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Fine Representation of Hessian of Convex Functions and Ricci Tensor on RCD Spaces

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Abstract

It is known that on RCD spaces one can define a distributional Ricci tensor $\mathbf{Ric}$. Here we give a fine description of this object by showing that it admits the polar decomposition

$$\mathbf{Ric} = \omega\, |\mathbf{Ric}|$$

for a suitable non-negative measure $|\mathbf{Ric}|$ and unitary tensor field ω. The regularity of both the mass measure and of the polar vector are also described. The representation provided here allows to answer some open problems about the structure of the Ricci tensor in such singular setting. Our discussion also covers the case of Hessians of convex functions and, under suitable assumptions on the base space, of the Sectional curvature operator.

Keywords RCD space · Fine representation · Ricci tensor · Convex function

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1 Introduction

A classical statement in modern analysis asserts that a positive distribution is a Radon measure. This fact extends to tensor-valued distributions so that, for instance, the distributional Hessian of a convex function on $\mathbb{R}^d$, that for trivial reasons is a symmetric non-negative matrix-valued distribution, can be represented by a matrix-valued measure. The proof for the tensor-valued case follows from the scalar-valued case simply by looking at the coordinates of the tensor. To put it differently, the fact that on $\mathbb{R}^d$ we can find an orthonormal basis of the tangent bundle made of smooth vectors allows to regard a tensor-valued distribution as a collection of scalar-valued ones and thus to transfer results valid in the latter case into the former one.

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The fact that positive functionals defined on a sufficiently large class of functions are represented by measures can be extended far beyond the Euclidean setting, up to at least locally compact spaces: this is the content of the Riesz–Daniell–Stone representation theorems. In this paper we are concerned with the tensor-valued case when the underlying space is an $\mathrm{RCD}(K, N)$ space. These classes of spaces, introduced in [20] after [3, 34, 36, 37] (see the surveys [1, 24] and references therein) are the non-smooth counterpart of Riemannian manifolds with Ricci curvature $\geq K$ and dimension $\leq N$. One of the key features of these spaces, and in fact the essence of the proposal in [20], is that calculus on them is built upon the notions of "Sobolev functions" and "integration by parts". As such, it is perhaps not surprising that distribution-like tensors appear frequently in the field. Let us mention three different instances when this occurs, where the relevant tensor is non-negative (or at least bounded from below):

i) The **Hessian of a convex function**. As observed in [31, 39], to a regular enough function f on an RCD space one can associate a suitable "distributional Hessian" that acts on sufficiently smooth vector fields: reformulating a bit the definition in [31], the Hessian of f is the map

$$\{\text{smooth vector fields}\} \ni X, Y$$
$$\mapsto \quad \mathrm{Hess}(f)(X, Y) := -\frac{1}{2}\int_{\mathsf{X}} \big(\mathrm{div}\, X \nabla f \cdot Y + \mathrm{div}\, Y \nabla f \cdot X + \nabla f \cdot \nabla(X \cdot Y)\big) \mathrm{d}\mathsf{m}$$

and it turns out, see [31, Theorem 7.1] that under suitable regularity assumptions on f we have

$$f \text{ is } \kappa\text{-convex} \qquad \Leftrightarrow \qquad \mathrm{Hess}(f)(X, X) \geq \kappa \quad \text{for every } X,$$

thus matching the Euclidean distributional characterization of convexity.

ii) The **Ricci curvature** of an $\mathrm{RCD}(K, \infty)$ space. As discussed in [22], one can use the Bochner identity to define what the Ricci tensor is in this low regularity setting, by putting

$$\{\text{smooth vector fields}\} \ni X, Y$$
$$\mapsto \quad \mathbf{Ric}(X, Y) := \Delta \frac{X \cdot Y}{2} + \left(\frac{1}{2} X \cdot \Delta_{\mathrm{H}} Y + \frac{1}{2} Y \cdot \Delta_{\mathrm{H}} X - \nabla X \cdot \nabla Y\right) \mathsf{m}$$

and it turns out that, see [22], in a suitable sense we have

$$\text{the space } (\mathsf{X}, \mathsf{d}, \mathsf{m}) \text{ is } \mathrm{RCD}(\kappa, \infty) \qquad \Leftrightarrow \qquad \mathbf{Ric}(X, X) \geq \kappa |X|^2 \mathsf{m} \quad \text{for every } X.$$

We add a couple of words about the notation used in the equation above, but we refer to [22] for the rigorous definitions. In particular, Δ is the distributional, or measure valued Laplacian and is defined via integration by parts as in the case of the L^2 Laplacian, but it is a signed measure instead of an L^2 function. Moreover, Δ_{H} is the Hodge Laplacian (we identify vector and covector freely, thanks to the Riesz's identification), and, in the smooth context, reads, as usual, as $\mathrm{d}\delta + \delta\mathrm{d}$.

iii) The **Sectional curvature** of an $\mathrm{RCD}(K, \infty)$ space. As discussed in [23], one can give a meaning to the full Riemann curvature tensor on a generic RCD space. In general, one cannot expect any sort of regularity on it, as the lower bound on the Ricci, encoded in the RCD assumption, cannot give any information on the full Riemann tensor. Still, this opens up the possibility of saying when is that the "sectional curvature of an RCD space is bounded from below". The geometric significance of this statement is still unknown.

In each of these cases, a better understanding of the relevant tensor is desirable and a first step in this direction is to comprehend whether the given bound from below forces it to be a measure-like object. To fix the ideas, let us discuss the case of the Ricci curvature: what one would like to know is whether the operator Ric described above can be represented via a sort of polar decomposition as

$$\mathbf{Ric} = \omega\,|\mathbf{Ric}|, \tag{1.1}$$

where $|\mathbf{Ric}|$ is a non-negative measure and ω is a tensor of norm 1 $|\mathbf{Ric}|$-a.e., meaning that the identity

$$\mathbf{Ric}(X, Y) = \omega \cdot (X \otimes Y)\,|\mathbf{Ric}|$$

holds as measures for any pair of sufficiently smooth vector fields X, Y. The main result of this manuscript is that, yes, a representation like Eq. 1.1 holds for the three tensors discussed above, see Theorem 1.2.

Few important remarks are in order (we shall discuss the case of the Ricci curvature, but similar comments are in place for the Hessian and the sectional curvature):

- A writing like that in the right hand side of Eq. 1.1 requires the tensor field ω to be $|\mathbf{Ric}|$-well-defined. In this respect notice that on one side on RCD spaces tensor fields can be well-defined up to Cap-null sets, where Cap is the 2-capacity (in some sense, thanks to the fact that one can speak about Sobolev vector fields - see [18]). On the other hand, the mass measure $|\mathbf{Ric}|$ is absolutely continuous with respect to Cap (because the distributional definition of Ricci tensor is continuous on the space of Sobolev vector fields). The combination of these two facts makes it possible to consider an expression as the one in Eq. 1.1. This perfect matching between the regularity achievable by ω and the sets that can actually be charged by $|\mathbf{Ric}|$ is far from being a coincidence.
- The construction of the polar decomposition as in Eq. 1.1 follows the same rough idea described at the beginning of the introduction: we would like to take a pointwise orthonormal basis $X_1, \ldots, X_n$ of sufficiently regular vector fields and then study the real valued functionals $\varphi \mapsto \int_{\mathsf{X}} \varphi \, \mathrm{d}\mathbf{Ric}(X_i, X_j)$. Clearly, even in a smooth Riemannian manifold one in general cannot find such a global orthonormal basis, but a first problem we encounter here is that such bases only exist on suitable Borel sets $A_k \subseteq \mathsf{X}$ (whose interior might in general be empty). This causes severe technical complications in handling the necessary localization arguments, see for instance the proof of Theorem 1.1.
- Related to the above there is the fact that the mass measure $|\mathbf{Ric}|$ turns out to be a σ-finite Borel measure that in general is *not* Radon. More precisely, on the sets $A_k \subseteq \mathsf{X}$ on which we have a pointwise orthonormal basis, the restriction of $|\mathbf{Ric}|$ is finite (whence σ-finiteness). However, in general it might very well be that there is some point $x \in \mathsf{X}$ such that every neighbourhood of x encounters infinitely many of the A_k's. This happens even in very simple examples such as the tip of a cone, as it is known, see [17] and reference therein, that at the tip of a cone every sufficiently regular vector field must vanish. In particular, it seems very unlikely that a direct application of the technique of [11] can be used in place of the ad-hoc argument that we employ in this manuscript. Indeed, it seems that we cannot use the Riesz's Theorem of [11], as we are not able to prove that the functional $(X, Y) \mapsto \mathbf{Ric}(X, Y)(\mathsf{X})$ is bounded with respect to the relevant norm on vector fields (and we believe that this issue can really happen, especially for what concerns the same discussion in the case of the Hessian). Nevertheless, once that one restricts integration to a set A_k, i.e. considers $(X, Y) \mapsto \mathbf{Ric}(X, Y)(A_k)$, it is possible to use the language of [11], but this possibility comes *a posteriori*, once we know that

$|\mathbf{Ric}| \llcorner A_k$ is a finite measure (and, to have this information, it seems necessary to argue as in this paper). As just discussed, this is not simply a matter of localization, but comes from the fact that some of the sets A_k may have empty interior.

To have a glance at the difficulties that prevent us from proving that $|\mathbf{Ric}|$ is a Radon measure, it may be useful to consider the following example, which should be compared to [18, Example 3.17]. Consider the $\mathrm{RCD}(0, 1)$ space $\mathsf{X} := [0, 1]$, endowed with the natural metric and measure. Assume that we have $A_0 = \{0\}$, $A_1 = (0, 1)$ and $A_2 = \{1\}$, so that $\mathsf{X} = A_0 \cup A_1 \cup A_2$. If we have three Radon measures μ_0, μ_1, μ_2 on A_0, A_1, A_2, respectively, it is in general false that $\mu_0 + \mu_1 + \mu_2$ is a Radon measure on X. To recover Radon measures, we have to consider compact sets of $\mathsf{X} \setminus \{0, 1\}$. In other words, it is certainly necessary to remove the "compactifying effect" of the boundary. Recalling [18, Example 3.17], it is clear that the decomposition of X in sets A_k given by Theorem 2.1 is certainly not better that the one proposed above, just look at the dimension change of the Cap-tangent module over the sets A_k (moreover, with a little more care, we see that the decomposition of X in sets A_k is much worse than the one described above, due to the vanishing of $|\nabla f|(0)$ for $f \in D(\Delta)$). Even though, after [12], boundaries of (non-collapsed) RCD spaces are well understood, we are not able to exploit this theory to give more precise information about the measures involved.

- Despite the above, for any pair of sufficiently regular vector fields X, Y we have $w \cdot (X \otimes Y) \in L^1(|\mathbf{Ric}|)$.
- Since $|\mathbf{Ric}|$ is not Radon, in constructing the representation Eq. 1.1, and more generally in understanding these distributional objects we have discussed, we cannot rely on the theory of local vector measures that we recently developed in [11].
- The representation Eq. 1.1 marks a clear step forward in the understanding of the Ricci tensor on RCD spaces, as what was previously manageable only via integration by parts - and thus required regularity of the vector fields involved - now is realized as a 0th-order object and thus has a more pointwise meaning. For instance, it allows to quickly solve a problem that was left open in [22]. The problem was as follows: suppose that X_i, X, Y, $i = 1, \ldots, n$, are smooth vector fields, that $f_i \in C_b(\mathsf{X})$ and that $\sum_i f_i X_i = X$. Can we conclude that $\sum_i f_i \mathbf{Ric}(X_i, Y) = \mathbf{Ric}(X, Y)$? One certainly expects the answer to be affirmative, but if the only definition of $\mathbf{Ric}$ involves integration by parts - as it was the case in [22] - then it is not clear how to conclude, given that in general $f_i X_i$ is not regular enough to justify the necessary computations. On the other hand, the representation Eq. 1.1 immediately allows to positively answer the question.

1.1 Statements of the Main Results

We now state the main results of this paper. These results aim at addressing points i), ii) and iii) raised in the first part of the introduction. We refer to the main part of the manuscript for the relevant definitions, to avoid overloading this introduction. In particular, we will need Definition 3.2 and Definition 3.6 for what concerns the Hessian, Theorem 3.10, which is [22, Theorem 3.6.7], for the Ricci tensor, and finally the definition of Riemann tensor as recalled at the beginning of section 3.4 (that is [23]) with Definition 3.17 for the Riemann tensor.

Theorem 1.1 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and* $f \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$ *satisfying* $\mathrm{Hess} f \geq \kappa$, *for some* $\kappa \in \mathbb{R}$. *Then,* $f \in D(\mathbf{Hess})$, *say* $\mathbf{Hess} f = \nu_f |\mathbf{Hess} f|$. *Moreover, we have that*

$$\nu_f |\mathbf{Hess} f| \geq \kappa g \mathsf{m},$$

in the sense that for every $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$*, it holds, as measures,*

$$v \otimes v \cdot \nu_f |\mathbf{Hess} f| \geq \kappa |v|^2 \mathsf{m}.$$

Finally, if in addition $f \in \mathrm{H}^{1,2}(\mathsf{X})$*, then for every* $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$*, we have that* $X \otimes Y \cdot \nu_f \in \mathrm{L}^1(|\mathbf{Hess} f|)$*, in particular, Eq. 3.4 holds for every* $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ *and* $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$*. More precisely, we have the explicit bound, for every* $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$

$$\int_{\mathsf{X}} |X \otimes X \cdot \nu_f| \,\mathrm{d}|\mathbf{Hess} f| \leq \int_{\mathsf{X}} -\mathrm{div}\, X \nabla f \cdot X - \nabla f \cdot \nabla(\tfrac{1}{2}|X|^2) + 2\kappa^- |X|^2 \,\mathrm{d}\mathsf{m}.$$

Theorem 1.2 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then there exists a unique* σ*-finite measure* $|\mathbf{Ric}|$ *that satisfies* $|\mathbf{Ric}| \ll \mathrm{Cap}$ *and a unique, up to* $|\mathbf{Ric}|$*-a.e. equality, symmetric tensor field* $\omega \in \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 2}\mathsf{X})$ *with* $|\omega| = 1$ $|\mathbf{Ric}|$*-a.e. such that* $\mathbf{Ric} = \omega |\mathbf{Ric}|$*, in the sense that for every* $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *we have that* $X \otimes Y \cdot \omega \in \mathrm{L}^1(|\mathbf{Ric}|)$ *and it holds that, as measures,*

$$X \otimes Y \cdot \omega |\mathbf{Ric}| = \mathbf{Ric}(X, Y).$$

Moreover

$$\omega |\mathbf{Ric}| \geq K \mathsf{g} \mathsf{m},$$

in the sense that for every $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$*, it holds, as measures,*

$$v \otimes v \cdot \omega |\mathbf{Ric}| \geq K |v|^2 \mathsf{m}.$$

Theorem 1.3 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space with sectional curvature bounded below by* κ*, for some* $\kappa \in \mathbb{R}$*. Then there exists a unique* σ*-finite measure* $|\mathbf{Riem}|$ *that satisfies* $|\mathbf{Riem}| \ll \mathrm{Cap}$ *and a unique, up to* $|\mathbf{Riem}|$*-a.e. equality, tensor field* $\nu \in \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X})$ *with* $|\nu| = 1$ $|\mathbf{Riem}|$*-a.e. such that for every* $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ *we have that* $X \otimes Y \otimes Z \otimes W \cdot \nu \in \mathrm{L}^1(|\mathbf{Riem}|)$ *and it holds*

$$\int_{\mathsf{X}} f X \otimes Y \otimes Z \otimes W \cdot \nu |\mathbf{Riem}| = \boldsymbol{\mathcal{R}}(X, Y, Z, W)(f) \qquad \text{for every } f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}).$$

For every $v, w \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$*,*

$$v \otimes w \otimes w \otimes v \cdot \nu |\mathbf{Riem}| \geq \kappa |v \wedge w|^2 \mathsf{m}.$$

The tensor field ν *has the following symmetries. Let* $\mathcal{I}, \mathcal{J}, \mathcal{K} : \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X}) \to \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X})$ *be the linear maps characterized as follows*

$$\begin{aligned}
\mathcal{I}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) &:= v_2 \otimes v_1 \otimes v_3 \otimes v_4\\
\mathcal{J}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) &:= v_3 \otimes v_4 \otimes v_1 \otimes v_2\\
\mathcal{K}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) &:= v_2 \otimes v_3 \otimes v_1 \otimes v_4.
\end{aligned}$$

Then, with respect to $|\mathbf{Riem}|$*-a.e. equality,*

$$\begin{aligned}
\mathcal{I}(\nu) &= -\nu\\
\mathcal{J}(\nu) &= \nu\\
\nu + \mathcal{K}(\nu) + \mathcal{K}^2(\nu) &= 0.
\end{aligned}$$

2 Preliminaries

2.1 RCD Spaces

In this note we are going to consider RCD spaces, which we now briefly introduce. An $\mathrm{RCD}(K, N)$ space is an infinitesimally Hilbertian [20] metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ satisfying a lower Ricci curvature bound and an upper dimension bound (meaningful if $N < \infty$) in a synthetic sense according to [34, 36, 37], see [1, 24, 40] and references therein. We assume the reader to be familiar with this material. Whenever we write $\mathrm{RCD}(K, N)$, we implicitly assume that $N < \infty$, unless otherwise stated.

Also, we assume that the reader is familiar with the calculus developed on this kind of non-smooth structures ([20, 22], see also [21, 25]): in particular, we assume familiarity with Sobolev spaces (and heat flow), with the notions of (co)tangent module and its tensor and exterior products, which are the non-smooth analogue of the space of sections of the (tensor/exterior product of the) tangent bundle (see also Section 2.2.1 and 2.2.2), and with the notions of divergence, Laplacian, Hessian and covariant derivative, together with their properties.

Throughout this work, we are going to use several function spaces, as well as operators. This material can be found in [22] (or [25]), however, we recall some of the notions that we are going to use more frequently, for the readers' convenience. First, the subscript "bs" has to be understood as "with bounded support", whereas "loc" means locally, in the sense that the given property has to hold in an open neighbourhood of each point (for spaces in which bounded sets are relatively compact, this means that the property has to hold on open sets with compact closure). We adopt the standard notation for the Lebesgue space L^p and space of Lipschitz functions LIP. $\mathrm{H}^{1,2}(\mathsf{X})$ is the space of Sobolev functions, whereas § is the Sobolev class, i.e. those functions f non necessarily integrable, but whose gradient is in L^2. The spaces $\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X})$ and $\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ are the closure of test objects in $\mathrm{L}^2(T\mathsf{X})$ with respect to the norm induced by the covariant derivative ∇_{C} (that we sometime simply denote by ∇) and the operator (d, δ), respectively (i.e. the topologies $\mathrm{W}^{1,2}_{\mathrm{C}}(T\mathsf{X})$ and $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$), see [22, Definition 3.4.3 and Definition 3.5.13].

We give now our working definition for the space of test functions and test vector fields. Following [22, 35] (with the additional request of an L^∞ bound on the Laplacian), we define the vector space of test functions on an $\mathrm{RCD}(K, \infty)$ space as

$$\mathrm{TestF}(\mathsf{X}) := \{f \in \mathrm{LIP}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}) \cap D(\Delta) : \Delta f \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})\},$$

and the vector space of test vector fields as

$$\mathrm{TestV}(\mathsf{X}) := \left\{ \sum_{i=1}^n f_i \nabla g_i : f_i \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}),\, g_i \in \mathrm{TestF}(\mathsf{X}) \right\}.$$

To be precise, the original definition of $\mathrm{TestV}(\mathsf{X})$ given by the second named author was slightly different. However, when using test vector fields to define regular subsets of vector fields such as $\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X})$ and $\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, the two definitions produce the same subspaces, see for example the proofs of [9, Lemma 4.3 and Lemma 4.4]. The advantage of working with this slightly more general class lies in the fact that

$$\frac{1}{1 \vee |v|} v \in \mathrm{TestV}(\mathsf{X}) \qquad \text{for every } v \in \mathrm{TestV}(\mathsf{X}),$$

whereas the drawback is that for $v \in \mathrm{TestV}(\mathsf{X})$, in general we do not have $\mathrm{div}(v) \in \mathrm{L}^\infty(\mathsf{m})$. Nevertheless, we are still going to need the classical definition of test vector fields (used, in particular in the references [22, 23] for what concerns Ricci and Riemann tensors): we call such space $\mathcal{V}$, i.e.

$$\mathcal{V} := \left\{ \sum_{i=1}^{n} f_i \nabla g_i : f_i, g_i \in \mathrm{TestF}(\mathsf{X}) \right\}. \tag{2.1}$$

For future reference, we recall here [9, Lemma 4.3].

Lemma 2.1 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space,* $X \in \mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ *and* $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$. *Then* $fX \in \mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *and*

$$\begin{aligned} \mathrm{div}(fX) &= \nabla f \cdot X + f \mathrm{div}\, X, \\ \mathrm{d}(fX) &= \nabla f \wedge X + f \mathrm{d}X. \end{aligned}$$

If moreover $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, *then* $fX \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$.

The following calculus lemma will serve as a key tool in proving, in a certain sense, a strong locality property of some measures.

Lemma 2.2 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$. *Take* $\{\tilde{\varphi}_n\}_n \subseteq \mathrm{LIP}_{\mathrm{b}}(\mathbb{R})$ *defined by*

$$\tilde{\varphi}_n(x) := \begin{cases} 1 & \text{if } x \le 0, \\ 1 - nx & \text{if } 0 < x < n^{-1}, \\ 0 & \text{if } x \ge n^{-1}. \end{cases}$$

Let then $\varphi_n := \tilde{\varphi}_n \circ |X|$. *Then* $\varphi_n \in \mathrm{H}^{1,2}(\mathsf{X})$, $\|\varphi_n\|_{\mathrm{L}^\infty(\mathsf{m})} \le 1$ *and* $\varphi_n X \to 0$ *in the* $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *topology.*

Proof By [18, Lemma 2.5], $\varphi_n \in \mathrm{H}^{1,2}(\mathsf{X})$ for every n with

$$|\nabla \varphi_n| = |\tilde{\varphi}_n'| \circ |X| \, |\nabla |X|| \le n \chi_{\{|X| \in (0, n^{-1})\}} |\nabla X| \qquad \mathsf{m}\text{-a.e.}$$

In particular,

$$|\nabla \varphi_n||X| \le \chi_{\{|X| \in (0, n^{-1})\}} |\nabla X| \qquad \mathsf{m}\text{-a.e.} \tag{2.2}$$

Notice that integrating by parts and using standard approximation arguments, taking into account Eq. 2.2 (which also gives the membership of the right hand sides to the relevant spaces) we have

$$\begin{aligned} \mathrm{div}(\varphi_n X) &= \nabla \varphi_n \cdot X + \varphi_n \mathrm{div}\, X \in \mathrm{L}^2(\mathsf{m}) \\ \mathrm{d}(\varphi_n X) &= \nabla \varphi_n \wedge X + \varphi_n \mathrm{d}\, X \in \mathrm{L}^2(\Lambda^2 T^* \mathsf{X}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{div}(\varphi_n X) &\to \chi_{\{|X|=0\}} \mathrm{div}\, X && \text{in } \mathrm{L}^2(\mathsf{m}) \\ \mathrm{d}(\varphi_n X) &\to \chi_{\{|X|=0\}} \mathrm{d}X && \text{in } \mathrm{L}^2(\Lambda^2 T^* \mathsf{X}). \end{aligned}$$

Now, by dominated convergence, $\varphi_n X \to 0$ in $\mathrm{L}^2(T\mathsf{X})$ so that by the closure of the operators div and d (see [22, Theorem 3.5.2]), it holds that $\varphi_n X \to 0$ in $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$. □

Remark 2.3 Inspecting the proof of Lemma 2.2, we see that if $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(K, \infty)$ space and $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, then $\mathrm{div}\, X = 0$ m-a.e. on $\{X = 0\}$ and similarly $\mathrm{d}X = 0$ m-a.e. on $\{X = 0\}$. ■

2.2 Cap-Modules

In this subsection we recall the basic theory of Cap-modules for RCD spaces. We assume familiarity with the definition of capacitary modules, quasi-continuous functions and vector fields and related material in [18]. A summary of the material we will use can be found in [13, Section 1.3]. For the reader's convenience, we write the results that we will need most frequently.

First, we recall that exploiting Sobolev functions, we define the 2-capacity (to which we shall simply refer as capacity) of any set $A \subseteq \mathsf{X}$ as

$$\mathrm{Cap}(A) := \inf \left\{ \|f\|^2_{\mathrm{H}^{1,2}(\mathsf{X})} : f \in \mathrm{H}^{1,2}(\mathsf{X}),\ f \geq 1 \ \mathsf{m}\text{-a.e. on some neighbourhood of } A \right\}. \tag{2.3}$$

An important object will be the one of fine tangent module, as follows (QCR stands for "quasi continuous representative").

Theorem 2.4 ([18, Theorem 2.6]) *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. Then there exists a unique pair* $(\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X}), \bar\nabla)$*, where* $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *is a* $\mathrm{L}^0(\mathrm{Cap})$*-normed* $\mathrm{L}^0(\mathrm{Cap})$*-module and* $\bar\nabla : \mathrm{TestF}(\mathsf{X}) \to \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *is a linear operator such that:*

i) $|\bar\nabla f| = \mathrm{QCR}(|\nabla f|)$ Cap-*a.e. for every* $f \in \mathrm{TestF}(\mathsf{X})$,
ii) the set $\left\{\sum_n \chi_{E_n} \bar\nabla f_n\right\}$*, where* $\{f_n\}_n \subseteq \mathrm{TestF}(\mathsf{X})$ *and* $\{E_n\}_n$ *is a Borel partition of* X *is dense in* $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$.

Uniqueness is intended up to unique isomorphism, this is to say that if another pair $(\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})', \bar\nabla')$ *satisfies the same properties, then there exists a unique module isomorphism* $\Phi : \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X}) \to \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})'$ *such that* $\Phi \circ \bar\nabla = \bar\nabla'$*. Moreover,* $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *is a Hilbert module that we call capacitary tangent module.*

Notice that we can, and will, extend the map QCR [18] from $\mathrm{H}^{1,2}(\mathsf{X})$ to $\mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ by a locality argument. Also, we often omit to write the map QCR. We define

$$\mathrm{Test}\bar{\mathrm{V}}(\mathsf{X}) := \left\{ \sum_{i=1}^n \mathrm{QCR}(f_i)\bar\nabla g_i : f_i \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}),\ g_i \in \mathrm{TestF}(\mathsf{X}) \right\} \subseteq \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X}).$$

We define also the vector subspace of quasi-continuous vector fields, $\mathcal{QC}(T\mathsf{X})$, as the closure of $\mathrm{Test}\bar{\mathrm{V}}(\mathsf{X})$ in $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$.

Recall now that as $\mathsf{m} \ll \mathrm{Cap}$, we have a natural projection map

$$\mathrm{Pr} : \mathrm{L}^0(\mathrm{Cap}) \to \mathrm{L}^0(\mathsf{m}) \qquad \text{defined as} \qquad [f]_{\mathrm{L}^0(\mathrm{Cap})} \mapsto [f]_{\mathrm{L}^0(\mathsf{m})}$$

where $[f]_{\mathrm{L}^0(\mathrm{Cap})}$ (resp. $[f]_{\mathrm{L}^0(\mathsf{m})}$) denotes the Cap (resp. m) equivalence class of f. It turns out that Pr, restricted to the set of quasi-continuous functions, is injective ([18, Proposition 1.18]). We have the following projection map $\bar{\mathrm{Pr}}$, given by [18, Proposition 2.9 and Proposition 2.13], that plays the role of Pr on vector fields.

Proposition 2.5 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. There exists a unique linear continuous map*

$$\bar{\mathrm{Pr}} : \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X}) \to \mathrm{L}^0(T\mathsf{X})$$

that satisfies

i) $\bar{\mathrm{Pr}}(\bar\nabla f) = \nabla f$ *for every* $f \in \mathrm{TestF}(\mathsf{X})$, *ii)* $\bar{\Pr}(gv) = \Pr(g)\bar{\Pr}(v)$ *for every* $g \in \mathrm{L}^0(\mathrm{Cap})$ *and* $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$.

Moreover, for every $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$,

$$|\bar{\Pr}(v)| = \Pr(|v|) \qquad \mathsf{m}\text{-}a.e.$$

and $\bar{\Pr}$, *when restricted to the set of quasi-continuous vector fields, is injective.*

Notice that $\bar{\Pr}(\mathrm{Test}\bar{\mathrm{V}}(\mathsf{X})) = \mathrm{TestV}(\mathsf{X})$. When there is be no ambiguity, we omit to write the map $\bar{\Pr}$.

Theorem 2.6 ([18, Theorem 2.14 and Proposition 2.13]) *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. Then there exists a unique map* $\mathrm{Q\bar{C}R} : \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \to \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *such that*

i) $\mathrm{Q\bar{C}R}(v) \in \mathcal{QC}(T\mathsf{X})$ *for every* $v \in \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X})$,
ii) $\bar{\Pr} \circ \mathrm{Q\bar{C}R}(v) = v$ *for every* $v \in \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X})$.

Moreover, $\mathrm{Q\bar{C}R}$ *is linear and satisfies*

$$|\mathrm{Q\bar{C}R}(v)| = \mathrm{QCR}(|v|) \qquad \mathrm{Cap}\text{-}a.e.\ \textit{for every}\ v \in \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}),$$

so that $\mathrm{Q\bar{C}R}$ *is continuous.*

We will often omit to write the $\mathrm{Q\bar{C}R}$ operator for simplicity of notation (but it will be clear from the context when we need the fine representative). This should cause no ambiguity thanks to the fact that

$$\mathrm{Q\bar{C}R}(gv) = \mathrm{QCR}(g)\mathrm{Q\bar{C}R}(v) \qquad \text{for every } g \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}) \text{ and } v \in \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X}). \tag{2.4}$$

This can be proved easily by locality and using the fact that the continuity of the map QCR implies that $\mathrm{QCR}(g)\mathrm{Q\bar{C}R}(v)$ as above is quasi-continuous and the injectivity of the map $\bar{\Pr}$ restricted the set of quasi-continuous vector fields yields the conclusion.

The following theorem, that is [13, Section 1.3], will be crucial in the construction of modules tailored to particular measures (see [9, Theorem 4.10] for an explicit proof of this result).

Theorem 2.7 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be a metric measure space and let* μ *be a Borel measure finite on balls such that* $\mu \ll \mathrm{Cap}$. *Let also* $\mathcal{M}$ *be a* $\mathrm{L}^0(\mathrm{Cap})$*-normed* $\mathrm{L}^0(\mathrm{Cap})$*-module. Define the natural (continuous) projection*

$$\pi_\mu : \mathrm{L}^0(\mathrm{Cap}) \to \mathrm{L}^0(\mu).$$

We define an equivalence relation $\sim_\mu$ *on* $\mathcal{M}$ *as*

$$v \sim_\mu w \textit{ if and only if } |v - w| = 0 \qquad \mu\text{-}a.e.$$

Define the quotient module $\mathcal{M}^0_\mu := \mathcal{M}/\sim_\mu$ *with the natural (continuous) projection*

$$\bar{\pi}_\mu : \mathcal{M} \to \mathcal{M}^0_\mu.$$

Then $\mathcal{M}^0_\mu$ *is a* $\mathrm{L}^0(\mu)$*-normed* $\mathrm{L}^0(\mu)$*-module, with the pointwise norm and product induced by the ones of* $\mathcal{M}$*: more precisely, for every* $v \in \mathcal{M}$ *and* $g \in \mathrm{L}^0(\mathrm{Cap})$,

$$\begin{cases} |\bar{\pi}_\mu(v)| := \pi_\mu(|v|), \\ \pi_\mu(g)\bar{\pi}_\mu(v) := \bar{\pi}_\mu(gv). \end{cases} \tag{2.5}$$

If $p \in [1, \infty]$, we set

$$\mathcal{M}^p_\mu := \left\{ v \in \mathcal{M}^0_\mu : |v| \in \mathrm{L}^p(\mu) \right\},$$

that is a $\mathrm{L}^p(\mu)$-normed $\mathrm{L}^\infty(\mu)$-module. Moreover, if $\mathcal{M}$ is a Hilbert module, also $\mathcal{M}^0_\mu$ and $\mathcal{M}^2_\mu$ are Hilbert modules.

Similarly as for $\mathrm{Q\bar{C}R}$, we often omit to write the $\bar{\pi}_\mu$ operator (and also the π_μ operator) for simplicity of notation (but it will be clear from the context when we need the fine representative). Again, this should cause no ambiguity thanks to Eqs. 2.4 and 2.5.

The following lemma, that is [13, Lemma 2.7], provides us with the density of test vector fields in quotient tangent modules.

Lemma 2.8 *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space and let μ be a finite Borel measure such that $\mu \ll \mathrm{Cap}$. Then $\mathrm{TestV}_\mu(\mathsf{X})$ is dense in $\mathrm{L}^p_\mu(T\mathsf{X})$ for every $p \in [1, \infty)$.*

In what follows, with a little abuse, we often write, for $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$, $v \in D(\mathrm{div})$ if and only if $\bar{\mathrm{Pr}}(v) \in D(\mathrm{div})$ and, if this is the case, $\mathrm{div}\, v = \mathrm{div}(\bar{\mathrm{Pr}}(v))$. Similar notation will be used for other operators acting on subspaces of $\mathrm{L}^0(T\mathsf{X})$.

2.2.1 Tensor Product

In this subsection we study the tensor product of normed modules. We focus on Cap-modules, as in these spaces we are going to find the main objects of this note. Fix then $n \in \mathbb{N}$, $n \geq 1$. We assume that $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(K, \infty)$ space, even though this is clearly not always needed. Just for the sake of notation, we set

$$\mathrm{L}^2(T^{\otimes n}\mathsf{X}) := \mathrm{L}^2(T\mathsf{X})^{\otimes n}.$$

Let now $\mathcal{C} \subseteq \mathrm{L}^2(T\mathsf{X})$ be a subspace. We define $\mathcal{C}^{\otimes n}$ as the vector subspace of the Hilbert $\mathrm{L}^2(\mathsf{m})$-normed $\mathrm{L}^\infty(\mathsf{m})$-module $\mathrm{L}^2(T^{\otimes n}\mathsf{X})$ that consists of finite sums of decomposable tensors of the type $v_1 \otimes \cdots \otimes v_n$ where $v_1, \ldots, v_n \in \mathcal{C}$, endowed with the structure of module (included the pointwise norm) induced by the one of $\mathrm{L}^2(T^{\otimes n}\mathsf{X})$. Notice that we can equivalently define $\mathcal{C}^{\otimes n}$ as follows. First, we consider the multilinear map

$$(v_1, \ldots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n \in \mathrm{L}^2(T^{\otimes n}\mathsf{X}) \qquad \text{if } v_1 \ldots, v_n \in \mathcal{C}$$

that factorizes to a well defined linear map

$$\mathcal{C} \otimes^{\mathrm{alg}}_{\mathbb{R}} \cdots \otimes^{\mathrm{alg}}_{\mathbb{R}} \mathcal{C} \to \mathrm{L}^2(T^{\otimes n}\mathsf{X})$$

and see that $\mathcal{C}^{\otimes n}$ coincides with the image of this map. Here, $\otimes^{\mathrm{alg}}_{\mathbb{R}}$ denotes the algebraic tensor product of real vector spaces. Notice that, unless we are in pathological cases, the map we have just defined is not injective. This is equivalent to the fact that not every map defined on $\mathcal{C} \otimes^{\mathrm{alg}}_{\mathbb{R}} \cdots \otimes^{\mathrm{alg}}_{\mathbb{R}} \mathcal{C}$ induces a map defined on $\mathcal{C}^{\otimes n}$, in general.

Let now $\mathcal{M}$ be an Hilbert $\mathrm{L}^0(\mathrm{Cap})$-normed $\mathrm{L}^0(\mathrm{Cap})$-module. We define the $\mathrm{L}^0(\mathrm{Cap})$-normed $\mathrm{L}^0(\mathrm{Cap})$-module $\mathcal{M}^{\otimes n}$ repeating the construction done to define the tensor product of $\mathrm{L}^0(\mathsf{m})$-normed $\mathrm{L}^0(\mathsf{m})$-modules in [25, Subsection 3.2.2] (originally of [22]), that is endowing the algebraic tensor product

$$\mathcal{M} \otimes^{\mathrm{alg}}_{\mathrm{L}^0(\mathrm{Cap})} \cdots \otimes^{\mathrm{alg}}_{\mathrm{L}^0(\mathrm{Cap})} \mathcal{M}$$

with the pointwise Hilbert-Schmidt norm and then taking the completion with respect to the induced distance.

If μ is a Borel measure finite on balls and such that $\mu \ll \mathrm{Cap}$, we set

$$\mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X}) := \mathrm{L}^p_\mu(T\mathsf{X})^{\otimes n} \qquad \text{for } p \in \{0\} \cup [1, \infty],$$

where the right hand side is given by Theorem 2.7.

Remark 2.9 Let μ be a Borel measure, finite on balls, such that $\mu \ll \mathrm{Cap}$. Let also $\mathcal{M}$ be an Hilbert $\mathrm{L}^0(\mathrm{Cap})$-normed $\mathrm{L}^0(\mathrm{Cap})$-module. Then, using the notation of Theorem 2.7, we have a canonical isomorphism

$$(\mathcal{M}^{\otimes n})^0_\mu \cong (\mathcal{M}^0_\mu)^{\otimes n}.$$

This isomorphism is obtained using the map induced by the well defined multilinear map

$$(\mathcal{M}^0_\mu)^n \ni ([v_1]_{\sim_\mu}, \dots, [v_n]_{\sim_\mu}) \mapsto [(v_1 \otimes \cdots \otimes v_n)]_{\sim_\mu} \in (\mathcal{M}^{\otimes n})^0_\mu$$

and noticing that such map turns out to be an isometry with dense image between complete spaces. Therefore we also have the canonical inclusion

$$(\mathcal{M}^{\otimes n})^p_\mu \cong \left\{ v \in (\mathcal{M}^0_\mu)^{\otimes n} : |v| \in \mathrm{L}^p(\mu) \right\} \qquad \text{if } p \in [1, \infty].$$

In particular, with the obvious interpretation for $\mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes n}\mathsf{X})$,

$$(\mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes n}\mathsf{X}))^0_\mu \cong \mathrm{L}^0_\mu(T^{\otimes n}\mathsf{X})$$

so that

$$(\mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes n}\mathsf{X}))^p_\mu \cong \mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X}) \qquad \text{if } p \in [1, \infty],$$

where $\mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X}) := \left\{ v \in \mathrm{L}^0_\mu(T^{\otimes n}\mathsf{X}) : |v| \in \mathrm{L}^p(\mu) \right\}$. ■

Now we consider $\mathrm{TestV}(\mathsf{X})^{\otimes n}$. Notice that $\mathrm{TestV}(\mathsf{X})^{\otimes n}$ is a module over the ring $\mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ in the algebraic sense and, by Lemma 2.11 below,

$$\frac{1}{1 \vee |v|} v \in \mathrm{TestV}(\mathsf{X})^{\otimes n} \qquad \text{for every } v \in \mathrm{TestV}(\mathsf{X})^{\otimes n}.$$

By the following lemma (with $\mu = \mathsf{m}$), $\mathrm{TestV}(\mathsf{X})^{\otimes n}$ is dense in $\mathrm{L}^2(T^{\otimes n}\mathsf{X})$, in particular, $\mathrm{TestV}(\mathsf{X})^{\otimes n}$ generates in the sense of modules $\mathrm{L}^2(T^{\otimes 2}\mathsf{X})$. The following lemma is proved with an approximation argument as in [25, Lemma 3.2.21], and is a generalization of Lemma 2.8.

Lemma 2.10 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* μ *be a finite Borel measure such that* $\mu \ll \mathrm{Cap}$. *Then* $\mathrm{TestV}(\mathsf{X})^{\otimes n}$ *is dense in* $\mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X})$ *for every* $p \in [1, \infty)$.

Proof Fix $p \in [1, \infty)$ and $w \in \mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X})$. This is to say that $|w| \in \mathrm{L}^p(\mu)$ and we can find a sequence of tensors

$$\{w^k\}_k \subseteq \mathrm{L}^0_\mu(T\mathsf{X}) \otimes^{\mathrm{alg}}_{\mathrm{L}^0(\mu)} \cdots \otimes^{\mathrm{alg}}_{\mathrm{L}^0(\mu)} \mathrm{L}^0_\mu(T\mathsf{X})$$

such that $\left| w - w^k \right| \to 0$ in $\mathrm{L}^0(\mu)$. We can assume that $\{|w^k|\}_k \subseteq \mathrm{L}^p(\mu)$ is a bounded sequence, up to replacing w^k with $\frac{|w|}{|w^k|} w^k$. In this case, by dominated convergence, we see that $w^k \to w$ in $\mathrm{L}^p_\mu(T^{\otimes n}\mathsf{X})$. As a consequence of this discussion, an orthonormalization procedure and a truncation argument, we see that we can reduce ourselves to the case $w = w_1 \otimes \cdots \otimes w_n$ where $w_i \in \mathrm{L}^\infty_\mu(T^{\otimes n}\mathsf{X})$ for every i. Then we can conclude iterating Lemma 2.8. □

Lemma 2.11 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* $v \in (\mathrm{H}^{1,2}_C(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\otimes n}$. *Then* $|v| \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$.

Proof Fix $v = \sum_{i=1}^m v_1^i \otimes \cdots \otimes v_n^i \in (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\otimes n}$, where $\{v_j^i\} \subseteq \mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, so that there exist $H_v \in \mathbb{R}$ and $h_v \in \mathrm{L}^2(\mathsf{m})$ such that for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$ it holds

$$|v_j^i| \le H_v \qquad \mathsf{m}\text{-a.e.}$$
$$|\nabla v_j^i| \le h_v \qquad \mathsf{m}\text{-a.e.}$$

Following a standard argument as e.g. in the proof of [18, Lemma 2.5] it is enough to show that $|\nabla|v|^2| \le g_v|v|$ m-a.e. for some $g_v \in \mathrm{L}^2(\mathsf{m})$. For $Z \in \mathrm{L}^0(T\mathsf{X})$ we define $\nabla_Z v$ as in the discussion right above [22, Proposition 3.4.6], that is the unique vector field in $\mathrm{L}^0(T\mathsf{X})$ such that for every $Y \in \mathrm{L}^0(T\mathsf{X})$

$$\nabla_Z v \cdot Y = \nabla v \cdot Z \otimes Y \qquad \mathsf{m}\text{-a.e.}$$

Clearly, $|\nabla_Z v| \le |\nabla v||Z|$. If $Z \in \mathrm{L}^0(T\mathsf{X})$, we compute, by [22, Proposition 3.4.6 i)],

$$\nabla(v \cdot v) \cdot Z = \sum_{i,j=1}^m \nabla\left(\prod_{k=1}^n v_k^i \cdot v_k^j\right) \cdot Z = 2\sum_{i,j=1}^m \sum_{h=1}^n (\nabla_Z v_h^i) \cdot v_h^j \prod_{\substack{k=1\\k\neq h}}^n v_k^i \cdot v_k^j$$
$$= 2\left(\sum_{i=1}^m \sum_{h=1}^n v_1^i \otimes \cdots \otimes v_{h-1}^i \otimes \nabla_Z v_h^i \otimes v_{h+1}^i \otimes \cdots \otimes v_n^i\right) \cdot \left(\sum_{j=1}^m v_1^j \otimes \cdots \otimes v_n^j\right).$$

Now we have finished, as, by the arbitrariness of $Z \in \mathrm{L}^0(T\mathsf{X})$, the equality above implies that

$$|\nabla|v|^2| \le 2mnH_v^{n-1}h_v|v| \qquad \mathsf{m}\text{-a.e.}$$

□

We consider now the multilinear map

$$(\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^n \ni (v_1, \ldots, v_n) \mapsto \bar{\mathrm{QCR}}(v_1) \otimes \cdots \otimes \bar{\mathrm{QCR}}(v_n) \in \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes n}\mathsf{X}) \tag{2.6}$$

and we notice that (the left hand side is well defined thanks to Lemma 2.11 - but as there is a squared norm here, this is indeed trivial)

$$\mathrm{QCR}\left(\left|\sum_{i=1}^m v_1^i \otimes \cdots \otimes v_n^i\right|^2\right) = \left|\sum_{i=1}^m \bar{\mathrm{QCR}}(v_1^i) \otimes \cdots \otimes \bar{\mathrm{QCR}}(v_n^i)\right|^2 \qquad \mathrm{Cap}\text{-a.e.} \tag{2.7}$$

so that the map in Eq. 2.6 induces a map $\bar{\mathrm{QCR}} : (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\otimes n} \to \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes n}\mathsf{X})$ that satisfies, thanks to Eq. 2.7,

$$\mathrm{QCR}(|v|) = \left|\bar{\mathrm{QCR}}(v)\right| \qquad \mathrm{Cap}\text{-a.e. for every } v \in (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\otimes n}.$$

As usual, we often omit to write the maps $\bar{\mathrm{QCR}}$ and QCR.

2.2.2 Exterior Power

Much like the previous section dealt with tensor product, in this section we deal with exterior power (see [22]). As the arguments are mostly identical, we are going just to sketch the key ideas and we will assume familiarity with the related theory. We assume again that $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a $\mathrm{RCD}(K, \infty)$ space. Similarly to what done before, we set

$$\mathrm{L}^2(T^{\wedge n}\mathsf{X}) := \mathrm{L}^2(T\mathsf{X})^{\wedge n}$$

and define, for $\mathcal{C} \subseteq L^2(T\mathsf{X})$, $\mathcal{C}^{\wedge n}$ as before.

Let now $\mathcal{M}$ be an Hilbert $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module. We define the $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module $\mathcal{M}^{\wedge n}$ as the quotient of $\mathcal{M}^{\otimes n}$ with respect to the closure of the subspace generated by elements of the form $v_1 \otimes \cdots \otimes v_n$, where $v_1, \ldots, v_n \in \mathcal{M}$ are such that $v_i = v_j$ for at least two different indices i and j. This definition is the trivial adaptation of [22] to our context and it is possible to prove that $\mathcal{M}^{\wedge n}$ is indeed an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module and that its scalar product is characterized by

$$v_1 \wedge \cdots \wedge v_n \cdot w_1 \wedge \cdots \wedge w_n = \det(v_i \cdot w_j) \qquad \mathrm{Cap}\text{-a.e.}$$

Similarly to what done before, if μ is a Borel measure finite on balls such that $\mu \ll \mathrm{Cap}$, we set

$$L^p_\mu(T^{\wedge n}\mathsf{X}) := L^p_\mu(T\mathsf{X})^{\wedge n} \qquad \text{for } p \in \{0\} \cup [1, \infty].$$

Remark 2.12 Now we adapt Remark 2.9. Let μ be a Borel measure, finite on balls, such that $\mu \ll \mathrm{Cap}$. Let also $\mathcal{M}$ be an Hilbert $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module. Then, using the notation of Theorem 2.7, we have a canonical isomorphism

$$(\mathcal{M}^{\wedge n})^0_\mu \cong (\mathcal{M}^0_\mu)^{\wedge n}.$$

This isomorphism is obtained using the map induced by the well defined multilinear map

$$(\mathcal{M}^0_\mu)^n \ni ([v_1]_{\sim_\mu}, \ldots, [v_n]_{\sim_\mu}) \mapsto [(v_1 \wedge \cdots \wedge v_n)]_{\sim_\mu} \in (\mathcal{M}^{\wedge n})^0_\mu$$

and noticing that such map turns out to be an isometry with dense image between complete spaces. Therefore we also have the canonical inclusion

$$(\mathcal{M}^{\wedge n})^p_\mu \cong \left\{v \in (\mathcal{M}^0_\mu)^{\wedge n} : |v| \in L^p(\mu)\right\} \qquad \text{if } p \in [1, \infty].$$

In particular, then, with the obvious interpretation for $L^0_{\mathrm{Cap}}(T^{\wedge n}\mathsf{X})$,

$$(L^0_{\mathrm{Cap}}(T^{\wedge n}\mathsf{X}))^0_\mu \cong L^0_\mu(T^{\wedge n}\mathsf{X})$$

so that

$$(L^0_{\mathrm{Cap}}(T^{\wedge n}\mathsf{X}))^p_\mu \cong L^p_\mu(T^{\wedge n}\mathsf{X}) \qquad \text{if } p \in [1, \infty],$$

where $L^p_\mu(T^{\wedge n}\mathsf{X}) := \left\{v \in L^0_\mu(T^{\wedge n}\mathsf{X}) : |v| \in L^p(\mu)\right\}$. ■

Now we consider $\mathrm{TestV}(\mathsf{X})^{\wedge n}$, that is a module over the ring $S^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$ in the algebraic sense and, by Lemma 2.14 below,

$$\frac{1}{1 \vee |v|} v \in \mathrm{TestV}(\mathsf{X})^{\wedge n} \qquad \text{for every } v \in \mathrm{TestV}(\mathsf{X})^{\wedge n}.$$

As a consequence of of Lemma 2.10, we have the following result.

Lemma 2.13 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* μ *be a finite Borel measure such that* $\mu \ll \mathrm{Cap}$. *Then* $\mathrm{TestV}(\mathsf{X})^{\wedge n}$ *is dense in* $L^p_\mu(T^{\wedge n}\mathsf{X})$ *for every* $p \in [1, \infty)$.

The following result corresponds to Lemma 2.11.

Lemma 2.14 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* $v \in (H^{1,2}_C(T\mathsf{X}) \cap L^\infty(\mathsf{m}))^{\wedge n}$. *Then* $|v| \in H^{1,2}(\mathsf{X}) \cap L^\infty(\mathsf{m})$.

Proof The proof is very similar to the one of Lemma 2.11, we simply sketch the key computation for the sake of completeness. Take $v \in (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\wedge n}$, say $v = \sum_{i=1}^m v_1^i \wedge \cdots \wedge v_n^i$, where $\{v_j^i\} \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$. Take $H_v \in \mathbb{R}$ and $h_v \in \mathrm{L}^2(\mathsf{m})$ as in the proof of Lemma 2.11. We will prove that

$$|\nabla |v|^2| \le 2mnH_v^{n-1} h_v |v| \qquad \mathsf{m}\text{-a.e.}$$

and thus the proof will be concluded as in Lemma 2.11. We take $Z \in \mathrm{L}^0(T\mathsf{X})$ and we compute (here S_n denotes the symmetric group)

$$\begin{aligned}
\nabla(v \cdot v) \cdot Z &= \sum_{i,j=1}^m \nabla \det((v_k^i \cdot v_h^j)_{h,k}) \cdot Z = \sum_{i,j=1}^m \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \nabla \left(\prod_{k=1}^n v_k^i \cdot v_{\sigma(k)}^j \right) \cdot Z \\
&= 2 \sum_{i,j=1}^m \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \sum_{h=1}^n (\nabla_Z v_h^i) \cdot v_{\sigma(h)}^j \prod_{\substack{k=1 \\ k \neq h}}^n v_k^i \cdot v_{\sigma(k)}^j \\
&= 2 \left(\sum_{i=1}^m \sum_{h=1}^n v_1^i \wedge \cdots \wedge v_{h-1}^i \wedge \nabla_Z v_h^i \wedge v_{h+1}^i \wedge \cdots \wedge v_n^i \right) \cdot \left(\sum_{j=1}^m v_1^j \wedge \cdots \wedge v_n^j \right)
\end{aligned}$$

so that we have proved the claim. □

As before, we consider the multilinear map defined by Eq. 2.6 and we notice that (the left hand side is well defined thanks to Lemma 2.14 - but as there is a squared norm here, this is indeed trivial)

$$\mathrm{QCR}\left(\left| \sum_{i=1}^m v_1^i \wedge \cdots \wedge v_n^i \right|^2 \right) = \left| \sum_{i=1}^m \bar{\mathrm{QCR}}(v_1^i) \wedge \cdots \wedge \bar{\mathrm{QCR}}(v_n^i) \right|^2 \qquad \text{Cap-a.e.} \tag{2.8}$$

so that the map in Eq. 2.6 induces a map $\bar{\mathrm{QCR}} : (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\wedge n} \to \mathrm{L}^0_{\mathrm{Cap}}(T^{\wedge n}\mathsf{X})$ that satisfies, thanks to Eq. 2.8,

$$\mathrm{QCR}(|v|) = \left|\bar{\mathrm{QCR}}(v)\right| \qquad \text{Cap-a.e. for every } v \in (\mathrm{H}^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m}))^{\wedge n}.$$

As usual, we often omit to write the maps $\bar{\mathrm{QCR}}$ and QCR.

3 Main Part

3.1 Decomposition of the Tangent Module

The following theorem provides us with a dimensional decomposition of the Cap-tangent module, along with an orthonormal basis made of "smooth" vector fields of the Cap-tangent module on every element of the induced partition. This will be the first step towards the construction of the most relevant objects of this note. Notice that one should not expect the relevant dimension to be unique: in a smooth manifold of dimension n with boundary, the Cap-tangent module sees the boundary, thus it has dimension n in the interior of the manifold and dimension $n-1$ at the boundary. It is unclear if the situation on RCD spaces can be more complicated than that.

In view of the theorem below, recall that the essential dimension of an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, after [14], is the unique integer $n \le N$ such that at m-a.e. x, the tangent at x is $\mathbb{R}^n$.

Theorem 2.1 ([10, Theorem 3]) *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n. *Then there exists a partition of* X *made of countably many bounded Borel sets* $\{A_k\}_{k\in\mathbb{N}}$ *such that for every* k *there exist* $n(k)$ *with* $0\le n(k)\le n$ *and* $\left\{v_1^k,\dots,v_{n(k)}^k\right\}\subseteq \mathrm{Test}\bar{\mathrm{V}}(\mathsf{X})$ *with bounded support which is an orthonormal basis of* $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *on* A_k, *in the sense that*

$$v_i\cdot v_j=\delta_i^j\qquad \text{Cap-}a.e.\ on\ A_k$$

and for every $v\in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *there exist* $g_1,\dots,g_{n(k)}\in \mathrm{L}^0(\mathrm{Cap})$ *such that*

$$v=\sum_{i=1}^{n(k)} g_i v_i^k\qquad \text{Cap-}a.e.\ on\ A_k,$$

where, in particular,

$$g_i=v\cdot v_i^k\qquad \text{Cap-}a.e.\ on\ A_k.$$

Here we implicitly state that if $n(k)=0$ *then for every* $v\in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ *we have* $v=0$ Cap-*a.e. on* A_k.

3.2 Hessian

3.2.1 Convexity

In the following definition we restrict ourselves to the case of RCD spaces. This restriction is clearly unnecessary for items (1) and (2), however, we preferred this formulation for the sake of simplicity, taken into account that all the results of this note are in the framework of RCD spaces.

Definition 3.2 Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $f:\mathsf{X}\to(-\infty,+\infty]$. Let also $\kappa\in\mathbb{R}$. We say that

(1) f is weakly κ geodesically convex if for every $x_0,x_1\in\mathsf{X}$, there exists a constant speed geodesic $\gamma:[0,1]\to\mathsf{X}$ joining x_0 to x_1 satisfying

$$f(\gamma(t))\le(1-t)f(\gamma(0))+tf(\gamma(1))-\frac{\kappa}{2}t(1-t)\mathsf{d}(x_0,x_1)^2\qquad\text{for every } t\in[0,1]. \tag{3.1}$$

(2) f is strongly κ geodesically convex if for every $x_0,x_1\in\mathsf{X}$, for every constant speed geodesic $\gamma:[0,1]\to\mathsf{X}$ joining x_0 to x_1, Eq. 3.1 holds.

If moreover $f\in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$, we say that

(3) $\mathrm{Hess} f\ge\kappa$ if for every $h,g\in\mathrm{TestF}_{\mathrm{bs}}(\mathsf{X})$ with $h\ge 0$ m-a.e.

$$\int_{\mathsf{X}}-\mathrm{div}(h\nabla g)\nabla f\cdot\nabla g-\frac{1}{2}h\nabla f\cdot\nabla(|\nabla g|^2)\mathrm{d}\mathsf{m}\ge\kappa\int_{\mathsf{X}}|\nabla g|^2h\mathrm{d}\mathsf{m}. \tag{3.2}$$

Remark 3.3 Notice that if moreover $f\in\mathrm{H}^{1,2}(\mathsf{X})$ and the space is locally compact, then item (3) of the definition above implies that Eq. 3.2 holds for every $h\in \mathrm{S}^2(\mathsf{X})\cap\mathrm{L}^\infty(\mathsf{m})$ and $g\in\mathrm{TestF}(\mathsf{X})$. This follows from an approximation argument, taking into account also the existence of good cut-off functions as in [5, Lemma 6.7] together with the algebra property of test functions. ■

Some implications among the various items of the previous definition have already been extensively studied in the literature, see e.g. [19, 26, 30, 31, 33, 38] for similar statements. Notice that (2) $\Rightarrow$ (1) is trivially satisfied in geodesic spaces. The implication (1) $\Rightarrow$ (3), recalled in Proposition 3.5 below, is particularly important in the sequel, as it motivates Theorem 1.1, one of the main results of this note. For the proof of Proposition 3.5 we are going to follow the lines of the proof of [31, Theorem 7.1]. As we are going to work with weaker regularity assumptions, we give the details anyway. Indeed, the fact that we do not assume $f \in D(\mathrm{Hess})$ forces us to proceed through a delicate approximation argument. Finally, under additional regularity assumptions, e.g. $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a locally compact $\mathrm{RCD}(K, \infty)$ space and $f \in \mathrm{TestF}(\mathsf{X})$, it turns out that items (1), (2) and (3) are all equivalent (see [31] and [38]). The equivalence of these notions of convexity is expected to hold even under weaker assumptions on f (see, in this direction, Proposition 3.5) but this investigation (in particular (3) $\Rightarrow$ (1)) is beyond the scope of this note.

Remark 3.4 The implication (3) $\Rightarrow$ (1) seems anything but trivial, if one does not assume that $f \in D(\mathrm{Hess})$. Indeed, one could hope to follow [31] and start by proving that $(\mathsf{X}, \mathsf{d}, e^{-f}\mathsf{m})$ is $\mathrm{RCD}(K+\kappa, \infty)$ whenever $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is $\mathrm{RCD}(K, \infty)$ and $\mathrm{Hess} f \geq \kappa$. In this context, the natural way to verify the $\mathrm{RCD}(K+\kappa, \infty)$ condition is via the Eulerian point of view, i.e. via the weak Bochner inequality. However, in order to so, we would have wanted to exploit an approximation argument, to plug in the weak Bochner inequality for $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ and the fact that $\mathrm{Hess} f \geq \kappa$ and such approximation argument seems to require that the heat flow on $(\mathsf{X}, \mathsf{d}, e^{-f}\mathsf{m})$ maps regular enough functions to Lipschitz functions, and we were not able to prove this fact (that we remark is linked with the RCD condition). ■

Proposition 3.5 *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space and let $f \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$ be a continuous and weakly κ geodesically convex function, for some $\kappa \in \mathbb{R}$. Assume moreover that f is bounded from below and locally bounded from above, in the sense that f is bounded from above on every bounded subset of X. Then $\mathrm{Hess} f \geq \kappa$.*

Proof As remarked above, we follow the proof of [31]. Define $\tilde{\mathsf{m}} := e^{-f}\mathsf{m}$. When we want to stress that an object is relative to the space $(\mathsf{X}, \mathsf{d}, \tilde{\mathsf{m}})$, we add the symbol $\tilde{}$.

Step 1. We show that $(\mathsf{X}, \mathsf{d}, \tilde{\mathsf{m}})$ is an $\mathrm{RCD}(K+\kappa, \infty)$ space, following [3, Proposition 6.19], which builds upon [36, Proposition 4.14] and [2, Lemma 4.11].

To show $(K+\kappa)$-convexity of $\mathrm{Ent}_{\tilde{\mathsf{m}}}$, first notice that the continuity of f and classical measurable selection arguments (e.g. [8, 6.9.13]) grant that there exists a $\mathsf{m}\otimes\mathsf{m}$-measurable map $\Gamma : \mathsf{X}\times\mathsf{X} \to \mathrm{Geod}(\mathsf{X})$ such that for $\mathsf{m}\otimes\mathsf{m}$-a.e. $(x_0, x_1) \in \mathsf{X}\times\mathsf{X}$, $\Gamma(x_0, x_1)$ is a geodesic joining x_0 to x_1 satisfying Eq. 3.1. Also ([15, Theorem 3.2]), the $\mathrm{RCD}(K, \infty)$ assumption on $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ implies that $\mathrm{Ent}_{\mathsf{m}}$ is K-convex along every constant speed geodesic $\{\mu_t\}_t \subseteq \mathscr{P}_2(\mathsf{X})$. Then, given $\mu, \nu \in \mathscr{P}_2(\mathsf{X}) \cap D(\mathrm{Ent}_{\tilde{\mathsf{m}}})$ we can argue as in [36, Proposition 4.14], verifying $(K+\kappa)$-convexity of $\mathrm{Ent}_{\tilde{\mathsf{m}}}$ along the geodesic given by $\{(e_t \circ \Gamma)_*(\mu\otimes\nu)\}_t$.

Step 2. Notice that, as f is locally bounded, [2, Lemma 4.11] implies that $\phi \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$ if and only if $\phi \in \tilde{\mathrm{H}}^{1,2}_{\mathrm{loc}}(\mathsf{X})$ and, if this is the case

$$|\nabla\phi| = |\tilde{\nabla}\phi|.$$

Also, polarizing, we obtain that the $\cdot$ product between gradients is independent of the space, so that we will drop the $\tilde{}$ on gradients. Moreover, if $\phi \in \mathrm{H}^{1,2}(\mathsf{X})$, then $\phi \in \tilde{\mathrm{H}}^{1,2}(\mathsf{X})$ and, if $\phi \in \mathrm{TestF}_{\mathrm{bs}}(\mathsf{X})$, then $\phi \in D(\tilde{\Delta})$ and

$$\tilde{\Delta}\phi = \Delta\phi - \nabla f \cdot \nabla\phi. \tag{3.3}$$

By the equivalence result in [4] (see also [6, 19]), we know that if $k, g \in D(\tilde{\Delta})$ with $\tilde{\Delta}k \in \mathrm{L}^\infty(\mathsf{m})$, $\tilde{\Delta}g \in \tilde{\mathrm{H}}^{1,2}(\mathsf{X})$ and $k \in \mathrm{L}^\infty(\mathsf{m})$, $k \geq 0$,

$$\begin{aligned}(K+\kappa)\int_{\mathsf{X}}|\nabla g|^2 k\mathrm{d}\tilde{\mathsf{m}} &\leq \frac{1}{2}\int_{\mathsf{X}}|\nabla g|^2\tilde{\Delta}k\mathrm{d}\tilde{\mathsf{m}} - \int_{\mathsf{X}}(\nabla g\cdot\nabla\tilde{\Delta}g)k\mathrm{d}\tilde{\mathsf{m}}\\ &= \frac{1}{2}\int_{\mathsf{X}}|\nabla g|^2\tilde{\Delta}k\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}\tilde{\mathrm{div}}(k\nabla g)\tilde{\Delta}g\mathrm{d}\tilde{\mathsf{m}}\\ &= \frac{1}{2}\int_{\mathsf{X}}|\nabla g|^2\tilde{\Delta}k\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}\nabla k\cdot\nabla g\,\tilde{\Delta}g\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}k(\tilde{\Delta}g)^2\mathrm{d}\tilde{\mathsf{m}}.\end{aligned}$$

By an approximation argument based on the mollified heat flow (for the space $(\mathsf{X},\mathsf{d},\tilde{\mathsf{m}})$) on g, we can use what we just proved to show that if $g \in \mathrm{TestF}_{\mathrm{bs}}(\mathsf{X})$ and k is as above,

$$\begin{aligned}(K+\kappa)\int_{\mathsf{X}}|\nabla g|^2 k\mathrm{d}\tilde{\mathsf{m}} &\leq \frac{1}{2}\int_{\mathsf{X}}|\nabla g|^2\tilde{\Delta}k\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}\nabla k\cdot\nabla g\tilde{\Delta}g\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}k(\tilde{\Delta}g)^2\mathrm{d}\tilde{\mathsf{m}}\\ &= -\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla k\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}\nabla k\cdot\nabla g\tilde{\Delta}g\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}k(\tilde{\Delta}g)^2\mathrm{d}\tilde{\mathsf{m}}.\end{aligned}$$

Then, with an approximation argument based on the mollified heat flow on k, we have that if $g \in \mathrm{TestF}_{\mathrm{bs}}(\mathsf{X})$ and $k \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, it holds that

$$(K+\kappa)\int_{\mathsf{X}}|\nabla g|^2 k\mathrm{d}\tilde{\mathsf{m}} \leq -\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla k\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}\nabla k\cdot\nabla g\tilde{\Delta}g\mathrm{d}\tilde{\mathsf{m}} + \int_{\mathsf{X}}k(\tilde{\Delta}g)^2\mathrm{d}\tilde{\mathsf{m}}.$$

We choose then $k = he^f$ to obtain ($h \in \mathrm{TestF}_{\mathrm{bs}}(\mathsf{X})$), recalling Eq. 3.3,

$$\begin{aligned}(K+\kappa)\int_{\mathsf{X}}|\nabla g|^2 h\mathrm{d}\mathsf{m} &\leq -\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla h\mathrm{d}\mathsf{m} - \frac{1}{2}\int_{\mathsf{X}}(\nabla|\nabla g|^2\cdot\nabla f)h\mathrm{d}\mathsf{m}\\ &\quad + \int_{\mathsf{X}}\nabla h\cdot\nabla g(\Delta g - \nabla f\cdot\nabla g)\mathrm{d}\mathsf{m}\\ &\quad + \int_{\mathsf{X}}\nabla f\cdot\nabla g(\Delta g - \nabla f\cdot\nabla g)h\mathrm{d}\mathsf{m}\\ &\quad + \int_{\mathsf{X}}h(\Delta g - \nabla f\cdot\nabla g)^2\mathrm{d}\mathsf{m}\\ &= -\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla h\mathrm{d}\mathsf{m} - \frac{1}{2}\int_{\mathsf{X}}(\nabla|\nabla g|^2\cdot\nabla f)h\mathrm{d}\mathsf{m}\\ &\quad + \int_{\mathsf{X}}\mathrm{div}(h\nabla g)\Delta g\mathrm{d}\mathsf{m} - \int_{\mathsf{X}}\mathrm{div}(h\nabla g)\nabla g\cdot\nabla f\mathrm{d}\mathsf{m}.\end{aligned}$$

Step 3. Let now $\alpha > 0$. We repeat the same computation of **Step 2** but starting from the $\mathrm{RCD}(\alpha^2 K, \infty)$ space $(\mathsf{X}, \alpha^{-1}\mathsf{d}, \mathsf{m})$ and the κ convex function $\alpha^{-2}f$ (of course, with respect to $\alpha^{-1}\mathsf{d}$) to obtain (all the differential operators are with respect to $(\mathsf{X},\mathsf{d},\mathsf{m})$)

$$\begin{aligned}(\alpha^2 K+\kappa)\int_{\mathsf{X}}\alpha^2|\nabla g|^2 h\mathrm{d}\mathsf{m} &\leq -\alpha^4\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla h\mathrm{d}\mathsf{m} - \alpha^2\frac{1}{2}\int_{\mathsf{X}}\nabla|\nabla g|^2\cdot\nabla f h\mathrm{d}\mathsf{m}\\ &\quad + \alpha^4\int_{\mathsf{X}}\mathrm{div}(h\nabla g)\Delta g\mathrm{d}\mathsf{m} - \alpha^2\int_{\mathsf{X}}\mathrm{div}(h\nabla g)\nabla g\cdot\nabla f\mathrm{d}\mathsf{m}.\end{aligned}$$

Dividing this inequality by α^2 and letting $\alpha \searrow 0$ yields the claim. □

3.2.2 Measure Valued Hessian

In this section we state and prove the first main result of this note, namely Theorem 1.1. More precisely, we show that convex functions have, in a certain sense, a measure valued Hessian. In the Euclidean space, this is an immediate consequence of Riesz's Theorem for positive functionals and it implies that gradients of convex functions are vector fields of bounded variation. Hence we have that Hessian measures are absolutely continuous with respect to Cap, and this is the case even on RCD spaces. This absolute continuity allows us to build the measure valued Hessian on RCD spaces as product of a Cap-tensor field and a σ-finite measure that is absolutely continuous with respect to Cap. We remark that, as the decomposition of the Cap-tangent module given by Theorem 2.1 induces a decomposition of the space in Borel sets (not open ones), we are not able to prove that the total variation of the Hessian measure is a Radon measure.

Before dealing with the main theorem of this section, we define when a $\mathrm{H}^{1,2}(\mathsf{X})$ function has a measure valued Hessian (cf. [22, Definition 3.3.1]) and study a couple of basic calculus properties of this newly defined notion.

Definition 3.6 Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and $f\in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$. We write $f\in D(\mathbf{Hess})$, if there exists a σ-finite measure $|\mathbf{Hess} f|$ that satisfies $|\mathbf{Hess} f|\ll$ Cap and a symmetric tensor field $\nu_f\in \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 2}\mathsf{X})$ with $|\nu_f|=1$ $|\mathbf{Hess} f|$-a.e. such that

$$\mathbf{Hess} f=\nu_f|\mathbf{Hess} f|,$$

in the sense that for every $X,Y\in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap \mathrm{L}^\infty(T\mathsf{X})$, it holds that $X\otimes Y\cdot \nu_f\in \mathrm{L}^1_{\mathrm{loc}}(|\mathbf{Hess} f|)$ and, if $h\in \mathrm{H}^{1,2}(\mathsf{X})\cap \mathrm{L}^\infty(\mathsf{m})$ has bounded support,

$$\int_{\mathsf{X}} hX\otimes Y\cdot \nu_f \,\mathrm{d}|\mathbf{Hess} f|=-\int_{\mathsf{X}}\nabla f\cdot X\mathrm{div}(hY)+h\nabla_Y X\cdot\nabla f\,\mathrm{d}\mathsf{m}. \tag{3.4}$$

Proposition 3.7 *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and let* $f\in D(\mathbf{Hess})$. *Then the decomposition of* $\mathbf{Hess} f$ *is unique, in the sense that, adopting the same notation as in* Definition 3.6*, the measure* $|\mathbf{Hess} f|$ *is unique and the tensor field* ν_f *is unique, up to* $|\mathbf{Hess} f|$*-a.e. equality.*

Proof Assume that (ν,μ) and (ν',μ') are two pairs having the same properties of the pair $(\nu_f,|\mathbf{Hess} f|)$ as in Definition 3.6. We show that $\mu=\mu'$ and that $\nu=\nu'$ μ-a.e. By dominated convergence, we have that

$$\int_K X\otimes Y\cdot\nu\,\mathrm{d}\mu=\int_K X\otimes Y\cdot\nu'\,\mathrm{d}\mu'$$

for every $K\subseteq\mathsf{X}$ compact and $X,Y\in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap \mathrm{L}^\infty(T\mathsf{X})$. As both μ and μ' are σ-finite, we can partition X, up to subsets that are negligible with respect to both μ and μ', in a countable union of compact sets on which both μ and μ' are finite. Therefore, we prove that $\mu=\mu'$ and $\nu=\nu'$ μ-a.e. on a compact set K for which $\mu(K),\mu'(K)<\infty$ and this will be enough to conclude. Now, we write $\bar\nu:=\nu\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\mu')}$ and $\bar\nu':=\nu'\frac{\mathrm{d}\mu'}{\mathrm{d}(\mu+\mu')}$, so that we have, for every $X,Y\in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap \mathrm{L}^\infty(T\mathsf{X})$,

$$\int_K X\otimes Y\cdot\bar\nu\,\mathrm{d}(\mu+\mu')=\int_K X\otimes Y\cdot\bar\nu'\,\mathrm{d}(\mu+\mu').$$

Applying Lemma 2.10 to the space $\mathrm{L}^2_{(\mu+\mu')\llcorner K}(T^{\otimes 2}\mathsf{X})$, we deduce that $\bar\nu=\bar\nu'$, $(\mu+\mu')$-a.e. on K. As $|\nu|=1$ μ-a.e. and $|\nu'|=1$ μ'-a.e. we deduce that

$$\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\mu')}=\frac{\mathrm{d}\mu'}{\mathrm{d}(\mu+\mu')}\qquad(\mu+\mu')\text{-a.e. on } K,$$

which implies, as the two terms above sum to 1, that $\mu=\mu'$ on K. Using again that $\bar\nu=\bar\nu'$ $(\mu+\mu')$-a.e. on K, we deduce that also $\nu=\nu'$ μ-a.e. on K. □

Using the by now classical calculus tools on RCD spaces, the following proposition easily follows, starting from Eq. 3.4.

Proposition 3.8 *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then*

i) $D(\mathrm{Hess})\subseteq D(\mathbf{Hess})$. *Namely, if* $f\in D(\mathrm{Hess})$, *then* $\mathbf{Hess}f=\frac{\mathrm{Hess}f}{|\mathrm{Hess}f|}(|\mathrm{Hess}f|\mathsf{m})$;

ii) $D(\mathbf{Hess})$ *is a vector space. Namely, if* $f,g\in D(\mathbf{Hess})$, *say* $\mathbf{Hess}f=\nu_f|\mathbf{Hess}f|$, $\mathbf{Hess}g=\nu_g|\mathbf{Hess}g|$, *then* $f+g\in D(\mathbf{Hess})$ *and, setting*

$$\mu:=|\mathbf{Hess}f|+|\mathbf{Hess}g|,$$
$$\nu:=\nu_f\frac{\mathrm{d}|\mathbf{Hess}f|}{\mathrm{d}\mu}+\nu_g\frac{\mathrm{d}|\mathbf{Hess}g|}{\mathrm{d}\mu},$$

it holds that $\mathbf{Hess}(f+g)=\frac{\nu}{|\nu|}(|\nu|\mu)$;

iii) $D(\mathbf{Hess})\cap\mathrm{L}^\infty_{\mathrm{loc}}(\mathsf{m})$ *is closed under multiplication. Namely, if* $f,g\in D(\mathbf{Hess})\cap\mathrm{L}^\infty_{\mathrm{loc}}(\mathsf{m})$, *say* $\mathbf{Hess}f=\nu_f|\mathbf{Hess}f|$, $\mathbf{Hess}g=\nu_g|\mathbf{Hess}g|$, *then* $fg\in D(\mathbf{Hess})$ *and, setting*

$$\mu:=|\mathbf{Hess}f|+|\mathbf{Hess}g|+\mathsf{m},$$
$$\nu:=g\nu_f\frac{\mathrm{d}|\mathbf{Hess}f|}{\mathrm{d}\mu}+f\nu_g\frac{\mathrm{d}|\mathbf{Hess}g|}{\mathrm{d}\mu}+\big(\nabla f\otimes\nabla g+\nabla g\otimes\nabla f\big)\frac{\mathrm{d}\mathsf{m}}{\mathrm{d}\mu},$$

it holds that $\mathbf{Hess}(fg)=\frac{\nu}{|\nu|}(|\nu|\mu)$;

iv) $D(\mathbf{Hess})\cap\mathrm{L}^\infty_{\mathrm{loc}}(\mathsf{m})$ *is closed under post-composition with* C^2 *functions. Namely, if* $f\in D(\mathbf{Hess})$, *say* $\mathbf{Hess}f=\nu_f|\mathbf{Hess}f|$ *and* $\varphi\in C^2(\mathbb{R})$, *then* $\varphi\circ f\in D(\mathbf{Hess})$ *and, setting*

$$\mu:=|\mathbf{Hess}f|+\mathsf{m},$$
$$\nu:=\varphi'\circ f\nu_f\frac{\mathrm{d}|\mathbf{Hess}f|}{\mathrm{d}\mu}+\varphi''\circ f\nabla f\otimes\nabla f\frac{\mathrm{d}\mathsf{m}}{\mathrm{d}\mu},$$

it holds that $\mathbf{Hess}(\varphi\circ f)=\frac{\nu}{|\nu|}(|\nu|\mu)$.

Now we show that convex functions (recall Definition 3.2) have measure valued Hessian. As a notation, here and below, $a^-:=-a\vee 0$ for every $a\in\mathbb{R}$.

Theorem 3.9 (Theorem 1.1 restated) *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and* $f\in\mathrm{H}^{1,2}_{\mathrm{loc}}(\mathsf{X})$ *satisfying* $\mathrm{Hess}f\geq\kappa$, *for some* $\kappa\in\mathbb{R}$. *Then,* $f\in D(\mathbf{Hess})$, *say* $\mathbf{Hess}f=\nu_f|\mathbf{Hess}f|$. *Moreover, we have that*

$$\nu_f|\mathbf{Hess}f|\geq\kappa g\mathsf{m},$$

in the sense that for every $v\in\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$, *it holds, as measures,*

$$v\otimes v\cdot\nu_f|\mathbf{Hess}f|\geq\kappa|v|^2\mathsf{m}. \tag{3.5}$$

Finally, if in addition $f\in\mathrm{H}^{1,2}(\mathsf{X})$, *then for every* $X,Y\in\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})$, *we have that* $X\otimes Y\cdot\nu_f\in\mathrm{L}^1(|\mathbf{Hess}f|)$, *in particular, Eq. 3.4 holds for every* $h\in\mathrm{S}^2(\mathsf{X})\cap\mathrm{L}^\infty(\mathsf{m})$

and $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. *More precisely, we have the explicit bound, for every* $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$

$$\int_{\mathsf{X}} |X \otimes X \cdot \nu_f| \,\mathrm{d}|\mathbf{Hess} f| \leq \int_{\mathsf{X}} -\mathrm{div}\, X \nabla f \cdot X - \nabla f \cdot \nabla(\tfrac{1}{2}|X|^2) + 2\kappa^-|X|^2 \,\mathrm{dm}. \quad (3.6)$$

Proof We divide the proof in several steps.

Step 1. We define for $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ with bounded support and $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$,

$$\mathcal{G}_f(h, X, Y) := \int_{\mathsf{X}} -\frac{1}{2}\mathrm{div}(hX)\nabla f \cdot Y - \frac{1}{2}\mathrm{div}(hY)\nabla f \cdot X - \frac{1}{2}h\nabla f \cdot \nabla(X \cdot Y)\mathrm{dm}$$

and we write for simplicity

$$\mathcal{G}_f(h, X) := \mathcal{G}_f(h, X, X).$$

We shall frequently use the fact that for given f, h, Y as above

$$\begin{gathered}\text{the map } X \mapsto \mathcal{G}_f(h, X, Y) \in \mathbb{R} \text{ is continuous w.r.t. the } \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\text{-norm}\\ \text{on sets of vector fields with uniformly bounded } L^\infty\text{-norm,}\end{gathered} \quad (3.7)$$

and similarly for Y.

Notice that $\mathcal{G}_f(h, X, Y)$ equals the right hand side of Eq. 3.4 for $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, as a consequence of a simple approximation argument on f. Indeed, as h has bounded support, a locality argument shows that it is not restrictive to assume also $f \in \mathrm{H}^{1,2}(\mathsf{X})$, then we can approximate f in the $\mathrm{H}^{1,2}(\mathsf{X})$ topology with functions in $\mathrm{TestF}(\mathsf{X})$ and see the equality of the two quantities. This shows also that, as $\mathrm{Hess} f \geq \kappa$,

$$\mathcal{G}_f(h, \nabla g) \geq \kappa \int_{\mathsf{X}} h|\nabla g|^2\mathrm{dm} \quad (3.8)$$

if $h \in \mathrm{TestF}(\mathsf{X})$ is non negative and has bounded support and $g \in \mathrm{TestF}(\mathsf{X})$ (by approximation, Eq. 3.8 continues to hold if $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ is non negative and has bounded support). This argument also shows that

$$\sum_{i,j=1}^m \mathcal{G}_f(hf_i g_j, X_i, Y_j) = \sum_{i,j=1}^m \mathcal{G}_f(h, f_i X_i, g_j Y_j) \quad (3.9)$$

for every $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ with bounded support, $\{f_i\}_i, \{g_i\}_i \subseteq \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ and $\{X_i\}_i, \{Y_i\}_i \subseteq \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ (we are implicitly using Lemma 2.1). Clearly, $\mathcal{G}_f(h, \cdot, \cdot)$ is symmetric and $\mathbb{R}$-bilinear for every $h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$.

Step 2. By Eq. 3.8, the Riesz—Daniell—Stone Theorem yields that for every $g \in \mathrm{TestF}(\mathsf{X})$ there exists a unique Radon measure $\mu_{\nabla g}$ such that

$$\mathcal{G}_f(h, \nabla g) = \int_{\mathsf{X}} h\mathrm{d}\mu_{\nabla g} \qquad \text{for every } h \in \mathrm{LIP_{bs}}(\mathsf{X}). \quad (3.10)$$

Recalling Eq. 3.8, we have that

$$\mu_{\nabla g} \geq \kappa|\nabla g|^2\mathrm{dm}. \quad (3.11)$$

We show now that $\mu_{\nabla g} \ll \mathrm{Cap}$. Being $\mu_{\nabla g}$ a Radon measure, it is enough to show that if K is a compact set such that $\mathrm{Cap}(K) = 0$, then $\mu_{\nabla g}(K) = 0$. As K is compact, we can find a sequence $\{u_n\}_n \subseteq \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{LIP_b}(\mathsf{X})$ with $u_n = 1$ on a neighbourhood K, $0 \leq u_n \leq 1$ and $\|u_n\|_{\mathrm{H}^{1,2}(\mathsf{X})} \to 0$ (see e.g. [7, Lemma 5.4] or [9, Lemma 3.3]). Also, it is easy to see that we can assume with no loss of generality also that $\{u_n\}_n \subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ have uniformly bounded support. Therefore, using Eq. 3.11, dominated convergence and Eq. 3.10,

$$0 \le \mu_{\nabla g}(K) - \kappa \int_K |\nabla g|^2 \mathrm{dm} \le \int_{\mathsf{X}} u_n \mathrm{d}\mu_{\nabla g} - \kappa \int_{\mathsf{X}} u_n |\nabla g|^2 \mathrm{dm} \overset{Eq. 3.7}{\to} 0,$$

having used also that $\mathrm{m}(K)$=0 in the last step. In particular, $\mu_{\nabla g}(K) = \mu_{\nabla g}(K) - \kappa \int_K |\nabla g|^2 \mathrm{dm}$ and thus the above proves that $\mu_{\nabla g} \ll \mathrm{Cap}$, as claimed.

As $\mu_{\nabla g} \ll \mathrm{Cap}$, using dominated convergence, we can show that

$$\mathcal{G}_f(h, \nabla g) = \int_{\mathsf{X}} h \mathrm{d}\mu_{\nabla g} \qquad \text{for every } h \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathrm{m}) \text{ with bounded support}$$

where we implicitly take the quasi continuous representative of h. We define by polarization the signed Radon measure, absolutely continuous with respect to Cap,

$$\mu_{\nabla g_1, \nabla g_2} := \frac{1}{4}\left(\mu_{\nabla(g_1+g_2)} - \mu_{\nabla(g_1-g_2)}\right) \qquad \text{if } g_1, g_2 \in \mathrm{TestF}(\mathsf{X})$$

so that, by the properties of $(X, Y) \mapsto \mathcal{G}_f(h, X, Y)$,

$$\mathcal{G}_f(h, \nabla g_1, \nabla g_2) = \int_{\mathsf{X}} h \mathrm{d}\mu_{\nabla g_1, \nabla g_2} \qquad \text{for every } h \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathrm{m}) \text{ with bounded support.}$$

Notice that the map $(g_1, g_2) \mapsto \mu_{\nabla g_1, \nabla g_2}$ is symmetric and $\mathbb{R}$-bilinear by its very definition.

Step 3. We show that for every $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and $h \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathrm{m})$ non negative and with bounded support,

$$\mathcal{G}_f(h, X, X) \ge \kappa \int_{\mathsf{X}} h|X|^2 \mathrm{dm}. \tag{3.12}$$

By dominated convergence and [9, Lemma 4.2], we see that it is enough to assume $X \in \mathrm{TestV}(\mathsf{X})$, say

$$X = \sum_{i=1}^m f_i \nabla g_i \qquad \text{with } \{f_i\}_i \subseteq \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathrm{m}) \text{ and } \{g_i\}_i \subseteq \mathrm{TestF}(\mathsf{X}).$$

By the properties of the map $(X, Y) \mapsto \mathcal{G}_f(h, X, Y)$ (in particular, recall Eq. 3.9) we see that Eq. 3.12 will follow from

$$\sum_{i,j=1}^m \mathcal{G}_f(h f_i f_j, \nabla g_i, \nabla g_j) \ge \kappa \int_{\mathsf{X}} h|X|^2 \mathrm{dm}.$$

It suffices then to show then that, as measures,

$$\sum_{i,j=1}^m f_i f_j \mu_{\nabla g_i, \nabla g_j} \ge \kappa \sum_{i,j=1}^m f_i f_j \nabla g_i \cdot \nabla g_j \mathrm{m}.$$

By dominated convergence and localizing, we further reduce to the case in which $f_i = c_i \in \mathbb{R}$ for every $i = 1, \dots, m$, this is to say, as measures,

$$\sum_{i,j=1}^m c_i c_j \mu_{\nabla g_i, \nabla g_j} \ge \kappa \sum_{i,j=1}^m c_i c_j \nabla g_i \cdot \nabla g_j \mathrm{m},$$

that follows by Eq. 3.11 with $\sum_{i=1}^m c_i g_i$ in place of g.

Step 4. Building upon Eq. 3.12 and arguing as in **Step 2**, we can define the Radon measure $\mu_{X,Y}$ whenever $X, Y \in H^{1,2}_H(TX) \cap L^\infty(TX)$, such that

$$\int_X h \mathrm{d}\mu_{X,Y} = \mathcal{G}_f(h, X, Y) \qquad \text{for every } h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X}).$$

More precisely, we first define $\mu_{X,X}$ for $X \in H^{1,2}_H(TX) \cap L^\infty(TX)$ and then we define $\mu_{X,Y}$ for $X, Y \in H^{1,2}_H(TX) \cap L^\infty(TX)$ by polarization, taking into account that $\mathcal{G}_f(h, X, Y)$ is symmetric in X and Y.

By the properties of $(X, Y) \mapsto \mathcal{G}_f(h, X, Y)$ (in particular, recall Eq. 3.9) it follows that, if for some m and $l = 1, 2$,

$$X_l = \sum_{i=1}^m f_i^l \nabla g_i^l \qquad \text{with } \{f_i^l\}_i \subseteq S^2(\mathsf{X}) \cap L^\infty(\mathsf{m}) \text{ and } \{g_i^l\}_i \subseteq \mathrm{TestF}(\mathsf{X}),$$

then we have that

$$\mu_{X_1,X_2} = \sum_{i,j=1}^m f_i^1 f_j^2 \mu_{\nabla g_i^1, \nabla g_j^2}.$$

In particular, this definition is coherent with the one given in **Step 2**.

Recall that, by Eq. 3.12, if $X \in H^{1,2}_H(TX) \cap L^\infty(TX)$,

$$\mu_{X,X} \geq \kappa |X|^2 \mathsf{m}. \tag{3.13}$$

Also, as in **Step 2**, we show that

$$\mu_{X,Y} \ll \mathrm{Cap},$$

whenever $X, Y \in H^{1,2}_H(TX) \cap L^\infty(TX)$.

Step 5. We use now Theorem 2.1 to take a partition $\{A_k\}_{k\in\mathbb{N}}$ and, for any k, an orthonormal basis of $L^0_{\mathrm{Cap}}(TX)$ on A_k $v_1^k, \dots, v_{n(k)}^k$. Fix for the moment k and define, for $i, j = 1, \dots, n(k)$,

$$\mu_{i,j}^k := \mu_{v_i^k, v_j^k} \llcorner A_k \qquad \text{and} \qquad \mu^k := \sum_{i,j=1}^{n(k)} |\mu_{i,j}^k|.$$

Notice that this is a good definition as $\bar{\mathrm{Pr}}(v_i^k) \in H^{1,2}_H(TX)$ for every $i = 1, \dots, n(k)$ and that the measures above are finite signed measures absolutely continuous with respect to Cap.

We define

$$\tilde{\nu}_f^k := \sum_{i,j=1}^{n(k)} v_i^k \otimes v_j^k \frac{\mathrm{d}\mu_{i,j}^k}{\mathrm{d}\mu^k}$$

then

$$\nu_f^k := \frac{1}{|\tilde{\nu}_f^k|} \tilde{\nu}_f^k \qquad \text{and} \qquad |\mathbf{Hess}^k f| := |\tilde{\nu}_f^k| \mu^k.$$

Finally,

$$\nu_f := \sum_k \nu_f^k \qquad \text{and} \qquad |\mathbf{Hess} f| := \sum_k |\mathbf{Hess}^k f|.$$

Clearly, $|\mathbf{Hess} f|$ is a σ-finite measure, $|\mathbf{Hess}| \ll \mathrm{Cap}$, $|\nu_f| = 1$ $|\mathbf{Hess} f|$-a.e. and ν_f is symmetric.

Step 6. Let $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and $h \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ with bounded support. We verify that $X \otimes Y \cdot \nu_f \in \mathrm{L}^1_{\mathrm{loc}}(|\mathbf{Hess} f|)$ and that Eq. 3.4 holds. By polarization, there is no loss of generality in assuming that $X = Y$ and we can also assume, by linearity, that h is non negative. Notice indeed that the right hand side of Eq. 3.4 is equal to $\mathcal{G}_f(h, X, Y)$ (recall **Step 1**) and hence is symmetric in X, Y.

Consider the Borel partition $\{A_k\}_k$ as in Theorem 2.1. Assume for the moment that for every k, for every $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$,

$$\int_{A_k} h \mathrm{d}\mu_{X,X} = \int_{A_k} h X \otimes X \cdot \nu_f \mathrm{d}|\mathbf{Hess} f| \tag{3.14}$$

(notice that the restriction of $|\mathbf{Hess} f|$ to A_k is the finite measure $|\mathbf{Hess}^k f|$, so the right hand side is well defined). Then, it holds that

$$|X \otimes X \cdot \nu_f||\mathbf{Hess} f| = |\mu_{X,X}| \tag{3.15}$$

that yields local integrability. We can then compute, by dominated convergence and Eq. 3.14 (recall that h has bounded support),

$$\begin{aligned}\mathcal{G}_f(h, X, X) &= \int_{\mathsf{X}} h \mathrm{d}\mu_{X,X} = \sum_k \int_{A_k} h \mathrm{d}\mu_{X,X} \\ &= \sum_k \int_{A_k} h X \otimes X \cdot \nu_f \mathrm{d}|\mathbf{Hess} f| = \int_{\mathsf{X}} h X \otimes X \cdot \nu_f \mathrm{d}|\mathbf{Hess} f|,\end{aligned}$$

that is Eq. 3.4.

We show then Eq. 3.14. Fix k and recall the notation of **Step 6**. Notice that, considering the left hand side of Eq. 3.14, we have, by the very definition of ν_f and $|\mathbf{Hess} f|$, on A_k

$$X \otimes X \cdot \nu_f |\mathbf{Hess} f| = \sum_{i,j=1}^{n(k)} X \otimes X \cdot v_i^k \otimes v_j^k \mu_{i,j}^k = \sum_{i,j=1}^{n(k)} X \cdot v_i^k X \cdot v_j^k \mu^k_{v_i^k, v_j^k} = \mu_{\tilde{X},\tilde{X}},$$

where $\tilde{X} := \sum_{i=1}^{n(k)} (X \cdot v_i^k) v_i^k \in \mathrm{TestV}(\mathsf{X})$ satisfies then $\tilde{X} = X$ Cap-a.e. on A_k.

As we have reduced ourselves to check that $\mu_{X,X} \llcorner A_k = \mu_{\tilde{X},\tilde{X}} \llcorner A_k$, taking into account also the bilinearity of the map

$$(\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X}))^2 \ni (X, Y) \mapsto \mu_{X,Y},$$

we see that it is enough to show that

$$\text{for every } X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X}) \text{ we have } \qquad \mu_{X,Y} \llcorner \{|X| = 0\} = 0.$$

Let $\{\varphi_n\}_n \subseteq \mathrm{H}^{1,2}(\mathsf{X})$ be as in Lemma 2.2 for the vector field $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. We compute, if $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$, by Eq. 3.9,

$$\int_{\mathsf{X}} h\varphi_n \mathrm{d}\mu_{X,Y} = \mathcal{G}_f(h\varphi_n, X, Y) = \mathcal{G}_f(h, \varphi_n X, Y).$$

By the very definition, $\varphi_n(x) \searrow \chi_{\{|X|=0\}}$ Cap-a.e. so that, by dominated convergence,

$$\int_{\mathsf{X}} h\varphi_n \mathrm{d}\mu_{X,Y} \to \int_{\mathsf{X}} h\chi_{\{|X|=0\}} \mathrm{d}\mu_{X,Y}.$$

On the other hand, as $\varphi_n X \to 0$ in the $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ topology (Lemma 2.2), we have

$$\mathcal{G}_f(h, \varphi_n X, Y) \to 0$$

by Eq. 3.7. Therefore, for every $h \in \mathrm{LIP_{bs}}(\mathsf{X})$,

$$\int_{\mathsf{X}} h\chi_{\{|X|=0\}} \mathrm{d}\mu_{X,Y} = 0,$$

which means $\mu_{X,Y} \llcorner \{|X| = 0\} = 0$.

Step 7. We prove Eq. 3.5. By a locality argument, we reduce ourselves to the case $v \in \mathrm{L}^\infty_{\mathrm{Cap}}(T\mathsf{X})$. By density and dominated convergence, it is enough to show Eq. 3.5 for $v \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. By Eqs. 3.13 and 3.14 , if $v \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$,

$$v \otimes v \cdot \nu_f |\mathbf{Hess} f| = \mu_{v,v} \geq \kappa |v|^2 \mathsf{m},$$

that proves the claim.

Step 8. We prove the last claim. Again, we assume with no loss of generality that $X = Y$. It is enough to show that if moreover $f \in \mathrm{H}^{1,2}(\mathsf{X})$, then $X \otimes X \cdot \nu_f \in \mathrm{L}^1(|\mathbf{Hess} f|)$, then the rest will follow from dominated convergence. The integrability follows from Eq. 3.15 if we show that $\mu_{X,X}$ is a finite signed measure. Inequality Eq. 3.13 implies that the measure $\mu_{X,X} - \kappa|X|^2\mathsf{m}$ is non negative, but now, using an immediate approximation argument and monotone convergence together with Eq. 3.4, we see that it is also finite. As $|X|^2\mathsf{m}$ is a finite measure, we see that $\mu_{X,X}$ is a finite signed measure and that Eq. 3.6 holds. □

3.3 Ricci Tensor

As done for the Hessian, we give a fine meaning the Ricci tensor defined in [22]. Namely, we represent the Ricci tensor as a product of a Cap-tensor field and a σ-finite measure that is absolutely continuous with respect to Cap. As in the proof of Theorem 1.1, we are going to use a version of Riesz representation Theorem for positive functionals, this time leveraging on the bound from below for the Ricci tensor ensured, in a synthetic way, by the definition of the RCD condition.

We recall now the distributional definition of the objects that we are going to need and, to this aim, we recall also that the definition of $\mathcal{V}$ is in Eq. 2.1.

Theorem 3.10 ([22, Theorem 3.6.7]) *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space. There exists a unique continuous map*

$$\mathbf{Ric} : \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})^2 \to \mathrm{Meas}(\mathsf{X})$$

such that for every $X, Y \in \mathcal{V}$ it holds

$$\mathbf{Ric}(X, Y) := \Delta \frac{X \cdot Y}{2} + \left(\frac{1}{2} X \cdot \Delta_{\mathrm{H}} Y + \frac{1}{2} Y \cdot \Delta_{\mathrm{H}} X - \nabla X \cdot \nabla Y\right) \mathsf{m}. \tag{3.16}$$

Such map is bilinear, symmetric and satisfies

$$\begin{aligned} &\mathbf{Ric}(X, X) \geq K|X|^2\mathsf{m}; \\ &\int_{\mathsf{X}} \mathrm{d}\mathbf{Ric}(X, Y) = \int_{\mathsf{X}} \mathrm{d}X \cdot \mathrm{d}Y + \delta X \cdot \delta Y - \nabla X \cdot \nabla Y \mathrm{d}\mathsf{m}; \\ &\|\mathbf{Ric}(X, Y)\|_{\mathrm{TV}} \leq 2\sqrt{\mathcal{E}_{\mathrm{H}}(X) + K^-\|X\|^2_{\mathrm{L}^2(T\mathsf{X})}}\sqrt{\mathcal{E}_{\mathrm{H}}(Y) + K^-\|Y\|^2_{\mathrm{L}^2(T\mathsf{X})}}; \end{aligned} \tag{3.17}$$

for every $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$.

Theorem 3.11 (Theorem 1.2 restated) *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then there exists a unique* σ*-finite measure* $|\mathbf{Ric}|$ *that satisfies* $|\mathbf{Ric}| \ll \mathrm{Cap}$ *and a unique, up to* $|\mathbf{Ric}|$*-a.e. equality, symmetric tensor field* $\omega \in \mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 2}\mathsf{X})$ *with* $|\omega| = 1$ $|\mathbf{Ric}|$*-a.e. such that* $\mathbf{Ric} = \omega|\mathbf{Ric}|$*, in the sense that for every* $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *we have that* $X \otimes Y \cdot \omega \in \mathrm{L}^1(|\mathbf{Ric}|)$ *and it holds that, as measures,*

$$X \otimes Y \,\cdot\, \omega|\mathbf{Ric}| = \mathbf{Ric}(X, Y). \tag{3.18}$$

Moreover

$$\omega|\mathbf{Ric}| \geq K \mathsf{g}\mathsf{m},$$

in the sense that for every $v \in \mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$*, it holds, as measures,*

$$v \otimes v \,\cdot\, \omega|\mathbf{Ric}| \geq K|v|^2\mathsf{m}. \tag{3.19}$$

Proof We divide the proof in several steps.

Step 1. Uniqueness follows as in the proof of Proposition 3.7, by a localized version of Lemma 2.10.

Step 2. We remark that for every $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, it holds $|\mathbf{Ric}(X, Y)| \ll \mathrm{Cap}$, as a an immediate consequence of Eq. 3.16 together with a density and continuity argument (notice that it is enough to show that $\mathbf{Ric}(X, Y)(K) = 0$ whenever K is a compact set with $\mathrm{Cap}(K) = 0$).

Step 3. We proceed now as in **Step 5** of the proof of Theorem 1.1. In particular, we use Theorem 2.1 to take a partition of X, $\{A_k\}_k$. We fix for the moment k and take, (following Theorem 2.1), an orthonormal basis of $\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})$ on A_k, $v^k_1, \dots, v^k_{n(k)}$. Define, for $i, j = 1, \dots, n(k)$,

$$\mu^k_{i,j} := \mathbf{Ric}(v^k_i, v^k_j) \llcorner A_k \qquad \text{and} \qquad \mu^k := \sum_{i,j=1}^{n(k)} |\mu^k_{i,j}|.$$

We define

$$\tilde{\omega}^k := \sum_{i,j=1}^{n(k)} v^k_i \otimes v^k_j \frac{\mathrm{d}\mu^k_{i,j}}{\mathrm{d}\mu^k}$$

then

$$\omega^k := \frac{1}{|\tilde{\omega}^k|}\tilde{\omega}^k \qquad \text{and} \qquad |\mathbf{Ric}^k| := |\tilde{\omega}^k|\mu^k.$$

Finally,

$$\omega := \sum_k \omega^k \qquad \text{and} \qquad |\mathbf{Ric}| := \sum_k |\mathbf{Ric}^k|.$$

Clearly, $|\mathbf{Ric}|$ is a σ-finite measure, $|\omega| = 1$ $|\mathbf{Ric}|$-a.e. and ω is symmetric.

Step 4. Let $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$. We verify that $X \otimes Y \,\cdot\, \omega \in \mathrm{L}^1(|\mathbf{Ric}|)$ and that Eq. 3.18 holds. This will be similar to **Step 6** of the proof of Theorem 1.1 and we keep the same notation. By polarization, there is no loss of generality in assuming that $X = Y$ and it is enough to show that for every k, as measures,

$$\mathbf{Ric}(X, X) \llcorner A_k = X \otimes X \,\cdot\, \omega|\mathbf{Ric}| \llcorner A_k \tag{3.20}$$

(notice that the left hand side of Eq. 3.20 is a finite signed measure).

Assume for the moment that also $X \in \mathrm{L}^\infty(T\mathsf{X})$. Notice that Eq. 3.20 also yields integrability (still in the case $X \in \mathrm{L}^\infty(T\mathsf{X})$), as it shows that

$$\|X \otimes X \,\cdot\, \omega\|_{\mathrm{L}^1(|\mathbf{Ric}|)} = \|\mathbf{Ric}(X, X)\|_{\mathrm{TV}}. \tag{3.21}$$

Also, by Eq. 3.21, we see that the additional assumption $X \in \mathrm{L}^\infty(T\mathsf{X})$ is not restrictive: if $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, we can find a sequence $\{X_n\}_n \subseteq \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ with $X_n \to X$ in the $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ topology. For example, we can define

$$X_n := \frac{n}{n \vee |X|} X$$

$\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X}) \ni X_n \to X$ in the $\mathrm{W}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ topology thanks to the calculus rules of Lemma 2.1 and the computation

$$\left|\nabla\left(\frac{n}{n \vee |X|}\right)\right| |X| = \chi_{\{|X| \geq n\}} n \frac{|\nabla|X||}{|X|^2} |X| \leq \chi_{\{|X| \geq n\}} |\nabla|X|| \to 0 \qquad \text{in } \mathrm{L}^2(\mathfrak{m}).$$

Then, by the continuity of $\mathbf{Ric}$ and Eq. 3.21, the sequence $\{X_n \otimes X_n \cdot \omega\}_n \subseteq \mathrm{L}^1(|\mathbf{Ric}|)$ is a Cauchy sequence, whose limit coincides then with $X \otimes X \cdot \omega$ so that this implies the general case.

We prove now Eq. 3.20, under the additional assumption $X \in \mathrm{L}^\infty(T\mathsf{X})$. We first remark that it holds

$$f\mathbf{Ric}(X, \cdot) = \mathbf{Ric}(fX, \cdot) \qquad \text{if } f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathfrak{m}).$$

This is a consequence of [22, Proposition 3.6.9] together with an approximation argument, see Lemma 2.1 (here we use that $X \in \mathrm{L}^\infty(T\mathsf{X})$). Therefore, with the same computations of **Step 6** of the proof of Theorem 1.1, we see that

$$X \otimes X \cdot \omega |\mathbf{Ric}| \llcorner A_k = \mathbf{Ric}(\tilde{X}, \tilde{X}) \llcorner A_k$$

where $\tilde{X} \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ is such that $\tilde{X} = X$ Cap-a.e. on A_k, so that Eq. 3.20 reduces to the locality relation

$$\mathbf{Ric}(X, X) \llcorner A_k = \mathbf{Ric}(\tilde{X}, \tilde{X}) \llcorner A_k$$

whenever $X, \tilde{X} \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ are such that $\tilde{X} = X$ Cap-a.e. on A_k.

By bilinearity, we can just show that if $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ is such that $X = 0$ Cap-a.e. on A_k, then $\mathbf{Ric}(X, Y) \llcorner A_k = 0$ for every $Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. Let $\{\varphi_n\}_n \subseteq \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathfrak{m})$ be as in Lemma 2.2 for the vector field $X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and let $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ (notice that $\varphi_n X \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ by Lemma 2.1). We know that

$$\int_{\mathsf{X}} h\varphi_n \mathrm{d}\mathbf{Ric}(X, Y) = \int_{\mathsf{X}} h \mathrm{d}\mathbf{Ric}(\varphi_n X, Y).$$

By Lemma 2.2 and the continuity of the map $\mathbf{Ric}$, the right hand side of the equation above converges to 0, whereas the left hand side converges to (as in **Step 6** of the proof of Theorem 1.1)

$$\int_{\mathsf{X}} h\chi_{\{|X|=0\}} \mathrm{d}\mathbf{Ric}(X, Y).$$

This is to say that for every $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$

$$\int_{\mathsf{X}} h\chi_{\{|X|=0\}} \mathrm{d}\mathbf{Ric}(X, Y) = 0$$

which means that $\mathbf{Ric}(X, Y) \llcorner \{|X| = 0\} = 0$, whence the claim.

Step 5. Inequality Eq. 3.19 follows by an approximation argument as in **Step 8** of the proof of Theorem 1.1, Eqs. 3.18 and 3.17. $\square$

At the very end of [22], it has been asked how one may enlarge the domain of definition of the map **Ric**, and, towards this extension, whether

$$\sum_i f_i \mathbf{Ric}(X_i, Y) = \mathbf{Ric}\left(\sum_i f_i X_i, Y\right) \tag{3.22}$$

whenever $X_1, \ldots, X_n, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ and $f_i \in C_{\mathrm{b}}(\mathsf{X})$. It seems that basic algebraic manipulations based on the formulas involving the Ricci curvature as in Theorem 3.10 do not imply this fact. However, exploiting Theorem 1.2, we immediately have an affirmative result to this question, at least in the finite dimensional case, and we record this result in the following proposition. More generally, Theorem 1.2 gives a natural way to enlarge the domain of definition of the map **Ric**.

Proposition 3.12 *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Let* $X, Y, X_1, \ldots, X_n \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *and let* $f_1, \ldots, f_n \in C_{\mathrm{b}}(\mathsf{X})$ *such that* $X = \sum_{i=1}^n f_i X_i$. *Then*

$$\sum_{i=1}^n f_i \mathbf{Ric}(X_i, Y) = \mathbf{Ric}(X, Y).$$

Notice also that an immediate consequence of Theorem 1.2 (the point is **Step 4** of its proof, which relies on an approximation argument based on Lemma 2.2) is that for every $X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})$, $\mathbf{Ric}(X, Y) \llcorner \{|X| = 0\} = 0$, thus providing a different proof of the implication 3) $\Rightarrow$ 1) of [27, Proposition 3.7] (see the comments at [27, Pag. 3 and Pag. 4]).

Remark 3.13 Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. According to [28, Definition 4.2], we can define the modified N-Ricci tensor as

$$\mathbf{Ric}_N(X, Y) := \mathbf{Ric}(X, Y) - R_N(X, Y) \qquad \text{for every } X, Y \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}),$$

where

$$R_N(X, Y) := \begin{cases} \dfrac{1}{N-n}(\mathrm{tr}(\nabla X) - \mathrm{div} X)(\mathrm{tr}(\nabla Y) - \mathrm{div} Y) & \text{if } n < N, \\ 0 & \text{if } n \geq N. \end{cases}$$

In particular, notice that $\mathbf{Ric} = \mathbf{Ric}_\infty$.

Now, [28, Theorem 4.3] proves that still $\mathbf{Ric}_N \geq K$, thus improving Eq. 3.17 in the finite dimensional case. Hence, we can follow the proof of Theorem 1.2, and prove the analogue of Theorem 1.2 with $\mathbf{Ric}_N$ in place of **Ric**.

Exploiting to the representation of $\mathbf{Ric} = \omega|\mathbf{Ric}|$ given by Theorem 1.2 (or the analogue representation for $\mathbf{Ric}_N$), we can easily give a meaning to the trace the “tensor” measure **Ric**. However we *do not* expect that the trace of this polar measure has a meaning to represent the scalar curvature, if one does not add artificial correction terms (cf. the characterization of the scalar curvature on smoothable Alexandrov spaces in [32]). Indeed, already in the setting of a smooth weighted Riemannian manifold $(M, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g)$, $\mathbf{Ric}_N$ represents the modified Bakry-Émery N-Ricci curvature tensor, defined as

$$\mathrm{Ric}_N := \begin{cases} \mathrm{Ric}_g + \mathrm{Hess}_g(V) - \frac{\mathrm{d}V \otimes \mathrm{d}V}{N-n} & \text{if } N > n, \\ \mathrm{Ric}_g & \text{if } N = n \text{ and } V \text{ is constant}, \\ -\infty & \text{otherwise}. \end{cases} \tag{3.23}$$

Nevertheless, even if we restrict ourselves to non-collapsed spaces [16], which play the role of “unweighted” spaces, we can see that looking at the scalar curvature as trace of **Ric** is not yet meaningful: for example **Ric** vanishes on sets of 0 capacity, whereas the scalar curvature such behaviour is not expected.

For example, consider a cut-cone $C \subseteq \mathbb{R}^3$, which can be obtained by gluing along the edges a sector of the unit ball in $\mathbb{R}^2$ corresponding to an angle $\alpha \in (0, 2\pi)$. Endowed with its natural metric and measure, this is an RCD(0, 2) space. Also, such surface is locally flat in its interior away from the tip. Moreover, assume that we have something like Gauss–Bonnet Theorem: if C were smooth, then

$$\int_{C^\circ} K \mathrm{d}\sigma + \int_{\partial C} k_g \mathrm{d}s = 2\pi\chi(C) = 2\pi,$$

where the first term is the surface integral of the scalar curvature (in the interior of C), the second term the boundary integral of the geodesic curvature of the boundary, and the third term the Euler characteristic of the cone. Notice that C is singular only near its tip, so we expect all the terms except the first one to remain unchanged. We can replace then the first term by

$$\int_{C^\circ} \mathrm{tr}(\omega)\mathrm{d}|\mathbf{Ric}|,$$

which should be the non-smooth analogue of the area integral of the scalar curvature. By the (capacitary) discussion above, this term vanishes, so that we would have

$$\alpha = \int_{\partial C} k_g \mathrm{d}s = 2\pi,$$

which is clearly wrong. In this particular example, we see also which term should be added to the scalar curvature to recover the validity of the Gauss–Bonnet Theorem.

Along the same line of though, we can consider the surface of a tetrahedron T, which is again an RCD(0, 2) space, but this time we have no boundary. Taken any point of T which lies on an edge of T, but not on one of its vertices, we see that T is locally isometric to the plane near x. As also the faces of T are clearly flat, every notion of scalar curvature that we obtain from taking the trace of **Ric**, should be concentrated on the vertices, hence vanish, by a capacitary argument as above. As T has Euler characteristic 2, we see that also in this case one should add correction terms, to have a formula as the one in the statement of the Gauss–Bonnet Theorem. ■

3.4 Riemann Tensor

As done for Hessian and Ricci tensor, now we provide a representation for the Riemann curvature tensor defined in [23] as the product of a Cap-tensor field and a σ-finite measure that is absolutely continuous with respect to Cap. In order to do so, we again employ Riesz's representation Theorem for positive functionals, and hence we have to impose that the tensor representing the sectional curvature is bounded from below (hence, we will add the assumption of a bound on the distributional sectional curvature). Then, by standard algebra, we recover the full Riemann tensor out of the sectional curvatures.

We follow [23] to define $\nabla_X Y$, $[X, Y]$, $\mathbf{R}(X, Y)(Z)$ and $\mathcal{R}(X, Y, Z, W)$ on an RCD(K, ∞) space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. Even though we assume familiarity with [23], we recall briefly the (distributional) definitions. First, recall the definition of $\mathcal{V}$ in Eq. 2.1. Then we have what follows.

(1) **Distributional covariant derivative.** If $X, Y \in \mathrm{L}^2(T\mathsf{X})$ with $X \in D(\mathrm{div})$ and at least one of X, Y is in $\mathrm{L}^\infty(T\mathsf{X})$, then

$$\mathbf{\nabla}_X Y(W) := -\int_{\mathsf{X}} \nabla_X W \cdot Y + Y \cdot W \mathrm{div} X \mathrm{d}\mathsf{m} \qquad \text{for every } W \in \mathcal{V}.$$

(2) **Distributional Lie bracket.** If $X, Y \in D(\mathrm{div}) \cap \mathrm{L}^\infty(T\mathsf{X})$,

$$\mathbf{[}X, Y\mathbf{]} := \mathbf{\nabla}_X Y - \mathbf{\nabla}_Y X.$$

(3) **Distributional curvature tensor.** If $X, Y, Z \in \mathcal{V}$, then

$$\mathbf{R}(X, Y)(Z) := \mathbf{\nabla}_X(\nabla_Y Z) - \mathbf{\nabla}_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

(4) **Distributional Riemann tensor.** If $X, Y, Z, W \in \mathcal{V}$, then

$$\boldsymbol{\mathcal{R}}(X, Y, Z, W)(f) := (\mathbf{R}(X, Y)(Z))(f W) \qquad \text{for every } f \in \mathrm{TestF}(\mathsf{X}).$$

It is clear that the distributional covariant derivative and the distributional Lie bracket coincide with the covariant derivative $\nabla_X Y$ and the Lie bracket $[X, Y] := \nabla_X Y - \nabla_Y X$, whenever both the objects are defined. We are going to exploit this property throughout.

Remark 3.14 We want to extend the definition of $\boldsymbol{\mathcal{R}}(X, Y, Z, W)$ to the case $X, Y, Z, W \in \mathrm{TestV}(\mathsf{X})$ do not necessarily belong to $\mathcal{V}$. Clearly, as $\mathrm{TestV}(\mathsf{X}) \subseteq D(\mathrm{div}) \cap \mathrm{L}^\infty(T\mathsf{X})$, the first two terms $\mathbf{\nabla}_X(\nabla_Y Z)(f W) - \mathbf{\nabla}_Y(\nabla_X Z)(f W)$ are still well defined. Also the third term $\nabla_{[X,Y]} Z$ makes sense for this choice of vector fields. Notice that in [23], $\mathbf{\nabla}_{\mathbf{[}X,Y\mathbf{]}} Z$ was used instead of $\nabla_{[X,Y]} Z$. This clearly makes no substantial difference, but allows us to drop the request $\mathbf{[}X, Y\mathbf{]} = [X, Y] \in D(\mathrm{div})$, which is not granted if $X, Y \in \mathrm{TestV}(\mathsf{X})$ do not necessarily belong to $\mathcal{V}$, cf. [23, Lemma 2.4].

Following the same lines, we see that $\boldsymbol{\mathcal{R}}(X, Y, Z, W)$ makes sense for every $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and then $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$. ■

Remark 3.15 We do some trivial algebraic manipulations in order to deal with the quantity $\boldsymbol{\mathcal{R}}(X, Y, Z, W)(f)$, for $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$. We have that

$$\begin{aligned}
\boldsymbol{\mathcal{R}}(X, Y, Z, W)(f) = &-\int_{\mathsf{X}} \nabla_X(f W) \cdot \nabla_Y Z + \nabla_Y Z \cdot (f W)\mathrm{div}(X)\mathrm{d}\mathsf{m} \\
&+ \int_{\mathsf{X}} \nabla_Y(f W) \cdot \nabla_X Z + \nabla_X Z \cdot (f W)\mathrm{div}(Y)\mathrm{d}\mathsf{m} \\
&+ \int_{\mathsf{X}} f \nabla Z(\nabla_X Y - \nabla_Y X, W) \\
= &- \int_{\mathsf{X}} f \nabla W(X, \nabla_Y Z) + \nabla f \cdot X \nabla Z(Y, W) + f \nabla Z(Y, W)\mathrm{div}(X)\mathrm{d}\mathsf{m} \\
&+ \int_{\mathsf{X}} f \nabla W(Y, \nabla_X Z) + \nabla f \cdot Y \nabla Z(X, W) + f \nabla Z(X, W)\mathrm{div}(Y)\mathrm{d}\mathsf{m} \\
&+ \int_{\mathsf{X}} f \nabla Z(\nabla_X Y, W) - \int_{\mathsf{X}} f \nabla Z(\nabla_Y X, W)\mathrm{d}\mathsf{m}.
\end{aligned}$$ ■

In the sequel, we will **tacitly extend the definition** of $\boldsymbol{\mathcal{R}}(X, Y, Z, W)$ to the case $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, according to Remark 3.14 and Remark 3.15.

For future reference, we recall here [23, Proposition 2.7]. Notice that an immediate approximation argument (recall Remark 3.15) allows us to extend the claim to the slightly larger class of vector fields and functions that we are considering.

Proposition 3.16 (Symmetries of the curvature) *For any* $X, Y, Z, W \in H^{1,2}_H(T\mathsf{X}) \cap L^\infty(T\mathsf{X})$ *and* $f \in S^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$ *it holds:*

$$\begin{aligned}
\mathcal{R}(X, Y, Z, W) &= -\mathcal{R}(Y, X, Z, W) = \mathcal{R}(Z, W, X, Y),\\
\mathbf{R}(X, Y)(Z) + \mathbf{R}(Y, Z)(X) &+ \mathbf{R}(Z, X)(Y) = 0,\\
f\mathcal{R}(X, Y, Z, W) = \mathcal{R}(fX, Y, Z, W) &= \mathcal{R}(X, fY, Z, W) = \mathcal{R}(X, Y, fZ, W) = \mathcal{R}(X, Y, Z, fW).
\end{aligned}$$

The following definition has been implicitly proposed in [23, Conjecture 1.1].

Definition 3.17 Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space. We say that $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ has sectional curvature bounded below by κ, for some $\kappa \in \mathbb{R}$, if for every $X, Y \in \mathrm{TestV}(\mathsf{X})$ and $f \in \mathrm{TestF}(\mathsf{X})$, $f \geq 0$, it holds

$$\mathcal{R}(X, Y, Y, X)(f) \geq \kappa \int_{\mathsf{X}} f|X \wedge Y|^2 \mathrm{d}\mathsf{m}.$$

Remark 3.18 It would be interesting to analyse the links between sectional curvature bounds in the sense of Definition 3.17 and in the sense of Alexandrov. This question is the content of [23, Conjecture 1.1]. ■

Remark 3.19 It is well known that sectional curvatures (i.e. $\mathcal{R}(X, Y, Y, X)$) are sufficient to identify a unique full Riemann curvature tensor $\mathcal{R}(X, Y, Z, W)$. We write here an explicit expression, as we are going to need it in the sequel. For $X, Y \in H^{1,2}_H(T\mathsf{X}) \cap L^\infty(T\mathsf{X})$, set $\mathcal{K}(X, Y) := \mathcal{R}(X, Y, Y, X)$. We claim that

$$\begin{aligned}
6\mathcal{R}(X, Y, Z, W) = \mathcal{K}(X+W, Y+Z) &- \mathcal{K}(X+W, Y) - \mathcal{K}(X+W, Z) - \mathcal{K}(Y+Z, X)\\
&- \mathcal{K}(Y+Z, W) + \mathcal{K}(X, Z) + \mathcal{K}(W, Y) - \mathcal{K}(Y+W, X+Z)\\
&+ \mathcal{K}(Y+W, X) + \mathcal{K}(Y+W, Z) + \mathcal{K}(X+Z, Y) + \mathcal{K}(X+Z, W)\\
&- \mathcal{K}(Y, Z) - \mathcal{K}(W, X).
\end{aligned}$$

thebibliography claim follows by algebraic manipulation, by Proposition 3.16. See e.g. [29, Lemma 4.3.3] for the expression. ■

Theorem 3.20 (Theorem 1.3 restated) *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space with sectional curvature bounded below by* κ, *for some* $\kappa \in \mathbb{R}$. *Then there exists a unique* σ*-finite measure* $|\mathbf{Riem}|$ *that satisfies* $|\mathbf{Riem}| \ll \mathrm{Cap}$ *and a unique, up to* $|\mathbf{Riem}|$*-a.e. equality, tensor field* $\nu \in L^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X})$ *with* $|\nu| = 1$ $|\mathbf{Riem}|$*-a.e. such that for every* $X, Y, Z, W \in H^{1,2}_H(T\mathsf{X}) \cap L^\infty(T\mathsf{X})$ *we have that* $X \otimes Y \otimes Z \otimes W \cdot \nu \in L^1(|\mathbf{Riem}|)$ *and it holds*

$$\int_{\mathsf{X}} f X \otimes Y \otimes Z \otimes W \cdot \nu |\mathbf{Riem}| = \mathcal{R}(X, Y, Z, W)(f) \quad \text{for every } f \in S^2(\mathsf{X}) \cap L^\infty(\mathsf{m}). \tag{3.24}$$

For every $v, w \in L^0_{\mathrm{Cap}}(T\mathsf{X})$,

$$v \otimes w \otimes w \otimes v \cdot \nu|\mathbf{Riem}| \geq \kappa |v \wedge w|^2 \mathsf{m}.$$

The tensor field ν has the following symmetries. Let $\mathcal{I},\mathcal{J},\mathcal{K}:\mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X})\to\mathrm{L}^0_{\mathrm{Cap}}(T^{\otimes 4}\mathsf{X})$ be the linear maps characterized as follows

$$\begin{aligned}\mathcal{I}(v_1\otimes v_2\otimes v_3\otimes v_4)&:=v_2\otimes v_1\otimes v_3\otimes v_4\\ \mathcal{J}(v_1\otimes v_2\otimes v_3\otimes v_4)&:=v_3\otimes v_4\otimes v_1\otimes v_2\\ \mathcal{K}(v_1\otimes v_2\otimes v_3\otimes v_4)&:=v_2\otimes v_3\otimes v_1\otimes v_4.\end{aligned}$$

Then, with respect to $|\mathbf{Riem}|$-a.e. equality,

$$\begin{aligned}\mathcal{I}(\nu)&=-\nu\\ \mathcal{J}(\nu)&=\nu\\ \nu+\mathcal{K}(\nu)+\mathcal{K}^2(\nu)&=0.\end{aligned}$$

Proof We divide the proof in several steps.

Step 1. Uniqueness follows as in the proof of Proposition 3.7, by a localized version of Lemma 2.10.

Step 2. Notice that if $X,Y\in\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})$ and $f\in\mathrm{S}^2(\mathsf{X})\cap\mathrm{L}^\infty(\mathsf{m})$, $f\geq 0$ then

$$\boldsymbol{\mathcal{R}}(X,Y,Y,X)(f)\geq\kappa\int_{\mathsf{X}}f|X\wedge Y|^2\mathrm{d}\mathsf{m},$$

thanks to an approximation argument that exploits the computations of Remark 3.15 and [9, Lemma 4.2].

Therefore, Riesz–Daniell—Stone Theorem yields that for every $X,Y\in\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})$ there exists a unique Radon measure $\mu_{X,Y}$ such that

$$\boldsymbol{\mathcal{R}}(X,Y,Y,X)(f)=\int_{\mathsf{X}}f\,\mathrm{d}\mu_{X,Y}\qquad\text{for every } f\in\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X}).$$

Clearly, for every $X,Y\in\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})$, $\mu_{X,Y}\geq\kappa|X\wedge Y|^2\mathrm{d}\mathsf{m}$ and the assignment $\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})\ni(X,Y)\mapsto\mu_{X,Y}$ is symmetric, by the symmetries of $\boldsymbol{\mathcal{R}}$ (Proposition 3.16). Also, we can prove, following **Step 2** of the proof of Theorem 1.1 that $\mu_{X,Y}\ll\mathrm{Cap}$, so that (see **Step 2** of the proof of Theorem 1.1 again and use an approximation argument based on the computations of Remark 3.15)

$$\boldsymbol{\mathcal{R}}(X,Y,Y,X)(f)=\int_{\mathsf{X}}f\,\mathrm{d}\mu_{X,Y}\qquad\text{for every } f\in\mathrm{S}^2(\mathsf{X})\cap\mathrm{L}^\infty(\mathsf{m}),$$

where we implicitly take the quasi continuous representative of f. This expression, together with the positivity of $\mu_{X,Y}-\kappa|X\wedge Y|^2$ yields that $\mu_{X,Y}$ is indeed a finite measure.

Step 3. We notice that by Remark 3.19 and by **Step 2**, if $X,Y,Z,W\in\mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap\mathrm{L}^\infty(T\mathsf{X})$ then the map $\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})\ni f\mapsto\boldsymbol{\mathcal{R}}(X,Y,Z,W)(f)$ is induced by a finite measure $\mu_{X,Y,Z,W}$, i.e.

$$\boldsymbol{\mathcal{R}}(X,Y,Z,W)(f)=\int_{\mathsf{X}}f\,\mathrm{d}\mu_{X,Y,Z,W}\qquad\text{for every } f\in\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X}),$$

where

$$\begin{aligned}\mu_{X,Y,Z,W}=\frac{1}{6}\Big(&\mu_{X+W,Y+Z}-\mu_{X+W,Y}-\mu_{X+W,Z}-\mu_{Y+Z,X}-\mu_{Y+Z,W}+\mu_{X,Z}+\mu_{W,Y}\\ &-\mu_{Y+W,X+Z}+\mu_{Y+W,X}+\mu_{Y+W,Z}+\mu_{X+Z,Y}+\mu_{X+Z,W}-\mu_{Y,Z}-\mu_{W,X}\Big).\end{aligned}$$

Clearly, still $\mu_{X,Y,Z,W} \ll \mathrm{Cap}$, hence, the equations above continue to hold even if only $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$. Also, for $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, and $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$

$$f\mu_{X,Y,Z,W} = \mu_{fX,Y,Z,W} = \mu_{X,fY,Z,W} = \mu_{X,Y,fZ,W} = \mu_{X,Y,Z,fW} \tag{3.25}$$

by Proposition 3.16.

Step 4. This is similar to **Step 5** of the proof of Theorem 1.1, we use the same notation. Let then $\{A_k\}$ and $\{v^k_i\}$ be as in **Step 5** of the proof of Theorem 1.1, building upon Theorem 2.1. Fix for the moment k and define, for $i, j, l, m = 1, \dots, n(k)$,

$$\mu^k_{i,j,l,m} := \mu_{v^k_i + v^k_j, v^k_l + v^k_m} \llcorner A_k \quad \text{and} \quad \mu^k_{i,j,l} := \mu_{v^k_i + v^k_j, v^k_l} \llcorner A_k \quad \text{and} \quad \mu^k_{i,j} := \mu_{v^k_i, v^k_j} \llcorner A_k$$

and also

$$\mu^k := \sum_{i,j,l,m=1}^{n(k)} \left(|\mu^k_{i,j,l,m}| + |\mu^k_{i,j,l}| + |\mu^k_{i,j}| \right).$$

Now we define

$$\tilde\rho^k_{i,j,l,m} := \frac{\mathrm{d}\mu_{v^k_i, v^k_j, v^k_l, v^k_m} \llcorner A_k}{\mathrm{d}\mu^k},$$

notice that $\mu_{v^k_i, v^k_j, v^k_l, v^k_m} \ll \mu^k$ by construction for every $i, j, l, m = 1, \dots n(k)$. Set also

$$\tilde\nu^k := \sum_{i,j,l,m=1}^{n(k)} v^k_i \otimes v^k_j \otimes v^k_l \otimes v^k_m \tilde\rho^k_{i,j,l,m}.$$

and

$$\nu^k := \frac{1}{|\tilde\nu^k|}\tilde\nu^k \qquad \text{and} \qquad |\mathbf{Riem}^k| := |\tilde\nu^k|\mu^k.$$

Finally

$$\nu := \sum_k \nu^k \qquad \text{and} \qquad |\mathbf{Riem}| := \sum_k |\mathbf{Riem}^k|.$$

Clearly, $|\mathbf{Riem}|$ is a σ-finite measure, $|\mathbf{Riem}| \ll \mathrm{Cap}$ and $|\nu| = 1$ $|\mathbf{Riem}|$-a.e.

Step 5. We claim that

$$X \otimes Y \otimes Z \otimes W \cdot \nu^k \mathrm{d}|\mathbf{Riem}^k| = \mu_{X,Y,Z,W} \llcorner A_k$$

for every $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$ and for every k. Recall that $\int_\mathsf{X} f \mathrm{d}\mu_{X,Y,Z,W} = \boldsymbol{\mathcal{R}}(X, Y, Z, W)(f)$ for every $f \in \mathrm{S}^2(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$, so that the claim will imply Eq. 3.24 and also the fact that

$$|X \otimes Y \otimes Z \otimes W \cdot \nu^k| \mathrm{d}|\mathbf{Riem}^k| = |\mu_{X,Y,Z,W}| \llcorner A_k,$$

so that, being $\mu_{X,Y,Z,W}$ a finite measure, $X \otimes Y \otimes Z \otimes W \cdot \nu^k \in \mathrm{L}^1(|\mathbf{Riem}|)$.

Fix k and take then $X, Y, Z, W \in \mathrm{H}^{1,2}_{\mathrm{H}}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. We write $\tilde X := \sum_{i=1}^{n(k)} X^i v^k_i$, for $X^i := X \cdot v^k_i$ and similarly for Y, W, Z. Notice $X^i, Y^i, Z^i, W^i \in \mathrm{H}^{1,2}(\mathsf{X}) \cap \mathrm{L}^\infty(\mathsf{m})$ and $\tilde X, \tilde Y, \tilde Z, \tilde W \in \mathrm{TestV}(\mathsf{X})$. Notice that these newly defined functions and vector fields depend on k, but as we are working for a fixed k, we do not make this dependence explicit. We compute, on A_k,

$$X \otimes Y \otimes Z \otimes W \cdot \nu^k \mathrm{d}|\mathbf{Riem}^k| = \sum_{i,j,l,m=1}^{n(k)} X \otimes Y \otimes Z \otimes W \cdot v^k_i \otimes v^k_j \otimes v^k_l \otimes v^k_m \tilde\rho^k_{i,j,l,m} \mathrm{d}\mu^k$$

$$
\begin{aligned}
&= \sum_{i,j,l,m=1}^{n(k)} X \otimes Y \otimes Z \otimes W \cdot v_i^k \otimes v_j^k \otimes v_l^k \otimes v_m^k \mathrm{d}\mu_{v_i^k, v_j^k, v_l^k, v_m^k} \llcorner A_k \\
&= \sum_{i,j,l,m=1}^{n(k)} X^i Y^j Z^l W^m \mathrm{d}\mu_{v_i^k, v_j^k, v_l^k, v_m^k} \llcorner A_k = \sum_{i,j,l,m=1}^{n(k)} \mathrm{d}\mu_{X^i v_i^k, Y^j v_j^k, Z^l v_l^k, W^m v_m^k} \llcorner A_k \\
&= \mu_{\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}} \llcorner A_k,
\end{aligned}
$$

where the next to last equality is due to Eq. 3.25. We verify now that $\mu_{\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}} \llcorner A_k = \mu_{X,Y,Z,W} \llcorner A_k$, which will conclude the proof of the claim. This will be similar to **Step 6** of the proof of Theorem 1.1.

By multi-linearity and Proposition 3.16, it is enough to show that for every $X, Y, Z, W \in \mathrm{H}_{\mathrm{H}}^{1,2}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$, then $\mu_{X,Y,Z,W} \llcorner \{|X| = 0\} = 0$. We take $\{\varphi_n\}_n$ be as in Lemma 2.2 for the vector field $X \in \mathrm{H}_{\mathrm{H}}^{1,2}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$. We take also $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ and we compute (recall Lemma 2.1)

$$
\int_{\mathsf{X}} h \varphi_n \mathrm{d}\mu_{X,Y,Z,W} = \int_{\mathsf{X}} h \mathrm{d}\mu_{\varphi_n X,Y,Z,W} = \boldsymbol{\mathcal{R}}(\varphi_n X, Y, Z, W)(h).
$$

By Lemma 2.2 and the expression for the map $\boldsymbol{\mathcal{R}}$ in Remark 3.15, the right hand side of the equation above converges to 0, whereas the left hand side converges to

$$
\int_{\mathsf{X}} h \chi_{\{|X|=0\}} \mathrm{d}\mu_{X,Y,Z,W}.
$$

This is to say that for every $h \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$

$$
\int_{\mathsf{X}} h \chi_{\{|X|=0\}} \mathrm{d}\mu_{X,Y,Z,W} = 0
$$

whence the claim.

Step 6. By approximation (Lemma 2.10), it is enough to show the claim for $X, Y \in \mathrm{TestV}(\mathsf{X})$. Then the claim follows from Eq. 3.24 and the assumption on the bound from below for the sectional curvature.

Step 7. The symmetries claimed follow from Proposition 3.16. We prove, for example, the first one. It is enough to show that $\nu + \mathcal{I}(\nu) = 0$ with respect to $|\mathbf{Riem}|$-a.e. equality. Now, if $X, Y, Z, W \in \mathrm{H}_{\mathrm{H}}^{1,2}(T\mathsf{X}) \cap \mathrm{L}^\infty(T\mathsf{X})$, then

$$
\begin{aligned}
&\int_{\mathsf{X}} X \otimes Y \otimes Z \otimes W \cdot (\nu + \mathcal{I}(\nu)) |\mathbf{Riem}| \\
&\quad = \int_{\mathsf{X}} X \otimes Y \otimes Z \otimes W \cdot \nu |\mathbf{Riem}| + \int_{\mathsf{X}} \mathcal{I}(X \otimes Y \otimes Z \otimes W) \cdot \nu |\mathbf{Riem}| \\
&\quad = \boldsymbol{\mathcal{R}}(X, Y, Z, W)(1) + \boldsymbol{\mathcal{R}}(Y, X, Z, W)(1) = 0,
\end{aligned}
$$

so that the claim follows from Lemma 2.10. □

Remark 3.21 Notice that, thanks to its symmetries, the tensor field ν of Theorem 1.3 can be seen as an element of $\left(\mathrm{L}^0_{\mathrm{Cap}}(T\mathsf{X})^{\wedge 2}\right)^{\otimes 2}$. ■

Remark 3.22 Comparing the main results of Section 3.3 and Section 3.4, we may wonder whether **Ric** is linked to the trace of $\boldsymbol{\mathcal{R}}$. By what remarked in Remark 3.13, we see that this question makes sense only on non-collapsed spaces. However, the non-smooth structure of the space, in particular, the lack of a third order calculus and charts defined on open sets, prevent us from giving an easy proof of this fact. ■

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