Research Article

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The notions of inertial balanced viscosity and inertial virtual viscosity solution for rate-independent systems

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Abstract: The notion of inertial balanced viscosity (IBV) solution to rate-independent evolutionary processes is introduced. Such solutions are characterized by an energy balance where a suitable, rate-dependent, dissipation cost is optimized at jump times. The cost is reminiscent of the limit effect of small inertial terms. Therefore, this notion proves to be a suitable one to describe the asymptotic behavior of evolutions of mechanical systems with rate-independent dissipation in the limit of vanishing inertia and viscosity. It is indeed proved, in finite dimension, that these evolutions converge to IBV solutions. If the viscosity operator is neglected, or has a nontrivial kernel, the weaker notion of inertial virtual viscosity (IVV) solutions is introduced, and the analogous convergence result holds. Again in a finite-dimensional context, it is also shown that IBV and IVV solutions can be obtained via a natural extension of the minimizing movements algorithm, where the limit effect of inertial terms is taken into account.

Keywords: Inertial balanced viscosity solutions, inertial virtual viscosity solutions, rate-independent systems, vanishing inertia and viscosity limit, minimizing movements scheme, variational methods

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1 Introduction

Rate-independent evolutions frequently occur in physics and mechanics when the problem under consideration presents such small rate-dependent effects, as inertia or viscosity, that can be neglected. Several applications can be (formally) modelled by the doubly nonlinear differential inclusion

$$\begin{cases} \partial\mathcal{R}(\dot u(t)) + D_x\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } X^*, \text{ for a.e. } t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathcal{R}$ is a rate-independent dissipation potential, while $\mathcal{E}$ is a time-dependent potential energy. In this paper, we limit ourselves to the case of a finite-dimensional normed space X, although we plan to extend the whole analysis to the infinite-dimensional context, where several difficulties concerning weak topologies

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and nonsmoothness of $\mathcal{E}$ naturally arise (we refer to the monograph [22] for more details, also compare [19] with [20]).

If the driving energy $\mathcal{E}$ is nonconvex, continuous solutions to (1.1) are not expected to exist, and thus in the past decades huge efforts have been spent to develop weak notions of solution capable of describing the behavior of the system at jumps. A first attempt can be found in the notion of *energetic solution* [23, 24] based on a global stability condition together with an energy balance which must hold for every $t \in [0, T]$:

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, x) + \mathcal{R}(x - u(t)) \quad \text{for every } x \in X, \tag{GS}$$

$$\mathcal{E}(t, u(t)) + V_{\mathcal{R}}(u; 0, t) = \mathcal{E}(0, u_0) + \int_0^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r. \tag{EB}$$

Condition (GS) actually turns out to be still too restrictive in the nonconvex case, where a local minimality condition would be preferable. Starting from this consideration, in [18–20] the notion of *balanced viscosity solution* has been introduced and analyzed; see also the recent paper [25]. The idea of Mielke, Rossi and Savaré [18–20] relies on the fact that physical solutions to (1.1) should arise as the vanishing-viscosity limit of a richer and more natural viscous problem

$$\begin{cases} \varepsilon \mathbb{V}\dot{u}^\varepsilon(t) + \partial\mathcal{R}(\dot{u}^\varepsilon(t)) + D_x\mathcal{E}(t, u^\varepsilon(t)) \ni 0 \quad \text{in } X^*, \ \text{for a.e. } t \in [0, T], \\ u^\varepsilon(0) = u_0^\varepsilon, \end{cases} \tag{1.2}$$

as the parameter $\varepsilon \to 0$. Here, $\mathbb{V}$ denotes a symmetric positive-definite linear operator modelling viscosity. Actually, in [19, 20] more general viscous potentials are considered. The resulting evolution, called balanced viscosity (BV) solution, turns out to satisfy a local stability condition together with an augmented energy balance:

$$-D_x\mathcal{E}(t, u(t)) \in \partial\mathcal{R}(0) \qquad \text{for a.e. } t \in [0, T], \tag{LS}$$

$$\mathcal{E}(t, u(t)) + V_{\mathcal{R}}^{\mathbb{V}}(u; 0, t) = \mathcal{E}(0, u_0) + \int_0^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r \quad \text{for all } t \in [0, T]. \tag{EB$^{\mathbb{V}}$}$$

While in (EB) the classical total variation (actually, $\mathcal{R}$-variation, see Definition 2.3) controls both the continuous part u_{co} of the evolution and the jump part, as it holds (see also (2.13))

$$V_{\mathcal{R}}(u; 0, t) = V_{\mathcal{R}}(u_{\mathrm{co}}; 0, t) + \sum_{r \in J_u \cap [0,t]} (\mathcal{R}(u(r) - u^-(r)) + \mathcal{R}(u^+(r) - u(r))),$$

in (EB$^{\mathbb{V}}$) the jump part of the “viscous” variation involves a more complicated cost function (a Finsler distance) which takes into account the original presence of viscosity:

$$V_{\mathcal{R}}^{\mathbb{V}}(u; 0, t) = V_{\mathcal{R}}(u_{\mathrm{co}}; 0, t) + \sum_{r \in J_u \cap [0,t]} (c_r^{\mathbb{V}}(u^-(r), u(r)) + c_r^{\mathbb{V}}(u(r), u^+(r))).$$

At time $t \in [0, T]$, the viscous cost function is obtained as

$$c_t^{\mathbb{V}}(u_1, u_2) := \inf\Big\{ \int_0^1 p_{\mathbb{V}}(\dot{v}(r), -D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r \mid v \in W^{1,\infty}(0, 1; X),\ v(0) = u_1,\ v(1) = u_2 \Big\}, \tag{1.3}$$

where $p_{\mathbb{V}}$, called *vanishing-viscosity contact potential*, is a suitable density arising from both the viscous and the rate-independent dissipation (see also Definition 3.1).

In [19, 20], it is also shown that BV solutions can be obtained as the limit of time discrete approximations, solutions to the recursive discrete-in-time variational incremental scheme with time step τ:

$$u_{\tau,\varepsilon}^k \in \operatorname*{arg\,min}_{x \in X} \Big\{ \frac{\varepsilon}{2\tau} \|x - u_{\tau,\varepsilon}^{k-1}\|_{\mathbb{V}}^2 + \mathcal{R}(x - u_{\tau,\varepsilon}^{k-1}) + \mathcal{E}(t^k, x) \Big\}, \quad k = 1, \ldots, \frac{T}{\tau}, \tag{1.4}$$

when sending simultaneously ε and τ to 0 with also $\tau/\varepsilon \to 0$.

Although the notion of BV solution turned out to be extremely powerful in applications (see, for instance, [2, 6, 12, 21]), it still lacks inertial terms, which are however essential in the description of real world phenomena, as stated by the second principle of dynamics.

In this paper, we thus present a novel notion of solution which takes into account this feature. Our starting point is augmenting (1.2) with an inertial term

$$\begin{cases} \varepsilon^2 \mathbb{M}\ddot{u}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{u}^\varepsilon(t) + \partial\mathcal{R}(\dot{u}^\varepsilon(t)) + D_x\mathcal{E}(t, u^\varepsilon(t)) \ni 0 \quad \text{in } X^*, \text{ for a.e. } t \in [0, T], \\ u^\varepsilon(0) = u_0^\varepsilon, \quad \dot{u}^\varepsilon(0) = u_1^\varepsilon, \end{cases} \tag{1.5}$$

and then sending $\varepsilon \to 0$, namely performing a *vanishing-inertia and viscosity* argument. The symmetric positive-definite linear operator $\mathbb{M}$ appearing in (1.5) represents masses. Its presence allows to consider also the case of null viscosity, i.e. $\mathbb{V} = 0$, or more generally of a positive-semidefinite linear operator. We point out that such a limit procedure is also known as *slow-loading* limit, since (1.5) comes from a dynamic problem with slow data after a reparametrization of time. We refer the interested reader to [10], or to [22, Chapter 5] for a more detailed explanation.

This approach has already been adopted for concrete models (in infinite dimension) in [7, 8, 14, 15, 17, 26, 29], all in the case of convex, or even quadratic, energies $\mathcal{E}$. An abstract analysis has been performed in [10], in finite dimension and always under convexity assumptions. Hence, the results contained in this paper on the one hand represent an extension to nonconvex energies of the ones presented in [10] (see Remark 2.2), and on the other hand put the basis for an abstract investigation in infinite dimension, where nonconvex problems are common in applications (we refer again to [2, 6, 12, 21, 22]).

In our nonconvex setting, the limiting evolution of (1.5) provides the novel notion of *inertial balanced viscosity (IBV) solution* (we refer to the discussion in Section 3 for the subtler notion of *inertial virtual viscosity (IVV) solution*, arising when the viscosity operator $\mathbb{V}$ is not positive-definite) for the rate-independent system (1.1), namely a function satisfying the same local stability condition (LS) as BV solutions, together with an energy balance in which the cost at jump points is sensitive of the presence of inertia:

$$-D_x\mathcal{E}(t, u(t)) \in \partial\mathcal{R}(0) \quad \text{for a.e. } t \in [0, T], \tag{LS}$$

$$\mathcal{E}(t, u^+(t)) + V_\mathcal{R}(u_{\mathrm{co}}; s, t) + \sum_{r \in J_u^e \cap [s,t]} c_r^{\mathbb{M},\mathbb{V}}(u^-(r), u^+(r)) = \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r, \tag{EB$^{\mathbb{M},\mathbb{V}}$}$$

for every $0 \le s \le t \le T$, where now the cost turns out to be

$$c_t^{\mathbb{M},\mathbb{V}}(u_1, u_2) := \inf\Big\{ \int_{-N}^{N} p_\mathbb{V}(\dot{v}(r), -\mathbb{M}\ddot{v}(r) - D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r \mid N \in \mathbb{N},\ v \in V_{u_1,u_2}^{\mathbb{M},N} \Big\},$$

where

$$V_{u_1,u_2}^{\mathbb{M},N} := \{ v \in W^{2,\infty}(-N, N; X) \mid v(-N) = u_1,\ v(N) = u_2,\ \mathbb{M}\dot{v}(\pm N) = 0,\ \operatorname*{ess\,sup}_{r\in[-N,N]} \|\mathbb{M}\ddot{v}(r)\|_* \le \overline{C} \}.$$

Notice that, differently than in the vanishing-viscosity case, the dissipation cost we consider is no longer invariant under a time-reparametrization, due to the presence of the second-order term $\mathbb{M}\ddot{v}(r)$ inside the integral. This prevents an easy generalization of the notion of *parametrized BV solution*, again introduced in [19, 20], to the inertial setting: such a notion is indeed build starting from a suitable viscous-reparametrization of the time variable. Furthermore, the rate-dependent nature of the cost forces one to consider minimization problems on an asymptotically infinite time horizon and take the infimum over them. We also refer to [1, 31] for a similar analysis with no rate-independent dissipation, namely considering $\mathcal{R} = 0$, where an analogous notion of solution was developed. As it happened in [31], we prefer to consider a notion of solution which does not depend on the chosen representative of u in its Lebesgue class, which is done by considering left and right limits only in the energy balance.

The proposed notion of solution is indeed a suitable one to extend the results of [19, 20] to a context where the limiting effect of inertial terms is taken into consideration. We show this in our main result, Theorem 3.10, which fulfills a twofold goal. First, we show convergence of the solutions u^ε of (1.5) to an inertial balanced viscosity solution of (1.1), under a quite general set of assumptions on the energy $\mathcal{E}$, which includes the one considered in [10]. Secondly, we also prove that IBV solutions can be obtained via a natural extension of the minimizing movements algorithm (1.4), namely

$$u^k_{\tau,\varepsilon} \in \arg\min_{x\in X}\Big\{\frac{\varepsilon^2}{2\tau^2}\|x - 2u^{k-1}_{\tau,\varepsilon} + u^{k-2}_{\tau,\varepsilon}\|^2_{\mathbb{M}} + \frac{\varepsilon}{2\tau}\|x - u^{k-1}_{\tau,\varepsilon}\|^2_{\mathbb{V}} + \mathcal{R}(x - u^{k-1}_{\tau,\varepsilon}) + \mathcal{E}(t^k, x)\Big\}, \tag{1.6}$$

by sending both τ and ε to 0. Differently from (1.4), for technical reasons we need to strengthen the rate of convergence requiring τ/ε^2 to be bounded. Furthermore, we have to require $\mathcal{E}$ to be Λ-convex (assumption (E5) in Section 2). Such a condition, which amounts to require that the sum of $\mathcal{E}$ with a suitably large quadratic perturbation is convex, is quite typical in the analysis of such approximation schemes (see [4]) and complies with many relevant applications. It actually allows one to have precise estimates on some rest terms in the energy balance, which arise from the iterative minimization schemes.

Plan of this paper. In Section 2, we fix the main notation and list the main assumptions of this paper. We also recall some basic properties of functions of bounded $\mathcal{R}$-variation (Section 2.2). In Section 3, we introduce the notions of inertial balanced viscosity and inertial virtual viscosity solution. We also define the contact potentials (Section 3.1) and the regularized contact potentials (Section 3.2), while in Section 3.3 we introduce the inertial cost function which will characterize the description of the jumps. Section 4 contains the first characterization of the IBV and IVV solutions as the slow-loading limit as $\varepsilon \to 0$ of dynamical solutions to (1.5). Finally, in Section 5, we derive these solutions as the limit of the time-discrete incremental variational scheme (1.6) as τ and ε go simultaneously to 0.

2 Notation and setting

Let $(X, \|\cdot\|)$ be a finite-dimensional normed vector space. We denote by $(X^*, \|\cdot\|_*)$ the topological dual of X and by $\langle w, v\rangle$ the duality product between $w \in X^*$ and $v \in X$. For $R > 0$, we denote by B_R the open ball in X of radius R centered at the origin, and by $\overline{B_R}$ its closure.

Given any symmetric positive-semidefinite linear operator $\mathbb{Q}\colon X \to X^*$, we introduce the induced (Hilbertian) seminorm

$$|x|_{\mathbb{Q}} := \langle \mathbb{Q}x, x\rangle^{\frac{1}{2}},$$

and we denote with a capital letter $Q \geq 0$ a nonnegative constant satisfying

$$0 \leq |x|^2_{\mathbb{Q}} \leq Q\|x\|^2 \quad \text{for every } x \in X.$$

We point out that such a constant Q exists, since in finite dimension any linear operator is necessarily continuous. The least Q that may be chosen here is the operator norm of $\mathbb{Q}$, denoted by $\|\mathbb{Q}\|_{\mathrm{op}}$.

If $\mathbb{Q}$ is positive-definite, the induced seminorm is actually a norm, denoted by $\|\cdot\|_{\mathbb{Q}}$, and, up to possibly enlarging the constant Q, there holds

$$\frac{1}{Q}\|x\|^2 \leq \|x\|^2_{\mathbb{Q}} \leq Q\|x\|^2 \quad \text{for every } x \in X.$$

Furthermore, the inverse operator $\mathbb{Q}^{-1}\colon X^* \to X$ induces in X^* the norm

$$\|w\|_{\mathbb{Q}^{-1}} := \langle w, \mathbb{Q}^{-1}w\rangle,$$

which is dual to $\|\cdot\|_{\mathbb{Q}}$ and thus satisfies

$$|\langle w, v\rangle| \leq \|v\|_{\mathbb{Q}}\|w\|_{\mathbb{Q}^{-1}} \quad \text{for every } w \in X^* \text{ and } v \in X.$$

We briefly recall some basic definitions in convex analysis (see, for instance, [27]). Given a proper, convex, lower semicontinuous function $f\colon X \to (-\infty, +\infty]$, its (convex) subdifferential $\partial f\colon X \rightrightarrows X^*$ at a point $v \in X$ is defined by

$$\partial f(v) = \{w \in X^* \mid f(z) \geq f(v) + \langle w, z - v\rangle \text{ for every } z \in X\}.$$

Notice that if $f(v) = +\infty$, then from the very definition it turns out that $\partial f(v) = \emptyset$.

The Fenchel conjugate of f is the convex, lower semicontinuous function

$$f^*\colon X^* \to (-\infty, +\infty], \quad \text{defined by} \quad f^*(w) := \sup_{v\in X}\{\langle w, v\rangle - f(v)\},$$

and for every $w \in X^*$ and $v \in X$ it satisfies

$$f^*(w) + f(v) \geq \langle w, v\rangle, \quad \text{with equality if and only if} \quad w \in \partial f(v). \tag{2.1}$$

Given a subset $A \subset X$, we denote with $\chi_A\colon X \to [0, +\infty]$ its characteristic function, defined by

$$\chi_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

2.1 Main assumptions

Below, we list the main assumptions we will use throughout the paper.

In the dynamic problem (1.5), the inertial term is described by a

$$\text{symmetric positive-definite linear operator } \mathbb{M}\colon X \to X^*, \tag{2.2}$$

which represents a *mass distribution*.

The possible presence of *viscosity* is also considered by introducing the

$$\text{symmetric positive-semidefinite linear operator } \mathbb{V}\colon X \to X^*. \tag{2.3}$$

In particular, in our analysis we also include the case $\mathbb{V} \equiv 0$ (for which $|x|_{\mathbb{V}} \equiv 0$), corresponding to the absence of viscous friction forces.

Both the rate-independent problem (1.1) and the dynamic problem (1.5) are damped by a *rate-independent dissipation potential* $\mathcal{R}\colon X \to [0, +\infty)$, which models for instance dry friction. We make the following assumption:

(R1) The function $\mathcal{R}$ is coercive, convex and positively homogeneous of degree one.

Assumption (R1) implies subadditivity, namely

$$\mathcal{R}(v_1 + v_2) \leq \mathcal{R}(v_1) + \mathcal{R}(v_2) \quad \text{for every } v_1, v_2 \in X,$$

and the existence of two positive constants $\alpha^* \geq \alpha_* > 0$ for which

$$\alpha_*\|v\| \leq \mathcal{R}(v) \leq \alpha^*\|v\| \quad \text{for every } v \in X. \tag{2.4}$$

This means that $\mathcal{R}$ fails to be a norm only for the lack of symmetry.

Furthermore, since $\mathcal{R}$ is one-homogeneous, for every $v \in X$ its subdifferential $\partial\mathcal{R}(v)$ can be characterized by

$$\partial\mathcal{R}(v) = \{w \in \partial\mathcal{R}(0) \mid \langle w, v\rangle = \mathcal{R}(v)\} \subseteq \partial\mathcal{R}(0) =: K^*. \tag{2.5}$$

By (2.4), we notice that there holds

$$K^* \subseteq \overline{B_{\alpha^*}}. \tag{2.6}$$

It is also well known (see, e.g., [27]) that K^* coincides with the proper domain of the Fenchel conjugate $\mathcal{R}^*$ of $\mathcal{R}$; indeed, it actually holds $\mathcal{R}^* = \chi_{K^*}$.

We finally consider the driving *potential energy* $\mathcal{E}\colon [0,T]\times X \to [0,+\infty)$, which we assume to possess the following properties:

(E1) $\mathcal{E}(\cdot, u)$ is absolutely continuous in $[0,T]$ for every $u \in X$.

(E2) $\mathcal{E}(t,\cdot)$ is differentiable for every $t \in [0,T]$, and the differential $D_x\mathcal{E}$ is continuous from $[0,T]\times X$ to X^*.

(E3) For a.e. $t \in [0,T]$ and for every $u \in X$, it holds

$$|\partial_t \mathcal{E}(t,u)| \le a(\mathcal{E}(t,u))b(t),$$

where $a\colon [0,+\infty) \to [0,+\infty)$ is nondecreasing and continuous, while $b \in L^1(0,T)$ is nonnegative.

(E4) For every $R>0$, there exists a nonnegative function $c_R \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and for every $u_1, u_2 \in B_R$ it holds

$$|\partial_t \mathcal{E}(t,u_2) - \partial_t \mathcal{E}(t,u_1)| \le c_R(t)\|u_2 - u_1\|.$$

We point out that the prototypical example of potential energy

$$\mathcal{E}(t,u) = \mathcal{U}(u) - \langle \ell(t), u\rangle, \tag{2.7}$$

with $\mathcal{U} \in C^1(X)$ superlinear and $\ell \in W^{1,1}(0,T;X^*)$, fulfils all previous assumptions.

As mentioned in [10], under these hypotheses one can prove that $\mathcal{E}$ is a continuous map, and that $t \mapsto \mathcal{E}(t,u(t))$ is absolutely continuous (resp. of bounded variation) if u is absolutely continuous (resp. of bounded variation).

Remark 2.1. Thanks to (E4), it is easy to see that $D_x\mathcal{E}(\cdot, u)$ is absolutely continuous in $[0,T]$ for every $u \in X$:

$$\begin{aligned}\|D_x\mathcal{E}(t,u) - D_x\mathcal{E}(s,u)\|_* &= \langle D_x\mathcal{E}(t,u) - D_x\mathcal{E}(s,u), v\rangle \\ &= \lim_{h\to 0} \frac{\mathcal{E}(t,u+hv) - \mathcal{E}(t,u) - \mathcal{E}(s,u+hv) + \mathcal{E}(s,u)}{h} \\ &\le \liminf_{h\to 0} \frac{1}{h}\int_s^t |\partial_t\mathcal{E}(r,u+hv) - \partial_t\mathcal{E}(r,u)|\,\mathrm{d}r \\ &\le \int_s^t c_R(r)\,\mathrm{d}r,\end{aligned}$$

where $v \in X$ is a suitable unit vector at which the dual norm is attained, and R can be chosen for instance equal to $\|u\|+1$. This last property will be used in Proposition 4.5 in order to apply a chain-rule formula for functions of bounded variation (see [5]).

Remark 2.2. All applications that are presented in [10, Section 7], basically regarding masses connected with springs, are described by adding together elastic quadratic energies of the form

$$\mathcal{E}(t,u) = \frac{k}{2}(u - \ell(t))^2,$$

with $k>0$ and $\ell \in W^{1,1}(0,T;\mathbb{R})$. This specific form simply is the second order expansion of the real elastic potential energy of the springs

$$\mathcal{E}(t,u) = k(1 - \cos(u - \ell(t))),$$

which is of course nonconvex. It is however straightforward to check that it satisfies (E1)–(E4) (and actually also (E3') and (E5) below), and thus it can be included within the framework of this paper.

In Section 5, where we deal with the discrete approximation of IBV and IVV solutions, in addition to the previous assumptions, we need to require the following assumptions:

(E3') The energy $\mathcal{E}$ fulfils (E3) with the particular choice $a(y) = y + a_1$ for some $a_1 \ge 0$.

(E5) $\mathcal{E}(t,\cdot)$ is Λ-convex for every $t \in [0,T]$; i.e., there exists $\Lambda > 0$ such that for every $t \in [0,T]$, $u_1, u_2 \in X$ and every $\theta \in (0,1)$ it holds

$$\mathcal{E}(t,(1-\theta)u_1 + \theta u_2) \le (1-\theta)\mathcal{E}(t,u_1) + \theta\mathcal{E}(t,u_2) + \frac{\Lambda}{2}\theta(1-\theta)\|u_1 - u_2\|^2_{\mathbb{I}}$$

for some symmetric positive-definite linear operator $\mathbb{I}\colon X \to X^*$.

We notice that by (E3') and Gronwall's lemma, we can infer

$$\mathcal{E}(t,u) + a_1 \leq (\mathcal{E}(s,u) + a_1) e^{\int_s^t b(r)\,\mathrm{d}r} \quad \text{for every } 0 \leq s \leq t \leq T,$$

whence

$$|\partial_t \mathcal{E}(t,u)| \leq (\mathcal{E}(s,u) + a_1) b(t) e^{\int_s^t b(r)\,\mathrm{d}r} \quad \text{for every } 0 \leq s \leq t \leq T. \tag{2.8}$$

It is also easy to check that (E5) implies

$$\langle D_x \mathcal{E}(t,u_1), u_2 - u_1 \rangle \leq \mathcal{E}(t,u_2) - \mathcal{E}(t,u_1) + \frac{\Lambda}{2}\|u_1 - u_2\|_{\mathbb{I}}^2 \quad \text{for every } t \in [0,T],\ u_1, u_2 \in X. \tag{2.9}$$

Indeed, by using the mean value theorem, for some $\zeta \in [0,1]$ we have

$$\begin{aligned} \theta\Big[\mathcal{E}(t,u_2) - \mathcal{E}(t,u_1) + \frac{\Lambda}{2}(1-\theta)\|u_1 - u_2\|_{\mathbb{I}}^2\Big] &\geq \mathcal{E}(t,(1-\theta)u_1 + \theta u_2) - \mathcal{E}(t,u_1) \\ &= \theta\langle D_x \mathcal{E}(t, u_1 + \zeta\theta(u_1 - u_2)), u_2 - u_1\rangle, \end{aligned}$$

whence (2.9) follows up to simplifying θ in both sides and then letting $\theta \to 0$.

We finally point out that an energy $\mathcal{E}$ as in (2.7) always complies with (E3'), while it fulfils (E5) if in addition $\mathcal{U}$ is Λ-convex.

2.2 Functions of bounded $\mathcal{R}$-variation

We recall here a suitable generalization of functions of bounded variation useful to deal with functions satisfying (R1).

Definition 2.3. Let a function $f : [a,b] \to X$ be given. Then we define the *pointwise* $\mathcal{R}$*-variation* of f in $[s,t]$, with $a \leq s < t \leq b$, by

$$V_{\mathcal{R}}(f; s, t) := \sup\Big\{\sum_{k=1}^{n} \mathcal{R}(f(t_k) - f(t_{k-1})) \mid s = t_0 < t_1 < \dots < t_{n-1} < t_n = t\Big\}.$$

We also set $V_{\mathcal{R}}(f; t, t) := 0$ for every $t \in [a,b]$.

We say that f is a *function of bounded* $\mathcal{R}$*-variation* in $[a,b]$, and we write $f \in \mathrm{BV}_{\mathcal{R}}([a,b];X)$, if its $\mathcal{R}$-variation in $[a,b]$ is finite; i.e., $V_{\mathcal{R}}(f; a, b) < +\infty$.

Notice that, by virtue of (2.4), we have $f \in \mathrm{BV}_{\mathcal{R}}([a,b];X)$ if and only if $f \in \mathrm{BV}([a,b];X)$ in the classical sense. In particular, $f \in \mathrm{BV}_{\mathcal{R}}([a,b];X)$ is regulated, i.e., it admits left and right limits at every $t \in [a,b]$:

$$f^+(t) := \lim_{t_j \searrow t} f(t_j) \quad \text{and} \quad f^-(t) := \lim_{t_j \nearrow t} f(t_j),$$

with the convention $f^-(a) := f(a)$ and $f^+(b) := f(b)$. Moreover, its pointwise jump set J_f is at most countable.

It is well known (see, e.g., [3]) that f can be uniquely decomposed as follows:

$$f = f_{\mathcal{L}} + f_{\mathrm{Ca}} + f_J, \tag{2.10}$$

with $f_{\mathcal{L}}$ being an absolutely continuous function, f_{Ca} a continuous Cantor-type function, and f_J a jump function. If we denote by f' the distributional derivative of $f \in \mathrm{BV}_{\mathcal{R}}([a,b];X)$, and recall that f' is a Radon vector measure with finite total variation $|f'|$, it follows that f' can be decomposed into the sum of the three mutually singular measures

$$f' = f'_{\mathcal{L}} + f'_{\mathrm{Ca}} + f'_{\mathrm{J}}, \quad f'_{\mathcal{L}} = \dot{f}\mathcal{L}^1, \quad f'_{\mathrm{co}} := f'_{\mathcal{L}} + f'_{\mathrm{Ca}}. \tag{2.11}$$

In (2.11), $f'_{\mathcal{L}}$ is the absolutely continuous part with respect to the Lebesgue measure $\mathcal{L}^1$, whose Lebesgue density $\dot{f}$ is the usual pointwise ($\mathcal{L}^1$-a.e. defined) derivative, f'_{J} is the jump part concentrated on the essential jump set of f, i.e.

$$J_f^{\mathrm{e}} := \{t \in [a,b] \mid f^+(t) \neq f^-(t)\} \subseteq J_f,$$

and f'_{Ca} is the Cantor part such that $f'_{\mathrm{Ca}}(\{t\}) = 0$ for every $t \in [0, T]$. The measure f'_{co} is the diffuse part of the measure, and does not charge J_f. The functions $f_{\mathcal{L}}, f_{\mathrm{Ca}}$ and f_J in (2.10) are exactly the distributional primitives of the measures $f'_{\mathcal{L}}, f'_{\mathrm{Ca}}$ and f'_J in (2.11). We will use the notation f_{co} to denote the continuous part of f, that is, $f_{\mathrm{co}} = f_{\mathcal{L}} + f_{\mathrm{Ca}}$.

We also remark that, for $a \le s \le t \le b$, the function $V_{\mathcal{R}}(f; s, t)$ is monotone in both entries, and hence it makes sense to consider the limits $V_{\mathcal{R}}(f; s-, t+)$. The following formula (see, for instance, [19, Section 2]) relates $V_{\mathcal{R}}(f; s-, t+)$ with the distributional derivative of f, up to the jump part which is depending on the pointwise behavior of f. Setting $\lambda = \mathcal{L}^1 + |f'_{\mathrm{Ca}}|$, it namely holds

$$V_{\mathcal{R}}(f; s-, t+) = \int_s^t \mathcal{R}\Big(\frac{\mathrm{d}f'_{\mathrm{co}}}{\mathrm{d}\lambda}(r)\Big)\,\mathrm{d}\lambda(r) + \sum_{r \in J_f \cap [s,t]} \big(\mathcal{R}(f^+(r) - f(r)) + \mathcal{R}(f(r) - f^-(r))\big), \tag{2.12}$$

where $\frac{\mathrm{d}f'_{\mathrm{co}}}{\mathrm{d}\lambda}$ is the Radon-Nikodym derivative. Observe that, by the positive one-homogeneity of $\mathcal{R}$, actually any measure ν such that $f'_{\mathrm{co}} \ll \nu$ can replace λ in the integral term on the right-hand side.

It follows from (2.12) that the continuous part of the $\mathcal{R}$-variation of f agrees with the $\mathcal{R}$-variation of f_{co} and satisfies

$$V_{\mathcal{R}}(f_{\mathrm{co}}; s, t) = \int_s^t \mathcal{R}\Big(\frac{\mathrm{d}f'_{\mathrm{co}}}{\mathrm{d}\lambda}(r)\Big)\,\mathrm{d}\lambda(r). \tag{2.13}$$

We finally notice that, by dropping the pointwise value of f at jump points (by the subadditivity of $\mathcal{R}$), and only considering the essential jumps, we are led to the so-called essential $\mathcal{R}$-variation

$$\mathcal{R}(f')([s,t]) := V_{\mathcal{R}}(f_{\mathrm{co}}; s, t) + \sum_{r \in J_f^{\mathrm{e}} \cap [s,t]} \mathcal{R}(f^+(r) - f^-(r)) \le V_{\mathcal{R}}(f; s-, t+). \tag{2.14}$$

The term $\mathcal{R}(f')$ actually defines a Radon measure (see [11]), which generalizes the concept of total variation $|f'|$ (corresponding to the particular choice $\mathcal{R}(\cdot) = \|\cdot\|$).

3 Inertial balanced viscosity and inertial virtual viscosity solutions

In this section, we rigorously introduce the notions of inertial balanced viscosity and inertial virtual viscosity solution. We also state our main result, see Theorem 3.10, postponing its proof to the forthcoming sections.

As in the vanishing-viscosity approach of [19], the starting point consists in an alternative formulation of the dynamic problem (1.5) based on the so–called De Giorgi's energy-dissipation principle (see the pioneering work [9] and other applications in [16, 28]). Roughly speaking, the idea is to keep together all dissipative terms appearing in the dynamic model, namely viscosity and rate-independent dissipation; we are thus led to consider the functional

$$\mathcal{R}_\varepsilon(v) := \mathcal{R}(v) + \frac{\varepsilon}{2}|v|^2_{\mathbb{V}}. \tag{3.1}$$

It is then easy to check (see, e.g., [19, p. 47]) that the subdifferential of $\mathcal{R}_\varepsilon$ is explicitly given by

$$\partial\mathcal{R}_\varepsilon(v) = \partial\mathcal{R}(v) + \varepsilon\mathbb{V}v,$$

so the dynamic problem (1.5) can be rewritten as

$$\partial\mathcal{R}_\varepsilon(\dot u^\varepsilon(t)) \ni -\varepsilon^2\mathbb{M}\ddot u^\varepsilon(t) - D_x\mathcal{E}(t, u^\varepsilon(t)) =: w^\varepsilon(t) \quad \text{for a.e. } t \in [0, T]. \tag{3.2}$$

By using (2.1), and exploiting the classical chain-rule formula for $\mathcal{E}$, one obtains that the dynamic problem (3.2) is actually equivalent to the augmented energy balance

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\dot u^\varepsilon(t)\|^2_{\mathbb{M}} + \mathcal{E}(t, u^\varepsilon(t)) + \int_s^t \mathcal{R}_\varepsilon(\dot u^\varepsilon(r)) + \mathcal{R}^*_\varepsilon(w^\varepsilon(r))\,\mathrm{d}r \\ = \frac{\varepsilon^2}{2}\|\dot u^\varepsilon(s)\|^2_{\mathbb{M}} + \mathcal{E}(s, u^\varepsilon(s)) + \int_s^t \partial_t\mathcal{E}(r, u^\varepsilon(r))\,\mathrm{d}r \quad \text{for every } 0 \le s \le t \le T.\end{aligned} \tag{3.3}$$

We point out that in our case of additive viscosity (3.1), the Fenchel conjugate $\mathcal{R}_\varepsilon^*$ can be explicitly computed by means of the inf-sup convolution formula (see, e.g., [27, Section 12]) and turns out to be

$$\mathcal{R}_\varepsilon^*(w) = \begin{cases} \dfrac{1}{2\varepsilon} \inf\limits_{\substack{z\in K^* \\ w-z\in(\ker \mathbb{V})^\perp}} \langle w-z, \mathbb{V}'(w-z)\rangle & \text{if } w \in K^* + (\ker \mathbb{V})^\perp, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

where

$$(\ker \mathbb{V})^\perp = \{w \in X^* \mid \langle w, v\rangle = 0 \text{ for every } v \in \ker \mathbb{V}\}$$

is the *annihilator* of $\ker \mathbb{V}$ and $\mathbb{V}' : (\ker \mathbb{V})^\perp \to X$ is the inverse of the operator $\mathbb{V}$ restricted to (the identification of) $(\ker \mathbb{V})^\perp$ (in X).

In particular, in the two extreme situations $\mathbb{V} = 0$ and being $\mathbb{V}$ positive-definite, we get respectively

$$\mathcal{R}_\varepsilon^*(w) = \mathcal{R}^*(w) = \chi_{K^*}(w), \quad \mathcal{R}_\varepsilon^*(w) = \frac{1}{2\varepsilon}\operatorname{dist}^2_{\mathbb{V}^{-1}}(w, K^*),$$

where

$$\operatorname{dist}_{\mathbb{V}^{-1}}(w, K^*) := \inf_{z\in K^*} \|w-z\|_{\mathbb{V}^{-1}},$$

denotes the distance from K^*, measured with respect to the norm $\|\cdot\|_{\mathbb{V}^{-1}}$.

3.1 Contact potentials

The energy balance (3.3) naturally leads to the introduction of a so-called (viscous) contact potential associated to $\mathbb{V}$. In the spirit of [19] and taking into account (3.4), we thus give the following definition.

Definition 3.1. The (viscous) *contact potential* related to the viscosity operator $\mathbb{V}$ is the map

$$p_\mathbb{V} : X \times X^* \to [0, +\infty]$$

defined by

$$p_\mathbb{V}(v, w) := \inf_{\varepsilon>0}(\mathcal{R}_\varepsilon(v) + \mathcal{R}_\varepsilon^*(w)) = \begin{cases} \mathcal{R}(v) + |v|_\mathbb{V} \inf\limits_{\substack{z\in K^* \\ w-z\in(\ker \mathbb{V})^\perp}} |w-z|_{\mathbb{V}'} & \text{if } w \in K^* + (\ker \mathbb{V})^\perp, \\ +\infty & \text{otherwise.} \end{cases}$$

In the two extreme situations $\mathbb{V} = 0$ and $\mathbb{V}$ being positive-definite, we get respectively

$$p_0(v, w) = \mathcal{R}(v) + \chi_{K^*}(w), \quad p_\mathbb{V}(v, w) = \mathcal{R}(v) + \|v\|_\mathbb{V} \operatorname{dist}_{\mathbb{V}^{-1}}(w, K^*). \tag{3.5}$$

Therefore, in the positive-definite case we retrieve the vanishing-viscosity contact potential defined in [19].

By the explicit formula, we easily infer the following properties for the contact potential $p_\mathbb{V}$:

(i) $p_\mathbb{V}(\cdot, w)$ is positively one-homogeneous and convex for every $w \in X^*$.
(ii) $p_\mathbb{V}(v, \cdot)$ is convex for every $v \in X$.
(iii) $p_\mathbb{V}(v, w) \geq \max\{\mathcal{R}(v), \langle w, v\rangle\}$ for every $v \in X$ and $w \in X^*$.
(iv) $p_\mathbb{V}(0, w) = \chi_{K^*+(\ker \mathbb{V})^\perp}(w)$ and $p_\mathbb{V}(v, 0) = \mathcal{R}(v)$.

Furthermore, we also observe the following property:

(v) $p_\mathbb{V}(\cdot, w)$ is symmetric for every $w \in X^*$ if and only if $\mathcal{R}$ is symmetric.

At this stage, a warning is mandatory: we point out that our potential $p_\mathbb{V}$ in general can take the value $+\infty$, due to the semidefiniteness of the viscosity operator $\mathbb{V}$. This feature does not appear in [19], where indeed a full viscosity is always present and the contact potential is continuous and finite. This difference will create serious issues in the forthcoming analysis, leading to the original notion of inertial virtual viscosity; we are thus led to couple $p_\mathbb{V}$ with a "regularized" contact potential p, as follows.

Definition 3.2. We say that a *continuous* map $p\colon X\times X^*\to[0,+\infty)$ is a *regularized contact potential* with respect to $p_{\mathbb{V}}$, and we write $p\in\mathrm{RCP}_{\mathbb{V}}$, if the following conditions hold:
(i) $p(\cdot\,,w)$ is positively one-homogeneous for every $w\in X^*$.
(ii) $p(v,\cdot\,)$ is convex for every $v\in X$.
(iii) $\max\{\mathcal{R}(v),\langle w,v\rangle\}\le p(v,w)\le p_{\mathbb{V}}(v,w)$ for every $v\in X$ and $w\in X^*$.
(iv) There exists a positive constant $L>0$ such that

$$|p(v,w_1)-p(v,w_2)|\le L\|v\|\|w_1-w_2\|_*\quad\text{for every } v\in X \text{ and } w_1,w_2\in X^*.$$

Remark 3.3. In the case that $\mathbb{V}$ is positive-definite, the contact potential $p_{\mathbb{V}}$ itself belongs to $\mathrm{RCP}_{\mathbb{V}}$. This easily follows by the explicit form (3.5). Observe that in this case $p_{\mathbb{V}}$ takes only finite values.

Notice that in the above definition we are not requiring the convexity of p with respect to the variable v. Instead, the convexity in the second variable, i.e. property (ii), will be crucial in Proposition 3.11.

We also point out that the main property of regularized contact potentials, missing in general for $p_{\mathbb{V}}$, is the weighted Lipschitzianity (iv) with respect to the second variable: this will be heavily used in Proposition 4.6.

We finally observe that by (iii) and (iv), any $p\in\mathrm{RCP}_{\mathbb{V}}$ satisfies

$$p(v,w)\le|p(v,w)-p(v,0)|+p(v,0)\le L\|v\|\|w\|_*+p_{\mathbb{V}}(v,0)=L\|v\|\|w\|_*+\mathcal{R}(v)\le(\alpha^*+L\|w\|_*)\|v\|,\tag{3.6}$$

where we exploited (2.4). In particular, it holds

$$p(0,w)=0\quad\text{for every } w\in X^*.$$

3.2 Parametrized families of regularized contact potentials

With the following result, we show that a whole family of regularized contact potentials can be constructed by means of a suitable version of the Yosida transform.

For every $\lambda\ge1$ and every symmetric positive-definite linear operator $\mathbb{U}\colon X\to X^*$, we define the function $p_{\mathbb{V}}^{\lambda,\mathbb{U}}\colon X\times X^*\to[0,+\infty)$ by

$$p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v,w):=\inf_{\eta\in X^*}\{p_{\mathbb{V}}(v,\eta)+\lambda\|v\|_{\mathbb{U}}\|w-\eta\|_{\mathbb{U}^{-1}}\},\qquad v\in X,\ w\in X^*.\tag{3.7}$$

Proposition 3.4. *Let $\lambda\ge1$ and let $\mathbb{U}$ be a symmetric positive-definite linear operator. Then $p_{\mathbb{V}}^{\lambda,\mathbb{U}}\in\mathrm{RCP}_{\mathbb{V}}$. Furthermore, for every $v\ne0$, one has*

$$p_{\mathbb{V}}(v,w)=\sup_{\lambda\ge1}p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v,w)=\lim_{\lambda\to+\infty}p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v,w).\tag{3.8}$$

If in addition $\mathcal{R}$ is symmetric, then for every $w\in X^*$ the function $p_{\mathbb{V}}^{\lambda,\mathbb{U}}(\cdot\,,w)$ is symmetric as well.

Proof. We first notice that $p_{\mathbb{V}}^{\lambda,\mathbb{U}}$ has nonnegative finite values since $p_{\mathbb{V}}$ is not identically $+\infty$ and it is nonnegative. Moreover, (3.8) is a standard property of the Yosida transform (notice that $\lambda\|v\|_{\mathbb{U}}>0$ if $v\ne0$). Also, if $\mathcal{R}$ is symmetric, the symmetry of $p_{\mathbb{V}}^{\lambda,\mathbb{U}}(\cdot\,,w)$ is a straightforward byproduct of (3.7) since in this case $p_{\mathbb{V}}(\cdot\,,w)$ is symmetric.

Now, we have to confirm properties (i)–(iv) of Definition 3.2. Property (i) follows by the one-homogeneity of $p_{\mathbb{V}}(\cdot\,,\eta)$ and of the norm.

Property (ii) follows since the Yosida transform of a convex function is convex, and $p_{\mathbb{V}}(v,\cdot\,)$ is convex.

The right inequality in (iii) is obtained by choosing $\eta=w$ in the definition of $p_{\mathbb{V}}^{\lambda,\mathbb{U}}$, while the left one follows from the fact that $p_{\mathbb{V}}(v,\cdot\,)\ge\mathcal{R}(v)$ combined with the simple inequality

$$\langle w,v\rangle=\langle\eta,v\rangle+\langle w-\eta,v\rangle\le p_{\mathbb{V}}(v,\eta)+\lambda\|v\|_{\mathbb{U}}\|w-\eta\|_{\mathbb{U}^{-1}}.$$

Property (iv) is again a straightforward consequence of the Yosida transform: one can choose

$$L=\lambda\sqrt{\|\mathbb{U}\|_{\mathrm{op}}\|\mathbb{U}^{-1}\|_{\mathrm{op}}}.$$

We are only left to prove that $p_{\mathbb{V}}^{\lambda,\mathbb{U}}$ is continuous. We first observe that, thanks to (iv), it is enough to prove that $p_{\mathbb{V}}^{\lambda,\mathbb{U}}(\cdot, w)$ is continuous for every fixed $w \in X^*$. The continuity in $v = 0$ follows easily by (3.6); if $v \neq 0$, we need to do more work. We make the following claims.

Claim 1. There exists a positive constant $C_1 > 0$ such that

$$p_{\mathbb{V}}(v_1, w) \leq p_{\mathbb{V}}(v_2, w) + C_1(1 + \|w\|_*)\|v_1 - v_2\| \quad \text{for every } v_1, v_2 \in X \text{ and } w \in X^*. \tag{3.9}$$

Claim 2. If $v \neq 0$, then there exists a positive constant $C_2 > 0$ such that

$$p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v, w) = \inf_{\substack{\eta \in X^* \\ \|\eta\|_* \leq C_2 \rho(\|v\|, \|w\|_*)}} \{p_{\mathbb{V}}(v, \eta) + \lambda \|v\|_{\mathbb{U}} \|w - \eta\|_{\mathbb{U}^{-1}}\},$$

where $\rho(\|v\|, \|w\|_*) := 1 + \|w\|_* + 1/\|v\|$.

Claim 3. There exists a positive constant $C_3 > 0$ such that

$$|p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v_1, w) - p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v_2, w)| \leq C_3 \max\{\rho(\|v_1\|, \|w\|_*), \rho(\|v_2\|, \|w\|_*)\}\|v_1 - v_2\|$$

for every $v_1, v_2 \in X \setminus \{0\}$ and $w \in X^*$.

From Claim 3, we easily deduce the continuity of $p_{\mathbb{V}}^{\lambda,\mathbb{U}}(\cdot, w)$ in $v \neq 0$, and thus we only need to prove its validity.

We start with Claim 1, and we observe that it is enough to prove it for $w \in K^* + (\ker \mathbb{V})^\perp$. By exploiting the subadditivity of $\mathcal{R}$ together with (2.4) and (2.6), we easily obtain

$$\begin{aligned}
p_{\mathbb{V}}(v_1, w) &= \mathcal{R}(v_1) + |v_1|_{\mathbb{V}} \inf_{\substack{z \in K^* \\ w - z \in (\ker \mathbb{V})^\perp}} |w - z|_{\mathbb{V}'} \\
&\leq \mathcal{R}(v_2) + \alpha^* \|v_1 - v_2\| + \big(|v_2|_{\mathbb{V}} + \sqrt{V}\|v_1 - v_2\|\big) \inf_{\substack{z \in K^* \\ w - z \in (\ker \mathbb{V})^\perp}} |w - z|_{\mathbb{V}'} \\
&\leq \mathcal{R}(v_2) + |v_2|_{\mathbb{V}} \inf_{\substack{z \in K^* \\ w - z \in (\ker \mathbb{V})^\perp}} |w - z|_{\mathbb{V}'} + \big(\alpha^* + \sqrt{VV'}\|w\|_* + \sqrt{VV'}\alpha^*\big)\|v_1 - v_2\| \\
&= p_{\mathbb{V}}(v_2, w) + \big(\alpha^* + \sqrt{VV'}\|w\|_* + \sqrt{VV'}\alpha^*\big)\|v_1 - v_2\|,
\end{aligned}$$

and Claim 1 is proved.

To prove Claim 2, it is enough to show that an infimizing sequence $\{\eta_j\}_{j \in \mathbb{N}}$ for $p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v, w)$ is uniformly bounded by $\rho(\|v\|, \|w\|_*)$, up to a multiplicative constant. Being an infimizing sequence, η_j satisfies

$$1 + p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v, w) \geq p_{\mathbb{V}}(v, \eta_j) + \lambda \|v\|_{\mathbb{U}} \|w - \eta_j\|_{\mathbb{U}^{-1}} \geq \|v\|_{\mathbb{U}} \|w - \eta_j\|_{\mathbb{U}^{-1}}.$$

Thus, by using (3.6), we infer

$$\begin{aligned}
\|\eta_j\|_* &\leq \|w\|_* + C\|w - \eta_j\|_{\mathbb{U}^{-1}} \\
&\leq \|w\|_* + C\frac{1 + p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v, w)}{\|v\|} \\
&\leq \|w\|_* + C\frac{1 + (\alpha^* + L\|w\|_*)\|v\|}{\|v\|} \\
&\leq C\rho(\|v\|, \|w\|_*).
\end{aligned}$$

We now need to prove Claim 3. To this end, we take $\eta \in X^*$ such that $\|\eta\|_* \leq C_2\rho(\|v_2\|, \|w\|_*)$ and, exploiting Claim 1, we estimate

$$\begin{aligned}
p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v_1, w) &\leq p_{\mathbb{V}}(v_1, \eta) + \lambda \|v_1\|_{\mathbb{U}} \|w - \eta\|_{\mathbb{U}^{-1}} \\
&\leq p_{\mathbb{V}}(v_2, \eta) + C_1(1 + \|w\|_*)\|v_1 - v_2\| + \lambda \|v_2\|_{\mathbb{U}} \|w - \eta\|_{\mathbb{U}^{-1}} + \lambda \|w - \eta\|_{\mathbb{U}^{-1}} \|v_1 - v_2\|_{\mathbb{U}} \\
&\leq p_{\mathbb{V}}(v_2, \eta) + \lambda \|v_2\|_{\mathbb{U}} \|w - \eta\|_{\mathbb{U}^{-1}} + C(1 + \|w\|_* + \|\eta\|_*)\|v_1 - v_2\| \\
&\leq p_{\mathbb{V}}(v_2, \eta) + \lambda \|v_2\|_{\mathbb{U}} \|w - \eta\|_{\mathbb{U}^{-1}} + C\rho(\|v_2\|, \|w\|_*)\|v_1 - v_2\|.
\end{aligned}$$

By using Claim 2, from the above inequality we deduce

$$p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v_1, w) \le p_{\mathbb{V}}^{\lambda,\mathbb{U}}(v_2, w) + C\rho(\|v_2\|, \|w\|_*)\|v_1 - v_2\|.$$

By interchanging the roles of v_1 and v_2, we thus complete the proof of Claim 3, and we conclude the proof of the proposition. □

We notice that in the case that $\mathbb{V}$ is positive-definite, the contact potential $p_{\mathbb{V}}$ coincides with its Yosida approximation if we choose $\mathbb{U} = \mathbb{V}$:

$$p_{\mathbb{V}}^{\lambda,\mathbb{V}}(v, w) = p_{\mathbb{V}}(v, w) \quad \text{for every } \lambda \ge 1,\ v \in X \text{ and } w \in X^*.$$

Indeed, by means of the explicit formula (3.5), it holds

$$\begin{aligned} p_{\mathbb{V}}(v, w) &\ge p_{\mathbb{V}}^{\lambda,\mathbb{V}}(v, w) \\ &\ge p_{\mathbb{V}}^{1,\mathbb{V}}(v, w) \\ &= \inf_{\eta \in X^*} \{p_{\mathbb{V}}(v, \eta) + \|v\|_{\mathbb{V}}\|w - \eta\|_{\mathbb{V}^{-1}}\} \\ &= \inf_{\eta \in X^*} \{\mathcal{R}(v) + \|v\|_{\mathbb{V}}(\operatorname{dist}_{\mathbb{V}^{-1}}(\eta, K^*) + \|w - \eta\|_{\mathbb{V}^{-1}})\} \\ &= \mathcal{R}(v) + \|v\|_{\mathbb{V}} \inf_{\eta \in X^*} \{\operatorname{dist}_{\mathbb{V}^{-1}}(\eta, K^*) + \|w - \eta\|_{\mathbb{V}^{-1}}\} \\ &\ge \mathcal{R}(v) + \|v\|_{\mathbb{V}} \operatorname{dist}_{\mathbb{V}^{-1}}(w, K^*) \\ &= p_{\mathbb{V}}(v, w), \end{aligned}$$

where the last inequality is a simple byproduct of the triangle inequality. This fact corroborates Remark 3.3.

In the opposite situation $\mathbb{V} = 0$, it is not difficult to see that the Yosida transform takes a more explicit form:

$$p_0^{\lambda,\mathbb{U}}(v, w) = \mathcal{R}(v) + \lambda\|v\|_{\mathbb{U}} \operatorname{dist}_{\mathbb{U}^{-1}}(w, K^*). \tag{3.10}$$

Compare this last formula with (3.5), the case of $\mathbb{V}$ being positive-definite.

3.3 The inertial energy-dissipation cost

Once the notion of contact potential has been developed, we are in a position to rigorously introduce the cost function which will govern the jump transient of IBV and IVV solutions. The crucial difference with respect to the vanishing-viscosity cost of BV solutions [19] is its rate-dependent nature, caused by the term $\mathbb{M}\ddot{v}$ inside the integral which is reminiscent of the original inertial effects.

Definition 3.5. For every $t \in [0, T]$ and $u_1, u_2 \in X$, we define the *inertial energy-dissipation cost* related to $p \in \mathrm{RCP}_{\mathbb{V}} \cup \{p_{\mathbb{V}}\}$ by

$$c_t^{\mathbb{M},p}(u_1, u_2) := \inf \left\{ \int_{-N}^{N} p(\dot{v}(r), -\mathbb{M}\ddot{v}(r) - D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r \mid N \in \mathbb{N},\ v \in V_{u_1,u_2}^{\mathbb{M},N} \right\}, \tag{3.11}$$

where

$$V_{u_1,u_2}^{\mathbb{M},N} := \{v \in W^{2,\infty}(-N, N; X) \mid v(-N) = u_1,\ v(N) = u_2,\ \mathbb{M}\dot{v}(\pm N) = 0,\ \operatorname*{ess\,sup}_{r \in [-N,N]} \|\mathbb{M}\ddot{v}(r)\|_* \le \overline{C}\}$$

denotes the class of the *admissible curves* and $\overline{C}$ is the constant of Proposition 4.2 and Corollary 5.4 (depending only on the data of the problem).

We also define the inertial cost directly related to the viscosity operator $\mathbb{V}$ by taking the supremum among the costs over all regularized contact potentials:

$$c_t^{\mathbb{M},\mathbb{V}}(u_1, u_2) := \sup_{p \in \mathrm{RCP}_{\mathbb{V}}} c_t^{\mathbb{M},p}(u_1, u_2).$$

Remark 3.6. We point out that in the case that $\mathbb{V}$ is positive-definite, Remark 3.3 yields

$$c_t^{\mathbb{M},\mathbb{V}}(u_1, u_2) = c_t^{\mathbb{M},p_\mathbb{V}}(u_1, u_2). \tag{3.12}$$

This is consistent with the vanishing-viscosity analysis performed in [19], in which the cost is (formally) equivalent to (3.12) by taking $\mathbb{M} \equiv 0$ (see (1.3)).

A relevant feature of the inertial cost is that the value $c_t^{\mathbb{M},p}(u_1, u_2)$ provides an upper bound for the energy gap $\mathcal{E}(t, u_1) - \mathcal{E}(t, u_2)$ for every $p \in \mathrm{RCP}_\mathbb{V}$, as shown with the following proposition.

Proposition 3.7. *For every $t \in [0, T]$ and $u_1, u_2 \in X$, we have*

$$\mathcal{E}(t, u_1) - \mathcal{E}(t, u_2) \leq \inf_{p\in\mathrm{RCP}_\mathbb{V}} c_t^{\mathbb{M},p}(u_1, u_2).$$

Proof. Fix $p \in \mathrm{RCP}_\mathbb{V}$ and let $N \in \mathbb{N}$ and $v \in V_{u_1,u_2}^{\mathbb{M},N}$. From the fundamental theorem of calculus and property (iii) of regularized contact potentials, we deduce

$$\begin{aligned}\mathcal{E}(t, u_1) - \mathcal{E}(t, u_2) &= \mathcal{E}(t, v(-N)) - \mathcal{E}(t, v(N)) + \frac{1}{2}\|\dot v(-N)\|_\mathbb{M}^2 - \frac{1}{2}\|\dot v(N)\|_\mathbb{M}^2 \\ &= \int_{-N}^{N} \langle -\mathbb{M}\ddot v(r) - D_x\mathcal{E}(t, v(r)), \dot v(r)\rangle \,\mathrm{d}r \\ &\leq \int_{-N}^{N} p(\dot v(r), -\mathbb{M}\ddot v(r) - D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r,\end{aligned}$$

and the assertion follows by the arbitrariness of v, N and p. □

With the notion of inertial energy-dissipation cost at hand, we can give the definition of inertial virtual viscosity and inertial balanced viscosity solutions.

Definition 3.8. We say that a function $u \in \mathrm{BV}_\mathcal{R}([0, T]; X)$ is an *inertial virtual viscosity* (IVV) solution to the rate-independent system (1.1), related to $\mathbb{M}$ and $\mathbb{V}$, if it complies both with the local stability condition

$$-D_x\mathcal{E}(t, u(t)) \in K^* \quad \text{for every } t \in [0, T] \setminus J_u, \tag{3.13}$$

and the energy balance

$$\mathcal{E}(t, u^+(t)) + V_\mathcal{R}(u_{\mathrm{co}}; s, t) + \sum_{r\in J_u^e\cap[s,t]} c_r^{\mathbb{M},\mathbb{V}}(u^-(r), u^+(r)) = \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r \tag{3.14}$$

for every $0 \leq s \leq t \leq T$.

If $\mathbb{V}$ is positive-definite, in which case (3.14) is satisfied with $c^{\mathbb{M},p_\mathbb{V}}$ in place of $c^{\mathbb{M},\mathbb{V}}$ (see (3.12)), we say that u is an *inertial balanced viscosity* (IBV) solution.

Remark 3.9. By Proposition 3.7, we deduce that for any IVV solution there holds

$$c_t^{\mathbb{M},\mathbb{V}}(u^-(t), u^+(t)) = \sup_{p\in\mathrm{RCP}_\mathbb{V}} c_t^{\mathbb{M},p}(u^-(t), u^+(t)) = \inf_{p\in\mathrm{RCP}_\mathbb{V}} c_t^{\mathbb{M},p}(u^-(t), u^+(t)) \quad \text{for every } t \in [0, T],$$

and thus in (3.14) we can actually replace $c^{\mathbb{M},\mathbb{V}}$ with $c^{\mathbb{M},p}$ for an arbitrary $p \in \mathrm{RCP}_\mathbb{V}$.

It is however not clear whether we can replace it with the cost related to the contact potential $p_\mathbb{V}$ itself (i.e. $c^{\mathbb{M},p_\mathbb{V}}$) in the case of a generic semidefinite operator $\mathbb{V}$, despite Proposition 3.4 shows that $p_\mathbb{V}$ can always be approximated (except for $v = 0$) by suitable regularized contact potentials.

The term “virtual” in the definition of IVV solutions is motivated by the presence of the Yosida-type potentials (3.7) inside the set of regularized contact potentials $\mathrm{RCP}_\mathbb{V}$. They are indeed constructed by means of a symmetric positive-definite linear operator $\mathbb{U}$, which plays the role of a virtual viscosity, since a priori it is not present in the problem under study (see in particular (3.10)). The term “balanced” for IBV solutions is instead inherited from [19].

The main result of this paper can now be stated as follows.

Theorem 3.10. *Let $\mathbb{M}$ and $\mathbb{V}$ satisfy* (2.2) *and* (2.3) *and assume (E1)–(E4) and (R1). Let $u_0^\varepsilon \to u_0$, $\varepsilon u_1^\varepsilon \to 0$. Then the following two assertions hold true:*

(i) *For every sequence $\varepsilon_j \to 0$, there exists a subsequence (not relabelled) along which the sequence of dynamic solutions u^{ε_j} to* (1.5) *pointwise converges to an inertial virtual viscosity solution of the rate-independent system* (1.1).

(ii) *Assume in addition (E3') and (E5). Then, for every sequence $(\tau_j, \varepsilon_j) \to (0, 0)$ satisfying*

$$\sup_{j\in\mathbb{N}} \frac{\tau_j}{\varepsilon_j^2} < +\infty,$$

there exists a subsequence (not relabelled) along which the sequence of piecewise affine interpolants $\widehat{u}_{\tau_j,\varepsilon_j}$, defined in (5.14) *and coming from the minimizing movements scheme* (5.1), *pointwise converges to an inertial virtual viscosity solution of the rate-independent system* (1.1).

In both cases, the limit function is an inertial balanced viscosity solution if $\mathbb{V}$ is positive-definite.

The proof of part (i) is carried out in Section 4, while part (ii) is proved in Section 5. The rest of this section is devoted to the main properties of the inertial cost.

Proposition 3.11. *Fix $t \in [0, T]$, $u_1, u_2 \in X$ and $p \in \mathrm{RCP}_{\mathbb{V}}$. Then the inertial energy-dissipation cost related to p can be computed as follows:*

$$c_t^{\mathbb{M},p}(u_1, u_2) = \lim_{N\to+\infty} \min_{v\in V^{\mathbb{M},N}_{u_1,u_2}} \int_{-N}^{N} p(\dot v(r), -\mathbb{M}\ddot v(r) - D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r. \tag{3.15}$$

Proof. For a fixed $N \in \mathbb{N}$, let $\{v_j\}_{j\in\mathbb{N}} \subseteq V^{\mathbb{M},N}_{u_1,u_2}$ be an infimizing sequence for

$$\inf_{v\in V^{\mathbb{M},N}_{u_1,u_2}} \int_{-N}^{N} p(\dot v(r), -\mathbb{M}\ddot v(r) - D_x\mathcal{E}(t, v(r)))\,\mathrm{d}r. \tag{3.16}$$

By the definition of $V^{\mathbb{M},N}_{u_1,u_2}$, especially from the bound on the second derivative, we deduce that, up to a not relabelled subsequence, it holds

$$v_j \rightharpoonup v \quad \text{weakly in } W^{2,2}(-N, N; X), \text{ for some } v \in V^{\mathbb{M},N}_{u_1,u_2}.$$

For the sake of clarity, we introduce the following notation:

$$w_j := -\mathbb{M}\ddot v_j - D_x\mathcal{E}(t, v_j), \qquad w := -\mathbb{M}\ddot v - D_x\mathcal{E}(t, v),$$

and we notice that

$$w_j \rightharpoonup w \quad \text{weakly in } L^2(-N, N; X^*).$$

By Definition 3.2 (ii) and (iv), we observe that the map $w \mapsto \int_{-N}^{N} p(\dot v(r), w(r))\,\mathrm{d}r$ is convex and strongly continuous in $L^2(-N, N; X^*)$, and thus weakly lower semicontinuous. Hence, we get

$$\begin{aligned}\int_{-N}^{N} p(\dot v(r), w(r))\,\mathrm{d}r &\le \liminf_{j\to+\infty} \int_{-N}^{N} p(\dot v(r), w_j(r))\,\mathrm{d}r \\ &\le \liminf_{j\to+\infty} \int_{-N}^{N} p(\dot v_j(r), w_j(r))\,\mathrm{d}r + \limsup_{j\to+\infty} \int_{-N}^{N} |p(\dot v_j(r), w_j(r)) - p(\dot v(r), w_j(r))|\,\mathrm{d}r.\end{aligned}$$

Since $\{v_j\}_{j\in\mathbb{N}}$ is an infimizing sequence, we conclude that the minimum in (3.16) is attained if we prove that the last term in the above estimate vanishes. To this end, we first notice that for almost every $r \in [-N, N]$ the sequence $(\dot v_j(r), w_j(r))$ is contained in a compact subset K of $X \times X^*$; then let ω be a modulus of continuity of p in K. We thus obtain

$$\limsup_{j\to+\infty}\int_{-N}^{N}|p(\dot v_j(r),w_j(r))-p(\dot v(r),w_j(r))|\,\mathrm{d}r\le\limsup_{j\to+\infty}\int_{-N}^{N}\omega(\|\dot v_j(r)-\dot v(r)\|)\,\mathrm{d}r,$$

which vanishes since $\dot v_j\rightharpoonup\dot v$ weakly in $W^{1,2}(-N,N;X)$, and thus strongly in $C^0([-N,N];X)$.

To obtain formula (3.15), we simply notice that the map

$$N\mapsto\min_{v\in V^{\mathbb{M},N}_{u_1,u_2}}\int_{-N}^{N}p(\dot v(r),-\mathbb{M}\ddot v(r)-D_x\mathcal{E}(t,v(r)))\,\mathrm{d}r$$

is nonincreasing. Indeed, if $N\le M$, any minimizer v_N in $[-N,N]$ can be trivially extended constant to $[-M,M]$, and thus obtaining a competitor in $V^{\mathbb{M},M}_{u_1,u_2}$ (we recall that $p(0,w)=0$ for every $w\in X^*$). Hence, (3.15) follows by the very definition of the inertial energy-dissipation cost (3.11). □

With the following proposition, we prove that the cost function $c_t^{\mathbb{M},p}$ is a (possibly asymmetric) distance. We point out that in the vanishing-viscosity setting of [19], this distance is induced by a Finsler metric $F(u,\dot u)$; in our case, the presence of inertia destroys this additional structure.

Proposition 3.12. *For every $t\in[0,T]$, $u_1,u_2,u_3\in X$ and $p\in\mathrm{RCP}_{\mathbb{V}}$, the following assertions hold:*

$$c_t^{\mathbb{M},p}(u_1,u_2)=0\quad\textit{if and only if}\quad u_1=u_2,\tag{3.17}$$

$$c_t^{\mathbb{M},p}(u_1,u_2)\le c_t^{\mathbb{M},p}(u_1,u_3)+c_t^{\mathbb{M},p}(u_3,u_2).\tag{3.18}$$

If in addition $p(\cdot,w)$ is symmetric for every $w\in X^*$ (see, for instance, Proposition 3.4), then the cost is symmetric, i.e.:

$$c_t^{\mathbb{M},p}(u_1,u_2)=c_t^{\mathbb{M},p}(u_2,u_1).\tag{3.19}$$

Proof. We start by proving (3.17). If $u_1=u_2$, then the constant function is an admissible competitor. Thus, $c_t^{\mathbb{M},p}(u_1,u_2)=0$ since $p(0,w)=0$. On the other hand, if $u_1\ne u_2$, then for every $N\in\mathbb{N}$ and $v\in V^{\mathbb{M},N}_{u_1,u_2}$, by exploiting Definition 3.2 (iii), Jensen's inequality and the one-homogeneity of $\mathcal{R}$, we have

$$\int_{-N}^{N}p(\dot v(r),-\mathbb{M}\ddot v(r)-D_x\mathcal{E}(t,v(r)))\,\mathrm{d}r\ge\int_{-N}^{N}\mathcal{R}(\dot v(r))\,\mathrm{d}r\ge\mathcal{R}\Big(\int_{-N}^{N}\dot v(r)\,\mathrm{d}r\Big)=\mathcal{R}(u_2-u_1).$$

Hence, we get

$$c_t^{\mathbb{M},p}(u_1,u_2)\ge R(u_2-u_1)>0,$$

and (3.17) is proved.

To show the validity of (3.18), we fix $N_1,N_2\in\mathbb{N}$ and $v_1\in V^{\mathbb{M},N_1}_{u_1,u_3}$, $v_2\in V^{\mathbb{M},N_2}_{u_3,u_2}$. It is then easy to see that the concatenation

$$v_3(s):=\begin{cases}v_1(s+N_2) & \text{if } s\in[-N_1-N_2,N_1-N_2],\\ v_2(s-N_1) & \text{if } s\in(N_1-N_2,N_1+N_2],\end{cases}$$

belongs to $V^{\mathbb{M},N_1+N_2}_{u_1,u_2}$, and that there holds

$$\begin{aligned}&\int_{-N_1-N_2}^{N_1+N_2}p(\dot v_3(s),-\mathbb{M}\ddot v_3(s)-D_x\mathcal{E}(t,v_3(s)))\,\mathrm{d}s\\&\quad=\int_{-N_1}^{N_1}p(\dot v_1(r),-\mathbb{M}\ddot v_1(r)-D_x\mathcal{E}(t,v_1(r)))\,\mathrm{d}r+\int_{-N_2}^{N_2}p(\dot v_2(r),-\mathbb{M}\ddot v_2(r)-D_x\mathcal{E}(t,v_2(r)))\,\mathrm{d}r.\end{aligned}$$

This yields (3.18).

To prove (3.19), we fix $N \in \mathbb{N}$ and $v \in V^{\mathbb{M},N}_{u_2,u_1}$, and we observe that the backward function $\check{v}(r) := v(-r)$ belongs to $V^{\mathbb{M},N}_{u_1,u_2}$. Then, by exploiting the symmetry of $p(\cdot\,, w)$, we get

$$\begin{aligned} c_t^{\mathbb{M},p}(u_1,u_2) &\le \int_{-N}^{N} p(\dot{\check{v}}(r), -\mathbb{M}\ddot{\check{v}}(r) - D_x\mathcal{E}(t,\check{v}(r)))\,\mathrm{d}r \\ &= \int_{-N}^{N} p(-\dot{v}(r), -\mathbb{M}\ddot{v}(r) - D_x\mathcal{E}(t,v(r)))\,\mathrm{d}r \\ &= \int_{-N}^{N} p(\dot{v}(r), -\mathbb{M}\ddot{v}(r) - D_x\mathcal{E}(t,v(r)))\,\mathrm{d}r. \end{aligned}$$

This implies

$$c_t^{\mathbb{M},p}(u_1,u_2) \le c_t^{\mathbb{M},p}(u_2,u_1),$$

and, by interchanging the roles of u_1 and u_2, we conclude. □

4 Continuous slow-loading limit

The aim of this section is to show that inertial balanced (and virtual) viscosity solutions can be obtained as slow-loading limit (i.e. as $\varepsilon \to 0$) of dynamical solutions u^ε to (1.5). Namely, we prove Theorem 3.10 (i). Hence, here we are assuming (E1)–(E4) and (R1).

4.1 Known results

First, we briefly recall the known results proved in [10]. In particular, for the existence of solutions to (1.5), we refer to [10, Theorem 3.8], where the problem is considered under more general assumptions.

Theorem 4.1. *For every pair of initial data $(u_0^\varepsilon, u_1^\varepsilon) \in X \times X$, there exists at least one solution $u^\varepsilon \in W^{2,\infty}(0,T;X)$ to the differential inclusion* (1.5)*. Moreover, the following energy identity holds:*

$$\begin{aligned} &\frac{\varepsilon^2}{2}\|\dot{u}^\varepsilon(t)\|^2_{\mathbb{M}} + \mathcal{E}(t,u^\varepsilon(t)) + \int_s^t \mathcal{R}(\dot{u}^\varepsilon(r))\,\mathrm{d}r + \varepsilon\int_s^t |\dot{u}^\varepsilon(r)|^2_{\mathbb{V}}\,\mathrm{d}r \\ &\qquad = \frac{\varepsilon^2}{2}\|\dot{u}^\varepsilon(s)\|^2_{\mathbb{M}} + \mathcal{E}(s,u^\varepsilon(s)) + \int_s^t \partial_t\mathcal{E}(r,u^\varepsilon(r))\,\mathrm{d}r \end{aligned} \tag{4.1}$$

for every $0 \le s \le t \le T$.

Proposition 4.2. *Let u^ε be a solution to problem* (1.5)*, and assume u_0^ε, $\varepsilon u_1^\varepsilon$ to be uniformly bounded. Then there exists a positive constant $\overline{C} > 0$ such that for every $\varepsilon > 0$ the following a priori bounds hold:*

$$\max_{t\in[0,T]} \|u^\varepsilon(t)\| \le \overline{C}, \tag{4.2}$$

$$\max_{t\in[0,T]} \varepsilon\|\dot{u}^\varepsilon(t)\|_{\mathbb{M}} \le \overline{C}, \tag{4.3}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \varepsilon^2\|\mathbb{M}\ddot{u}^\varepsilon(t)\|_* \le \overline{C}, \tag{4.4}$$

$$\int_0^T \mathcal{R}(\dot{u}^\varepsilon(r))\,\mathrm{d}r \le \overline{C}. \tag{4.5}$$

Proof. See [10, Corollary 3.4] for (4.2), (4.3) and (4.5). The uniform bound (4.4) follows by exploiting the differential inclusion solved by u^ε together with (2.6). □

With the a priori bounds of Proposition 4.2 at hand, an argument based on Helly's selection theorem provides the existence of a convergent subsequence of dynamic solutions u^ε. This is the content of the following proposition, whose proof is given in [10, Theorem 6.1] and is thus omitted here.

Proposition 4.3. *Let u_0^ε and $\varepsilon u_1^\varepsilon$ be uniformly bounded. Then for every sequence $\varepsilon_j \to 0$, there exist a subsequence (not relabelled) and a function $u \in \mathrm{BV}_{\mathcal{R}}([0;T];X)$ such that the following assertions hold:*
(i) $u^{\varepsilon_j}(t) \to u(t)$ *for every* $t \in [0, T]$.
(ii) $V_{\mathcal{R}}(u; s, t) \le \liminf_{j\to+\infty} \int_s^t \mathcal{R}(\dot u^{\varepsilon_j}(r))\,\mathrm{d}r$ *for every* $0 \le s \le t \le T$.
(iii) $\varepsilon_j \|\dot u^{\varepsilon_j}(t)\|_{\mathbb{M}} \to 0$ *for every* $t \in (0, T] \setminus J_u$, *where* J_u *is the jump set of* u.

In addition, arguing as for [10, Propositions 6.2-6.3], it can be proven that the limit evolution u above complies with the local stability condition (3.13) and a suitable energy inequality. We highlight that a function fulfilling such properties is usually called an *a.e. local solution* to the rate-independent system (1.1) (see, for instance, [22, Chapter 3]).

Proposition 4.4. *Let u_0^ε and $\varepsilon u_1^\varepsilon$ be uniformly bounded, and let u be as in Proposition 4.3. Then the inequality*

$$\int_s^t \mathcal{R}(v) + \langle D_x\mathcal{E}(r, u(r)), v\rangle \,\mathrm{d}r \ge 0$$

holds for every $v \in X$ and for every $0 \le s \le t \le T$. In particular, the left and right limits of u are locally stable; i.e., they fulfill the inequalities

$$\begin{aligned} &\mathcal{R}(v) + \langle D_x\mathcal{E}(t, u^-(t)), v\rangle \ge 0 \quad \text{for every } v \in X \text{ and for every } t \in (0, T],\\ &\mathcal{R}(v) + \langle D_x\mathcal{E}(t, u^+(t)), v\rangle \ge 0 \quad \text{for every } v \in X \text{ and for every } t \in [0, T], \end{aligned} \tag{4.6}$$

or equivalently

$$\begin{aligned} -D_x\mathcal{E}(t, u^-(t)) \in K^* \quad &\text{for every } t \in (0, T],\\ -D_x\mathcal{E}(t, u^+(t)) \in K^* \quad &\text{for every } t \in [0, T]. \end{aligned}$$

Moreover, the energy inequality

$$\mathcal{E}(t, u^+(t)) + V_{\mathcal{R}}(u; s-, t+) \le \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r \tag{4.7}$$

holds for every $0 < s \le t \le T$. If in addition $\varepsilon u_1^\varepsilon \to 0$, then (4.7) holds true also for $s = 0$.

It is worth mentioning that, under the additional assumption of (uniform) convexity on the energy $\mathcal{E}$, Gidoni and Riva [10] were able to deduce that the limit function u is continuous and that (4.7) is actually an energy equality. They thus obtained in the limit an energetic solution of the rate-independent problem.

4.2 Characterization of the energy loss at jumps

In the nonconvex setting, continuity of the limit function is no more reasonable, and hence the gap of the energy in (4.7) has to be characterized. This first proposition shows that, as expected, the peculiar behavior of the limit function u is restricted to its (essential) jump set.

Proposition 4.5. *Let $u_0^\varepsilon \to u_0$, $\varepsilon u_1^\varepsilon \to 0$ and u be as in Proposition 4.3. Then there exists a positive Radon measure μ such that for every $0 \le s \le t \le T$ there holds*

$$\mathcal{E}(t, u^+(t)) + V_{\mathcal{R}}(u_{\mathrm{co}}; s, t) + \sum_{r\in J_u^{\mathrm{e}}\cap[s,t]} \mu(\{r\}) = \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r.$$

In particular, $\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) = \mu(\{t\}) \ge 0$ for every $t \in J_u^{\mathrm{e}}$.

Proof. By reasoning as in [31, Theorem 5.4], it is easy to see that the map

$$t \mapsto \mathcal{E}(t, u^+(t)) - \int_0^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r$$

is nonincreasing; it essentially follows from the energy balance (4.1) by dropping the dissipated energy (i.e. the terms with $\mathcal{R}(\cdot)$ and $|\cdot|_{\mathbb{V}}$) and controlling the kinetic energy in the limit $\varepsilon \to 0$ by means of Proposition 4.3 (iii). This implies the existence of a positive Radon measure μ for which

$$\mathcal{E}(t, u^+(t)) + \mu([s,t]) = \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r \quad \text{for every } 0 \le s \le t \le T. \tag{4.8}$$

This in particular yields that the distributional derivative of $t \mapsto \mathcal{E}(t, u(t))$, denoted by $\mathcal{E}(\cdot, u(\cdot))'$, fulfils the relation

$$\mathcal{E}(\cdot, u(\cdot))' = -\mu + \partial_t \mathcal{E}(\cdot, u(\cdot))\mathcal{L}^1. \tag{4.9}$$

On the other hand, by the chain-rule formula in BV (see, for instance, [5, Theorem 4.1] and recall Remark 2.1), it holds

$$\begin{aligned}\mathcal{E}(\cdot, u(\cdot))' = \partial_t \mathcal{E}(\cdot, u(\cdot))\mathcal{L}^1 + \langle D_x \mathcal{E}(\cdot, u(\cdot)), \dot{u}(\cdot)\rangle \mathcal{L}^1 + \Big\langle D_x \mathcal{E}(\cdot, u(\cdot)), \frac{\mathrm{d}u'_{\mathrm{Ca}}}{\mathrm{d}\lambda}(\cdot)\Big\rangle \lambda \\ + [\mathcal{E}(\cdot, u^+(\cdot)) - \mathcal{E}(\cdot, u^-(\cdot))]\mathcal{H}^0 \llcorner J_u^{\mathrm{e}},\end{aligned} \tag{4.10}$$

where $\lambda = \mathcal{L}^1 + |u'_{\mathrm{Ca}}|$ and we recall $u' = u'_{\mathrm{co}} + u'_{\mathrm{J}} = \dot{u}\mathcal{L}^1 + u'_{\mathrm{Ca}} + u'_{\mathrm{J}}$ (see (2.11)).

By combining (4.9) and (4.10), we obtain

$$\begin{aligned}\mu &= -\Big\langle D_x \mathcal{E}(\cdot, u(\cdot)), \dot{u}(\cdot)\frac{\mathrm{d}\mathcal{L}^1}{\mathrm{d}\lambda}(\cdot) + \frac{\mathrm{d}u'_{\mathrm{Ca}}}{\mathrm{d}\lambda}(\cdot)\Big\rangle \lambda - [\mathcal{E}(\cdot, u^+(\cdot)) - \mathcal{E}(\cdot, u^-(\cdot))]\mathcal{H}^0 \llcorner J_u^{\mathrm{e}} \\ &= -\Big\langle D_x \mathcal{E}(\cdot, u(\cdot)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(\cdot)\Big\rangle \lambda - [\mathcal{E}(\cdot, u^+(\cdot)) - \mathcal{E}(\cdot, u^-(\cdot))]\mathcal{H}^0 \llcorner J_u^{\mathrm{e}}.\end{aligned}$$

The latter equality yields

$$\frac{\mathrm{d}\mu_{\mathrm{co}}}{\mathrm{d}\lambda}(t) = -\Big\langle D_x \mathcal{E}(t, u(t)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(t)\Big\rangle \quad \text{for } \lambda\text{-a.e. } t \in [0,T], \tag{4.11}$$

where we define

$$\mu_{\mathrm{co}} := \mu + [\mathcal{E}(\cdot, u^+(\cdot)) - \mathcal{E}(\cdot, u^-(\cdot))]\mathcal{H}^0 \llcorner J_u^{\mathrm{e}}.$$

By choosing $v = \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(t)$ in the local stability condition (4.6), and using (4.11), we deduce that

$$\mathcal{R}\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(t)\Big) \ge -\Big\langle D_x \mathcal{E}(t, u(t)), \frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(t)\Big\rangle = \frac{\mathrm{d}\mu_{\mathrm{co}}}{\mathrm{d}\lambda}(t) \quad \text{for } \lambda\text{-a.e. } t \in [0,T].$$

By integrating the above inequality in $[s,t] \subseteq [0,T]$, and recalling (2.13), we finally get

$$V_{\mathcal{R}}(u_{\mathrm{co}}; s, t) = \int_s^t \mathcal{R}\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(r)\Big)\,\mathrm{d}\lambda(r) \ge \int_s^t \frac{\mathrm{d}\mu_{\mathrm{co}}}{\mathrm{d}\lambda}(r)\,\mathrm{d}\lambda(r) = \mu([s,t] \setminus J_u^{\mathrm{e}}). \tag{4.12}$$

To obtain the reverse inequality, we combine (4.8) and (4.7), and use (2.12) and (2.14), to get

$$\mu([s,t]) = \mathcal{E}(s, u^-(s)) - \mathcal{E}(t, u^+(t)) + \int_s^t \partial_t \mathcal{E}(r, u(r))\,\mathrm{d}r \ge V_{\mathcal{R}}(u; s-, t+) \ge \mathcal{R}(u')([s,t]).$$

Since both μ and $\mathcal{R}(u')$ are Radon measures, the above inequality implies

$$\mu(B) \ge \mathcal{R}(u')(B) \quad \text{for every Borel set } B \subseteq [0,T].$$

In particular, we deduce

$$\mu([s,t]\setminus J_u^{\mathrm{e}}) \geq \mathcal{R}(u')([s,t]\setminus J_u^{\mathrm{e}}) = \int_s^t \mathcal{R}\Big(\frac{\mathrm{d}u'_{\mathrm{co}}}{\mathrm{d}\lambda}(r)\Big)\,\mathrm{d}\lambda(r) = V_{\mathcal{R}}(u_{\mathrm{co}}; s, t). \tag{4.13}$$

By joining (4.12) with (4.13), we finally obtain

$$\mu([s,t]) = \mu([s,t]\setminus J_u^{\mathrm{e}}) + \mu(J_u^{\mathrm{e}}\cap[s,t]) = V_{\mathcal{R}}(u_{\mathrm{co}}; s, t) + \sum_{r\in J_u^{\mathrm{e}}\cap[s,t]} \mu(\{r\}),$$

and we conclude. □

Thanks to Proposition 3.7, we already know that the inertial cost $c_t^{\mathbb{M},p}(u^-(t), u^+(t))$ is an upper bound for $\mu(\{t\})$ for every $p \in \mathrm{RCP}_{\mathbb{V}}$. We now prove that it is a lower bound as well, thus concluding the proof of Theorem 3.10 (i).

Proposition 4.6. *Let $u_0^\varepsilon \to u_0$, $\varepsilon u_1^\varepsilon \to 0$ and u be as in Proposition 4.3. Then for every $t \in [0,T]$, it holds*

$$\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) \geq \sup_{p\in\mathrm{RCP}_{\mathbb{V}}} c_t^{\mathbb{M},p}(u^-(t), u^+(t)). \tag{4.14}$$

Proof. Let u^{ε_j} be the subsequence obtained in Proposition 4.3. We restrict to the case $t \in J_u^{\mathrm{e}}$, since for any $t \in [0,T]\setminus J_u^{\mathrm{e}}$ inequality (4.14) holds as a trivial equality in view of (3.17). If $t = 0$, we convene that the function u^{ε_j} is extended to a left neighborhood of 0 with an affine function of constant slope $u_1^{\varepsilon_j}$. Reasoning as in [31, Proposition 5.8], by a diagonal argument, we can find sequences $t_j^- \nearrow t$ and $t_j^+ \searrow t$ and a (further) subsequence, still denoted by ε_j, such that

$$u^{\varepsilon_j}(t_j^-) \to u^-(t), \qquad u^{\varepsilon_j}(t_j^+) \to u^+(t), \tag{4.15}$$

$$\varepsilon_j \dot u^{\varepsilon_j}(t_j^-) \to 0, \qquad \varepsilon_j \dot u^{\varepsilon_j}(t_j^+) \to 0, \tag{4.16}$$

as $j \to +\infty$.

By exploiting (3.3) and from the definition of the contact potential $p_{\mathbb{V}}$, we thus infer

$$\begin{aligned}
&\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t))\\
&\quad= \lim_{j\to+\infty}\Big[\mathcal{E}(t_j^-, u^{\varepsilon_j}(t_j^-)) - \mathcal{E}(t_j^+, u^{\varepsilon_j}(t_j^+)) + \frac{\varepsilon_j^2}{2}\|\dot u^{\varepsilon_j}(t_j^-)\|_{\mathbb{M}}^2 - \frac{\varepsilon_j^2}{2}\|\dot u^{\varepsilon_j}(t_j^+)\|_{\mathbb{M}}^2 + \int_{t_j^-}^{t_j^+} \partial_t\mathcal{E}(r, u^{\varepsilon_j}(r))\,\mathrm{d}r\Big]\\
&\quad= \lim_{j\to+\infty}\int_{t_j^-}^{t_j^+} \mathcal{R}_{\varepsilon_j}(\dot u^{\varepsilon_j}(r)) + \mathcal{R}^*_{\varepsilon_j}(w^{\varepsilon_j}(r))\,\mathrm{d}r\\
&\quad\geq \limsup_{j\to+\infty}\int_{t_j^-}^{t_j^+} p_{\mathbb{V}}(\dot u^{\varepsilon_j}(r), w^{\varepsilon_j}(r))\,\mathrm{d}r.
\end{aligned}$$

We now take any $p \in \mathrm{RCP}_{\mathbb{V}}$, and from (iii) in Definition 3.2 we can continue the previous inequality, getting

$$\begin{aligned}
\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) &\geq \limsup_{j\to+\infty}\int_{t_j^-}^{t_j^+} p(\dot u^{\varepsilon_j}(r), w^{\varepsilon_j}(r))\,\mathrm{d}r\\
&= \limsup_{j\to+\infty}\int_{t_j^-}^{t_j^+} p\big(\dot u^{\varepsilon_j}(r), -\varepsilon_j^2\mathbb{M}\ddot u^{\varepsilon_j}(r) - D_x\mathcal{E}(r, u^{\varepsilon_j}(r))\big)\,\mathrm{d}r.
\end{aligned} \tag{4.17}$$

Then, by using the Lipschitzianity of p in the second variable (property (iv) in Definition 3.2), we notice that

$$\int_{t_j^-}^{t_j^+} \big|p(\dot u^{\varepsilon_j}(r), -\varepsilon_j^2 \mathbb{M}\ddot u^{\varepsilon_j}(r) - D_x\mathcal{E}(r, u^{\varepsilon_j}(r))) - p(\dot u^{\varepsilon_j}(r), -\varepsilon_j^2 \mathbb{M}\ddot u^{\varepsilon_j}(r) - D_x\mathcal{E}(t, u^{\varepsilon_j}(r)))\big|\,\mathrm{d}r$$
$$\leq L \int_{t_j^-}^{t_j^+} \|\dot u^{\varepsilon_j}(r)\| \|D_x\mathcal{E}(t, u^{\varepsilon_j}(r)) - D_x\mathcal{E}(r, u^{\varepsilon_j}(r))\|_*\,\mathrm{d}r.$$

If we denote by ω a modulus of continuity for $D_x\mathcal{E}$ on $[0,T]\times B_{\overline{C}}$, where $\overline{C}$ is the constant of Proposition 4.2, we can bound the last term in the above inequality by

$$L\omega(\max\{|t_j^+ - t|, |t - t_j^-|\}) \int_{t_j^-}^{t_j^+} \|\dot u^{\varepsilon_j}(r)\|\,\mathrm{d}r,$$

which vanishes as $j \to +\infty$, thanks to the uniform bound (4.5).

This means that we can freeze the time t in $D_x\mathcal{E}$ of (4.17), getting

$$\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) \geq \limsup_{j\to+\infty} \int_{t_j^-}^{t_j^+} p(\dot u^{\varepsilon_j}(r), -\varepsilon_j^2 \mathbb{M}\ddot u^{\varepsilon_j}(r) - D_x\mathcal{E}(t, u^{\varepsilon_j}(r)))\,\mathrm{d}r. \tag{4.18}$$

Following [31, Proposition 5.8], we now set

$$v_j(\tau) := u^{\varepsilon_j}(\varepsilon_j\tau + t_j^-) \quad \text{for every } \tau \in [0, \sigma_j], \tag{4.19}$$

where we denoted by σ_j the ratio

$$\frac{t_j^+ - t_j^-}{\varepsilon_j}.$$

Then, through the change of variables $r = \varepsilon_j\tau + t_j^-$, and recalling the one-homogeneity of p with respect to the first variable, we obtain

$$\int_{t_j^-}^{t_j^+} p(\dot u^{\varepsilon_j}(r), -\varepsilon_j^2 \mathbb{M}\ddot u^{\varepsilon_j}(r) - D_x\mathcal{E}(t, u^{\varepsilon_j}(r)))\,\mathrm{d}r = \int_0^{\sigma_j} p(\dot v_j(\tau), -\mathbb{M}\ddot v_j(\tau) - D_x\mathcal{E}(t, v_j(\tau)))\,\mathrm{d}\tau.$$

We also notice that (4.15)–(4.16) can be re-read for v_j as

$$\begin{cases} \|v_j(0) - u^-(t)\| \to 0, & \|v_j(\sigma_j) - u^+(t)\| \to 0, \\ \|\dot v_j(0)\| \to 0, & \|\dot v_j(\sigma_j)\| \to 0, \end{cases} \tag{4.20}$$

as $j \to +\infty$.

We now introduce the functions

$$g(x) = 3x^2 - 2x^3, \quad h(x) = -x^2(1-x), \qquad x \in [0,1],$$

and the competitor

$$\tilde v_j(\tau) = \begin{cases} u^-(t), & \tau \leq -1, \\ u^-(t) + g(\tau+1)(v_j(0) - u^-(t)) + h(\tau+1)\dot v_j(0), & \tau \in [-1, 0], \\ v_j(\tau), & \tau \in [0, \sigma_j], \\ v_j(\sigma_j) + g(1+\sigma_j-\tau)(u^+(t) - v_j(\sigma_j)) - h(1+\sigma_j-\tau)\dot v_j(\sigma_j), & \tau \in [\sigma_j, \sigma_j+1], \\ u^+(t), & \tau \geq \sigma_j + 1. \end{cases}$$

For the sake of clarity, we denote by $\alpha_j(\tau)$ and $\beta_j(\tau)$ the expressions of $\tilde{v}_j$ in $[-1,0]$ and $[\sigma_j, \sigma_j+1]$, respectively, and we notice that, by (4.20), α_j and β_j are uniformly bounded and there holds

$$\lim_{j\to+\infty}\Big(\max_{\tau\in[-1,0]}(\|\dot{\alpha}_j(\tau)\|+\|\ddot{\alpha}_j(\tau)\|)+\max_{\tau\in[\sigma_j,\sigma_j+1]}(\|\dot{\beta}_j(\tau)\|+\|\ddot{\beta}_j(\tau)\|)\Big)=0. \tag{4.21}$$

Fix now an arbitrary $N_j \in \mathbb{N}$ with $2N_j - 1 > \sigma_j + 1$. Recalling that $p(0, w) = 0$, we observe that

$$\begin{aligned}&\int_{-1}^{2N_j-1} p\big(\dot{\tilde{v}}_j(\tau), -\mathbb{M}\ddot{\tilde{v}}_j(\tau) - D_x\mathcal{E}(t, \tilde{v}_j(\tau))\big)\,\mathrm{d}\tau\\ &\quad= \int_0^{\sigma_j} p\big(\dot{v}_j(\tau), -\mathbb{M}\ddot{v}_j(\tau) - D_x\mathcal{E}(t, v_j(\tau))\big)\,\mathrm{d}\tau + \int_{-1}^{0} p\big(\dot{\alpha}_j(\tau), -\mathbb{M}\ddot{\alpha}_j(\tau) - D_x\mathcal{E}(t, \alpha_j(\tau))\big)\,\mathrm{d}\tau\\ &\qquad+ \int_{\sigma_j}^{\sigma_j+1} p\big(\dot{\beta}_j(\tau), -\mathbb{M}\ddot{\beta}_j(\tau) - D_x\mathcal{E}(t, \beta_j(\tau))\big)\,\mathrm{d}\tau.\end{aligned}$$

By means of (3.6) and (4.21), it is easy to see that both terms in the last line above vanish as $j \to +\infty$. This allows us to continue (4.18), getting

$$\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) \ge \limsup_{j\to+\infty} \int_{-1}^{2N_j-1} p\big(\dot{\tilde{v}}_j(\tau), -\mathbb{M}\ddot{\tilde{v}}_j(\tau) - D_x\mathcal{E}(t, \tilde{v}_j(\tau))\big)\,\mathrm{d}\tau. \tag{4.22}$$

With the time translation $\hat{v}_j(s) = \tilde{v}_j(s + N_j - 1)$, we finally construct a function belonging to $V^{\mathbb{M},N_j}_{u^-(t),u^+(t)}$ (indeed, notice that the bound on the second derivative follows from (4.4), (4.19) and (4.21)). From (4.22), we thus get

$$\begin{aligned}\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) &\ge \limsup_{j\to+\infty} \int_{-1}^{2N_j-1} p\big(\dot{\tilde{v}}_j(\tau), -\mathbb{M}\ddot{\tilde{v}}_j(\tau) - D_x\mathcal{E}(t, \tilde{v}_j(\tau))\big)\,\mathrm{d}\tau\\ &= \limsup_{j\to+\infty} \int_{-N_j}^{N_j} p\big(\dot{\hat{v}}_j(s), -\mathbb{M}\ddot{\hat{v}}_j(s) - D_x\mathcal{E}(t, \hat{v}_j(s))\big)\,\mathrm{d}s\\ &\ge c_t^{\mathbb{M},p}(u^-(t), u^+(t)),\end{aligned}$$

and by the arbitrariness of $p \in \mathrm{RCP}_{\mathbb{V}}$, we conclude. □

5 Incremental minimization scheme

This last section is devoted to the proof of Theorem 3.10 (ii): namely, we show that IBV and IVV solutions can also be obtained as limits of time-discrete solutions when ε and the time step τ vanish simultaneously (with a certain rate). For this, in addition to the assumptions of the previous section, we need to require (E3’) and (E5).

Let $T > 0$ and let $\tau \in (0, 1)$ be a fixed time step such that $\frac{T}{\tau} \in \mathbb{N}$. We consider the corresponding induced partition $\Pi_\tau := \{t^k\}_k$ of the time-interval $[0, T]$, defined by $t^k := k\tau$ where $k = 0, 1, \dots, \frac{T}{\tau}$. For future use, we also define $t^{-1} := -\tau$ and we set $\mathcal{K}_\tau := \{1, \dots, \frac{T}{\tau}\}$ and $\mathcal{K}^0_\tau := \mathcal{K}_\tau \cup \{0\}$.

We construct a recursive sequence $\{u^k_{\tau,\varepsilon}\}_{k\in\mathcal{K}_\tau}$ by solving the following iterated minimum problem à la minimizing movements:

$$u^k_{\tau,\varepsilon} \in \operatorname*{arg\,min}_{x\in X} \mathcal{F}_{\tau,\varepsilon}(t^k, x, u^{k-1}_{\tau,\varepsilon}, u^{k-2}_{\tau,\varepsilon}), \quad k \in \mathcal{K}_\tau, \tag{5.1a}$$

with initial conditions

$$u^0_{\tau,\varepsilon} := u^\varepsilon_0, \quad u^{-1}_{\tau,\varepsilon} := u^\varepsilon_0 - \tau u^\varepsilon_1, \tag{5.1b}$$

where

$$\mathcal{F}_{\tau,\varepsilon}(t^k, x, u^{k-1}_{\tau,\varepsilon}, u^{k-2}_{\tau,\varepsilon}) := \frac{\varepsilon^2}{2\tau^2}\|x - 2u^{k-1}_{\tau,\varepsilon} + u^{k-2}_{\tau,\varepsilon}\|^2_{\mathbb{M}} + \frac{\varepsilon}{2\tau}|x - u^{k-1}_{\tau,\varepsilon}|^2_{\mathbb{V}} + \mathcal{R}(x - u^{k-1}_{\tau,\varepsilon}) + \mathcal{E}(t^k, x) + \frac{\Lambda_{\mathbb{V}}}{4}\|x - u^{k-1}_{\tau,\varepsilon}\|^2_{\mathbb{I}}$$

and

$$\Lambda_{\mathbb{V}} := \begin{cases} 0 & \text{if } \mathbb{V} \text{ is positive-definite,} \\ \Lambda & \text{otherwise,} \end{cases} \tag{5.2}$$

with Λ and $\mathbb{I}$ from (E5).

The addition in the functional $\mathcal{F}_{\tau,\varepsilon}$ of the last fictitious viscous term, which by the definition (5.2) of $\Lambda_{\mathbb{V}}$ is present only if $\mathbb{V}$ is not positive-definite, is needed to deal with the Λ-convexity assumption (E5). If $\mathbb{V}$ is positive-definite and the ratio $\frac{\varepsilon}{\tau}$ is large enough (see (5.5)), the second term will be enough to keep the Λ-convexity under control.

We observe that the existence of a minimum in (5.1a) follows easily from the direct method. Furthermore, if $\frac{\varepsilon^2}{\tau^2}$ is large enough (this is the case under the assumption (5.27) needed to conclude the whole argument), the minimum is unique by strict convexity of the functional.

By defining

$$v^k_{\tau,\varepsilon} := \frac{u^k_{\tau,\varepsilon} - u^{k-1}_{\tau,\varepsilon}}{\tau},$$

we notice that the Euler–Lagrange equation solved by $u^k_{\tau,\varepsilon}$ reads as

$$\varepsilon^2\mathbb{M}\frac{v^k_{\tau,\varepsilon} - v^{k-1}_{\tau,\varepsilon}}{\tau} + \varepsilon\mathbb{V}v^k_{\tau,\varepsilon} + \partial\mathcal{R}(v^k_{\tau,\varepsilon}) + D_x\mathcal{E}(t^k, u^k_{\tau,\varepsilon}) + \frac{\Lambda_{\mathbb{V}}}{2}\tau\mathbb{I}v^k_{\tau,\varepsilon} \ni 0. \tag{5.3}$$

We also observe that by (5.1b), one has $v^0_{\tau,\varepsilon} = u^\varepsilon_1$. Thus, in the limit as $\tau \to 0$ with ε fixed, we formally (but this could actually be made rigorous, see for instance [28]) recover the dynamic problem (1.5).

In order to enlighten the notation, from now on we will drop the dependence on τ, ε in $u^k_{\tau,\varepsilon}$ and $v^k_{\tau,\varepsilon}$, and we will simply write u^k and v^k.

As in the continuous counterpart developed in Section 4, the first step in the analysis consists in finding uniform a priori estimates, which usually follows by combining an energy inequality together with Grönwall's lemma. In the discrete setting, we employ the following version of the discrete Grönwall's inequality, whose proof can be found for instance in [13, Appendix A].

We want to stress that here and henceforth we adopt the convention that an empty sum is equal to 0.

Lemma 5.1 (Grönwall). *Let $\{y^n\}_{n\in\mathbb{N}}$ and $\{f^n\}_{n\in\mathbb{N}}$ be two nonnegative sequences and let $c \geq 0$. If*

$$y^n \leq c + \sum_{k=1}^{n-1} f^k y^k \quad \text{for every } n \in \mathbb{N},$$

then one has

$$y^n \leq c \exp\Big(\sum_{k=1}^{n-1} f^k\Big) \quad \text{for every } n \in \mathbb{N}.$$

Proposition 5.2. *For every $m, n \in \mathcal{K}^0_\tau$ with $m \leq n$, the following discrete energy inequality holds true:*

$$\frac{\varepsilon^2}{2}\|v^n\|^2_{\mathbb{M}} + \mathcal{E}(t^n, u^n) + \sum_{k=m+1}^{n} \tau\mathcal{R}(v^k) + \sum_{k=m+1}^{n} \tau\Big(\varepsilon|v^k|^2_{\mathbb{V}} - \frac{\Lambda - \Lambda_{\mathbb{V}}}{2}\tau\|v^k\|^2_{\mathbb{I}}\Big) \leq \frac{\varepsilon^2}{2}\|v^m\|^2_{\mathbb{M}} + \mathcal{E}(t^m, u^m) + \sum_{k=m+1}^{n} \int_{t^{k-1}}^{t^k} \partial_t\mathcal{E}(r, u^{k-1})\,\mathrm{d}r. \tag{5.4}$$

Furthermore, if u^ε_0 and $\varepsilon u^\varepsilon_1$ are uniformly bounded and

$$\frac{\tau}{\varepsilon} \leq \frac{2}{\Lambda VI} \quad \text{if } \mathbb{V} \text{ is positive-definite,} \tag{5.5}$$

then there exists $C > 0$, independent of ε and τ, such that

$$\frac{\varepsilon^2}{2}\|v^n\|^2_{\mathbb{M}} + \mathcal{E}(t^n, u^n) + \sum_{k=1}^{n} \tau\mathcal{R}(v^k) \le C \tag{5.6}$$

for every $n \in \mathcal{K}^0_\tau$.

Proof. Testing (5.3) by τv^k, from (2.5), (2.9) and from the fact that

$$\frac{\|x\|^2_{\mathbb{M}}}{2} - \frac{\|y\|^2_{\mathbb{M}}}{2} = \frac{\|x-y\|^2_{\mathbb{M}}}{2} - \langle \mathbb{M}(y-x), y\rangle \ge -\langle \mathbb{M}(y-x), y\rangle,$$

we deduce

$$\begin{aligned}
\tau\mathcal{R}(v^k) &= -\Big\langle \varepsilon^2\mathbb{M}\frac{v^k - v^{k-1}}{\tau} + \varepsilon\mathbb{V}v^k + D_x\mathcal{E}(t^k, u^k) + \frac{\Lambda_{\mathbb{V}}}{2}\mathbb{I}\tau v^k, \tau v^k\Big\rangle \\
&= -\varepsilon\tau|v^k|^2_{\mathbb{V}} - \varepsilon^2\langle \mathbb{M}(v^k - v^{k-1}), v^k\rangle + \langle D_x\mathcal{E}(t^k, u^k), u^{k-1} - u^k\rangle - \frac{\Lambda_{\mathbb{V}}}{2}\|u^k - u^{k-1}\|^2_{\mathbb{I}} \\
&\le -\varepsilon\tau|v^k|^2_{\mathbb{V}} + \frac{\varepsilon^2}{2}\|v^{k-1}\|^2_{\mathbb{M}} - \frac{\varepsilon^2}{2}\|v^k\|^2_{\mathbb{M}} - \mathcal{E}(t^k, u^k) + \mathcal{E}(t^k, u^{k-1}) + \frac{\Lambda - \Lambda_{\mathbb{V}}}{2}\tau^2\|v^k\|^2_{\mathbb{I}}.
\end{aligned}$$

Subtracting $\mathcal{E}(t^{k-1}, u^{k-1})$ from both sides, rearranging the terms and summing upon $k = m, \dots, n$, we obtain (5.4).

We now come to the proof of (5.6). We first notice that, defining

$$\begin{aligned}
y^n &:= \frac{\varepsilon^2}{2}\|v^n\|^2_{\mathbb{M}} + \mathcal{E}(t^n, u^n) + \sum_{k=1}^{n}\tau\mathcal{R}(v^k) + a_1 \quad \text{if } n \in \mathcal{K}_\tau, \\
y^0 &:= \frac{\varepsilon^2}{2}\|u_1^\varepsilon\|^2_{\mathbb{M}} + \mathcal{E}(0, u_0^\varepsilon) + a_1,
\end{aligned}$$

where a_1 is the constant appearing in (E3'), from (5.5) it holds

$$y^n \le y^0 + \sum_{k=1}^{n}\int_{t^{k-1}}^{t^k} \partial_t\mathcal{E}(r, u^{k-1})\,\mathrm{d}r \quad \text{for every } n \in \mathcal{K}_\tau. \tag{5.7}$$

We indeed observe that the term $\varepsilon|v^k|^2_{\mathbb{V}} - \frac{\Lambda-\Lambda_{\mathbb{V}}}{2}\tau\|v^k\|^2_{\mathbb{I}}$ in (5.4) is nonnegative: if $\mathbb{V}$ is not positive-definite, it reduces to $\varepsilon\tau|v^k|^2_{\mathbb{V}}$; otherwise, we exploit (5.5):

$$\varepsilon\|v^k\|^2_{\mathbb{V}} - \frac{\Lambda}{2}\tau\|v^k\|^2_{\mathbb{I}} \ge \Big(\frac{\varepsilon}{V} - \frac{\Lambda}{2}I\tau\Big)\|v^k\|^2 \ge 0.$$

Thanks to (2.8), we now have

$$\int_{t^{k-1}}^{t^k} \partial_t\mathcal{E}(r, u^{k-1})\,\mathrm{d}r \le (\mathcal{E}(t^{k-1}, u^{k-1}) + a_1)\int_{t^{k-1}}^{t^k} b(r)e^{\int_{t^{k-1}}^{r} b(s)\,\mathrm{d}s}\,\mathrm{d}r \le (e^{\int_{t^{k-1}}^{t^k} b(r)\,\mathrm{d}r} - 1)y^{k-1},$$

and thus from (5.7) we infer

$$y^n \le y^0 e^{\int_0^\tau b(r)\,\mathrm{d}r} + \sum_{k=1}^{n-1}(e^{\int_{t^k}^{t^{k+1}} b(r)\,\mathrm{d}r} - 1)y^k \quad \text{for every } n \in \mathcal{K}_\tau.$$

Hence, by means of Lemma 5.1, we get

$$y^n \le y^0 e^{\int_0^\tau b(r)\,\mathrm{d}r}\exp\Big(\sum_{k=1}^{n-1}(e^{\int_{t^k}^{t^{k+1}} b(r)\,\mathrm{d}r} - 1)\Big) \quad \text{for every } n \in \mathcal{K}_\tau. \tag{5.8}$$

By defining $B := \int_0^T b(r)\,\mathrm{d}r$ and recalling the elementary inequality

$$e^x - 1 \le \frac{e^B - 1}{B}x \quad \text{for every } x \in [0, B],$$

from (5.8) we finally obtain

$$y^n \le y^0 e^{e^B - 1} \quad \text{for every } n \in \mathcal{K}_\tau.$$

Since y^0 is uniformly bounded by assumption, we conclude. □

Corollary 5.3. *Assume that u_0^ε and $\varepsilon u_1^\varepsilon$ are uniformly bounded and assume* (5.5). *Then the following uniform bounds hold for every $n \in \mathcal{K}_\tau$:*

$$\|u^n\| \le C, \tag{5.9}$$

$$\varepsilon \|v^n\|_{\mathbb{M}} \le C, \tag{5.10}$$

$$\varepsilon^2 \Big\| \mathbb{M} \frac{v^n - v^{n-1}}{\tau} \Big\|_* \le C, \tag{5.11}$$

$$\sum_{k=1}^{n} \tau \mathcal{R}(v^k) \le C. \tag{5.12}$$

Proof. The bounds (5.10) and (5.12) can be easily inferred from (5.6). We then prove (5.9). Let $n \in \mathcal{K}_\tau$ be fixed. Then, with (5.12), (2.4) and the triangle inequality, we have

$$\|u^n\| \le \|u_0^\varepsilon\| + \sum_{k=1}^{n} \|u^k - u^{k-1}\| = \|u_0^\varepsilon\| + \sum_{k=1}^{n} \tau \|v^k\| \le C.$$

Lastly, (5.11) can be obtained from the Euler-Lagrange equation (5.3) taking into account (2.6), (E2) and (5.10). □

5.1 The main interpolants

Once the discrete bounds are obtained, in order to retrieve the continuous framework we need to introduce suitable interpolants of the discrete-in-time sequence $\{u^k\}_{k \in \mathcal{K}_\tau^0}$. First, we denote by $\overline{u}_{\tau,\varepsilon}$ (resp., $\underline{u}_{\tau,\varepsilon}$) the left-continuous (resp., right-continuous) piecewise constant interpolant of $\{u^k\}_{k \in \mathcal{K}_\tau^0}$, defined by

$$\overline{u}_{\tau,\varepsilon}(t) := u^k \quad \text{for } t \in (t^{k-1}, t^k], \qquad \underline{u}_{\tau,\varepsilon}(t) := u^{k-1} \quad \text{for } t \in [t^{k-1}, t^k), \qquad k \in \mathcal{K}_\tau^0, \tag{5.13}$$

respectively. We denote by $\widehat{u}_{\tau,\varepsilon}$ the piecewise affine interpolant of $\{u^k\}_{k \in \mathcal{K}_\tau^0}$, defined by

$$\widehat{u}_{\tau,\varepsilon}(t) := \frac{u^k - u^{k-1}}{\tau}(t - t^{k-1}) + u^{k-1} = v^k (t - t^{k-1}) + u^{k-1} \quad \text{for } t \in (t^{k-1}, t^k],\ k \in \mathcal{K}_\tau^0. \tag{5.14}$$

Since in the definition of the inertial cost (3.11) a second derivative is present, we also need to keep track of its discrete counterpart $\frac{v^k - v^{k-1}}{\tau}$. This is done by finally introducing the function $\widetilde{u}_{\tau,\varepsilon}$ such that $\widetilde{u}_{\tau,\varepsilon}(0) = u_\varepsilon^0$ and whose first derivative is the piecewise affine interpolant of $\{v^k\}_{k \in \mathcal{K}_\tau^0}$, namely

$$\begin{cases} \widetilde{u}_{\tau,\varepsilon}(t) := u_\varepsilon^0 + \displaystyle\int_0^t \dot{\widetilde{u}}_{\tau,\varepsilon}(r)\, \mathrm{d}r & \text{for } t \in [0, T], \\ \dot{\widetilde{u}}_{\tau,\varepsilon}(t) := \dfrac{v^k - v^{k-1}}{\tau}(t - t^{k-1}) + v^{k-1} & \text{for } t \in (t^{k-1}, t^k],\ k \in \mathcal{K}_\tau. \end{cases} \tag{5.15}$$

Notice indeed that $\widetilde{u}_{\tau,\varepsilon}$ is in $W^{2,\infty}(0, T; X)$ with

$$\ddot{\widetilde{u}}_{\tau,\varepsilon}(t) = \frac{v^k - v^{k-1}}{\tau} \quad \text{for } t \in (t^{k-1}, t^k),\ k \in \mathcal{K}_\tau.$$

Thus, thanks to (5.11), the function $\widetilde{u}_{\tau,\varepsilon}$ is the correct “discrete” counterpart of the continuous dynamic solution u^ε to (1.5).

For any $t \in (-\tau, T]$, we also denote by $\mathfrak{t}_\tau$ the least point of the partition Π_τ which is greater or equal to t; i.e., it is defined by

$$\mathfrak{t}_\tau := \min\{r \in \Pi_\tau : r \ge t\}. \tag{5.16}$$

Note that $\mathfrak{t}_\tau \searrow t$ as $\tau \to 0$ (for $t \in [0, T]$).

We finally define a piecewise constant interpolant of the values $\mathcal{E}(\cdot, u)$, setting for every $u \in X$,

$$\mathcal{E}_\tau(t, u) := \mathcal{E}(t^k, u) \quad \text{if } t \in (t^{k-1}, t^k],\ k \in \mathcal{K}_\tau^0.$$

From assumptions (E1) and (E2), we deduce that, in the limit as $\tau \to 0$,

$$\mathcal{E}_\tau(t,u) \to \mathcal{E}(t,u) \quad \text{and} \quad D_x\mathcal{E}_\tau(t,u) \to D_x\mathcal{E}(t,u), \tag{5.17}$$

uniformly with respect to $(t,u) \in [0,T] \times \overline{B_R}$.

In terms of interpolants, the energy inequality (5.4) can be rewritten as

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\dot{\hat{u}}_{\tau,\varepsilon}(t)\|^2_{\mathbb{M}} + \mathcal{E}_\tau(t,\overline{u}_{\tau,\varepsilon}(t)) + \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau} \mathcal{R}(\dot{\hat{u}}_{\tau,\varepsilon}(r))\,\mathrm{d}r + \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau}\Big(\varepsilon|\dot{\hat{u}}_{\tau,\varepsilon}(r)|^2_{\mathbb{V}} - \frac{\Lambda-\Lambda_{\mathbb{V}}}{2}\tau\|\dot{\hat{u}}_{\tau,\varepsilon}(r)\|^2_{\mathbb{I}}\Big)\,\mathrm{d}r \\ \le \frac{\varepsilon^2}{2}\|\dot{\hat{u}}_{\tau,\varepsilon}(s)\|^2_{\mathbb{M}} + \mathcal{E}_\tau(s,\overline{u}_{\tau,\varepsilon}(s)) + \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau} \partial_t\mathcal{E}(r,\underline{u}_{\tau,\varepsilon}(r))\,\mathrm{d}r\end{aligned}$$

for every $s,t \in (-\tau,T] \setminus \Pi_\tau$ with $s \le t$. Furthermore, Proposition 5.2 and the subsequent Corollary 5.3 can be re-read as follows.

Corollary 5.4. *Assume that u_0^ε and $\varepsilon u_1^\varepsilon$ are uniformly bounded and assume* (5.5). *Then there exists $C > 0$, independent of τ and ε, such that*

$$\frac{\varepsilon^2}{2}\|\dot{\hat{u}}_{\tau,\varepsilon}(t)\|^2_{\mathbb{M}} + \mathcal{E}_\tau(t,\overline{u}_{\tau,\varepsilon}(t)) + \int_0^{\mathfrak{t}_\tau} \mathcal{R}(\dot{\hat{u}}_{\tau,\varepsilon}(r))\,\mathrm{d}r \le C$$

for every $t \in (-\tau,T] \setminus \Pi_\tau$.

Moreover, up to enlarging the constant $\overline{C}$ appearing in Proposition 4.2, there holds

$$\begin{aligned}\max_{t\in[0,T]}\|\overline{u}_{\tau,\varepsilon}(t)\| &\le \overline{C},\\ \max_{t\in[0,T]\setminus\Pi_\tau} \varepsilon\|\dot{\hat{u}}_{\tau,\varepsilon}(t)\|_{\mathbb{M}} &\le \overline{C},\\ \max_{t\in[0,T]\setminus\Pi_\tau} \varepsilon^2\|\mathbb{M}\ddot{\tilde{u}}_{\tau,\varepsilon}(t)\|_* &\le \overline{C},\\ \int_0^T \mathcal{R}(\dot{\hat{u}}_{\tau,\varepsilon}(r))\,\mathrm{d}r &\le \overline{C}.\end{aligned} \tag{5.18}$$

The next proposition shows that the mismatch between the many interpolants defined above can be bounded by suitable ratios of the parameters τ and ε.

Proposition 5.5. *Assume that u_0^ε and $\varepsilon u_1^\varepsilon$ are uniformly bounded and assume* (5.5). *Then we have*

$$\max_{t\in[0,T]}\{\|\overline{u}_{\tau,\varepsilon}(t) - \underline{u}_{\tau,\varepsilon}(t)\| + \|\overline{u}_{\tau,\varepsilon}(t) - \hat{u}_{\tau,\varepsilon}(t)\| + \|\tilde{u}_{\tau,\varepsilon}(t) - \hat{u}_{\tau,\varepsilon}(t)\|\} \le C\frac{\tau}{\varepsilon}. \tag{5.19}$$

Moreover, it holds

$$\max_{t\in[0,T]\setminus\Pi_\tau}\|\dot{\tilde{u}}_{\tau,\varepsilon}(t) - \dot{\hat{u}}_{\tau,\varepsilon}(t)\| \le C\frac{\tau}{\varepsilon^2}. \tag{5.20}$$

Proof. We first notice that, by virtue of (5.10) and since $\varepsilon u_1^\varepsilon$ is uniformly bounded, one has

$$\max_{k\in\mathcal{K}_\tau^0}\|u^k - u^{k-1}\| \le C\frac{\tau}{\varepsilon}. \tag{5.21}$$

Thus, let $t \in (t^{k-1}, t^k]$ for some $k \in \mathcal{K}_\tau^0$. Then there holds

$$\begin{aligned}\|\overline{u}_{\tau,\varepsilon}(t) - \underline{u}_{\tau,\varepsilon}(t)\| &\le \|u^k - u^{k-1}\|,\\ \|\overline{u}_{\tau,\varepsilon}(t) - \hat{u}_{\tau,\varepsilon}(t)\| &= \|u^k - u^{k-1}\|\frac{\tau-(t-t^{k-1})}{\tau} \le \|u^k - u^{k-1}\|.\end{aligned}$$

In order to deal with the last term in (5.19), we observe that

$$\dot{\tilde{u}}_{\tau,\varepsilon}(t) - \dot{\hat{u}}_{\tau,\varepsilon}(t) = \frac{v^k - v^{k-1}}{\tau}(t - t^{k-1} - \tau) \quad \text{for } t \in (t^{k-1}, t^k),\ k \in \mathcal{K}_\tau, \tag{5.22}$$

and thus

$$\begin{aligned}
\tilde{u}_{\tau,\varepsilon}(t) - \hat{u}_{\tau,\varepsilon}(t) &= \int_0^t (\dot{\tilde{u}}_{\tau,\varepsilon}(r) - \dot{\hat{u}}_{\tau,\varepsilon}(r))\,\mathrm{d}r \\
&= \frac{v^k - v^{k-1}}{\tau}\int_{t^{k-1}}^{t} (r - t^{k-1} - \tau)\,\mathrm{d}r + \sum_{i=1}^{k-1}\frac{v^i - v^{i-1}}{\tau}\int_{t^{i-1}}^{t^i}(r - t^{i-1} - \tau)\,\mathrm{d}r \\
&= \frac{v^k - v^{k-1}}{2\tau}(t - t^{k-1})(t - t^{k-1} - 2\tau) - \tau\sum_{i=1}^{k-1}\frac{v^i - v^{i-1}}{2} \\
&= \frac{v^k - v^{k-1}}{2\tau}(t - t^{k-1})(t - t^{k-1} - 2\tau) + \frac{\tau u_1^\varepsilon}{2} - \frac{\tau v^{k-1}}{2} \\
&= \frac{v^k}{2}\Big(\frac{t - t^{k-1}}{\tau}\Big)(t - t^{k-1} - 2\tau) - \frac{v^{k-1}}{2}\frac{(t - t^{k-1} - \tau)^2}{\tau} + \frac{\tau u_1^\varepsilon}{2}.
\end{aligned}$$

Since $t - t^{k-1} \in (0, \tau)$, we now get

$$2\|\tilde{u}_{\tau,\varepsilon}(t) - \hat{u}_{\tau,\varepsilon}(t)\| \leq \frac{\tau}{\varepsilon}\|\varepsilon u_1^\varepsilon\| + \tau\|v^k\| + \tau\|v^{k-1}\| = \frac{\tau}{\varepsilon}\|\varepsilon u_1^\varepsilon\| + \|u^k - u^{k-1}\| + \|u^{k-1} - u^{k-2}\|,$$

and assertion (5.19) follows from (5.21).

From (5.22), the bound (5.20) easily follows by means of (5.11). □

We are now in a position to prove the analogues of Propositions 4.3–4.5 for the sequence of piecewise affine interpolants $\hat{u}_{\tau,\varepsilon}$.

Proposition 5.6. *Let u_0^ε and $\varepsilon u_1^\varepsilon$ be uniformly bounded and assume that* (5.5) *holds. Then, for every sequence* $(\tau_j, \varepsilon_j) \to (0, 0)$, *there exists a subsequence (not relabelled) and a function* $u \in \mathrm{BV}_{\mathcal{R}}([0, T]; X)$ *such that the following assertions hold:*

(i) $\hat{u}_{\tau_j,\varepsilon_j}(t) \to u(t)$ *for every* $t \in [0, T]$.

(ii) *For every* $0 \leq s \leq t \leq T$,

$$V_{\mathcal{R}}(u; s, t) \leq \liminf_{j\to+\infty}\int_s^t \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r.$$

(iii) $\varepsilon_j\|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(t)\|_{\mathbb{M}} \to 0$ *for a.e.* $t \in [0, T]$.

If in addition $\frac{\tau_j}{\varepsilon_j} \to 0$, *then also the following assertions hold:*

(ii') *For every* $0 \leq s \leq t \leq T$,

$$V_{\mathcal{R}}(u; s, t) \leq \liminf_{j\to+\infty}\int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r.$$

(iii') $\varepsilon_j\|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(t)\|_{\mathbb{M}} \to 0$ *for every* $t \in (0, T] \setminus (J_u \cup N)$, *where* $N = \bigcup_{j\in\mathbb{N}}\Pi_{\tau_j}$.

Proof. We can argue as in [10, Theorem 6.1]. In view of the a priori bounds of Corollary 5.4 and (2.4), the sequence $\{\hat{u}_{\tau_j,\varepsilon_j}\}_{j\in\mathbb{N}}$ is uniformly equibounded with uniformly equibounded variation. Then, by virtue of Helly's selection theorem, there exist a subsequence and a function $u \in \mathrm{BV}([0, T]; X)$ complying with (i). Furthermore, with [10, Proposition 4.11 and Lemma 4.12], we get $u \in \mathrm{BV}_{\mathcal{R}}([0, T]; X)$ and assertion (ii).

By virtue of Corollary 5.4 and (2.4), we also have

$$\lim_{j\to+\infty}\varepsilon_j\int_0^T\|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r = 0,$$

whence (iii) follows, up to possibly passing to a further subsequence.

To obtain (ii'), it is enough to observe that

$$\int_s^t \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r = \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r + \int_s^{\mathfrak{s}_{\tau_j}} \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r - \int_t^{\mathfrak{t}_{\tau_j}} \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r,$$

and to notice that the last two terms vanish as $j \to +\infty$, since

$$\int_s^{\mathfrak{s}_{\tau_j}} \mathcal{R}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r \le C \int_s^{\mathfrak{s}_{\tau_j}} \|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|_{\mathbb{M}}\,\mathrm{d}r \le C\frac{\mathfrak{s}_{\tau_j} - s}{\varepsilon_j} \le C\frac{\tau_j}{\varepsilon_j},$$

and the same holds for the other one.

The proof of (iii') follows exactly as in [10, Theorem 6.1], by using (5.26) and recalling that thanks to (5.19) we also have $\overline{u}_{\tau_j,\varepsilon_j}(t) \to u(t)$ for every $t \in [0, T]$. □

Proposition 5.7. *Let u_0^ε and $\varepsilon u_1^\varepsilon$ be uniformly bounded, and let u be the limit function obtained in Proposition 5.6 from a subsequence satisfying*

$$\lim_{j\to+\infty} \varepsilon_j = \lim_{j\to+\infty} \frac{\tau_j}{\varepsilon_j} = 0.$$

Then the inequality

$$\int_s^t \mathcal{R}(v) + \langle D_x\mathcal{E}(r, u(r)), v\rangle\,\mathrm{d}r \ge 0 \tag{5.23}$$

holds for every $v \in X$ and for every $0 \le s \le t \le T$. In particular, the left and right limits of u are locally stable; i.e., they fulfill the inclusions

$$-D_x\mathcal{E}(t, u^-(t)) \in K^* \quad \text{for every } t \in (0, T],$$
$$-D_x\mathcal{E}(t, u^+(t)) \in K^* \quad \text{for every } t \in [0, T].$$

Moreover, if in addition $\varepsilon u_1^\varepsilon \to 0$, there exists a positive Radon measure μ such that for every $0 \le s \le t \le T$ there holds

$$\mathcal{E}(t, u^+(t)) + V_{\mathcal{R}}(u_{\mathrm{co}}; s, t) + \sum_{r\in J_u^{\mathrm{e}}\cap[s,t]} \mu(\{r\}) = \mathcal{E}(s, u^-(s)) + \int_s^t \partial_t\mathcal{E}(r, u(r))\,\mathrm{d}r.$$

In particular,

$$\mathcal{E}(t, u^-(t)) - \mathcal{E}(t, u^+(t)) = \mu(\{t\}) \ge 0 \quad \text{for every } t \in J_u^{\mathrm{e}}.$$

Proof. We only prove (5.23), the remaining assertions being as in [10, Propositions 6.2 and 6.3] and in Proposition 4.5, exploiting Proposition 5.6 (ii') and (iii'). From (5.3) and (2.5), for every $v \in X$ and $k \in \mathcal{K}_{\tau_j}$ we have

$$0 \le \mathcal{R}(v) + \Big\langle \varepsilon_j^2\mathbb{M}\frac{v^k - v^{k-1}}{\tau_j} + \varepsilon_j\mathbb{V}v^k + D_x\mathcal{E}(t^k, u^k) + \frac{\Lambda_{\mathbb{V}}}{2}\tau_j\mathbb{I}v^k, v\Big\rangle.$$

By multiplying both sides by τ_j and summing over k, for every $m, n \in \mathcal{K}_{\tau_j}^0$ with $m \le n$ we now obtain

$$0 \le \varepsilon_j^2\langle\mathbb{M}(v^n - v^m), v\rangle + \sum_{k=m+1}^{n} \tau_j\Big(\mathcal{R}(v) + \langle D_x\mathcal{E}(t^k, u^k), v\rangle + \varepsilon_j\langle\mathbb{V}v^k, v\rangle + \frac{\Lambda_{\mathbb{V}}}{2}\tau_j\langle\mathbb{I}v^k, v\rangle\Big),$$

namely for every $0 \le s \le t \le T$ it holds

$$0 \le \varepsilon_j^2\langle\mathbb{M}(\dot{\hat{u}}_{\tau_j,\varepsilon_j}(\mathfrak{t}_{\tau_j}) - \dot{\hat{u}}_{\tau_j,\varepsilon_j}(\mathfrak{s}_{\tau_j})), v\rangle + \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \mathcal{R}(v) + \langle D_x\mathcal{E}_{\tau_j}(r, \overline{u}_{\tau_j,\varepsilon_j}(r)), v\rangle\,\mathrm{d}r$$
$$+ \varepsilon_j \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \langle\mathbb{V}\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r), v\rangle\,\mathrm{d}r + \frac{\Lambda_{\mathbb{V}}}{2}\tau_j \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \langle\mathbb{I}\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r), v\rangle\,\mathrm{d}r.$$

Passing to the limit as $j \to +\infty$, by Corollary 5.3 (ii) we have that

$$\lim_{j\to+\infty} \varepsilon_j^2 |\langle \mathbb{M}\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(\mathfrak{t}_{\tau_j}), v\rangle| = \lim_{j\to+\infty} \varepsilon_j^2 |\langle \mathbb{M}\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(\mathfrak{s}_{\tau_j}), v\rangle| = 0.$$

From Corollary 5.4, (2.4) and the Cauchy–Schwarz inequality, we also have

$$\Big|\varepsilon_j \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \langle \mathbb{V}\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r), v\rangle \,\mathrm{d}r\Big| \le C\|v\|\varepsilon_j \int_0^T \|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\| \,\mathrm{d}r \to 0,$$

and a similar argument shows that the last term in the inequality above vanishes as well as $j \to +\infty$.

We conclude observing that (5.17) and (5.19) allow us to use the dominated convergence theorem, getting

$$\lim_{j\to+\infty} \int_{\mathfrak{s}_{\tau_j}}^{\mathfrak{t}_{\tau_j}} \mathcal{R}(v) + \langle D_x\mathcal{E}_{\tau_j}(r, \overline{u}_{\tau_j,\varepsilon_j}(r)), v\rangle \,\mathrm{d}r = \int_s^t \mathcal{R}(v) + \langle D_x\mathcal{E}(r, u(r)), v\rangle \,\mathrm{d}r. \qquad \square$$

5.2 The convergence result

As already done in the time-continuous setting in (3.3), we now rephrase the energy inequality (5.4) in terms of De Giorgi's principle. For simplicity, we set

$$w^k := -\varepsilon^2\mathbb{M}\frac{v^k - v^{k-1}}{\tau} - D_x\mathcal{E}(t^k, u^k) - \frac{\Lambda_{\mathbb{V}}}{2}\tau\mathbb{I}v^k, \quad k \in \mathcal{K}_\tau.$$

Recalling the definitions of the interpolants $\overline{u}_{\tau,\varepsilon}$ (see (5.13)) and $\tilde{u}_{\tau,\varepsilon}$ (see (5.15)), for $t \in [0, T] \setminus \Pi_\tau$ we define

$$\overline{w}_{\tau,\varepsilon}(t) := -\varepsilon^2\mathbb{M}\ddot{\tilde{u}}_{\tau,\varepsilon}(t) - D_x\mathcal{E}_\tau(t, \overline{u}_{\tau,\varepsilon}(t)) - \frac{\Lambda_{\mathbb{V}}}{2}\tau\mathbb{I}\dot{\hat{u}}_{\tau,\varepsilon}(t),$$
$$\tilde{w}_{\tau,\varepsilon}(t) := -\varepsilon^2\mathbb{M}\ddot{\tilde{u}}_{\tau,\varepsilon}(t) - D_x\mathcal{E}(t, \tilde{u}_{\tau,\varepsilon}(t)).$$

Then, by virtue of Corollary 5.4 and Proposition 5.5, if $\frac{\tau}{\varepsilon}$ is bounded, we deduce

$$\max_{t\in[0,T]\setminus\Pi_\tau} (\|\overline{w}_{\tau,\varepsilon}(t)\|_* + \|\tilde{w}_{\tau,\varepsilon}(t)\|_*) \le C. \tag{5.24}$$

Furthermore, thanks to the continuity of $D_x\mathcal{E}$, if $\frac{\tau}{\varepsilon} \to 0$, we also have

$$\lim_{(\varepsilon,\tau)\to(0,0)} \max_{t\in[0,T]\setminus\Pi_\tau} \|\overline{w}_{\tau,\varepsilon}(t) - \tilde{w}_{\tau,\varepsilon}(t)\|_* = 0. \tag{5.25}$$

From (5.3) and recalling that using (2.1) it holds

$$\mathcal{R}_\varepsilon(v^k) + \mathcal{R}_\varepsilon^*(w^k) = \langle w^k, v^k\rangle,$$

inequality (5.4) can be rewritten, arguing in a similar way, as

$$\frac{\varepsilon^2}{2}\|v^n\|_{\mathbb{M}}^2 + \mathcal{E}(t^n, u^n) + \sum_{k=m+1}^n \tau(\mathcal{R}_\varepsilon(v^k) + \mathcal{R}_\varepsilon^*(w^k))$$
$$\le \frac{\varepsilon^2}{2}\|v^m\|_{\mathbb{M}}^2 + \mathcal{E}(t^m, u^m) + \sum_{k=m+1}^n \int_{t^{k-1}}^{t^k} \partial_t\mathcal{E}(r, u^{k-1}) \,\mathrm{d}r + \frac{\Lambda - \Lambda_{\mathbb{V}}}{2}\tau^2 \sum_{k=m+1}^n \tau\|v^k\|_{\mathbb{I}}^2,$$

and thus, in terms of interpolants, as

$$\frac{\varepsilon^2}{2}\|\dot{\hat{u}}_{\tau,\varepsilon}(t)\|_{\mathbb{M}}^2 + \mathcal{E}_\tau(t, \overline{u}_{\tau,\varepsilon}(t)) + \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau} \mathcal{R}_\varepsilon(\dot{\hat{u}}_{\tau,\varepsilon}(r)) + \mathcal{R}_\varepsilon^*(\overline{w}_{\tau,\varepsilon}(r)) \,\mathrm{d}r$$
$$\le \frac{\varepsilon^2}{2}\|\dot{\hat{u}}_{\tau,\varepsilon}(s)\|_{\mathbb{M}}^2 + \mathcal{E}_\tau(s, \overline{u}_{\tau,\varepsilon}(s)) + \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau} \partial_t\mathcal{E}(r, \underline{u}_{\tau,\varepsilon}(r)) \,\mathrm{d}r + \frac{\Lambda - \Lambda_{\mathbb{V}}}{2}\tau^2 \int_{\mathfrak{s}_\tau}^{\mathfrak{t}_\tau} \|\dot{\hat{u}}_{\tau,\varepsilon}(r)\|_{\mathbb{I}}^2 \,\mathrm{d}r \tag{5.26}$$

for every $s, t \in (-\tau, T] \setminus \Pi_\tau$ with $s \le t$.

To conclude the proof of Theorem 3.10 (ii), we only have to confirm the validity of an analogue of Proposition 4.6 for the function u obtained with Proposition 5.6. For this, we will need to reinforce the assumption $\frac{\tau}{\varepsilon} \to 0$ by requiring, in addition, that $\frac{\tau}{\varepsilon^2}$ is uniformly bounded (see (5.27) below), in order to exploit (5.20).

Proposition 5.8. *Let $u_0^\varepsilon \to u_0$, $\varepsilon u_1^\varepsilon \to 0$ and let u be the limit function obtained in Proposition 5.6 from a subsequence satisfying*

$$\lim_{j\to+\infty} \varepsilon_j = 0 \quad \textit{and} \quad \sup_{j\in\mathbb{N}} \frac{\tau_j}{\varepsilon_j^2} < +\infty. \tag{5.27}$$

Then for every $t \in [0,T]$, it holds

$$\mathcal{E}(t,u^-(t)) - \mathcal{E}(t,u^+(t)) \geq \sup_{p\in\mathrm{RCP}_{\mathbb{V}}} c_t^{\mathbb{M},p}(u^-(t),u^+(t)). \tag{5.28}$$

Proof. As already remarked in the proof of Proposition 4.6, it will suffice to prove (5.28) in the case $t \in J_u^{\mathrm{e}}$. By arguing as in [30, Proposition 5.9], taking into account Proposition 5.6, by a diagonal argument we may assume that there are two sequences $t_j^- \nearrow t$ and $t_j^+ \searrow t$ such that

$$\lim_{j\to+\infty} \|\widehat{u}_{\tau_j,\varepsilon_j}(t_j^-) - u^-(t)\| + \|\widehat{u}_{\tau_j,\varepsilon_j}(t_j^+) - u^+(t)\| = 0$$

and

$$\lim_{j\to+\infty} \varepsilon_j \dot{\widehat{u}}_{\tau_j,\varepsilon_j}(t_j^-) = \lim_{j\to+\infty} \varepsilon_j \dot{\widehat{u}}_{\tau_j,\varepsilon_j}(t_j^+) = 0. \tag{5.29}$$

By exploiting (5.27), Proposition 5.5 yields as a byproduct

$$\begin{cases} \lim\limits_{j\to+\infty} \|\overline{u}_{\tau_j,\varepsilon_j}(t_j^-) - u^-(t)\| + \|\overline{u}_{\tau_j,\varepsilon_j}(t_j^+) - u^+(t)\| = 0, \\ \lim\limits_{j\to+\infty} \|\widetilde{u}_{\tau_j,\varepsilon_j}(t_j^-) - u^-(t)\| + \|\widetilde{u}_{\tau_j,\varepsilon_j}(t_j^+) - u^+(t)\| = 0 \end{cases} \tag{5.30}$$

and

$$\lim_{j\to+\infty} \varepsilon_j \dot{\widetilde{u}}_{\tau_j,\varepsilon_j}(t_j^-) = \lim_{j\to+\infty} \varepsilon_j \dot{\widetilde{u}}_{\tau_j,\varepsilon_j}(t_j^+) = 0. \tag{5.31}$$

The continuity of $\mathcal{E}$ together with (5.17) and (5.30) now implies that

$$\lim_{j\to+\infty} \mathcal{E}_{\tau_j}(t_j^-, \overline{u}_{\tau_j,\varepsilon_j}(t_j^-)) - \mathcal{E}_{\tau_j}(t_j^+, \overline{u}_{\tau_j,\varepsilon_j}(t_j^+)) = \mathcal{E}(t,u^-(t)) - \mathcal{E}(t,u^+(t)). \tag{5.32}$$

For a lighter exposition, with a little abuse of notation we denote by t_j^- and t_j^+ the least points of the partition Π_{τ_j} which are greater than or equal to t_j^- and t_j^+, respectively (see (5.16)). By exploiting (5.26), (5.29), (5.32), and from the definition of the contact potential $p_{\mathbb{V}}$, we thus infer

$$\begin{aligned} \mathcal{E}(t,u^-(t)) - \mathcal{E}(t,u^+(t)) &= \lim_{j\to+\infty} \Big[\frac{\varepsilon_j^2}{2} \|\dot{\widehat{u}}_{\tau_j,\varepsilon_j}(t_j^-)\|_{\mathbb{M}}^2 + \mathcal{E}_{\tau_j}(t_j^-, \overline{u}_{\tau_j,\varepsilon_j}(t_j^-)) + \int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+} \partial_t \mathcal{E}(r, \underline{u}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r \\ &\qquad + \frac{\Lambda - \Lambda_{\mathbb{V}}}{2} \tau^2 \int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+} \|\dot{\widehat{u}}_{\tau_j,\varepsilon_j}(r)\|_{\mathbb{I}}^2\,\mathrm{d}r - \frac{\varepsilon_j^2}{2} \|\dot{\widehat{u}}_{\tau_j,\varepsilon_j}(t_j^+)\|_{\mathbb{M}}^2 - \mathcal{E}_{\tau_j}(t_j^+, \overline{u}_{\tau_j,\varepsilon_j}(t_j^+)) \Big] \\ &\geq \limsup_{j\to+\infty} \int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+} \mathcal{R}_{\varepsilon_j}(\dot{\widehat{u}}_{\tau_j,\varepsilon_j}(r)) + \mathcal{R}_{\varepsilon_j}^*(\overline{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r \\ &\geq \limsup_{j\to+\infty} \int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+} p_{\mathbb{V}}(\dot{\widehat{u}}_{\tau_j,\varepsilon_j}(r), \overline{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r. \end{aligned}$$

Taking into account (3.9), we can continue the above inequality, getting

$$\begin{aligned}&\mathcal{E}(t,u^-(t))-\mathcal{E}(t,u^+(t))\\&\quad\geq\limsup_{j\to+\infty}\Big[\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}p_{\mathbb{V}}(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\overline{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r-C\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}(1+\|\overline{w}_{\tau_j,\varepsilon_j}(r)\|_*)\|\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r)-\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r\Big].\end{aligned}$$

Thanks to (5.20), (5.24) and the assumption (5.27), we easily obtain

$$\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}(1+\|\overline{w}_{\tau_j,\varepsilon_j}(r)\|_*)\|\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r)-\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r\leq C\frac{\tau_j}{\varepsilon_j^2}(\mathrm{t}_j^+-\mathrm{t}_j^-)\leq C(\mathrm{t}_j^+-\mathrm{t}_j^-)\to 0. \tag{5.33}$$

Thus, also taking any $p\in\mathrm{RCP}_{\mathbb{V}}$, we get

$$\mathcal{E}(t,u^-(t))-\mathcal{E}(t,u^+(t))\geq\limsup_{j\to+\infty}\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}p_{\mathbb{V}}(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\overline{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r\geq\limsup_{j\to+\infty}\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}p(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\overline{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r.$$

By exploiting property (iv) of Definition 3.2 and using (5.18) and (5.25), we deduce

$$\begin{aligned}&\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}\big|p(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\overline{w}_{\tau_j,\varepsilon_j}(r))-p(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\widetilde{w}_{\tau_j,\varepsilon_j}(r))\big|\,\mathrm{d}r\\&\quad\leq L\max_{r\in[0,T]\setminus\Pi_\tau}\|\overline{w}_{\tau_j,\varepsilon_j}(r)-\widetilde{w}_{\tau_j,\varepsilon_j}(r)\|_*\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}\|\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r\\&\quad\leq L\max_{r\in[0,T]\setminus\Pi_\tau}\|\overline{w}_{\tau_j,\varepsilon_j}(r)-\widetilde{w}_{\tau_j,\varepsilon_j}(r)\|_*\Big(\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}\|\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r)-\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r+\int_0^T\|\dot{\hat{u}}_{\tau_j,\varepsilon_j}(r)\|\,\mathrm{d}r\Big)\to 0.\end{aligned}$$

We indeed notice that the first term within the brackets is bounded (it actually vanishes) by arguing as in (5.33).

Therefore, we finally obtain

$$\begin{aligned}\mathcal{E}(t,u^-(t))-\mathcal{E}(t,u^+(t))&\geq\limsup_{j\to+\infty}\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}p(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),\widetilde{w}_{\tau_j,\varepsilon_j}(r))\,\mathrm{d}r\\&=\limsup_{j\to+\infty}\int_{\mathrm{t}_j^-}^{\mathrm{t}_j^+}p(\dot{\tilde{u}}_{\tau_j,\varepsilon_j}(r),-\varepsilon_j^2\mathbb{M}\ddot{\tilde{u}}_{\tau_j,\varepsilon_j}(r)-D_x\mathcal{E}(r,\tilde{u}_{\tau_j,\varepsilon_j}(r)))\,\mathrm{d}r.\end{aligned}$$

The rest of the proof follows closely the argument of Proposition 4.6 from (4.17) on, with $\tilde{u}_{\tau_j,\varepsilon_j}$ in place of u^{ε_j}, by exploiting (5.30) and (5.31). Thus, we omit the details. □

5.3 An enhanced version of the scheme

We conclude by proposing a slightly modified discrete algorithm which allows us to get rid of the assumption that

$$\frac{\tau}{\varepsilon^2}\quad\text{is bounded,}$$

needed for Proposition 5.8. To describe it, we consider an additional parameter $\delta \in [0, 1)$, in (5.1a) we replace ε by $\sqrt{\varepsilon^2 + \delta}$, and we carefully adjust the initial velocity; namely we consider the following incremental variational scheme:

$$\begin{cases} u^k_{\tau,\varepsilon,\delta} \in \operatorname*{arg\,min}_{x \in X} \mathcal{F}_{\tau,\sqrt{\varepsilon^2+\delta}}(t^k, x, u^{k-1}_{\tau,\varepsilon,\delta}, u^{k-2}_{\tau,\varepsilon,\delta}), & k \in \mathcal{K}_\tau, \\ u^0_{\tau,\varepsilon,\delta} := u^\varepsilon_0, \quad u^{-1}_{\tau,\varepsilon,\delta} := u^\varepsilon_0 - \tau \dfrac{\varepsilon}{\sqrt{\varepsilon^2 + \delta}} u^\varepsilon_1. \end{cases} \tag{5.34}$$

For $\delta = 0$, we easily recover the original scheme (5.1).

Since the only change with respect to previous sections is the replacement of ε by $\sqrt{\varepsilon^2 + \delta}$, all results still hold true if in the statements one performs the same replacement (without touching the initial data $u^\varepsilon_0, u^\varepsilon_1$). In particular, Theorem 3.10 (ii) can be rewritten as follows.

Theorem 5.9. *Let* $\mathbb{M}$ *and* $\mathbb{V}$ *satisfy* (2.2) *and* (2.3) *and assume (E1)–(E5), (E3') and (R1). Let* $u^\varepsilon_0 \to u_0$ *and* $\varepsilon u^\varepsilon_1 \to 0$. *Then for every sequence* $(\tau_j, \varepsilon_j, \delta_j) \to (0, 0, 0)$ *satisfying*

$$\sup_{j \in \mathbb{N}} \frac{\tau_j}{\varepsilon_j^2 + \delta_j} < +\infty, \tag{5.35}$$

there exists a subsequence (not relabelled) along which the sequence of piecewise affine interpolants $\widehat{u}_{\tau_j,\varepsilon_j,\delta_j}$ *pointwise converges to an inertial virtual viscosity solution of the rate-independent system* (1.1).

Furthermore, the limit function is an inertial balanced viscosity solution if $\mathbb{V}$ *is positive-definite.*

The advantage of condition (5.35) is that it is automatically satisfied by any sequence δ_j for which

$$\sup_{j \in \mathbb{N}} \frac{\tau_j}{\delta_j} < +\infty, \tag{5.36}$$

and thus it permits to separate the vanishing rates of τ_j and ε_j, which can be completely unrelated.

We finally notice that the simplest choice of $\delta = \tau$ in (5.34) trivially fulfils (5.36) along any subsequence, and thus allows to obtain Theorem 5.9 without really adding a further parameter.

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