sum_{\theta_2,\theta_3=\pm}\mathcal{M}_{\theta_1\theta_2}(b\alpha_1,b\alpha;b\alpha_0)\mathcal{Q}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2})b^{-\frac{1}{2}+\theta_2 b\alpha_{\theta_2}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta_3)}\left(\alpha_{2,1}\;\alpha_{\theta_2}\;\genfrac{}{}{0pt}{}{\alpha_1}{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right) .\end{aligned} \tag{3.3.119}$$

Diagrammatically we have



(3.3.120)

where we have suppressed the arguments of the connection coefficients for brevity.

### Semiclassical limit

The story works the same way here as for the confluent case. In the semiclassical limit the BPZ equation becomes

$$\left(\partial_z^2+\frac{u-\frac{1}{2}+a_1^2+a_0^2}{z(z-1)}+\frac{\frac{1}{4}-a_1^2}{(z-1)^2}+\frac{\frac{1}{4}-a_0^2}{z^2}-\frac{L^2}{4z}\right){}_{\frac{1}{2}}\mathfrak{F}(z)=0\,, \tag{3.3.121}$$

for *any* semiclassical block. Here $u$ is given by

$$u=\lim_{b\to 0}b^2\Lambda^2\partial_{\Lambda^2}\log{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2\right)=\frac{1}{4}-a^2+\mathcal{O}(L^2) \tag{3.3.122}$$

by the same argument as before. The finite semiclassical conformal blocks are defined by normalizing by the same block without the degenerate field insertion, i.e. the semiclassical block for $z\sim 0$ is

$${}_{\frac{1}{2}}\mathcal{F}\left(a\;{}^{a_1}\;a_{0\theta}\;\overset{a_{2,1}}{\underset{a_0}{}};L^2,z\right)=\lim_{b\to 0}\frac{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;\overset{\alpha_{2,1}}{\underset{\alpha_0}{}};\Lambda^2,z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;\overset{\alpha_0}{\underset{\alpha_1}{}};\Lambda^2\right)}=e^{-\frac{\theta}{2}\partial_{a_0}F}z^{\frac{1}{2}+\theta a_0}(1+\mathcal{O}(L^2,z))\,. \tag{3.3.123}$$

Here $F=\lim_{b\to 0}b^2\log\left[\Lambda^{-2\Delta}{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2\right)\right]$.

$$\begin{aligned}{}_{\frac{1}{2}}\mathcal{F}\left(a\;{}^{a_0}\;a_{1\theta}\;\overset{a_{2,1}}{\underset{a_1}{}};-L^2,1-z\right)&=\lim_{b\to 0}\frac{e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;\overset{\alpha_{2,1}}{\underset{\alpha_1}{}};e^{-i\pi}\Lambda^2,1-z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2\right)}=\\&=\lim_{b\to 0}\frac{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_0}\;\alpha_{1\theta}\;\overset{\alpha_{2,1}}{\underset{\alpha_1}{}};e^{-i\pi}\Lambda^2,1-z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;\overset{\alpha_0}{\underset{\alpha_1}{}};e^{-i\pi}\Lambda^2\right)}=e^{-\frac{\theta}{2}\partial_{a_1}F}(1-z)^{\frac{1}{2}+\theta a_1}(1+\mathcal{O}(L^2,1-z))\,.\end{aligned} \tag{3.3.124}$$

In the deep irregular region we define the semiclassical block as

$${}_{\frac{1}{2}}\mathcal{E}^{(\theta)}\left(a_{2,1}\;a\;\overset{a_1}{\underset{a_0}{}};L^2,\tfrac{1}{L\sqrt{z}}\right)=\lim\nolimits_{b\to 0}b^{-\frac{1}{2}}\frac{{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\;\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\mu\,\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2\right)}=(L\sqrt{z})^{-\frac{1}{2}}e^{\theta L\sqrt{z}}(1+\mathcal{O}(L^2,\tfrac{1}{L\sqrt{z}}))\,. \tag{3.3.125}$$

All these blocks satisfy the same equation (3.3.121). As for the confluent case, in the connection formula between 1 and $\infty$ we have four different $\mathfrak{E}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\;\alpha_{\theta'}\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2,\frac{1}{\Lambda z}\right)\sim(\Lambda^2)^{\theta' a}e^{-\frac{\theta'}{2}\partial_a F}{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\;\alpha\;\overset{\alpha_1}{\underset{\alpha_0}{}};\Lambda^2,\frac{1}{\Lambda z}\right),\quad\text{as } b\to 0\,, \tag{3.3.126}$$

as in (3.3.34). Now that we have defined the semiclassical conformal blocks, we state the connection formulae. The connection formula from 0 to 1 (3.3.111) reduces trivially in the semiclassical limit to

$${}_{\frac{1}{2}}\mathcal{F}\left(a\;{}^{a_1}\;a_{0\theta}\;\overset{a_{2,1}}{\underset{a_0}{}};L^2,z\right)=\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(a_0,a_1;a)\,{}_{\frac{1}{2}}\mathcal{F}\left(a\;{}^{a_0}\;a_{1\theta}\;\overset{a_{2,1}}{\underset{a_1}{}};-L^2,1-z\right)\,. \tag{3.3.127}$$

The connection formula from 1 to $\infty$ (3.3.119) becomes

$$
{}_{\frac{1}{2}}\mathcal{F}\left(a\ {}^{a_0}\ a_{1\theta}\ \begin{matrix}a_{2,1}\\ a_1\end{matrix};-L^2,1-z\right)=\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{Q}_{(-\sigma)\theta'}(a)L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right){}_{\frac{1}{2}}\mathcal{E}^{(\theta')}\left(a_{2,1}\ a\ \begin{matrix}a_1\\ a_0\end{matrix};L^2,\tfrac{1}{L\sqrt{z}}\right), \tag{3.3.128}
$$

with connection coefficients[11]

$$
\begin{aligned}
&\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{Q}_{(-\sigma)\theta'}(a)L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}=\\
=&\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_1)2^{-2\sigma a}L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-2\sigma a\right)}}{\sqrt{2\pi}\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a-a_0\right)}\,.
\end{aligned} \tag{3.3.129}
$$

### 3.3.4 Doubly confluent conformal blocks

**General case**

Via a further collision limit we reach a correlator that solves the BPZ equation

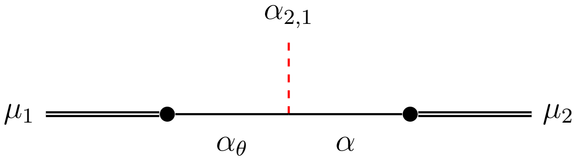
$$
\left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z+\frac{\mu_1\Lambda_1}{z}-\frac{\Lambda_1^2}{4}+\frac{\Lambda_2\partial_{\Lambda_2}}{z^2}+\frac{\mu_2\Lambda_2}{z^3}-\frac{\Lambda_2^2}{4z^4}\right)\langle\mu_1,\Lambda_1|\Phi(z)|\mu_2,\Lambda_2\rangle=0\,. \tag{3.3.130}
$$

This correlator can be expanded in the intermediate region $\Lambda_2\ll z\ll\Lambda_1^{-1}$ and near the two irregular singularities, that is either $z\gg\Lambda_1^{-1}\gg 1$ or $z\ll\Lambda_2\ll 1$. Note that in (3.3.130) one of the three parameters $\Lambda_1,\Lambda_2,z$ is redundant. Indeed the conformal blocks will only depend on two ratios. The conformal blocks in these regions can easily be computed as a collision limit. Explicitly, in the intermediate region $\Lambda_2\ll z\ll\Lambda_1^{-1}$

$$
{}_1\mathfrak{F}_1\left(\mu_1\ \alpha_\theta\ {}^{\alpha_{2,1}}\alpha\ \mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)=\Lambda_1^{\Delta_\theta}\Lambda_2^{\Delta}z^{\frac{bQ}{2}+\theta b\alpha}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{F}}\left(\mu_1\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ \begin{matrix}\frac{\eta-\mu_2}{2}\\ \frac{\eta+\mu_2}{2}\end{matrix};\Lambda_1 z,\frac{\Lambda_2}{z\eta}\right)\,. \tag{3.3.131}
$$

This conformal block is the result of the projection of the Whittaker module $|\mu_2,\Lambda_2\rangle$ on a Verma module $\Delta$ and of $\langle\mu_1,\Lambda_1|$ on $\Delta_\theta$. We represent this block by the diagram

$$
{}_1\mathfrak{F}_1\left(\mu_1\ \alpha_\theta\ {}^{\alpha_{2,1}}\alpha\ \mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)=
$$



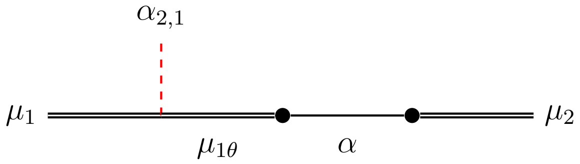
. (3.3.132)

The expansion near the irregular singularity at infinity can be obtained by colliding in (3.3.49) the insertions far from the Whittaker state in the confluent conformal block. This gives

$$
{}_1\mathfrak{D}_1\left(\mu_1\ {}^{\alpha_{2,1}}\ \mu_{1\theta}\ \alpha\ \mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1 z}\right)=e^{\theta b\Lambda_1 z/2}\Lambda_1^{\Delta+\Delta_{2,1}}\Lambda_2^{\Delta}\left(\Lambda_1 z\right)^{-\theta b\mu_1+\frac{b^2}{2}}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{D}}\left(\mu_1\ {}^{\alpha_{2,1}}\ \mu_{1\theta}\ \alpha\ \begin{matrix}\frac{\eta-\mu_2}{2}\\ \frac{\eta+\mu_2}{2}\end{matrix};\tfrac{\Lambda_1\Lambda_2}{\eta},\tfrac{1}{\Lambda_1 z}\right). \tag{3.3.133}
$$

We represent this block diagrammatically by

$$
{}_1\mathfrak{D}_1\left(\mu_1\ {}^{\alpha_{2,1}}\ \mu_{1\theta}\ \alpha\ \mu_2;\Lambda_1\Lambda_2,\frac{1}{\Lambda_1 z}\right)=
$$



. (3.3.134)

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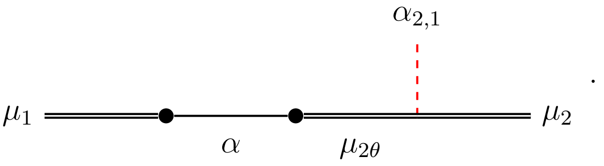
[11] Note that the Gamma functions in the denominator precisely correspond to the one-loop factors of the two hypermultiplets of the corresponding AGT dual gauge theory.

Finally, the expansion near the irregular singularity at zero is easily obtained from (3.3.133) by exchanging $\Lambda_1$ and $\Lambda_2$ and sending $z \to 1/z$, up to a Jacobian. The corresponding conformal block is

$$z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2 \;{}^{\alpha_{2,1}}\; \mu_{2\theta}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right) = \tag{3.3.135}$$



.

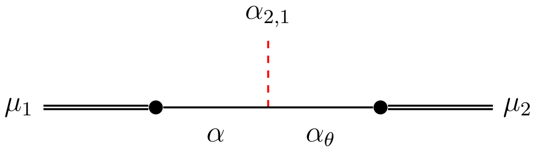
Expanding now the correlator first near 0 and then in the intermediate region, crossing symmetry implies

$$\begin{aligned}\langle\mu_1,\Lambda_1|\Phi(z)|\mu_2,\Lambda_2\rangle &= \int d\alpha\, G_\alpha^{-1}C_{\mu_1\alpha}G_\alpha^{-1}\sum_{\theta=\pm}B^{\mu_{2\theta}}_{\alpha_{2,1},\mu_2}C_{\mu_{2\theta}\alpha}\left|z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right)\right|^2 = \\ &= \int d\alpha\, G_\alpha^{-1}C_{\mu_1\alpha}\sum_{\theta'=\pm}C^{\alpha_{\theta'}}_{\alpha_{2,1}\alpha}C_{\mu_2\alpha_{\theta'}}\left|{}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;{}^{\alpha_{2,1}}\alpha_{\theta'}\;\mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)\right|^2 .\end{aligned} \tag{3.3.136}$$

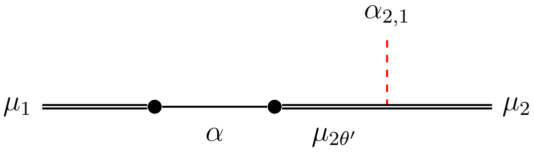
We recognize this condition from (3.2.21), and we can readily write down the solution to the connection problem:

$$b^{\theta b\alpha}{}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;{}^{\alpha_{2,1}}\alpha_\theta\;\mu_2;\Lambda_1 z,\tfrac{\Lambda_2}{z}\right) = \textstyle\sum_{\theta'=\pm}b^{-\frac{1}{2}-\theta' b\mu_2}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu_2)z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta'}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\tfrac{z}{\Lambda_2}\right). \tag{3.3.137}$$

In diagrams:



$$= \textstyle\sum_{\theta'=\pm}\mathcal{N}_{\theta\theta'}$$

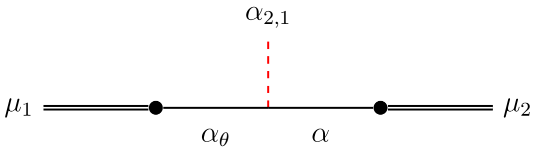


. (3.3.138)

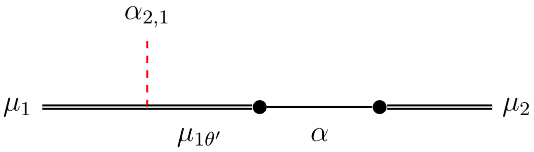
A similar argument works for the connection between the intermediate region and infinity. We obtain

$$b^{\theta b\alpha}{}_1\mathfrak{F}_1\left(\mu_1\;\alpha_\theta\;{}^{\alpha_{2,1}}\alpha\;\mu_2;\Lambda_1 z,\tfrac{\Lambda_2}{z}\right) = \textstyle\sum_{\theta'=\pm}b^{-\frac{1}{2}-\theta' b\mu_1}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu_1){}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta'}\;\alpha\;\mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1 z}\right). \tag{3.3.139}$$

Or, diagrammatically:



$$= \textstyle\sum_{\theta'=\pm}\mathcal{N}_{\theta\theta'}$$

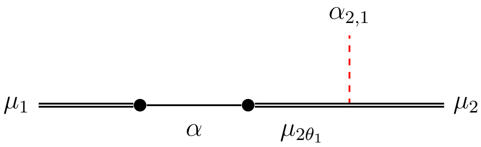


. (3.3.140)

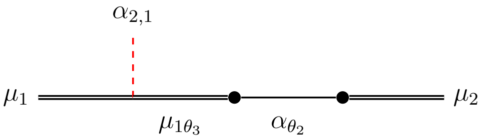
Concatenating the previous connection formulae we can connect 0 directly with $\infty$ as follows

$$\begin{aligned}&b^{-\frac{1}{2}-\theta_1 b\mu_2}z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta_1}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right) = \\ &= \sum_{\theta_2,\theta_3=\pm}b^{\theta_2 b\alpha}\mathcal{N}^{-1}_{\theta_1\theta_2}(b\mu_2,b\alpha)b^{-\frac{1}{2}+\theta_2 b\alpha-\theta' b\mu_1}\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2},b\mu_1){}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta_3}\;\alpha_{\theta_2}\;\mu_2;\Lambda_1\Lambda_2,\frac{1}{\Lambda_1 z}\right).\end{aligned} \tag{3.3.141}$$

In diagrams:



$$= \textstyle\sum_{\theta_2,\theta_3=\pm} \mathcal{N}^{-1}_{\theta_1\theta_2}\mathcal{N}_{(-\theta_2)\theta_3}$$



. (3.3.142)

### Semiclassical limit

Let us now consider the semiclassical limit of the doubly confluent conformal blocks. Once again, the divergence as $b \to 0$ is expected to exponentiate, that is

$$z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2 \ {}^{\alpha_{2,1}} \ \mu_{2\theta} \ \alpha \ \mu_1; \Lambda_1\Lambda_2, \tfrac{z}{\Lambda_2}\right) = z^{-2\Delta_{2,1}} e^{\frac{\theta b \Lambda_2}{2z}} \Lambda_2^{\Delta+\Delta_{2,1}} \Lambda_1^{\Delta} \left(\tfrac{\Lambda_2}{z}\right)^{-\theta b \mu_2 + \frac{b^2}{2}} \exp\left(b^{-2}F\left(L_1L_2\right) + W\left(L_1L_2, zL_2^{-1}\right)\right) , \tag{3.3.143}$$

where $F$ is the classical conformal block defined by

$${}_1\mathfrak{F}_1\left(\mu_1 \ \alpha \ \mu_2, \Lambda_1\Lambda_2\right) = \left(\Lambda_1\Lambda_2\right)^{\Delta} \exp\left(b^{-2}F + \mathcal{O}(b^0)\right) , \tag{3.3.144}$$

and the ${}_1\mathfrak{F}_1$ block is given by

$$\langle \mu_1, \Lambda_1 | \mu_2, \Lambda_2 \rangle = \int d\alpha \, C_{\mu_1\alpha} C_{\mu_2\alpha} \left|{}_1\mathfrak{F}_1\left(\mu_1 \ \alpha \ \mu_2, \Lambda_1\Lambda_2\right)\right|^2 . \tag{3.3.145}$$

We define the semiclassical block near zero to be

$$z\,{}_1\mathcal{D}_1\left(m_2 \ {}^{a_{2,1}} \ m_{2\theta} \ a \ m_1; L_1L_2, \frac{z}{L_2}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta b \mu_2 + \frac{b^2}{2}} z^{-2\Delta_{2,1}} \frac{{}_1\mathfrak{D}_1\left(\mu_2 \ {}^{\alpha_{2,1}} \ \mu_{2\theta} \ \alpha \ \mu_1; \Lambda_1\Lambda_2, \frac{z}{\Lambda_2}\right)}{{}_1\mathfrak{F}_1\left(\mu_1 \ \alpha \ \mu_2, \Lambda_1\Lambda_2\right)} , \tag{3.3.146}$$

The semiclassical blocks satisfy the equation

$$\left(\partial_z^2 + \frac{m_1L_1}{z} - \frac{L_1^2}{4} + \frac{u}{z^2} + \frac{m_2L_2}{z^3} - \frac{L_2^2}{4}\frac{1}{z^4}\right) z\,{}_1\mathcal{D}_1\left(m_2 \ {}^{a_{2,1}} \ m_{2\theta} \ a \ m_1; L_1L_2, \frac{z}{L_2}\right) = 0 , \tag{3.3.147}$$

with the $u$ parameter defined as usual to be the leftover of the $\Lambda_2$ derivative, that is

$$u = \frac{1}{4} - a^2 + L_2\partial_{L_2}F\left(L_1L_2\right) . \tag{3.3.148}$$

Similarly, the semiclassical block near the irregular singularity at infinity is defined to be

$${}_1\mathcal{D}_1\left(m_1 \ {}^{a_{2,1}} \ m_{1\theta} \ a \ m_2; L_1L_2, \frac{1}{L_1 z}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta b \mu_1 + \frac{b^2}{2}} \frac{{}_1\mathfrak{D}_1\left(\mu_1 \ {}^{\alpha_{2,1}} \ \mu_{1\theta} \ \alpha \ \mu_2; \Lambda_1\Lambda_2, \frac{1}{\Lambda_1 z}\right)}{{}_1\mathfrak{F}_1\left(\mu_1 \ \alpha \ \mu_2, \Lambda_1\Lambda_2\right)} , \tag{3.3.149}$$

and satisfies the same equation (3.3.147). In equation (3.3.141) 4 different blocks near infinity appear in the RHS. However they collapse to two of them in the semiclassical limit as in the previous cases. That is,

$${}_1\mathfrak{D}_1\left(\mu_1 \ {}^{\alpha_{2,1}} \ \mu_{1\theta} \ \alpha_{\theta'} \ \mu_2; \Lambda_1\Lambda_2, \tfrac{1}{\Lambda_1 z}\right) \sim \left(\Lambda_1\Lambda_2\right)^{\theta' a} e^{-\frac{\theta'}{2}\partial_a F}\,{}_1\mathfrak{D}_1\left(\mu_1 \ {}^{\alpha_{2,1}} \ \mu_{1\theta} \ \alpha \ \mu_2; \Lambda_1\Lambda_2, \tfrac{1}{\Lambda_1 z}\right) , \quad \text{as } b \to 0 , \tag{3.3.150}$$

as in (3.3.34). Finally, the connection formula (3.3.141) in the semiclassical limit becomes

$$z\,{}_1\mathcal{D}_1\left(m_2\;{}^{a_{2,1}}\;m_{2\theta}\;a\;m_1;L_1L_2,\frac{z}{L_2}\right)=$$
$$=\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{N}^{-1}_{\theta\sigma}(m_2,a)\mathcal{N}_{(-\sigma)\theta'}(a,m_1)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}\right){}_1\mathcal{D}_1\left(m_1\;{}^{a_{2,1}}\;m_{1\theta'}\;a\;m_2;L_1L_2,\frac{1}{L_1z}\right), \tag{3.3.151}$$

where explicitly the connection coefficients read

$$\sum_{\sigma=\pm}\mathcal{N}^{-1}_{\theta\sigma}(m_2,a)\mathcal{N}_{(-\sigma)\theta'}(a,m_1)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}=$$
$$=\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}}{\Gamma\left(\frac{1}{2}+\theta m_2-\sigma a\right)\Gamma\left(\frac{1}{2}-\theta' m_1-\sigma a\right)}e^{i\pi\left(\frac{1+\theta}{2}\right)\left(-\frac{1}{2}-m_2-\sigma a\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m_1-\sigma a\right)}, \tag{3.3.152}$$

### 3.3.5 Reduced doubly confluent conformal blocks

**General case**

Consider the correlation function

$$\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle\,, \tag{3.3.153}$$

which solves the BPZ equation

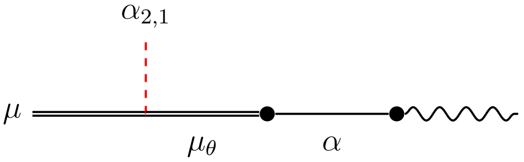
$$\left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z+\frac{\mu\Lambda_1}{z}-\frac{\Lambda_1^2}{4}+\frac{\Lambda_2^2\partial_{\Lambda_2^2}}{z^2}-\frac{\Lambda_2^2}{4z^3}\right)\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle=0\,. \tag{3.3.154}$$

One of the parameters among $\Lambda_1,\Lambda_2,z$ is redundant and can be set to an arbitrary value via a rescaling. We keep them all generic for convenience. We have three different conformal blocks, corresponding to the expansion of $z$ near the two irregular singular points, and for $z$ in the intermediate region. The block for $z\sim\infty$ is given by the decoupling limit of the corresponding doubly confluent block (3.3.133):

$${}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\;{}^{\alpha_{2,1}}\;\mu_\theta\;\alpha\,;\Lambda_1\Lambda_2^2,\tfrac{1}{\Lambda_1z}\right)=e^{\theta b\Lambda_1z/2}\Lambda_1^{\Delta+\Delta_{2,1}}(\Lambda_2^2)^{\Delta}\left(\Lambda_1z\right)^{-\theta b\mu+\frac{b^2}{2}}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{D}}_1\left(\mu\;{}^{\alpha_{2,1}}\;\mu_\theta\;\alpha\;\eta;-\tfrac{\Lambda_1\Lambda_2^2}{4\eta},\tfrac{1}{\Lambda_1z}\right). \tag{3.3.155}$$

Equivalently, this block can be computed by doing the OPE $\langle\mu,\Lambda_1|\Phi(z)$, projecting the result onto the Verma module $\Delta_\alpha$ and contracting the result with $|\Lambda_2^2\rangle$. We denote it diagrammatically by

$${}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\;{}^{\alpha_{2,1}}\;\mu_\theta\;\alpha\,;\Lambda_1\Lambda_2^2,\frac{1}{\Lambda_1z}\right)=$$



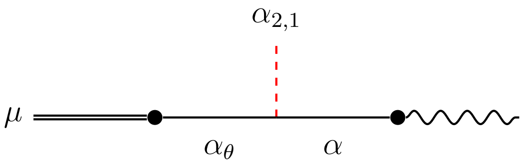
. (3.3.156)

Also for the intermediate region $\Lambda_2^2\ll z\ll\frac{1}{\Lambda_1}$ we have a closed form expression, given by

$${}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\;\alpha_\theta\;{}^{\alpha_{2,1}}\;\alpha\;;\Lambda_1z,\frac{\Lambda_2^2}{z}\right)=\Lambda_1^{\Delta_\theta}(\Lambda_2^2)^{\Delta}z^{\frac{bQ}{2}+\theta b\alpha}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{F}}_1\left(\mu\;\alpha_\theta\;{}^{\alpha_{2,1}}\;\alpha\;\eta;\Lambda_1z,-\frac{\Lambda_2^2}{4\eta z}\right). \tag{3.3.157}$$

This conformal block can also be computed directly by projecting $|\Lambda_2^2\rangle$ onto the Verma module $\Delta$, then doing the OPE of $\Phi(z)$ term by term with the resulting expansion and then contracting with $\langle\mu,\Lambda_1|$. In diagrams

$${}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right)= \qquad . \tag{3.3.158}$$

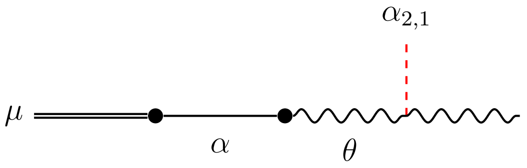


For the expansion around the irregular singular point of half rank no explicit, closed form expression is known to us. In any case one can calculate the expansion iteratively via other methods as for (3.3.109). We denote the corresponding conformal block in this region, where $z\ll\Lambda_2^2$ and $\Lambda_1\Lambda_2^2\ll 1$ by

$${}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\sim e^{\theta b\Lambda_2/\sqrt{z}}\left(\frac{\sqrt{z}}{\Lambda_2}\right)^{-\frac{1}{2}-b^2} z^{-2\Delta_{2,1}}\Lambda_1^{\Delta}(\Lambda_2^2)^{\Delta_{2,1}+\Delta}\left[1+\mathcal{O}\left(\frac{\sqrt{z}}{\Lambda_2},\Lambda_1\Lambda_2^2\right)\right]. \tag{3.3.159}$$

Diagrammatically,

$${}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)= \qquad . \tag{3.3.160}$$



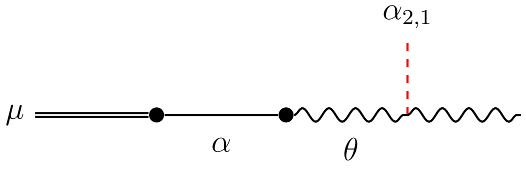
To connect 0 with the intermediate region we decompose

$$\begin{aligned}\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle &= \int d\alpha\, C_{\mu\alpha}G_\alpha^{-1}\sum_{\theta=\pm} C_\alpha B_{\alpha_{2,1}}\left|{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\right|^2 = \\ &= \int d\alpha\, C_{\mu\alpha}G_\alpha^{-1}\sum_{\theta'=\pm} C_{\alpha_{\theta'}}C^{\alpha_{\theta'}}_{\alpha_{2,1},\alpha}\left|{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right)\right|^2 .\end{aligned} \tag{3.3.161}$$

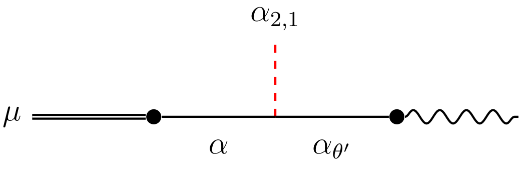
We recognize this constraint from (3.2.36). Its solution is

$$b^{-\frac{1}{2}}{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)=\sum_{\theta'=\pm} b^{2\theta' b\alpha}\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha)\,{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right). \tag{3.3.162}$$

In diagrams we write



$$=\sum_{\theta'=\pm}\mathcal{Q}^{-1}_{\theta\theta'} \qquad . \tag{3.3.163}$$



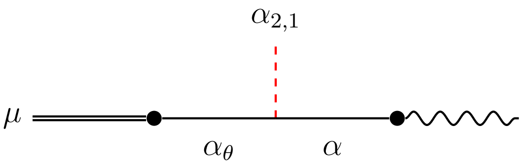
Instead, to connect from the intermediate region to $\infty$ we decompose

$$\begin{aligned}\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle &= \int \mathrm{d}\alpha\, C_\alpha G_\alpha^{-1}\sum_{\theta=\pm} C_{\mu\alpha_\theta}C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left|{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right)\right|^2 = \\ &= \int \mathrm{d}\alpha\, C_\alpha G_\alpha^{-1}\sum_{\theta'=\pm} C_{\mu_{\theta'}\alpha}B^{\mu_{\theta'}}_{\alpha_{2,1}\mu}\left|{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\,;\Lambda_1\Lambda_2^2,\frac{1}{\Lambda_1 z}\right)\right|^2 .\end{aligned} \tag{3.3.164}$$

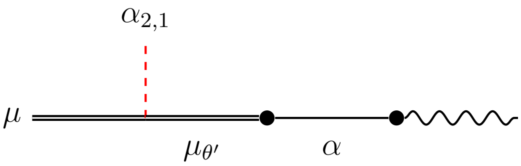
This is just the same constraint as for the Whittaker functions (3.2.21). The solution is

$$b^{\theta b\alpha}{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ;\Lambda_1 z, \frac{\Lambda_2^2}{z}\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha, b\mu)\,{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\,;\Lambda_1\Lambda_2^2, \frac{1}{\Lambda_1 z}\right). \tag{3.3.165}$$

Diagrammatically



$$= \sum_{\theta'=\pm} \mathcal{N}_{\theta\theta'}$$

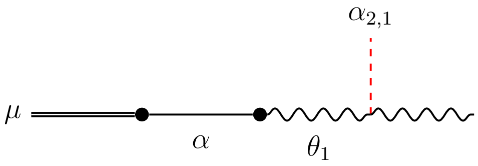


(3.3.166)

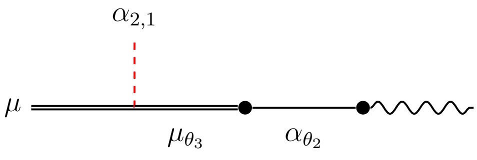
To connect from 0 to $\infty$ we just need to concatenate the two connection formulae above to obtain

$$b^{-\frac{1}{2}}{}_1\mathfrak{E}^{(\theta_1)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2, \tfrac{\sqrt{z}}{\Lambda_2}\right) = \textstyle\sum_{\theta_2,\theta_3=\pm} b^{2\theta_2 b\alpha}\mathcal{Q}^{-1}_{\theta_1\theta_2}(b\alpha)b^{-\frac{1}{2}+\theta_2 b\alpha-\theta_3 b\mu}\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2}, b\mu)\,{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta_3}\ \alpha_{\theta_2}\,;\Lambda_1\Lambda_2^2, \tfrac{1}{\Lambda_1 z}\right). \tag{3.3.167}$$

In diagrams



$$= \textstyle\sum_{\theta_2,\theta_3=\pm} \mathcal{Q}^{-1}_{\theta_1\theta_2}\mathcal{N}_{(-\theta_2)\theta_3}$$



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(3.3.168)

### Semiclassical limit

The BPZ equation in this limit becomes

$$\left(\partial_z^2 - \frac{L_1^2}{4} + \frac{mL_1}{z} + \frac{u}{z^2} - \frac{L_2^2}{4z^3}\right){}_1\mathfrak{F}_{\frac{1}{2}} = 0\,. \tag{3.3.169}$$

for *any* semiclassical block. Here $u$ is given by

$$u = \lim_{b\to 0} b^2\Lambda_2^2\partial_{\Lambda_2^2}\log{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1L_2^2)\,, \tag{3.3.170}$$

where ${}_1\mathfrak{F}_{\frac{1}{2}}\,(\mu\,\alpha;\Lambda_1\Lambda_2^2)$ is the conformal block corresponding to $\langle\mu,\Lambda_1|\Lambda_2^2\rangle$ with intermediate momentum $\alpha$. The finite semiclassical conformal blocks are defined as before by normalizing by the same block without the degenerate field insertion, i.e. for $z\sim 0$

$${}_1\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(m\,a\,a_{2,1}\,;L_1L_2^2, \tfrac{\sqrt{z}}{L_2}\right) = \lim_{b\to 0} b^{-\frac{1}{2}}\frac{{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2, \frac{\sqrt{z}}{\Lambda_2}\right)}{{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)} \sim e^{\theta L_2/\sqrt{z}}L_2^{-\frac{1}{2}}z^{\frac{3}{4}}(1+\mathcal{O}(L_1L_2^2, \sqrt{z}/L_2)) \tag{3.3.171}$$

For $z\sim\infty$ instead we have

$$\begin{aligned}{}_1\mathcal{D}_{\frac{1}{2}}\left(m\ {}^{a_{2,1}}\ m_\theta\ a\,;L_1L_2^2, \frac{1}{L_1 z}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta m}\frac{{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\,;\Lambda_1\Lambda_2^2, \frac{1}{\Lambda_1 z}\right)}{{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)} \sim \\ \sim e^{-\frac{\theta}{2}\partial_m F}e^{\theta L_1 z/2}L_1^{-\frac{1}{2}-\theta m}z^{-\theta m}(1+\mathcal{O}(L_1L_2^2, 1/L_1 z))\,.\end{aligned} \tag{3.3.172}$$

Here

$$F = \lim_{b\to 0} b^2 \log \left[ (\Lambda_1\Lambda_2^2)^{-\Delta}\, {}_1\mathfrak{F}_{\frac{1}{2}} \left( \mu\, \alpha ; \Lambda_1\Lambda_2^2 \right) \right] . \tag{3.3.173}$$

Both these blocks satisfy the same BPZ equation (3.3.169). Analogously to the previous confluences, in the connection formula between 0 and $\infty$ we have four different $\mathfrak{D}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$${}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu^{\alpha_{2,1}}\ \mu_\theta\ \alpha_{\theta'} ; \Lambda_1\Lambda_2^2, \frac{1}{\Lambda_1 z}\right) \sim \left(\Lambda_1\Lambda_2^2\right)^{\theta' a} e^{-\frac{\theta'}{2}\partial_a F}\, {}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu^{\alpha_{2,1}}\ \mu_\theta\ \alpha ; \Lambda_1\Lambda_2^2, \frac{1}{\Lambda_1 z}\right), \quad \text{as } b \to 0\,, \tag{3.3.174}$$

as in (3.3.34). Now that we have defined the semiclassical conformal blocks, we state the connection formula. (3.3.167) in the semiclassical limit becomes

$${}_1\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(m\, a\, a_{2,1} ; L_1L_2^2, \tfrac{\sqrt{z}}{L_2}\right) = \textstyle\sum_{\theta'} \left(\sum_{\sigma=\pm} \mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{N}_{(-\sigma)\theta'}(a,m) \left(L_1L_2^2\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}\right) {}_1\mathcal{D}_{\frac{1}{2}}\left(m^{a_{2,1}}\ m_{\theta'}\ a ; L_1L_2^2, \tfrac{1}{L_1 z}\right) . \tag{3.3.175}$$

With connection coefficients[12]

$$\begin{aligned} &\sum_{\sigma=\pm} \mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{N}_{(-\sigma)\theta'}(a,m) \left(L_1L_2^2\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} = \\ &= \frac{1}{\sqrt{2\pi}} \sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)}{\Gamma\left(\frac{1}{2} - \theta' m - \sigma a\right)} \left(\frac{L_1L_2^2}{4}\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\sigma a\right)} e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m-\sigma a\right)} . \end{aligned} \tag{3.3.176}$$

Note that the factors of $b$ appearing in (3.3.167) precisely combine with all the factors of $\Lambda_1, \Lambda_2$ to give the finite $L_1, L_2$.

### 3.3.6 Doubly reduced doubly confluent conformal blocks

#### General case

Decoupling the last mass we land on the last correlator of our interest, which solves the BPZ equation

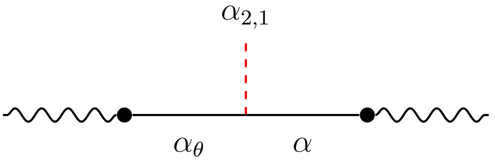
$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z - \frac{\Lambda_1^2}{4}\frac{1}{z} + \frac{\Lambda_2^2\partial_{\Lambda_2^2}}{z^2} - \frac{\Lambda_2^2}{4}\frac{1}{z^3}\right) \langle \Lambda_1^2 | \Phi(z) | \Lambda_2^2 \rangle = 0\,, \tag{3.3.177}$$

Again, one of the parameters among $\Lambda_1, z, \Lambda_2$ is redundant and can be set to an arbitrary value via a rescaling. We keep them generic for convenience. We can decompose the above correlator into conformal blocks in three different regions, that is for $z \ll \Lambda_2^2 \ll 1$, $z \gg \Lambda_1^{-2} \gg 1$, or for $z$ in the intermediate region $\Lambda_2^2 \ll z \ll \Lambda_1^{-2}$. The conformal block in the intermediate region is again a block that can be expressed as a collision limit

$${}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha ; \Lambda_1^2 z, \frac{\Lambda_2^2}{z}\right) = (\Lambda_1^2)^{\Delta_\theta} (\Lambda_2^2)^{\Delta} z^{\frac{bQ}{2}+\theta b\alpha} \lim_{\eta\to\infty} {}_1\widetilde{\mathfrak{F}}_{\frac{1}{2}}\left(\eta\ \alpha_\theta{}^{\alpha_{2,1}}\ \alpha\ ; \frac{-\Lambda_1^2}{4\eta} z, \frac{\Lambda_2^2}{z}\right) . \tag{3.3.178}$$

This conformal block can also be computed directly by projecting $|\Lambda_2^2\rangle$ onto the Verma module $\Delta$, then doing the OPE of $\Phi(z)$ term by term with the resulting expansion and then contracting with $\langle\Lambda_1^2|$. In diagrams we represent it by

$${}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha ; \Lambda_1^2 z, \frac{\Lambda_2^2}{z}\right) =$$



. (3.3.179)

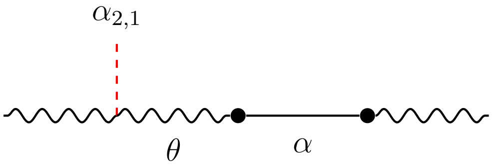
---

[12] Note that the Gamma functions in the denominator precisely correspond to the one-loop factor of the single hypermultiplet of the corresponding AGT dual gauge theory.

The block corresponding to the expansion for $z \gg \Lambda_1^{-2}$

$${}_{\frac12}\mathfrak{E}^{(\theta)}_{\frac12}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\sim(\Lambda_1^2)^{\Delta_{2,1}+\Delta}(\Lambda_2^2)^{\Delta}(\Lambda_1\sqrt{z})^{\frac12+b^2}e^{\theta b\Lambda_1\sqrt{z}}\left[1+\mathcal{O}\left(\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\right]$$

$=$

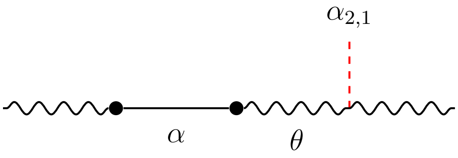


, (3.3.180)

and similarly for the expansion for $z \ll \Lambda_2^2$

$$z^{-2\Delta_{2,1}}{}_{\frac12}\mathfrak{E}^{(\theta)}_{\frac12}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\sim(\Lambda_1^2)^{\Delta}(\Lambda_2^2)^{\Delta_{2,1}+\Delta}z^{-2\Delta_{2,1}}\left(\frac{\sqrt{z}}{\Lambda_2}\right)^{-\frac12-b^2}e^{\theta b\Lambda_2/\sqrt{z}}\left[1+\mathcal{O}\left(\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\right]$$

$=$



. (3.3.181)

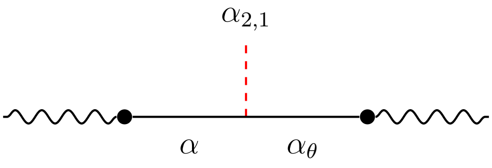
To connect the intermediate region with $z \sim 0$ we decompose the correlator as

$$\begin{aligned}\langle\Lambda_1^2|\Phi(z)|\Lambda_2^2\rangle&=\int d\alpha\,C_\alpha G_\alpha^{-1}\sum_{\theta=\pm}C^{\alpha_\theta}_{\alpha_{2,1},\alpha}C_{\alpha_\theta}\left|{}_{\frac12}\mathfrak{F}_{\frac12}\left(\alpha\,\alpha_{2,1}\,\alpha_\theta\,;\Lambda_1^2 z,\frac{\Lambda_2^2}{z}\right)\right|^2=\\&=\int d\alpha\,C_\alpha G_\alpha^{-1}\sum_{\theta'=\pm}A_{-\frac{b}{2}}C_\alpha\left|z^{-2\Delta_{2,1}}{}_{\frac12}\mathfrak{E}^{(\theta')}_{\frac12}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\right|^2.\end{aligned}\tag{3.3.182}$$

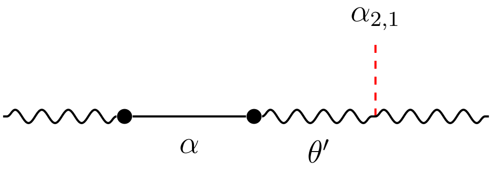
This is the same constraint as in (3.2.36). Therefore the connection formula is

$$b^{2\theta b\alpha}{}_{\frac12}\mathfrak{F}_{\frac12}\left(\alpha\,\alpha_{2,1}\,\alpha_\theta\,;\Lambda_1^2 z,\frac{\Lambda_2^2}{z}\right)=\sum_{\theta'=\pm}b^{-\frac12}\mathcal{Q}_{\theta\theta'}(b\alpha)z^{-2\Delta_{2,1}}{}_{\frac12}\mathfrak{E}^{(\theta')}_{\frac12}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right),\tag{3.3.183}$$

Diagrammatically



$$=\sum_{\theta'=\pm}\mathcal{Q}_{\theta\theta'}$$

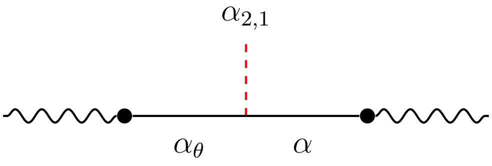


. (3.3.184)

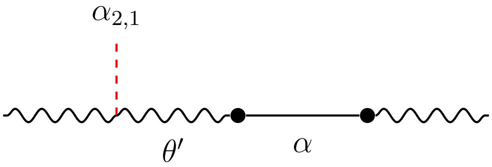
Similarly, the connection formula between the intermediate region and $\infty$ is

$$b^{2\theta b\alpha}{}_{\frac12}\mathfrak{F}_{\frac12}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha\,;\Lambda_1^2 z,\frac{\Lambda_2^2}{z}\right)=\sum_{\theta'=\pm}b^{-\frac12}\mathcal{Q}_{\theta\theta'}(b\alpha){}_{\frac12}\mathfrak{E}^{(\theta')}_{\frac12}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right).\tag{3.3.185}$$

In diagrams:



$$=\sum_{\theta'=\pm}\mathcal{Q}_{\theta\theta'}$$

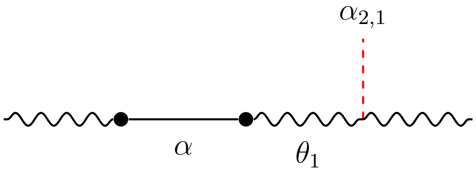


. (3.3.186)

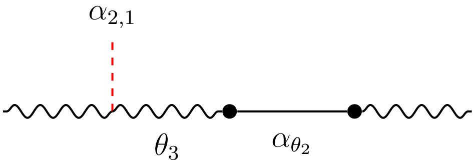
As in the previous cases, we can easily obtain a connection formula connecting the two irregular singularities, namely

$$b^{-\frac{1}{2}}z^{-2\Delta_{2,1}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta_1)}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\tfrac{\sqrt{z}}{\Lambda_2}\right)=\textstyle\sum_{\theta_2,\theta_3=\pm}b^{2\theta_2 b\alpha}\mathcal{Q}^{-1}_{\theta_1\theta_2}(b\alpha)b^{-\frac{1}{2}+2\theta_2 b\alpha}\mathcal{Q}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2})\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta_3)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha_{\theta_2}\,;\Lambda_1^2\Lambda_2^2,\tfrac{1}{\Lambda_1\sqrt{z}}\right)\,. \tag{3.3.187}$$

Diagrammatically:



$$= \textstyle\sum_{\theta_2,\theta_3=\pm}\mathcal{Q}^{-1}_{\theta_1\theta_2}\mathcal{Q}_{(-\theta_2)\theta_3}$$



$$. \tag{3.3.188}$$

### Semiclassical limit

The BPZ equation in this limit becomes

$$\left(\partial_z^2-\frac{L_1^2}{4z}+\frac{u}{z^2}-\frac{L_2^2}{4z^3}\right){}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}=0\,. \tag{3.3.189}$$

for *any* semiclassical block. Here $u$ is given by

$$u=\lim_{b\to 0}b^2\Lambda_2^2\partial_{\Lambda_2^2}\log{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)=\frac{1}{4}-a^2+\mathcal{O}(L_1^2L_2^2)\,, \tag{3.3.190}$$

where ${}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)$ is the conformal block corresponding to $\langle\Lambda_1^2|\Lambda_2^2\rangle$ with intermediate momentum $\alpha$. The finite semiclassical conformal blocks are defined as before by normalizing by the same block without the degenerate field insertion, i.e. for $z\sim 0$

$$z\,{}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;L_1^2L_2^2,\tfrac{\sqrt{z}}{L_2}\right)=\lim\nolimits_{b\to 0}b^{-1/2}\frac{z^{-2\Delta_{2,1}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)}{{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)}\sim e^{\theta L_2/\sqrt{z}}L_2^{-1/2}z^{3/4}(1+\mathcal{O}(L_1^2L_2^2,\sqrt{z}/L_2))\,. \tag{3.3.191}$$

For $z\sim\infty$ instead we have

$${}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a_{2,1}\,a\,;L_1^2L_2^2,\tfrac{1}{L_1\sqrt{z}}\right)=\lim\nolimits_{b\to 0}b^{-1/2}\frac{{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)}{{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)}\sim e^{\theta L_1\sqrt{z}}L_1^{-1/2}z^{1/4}(1+\mathcal{O}(L_1^2L_2^2,1/L_1\sqrt{z}))\,. \tag{3.3.192}$$

Here

$$F=\lim_{b\to 0}b^2\log\left[(\Lambda_1^2\Lambda_2^2)^{-\Delta}\,{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)\right]\,. \tag{3.3.193}$$

Both these blocks satisfy the same BPZ equation (3.3.189). Analogously to the previous confluences, in the connection formula between 0 and $\infty$ we have four different $\mathfrak{E}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha_{\theta'}\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\sim\left(\Lambda_1^2\Lambda_2^2\right)^{\theta' a}e^{-\frac{\theta'}{2}\partial_a F}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\,,\quad\text{as } b\to 0\,, \tag{3.3.194}$$

as in (3.3.34). Now that we have defined the semiclassical conformal blocks, we state the connection formula. (3.3.187) in the semiclassical limit becomes

$$z\,{}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;L_1^2L_2^2,\tfrac{\sqrt{z}}{L_2}\right)=\textstyle\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{Q}_{(-\sigma)\theta'}(a)\left(L_1L_2\right)^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right){}_{\frac{1}{2}}\mathcal{E}^{(\theta')}_{\frac{1}{2}}\left(a_{2,1}\,a\,;L_1^2L_2^2,\tfrac{1}{L_1\sqrt{z}}\right)\,. \tag{3.3.195}$$

With connection coefficients[13]

$$
\begin{aligned}
&\sum_{\sigma=\pm} \mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{Q}_{(-\sigma)\theta'}(a)\left(L_1L_2\right)^{2\sigma a} e^{-\frac{\sigma}{2}\partial_a F} = \\
=&\frac{1}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\left(\frac{L_1L_2}{4}\right)^{2\sigma a} e^{-\frac{\sigma}{2}\partial_a F} e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\sigma a\right)} e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-2\sigma a\right)} \,.
\end{aligned}
\tag{3.3.196}
$$

Note that the factors of $b$ appearing in (3.3.187) precisely combine with all the factors of $\Lambda_1, \Lambda_2$ to give the finite $L_1, L_2$.

## 3.4 Heun equations, confluences and connection formulae

In this section we derive the explicit connection formulae for Heun functions and its confluences by identifying the semi-classical conformal blocks with the Heun functions and using the results so far obtained.

### 3.4.1 The Heun equation

In the following we identify the semiclassical BPZ equation (3.3.28) with Heun's equation via a dictionary between the relevant parameters. Moreover, we establish a precise relation between the Heun functions and the semiclassical *regular* conformal blocks. This is further used to obtain explicit formulae for the relevant connection coefficients. WLOG, we focus on the case $t \sim 0$. The connection formulae for $t \sim 1$, $t \sim \infty$ can be easily derived by matching the Heun equation and its local solutions with the corresponding semiclassical BPZ equations and the associated semiclassical conformal blocks.

#### The dictionary

Let us start giving the dictionary with CFT. The Heun equation reads

$$
\begin{aligned}
&\left(\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}\right) w(z) = 0\,, \\
&\alpha + \beta + 1 = \gamma + \delta + \epsilon\,,
\end{aligned}
\tag{3.4.1}
$$

where the condition $\alpha + \beta + 1 = \gamma + \delta + \epsilon$ ensures that the exponents of the local solutions at infinity are given by $\alpha, \beta$. Here and in the following we restrict to generic values of the parameters. Define $w(z) = P_4(z)\psi(z)$ with

$$
P_4(z) = z^{-\gamma/2}(1-z)^{-\delta/2}(t-z)^{-\epsilon/2}\,. \tag{3.4.2}
$$

$\psi(z)$ then satisfies the Heun equation in normal form, which is easily compared with the semi-classical BPZ equation (3.3.28). We get $2^4 = 16$ dictionaries corresponding to the $(\mathbb{Z}_2)^4$ sym-

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[13] Note also that there are no Gamma functions in the denominator corresponding to the fact that we have no hypermultiplets in the corresponding AGT dual gauge theory.

metry associated to flipping the signs of the momenta. We choose the following:

$$\begin{aligned} a_0 &= \frac{1-\gamma}{2}\,, \\ a_1 &= \frac{1-\delta}{2}\,, \\ a_t &= \frac{1-\epsilon}{2}\,, \\ a_\infty &= \frac{\alpha-\beta}{2}\,, \\ u^{(0)} &= \frac{-2q+2t\alpha\beta+\gamma\epsilon-t(\gamma+\delta)\epsilon}{2(t-1)}\,. \end{aligned} \tag{3.4.3}$$

The inverse dictionary is

$$\begin{aligned} \alpha &= 1-a_0-a_1-a_t+a_\infty\,, \\ \beta &= 1-a_0-a_1-a_t-a_\infty\,, \\ \gamma &= 1-2a_0\,, \\ \delta &= 1-2a_1\,, \\ \epsilon &= 1-2a_t\,, \\ q &= \frac{1}{2}+t(a_0^2+a_t^2+a_1^2-a_\infty^2)-a_t-a_1t+a_0(2a_t-1+t(2a_1-1))+(1-t)u^{(0)}\,. \end{aligned} \tag{3.4.4}$$

The two linearly independent solutions for $z \sim 0$ of (3.3.28) are related by $a_0 \to -a_0$. This corresponds to the identification of the two linearly independent solutions of (3.4.1) for $z \sim 0$ as

$$\begin{aligned} w_-^{(0)}(z) &= \mathrm{HeunG}\left(t,q,\alpha,\beta,\gamma,\delta,z\right)\,, \\ w_+^{(0)}(z) &= z^{1-\gamma}\mathrm{HeunG}\left(t,q-(\gamma-1)(t\delta+\epsilon),\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,\delta,z\right)\,, \end{aligned} \tag{3.4.5}$$

where by definition

$$\mathrm{HeunG}\left(t,q,\alpha,\beta,\gamma,\delta,z\right) = 1+\frac{q}{t\gamma}z+\mathcal{O}(z^2)\,. \tag{3.4.6}$$

The Heun function can be identified with the semiclassical conformal blocks introduced before. In particular comparing with (3.3.25) we get the two solutions

$$\begin{aligned} w_-^{(0)}(z) &= P_4(z)\,t^{\frac{1}{2}-a_t-a_0}e^{-\frac{1}{2}\partial_{a_0}F(t)}\mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix}\, a \,{}^{a_t}\, a_{0-}\begin{matrix} a_{2,1} \\ a_0 \end{matrix}; t, \frac{z}{t}\right)\,, \\ w_+^{(0)}(z) &= P_4(z)\,t^{\frac{1}{2}-a_t+a_0}e^{\frac{1}{2}\partial_{a_0}F(t)}\mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix}\, a \,{}^{a_t}\, a_{0+}\begin{matrix} a_{2,1} \\ a_0 \end{matrix}; t, \frac{z}{t}\right)\,. \end{aligned} \tag{3.4.7}$$

Note that HeunG is an expansion in $z$, while the semiclassical conformal blocks are expanded both in $z$ and $t$. To match the two expansions one has to express the accessory parameter $q$ in terms of the Floquet exponent $a$ as a series in $t$. This can be done substituting the dictionary as explained in Appendix C.

The solutions for $z \sim t$ are given by

$$\begin{aligned} w_-^{(t)}(z) &= \mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t\alpha\beta}{1-t},\alpha,\beta,\epsilon,\delta,\frac{z-t}{1-t}\right)\,, \\ w_+^{(t)}(z) &= (t-z)^{1-\epsilon}\mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t\alpha\beta}{1-t}-(\epsilon-1)\left(\frac{t}{t-1}\delta+\gamma\right),\alpha+1-\epsilon,\beta+1-\epsilon,2-\epsilon,\delta,\frac{z-t}{1-t}\right)\,. \end{aligned} \tag{3.4.8}$$

Comparing with the semiclassical blocks (3.3.29) we get

$$\begin{aligned} w_-^{(t)}(z) &= P_4(z)\, t^{\frac{1}{2}-a_0-a_t}(1-t)^{\frac{1}{2}-a_1} e^{-\frac{1}{2}\partial_{a_t}F(t)} \left( (t-1)^{\frac{1}{2}} \mathcal{F}\left( {}^{a_1}_{a_\infty}\, a^{\,a_0}\, a_{t-}\, {}^{a_{2,1}}_{a_t}; \frac{t}{t-1}, \frac{t-z}{t} \right) \right) , \\ w_+^{(t)}(z) &= P_4(z)\, t^{\frac{1}{2}-a_0+a_t}(1-t)^{\frac{1}{2}-a_1} e^{\frac{1}{2}\partial_{a_t}F(t)} \left( (t-1)^{\frac{1}{2}} \mathcal{F}\left( {}^{a_1}_{a_\infty}\, a^{\,a_0}\, a_{t+}\, {}^{a_{2,1}}_{a_t}; \frac{t}{t-1}, \frac{t-z}{t} \right) \right) . \end{aligned} \tag{3.4.9}$$

The two solutions for $z \sim 1$ read

$$\begin{aligned} w_-^{(1)}(z) &= \left(\frac{z-t}{1-t}\right)^{-\alpha} \text{HeunG}\left(t, q+\alpha(\delta-\beta), \alpha, \delta+\gamma-\beta, \delta, \gamma, t\frac{1-z}{t-z}\right) , \\ w_+^{(1)}(z) &= \left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta} (1-z)^{1-\delta} \text{HeunG}\left(t, q-\alpha(\beta+\delta-2)+(\delta-1)(\alpha+\beta-1-t\gamma), \alpha+1-\delta, 1+\gamma-\beta, 2-\delta, \gamma, t\frac{1-z}{t-z}\right) , \end{aligned} \tag{3.4.10}$$

and matching with (3.3.37) gives

$$\begin{aligned} w_-^{(1)}(z) &= P_4(z) e^{\pm i\pi(a_1+a_t)} (1-t)^{\frac{1}{2}-a_t} e^{-\frac{1}{2}\partial_{a_1}F(t)} \left( (t(1-t))^{-\frac{1}{2}} (t-z) \mathcal{F}\left( {}^{a_0}_{a_t}\, a^{\,a_\infty}\, a_{1-}\, {}^{a_{2,1}}_{a_1}; t, \frac{1-z}{t-z} \right) \right) \\ w_+^{(1)}(z) &= P_4(z) e^{\pm i\pi(-a_1+a_t)} (1-t)^{\frac{1}{2}-a_t} e^{\frac{1}{2}\partial_{a_1}F(t)} \left( (t(1-t))^{-\frac{1}{2}} (t-z) \mathcal{F}\left( {}^{a_0}_{a_t}\, a^{\,a_\infty}\, a_{1+}\, {}^{a_{2,1}}_{a_1}; t, \frac{1-z}{t-z} \right) \right) . \end{aligned} \tag{3.4.11}$$

The ± ambuiguity in the overall phase depends on the choice of branch corresponding to

$$P_4(z) \mathcal{F}\left( {}^{a_0}_{a_t}\, a^{\,a_\infty}\, a_{1\theta}\, {}^{a_{2,1}}_{a_1}; t, \frac{1-z}{t-z} \right) \propto (t-1)^{\theta a_1 + a_t} = e^{\pm i\pi(\theta a_1 + a_t)} (1-t)^{\theta a_1 + a_t} . \tag{3.4.12}$$

Finally, the two solutions near $z \sim \infty$ are given by

$$\begin{aligned} w_+^{(\infty)}(z) &= z^{-\alpha} \text{HeunG}\left(t, q-\alpha\beta(1+t)+\alpha(\delta+t\epsilon), \alpha, \alpha-\gamma+1, \alpha-\beta+1, \alpha+\beta+1-\gamma-\delta, \frac{t}{z}\right) , \\ w_-^{(\infty)}(z) &= z^{-\beta} \text{HeunG}\left(t, q-\alpha\beta(1+t)+\beta(\delta+t\epsilon), \beta, \beta-\gamma+1, \beta-\alpha+1, \alpha+\beta+1-\gamma-\delta, \frac{t}{z}\right) . \end{aligned} \tag{3.4.13}$$

Comparing with (3.3.31) we get

$$\begin{aligned} w_+^{(\infty)}(z) &= P_4(z) e^{\pm i\pi(1-a_1-a_t)} e^{\frac{1}{2}\partial_{a_\infty}F(t)} \left( t^{-\frac{1}{2}} z \mathcal{F}\left( {}^{a_t}_{a_0}\, a^{\,a_1}\, a_{\infty+}\, {}^{a_{2,1}}_{a_\infty}; t, \frac{1}{z} \right) \right) , \\ w_-^{(\infty)}(z) &= P_4(z) e^{\pm i\pi(1-a_1-a_t)} e^{-\frac{1}{2}\partial_{a_\infty}F(t)} \left( t^{-\frac{1}{2}} z \mathcal{F}\left( {}^{a_t}_{a_0}\, a^{\,a_1}\, a_{\infty-}\, {}^{a_{2,1}}_{a_\infty}; t, \frac{1}{z} \right) \right) , \end{aligned} \tag{3.4.14}$$

where again the ± in the phase depends on the choice of branch corresponding to

$$P_4(z) = z^{-\frac{1}{2}+a_0}(1-z)^{-\frac{1}{2}+a_1}(t-z)^{-\frac{1}{2}+a_t} = e^{\mp i\pi(1-a_1-a_t)} z^{-\frac{1}{2}+a_0}(z-1)^{-\frac{1}{2}+a_1}(z-t)^{-\frac{1}{2}+a_t} . \tag{3.4.15}$$

### Connection formulae

Finally we are in the position to give the connection formulae for the Heun function. Let us start with $z \sim 0$ and $z \sim t$. The corresponding connection formula can be read off from (3.3.30), which in the Heun notation reads

$$w_-^{(0)}(z) = \frac{\Gamma(1-\epsilon)\Gamma(\gamma) e^{\frac{1}{2}\left(\partial_{a_t}-\partial_{a_0}\right)F}}{\Gamma\left(\frac{1+\gamma-\epsilon}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\epsilon}{2}-a(q)\right)} (1-t)^{-\frac{\delta}{2}} w_-^{(t)}(z) + \frac{\Gamma(\epsilon-1)\Gamma(\gamma) e^{\frac{1}{2}\left(-\partial_{a_t}-\partial_{a_0}\right)F}}{\Gamma\left(\frac{-1+\gamma+\epsilon}{2}+a(q)\right)\Gamma\left(\frac{-1+\gamma+\epsilon}{2}-a(q)\right)} t^{\epsilon-1}(1-t)^{-\frac{\delta}{2}} w_+^{(t)}(z) , \tag{3.4.16}$$

for the other solution one finds

$$w_+^{(0)}(z)=\frac{\Gamma(1-\epsilon)\Gamma(2-\gamma)e^{\frac{1}{2}\left(\partial_{a_t}+\partial_{a_0}\right)F}}{\Gamma\left(1+\frac{1-\gamma-\epsilon}{2}+a(q)\right)\Gamma\left(1+\frac{1-\gamma-\epsilon}{2}-a(q)\right)}t^{1-\gamma}(1-t)^{-\frac{\delta}{2}}w_-^{(t)}(z)+\frac{\Gamma(\epsilon-1)\Gamma(\gamma)e^{\frac{1}{2}\left(-\partial_{a_t}+\partial_{a_0}\right)F}}{\Gamma\left(\frac{1-\gamma+\epsilon}{2}+a(q)\right)\Gamma\left(\frac{1-\gamma+\epsilon}{2}-a(q)\right)}t^{\epsilon-\gamma}(1-t)^{-\frac{\delta}{2}}w_+^{(t)}(z)\,. \tag{3.4.17}$$

Here $a(q)$ has to be computed inverting the relation (3.3.27) and substituting the dictionary as shown explicitly in Appendix C, formula (C.1.13). The result to first order is

$$\begin{aligned}a(q)=&\frac{1}{16}\sqrt{3-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2)}\times\\ \times&\left(8-\frac{4(-1+2q-\epsilon(\gamma+\epsilon-2))(-3+4q+(\alpha-\beta)-\gamma^2-\delta(\delta-2)-2\gamma(\epsilon-1)-\epsilon(\epsilon-2))}{(3-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2))(2-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2))}t\right)+\mathcal{O}(t^2)\,.\end{aligned} \tag{3.4.18}$$

In Appendix C we also explain how to compute the classical conformal block $F$ and its derivatives (see formula (C.1.10)). For example, to first order

$$\partial_{a_t}F(t)=\frac{\left(4a(q)^2-\alpha^2+2\alpha\beta-\beta^2-2\delta+\delta^2\right)(1-\epsilon)}{2-8a(q)^2}t+\mathcal{O}(t^2)\,. \tag{3.4.19}$$

The connection formula for $w_+^{(0)}(z)$ can be obtained from (3.4.16) by multiplying by $z^{1-\gamma}$, substituting

$$q\to q-(\gamma-1)(t\delta+\epsilon),\ \alpha\to\alpha+1-\gamma,\ \beta\to\beta+1-\gamma,\ \gamma\to 2-\gamma \tag{3.4.20}$$

as in (3.4.5), and noting that

$$\begin{aligned}&\mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t\alpha\beta}{1-t},\alpha,\beta,\epsilon,\delta,\frac{z-t}{1-t}\right)=\\ &=\left(\frac{z}{t}\right)^{1-\gamma}\mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t(\alpha+1-\gamma)(\beta+1-\gamma)-(\gamma-1)(t\delta+\epsilon)}{1-t},\alpha+1-\gamma,\beta+1-\gamma,\epsilon,\delta,\frac{z-t}{1-t}\right)\,.\end{aligned} \tag{3.4.21}$$

Similarly, the connection formula from $z\sim 0$ to $z\sim\infty$ can be read off from (3.3.35), and gives

$$\begin{aligned}w_-^{(0)}(z)=&\left(\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\Gamma(\gamma)\Gamma(\beta-\alpha)t^{\frac{\gamma+\epsilon-1}{2}-\sigma a(q)}e^{-\frac{1}{2}\left(\partial_{a_0}-\partial_{a_\infty}+\sigma\partial_a\right)F}e^{i\pi\left(\frac{\delta+\gamma}{2}\right)}}{\Gamma\left(\frac{\gamma-\epsilon+1}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\epsilon-1}{2}-\sigma a(q)\right)\Gamma\left(1+\frac{\beta-\alpha-\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\beta-\alpha+\delta}{2}-\sigma a(q)\right)}\right)w_+^{(\infty)}(z)+\\ &+\left(\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\Gamma(\gamma)\Gamma(\alpha-\beta)t^{\frac{\gamma+\epsilon-1}{2}-\sigma a(q)}e^{-\frac{1}{2}\left(\partial_{a_0}-\partial_{a_\infty}+\sigma\partial_a\right)F}e^{i\pi\left(\frac{\delta+\gamma}{2}\right)}}{\Gamma\left(\frac{\gamma-\epsilon+1}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\epsilon-1}{2}-\sigma a(q)\right)\Gamma\left(1+\frac{\alpha-\beta-\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\alpha-\beta+\delta}{2}-\sigma a(q)\right)}\right)w_-^{(\infty)}(z)\,.\end{aligned} \tag{3.4.22}$$

Let us conclude the section by giving the connection formulae from 1 to infinity. This can be derived from (3.3.20), and gives

$$\begin{aligned}w_-^{(1)}(z)=&-(1-t)^{\frac{1}{2}-a_t}\frac{\Gamma(\beta-\alpha)\Gamma(\delta)e^{-\frac{1}{2}\left(\partial_{a_1}+\partial_{a_\infty}\right)F(t)}}{\Gamma\left(\frac{\delta-\alpha+\beta}{2}+a(q)\right)\Gamma\left(\frac{\delta-\alpha+\beta}{2}-a(q)\right)}w_+^{(\infty)}(z)+\\ &-(1-t)^{\frac{1}{2}-a_t}\frac{\Gamma(\alpha-\beta)\Gamma(\delta)e^{-\frac{1}{2}\left(\partial_{a_1}-\partial_{a_\infty}\right)F(t)}}{\Gamma\left(\frac{\delta+\alpha-\beta}{2}+a(q)\right)\Gamma\left(\frac{\delta+\alpha-\beta}{2}-a(q)\right)}w_-^{(\infty)}(z)\,.\end{aligned} \tag{3.4.23}$$

The connection formulae involving the other solutions can be read off from the previous ones, and the formulae involving different pairs of points can be similarly derived by considering the corresponding semiclassical conformal blocks. We conclude by stressing again that the connection formulae involving different regions in the $t-$plane are completely analogous to the previous ones, since all the singularities are regular. This will not be the case in the following.

### 3.4.2 The confluent Heun equation

#### The dictionary

Here we establish the dictionary between our results of section 3.3.2 on confluent conformal blocks and the confluent Heun equation (CHE) in standard notation, which reads

$$\frac{d^2w}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)\frac{dw}{dz}+\frac{\alpha z-q}{z(z-1)}w=0\,. \tag{3.4.24}$$

By defining $w(z)=P_3(z)\psi(z)$ with $P_3(z)=e^{-\epsilon z/2}z^{-\gamma/2}(1-z)^{-\delta/2}$, we get rid of the first derivative and bring the equation to normal form, which can easily be compared with the semiclassical BPZ equation (3.3.82). We can read off the dictionary between the CFT parameters and the parameters of the CHE:

$$\begin{aligned}
&a_0=\frac{1-\gamma}{2}\,,\\
&a_1=\frac{1-\delta}{2}\,,\\
&m=\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}\,,\\
&L=\epsilon\,,\\
&u=\frac{1}{4}-q+\alpha-\frac{(\gamma+\delta-1)^2}{4}-\frac{\delta\epsilon}{2}\,,
\end{aligned} \tag{3.4.25}$$

where

$$u=\lim_{b\to0}b^2\Lambda\partial_\Lambda\log{}_1\mathfrak{F}\left(\mu\,\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda\right)=\frac{1}{4}-a^2+\mathcal{O}(L) \tag{3.4.26}$$

as in (3.3.82). This relation can then be inverted to find $a$ in terms of the parameters of the CHE: we denote this by $a(q)$. We write the solutions to the CHE in standard form in the notation of Mathematica, and their relation to the conformal blocks used before. We focus first on the blocks given as an expansion for small $L$. Then, near $z=0$ we have the two linearly independent solutions

$$\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)\,,\\
&z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta),\alpha+(1-\gamma)\epsilon,2-\gamma,\delta,\epsilon;z\right)\,,
\end{aligned} \tag{3.4.27}$$

where the confluent Heun function has the following expansion around $z=0$:

$$\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)=1-\frac{q}{\gamma}z+\mathcal{O}(z^2)\,. \tag{3.4.28}$$

Comparing with the semiclassical conformal blocks in 3.3.2 we identify

$$\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)=P_3(z)e^{-\frac{1}{2}\partial_{a_0}F}{}_1\mathcal{F}\left(m\;a\,{}^{a_1}\,a_{0-}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};L,z\right)\,,\\
&z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta),\alpha+(1-\gamma)\epsilon,2-\gamma,\delta,\epsilon;z\right)=P_3(z)e^{\frac{1}{2}\partial_{a_0}F}{}_1\mathcal{F}\left(m\;a\,{}^{a_1}\,a_{0+}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};L,z\right)\,,
\end{aligned} \tag{3.4.29}$$

where

$$F=\lim_{b\to0}b^2\log\left[\Lambda^{-\Delta}{}_1\mathfrak{F}\left(\mu\,\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda\right)\right]\,. \tag{3.4.30}$$

Doing a Möbius transformation $z \to 1 - z$ we obtain solutions around $z = 1$, which being a regular singularity can again be written in terms of HeunC. This amounts to sending $\gamma \to \delta$, $\delta \to \gamma$, $\epsilon \to -\epsilon$, $\alpha \to -\alpha$, $q \to q - \alpha$. The two solutions are therefore

$$\begin{aligned} &\mathrm{HeunC}(q-\alpha, -\alpha, \delta, \gamma, -\epsilon; 1-z)\,, \\ &(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma), -\alpha-(1-\delta)\epsilon, 2-\delta, \gamma, -\epsilon; 1-z\right)\,. \end{aligned} \tag{3.4.31}$$

Again, comparing with the semiclassical conformal blocks in 3.3.2, we identify

$$\begin{aligned} &\mathrm{HeunC}(q-\alpha, -\alpha, \delta, \gamma, -\epsilon; 1-z) = P_3(z)e^{-\frac{1}{2}\partial_{a_1}F}{}_1\mathcal{F}\left(-m\ a \overset{a_0}{}\ a_{1-} \genfrac{}{}{0pt}{}{a_{2,1}}{a_1}; L, 1-z\right)\,, \\ &(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma), -\alpha-(1-\delta)\epsilon, 2-\delta, \gamma, -\epsilon; 1-z\right) = \\ &\quad = P_3(z)e^{\frac{1}{2}\partial_{a_1}F}{}_1\mathcal{F}\left(-m\ a \overset{a_0}{}\ a_{1+} \genfrac{}{}{0pt}{}{a_{2,1}}{a_1}; L, 1-z\right)\,. \end{aligned} \tag{3.4.32}$$

Around the irregular singular point $z = \infty$, we write the solutions in terms of a different function $\mathrm{HeunC}_\infty$:

$$\begin{aligned} &z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q, \alpha, \gamma, \delta, \epsilon; z^{-1}) \\ &e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon, \alpha-\epsilon(\gamma+\delta), \gamma, \delta, -\epsilon; z^{-1})\,, \end{aligned} \tag{3.4.33}$$

where the function $\mathrm{HeunC}_\infty$ has a simple asymptotic expansion around $z = \infty$:

$$\mathrm{HeunC}_\infty(q, \alpha, \gamma, \delta, \epsilon; z^{-1}) \sim 1 + \frac{\alpha^2 - (\gamma+\delta-1)\alpha\epsilon + (\alpha-q)\epsilon^2}{\epsilon^3}z^{-1} + \mathcal{O}(z^{-2})\,. \tag{3.4.34}$$

Comparing with the semiclassical conformal blocks we identify

$$\begin{aligned} &z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q, \alpha, \gamma, \delta, \epsilon; z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{\frac{1}{2}\partial_m F}L^{\frac{1}{2}+m}{}_1\mathcal{D}\left(m^{a_{2,1}}\ m_+\ a \genfrac{}{}{0pt}{}{a_1}{a_0}; L, \frac{1}{z}\right) \\ &e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon, \alpha-\epsilon(\gamma+\delta), \gamma, \delta, -\epsilon; z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{-\frac{1}{2}\partial_m F}L^{\frac{1}{2}-m}{}_1\mathcal{D}\left(m^{a_{2,1}}\ m_-\ a \genfrac{}{}{0pt}{}{a_1}{a_0}; L, \frac{1}{z}\right)\,. \end{aligned} \tag{3.4.35}$$

The phase $e^{\mp\frac{i\pi\delta}{2}}$ comes from the fact that near $z = \infty$

$$P_3(z) \sim e^{-\epsilon z/2}z^{-\gamma/2}(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}e^{-\epsilon z/2}z^{-\gamma/2-\delta/2}\,. \tag{3.4.36}$$

The second solution around $z = \infty$ can be found by using the manifest symmetry $(m, L) \to (-m, -L)$ of the semiclassical BPZ equation which according to the dictionary gives the symmetry $(q, \alpha, \epsilon) \to (q - \gamma\epsilon, \alpha - \epsilon(\gamma+\delta), -\epsilon)$ of the CHE in normal form.
For the large-$L$ blocks the story is analogous. The dictionary (3.4.25) is the same, up to the substitution

$$u \to u_D = \mathrm{lim}_{b\to 0}\, b^2\Lambda\partial_\Lambda \log {}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{}\ \mu'\ \alpha_0; \tfrac{1}{\Lambda}\right) = -(m'-m)L + \tfrac{1}{4} - a_0^2 + 2m'(m'-m) + \mathcal{O}(L^{-1})\,. \tag{3.4.37}$$

This relation can be inverted to find $m'$ in terms of the parameters of the CHE. We will call this $m'(q)$. With this dictionary we can identify solutions of the CHE with conformal blocks as follows: near $z = 0$ we have

$$\begin{aligned} &\mathrm{HeunC}(q, \alpha, \gamma, \delta, \epsilon; z) = P_3(z)e^{-\frac{1}{2}\partial_{a_0}F_D}\,{}_1\mathcal{D}\left(m^{a_1}\ m'\ a_{0-} \genfrac{}{}{0pt}{}{a_{2,1}}{a_0}; \frac{1}{L}, Lz\right)\,, \\ &z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta), \alpha+(1-\gamma)\epsilon, 2-\gamma, \delta, \epsilon; z\right) = P_3(z)e^{\frac{1}{2}\partial_{a_0}F_D}\,{}_1\mathcal{D}\left(m^{a_1}\ m'\ a_{0+} \genfrac{}{}{0pt}{}{a_{2,1}}{a_0}; \frac{1}{L}, Lz\right)\,, \end{aligned} \tag{3.4.38}$$

with $F_D$ given in (3.3.93). Near $z = 1$ we have

$$\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z) = P_3(z)e^{-\frac{1}{2}\partial_{a_1}F_D}\,{}_1\mathcal{D}\left( -m \,{}^{a_0}\, m' - m \; a_{1-} \, \overset{a_{2,1}}{a_1} ; \frac{1}{L}, L(1-z)\right) , \\
&(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma), -\alpha-(1-\delta)\epsilon, 2-\delta,\gamma,-\epsilon;1-z\right) = \\
&\qquad = P_3(z)e^{\frac{1}{2}\partial_{a_1}F_D}\,{}_1\mathcal{D}\left( -m \,{}^{a_0}\, m' - m \; a_{1+} \, \overset{a_{2,1}}{a_1} ; \frac{1}{L}, L(1-z)\right) .
\end{aligned} \tag{3.4.39}$$

While near $z = \infty$ we have

$$\begin{aligned}
&z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{L/2}e^{\frac{1}{2}\partial_m F_D}L^{\frac{1}{2}-(m'-m)}{}_1\mathcal{D}\left( m \;{}^{a_{2,1}}\; m_+ \;{}^{a_1}\; m' \; a_0 ; \frac{1}{L},\frac{1}{z}\right) \\
&e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{-L/2}e^{-\frac{1}{2}\partial_m F_D}L^{\frac{1}{2}+(m'-m)}{}_1\mathcal{D}\left( m \;{}^{a_{2,1}}\; m_+ \;{}^{a_1}\; m' \; a_0 ; \frac{1}{L},\frac{1}{z}\right) .
\end{aligned} \tag{3.4.40}$$

As the careful reader should have noticed, we identify the small-$L$ and large-$L$ conformal blocks with the same confluent Heun functions. The only difference is in the expansion of the accessory parameter: in one case it is given in terms of the Floquet exponent $a$ as an expansion in $L$, and in the other case in terms of the parameter $m'$ as an expansion in $L^{-1}$.

### Connection formulae

The connection formula between $z = 0, 1$ written in (3.3.91) for the semiclassical conformal blocks can now be restated as:

$$\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z) = \frac{\Gamma(1-\delta)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F+\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1+\gamma-\delta}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\delta}{2}-a(q)\right)}\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z) + \\
&+\frac{\Gamma(\delta-1)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F-\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{\gamma+\delta-1}{2}+a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-a(q)\right)}(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right) .
\end{aligned} \tag{3.4.41}$$

The quantities $a(q)$ and $F$ can be computed as explained in Appendix C.
The connection formula between $z = 1, \infty$ written in (3.3.89) reads in terms of confluent Heun functions:

$$\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z) = \\
&= \left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\epsilon^{-\frac{1}{2}-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F+\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F(a)}}{\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)\Gamma\left(\frac{1+\gamma+\delta}{2}-\frac{\alpha}{\epsilon}-\sigma a(q)\right)}\right)\times \\
&\times z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z) + \\
&+\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\epsilon^{-\frac{1}{2}+\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F+\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F(a)}}{\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)\Gamma\left(\frac{1-\gamma-\delta}{2}+\frac{\alpha}{\epsilon}-\sigma a(q)\right)}\right)\times \\
&\times e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z) .
\end{aligned} \tag{3.4.42}$$

Here the phase ambiguity comes from (3.4.35), i.e. corresponds to the choice $(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}z^{-\delta/2}$. A similar expression can be found connecting $z = 0$ and $\infty$. All connection coefficients given above are calculated in a series expansion in $L$. Therefore they are not valid for large $L$ and in that case one has to use different connection formulae, which are derived in section 3.3.2 for the large-$L$ semiclassical conformal blocks. Here we restate those results in the language of Heun functions. The connection formula from $z = 0$ to $z = 1$, valid for large $L$ is given by

$$\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z) = \\
&=\left(\sum_{\sigma=\pm}\frac{\Gamma(\gamma)\Gamma(1-\delta)e^{\frac{\sigma}{2}\epsilon}\epsilon^{-\sigma\left(2m'(q)-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}\right)}e^{-\frac{1}{2}\partial_{a_0}F_D+\frac{1}{2}\partial_{a_1}F_D-\frac{\sigma}{2}\partial_{m'}F_D}e^{i\pi\left(\frac{1-\sigma}{2}\right)\left(\frac{\alpha}{\epsilon}-\delta-2m'(q)\right)}}{\Gamma\left(\frac{\gamma}{2}-\sigma m'(q)\right)\Gamma\left(1-\frac{\delta}{2}-\sigma\left(m'(q)-\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}\right)\right)}\right)\times \\
&\quad\times \mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)+ \\
&+\left(\sum_{\sigma=\pm}\frac{\Gamma(\gamma)\Gamma(\delta-1)e^{\frac{\sigma}{2}\epsilon}\epsilon^{-\sigma\left(2m'(q)-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}\right)}e^{-\frac{1}{2}\partial_{a_0}F_D-\frac{1}{2}\partial_{a_1}F_D-\frac{\sigma}{2}\partial_{m'}F_D}e^{i\pi\left(\frac{1-\sigma}{2}\right)\left(\frac{\alpha}{\epsilon}-2m'(q)-1\right)}}{\Gamma\left(\frac{\gamma}{2}-\sigma m'(q)\right)\Gamma\left(\frac{\delta}{2}-\sigma\left(m'(q)-\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}\right)\right)}\right)\times \\
&\quad\times \mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right)\,,
\end{aligned} \tag{3.4.43}$$

where the quantities $m'(q)$ and $F_D$ are computed as explained in Appendix C.
The connection formula from $z=1$ to $\infty$ is simpler and reads

$$\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z) = \\
&=e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F_D-\frac{1}{2}\partial_m F_D}\epsilon^{-\frac{1}{2}-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta-\epsilon}{2}+m'(q)}\frac{\Gamma(\delta)e^{i\pi\left(\frac{\alpha}{\epsilon}-\frac{\gamma}{2}-m'(q)\right)}}{\Gamma\left(-\frac{\alpha}{\epsilon}+\frac{\gamma}{2}+\delta+m'(q)\right)}z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1})+ \\
&+e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F_D+\frac{1}{2}\partial_m F_D}\epsilon^{-\frac{1}{2}+\frac{\alpha}{\epsilon}-\frac{\gamma+\delta-\epsilon}{2}-m'(q)}\frac{\Gamma(\delta)}{\Gamma\left(\frac{\alpha}{\epsilon}-\frac{\gamma}{2}+m'(q)\right)}e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1})\,.
\end{aligned} \tag{3.4.44}$$

### 3.4.3 The reduced confluent Heun equation

**The dictionary**

Here we establish the dictionary between our results of section 3.3.3 on reduced confluent conformal blocks the reduced confluent Heun equation (RCHE) in standard notation, which reads

$$\frac{d^2w}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}\right)\frac{dw}{dz}+\frac{\beta z-q}{z(z-1)}w=0\,. \tag{3.4.45}$$

This is of course just the CHE specialized to[14] $\epsilon=0$. The interesting difference with respect to the CHE is the behaviour for $z\to\infty$, which is no longer controlled by $\epsilon$ and the degree of the singularity gets lowered to 1/2. By defining $w(z)=P_2(z)\psi(z)$ with $P_2(z)=z^{-\gamma/2}(1-z)^{-\delta/2}$, we pass to the normal form which is easily compared with the semiclassical BPZ equation (3.3.121). The dictionary between the CFT parameters and the parameters of the RCHE reads:

$$\begin{aligned}
a_0 &= \frac{1-\gamma}{2}\,, \\
a_1 &= \frac{1-\delta}{2}\,, \\
L &= 2i\sqrt{\beta}\,, \\
u &= \frac{1}{4}-q+\beta-\frac{(\gamma+\delta-1)^2}{4}\,,
\end{aligned} \tag{3.4.46}$$

where

$$u=\lim_{b\to 0}b^2\Lambda^2\partial_{\Lambda^2}\log{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda^2\right)=\frac{1}{4}-a^2+\mathcal{O}(L^2) \tag{3.4.47}$$

as in (3.3.121). This relation can then be inverted to find $a$ in terms of the parameters of the RCHE: we denote this by $a(q)$. We therefore infer the relation between the solutions of the

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[14] This corresponds to the usual decoupling limit $m\to\infty, L\to 0$ such that $mL$ remains finite.

RCHE in standard form and the conformal blocks defined before. Near $z = 0$ we have the following two linearly independent solutions to the RCHE in standard form (3.4.45):

$$\begin{aligned}&\mathrm{HeunRC}(q,\beta,\gamma,\delta;z)\,,\\&z^{1-\gamma}\mathrm{HeunRC}\left(q-(1-\gamma)\delta,\beta,2-\gamma,\delta;z\right)\,,\end{aligned}\tag{3.4.48}$$

where

$$F=\lim_{b\to0}b^2\log\left[\Lambda^{-2\Delta}{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda^2\right)\right]\,.\tag{3.4.49}$$

Since HeunRC is nothing else than HeunC with $\epsilon = 0$, it has the following expansion around $z = 0$:

$$\mathrm{HeunRC}(q,\beta,\gamma,\delta;z)=1-\frac{q}{\gamma}z+\mathcal{O}(z^2)\,.\tag{3.4.50}$$

Comparing with the conformal blocks in subsection 3.3.3 we identify

$$\begin{aligned}\mathrm{HeunRC}(q,\beta,\gamma,\delta;z)&=P_2(z)e^{-\frac{1}{2}\partial_{a_0}F}{}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_1}\,a_{0-}\,{}^{a_{2,1}}_{a_0};L^2,z\right)\,,\\z^{1-\gamma}\mathrm{HeunRC}\left(q-(1-\gamma)\delta,\beta,2-\gamma,\delta;z\right)&=P_2(z)e^{\frac{1}{2}\partial_{a_0}F}{}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_1}\,a_{0+}\,{}^{a_{2,1}}_{a_0};L^2,z\right)\,,\end{aligned}\tag{3.4.51}$$

Doing a Möbius transformation $z \to 1 - z$ we obtain the solutions around $z = 1$. Since this is a regular singularity the solution can again be written in terms of HeunRC. This amounts to sending $\gamma \to \delta$, $\delta \to \gamma$, $\beta \to -\beta$, $q \to q - \beta$. The two solutions are therefore

$$\begin{aligned}&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)\,,\\&(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right)\,.\end{aligned}\tag{3.4.52}$$

Comparig with the conformal blocks we identify

$$\begin{aligned}&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)=P_2(z)e^{-\frac{1}{2}\partial_{a_1}F}{}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_0}\,a_{1-}\,{}^{a_{2,1}}_{a_1};-L^2,1-z\right)\,,\\&(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right)=\\&\quad=P_2(z)e^{\frac{1}{2}\partial_{a_1}F}{}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_0}\,a_{1+}\,{}^{a_{2,1}}_{a_1};-L^2,1-z\right)\,.\end{aligned}\tag{3.4.53}$$

The new behaviour arises for $z \to \infty$, where we write the solutions in terms of another function $\mathrm{HeunRC}_\infty$:

$$\begin{aligned}&e^{2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})\\&e^{-2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}})\,.\end{aligned}\tag{3.4.54}$$

The function $\mathrm{HeunRC}_\infty$ has a simple asymptotic expansion around $z = \infty$:

$$\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})\sim1-\frac{q-\beta+\left(\frac{\gamma+\delta}{2}-\frac{3}{4}\right)\left(\frac{\gamma+\delta}{2}-\frac{1}{4}\right)}{i\sqrt{\beta}}z^{-\frac{1}{2}}+\mathcal{O}(z^{-1})\,.\tag{3.4.55}$$

Comparing with the conformal blocks we identify

$$\begin{aligned}&e^{2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})=e^{\mp\frac{i\pi\delta}{2}}P_2(z)L^{\frac{1}{2}}{}_{\frac{1}{2}}\mathcal{E}^{(+)}\left(a_{2,1}\;a\,{}^{a_1}_{a_0};L^2,\frac{1}{L\sqrt{z}}\right)\\&e^{-2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}})=e^{\mp\frac{i\pi\delta}{2}}P_2(z)L^{\frac{1}{2}}{}_{\frac{1}{2}}\mathcal{E}^{(-)}\left(a_{2,1}\;a\,{}^{a_1}_{a_0};L^2,\frac{1}{L\sqrt{z}}\right)\,.\end{aligned}\tag{3.4.56}$$

Note that due to the nature of the rank 1/2 singularity at infinity, the expansion is in inverse powers of $\sqrt{z}$. The phase $e^{\mp\frac{i\pi\delta}{2}}$ comes from the fact that near $z=\infty$

$$P_2(z) \sim z^{-\gamma/2}(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}} z^{-\gamma/2-\delta/2}\,. \tag{3.4.57}$$

The second solution around $z=\infty$ can be found by using the manifest symmetry $L \to -L$ of the BPZ equation which according to the dictionary gives the symmetry $\beta \to e^{2\pi i}\beta$ of the RCHE in normal form.

**Connection formulae**

The connection formula between $z = 0, 1$ written in (3.3.127) for the semiclassical conformal blocks can now be restated as:

$$\begin{aligned}\mathrm{HeunRC}(q,\beta,\gamma,\delta;z) &= \frac{\Gamma(1-\delta)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F+\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1+\gamma-\delta}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\delta}{2}-a(q)\right)}\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)+\\ &+\frac{\Gamma(\delta-1)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F-\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{\gamma+\delta-1}{2}+a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-a(q)\right)}(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right)\,,\end{aligned} \tag{3.4.58}$$

where the quantities $a(q)$ and $F$ are computed as explained in Appendix C.
The connection formula between $z = 1, \infty$ written in (3.3.128) reads

$$\begin{aligned}&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z) =\\ &=\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\left(e^{i\pi}\beta\right)^{-\frac{1}{4}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F-\frac{\sigma}{2}\partial_a F}}{2\sqrt{\pi}\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)}\right)e^{2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})+\\ &+\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\left(e^{-i\pi}\beta\right)^{-\frac{1}{4}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F-\frac{\sigma}{2}\partial_a F}}{2\sqrt{\pi}\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)}\right)e^{-2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}})\,.\end{aligned} \tag{3.4.59}$$

Here the phase ambiguity comes from (3.4.56), i.e. corresponds to the choice $(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}z^{-\delta/2}$. A similar expression can be found connecting $z = 0$ and $\infty$.

### 3.4.4 The doubly confluent Heun equation

**The dictionary**

The doubly confluent Heun equation (DCHE) reads

$$\left(\frac{d^2}{dz^2}+\frac{\delta+\gamma z+z^2}{z^2}\frac{d}{dz}+\frac{\alpha z-q}{z^2}\right)w(z)=0\,. \tag{3.4.60}$$

Again putting the DCHE in its normal form via the substitution $w(z) = \tilde{P}_2(z)\psi(z)$ with

$$\tilde{P}_2(z) = e^{\frac{1}{2}\left(\frac{\delta}{z}-z\right)}z^{-\frac{\gamma}{2}} \tag{3.4.61}$$

we find the $2^2 = 4$ different dictionaries with (3.3.147) corresponding to the $\mathbb{Z}_2^2$ symmetries $(m_i, L_i) \to (-m_i, -L_i)$ for $i = 1, 2$. For brevity we only write one of them, namely

$$\begin{aligned}L_1 &= 1\,,\\ L_2 &= \delta\,,\\ m_1 &= \frac{1}{2}\left(2\alpha-\gamma\right)\,,\\ m_2 &= 1-\frac{\gamma}{2}\,,\\ u &= \frac{1}{4}\left(-4q+2\gamma-\gamma^2-2\delta\right)\,.\end{aligned} \tag{3.4.62}$$

and the inverse dictionary is

$$\begin{aligned}
&\alpha = 1 + m_1 - m_2 \,, \\
&\delta = L_2 \,, \\
&\gamma = 2(1 - m_2) \,, \\
&q = -\frac{1}{2}\left(L_2 + 2u + 2m_2(m_2 - 1)\right) \,, \\
&L_1 = 1 \,.
\end{aligned} \tag{3.4.63}$$

We denote the two solutions of the DCHE near the irregular singularity at zero as

$$\begin{aligned}
&\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) \,, \\
&e^{\frac{\delta}{z}} z^{2-\gamma}\text{HeunDC}\left(\delta + q + \gamma - 2, \alpha - \gamma + 2, \gamma, -\delta, z\right) \,,
\end{aligned} \tag{3.4.64}$$

where HeunDC has the following asymptotic expansion around $z = 0$:

$$\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) \sim 1 + \frac{q}{\delta} z + \frac{q(q-\gamma) - \alpha\delta}{2\delta^2} z^2 + \mathcal{O}(z^3) \,. \tag{3.4.65}$$

Comparing with the semiclassical block (3.3.146) we get

$$\begin{aligned}
&\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) = \tilde{P}_2(z) L_2^{\frac{1}{2} - m_2} e^{-\frac{1}{2}\partial_{m_2} F} \left( z \, {}_1\mathcal{D}_1\left( m_2 \, {}^{a_{2,1}} \; m_{2-} \; a \; m_1; L_2, \frac{z}{L_2} \right) \right) \,, \\
&\text{HeunDC}\left(\delta + q + \gamma - 2, \alpha - \gamma + 2, \gamma, -\delta, z\right) = \tilde{P}_2(z) L_2^{\frac{1}{2} + m_2} e^{\frac{1}{2}\partial_{m_2} F} \left( z \, {}_1\mathcal{D}_1\left( m_2 \, {}^{a_{2,1}} \; m_{2+} \; a \; m_1; L_2, \frac{z}{L_2} \right) \right) \,.
\end{aligned} \tag{3.4.66}$$

The solutions near the irregular singularity at infinity are given by

$$\begin{aligned}
&z^{-\alpha}\text{HeunDC}\left(q - \alpha(\alpha + 1 - \gamma), \alpha, 2(\alpha + 1) - \gamma, \delta, -\frac{\delta}{z}\right) \,, \\
&e^{-z} z^{\alpha - \gamma}\text{HeunDC}\left(q + \delta + (\gamma - \alpha)(\alpha - 1), \gamma - \alpha, -2(\alpha - 1) + \gamma, -\delta, -\frac{\delta}{z}\right) \,.
\end{aligned} \tag{3.4.67}$$

Comparing with the semiclassical block (3.3.149) we find

$$\begin{aligned}
&\text{HeunDC}\left(q - \alpha(\alpha + 1 - \gamma), \alpha, 2(\alpha + 1) - \gamma, \delta, -\frac{\delta}{z}\right) = \tilde{P}_2(z) e^{\frac{1}{2}\partial_{m_1} F} {}_1\mathcal{D}_1\left( m_1 \, {}^{a_{2,1}} \; m_{1+} \; a \; m_2; L_2, \frac{1}{z} \right) \,, \\
&\text{HeunDC}\left(q + \delta + (\gamma - \alpha)(\alpha - 1), \gamma - \alpha, -2(\alpha - 1) + \gamma, -\delta, -\frac{\delta}{z}\right) = \tilde{P}_2(z) e^{-\frac{1}{2}\partial_{m_1} F} {}_1\mathcal{D}_1\left( m_1 \, {}^{a_{2,1}} \; m_{1-} \; a \; m_2; L_2, \frac{1}{z} \right) \,.
\end{aligned} \tag{3.4.68}$$

### Connection formulae

In this case the only connection formula is the one between zero and infinity. This can be obtained from equation (3.3.151) and reads

$$\begin{aligned}
\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) = &\left( \sum_{\sigma = \pm} \frac{\Gamma\left(-2\sigma a\right) \Gamma\left(1 - 2\sigma a\right) \delta^{-\frac{1}{2} + \frac{\gamma}{2} + \sigma a}}{\Gamma\left(\frac{1}{2} - \left(1 - \frac{\gamma}{2}\right) - \sigma a\right) \Gamma\left(\frac{1}{2} - \frac{2\alpha - \gamma}{2} - \sigma a\right)} \right) \times \\
&\times e^{\frac{1}{2}\left(-\partial_{m_1} - \partial_{m_2} - \sigma \partial_a\right) F} z^{-\alpha}\text{HeunDC}\left(q - \alpha(\alpha + 1 - \gamma), \alpha, 2(\alpha + 1) - \gamma, \delta, -\frac{\delta}{z}\right) + \\
+ &\left( \sum_{\sigma = \pm} \frac{\Gamma\left(-2\sigma a\right) \Gamma\left(1 - 2\sigma a\right) \delta^{-\frac{1}{2} + \frac{\gamma}{2} + \sigma a} e^{i\pi\left(\frac{1+\gamma}{2} - \alpha - \sigma a\right)}}{\Gamma\left(\frac{1}{2} - \left(1 - \frac{\gamma}{2}\right) - \sigma a\right) \Gamma\left(\frac{1}{2} + \frac{2\alpha - \gamma}{2} - \sigma a\right)} e^{\frac{1}{2}\left(\partial_{m_1} - \partial_{m_2} - \sigma \partial_a\right) F} \right) \times \\
&\times e^{-z} z^{\alpha - \gamma}\text{HeunDC}\left(q + \delta + (\gamma - \alpha)(\alpha - 1), \gamma - \alpha, -2(\alpha - 1) + \gamma, -\delta, -\frac{\delta}{z}\right) \,,
\end{aligned} \tag{3.4.69}$$

### 3.4.5 The reduced doubly confluent Heun equation

#### The dictionary

Here we establish the dictionary between our results of section 3.3.5 on reduced doubly confluent conformal blocks and the reduced doubly confluent Heun equation (RDCHE) in the standard form, which reads

$$\frac{d^2w}{dz^2} - \frac{dw}{dz} + \frac{\beta z - q + \epsilon z^{-1}}{z^2} w = 0 \,. \tag{3.4.70}$$

By defining $w(z) = e^{z/2}\psi(z)$ we get rid of the first derivative and bring the equation to the normal form which is to be compared with the semiclassical BPZ equation (3.3.169). The resulting dictionary between the CFT parameters and the parameters of the RDCHE is

$$\begin{aligned} L_1 &= 1 \,, \\ L_2 &= 2i\sqrt{\epsilon} \,, \\ m &= \beta \,, \\ u &= -q \,. \end{aligned} \tag{3.4.71}$$

The fact that $L_1 = 1$ is of course consistent with the fact that it is a redundant parameter. Here

$$u = \lim_{b\to 0} b^2 \Lambda_2^2 \partial_{\Lambda_2^2} \log {}_1\mathfrak{F}_{\frac{1}{2}} \left(\mu\, \alpha; \Lambda_1 \Lambda_2^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1 L_2^2) \tag{3.4.72}$$

as in (3.3.169). This relation can then be inverted to find $a$ in terms of the parameters of the RDCHE: we denote this by $a(q)$. We can now write the solutions to the RDCHE in standard form and their relation to the conformal blocks by comparison. Near $z = 0$ we denote the two linearly independent solutions to the RDCHE in standard form (3.4.70) by:

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}} z^{3/4} \mathrm{HeunRDC}_0(q, \beta, \epsilon; \sqrt{z}) \,, \\ &e^{-2i\sqrt{\epsilon/z}} z^{3/4} \mathrm{HeunRDC}_0(q, \beta, e^{2\pi i}\epsilon; \sqrt{z}) \,. \end{aligned} \tag{3.4.73}$$

The two solutions are related by the manifest symmetry $L_2 \to -L_2$ of the BPZ equation which according to the dictionary (3.4.71) gives the symmetry $\epsilon \to e^{2\pi i}\epsilon$ of the RDCHE in normal form. The function $\mathrm{HeunRDC}_0$ has the following asymptotic expansion around $z = 0$:

$$\mathrm{HeunRDC}_0(q, \beta, \epsilon; \sqrt{z}) \sim 1 - \frac{\frac{3}{16} + q}{i\sqrt{\epsilon}} \sqrt{z} + \mathcal{O}\left(z\right) . \tag{3.4.74}$$

Note again that due to the presence of a rank 1/2 singularity, the expansion is in powers of $\sqrt{z}$. Comparing with the semiclassical conformal blocks in 3.3.5 we identify

$$\begin{aligned} e^{2i\sqrt{\epsilon/z}} z^{3/4} \mathrm{HeunRDC}_0(q, \beta, \epsilon; \sqrt{z}) &= e^{z/2} L_2^{\frac{1}{2}} {}_1\mathcal{E}_{\frac{1}{2}}^{(+)} \left( m\, a\, a_{2,1} \,; L_2^2, \frac{\sqrt{z}}{L_2} \right) , \\ e^{-2i\sqrt{\epsilon/z}} z^{3/4} \mathrm{HeunRDC}_0(q, \beta, e^{2\pi i}\epsilon; \sqrt{z}) &= e^{z/2} L_2^{\frac{1}{2}} {}_1\mathcal{E}_{\frac{1}{2}}^{(-)} \left( m\, a\, a_{2,1} \,; L_2^2, \frac{\sqrt{z}}{L_2} \right) . \end{aligned} \tag{3.4.75}$$

For $z \sim \infty$ instead we have the two solutions

$$\begin{aligned} &z^{\beta} \mathrm{HeunRDC}_\infty(q, \beta, \epsilon; z^{-1}) \,, \\ &e^{z} z^{-\beta} \mathrm{HeunRDC}_\infty(q, -\beta, -\epsilon; -z^{-1}) \,. \end{aligned} \tag{3.4.76}$$

The function $\mathrm{HeunRDC}_\infty(q, \beta, \epsilon; z^{-1})$ has the following asymptotic expansion around $z = \infty$:

$$\mathrm{HeunRDC}_\infty(q, \beta, \epsilon; z^{-1}) \sim 1 + (q + \beta - \beta^2) z^{-1} + \mathcal{O}\left(z^{-2}\right) . \tag{3.4.77}$$

Comparing with the semiclassical conformal blocks we identify

$$\begin{aligned}
z^{\beta}\mathrm{HeunRDC}_{\infty}(q,\beta,\epsilon;z^{-1}) &= e^{z/2}e^{-\frac{1}{2}\partial_m F}\,{}_1\mathcal{D}_{\frac{1}{2}}\left(m\;{}^{a_{2,1}}\;m_-\;a\,;L_2^2,\frac{1}{z}\right),\\
e^{z}z^{-\beta}\mathrm{HeunRDC}_{\infty}(q,-\beta,-\epsilon;-z^{-1}) &= e^{z/2}e^{\frac{1}{2}\partial_m F}\,{}_1\mathcal{D}_{\frac{1}{2}}\left(m\;{}^{a_{2,1}}\;m_+\;a\,;L_2^2,\frac{1}{z}\right).
\end{aligned} \tag{3.4.78}$$

These solutions are related by the symmetry $(m, L_1) \to (-m, -L_1)$ of the semiclassical BPZ equation. Notice that one can rescale the BPZ equation such that it only depends on the combination $L_1 z$ and the coefficient of the cubic pole is $-L_1 L_2^2/4$. By setting $L_1 = 1$ according to the dictionary with the RDCHE, the above symmetry descends to the symmetry $(\beta, \epsilon, z) \to (-\beta, -\epsilon, -z)$ of the RDCHE in normal form. Furthermore, in the equation above

$$F = \lim_{b\to 0} b^2 \log\left[(\Lambda_1\Lambda_2^2)^{-\Delta}\,{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)\right] \tag{3.4.79}$$

as in (3.3.172).

**Connection formulae**

The connection formula between $z = 0$ and $\infty$ written in (3.3.175) for the semiclassical conformal blocks can now be restated as:

$$\begin{aligned}
&e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunRDC}_0(q,\beta,\epsilon;\sqrt{z}) =\\
&=\left(\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\beta-\sigma a(q)\right)}\epsilon^{\frac{1}{4}+\sigma a(q)}e^{\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F}e^{-i\pi\left(\frac{1}{4}+\sigma a(q)\right)}e^{i\pi\left(\frac{1}{2}-\beta-\sigma a(q)\right)}\right) z^{\beta}\mathrm{HeunRDC}_{\infty}(q,\beta,\epsilon;z^{-1})+\\
&+\left(\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\beta-\sigma a(q)\right)}\epsilon^{\frac{1}{4}+\sigma a(q)}e^{\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F}e^{-i\pi\left(\frac{1}{4}+\sigma a(q)\right)}\right) e^{z}z^{-\beta}\mathrm{HeunRDC}_{\infty}(q,-\beta,-\epsilon;-z^{-1}),
\end{aligned} \tag{3.4.80}$$

where the quantities $a(q)$ and $F$ are computed as explained in Appendix C.

### 3.4.6 The doubly reduced doubly confluent Heun equation

**The dictionary**

Here we establish the dictionary between our results of section 3.3.6 on doubly reduced doubly confluent conformal blocks and the corresponding Heun equation (DRDCHE) which reads

$$\frac{d^2w}{dz^2} + \frac{z - q + \epsilon z^{-1}}{z^2}w = 0\,. \tag{3.4.81}$$

This already takes the normal form of the semiclassical BPZ equation (3.3.189) and we immediately read off the dictionary:

$$\begin{aligned}
L_1 &= 2i\,,\\
L_2 &= 2i\sqrt{\epsilon}\,,\\
u &= -q\,,
\end{aligned} \tag{3.4.82}$$

where

$$u = \lim_{b\to 0} b^2\Lambda_2^2\partial_{\Lambda_2^2}\log{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1^2L_2^2) \tag{3.4.83}$$

as in (3.3.189). This relation can be inverted to find $a$ in terms of the parameters of the DRDCHE: we denote this by $a(q)$. Near $z = 0$ we denote the two linearly independent solutions to (3.4.81) by

$$\begin{aligned}
&e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunDRDC}(q,\epsilon;\sqrt{z})\,,\\
&e^{-2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunDRDC}(q,e^{2\pi i}\epsilon;\sqrt{z})\,.
\end{aligned} \tag{3.4.84}$$

The DRDC Heun function has a simple asymptotic expansion around $z=0$:

$$\mathrm{HeunDRDC}(q,\epsilon;\sqrt{z})\sim 1-\frac{\frac{3}{16}+q}{i\sqrt{\epsilon}}\sqrt{z}+\mathcal{O}(z)\,. \tag{3.4.85}$$

Note that in the expansion, $z$ appears with a square-root, and therefore mapping $z\to e^{2\pi i}z$ gives another solution. Comparing with the semiclassical conformal blocks in subsection 3.3.6, we identify

$$\begin{aligned} e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunDRDC}(q,\epsilon;\sqrt{z}) &= zL_2^{1/2}\,{}_{\frac{1}{2}}\mathcal{E}^{(+)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;-4L_2^2,\frac{\sqrt{z}}{L_2}\right)\,,\\ e^{-2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunDRDC}(q,e^{2\pi i}\epsilon;\sqrt{z}) &= zL_2^{1/2}\,{}_{\frac{1}{2}}\mathcal{E}^{(-)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;-4L_2^2,\frac{\sqrt{z}}{L_2}\right)\,. \end{aligned} \tag{3.4.86}$$

Around $z=\infty$ we have the two linearly independent solutions

$$\begin{aligned} &e^{2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}})\,,\\ &e^{-2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}})\,, \end{aligned} \tag{3.4.87}$$

which we identify with the conformal blocks

$$\begin{aligned} e^{2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}}) &= \sqrt{2i}\,{}_{\frac{1}{2}}\mathcal{E}^{(+)}_{\frac{1}{2}}\left(a_{2,1}\,a\,;-4L_2^2,\frac{1}{2i\sqrt{z}}\right)\,,\\ e^{-2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}}) &= \sqrt{2i}\,{}_{\frac{1}{2}}\mathcal{E}^{(-)}_{\frac{1}{2}}\left(a_{2,1}\,a\,;-4L_2^2,\frac{1}{2i\sqrt{z}}\right)\,. \end{aligned} \tag{3.4.88}$$

### Connection formulae

The connection formula (3.3.195) from 0 to $\infty$ in terms of the DRDC Heun functions is

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunDRDC}(q,\epsilon;\sqrt{z})=\\ &=\left(\frac{-i}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\epsilon^{\frac{1}{4}+\sigma a(q)}e^{-\frac{\sigma}{2}\partial_a F}\right)e^{2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}})+\\ &+\left(\frac{1}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\epsilon^{\frac{1}{4}+\sigma a(q)}e^{-\frac{\sigma}{2}\partial_a F}e^{-2\pi i\sigma a(q)}\right)e^{-2i\sqrt{z}}z^{1/4}\mathrm{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}})\,, \end{aligned} \tag{3.4.89}$$

where the quantities $a(q)$ and $F$ are computed as explained in Appendix C.

# Chapter 4

# Heun equations in gravity: perturbations of Kerr black holes

## 4.1 Introduction and outlook

In this chapter we apply the exact results for the connection coefficients of the Heun equations derived in the previous chapter to linear perturbations of a Kerr black hole. The wave equation satisfied by such perturbations is separable, and reduces to two ODEs for its radial and angular dependence. The radial equation is the celebrated Teukoslky equation [107]. Both radial and angular ODEs have the same singularity structure of the confluent Heun equation analyzed in the previous section. This allows us to derive exact expression for relevant physical observables in terms of Liouville CFT objects.

A similar perspective has been analyzed in [108, 109, 110, 111] where it was suggested that some physical properties of black holes, such as their greybody factor and quasinormal modes, can be studied in a particular regime in terms of Painlevé equations. Numerical checks appeared in [112, 113]. A decisive step forward in the quasinormal mode problem has been taken in [77], where a different approach making use of the Seiberg-Witten quantum curve of an appropriate supersymmetric gauge theory has been advocated to justify their spectrum and whose evidence was also supported by comparison with numerical analysis of the gravitational equation (see also [114, 115] for further developments). This view point has been further analysed in [83, 84], where the context is widely generalized to D-branes and other types of gravitational backgrounds in various dimensions. From the AGT dual CFT$_2$ viewpoint, the gauge theoretical approach corresponds to the large Virasoro central charge limit, instead of the $c = 1$ limit relevant for Painlevé equations. It would be interesting to explore the relation between the $c = 1$ and $c = \infty$ approaches (see [116] for recent interesting developments).

The large $c$ CFT theoretical approach has the advantage of being well suited for computing connection coefficients, as it has been done in the previous chapter.

In what follows we perform the study of the greybody factor of the Kerr black hole at finite frequency for which we give an exact formula. This reduces to the well-known result of Maldacena and Strominger [117] in the zero frequency limit and in the semiclassical regime reproduces the results computed via standard WKB approximation in [118].

By using the explicit solution of the connection problem, we also provide a proof of the exact quantization of Kerr black hole quasinormal modes proposed in [77]. By solving the angular Teukolsky equation, we prove the analogue quantization condition on the corresponding parameters of the spin-weighted spheroidal harmonics.

Finally, we discuss the use of the precise asymptotics of our solution to determine the tidal deformation profile in the far away region of the Kerr black hole and compare it to recent results on the associated Love numbers in the static [119] and quasi-static [120, 121] regimes. We observe that our method naturally distinguishes the source and response terms in the solution without needing analytic continuation in the angular momentum [122, 123] and provides an alternative regularization procedure for the computation of static Love numbers.

## 4.2 Perturbations of Kerr black holes

The Kerr metric describes the spacetime outside of a stationary, rotating black hole in asymptotically flat space. In Boyer-Lindquist coordinates it reads:

$$\begin{aligned} ds^2 = &- \left(\frac{\Delta - \mathrm{a}^2\sin^2\theta}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2+\mathrm{a}^2)^2 - \Delta \mathrm{a}^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 \\ &- \frac{2\mathrm{a}\sin^2\theta(r^2+\mathrm{a}^2-\Delta)}{\Sigma} dt\, d\phi\,, \end{aligned} \tag{4.2.1}$$

where

$$\Sigma = r^2 + \mathrm{a}^2\cos^2\theta\,, \quad \Delta = r^2 - 2Mr + \mathrm{a}^2\,. \tag{4.2.2}$$

The horizons are given by the roots of $\Delta$:

$$r_\pm = M \pm \sqrt{M^2 - \mathrm{a}^2}\,. \tag{4.2.3}$$

Two other relevant quantities are the Hawking temperature and the angular velocity at the horizon:

$$T_H = \frac{r_+ - r_-}{8\pi M r_+}\,, \quad \Omega = \frac{\mathrm{a}}{2Mr_+}\,. \tag{4.2.4}$$

Perturbations of the Kerr metric by fields of spin $s = 0, -1, -2$ are described by the Teukolsky equation [107], who found that an Ansatz of the form

$$\Phi_s = e^{im\phi - i\omega t} S_{\lambda,s}(\theta, \mathrm{a}\omega) R_s(r)\,. \tag{4.2.5}$$

permits a separation of variables of the partial differential equation. One gets[1] the following equations for the radial and the angular part (see for example [18] eq.25):

$$\begin{aligned} &\Delta\frac{d^2R}{dr^2} + (s+1)\frac{d\Delta}{dr}\frac{dR}{dr} + \left(\frac{K^2 - 2is(r-M)K}{\Delta} - \Lambda_{\lambda,s} + 4is\omega r\right) R = 0\,, \\ &\partial_x(1-x^2)\partial_x S_\lambda + \left[(cx)^2 + \lambda + s - \frac{(m+sx)^2}{1-x^2} - 2csx\right] S_\lambda = 0\,. \end{aligned} \tag{4.2.6}$$

Here $x = \cos\theta$, $c = \mathrm{a}\omega$ and

$$K = (r^2+\mathrm{a}^2)\omega - \mathrm{a}m, \quad \Lambda_{\lambda,s} = \lambda + \mathrm{a}^2\omega^2 - 2\mathrm{a}m\omega\,. \tag{4.2.7}$$

$\lambda$ has to be determined as the eigenvalue of the angular equation with suitable boundary conditions imposing regularity at $\theta = 0, \pi$. In general no closed-form expression is known, but for small $\mathrm{a}\omega$ it is given by $\lambda = \ell(\ell+1) - s(s+1) + \mathcal{O}(\mathrm{a}\omega)$. We give a way to calculate it to arbitrary order in $\mathrm{a}\omega$ in subsection 4.4.3.

For later purposes it is convenient to write both equations in the form of a Schrödinger equation. For the radial equation we define

$$z = \frac{r - r_-}{r_+ - r_-}\,, \quad \psi(z) = \Delta(r)^{\frac{s+1}{2}} R(r)\,. \tag{4.2.8}$$

---

[1] Dropping the $_s$ subscript to ease the notation

With this change of variables the inner and outer horizons are at $z = 0$ and $z = 1$, respectively, and $r \to \infty$ corresponds to $z \to \infty$. We obtain the differential equation

$$\frac{d^2\psi(z)}{dz^2} + V_r(z)\psi(z) = 0 \tag{4.2.9}$$

with potential

$$V_r(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i^r z^i \,. \tag{4.2.10}$$

The coefficients $\hat{A}_i^r$ depend on the parameters of the black hole and the frequency, spin and angular momentum of the perturbation. Their explicit expression is given in Appendix E.
For the angular part instead we define

$$z = \frac{1+x}{2}\,, \quad y(z) = \sqrt{1-x^2}\frac{S_\lambda}{2}\,. \tag{4.2.11}$$

After this change of variables, $\theta = 0$ corresponds to $z = 1$, and $\theta = \pi$ to $z = 0$. The equation now reads

$$\frac{d^2y(z)}{dz^2} + V_{ang}(z)y(z) = 0\,, \tag{4.2.12}$$

with potential

$$V_{ang}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i^\theta z^i \,. \tag{4.2.13}$$

Again, we give the explicit expressions of the coefficients $\hat{A}_i^\theta$ in Appendix E. When written as Schrödinger equations, it is evident that the radial and angular equations share the same singularity structure. They both have two regular singular points at $z = 0, 1$ and an irregular singular point of Poincaré rank one at $z = \infty$. This is precisely the confluent Heun equation in its normal form. Accordingly, parameters in $V_r$ and $V_{ang}$ can be matched with the ones appearing in equation (3.3.82).

## 4.3 The dictionary with CFT

We now match parameters in the radial and angular equation with the ones appearing in the semiclassical BPZ equation (3.3.82). Writing down a dictionary between Liouville CFT and the gravitational problem will allow us to compute exact expression for observables related to perturbations of the Kerr background.

### 4.3.1 The radial dictionary

Comparing (4.2.10) with (3.3.82) we find the following eight dictionaries between the parameters of the radial equation in the black hole problem and the CFT[2]:

$$\begin{aligned}
&u = -\lambda - s(s+1) + 8M^2\omega^2 - \mathrm{a}^2\omega^2 + (2M\omega^2 + is\omega)(r_+ - r_-)\,,\\
&\theta a_0 = -i\frac{\omega - m\Omega}{4\pi T_H} + 2\mathrm{i}M\omega + \frac{s}{2}\,,\\
&\theta' a_1 = -i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2}\,,\\
&\theta'' m_3 = 2iM\omega - s\,,\\
&\theta'' L = -2i\omega(r_+ - r_-)\,.
\end{aligned} \tag{4.3.1}$$

---

[2] We call here $m_3 = b\mu$ the semiclassical parameter of the irregular field instead of $m$ as in the previous chapter to avoid confusion with the projection of the angular momentum $m$.

where $\theta, \theta', \theta'' = \pm 1$. We will make the following choice for the dictionary from now on:

$$\boxed{\begin{aligned}
&u = -\lambda - s(s+1) + 8M^2\omega^2 - \mathrm{a}^2\omega^2 + (2M\omega^2 + is\omega)(r_+ - r_-)\,,\\
&a_0 = -i\frac{\omega - m\Omega}{4\pi T_H} + 2\mathrm{i}M\omega + \frac{s}{2}\,,\\
&a_1 = -i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2}\,,\\
&m_3 = 2iM\omega - s\,,\\
&L = -2i\omega(r_+ - r_-)\,.
\end{aligned}} \tag{4.3.2}$$

which corresponds to $\theta = \theta' = \theta'' = +1$. These $8 = 2^3$ dictionaries reflect the symmetries of the equation, which is invariant independently under $a_0 \to -a_0$, $a_1 \to -a_1$ and $(m_3, \Lambda) \to -(m_3, \Lambda)$. Using AGT this dictionary gives the following masses in the gauge theory (see Appendix C for details):

$$\begin{aligned}
m_1 &= a_0 + a_1 = -i\frac{\omega - m\Omega}{2\pi T_H} + 2iM\omega\,,\\
m_2 &= a_1 - a_0 = -2iM\omega - s\,,\\
m_3 &= 2iM\omega - s\,.
\end{aligned} \tag{4.3.3}$$

This is consistent with what has been found in [77]. For $s = 0$ the values are purely imaginary and correspond to physical Liouville momenta. For $s \neq 0$ the conformal block gets analytically continued.

### 4.3.2 The angular dictionary

Comparing instead with the $\hat{A}_i^\theta$ in (E.0.4) we find the following eight dictionaries between the parameters of the angular equation in the black hole problem and the CFT:

$$\begin{aligned}
&u = -c^2 - s(s+1) + 2cs - \lambda\,,\\
&\theta a_0 = \left(-\frac{m-s}{2}\right)\,,\\
&\theta' a_1 = \left(-\frac{m+s}{2}\right)\,,\\
&\theta'' m_3 = -s\,,\\
&\theta'' L = 4c\,,
\end{aligned} \tag{4.3.4}$$

where again $\theta, \theta', \theta'' = \pm 1$ and our choice from here on will be $\theta = \theta' = \theta'' = +1$, i.e.:

$$\boxed{\begin{aligned}
&u = -c^2 - s(s+1) + 2cs - \lambda\,,\\
&a_0 = \left(-\frac{m-s}{2}\right)\,,\\
&a_1 = \left(-\frac{m+s}{2}\right)\,,\\
&m_3 = s\,,\\
&L = 4c\,,
\end{aligned}} \tag{4.3.5}$$

Using AGT this dictionary gives the following masses in the gauge theory (see Appendix C for details):

$$\begin{aligned}
m_1 &= a_0 + a_1 = -m\,,\\
m_2 &= a_1 - a_0 = -s\,,\\
m_3 &= s\,.
\end{aligned} \tag{4.3.6}$$

Again consistently with [77].

## 4.4 Applications to the black hole problem

There are several interesting physical quantities in the black hole problem which are governed by the Teukolsky equation. Having the explicit expression for the connection coefficients allows us to compute them exactly. We turn to this now.

### 4.4.1 The greybody factor

While all our analysis has been for classical black holes, it is known that quantum black holes emit thermal radiation from their horizons [124]. However, the spacetime outside of the black hole acts as a potential barrier for the emitted particles, so that the emission spectrum as measured by an observer at infinity is no longer thermal, but is given by $\frac{\sigma(\omega)}{\exp\frac{\omega-m\Omega}{T_H}-1}$, where $\sigma(\omega)$ is the so-called greybody factor. Incidentally, it is the same as the absorption coefficient of the black hole, which tells us the ratio of a flux of particles incoming from infinity which penetrates the potential barrier and is absorbed by the black hole [124] [125]. More precisely, the radial equation with $s = 0$ has a conserved flux, given by the "probability flux" when written as a Schrödinger equation: $\phi = \mathrm{Im}\psi^\dagger(z)\partial_z\psi(z)$ for $z$ on the real line. The absorption coefficient is then defined as the ratio between the flux $\phi_{abs}$ absorbed by the black hole (ingoing at the horizon) and the flux $\phi_{in}$ incoming from infinity. For non-zero spin, the potential (4.2.10) becomes complex, and the flux is no longer conserved. In that case the absorption coefficient can be computed using energy fluxes [126], but for simplicity we stick here to $s = 0$.

**The exact result**

On physical grounds we impose the boundary condition that there is only an ingoing wave at the horizon:

$$R(r\to r_+) \sim (r-r_+)^{-i\frac{\omega-m\Omega}{4\pi T_H}}\,, \tag{4.4.1}$$

so the wavefunction near the horizon is given by

$$\psi(z) = (1-z)^{\frac{1}{2}+a_1}\left(1+\mathcal{O}(z-1)\right)\,, \tag{4.4.2}$$

with $a_2 = -i\frac{\omega-m\Omega}{4\pi T_H}$ and recall that the time-dependent part goes like $e^{-i\omega t}$. This boundary condition is independent of whether $\omega - m\Omega$ is positive or negative: an observer near the horizon always sees an ingoing flux into the horizon, but when $\omega - m\Omega < 0$ it is outgoing according to an observer at infinity. This phenomenon is known as superradiance [127]. In any case, this gives the flux

$$\phi_{abs} = \mathrm{Im}a_1 \tag{4.4.3}$$

ingoing at the horizon. Using our connection formula (3.3.89), we find that near infinity the wavefunction behaves as

$$\psi(z) = (z-1)^{\frac{1}{2}+a_1}\left(1+\mathcal{O}(1-z)\right) = C^-_{\ell m\omega}e^{\frac{Lz}{2}}z^{-m_3}\left(1+\mathcal{O}(z^{-1})\right) + C^+_{\ell m\omega}e^{-\frac{Lz}{2}}z^{m_3}\left(1+\mathcal{O}(z^{-1})\right) \tag{4.4.4}$$

where

$$C^\pm_{\ell m\omega} = \sum_{\sigma=\pm}\frac{(\mp L)^{-\frac{1}{2}\pm m_3+\sigma a}e^{-\frac{\sigma}{2}\partial_a F\pm\frac{1}{2}\partial_{m_3}F}\Gamma\left(1-2\sigma a\right)\Gamma\left(-2\sigma a\right)\Gamma\left(1+2a_1\right)}{\Gamma\left(\frac{1}{2}+a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+a_1-\sigma a-a_0\right)\Gamma\left(\frac{1}{2}-\sigma a\pm m_3\right)}\,. \tag{4.4.5}$$

At infinity, the ingoing part of the wave is easy to identify: recalling that $\Lambda = -2i\omega(r_+ - r_-)$ it corresponds to the positive sign in the exponential. So the flux incoming from infinity is

$$\phi_{in} = \left(\mathrm{Im}\frac{L}{2}\right)\left|C^{-}_{\ell m\omega}\right|^2 = -\frac{1}{2}\left|\sum_{\sigma=\pm}\frac{L^{-m_3+\sigma a}e^{-\frac{\sigma}{2}\partial_a F-\frac{1}{2}\partial_{m_3}F}\Gamma\left(1-2\sigma a\right)\Gamma\left(-2\sigma a\right)\Gamma\left(1+2a_1\right)}{\Gamma\left(\frac{1}{2}+a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+a_1-\sigma a-a_0\right)\Gamma\left(\frac{1}{2}-\sigma a-m_3\right)}\right|^2 . \tag{4.4.6}$$

The minus sign comes from the fact that we have simplified $\Lambda$ and we have $\mathrm{Im}L = -|L|$. Note that also the flux at the horizon is negative (for non-superradiant modes). So the full absorption coefficient/greybody factor, defined as the flux going into the horizon normalized by the flux coming in from infinity is:

$$\sigma = \frac{\phi_{abs}}{\phi_{in}} = \frac{-\mathrm{Im}2a_1}{\left|\sum_{\sigma=\pm}\frac{L^{-m_3+\sigma a}e^{-\frac{\sigma}{2}\partial_a F-\frac{1}{2}\partial_{m_3}F}\Gamma\left(1-2\sigma a\right)\Gamma\left(-2\sigma a\right)\Gamma\left(1+2a_1\right)}{\Gamma\left(\frac{1}{2}+a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+a_1-\sigma a-a_0\right)\Gamma\left(\frac{1}{2}-\sigma a-m_3\right)}\right|^2} . \tag{4.4.7}$$

This is the exact result, given as a power series in L. Substituting the dictionary with gravity we get

$$\boxed{\begin{aligned}&\sigma = \frac{\phi_{abs}}{\phi_{in}} = \frac{\omega - m\Omega}{2\pi T_H}\times\\ &\times\left|\frac{\Gamma(1+2a)\Gamma(2a)\Gamma(1-i\frac{\omega-m\Omega}{2\pi T_H})(-2i\omega(r_+-r_-))^{-a-2iM\omega}e^{-i\omega(r_+-r_-)}e^{-\frac{1}{2}\partial_a F+\frac{1}{2}\partial_{m_3}F}}{\Gamma\left(\frac{1}{2}-2iM\omega+a\right)\Gamma\left(\frac{1}{2}-i\frac{\omega-m\Omega}{2\pi T_H}+2iM\omega+a\right)\Gamma\left(\frac{1}{2}-2iM\omega+a\right)}+(a\to -a)\right|^{-2} .\end{aligned}} \tag{4.4.8}$$

Here $F(L, a_0, a_1, m_3)$ is the instanton part of the NS free energy. We stress that to write this result fully in terms of the parameters of the black hole problem using the dictionary (4.3.2), one has to invert the Matone relation $u = \frac{1}{4} - a^2 + L\partial_L F$ to obtain $a(u)$, which can be done order by order in $L$. In the literature, the absorption coefficient for Kerr black holes has been calculated using various approximations. As a consistency check, we show that our result reproduces the known results in the appropriate regimes.

### Comparison with asymptotic matching

In [117], the absorption coefficient is calculated via an asymptotic matching procedure. They work in a regime in which $\mathrm{a}\omega \ll 1$ such that the angular eigenvalue $\lambda \approx \ell(\ell+1)$, and solve the Teukolsky equation for $s = 0$ asymptotically in the regions near and far from the outer horizon. Then one also takes $M\omega \ll 1$ such that there exists an overlap between the far and near regions and one can match the asymptotic solutions. For us these limits imply that also $|L| = 4\omega\sqrt{M^2 - \mathrm{a}^2} \ll 1$, so we expand our exact transmission coefficient to lowest order in $\mathrm{a}\omega$, $M\omega$ and $L$. Since from the dictionary (4.3.2) $u = \frac{1}{4} - a^2 + \mathcal{O}(L) = \frac{1}{4} + \ell(\ell+1) + \mathcal{O}(\mathrm{a}\omega, M\omega)$, in this limit we have $a = \ell + \frac{1}{2}$. Then the second term in the denominator of (4.4.7) which contains $L^a$ vanishes for $L \to 0$ while the first one survives and passes to the numerator. The NS instanton partition function $F$ also vanishes, $F(L \to 0) = 0$. (4.4.8) then becomes

$$\sigma \approx \frac{\omega - m\Omega}{2\pi T_H}(2\omega(r_+ - r_-))^{2\ell+1}\left|\frac{\Gamma\left(\ell+1\right)\Gamma\left(\ell+1-i\frac{\omega-m\Omega}{2\pi T_H}\right)\Gamma\left(\ell+1\right)}{\Gamma(2\ell+2)\Gamma(2\ell+1)\Gamma(1-i\frac{\omega-m\Omega}{2\pi T_H})}\right|^2 . \tag{4.4.9}$$

Using the relation $\frac{\Gamma(\ell+1)}{\Gamma(2\ell+2)} = \frac{\sqrt{\pi}}{2^{2\ell+1}\Gamma(\ell+\frac{3}{2})}$ (and sending $i \to -i$ inside the modulus squared) we reduce precisely to the result of [117] (eq. 2.29):

$$\boxed{\sigma \approx \frac{\omega - m\Omega}{2T_H} \frac{(r_+ - r_-)^{2\ell+1}\omega^{2\ell+1}}{2^{2\ell+1}} \left| \frac{\Gamma(\ell+1)\Gamma\left(\ell+1+i\frac{\omega-m\Omega}{2\pi T_H}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma(2\ell+1)\Gamma\left(1+i\frac{\omega-m\Omega}{2\pi T_H}\right)} \right|^2} \,, \tag{4.4.10}$$

which is valid for $M\omega, \mathrm{a}\omega \ll 1$.

### Comparison with semiclassics

We now show that the exact absorption coefficient reduces to the semiclassical result obtained via a standard WKB analysis of the equation

$$\epsilon_1^2 \partial_z^2 \psi(z) + V(z)\psi(z) = 0\,. \tag{4.4.11}$$

where we have introduced the small parameter $\epsilon_1$ which plays the role of the Planck constant to keep track of the orders in the expansion. For the Teukolsky equation (which has $\epsilon_1 = 1$) the semiclassical regime is the regime in which $\ell \gg 1$. Following [118], we also take $M\omega \ll 1$ and $s = 0$ such that there are two zeroes of the potential between the outer horizon and infinity for real values of $z$ which we denote by $z_1$ and $z_2$ with $z_2 > z_1$, between which there is a potential barrier for the particle ($V(z)$ becomes negative, notice the "wrong sign" in front of the second derivative). Without these extra conditions, the potential generically becomes complex, or does not form a barrier. The main difference with the regime used for the asymptotic matching procedure in the previous section is that there we worked to leading order in $M\omega, a\omega$. Now we still assume them to be small but keep all orders, while working to first subleading order in $\epsilon_1$.

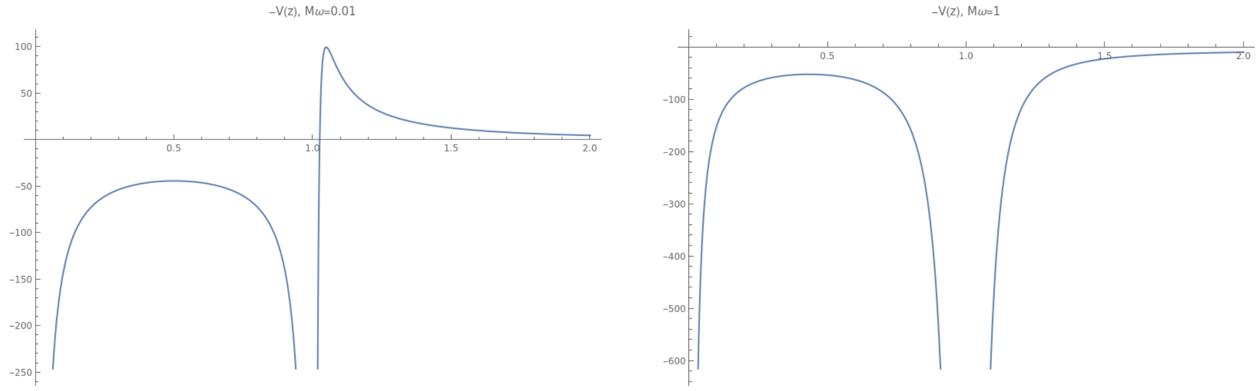


Figure 4.1: Forms of the potential $-V(z)$ for $M = 1$, $a = 0.5$, $\lambda = 10$, $m = 0\,, s = 0$, and $\omega = 0.01$ (left) and $\omega = 1$ (right). We see that for $M\omega$ not small enough, the potential does not form a barrier.

The standard WKB solutions are

$$\psi(z) \propto V(z)^{-\frac{1}{4}} \exp\left( \pm \frac{i}{\epsilon_1} \int_{z_*}^{z} \sqrt{V(z')} dz' \right)\,, \tag{4.4.12}$$

where $z_*$ is some arbitrary reference point, usually taken to be a turning point of the potential, here corresponding to a zero. The absorption coefficient is given by the transmission coefficient from infinity to the horizon and captures the tunneling amplitude through this potential barrier. It is simply given by

$$\sigma \approx \exp\left( \frac{2i}{\epsilon_1} \int_{z_1}^{z_2} \sqrt{V(z')} dz' \right) = \exp\left( -\frac{2}{\epsilon_1} \int_{z_1}^{z_2} \sqrt{|V(z')|} dz' \right)\,. \tag{4.4.13}$$

On the other hand it is known that in the semiclassical limit the potential of the BPZ equation reduces to the Seiberg-Witten differential of the AGT dual gauge theory [69], which for us is $SU(2)$ gauge theory with $N_f = 3$: $V(z) \to -\phi_{SW}^2(z)$. The integral between the two zeroes then corresponds to half a B-cycle, so we identify

$$\boxed{\sigma \approx \exp\left(-\frac{2}{\epsilon_1}\int_{z_1}^{z_2} \phi_{SW}(z')dz'\right) = \exp\left(-\frac{1}{\epsilon_1}\oint_B \phi_{SW}(z')dz'\right) =: \exp\left(-\frac{a_D}{\epsilon_1}\right) \,,} \tag{4.4.14}$$

where we have chosen an orientation of the B-cycle. Our exact absorption coefficient reduces to this expression in the semiclassical limit $\epsilon_1 \to 0$.

### 4.4.2 Quantization of quasinormal modes

With the explicit expression of the connection matrix (3.3.89) in our hands we can extract the quantization condition for the quasinormal modes. The correct boundary conditions for quasinormal modes is only an ingoing wave at the horizon and only an outgoing one at infinity (see e.g. [18], eq. (80)), that is

$$\begin{aligned} &R_{\mathrm{QNM}}(r \to r_+) \sim (r - r_+)^{-i\frac{\omega - m\Omega}{4\pi T_H} - s} \\ &R_{\mathrm{QNM}}(r \to \infty) \sim r^{-1-2s+2iM\omega} e^{i\omega r} \,. \end{aligned} \tag{4.4.15}$$

In terms of the function $\psi(z)$ satisfying the Teukolsky equation in Schrödinger form:

$$\begin{aligned} &\psi_{\mathrm{QNM}}(z \to 1) \sim (z-1)^{\frac{1}{2}+a_1} \,, \\ &\psi_{\mathrm{QNM}}(z \to \infty) \sim e^{-Lz/2}\,(Lz)^{-m_3} \,. \end{aligned} \tag{4.4.16}$$

However, imposing the ingoing boundary condition at the horizon and using the connection formula, we get that near infinity

$$\psi_{\mathrm{QNM}}(z \to \infty) = (z-1)^{\frac{1}{2}+a_1}\left(1 + \mathcal{O}(1-z)\right) = C^-_{\ell m\omega} e^{\frac{Lz}{2}} z^{-m_3}\left(1 + \mathcal{O}(z^{-1})\right) + C^+_{\ell m\omega} e^{-\frac{Lz}{2}} z^{m_3}\left(1 + \mathcal{O}(z^{-1})\right) \tag{4.4.17}$$

which contains both an ingoing an an outgoing wave at infinity. In order to impose the correct boundary condition (4.4.16) we need to impose that the coefficient of the ingoing wave vanishes:

$$C^-_{\ell m\omega_n} = \sum_{\sigma=\pm} \frac{L^{-\frac{1}{2}-m_3+\sigma a} e^{-\frac{\sigma}{2}\partial_a F - \frac{1}{2}\partial_{m_3} F}\,\Gamma\left(1-2\sigma a\right)\Gamma\left(-2\sigma a\right)\Gamma\left(1+2a_1\right)}{\Gamma\left(\frac{1}{2}+a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+a_1-\sigma a-a_0\right)\Gamma\left(\frac{1}{2}-\sigma a-m_3\right)} = 0 \tag{4.4.18}$$

This gives

$$\sum_\sigma \frac{L^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}\,\Gamma\left(1-2\sigma a\right)\Gamma\left(-2\sigma a\right)}{\Gamma\left(\frac{1}{2}+m_1-\sigma a\right)\Gamma\left(\frac{1}{2}+m_2-\sigma a\right)\Gamma\left(\frac{1}{2}-m_3-\sigma a\right)} = 0 \tag{4.4.19}$$

Note that

$$\begin{aligned} &\frac{\Gamma\left(2a\right)\Gamma\left(1+2a\right)}{\Gamma\left(-2a\right)\Gamma\left(1-2a\right)}\frac{\Gamma\left(\frac{1}{2}-m_3-a\right)}{\Gamma\left(\frac{1}{2}-m_3+a\right)}\prod_{i=1}^{2}\frac{\Gamma\left(\frac{1}{2}+m_i-a\right)}{\Gamma\left(\frac{1}{2}+m_i+a\right)} = e^{-i\pi}\left(\frac{\Gamma\left(1+2a\right)}{\Gamma\left(1-2a\right)}\right)^2 \frac{\Gamma\left(\frac{1}{2}-m_3-a\right)}{\Gamma\left(\frac{1}{2}-m_3+a\right)}\prod_{i=1}^{2}\frac{\Gamma\left(\frac{1}{2}+m_i-a\right)}{\Gamma\left(\frac{1}{2}+m_i+a\right)} = \\ &= \exp\left[-i\pi + 2\log\frac{\Gamma\left(1+2a\right)}{\Gamma\left(1-2a\right)} + \log\frac{\Gamma\left(\frac{1}{2}-m_3-a\right)}{\Gamma\left(\frac{1}{2}-m_3+a\right)} + \sum_{i=1}^{2}\log\frac{\Gamma\left(\frac{1}{2}+m_i-a\right)}{\Gamma\left(\frac{1}{2}+m_i+a\right)}\right] \,. \end{aligned} \tag{4.4.20}$$

Including also the $L$ factor, we identify the exponent with (see conventions in [77])

$$-2a\log L + 2\log\frac{\Gamma\left(1+2a\right)}{\Gamma\left(1-2a\right)} + \log\frac{\Gamma\left(\frac{1}{2}-m_3-a\right)}{\Gamma\left(\frac{1}{2}-m_3+a\right)} + \sum_{i=1}^{2}\log\frac{\Gamma\left(\frac{1}{2}+m_i-a\right)}{\Gamma\left(\frac{1}{2}+m_i+a\right)} = \partial_a \mathcal{F}^{1-\mathrm{loop}} \,. \tag{4.4.21}$$

The instanton and one loop part combine to give the full NS free energy, and hence (4.4.18) can be conveniently rewritten as

$$1-e^{\partial_a\mathcal{F}}=0\Rightarrow\partial_a\mathcal{F}=2\pi i n\,,n\in\mathbb{Z}\,. \tag{4.4.22}$$

To solve for the quasinormal mode frequencies, we need to invert the relation $u=\frac{1}{4}-a^2-L\partial_L F$ to obtain $a(u)$. Then the quantization condition for the quasinormal mode frequencies that we have derived reads

$$\boxed{\partial_a\mathcal{F}\left(-2i\omega(r_+-r_-),a(u),-i\frac{\omega-m\Omega}{2\pi T_H}+2iM\omega,-2iM\omega-s,-2iM\omega+s,1\right)=2\pi i n\,,n\in\mathbb{Z}\,,} \tag{4.4.23}$$

This gives an equation that is solved for a discrete set of $\omega_n$, in agreement with [77][3].

### 4.4.3 Angular quantization

Yet another application of the connection formulae is the computation of the angular eigenvalue $\lambda$. To this end, we impose regularity of the angular eigenfunctions at $z=0,1$. According to the angular dictionary (4.3.5),

$$\frac{1\pm 2a_0}{2}=\frac{1}{2}\mp\frac{m-s}{2}\,,\ \frac{1\pm 2a_1}{2}=\frac{1}{2}\mp\frac{m+s}{2}\,, \tag{4.4.24}$$

therefore, according to (4.2.11) the behavior of $S_\lambda$ as $z\to 0$ is given by

$$S_\lambda(z\to 0)\propto z^{\mp\frac{m-s}{2}}\,. \tag{4.4.25}$$

Since $\lambda_{s,m}=\lambda^*_{s,-m}$, $\lambda_{-s,m}=\lambda_{s,m}+2s$ [128], we can restrict without loss of generality to the case $m,-s\geq 0$. Regularity of $S_\lambda$ as $z\to 0$ requires the boundary condition

$$y_{m>s}(z\to 0)=\simeq z^{\frac{1}{2}+\frac{m-s}{2}}\,. \tag{4.4.26}$$

Therefore near $z\to 1$,

$$e^{\frac{1}{2}\partial_{a_0}F}y_{m>s}(z\to 1)\simeq\frac{e^{\frac{1}{2}\partial_{a_1}F}\Gamma(-m-s)\Gamma(1+m-s)}{\Gamma(\frac{1}{2}-a-s)\Gamma(\frac{1}{2}+a-s)}(1-z)^{\frac{1}{2}+\frac{m+s}{2}}+\frac{e^{-\frac{1}{2}\partial_{a_0}F}\Gamma(m+s)\Gamma(1+m-s)}{\Gamma(\frac{1}{2}-a+m)\Gamma(\frac{1}{2}+a+m)}(1-z)^{\frac{1}{2}-\frac{m+s}{2}}\,. \tag{4.4.27}$$

Let us start by assuming $m+s>0$. Then the second term in (4.4.27) has a pole at $z=1$ for generic values of $a$, and the first gamma function is divergent as it stands. However both divergences are cured by imposing that

$$a=\ell+\frac{1}{2}\,, \tag{4.4.28}$$

for some positive integer $\ell\geq m\geq -s$. Analogously if $m+s\leq 0$, regularity is ensured by imposing $a=\ell+\frac{1}{2}$ with $\ell\geq m\geq -s$. Therefore in general the quantization condition for the angular eigenvalue is

$$a(u)=\ell+\frac{1}{2}\,,\ \ell\geq\max(m\,,-s)\,. \tag{4.4.29}$$

Again, $a$ is obtained by inverting the expression $u=\frac{1}{4}-a^2+\Lambda\partial_\Lambda\mathcal{F}^{\text{inst}}$ order by order in $\Lambda$. Let us denote by

$$\lambda_0=\lambda(\Lambda=0)=\ell(\ell+1)-s(s+1)\,. \tag{4.4.30}$$

Then the above quantization condition for the angular eigenvalue $\lambda$ can be more conveniently written as

$$\lambda-\lambda_0=2cs-c^2+L\partial_L F\,, \tag{4.4.31}$$

which is in agreement with the result already obtained in [77].

---

[3] In order to match with [77], it is important to notice that they use the variable $-ia$ instead of $a$, and a sign difference in the definition of $m_3$. Moreover, their $\partial_a\mathcal{F}$ is shifted by a factor of $-i\pi$ with respect to ours.

### 4.4.4 Love numbers

Applying an external gravitational field to a self-gravitating body generically causes it to deform, much in the same way as an external electric field polarizes a dielectric material. The response of the body to the external gravitational tidal field is captured by the so-called tidal response coefficients or Love numbers, named after A. E. H. Love who first studied them in the context of the Earth's response to the tides [129]. In general relativity, the tidal response coefficients are generally complex, and the real part captures the conservative response of the body, whereas the imaginary part captures dissipative effects. There is some naming ambiguity where sometimes only the real, conservative part is called the Love number, whereas sometimes the full complex response coefficient is called Love number. For us the Love number will be the full complex response coefficient. For four-dimensional Kerr black holes, the conservative (real part) of the response coefficient to static external perturbations has been found to vanish [119, 121]. Moreover, Love numbers are measurable quantities that can be probed with gravitational wave observations [130, 131]. Using our conformal field theory approach to the Teukolsky equation we compute the Love number of a slowly rotating Kerr black hole at linear order in the frequency of the perturbation. The extension of our computation to higher orders is straightforward.

**Definition of Love number and the intermediate region**

For the definition of Love numbers we follow [121] and [119], to which we refer for a more complete introduction. In the case of a static external perturbation ($\omega = 0$), one imposes the ingoing boundary condition on the radial part of the perturbing field at the horizon, which then behaves near infinity as

$$\begin{aligned} R(r \to \infty) &= A r^{\ell-s}(1+\mathcal{O}(r^{-1})) + B r^{-\ell-s-1}(1+\mathcal{O}(r^{-1})) \\ &= A r^{\ell-s}\left[(1+\mathcal{O}(r^{-1})) + k^{(s)}_{\ell m}\left(\frac{r}{r_+ - r_-}\right)^{-2\ell-1}(1+\mathcal{O}(r^{-1}))\right] \end{aligned} \tag{4.4.32}$$

for some constants $A$ and $B$. The Love number $k^{(s)}_{\ell m}$ is then defined as the coefficient of $(r/(r_+ - r_-))^{-2\ell-1}$ (note that this differs from the definition in [121] where they define it as the coefficient of $(r/2M)^{-2\ell-1}$ instead). In the non-static case however, the definition of Love number is less clear, since the behaviour of the radial function at infinity is now qualitatively different from (4.4.32): it is oscillatory (cf. (4.4.4)) due to the term $\propto \omega^2$ in the potential (E.0.3). For small frequencies we can however define an intermediate regime $r \gg M$, $r\omega \ll 1$ in which the multipole expansion (4.4.32) is still valid and we can read off the Love numbers in the same way as in the static case. Recall the Teukolsky equation written as a Schrödinger equation:

$$\frac{d^2\psi(z)}{dz^2} + V_{CFT}(z)\psi(z) = 0 \tag{4.4.33}$$

with the potential

$$V_{CFT}(z) = \frac{u - \frac{1}{2} + a_0^2 + a_1^2}{z(z-1)} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{m_3 L}{z} - \frac{L^2}{4} \,. \tag{4.4.34}$$

The intermediate regime corresponds to $z \gg 1$, $Lz \ll 1$. Expanding in these variables the potential reads:

$$\frac{V_{CFT}(z)}{L^2} = \frac{u}{L^2 z^2}\left(1 + \mathcal{O}(z^{-1}, Lz)\right) \,. \tag{4.4.35}$$

We see that in this regime the leading term in the potential is the one $\propto 1/z^2$, and the multipole expansion holds. In a sense we are taking $z$ to be big enough to be far from the horizon, but not so far as to reach the oscillatory region at infinity, as already mentioned in [120]. In the static case this intermediate region where the multipole expansion is valid extends all the way to infinity. This is the so called near zone approximation (see for example [132]). On the CFT side, this is what we called intermediate region in the previous chapter, and the corresponding conformal block is given by (3.3.48). For the sake of concreteness, here we will compute the relevant conformal block explicitly. To do so we need to project the irregular state in the module $|\Delta\rangle$, where $\Delta \sim \frac{1}{b^2}\left(\frac{1}{4}-a^2\right)$ is fixed by the dictionary and the Matone relation. For later convenience we define the projected state

$$\langle \Delta, \Lambda, \mu| = \langle \Lambda, \mu|\Pi_\Delta\,, \tag{4.4.36}$$

where $\Pi_\Delta$ is the projector on the Verma module $|\Delta\rangle$. Recall that $b\mu = m_3$ as $b$ goes to zero.

### 4.4.5 Slowly rotating Kerr Love numbers

Let us compute the Kerr Love numbers up to first order in $M\omega \sim M\Omega$. In order to do this we have to consider only the first instanton correction since $\Lambda \propto M\omega$. The wavefunction up to one instanton can be derived from the conformal blocks in the intermediate regime. Schematically,

$$\psi(z) \sim \frac{\langle \Delta, \Lambda, \mu|\phi(z)V_1(1)|\Delta_0\rangle}{\langle \Delta, \Lambda, \mu|V_1(1)|\Delta_0\rangle} \simeq \frac{\left(\langle\Delta| + \frac{\mu\Lambda}{2\Delta}\langle\Delta|L_1\right)\phi(z)V_1(1)|\Delta_0\rangle}{\left(\langle\Delta| + \frac{\mu\Lambda}{2\Delta}\langle\Delta|L_1\right)V_1(1)|\Delta_0\rangle}\,. \tag{4.4.37}$$

The key observation is that the conformal blocks of the 4 point function $\langle\Delta|\phi(z)V_1(1)|\Delta_0\rangle$ are just given by hypergeometric functions. Imposing ingoing boundary condition at the horizon, this gives the following wavefunction in the intermediate regime:

$$\begin{aligned}\psi(z) = &\left[1 + \frac{m_3 L}{\frac{1}{2}-2a^2}\left(\left(1-\frac{1}{z}\right)\partial_{1/z} + z - \frac{1}{2}\right)\right]\sum_{\theta=\pm} M_{a_1+a_\theta} z^{\frac{1}{2}-\theta a}\left(1-\frac{1}{z}\right)^{\frac{1}{2}+a_1} \times \\ &\times {}_2F_1\left(\frac{1}{2}+a_1+\theta a - a_0, \frac{1}{2}+a_1+\theta a + a_0; 1+2\theta a; \frac{1}{z}\right) + \mathcal{O}\left(\Lambda^2\right)\,.\end{aligned} \tag{4.4.38}$$

Here $M_{a_1+a_\theta}$ are just hypergeometric connection coefficients. Note that the first instanton contributes at this order only if $s \neq 0$ since for zero spin $m_3 L \sim \mathcal{O}(M^2\omega^2)$. For a slowly rotating black hole the connection coefficients start with $\mathcal{O}((M\omega)^0) = \mathcal{O}((M\Omega)^0)$ terms. Indeed substituting the dictionary we find

$$\begin{aligned} M_{a_1+a_+} &= \frac{\Gamma(-1-2\ell-2\Delta\ell)\Gamma(1-2i\frac{\omega-m\Omega}{4\pi T_H}-s)}{\Gamma(-\ell-\Delta\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(-\ell-\Delta\ell-2iM\omega-s)} = \\ &= \frac{\ell!(\ell+s)!}{(2\ell+1)!}(-1)^{s+1}\frac{(2iM\omega)(-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{2\Delta\ell} + \mathcal{O}(M\omega)\,, \\ M_{a_1+a_-} &= \frac{\Gamma(1+2\ell)\Gamma(1-s)}{\Gamma(\ell+1)\Gamma(\ell-s+1)} + \mathcal{O}(M\omega)\,, \end{aligned} \tag{4.4.39}$$

where $a = \ell + 1/2 + \Delta\ell$. It turns out that the first correction to $a$ vanishes, so $\Delta\ell \sim \mathcal{O}(M^2\omega^2)$. Also note that all the Gamma functions are finite since $s \leq 0$. Plugging in the dictionary and expanding the hypergeometrics gives

$$
\begin{aligned}
&{}_2F_1\left(\frac{1}{2}+a_1+a-a_0,\frac{1}{2}+a_1+a+a_0;1+2a;\frac{1}{z}\right)\simeq {}_2F_1\left(1+\ell-s-2iM\omega,1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega;2+2\ell;\frac{1}{z}\right)\,,\\
&{}_2F_1\left(\frac{1}{2}+a_1-a-a_0,\frac{1}{2}+a_1-a+a_0;1-2a;\frac{1}{z}\right)\simeq \sum_{n=0}^{2\ell}\frac{(-\ell-s-2iM\omega)_{(n)}(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!}+\\
&+\frac{\Gamma(-2\ell-2\Delta\ell)\Gamma(1+\ell-s-2iM\omega)\Gamma(1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{\Gamma(-\ell-s-2iM\omega)\Gamma(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(2\ell+2)}z^{-2\ell-1}\times\\
&\times {}_2F_1\left(1+\ell-s-2iM\omega,1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega;2+2\ell;\frac{1}{z}\right)\,.
\end{aligned}
\tag{4.4.40}
$$

Note that

$$
\frac{\Gamma(-2\ell-2\Delta\ell)\Gamma(1+\ell-s-2iM\omega)\Gamma(1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{\Gamma(-\ell-s-2iM\omega)\Gamma(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(2\ell+2)}M_{a_1+a_-}=-M_{a_1+a_+}+\mathcal{O}\left((M\omega)^2\right)\,,
\tag{4.4.41}
$$

therefore at this order the hypergeometrics simplify one against the other up to a finite polynomial, hence

$$
\begin{aligned}
&\psi(z)=\left[1+\frac{m_3L}{\frac{1}{2}-2a^2}\left(\left(1-\frac{1}{z}\right)\partial_{1/z}+z-\frac{1}{2}\right)\right]\times\\
&\times M_{a_1+a_-}z^{\frac{1}{2}+a}\left(1-\frac{1}{z}\right)^{\frac{1}{2}+a_1}\sum_{n=0}^{2\ell}\frac{(-\ell-s-2iM\omega)_{(n)}(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!}+\mathcal{O}(M^2\omega^2)\,.
\end{aligned}
\tag{4.4.42}
$$

The radial wavefunction is given by

$$
R(r)=\Delta^{-\frac{s+1}{2}}(r)\psi(z)\,,
\tag{4.4.43}
$$

where

$$
z=\frac{r}{2M}+\mathcal{O}\left(M^2\Omega^2\right)\,,\ \Delta(r)^{-\frac{s+1}{2}}=(r_+-r_-)^{-s-1}z^{-s-1}\left(1-\frac{1}{z}\right)^{-\frac{s+1}{2}}\,.
\tag{4.4.44}
$$

To find the Love numbers, we need the ratio between the coefficient of $r^{-\ell-s-1}$ (the response) and the coefficient of $r^{\ell-s}$ (the source). The term coming from the first instanton in (4.4.42) will not contribute at this order. Indeed this term gives

$$
\begin{aligned}
&\psi(z)\supset\frac{-4iM^2\omega s}{\ell(\ell+1)}\left(\left(1-\frac{1}{z}\right)\partial_{1/z}+z-\frac{1}{2}\right)M_{a_2+a_-}z^{\ell+1}\left(1-\frac{1}{z}\right)^{\frac{1-s}{2}}\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!}+\mathcal{O}(M^2\omega^2)=\\
&=\frac{-4iM^2\omega s}{\ell(\ell+1)}M_{a_2+a_-}z^{\ell+1}\left(1-\frac{1}{z}\right)^{\frac{1-s}{2}}\left(-z\ell+\frac{2\ell+s}{2}+\left(1-\frac{1}{z}\right)\partial_{1/z}\right)\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!}+\mathcal{O}(M^2\omega^2)\,,
\end{aligned}
\tag{4.4.45}
$$

where by $\psi(z)\supset\dots$ we mean that we are considering only a subset of the terms appearing in $\psi(z)$. After taking into account the factor of $\Delta$ from (4.4.43), one sees that this contribution to $R(r)$ does not contain the power that we are interested in. Focusing on the zero instanton contribution, the $(1-1/z)$ prefactor has an $\mathcal{O}(M\omega)$ term in the exponent that has to be expanded, resulting in

$$
R(r)\supset i\frac{\omega-m\Omega}{4\pi T_H}\frac{M_{a_1+a_-}}{(r_+-r_-)^{s+1}}\frac{r^{\ell-s}}{((2M)^{\ell+1}}\left(1+s\frac{2M}{r}+\frac{s(s+1)}{2}\left(\frac{2M}{r}\right)^2\right)\sum_{k=1}^{\infty}\sum_{n=0}^{2\ell}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{\left(\frac{r}{2M}\right)^{-n-k}}{n!k}\,.
\tag{4.4.46}
$$

This term contains the correct power, with coefficient

$$R(r) \supset \frac{M_{a_1+a_-}}{(r_+ - r_-)^{s+1}} \frac{r^{\ell-s}}{((2M)^{\ell+1}} i\frac{\omega - m\Omega}{4\pi T_H} \left(\frac{r}{2M}\right)^{-2\ell-1} \left( \sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell+1-n)} + \right.$$
$$\left. + s\sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell-n)} + \frac{s(s+1)}{2} \sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell-1-n)} \right) . \tag{4.4.47}$$

A surprising identity reveals that

$$\sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell+1-n)} + s\sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell-n)} + \frac{s(s+1)}{2} \sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)} n!(2\ell-1-n)} =$$
$$= \frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!} (-1)^s \,, \tag{4.4.48}$$

therefore

$$R(r) \supset \frac{M_{a_1+a_-}}{(r_+ - r_-)^{s+1}} \frac{r^{\ell-s}}{(2M)^{\ell+1}} \left[ 1 + i\frac{\omega - m\Omega}{4\pi T_H} \left(\frac{r}{2M}\right)^{-2\ell-1} \frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!} (-1)^s \right] . \tag{4.4.49}$$

Noticing that $1/4\pi T_H \simeq 2M$ finally gives the Love number

$$k^s_{a\,,m} = 2iM\,(\omega - m\Omega)\,(-1)^s \frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!} + \mathcal{O}(M^2\omega^2, M^2\Omega^2, M^2\omega\Omega) \,. \tag{4.4.50}$$

This result matches with formula (6.17) in [121]. Note that the Love number remains purely imaginary for a small frequency perturbation, and that it vanishes in the case of a static perturbation of a Schwarzschild black hole.

# Chapter 5

# Heun equations in holography: perturbations of AdS black holes

## 5.1 Introduction

In this chapter we study the thermal two-point function in a holographic four-dimensional CFT[1] [13, 134, 135] using the connection coefficients of Heun functions derived in chapter 3. Above the Hawking-Page transition [14] this observable is computed by studying the wave equation on the AdS-Schwarzschild background [15]. Finite temperature dynamics of CFTs is particularly rich in $d > 2$, where propagation of energy is not fixed by symmetries. On the gravity side, this is related to the presence of a propagating graviton in the spectrum of the theory, namely gravity waves.[2] On the field theory side, it is due to the fact that conformal symmetry is finite-dimensional in $d > 2$. This richness comes at a price that even for the simplest finite temperature observables no explicit solutions are available in $d > 2$.[3] In this chapter we provide the first example of such an explicit result. The thermal two-point function is computed by studying the wave equation on the black hole background [103, 140, 141]. This equation is of the Heun type [8, 4, 9], and the retarded two-point function is given in terms of its connection coefficients. We reproduce the available perturbative results from the literature [142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 137] and make new predictions.

## 5.2 Holographic two-point function at finite temperature

### 5.2.1 Black hole

We consider a holographic conformal field theory at finite temperature. Above the Hawking-Page transition [14], this theory is dual to a black hole in AdS [15]. In this paper we will specialize to the case of $AdS_5$, where the black hole metric is

$$ds^2 = -f(r)\, dt^2 + f(r)^{-1}\, dr^2 + r^2 d\Omega_3^2. \tag{5.2.1}$$

---

[1] We consider a finite-temperature CFT on the sphere, $S^1_\beta \times S^3$, and on the plane, $S^1_\beta \times \mathbb{R}^3$. The former is related to the black hole geometry, and the latter to the black brane. The requirement of being holographic implies a large CFT central charge ($c_T \gg 1$), and a large gap in the spectrum of higher spin single trace operators ($\Delta_{\text{gap}} \gg 1$) [133].

[2] Another characteristic feature of black holes in $d > 2$ is the existence of stable orbits [37, 136, 137].

[3] Here we refer to the black hole phase. For the thermal AdS phase some explicit results exist [138]. They are also available in $d \leq 2$, see e.g. [139].

Setting the AdS radius to 1, the redshift factor takes the form

$$\begin{aligned} f(r) &= r^2 + 1 - \frac{\mu}{r^2} \\ &\equiv \left(1 - \frac{R_+^2}{r^2}\right)(r^2 + R_+^2 + 1), \end{aligned} \tag{5.2.2}$$

where the Schwarzschild radius is given by

$$R_+ = \sqrt{\frac{\sqrt{1+4\mu} - 1}{2}}. \tag{5.2.3}$$

The dimensionless parameter $\mu$ is related to the black hole mass $M$ by

$$\mu = \frac{8 G_N M}{3\pi} \ . \tag{5.2.4}$$

We are interested in the two-point function of a scalar operator $\mathcal{O}(x)$ with dimension $\Delta$, dual to a massive scalar $\phi$ in the bulk with mass [156]

$$m = \sqrt{\Delta(\Delta - 4)}. \tag{5.2.5}$$

In order to compute this two-point function, we need to solve the wave equation on the black hole background,

$$(\Box - m^2)\phi = 0. \tag{5.2.6}$$

Expanding the solution into Fourier modes, we have

$$\phi(t, r, \Omega) = \int d\omega \sum_{\ell, \vec{m}} e^{-i\omega t} Y_{\ell \vec{m}}(\Omega) \psi_{\omega\ell}(r). \tag{5.2.7}$$

Our conventions for spherical harmonics $Y_{\ell\vec{m}}$ can be found in Appendix A of [24]. The wave equation then takes the form (see [18] and references there)

$$\left(\frac{1}{r^3}\partial_r(r^3 f(r)\partial_r) + \frac{\omega^2}{f(r)} - \frac{\ell(\ell+2)}{r^2} - \Delta(\Delta - 4)\right)\psi_{\omega\ell} = 0. \tag{5.2.8}$$

We are interested in the retarded Green's function, and therefore we impose ingoing boundary conditions on the solution $\phi$ at the horizon,

$$\psi^{\text{in}}_{\omega\ell}(r) = (r - R_+)^{-\frac{i\omega}{2}\frac{R_+}{2R_+^2+1}} + \ldots \tag{5.2.9}$$

The solution $\psi^{\text{in}}$ behaves near the AdS boundary $r \to \infty$ as

$$\psi^{\text{in}}_{\omega\ell}(r) = \mathcal{A}(\omega, \ell)(r^{\Delta - 4} + \ldots) + \mathcal{B}(\omega, \ell)(r^{-\Delta} + \ldots). \tag{5.2.10}$$

The two-point function is then the ratio of the response $\mathcal{B}(\omega, \ell)$ to the source $\mathcal{A}(\omega, \ell)$ [103],

$$G_R(\omega, \ell) = \frac{\mathcal{B}(\omega, \ell)}{\mathcal{A}(\omega, \ell)}. \tag{5.2.11}$$

Our conventions for the thermal two-point function in the CFT dual are collected in Appendix F. The wave equation takes a particularly convenient form under the transformations

$$z = \frac{r^2}{r^2 + R_+^2 + 1} \ , \tag{5.2.12}$$

$$\psi_{\omega\ell}(r) = \left(r^3 f(r) \frac{dz}{dr}\right)^{-1/2} \chi_{\omega\ell}(z) \ . \tag{5.2.13}$$

We then obtain Heun's differential equation in normal form,

$$\left(\partial_z^2+\frac{\frac{1}{4}-a_1^2}{(z-1)^2}-\frac{\frac{1}{2}-a_0^2-a_1^2-a_t^2+a_\infty^2+u}{z(z-1)}+\frac{\frac{1}{4}-a_t^2}{(z-t)^2}+\frac{u}{z(z-t)}+\frac{\frac{1}{4}-a_0^2}{z^2}\right)\chi_{\omega\ell}(z)=0. \tag{5.2.14}$$

With parameters

$$\boxed{\begin{aligned}
&t=\frac{R_+^2}{2R_+^2+1}\,,\quad a_0=0\,,\quad a_t=\frac{i\omega}{2}\frac{R_+}{2R_+^2+1}\\
&a_1=\frac{\Delta-2}{2}\,,\quad a_\infty=\frac{\omega}{2}\frac{\sqrt{R_+^2+1}}{2R_+^2+1}\,,\\
&u=-\frac{\ell(\ell+2)+2(2R_+^2+1)+R_+^2\Delta(\Delta-4)}{4(R_+^2+1)}+\frac{R_+^2}{1+R_+^2}\frac{\omega^2}{4(2R_+^2+1)}\,.
\end{aligned}} \tag{5.2.15}$$

The purely ingoing solution behaves near the black hole horizon as

$$\chi_{\omega\ell}^{\text{in}}(z)=(t-z)^{\frac{1}{2}-a_t}+\ldots\,. \tag{5.2.16}$$

Close to the AdS boundary it takes the form

$$\chi_{\omega\ell}^{\text{in}}(z)\propto\mathcal{A}(\omega,\ell)\Big(\frac{1-z}{1+R_+^2}\Big)^{\frac{1}{2}-a_1}+\mathcal{B}(\omega,\ell)\Big(\frac{1-z}{1+R_+^2}\Big)^{\frac{1}{2}+a_1}+\ldots\,.$$

The problem of finding *the response function* (5.2.11) reduces to finding the *connection formulae* for the Heun function.

### 5.2.2 Black brane

The black brane is dual to CFT on $S^1\times\mathbb{R}^3$, and can be obtained by taking the high-temperature limit $T\to\infty$ of the black hole, while keeping $\frac{\omega}{T}\equiv\hat{\omega}$ and $\frac{\ell}{T}\equiv|\mathbf{k}|$ fixed. Here $\hat{\omega}$ and $\mathbf{k}$ are the dimensionless energy and three-momentum of the resulting theory on $S^1\times\mathbb{R}^3$ in units of temperature. Recall that for the AdS-Schwarzschild black hole [15]

$$T=\frac{1}{\sqrt{2}\pi}\sqrt{\frac{1+4\mu}{\sqrt{1+4\mu}-1}}, \tag{5.2.17}$$

and the high-temperature limit corresponds to $\mu\to\infty$.

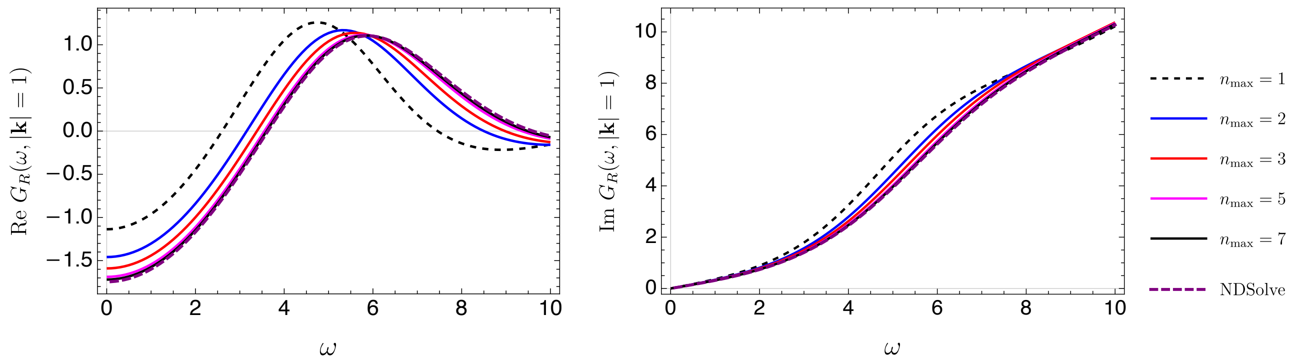
In this way we get the map between the gauge theory and gravity parameters for the black brane (to avoid clutter we switch from $\hat{\omega}$ to $\omega$), see 5.2.18.

$$\boxed{\begin{aligned}
&t=\frac{1}{2}\,,\quad a_0=0\,,\quad a_t=\frac{i\omega}{4\pi}\\
&a_1=\frac{\Delta-2}{2}\,,\quad a_\infty=\frac{\omega}{4\pi}\,,\\
&u=\frac{\omega^2-2\mathbf{k}^2}{8\pi^2}-\frac{1}{4}(\Delta-2)^2..
\end{aligned}} \tag{5.2.18}$$

Finally, we define the two-point function as follows,

$$G_R^{\text{brane}}(\omega,|\mathbf{k}|)=\lim_{T\to\infty}\frac{G_R(\omega T,|\mathbf{k}|T)}{T^{4a_1}}, \tag{5.2.19}$$

see Appendix G for the detailed derivation.



(a) (b)

Figure 5.1: We plot the retarded two-point function $G_R^{\text{brane}}(\omega, |\mathbf{k}|)$, given by (5.3.5) and (5.3.7), for $|\mathbf{k}| = 1$, $\Delta = 5/2$, as a function of $\omega$ and the maximal number of instantons $n_{\text{max}}$ in the truncated sum (5.3.8). a) The real part of the retarded two-point function Re $G_R^{\text{brane}}(\omega, 1)$. b) The imaginary part of the retarded two-point function Im $G_R^{\text{brane}}(\omega, 1)$. We set $T = 1$. We also compare our results with the direct numerical solution of the differential equation (we used NDSolve in Mathematica), see e.g. [157], and find beautiful agreement between the two methods. An analogous plot can be generated for the $|\mathbf{k}|$-dependence as well, and again we observed perfect agreement between our formulas and the direct numerical solution of the differential equation.

## 5.3 Exact thermal two-point function

From the gauge theory point of view, the parameters $a_0, a_1, a_t, a_\infty$ are related to the masses of the hypermultiplets, $t \sim e^{-1/g^2_{\text{YM}}}$ is the instanton counting parameter, and $u$ parameterizes the moduli space of vacua. The latter is related to the VEV $a$ of the scalar in the vector multiplet via the (quantum) Matone relation [158, 159]

$$u = -a^2 + a_t^2 - \frac{1}{4} + a_0^2 + t\partial_t F, \tag{5.3.1}$$

where $F$ is the instanton part of the NS free energy defined in appendix C. The dictionary (5.3.1) requires a careful treatment close to the points $2a = \mathbb{Z}$, where the NS function exhibits non-analyticity, see e.g. [160, 161]. We leave a more detailed discussion of this region for future work.

In particular this hidden connection between Heun's equation and supersymmetric gauge theory makes it possible to compute the connection coefficients $\mathcal{A}$ and $\mathcal{B}$ in (5.2.10) using the NS free energy, as done in [78].

Let

$$\chi_{\omega\ell}^{(t),\text{in}}(z) = (t - z)^{\frac{1}{2} - a_t} + \ldots \tag{5.3.2}$$

be the ingoing solution[4] of the wave equation (5.2.14) at the horizon ($z \sim t$) and let

$$\chi_{\omega\ell}^{(1),\pm}(z) = (1 - z)^{\frac{1}{2} \pm a_1} + \ldots \tag{5.3.3}$$

be the two independent solutions at infinity ($z \sim 1$). The connection formula reads

$$\begin{aligned} \chi_{\omega\ell}^{(t),\text{in}}(z) = \sum_{\theta'=\pm} \Big( \sum_{\sigma=\pm} \mathcal{M}_{-\sigma}(a_t, a; a_0) \mathcal{M}_{(-\sigma)\theta'}(a, a_1; a_\infty) t^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} \Big) t^{\frac{1}{2} - a_0 - a_t} (1 - t)^{a_t - a_1} \\ e^{\frac{1}{2}\left(-\partial_{a_t} - \theta' \partial_{a_1}\right) F} \chi_{\omega\ell}^{(1),\theta'}(z) \ , \end{aligned} \tag{5.3.4}$$

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[4] Here we have chosen the ingoing solution since we are interested in computing the retarded Green's function. Alternatively, the advanced Green's function can be computed by choosing the outgoing solution, resulting in a minor modification of (5.3.5).

where

$$\mathcal{M}_{\theta\theta'}(\alpha_0,\alpha_1;\alpha_2)=\frac{\Gamma(-2\theta'\alpha_1)}{\Gamma\left(\frac{1}{2}+\theta\alpha_0-\theta'\alpha_1+\alpha_2\right)}\frac{\Gamma(1+2\theta\alpha_0)}{\Gamma\left(\frac{1}{2}+\theta\alpha_0-\theta'\alpha_1-\alpha_2\right)},$$

and $F$ is the instanton part of the NS free energy.

The exact formula for the retarded two-point function (5.2.11) then reads

$$\boxed{G_R(\omega,\ell)=\left(1+R_+^2\right)^{2a_1}e^{-\partial_{a_1}F}\frac{\sum_{\sigma'=\pm}\mathcal{M}_{-\sigma'}(a_t,a;a_0)\mathcal{M}_{(-\sigma')+}(a,a_1;a_\infty)t^{\sigma' a}e^{-\frac{\sigma'}{2}\partial_a F}}{\sum_{\sigma=\pm}\mathcal{M}_{-\sigma}(a_t,a;a_0)\mathcal{M}_{(-\sigma)-}(a,a_1;a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}} \qquad (5.3.5)$$

where the parameters $t,a_0,a_t,a_1,a_\infty,u$ were defined in terms of $\omega,\ell$ and the mass of the black hole $\mu$ in (5.2.15). The instanton part of the free energy $F$ depends on all parameters, $F(t,a,a_0,a_t,a_1,a_\infty)$. Finally, we can eliminate $a$ from the problem using the Matone relation (5.3.1). In this way the right hand side of (5.3.5) is fully fixed in terms of $\omega,\ell$ and $\mu$.

Based on general grounds, $G_R(\omega,\ell)$ should be analytic in the upper half-plane (causality), it satisfies Im $G_R(\omega,\ell)=-$Im $G_R(-\omega,\ell)$ (KMS), and finally Im $G_R(\omega,\ell)\geq 0$ for $\omega>0$ (unitarity), see e.g. appendix B in [162]. In fact from the standard dispersive representation of $G_R(\omega,\ell)$ it follows that

$$[G_R(-\omega,\ell)]^*=G_R(\omega,\ell),\quad \omega\in\mathbb{R}. \qquad (5.3.6)$$

In this paper we mostly limit our analysis to $\omega\in\mathbb{R}$ and it is easy to check that (5.3.5) indeed satisfies (5.3.6). The argument for this goes as follows. First, we notice that for real $\omega$ and $\ell$, the relevant $a$ is either purely imaginary or purely real. Second, we notice that (5.3.5) is invariant under the change $a\to\pm a$, $a_\infty\to\pm a_\infty$. Finally, the instanton partition function for real $t$ is a real analytic function of its parameters, $F^*(a,a_0,a_t,a_1,a_\infty)=F(a^*,a_0^*,a_t^*,a_1^*,a_\infty^*)$. The property (5.3.6) then follows.

For the black brane, upon taking the limit (5.2.19) the result takes the form

$$G_R^{\text{brane}}(\omega,|\mathbf{k}|)=\pi^{4a_1}\frac{G_R(\omega,\ell)}{(1+R_+^2)^{2a_1}}, \qquad (5.3.7)$$

where $G_R(\omega,\ell)$ is taken from (5.3.5), but $a_i$, $t$, and $u$ are now mapped to $(\omega,\mathbf{k})$ according to (5.2.18). In (5.3.7) the temperature for the theory on $S^1\times\mathbb{R}^3$ is set to 1.

The exact expressions presented above involve in a crucial way the NS free energy. As explained in Appendix C, the NS free energy is computed as a (convergent) series expansion in the instanton counting parameter $t$,

$$F=\sum_{n\geq 1}^{\infty}c_n(a,a_0,a_t,a_1,a_\infty)t^n\ . \qquad (5.3.8)$$

The coefficients $c_n(a,a_0,a_t,a_1,a_\infty)$ in this series have a precise combinatorial definition in terms of Young diagrams. Hence in principle we can determine all of them. Given (5.3.8) one can straightforwardly solve the Matone relation (5.3.1) as a series in $t$ as well.

We can also write the above equation in a compact way by using the full NS free energy $F^{\text{NS}}$, which is the sum of the instanton part $F$, the one-loop part $F^{1-\text{loop}}$, and the classical term $F^{\text{p}}=-2a\log t$. The formula becomes

$$G_R(\omega,\ell)=(1+R_+^2)^{2a_1}\frac{\Gamma(-2a_1)}{\Gamma(2a_1)}\frac{\mathcal{G}(t,a,a_0,a_1,a_\infty,a_t)}{\mathcal{G}(t,a,a_0,-a_1,a_\infty,a_t)} \qquad (5.3.9)$$

with

$$\mathcal{G}(t, a, a_0, a_1, a_\infty, a_t) = e^{-\frac{1}{2}\partial_{a_1}F^{\rm NS}} \sinh\left(\frac{\partial_a F^{\rm NS}}{2}\right) \,. \tag{5.3.10}$$

This is the typical form of the Fredholm determinant in this class of theories [163, eq. 8.12], [164, eq. 5.6], see also [165, 166]. Note that the result for the two-point function has the following simple property: under $\Delta \to 4 - \Delta$ we have $G_R \to \frac{1}{G_R}$. This property is manifest in (5.3.9) after noticing that under this transformation $a_1 \to -a_1$. It is also expected on general grounds because sending $\Delta \to 4 - \Delta$ switches the boundary conditions [167], so that the source and response are interchanged.

One case where the exact Green's function (5.3.5) becomes analytically tractable is the limit where $\ell$ is the only large parameter. On the gauge theory side this means that the VEV of the scalar $a$ is much larger than all other parameters. In this limit one can use Zamolodchikov's formula for the Virasoro conformal blocks [168] and the AGT correspondence [69] to show that [169]

$$\begin{aligned} F &= a^2\left(\log\frac{t}{16} + \pi\frac{K(1-t)}{K(t)}\right) \\ &+ \left(a_1^2 + a_t^2 - \frac{1}{4}\right)\log(1-t) \\ &+ 2\left(a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - \frac{1}{4}\right)\log\left(\frac{2}{\pi}K(t)\right) + \mathcal{O}(a^{-2})\,. \end{aligned} \tag{5.3.11}$$

Here $K(t)$ is the complete elliptic integral of the first kind. Solving the Matone relation (5.3.1) for $a$, we find

$$a = -\frac{(\ell+1)\sqrt{1-2t}K(t)}{\pi} + \mathcal{O}(\ell^{-1}). \tag{5.3.12}$$

In Appendix J we use (5.3.12) to show that the imaginary part of $G_R$ is exponentially small at large $\ell$.

Let us conclude this section with a practical comment. When doing the actual computations we truncate the series in $t$ at some maximal instanton number $n_{\rm max}$. Given $n_{\rm max}$ and the corresponding $F^{n_{\rm max}}$, we then solve (5.3.1) for $a$ as a function of $u$ perturbatively in $t$. This step requires solving a linear equation at every new order in $t$. Finally, we plug both $F^{n_{\rm max}}$ and $a^{n_{\rm max}}(u)$ in (5.3.9) and evaluate $G_R^{n_{\rm max}}(\omega, \ell)$. We present an example of this procedure for $n_{\rm max} \leq 7$ and the case of the black brane in figure 5.1.[5] We find a beautiful agreement between our result and the direct numerical solution of the wave equation.

With the methods we used, going to higher $n_{\rm max}$ gets computationally costly rather quickly. For example, in the case of the $N_f = 4$ theory that we are interested in, going beyond 5-10 instantons appears challenging on a laptop. Hence to fully exploit the power of our method it would be important to identify the range of parameters for which $G_R(\omega, \ell)$ can be reliably computed with a few instantons. It would also be desirable to develop a more efficient way of computing the NS functions (either analytically or numerically).[6]

## 5.4 Relation to the heavy-light conformal bootstrap

The thermal two-point function computed in the previous section is directly related to the four-point correlation function of local operators $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ [171, 172]. Here $\mathcal{O}_L$ is the light or

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[5] Alternatively, we can use (5.3.5) to compute $G_R(\omega, a)$ and we can use (5.3.1) to evaluate the map $\ell(\omega, a)$ (or $\mathbf{k}(\omega, a)$). This is possible because the dependence on spin $\ell$ (or momentum $\mathbf{k}$) enters the problem only through the parameter $u$, which does not appear in the exact formula (5.3.5).

[6] For example using TBA-like techniques as in [170] and references there.

probe operator of dimension $\Delta_L$ from the previous section,[7] and $\mathcal{O}_H$ is a heavy operator with $\Delta_H \sim c_T$ that is dual to a black hole microstate, where $c_T$ parameterizes the two-point function of canonically normalized stress tensors. For the precise relationship between $\mu \sim \frac{\Delta_H}{c_T}$, $\Delta_H$ and $c_T$ see e.g. [142].

More precisely, we define the four-point function as follows

$$G(z, \bar{z}) \equiv \langle \mathcal{O}_H(0)\mathcal{O}_L(z, \bar{z})\mathcal{O}_L(1, 1)\mathcal{O}_H(\infty)\rangle, \tag{5.4.1}$$

where all operators for simplicity are taken to be real scalars. The insertion at infinity is given by $\mathcal{O}_H(\infty) = \lim_{x_4 \to \infty} |x_4|^{2\Delta_H}\mathcal{O}_H(x_4)$. We also used conformal symmetry to put all four operators in a two-dimensional plane with coordinate $z = x^1 + ix^2$.

We choose the normalization of operators such that in the short distance limit $z, \bar{z} \to 1$ we have

$$G(z, \bar{z}) = \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}} + \dots \; . \tag{5.4.2}$$

This four-point function admits an OPE expansion in various channels, see e.g. [173]. We focus on the heavy-light channel, in which the expansion of the four-point function takes the form

$$G(z, \bar{z}) = \sum_{\mathcal{O}_{\Delta,\ell}} \lambda^2_{H,L,\mathcal{O}_{\Delta,\ell}} \frac{g^{\Delta_{H,L},-\Delta_{H,L}}_{\Delta,\ell}(z, \bar{z})}{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}}, \tag{5.4.3}$$

where $\Delta_{H,L} \equiv \Delta_H - \Delta_L$, and $\lambda_{H,L,\mathcal{O}_{\Delta,\ell}} \in \mathbb{R}$ are the three-point functions. Finally, the expressions for the conformal blocks $g^{\Delta_{H,L},-\Delta_{H,L}}_{\Delta,\ell}(z, \bar{z})$ can be found for example in [174, 175].

We next consider the $\Delta_H, c_T \to \infty$ limit of the expansion of $G(z, \bar{z})$ above with $\mu = \frac{160}{3}\frac{\Delta_H}{c_T}$ kept fixed. In this limit the spectrum of operators becomes effectively continuous and the contribution of descendants is suppressed [173].[8] Specializing to $d = 4$, we get the following expression for the OPE expansion,

$$G(z, \bar{z}) = \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} d\omega \, g_{\omega,\ell} (z\bar{z})^{\frac{\omega-\Delta-\ell}{2}} \frac{z^{\ell+1} - \bar{z}^{\ell+1}}{z - \bar{z}} \; , \tag{5.4.4}$$

where we introduced $\omega = \Delta'_H - \Delta_H$, and $g_{\omega,\ell}$ for the product of the three-point functions $\lambda^2_{H,L,\mathcal{O}_{\Delta_{H'},\ell}}$ and the density of primaries. Thanks to unitarity we have $g_{\omega,\ell} \geq 0$ and KMS symmetry implies that

$$g_{-\omega,\ell} = e^{-\beta\omega} g_{\omega,\ell} \; . \tag{5.4.5}$$

We can now state the precise relationship between the heavy-light four-point function and the thermal two-point function [137],

$$g_{\omega,\ell} = \frac{\ell+1}{2\pi(\Delta_L - 1)(\Delta_L - 2)} \frac{\text{Im } G_R(\omega, \ell)}{1 - e^{-\beta\omega}} \; , \tag{5.4.6}$$

where $\beta$ and $\Delta_H$ are related in the standard way, $\beta = \frac{\partial S(\Delta_H)}{\partial \Delta_H}$. In this formula $S(\Delta_H)$ is the effective density of primaries of dimension $\Delta_H$. This relation is the combination of the eigenstate thermalization hypothesis [176, 177, 171, 172] and the standard relations between

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[7] In this section we switch from $\Delta$ to $\Delta_L$ to make the distinction between the light and heavy operators more obvious.

[8] This requires an extra assumption on which operators dominate the OPE, see e.g. the discussion in [137].

various thermal two-point functions [162]. The factor $\ell+1$ originates from summing over $\vec{m}$ of the spherical harmonics $Y_{\ell\vec{m}}$, see Appendix A of [24] for details.

There is a natural limit in which the general expression (5.4.6) simplifies: it is the large spin limit $\ell \to \infty$. As explained in detail in [136, 137], in this limit the relevant states are orbits which are stable perturbatively in $\frac{1}{\ell}$. These states manifest themselves in $G_R(\omega,\ell)$ as poles (also known as quasi-normal modes) with imaginary part which is non-perturbative in spin $\ell$. Therefore, perturbatively in $\ell$, Im $G_R(\omega,\ell)$ effectively becomes the sum of $\delta(|\omega|-\omega_{n\ell})$, where $\omega_{n\ell}=\Delta_L+\ell+2n+\gamma_{n\ell}$ and $\gamma_{n\ell}\to 0$ at large spin. Notice that for $|\omega|\sim\ell$, $[(1-e^{-\beta\omega})^{-1}]_{\text{pert}}$ becomes a step function $\theta(\omega)$, and in this way $g_{\omega,\ell}$ reduces at large spin to the expected sum over heavy-light double-twist operators $\mathcal{O}_H\Box^n\partial^\ell\mathcal{O}_L$ .

We can summarize this as follows

$$\begin{aligned} g^{\text{pert}}_{\omega,\ell} &= \theta(\omega)\frac{\ell+1}{2\pi(\Delta_L-1)(\Delta_L-2)}\text{Im}\, G_R^{\text{pert}}(\omega,\ell) \\ &= \sum_{n=0}^{\infty} c_{n\ell}\,\delta(\omega-\omega_{n\ell}) \ , \end{aligned} \tag{5.4.7}$$

where the relation holds for all the terms which contribute as powers at large spin $\ell$, namely $\frac{1}{\ell^{\#}}$. We signified this by writing Im $G_R^{\text{pert}}(\omega,\ell)$ (see also 5.5 for a more precise definition). Here $c_{n\ell}$ is the square of the OPE coefficients of double-twist operators. In writing (5.4.7) we also used the fact that at fixed $\omega$, Im $G_R(\omega,\ell)$ is nonperturbative in spin at large $\ell$.[9] We establish this fact in Appendix J.

The large spin expansion of the heavy-light four-point function was actively explored in the last few years [142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155]. One of the basic observations of these works is that in $d>2$ the effective expansion parameter is $\frac{\mu}{\ell^{\frac{d-2}{2}}}$. We can therefore equivalently study the small $\mu$ expansion of the exact results. This is what we do in the next section.

## 5.5 Small $\mu$ expansion

In the previous section we explained how to compute the dimensions and OPE data of heavy-light double-twist operators using the exact two-point function (5.3.5). Now we would like to carry out this procedure perturbatively in $1/\ell$. Note that the expected perturbative parameter is $\frac{\mu}{\ell}$ [142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155], so that instead of taking the large spin limit, we can equivalently consider the limit of small black holes. This is a natural limit from the point of view of the Nekrasov-Shatashvili functions, which are defined as a perturbative expansion in $t\sim\mu$ for small $\mu$.

### 5.5.1 Exact quantization condition and residues

In the small $\mu$ and large spin expansion, the Green's function (5.3.5) simplifies considerably. To see this, note that at small $\mu$ the Matone relation (5.3.1) becomes

$$a=\pm\frac{\ell+1}{2}+\mathcal{O}(\mu), \tag{5.5.1}$$

where we plugged in the dictionary from Table 5.2.15. Since the Green's function is invariant under $a\to -a$, it does not matter what sign we pick in (5.5.1). Choosing the minus sign

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[9] In principle, non-perturbative in spin effects are accessible to the light-cone bootstrap [178] thanks to the Lorentzian inversion formula [179, 180, 181]. However, such effects have not been yet explored in the context of the heavy-light bootstrap.

in (5.5.1), the ratio of the $\sigma = -1$ term to the $\sigma = 1$ term in both the numerator and the denominator of (5.3.5) scales as $\mu^{\ell+1}$, which is exponentially small in spin. Neglecting this nonperturbative correction, we find

$$G_R^{\rm pert}(\omega, \ell) = (1 + R_+^2)^{2a_1} e^{-\partial_{a_1} F} \\ \frac{\Gamma(-2a_1)\Gamma(1/2 - a + a_1 - a_\infty)\Gamma(1/2 - a + a_1 + a_\infty)}{\Gamma(2a_1)\Gamma(1/2 - a - a_1 - a_\infty)\Gamma(1/2 - a - a_1 + a_\infty)}. \tag{5.5.2}$$

In a sense, this expression is a generalization of the semi-classical Virasoro vacuum block [182, 183] to $d = 4$. Indeed, via (5.4.7) it encodes the contribution of the identity and multi-stress tensor contributions in the light-light channel, schematically $\mathcal{O}_L \times \mathcal{O}_L \sim 1 + T + T^2 + ...$ . The effects non-perturbative in spin (which are intimately related to the presence of the black hole horizon) are, on the other hand, encoded in the contribution of the double-twist operators $\mathcal{O}_L \times \mathcal{O}_L \sim \mathcal{O}_L \Box^n \partial^\ell \mathcal{O}_L$.

We can now explicitly read off the poles and residues of (5.5.2). There are poles in the function $\Gamma(1/2 - a + a_1 - a_\infty)$ at positive energies $\omega = \omega_{n\ell}$, which are nothing but the dimensions of the double-twist operators. The locations of these poles are determined by the following quantization condition,

$$\omega_{n\ell}: \quad n = a + a_\infty - a_1 - 1/2, \quad n \geq 0. \tag{5.5.3}$$

Geometrically this corresponds to the quantization of the quantum A-period associated to the Seiberg-Witten geometry. The relation (5.5.3) implicitly defines the scaling dimensions of the double-twist operators $\omega_{n,\ell}$ via the black hole to gauge theory dictionary in (5.2.15), along with the Matone relation (5.3.1). Computing the residues of the two-point function (5.5.2) and using (5.4.7) and (5.2.18) then gives

$$c_{n\ell} = \frac{(\ell+1)\Gamma(\Delta + n - 1)\Gamma(2a_\infty - n)}{\Gamma(\Delta)\Gamma(\Delta - 1)\Gamma(n+1)\Gamma(2a_\infty - n - \Delta + 2)} \times \frac{(1 + R_+^2)^{\Delta - 2} e^{-\partial_{a_1} F}}{2} \left( \frac{d(a + a_\infty)}{d\omega} \right)^{-1} \Bigg|_{\omega = \omega_{n\ell}} . \tag{5.5.4}$$

Note that, since $F$ is defined by a power series in $\mu$ whose coefficients are rational functions, it is straightforward to invert (5.5.3) to any desired order in $\mu$ by perturbing around the $\mu = 0$ result. In this sense, (5.5.3) and (5.5.4) represent an exact solution for the bootstrap data.

### 5.5.2 Anomalous dimensions and OPE data

To organize the perturbative series, let us define

$$\omega_{n\ell} = \omega_{n\ell}^{(0)} + \sum_{i=1}^{\infty} \mu^i \gamma_{n\ell}^{(i)} , \\ c_{n\ell} = c_{n\ell}^{(0)} \left( 1 + \sum_{i=1}^{\infty} \mu^i c_{n\ell}^{(i)} \right) . \tag{5.5.5}$$

We then plug these expansions into (5.5.3) and (5.5.2), using the dictionary in (5.2.15), the Matone relation (5.3.1), and the definitions in Appendix C. At zeroth order in $\mu$, we reproduce the OPE coefficients in generalized free field theory, see e.g. [143, 153],

$$\omega_{n\ell}^{(0)} = \Delta + \ell + 2n , \tag{5.5.6}$$

$$c_{n\ell}^{(0)} = \frac{(\ell+1)\Gamma(\Delta + n - 1)\Gamma(\Delta + n + \ell)}{\Gamma(\Delta)\Gamma(\Delta - 1)\Gamma(n+1)\Gamma(n + \ell + 2)}, \tag{5.5.7}$$

namely we have the following identity

$$\sum_{n,\ell=0}^{\infty} c_{n\ell}^{(0)}(z\bar{z})^{\frac{\omega_{n\ell}^{(0)}-\Delta-\ell}{2}} \frac{z^{\ell+1}-\bar{z}^{\ell+1}}{z-\bar{z}} = \frac{1}{(1-z)^{\Delta}(1-\bar{z})^{\Delta}} \,. \tag{5.5.8}$$

Now let us go to first order in $\mu$. We find

$$\gamma_{n\ell}^{(1)} = -\frac{\Delta^2+\Delta(6n-1)+6n(n-1)}{2(\ell+1)} \,, \tag{5.5.9}$$

$$\begin{aligned} c_{n\ell}^{(1)} &= \frac{1}{2}\bigg(3(\Delta-2) - \frac{3(\Delta+2n-1)}{\ell+1} + \\ &(3(\ell+2n+\Delta)-2\gamma_1)(\psi^{(0)}(2+\ell+n)-\psi^{(0)}(\Delta+\ell+n))\bigg) \,, \end{aligned} \tag{5.5.10}$$

where $\psi^{(m)}(x) = d^{m+1}\log\Gamma(x)/dx^{m+1}$ is the polygamma function of order $m$. These results agree with the light-cone bootstrap computations [153, 155, 146].

At second order $\mathcal{O}(\mu^2)$ the answers become more complicated, and are displayed explicitly in Appendix H. Already at this order only $\mathcal{O}(1/\ell^2)$ results are available in the literature, which is the leading term in the large spin expansion. We find complete agreement with the result of [153].

At $k$-th order $\mathcal{O}(\mu^k)$ we find the following structure

$$\gamma_{n\ell}^{(k)} = \sum_{j=0}^{2k+1} R_j^{(k)}(n,\ell)\Delta^j, \tag{5.5.11}$$

where $R_j^{(k)}(n,\ell)$ are polynomials of degree $k-j$ in $n$ and are meromorphic functions of $\ell$. The singularities occur at $\ell_{\text{sing}} \in \mathbb{Z}$ and $-k-1 \leq \ell_{\text{sing}} \leq k-1$. These singularities are however spurious and occur because for $\ell < k$ it is not justified to drop the $\sigma = -1$ term when going from (5.3.5) to (5.5.2).

For the three-point functions $c_{n\ell}^{(k)}$ the structure is very similar, the main difference being that the analogs of $R_j^{(k)}(n,\ell)$ can also depend on $\psi^{(m)}(\Delta+n+\ell)-\psi^{(m)}(2+n+\ell)$ with $m \leq k-1$

### 5.5.3 The imaginary part of quasi-normal modes

Until now, in computing the position of the poles of $G_R(\omega,\ell)$, we have neglected the imaginary part, which is exponentially suppressed at large spin.[10] This exponential suppression of the imaginary part means that the large spin quasinormal modes thermalize very slowly, so they give the leading contribution to the late time Green's function to leading order in the $1/c_T$ expansion.

Let us now compute the leading behavior of the imaginary part, for which we must consider the exact Green's function (5.3.5). In the large spin expansion, the numerator of (5.3.5) is finite, so the poles arise when the denominator vanishes. Therefore we must solve

$$0 = \sum_{\sigma=\pm} \mathcal{M}_{-\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)-}(a, a_1, a_\infty) t^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}. \tag{5.5.12}$$

We make an ansatz

$$\text{Im}\ \omega_{n\ell} = i\sum_{k=1}^{\infty} f_{n\ell}^{(k)} \mu^{\ell+1/2+k}, \tag{5.5.13}$$

---

[10] Physically, this is related to the fact that classically stable orbits can decay quantum-mechanically due to tunneling, see e.g. [37].

where $f_{n\ell}^{(k)}$ are real. Note that the imaginary part behaves as $\mu^\ell$ at large $\ell$, as expected from the tunneling calculation in [137]. The first contribution to the imaginary part is at order $\mu^{\ell+3/2}$, which is consistent with numerical evidence [184]. As shown in Appendix I, the explicit form of the leading contribution to the imaginary part is

$$f_{n\ell}^{(1)} = -\frac{2^{-4\ell}\pi^2}{(\ell+1)^2}\omega_{n\ell}^{(0)}\frac{\Gamma(\Delta+n+\ell)}{\Gamma(\Delta+n-1)}\frac{\Gamma(n+\ell+2)}{\Gamma(n+1)\Gamma(\frac{\ell+1}{2})^4}. \tag{5.5.14}$$

It should be possible to check this expression using the techniques of [185]. Note that Im $\omega_{n\ell} < 0$ as expected from causality.

# Chapter 6

# Conclusions and further directions

The main result of this thesis is the computation of the connection coefficients of Heun functions and their confluences. In deriving such results, we performed a detailed study of regular and irregular Liouville conformal blocks. In particular, we derived the structure constants involving a class of irregular states (see appendix A.1) and the connection formulas of degenerate conformal blocks that in the semiclassical limit solve the Heun equation and its confluences. Crucially for applications, we give concrete and explicit expressions for such connection coefficients in terms of instanton partitions functions of gauge theories which are AGT dual to Liouville CFT in their so NS limit. In this respect, there are a number of open questions which are left for further investigations.

- In this thesis we restricted to Heun functions. The generalization to more general Fuchsian ODEs, and accordingly $n-$point conformal blocks, can be done along the same lines as the ones we have been following. This produces explicit connection formulae for $n-$point Fuchsian systems in terms of Gamma functions and Nekrasov partition functions of linear quiver gauge theories.

- In the previous chapters we considered the class of confluences producing irregular singularities up to Poincaré rank one. This is implied by the fact that their gauge theory description can be given in a weakly coupled frame. It would be interesting to extend our analysis to higher rank singularities. These are related to Argyres-Douglas points in the gauge theory.

- All our discussion involved 2nd order ODEs. By considering BPZ equations corresponding to higher level degenerate vertices, one can extend our analysis to higher order linear Fuchsian ODEs.

- The uplift to q-difference equations can also be considered. This corresponds to consider q-Virasoro blocks and supersymmetric gauge theories in five dimensions [186]. This is related to q-Painlevé equations and topological strings [187, 188].

In order to test the applicability of our previous results, in chapter 4 we applied our connection coefficients to a concrete problem: perturbations on a Kerr background. We obtained a closed form result for the Kerr greybody factor, and gave a novel perspective on quasinormal quantization and Love numbers of these backgrounds. Since Kerr black holes are astrophysical objects, any new analytic result of this kind is of great interest for a variety of reasons. On the phenomenological side this allows precise tests of general relativity and possible deviations from it. From a different point of view, analytic control on the dynamics of astrophysical black holes could shed new light on their intrinsic theoretical properties. To what extent, for example, scattering off a Kerr black hole is controlled by a conformal field theory as proposed in

[189]. The recent excitement about vanishing of Love numbers for 4d black holes proves that 50 years after its discovery we still have a lot to learn from the Teukolsky equation, and a renewed analytic control on its solutions can help to uncover its secrets. Some open questions regarding this problem are the following.

- In chapter 4 we derived a number of exact results, but since our main goal was to test our novel method, most of our effort was directed in matching our new formulas with previous computations. But our formulas extend previous result: is there something new that we can learn?

- It is suggestive that perturbations of a Kerr background can be described in terms of a CFT. As mentioned above, although in a very different circle of ideas, a link of holographic type between CFT$_2$ and Kerr black hole physics emerged in the last years since [189]. It would be very interesting to find whether the mathematical structure behind the solution of the Kerr black hole radiation problem we present in this thesis could have a clear interpretation in the context of the Kerr/CFT correspondence.

- From the CFT$_2$ perspective, the Teukolsky equation arises in the semiclassical limit of Liouville field theory. An intriguing question to investigate is whether the quantum corrections in CFT$_2$ can have a physical interpretation in the black hole description. In principle, this could be related to quantum gravitational corrections or more generally to some deviations from General Relativity, which will affect the physical properties of the black hole's gravitational field.

- A further possible application of the method presented in this paper is the study of the physics of the last stages of coalescence of compact objects with the Zerilli function [190], see [191] for recent developments. The corresponding potential displays a fifth order singularity which can be engineered with a higher irregular state, corresponding to Argyres-Douglas SCFT in gauge theory [192]. Let us remark that the CFT$_2$ methods extend beyond the equivariant localisation results in gauge theory, making it possible to quantitatively study higher order singularities [74].

Finally in chapter 5 we have computed the holographic thermal scalar two-point function $\langle \mathcal{O}\mathcal{O}\rangle_\beta$. Via the AdS/CFT correspondence, the problem reduces to the study of wave propagation on the AdS-Schwarzschild background. To solve the problem we used the connection between the wave equation on the AdS-Schwarzschild background and Liouville CFT. The result for the two-point function for a four-dimensional holographic CFT on $S^1 \times S^3$ dual to a black hole geometry is the formula (5.3.5). For a holographic CFT on $S^1 \times \mathbb{R}^3$ dual to a black brane geometry the result is (5.3.7). We analyzed the exact formulas numerically in different regimes, matching and extending previous results. We also analyzed the exact formulas numerically by truncating the instanton sum to some finite value.

This work only embarks upon an exploration of a fascinating connection between finite-temperature correlators and supersymmetric gauge theories. There are many future directions to explore and we end our paper with naming an obvious few.

- In chapter 5 we have restricted our analysis to $d = 4$ and a black hole with zero charge and spin. It would be very interesting to generalize our analysis to general $d$, and to consider spinning and charged black holes, as well as spinning and charged probes. In the latter case, considering the two-point function of conserved currents $\langle J_\mu J_\nu\rangle_\beta$ and stress-energy tensors $\langle T_{\mu\nu} T_{\rho\sigma}\rangle_\beta$ is particularly interesting due to their relation to transport and hydrodynamics, see e.g. [193, 194, 195]. The corresponding stress-tensor OPE expansion was analyzed in [196].

- Another obvious extension is to consider thermal higher-point functions, e.g. the out-of-time-ordered four-point function [197], as well as to study gravitational loop effects for the two-point function [198]. In the bulk such computations correspond to going beyond linear order, and they require knowledge of the bulk-to-boundary and bulk-to-bulk propagators on the black hole background. In the language of [78] these are given in terms of the Virasoro conformal blocks and via the AGT correspondence can be again expressed in terms of the instanton partition functions.

- From the point of view of conformal bootstrap our results concern the heavy-heavy-light-light four-point function viewed from the heavy-light channel, see section 5.4. In the same sense the all-order formula (5.5.2) solves the light-cone bootstrap in the heavy-light channel. Intriguing structures have been recently observed in the light-light channel [145, 147, 150], which is related to our work by crossing. It would be very interesting to bridge the results of our work and these recent developments.

- At zero temperatures there is a simple correspondence between perturbative solutions to crossing equations and effective field theories in AdS [133]. A similar connection was explored in [138] in the thermal AdS phase, thanks to the fact that the relevant "unperturbed" finite temperature generalized free field solution is explicitly known, see e.g. [199]. An exciting problem in this context is to understand a similar connection between crossing and effective field theories in AdS in the black hole phase. Here our exact formula provides an unperturbed seed solution, around which perturbations can be studied. It would be very interesting to explore this possibility and more generally explore consistency of holographic conformal field theories at finite temperatures.

# Appendix A

# DOZZ factors and irregular generalizations

## A.1 Regular case

We use conventions where $\Delta = \frac{Q^2}{4} - \alpha^2$, i.e. physical range of the momentum is $\alpha \in i\mathbb{R}^+$. The formula proposed by DOZZ for the Liouville three-point function is then [95, 96]

$$\begin{aligned}&\langle \Delta_1 | V_2(1) | \Delta_3 \rangle = C_{\alpha_1\alpha_2\alpha_3} = \\ &= \frac{\Upsilon_b'(0)\Upsilon_b(Q+2\alpha_1)\Upsilon_b(Q+2\alpha_2)\Upsilon_b(Q+2\alpha_3)}{\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1-\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}-\alpha_1+\alpha_2+\alpha_3)}\,.\end{aligned} \tag{A.1.1}$$

We neglect the dependence on the cosmological constant since its value is arbitrary and is not needed for the following discussion. We will not define the special function $\Upsilon_b$ and state all its remarkable properties, instead we refer to [97]. The most important property for us is the functional relation

$$\Upsilon_b(x+b) = \gamma(bx) b^{1-2bx} \Upsilon_b(x)\,, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{A.1.2}$$

The normalization of the states is obtained from the three-point function by taking the operator in the middle to be the identity operator, i.e. with $\Delta = 0$ which in our conventions means $\alpha = -\frac{Q}{2}$. One finds

$$\lim_{\epsilon\to 0} C_{\alpha_1, -\frac{Q}{2}+\epsilon, \alpha_2} = 2\pi\delta(\alpha_1 - \alpha_2) G_{\alpha_1}\,, \tag{A.1.3}$$

with the two-point function $G_\alpha$ given by

$$G_\alpha = \frac{\Upsilon_b(2\alpha+Q)}{\Upsilon_b(2\alpha)}\,. \tag{A.1.4}$$

We use it to raise and lower indices: For example, OPE coefficients are given by

$$C^{\alpha_1}_{\alpha_2\alpha_3} = G^{-1}_{\alpha_1} C_{\alpha_1\alpha_2\alpha_3}\,. \tag{A.1.5}$$

We will be interested in the case where one of the fields is the degenerate field $\Phi_{2,1}$ with $\alpha_{2,1} = -\frac{2b+b^{-1}}{2}$, corresponding to $\Delta_{2,1} = -\frac{1}{2} - \frac{3b^2}{4}$. The fusion rules in this case impose that only two Verma modules appear in the OPE of this field with a primary:

$$\Phi_{2,1}(z)|\Delta\rangle = \sum_{\theta=\pm} z^{\frac{bQ}{2}+\theta b\alpha} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} |\Delta_\theta\rangle \left(1 + \mathcal{O}(z)\right)\,, \tag{A.1.6}$$

with

$$\alpha_\pm = \alpha \pm \left(-\frac{b}{2}\right)\,, \quad \Delta_\pm = \Delta_{\alpha_\pm} = \Delta \pm b\alpha - \frac{b^2}{4}\,. \tag{A.1.7}$$

Since the degenerate field is not in the physical spectrum, i.e. $\alpha_{2,1} \notin i\mathbb{R}^+$, the OPE coefficients $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$ have to be computed by analytic continuation of the DOZZ formula. This is tricky and is most easily performed by considering a four-point function, where the intermediate momentum is integrated over. During the analytic continuation one picks up residues of poles that cross the integration contour, and this in fact automatically imposes the fusion rules. In any case, the result is [200]:

$$C^{\alpha_+}_{\alpha_{2,1},\alpha} = 1\,, \quad C^{\alpha_-}_{\alpha_{2,1},\alpha} = b^{2bQ}\frac{\gamma(2b\alpha)}{\gamma(bQ+2b\alpha)}\,. \tag{A.1.8}$$

## A.2 Rank 1

In section 3.2.2 we introduced the rank 1 irregular state, which can be given as a confluence limit of primary operators (here we consider only the chiral half):

$$\langle \mu, \Lambda| \propto \lim_{\eta\to\infty} t^{\Delta_t - \Delta}\langle\Delta|V_t(t) \tag{A.2.1}$$

with

$$\Delta = \frac{Q^2}{4} - \alpha^2\,, \quad \alpha = -\frac{\eta+\mu}{2}\,, \quad \Delta_t = \frac{Q^2}{4} - \alpha_t^2\,, \quad \alpha_t = \frac{\eta-\mu}{2}\,, \quad t = \frac{\eta}{\Lambda}\,. \tag{A.2.2}$$

This reproduces the desired Ward identities for the irregular state. To determine its normalization, we perform the collision limit on a (chiral+antichiral) three-point function, keeping track of the DOZZ factors. Although irrelevant for the Ward identities, the signs of $\alpha, \alpha_t$ in (A.2.2) are crucial now. We find

$$\lim_{\eta\to\infty}(t\bar{t})^{\Delta_t-\Delta}\langle\Delta|V_t(t,\bar{t})|\Delta_0\rangle = (\Lambda\bar{\Lambda})^{\Delta_0}\lim_{\eta\to\infty}\eta^{-2\Delta_0}C_{-\frac{\eta+\mu}{2},\frac{\eta-\mu}{2},\alpha_0}\,. \tag{A.2.3}$$

Note that consistently with the main text, we consider the chiral and antichiral parts formally as independent and distinguish them by letting the "complex conjugation" formally act only on the coordinates $t, \Lambda$ and not on the momenta $\alpha_0, \mu, \eta$. The asymptotic behaviour of the $\Upsilon_b$ function, valid for large imaginary $x$ is:

$$\log\Upsilon_b\left(\frac{Q}{2}+x\right) = -\frac{1}{2}\Delta_x\log\Delta_x + \frac{1+Q^2}{12}\log\Delta_x + \frac{3}{2}\Delta_x + \mathcal{O}(x^0)\,. \tag{A.2.4}$$

We therefore find the following asymptotic behaviour of the DOZZ factor:

$$C_{-\frac{\eta+\mu}{2},\frac{\eta-\mu}{2},\alpha_0} \sim (-\eta^2)^{\Delta_0-\mu(Q-\mu)}\frac{\Upsilon_b(Q+2\alpha_0)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha_0)\Upsilon_b(\frac{Q}{2}+\mu-\alpha_0)}\,. \tag{A.2.5}$$

This suggests that we get a finite limit in (320) if we substract the factor of $(-\eta^2)^{-\mu(Q-\mu)}$ by hand. This can also be achieved by changing the power of $t$ that we substract in the definition (A.2.1), but this would change the $L_0$-action on the irregular state, which we avoid. It is however precisely what is done in [75]. In any case, we find the following normalization of the irregular state:

$$\langle\mu,\Lambda|\Delta_0\rangle = \lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}|t|^{2\Delta_t-2\Delta}\langle\Delta|V_t(t,\bar{t})|\Delta_0\rangle = |\Lambda|^{2\Delta_0}C_{\mu\alpha_0}\,, \tag{A.2.6}$$

with normalization function

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha)\Upsilon_b(\frac{Q}{2}+\mu-\alpha)}\,. \tag{A.2.7}$$

The choice of the branch for the phase is consistent with the result found in B.1.
In the text we also consider a different kind of collision limit, which reproduces the OPE between a primary operator and the irregular state. Performing this collision limit while keeping track of the DOZZ factors, we can extract the corresponding irregular OPE coefficient. In particular, consider the following correlation function, which we expand for large $\Lambda$:

$$\langle\mu,\Lambda|V_1(1)|\Delta_0\rangle = \int d\mu' B^{\mu'}_{\mu\alpha_1} C_{\mu'\alpha_0} \left|{}_1\mathfrak{D}\left(\mu\;{}^{\alpha_1}\;\mu'\,\alpha_0\,;\frac{1}{\Lambda}\right)\right|^2\,. \tag{A.2.8}$$

Here $B^{\mu'}_{\mu\alpha_1}$ is the OPE coefficient corresponding to the OPE between the irregular state and $V_1$, $C_{\mu'\alpha_0}$ is the normalization function defined above and ${}_1\mathfrak{D}$ is just the corresponding conformal block. Following [106], we can express an irregular three-point function equivalently as a limit of a regular four-point function:

$$\begin{aligned}\langle\mu,\Lambda|V_1(1)|\Delta_0\rangle &= \lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}\int d\mu' C^{\alpha(\eta)}_{\alpha_\infty(\eta),\alpha_1}C_{\alpha(\eta),\alpha_t(\eta),\alpha_0}\times\\ &\times\left|e^{-(\mu'-\mu)\Lambda}\left(-\frac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(\frac{\Lambda}{\eta}\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty(\eta)\end{matrix}\,\alpha(\eta)\,\begin{matrix}\alpha_t(\eta)\\ \alpha_0\end{matrix};\frac{\eta}{\Lambda}\right)\right|^2,\end{aligned} \tag{A.2.9}$$

with

$$\alpha_\infty(\eta) = -\frac{\eta+\mu}{2}\,,\quad \alpha_t(\eta) = \frac{\eta-\mu}{2}\,,\quad \alpha(\eta) = -\frac{\eta-\mu}{2}-\mu'\,. \tag{A.2.10}$$

Several comments are in order: First, notice that in line with the definition of the irregular state we have multiplied by the same factors of $(-\eta^2)^{\mu(Q-\mu)}$ and $\left(\Lambda\bar{\Lambda}/\eta^2\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}$ as in (A.2.6). Second, the remaining factors which we have put by hand are equal to 1 in the limit:

$$\lim_{\eta\to\infty}e^{-(\mu'-\mu)\Lambda}\left(-\tfrac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(1-\tfrac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} = \lim_{\eta\to\infty}e^{-(\mu'-\mu)\Lambda}\left(1-\tfrac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} = 1\,. \tag{A.2.11}$$

Therefore all the factors that we put by hand are the same as if we had computed (A.2.9) by doing the OPE between $V_1$ and $|\Delta_0\rangle$ instead of between $\langle\mu,\Lambda|$ and $V_1$. This ensures crossing symmetry of the irregular three-point function. Furthermore, the factors inside the modulus square in the limit give the irregular conformal block up to an overall divergence, i.e.:

$$\begin{aligned}&e^{-(\mu'-\mu)\Lambda}\left(-\frac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(\frac{\Lambda}{\eta}\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty(\eta)\end{matrix}\,\alpha(\eta)\,\begin{matrix}\alpha_t(\eta)\\ \alpha_0\end{matrix};\frac{\eta}{\Lambda}\right)\longrightarrow\\ &\longrightarrow\eta^{-\Delta_0-\Delta_1-2\mu'(\mu'-\mu)}{}_1\mathfrak{D}\left(\mu\;{}^{\alpha_1}\;\mu'\,\alpha_0\,;\frac{1}{\Lambda}\right)\,,\quad \text{as}\,\eta\to\infty\,.\end{aligned} \tag{A.2.12}$$

This leaves us with

$$\begin{aligned}&\lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}(\eta^2)^{-\Delta_0-\Delta_1-2\mu'(\mu'-\mu)}C^{\alpha(\eta)}_{\alpha_\infty(\eta),\alpha_1}C_{\alpha(\eta),\alpha_t(\eta),\alpha_0} =\\ &=\frac{e^{-i\pi(\Delta_1+2\mu'(\mu'-\mu))}\Upsilon_b(Q+2\alpha_1)}{\Upsilon_b(\frac{Q}{2}+\mu'-\mu-\alpha_1)\Upsilon_b(\frac{Q}{2}+\mu'-\mu+\alpha_1)}\frac{e^{-i\pi\Delta_0}\Upsilon_b(Q+2\alpha_0)}{\Upsilon_b(\frac{Q}{2}+\mu'+\alpha_0)\Upsilon_b(\frac{Q}{2}+\mu'-\alpha_0)}\,,\end{aligned} \tag{A.2.13}$$

which remarkably has a finite limit. We recognize $C_{\mu'\alpha_0}$ and therefore we can identify

$$B^{\mu'}_{\mu\alpha_1} = \frac{e^{-i\pi(\Delta_1+2\mu'(\mu'-\mu))}\Upsilon_b(Q+2\alpha_1)}{\Upsilon_b(\frac{Q}{2}+\mu'-\mu-\alpha_1)\Upsilon_b(\frac{Q}{2}+\mu'-\mu+\alpha_1)}\,. \tag{A.2.14}$$

Specializing this formula to the case when $V_1$ is a degenerate field is again tricky and involves analytic continuation. It is simpler to perform the collision limit again. The fusion rules now imply that $\alpha(\eta) = \alpha_\infty(\eta) \pm (-b/2)$, i.e. $\mu' = \mu_\pm = \mu \pm (-b/2)$. Performing the collision limit using the degenerate OPE coefficients A.1.8 one finds

$$B^{\mu_\theta}_{\mu\alpha_{2,1}} = e^{i\pi\left(\frac{1}{2}+\theta b\mu+\frac{b^2}{4}\right)}\,, \tag{A.2.15}$$

in agreement with the result (B.1.17).

## A.3 Rank 1/2

Unfortunately, for the rank 1/2 state the situation is not as nice. It is clear that if we decouple another mass, the normalization function $C_{\mu\alpha}$ will diverge badly, since there are no $\Upsilon_b$-functions in the numerator to compensate the divergence of the denominator. Indeed, it behaves as

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha)\Upsilon_b(\frac{Q}{2}+\mu-\alpha)} \to \text{const.} \times e^{3\mu^2}(-\mu^2)^{-\frac{1+Q^2}{6}-\mu^2+\Delta}e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)\,, \quad \text{as } \mu\to\infty\,. \tag{A.3.1}$$

The constant comes from the $\mathcal{O}(x^0)$ term in the expansion of the $\Upsilon_b$-function (A.2.4). We neglect it in the following/consider it substracted by hand. This suggests we define

$$\langle\Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta}C_\alpha = \lim_{\mu\to\infty} e^{-3\mu^2}(-\mu^2)^{\frac{1+Q^2}{6}+\mu^2}\langle -\frac{\Lambda^2}{4\mu}|\Delta\rangle = |\Lambda^2|^{2\Delta}2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,, \tag{A.3.2}$$

where the factor of $-\frac{1}{4}$ is needed to reproduce the Ward identity $\langle\Lambda^2|L_1 = -\frac{\Lambda^2}{4}\langle\Lambda^2|$. This gives the normalization function for the rank 1/2 state as

$$C_\alpha = 2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,, \tag{A.3.3}$$

in agreement with the result (B.2.13). Since no collision limit is known that reproduces the OPE between a primary and the rank 1/2 state, we cannot determine the corresponding OPE coefficient in the way we did in the previous section for the rank 1 state. For the case of a degenerate field however, we determine the OPE coefficient in Appendix B.2.

# Appendix B

# Irregular OPEs

## B.1 Rank 1

The form of the (chiral) OPE of a general vertex operator with the irregular state introduced in section 3.2.2 is fixed by the Ward identities to be:

$$\langle \mu, \Lambda | V^{\Delta}_{\mu,\mu'}(z) = \sum_{k=0}^{\infty} z^{2\mu'(\mu'-\mu)-k} \Lambda^{\Delta+2\mu'(\mu'-\mu)} e^{-(\mu'-\mu)\Lambda z} \langle \mu', \Lambda; k | \,. \tag{B.1.1}$$

Here $V^{\Delta}_{\mu,\mu'}(z)$ is a vertex operator of weight $\Delta$ which maps from the Whittaker module specified by $(\mu, \Lambda)$, to the module specified by $(\mu', \Lambda)$. Furthermore $\langle \mu', \Lambda; k|$ are the ("generalized") descendants of the irregular state. They take the form

$$\langle \mu', \Lambda; k | = \sum c_{ijY} \Lambda^{-i} \partial^{j}_{\Lambda} \langle \mu', \Lambda | L_Y \,, \tag{B.1.2}$$

where $c_{ijY}$ are coefficients fixed by the Ward identities and the sum runs over $i, j \geq 0$ and all Young diagrams $Y$ such that $i + j + |Y| = k$. Furthermore we normalize $\langle \mu', \Lambda; 0| \equiv \langle \mu', \Lambda|$. We then write the full (chiral+antichiral) OPE between the irregular state and a degenerate field as

$$\langle \mu, \Lambda | \Phi(z) = \sum_{\theta=\pm} B^{\mu_\theta}_{\mu,\alpha_{2,1}} \left| \sum_{k=0}^{\infty} e^{\theta b \Lambda z/2} \Lambda^{-\theta b\mu + \Delta_{2,1} + \frac{b^2}{2}} z^{-\theta b \mu + \frac{b^2}{2} - k} \right|^2 \langle \mu_\theta, \Lambda; k, \bar{k} | \,, \tag{B.1.3}$$

where $B^{\mu_\theta}_{\mu,\alpha_{2,1}}$ are the corresponding irregular OPE coefficients. We have anticipated the fact that for the OPE with the degenerate field $\mu' = m_{\pm} = \mu \pm \frac{-b}{2}$ as will be shown later from the BPZ equation. Furthermore we now have both chiral and antichiral descendants which we label by $k$ and $\bar{k}$, respectively.

We want to determine the irregular OPE coefficients $B$ and the normalization function $C$ introduced in (3.2.12). To this end consider the correlation function

$$\langle \mu, \Lambda | \Phi(z) | \Delta \rangle \,. \tag{B.1.4}$$

We can decompose it into irregular conformal blocks doing the OPE left or right as

$$\langle \mu, \Lambda | \Phi(z) | \Delta \rangle = \sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\mu\alpha_\theta} \left| {}_1\mathfrak{F}\left( \mu \, \alpha_\theta \, {\alpha_{2,1} \atop \alpha}; \Lambda z \right) \right|^2 = \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\alpha_{2,1},\mu} C_{\mu_{\theta'}\alpha} \left| {}_1\mathfrak{D}\left( \mu \, {}^{\alpha_{2,1}} \, \mu_{\theta'} \, \alpha; \frac{1}{\Lambda z} \right) \right|^2 . \tag{B.1.5}$$

Here $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$ is just the usual (regular) OPE coefficient given in terms of the DOZZ formula, $B$ is the irregular OPE coefficient to be determined, and $C_{\mu\alpha}$ is the normalization function of the irregular state, to be determined also. It is defined by

$$\langle \mu, \Lambda | \Delta \rangle = |\Lambda|^{2\Delta} C_{\mu\alpha} \,. \tag{B.1.6}$$

To determine $B$ and $C$ we use the BPZ equation

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} + \frac{\mu\Lambda}{z} - \frac{\Lambda^2}{4}\right)\langle\mu,\Lambda|\Phi(z)|\Delta\rangle = 0\,. \tag{B.1.7}$$

This equation can be solved exactly and has the two solutions $z^{\frac{b^2}{2}}M_{b\mu,\pm b\alpha}(b\Lambda z)$, where $M$ denotes the Whittaker function. It has a simple expansion around $z \sim 0$:

$$M_{b\mu,b\alpha}(b\Lambda z) = (b\Lambda z)^{\frac{1}{2}+b\alpha}\left(1 + \mathcal{O}(b\Lambda z)\right)\,. \tag{B.1.8}$$

Comparing this expansion with the leading term in the OPE between $\Phi(z)$ and $|\Delta\rangle$ we can identify

$${}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,{}^{\alpha_{2,1}}_{\alpha};\Lambda z\right) = \Lambda^{\Delta_\theta}z^{\frac{b^2}{2}}(b\Lambda)^{-\frac{1}{2}-\theta b\alpha}M_{b\mu,\theta b\alpha}(b\Lambda z)\,. \tag{B.1.9}$$

On the other hand, there exist two other solutions to the BPZ equation which have a simple expansion around $z \sim \infty$, namely the Whittaker $W$ functions $W_{\pm b\mu,b\alpha}(\pm b\Lambda z)$. They have an asymptotic expansion at $\infty$ given by

$$W_{b\mu,b\alpha}(b\Lambda z) \sim e^{-b\Lambda z/2}(b\Lambda z)^{b\mu}\left(1 + \mathcal{O}((b\Lambda z)^{-1})\right)\,, \tag{B.1.10}$$

valid in the Stokes sector $|\arg(b\Lambda z)| < \frac{3\pi}{2}$. An important fact is that this function is invariant under $\alpha \to -\alpha$. We see that the expansion of the Whittaker $W$ function (times the factor $z^{b^2/2}$) has exactly the form of the OPE between the irregular state and the degenerate field, with

$$\mu' = \mu_\pm = \mu \pm \left(-\frac{b}{2}\right)\,. \tag{B.1.11}$$

(Note that with this convention, $\mu_\pm$ corresponds to $W_{\mp b\mu,b\alpha}(\mp b\Lambda z)$. This may seem confusing but we like to keep the expression $\mu_\pm$ analogous to the fusion rules with a regular state which give $\alpha_\pm = \alpha \pm \frac{-b}{2}$).

Comparing the expansion of the $W$ function with the irregular OPE (B.1.3), we can identify

$$\begin{aligned}
{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_+\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}}e^{-i\pi b\mu}b^{b\mu}(\Lambda z)^{\frac{b^2}{2}}W_{-b\mu,b\alpha}(e^{-i\pi}b\Lambda z)\,,\\
{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_-\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}}b^{-b\mu}(\Lambda z)^{\frac{b^2}{2}}W_{b\mu,b\alpha}(b\Lambda z)\,.
\end{aligned} \tag{B.1.12}$$

For simplicity we focus on the branch specified by $-\Lambda = e^{-i\pi}\Lambda$ and use the asymptotic expansion (B.1.10) for both $b\Lambda z$ and $e^{-i\pi}b\Lambda z \to \infty$. This is valid for $-\frac{\pi}{2} < \arg(b\Lambda z) < \frac{3\pi}{2}$. The modulus squared has to be understood as acting by sending $\Lambda z \to \bar{\Lambda}\bar{z}$ and correspondingly $e^{-i\pi}\Lambda z \to e^{+i\pi}\bar{\Lambda}\bar{z}$. Since we have assumed $-\frac{\pi}{2} < \arg(b\Lambda z) < \frac{3\pi}{2}$, we also have $-\frac{\pi}{2} < \arg(e^{i\pi}b\bar{\Lambda}\bar{z}) < \frac{3\pi}{2}$, so all the asymptotic expansions are in their domain of validity. Similar expressions hold in the other Stokes sectors.

We can now restate the crossing symmetry condition (B.1.5) in terms of Whittaker functions and use the known connection formulae for them (see https://dlmf.nist.gov/13.14) to determine the normalization function $C$ and the OPE coefficient $B$. We have

$$M_{\kappa,\mu}(z) = \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\kappa+\mu\right)}e^{i\pi\left(\frac{1}{2}-\kappa+\mu\right)}W_{\kappa,\mu}(z) + \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}-\kappa+\mu\right)}e^{-i\pi\kappa}W_{-\kappa,\mu}(e^{-i\pi}z)\,. \tag{B.1.13}$$

Plugging this into (B.1.5) using the identifications of the conformal blocks with the Whittaker functions we obtain the condition

$$\begin{aligned}\langle\mu,\Lambda|\Phi(z)|\Delta\rangle &= |\Lambda|^{2\Delta+2\Delta_{2,1}+b^2}\sum_{\theta=\pm} b^{-1-2\theta b\alpha}C^{\alpha_\theta}_{\alpha_{2,1},\alpha}C_{\mu\alpha_\theta}\Gamma(1+2\theta b\alpha)^2\times\\ &\times\left|\frac{e^{i\pi\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}}{\Gamma\left(\frac{1}{2}+b\mu+\theta b\alpha\right)}z^{\frac{b^2}{2}}W_{b\mu,b\alpha}(b\Lambda z)+\frac{e^{-i\pi b\mu}}{\Gamma\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}z^{\frac{b^2}{2}}W_{-b\mu,b\alpha}(e^{-i\pi}b\Lambda z)\right|^2=\\ &=|\Lambda|^{2\Delta+2\Delta_{2,1}+b^2}B^{\mu+}_{\alpha_{2,1},\mu}C_{\mu+\alpha}\left|e^{-i\pi b\mu}b^{b\mu}z^{\frac{b^2}{2}}W_{-b\mu,b\alpha}(e^{-i\pi}b\Lambda z)\right|^2+\\ &+|\Lambda|^{2\Delta+2\Delta_{2,1}+b^2}B^{\mu-}_{\alpha_{2,1},\mu}C_{\mu-,\alpha}\left|b^{-b\mu}z^{\frac{b^2}{2}}W_{b\mu,b\alpha}(b\Lambda z)\right|^2\,,\end{aligned}\tag{B.1.14}$$

where we have used the fact that $W_{\kappa,-\mu}(z)=W_{\kappa,\mu}(z)$. Using the expression (A.1.8) for the coefficients $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$, the cancellation of the cross-terms in the modulus squared gives the following functional equation for $C_{\mu\alpha}$:

$$\frac{C_{\mu\alpha_+}}{C_{\mu\alpha_-}}=e^{-2\pi i b\alpha}b^{2bQ+4b\alpha}\frac{\gamma(-2b\alpha)\gamma\left(\frac{1}{2}+b\mu+b\alpha\right)}{\gamma(bQ+2b\alpha)\gamma\left(\frac{1}{2}+b\mu-b\alpha\right)}\,,\tag{B.1.15}$$

which is solved in terms of the usual $\Upsilon_b$-function:

$$C_{\mu\alpha}=\frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b\left(\frac{Q}{2}+\mu+\alpha\right)\Upsilon_b\left(\frac{Q}{2}+\mu-\alpha\right)}\,,\tag{B.1.16}$$

up to normalization and a periodic function of $\alpha$ with period $b$. We see however that the minimal choice is consistent with the result obtained by the collision limit in A.2. Once we know the expression for $C_{\mu\alpha}$,we can compute the irregular OPE coefficients $B^{\mu\pm}_{\alpha_{2,1},\mu}$ from the diagonal terms in (B.1.14). The result is

$$B^{\mu\pm}_{\alpha_{2,1},\mu}=e^{i\pi\left(\frac{1}{2}\pm b\mu+\frac{b^2}{4}\right)}\,.\tag{B.1.17}$$

Again, we find that this is in agreement with the result found by the collision limit in A.2. For completeness, let us write the connection formula for the conformal blocks $\mathfrak{F}$ and $\mathfrak{D}$, which solves the crossing symmetry constraint (B.1.5). Using the identification of the conformal blocks with the Whittaker functions with the correct prefactors we find

$$b^{\theta b\alpha}{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,\genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha};\Lambda z\right)=\sum_{\theta'=\pm}b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu){}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right)\,,\tag{B.1.18}$$

with irregular connection coefficients

$$\mathcal{N}_{\theta\theta'}(b\alpha,b\mu)=\frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}\,.\tag{B.1.19}$$

The inverse relation is

$$b^{-\frac{1}{2}-\theta b\mu}{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_\theta\,\alpha;\frac{1}{\Lambda z}\right)=\sum_{\theta'=\pm}b^{\theta' b\alpha}\mathcal{N}^{-1}_{\theta\theta'}(b\mu,b\alpha){}_1\mathfrak{F}\left(\mu\,\alpha_{\theta'}\,\genfrac{}{}{0pt}{}{\alpha_{2,1}}{\alpha};\Lambda z\right)\,,\tag{B.1.20}$$

with

$$\mathcal{N}^{-1}_{\theta\theta'}(b\mu,b\alpha)=\frac{\Gamma(-2\theta' b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\mu-\theta' b\alpha\right)}e^{i\pi\left(\frac{1+\theta}{2}\right)\left(-\frac{1}{2}-b\mu-\theta' b\alpha\right)}\,.\tag{B.1.21}$$

As a final remark, note that the Whittaker $W$-functions have a non-trivial monodromy around $\infty$. However, since for the correlator we considered, the monodromy around 0 and $\infty$ is the same, and by construction we have no monodromy around 0, the combination of $W$-functions appearing in the correlator expanded for large $\Lambda z$ is precisely such that the monodromy cancels. This can be checked also purely locally by carefully using the asymptotic expansions of the $W$-functions and its Stokes sectors. In particular, any other correlator involving this irregular state will have the same asymptotic behaviour and thus the normalization function $C_{\mu\alpha}$ ensures also the absence of monodromies for any other correlator.

## B.2 Rank 1/2

Let us repeat the same arguments for the rank 1/2 irregular state introduced in section 3.2.3. The (chiral) OPE between the irregular state and the degenerate field is fixed by the Ward identities to be:

$$\langle\Lambda^2|\Phi_{\Lambda,\pm}(z) = \sum_{k=0}^{\infty}(\Lambda^2)^{-\frac{1}{4}-\frac{b^2}{4}} z^{\frac{1}{4}+\frac{b^2}{2}-\frac{k}{2}} e^{\pm b\Lambda\sqrt{z}} \langle\Lambda^2; \frac{k}{2}| \,. \tag{B.2.1}$$

Here $\langle\Lambda^2; \frac{k}{2}|$ are the ("generalized") descendants of the irregular state. They take the form

$$\langle\Lambda^2; \frac{k}{2}| = \sum c_{ijY}\Lambda^{-i}\partial_\Lambda^j\langle\Lambda^2|L_Y \,, \tag{B.2.2}$$

where $c_{ijY}$ are coefficients fixed by the Ward identities and the sum runs over $i, j \geq 0$ and all Young diagrams $Y$ such that $i + j + 2|Y| = k$. In particular, note that only the integer descendants (i.e. $k \in 2\mathbb{Z}$) can contain Virasoro generators $L_Y$. Furthermore we normalize $\langle\Lambda^2; 0| \equiv \langle\Lambda^2|$. Since both $z$-behaviours in (B.2.1) given by $\pm$ live in the same Bessel module specified by $\Lambda$, there is no canonical way of choosing a basis of solutions, in contrast to the rank 1 case. This ambiguity does not affect the physical correlator, since we have to sum over both solutions with the corresponding OPE coefficients. Changing the basis of conformal blocks changes the OPE coefficients in a way that the physical correlator is invariant. Consider the following correlation function involving the rank 1/2 state:

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle \,. \tag{B.2.3}$$

We can decompose it into conformal blocks by doing the OPE left and right:

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle = \textstyle\sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\alpha_\theta} \left|{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha;\, \Lambda\sqrt{z}\right)\right|^2 = \textstyle\sum_{\theta'=\pm} B_{\alpha_{2,1}} C_\alpha \left|{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\, \alpha;\, \frac{1}{\Lambda\sqrt{z}}\right)\right|^2 . \tag{B.2.4}$$

Here $C_\alpha$ is the normalization function of the irregular state, defined by

$$\langle\Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta} C_\alpha \,, \tag{B.2.5}$$

which is to be determined. We also want to determine the irregular OPE coefficient $B_{\alpha_{2,1}}$. To do so, consider the BPZ equation that the correlator obeys:

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} - \frac{\Lambda^2}{4z}\right)\langle\Lambda^2|\Phi(z)|\Delta\rangle = 0 \,. \tag{B.2.6}$$

Solving this differential equation one identifies the conformal block corresponding to the expansion near 0 with a modified Bessel function:

$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha;\, \Lambda\sqrt{z}\right) = \Gamma(1 + 2\theta b\alpha)\Lambda^{2\Delta_\theta}\left(\frac{b\Lambda}{2}\right)^{-2\theta b\alpha} z^{\frac{bQ}{2}} I_{2\theta b\alpha}(b\Lambda\sqrt{z}) \,. \tag{B.2.7}$$

The prefactors are fixed by looking at the OPE between $\Phi$ and $|\Delta\rangle$ and using the expansion of the Bessel function:

$$I_{2\theta b\alpha}(b\Lambda\sqrt{z}) = \frac{(b\Lambda\sqrt{z}/2)^{2\theta b\alpha}}{\Gamma(1+2\theta b\alpha)}\left(1+\mathcal{O}(b\Lambda\sqrt{z})\right) . \tag{B.2.8}$$

On the other hand there are two other solutions to the BPZ equation given by the modified Bessel functions of the second kind $K_{2b\alpha}(\pm b\Lambda\sqrt{z})$. They have a nice behaviour at $\infty$, given by the asymptotic formula

$$K_{2b\alpha}(b\Lambda\sqrt{z}) \sim \sqrt{\frac{\pi}{2b\Lambda\sqrt{z}}}e^{-b\Lambda\sqrt{z}}(1+\mathcal{O}((b\Lambda\sqrt{z})^{-1}))\,. \tag{B.2.9}$$

Furthermore $K_{2b\alpha}(b\Lambda\sqrt{z}) = K_{-2b\alpha}(b\Lambda\sqrt{z})$. This expansion has precisely the form of the OPE between the irregular state and the degenerate field (B.2.1). We can therefore identify the necessary prefactors and defi1ne the irregular conformal blocks for $z\sim\infty$:

$$\begin{aligned}
{}_{\frac{1}{2}}\mathfrak{E}^{(+)}\left(\alpha_{2,1}\,\alpha;\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}e^{-\frac{i\pi}{2}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(e^{-i\pi}b\Lambda\sqrt{z})\,,\\
{}_{\frac{1}{2}}\mathfrak{E}^{(-)}\left(\alpha_{2,1}\,\alpha;\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(b\Lambda\sqrt{z})\,.
\end{aligned} \tag{B.2.10}$$

We can now restate the crossing symmetry condition (B.2.4) in terms of Bessel functions and use the known connection formulae for them (see e.g. dlmf.nist.gov/10.27) to determine the normalization function $C$ and the OPE coefficient $B_{\alpha_{2,1}}$. We have

$$I_\nu(z) = \frac{i}{\pi}e^{i\pi\nu}K_\nu(z) - \frac{i}{\pi}K_\nu(e^{-i\pi}z)\,. \tag{B.2.11}$$

Plugging this formula into (B.2.4) using the identifications between the conformal blocks and Bessel functions, one finds that the vanishing of the cross-terms gives the condition

$$\frac{C_{\alpha_+}}{C_{\alpha_-}} = 2^{-8b\alpha}b^{2bQ+8b\alpha}e^{-4\pi i b\alpha}\frac{\gamma(-2b\alpha)}{\gamma(bQ+2b\alpha)}\,. \tag{B.2.12}$$

We take the simplest solution, namely

$$C_\alpha = 2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,. \tag{B.2.13}$$

This is in agreement with the result found in A.3. Once we have the expression for $C_\alpha$, we can compute the irregular OPE coefficients from the diagonal terms of the crossing symmetry condition. The result is

$$B_{\alpha_{2,1}} = 2^{b^2}e^{\frac{i\pi bQ}{2}}\,. \tag{B.2.14}$$

We see that the OPE coefficients are independent of ±, which is a reflection of the fact that we have a symmetry rotating the basis of conformal blocks into each other and leaving the physical correlator invariant.

For completeness, let us write also the connection formula for the irregular conformal blocks:

$$b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \sum_{\theta'=\pm}b^{-\frac{1}{2}}\mathcal{Q}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\frac{1}{\Lambda\sqrt{z}}\right)\,. \tag{B.2.15}$$

with irregular connection coefficients

$$\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}}\Gamma(1+2\theta b\alpha)e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)}\,. \tag{B.2.16}$$

The inverse relation is

$$b^{-\frac{1}{2}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\,\alpha;\frac{1}{\Lambda\sqrt{z}}\right)=\sum_{\theta'=\pm}b^{2\theta' b\alpha}\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_{\theta'}\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right)\,. \tag{B.2.17}$$

with irregular connection coefficients

$$\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha)=\frac{2^{-2\theta' b\alpha}}{\sqrt{2\pi}}\Gamma(-2\theta' b\alpha)e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\theta' b\alpha\right)}\,. \tag{B.2.18}$$

# Appendix C

# Classical conformal blocks and accessory parameters

In this Appendix we give explicit combinatorial expressions for the classical conformal blocks used in the main text.

## C.1 The regular case

Let us start with the case of regular conformal blocks. Via the AGT correspondence [69] the four-point regular conformal block is given by

$$\begin{aligned}&\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t \\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{matrix}; t\right) = t^{\Delta-\Delta_t-\Delta_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)}\times \\ &\times \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_\infty\right) ,\end{aligned} \tag{C.1.1}$$

where the sum runs over all pairs of Young diagrams $(Y_1, Y_2)$. We denote the size of the pair $|\vec{Y}| = |Y_1| + |Y_2|$, and [100, 101]

$$\begin{aligned}&z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\mu\right) = \prod_{k=1,2}\prod_{(i,j)\in Y_k}\left(\alpha_k+\mu+b^{-1}\left(i-\frac{1}{2}\right)+b\left(j-\frac{1}{2}\right)\right) , \\ &z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) = \prod_{k,l=1,2}\prod_{(i,j)\in Y_k} E^{-1}\left(\alpha_k-\alpha_l, Y_k, Y_l, (i,j)\right) \prod_{(i',j')\in Y_l}\left(Q-E\left(\alpha_l-\alpha_k, Y_l, Y_k, (i',j')\right)\right)^{-1} , \\ &E\left(\alpha, Y_1, Y_2, (i,j)\right) = \alpha - b^{-1}L_{Y_2}((i,j)) + b\left(A_{Y_1}((i,j))+1\right) .\end{aligned} \tag{C.1.2}$$

Here $L_Y((i,j)), A_Y((i,j))$ denote respectively the leg-length and the arm-length of the box at the site $(i,j)$ of the diagram $Y$. If we denote a Young diagram as $Y = (\nu_1' \geq \nu_2' \geq \dots)$ and its transpose as $Y^T = (\nu_1 \geq \nu_2 \geq \dots)$, then $L_Y$ and $A_Y$ read

$$A_Y(i,j) = \nu_i' - j\ ,\ \ L_Y(i,j) = \nu_j - i\,. \tag{C.1.3}$$

Note that they can be negative if the box $(i,j)$ are the coordinates of a box outside the diagram. Also, the previous formulae has to be evaluated at $\vec{\alpha} = (\alpha_1, \alpha_2) = (\alpha, -\alpha)$. Comparing (C.1.1) with (3.3.24) we find the explicit expression for the classical conformal block $F$:

$$F(t) = \lim_{b\to 0} b^2 \log\left[(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)} \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_\infty\right)\right] . \tag{C.1.4}$$

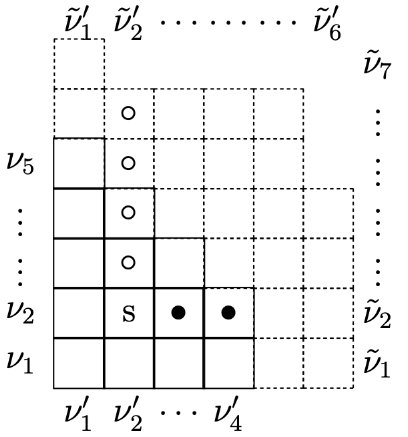

Figure C.1: Arm length $A_{\tilde{Y}}(s) = 4$ (white circles) and leg length $L_Y(s) = 2$ (black dots) of a box at the site $s = (2, 2)$ for the pair of superimposed diagrams $Y$ (solid lines) and $\tilde{Y}$ (dotted lines).

This turns into a combinatorial expression of the $u$ parameter defined as

$$u^{(0)} = \lim_{b\to 0} b^2 t\partial_t \log \mathfrak{F}\left(\begin{matrix} \alpha_1 & & \alpha_t \\ & \alpha & ;t \\ \alpha_\infty & & \alpha_0 \end{matrix}\right) = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F(t) \tag{C.1.5}$$

in terms of the intermediate momentum $\alpha$. After substituting the dictionary with the Heun equation this gives a combinatorial expression of the accessory parameter $q$ in terms of the Floquet exponent $a = b\alpha$. Inverting this relation order by order in $t$ allows us to compute the connection coefficients in terms of the accessory parameter. Let us carry out explicitly a first order computation for the sake of clarity. At one instanton the relevant pairs of Young diagrams are $\vec{Y} = ((1), (0))$ and $\vec{Y} = ((0), (1))$. The various contributions give

$$\begin{aligned} z_{\text{hyp}}\left(\vec{\alpha}, ((1), (0)), \mu\right) &= \frac{Q}{2} + \alpha + \mu \,, \\ z_{\text{hyp}}\left(\vec{\alpha}, ((0), (1)), \mu\right) &= \frac{Q}{2} - \alpha + \mu \,, \end{aligned} \tag{C.1.6}$$

and since $A_{(0)}(i = 1, j = 1) = L_{(0)}(i = 1, j = 1) = -1$ and $A_{(1)}(i = 1, j = 1) = L_{(1)}(i = 1, j = 1) = 0$,

$$\begin{aligned} &E\left(0, (1), (1), (i = 1, j = 1)\right) = b \,, \\ &E\left(2\alpha, (1), (0), (i = 1, j = 1)\right) = Q + 2\alpha \,, \end{aligned} \tag{C.1.7}$$

therefore

$$\begin{aligned} z_{\text{vec}}\left(\vec{\alpha}, ((1), (0))\right) &= \prod_{l=1,2} E^{-1}\left(\alpha - \alpha_l, (1), Y_l, (i = 1, j = 1)\right) \prod_{k=1,2} \left(Q - E\left(\alpha - \alpha_k, (1), Y_k, (i' = 1, j' = 1)\right)\right)^{-1} \\ &= \frac{1}{-2\alpha\left(Q + 2\alpha\right)} \,, \\ z_{\text{vec}}\left(\vec{\alpha}, ((0), (1))\right) &= \prod_{l=1,2} E^{-1}\left(-\alpha - \alpha_l, (1), Y_l, (i = 1, j = 1)\right) \prod_{k=1,2} \left(Q - E\left(-\alpha - \alpha_k, (1), Y_k, (i' = 1, j' = 1)\right)\right)^{-1} \\ &= \frac{1}{2\alpha\left(Q - 2\alpha\right)} \,. \end{aligned} \tag{C.1.8}$$

Note that and that every time $(i, j)$ have to run into an empty diagram, the corresponding term contributes with 1. Finally, substituting the previous results in (C.1.4) we get

$$F(t) = \frac{\left(\frac{1}{4} - a^2 - a_1^2 + a_\infty^2\right)\left(\frac{1}{4} - a^2 - a_t^2 + a_0^2\right)}{\frac{1}{2} - 2a^2} t + \mathcal{O}(t^2) \,. \tag{C.1.9}$$

In the main text we will need the derivatives of $F$ expressed in terms of Heun parameters. For example,

$$\partial_{a_t} F(t) = \frac{\left(4a^2-\alpha^2+2\alpha\beta-\beta^2-2\delta+\delta^2\right)(1-\epsilon)}{2-8a^2} t + \mathcal{O}(t^2)\,. \tag{C.1.10}$$

Moreover,

$$u^{(0)} = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + \frac{\left(\frac{1}{4}-a^2-a_1^2+a_\infty^2\right)\left(\frac{1}{4}-a^2-a_t^2+a_0^2\right)}{\frac{1}{2}-2a^2} t + \mathcal{O}(t^2)\,. \tag{C.1.11}$$

Note that the relation between $u^{(0)}$ and $a$ is quadratic at $t=0$, therefore we will have two solutions for $a(u^{(0)})$:

$$a = \pm\sqrt{-\frac{1}{4} - u^{(0)} + a_t^2 + a_0^2}\left(1 - \frac{\left(-1+2a_0^2+2a_1^2-2a_\infty^2+2a_t^2-2u^{(0)}\right)\left(-1+4a_t^2-2u^{(0)}\right)}{2\left(-1+4a_0^2+4a_t^2-4u^{(0)}\right)\left(-1+2a_0^2+2a_t^2-2u^{(0)}\right)} t + \mathcal{O}(t^2)\right)\,. \tag{C.1.12}$$

Substituting the dictionary (3.4.3) we obtain

$$a = \pm\frac{1}{2}\sqrt{(\alpha+\beta-\delta)^2-4q} \mp \frac{t\left(\delta(q(\alpha+\beta+1)-\gamma(\alpha\beta+q)) + (q-\alpha\beta)(2q-\gamma(\alpha+\beta-1)) + \delta^2(-q)\right)}{\sqrt{(\alpha+\beta-\delta)^2-4q}\,(4q-(\alpha+\beta-\delta-1)(\alpha+\beta-\delta+1))} + O\left(t^2\right)\,. \tag{C.1.13}$$

Note that that all the connection formulae near the various singularity are all symmetric under $a \to -a$. The sign has to be carefully chosen only when connecting to the intermediate region. Finally, we are in the position to expand the connection coefficients. For example, one would have, choosing the lower sign in $a$,

$$\begin{aligned}
&\Gamma\left(\frac{1+\gamma-\epsilon}{2}+a\right) \simeq \Gamma\left(\frac{1+\gamma-\epsilon-\sqrt{-4q+(\alpha+\beta-\delta)^2}}{2}\right) \times \\
&\times\left(1+\frac{t\left(\delta(q(\alpha+\beta+1)-\gamma(\alpha\beta+q)) + (q-\alpha\beta)(2q-\gamma(\alpha+\beta-1)) + \delta^2(-q)\right)\psi_0\left(\frac{1+\gamma-\epsilon-\sqrt{-4q+(\alpha+\beta-\delta)^2}}{2}\right)}{\sqrt{(\alpha+\beta-\delta)^2-4q}\,(4q-(\alpha+\beta-\delta-1)(\alpha+\beta-\delta+1))}\right),
\end{aligned} \tag{C.1.14}$$

where $\psi_0$ is the Digamma function.

## C.2 The confluent case

In order to discuss the confluent classical conformal block, let us write the four-point conformal block appearing in (3.3.39), that is

$$\begin{aligned}
&\mathfrak{F}\left(\begin{matrix}\alpha_t & & \alpha_1\\ & \alpha & \\ \alpha_\infty & & \alpha_0\end{matrix};\frac{1}{t}\right) = t^{-\Delta+\Delta_1+\Delta_0}(1-t^{-1})^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)} \times \\
&\times \sum_{\vec{Y}} t^{-|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_\infty\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right)\,.
\end{aligned} \tag{C.2.1}$$

Note that in the decoupling limit (3.3.45), that is

$$\alpha_t+\alpha_\infty = -\mu\,,\ \alpha_t-\alpha_\infty = \eta\,,\ t = \frac{\Lambda}{\eta}\,, \tag{C.2.2}$$

where then $\eta \to \infty$,

$$\begin{aligned}
&z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t-\alpha_\infty\right) \sim (\alpha_t-\alpha_\infty)^{2|\vec{Y}|} \sim \left(\frac{\Lambda}{t}\right)^{2|\vec{Y}|}\,, \\
&z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\alpha_\infty\right) = z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu\right)\,, \\
&(1-t^{-1})^{2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)} \sim e^{-\left(\frac{Q}{2}+\alpha_1\right)\Lambda}\,.
\end{aligned} \tag{C.2.3}$$

Therefore the confluent 3-point function (3.3.80) has the following combinatorial expression

$$ {}_1\mathfrak{F}\left(\mu\,\alpha\,\begin{matrix}\alpha_1\\ \\ \alpha_0\end{matrix};\Lambda\right) = \Lambda^{\Delta} e^{\left(\frac{Q}{2}+\alpha_1\right)\Lambda} \textstyle\sum_{\vec{Y}} \Lambda^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right) . \tag{C.2.4} $$

As for the previous case, this turns into a combinatorial expression of the $u$ parameter defined in equation 3.3.83 in terms of the intermediate momentum $a$, that after substituting the dictionary with the CHE gives a combinatorial expression for the accessory parameter in terms of the Floquet exponent. Again, inverting this relation is useful for computing the explicit connection coefficients. Similarly we can give an explicit expression of the classical conformal block for big $\Lambda$ appearing in (3.3.93), that is

$$ \begin{aligned} {}_1\mathfrak{D}\left(\mu\,\overset{\alpha_1}{\phantom{x}}\,\mu'\,\alpha_0;\frac{1}{\Lambda}\right) &= \lim_{\eta\to\infty} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)} e^{-(\mu'-\mu)\Lambda} \left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')-\left(\frac{Q}{2}+\alpha_1\right)(Q+\eta-\mu)} \times \\ &\times \sum_{\vec{Y}} \left(\frac{\eta}{\Lambda}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}(\eta),\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha}(\eta),\vec{Y},\frac{\eta-\mu}{2}+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha}(\eta),\vec{Y},\alpha_1+\theta\frac{-\eta-\mu}{2}\right) , \end{aligned} \tag{C.2.5} $$

where

$$ \vec{\alpha}(\eta) = \left(-\frac{\eta-\mu}{2}-\mu', \frac{\eta-\mu}{2}+\mu'\right) . \tag{C.2.6} $$

Again, this gives an explicit expression of the classical conformal block $F_D(L^{-1})$ recalling that

$$ {}_1\mathfrak{D}\left(\mu\,\overset{\alpha_1}{\phantom{x}}\,\mu'\,\alpha_0;\frac{1}{\Lambda}\right) = e^{-(\mu'-\mu)\Lambda} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)} e^{\frac{1}{b^2}\left(F_D(L^{-1})+\mathcal{O}(b^2)\right)} . \tag{C.2.7} $$

## C.3 The reduced confluent case

To obtain the reduced confluent classical block we decouple the momentum $\mu$ starting from (C.2.1) as follows

$$ \Lambda = -\frac{\Lambda_1\Lambda_2}{4\mu} , \text{ as } \mu\to\infty . \tag{C.3.1} $$

This gives

$$ {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,\begin{matrix}\alpha_1\\ \\ \alpha_0\end{matrix};\Lambda^2\right) = \Lambda^{2\Delta} \sum_{\vec{Y}} \left(\frac{\Lambda^2}{4}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right) . \tag{C.3.2} $$

This gives for the classical conformal blocks

$$ F(L^2) = \lim_{b\to 0} b^2 \log \sum_{\vec{Y}} \left(\frac{\Lambda^2}{4}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right) . \tag{C.3.3} $$

## C.4 The doubly confluent case

Let us consider the following decoupling limit of (C.2.1):

$$ \alpha_1+\alpha_0 = -\mu_2 , \; \alpha_1-\alpha_0 = \eta , \; \Lambda \to \frac{\Lambda_1\Lambda_2}{\eta} , \text{ as } \eta\to\infty . \tag{C.4.1} $$

This gives

$$ {}_1\mathfrak{F}_1\left(\mu_1\ \alpha\ \mu_2, \Lambda_1\Lambda_2\right) = \left(\Lambda_1\Lambda_2\right)^{\Delta} e^{\frac{\Lambda_1\Lambda_2}{2}} \sum_{\vec{Y}} \left(\Lambda_1\Lambda_2\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu_1\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu_2\right)\ , \tag{C.4.2} $$

and

$$ F(L_1L_2) = \lim_{b\to 0} b^2 \log\left[e^{\frac{\Lambda_1\Lambda_2}{2}} \sum_{\vec{Y}} \left(\Lambda_1\Lambda_2\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu_1\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu_2\right)\right]\ . \tag{C.4.3} $$

## C.5 The reduced doubly confluent case

We now decouple $\mu_2$ in (C.4.2) as follows

$$ \Lambda_2 \to -\frac{\Lambda_2^2}{4\mu_2}\ ,\ \text{as } \mu_2 \to \infty\ . \tag{C.5.1} $$

Again,

$$ {}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha; \Lambda_1\frac{\Lambda_2^2}{4}\right) = \left(\Lambda_1\Lambda_2^2\right)^{\Delta} \sum_{\vec{Y}} \left(\Lambda_1\Lambda_2^2\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu\right)\ . \tag{C.5.2} $$

Therefore the corresponding classical conformal block gives

$$ F(L_1L_2^2) = \lim_{b\to 0} b^2 \log \sum_{\vec{Y}} \left(\Lambda_1\frac{\Lambda_2^2}{4}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu\right)\ . \tag{C.5.3} $$

## C.6 The doubly reduced doubly confluent case

Decoupling the last momentum $\mu$ in (C.5.2) by setting

$$ \Lambda_1 \to -\frac{\Lambda_1^2}{4\mu_1}\ ,\ \text{as } \mu \to \infty \tag{C.6.1} $$

gives

$$ {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha; \Lambda_1^2\Lambda_2^2\right) = \left(\Lambda_1^2\Lambda_2^2\right)^{\Delta} \sum_{\vec{Y}} \left(\frac{\Lambda_1^2\Lambda_2^2}{16}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right)\ . \tag{C.6.2} $$

The corresponding classical conformal block gives

$$ F(L_1^2L_2^2) = \lim_{b\to 0} b^2 \log\left[\sum_{\vec{Y}} \left(\frac{\Lambda_1^2\Lambda_2^2}{16}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right)\right]\ . \tag{C.6.3} $$

# Appendix D

# Combinatorial formula for the degenerate 5-point block

As for the four-point blocks in the previous Appendix, we give an explicit combinatorial expression for the degenerate 5-point conformal block introduced in section 3.3.1 via the AGT correspondence. It can be computed as the partition function of $\mathcal{N}=2$ gauge theory with four flavours and a surface operator, or equivalently as a quiver gauge theory with specific masses fixed by the fusion rules of the degenerate field. Using the representation as a quiver gauge theory we find

$$\mathfrak{F}\begin{pmatrix} \alpha_1 & & \alpha_t & & \alpha_{2,1} \\ & \alpha & & \alpha_{0\theta} & ;t,\frac{z}{t} \\ \alpha_\infty & & & & \alpha_0 \end{pmatrix} = t^{\Delta-\Delta_t-\Delta_{0\theta}} z^{\frac{bQ}{2}+\theta b\alpha_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}-\alpha_t)}\left(1-\frac{z}{t}\right)^{-2(\frac{Q}{2}+\alpha_t)(\frac{Q}{2}+\alpha_{2,1})}(1-z)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_{2,1})}\times$$
$$\times\sum_{\vec{Y},\vec{W}} t^{|\vec{Y}|}\left(\frac{z}{t}\right)^{|\vec{W}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) z_{\text{vec}}\left(\vec{\alpha_{0\theta}},\vec{W}\right)\prod_{\sigma=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\sigma\alpha_\infty\right) z_{\text{hyp}}\left(\vec{\alpha_{0\theta}},\vec{W},\alpha_{2,1}+\sigma\alpha_0\right) z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\alpha_{0\theta}},\vec{W};\alpha_t\right), \tag{D.0.1}$$

where the sum runs over two pairs of Young diagrams $\vec{Y}=(Y_1,Y_2)$ and $\vec{W}=(W_1,W_2)$. $\vec{\alpha_{0\theta}}$ has to be understood as $(\alpha_{0\theta},-\alpha_{0\theta})$ and we recall that $\alpha_{2,1}=-\frac{2b+b^{-1}}{2}$. Furthermore $z_{\text{vec}}$ and $z_{\text{hyp}}$ are defined as in (C.1.2). The new ingredient is the contribution of a bifundamental, defined as

$$z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\beta},\vec{W};\alpha_t\right)=$$
$$=\prod_{k,l=1,2}\prod_{(i,j)\in Y_k}\left[E\left(\alpha_k-\beta_l,Y_k,W_l,(i,j)\right)-\left(\frac{Q}{2}+\alpha_t\right)\right]\prod_{(i',j')\in W_l}\left[Q-E\left(\beta_l-\alpha_k,W_l,Y_k,(i',j')\right)-\left(\frac{Q}{2}+\alpha_t\right)\right], \tag{D.0.2}$$

with $E$ as in (C.1.2).

Since all other conformal blocks are defined in terms of this degenerate 5-point block, the expression (D.0.1) can be used to compute any other block. In particular one can verify explicitly that the various confluence limits are finite.

# Appendix E

# The radial and angular potentials

Both the radial and angular part of the Teukolsky equation can be written as a Schrödinger equation:

$$\frac{d^2\psi(z)}{dz^2} + V(z)\psi(z) = 0 \tag{E.0.1}$$

with potential

$$V(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i z^i \,. \tag{E.0.2}$$

For the radial part, the coefficients are given by

$$\begin{aligned}
\hat{A}_0^r &= \frac{\mathrm{a}^2(1-m^2) - M^2 + 4\mathrm{a}mM\omega(M-\sqrt{M^2-\mathrm{a}^2}) + 4M^2\omega^2(a^2-2M^2) + 8M^3\sqrt{M^2-\mathrm{a}^2}\omega^2}{4(\mathrm{a}^2-M^2)} + \\
&+ (is)\frac{\mathrm{a}m\sqrt{M^2-\mathrm{a}^2} - 2\mathrm{a}^2M\omega + 2M^2\omega(M-\sqrt{M^2-\mathrm{a}^2})}{2(\mathrm{a}^2-M^2)} - \frac{s^2}{4}\,, \\
\hat{A}_1^r &= \frac{4\mathrm{a}^2\lambda - 4M^2\lambda + (8\mathrm{a}mM\omega + 16\mathrm{a}^2M\omega^2 - 32M^3\omega^2)\sqrt{M^2-\mathrm{a}^2} + 4\mathrm{a}^4\omega^2 - 36\mathrm{a}^2M^2\omega^2 + 32M^4\omega^2}{4(\mathrm{a}^2-M^2)} + \\
&+ (is)\left(-i + \frac{(2\mathrm{a}^2\omega - \mathrm{a}m)\sqrt{M^2-\mathrm{a}^2}}{\mathrm{a}^2-M^2}\right) + s^2\,, \\
\hat{A}_2^r &= -\lambda - 5\mathrm{a}^2\omega^2 + 12M^2\omega^2 - 12M\omega^2\sqrt{M^2-\mathrm{a}^2} + (is)(i - 6\omega\sqrt{M^2-\mathrm{a}^2}) - s^2\,, \\
\hat{A}_3^r &= 8\mathrm{a}^2\omega^2 - 8M^2\omega^2 + 8M\omega^2\sqrt{M^2-\mathrm{a}^2} + (is)4\omega\sqrt{M^2-\mathrm{a}^2}\,, \\
\hat{A}_4^r &= 4(M^2-\mathrm{a}^2)\omega^2\,,
\end{aligned} \tag{E.0.3}$$

while for the angular part they are

$$\begin{aligned}
\hat{A}_0^\theta &= -\frac{1}{4}(-1+m-s)(1+m-s)\,, \\
\hat{A}_1^\theta &= c^2 + s + 2cs - ms + s^2 + \lambda\,, \\
\hat{A}_2^\theta &= -s - (c+s)(5c+s) - \lambda\,, \\
\hat{A}_3^\theta &= 4c(2c+s)\,, \\
\hat{A}_4^\theta &= -4c^2\,.
\end{aligned} \tag{E.0.4}$$

# Appendix F

# Conventions for boundary correlators

Here we collect our conventions for various thermal two-point functions. Let us start with the case of the black hole. This is dual to a holographic CFT on $S^1 \times S^3$, with the radius of $S^1$ being $\beta$ and the radius of $S^3$ set to 1. We have for the retarded two-point function

$$i\theta(t)\langle[\mathcal{O}(t,\vec{n}),\mathcal{O}(0,\vec{n}')]\rangle_\beta = \frac{1}{4\pi(\Delta-1)(\Delta-2)}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\sum_{\ell=0}^{\infty}(\ell+1)G_R(\omega,\ell)\frac{\sin(\ell+1)\theta}{\sin\theta}, \tag{F.0.1}$$

where $\vec{n}\cdot\vec{n}' = \cos\theta$ and $\vec{n}^2 = \vec{n}'^2 = 1$, so that $\vec{n},\vec{n}' \in S^3$. $G_R(\omega,\ell)$ is given by (5.3.5). We also used for partial waves $C_\ell^{(1)}(\cos\theta) = \frac{\sin(\ell+1)\theta}{\sin\theta}$.

For the Euclidean two-point function we have

$$\langle\mathcal{O}(\tau,\vec{n})\mathcal{O}(0,\vec{n}')\rangle_\beta = \int_{-\infty}^{\infty} d\omega\, e^{-\omega\tau}\sum_{\ell=0}^{\infty} g_{\omega,\ell}\frac{\sin(\ell+1)\theta}{\sin\theta}, \quad 0<\tau<\beta, \tag{F.0.2}$$

where $g_{\omega,\ell}$ is given in (5.4.6) and $\tau$ is the Euclidean time. KMS symmetry or invariance under $\tau \to \beta-\tau$ holds thanks to (5.4.5). We normalize the operators such that the unit operator contributes as $\frac{e^{-\tau\Delta}}{([1-e^{-\tau+i\theta}][1-e^{-\tau-i\theta}])^\Delta}$. The Wightman function can be obtained through Wick rotation by taking $\tau \to \epsilon + it$ and then $\epsilon \to 0$.

For the black brane, or holographic CFT on $S^1 \times \mathbb{R}^{d-1}$ with the radius of $S^1$ set to 1, we have for the retarded two-point function

$$i\theta(t)\langle[\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0,0)]\rangle_{\beta=1} = \frac{1}{(4\pi)^2(\Delta-1)(\Delta-2)}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\int_{-\infty}^{\infty} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}G_R^{\mathrm{brane}}(\omega,\mathbf{k}). \tag{F.0.3}$$

$G_R^{\mathrm{brane}}(\omega,\mathbf{k})$ is given by (5.3.7).

For the Euclidean two-point function we have

$$\langle\mathcal{O}(\tau,\mathbf{x})\mathcal{O}(0,0)\rangle_{\beta=1} = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\, e^{-\omega\tau}\int_{-\infty}^{\infty} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}g_{\omega,\mathbf{k}}, \quad 0<\tau<1, \tag{F.0.4}$$

where $\tau$ is the Euclidean time and $g_{\omega,\mathbf{k}}$ is given by (G.0.2). We normalize operators such that the unit operator contributes as $\frac{1}{(\tau^2+\mathbf{x}^2)^\Delta}$. KMS symmetry or invariance under $\tau \to 1-\tau$ holds thanks to (G.0.5). The Wightman function can be obtained through Wick rotation by taking $\tau \to \epsilon + it$ and then $\epsilon \to 0$.

# Appendix G

# From black hole to black brane

Let us describe in a bit more detail the infinite temperature limit that takes us from the black hole to the black brane. This is one example of the so-called macroscopic limits considered in [173] and we simply apply the formulas of that paper to our case.

First of all, we introduce the limiting retarded two-point function as follows,

$$G_R^{\text{brane}}(\omega, |\mathbf{k}|) = \lim_{T\to\infty} \frac{G_R(\omega T, |\mathbf{k}|T)}{T^{4a_1}}, \tag{G.0.1}$$

where $G_R(\omega, |\mathbf{k}|)$ is the retarded thermal two-point function for a CFT on $S^1 \times \mathbb{R}^3$ with $(\omega, |\mathbf{k}|)$ measured in units of temperature on $S^1$. Let us also introduce

$$g_{\omega,\mathbf{k}}^{\text{brane}} = \frac{1}{2\pi(\Delta-1)(\Delta-2)} \frac{\text{Im } G_R^{\text{brane}}(\omega, |\mathbf{k}|)}{1-e^{-\omega}}. \tag{G.0.2}$$

At the level of the two-point function we consider the following limit

$$G^{\text{brane}}(w, \bar{w}) \equiv \lim_{T\to\infty} T^{-2\Delta} G\Big(z = 1 - \frac{w}{T}, \bar{z} = 1 - \frac{\bar{w}}{T}\Big). \tag{G.0.3}$$

Plugging this formula in the OPE expansion (5.4.4) we get

$$\begin{aligned} G^{\text{brane}}(w, \bar{w}) &= \lim_{T\to\infty} T^{-4} \int_0^\infty d|\mathbf{k}||\mathbf{k}| \times T^2 \int_{-\infty}^\infty d\omega \times T\, g_{\omega,\mathbf{k}} e^{-\frac{(w+\bar{w})}{2}(\omega-|\mathbf{k}|)} \frac{e^{-w|\mathbf{k}|} - e^{-\bar{w}|\mathbf{k}|}}{\bar{w}-w} \times T \\ &= \int_0^\infty d|\mathbf{k}||\mathbf{k}| \int_{-\infty}^\infty d\omega\, g_{\omega,\mathbf{k}} e^{-\frac{(w+\bar{w})}{2}(\omega-|\mathbf{k}|)} \frac{e^{-w|\mathbf{k}|} - e^{-\bar{w}|\mathbf{k}|}}{\bar{w}-w}\ , \end{aligned} \tag{G.0.4}$$

where we converted the sum to an integral, $\sum_\ell \to T \int d|\mathbf{k}|$.

The KMS symmetry becomes

$$g_{-\omega,\mathbf{k}} = e^{-\omega} g_{\omega,\mathbf{k}}. \tag{G.0.5}$$

We next consider the two-point function on $S^1 \times \mathbb{R}^{d-1}$,

$$\langle \mathcal{O}(\tau, \mathbf{x}) \mathcal{O}(0,0) \rangle_\beta = G^{\text{brane}}\Big(\tau + i|\mathbf{x}|, \tau - i|\mathbf{x}|\Big). \tag{G.0.6}$$

In terms of these variables we get

$$\begin{aligned} \langle \mathcal{O}(\tau, \mathbf{x}) \mathcal{O}(0,0) \rangle_\beta &= \int_0^\infty d|\mathbf{k}|\, |\mathbf{k}| \int_{-\infty}^\infty d\omega\, g_{\omega,\mathbf{k}} e^{-\omega\tau} \frac{\sin |\mathbf{k}||\mathbf{x}|}{|\mathbf{x}|} \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty d^3\mathbf{k} \int_{-\infty}^\infty d\omega\, e^{i\mathbf{k}\cdot\mathbf{x}} e^{-\omega\tau} g_{\omega,\mathbf{k}}. \end{aligned} \tag{G.0.7}$$

The result is indeed invariant under KMS symmetry $\tau \to 1-\tau$ (recall that we have set $\beta = 1$). By analytically continuing to Lorentzian time we see that $g_{\omega,\mathbf{k}}$ is the Fourier transform of the Wightman two-point function.

Note that taking the limit (G.0.3) does not change the normalization of the scalar operator, since

$$\lim_{T\to\infty} T^{-2\Delta}\frac{1}{\Big(1-(1-\frac{w}{T})\Big)^{\Delta}\Big(1-(1-\frac{\bar{w}}{T})\Big)^{\Delta}} = \frac{1}{(w\bar{w})^{\Delta}} = \frac{1}{(\tau^2+\mathbf{x}^2)^{\Delta}}. \tag{G.0.8}$$

In other words if the operator was unit-normalized it will continue to be unit-normalized after taking the limit.

Let us finish with a few formulas for the vacuum correlators. In Fourier space, the vacuum Wightman two-point function $\langle \mathcal{O}(t,\mathbf{x})\mathcal{O}(0,0)\rangle_0 = \frac{1}{(-(t-i\epsilon)^2+\mathbf{x}^2)^{\Delta}}$ takes the form

$$\int_{-\infty}^{\infty} dt\, d^3\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\frac{1}{(-(t-i\epsilon)^2+\mathbf{x}^2)^{\Delta}} = \theta(\omega)\theta(\omega^2-\mathbf{k}^2)\frac{2\pi^3}{\Gamma(\Delta)\Gamma(\Delta-1)}\Big(\frac{\omega^2-\mathbf{k}^2}{4}\Big)^{\Delta-2}. \tag{G.0.9}$$

It is expected that (G.0.9) controls the large $\omega$ asymptotics of the thermal correlators [201, 202].

From (G.0.7) we get that

$$g_{\omega,\mathbf{k}} = \lim_{\epsilon\to 0}\frac{1}{4\pi^3}\int_{-\infty}^{\infty} d^3\mathbf{k}\int_{-\infty}^{\infty} d\omega\, e^{-i\mathbf{k}\cdot\mathbf{x}}e^{it\omega}\langle \mathcal{O}(\epsilon+it,\mathbf{x})\mathcal{O}(0,0)\rangle_{\beta}. \tag{G.0.10}$$

Formulas (G.0.9), (G.0.5) together with (G.0.2) imply that

$$\lim_{|\omega|\gg 1, |\omega|\gg|\mathbf{k}|} \mathrm{Im}\; G_R^{\text{brane}}(\omega,|\mathbf{k}|) \simeq -\sin\pi\Delta\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\mathrm{sign}(\omega)\Big(\frac{|\omega|}{2}\Big)^{2(\Delta-2)}. \tag{G.0.11}$$

Via dispersion relations for $G_R^{\text{brane}}(\omega,|\mathbf{k}|)$ this leads to the following asymptotic behavior for the real part,

$$\lim_{|\omega|\gg 1, |\omega|\gg|\mathbf{k}|} \mathrm{Re}\; G_R^{\text{brane}}(\omega,|\mathbf{k}|) \simeq \cos\pi\Delta\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\Big(\frac{|\omega|}{2}\Big)^{2(\Delta-2)}, \tag{G.0.12}$$

where everywhere we tacitly assumed that $\Delta$ is not an integer. For the black hole case ($t<\frac{1}{2}$) we get in the same way

$$\lim_{|\omega|/T\gg 1,\ell} G_R(\omega,\ell) \simeq e^{-\pi i\Delta\mathrm{sign}(\omega)}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\Big(\frac{|\omega|}{2}\Big)^{2(\Delta-2)}. \tag{G.0.13}$$

We can also derive the large $\omega$ and fixed $\ell$ behavior of the Green's function directly from our exact expression (5.3.5). Let us start with the black hole case. By solving the Matone relation (5.3.1) order by order in the instanton expansion, one finds in this limit $\partial_a F = ic_1(t)\omega + \mathcal{O}(\omega^0)$, $\partial_{a_1}F = c_3(t)(\Delta-2)+\mathcal{O}(\omega^{-1})$, and $a = ic_2(t)\omega + \mathcal{O}(\omega^0)$, with $c_i(t)\in\mathbb{R}$. Since the Green's function (5.3.5) is invariant under $a\to -a$, we can choose $c_2>0$ without loss of generality. With this specification, the $\sigma=1$ term in (5.3.5) dominates over the $\sigma=-1$ term. Expanding the gamma functions at large $\omega$ and using the dictionary in Table 5.2.15, we find

$$\begin{aligned} G_R(\omega,\ell) &\approx (1+R_+^2)^{2a_1}e^{-\partial_{a_1}F}\frac{\Gamma(-2a_1)}{\Gamma(2a_1)}(a_\infty - a)^{2a_1}(-a-a_\infty)^{2a_1} \\ &\approx \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\left(\frac{|\omega|}{2}\right)^{2(\Delta-2)} e^{-\pi i\Delta\mathrm{sign}(\omega)}\Big(c(t)\Big)^{\Delta-2}, \end{aligned} \tag{G.0.14}$$

where

$$c(t) = \frac{e^{-c_3(t)}(1-t)(4c_2(t)^2 + 2t^2 - 3t + 1)}{1-2t}. \tag{G.0.15}$$

The OPE predicts that $c(t) = 1$.

We do not have complete analytic control over the constants $c_2(t)$ and $c_3(t)$, but we checked that (G.0.15) approaches 1 by computing the first few orders in the instanton expansion, see (G.1). Hence we recover (G.0.13). The black brane results (G.0.11) and (G.0.12) correspond to $t \to \frac{1}{2}$ in (G.1).

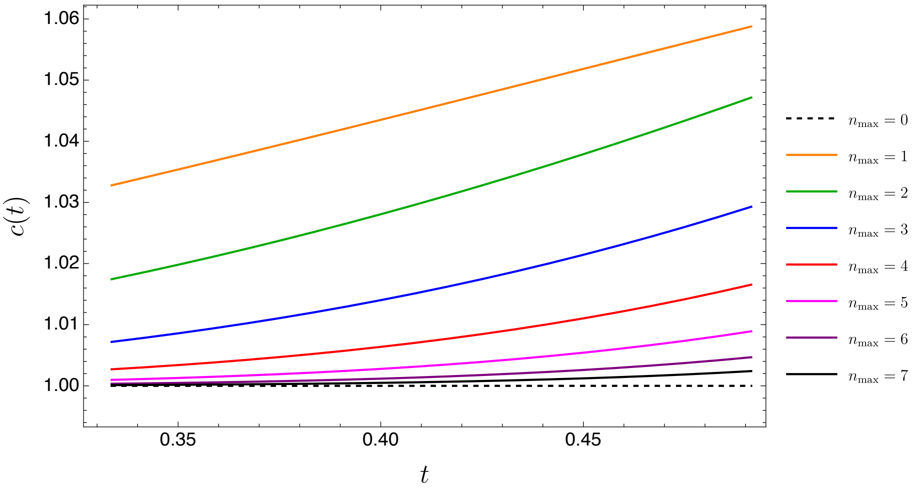

Figure G.1: $c(t)$ defined in (G.0.15) as a function of the black hole mass (here parameterized by $t$), and the maximum instanton number $n_{\max}$. Based on the OPE we expect that $c(t)$ is independent of $t$ and is equal to 1.

# Appendix H

# $\mathcal{O}(\mu^2)$ OPE data of double-twist operators

Here we display the results for the OPE data at order $\mu^2$. These expressions are in full agreement with [153] at order $1/\ell^2$, and provide new predictions at higher orders in $1/\ell$. We find

$$
\begin{aligned}
\gamma^{(2)}_{n\ell} = &-\frac{\big((\Delta-1)\Delta+6(\Delta-1)n+6n^2\big)^2}{8(\ell+1)^3} - \frac{n(\Delta+n-2)(\Delta+2n-2)^2}{2(\ell+2)} - \frac{(n+1)(\Delta+n-1)(\Delta+2n)^2}{2\ell} \\
&+ \frac{(\Delta-1)\Delta(8\Delta+1)+65n^4+130(\Delta-1)n^3+(3\Delta(27\Delta-43)+133)n^2+(\Delta-1)\left(16\Delta^2+\Delta+68\right)n}{16(\ell+1)} \\
&- \frac{(n-1)n(\Delta+n-3)(\Delta+n-2)}{32(\ell+3)} - \frac{(n+1)(n+2)(\Delta+n-1)(\Delta+n)}{32(\ell-1)}, \\
c^{(2)}_{n\ell} = &\,\frac{1}{8}(\Delta-2)(9\Delta-44) - \frac{(2n+3)(\Delta+n-1)(\Delta+n)}{32(\ell-1)} - \frac{3(\Delta+2n-1)\left((\Delta-1)\Delta+6n^2+6(\Delta-1)n\right)}{4(\ell+1)^3} \\
&+ \frac{(\Delta+2n-1)\left(\Delta(16\Delta-71)+130n^2+130(\Delta-1)n+212\right)}{32(\ell+1)} - \frac{(\Delta+n-1)(\Delta+2n)(\Delta+4n+2)}{2\ell} \\
&- \frac{(n-1)n(2\Delta+2n-5)}{32(\ell+3)} - \frac{n(\Delta+2n-2)(3\Delta+4n-6)}{2(\ell+2)} + \frac{1}{4}(\psi^{(0)}(n+\ell+2)-\psi^{(0)}(n+\ell+\Delta)) \\
&\times \Big(9\Delta^2 - \frac{89\Delta}{2} + \frac{\big((\Delta-1)\Delta+6n^2+6(\Delta-1)n\big)^2}{2(\ell+1)^3} - \frac{3(\Delta+2n-1)\left((\Delta-1)\Delta+6n^2+6(\Delta-1)n\right)}{(\ell+1)^2} \\
&+ \frac{(\Delta-1)(\Delta(4\Delta-73)+36)-65n^4-130(\Delta-1)n^3+(3(67-27\Delta)\Delta-493)n^2-(\Delta-1)(\Delta(16\Delta-71)+428)n}{4(\ell+1)} \\
&+ (18\Delta-89)n + \frac{2n(\Delta+n-2)(\Delta+2n-2)^2}{\ell+2} + \frac{2(n+1)(\Delta+n-1)(\Delta+2n)^2}{\ell} + \frac{(n-1)n(\Delta+n-3)(\Delta+n-2)}{8(\ell+3)} \\
&+ \frac{(n+1)(n+2)(\Delta+n-1)(\Delta+n)}{8(\ell-1)} + \left(9\Delta-\frac{71}{2}\right)\ell+9\Big) + \big(\Delta(\Delta+2)+6n^2+6n(\Delta+\ell)+3\ell^2+3(\Delta+1)\ell\big)^2 \\
&\times \frac{(\psi^{(0)}(n+\ell+2)-\psi^{(0)}(n+\ell+\Delta))^2+\psi^{(1)}(n+\ell+\Delta)-\psi^{(1)}(n+\ell+2)}{8(\ell+1)^2}\,.
\end{aligned}
$$

# Appendix I

# The imaginary part of quasi-normal modes

In this appendix we spell out some details for the computation of (5.5.14). The condition for a pole in $G_R(\omega, \ell)$ follows from (5.3.5) and reads

$$t^{-2a} e^{\partial_a F} \left( \frac{\Gamma(2a)\Gamma\left(-a - a_t + \frac{1}{2}\right)}{\Gamma(-2a)\Gamma\left(a - a_t + \frac{1}{2}\right)} \right)^2 - \frac{\Gamma\left(a + a_1 - a_\infty + \frac{1}{2}\right)\Gamma\left(a + a_1 + a_\infty + \frac{1}{2}\right)}{\Gamma\left(-a + a_1 - a_\infty + \frac{1}{2}\right)\Gamma\left(-a + a_1 + a_\infty + \frac{1}{2}\right)} = 0 \ . \tag{I.0.1}$$

By using the ansatz (5.5.13) as well as the dictionary in Table 5.2.15 and the perturbative solution for the real part (5.5.5), we obtain

$$\begin{aligned} &\mathrm{Im}\left( \frac{\Gamma\left(a + a_1 - a_\infty + \frac{1}{2}\right)\Gamma\left(a + a_1 + a_\infty + \frac{1}{2}\right)}{\Gamma\left(-a + a_1 - a_\infty + \frac{1}{2}\right)\Gamma\left(-a + a_1 + a_\infty + \frac{1}{2}\right)} \right) = \qquad \text{(I.0.2)} \\ &= \mu^{\ell+1/2} \left( \frac{\Gamma(n+1)\Gamma(n+\Delta-1)}{\Gamma(\ell+n+2)\Gamma(\ell+n+\Delta)} \frac{(-1)^\ell}{3\omega_{n\ell}^{(0)} - 2\gamma_{n\ell}^{(1)}} f_{n\ell}^{(1)} + \mathcal{O}(\mu) \right) \\ &\mathrm{Im}\left( t^{-2a} e^{\partial_a F} \left( \frac{\Gamma(2a)\Gamma\left(-a - a_t + \frac{1}{2}\right)}{\Gamma(-2a)\Gamma\left(a - a_t + \frac{1}{2}\right)} \right)^2 \right) = -\mu^{\ell+1/2} \left( \frac{\Gamma\left(\frac{\ell}{2}+1\right)^4}{\Gamma(\ell+1)^2\Gamma(\ell+2)^2} \frac{(-1)^\ell \omega_{n\ell}^{(0)}}{3\omega_{n\ell}^{(0)} - 2\gamma_{n\ell}^{(1)}} + \mathcal{O}(\mu) \right) \ , \end{aligned} \tag{I.0.3}$$

leading to (5.5.14).

# Appendix J

# The large $\ell$/large k, fixed $\omega$ limit

Using the asymptotic behavior (5.3.12), we can investigate the behavior of $G_R$ at large $\ell$. We start with the real part of $G_R$, for which the leading behavior comes from the $\sigma = 1$ terms in (5.3.5). Expanding at large $a$, we find

$$\text{Re } G_R(\omega, \ell) \approx (1 + R_+^2)^{\Delta - 2} \frac{\Gamma(2 - \Delta)}{\Gamma(\Delta - 2)} e^{-\partial_{a_1} F} (-a)^{4a_1} \approx \frac{\Gamma(2 - \Delta)}{\Gamma(\Delta - 2)} \left(\frac{\ell}{2}\right)^{2(\Delta - 2)} \tag{J.0.1}$$

Note that this is independent of the temperature.

Now let us turn to the imaginary part. The leading contribution comes from expanding to first order in the $\sigma = -1$ term in both the numerator and denominator of (5.3.5). We find

$$\begin{aligned} \text{Im } G_R(\omega, \ell) &\approx -\frac{2(1 + R_+^2)^{2a_1} e^{\partial_a F - \partial_{a_1} F} t^{-2a} \sin(2\pi a) \sin(2\pi a_1)}{\cos(2\pi(a - a_1)) + \cos(2\pi a_\infty)} \\ &\quad \times \frac{\Gamma(2a)^2 \Gamma(-2a_1) \Gamma\left(\frac{1}{2} - a + a_1 - a_\infty\right) \Gamma\left(\frac{1}{2} - a + a_1 + a_\infty\right)}{\Gamma(-2a)^2 \Gamma(2a_1) \Gamma\left(\frac{1}{2} + a - a_1 - a_\infty\right) \Gamma\left(\frac{1}{2} + a - a_1 + a_\infty\right)} \text{Im} \left( \frac{\Gamma\left(\frac{1}{2} - a - a_t\right)^2}{\Gamma\left(\frac{1}{2} + a - a_t\right)^2} \right) \\ &\approx -\frac{\Gamma(-2a_1)}{\Gamma(2a_1)} (1 + R_+^2)^{2a_1} e^{\partial_a F - \partial_{a_1} F} t^{-2a} 2^{8a+1} (-a)^{4a_1} \sin(2\pi a_1) \sinh(2\pi |a_t|), \end{aligned} \tag{J.0.2}$$

where in the second equality we took the large $a$ limit. Plugging in the asymptotic behavior (5.3.12) and the dictionary given in Table 5.2.18 gives

$$\text{Im } G_R(\omega, \ell) \approx \frac{2\pi \sinh(\pi\omega\sqrt{t(1 - 2t)})}{\Gamma(\Delta - 1)\Gamma(\Delta - 2)} \left(\frac{\ell}{2}\right)^{2(\Delta - 2)} \exp\left(-2(\ell + 1)\sqrt{1 - 2t} K(1 - t)\right). \tag{J.0.3}$$

We see that the imaginary part decays exponentially with spin.

To compute the large $|\mathbf{k}|$ behavior for the black brane, we can take the infinite temperature limit of (J.0.1) and (J.0.3). Using the definition (5.2.19) of the brane two-point function, we find

$$G_R^{\text{brane}}(\omega, |\mathbf{k}|) \approx \frac{\Gamma(2 - \Delta)}{\Gamma(\Delta - 2)} \left(\frac{|\mathbf{k}|}{2}\right)^{2(\Delta - 2)} + i \frac{2\pi \sinh\left(\frac{\omega}{2}\right)}{\Gamma(\Delta - 1)\Gamma(\Delta - 2)} \left(\frac{|\mathbf{k}|}{2}\right)^{2(\Delta - 2)} \exp\left(-\sqrt{\frac{\pi}{2}} \frac{|\mathbf{k}|}{\Gamma\left(\frac{3}{4}\right)^2}\right). \tag{J.0.4}$$

The rate of exponential decay of the imaginary part matches the result from [103].

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