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Aspects of (generalized) symmetries

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Chi cerca trova, chi ricerca ritrova

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Abstract

In this thesis, we present various aspects of generalized symmetries in quantum field theory and holography. After a brief introduction to the subject, we analyze various examples in which the symmetry structure is quite peculiar and extends beyond the standard framework of global symmetries as group-like transformations of local operators. The modern approach to this subject relies on the correspondence between symmetries and topological operators within a given quantum theory. In the first part of this dissertation, we analyze theories in which the set of topological operators can extend beyond groups (also known as non-invertible symmetries) and explore situations in which a symmetry broken by a deformation can re-emerge after ensemble averaging. In the second part of the thesis, we examine how special non-invertible symmetries arise in the holographic duals of certain supersymmetric quantum field theories. This holographic understanding proves to be useful in comprehending the intricate structure of these particular symmetries, revealing properties that might be challenging to grasp solely from a quantum field theory perspective.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following works:

- Andrea Antinucci, Giovanni Galati, Giovanni Rizi *On continuous 2-category symmetries and Yang-Mills theory*, JHEP 12 (2022) 061, [arXiv:2206.05646]

- Andrea Antinucci, Francesco Benini, Christian Copetti, Giovanni Galati, Giovanni Rizi, *The holography of non-invertible self-duality symmetries*, [arXiv:2210.09146]

- Andrea Antinucci, Christian Copetti, Giovanni Galati, Giovanni Rizi, *"Zoology" of non-invertible duality defects: the view from class* $\mathcal{S}$, [arXiv:2212.09549]

- Andrea Antinucci, Giovanni Galati, Giovanni Rizi, Marco Serone *Symmetries and topological operators, on average*, [arXiv:2305.08911]

I also coauthored these works, which are not part of this thesis:

- Giovanni Galati, Marco Serone, *Cancellation of IR divergences in 3d Abelian gauge theories* JHEP 02 (2022) 123, [arXiv:2111.02124]

- Andrea Antinucci, Francesco Benini, Christian Copetti, Giovanni Galati, Giovanni Rizi, *Anomalies of non-invertible self-duality symmetries: fractionalization and gauging*, [arXiv:2308.11707]

Contents

Chapter 1

Introduction

Quantum field theory (QFT) stands as an exemplary conceptual framework, offering a wealth of theoretical and phenomenological tools for the understanding of a huge variety of physical phenomena, ranging from high energy to condensed matter physics. Additionally, it is widely believed that it shares strong interconnections with theories of Quantum gravity due to the well known Gauge/Gravity duality. It is therefore one of the main goals of theoretical physicists to get a deeper and more complete understanding of QFT in its full generality.

Global symmetries constitute an indispensable tool for studying physical systems, especially when their dynamics cannot be analyzed using exact techniques. The relevance of the concept of symmetry in quantum systems dates back to Wigner [1], who showed that a symmetry group G is realized by (anti)linear and (anti)unitary operators U_g on the Hilbert space, labeled by $g \in G$ and commuting with the Hamiltonian. In *local* quantum field theory global symmetry is the main tool. In particular it organizes the spectrum in representations of G, hinting which QFT can describe a given physical phenomenon. Global symmetries and their anomalies are also among the few intrinsic and renormalization group (RG) flow invariant properties [2, 3], imposing selection rules on correlation functions as well as constraints in strongly coupled theories. For instance, along with the RG flow, all the operators compatible with the global symmetries are generated by quantum effects, so that the full classification of the global symmetries of a model is a powerful tool to have control over the flow. This is the classic notion of *naturalness* [2]. Moreover their fate along the RG flow fully characterizes the long distance physics of a particular high energy system independently on its short distance details. Such concept is usually referred to as the *Ginzburg-Landau paradigm*.

Nevertheless, in some models such standard notion of symmetry is not enough to fully characterize their infrared (IR) behavior, so that they are naively outside the standard Ginzburg-Landau framework. Very familiar instances of this case are non-Abelian gauge theories, where the confining/de-confining phases are not described by some standard symmetry preserving or breaking pattern. Moreover in some other systems there are evidences against the generation, along the RG flow, of operators which do not violate the known symmetries of the theory evading the standard notion of naturalness. Such examples highly suggest that we should enlarge the *category* of what we want to call global symmetry in order to unify all such examples in a general framework.

For these reasons, the idea of symmetry has recently been made more precise and intrinsic through the notion of *topological operators* [4]. These are extended operators or defects in Quantum Field Theory (QFT) supported on co-dimension 1 submanifolds in space-time and labeled by a group element $g \in G$. Their dependence is purely topological: small deformations of the support do not change the correlation functions, but when they pass a charged operator, it undergoes a symmetry transformation. The topological nature implies that, in any quantization scheme, if they are placed on a space-like slice, they become operators on the Hilbert space commuting with the Hamiltonian, recovering the standard Wigner classification. Indeed it is really the topological nature of these operators which can replace the usual notion of symmetry, yielding by itself to their RG invariance, selection rules, anomalies, and the notion of naturalness. However in a given QFT there are more topological operators that those supported on a co-dimension 1 submanifold and labeled by $g \in G$. From this point of view it is therefore natural to generalize our definition of symmetries to include also more general topological operators which are, for instance, supported on higher co-dimensional manifold, leading to the notion of higher p-form symmetries, or more drastically not labeled by group elements and not unitary, leading to the notion of non-invertible symmetries, thus finding a way out from the Wigner paradigm. In the recent years a great effort was made on study the physical realization and the dynamical consequences of such generalized symmetries on physical systems finding that the previously mentioned puzzles can be understood only taking into account such general definition. To have a feeling of the extensive body of work that has emerged during these years see *e.g.* this (not-complete) list of references [5–127].

Global symmetries are a key ingredient also in the context of AdS/CFT correspondence and in quantum gravity. Indeed it is highly believed that in the full

theory of quantum gravity no global symmetries are present due to their incompatibility with gravitational processes. In the standard holographic dictionary, boundary global symmetries correspond to bulk gauge symmetries which are a redundancy of the system rather than an actual physical symmetry. Also in this case however (super)gravity theories have more gauge fields with respect to the ones corresponding to standard boundary global symmetries. This is the case, for instance, of the NS and RR 2-form potentials of type IIB string theory on asymptotically $AdS_5 \times S_5$ space. Also in this case the generalized notion of symmetries in the sense of topological operators resolves the tension by enlarging the standard dictionary, for instance by connecting higher-forms gauge fields in the bulk to higher-form global symmetries in the boundary.

The goal of this thesis is to explore in more details this correspondence between symmetries and topological operators, mostly focusing on the subtler case of *non-invertible* symmetries. This dissertation is conceptually divided into two parts. The first one analyzes symmetries in the context of QFT and is based on [29, 128] while the second explores the holographic origin of non-invertible symmetries, by explaining how the dictionary works in such exotic cases, and is based on [42, 43]. In particular, the content of the chapters is organized as follows:

Chapter 2 is a pedagogical introduction to the field of generalized symmetries. We mostly focus on motivating the correspondence between symmetries and topological operators in Quantum Field Theory. We introduce the notion of higher-form and non-invertible symmetries and we provide some examples in which such symmetries arise and which may be useful for the rest of the thesis. At the end we provide a brief introduction to the generic framework of category theory which is believed to describe symmetries in QFT in their full generality.

Chapter 3 is a transcription of the original works [29, 128]. In [29] we analyze categorical symmetries arising in 4d gauge theories. In particular we show that $U(1)^{N-1} \rtimes S_N$ gauge theory has a global continuous 2-category symmetry which is a generalization of the electric and magnetic 1-form symmetry of Maxwell theory. We study in great details the fusion algebra of those symmetries, unveiling some interesting properties related to the existence of the so-called condensation defects of the theory. We prove that these symmetries are parameterized by a continuous parameter $\alpha \in U(1)^{N-1}/S_N$ which coincides with the conjugacy classes of the non-Abelian group $SU(N)$. By studying the spectrum of local and extended operators

of this theory, we find a mapping with gauge invariant operators of 4d $SU(N)$ Yang-Mills theory. In particular, the largest group-like subcategory of the non-invertible symmetries of the $U(1)^{N-1} \rtimes S_N$ theory is a $\mathbb{Z}_N^{(1)}$ 1-form symmetry, acting on the Wilson lines in the same way as the center symmetry of Yang-Mills theory does. In Section 3.1.3 we argue that the $U(1)^{N-1} \rtimes S_N$ gauge theory has a relation with the ultraviolet limit of $SU(N)$ Yang-Mills theory in which all Gukov-Witten operators become topological, and form a continuous non-invertible 2-category symmetry, broken down to the center symmetry by the RG flow.

In [128] we study symmetries which are broken by some deformation parameterized by a coupling h of a given system but which re-emerge after quenched average. When the coupling h is space-time dependent, such systems usually describe some statistical models with impurities or disorder assumed to be random. When instead h is constant, the corresponding systems are closely related to higher dimensional quantum gravity systems. In these cases the lack of factorization in the bulk, due to Euclidean wormhole configurations, is mapped to the lack of factorization in averaged observables of the boundary. In these situations, symmetries may be broken by the random interactions, but they can re-emerge when we look at averaged observables. In this work we analyze if the existence of selection rules on some observables of the theory, automatically implies the presence of topological operators generating the symmetry. When h is space-dependent, the topological operator indeed exists and such symmetries emerging after average can be coupled to external backgrounds and can be gauged, like ordinary symmetries in QFTs. When instead h is constant the symmetry operator is not purely codimension-1, it can be defined only on homologically trivial cycles and on connected spaces. Selection rules for average correlators exist, yet such symmetries cannot be coupled to background gauge fields in ordinary ways and cannot be gauged. Such exotic example emphasizes the conceptual distinction between symmetries and selection rules, which is sometimes confused in more standard cases.

Chapter 4 is a transcription of the original works [42, 43]. In [43] we study how non-invertible symmetries arise in the bulk gravitational dual of a boundary SCFT. We focus on the paradigmatic example of $\mathcal{N} = 4$ super Yang-Mills theories with gauge algebra $\mathfrak{su}(N)$. The theory is known to have non-invertible duality and triality defects at particular points of its conformal manifold. At these points in the gravitational moduli space, the gauged $SL(2, \mathbb{Z})$ duality symmetry of type IIB string theory is spontaneously broken to a finite subgroup G, giving rise to a discrete

emergent G gauge field. After reduction on the internal manifold, the low-energy physics is dominated by an interesting 5d Chern-Simons theory, further gauged by G, that we analyze and which gives rise to the self-duality defects in the boundary theory. We therefore confirm the standard dictionary which relates boundary global symmetries with bulk gauge fields.

In [42] we extend the above analysis to theories of class $\mathcal{S}$ obtained by the dimensional reduction of the 6d $\mathcal{N} = (2,0)$ theory of A_{N-1} type on a Riemann surface Σ_g without punctures. The duality symmetries in these cases arise when the Riemann surface Σ_g is invariant under some nontrivial automorphism group G, subgroup of the full set of large diffeomorphisms. We discuss the properties of such non-invertible duality symmetries and provide two ways to compute their fusion algebra: either using discrete topological manipulations or a 5d TQFT description, very similar to the one obtained in the previous case of $\mathcal{N} = 4$ SYM.

Chapter 2

A brief tour on categorical symmetries

The aim of this chapter is to provide a concise yet comprehensive exploration of the relationship between global symmetries in quantum field theory and topological operators. We begin by establishing a mapping between ordinary global symmetries and topological operators, shedding light on their interplay. Then we delve into several generalizations that expand upon this correspondence. In particular we review the definition and the applications of higher-form symmetries and we introduce the notion of non-invertible symmetries. Finally, we present a comprehensive framework that encapsulates all such generalized symmetries, expressing them in the more precise and mathematically rigorous language of category theory.

2.1 Symmetries as topological operators

Let us start by considering a physical system described by a collection of fields Φ_i with (classical) dynamics controlled by the action $S(\Phi_i)$. The *classical* way to look at symmetries is to find global transformations on the fields $\Phi_i \to \Phi_i' = U(\Phi_i)$ which leave the action invariant. Obviously, the trivial action, which we can also call identity transformation and which corresponds to $U = 1$, is always a symmetry of the system. Moreover, there is always an inverse map which brings back the fields to their original values. Because of this, the set of maps U, together with their composition, are isomorphic to a group $G_{symm.}$ which uniquely characterizes the structure of the allowed symmetry transformations. It is therefore very common to identify the symmetries of a (classical) system with the corresponding groups.

When all the transformations can be done infinitesimally closed to the identity, the symmetry group is continuous and the Noether theorem ensures the existence of a conserved current [1] $\partial^\mu J_\mu(x) = 0$ and the corresponding Noether charge

$$Q(t) := \int d^3x J_0(x)\,. \tag{2.1.1}$$

The conservation of the current implies that $Q(t)$ is actually a constant, namely $\frac{dQ(t)}{dt} = 0$. In this infinitesimal regime we can identify the symmetry action as

$$U(\Phi_i) = (1 + i\epsilon Q)(\Phi_i) \tag{2.1.2}$$

where $\epsilon << 1$ is the infinitesimal parameter which controls the expansion and $Q(\Phi_i) = \{Q, \Phi_i\}$ is the Poisson bracket operation. The corresponding finite symmetry transformation is implemented by the exponential map

$$U_g = e^{i\epsilon Q} \qquad g = e^{i\epsilon} \in G\,. \tag{2.1.3}$$

At the Quantum level the notion of symmetry can be extended to be an operation on states $|\psi_i\rangle \to |\psi_i'\rangle$ which preserves transition amplitudes [2]. Wigner's theorem implies that such operations must be implemented by unitary and linear (or antiunitary and antilinear) operators U_g which commute with the Hamiltonian H. The action on states is

$$|\psi_i\rangle \to U_g|\psi_i\rangle \tag{2.1.4}$$

and equivalently we can define the symmetry action on operators as

$$U_g \mathcal{O} U_g^{-1} = \mathcal{O}'\,. \tag{2.1.5}$$

In the path integral formulation we can still talk about classical action and classical fields transformations which leave it invariant. However, such classical values of the fields are just the on-shell approximations of the full quantum theory. Unlike the classical solutions derived from equations of motion, the path integral includes integration over all quantum field fluctuations. Consequently, it is not a priori evident that transformations preserving the classical action will exhibit any physical consequences when extended to the quantum level.

[1] For simplicity, in the following we avoid to write the group indices.

[2] Such transformations are actually defined modulo an overall phase since states in the Hilbert space are rays. Such property is crucial to characterize anomalies in quantum mechanics.

However, crucially, the Noether theorem can be uplifted to the quantum level through the usage of the Ward-Takahashi identities (WI). Because of this the classical predictions due to symmetry considerations are robust to quantum fluctuations. Given a QFT described by the generating functional

$$Z[K^i] = \int [D\Phi] e^{-S[\Phi] + \int d^d x K^i \mathcal{O}_i} \tag{2.1.6}$$

where $\mathcal{O}_i$ and K^i are respectively operators and their external sources, if the classical action is invariant under some continuous symmetry group G, standard functional methods imply the following WI between correlators

$$\langle \partial^\mu J_\mu(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \sum_{i=1}^{n} \delta^{(d)}(x - x_i) \langle \mathcal{O}_1(x_1) \cdots \delta\mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle \,, \tag{2.1.7}$$

where $\delta\mathcal{O}_i(x_i)$ is the infinitesimal variation of the operator $\mathcal{O}_i(x_i)$ under the action of G and J_μ is the Noether current[3]. Such relations are the quantum analogue of the classical conservation equations $\partial^\mu J_\mu(x) = 0$. At the quantum level (2.1.7) implies that $\partial^\mu J_\mu$ is zero up to contact terms, i.e. is a redundant operator of the theory. The conservation equation (2.1.7) allows us to construct a conserved charge exactly as in the classical case

$$Q(t) = \int_{M_t^{(d-1)}} J_0(x) d^{d-1}x \tag{2.1.8}$$

where $M_t^{(d-1)}$ is a spacial slice of the spacetime. (2.1.7) implies that $Q(t)$ is conserved, namely it commutes with the Hamiltonian.

However, a more relativistic way of manipulating (2.1.7) is to define the charge operator $Q[\Sigma^{(d-1)}]$ by integrating on a *generic* d-dimensional manifold $D^{(d)}$ with an (oriented) boundary $\Sigma^{(d-1)}$,

$$Q[\Sigma^{(d-1)}] = \int\limits_{D^{(d)}} \partial^\mu J_\mu = \int\limits_{\Sigma^{(d-1)}} J_\mu n^\mu \,. \tag{2.1.9}$$

From (2.1.7) it follows that $Q[\Sigma^{(d-1)}]$ is *topological*, namely it does not depend on small deformations of its support $\Sigma^{(d-1)}$ if such deformations do not cross charged

[3]In (2.1.7) we assumed the path integral measure to be invariant under the symmetry transformation. If this is not the case the symmetry is broken by quantum effects and the WI are not valid anymore. This kind of symmetries are called ABJ anomalous.

operators [4]. The topological property of $Q[\Sigma^{(d-1)}]$ is the Poincaré invariant way to declare that it is conserved.

From the infinitesimal transformation $\delta\mathcal{O}$ one can produce a Ward-Takahashi identity which involves a finite symmetry transformation, labelled by a group element $g \in G$. The operator which implements such transformation is the exponential of the Noether charge

$$U_g[\Sigma^{(d-1)}] = e^{i\alpha Q[\Sigma^{d-1)}]} \qquad , \qquad g = e^{i\alpha} \tag{2.1.10}$$

which satisfies the finite version of the WI

$$\langle U_g[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = R_1(g)\cdots R_n(g)\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{2.1.11}$$

where for simplicity of notation we have chosen $\Sigma^{(d-1)}$ as a surface which surrounds all the points x_i and $(R_i(g)\mathcal{O}_i)(x_i)$ is the transformed operator according to its representation R_i under G.

Let us now comment on the properties of the operator $U_g[\Sigma^{(d-1)}]$. The topological nature of the Noether current is inherited by the exponentiated operator, as evident from the WI (2.1.11). Since $U_g[\Sigma^{(d-1)}]$ is labeled by a group element, we can construct an isomorphism which relates such topological operators and the group G. The product structure of the group $g_1g_2 = g_3$ corresponds to fusing two topological operators, namely collapsing them into the same surface $\Sigma^{(d-1)}$. Usually in QFT, the operation of fusing two or more operators at the same point produces UV divergences. In this case however, the topological nature of such operators ensures that the resulting product is finite and corresponds to the operator implementing the group action $g_1g_2 = g_3$. Since it is a unitary operator, we can regard its inverse as the conjugate operator $U_g^\dagger[\Sigma^{(d-1)}] = U_{g^{-1}}[\Sigma^{(d-1)}]$. From its definition (2.1.10) it is clear that the conjugate operator corresponds to the operator $U[\Sigma'^{(d-1)}]_g$ inserted on the surface $\Sigma'^{(d-1)}$ with opposite orientation.

For continuous symmetries we can equivalently discuss the hermitian charge operator $Q[\Sigma^{(d-1)}]$ or the corresponding unitary operator $U_g[\Sigma^{(d-1)}]$, both are topological and enforce equivalent constraints on the theory. The advantage of using the exponentiated operator $U_g[\Sigma^{(d-1)}]$ is that in (2.1.11) we do not need to define the infinitesimal transformation $\delta\mathcal{O}$ so that the generalization to finite symmetries

[4] The definition (2.1.9) can be straightforwardly extended to the case when $\Sigma^{(d-1)}$ is not a boundary of a d-dimensional manifold by simply defining $Q[\Sigma^{(d-1)}] = \int_\Sigma *J$.

is straightforward. In this case however, no current and the corresponding charge operator exists but only the unitary symmetry operator[5].

Symmetries: not just selection rules Given the proper definition of symmetry in quantum field theory, we can start to ask what the presence of such symmetries implies in a given physical system. The first more obvious implication is the presence of some selection rules on the observables of the theory. Indeed by integrating (2.1.7) on the full spacetime, or equivalently choosing $\Sigma^{(d-1)}$ to be trivial in (2.1.11), we can prove *exactly*, namely at the non-perturbative level, that

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = R_1(g)\cdots R_n(g)\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{2.1.12}$$

which implies

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = 0 \tag{2.1.13}$$

unless the tensor product of representations $R_1(g)\cdots R_n(g)$ contains the trivial one[6]. However, in QFT, symmetries are more than selection rules. When $d>1$, a symmetry operator extended along the time direction imposes twisted boundary conditions on the states of the theory and therefore it defines a new twisted Hilbert space of the theory $\mathcal{H}_g$ labeled by the group element g. We refer to such configuration as a symmetry *defect*, in contrast to the case in which $U_g[\Sigma^{(d-1)}]$ is placed on a spatial slice and acts as an operator on the states of the untwisted Hilbert space. Moreover, due to their topological nature, symmetries are RG-protected quantities. Therefore any charged operator cannot be generated along the RG flow and the IR spectrum of the theory must be organized in representations of the symmetry group G. For instance, mixing between operators with different charges is forbidden.

Even more importantly we can turn on background fields for such symmetries. In the case of continuous symmetries, this is achieved by adding to the action the minimal coupling

$$S[\Phi] \to S[\Phi] + \int A\wedge *J \tag{2.1.14}$$

where A is a 1-form background gauge field with gauge transformations $A^\lambda = A + d\lambda$. The conservation equation $\partial^\mu J_\mu = 0$ implies the gauge invariance of this coupling on spacetime manifolds without boundaries. When A is flat with $dA=0$, we can

[5]When the discrete symmetry is abelian it is convenient to embed it into a $U(1)$ symmetry in order to get an explicit expression for the symmetry operator in terms of its Noether current.

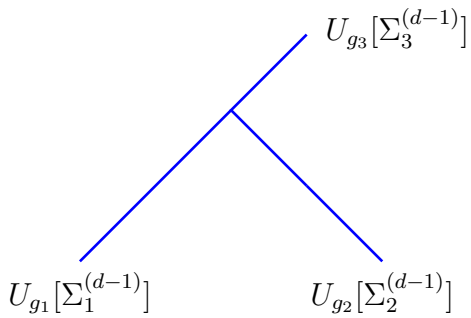
[6]Such condition is the non-abelian analogue of charge conservation.

rewrite the minimal coupling expression using Poincaré duality (see Appendix 2.3.1 for more details). This allows us to express it as

$$\int_{X^{(d)}} A \wedge *J = \int_{PD(A)} *J = Q[PD(A)] \tag{2.1.15}$$

where $PD(A)$ is a network of topological $(d-1)$ surfaces, Poincaré dual to the flat gauge field A. This establishes a mapping between flat gauge fields and specific networks of topological symmetry operators $U[\Sigma^{(d-1)}]$. The flatness condition of the gauge field is then mapped to the consistency condition of the group law at each junction of the defect within the network, while the gauge invariance is mapped to the topological nature of the network itself:

$$A \in \underbrace{H^1(X^{(d)}, G)}_{\substack{A \sim A + d\lambda \\ dA = 0}} \quad \Leftrightarrow$$



$$\begin{cases} \Sigma_{1,2,3}^{(d-1)} \sim \text{topological} \\ g_3 = g_1 g_2 \end{cases}$$

Even if topological changes of the network should produce equivalent configurations, the partition function coupled to such backgrounds can shift by a background dependent phase

$$Z[A^\lambda] = e^{\int \alpha(A,\lambda)} Z[A]\,. \tag{2.1.16}$$

When such ambiguity occurs we say that the symmetry is 't Hooft anomalous. Despite their names, 't Hooft anomalies are features rather than a bug of the theory. In the absence of background gauge fields, the symmetry remains well-defined, and its other properties continue to hold. By looking at associativity conditions, it is easy to prove that the anomalous phase α is a discrete quantity classified by elements of the group cohomology $H^{d+1}(G, U(1))$ [7]. The discrete nature of this quantity implies that it is constant along the RG flow. Therefore the IR physics, described only by the lightest modes of the spectrum, must reproduce the same anomaly somehow. This immediately implies that an anomalous theory cannot be trivially gapped, namely

[7] This condition is the generalized version of Wess-Zumino consistency conditions and it is valid only for bosonic symmetries. For fermionic symmetries, anomalies are classified by the Bordism groups.

gapped with a non-degenerate vacuum, in the deep IR. This condition is commonly referred to as 't Hooft anomaly matching.

When the anomalous phase $\alpha(A, \lambda)$ vanishes, we can gauge the symmetry. In the case of continuous symmetries, this procedure is achieved by adding the minimal coupling and functional of the background gauge field A (such as its kinetic term $\int F \wedge *F$) and subsequently performing the path integral over A. When instead the symmetry is discrete the gauging can be done equivalently by inserting a fine enough mesh of symmetry operators and then sum over all the possible in-equivalent insertions (see Appendix 2.3.1 for some details). Following the previous discussion this is equivalent to sum over all the possible background gauge fields $A \in H^1(X^{(d)}, G)$. This operation is equivalent to modding out all the gauge-variant states (*i.e.* states charged under the symmetry) as well as introducing all the gauge-invariant states belonging to the twisted Hilbert space[8].

2.1.1 Higher-form symmetries

The above recap about symmetries in QFT is quite standard and well-known. However, it is crucial to understand the more exotic generalizations. Along with the previous discussion about the properties enjoyed by symmetric systems, the crucial one was the map between symmetries and topological operators supported on some co-dimension 1 surface embedded in the spacetime. Indeed it is the topological nature of symmetries which replaces the classical notion of conserved quantities and implies so many constraints on the dynamics of the theory. It is therefore natural to identify the entire set of topological operators of a given theory with its symmetry structure. However, QFTs have more topological operators with respect to the ones already described. For instance, they can be supported on higher codimensional submanifolds $\Sigma^{(d-p-1)}$. We define a symmetry corresponding to such operators as a *p-form symmetry*

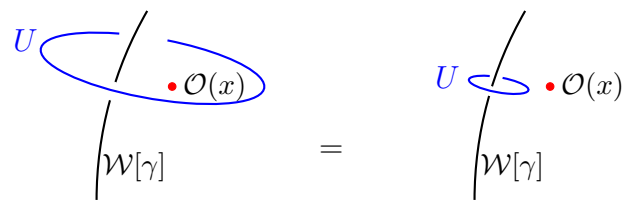
$$\text{p-form symmetry} \quad \Longleftrightarrow \quad U[\Sigma^{(d-p-1)}]\,. \tag{2.1.17}$$

Let us describe the generic features of such higher-form symmetries.

1. Because of their higher codimensional support, local operators are transparent with respect to those symmetry operators. However the latter have a

[8]In 2d discrete gauging is also referred to as orbifold operation.

natural action on p-dimensional extended operators [9]. For instance in three-dimensional ambient space, a 1-form symmetry generator is a topological line acting on charged lines but without acting on any local operators:



2. As in the case of 0-form symmetries, if we select only unitary symmetry operators, there is a precise isomorphism between the set of topological operators supported on submanifolds $\Sigma^{(d-p-1)}$ and a group G. In the following, we denote a p-form symmetry described by the group G as $G^{(p)}$.

3. Between two symmetry operators $U_{g_1}[\Sigma^{(d-p-1)}]$ and $U_{g_2}[\Sigma'^{(d-p-1)}]$ there is no preferred order if $p > 0$ since we can continuously deform their support without fusing them. Therefore the two operators must commute and the symmetry group must be Abelian.

4. Exactly as before, a network of topological defects can be identified, by using Poincaré duality, with a flat background gauge field $B \in H^{p+1}(X^{(d)}, G)$. Notice that, consistently with comment 3, there is not any consistent definition of $H^{p+1}(X^{(d)}, G)$ for a non-Abelian group G and $p > 0$. In this case bosonic 't Hooft anomalies are classified by $H^{d+1}(B^{p+1}G, U(1))$ where $B^{p+1}G$ is the Eilenberg-Mac Lane space of G [129]. Physically, this cohomology group classifies all the possible topological actions constructed out of the gauge field for the p-form symmetry $G^{(p)}$ (see Appendix 2.3.1 for some details on topological actions).

1-form symmetries in 4d gauge theories. Higher-form symmetries are ubiquitous in gauge theories and they provide a nice and clean organization of their global properties. A pure gauge theory with gauge group G has an electric 1-form symmetry labeled by its center $Z(G)$ and acting on the Wilson lines of the theory,

[9] However, generically, they can also act on q-dimensional extended operators with $q > p$. For instance 0-form symmetries, such as charge conjugation, naturally act on all (local and extended) operators of the theory. See *e.g.* [53] for a very recent treatment of this phenomenon in terms of higher-representation theory.

giving them a charge equal to the N-ality of the corresponding representation (see appendix 2.3.2). Concretely, electric 1-form symmetries act on the principal bundle defined by the gauge theory by shifting the transition functions $g_{ij} \in SU(N)$ but without acting on the gauge connections A_i. From this point of view, it is easy to see that, in order to not spoil the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ (imposed on the triple intersections) such transformations must be valued in the center of the gauge group.

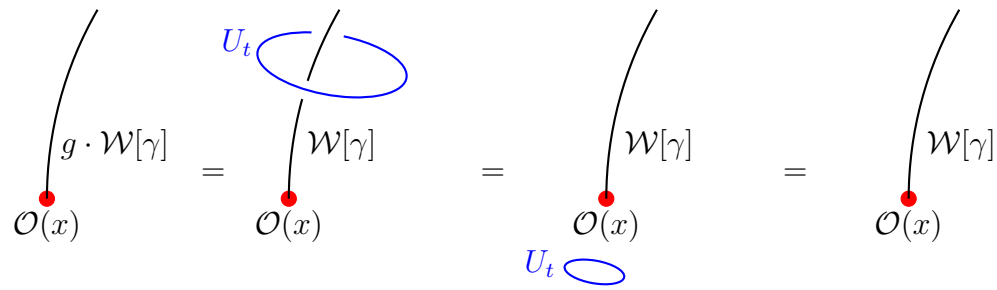
Such theories enjoy also a magnetic $(d-3)$-form symmetry labeled by $\pi_1(G)$ and acting on the 't Hooft lines in a similar way. We emphasize that the correct higher symmetry structure of gauge theories depends on their symmetry *group* rather than the corresponding algebra. Therefore they constrain and organize the spectrum of extended operators while the local dynamics is blind to the different higher symmetry structure of the theory [62]. For instance, we can consider pure Yang-Mills (YM) theories in 4d with gauge algebra $\mathfrak{su}(2)$. The possible gauge groups compatible with this local dynamics are $SU(2)$ and $SO(3)$. Since

$$Z(SU(2)) = \pi_1(SO(3)) = \mathbb{Z}_2 \quad , \quad Z(SO(3)) = \pi_1(SU(2)) = 0 \tag{2.1.18}$$

the two theories exhibit distinct 1-form symmetries: while $SU(2)$ YM theory possesses an electric $\mathbb{Z}_2^{(1)}$ 1-form symmetry, $SO(3)$ has a magnetic one. In the $SU(2)$ theory there exist Wilson lines of N-ality $(0,1)$, indicating the different charges carried by these lines, but 't Hooft lines just with 0 N-ality, which therefore are uncharged under the 1-form symmetry. On the other hand, in the $SO(3)$ theory (with no θ angle), the spectrum of line operators differs. Wilson lines only possess 0 N-ality, while 't Hooft lines can carry both $(0,1)$ charges. The presence of a θ angle introduces an important distinction between the two theories. In the case of $SO(3)$, the periodicity of θ is 4π instead of 2π. Therefore the theory with $\theta = 2\pi$ exhibits different line operators compared to the one with $\theta = 0$. In this scenario, the charged lines are dyonic and can be understood as a superposition of a Wilson line and a 't Hooft line. These two theories, with the same gauge group but a different spectrum of lines, are usually called $SO(3)_+$ and $SO(3)_-$ respectively.

If we include dynamical matter, the screening procedure on the lines of the theory generically breaks a subgroup of the entire higher-form symmetry. In particular, lines in the representation with the same N-ality of the dynamical matter can end on it and therefore the corresponding 1-form symmetry generator cannot be topological

anymore:



The topological operators generating the electric 1-form symmetry are the Gukov-Witten (GW) operators $U_g[\Sigma^{(2)}]$ [130] corresponding to the center of the group and defined abstractly by path-integrating over the G gauge connection A such that

$$P\exp\left(i\int_\gamma A\right) = g \in Z(G) \qquad \text{if} \;\; Lk(\gamma, \Sigma^{(2)}) \neq 0\,, \tag{2.1.19}$$

where Pexp is the path-ordered exponential and $Lk(\gamma, \Sigma^{(2)})$ is the linking number between the latter and the surface $\Sigma^{(2)}$. For reasons that will become clear in section 3.1, we emphasize that Yang-Mills theories have more GW operators with respect to the ones corresponding to their center. In particular, they are labeled by the conjugacy classes of the gauge group. However, along the RG flow only the ones in the center are topological and generate a symmetry of the theory.

The presence of those symmetries in gauge theories is crucial in order to properly define their IR phases following the standard framework of Landau-Ginzburg. Indeed in this language, the confining phase of Yang-Mills theory is better defined as symmetry preserving phase for its electric 1-form symmetry, whose order parameter is a large Wilson loop operator. On the contrary, its deconfined phase corresponds to the spontaneous symmetry breaking of the same symmetry.

A deep understanding of higher-form symmetries in quantum field theory has yielded significant breakthroughs across various contexts. Notably, it has provided evidence for the double vacuum degeneracy of $SU(N)$ Yang-Mills theories at $\theta = \pi$ thanks to a mixed 't Hooft anomaly between its electric 1-form symmetry and time-reversal symmetry [131]. In the realm of condensed matter they play a crucial role in order to classify topological phases of matter where particular gapped systems (a prototypical example is the toric code) enjoy different topological excitations depending on the topology of the ambient spacetime manifold.

2.1.2 Non-invertible symmetries

Let us now delve deeper into our examination of the set of all possible topological operators within a given quantum system. In the previous subsection, we extended our definition of symmetry beyond conventional topological operators confined to co-dimension one submanifolds. Instead, we considered a broader set of symmetry operators supported on higher-codimensional manifolds, which act on higher-dimensional charged objects. However, we still assumed that these operators are unitary, so their orientation reversal corresponds to their inverse. This assumption is crucial for constructing a correspondence between the entire set of topological operators and a particular group G. Therefore, it's natural to inquire whether this condition can be relaxed in order to find symmetry operators that do not correspond to any group, particularly those that lack an inverse. Due to this latter property, the symmetry associated with such operators is usually termed a *non-invertible symmetry*. These exotic symmetries have been known to exist in 2d CFTs for several years (see [5, 6, 10, 132] for works in this direction). In 2d theories, non-invertible topological defect lines (TDLs) are ubiquitous, and an example of this kind is already present in one of the simpler 2-dimensional models, namely the Ising CFT. Several approaches can be used to understand the existence of such non-invertible symmetries in this CFT[10]. Such system has 3 different TDLs which we dub $\{1, \eta, \mathcal{N}\}$ whose fusion rules read

$$\eta \times \eta = 1\,, \qquad \eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}\,, \qquad \mathcal{N} \times \mathcal{N} = 1 + \eta\,. \tag{2.1.20}$$

While the subset of lines $\{1, \eta\}$ follow a $\mathbb{Z}_2$ group-like fusion rule, the entire set of TDLs is incompatible with a group structure and in particular the line $\mathcal{N}$ does not have an inverse. Physically, the line $\mathcal{N}$ is the manifestation of the Kramers-Wannier self-duality of the Ising CFT which relates this theory with its $\mathbb{Z}_2$-gauged orbifold. Similar exotic relations between TDLs appear in various rational CFTs and, as emphasized before, due to their topological nature, they highly constrain the dynamics of the theory exactly as an ordinary, group-like, symmetry would do. For instance, constraints on 2d RG flows implied by this type of symmetries were found and analyzed recently in [14] while the apparent lack of naturalness observed in 2d adjoint QCD was resolved by looking at non-invertible symmetries in [93].

It is then natural to ask if such particular non-invertible symmetries are present in higher-dimensional QFTs or they are just a peculiar property of 2d systems.

[10] For a nice review on the subject, see *e.g.* [133].

The affirmative answer to such a question is much more recent with respect to its 2-dimensional counterpart and the discovery of non-invertible symmetries in $d > 2$ took place in recent years starting from the seminal papers [120,134] . By examining the distinct characteristics exhibited by non-invertible TDLs in two-dimensional systems, one can explore the possibility of extending their origins to higher-dimensional theories. Therefore it has been recognized that certain non-invertible topological operators are always present in some specific theory and they can be constructed by following some definite procedure. This observation implies that particular properties inherent to a given system are naturally deduced from kinematical symmetry arguments rather than relying on more intricate dynamical considerations. Possible constructions of non-invertible symmetries which we will discuss during this thesis are in order.

- **Condensation defects.** This is the most universal way of obtaining non-invertible symmetries. Given a generic theory with a non-anomalous higher-form invertible symmetry $G^{(p)}$, we can construct a new defect by gauging $G^{(p)}$ only on a submanifold $\Sigma^{(q)}$ of the ambient spacetime [23,135]. Such procedure (usually called higher gauging procedure) defines a new topological operator $\mathcal{C}_{G^{(p)}}[\Sigma^{(q)}]$ of the theory which generically follows non-invertible fusion rules:

$$\mathcal{C}_{G^{(p)}}[\Sigma^{(q)}] \times \mathcal{C}_{G^{(p)}}[\Sigma^{(q)}] = a[\Sigma^{(q)}]\mathcal{C}_{G^{(p)}}[\Sigma^{(q)}]\,, \tag{2.1.21}$$

  where $a[\Sigma^{(q)}]$ is a complex number which depends on the support $\Sigma^{(q)}$[11]. This type of non-invertible operators does not produce any new constraints on the dynamics of the theory with respect to the invertible symmetry $G^{(p)}$ generating them. However, they are usually ubiquitous in other more interesting examples of generalized symmetries and they are usually a signal of their non-invertible nature.

- **Duality defects.** This type of construction mimic the $\mathcal{N}$ defect of the Ising CFT which describe its self-duality under discrete gauging. Such symmetries are present whenever a QFT $\mathcal{T}$ is self-dual under a discrete gauging of a global symmetry G, *i.e.* whenever there exists an isomorphism I_g which relates $\mathcal{T}$ with $\mathcal{T}/G$. Whenever we gauge a discrete symmetry in a d-dimensional theory, the resulting theory enjoys a $(d-p-1)$-form symmetry generated by the

[11] Such coefficient can also be interpreted as the partition function of a q-dimensional TQFT evaluated on the manifold $\Sigma^{(q)}$.

(topological) Wilson lines of the discrete gauge field. Therefore a necessary condition for the isomorphism to exists is that $d = 2n$ and that G is a $(n-1)$-form symmetry.

When such $\mathcal{T}$ exists, we can construct a topological defect $\mathfrak{D}_g$ of codimension one which implements the discrete gauging operation together with the isomorphism only on half space [120]. Due to the gauging procedure implemented by the defect, the fusions of $\mathfrak{D}_g$ are generically non-invertible[12]. Examples of theories with these symmetries are Maxwell or $\mathcal{N} = 4$ super Yang-Mills (SYM). In the latter case we will analyze the holographic dual of such duality defect in section 4.1.

- **Non-invertible symmetries from mixed 't Hooft anomalies.** Let us consider a four-dimensional theory with two invertible symmetries $G_1^{(0)}$ and $G_2^{(p_2)}$, linked by a mixed 't Hooft anomaly. If one gauges the $G_2^{(p_2)}$ symmetry, the $G_1^{(0)}$ defects become ill-defined because of the anomaly, but in some cases they can be made well-defined by stacking them with suitable 3d TQFTs coupled to the dynamical $G_2^{(p_2)}$ gauge field [119]. The resulting defects have non-invertible fusion laws due to the stacking rules of the 3d TQFTs. Such construction works for both discrete and continuous Abelian symmetries and prototypical examples are $\mathcal{N} = 1$ SYM with gauge group $PSU(N)$ and massless QED with one Dirac fermion. Interestingly such non-invertible symmetries are always duality defects, namely they can also be constructed as described in the previous bullet. However, the opposite is not true: there exist duality defects which are not of this type. We call this type of duality defects *intrisically non-invertible* in order to emphasize that they do not have an invertible origin.

- **Gauging a non-normal subgroup.** These are again non-invertible symmetries arising after some manipulations on a theory with just invertible symmetries. In this sense, they are non-intrinsically non-invertible as the ones coming from 't Hooft anomalies. This construction consists in gauging a discrete invertible 0-form symmetry $G^{(0)}$ which acts as an automorphism of the set of generators $\{U_h\}_{h\in H}$ of a discrete 1-form symmetry $H^{(1)}$ [24]. The basic idea is the

[12] In some particular cases it can happen that the gauging procedure is not necessary since the duality G does not change the global structure of the theory. In this case the theory has an ordinary 0-form invertible symmetry rather than a non-invertible one.

following. The generators of $H^{(1)}$ fall into various orbits $\mathcal{O}_{[h]} = \{U_{g\cdot h} \mid g \in G\}$ for the action $g: \ h \to g \cdot h$ of $G^{(0)}$ on $H^{(1)}$. After gauging $G^{(0)}$ most of the U_h are no longer gauge invariant. However, instead of throwing them away, we get new indecomposable objects labeled by the orbits $[h]$, each one being the sum of the objects in the corresponding orbit

$$\widehat{U}_{[h]} = \bigoplus \{U_{h'} \mid [h'] = [h]\} \tag{2.1.22}$$

up to a normalization factor. These new objects have *not* group-like fusion rules and we will expand on their properties in section 3.1.

2.2 The big picture: Symmetries and category theory

In the previous discussion, we have described some procedures that one can use in order to construct non-invertible symmetries in higher dimensions. Even if these methods have the power to produce a lot of interesting examples, they do not fully reveal the underlying structure of these symmetries. A comprehensive understanding of the language of group theory was essential to comprehend the various powers and implications of vanilla invertible symmetries. This includes the classification of anomalies, the arrangement of the spectrum into representations, and various other consequences. Consequently, in order to describe the complete set of generalized symmetries in QFT, it becomes crucial to determine the appropriate language to employ. In two dimensions the right formalism to use in order to describe the full set of TDLs is the one of fusion category, which we now briefly review (see [136] for a mathematical-oriented introduction to the subject and [13] for a more physically-oriented one).

Symmetries in 2d and fusion categories. A fusion category $\mathcal{C}$ is a particular type of abelian category composed by a finite set of objects $L_x \in \mathcal{C}$ (and morphisms between them) equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object 1 and a (natural) isomorphism

$$F_{x,y,z} : (L_x \otimes L_y) \otimes L_z \to L_x \otimes (L_y \otimes L_z) \tag{2.2.1}$$

called associativity constraint and constrained by the pentagon axiom

$$
\begin{tikzcd}
 & ((X \times Y) \times Z) \times W \arrow[dl, "F_{X,Y,Z}"'] \arrow[dr, "F_{X\times Y,Z,W}"] & \\
(X \times (Y \times Z)) \times W \arrow[d, "F_{X,Y\times Z,W}"] & & (X \times Y) \times (Z \times W) \arrow[d, "F_{X,Y,Z\times W}"] \\
X \times ((Y \times Z) \times W) \arrow[rr, "F_{Y,Z,W}"] & & X \times (Y \times (Z \times W))
\end{tikzcd}
\tag{2.2.2}
$$

The abelian structure means that one can construct finite direct sums of objects, while the morphisms from L_x to L_y are assumed to form a $\mathbb{C}-$vector space $\mathrm{Hom}(L_a, L_b)$. By defining a *simple* object L_a as the one which cannot be decomposed as a direct sum of others, the tensor product of two simple objects can be uniquely decomposed as

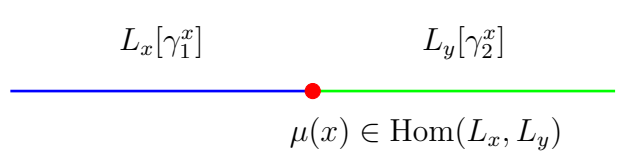
$$
L_a \otimes L_b = \sum_c f_{ab}^c L_c \tag{2.2.3}
$$

where the coefficients $f_{ab}^c \in \mathbb{Z}_+$ are the dimensions of $\mathrm{Hom}(L_a \otimes L_b, L_c)$.

Such categorical structure naturally encodes all the properties of TDLs in two-dimensional theories. Let us then describe the isomorphism between these two concepts. Evidently, the full set of (oriented) topological line operators of the theory can be identified as objects of a fusion category $\mathcal{C}$. The right normalization of those lines which ensures that they are also defect of the theory (remember the discussion after (2.1.13)) implies that we are not allowed to rescale them with any complex number. This property provides the identification of the simple lines of the theory. Direct sums of lines $L_c = L_a + L_b$ are intended as sums when inserted inside correlators,

$$
\langle L_c \cdots \rangle = \langle L_a \cdots \rangle + \langle L_b \cdots \rangle \,, \tag{2.2.4}
$$

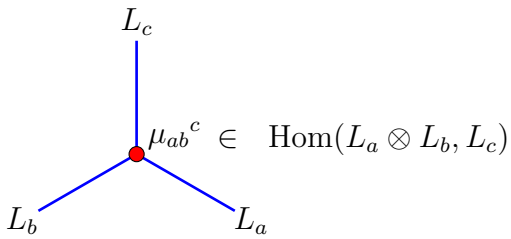
which gives the abelian structure of the category. The morphisms between the objects of $\mathcal{C}$ correspond to topological local operators living between the lines:



(2.2.5)

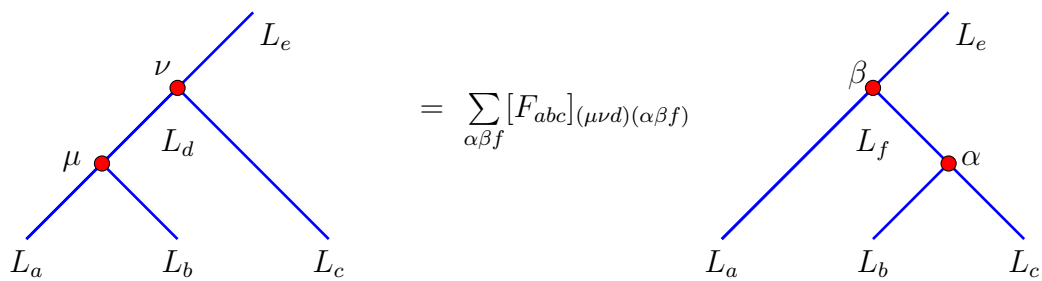
while endomorphisms are topological operators between a line to itself. One can show that all the endomorphisms of simple lines are proportional to the identity.

The tensor product $\otimes$ corresponds to the fusion of two TDLs, as previously described and $\mathrm{Hom}(L_a \otimes L_b, L_c)$ can be interpreted as three-valent junctions between $L_{a,b,c}$



. (2.2.6)

In particular, for any simple line L_c in the fusion channel of $L_a \otimes L_b$, there exist morphisms $\mu_{ab}{}^c$. The associator constraint relates two different configurations of defects



inserted in the spacetime and F is a complex number, usually called F-symbol (or 6j symbol), which implements the natural isomorphism and which must be compatible with the pentagon equations described before. When the symmetry is a vanilla invertible symmetry, the two configurations above correspond, by Poincaré duality, to two gauge fields related by a gauge transformation. Therefore the F-symbol is the lack of the gauge invariance of partition function coupled to a mesh of symmetry defects (or equivalently to a background gauge field) which is the signal of a 't Hooft anomaly. The pentagon equations, in this particular example, impose constraints on the set of F-symbols, restricting them to belong within the cohomology group $H^3(G, U(1))$. This aligns with the anticipated classification of 't Hooft anomalies in the two-dimensional scenario. In the case of most general non-invertible symmetries, Poincaré duality does not hold anymore. Nevertheless, we can still rely on the homological framework associated with the aforementioned network of defects. While we won't delve into the technicalities of defining 't Hooft anomalies in this introductory context, it is evident that the F-symbols and their corresponding pentagon equations serve as a generalization of the 't Hooft anomaly concept. Therefore we have found that *categorical symmetries naturally encode the specification of their anomaly.* In

the above case of group-like symmetries the fusion category describing them needs the specification of the group G and its anomaly $\alpha \in H^3(G, U(1))$ and it is usually called Vec_G^α fusion category. A more non-trivial example is the already mentioned Kramers-Wannier duality line of the Ising CFT. The fusion category described the symmetry structure of Ising is dubbed Tambara-Yamagami fusion category [137]. Besides the fusion rules (2.1.20) which are completely determined by the group $\mathbb{Z}_2$ generated by η, the Tambara-Yamagami fusion category has two non-trivial data: a non-degenerate symmetric bicharacter $\gamma : \mathbb{Z}_2 \times \mathbb{Z}_2 \to U(1)$ and the Frobenius-Schur indicator $[\epsilon] \in H^3\big(\mathbb{Z}_2, U(1)\big) \cong \mathbb{Z}_2$, which completely determine the F-symbols.

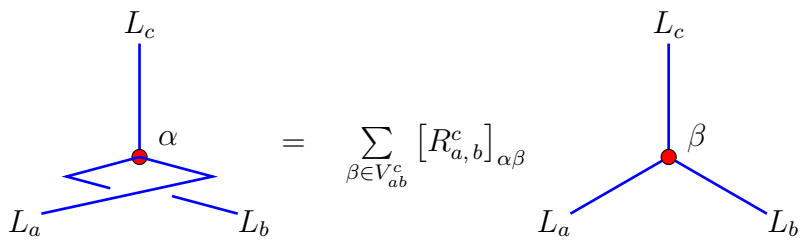
Even within this general (and by no means exhaustive) introduction to fusion categories, we can already appreciate their inherent power. In this language, the concept of 't Hooft anomalies and gauging emerges naturally as a part of the *symmetry structure*, rather than an additional property possessed by a symmetry. Indeed in the case of invertible symmetries, the pentagon equations are *always* satisfied by a trivial choice of the F-symbols, implying the existence of an invertible symmetry that is free from anomalies. However, when considering more general fusion categories, this is no longer the case. Fusion algebras may arise that are incompatible with trivial F-symbols and trivial 't Hooft anomalies [13]. What is even more surprising is that certain fusion algebras may be fundamentally incompatible with any choice of F-symbols, indicating the non-existence of a symmetry with that specific fusion structure.

Symmetries in $d > 2$ and higher categories. We now want to describe what is the general symmetry structure in higher dimensions. The natural generalization of the two-dimensional case is the language of fusion $(d-1)$-categories consisting of a set objects $U^{(d-1)}_{a,b,\cdots}$, 1-morphisms between them $U^{d-2}_\alpha : U^{(d-1)}_a \to U^{(d-1)}_b$, 2-morphisms between 1-morphisms $U^{(d-3)}_{m,n,\cdots} : U^{(d-2)}_\alpha \Rightarrow U^{(d-2)}_\beta$ and so on until $(d-1)$-morphisms assumed to form a $\mathbb{C}$-vector space. From a physical perspective, the objects within this category correspond to $(d-1)$-dimensional topological operators present in the theory. On the other hand, p-morphisms represent operators of dimensions $(d-1-p)$ that lives between two $(p-1)$-morphisms. Notably, the p-endomorphisms of the identity operator constitute a collection of genuine $(d-p-1)$-dimensional topological operators within the theory. Consequently, they form a $(d-p-1)$-

[13] Even if F-symbols and anomalies are related, they are not the same. In particular, there can be fusion categories with non-trivial F-symbols which are anomaly free, see *e.g.* [15].

subcategory, which stands independently inside the full symmetry structure [14].

Opposite to the two-dimensional case, there is still not a fully understood picture of such formalism so we cannot use a well-defined and established language in order to study physical systems with generalized symmetries. Indeed going up with both the ambient spacetime dimensions and the dimensionality of the topological operators, it is not hard to expect that the symmetry structure becomes complicated. Just considering the same topological lines, already described before, in three spacetime dimensions induces a further complication. In three dimensions lines can braid between each other and this induces a new corresponding categorical data usually called R-symbol defined as



(2.2.7)

Similarly to the F-symbols, the entire set of R-symbols must satisfy consistency conditions which highly constrain their values. The corresponding mathematical framework describing topological lines that can braid between each other is usually called Modular Tensor Categories (MTCs) (see e.g. [138] for a nice physical review on the subject).

Even if more complicated situations (involving for instance topological surfaces in arbitrary dimensions) are much harder to describe, the proliferation of physical systems equipped with such generalized symmetries is now a perfect laboratory to explore the underlying mathematical structure of such formalism and it offers an opportunity to gain insights into both new and familiar properties exhibited by theories possessing this rich symmetrical structure.

The aim of this thesis is to work in this direction from a physical viewpoint, with the goal of unravelling the general patterns underlying generalized symmetries and their correspondence with topological operators. Through the study of specific examples and the identification of shared characteristics arising from distinct setups, we will try to uncover a partial understanding of these phenomena. By looking into concrete scenarios, we can better understand the principles that govern these

[14] See *e.g.* [24] for more details on this physical picture about higher categories.

generalized symmetries, which can be useful for broader insights into their nature and implications.

2.3 Appendices

2.3.1 Appendix A: Gauge fields, topological defects and Poincaré duality

In this appendix, we want to make sharper the relation claimed in the text between a network of topological operators and flat gauge fields. With a network of topological defects we mean a consistent mesh of extended topological operators inserted in the space-time equipped with multi-valent junctions while a flat gauge field is a gauge field with a trivial curvature. To construct a bundle for such connections one can use simplicial calculus (see *e.g.* [139] for a formal introduction to the subject). To this aim we choose a simplicial triangulation of the n dimensional spacetime X made by vertices or 0-simplices p_i with an arbitrary ordering for the index i, edges or 1-simplices p_{ij} (with $i < j$) connecting the vertices p_i and p_j, 2-simplices p_{ijk} (with $i < j < k$) bounded by p_{ij}, p_{jk} and p_{ik}, and so on. The n dimensional space-time X will be defined as the union of all the n-simplices.

Given a triangulation, a gauge field a for the discrete p-form symmetry G (possibly non-abelian if $p = 0$) is a $(p+1)$-cochain $a \in C^{(p+1)}(X, G)$ that assigns an element $a_{ij} \in G$ to each 1-simplex p_{ij} (with $i < j$), with the constraint that $da = 0$. Using additive notation for the group G (for simplicity we consider G an abelian group) and choosing $p = 0$, we define the differential as

$$(da)_{ijk} = a_{jk} - a_{ik} + a_{ij} \qquad \text{with } i < j < k\,. \tag{2.3.1}$$

Therefore, gauge transformations are defined as

$$a_{ij} \mapsto a_{ij} + (d\lambda)_{ij} \qquad \text{where } d\lambda_{ij} = \lambda_j - \lambda_i\,, \tag{2.3.2}$$

and $\lambda \in C^0(X, G)$ in a 0-cochain. The flatness condition, together with its gauge transformations, implies that a is actually an element of the (simplicial) cohomology group $H^1(X, G)$ whose representatives are usually called cocycles. An analogous definition holds for p-cochains and p-cocycles, thus defining gauge fields for higher-form symmetries.

Topological actions. Given p-cocycles valued in abelian groups, we can define products between them, dubbed as cup products. They are defined as

$$(f \cup g)_{i_0,\cdots i_{p+q}} = f_{i_0\cdots i_p} \cdot g_{i_p\cdots i_q}\,, \tag{2.3.3}$$

where $f \in C^p(X,G)$, $g \in C^q(X,G)$ and $\cdot$ is a bilinear pairing from $G \times G$ to G [15]. This definition implies that the cup product is a bilinear function of its entries. More generically we can define cup products between cochains in $C^p(X,G)$ and $C^q(X,H)$ if there exists a bilinear map $G \times H \to K$ for a third abelian group K. A very common situation is the case when H is the Pontryagin dual of G (*i.e.* the group of characters for G) and K is $U(1)$.

Crucially the differential operator d satisfies the usual Leibniz identity with respect to such cup products, *i.e.*

$$d(f \cup g) = df \cup g + (-)^p f \cup dg\,. \tag{2.3.4}$$

Consequently, if both f and g are closed, it follows that the composition $f \cup g$ will also maintain the closeness condition. Such property allows us to construct higher q-cocycles given just a set of p-cocycles with $p < q$. In particular, in the case $p+q = d$, such cocycles are also called topological actions since they produce gauge invariant topological terms that one can add to a given action. In general, there can be different maps from $H^p(X,G) \times H^q(X,H) \to H^{p+q}(X,K)$ which are not cup products and not bilinear. Such maps can also produce different topological terms. Let us give an example of topological terms arising from discrete gauge fields. To this aim, let us consider a 4d theory with a 1-form gauge symmetry $\mathbb{Z}_N^{(1)}$ with odd N and the corresponding dynamical 2-cocycle $b \in H^2(X,\mathbb{Z}_N)$. In this case, we can add to the action a topological term of the form $\frac{2\pi i k}{N}\int b \cup b$ for some $k = 0, \cdots N-1$. Such term can be seen as the discrete analogue of the standard θ-term that one can add in a gauge theories [16].

Simplicial homology and Poincaré duality. A network of p dimensional topological symmetry operators on X can be represented as a p-chain $\Sigma \in C_p(X,G)$. In

[15] For instance when $G = \mathbb{Z}_N$ this pairing is simply the product in $\mathbb{Z}$ of two elements $a, b \in \mathbb{Z}_N$ modulo N.

[16] When instead N is even, it turns out that the above expression does not produce all the possible topological actions of the theory. Indeed there exist a map $\mathfrak{P}$, dubbed Pontryagin square, from $H^2(X,\mathbb{Z}_N)$ to $H^4(X,\mathbb{Z}_{2N})$ (see *e.g.* [140]) which captures all the possibilities.

the space of p-chains, we can define the boundary operator $\partial : C_p \to C_{p-1}$ which geometrically represents the boundary of the mesh (taken with appropriate orientation). Imposing $\partial\Sigma = 0$ is equivalent to asking that the boundary of the network corresponds to the trivial element of G, *i.e.* the group law is satisfied in every multi-valent junction of the mesh. Since $\partial^2 = 0$ we can define the homology group $H_p(X, G)$ as the closed chains, modulo exact ones. Elements in the homology group $H_p(X, G)$ are topological p-dimensional surfaces labeled by a group element of G.

Crucially there exists an isomorphisms between $H_k(X, G)$ and $H^{(d-k)}(X, G)$ called *Poincaré duality* which holds for any d dimensional oriented closed manifolds X. The proof of such duality lies in the definition of the dual triangulation which relates p_i simplices with dual $\hat{p}_{d-i}$ ones. Instead of running through the formal proof of this statement let us try to give a physical intuition of such duality. Given a p-cochain $b \in H^p(X, G)$ we can always integrate it over a p-chain $\Sigma \in H_p(X, G)$ [17]. The resulting integral $\int_\Sigma a$ can be rewritten as an integral over the full spacetime by introducing a $(d-p)$-cochain $PD(\Sigma) \in H^{(d-p)}(X, G)$ which has support only on Σ. Namely

$$\int_\Sigma b =: \int_X b \cup PD(\Sigma)\,. \tag{2.3.5}$$

$PD(\Sigma)$ is explicitly the Poincaré dual of Σ. The fact that such cochain exists is the non-trivial statement of Poincaré duality.

Poincaré duality is particularly useful because it relates flat gauge fields, *i.e.* cocycle in simplicial cohomology, with a network of symmetry defects, *i.e.* elements in simplicial homology. This is particularly useful since it is often the case that the homological picture is more accessible with respect to the cohomological one.

Gauging discrete symmetries

Now that we have the precise map between gauge fields and a network of topological defects, we can discuss the gauging of discrete (higher-form) symmetries using both formalisms. Analogous to the standard continuous case, gauging a discrete symmetry is achieved by turning on a background gauge field and then summing over all its inequivalent values. For instance, given a d-dimensional QFT $\mathcal{T}$ with a discrete p-form symmetry $G^{(p)}$ and described by the partition function $Z[A]$ with $A \in H^{p+1}(X, G)$, the gauge theory $\mathcal{T}/G$ is defined by the partition function

[17] Such integral is defined as the sum of $b_{i_0,\cdots,i_p}$ over all the p-simplices intersecting Σ.

$$Z_{\mathcal{T}/G} \propto \sum_{[a]\in H^{p+1}(X,G)} Z[a], \tag{2.3.6}$$

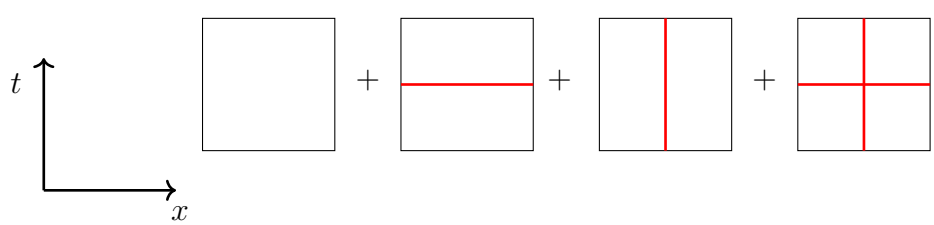
where the proportionality factor is completely specified by G and X.

By Poincaré duality, $Z[a]$ is equivalent to a correlator with an insertion of topological symmetry defects for $G^{(p)}$ on the Poincaré dual network of a. Equation (2.3.6) is equivalent to

$$Z_{\mathcal{T}/G} \propto \sum_{[\gamma]\in H_{d-p-1}(X,G)} \langle U[\gamma]\rangle. \tag{2.3.7}$$

We want to show, using a very simple and standard example, that such a presentation of the gauging procedure is actually equivalent to what we usually expect. Let us consider a 2D QFT with discrete $\mathbb{Z}_2^{(0)}$ symmetry. The Hilbert space of this theory can be split into charged and uncharged states under the $\mathbb{Z}_2$ symmetry. We can also define the twisted Hilbert space by imposing twisted periodicity conditions for the states constructed on $S^1 \times \mathbb{R}$. The gauged theory $\mathcal{T}/\mathbb{Z}_2$ has a Hilbert space constructed by taking the uncharged states of the untwisted and twisted Hilbert spaces of $\mathcal{T}$.

Let us show that (2.3.7) exactly produces such a Hilbert space. Considering the 2D manifold as a torus, we have 4 homologically inequivalent networks of symmetry defects, so the sum in (2.3.7) has 4 terms. Pictorially, we have:



where the red line represents the generator U_{-1} of the $\mathbb{Z}_2$ 0-form symmetry. In this instance, the normalization factor in (2.3.7) is $\frac{1}{2}$. We notice that within the first two terms of (2.3.1), when we quantize the theory, the operators placed along the spatial direction induce an action on the Hilbert space:

$$|\psi\rangle \to \frac{1+U_{-1}}{2}|\psi\rangle, \tag{2.3.8}$$

effectively projecting the space into the uncharged sector. In the last two terms, the presence of a non-trivial line along the time direction twists the periodicity conditions of the states, giving a twisted Hilbert space. In this space, operators

along the spatial dimensions continue to project the states into the uncharged sector. Consequently, the configuration (2.3.1) faithfully replicates the well-established characteristics of the gauging procedure.

2.3.2 Appendix B: 1-form symmetries in gauge theories

In this appendix, we want to explicitly show that Yang-Mills theories have an electric 1-form symmetry labeled by the center of the group. To properly define a gauge theory globally on a manifold X we need to define its connections on a principal bundle. To this aim, we have to cover X with a good enough collection of open patches U_i which are contractible. On each patch, we can then define a connection A_i which is now a 1-form valued on the Lie group $\mathfrak{g}$ of G. To get a consistent bundle we should impose that on the double intersection $U_{ij} = U_i \cap U_j$ there exists a transition function $g_{ij} : U_{ij} \to G$ constrained by

$$\begin{aligned} &g_{ji} = g_{ij}^{-1} \\ &g_{ij} g_{jk} g_{ki} = 1 \qquad \text{on } U_{ijk} \equiv U_i \cap U_j \cap U_k \end{aligned} \tag{2.3.9}$$

and such that

$$A_j = g_{ij}^{-1} A_i g_{ij} - i g_{ij}^{-1} d g_{ij} \,. \tag{2.3.10}$$

In this language a gauge transformation acts on both A_i and g_{ij} as

$$\begin{aligned} &A_i \to \lambda_i^{-1} A_i \lambda_i - i \lambda_i^{-1} d\lambda_i \\ &g_{ij} \to \lambda_j^{-1} g_{ij} \lambda_i \end{aligned} \tag{2.3.11}$$

for some functions $\lambda_i : U_i \to G$. Let us now consider the following transformation

$$\begin{aligned} &A_i \to A_i \\ &g_{ij} \to \lambda_{ij} g_{ij} \,, \end{aligned} \tag{2.3.12}$$

for some $\lambda_{ij} \in G$. This is clearly a symmetry of the action (which does not depend on the transition functions). However, in order to be a symmetry of the partition function, we have to impose that the constraints (2.3.10),(2.3.9) are not spoiled by such transformation. This implies that λ_{ij} must commute with every elements of G, *i.e.* they are valued in the center $Z(G)$ and they must satisfy the cocycle condition

$$\lambda_{ij} \lambda_{jk} \lambda_{ki} = 1 \qquad \text{if } U_{ijk} \equiv U_i \cap U_j \cap U_k \neq \emptyset \,. \tag{2.3.13}$$

For such values of λ_{ij}, (2.3.12) is a symmetry of the theory. To see if such symmetry is trivial or not, we should find some gauge invariant operator that transforms

under (2.3.12). Since the local connection A_i is invariant, any gauge invariant local operator of the theory will be uncharged under this symmetry. However, we can consider extended operators, *i.e.* Wilson lines. For instance, the fundamental Wilson line is defined as

$$W_\gamma = Tr_F(Pe^{i\int_{\gamma_i} A_i} g_{ij} Pe^{i\int_{\gamma_j} A_j} g_{jk} \cdots Pe^{i\int_{\gamma_l} A_l} g_{li}) \tag{2.3.14}$$

where γ is a line passing through the patches $U_{i,j,k,\cdots,l}$, γ_p is the piece of γ in the patch U_p and the transition functions are present in order to achieve the gauge invariance. If γ is a non-contractible cycle in X, then $\lambda_{ij}\lambda_{jk}\cdots\lambda_{li} \neq 1$ and such Wilson line transforms under (2.3.12) as

$$W_\gamma \to \phi(\gamma) W_\gamma = \lambda_{ij}\lambda_{jk}\cdots\lambda_{li} W_\gamma \neq W_\gamma \,. \tag{2.3.15}$$

Therefore (2.3.12) defines a 1-form symmetry of Yang-Mills theory acting on the Wilson lines.

Let us now add a fundamental matter Φ in the theory. As with all the other fields, Φ takes different values in every patch of X and we should impose

$$\Phi_j = g_{ij}\Phi_i \,, \tag{2.3.16}$$

which is compatible with the gauge transformations (2.3.11) if we impose $\Phi_i \to \lambda_i^{-1}\Phi_i$. However, we immediately see that the 1-form symmetry transformations (2.3.12) do not leave (2.3.16) invariant so that the symmetry is explicitly broken.

Chapter 3

Generalized symmetries in Quantum Field Theories

In this chapter, we delve into the analysis of generalized symmetries and topological operators in the broad context of Quantum Field Theory. Section 3.1 is based on [29], where we study higher-categorical (usually referred to as non-invertible) symmetries that arise in 4d gauge theories. In section 3.2, based on [128], we investigate symmetries in disordered theories, where a 0-form global symmetry of a QFT is explicitly broken by a random coupling h, but it re-emerges after a quenched average.

3.1 Categorical symmetries in Yang-Mills theories

3.1.1 Introduction and summary of the results

In the previous section we have discussed particular procedures one can perform in order to obtain theories equipped with non-invertible symmetries. One particular example is the one of gauging a non-normal subgroup of a given invertible symmetry group. This way of obtaining non-invertible symmetries by gauging automorphisms can produce a large number of examples [24], which is a very interesting "data-base" of higher category symmetries, potentially also for mathematicians.

In this section, based on [29], we provide an instance in which the gauging procedure is very natural, and is in some sense built-in. This is the case of the Weyl group W_G in 4d G Yang-Mills (YM) theory. If we denote by $N\left(U(1)^r\right) \subset G$ the normalizer of the Cartan torus $U(1)^r$ in G, the Weyl group is the quotient $W_G =$

$N\left(U(1)^r\right)/U(1)^r$ and the normalizer can be written as $U(1)^r \rtimes W_G$[1]. Therefore W_G is automatically gauged in the G YM theory. Does this produce a non-invertible symmetry? Strictly speaking the answer is no, basically because there is no theory producing G YM theory upon gauging W_G. However, if we go to high energy where the theory becomes free[2], a partial fixing of the non-Abelian gauge invariance is achieved by looking at the gauge theory for the Cartan torus $U(1)^r$ [141] (see section 3.1.3 for a detailed discussion). Here the Weyl group appears as a global 0-form symmetry, and thus we need to gauge it to obtain a theory related to the UV limit of YM theory. We are led to look at the theory with gauge group given by the normalizer $U(1)^r \rtimes W_G$ of the Cartan torus. The Abelian gauge theory $U(1)^r$ has continuous 1-form symmetries on which the Weyl group acts as an automorphism[3]. Thus we are precisely in the condition described above, except that the 1-form symmetry is continuous. Then the $U(1)^r \rtimes W_G$ gauge theory is expected to have 1-form continuous non-invertible symmetries, described by a 2-category. We will focus on the case $G = SU(N)$, so we consider the $U(1)^{N-1} \rtimes S_N$ gauge theory in 4d. The 3d analog of this theory has been constructed on the lattice and with a different goal in [22], where it has been dubbed *semi-Abelian theory*. In that paper it is also pointed out that there are non-invertible symmetries. However, their fusion rules have not been computed, and only a subset of these symmetries has been discussed. In particular, even if it is pointed out that the general topological operators are parametrized by $N-1$ parameters, the ones studied in [22] depends only on one compact variable. On the other hand, we will see that the parameter space of the non-invertible symmetry is $U(1)^{N-1}/S_N$, and that the fusion rules are

$$\mathcal{T}(\boldsymbol{\alpha}) \otimes \mathcal{T}(\boldsymbol{\beta}) = \sum_{\sigma \in H_{\boldsymbol{\alpha}} \backslash S_N / H_{\boldsymbol{\beta}}} f^{\sigma}_{\alpha\beta} \ \mathcal{T}\left(\boldsymbol{\alpha} + \mathfrak{S}^{\vee}_{\sigma} \cdot \boldsymbol{\beta}\right) \tag{3.1.2}$$

[1]More precisely the normalizer fits in the short exact sequence

$$0 \to U(1)^r \to N\left(U(1)^r\right) \to W_G \to 0 \tag{3.1.1}$$

defined by an action $\rho : W_G \to Aut\left(U(1)^r\right)$ and a non-trivial cocycle $e \in H^2_{\rho}\left(W_G, U(1)^r\right)$. As previously emphasized, we are mainly interested in the action of ρ which specify how the zero-form symmetry acts on the generators of the one-form symmetry. Therefore the role of the cocycle e in our discussion is marginal and we will neglect its effect in the following. It would be interesting to analyze its role in a future work.

[2]With an abuse of terminology by *free* in the UV we will always mean *weakly coupled*.

[3]As pointed out in [142,143], the interplay between the 1-form and the 0-form symmetry is fully specified only after we specify a symmetry fractionalization class in $H^2_{\rho}\left(W_G, (1)^r\right)$. This class is given by the cocycle e which gives the extension $N\left(U(1)^r\right))$ of $U(1)^r$ by W_G.

where

$$f^{\sigma}_{\alpha\beta} = \frac{|H_{\boldsymbol{\alpha}+\mathfrak{S}^{\vee}_{\sigma}\cdot\boldsymbol{\beta}}|}{|H_{\boldsymbol{\alpha}} \cap \sigma H_{\boldsymbol{\beta}} \sigma^{-1}|}. \tag{3.1.3}$$

Here $\mathfrak{S}^{\vee}_{\sigma}$ is the relevant action of the permutation group S_N on the labels $\boldsymbol{\alpha} \in U(1)^{N-1}$, while $H_{\boldsymbol{\alpha}} \subset S_N$ denotes the stabilizer for this action. One of our main results is that, while the fusions above hold when the defect does not contain non-trivial 1-cycles, on a general topology we have to modify the formula above by including the condensations

$$\mathcal{T}(\boldsymbol{\alpha})[\Sigma] \otimes \mathcal{T}(\boldsymbol{\beta})[\Sigma] = \sum_{\sigma \in H_{\boldsymbol{\alpha}} \backslash S_N / H_{\boldsymbol{\beta}}} f^{\sigma}_{\alpha\beta}\, P_{\mathrm{Rep}\left((H_{\boldsymbol{\alpha}} \cap H_{\boldsymbol{\beta}})^{\perp}\right)} \otimes \mathcal{T}\left(\boldsymbol{\alpha} + \mathfrak{S}^{\vee}_{\sigma} \cdot \boldsymbol{\beta}\right)[\Sigma]\,. \tag{3.1.4}$$

The operator $P_{\mathrm{Rep}\left((H_{\boldsymbol{\alpha}} \cap H_{\boldsymbol{\beta}})^{\perp}\right)}$ coincide, up to a normalization, with a condensation defect, and it is a projector in the sense that it squares to itself.

Notice that $U(1)^{N-1}/S_N$ coincides with the set of conjugacy classes of $SU(N)$ also labeling the Gukov-Witten (GW) operators of $SU(N)$ YM theory [130,144,145]. In the full YM theory, only the GW labeled by central elements are topological, and generate the 1-form center symmetry. We propose that *all* the GW operators in YM theory become topological at high energy and form a non-invertible symmetry, broken down to the center symmetry by the RG flow. That the $SU(N)$ YM theory at high energy has non-invertible symmetries has been recently observed from a different point of view also in [146]. The fact that these two distinct arguments agree is reassuring. Moreover, in that paper, the fusion rules have been computed only in the $N = 2$ case, and they agree with those of the $U(1) \rtimes S_2$ gauge theory[4]. It is reasonable that by applying the methods of [146] for any N one gets the fusion rules which we compute in the $U(1)^{N-1} \rtimes S_N$ theory, thus confirming that the symmetry found in that paper is really the same discussed here. We leave this interesting problem for future work. In 2d YM theories it was already pointed out in [98] that all the GW operators are topological, and they form a non-invertible symmetry at all energy scales. This conclusion is peculiar of 2d YM theory since the theory is quasi-topological. In $d > 2$ this is obviously not true and indeed this symmetry exists only in the UV limit.

[4]The coefficients appearing are different, but this has to do with different choices of normalization. However, as we will see, with our normalization the fusion coefficients turn out to be always integers and as we point out in the main text, this is important since they count the number of 1-morphisms up to endomorphisms.

The connection between the UV symmetries of YM theory and those of the $U(1)^{N-1} \rtimes S_N$ gauge theory is important because the second case is much more under control, and we are able to discuss the 2-categorical structure of this symmetry (section 3.1.2), which indeed was not analyzed before. The analysis of this structure is the bulk of this section. In our examples, we find several properties which we believed to be general aspects of 2-category symmetries. For instance, we argue that the almost universal presence of condensation defects on the right-hand side of the fusion rules is what distinguishes the global fusion rules (those obtained on general manifolds) from the local ones, which are true only if the defects are topologically trivial. Moreover, we find that the fusion coefficients are always positive integer numbers. We interpret these numbers as counting the 1-morphisms up to possible endomorphisms. This is an important difference with respect to fusion 1-category symmetries, in which the indecomposable objects cannot have non-trivial endomorphisms, and therefore the fusion coefficients are directly counting the morphisms living at the junctions. Moreover, while in fusion 1-categories the morphisms form a vector space, and therefore these numbers are the dimensions of these vector spaces, in fusion 2-categories the 1-morphsims form by itself a category, and the numbers are better interpreted as quantum dimensions. Finally, after understanding the map between the extended gauge invariant operators of YM theory and the $U(1)^{N-1} \rtimes S_N$ gauge theory, we are able to determine how this non-invertible symmetry acts on line operators which are compatible with the known results when we restrict to the group-like subcategory $\mathbb{Z}_N^{(1)}$ corresponding to the center.

The rest of the section is organized as follows. In section 3.1.2 we study the $U(1)^{N-1} \rtimes S_N$ gauge theory, by analyzing the full spectrum of gauge-invariant operators, finding the continuous non-invertible 1-form symmetries. The intricate 2-categorical structure of this symmetry is analyzed in section 3.1.2, where we also explain the connection, in our specific example, between the concept of *global fusions* introduced in [24] and the *higher condensation defects* constructed in [23]. Then section 3.1.3 is devoted to the connection between the $U(1)^{N-1} \rtimes S_N$ gauge theory and $SU(N)$ YM theory at high energy. After a path integral argument, we show a mapping among extended gauge invariant observables of the two theories. Then we identify the center symmetry of $SU(N)$ YM theory with a discrete subset of topological surface operators of the $U(1)^{N-1} \rtimes S_N$ theory, by showing that they give rise to the same Ward identities with the Wilson line operators. We also discuss how all the possible choices of the global structure of the YM theory are obtained from the point of view of the $U(1)^{N-1} \rtimes S_N$ gauge theory. We conclude in section

3.1.4 with a discussion on possible future directions.

3.1.2 The 4d $U(1)^{N-1} \rtimes S_N$ Gauge Theory

This section is devoted to the $U(1)^{N-1} \rtimes S_N$ gauge theory on its own. We show that the theory has non-invertible 1-form symmetries labeled by continuous parameters valued in $U(1)^{N-1}/S_N$. This non-invertible symmetry is described by a 2-category which we study in detail, discovering an interesting mathematical structure.

Abelian Gauge Theory

We start with a free Abelian gauge theory with gauge group $U(1)^{N-1}$. The definition of the theory is encoded in the choice of the spectrum of Wilson line operators, namely an $N-1$ dimensional lattice. A way to make this explicit is by exhibiting a basis for the gauge fields $\mathcal{A}_{i=1,...,N-1}$ in which the Wilson lines have integer charges. This is a choice of a symmetric positive definite $(N-1) \times (N-1)$ matrix $Q_{ij}^{(N-1)}$ such that the action is

$$S = \frac{1}{2e^2} \int d^4x \, Q_{ij}^{(N-1)} \mathcal{F}_i \wedge *\mathcal{F}_j \tag{3.1.5}$$

where $\mathcal{F}_i = \, d\mathcal{A}_i$. Then the most general Wilson line is

$$\mathcal{W}(\boldsymbol{n})[\gamma] = \mathcal{W}(n_1, ..., n_{N-1})[\gamma] = \prod_{i=1}^{N-1} \mathcal{W}_i[\gamma]^{n_i} \ , \qquad \mathcal{W}_i[\gamma]^{n_i} := \exp\left(i n_i \oint_\gamma \mathcal{A}_i \right) \tag{3.1.6}$$

where $\boldsymbol{n} = (n_1, ..., n_{N-1}) \in \mathbb{Z}^{N-1}$. To make the action of the S_N 0-form symmetry explicit we define the theory by demanding that, upon introducing $\mathcal{A}_N = -\mathcal{A}_1 - ... - \mathcal{A}_{N-1}$ the action takes the form[5]

$$S = \frac{1}{2e^2} \int d^4x \left(\mathcal{F}_1^2 + ... \mathcal{F}_N^2 \right) = \frac{1}{2e^2} \int d^4x \left(\sum_{i=1}^{N-1} 2\mathcal{F}_i^2 + \sum_{i<j}^{N-1} 2\mathcal{F}_i \mathcal{F}_j \right) = \frac{1}{2e^2} \int d^4x Q_{ij}^{(N-1)} \mathcal{F}_i \mathcal{F}_j \tag{3.1.7}$$

thus defining the quadratic form $Q^{(N-1)}$ as

$$Q_{ij}^{(N-1)} = 1 + \delta_{ij} \ , \text{ with } \qquad \det(Q^{(N-1)}) = N \ , \quad \left(Q^{(N-1)}\right)_{ij}^{-1} = \frac{-1 + N\delta_{ij}}{N}. \tag{3.1.8}$$

[5]Here $\mathcal{F}_i \mathcal{F}_j$ means $\mathcal{F}_i \wedge *\mathcal{F}_j$.

The S_N symmetry permutes the connections $\mathcal{A}_{i=1,...,N}$. On the $N-1$ field strengths $\mathcal{F}_1, ...\mathcal{F}_{N-1}$ it acts in the *standard representation* of S_N, which we denote by $\mathfrak{S}$ (see appendix 3.3.1). This obviously induces also an action of S_N on the Wilson lines, which is conveniently rewritten as an action on the charges:

$$\sigma\cdot\mathcal{W}(\boldsymbol{n}) = \exp\left(i\oint\sum_{j=1}^{N-1} n_j\mathfrak{S}_\sigma\left(\mathcal{A}_j\right)\right) = \exp\left(i\oint\sum_{j=1}^{N-1}\mathfrak{S}^\vee_{\sigma^{-1}}(n_j)\mathcal{A}_j\right) = \mathcal{W}\left(\mathfrak{S}^\vee_{\sigma^{-1}}\cdot\boldsymbol{n}\right) . \tag{3.1.9}$$

We have introduced $\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{n} = (\mathfrak{S}^\vee_\sigma(n_1), ..., \mathfrak{S}^\vee_\sigma(n_{N-1}))$, with $\mathfrak{S}^\vee_\sigma(n_i) = m_{\sigma(i)} - m_{\sigma(N)}$, where $n_i = m_i - m_N$. This is a dual representation of S_N on $N-1$ variables (see appendix 3.3.1).

We also have electric GW operators [130, 144] (for a review [145])

$$\mathcal{D}(\boldsymbol{\alpha})[\Sigma] = \mathcal{D}(\alpha_1, ..., \alpha_{N-1})[\Sigma] = \prod_{i=1}^{N-1}\mathcal{D}_i(\alpha_i)[\Sigma] \ , \quad \mathcal{D}_i(\alpha_i)[\Sigma] := \exp\left(i\alpha_i\int_\Sigma\frac{*\mathcal{F}_i}{e^2}\right). \tag{3.1.10}$$

The variables $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{N-1})$ parametrize an $(N-1)$-dimensional torus (the precise periodicity is shown below). These operators are the generators of the electric 1-form symmetry [4] $\left(U(1)^{(1)}_e\right)^{N-1}$. On the GW operators S_N acts as it does on the Wilson lines:

$$\sigma\cdot\mathcal{D}(\boldsymbol{\alpha}) = \mathcal{D}\left(\mathfrak{S}^\vee_{\sigma^{-1}}\cdot\boldsymbol{\alpha}\right). \tag{3.1.11}$$

The electric GW operators $\mathcal{D}(\boldsymbol{\alpha})$ have an action on the Wilson lines $\mathcal{W}(\boldsymbol{n})$ by linking, and a simple computation shows the following Ward identity

$$\mathcal{D}(\boldsymbol{\alpha})[\Sigma]\cdot\mathcal{W}(\boldsymbol{n})[\gamma] = \exp\left(i\ Lk(\Sigma,\gamma)\sum_{i,j=1}^{N-1}\alpha_i\left(Q^{(N-1)}\right)^{-1}_{ij} n_j\right)\mathcal{W}(\boldsymbol{n})[\gamma] \tag{3.1.12}$$

where $Lk(\Sigma,\gamma)$ denotes the linking number between Σ and γ. From this we deduce the periodicity $\alpha_i \sim \alpha_i + 2\pi w_j Q_{ji}$, $w_i \in \mathbb{Z}$. Equivalently, the variables

$$\beta_i := \alpha_i\left(Q^{N-1}\right)^{-1}_{ji}$$

are 2π periodic, thus parametrizing a torus $U(1)^{N-1}$.

An analogous discussion holds for the *'t Hooft lines* $\widetilde{\mathcal{W}}(\boldsymbol{n})$ and magnetic GW operators $\widetilde{\mathcal{D}}(\boldsymbol{\alpha})$. However the global structure we have chosen restricts the set of allowed 't Hooft lines by Dirac quantization conditions. We will discuss this in detail in section 3.1.3.

Warm Up: $N = 2$ Case

Before we face the general case, it is useful to study the baby example $N = 2$, which is simpler since $S_2 = \mathbb{Z}_2$ is Abelian, but captures several features of the general case. Indeed $U(1) \rtimes \mathbb{Z}_2 = O(2)$ and the model is known as the $O(2)$ gauge theory [147,148]. This subsection contains a review of discussions in [99] and [24], where the model has been shown to have non-invertible symmetries, but we also introduce new points, which we will expand on in the general case.

We start from the $U(1)$ Maxwell theory, in which $S_2 = \mathbb{Z}_2$ acts as charge conjugation by reversing the sign of the connection $\mathcal{A}$, we then gauge this symmetry obtaining the $U(1) \rtimes \mathbb{Z}_2$ theory. A class of operators of this theory consists in gauge invariant operators of the $U(1)$ theory. but we also have the $\mathbb{Z}_2$ Wilson line

$$\eta[\gamma] = e^{i \oint_\gamma a_2} \tag{3.1.13}$$

where $a_2 \in H^1(\mathcal{M}_4, \mathbb{Z}_2)$ is the dynamical $\mathbb{Z}_2$ gauge field. The η line is topological and generates the quantum 2-form symmetry $\widehat{\mathbb{Z}}_2^{(2)}$ as $\eta^2 = 1$.

Let us discuss the $\mathbb{Z}_2$ invariant combinations of operators of the original Abelian theory, which remain good operators after gauging. The local operators are all the even polynomials in the field strength. The Wilson lines $\mathcal{W}(n)$ of the Maxwell theory are labeled by one integer, their charge, and $\mathbb{Z}_2$ acts by reversing the sign of n. The Wilson line operators of the $O(2)$ gauge theory are obtained from those of $U(1)$ by summing over the $\mathbb{Z}_2$ orbits:

$$\mathcal{V}(n)[\gamma] = \mathcal{W}(n)[\gamma] + \mathcal{W}(-n)[\gamma] = e^{in \oint_\gamma \mathcal{A}} + e^{-in \oint_\gamma \mathcal{A}}. \tag{3.1.14}$$

We have a similar story for the electric GW operators. Imitating the well-known 3d procedure of [138] described in the introduction, we build the gauge-invariant surface operators by summing over the $\mathbb{Z}_2$ orbits. For reasons that will be clear in the following, we normalize the operators by dividing them by $|H_\alpha|$, where $H_\alpha \subset \mathbb{Z}_2$ is the stabilizer of α

$$\mathcal{T}(\alpha)[\Sigma] = \frac{1}{|H_\alpha|}\left(\mathcal{D}(\alpha)[\Sigma] + \mathcal{D}(-\alpha)[\Sigma]\right) = \frac{1}{|H_\alpha|}\left(e^{i\alpha \int_\Sigma \frac{*\mathcal{F}}{e^2}} + e^{-i\alpha \int_\Sigma \frac{*\mathcal{F}}{e^2}}\right). \tag{3.1.15}$$

In this case H_α can be either trivial (for $\alpha \neq 0, 2\pi$) or equal to $\mathbb{Z}_2$ (for $\alpha = 0, 2\pi$). The operators $\mathcal{T}(\alpha)$ are *indecomposable* objects after gauging, meaning that they cannot be written as direct sum of other objects. Note that with this normalization we always define the operators $\mathcal{T}(\alpha)[\Sigma]$ as the direct sum of $\mathcal{D}$ without any coefficient.

In particular

$$\begin{aligned} \mathcal{T}(\alpha)[\Sigma] &= \mathcal{D}(\alpha)[\Sigma] + \mathcal{D}(-\alpha)[\Sigma] \qquad \alpha \neq 0, \pi \\ \mathcal{T}(\alpha)[\Sigma] &= \mathcal{D}(\alpha)[\Sigma] \qquad \alpha = 0, \pi. \end{aligned} \tag{3.1.16}$$

With other normalizations we either get fractional coefficients (which are meaningless in a categorical language) or *decomposable* objects. This is the very reason for our choice of normalization, which we will keep also in the general N case.

Since $Q^{(1)} = 2$ in our normalization, $\mathcal{D}(\alpha)$ is parametrized by $\alpha \in [0, 4\pi)$. Then the manifold where α takes values in the $O(2)$ theory is $U(1)/\mathbb{Z}_2 = [0, 2\pi]$, which is singular since $\alpha = 0, 2\pi$ are fixed points of the $\mathbb{Z}_2$ action. The somewhat surprising fact is that, since these operators are topological, they can be regarded as the generator of a symmetry, even though $\mathcal{T}(\alpha)$ is not a unitary operator and does not satisfy a group law multiplication:

$$\mathcal{T}(\alpha) \otimes \mathcal{T}(\beta) = \frac{1}{|H_\alpha||H_\beta|}\Big(|H_{\alpha+\beta}|\mathcal{T}(\alpha+\beta) + |H_{\alpha-\beta}|\mathcal{T}(\alpha-\beta)\Big). \tag{3.1.17}$$

This is a non-invertible symmetry [13]. In the last few years these new type of symmetries have been analyzed extensively in 2d (for instance [13–15, 93, 110]), and very recently also in higher dimensions [23–25, 27, 28, 119, 120]. However most of the examples in the literature discuss *discrete* non-invertible symmetries, while the non-invertible symmetry of the $O(2)$ gauge theory, as well as the other cases we discuss here are *continuous* non-invertible symmetries. Until recently, these where believed to be very rare and exotic type of symmetries. One of our aims is to show that they can appear quite naturally, and they have some features similar to more common continuous symmetries.

Notice that there are exactly two values of α for which the fusion is group-like, namely the fixed points of the $\mathbb{Z}_2$ action $\alpha = 0, 2\pi$, for which

$$\mathcal{T}(2\pi) \otimes \mathcal{T}(2\pi) = \mathcal{T}(4\pi) = \mathcal{T}(0) = 1. \tag{3.1.18}$$

These are also the only two unitary operators. This shows that the large and continuous non-invertible symmetry contains an invertible $\mathbb{Z}_2^{(1)}$ 1-form symmetry, which is nothing but the center symmetry since $\mathcal{Z}(O(2)) = \mathbb{Z}_2$. It is also important to notice that in the fusion (3.1.17) the coefficients are always integer numbers. This is obvious when H_α and H_β are both either trivial or $\mathbb{Z}_2$. When instead $H_\alpha = 1$ but $H_\beta = \mathbb{Z}_2$ the $1/2$ factor is cancelled because $\mathcal{T}(\alpha+\beta) = \mathcal{T}(\alpha-\beta)$. We will show that an analogous mechanism takes place for general N. This fact is important

because the fusion coefficients have a meaning and must be integer numbers: when $\mathcal{T}(\gamma)$ appears in the fusion $\mathcal{T}(\alpha) \otimes \mathcal{T}(\beta)$ it means that there is a fusion category of 1-morphisms $\mathcal{T}(\alpha) \otimes \mathcal{T}(\beta) \to \mathcal{T}(\gamma)$, and the coefficient counts the number of simple lines in this category, or more precisely its *total quantum dimension*. However, since some objects have non-trivial endomorphisms, this counting is only up to these endomorphisms. We will expand on this point in the general case.

The non-unitarity of the GW operators $\mathcal{T}(\alpha)$ for $\alpha \neq 0, 2\pi$ reflects itself in the fact that the charges of Wilson lines are not phases, as follows from the generalized Ward identity

$$\mathcal{T}(\alpha)[\Sigma] \cdot \mathcal{V}(n)[\gamma] = \frac{2}{|H_{\boldsymbol{\alpha}}|} \cos\Big(Lk(\Sigma, \gamma)\, n\frac{\alpha}{2}\Big)\mathcal{V}(n)[\gamma]. \tag{3.1.19}$$

We get a phase only for $\alpha = 0, 2\pi$ in which the GW operators are group-like. This phase is $(-1)^n$ depending only on the parity of n. Notice that at generic values of α, different n's with the same parity give different charges.

Up to this point, the discussion was a bit naive and indeed was correct only in the case when Σ does not have non-trivial 1-cycles [24]. When we consider topologically non-trivial defects, we need to modify the discussion above and analyze in detail the 2-categorical structure of the non-invertible symmetries. To do this, we have to incorporate the dual 2-form symmetry $\mathbb{Z}_2^{(2)}$ arising from the gauging. This story will be more complicated in the general case $N > 2$ in which S_N is non-Abelian, so it is worth discussing the symmetry structure before in this simple example. Before gauging, the electric 1-form symmetry has a very simple 2-categorical structure: the indecomposable objects are $\{\mathcal{D}(\alpha)\}_{\alpha\in[0,4\pi)}$ and the category of 1-morphisms $\mathcal{D}(\alpha) \to \mathcal{D}(\beta)$ is empty unless $\alpha = \beta$, in which case it contains only the identity line. After gauging, we get one additional topological operator, namely the non-trivial $\mathbb{Z}_2$ Wilson line η, which does not affect the indecomposable objects but enters in the 1-morphisms. This is a sharp difference with respect to the 3d case of [138] in which by *dressing* the objects with η one gets new indecomposable objects. Naively, in 4d it seems that there are no further indecomposable objects, but we will explain shortly that this conclusion is wrong.

Since η is a bulk line, it exists as a 1-morphism $\eta : \mathcal{T}(0) \to \mathcal{T}(0)$, but also as a 1-morphism on the surface $\mathcal{T}(2\pi)$ on which it is non-trivial[6]. Notice an important

[6]This is because the operators $\mathcal{T}(0), \mathcal{T}(2\pi)$ were indecomposable objects also in the pre-gauged theory, and they do not see the $\mathbb{Z}_2$ symmetry. Therefore it is not required to put boundary conditions for the $\mathbb{Z}_2$ on the gauge field.

difference of higher category symmetries with respect to more standard fusion categories of topological defect lines in 2d [13,15]: even for indecomposable objects, the category of 1-endomorphism can contain non-trivial operators because there can be lower dimensional topological bulk defects which can be put on the objects without becoming trivial. As we will see, there are further interesting cases in which additional topological lines exist only stacked on a non-trivial surface. The surface operators $\mathcal{T}(\alpha)$, $\alpha \neq 0, 2\pi$ on the other hand absorb the Wilson line η. Therefore the only 1-endomorphism on them is the identity. This is because before gauging $\mathcal{D}(\alpha)$, $\alpha \neq 0, 2\pi$ is not invariant under $\mathbb{Z}_2$, so the precise definition of the gauge invariant defect $\mathcal{T}(\alpha) = \mathcal{D}(\alpha) + \mathcal{D}(-\alpha)$ requires to fix Dirichlet boundary conditions for the $\mathbb{Z}_2$ gauge field on the surface. We will call these kinds of objects *strongly simple*, following the terminology of [149].

The discussion above is crucial whenever Σ has non-contractible 1-cycles. When this is the case, the same line η can be non-trivial on the surface, and generates a 0-form symmetry $\mathbb{Z}_2$ on it. As suggested in [24], the *local* fusion rules (3.1.17) must be modified by generally gauging this 0-form symmetry on Σ, leading to the *global* fusion rules. We understand this gauging procedure as well as the necessary modification of the fusion by a different argument. One can use the 2-form symmetry $\mathbb{Z}_2^{(2)}$ in the bulk to construct one further topological surface operator by condensing the symmetry on a surface, as explained in detail in [23]:

$$\mathcal{C}[\Sigma] := \frac{1}{\sqrt{|H_1(\Sigma, \mathbb{Z}_2)|}} \sum_{\gamma \in H_1(\Sigma, \mathbb{Z}_2)} \eta[\gamma]. \tag{3.1.20}$$

Even if it is a surface operator, it has trivial action on lines because it is made of lower dimensional objects which cannot braid with lines. Notice that the condensation produces a dual 0-form $\mathbb{Z}_2$ symmetry living on the defect, which is generated by topological lines. The condensation defect is non-invertible, and its fusion was computed in [23] to be

$$\mathcal{C}[\Sigma] \otimes \mathcal{C}[\Sigma] = \mathcal{Z}(\mathbb{Z}_2; \Sigma)\mathcal{C}[\Sigma] \tag{3.1.21}$$

where $\mathcal{Z}(\mathbb{Z}_2; \Sigma) = \sqrt{|H_1(\Sigma, \mathbb{Z}_2)|}$ is the partition function of the 2d pure $\mathbb{Z}_2$ gauge theory on Σ. The fact that the fusion coefficients are not numbers, but partition functions of TQFT, seems to be a general feature of higher category symmetries, as pointed out in recent papers [23,25]. We will derive the same result from a different point of view in subsection 3.1.2, also generalizing to the case in which the symmetry that we condense to produce $\mathcal{C}[\Sigma]$ is non-invertible.

Having introduced the condensation defect, the gauging procedure on $\mathcal{T}(\alpha)[\Sigma]$ described in [24] in order to get the global fusion rule is nothing but stacking $\mathcal{C}[\Sigma]$ on $\mathcal{T}(\alpha)[\Sigma]$, up to a normalization coefficient:

$$\frac{\mathcal{T}(\alpha)[\Sigma]}{\mathbb{Z}_2} \equiv \frac{1}{\mathcal{Z}(\mathbb{Z}_2;\Sigma)}\mathcal{T}(\alpha)[\Sigma]\otimes\mathcal{C}[\Sigma]. \tag{3.1.22}$$

With this definition, using the fusion of $\mathcal{C}[\Sigma]$ with itself we see that for all the GW operators

$$\frac{1}{\mathcal{Z}(\mathbb{Z}_2;\Sigma)}\frac{\mathcal{T}(\alpha)[\Sigma]}{\mathbb{Z}_2}\otimes\mathcal{C}[\Sigma] = \frac{\mathcal{T}(\alpha)[\Sigma]}{\mathbb{Z}_2}. \tag{3.1.23}$$

The invariance of the GW with $\alpha \neq 0, 2\pi$ by stacking η is equivalent to

$$\mathcal{T}(\alpha)[\Sigma]\otimes\mathcal{C}[\Sigma] = \mathcal{Z}(\mathbb{Z}_2;\Sigma)\mathcal{T}(\alpha)[\Sigma] \;\Rightarrow\; \mathcal{T}(\alpha)[\Sigma]/\mathbb{Z}_2 = \mathcal{T}(\alpha)[\Sigma]. \tag{3.1.24}$$

The two equations above can be rephrased by introducing the *projector* $P_{\mathbb{Z}_2}$ which acts on surface operators as

$$P_{\mathbb{Z}_2} \equiv \frac{1}{\mathcal{Z}(\mathbb{Z}_2;\Sigma)}\mathcal{C}[\Sigma]. \tag{3.1.25}$$

This is a projector because $P_{\mathbb{Z}_2}^2 = P_{\mathbb{Z}_2}$, and we have $P_{\mathbb{Z}_2}\otimes\mathcal{T}(\alpha)[\Sigma] \equiv \mathcal{T}(\alpha)[\Sigma]/\mathbb{Z}_2$. Then (3.1.23) follows from $P_{\mathbb{Z}_2}^2 = P_{\mathbb{Z}_2}$, while (3.1.24) is just the statement that for the strongly simple objects $\alpha \neq 0, 2\pi$, $P_{\mathbb{Z}_2}\otimes\mathcal{T}(\alpha)[\Sigma] = \mathcal{T}(\alpha)[\Sigma]$. On the other hand, the topological operators $\mathcal{T}(0)[\Sigma]/\mathbb{Z}_2$, $\mathcal{T}(2\pi)[\Sigma]/\mathbb{Z}_2$ are further indecomposable objects[7]. This explains why there is not really a mismatch with respect to the 3d case: also in 4d, the defects associated with the short orbits come in different copies obtained by stacking the condensation defect on them. All these copies are connected by 1-morphisms, obtained by putting at the junction lines generating the dual symmetry of the condensed one[8]. This point will be generalized for $N > 2$, but the story will be more involved.

By having understood that to a $\mathbb{Z}_2$ surface operator of the ungauged theory there may correspond different defects of the gauged theory, the necessary modification

[7]The procedure of adding to the category all the defects obtained by condensations is known in category theory as *idempotent completion*, *Karoubi completion*, or *condensation completion* [135, 150]. Objects related among each other by condensation are said to be in the same Schur component, and they have non-trivial morphisms between them.

[8]In fusion higher category theory it is known that the simple objects connected by 1-morphisms are only those related among them by condensation [150].

of the fusion rules, roughly speaking, involves the choices of which of the copies of a given defect appears on the right-hand side. We can determine this by requiring consistency with the fusion with $P_{\mathbb{Z}_2}$: when the left-hand side of the fusion is $P_{\mathbb{Z}_2}$ invariant, also the right-hand side must be invariant. Whenever the local fusion does not have this property, we make it consistent by replacing the right-hand side with $P_{\mathbb{Z}_2}$(r.h.s.). This approach leads to the following modifications (here $\alpha \neq 0, \pi, 2\pi$):

$$\begin{aligned}
&\mathcal{T}(\alpha)[\Sigma] \otimes \mathcal{T}(2\pi - \alpha)[\Sigma] = 2\mathcal{T}(2\pi)[\Sigma]/\mathbb{Z}_2 + \mathcal{T}(2\alpha - 2\pi)[\Sigma] \\
&\mathcal{T}(\alpha)[\Sigma] \otimes \mathcal{T}(\alpha)[\Sigma] = 2\mathcal{T}(0)[\Sigma]/\mathbb{Z}_2 + \mathcal{T}(2\alpha)[\Sigma] \\
&\mathcal{T}(\pi)[\Sigma] \otimes \mathcal{T}(\pi)[\Sigma] = 2\mathcal{T}(0)[\Sigma]/\mathbb{Z}_2 + 2\mathcal{T}(2\pi)[\Sigma]/\mathbb{Z}_2
\end{aligned} \tag{3.1.26}$$

in agreement with the fusion rules found in [24], up to the coefficients in front of the defects on which $\mathbb{Z}_2$ is gauged. This difference boils down to a different normalization for the gauging procedure. Our choice is the one that, when generalized to $N > 2$, makes all the fusion coefficients to be positive integer numbers. This makes it possible to relate these coefficients with the total quantum dimensions of the fusion categories of 1-morphisms, made of topological defect lines at the junctions. Indeed in our case, these fusion categories are always categories of modules of finite groups, and they must have integer quantum dimensions equal to the order of the group.

$U(1)^{N-1} \rtimes S_N$ Gauge Theory

Now we construct the $U(1)^{N-1} \rtimes S_N$ gauge theory we are interested in, by gauging the 0-form symmetry S_N of the Abelian theory. The 3d analog of this theory has been discussed on the lattice in [22]. The toy example in the last subsection has several features of the general case, but there are many other interesting aspects for $N > 2$ which make the analysis more complicated. In section 3.1.3 we will show the connection between this theory and 4d $SU(N)$ YM theory. For this reason, we present the results in a way to make the comparison with the YM theory suitable.

The local operators are the S_N invariant combinations of those of the Abelian theory, namely all the symmetric polynomials in the $N-1$ variables $\mathcal{F}_{i=1,\dots,N-1}$. There are $N-1$ independent symmetric polynomials obtained by adding $\mathcal{F}_N = -\mathcal{F}_1 - \dots - \mathcal{F}_{N-1}$ and constructing the $N-1$ symmetric polynomials of degrees $2, 3, \dots, N$ in the N variables $\mathcal{F}_{i=1,\dots,N}$.

The Wilson lines are the minimal S_N invariant combinations of the Wilson lines $\mathcal{W}(\boldsymbol{n})$ of the Abelian theory $U(1)^{N-1}$. Recalling the action (3.1.9), we construct the

Wilson lines of the $U(1)^{N-1} \rtimes S_N$ theory by summing over the orbit of S_N

$$\mathcal{V}(\boldsymbol{n})[\gamma] = \sum_{\sigma \in S_N} \mathcal{W}\left(\mathfrak{S}_\sigma^\vee \cdot \boldsymbol{n}\right)[\gamma]. \tag{3.1.27}$$

Now we look at the electric GW operators and their action on the Wilson lines. These are the objects of a 2-category with non-trivial morphisms structure, coming from the dual non-invertible 2-form symmetry $\mathrm{Rep}(S_N)$ induced by the gauging of S_N. These new topological lines arise as 1-morphisms and play a crucial role in the global fusion. Since the problem is a bit intricate, we start at the local level by putting all the GW on surfaces without non-trivial 1-cycles. We will discuss the 2-category structure and the global fusions in the next subsection. The following discussion applies, *mutatis mutandis*, for the magnetic GW operators as well. The GW operators of the $U(1)^{N-1} \rtimes S_N$ theory are the minimal S_N invariant combination of GW operators of $U(1)^{N-1}$, and their construction is parallel to that for the S_N Wilson lines explained above. We normalize the GW dividing by $|H_{\boldsymbol{\alpha}}|$, where $H_{\boldsymbol{\alpha}} \subset S_N$ is the stabilizer of $\boldsymbol{\alpha}$:

$$\mathcal{T}(\boldsymbol{\alpha})[\Sigma] = \frac{1}{|H_{\boldsymbol{\alpha}}|} \sum_{\sigma \in S_N} \mathcal{D}\left(\mathfrak{S}_\sigma^\vee \cdot \boldsymbol{\alpha}\right)[\Sigma]. \tag{3.1.28}$$

These operators are topological and, with same reasons of the $N = 2$ case, with this normalization they are always defined as a sum of $\mathcal{D}$ operators without overcounting. By construction $\mathcal{T}\left(\mathfrak{S}_\sigma^\vee(\boldsymbol{\alpha})\right) = \mathcal{T}(\boldsymbol{\alpha})$, so that the parameter space of the GW operators is

$$U(1)^{N-1}/S_N. \tag{3.1.29}$$

This is a singular manifold since the S_N action on $U(1)^{N-1}$ has fixed points. It is easier to see this in the variables β_i introduced above, and we will do it shortly. For the time being we just emphasize that $U(1)^{N-1}/S_N$ coincide with the set of conjugacy classes of $SU(N)$, which labels also the (generically non-topological) GW operators of the $SU(N)$ YM theory [130,144,145]. This is a first clue of a connection between the $U(1)^{N-1} \rtimes S_N$ theory and $SU(N)$ YM theory which we explore in the next section. We will see that it is natural to identify $\mathcal{T}(\boldsymbol{\alpha})$ with the high energy limit of the GW operators of $SU(N)$ YM theory, which becomes topological in the ultraviolet and form a non-invertible symmetry, broken by the RG flow to the center 1-form symmetry $\mathbb{Z}_N^{(1)}$.

Let us look at the *local fusion* rules . From the definition (3.1.28) we get

$$\begin{aligned}\mathcal{T}(\boldsymbol{\alpha})\otimes\mathcal{T}(\boldsymbol{\beta}) &= \frac{1}{|H_{\boldsymbol{\alpha}}||H_{\boldsymbol{\beta}}|}\sum_{\sigma_1,\sigma_2\in S_N}\mathcal{D}\left(\mathfrak{S}^\vee_{\sigma_1}\cdot(\boldsymbol{\alpha}+\mathfrak{S}^\vee_{\sigma_1^{-1}\circ\sigma_2}\cdot\boldsymbol{\beta})\right)= \\ &= \frac{1}{|H_{\boldsymbol{\alpha}}||H_{\boldsymbol{\alpha}}|}\sum_{\sigma\in S_N}\sum_{\sigma_1\in S_N}\mathcal{D}\left(\mathfrak{S}^\vee_{\sigma_1}\cdot(\boldsymbol{\alpha}+\mathfrak{S}^\vee_{\sigma}\cdot\beta)\right)= \\ &= \frac{1}{|H_{\boldsymbol{\alpha}}||H_{\boldsymbol{\beta}}|}\sum_{\sigma\in S_N}|H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}|\mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right)\end{aligned} \tag{3.1.30}$$

showing that the symmetry generated by the GW operators is non-invertible. This formula is very implicit and does not make it clear the interpretation of the coefficients appearing. Indeed it is important to show that, as in the $N=2$ case, the fusion coefficients are always integer numbers, counting the total quantum dimension of the fusion category of 1-morphisms living at the junctions. We can massage the formula above as follows. Notice that for any $x\in H_{\boldsymbol{\alpha}}, y\in H_{\boldsymbol{\beta}}$ we have $\mathcal{T}\big(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\big)=\mathcal{T}\big(\boldsymbol{\alpha}+\mathfrak{S}^\vee_{x\sigma y}\cdot\boldsymbol{\beta}\big)$, and $x\sigma y$ are all the elements of the double coset $H_{\boldsymbol{\alpha}}\sigma H_{\boldsymbol{\beta}}$. Moreover S_N is the disjoint union of all the double cosets, labeled by elements of the double cosets space $H_{\boldsymbol{\alpha}}\backslash S_N/H_{\boldsymbol{\beta}}$. By choosing arbitrarily one element for each double coset the formula above can be rewritten as

$$\mathcal{T}(\boldsymbol{\alpha})\otimes\mathcal{T}(\boldsymbol{\beta})=\frac{1}{|H_{\boldsymbol{\alpha}}||H_{\boldsymbol{\beta}}|}\sum_{\sigma\in H_{\boldsymbol{\alpha}}\backslash S_N/H_{\boldsymbol{\beta}}}|H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}||H_{\boldsymbol{\alpha}}\sigma H_{\boldsymbol{\beta}}|\mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right)\ . \tag{3.1.31}$$

The order of the double coset $H_{\boldsymbol{\alpha}}\sigma H_{\boldsymbol{\beta}}$ is [151]

$$|H_{\boldsymbol{\alpha}}\sigma H_{\boldsymbol{\beta}}|=\frac{|H_{\boldsymbol{\alpha}}||H_{\boldsymbol{\beta}}|}{|H_{\boldsymbol{\alpha}}\cap\sigma H_{\boldsymbol{\beta}}\sigma^{-1}|} \tag{3.1.32}$$

from which we find

$$\mathcal{T}(\boldsymbol{\alpha})\otimes\mathcal{T}(\boldsymbol{\beta})=\sum_{\sigma\in H_{\boldsymbol{\alpha}}\backslash S_N/H_{\boldsymbol{\beta}}}f^\sigma_{\alpha\beta}\ \mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right) \tag{3.1.33}$$

where

$$f^\sigma_{\alpha\beta}=\frac{|H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}|}{|H_{\boldsymbol{\alpha}}\cap\sigma H_{\boldsymbol{\beta}}\sigma^{-1}|}\in\mathbb{Z}_+. \tag{3.1.34}$$

The fusion coefficients f^σ_{ab} appearing here are integers because $H_{\boldsymbol{\alpha}}\cap\sigma H_{\boldsymbol{\beta}}\sigma^{-1}$ is a subgroup of $H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}$. These numbers are counting the 1-morphisms living at the

junctions, up to the endomorphisms. We will shortly see how these numbers are related with the condensation defects that we need to add on right hand side to correct the fusion rules whenever the surface is topologically non-trivial.

Let us look at the Ward identities involving the GW $\mathcal{T}(\boldsymbol{\alpha})[\Sigma]$ and the Wilson lines linking once with Σ. Consider first a Wilson line $\mathcal{W}(\boldsymbol{n})$ of the $U(1)^{N-1}$ theory, and the action of $\mathcal{T}(\boldsymbol{\alpha})$ on it. By using (3.1.12) we obtain

$$\mathcal{T}(\boldsymbol{\alpha})\cdot\mathcal{W}(\boldsymbol{n})=\frac{1}{|H_{\boldsymbol{\alpha}}|}\sum_{\sigma\in S_N}\exp\left(i\sum_{i,j=1}^{N-1}\mathfrak{S}_\sigma^\vee(\alpha_i)\left(Q^{-1}\right)_{ij}n_j\right)\mathcal{W}(\boldsymbol{n})=\mathfrak{C}(\boldsymbol{\alpha},\boldsymbol{n})\mathcal{W}(\boldsymbol{n}). \tag{3.1.35}$$

To prove that the action on the Wilson lines $\mathcal{V}(\boldsymbol{n})$ is diagonal we need to show that $\mathfrak{C}(\boldsymbol{\alpha},\mathfrak{S}_\sigma^\vee\cdot\boldsymbol{n})=\mathfrak{C}(\boldsymbol{\alpha},\boldsymbol{n})$ for any $\sigma\in S_N$. We recall that from the definition of Q_{ij} we have $\mathfrak{S}_\sigma(\mathcal{F}_i)Q_{ij}\mathfrak{S}_\sigma(\mathcal{F}_j)=\mathcal{F}_iQ_{ij}\mathcal{F}_j$, implying that $\mathfrak{S}_\sigma^TQ\mathfrak{S}_\sigma=Q$. Then $Q^{-1}=\mathfrak{S}_\sigma^{-1}Q^{-1}\left(\mathfrak{S}_\sigma^T\right)^{-1}=\left(\mathfrak{S}_\sigma^\vee\right)^TQ^{-1}\mathfrak{S}_\sigma^\vee$ which implies $Q^{-1}\left(\mathfrak{S}_\sigma^\vee\right)^{-1}=\left(\mathfrak{S}_\sigma^\vee\right)^TQ^{-1}$, or $Q^{-1}\left(\mathfrak{S}_\sigma^\vee\right)=\left(\mathfrak{S}_{\sigma^{-1}}^\vee\right)^TQ^{-1}$. This gives us the desired invariance

$$\begin{aligned}\mathfrak{C}(\boldsymbol{\alpha},\mathfrak{S}_\sigma^\vee\cdot\boldsymbol{n})&=\frac{1}{|H_{\boldsymbol{\alpha}}|}\sum_{\sigma'\in S_N}\exp\left(i\boldsymbol{\alpha}^T\cdot\left(\mathfrak{S}_{\sigma'}^\vee\right)^TQ^{-1}\mathfrak{S}_\sigma^\vee\cdot\boldsymbol{n}\right)\\&=\frac{1}{|H_{\boldsymbol{\alpha}}|}\sum_{\sigma'\in S_N}\exp\left(i\boldsymbol{\alpha}^T\cdot\left(\mathfrak{S}_{\sigma^{-1}\sigma'}^\vee\right)^TQ^{-1}\cdot\boldsymbol{n}\right)=\mathfrak{C}(\boldsymbol{\alpha},\boldsymbol{n})\end{aligned} \tag{3.1.36}$$

which proves the following Ward identities

$$\begin{gathered}\mathcal{T}(\boldsymbol{\alpha})[\Sigma]\cdot\mathcal{V}(\boldsymbol{n})[\gamma]=\mathfrak{C}(\boldsymbol{\alpha},\boldsymbol{n})^{Lk(\Sigma,\gamma)}\mathcal{V}(\boldsymbol{n})[\gamma]\\\mathfrak{C}(\boldsymbol{\alpha},\boldsymbol{n})=\frac{1}{|H_{\boldsymbol{\alpha}}|}\sum_{\sigma\in S_N}\exp\left(i\sum_{i,j=1}^{N-1}\mathfrak{S}_\sigma^\vee(\alpha_i)\left(Q^{-1}\right)_{ij}n_j\right).\end{gathered} \tag{3.1.37}$$

Notice that for $N=2$ we have $\mathfrak{C}(\alpha,n)=\frac{2}{|H_\alpha|}\cos\left(n\frac{\alpha}{2}\right)$, as we obtained before.

The GW operators $\mathcal{T}(\boldsymbol{\alpha})[\Sigma]$ are the generator of a continuous non-invertible symmetry. However an interesting issue is the identification of the sub-category of group-like symmetries. Because the center of $U(1)^{N-1}\rtimes S_N$ is isomorphic to $\mathbb{Z}_N$ we already expect the discrete center symmetry $\mathbb{Z}_N^{(1)}$ to be embedded in the continuous non-invertible symmetry. In the $N=2$ case it was easy to see that $\mathbb{Z}_2$ is the maximal set of invertible unitary generators. We are going to show the same for any N, and we provide some interesting property of this center symmetry related with the action on the Wilson lines, to be compared with the non-invertible one.

This analysis is also interesting in view of the connection with $SU(N)$ YM theory in the next section, in which only the center symmetry $\mathbb{Z}_N$ remains as an unbroken symmetry along the RG flow.

From (3.1.30) we see that $T(\boldsymbol{\alpha})$ has group-like fusion only if $\boldsymbol{\alpha}$ is a fixed point of the Weyl group. The tricky point here is to properly account for the identifications on the parameters. It is convenient to work in the variables $\boldsymbol{\beta} = Q^{-1}\boldsymbol{\alpha}$ which are separately 2π periodic. S_N acts on $\boldsymbol{\alpha}$ with $\mathfrak{S}_\sigma^\vee$, thus we need to work out the action on $\boldsymbol{\beta}$. By definition

$$\beta_i = \sum_{j=1}^{N-1} \frac{-1 + N\delta_{ij}}{N} \alpha_j = \alpha_i - \frac{1}{N} \sum_{j=1}^{N-1} \alpha_j \tag{3.1.38}$$

Since the α_i transform in the $\mathfrak{S}^\vee$ representation we may write them as $\alpha_i = u_i - u_N$ where u_i transform in the N-dimensional natural representation. We then have

$$\beta_i = u_i - u_N - \frac{1}{N} \sum_{j=1}^{N-1} (u_j - u_N) = \left(1 - \frac{1}{N}\right) u_i - \frac{1}{N} \sum_{j \neq i}^{N} u_j \tag{3.1.39}$$

We now introduce an N-th variable

$$\beta_N = -\sum_{i=1}^{N-1} \beta_i = \left(1 - \frac{1}{N}\right) u_N - \frac{1}{N} \sum_{j \neq N} u_j. \tag{3.1.40}$$

Since the u_i are permuted by S_N it is clear that also the β_i, including β_N, are permuted, i.e. sit in the natural representation. By construction the sum of the β_i vanishes hence they transform in the standard $N-1$-dimensional representation. It is now easy to determine the fixed points. Clearly S_N contains a subgroup S_{N-1} which permutes the $N-1$ unconstrained β_i's, those must then be equal at the fixed point: $\beta_i = \beta$. The only remaining equation to solve is

$$\beta = -\sum_{i=1}^{N-1} \beta = -(N-1)\beta \bmod 2\pi \quad \Rightarrow \quad N\beta = 0 \bmod 2\pi \tag{3.1.41}$$

which is solved by the N-th roots of unity

$$\beta_* = \frac{2\pi k}{N} \qquad k = 0, .., N-1. \tag{3.1.42}$$

This shows that there are N fixed points. We can map them back to the original basis

$$\alpha_i = \sum_{j=1}^{N-1} Q_{ij}\beta_* = \sum_{j=1}^{N-1} (1 + \delta_{ij})\beta_* = N\beta_* = 2\pi k \qquad \forall i = 1, .., N-1. \tag{3.1.43}$$

We will denote this fixed points by $\boldsymbol{\alpha}_k$, $k = 0, ..., N-1$. The corresponding fusions are

$$\mathcal{T}(\boldsymbol{\alpha}_k)\,\mathcal{T}(\boldsymbol{\alpha}_l) = \mathcal{T}(\boldsymbol{\alpha}_{l+k}) \tag{3.1.44}$$

proving that these operators form a $\mathbb{Z}_N$ subgroup of the non-invertible symmetry. This construction shows that $\mathbb{Z}_N$ is the largest possible subcategory with group-like fusions.

Let us now see how this subgroup acts on the lines of the theory. By inserting $\boldsymbol{\alpha} = \boldsymbol{\alpha}_k$ in (3.1.37) we get

$$\begin{aligned}\mathfrak{C}(\boldsymbol{\alpha}_k, \boldsymbol{n}) &= \frac{1}{N!}\sum_{\sigma\in S_N}\exp\left(i\sum_{i,j=1}^{N-1}\mathfrak{S}^{\vee}_{\sigma}(\alpha_i)\left(Q^{-1}\right)_{ij}n_j\right) = \\ &= \frac{1}{N!}\sum_{\sigma\in S_N}\exp\left(i\sum_{i=1}^{N-1}\mathfrak{S}_{\sigma}(\beta_i)n_i\right) = \exp\left(\frac{2\pi i k}{N}\sum_{i=1}^{N-1}n_i\right).\end{aligned} \tag{3.1.45}$$

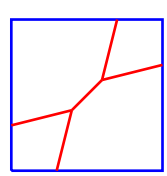
This shows that when we restrict to the $\mathbb{Z}_N$ subgroup of the non-invertible symmetry, the action on the Wilson line $\mathcal{V}(\boldsymbol{n})$ becomes group-like with a phase which is an N-root of unity with charge

$$|\boldsymbol{n}| := \sum_{i=1}^{N-1} n_i. \tag{3.1.46}$$

Higher Condensation and Global Fusion

When the GW operators are supported on surfaces Σ with non-trivial topology we are able to probe the full structure of the 2-category symmetry. An important role is played by the 1-morphisms, which are non-trivial due to the quantum 2-form symmetry arsing by the gauging of S_N, implying that there are indecomposable objects with non-trivial endomorphisms. For $N > 2$ the quantum symmetry is a discrete non-invertible symmetry $\mathrm{Rep}\,(S_N)$ and the analysis is more involved with respect to the $O(2)$ gauge theory. The higher condensation defect $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ must be constructed by gauging non-invertible lines on a surface. There is a well established definition of gauging in fusion categories described in [13], and fortunately for any discrete group G the fusion category $\mathrm{Rep}(G)$ in 2d can be fully gauged, thus defining the following condensation defect on Σ:

$$\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma] \quad = \tag{3.1.47}$$



Here the red line is the Frobenius algebra object of $\mathrm{Rep}\,(S_N)$ corresponding to the *regular* representation[9], and by $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ we mean a fine enough mesh of this object on Σ. On the defect there is a symmetry S_N [13]: the fusion category of 1-endomorphisms is the group S_N. Notice that the lines generating this symmetry are stacked on the defect and do not exist in the bulk. Below we will give an equivalent description of $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$, which turns out to be useful to compute the fusion with itself, allowing us to find the non-Abelian generalization of the fusions found in [23].

As in the $N = 2$ case, we determine the global fusion of $\mathcal{T}(\boldsymbol{\alpha})$ and $\mathcal{T}(\boldsymbol{\beta})$ by requiring consistency with the stacking of Wilson lines which are absorbed by $\mathcal{T}(\boldsymbol{\alpha})$ and $\mathcal{T}(\boldsymbol{\beta})$. For $N = 2$ this corresponded to the use of the projector $P_{\mathbb{Z}_2}$, and it was enough because $S_2 = \mathbb{Z}_2$ does not have non-trivial proper subgroups: $\alpha \in U(1)/\mathbb{Z}_2$ is either fixed or invariant under $\mathbb{Z}_2$. For $N > 2$ there are values $\boldsymbol{\alpha} \in U(1)^{N-1}$ which are stabilized by a non-trivial proper subgroup $H_{\boldsymbol{\alpha}} \subset S_N$. Then the fusion category of 1-endomorphisms $\mathcal{T}(\boldsymbol{\alpha}) \to \mathcal{T}(\boldsymbol{\alpha})$ is isomorphic to $\mathrm{Rep}\,(H_{\boldsymbol{\alpha}})$, meaning that the $H_{\boldsymbol{\alpha}}$ Wilson lines are not *absorbed* by $\mathcal{T}(\boldsymbol{\alpha})$ and can live on it as non-trivial lines. On the other hand there are S_N Wilson lines which are not $H_{\boldsymbol{\alpha}}$ Wilson lines, and these are absorbed by $\mathcal{T}(\boldsymbol{\alpha})$. This implies that the local fusion rules require modifications which cannot be seen by simply applying the projector $P_{\mathrm{Rep}(S_N)}$ corresponding to $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$. Indeed this projector condenses the full symmetry living on the defect:

$$P_{\mathrm{Rep}(S_N)} \otimes \mathcal{T}\,(\boldsymbol{\alpha})\,[\Sigma] = \mathcal{T}\,(\boldsymbol{\alpha})\,[\Sigma]/\mathrm{Rep}\,(H_{\boldsymbol{\alpha}})\,. \tag{3.1.48}$$

When $\mathcal{T}(\boldsymbol{\alpha})$ is a strongly simple object, namely $H_{\boldsymbol{\alpha}} = 1$, by using $P_{\mathrm{Rep}(S_N)}$ we can determine the correct fusion rules. On the other hand if $H_{\boldsymbol{\alpha}}$ is a non-trivial proper subgroup, using only $P_{\mathrm{Rep}(S_N)}$ we would miss the global fusion rules with $\mathcal{T}\,(\boldsymbol{\alpha})\,[\Sigma]$ appearing on the left hand side. We then need to construct the projector containing the maximal set of lines absorbed by $\mathcal{T}(\boldsymbol{\alpha})$. Before clarifying what does this mean and giving a general construction, we need to introduce the promised alternative definition of $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$.

The idea is that since the $\mathrm{Rep}(S_N)$ symmetry is obtained by the gauging of S_N, condensing it on Σ is the same as doing a step back before gauging S_N, removing Σ from the space-time manifold $\mathcal{M}$ and then gauging S_N in $\mathcal{M} - \Sigma$. We do so imposing Dirichlet boundary conditions $a|_\Sigma = 0$ on the surface for the S_N gauge field a. This construction produces the $U(1)^{N-1} \rtimes S_N$ theory with the insertion of

[9]For Abelian group the regular representation is just the sum of all the irreducible representations, and this generalized gauging procedure coincide with the standard one.

a condensation defect $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$. Notice that this picture is consistent with the presence of a dual S_N symmetry living on $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$: a co-dimension one defect of the 0-form global symmetry S_N in the $U(1)^{N-1}$ theory can intersect Σ on a line wrapping a cycle, then this defect is made transparent outside Σ by the gauging of S_N in $\mathcal{M}-\Sigma$, while the line on Σ remains as the generator of a 0-form symmetry on the condensation defect.

This way of presenting $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ may seem abstract, but it does not rely on the concept of gauging a Frobenius algebra object, and turns out to be useful to determine the fusion $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]\otimes\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$. For convenience we denote the defect constructed in this way by $\widetilde{\mathcal{C}}$, even if $\widetilde{\mathcal{C}}=\mathcal{C}$, to distinguish when we are thinking about the condensation defect in the standard or in the latter presentation. To compute $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]\otimes\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ the trick is to think one of the two supported on Σ and defined in the presentation $\widetilde{\mathcal{C}}$, while the other defined in the standard way $\mathcal{C}$ with the condensation of the algebra object $\mathcal{A}_{\mathrm{Rep}(S_N)}$ on a surface $\Sigma'=\Sigma+\delta\Sigma$, which lies inside the mesh of S_N defects in $\mathcal{M}-\Sigma$. When we send the displacement $\delta\Sigma$ to zero the mesh of $\mathcal{A}_{\mathrm{Rep}(S_N)}$ defining $\mathcal{C}$ enters into the "hole" Σ defining $\widetilde{\mathcal{C}}$ (see figure 3.1). The result is again $\widetilde{\mathcal{C}}_{\mathrm{Rep}(S_N)}[\Sigma]$ but with the hole Σ filled with a mesh of algebra objects implementing the higher gauging of $\mathrm{Rep}(S_N)$. Because of the Dirichlet boundary conditions this condensation does not speak with the S_N gauge field in the bulk, and it simply computes the partition function of the 2d pure $\mathrm{Rep}(S_N)$ gauge theory on Σ, denoted by $\mathcal{Z}\left(\mathrm{Rep}\left(S_N\right);\Sigma\right)$. Since $\widetilde{\mathcal{C}}=\mathcal{C}$ we get

$$\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]\otimes\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]=\mathcal{Z}\left(\mathrm{Rep}\left(S_N\right);\Sigma\right)\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma] \tag{3.1.49}$$

which can be thought of as a non-Abelian generalization of the results in [23]. The pure $\mathrm{Rep}(S_N)$ gauge theory is a theory with non-abelian 0-form symmetry S_N, which can be described explicitly in terms of commutative Frobenius algebras. In appendix 3.3.2 we provide some detail on this construction.

We check the correctness of this abstract procedure by repeating it in an Abelian case where S_N is replaced by $\mathbb{Z}_N$. This has the advantage that the dual symmetry is invertible and its higher gauging on $\Sigma'\subset\mathcal{M}-\Sigma$ can be done by simply coupling the defect to a background gauge field $b\in H^1\left(\Sigma',\mathbb{Z}_N\right)$ and summing over it. The coupling of b to the $\mathbb{Z}_N$ gauge field $a\in H^1(\mathcal{M}-\Sigma,\mathbb{Z}_N)$ is the standard one:

$$\exp\left(\int_{\Sigma'}a\cup b\right)\mathcal{Z}(\mathbb{Z}_N;\Sigma'). \tag{3.1.50}$$

By summing over b one gets the condensation defect in the standard presentation $\mathcal{C}$

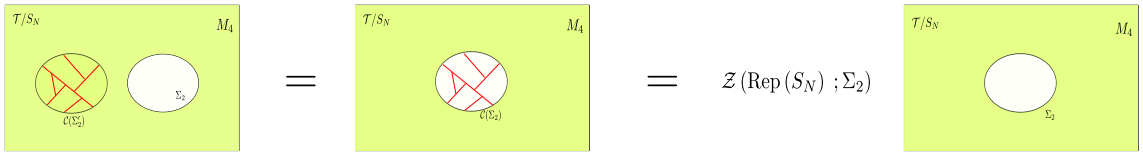


Figure 3.1: A pictorial representation of the fusion of the condensation defects. The green region is the one with a gauged S_N symmetry while the white one is the one in which such a symmetry is still global. The red lines represent a fine enough mesh of the algebra object representing the gauging of $\mathrm{Rep}(S_N)$ in the 2-dimensional surface Σ_2 or Σ_2'.

of [23]. On the other hand our alternative definition $\widetilde{\mathcal{C}}$ is formally the same in the Abelian and in the non-Abelian case, since it does not use the notion of gauging non-invertible symmetries. The insertion of $\mathcal{C}_{\mathbb{Z}_N}[\Sigma]\otimes\mathcal{C}_{\mathbb{Z}_N}[\Sigma]$ in a correlation function can be replaced by (3.1.50) in the same correlation function, computed in the theory where $\mathbb{Z}_N$ is gauged in $\mathcal{M}-\Sigma$ with Dirichlet boundary condition $a|_\Sigma = 0$, and then take the limit $\Sigma' \to \Sigma$. In this limit the exponential factor disappear because of the Dirichlet boundary condition. We remain with the partition function of the 2d $\mathbb{Z}_N$ gauge theory on Σ multiplying the correlation function computed in the theory with dynamical gauge field $a \in H^1(\mathcal{M}-\Sigma, \mathbb{Z}_N)$. This means

$$\mathcal{C}_{\mathbb{Z}_N}[\Sigma] \otimes \mathcal{C}_{\mathbb{Z}_N}[\Sigma] = \mathcal{Z}\left(\mathbb{Z}_N;\Sigma\right)\mathcal{C}_{\mathbb{Z}_N}[\Sigma] \tag{3.1.51}$$

which is the same fusion of [23].

The non-Abelian condensation defects we defined allow to construct the projector $P_{\mathrm{Rep}(S_N)}$ satisfying $P^2_{\mathrm{Rep}(S_N)} = P_{\mathrm{Rep}(S_N)}$. By using it we obtain the global fusions of the strongly simple GW operators, namely those with trivial stabilizers $H_{\boldsymbol{\alpha}} = H_{\boldsymbol{\beta}} = 1$

$$\begin{aligned}\mathcal{T}(\boldsymbol{\alpha})[\Sigma]\otimes\mathcal{T}(\boldsymbol{\beta})[\Sigma] &= \sum_{\sigma\in S_N}|H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}|P_{\mathrm{Rep}(S_N)}\otimes\mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right)[\Sigma] = \\ &= \sum_{\sigma\in S_N}|H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}|\frac{\mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right)[\Sigma]}{\mathrm{Rep}\left(H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}\right)}\end{aligned} \tag{3.1.52}$$

where we used that the projector on the right hand side implements the gauging of the full symmetry $\mathrm{Rep}\left(H_{\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}}\right)$ living on the GW $\mathcal{T}\left(\boldsymbol{\alpha}+\mathfrak{S}^\vee_\sigma\cdot\boldsymbol{\beta}\right)$.

As advertised before, when $H_{\boldsymbol{\alpha}}$ is a non-trivial proper subgroup of S_N we need the maximal projector absorbed by $\mathcal{T}(\boldsymbol{\alpha})$. A priory it is not obvious at all how to

define this projector. If $H_{\boldsymbol{\alpha}}$ is a normal subgroup, then $H_{\boldsymbol{\alpha}}^{\perp} = S_N/H_{\boldsymbol{\alpha}}$ is a group, and intuitively we need a projector $P_{\mathrm{Rep}(H_{\boldsymbol{\alpha}}^{\perp})}$ obtained from the condensation of $\mathrm{Rep}(H_{\boldsymbol{\alpha}}^{\perp})$ Wilson lines. However it is not obvious that this an allowed gauging in the category $\mathrm{Rep}(S_N)$ of bulk lines, and more seriously we would not know how to proceed when the stabilizer is not a normal subgroup[10]. Our definition of the relevant condensation defect absorbed by $\mathcal{T}(\boldsymbol{\alpha})$ is as follows. We start from the maximal condensation defect $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ and we recall that there is a quantum symmetry S_N living on it, which is very explicit in our presentation $\widetilde{\mathcal{C}}$ of this defect. Then for any subgroup $H_{\boldsymbol{\alpha}} \subset S_N$ we can gauge this smaller symmetry on the defect, which corresponds to remove the $\mathrm{Rep}(H_{\boldsymbol{\alpha}})$ Wilson line from the condensate, and generate a *new* higher-condensation defect which, with an abuse of notation, we denote with $\mathcal{C}_{\mathrm{Rep}(H_{\alpha}^{\perp})}[\Sigma]$. Notice that this construction matches nicely with the known fact that the Frobenius algebra objects of $\mathrm{Rep}(G)$ are in one-to-one correspondence with the subgroups of G [13].

From this defect we can construct the projector $P_{\mathrm{Rep}(H_{\boldsymbol{\alpha}}^{\perp})}$ for any $\boldsymbol{\alpha} \in U(1)^{N-1}/S_N$, and this is the maximal projector absorbed by $\mathcal{T}(\boldsymbol{\alpha})$. When we fuse two of these higher condensation defects for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we are essentially removing from the condensate $\mathcal{C}_{\mathrm{Rep}(S_N)}[\Sigma]$ all the lines which are lines of both $H_{\boldsymbol{\alpha}}$ and $H_{\boldsymbol{\beta}}$, while keeping all the others. This leads to the following algebra of projectors

$$P_{\mathrm{Rep}(H_{\boldsymbol{\alpha}}^{\perp})} \otimes P_{\mathrm{Rep}(H_{\boldsymbol{\beta}}^{\perp})} = P_{\mathrm{Rep}\left((H_{\boldsymbol{\alpha}} \cap H_{\boldsymbol{\beta}})^{\perp}\right)} \tag{3.1.53}$$

We can use this knowledge to compute the most general global fusion rules, by starting from the local one (3.1.33) and apply the projectors $P_{\mathrm{Rep}(H_{\boldsymbol{\alpha}}^{\perp})}$ and $P_{\mathrm{Rep}(H_{\boldsymbol{\beta}}^{\perp})}$ to both sides of the equation, which are absorbed by the left-hand side:

$$\mathcal{T}(\boldsymbol{\alpha})[\Sigma] \otimes \mathcal{T}(\boldsymbol{\beta})[\Sigma] = \sum_{\sigma \in H_{\boldsymbol{\alpha}} \backslash S_N / H_{\boldsymbol{\beta}}} f_{\alpha\beta}^{\sigma}\, P_{\mathrm{Rep}\left((H_{\boldsymbol{\alpha}} \cap H_{\boldsymbol{\beta}})^{\perp}\right)} \otimes \mathcal{T}\left(\boldsymbol{\alpha} + \mathfrak{S}_{\sigma}^{\vee} \cdot \boldsymbol{\beta}\right)[\Sigma] \quad . \tag{3.1.54}$$

It is a trivial exercise to check that this formula agrees with the global fusion of the $O(2)$ gauge theory. The general fusion rule above explains the meaning of the integer fusion coefficients coefficients f_{ab}^{σ}. These numbers are greater than one whenever they multiply an operator dressed with some condensation defect, and the number coincide with the quantum dimension of the algebra object condensed on the defect. This fact has a simple interpretation. The condensation produces

[10] For $N \geq 5$ there is only one non-trivial proper normal subgroup of S_N, namely the alternating group A_N.

a dual symmetry on the defect, and a junction among the two fused defects and any one of those appearing on the right can be constructed using any of the lines generating this dual symmetry, whose total quantum dimension is equal to that of the condensed algebra object.

We can look back at the $N = 2$ case and check that this discussion applies. A richer example is the case $N = 3$, and it is worth to discuss it here. Given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in U(1)^2/S_3$ there are four possible stabilizers:

- $\alpha_1 = \alpha_2 = 2\pi k$, $k = 0, 1, 2$ is fixed by the full group $H_{\boldsymbol{\alpha}} = S_N$.
- For $\alpha_1 = 2\pi k_1$, $\alpha_2 = 2\pi k_2$, $k_1, k_2 = 0, 1, 2$, $k_1 \neq k_2$ the stabilizer is $H_{\boldsymbol{\alpha}} = \mathbb{Z}_3$.
- For $\alpha_1 = \alpha_2 =: \alpha$, but $\alpha/2\pi \notin \mathbb{Z}$ the stabilizer is $H_{\boldsymbol{\alpha}} = \mathbb{Z}_2$
- In all the other cases the stabilizer is trivial.

When $\boldsymbol{\alpha} = (\alpha, \alpha)$, $\boldsymbol{\beta} = (\beta, \beta)$ are both stabilized by $\mathbb{Z}_2$, by using (3.1.33) the local fusion is[11]

$$\mathcal{T}(\alpha, \alpha) \otimes \mathcal{T}(\beta, \beta) = \mathcal{T}(\alpha, \alpha - \beta) + \frac{|H_{\boldsymbol{\alpha}+\boldsymbol{\beta}}|}{2} \mathcal{T}(\alpha + \beta, \alpha + \beta) \tag{3.1.55}$$

which is modified, at the global level, by gauging $\mathbb{Z}_3$. Since we are assuming $\alpha, \beta \notin 2\pi\mathbb{Z}$ the first term cannot be stabilized by $\mathbb{Z}_3$. Then the only non-trivial modification is when $\beta = 2\pi k - \alpha$, $\alpha/2\pi \notin \mathbb{Z}$, in which case the last term is central, and we get

$$\mathcal{T}(\alpha, \alpha) \otimes \mathcal{T}(2\pi k - \alpha, 2\pi k - \alpha) = \mathcal{T}(\alpha, 2\alpha - 2\pi k) + 3 \frac{\mathcal{T}(2\pi k, 2\pi k)}{\mathbb{Z}_3} \ . \tag{3.1.56}$$

Notice that, even if the last GW in the local fusion is a generator of the center which stabilizes the full S_3, the condensation defect dressing it in the global fusion is the one associated with $\mathbb{Z}_3$. There is a quantum $\mathbb{Z}_3$ symmetry on this defect, and the coefficient 3 is counting precisely its total quantum dimension.

[11] The fact that only two terms appear in the right hand side follows the Cauchy-Frobenius lemma

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1, h_2 \in H_2} |G_{h_1, h_2}|$$

where $G_{h_1,h_2} = \{g \in G \mid h_1 g h_2 = g\}$. Indeed $\mathbb{Z}_2 = \{1, s\} \subset S_3$, where $s = (213)$, and it is easy to see that $G_{11} = S_3$, $G_{1x} = G_{x1} = \emptyset$, $G_{xx} = \mathbb{Z}_2$.

Now we fuse two GW whose parameters $\boldsymbol{\alpha}_{a_1a_2} = (2\pi a_1, 2\pi a_2), \boldsymbol{\beta}_{b_1b_2} = (2\pi b_1, 2\pi b_2)$ have both stabilizer $\mathbb{Z}_3$. The local fusion is

$$\mathcal{T}(\boldsymbol{\alpha}_{a_1a_2})\otimes\mathcal{T}(\boldsymbol{\beta}_{b_1b_2}) = \frac{|H_{\boldsymbol{\alpha}_{a_1a_2}+\boldsymbol{\beta}_{b_1b_2}}|}{3}\mathcal{T}(\boldsymbol{\alpha}_{a_1a_2}+\boldsymbol{\beta}_{b_1b_2})+\frac{|H_{\boldsymbol{\alpha}_{a_1a_2}+\boldsymbol{\beta}_{b_2b_1}}|}{3}\mathcal{T}(\boldsymbol{\alpha}_{a_1a_2}+\boldsymbol{\beta}_{b_2b_1}) \tag{3.1.57}$$

which should be modified by applying $P_{\mathbb{Z}_2}$. Notice that it is impossible that both terms on the right hand side are stabilized by $\mathbb{Z}_2$, otherwise $a_1 = a_2, b_1 = b_2$. When the second term is stabilized by $\mathbb{Z}_2$ we get the global fusion rule

$$\mathcal{T}(2\pi a_1, 2\pi a_2) \otimes \mathcal{T}\left(2\pi b_1, 2\pi(a_2 + b_1 - a_1)\right) = \mathcal{T}(2\pi(a_1 + b_1), 2\pi(2a_2 + b_1 - a_1))+ \\ +2\frac{\mathcal{T}(2\pi(a_2 + b_1), 2\pi(a_2 + b_1))}{\mathbb{Z}_2} \tag{3.1.58}$$

The coefficient 2 in the last term has the same interpretation of the 3 in previous case, as the number of possible junctions.

Because $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, this example can be analyzed also with the technique of gauging sequentially $\mathbb{Z}_3$ and then $\mathbb{Z}_2$ as in [24], and one can check that we reproduce the same global fusions. On the other hand our method is more general since it does not assume that the group to be gauged is a semidirect product of Abelian factor, which is not true for S_N, $N \geq 5$. Nevertheless the computation is incredibly harder for $N > 3$, even if it is algorithmic.

We conclude this subsection with a general remark. The method we described to derive the global fusion rules in the $U(1)^{N-1} \rtimes S_N$ appears to be general in higher category symmetries. The difference between local and global fusion arises in this context because also indecomposable objects can have a non-trivial category of 1-endomorphisms, and one needs to require consistency of the fusions with the condensation of these symmetries generated by 1-endomorphisms. This takes the form of various projections obtained by fusing with the higher condensation defects introduced in [23]. As we have discussed, the determination of the full set of higher condensation defects of a given theory might be non-trivial. Nevertheless we propose that, at least for non-invertible symmetries induced by gauging, the only modification of the local fusions required when the defects have non-trivial topology are those coming from these consistency conditions. As a consequence, finding all the higher condensation defects of a theory allows to fully determine the global fusions. This proposal is motivated by the observation that the only difference arising when the defect is topologically non-trivial can be in the presence of lower dimensional

defects wrapping cycles. By definition, the higher condensation defects are precisely those which are made by lower dimensional objects[12]. Notice, for example, that the way in which the authors of [120] determined the fusion rules of the *duality defects*, after being aware of condensation defects, can be interpret as our method.

3.1.3 The Ultraviolet Limit of 4d Yang-Mills Theory

In this section we connect the $U(1)^{N-1} \rtimes S_N$ gauge theories to $SU(N)$ YM theories showing that *some* properties of the UV limit of the latter are nicely captured by the former. We will argue that a convenient way to analyze this relation boils down to choosing a particular gauge fixing, originally introduced in [141], in which the connection with the semi-Abelian theory is more manifest. We will show that all gauge invariant *scalar* operators of $SU(N)$ YM theory are matched by operators in the semi-Abelian theory. The relation we find implies that the global symmetries of the high energy YM theory are much larger than those of the full theory as they include much more topological operators which generate a non-invertible symmetry[13].

Naively one might say that the UV limit of $SU(N)$ YM theory is sharply different from the $U(1)^{N-1} \rtimes S_N$, since the latter is locally a theory of $N-1$ photons, while the former seems to be a theory of N^2-1 free gluons [14]. However in a non-Abelian theory there are much more gauge transformations than in a collection of Abelian ones, as for instance gluons can be rotated into each other, thus a UV description in terms N^2-1 photons is misleading as it does not account for all the redundancies. In order to introduce the general idea, it is useful to look at a toy example. Consider the matrix model of $N \times N$ hermitian matrices. Here it is clear that, by diagonalizing the matrices, we can reduce the initial N^2 degrees of freedom to only N at the price of introducing a potential among them related to the Vandermonde determinant[15] (for a review see e.g. [152]). This determinant is crucial in order to match all the

[12]It is worth noting that this procedure is reminiscent of the idempotent completion in higher categories introduced in [135]. It would be very interesting to draw a precise connection with the recent known mathematical results

[13]The same conclusion was argued for $SU(2)$ YM theory in [146] by directly doing the $g \to 0$ limit.

[14]Indeed the number of degrees of freedom of the two theories do not match; therefore also the would-be map between the gauge invariant operators of the two theories cannot be one-to-one.

[15]Note that in this diagonal gauge the Weyl group which permutes the eigenvalues is still a gauge symmetry.

calculable quantities of the original theory[16]. However the gauge invariant content of the theory is entirely captured by the N degrees of freedom and, in the free limit, the measure induced potential are turned off and become irrelevant to study *kinematical* properties of the original theory.

Inspired by this example we can argue that the UV limit of $SU(N)$ YM theory is related to the semi-Abelian gauge theory $U(1)^{N-1} \rtimes S_N$. In particular, even if the dynamics of the YM theory at arbitrary small coupling is *not* captured by the semi-Abelian theory, the equations of motion of the latter, together with the symmetry structure, carry over to the UV limit of the YM theory. In the next subsection we make this argument more precise. Then the other subsections are devoted to show that a subset of all the gauge invariant operators of the YM theory can be mapped to the ones of the semi-Abelian theory. Finally we will discuss how the possible global structures of the YM theory are captured by the free theory.

Yang-Mills theory

Consider the 4d $SU(N)$ YM theory

$$Z = \int DA e^{-\frac{1}{2g^2}\int d^4x Tr F\wedge *F} \tag{3.1.59}$$

where $F = dA + A\wedge A$ is an hermitian and traceless matrix transforming covariantly under $SU(N)$ gauge transformations $F \to \Omega^{-1}F\Omega$. We use the letters $i, j, ...$ for the generators h_i in the Cartan subalgebra, while $a, b, ...$ for the off-diagonal ones T_a. We use the non-Abelian gauge redundancy to choose a gauge in which the Lagrangian density is diagonal [17]. In this gauge the action of the theory becomes

$$S = \frac{1}{2g^2}\int d^4x \sum_{i,j=1}^{N-1} K_{ij} F_i \wedge *F_j \tag{3.1.60}$$

and all the complicated dynamics is then captured by the induced gauge fixing determinant.

[16]For instance the free energy computed directly from the matrix model is proportional to g^{N^2}, signal of the fact that the theory contains N^2 degrees of freedom. The theory described by the eigenvalues gives the same result only if the Vandermonde is taken into account.

[17]The idea to Abelianize a non-Abelian gauge theory using particular gauge fixing was introduced in [141]. In particular this method was made rigorous and it was used in [153] in order to solve G YM theories in 2d where these theories are quasi-topological and solvable.

The $(N-1)\times(N-1)$ matrix K_{ij} is the Killing form restricted to the Cartan subalgebra. It is useful to choose the Chevalley basis in which

$$h_i = 2\frac{\alpha_i^I H^I}{|\alpha_i|^2} \qquad K(H^I, H^J) = \delta^{I,J}. \tag{3.1.61}$$

so that (for simply-laced Lie algebras) the Killing form restricted to the Cartan subalgebra is the Cartan matrix

$$K_{ij} = \frac{2}{|\alpha_i|^2}\frac{2\alpha_i^I\alpha_j^J K(H^I, H^J)}{|\alpha_j|^2} = 2\frac{\alpha_i\cdot\alpha_j}{|\alpha_j|^2} = A_{ij}. \tag{3.1.62}$$

The residual gauge freedom is now described by the semi-Abelian gauge group $U(1)^{N-1}\rtimes S_N$, where $U(1)^{N-1}$ is the maximal torus of $SU(N)$ and S_N is the Weyl group. Its gauging reflects the freedom of defining the N eigenvalues F_i in different orders.

As opposed to the simpler case of the matrix model, this gauge fixing condition is now more complicated. In what follows we sketch how this procedure should be done, even if in order to analyze the kinematical properties of the UV theory (such as the symmetries) all the technicalities turn out not to be crucial.
In the YM path integral we integrate over the connections, not over the field strengths which are the objects transforming covariantly. However we can still do similar considerations. The gauge fixing condition which we want to impose is

$$(F\wedge *F)^a = 0. \tag{3.1.63}$$

Usually in the Faddeev-Popov procedure we do not resolve the δ-function corresponding to the gauge fixing, but instead we rewrite it as a gauge fixing term in the action. In this case however it is convenient to resolve the δ-function, so that the constraint is imposed directly in the action. This is because we do not want to preserve the full gauge covariant form of the action, but only the $U(1)^{N-1}$ one. Note that the connections A^a are not necessarily zero. There is an induced Faddeev-Popov determinant so that the gauge-fixed path integral looks like

$$Z = \int DA^i\ DA^a e^{-\frac{1}{2}\sum_{i,j}F_iF_jK_{ij}}\Delta(A^i, A^a). \tag{3.1.64}$$

In writing this we used the normalization in which the field strength is $F = dA + gA\wedge A$ so that in the $g=0$ limit we just get the Abelian kinetic term for the A^i connections[18]. When we write $\Delta(A^i, A^a) = e^{V(A^i, A^a)}$ and we integrate over A^a,

[18]Note that in this case also in $\Delta(A^i, A^a)$ there is a dependence on g.

we induce complicated non-local interactions among the Cartan gauge connections A^i. These interactions play the same role as the Vendermonde determinant, and in particular it will be crucial in order to match all the complicated dynamics of the non Abelian theory, precisely as in the matrix model. However these interactions are weighted by the gauge coupling g, and in the high energy limit are turned off. Therefore, for what concerns the analysis of topological operators and symmetries we can safely drop the non-local interactions at high energy and study the remaining theory, which is precisely the $U(1)^{N-1} \rtimes S_N$ gauge theory.

We want now to discuss an additional issue which concerns the global properties of the non Abelian theory. Indeed in the $SU(N)$ theory we have different instanton sectors labeled by the third homotopy group of the gauge group. When we fix the gauge we loose this information since the residual gauge symmetry has no non-trivial topological sectors. This means that this gauge fixing works only locally and it must be modified if we want to account the global properties of the theory [154]. However, in the $g = 0$ limit also in the $SU(N)$ theory all the non-trivial instanton sectors decouple and the lack of non-trivial topological sectors is no longer an issue.

The argument above suggests to look for a mapping between the gauge invariant operator of the $SU(N)$ YM theory and those of the semi-Abelian theory, in the following three subsections we show the precise correspondence. However not all operators of the non-Abelian theory have a natural map to ones of the free theory. Indeed gauge invariant operators carrying non trivial spin cannot be diagonalize using the gauge redundancy: since they have more components which do not commute between each-other it is not possible to diagonalize the entire Lorentz tensor in a covariant way. We want to stress that the matching of the gauge invariant scalar operators is independent of the gauge fixing procedure since it comes just from the freedom of applying gauge rotations on gauge invariant quantities. Instead, the power of these considerations is that, once we understand the map of operators, we will be able to extract some information about the UV limit of YM theory knowing the properties of the semi-Abelian one already discussed in the previous section. In the YM theory the extended operators that we are interested in are of two kinds:

- **Line operators.** These are the simplest kind of extended operators, supported on lines. In YM theory they are the Wilson operators

$$W_{\mathcal{R}}[\gamma] = Tr_{\mathcal{R}} P \exp\left(i \oint_{\gamma} A\right) \tag{3.1.65}$$

  labeled by an irreducible representation of the gauge group, as well as the 't

Hooft lines, defined as disorder operators [155] . These are also labeled by representations [156], but for gauge group $SU(N)$ only those with $N-$ality zero are *genuine line operators*, while the others require topological surfaces attached to them [62].

- **Surface operators.** These operators are supported on surfaces, which in 4d can link with lines $Lk(\Sigma, \gamma) \in \mathbb{Z}$. For this reason there is a crucial interplay between line and surface operators. When a surface operator is topological it is a generator of a 1-form symmetry, possibly non-invertible, and the charged objects are line operators. In 4d gauge theories the surface operators are known as GW operators [130, 144, 145]. They are labeled by the conjugacy classes of G parametrized by $U(1)^r/W_G$, but only those corresponding to the center $Z(G) \subset G$ are topological in the full theory, generating the center symmetry.

In the next three subsections we will discuss the matching of the extended operators between the $U(1)^{N-1} \rtimes S_N$ gauge theory and the UV effective description of YM theory. In the beginning we also clarify the relation between the various Lagrangians, in the various basis.

Relation between basis

As explained above, in the $g_{YM} \to 0$ limit we can reduce to the action

$$S = \frac{1}{2} \int d^4x \; K_{ij} F_i \wedge *F_j \tag{3.1.66}$$

where $K_{ij} = K(h_i, h_j)$ is the block of the Killing form relative to the Cartan subalgebra. This is an Abelian gauge theory with gauge group $U(1)^{N-1}$. As pointed out in section (3.1.2) the precise definition of this theory requires the choice of the global structure, which can be fixed declaring which of the transformations $A_i \to A_i + \lambda_i$ are gauge transformations, or, equivalently, specifying the spectrum of line operators. However here the choice is dictated by the global structure of the YM theory. Indeed in the Chevalley basis the eigenvalues of h_i on the weight states of any representations are the Dynkin labels, which must be integer numbers. These are precisely the charges of the Abelian Wilson lines written for the connection A_i in this basis. Therefore the global structure of the $U(1)^{N-1}$ theory we need is that in which, when the Killing form is the Cartan matrix, all the Wilson lines have integer charges.

At the global level this Abelian theory cannot be the correct UV description of YM theory, since it has a S_N 0-form global symmetry, which is instead gauged in

YM theory. The action of the permutation group on the $N-1$ field strengths is more evident in the basis defined by the quadratic form $Q_{ij}^{(N-1)}$ defined in (3.1.8), which we dub *symmetric basis*. Therefore it is worth to pause a bit to discuss the relation between the two basis of interests. We look for a matrix L such that

$$A_i = L_{ij}\mathcal{A}_j \quad \Rightarrow \quad L^T A^{(N-1)} L = Q^{(N-1)} \tag{3.1.67}$$

Where $A^{(N-1)}$ is the Cartan matrix of $\mathfrak{su}(N)$. We solve this constraint using the Cholesky decomposition for both $A^{(N-1)}$ and $Q^{(N-1)}$, namely $A^{(N-1)} = H^T H$, $Q^{(N-1)} = G^T G$, where H, G are upper triangular matrices. Then L is uniquely defined as $L = H^{-1}G$. It turns out that L is upper triangular, with all non-zero components equal to 1:

$$L_{ij} = \begin{cases} 1 & \text{if } \ i \leq j \\ 0 & \text{if } \ i > j \end{cases} \qquad \Rightarrow \quad A_i = \sum_{j \geq i} \mathcal{A}_j \tag{3.1.68}$$

Notice that $\det(L) = 1$, so $L \in GL_{N-1}(\mathbb{Z})$ is an automorphism of the lattice $\mathbb{Z}^{N-1}$. Thus with the global structure dictated by $SU(N)$ YM theory the charges of the Wilson lines are integers in both the Chevalley and symmetric basis.

Line Operators

Now we discuss the Wilson line operators of the $SU(N)$ YM theory

$$W_{\mathcal{R}} = Tr_{\mathcal{R}} \mathcal{P} \exp\left(i \oint_\gamma A \right) \tag{3.1.69}$$

labeled by an irreducible representation $\mathcal{R}$ of the gauge group $SU(N)$. In the full theory they are charged under the $\mathbb{Z}_N$ 1-form symmetry generated by the GW operators corresponding to conjugacy classes in the center $\mathbb{Z}_N \subset SU(N)$, and their charge is the $N-$ality of the representation $\mathcal{R}$.

We want to analyze the UV limit of the Wilson lines. Following the general philosophy that we have described, all the gauge covariant observables, without Lorentz indices,can be mapped to the Cartan torus by performing suitable gauge transformations. The holonomy

$$hol_\gamma[A] = \mathcal{P} \exp\left(i \oint_\gamma A \right)$$

indeed transform covariantly under $SU(N)$ gauge transformations. Its trace on an irreducible representation $\mathcal{R}$ gives the Wilson line $W_{\mathcal{R}}[\gamma]$. By decomposing the

representation in weight states $|\lambda\rangle$ labeled by their Dynkin labels $(\lambda_1, ..., \lambda_{N-1}) \in \mathbb{Z}^{N-1}$, the trace simply amounts to summing over these states. Since the off-diagonal components of the connections decouple in the UV limit this sum is particularly simple. We express the connection into the Chevalley basis as

$$A = A_i h_i \quad , \qquad h_i = 2\frac{\alpha_i^I H^I}{|\alpha_i|^2} \quad .$$

The eigenvalues of the Cartan generators in the Chevalley basis are just the Dynkin labels, thus we get a sum of Abelian Wilson lines of the Cartan torus $U(1)^{N-1}$, with charges given by the Dynkin labels. These combinations are always invariant for the action of the Weyl group S_N and then they correspond to a linear sum of the simple Wilson lines of the $U(1)^{N-1} \rtimes S_N$ gauge theory described in section (3.1.2). In order to define carefully this action we have to consider the symmetric basis $\mathcal{A}_i$, on which S_N acts naturally, and then change basis to the connections A_i in the Chevalley basis, in which the Wilson lines are easily written, using $A_i = L_{ij}\mathcal{A}_j$.

To prove the invariance of such lines under S_N we can adopt another point of view. The Wilson lines coincide formally with the characters of the associated representations

$$\chi(v) = \mathrm{Tr}_{\mathcal{R}} \prod_i v_i^{h_i} \tag{3.1.70}$$

where the product runs over the Chevalley basis and the fugacities v_i are generically complex variables. The Wilson line in representation $\mathcal{R}$ is given by an expression formally identical to the character where the fugacities have been replaced with the holonomies of the components of the gauge field in the Chevalley basis. This proves that the Wilson lines are always invariant under the Weyl group. Indeed the characters are generally defined as the trace of a generic group element in a given representation, as such they are only sensible to the conjugacy class of the element. In other words characters are complex-valued functions defined on the set of conjugacy classes which, for $SU(N)$, is given by $U(1)^{N-1}/S_N$. It follows that the characters written as Laurent polynomials in the $N-1$ variables corresponding to a maximal torus of $SU(N)$ must be well defined functions on the quotient space $U(1)^{N-1}/S_N$, thus they must be invariant under S_N[19].

Since this discussion is quite abstract we want to present some concrete examples on how to construct these lines for $SU(2)$ and $SU(3)$ YM theories. The reader convinced by the argument above may wish to skip these examples.

[19] As an aside notice that this point of view on the Wilson lines tells us that they fuse exactly as the associated representations of the group which is what should happen at $g_{YM} = 0$.

***SU*(2)**. The irreducible representations of $SU(2)$ are characterized by one positive integer $\lambda \in \mathbb{N}$, the Dynkin label of the highest weight state. The states have Dynkin labels $\lambda, \lambda-2, ..., -\lambda+2, -\lambda$. In the $g_{YM} \to 0$ limit the $SU(2)$ Wilson lines $W_\lambda^{SU(2)}$ decompose into a sum over the weight states of the Wilson lines $W(n) = W^n$ of the Abelian theory $U(1)$. In the Chevalley basis the charges n coincide with the Dynkin labels, and we get

$$W_\lambda^{SU(2)} = \sum_{k=0}^{\lambda} W^{\lambda-2k} \tag{3.1.71}$$

For $SU(2)$ the Chevalley basis and the symmetric one are the same, and indeed the Wilson lines above are manifestly $S_2 = \mathbb{Z}_2$ invariant, being a sum of lines $\mathcal{V}(n) = \mathcal{W}^n + \mathcal{W}^{-n}$.

***SU*(3).** The $SU(3)$ case is richer. The weight states in any irreducible representation are labeled by two Dynkin labels $(n_1, n_2) \in \mathbb{Z}^2$, which are the charges of the Wilson lines

$$W_1^{n_1} = \exp\left(in_1 \oint_\gamma A_1\right) , \qquad W_2^{n_2} = \exp\left(in_2 \oint_\gamma A_2\right) \tag{3.1.72}$$

of the $U(1)^2$ theory expressed in the Chevalley basis. The relation with the symmetric case is $A_1 = \mathcal{A}_1 + \mathcal{A}_2$, $A_2 = \mathcal{A}_2$, so that $W_1 = \mathcal{W}_1\mathcal{W}_2$, $W_2 = \mathcal{W}_2$ and $\mathcal{W}_1 = W_1 W_2^{-1}$, $\mathcal{W}_2 = W_2$. The action of S_3 on the Wilson lines in the symmetric basis is by simple permutations

$$(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) \to (\mathcal{W}_{\sigma(1)}, \mathcal{W}_{\sigma(2)}, \mathcal{W}_{\sigma(3)}) \ , \quad \sigma \in S_3$$

where we should remember that $\mathcal{W}_1\mathcal{W}_2\mathcal{W}_3 = 1$. Consider the UV Wilson line in the fundamental representation, whose weight states are $(1,0), (-1,1), (0,-1)$. The Dynkin labels coincide with the charges (n_1, n_2) of the Wilson lines in the Chevalley basis. Hence we have

$$W_{(1,0)}^{SU(3)} = W_1 + W_1^{-1}W_2 + W_2^{-1}. \tag{3.1.73}$$

We can easily check that this operator is S_3 invariant. Notice also that the terms above are all mapped into each other by the Weyl group. Indeed by rewriting the lines in the symmetric basis we have

$$W_{(1,0)}^{SU(3)} = \mathcal{W}_1^{-1} + \mathcal{W}_2^{-1} + \mathcal{W}_1\mathcal{W}_2 = \mathcal{V}(-1,0) = \mathcal{V}(0,-1) = \mathcal{V}(1,1) \tag{3.1.74}$$

namely a single Wilson line of the $U(1)^2 \rtimes S_3$ gauge theory. This property is clearly not true for all the representations of $SU(3)$.

It is worth considering also the anti-fundamental representation, whose weight states are $(0,1),(1,-1),(-1,0)$. The corresponding Wilson line is

$$W^{SU(3)}_{(0,1)} = W_2 + W_1 W_2^{-1} + W_1^{-1} \tag{3.1.75}$$

which is again S_3 invariant. Notice that we can obtain this Wilson line from the one in the fundamental by acting with

$$C \cdot W_1 = W_2 \ , \qquad C \cdot W_2 = W_1 .$$

The operator C is *charge conjugation*. At the level of the connections it exchanges $A_1 \leftrightarrow A_2$, thus leaving the Lagrangian $F_1^2 + F_2^2 - F_1 F_2$ invariant. However, as we have just seen, C can act non-trivially on gauge-invariant operators and therefore it is a global symmetry of the theory. This has to be contrasted with S_3 which leaves the action invariant, but acts trivially also on the gauge invariant operators. This is because the Weyl group S_3 is gauged in the YM theory, while charge conjugation is a 0-form global symmetry acting as an automorphism of the set of line operators.

Gukov-Witten Operators

The surface operators of YM theory, introduced by Gukov and Witten in [130,145], are of two types, electric and magnetic. Both types are labeled by conjugacy classes of the gauge group, namely points in $\boldsymbol{\alpha} \in U(1)^{N-1}/S_N$. The electric GW operators labeled by elements of the center $\mathbb{Z}_N \subset SU(N)$ are topological and generate the 1-form center symmetry $\mathbb{Z}_N^{(1)}$ acting on Wilson lines with charge given by the N-ality of the associated representation. In the semi-Abelian theory we similarly have electric and magnetic surface operators, denoted $\mathcal{T}(\boldsymbol{\alpha})$ and $\widetilde{\mathcal{T}}(\boldsymbol{\alpha})$ respectively. As we have seen these are labeled by $\boldsymbol{\alpha} \in U(1)^{N-1}/S_N$, thus exactly matching those of the $SU(N)$ theory.

A further confirmation that the surface operators of the semi-Abelian theory are related to those of YM theory comes from the action on Wilson lines. The center symmetry of $SU(N)$ is preserved along the RG flow hence must be present also in the deep ultraviolet and should be realized in the semi-Abelian theory. We have already shown that the largest invertible symmetry inside the 2-category describing the surface operators is $\mathbb{Z}_N^{(1)}$ and that these defects act on simple Wilson lines multiplying

them by a phase

$$\mathfrak{C}(\boldsymbol{\alpha}_k, \boldsymbol{n}) = \exp\left(\frac{2\pi i k}{N}|\boldsymbol{n}|\right). \tag{3.1.76}$$

To prove that this $\mathbb{Z}_N$ subgroup of the non-invertible symmetry corresponds to the one-form symmetry of the YM theory we need to check that the $SU(N)$ Wilson lines have definite charge proportional to the N-ality of the representation. Notice that a priory this is not obvious since the lines of $SU(N)$ are combinations of the lines of the semi-Abelian theory, and so for generic GW operator $\mathcal{T}(\boldsymbol{\alpha})$

$$\mathcal{T}(\boldsymbol{\alpha})\mathcal{W}^{SU(N)} \not\propto \mathcal{W}^{SU(N)}. \tag{3.1.77}$$

Actually this factorization occurs precisely for the GW operators generating the center symmetry $\mathbb{Z}_N$. In order to see this we have to rewrite the charge $|\boldsymbol{n}|$ appearing in (3.1.76) in the Chevalley basis. From $A_i = L_{ij}\mathcal{A}_i$ we get $n_i = L_{ji}q_j$, where q_j are the charges in the Chevalley basis. By noting that $\sum_i L_{ji} = j$ we obtain

$$|\boldsymbol{n}| = \sum_{i,j=1}^{N-1} L_{ji}q_j = \sum_j jq_j =: p \tag{3.1.78}$$

where $p = \sum_i iq_i \bmod N$ is precisely the N-ality of the weight state $(q_1, ..., q_{N-1})$. An $SU(N)$ Wilson line in representation $\mathcal{R}$ is a particular combination of simple S_N-invariant lines with charges given by the weights of $\mathcal{R}$. Since each weight of a weight system belongs to the same congruence class all terms in the $SU(N)$ Wilson line have same charge under the $\mathbb{Z}_N$ generators. Thus on $SU(N)$ Wilson lines the action of the invertible GW operators factorizes and assigns a charge exactly coinciding with the N-ality of the representation. Notice that we found this action only after implicitly imposing a global structure for the semi-Abelian theory dictated by choosing $SU(N)$ as the gauge group of YM theory. Other choices of global structure will lead to different group-like symmetries, this will be discussed in the next subsection.

Global Structures

For a gauge theory with Lie algebra $\mathfrak{g}$ we have different choices of global structures, corresponding to different choices of genuine line operators of the theory [62], which can be related by the gaugings of the center symmetry (or some subgroup of it) [157]. In this section we show that all the possible global structures of $\mathfrak{g} = su(N)$ YM theories are nicely matched in the $U(1)^{N-1} \rtimes S_N$ gauge theory[20].

[20] A similar idea is used in [158] in order to derive the possible choices of global structures of supersymmetric gauge theories from the infrared the Coulomb branch. Here we perform the

Dirac quantization condition and 't Hooft lines In 4d Maxwell theories the possible global structures are the solutions of the Dirac quantization condition. For a single Abelian gauge field the only compact global structure is $U(1)$ and the usual Dirac quantization condition imposes that the charges q and $\widetilde{q}$ of the Wilson and 't Hooft lines respectively must satisfy the condition $q\widetilde{q} \in \mathbb{Z}$. In the case of $U(1)^{N-1}$ Maxwell theory this condition is a straightforward generalization, if we consider the diagonal action

$$S = \frac{1}{2}\int d^4x \widehat{F}_i \wedge *\widehat{F}_i \tag{3.1.79}$$

we have

$$q_i\widetilde{q}_i \in \mathbb{Z}\ , \quad \forall i = 1, ..., N-1. \tag{3.1.80}$$

These charges however do not have an immediate interpretation in terms of the relation with $SU(N)$ YM theory. To have such interpretation we should work in the Chevalley basis (or the symmetric one) which is non-diagonal. By changing basis $\widehat{F}_i = R_{ij}F_j$, so that the action in the A_i variables is (3.1.66), then $K = R^T R$. By denoting with $n_i, \widetilde{n}_i$ the electric and magnetic charges in the basis with quadratic form K, we get

$$q_i = n_j(R^{-1})_{ji}\ , \quad \widetilde{q}_i = \widetilde{n}_j(R^{-1})_{ji}\ . \tag{3.1.81}$$

The Dirac condition (3.1.80) can now be written as (not summed over i)

$$q_i\widetilde{q}_i = n_j(R^{-1})_{ji}\widetilde{n}_k(R^{-1})_{ki} \in \mathbb{Z}. \tag{3.1.82}$$

Then by summing over i we get

$$n_i(K^{-1})_{ij}\widetilde{n}_j \in \mathbb{Z}. \tag{3.1.83}$$

A particular choice of the global structure in the $U(1)^{N-1} \rtimes S_N$ gauge theory will constraint the set of possible n_i, or equivalently the set of possible $\widetilde{n}_i$. Then the constraints on the other charges are completely fixed by (3.1.83). The 't Hooft lines of the $U(1)^{N-1} \rtimes S_N$ gauge theory are of the form

$$\mathcal{M}(\widetilde{\boldsymbol{n}}) = \sum_{\sigma \in S_N} \widetilde{\mathcal{W}}(\mathfrak{S}_\sigma^\vee \cdot \widetilde{\boldsymbol{n}}). \tag{3.1.84}$$

The UV limit of the $SU(N)$ 't Hooft lines are particular combinations of the $\mathcal{M}(\widetilde{\boldsymbol{n}})$ for various $\widetilde{\boldsymbol{n}} \in \mathbb{Z}^{N-1}$ such that the quantity

$$|\widetilde{\boldsymbol{n}}| = \sum_{i=1}^{N-1} \widetilde{n}_i \tag{3.1.85}$$

somewhat complementary analysis in the ultraviolet.

is fixed. As for the Wilson lines $|\widetilde{\boldsymbol{n}}|$ is the N-ality of the corresponding $SU(N)$ representation. By keeping this in mind we are ready to discuss the relation between the possible global structures of YM theory and those of the semi-Abelian theory.

Matching the global structures To match the $SU(N)$ global structure in the $U(1)^{N-1} \rtimes S_N$ theory we require the charges n_i of the Wilson lines in the Chevalley basis to be all possible integers. With this choice all the UV Wilson lines defined in section (3.1.3) are genuine line operators of the theory. Taking $K = Q$ in (3.1.83), and choosing only one n_i different than zero and equal to one we get the constraint

$$\widetilde{n}_i = Q_{ij} v_j \ , \qquad v_i \in \mathbb{Z} \tag{3.1.86}$$

for the charges of the 't Hooft line $\mathcal{H}(\widetilde{n}_1, \cdots, \widetilde{n}_{N-1})$. The condition (3.1.86) implies that

$$|\widetilde{\boldsymbol{n}}| = \sum_{i=1}^{N-1} \widetilde{n}_i = \sum_{i,j=1}^{N-1} Q_{ij} v_j = N \sum_i v_i \in N\mathbb{Z} \tag{3.1.87}$$

where we used $\sum_j Q_{ij} = N$. As expected only the 't Hooft lines with 0 N-ality are genuine line operators. Notice that in this case the invertible magnetic GW operators do not have charged operators, hence only the electric $\mathbb{Z}_N^{(1)}$ is non trivial. By exchanging the roles of $\boldsymbol{n}$ and $\tilde{\boldsymbol{n}}$ we immediately see that also the global structure of $PSU(N)$ can be reproduced in the semi-Abelian theory, in this case the electric $\mathbb{Z}_N^{(1)}$ invertible symmetry has no charged operator and the one-form symmetry $\mathbb{Z}_N^{(1)}$ of the theory is entirely generated by the invertible magnetic GW operators.

The $SU(N)$ and $PSU(N)$ theories are connected by the gauging of the center symmetry. We want to show that also in the UV theory the same conclusion is true. Indeed in the previous section we have shown that $U(1)^{N-1} \rtimes S_N$ posses a $\mathbb{Z}_N$ 1-form symmetry which can be gauged. The action of this group on the Wilson lines of the theory is presented in (3.1.3) and it is

$$\mathcal{T}(\boldsymbol{\alpha}_k) \cdot \mathcal{V}(\boldsymbol{n}) = e^{\frac{2\pi i k}{N}|\boldsymbol{n}|} \mathcal{V}(\boldsymbol{n}). \tag{3.1.88}$$

After gauging only the Wilson lines satisfying $|\boldsymbol{n}| = 0 \bmod N$ remain as good operators of the theory, matching the spectrum of Wilson lines in the $PSU(N)$ theory[21].

[21] We have also a different but equivalent way to gauge this symmetry. Indeed the GW operators generating $\mathbb{Z}_N$ are a subgroup of the $(U(1)_e^{(1)})^{N-1}$ symmetry of the $U(1)^{N-1}$ gauge theory before the S_N gauging. Then we can gauge this subgroup in this theory and then gauge the permutation symmetry in the resulting theory. As known, gauging a $\mathbb{Z}_N$ symmetry in a Maxwell theory simply changes the quantization conditions for the electric and magnetic charges and we can easily get the same result obtained in the main text.

Moreover since now we have eliminated some Wilson lines in the theory, the Dirac quantization condition for the genuine 't Hooft lines

$$n_i (Q^{-1})_{ij} \widetilde{n}_j \in \mathbb{Z} \tag{3.1.89}$$

implies that

$$n_i \in \frac{1}{N} Q_{ij} v_j \quad (v_i \in \mathbb{Z}) \tag{3.1.90}$$

which imposes that $|\widetilde{\boldsymbol{n}}| \in \mathbb{Z}$ as it should in $PSU(N)$. It is straightforward to check that gauging $\mathbb{Z}_l$ subgroups of the center symmetry one gets a spectrum of lines in the semi-Abelian theory which exactly matches the spectrum of the $SU(N)/\mathbb{Z}_l$ gauge theory.

3.1.4 Outlook

The main motivation of this section, based on [29] was studying the properties of the continuous non-invertible symmetries arising in the $U(1)^{N-1} \rtimes S_N$ gauge theories and make a connection with the UV limit of $SU(N)$ YM theory. In particular we have found that all the GW operators of the non-Abelian theories become topological in the deep UV and they describe a non-invertible symmetry which is broken to its group-like subcategory $\mathbb{Z}_N$ along the RG flow. Therefore this is one of the few examples in which the gauging of an automorphism is not an artificial mechanism introduced to produce non-invertible symmetries but instead comes naturally from physically interesting systems. In doing this we have analyzed extensively the symmetry, which forms a continuous 2-category with an intricate structure arising form the presence of topological lines, appearing as 1-morphisms. The fusion rules encodes information about these morphisms in the integer constants f_{ab}^{σ}, and in the presence of the condensation defects.

Even if we analyzed explicitly the $SU(N)$ gauge theory, it is easy to see that our results extend to any gauge group G. The theory encoding the symmetry structure in the ultraviolet is the $U(1)^r \rtimes W_G$ gauge theory, where r is the rank of $\mathfrak{g} = \text{Lie } G$ and W_G is the Weyl group. Then the fusion rules (3.1.33) as well as the action of the GW on line operators (3.1.37) are simply obtained by replacing S_N with W_G. Also the analysis of the condensation defects, the global fusions and the 2-categorical structure is conceptually identical for any gauge group G.

We conclude by proposing interesting open problems which arise naturally from our work, and also give qualitative ideas and suggestions about these issues.

Non-local currents, spontaneous symmetry breaking and anomalies. The first question concerns the properties of the continuous non-invertible symmetries studied in this section. Indeed it is natural to ask if such symmetries have conserved currents and if possible spontaneous symmetry breaking of continuous non-invertible symmetry would lead to Goldstone bosons. The existence of conserved currents can be derived from the known conserved currents of the $U(1)^{N-1}$ theory before the S_N gauging. In this theory we have the conserved 2-form current

$$j^i = F^i \tag{3.1.91}$$

where $i = 1, \cdots, N-1$, corresponding to the $(U(1)^{(1)})^{N-1}$ 1-form symmetry of the theory.

After the S_N gauging this operator is no longer gauge invariant and then it cannot be regarded as a good operator of the theory. However we can construct a gauge invariant non-genuine local operator attaching to F^i an S_N Wilson line in the $N-1$ standard representation

$$J = W_{S_N}(\gamma_x)^i F^i(x). \tag{3.1.92}$$

In the above equation γ_x is an infinite topological line which ends on x and then J is a good gauge invariant operator. The idea is that currents of non-invertible symmetries correspond to non-genuine local operators [110]. Note that however this new current is not conserved but is covariantly conserved with respect to S_N transformations, namely

$$D_{S_N} J = 0. \tag{3.1.93}$$

In particular the conserved current in ordinary invertible symmetries is the operator creating Goldstone particles from the vacuum when such a symmetry is spontaneously broken. In this case it would be interesting to understand what happens to these excitations and interpreting them from a generalized version of a Goldstone theorem[22].

Another interesting question is about the possible mixed 't Hooft anomaly between the electric and magnetic non-invertible symmetries possessed by the semi-Abelian gauge theory. Indeed before the S_N gauging the $U(1)^{N-1}$ gauge theory has such an anomaly between the invertible 1-form symmetries $(U(1)_e^{(1)})^{N-1}$ and $(U(1)_m^{(1)})^{N-1}$. This anomaly involves continuous symmetries and we expect it to be inherited by the non-invertible symmetries since a discrete gauging cannot cancel

[22]For a generalization of the Goldstone theorem in the case of *invertible* higher-form symmetries see e.g. [159, 160].

a continuous anomaly. However to study this anomaly we need to couple these symmetries to backgrounds (note that the $B_{e,m}$ backgrounds of the Abelian theory are not anymore gauge invariant) but a consistent definition of backgrounds for non-invertible symmetries is still an open problem.

Constraints on the RG flow of Yang-Mills theories. Perhaps the most important question regards possible implications of the UV emergent symmetries along the RG flow of YM theories. Indeed in a generic QFT, a symmetry possessed by the UV fixed point and broken by some relevant deformations affects the possible structure of the low energy effective theory. This is the case, for instance, of the quark mass perturbation in QCD which leads to mass terms in the chiral Lagrangian. In this case it would be interesting to study more carefully the deformation which breaks this non-invertible symmetry to the center symmetry of YM theory. In particular we expect that for instance correlation functions involving a GW operator and a Wilson line

$$\langle T(\boldsymbol{\alpha})^{SU(N)}[\Sigma_2]W_R(\boldsymbol{n})^{SU(N)}[\gamma]...\rangle, \tag{3.1.94}$$

which at $g \neq 0$ and $T(\alpha) \notin \mathbb{Z}_N$ depends on the relative position of the surface Σ_2 and the curve γ, when the surface is infinitesimally closed to γ, they approximately follow the topological action presented in the previous sections, with corrections of order $\Lambda_{YM}r$ where r parametrizes the distance between Σ_2 and γ[23].
We hope that other possible predictions can be achieved also when the issues presented in the first part of this section will be understood. In particular the presence of an anomaly before the deformation would suggest that the gap produced by the RG flow should go to zero in the limit in which the RG flow is never triggered. Indeed this is something believed to happen in YM theory since the gap is of order Λ_{YM}.

3.2 Symmetries and topological operators, on average

3.2.1 Introduction and summary of the results

In section 3.1.2, we highlighted the significant connection between global symmetries and the set of topological operators, which provides a compelling approach to effectively incorporate the predictive powers of global symmetries in a given QFT.

[23]For instance taking $\Sigma_2 = S^2$ surrounding γ then r is exaclty the radius of the sphere.

Some of the implications obtained by looking at global symmetries, such as selection rules on observables, the establishment of twisted Hilbert spaces, the introduction of background gauge fields for gauging and the identification of anomalies, emerge naturally from the use of topological operators enhancing our understanding and broadening the applicability of these concepts in QFT.

The aim of this section is to extend this formalization of symmetries to QFTs where the interactions are randomly distributed, for the case of 0-form global symmetries. We believe that a more systematic treatment of symmetries in QFTs of this kind can be useful, given the notorious difficulties in treating such systems. There are two relevant possibilities considered here.

1. The random couplings $h(x)$ vary in space and are distributed according to a probability functional $P[h]$.

2. The random couplings h are constant and drawn from a probability density $P(h)$.

Scenario 1. is relevant for statistical mechanical systems with impurities or disorder (for a review see [161]). There are two main variants of disorder QFT: *quenched* if the impurities are treated as external random sources and *annealed* if the impurities are taken dynamical. Physically the two situations depend on the time scale we are looking at. At extremely long time-scales, where the entire system reaches equilibrium, we should take the impurities dynamical. Since impurities have very long thermalizations time scales, quenching is useful for time-scales where the system essentially thermalizes, with the impurities taken fixed. In the quenched case, the properties of the QFT will of course depend on the impurities. If we assume that impurities are random, possible observables are taken by averaging over the impurities with the chosen distribution. In a lattice formulation an impurity is modelled by an interaction which is different at any site, and its presence is unpredictable. In the continuum limit it is often the case that we can describe such systems as the average over an ensemble of field theories where the coupling constants are space dependent. Particularly interesting is the case of the Ising model perturbed with a random magnetic field (dubbed as random field Ising model) [162] or with a random interaction between nearby spins (dubbed as random bond Ising model) [163]. See e.g. [164–167] for recent works on these models.

Scenario 2. is relevant for quantum gravity and has received significant attention lately. The connection between averaging and euclidean gravity path integrals

dates back to [168, 169] in association to Euclidean wormholes. In the context of the AdS/CFT correspondence [170–172], the connection has been invoked in [173] as a possible way to interpret from a boundary point of view the origin of interactions between disconnected components of a boundary theory induced by bulk Euclidean wormholes (factorization puzzle). Further elaborations with concrete examples appear in [174]. Ensemble averaging features also in the Sachdev-Ye-Kitaev (SYK) model [175–177]. A concrete connection has recently been made in [178], where it has been shown that the sum over geometries in Jackiw-Teitelboim gravity [179, 180] with n disconnected boundaries is dual to the ensemble average of an n-point correlation function in a matrix model. Other notable examples of ensemble averaging after [178] include averages over free compact bosons in 2d [181, 182] (see also e.g. [183–187] for related studies and generalizations), averages over OPE coefficients in effective 2d CFTs [188, 189], averages over the gauge coupling in 4d $\mathcal{N} = 4$ super Yang-Mills theory [190].

In both scenarios 1. and 2. we focus on correlation functions of local operators with *quenched disorder* averaging. These include averages of products of correlators, which are effectively independent observables. In disconnected spaces, when h is constant, also averaged single correlators can lead to averages of products of correlators, which is the mechanism leading to the factorization puzzle in the context of AdS/CFT. In order to distinguish scenario 1. from scenario 2. we dub the first as "quenched disorder" and the latter as "ensemble average", but it should be kept in mind that quenching is involved also in scenario 2.

We start from a *pure theory*, that is an ordinary QFT with no disorder, and deform it with a certain local interaction. In the quenched disorder case the strength of this interaction varies from point to point, while it is constant in ensemble average. In both cases the interaction can break part of all of the global symmetries of the pure system, so that each specific realization generically has less symmetries and less predictive power than the pure theory. On the other hand it has been noticed in several examples that symmetries of the pure systems can be recovered after the average on the coupling is taken into account. These statements are mostly based on the observation that the averaged system satisfies selection rules which are not enjoyed by the generic specific realization. Intuitively speaking, even if the random coupling breaks the symmetry, this re-emerges provided we average over all the ensemble in a sufficiently symmetric way (in a sense to be clarified). For simplicity, in what follows we refer to such symmetries as *disordered symmetries* and *averaged symmetries* respectively in the context of quenched disorder and ensemble average.

Note that this is distinct from the notion of emergent symmetries used in pure QFTs when a symmetry is approximately conserved in the IR. In the disorder case the symmetry is *exact* at all energy scales, but only on average.[24]

We will review these kind of arguments from a spurionic point of view at the beginning of section 3.2.2 for quenched disorder, and in section 3.2.5 for ensemble average, deriving under which condition the selection rules of the pure theory are satisfied after the average.

This is still an imprecise information since, as we emphasized, having a global symmetry is stronger than just observing the validity of some selection rule. This is crucial in order to get stronger dynamical constraints implied by 't Hooft anomalies, and eventually gauging the symmetry. Our goal is to clarify the sense in which these systems recover the symmetry, aiming to construct the analog of topological operators for both quenched disorder and ensemble average QFTs.

Sections 3.2.2, 3.2.3 and 3.2.4 focus on quenched disordered systems. We consider theories defined in the continuum and admitting a description in terms of an action (Hamiltonian) obtained from that of the pure theory S_0 by adding a local operator $\mathcal{O}_0(x)$ with a space-time dependent coupling $h(x)$:

$$S[h] = S_0 + \int d^d x\, h(x)\mathcal{O}_0(x) \ . \tag{3.2.1}$$

This is what we will call a specific realization. Correlation functions of local operators $\mathcal{O}_i$ for a given value of $h(x)$ are computed by a path integral:

$$\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle = \frac{\int \mathcal{D}\mu\, e^{-S[h]}\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)}{\int \mathcal{D}\mu\, e^{-S[h]}} \ . \tag{3.2.2}$$

Given a probability functional $P[h]$, a set of observables of the disordered system are the averaged correlation functions

$$\overline{\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle} = \int \mathcal{D}h\, P[h]\, \big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle \,, \tag{3.2.3}$$

or more generally the averages of products of correlators

$$\overline{\prod_{j=1}^{N} \Big\langle \mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{n_j}^{(j)}(x_{k_j}^{(j)})\Big\rangle} \ . \tag{3.2.4}$$

[24]We can also have emergent symmetries in both senses, namely emerging after average and in the IR. We will discuss this case in section 3.2.4.

The starting point for systematizing global symmetries of the disordered system which are not enjoyed by the specific realizations is to derive Ward identities for the averaged correlators. We do this in section 3.2.2, starting from the simplest case of continuous 0-form invertible symmetries. The Noether current J^μ associated to the symmetry of the pure theory is no longer conserved in the specific realizations if $\mathcal{O}_0(x)$ is charged under the symmetry. However we find that the *shifted current* in (3.2.37) leads to standard Ward identities (3.2.41) for averaged single correlators and the less standard identities (3.2.45) for averages of products of correlators.

In order to generalize our results to discrete symmetries, where a Noether current is unavailable, in section 3.2.2 we construct the symmetry operators, topological on average, which implements the finite group action. This is not as easy as in pure theories because of the disorder. The topological operator $\widetilde{U}_g$ is a complicated power series of integrated currents which however can be resummed to give the simple expression

$$\widetilde{U}_g = U_g \langle U_g \rangle^{-1} \,, \tag{3.2.5}$$

where U_g is the topological operator of the pure theory. Its action on averages of simple correlators is given in (3.2.58). In products of correlators (3.2.4) the operator $\widetilde{U}_g$ is topological on average only if inserted in all the correlators involved, as in (3.2.61). This characterizes intrinsically the disordered symmetries and implies somewhat exotic selection rules which are weaker with respect to symmetries not broken by the random interactions.

The Ward identities satisfied by $\widetilde{U}_g$, when the latter is supported on a compact surface $\Sigma^{(d-1)}$ are valid regardless of how the symmetry is realized on the vacuum. When the symmetry operator is well defined also on infinite surfaces the disordered symmetry is not spontaneously broken and implies selection rules. The same is not true for spontaneously broken symmetries, we will briefly discuss this situation in the final section 3.2.6.

Beyond selection rules, our analysis allows us to show that disordered symmetries (both continuous and discrete) can be coupled to external backgrounds, can be gauged, and can have 't Hooft anomalies, precisely like ordinary symmetries. We also argue that a symmetry of a pure system with a 't Hooft anomaly, when it reappears as disordered symmetry, enjoys the same 't Hooft anomaly thus implying the same constraints on the dynamics, and that a possible higher-group structure of the underlying 0-form symmetry with higher-form symmetries of the pure theory is recovered after average due to the topological nature of the higher-group structure.

Symmetry Protected Topological (SPT) phases [191], protected by what we denoted disordered symmetries, appeared already in condensed matter, see e.g. [192–199]. Our findings can possibly provide a different theoretical QFT-based framework for such phases of matter.

In section 3.2.3 the above results, derived directly from the disordered theory, are reproduced using the replica trick, the standard way to deal with theories of this kind. Disordered symmetries manifest themselves as standard symmetries in the replica theory, thus offering a conceptual different viewpoint on these kind of symmetries. Aside from providing a sanity check of the results, the replica theory allows us to also study another scenario: disordered symmetries emerging at long distances, discussed in section 3.2.4. The effect of the disorder can now lead to the more exotic selection rules (3.2.104) and (3.2.105). The phenomenon manifests in the replica theory as two irreducible representations of the replica symmetry transforming in different representations of the emergent disordered symmetry. As an interesting application of this result we consider the prime example of an emergent symmetry, conformal invariance, and we show that as a consequence of these exotic selection rules, a quenched disordered system can flow in the IR to a fixed point described by a Logarithmic conformal field theory (LogCFT) [200–204].

We analyze ensemble average in section 3.2.5. While the intuitive idea that the average restores the symmetry is still true, and selection rules apply (section 3.2.5), the status of the averaged symmetry is drastically different. A hint already comes from the replica trick: when applied with constant couplings, the replica theory is non-local, and even if the symmetry is manifest its Noether current is not a local operator. This is problematic for constructing a topological operator. Indeed, independently of the replica trick, we imitate the analysis done for disordered theories, and we get the exotic topological charge operator (3.2.134). This is not really a co-dimension one operator, since it depends both on a closed surface $\Sigma^{(d-1)}$ and on a filling region $D^{(d)}$ such that $\partial D^{(d)} = \Sigma^{(d-1)}$. In particular the operator cannot be supported on homologically non-trivial cycles. Crucially, the operator $\widehat{Q}$ implies selection rules, because the second term in (3.2.134), when inserted on average of correlation functions of local operators, vanishes when integrated over the full space. If the space manifold is connected, there are only two possible filling regions of a homologically trivial $\Sigma^{(d-1)}$, and $\widehat{Q}$ is independent of the choice. On the other hand on a *disconnected* space there are several choices of filling region $D^{(d)}$, and the charge operator does depend on these choices. Nevertheless, we do have selection rules for averages of correlators, if one takes into account all the connected

components of space, and we can construct operators (3.3.60) implementing the finite group action. In each connected component the selection rules can be violated, allowing the charges to escape from one component to the other. We have then the somewhat exotic situation of a 0-form symmetry in the sense of selection rules on correlation functions of local operators, but without having genuine topological operators (even after average). In contrast to ordinary symmetries and disordered symmetries in the quenched disordered case above, averaged symmetries cannot be coupled to background gauge fields in ordinary ways and hence cannot be gauged.

In section 3.2.5 we comment about the gravity interpretation of these results. Whenever the average theory admits a gravitational bulk dual, the local charge violation in presence of disconnected space has the natural interpretation in the bulk as charge violation induced by Euclidean wormholes configurations, as pointed out in [205–207]. The difficulty (impossibility) of gauging averaged boundary symmetries that we have found clarify why such symmetries cannot be identified with bulk gauge symmetries.

We conclude in section 3.2.6. In appendix 3.3.3 we work out some specific examples for concreteness, and in appendix 3.3.4 we explicitly construct the operator which implements the action of the group for averaged symmetries.

3.2.2 Symmetries in quenched disorder

In this section we study global 0-form symmetries in quenched disorder theories which arise only after the average. We start in section 3.2.2 by reviewing how Ward identities for ordinary 0-form symmetries are recasted in terms of topological operators in pure QFTs. We generalize the analysis to theories with quenched disorder in section 3.2.2 and construct the topological operator implementing the symmetry group action in section 3.2.2. We then discuss 't Hooft anomalies and gaugings for both continuous and discrete disordered symmetries in sections 3.2.2 and 3.2.2.

Pure theories and explicit symmetry breaking

Consider a standard d-dimensional Euclidean QFT described by the action S_0. If this theory is invariant under some continuous symmetry group G, correlation functions of local operators must satisfy the usual constraints imposed by the Ward-Takahashi

identities:

$$i\big\langle \partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle = \sum_{l=1}^{k}\delta^{(d)}(x-x_l)\big\langle \mathcal{O}_1(x_1)\dots\delta\mathcal{O}_l(x_l)\dots\mathcal{O}_k(x_k)\big\rangle\,. \tag{3.2.6}$$

Here $J_\mu(x)$ is the Noether current[25] and $\delta\mathcal{O}_l(x_l)$ is the transformation of the local operator $\mathcal{O}_l$ under the action of the Lie algebra of G. For instance if $G = U(1)$ and $\mathcal{O}_l$ has charge q_l, then $\delta\mathcal{O}_l = iq_l\mathcal{O}_l$. Integrating over the full space $X^{(d)}$, the left hand side of (3.2.6) vanishes if $X^{(d)}$ has no boundary and the symmetry is not spontaneously broken, and we get selection rules on the correlators.

The modern approach [4] to interpret the same constraints consists in associating global symmetries to co-dimension one *topological operators* $U_g[\Sigma^{(d-1)}]$, $g \in G$, namely extended operators supported on some $(d-1)$-dimensional closed surface $\Sigma^{(d-1)}$, which are invariant under continuous deformations of their support. In the case of continuous symmetries such topological operators are simply[26]

$$U_g[\Sigma^{(d-1)}] = e^{i\alpha Q[\Sigma^{(d-1)}]}\,, \tag{3.2.7}$$

where $Q[\Sigma^{(d-1)}] = \int_{\Sigma^{(d-1)}} J_\mu n^\mu$ is the Noether operator which measures the charge enclosed within the region $D^{(d)}$ delimited by $\Sigma^{(d-1)}$ with $\partial D^{(d)} = \Sigma^{(d-1)}$. We can then write integrated Ward identities. For instance, if $G = U(1)$ we have

$$\Big\langle Q[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = \chi(\Sigma^{(d-1)})\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle\,, \tag{3.2.8}$$

with

$$\chi(\Sigma^{(d-1)}) = \sum_{l, x_l\in D^{(d)}} q_l\,. \tag{3.2.9}$$

The integrated Ward identity can be iterated using the fact that $J^\mu(x)$ is uncharged with respect to $Q[\Sigma^{(d-1)}]$,[27] resulting in

$$\Big\langle Q^n[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = \chi^n(\Sigma^{(d-1)})\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle\,. \tag{3.2.10}$$

[25]For convenience we define the Noether current as $\delta S = i\int\epsilon(x)\partial^\mu J_\mu$. Notice that this has an extra factor of i with respect to the one obtained by Wick rotating the standard Minkowski current.

[26]In the following we suppress the group and algebra indices. In (3.2.7) the element $g \in G$ is the exponential of α valued in the dual of the Lie algebra of G.

[27]This is not true for non-abelian G. However with simple manipulations one can reach the same conclusion. Here we focus on the abelian case just for notational simplicity.

This implies that the exponentiated operators (3.2.7) satisfy

$$\Big\langle U_g[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = e^{i\alpha\chi(\Sigma^{(d-1)})}\big\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle\ , \quad g=e^{i\alpha}\ . \tag{3.2.11}$$

More generally the integrated Ward identities associated to a finite transformation $g\in G$ can be written as

$$\Big\langle U_g[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = \Big\langle \mathcal{O}'_1(x_1)U_g[\Sigma'_{d-1}]\mathcal{O}_2(x_2)\dots\mathcal{O}_k(x_k)\Big\rangle\,, \tag{3.2.12}$$

where $\mathcal{O}'_1(x_1)=(R_1(g)\cdot\mathcal{O}_1)(x_1)$ is the transformed operator according to its representation R_1 under G, $\Sigma^{(d-1)}$ is a surface linking with the point x_1 and $\Sigma'^{(d-1)}$ is its deformation across the point. The selection rules on correlation functions now follow from the fact that a topological operator $U_g[\Sigma^{(d-1)}]$ supported on a very big surface at infinity is trivial, but shrinking it to a point, U_g passes and transforms all the local operators. We then get

$$\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = R_1(g)\dots R_n(g)\cdot\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle\,, \tag{3.2.13}$$

which is the desired selection rule. A correlation function of local operators can be non-vanishing only if the direct product of representations contains the singlet representation. While $Q[\Sigma^{(d-1)}]$ and $U_g[\Sigma^{(d-1)}]$ enforce equivalent constraints on the theory, the advantage of using the exponentiated operator $U_g[\Sigma^{(d-1)}]$ is that in (3.2.12) we do not need to define the infinitesimal transformation $\delta\mathcal{O}$ so that the generalization to discrete symmetries is straightforward.

If we add a deformation of the pure theory which explicitly breaks G, the Ward identities (3.2.6) acquire a new term and, as expected, the operator $Q[\Sigma^{(d-1)}]$ (or equivalently $U_g[\Sigma^{(d-1)}]$) is no longer topological. For example, for $G=U(1)$ and a deformation described by the action (the term $h\mathcal{O}_0(x)$ is always paired with its hermitian conjugate, which we leave implicit)

$$S = S_0 + h\int d^dx\,\mathcal{O}_0(x)\,, \tag{3.2.14}$$

where $\mathcal{O}_0(x)$ is a local operator with charge q_0 under $U(1)$ and h is a coupling, we get

$$\begin{aligned} i\big\langle\partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle = &\sum_{l=1}^{k}\delta^{(d)}(x-x_l)\big\langle\mathcal{O}_1(x_1)\dots\delta\mathcal{O}_l(x_l)\dots\mathcal{O}_k(x_k)\big\rangle \\ &- ihq_0\big\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\mathcal{O}_0(x)\big\rangle\,. \end{aligned} \tag{3.2.15}$$

Integrating over an open region $D^{(d)}$ with boundary $\Sigma^{(d-1)}$ we have

$$\Big\langle Q[\Sigma^{(d-1)}]\mathcal{O}_1\dots\mathcal{O}_k\Big\rangle = \chi(\Sigma^{(d-1)})\langle\mathcal{O}_1\dots\mathcal{O}_k\rangle - hq_0\int_{D^{(d)}} d^dx\,\big\langle\mathcal{O}_1\dots\mathcal{O}_k\mathcal{O}_0(x)\big\rangle\,. \tag{3.2.16}$$

If the coupling h is irrelevant, at large distances and for sufficiently large surfaces $\Sigma^{(d-1)}$, the second term in the r.h.s. of (3.2.16) is suppressed with respect to the first one, and the operators $Q(\Sigma^{(d-1)})$ become approximately topological.[28] In this case we say that the symmetry G is *emergent* in the IR.

Quenched disorder and Ward identities

Theories with quenched disorder in the continuum limit can often be described starting from a pure theory S_0 and adding a perturbation like in (3.2.14) (see e.g. [208, 209]), where h is taken to be space-dependent (again we always implicitly pair up $h(x)\mathcal{O}_0(x)$ with its hermitian conjugate):

$$S[h] = S_0 + \int d^dx\, h(x)\mathcal{O}_0(x)\,. \tag{3.2.17}$$

The random coupling is sampled from a distribution $P[h]$ and we should think of an ensemble of systems, each member being described by the action (3.2.17). Note that the considerations above on the explicit breaking are valid, with minor modifications, for each member of the ensemble.

A relevant example (which we will extensively consider in the sections 3.2.3 and 3.2.4) is the case of white noise, where $P[h]$ is Gaussian

$$P[h] \propto \exp\Big(-\frac{1}{2v}\int d^dx\, h^2(x)\Big)\,, \tag{3.2.18}$$

parametrized by a coupling v which governs the width of the Gaussian distribution. Dimensional analysis fixes the dimension of v to be

$$[v] = d - 2\Delta_{\mathcal{O}_0}\,, \tag{3.2.19}$$

where $\Delta_{\mathcal{O}_0}$ is the classical scaling dimension of the operator $\mathcal{O}_0$. The disorder is classically irrelevant in the RG sense when

$$\Delta_{\mathcal{O}_0} > \frac{d}{2}\,. \tag{3.2.20}$$

[28] For a related discussion on approximate symmetries in the language of topological operators see [146].

The equation (3.2.20) is called Harris criterion [210]. If the disorder is classically relevant or marginal, it has an important effect on the IR dynamics. For instance, other fixed points could emerge, so called random fixed points, which can also have logarithmic behavior (see section 3.2.4), or we could have no fixed points at all. When (3.2.20) is satisfied, the IR behaviour of the system is unaffected by the impurities.

Like in the pure theory case, if the coupling $h(x)$ breaks a symmetry G and is irrelevant, then the symmetry G will appear as an emergent symmetry in the IR theory. On the other hand, in disordered theories symmetries might also appear on average, but *exactly*, namely at all energy scales, independently on the scaling dimension of $h(x)$. It is important to keep into account this distinction in the considerations that will follow. The latter case is the one that we will call *disordered symmetries*.

The observables we are interested in are averaged correlation functions of local operators defined as (we adopt here the notation of [209])

$$\overline{\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\rangle} = \int \mathcal{D}h\, P[h] \frac{\int \mathcal{D}\mu\, e^{-S[h]}\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)}{\int \mathcal{D}\mu\, e^{-S[h]}}\,, \tag{3.2.21}$$

where μ is the path integral measure and $P[h]$ is an arbitrary distribution, not necessarily of the form (3.2.18). Correlation functions can be obtained as usual by coupling each local operator $\mathcal{O}_i$ to an external source K_i and by taking functional derivatives with respect to the K_i's of the averaged generating functional $Z_D[K_i]$ defined as

$$Z_D[K_i] := \int \mathcal{D}h\, P[h] \frac{\int \mathcal{D}\mu\, e^{-S[h]+\int K_i\mathcal{O}_i}}{\int \mathcal{D}\mu\, e^{-S[h]}} = \int \mathcal{D}h\, P[h] \frac{Z[K_i,h]}{Z[0,h]}\,. \tag{3.2.22}$$

We can also define the disordered free energy $W_D[K_i]$ as

$$W_D[K_i] := \int \mathcal{D}h\, P[h] \log Z[K_i,h] = \int \mathcal{D}h\, P[h] W[K_i,h] = \overline{W[K_i,h]}\,, \tag{3.2.23}$$

that generates averages of *connected* correlation functions

$$\overline{\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\rangle_c} = \left.\frac{\delta^k W_D[K_i]}{\delta K_1(x_1)\dots\delta K_k(x_k)}\right|_{K_i=0}\,. \tag{3.2.24}$$

We stress that, unlike standard QFTs, in quenched disorder theories not all correlators can be determined from the connected ones and in particular

$$\overline{\langle \mathcal{O}_i(x)\rangle\langle \mathcal{O}_j(y)\rangle} \neq \overline{\langle \mathcal{O}_i(x)\rangle}\;\overline{\langle \mathcal{O}_j(y)\rangle}\,. \tag{3.2.25}$$

This is one of the crucial properties of disordered systems which will play an important role in the following. This motivates to introduce a more general generating functional

$$Z_D^{(N)}[K_i^{(1)}, ..., K_i^{(N)}] := \int \mathcal{D}h\, P[h] \prod_{j=1}^{N} \frac{Z[K_i^{(j)}, h]}{Z[0, h]} \tag{3.2.26}$$

whose functional derivatives produce the average of products of correlators. The generalization of (3.2.25) is

$$Z_D^{(N)}[K_i^{(1)}, ..., K_i^{(N)}] \neq \prod_{j=1}^{N} Z_D[K_i^{(j)}]\,. \tag{3.2.27}$$

Now suppose that the pure theory S_0 has some global 0-form invertible symmetry G. If the random deformation is $G-$invariant every realization of the system enjoys the symmetry, therefore G is a symmetry of the full disordered theory and it will show up in the averaged correlators. Indeed from the Ward identities of the theory in presence of a random source $h(x)$, by simply taking the average we immediately get the expected identities. This applies also to higher-form symmetries which cannot be broken by adding local operators to the action [4, 157].

If the random deformation breaks some or all of the symmetries of the pure theory, the story is more interesting. In this case we want to understand if and under which conditions the disordered theory still enjoys these symmetries. We start by considering an internal invertible continuous 0-form symmetry G, but our conclusions apply also in more general setups. In order to gain some intuition it is useful to use a spurionic argument. The path integral of the theory coupled to a random source $h(x)$ is

$$Z[h] = \int \mathcal{D}\mu \exp\left(-S_0 - \int h(x)\mathcal{O}_0(x)\right)\,. \tag{3.2.28}$$

Because of the explicit breaking the partition function obeys

$$Z[h] = Z[R_0^\vee(g)\cdot h], \qquad g \in G \tag{3.2.29}$$

where $\mathcal{O}_0$ transforms in representation R_0 of G, and $R_0^\vee$ is its transpose. Turning on sources K_i for operators of the pure theory we see that the generating functional satisfies

$$\begin{aligned} Z[K_i, h] &= \int \mathcal{D}\mu \exp\left(-S_0 - \int h(x)\mathcal{O}_0(x) + \int K_i(x)\mathcal{O}_i(x)\right) \\ &= Z[R_i^\vee(g)\cdot K_i, R_0^\vee(g)\cdot h]. \end{aligned} \tag{3.2.30}$$

Thus the correlators before averaging are not $G-$invariant but

$$\frac{\delta}{\delta K_1 \dots \delta K_n} Z[K_i, h]\Big|_{K_i=0} = R_1(g) \dots R_n(g) \cdot \frac{\delta}{\delta K_1 \dots \delta K_n} Z[K_i, R_0^\vee(g) \cdot h]\Big|_{K_i=0} . \tag{3.2.31}$$

This implies that

$$\begin{aligned} \overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle} &= \int \mathcal{D}h \, P[h] \frac{1}{Z[h]} \frac{\delta Z[K_i, h]}{\delta K_1 \dots \delta K_n}\Big|_{K_i=0} \\ &= R_1(g) \dots R_n(g) \cdot \int \mathcal{D}h \, P[h] \frac{1}{Z[h]} \frac{\delta Z[K_i, R_0(g)^\vee \cdot h]}{\delta K_1 \dots \delta K_n}\Big|_{K_i=0} . \end{aligned} \tag{3.2.32}$$

We can now change variable in the h-path integral, $R_0(g^{-1})^\vee \cdot h(x) \to h(x)$. Crucially, *if the probability measure $\mathcal{D}h \, P[h]$ is invariant*, the averaged correlator obeys the G selection rules

$$\overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle} = R_1(g) \dots R_n(g) \cdot \overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle} , \tag{3.2.33}$$

but only on average. For example, a space-dependent coupling breaks translations, but if $P[h]$ is translation-invariant (like e.g. in (3.2.18)), then momentum conservation is recovered on average.

Although the above spurion analysis is enough to determine selection rules, it does not provide the explicit form of the conserved currents and which Ward identities are satisfied (and how). The existence of topological operators is not even guaranteed and the common lore which identifies symmetries with topological defects needs a more detailed analysis in order to be verified. Let us then derive the form of Ward identities for disordered symmetries. For notational simplicity we focus on $G = U(1)$, but the analysis can be extended to any Lie group. Consider the generating functional $Z_D[K_i]$ defined in (3.2.22). The usual Ward identities are derived by changing variables in the path integral at the numerator, transforming all the fields with a space-time dependent $U(1)$ element $e^{i\epsilon(x)}$, so that

$$S_0 \to S_0 + i \int \epsilon(x) \partial_\mu J^\mu(x) , \tag{3.2.34}$$

J^μ being the Noether current. Here the symmetry is broken by $h(x)$ in any specific realization, nevertheless we can modify the standard procedure by changing variable also in the path integral at the denominator. Since $h(x)$ is space dependent, Poincaré invariance is explicitly broken in each specific realization and generally $\langle J^\mu \rangle \neq 0$.

This suggests that even if the symmetry is recovered on average the current must be modified somehow. The above-mentioned change of variable in both numerator an denominator, expanding at first order in $\epsilon(x)$ leads to

$$\int \mathcal{D}h\, P[h] \left(\big\langle -\partial_\mu J^\mu - q_0 h \mathcal{O}_0 + q_i K_i \mathcal{O}_i \big\rangle_K + \frac{Z[K_i,h]}{Z[0,h]} \big\langle \partial_\mu J^\mu + q_0 h \mathcal{O}_0 \big\rangle \right) = 0\,. \tag{3.2.35}$$

By taking functional derivatives with respect to the sources K_i and then setting them to zero we get

$$\overline{\big\langle \partial_\mu \widetilde{J}^\mu(x) \mathcal{O}_1(x_1) \cdots \big\rangle} + q_0 \overline{\big\langle h(x) \widetilde{\mathcal{O}}_0(x) \mathcal{O}_1(x_1) \cdots \big\rangle} = \sum_i q_i \delta^{(d)}(x - x_i) \overline{\big\langle \mathcal{O}_1(x_1) \cdots \big\rangle}\,, \tag{3.2.36}$$

where we introduced the shifted operators

$$\widetilde{J}^\mu(x) := J^\mu(x) - \langle J^\mu(x) \rangle\,, \qquad \widetilde{\mathcal{O}}_0(x) := \mathcal{O}_0(x) - \langle \mathcal{O}_0(x) \rangle\,. \tag{3.2.37}$$

The vacuum expectation values should be thought of as certain (generally non-local) functionals of $h(x)$, whose presence is important in the average.

Since $h(x)$ is integrated over all space-dependent configurations, the second term in (3.2.36) vanishes identically provided that the probability measure satisfies certain invariance conditions. Indeed we are allowed to perform the change of variable $h(x) \to e^{-iq_0\epsilon(x)} h(x)$ in the h path integral of (3.2.22), and if the probability measure is invariant under this formal transformation we obtain

$$q_0 \int \mathcal{D}h P[h] \left(\big\langle h \mathcal{O}_0 \big\rangle_{K_i} - \frac{Z[K_i,h]}{Z[0,h]} \big\langle h \mathcal{O}_0 \big\rangle \right) = 0\,. \tag{3.2.38}$$

Taking arbitrary functional derivatives with respect to the external sources and setting them to zero we find

$$q_0 \overline{\big\langle h(x) \widetilde{\mathcal{O}}_0(x) \mathcal{O}_1(x_1) \ldots \big\rangle} = 0\,, \tag{3.2.39}$$

which implies the vanishing of the second term in the left hand side of (3.2.36). By changing variables in the path integral, we also get the relation

$$\Big\langle \partial^\mu J_\mu(x) + q_0 h(x) \mathcal{O}_0(x) \Big\rangle = 0\,, \tag{3.2.40}$$

valid before averaging. We are now ready to discuss Ward identities. If $q_0 = 0$, namely the $U(1)$ symmetry is unbroken in any realization of the ensemble, plugging (3.2.40) in (3.2.36) leads to the averaged version of the ordinary Ward identities

(3.2.6). This is of course expected, given that (3.2.6) holds even before average in this case. More interestingly, for $q_0 \neq 0$, thanks to (3.2.39) we find the disordered Ward identities

$$i\overline{\left\langle \partial_\mu \widetilde{J}^\mu(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\right\rangle} = \sum_{i=1}^{k} iq_i\delta^{(d)}(x-x_i)\overline{\left\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\right\rangle} \,. \tag{3.2.41}$$

Several comments are in order.

- The relation we obtained has the same form of a standard Ward identity, but for a modified current $\widetilde{J}^\mu = J^\mu - \langle J^\mu \rangle$. The modification is proportional to the identity operator in any of the specific realization of the ensemble, and can be thought of as an $h-$dependent counterterm which restores the conservation in the disordered theory. Note that the Ward identities written as in (3.2.41) apply for arbitrary correlation functions of local operators which *do not* contain explicit powers of $h(x)$.

- Before averaging the current J^μ (as well as its shifted version $\widetilde{J}^\mu$) is sensitive to the UV renormalization of the theory, i.e. it acquires a non-vanishing anomalous dimension (in contrast to ordinary conserved currents in pure theories). A proper definition of J^μ would require a regularization of the theory and a choice of renormalization scheme. Luckily enough, if we are *only* interested in averaged correlators, we do not need to worry about these issues, since (3.2.41) guarantees that $\widetilde{J}^\mu$ is effectively conserved inside averaged correlators.

- The Ward identities (3.2.41) are valid independently of the behavior of the current at infinity. When the integral of $\partial_\mu \widetilde{J}^\mu$ over the full space diverges (this requires the space to be non compact) the disordered symmetry is spontaneously broken. We do not discuss spontaneous disordered symmetry breaking in detail here. We briefly comment on it in the conclusions. If the symmetry is not spontaneously broken the integral of $\partial_\mu \widetilde{J}^\mu$ over the full space vanishes. Thus (3.2.41) implies the selection rules we already derived from the spurionic argument. However (3.2.41) is a more refined constraint being a local conservation equation: local currents can be used to discuss 't Hooft anomalies and eventually gauging the symmetry, as we will see shortly. Moreover we will show in the next subsection that, with some modification with respect to the usual story, the conservation of $\widetilde{J}^\mu$ leads to topological operators as in the pure case.

- Since the random coupling $h(x)$ is space dependent, in every member of the ensemble translational symmetry is explicitly broken. The analysis above can be repeated for the stress-energy tensor $T^{\mu\nu}$, showing that also traslational invariance is recovered in a theory with quenched disorder, provided $P[h]$ is translational invariant.

With simple modifications we have a similar identity for any Lie group G:

$$i\overline{\left\langle \partial_\mu \widetilde{J}^\mu_a(x)\mathcal{O}_1(x_1)\cdots \right\rangle} = \sum_i \delta^{(d)}(x-x_i)\overline{\left\langle \mathcal{O}_1(x_1)\cdots r_i(T_a)\cdot \mathcal{O}_i(x_i)\cdots \right\rangle} \,. \quad (3.2.42)$$

Here T_a is a Lie algebra generator and r_i is the representation of the Lie algebra, induced by R_i, under which $\mathcal{O}_i$ transforms. A more general situation could take place, in which the disorder deformation does not break the full group, but leaves a subgroup $H \subset G$ unbroken. In this case any specific realization is $H-$symmetric, and thus the currents J^μ_α, with T_α generator of $\mathfrak{h} = \mathrm{Lie}(H)$, satisfy the standard Ward identity without the necessity of averaging. In particular $\langle \partial_\mu J^\mu_\alpha \rangle = 0$, even if the expectation value of the current itself is not necessarily vanishing due to the lack of Poincaré invariance. Even if G/H is generically not a group, the associated currents, which are not conserved in any specific realization, after the appropriate shift by their expectation values turn out to satisfy the Ward identity (3.2.42) in the disordered theory, and reconstruct the full group G.

Sometimes a 0-form symmetry G can form an higher-group structure with higher-form symmetries of the theory [61, 66, 140]. In this case G is not really a subgroup of the full symmetry structure, since the product of several $G-$elements can also produce an element of the higher-form symmetry. This kind of extension is classified by group-cohomology classes, the Postnikov classes: for instance in a 2-group, mixing G with a 1-form symmetry Γ, the relevant datum is a class $\beta \in H^3(BG, \Gamma)$, with BG the classifying space of G. The important thing is that this is a discrete datum and cannot change under continuous deformation. Suppose we add a disorder breaking G, and this re-emerges as a disordered symmetry. A natural question is whether the higher-group structure is also recovered. The answer is affirmative as a consequence of the discrete nature of this structure. Indeed the probability distributions $P[h]$ have some tunable continuous parameters, like v in the Gaussian case (3.2.18), such that the pure theory is recovered in some limit ($v \to 0$ in the Gaussian case). The cohomology class characterising the higher-group is discrete and cannot change with this continuous parameter. Since all these disordered theories are continuously connected to the pure one, the higher-group structure is unchanged.

Up to this point disordered symmetries seem to behave like ordinary global symmetries in pure theories. The difference arises by considering averages of products of correlators

$$\overline{\prod_{j=1}^{N}\left\langle \mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)})\right\rangle}\,. \tag{3.2.43}$$

Because of (3.2.27) these are independent correlators, and we do not expect them to satisfy Ward identities immediately implied by (3.2.41), or to be constrained by the usual selection rules. Let us consider the more general generating functional $Z_D^{(N)}[\{K_i^{(j)}\}]$ introduced in (3.2.26). With the same manipulations which led to (3.2.38), we get

$$q_0\sum_{j=1}^{N}\int\mathcal{D}hP[h]\left(\left(\left\langle h\mathcal{O}_0\right\rangle_{K_i^{(j)}}-\frac{Z[K_i^{(j)},h]}{Z[0,h]}\langle h\mathcal{O}_0\rangle\right)\prod_{l\neq j}\frac{Z[K_i^{(l)},h]}{Z[0,h]}\right)=0\,, \tag{3.2.44}$$

while the individual terms of the sum are generically non-vanishing. This implies that the only Ward identity we can prove from $Z_D^{(N)}[\{K_i^{(j)}\}]$ are obtained by changing variable in *all* the path integrals involved: if we try to change variables only in a subset of these path integrals, the extra term arising would be not be of the form (3.2.44), but the sum would be over that subset of indices. Repeating the steps above we obtain the Ward identities for averages of products of correlators:

$$\sum_{j=1}^{N}\overline{\langle\partial_\mu\widetilde{J}^\mu\mathcal{O}_1^{(j)}\cdots\mathcal{O}_{k_j}^{(j)}\rangle\left(\prod_{l\neq j}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle\right)}=\sum_{j=1}^{N}\sum_{i_j=1}^{k_j}q_{i_j}^{(j)}\delta^{(d)}(x-x_{i_j}^{(j)})\overline{\prod_{l=1}^{N}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle}\,. \tag{3.2.45}$$

These Ward identities imply weaker selection rules. For instance, the correlator

$$\overline{\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_{k_1}(x_{k_1})\rangle\langle\mathcal{O}_{k_1+1}(x_{k_1+1})\cdots\mathcal{O}_{k_1+k_2}(x_{k_1+k_2})\rangle} \tag{3.2.46}$$

can be non zero when $\sum_{i=1}^{k_1}q_i\neq 0$ and $\sum_{i=k_1+1}^{k_1+k_2}q_i\neq 0$, provided that $\sum_{i=1}^{k_1+k_2}q_i=0$.

In a theory with quenched disorder ordinary and disordered symmetries can be present at the same time, and we see that their different action shows up in looking at averages of products of correlators.

See appendix 3.3.3 for an explicit derivation of (3.2.41) for a two-point ($k=2$) function in a simple solvable model.

Topological operators for disordered symmetries

We now address the question of whether there exist topological symmetry operators implementing disordered symmetries, placing them in the general framework of [4].

This is important to e.g. generalize to discrete symmetries, coupling them to backgrounds and discuss non-perturbative anomalies. For notational simplicity we again focus on the $G = U(1)$ case, but all the considerations can be extended to any Lie group. We introduce the modified charge operator

$$\widetilde{Q}[\Sigma^{(d-1)}] = \int_{\Sigma^{(d-1)}} \widetilde{J}_\mu n^\mu = Q[\Sigma^{(d-1)}] - \left\langle Q[\Sigma^{(d-1)}] \right\rangle \tag{3.2.47}$$

which satisfies the integrated Ward identity

$$\overline{\left\langle \widetilde{Q}[\Sigma^{(d-1)}] \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \right\rangle} = \chi(\Sigma^{(d-1)}) \overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \rangle} \ , \tag{3.2.48}$$

with $\chi(\Sigma^{(d-1)})$ as in (3.2.9), as well as the generalization to arbitrary products

$$\sum_{j=1}^{N} \overline{\left\langle \widetilde{Q}[\Sigma^{(d-1)}] \mathcal{O}_1^{(j)} \cdots \mathcal{O}_{k_j}^{(j)} \right\rangle \left(\prod_{l \neq j} \langle \mathcal{O}_1^{(l)} \cdots \mathcal{O}_{k_l}^{(l)} \rangle \right)} = \chi(\Sigma^{(d-1)}) \overline{\prod_{l=1}^{N} \langle \mathcal{O}_1^{(l)} \cdots \mathcal{O}_{k_l}^{(l)} \rangle} \ . \tag{3.2.49}$$

The reason why the naive procedure of constructing the symmetry operator by exponentiating $\widetilde{Q}[\Sigma^{(d-1)}]$ does not work can be already understood at the second order: $\widetilde{Q}^2[\Sigma^{(d-1)}]$ does not measure the square of the total charge. Let Φ be a generic product of local operators.[29] We have

$$\overline{\langle \widetilde{Q}^2 \Phi \rangle} = \overline{\langle \widetilde{Q}\, Q \Phi \rangle} - \overline{\langle Q \rangle \langle \widetilde{Q} \Phi \rangle} = \chi \overline{\langle\, Q \Phi \rangle} - \chi \overline{\langle Q \rangle \langle \Phi \rangle} + \overline{\langle \widetilde{Q} Q \rangle \langle \Phi \rangle} = \chi^2 \overline{\langle\, \Phi \rangle} + \overline{\langle \widetilde{Q} Q \rangle \langle \Phi \rangle} \ . \tag{3.2.50}$$

In the second step we used both the Ward identity (3.2.48) and (3.2.49) with $N = 2$. We deduce that what measures the total charge square is not $\widetilde{Q}^2$ but

$$\widetilde{Q}_2 := \widetilde{Q}^2 - \langle \widetilde{Q} Q \rangle = Q^2 - 2\langle Q \rangle Q + 2\langle Q \rangle^2 - \langle Q^2 \rangle \ . \tag{3.2.51}$$

In order to construct the topological symmetry operator we need, for any $n \in \mathbb{N}$, an operator $\widetilde{Q}_n$ such that

$$\overline{\langle \widetilde{Q}_n \mathcal{O}_1 \cdots \mathcal{O}_k \rangle} = \chi^n \overline{\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle} \tag{3.2.52}$$

and then define the symmetry operators as

$$\widetilde{U}_g = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \widetilde{Q}_n \ , \qquad g = e^{i\alpha} \ . \tag{3.2.53}$$

[29] In order to avoid cluttering in the formulas, from now on we will adopt a lighter notation omitting often the support of local operators or indices.

To prove that such operators exist, and show how to compute them, we start from $\overline{\langle Q^n\Phi\rangle}$ (again Φ denotes a generic product of local operators), and we rewrite one Q as $\widetilde{Q}+\langle Q\rangle$, so that we can use a linear Ward identity for $\widetilde{Q}$, and we iterate until we eliminate all the Qs:

$$\begin{aligned}\overline{\langle Q^n\Phi\rangle} &= \overline{\langle\widetilde{Q}Q^{n-1}\Phi\rangle}+\overline{\langle Q\rangle\langle Q^{n-1}\Phi\rangle}=\chi\overline{\langle Q^{n-1}\Phi\rangle}+\overline{\langle Q\rangle\langle Q^{n-1}\Phi\rangle}\\ &= \chi^2\overline{\langle Q^{n-2}\Phi\rangle}+\chi\overline{\langle Q\rangle\langle Q^{n-2}\Phi\rangle}+\overline{\langle Q\rangle\langle Q^{n-1}\Phi\rangle}\\ &\vdots\\ &= \chi^n\overline{\langle\Phi\rangle}+\sum_{k=0}^{n-1}\chi^k\overline{\langle Q\rangle\langle Q^{n-k-1}\Phi\rangle}\,.\end{aligned} \tag{3.2.54}$$

The terms $\chi^k\overline{\langle Q\rangle\langle Q^{n-k-1}\Phi\rangle}$ can be managed as follows. We eliminate one χ by using the linear Ward identity for the averaged product of two correlators for $\widetilde{Q}$, which we then re-expand as $Q-\langle Q\rangle$:

$$\begin{aligned}\chi^k\overline{\langle Q\rangle\langle Q^{n-k-1}\Phi\rangle} &= \chi^{k-1}\Big(\overline{\langle\widetilde{Q}Q\rangle\langle Q^{n-k-1}\Phi\rangle}+\overline{\langle Q\rangle\langle\widetilde{Q}Q^{n-k-1}\Phi\rangle}\Big)\\ &= \chi^{k-1}\Big(\overline{\langle Q^2\rangle\langle Q^{n-k-1}\Phi\rangle}-2\overline{\langle Q\rangle^2\langle Q^{n-k-1}\Phi\rangle}+\overline{\langle Q\rangle\langle Q^{n-k}\Phi\rangle}\Big)\,.\end{aligned} \tag{3.2.55}$$

Then we eliminate an other χ from each term, again using the linear Ward identity, in some terms with the product of two correlators, in others with the product of three correlators. We continue in this way until we eliminate all the χs, and remain with a sum of averages of products of expectation values of $\langle Q^a\rangle$ for various a, and $\langle Q^b\Phi\rangle$ for a certain b, generally different for each term. This defines the operator $\widetilde{Q}_n$. For instance

$$\widetilde{Q}_3=Q^3-3\langle Q\rangle Q^2-3\langle Q^2\rangle Q+6\langle Q\rangle^2Q-\langle Q^3\rangle+6\langle Q\rangle\langle Q^2\rangle-6\langle Q\rangle^3\,. \tag{3.2.56}$$

While this seems very complicated, one can check until arbitrarily high order that the expansion can be beautifully resummed as

$$\widetilde{U}_g=\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\widetilde{Q}_n=e^{i\alpha Q}\Big\langle e^{i\alpha Q}\Big\rangle^{-1}\,, \tag{3.2.57}$$

where $\widetilde{Q}_0:=1$. Note that this is the only result consistent with $\big\langle\widetilde{U}_g\big\rangle=1$, which must be true by construction since $\langle\widetilde{Q}_n\rangle=0$ as a direct consequence of the Ward identities (3.2.52) satisfied by $\widetilde{Q}_n$ in absence of local operators.

The operator $\widetilde{U}_g$ in averaged correlators behaves as

$$\overline{\langle \widetilde{U}_g[\Sigma^{(d-1)}]\mathcal{O}_1\cdots\mathcal{O}_k\rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})}\overline{\langle \mathcal{O}_1\cdots\mathcal{O}_k\rangle} \tag{3.2.58}$$

and is hence a *topological symmetry operator, on average*. It satisfes the group law

$$\overline{\langle \widetilde{U}_g\widetilde{U}_h\Phi\rangle} = \overline{\langle \widetilde{U}_{gh}\Phi\rangle}\,, \tag{3.2.59}$$

Φ being an arbitrary product of local operators. As a consequence, the naive expectation that $e^{i\alpha Q}e^{i\beta Q} = e^{i(\alpha+\beta)Q}$ is wrong because of the disorder. Note that before averaging the operator $\widetilde{U}$ is subject to renormalization and its proper definition requires a choice of renormalization scheme. We do not need to keep track of these subtleties, however, because they are washed away after the average is taken.

We now consider how $\widetilde{U}_g$ behaves inside averages of products of correlators (3.2.43), extending (3.2.49) to finite symmetry actions. This is important because, as we mentioned, products of correlators is what really characterizes disordered symmetries with respect to ordinary ones, and we need the symmetry operator version of the criterion we discussed at the end of section 3.2.2. In principle one could explicitly construct the correct combination of charges $\widetilde{Q}_n$ entering the Ward identities using the results above. For example, in the average of products of two correlators, at quadratic order in the charges we have

$$\overline{\langle \widetilde{Q}_2\Phi_1\rangle\langle\Phi_2\rangle} + \overline{\langle\Phi_1\rangle\langle\widetilde{Q}_2\Phi_2\rangle} + 2\overline{\langle\widetilde{Q}_1\Phi_1\rangle\langle\widetilde{Q}_1\Phi_2\rangle} = \chi^2\overline{\langle\Phi_1\rangle\langle\Phi_2\rangle}\,, \tag{3.2.60}$$

$\Phi_{1,2}$ being two distinct generic products of local operators. Similarly for multiple products.

We claim that the correct Ward identities consist in inserting $\widetilde{U}_g$ in all the (un)factorized correlators under average:

$$\overline{\prod_{j=1}^{N}\left\langle \widetilde{U}_g[\Sigma^{(d-1)}]\mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)})\right\rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})}\overline{\prod_{j=1}^{N}\left\langle \mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)})\right\rangle}\,. \tag{3.2.61}$$

This can be checked by expanding both members in powers of α, which gives a series of Ward identities for the $\widetilde{Q}_n$'s. For example, for two correlators ($N=2$) we have

$$\sum_{l=0}^{k}\binom{k}{l}\overline{\left\langle\widetilde{Q}_l\Phi_1\right\rangle\left\langle\widetilde{Q}_{k-l}\Phi_2\right\rangle} = \chi^k\overline{\left\langle\Phi_1\right\rangle\left\langle\Phi_2\right\rangle}\,, \tag{3.2.62}$$

where $\chi = \chi_1 + \chi_2$ are the sum of charges of the local operators in the product $\Phi_{1,2}$ which are inside the support of the charge operators. Checking (3.2.62) directly is

cumbersome, but we can proceed as follows. We rewrite the last term appearing in (3.2.54) using (3.2.62) (assuming its validity) with $\Phi_1 = Q$ and $\Phi_2 = Q^{n-k-1}\Phi$.[30] In this way we get

$$\widetilde{Q}_n = Q^n - \sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{k}{l}\langle\widetilde{Q}_l Q\rangle\widetilde{Q}_{k-l}Q^{n-k-1} = Q\widetilde{Q}_{n-1} - \sum_{l=0}^{n-1}\binom{n-1}{l}\langle\widetilde{Q}_l Q\rangle\widetilde{Q}_{n-l-1}\ . \tag{3.2.63}$$

This is a recursion formula which determines $\widetilde{Q}_n$ in terms of all the $\widetilde{Q}_m$ for $m < n$, and it is equivalent to (3.2.62). It can be checked that computing the topological charges with this formula gives the same result as computing them directly from the linear Ward identities, proving in this way the validity of (3.2.61) and (3.2.62).

For averages of multiple correlators the group law (3.2.59) generalizes to

$$\overline{\prod_{j=1}^{N}\left\langle\widetilde{U}_g\widetilde{U}_h\Phi_j\right\rangle} = \overline{\prod_{j=1}^{N}\left\langle\widetilde{U}_{gh}\Phi_j\right\rangle}\ . \tag{3.2.64}$$

We are finally able to characterize disordered symmetries in full generality. These are symmetries of theories with quenched disorder implemented by symmetry operators $\widetilde{U}_g$, $g \in G$, which become topological after quenched average. They satisfy the identity (3.2.58) and the group law (3.2.59) as operator equations valid in any averaged correlator. Differently from ordinary global symmetries, in averages of products of correlators like (3.2.43) they are topological only if inserted in each factor of the product, and satisfy the generalized group law (3.2.64) inside averaged correlators. Disordered symmetries are symmetries of the pure system broken by the disorder but with a symmetric probability measure. It is then not surprising that $\widetilde{U}_g$ can be written in terms of the corresponding topological operator U_g of the pure system as

$$\widetilde{U}_g[\Sigma^{(d-1)}] = U_g[\Sigma^{(d-1)}]\left\langle U_g[\Sigma^{(d-1)}]\right\rangle^{-1}\ . \tag{3.2.65}$$

However the characterization above is intrinsic and does not require to know the pure system. The resummation of the series (3.2.53) into the compact expression (3.2.65) allows us to immediately generalize the analysis to more general groups G, including discrete ones where there is no current or charge operator available.

[30]Note that Φ can include integrated current operators J_μ, hence powers of charges Q, but *not* powers of $\widetilde{Q}$. The latter is still the integral of a local operator, but with an explicit dependence on $h(x)$, in which case the analysis does not apply.

't Hooft anomalies for continuous disordered symmetries

We examine in this and the next subsections some general properties of disordered symmetries. We will argue that the concept of 't Hooft anomalies, for both continuous and discrete symmetries, extends to this context. In particular we show that disordered symmetries inherit the anomaly of their pure counterpart. This is important because we can use anomalies to constraint the IR dynamics of quenched disordered theories, whose flow is generally extremely complicated. We start discussing 't Hooft anomalies for continuous disordered symmetries, postponing to section 3.2.2 the case of discrete symmetries.

A theory with a global symmetry can be coupled to a background gauge field A which acts as an external source for the conserved current J, and results in a partition function $Z[A]$. A 't Hooft anomaly arises whenever $Z[A]$ is not invariant under gauge transformations of the background (see e.g. [211] for a modern review). Denoting by A^λ the gauge transformed background, we have

$$Z[A^\lambda] = e^{i\int_{X^{(d)}} \alpha(\lambda,A)} Z[A]\,, \tag{3.2.66}$$

where the phase in the exponent is the t'Hooft anomaly, a functional depending on λ and A, which cannot be cancelled by local counterterms. Coupling to backgrounds for disordered symmetries is more subtle, because the symmetry is explicitly broken in any specific realization of the ensemble. If the symmetry is restored on average, however, a coupling to an external background becomes possible via the shifted current $\widetilde{J}$ defined in (3.2.37), namely we define

$$\overline{Z[A]} = \int \mathcal{D}h\, P[h] \int \mathcal{D}\mu e^{-S_0 - \int h(x)\mathcal{O}_0(x) + \int A_\mu \widetilde{J}^\mu}\,. \tag{3.2.67}$$

A 't Hooft anomaly for a disordered symmetry G can be defined in close analogy with the ordinary case (3.2.66):

$$\overline{Z[A^\lambda]} = e^{i\int_{X^{(d)}} \alpha(\lambda,A)} \overline{Z[A]}\,. \tag{3.2.68}$$

Anomalies (both continuous and discrete) are known to be invariant under RG flow thanks to their topological nature (typically associated to a Chern-Simons level taking value in a cohomology group, see e.g. [212,213]). In particular, the value of the anomaly cannot depend on possible continuous parameters entering in the disorder distribution $P[h]$, such as v in the Gaussian example (3.2.18). By adiabatically changing such parameters, we can make the distribution arbitrarily peaked around $h = 0$, in which case we effectively recover the pure theory.[31] We then expect

[31] For the gaussian case this is achieved by taking $v \to 0$.

that a 't Hooft anomaly (3.2.68) associated to a disordered symmetry G can only appear if the associated pure theory (before adding the disorder perturbation) had a 't Hooft anomaly for the same symmetry G. Moreover, the two anomalies must coincide. This can be easily verified for all anomalies which, from a path integral point of view, can be seen to derive from the non-invariance of the path integral measure [214]. Starting from the left hand side of (3.2.68) when λ is infinitesimal, we perform a change of variable in the path integral in $Z[A^\lambda]$, which corresponds to an x-dependent transformation under G such that $A^\lambda \to A$. As in pure theories, the non-invariance of the measure leads to the anomaly term. The derivative of the current coming from the action variation is cancelled by the explicit symmetry breaking term and we are left with the anomalous term only. Crucially, the latter does not depend on the disorder h and hence we immediately get the infinitesimal version of the right hand side of (3.2.68), where α is exactly the same as in the underlying pure theory. If the anomaly vanishes, the disordered symmetry can be gauged by making the gauge field A_μ in (3.2.67) dynamical.

We report in appendix 3.3.3 an example of matching of 't Hooft anomalies between the pure and the disorder theories using the replica trick, which will be introduced in section 3.2.3, for the case of the $U(1)$ chiral anomaly in four dimensions.

Discrete disordered symmetries: 't Hooft anomalies and gauging

The topological operators $U_g[\Sigma^{(d-1)}]$ are crucial to handle discrete symmetries for which there is no current. In pure theories the coupling to background gauge fields associated to a discrete symmetry group G can be achieved by modifying the path integral with the topological symmetry operators [4]. There are several equivalent ways to introduce a background gauge field for a discrete symmetry group G. One of these (see e.g. [140] for further details) consists in taking an atlas $\{U_i\}$ of the d-dimensional space $X^{(d)}$ and assigning group-valued connections $A_{ij} \in G$ on $U_i \cap U_j$ such that $A_{ij} = A_{ji}^{-1}$ and $A_{ij}A_{jk}A_{ki} = 1$ on triple intersections $U_i \cap U_j \cap U_k$. A codimension one symmetry operator $U_{g_p}[\Sigma_p^{(d-1)}]$ assigns $A_{ij} = g_p$ (or g_p^{-1} depending on its orientation) if $\Sigma_p^{(d-1)}$ has a non trivial intersection number with the line dual to $U_i \cap U_j$ and $A_{ij} = 1$ otherwise.[32] The resulting sets of connections A_{ij} defines a background gauge field for G and can be represented by a cohomology class $A \in H^1(X^{(d)}, G)$. The operators $U_{g_p}[\Sigma_p^{(d-1)}]$ can intersect in three-valent junctions

[32] In the dual triangulation the charts U_i are points, the intersections $U_i \cap U_j$ are lines, and so on.

of codimension two provided that

$$g_i g_j g_k = 1\,, \tag{3.2.69}$$

or also in higher multi-valued junctions. The configuration described above requires few choices, and one must check independence on those. Since the operators are topological local changes in their support are immaterial. We could also change the mesh locally near the junctions, which corresponds to resolve a multi-valent junction in three-valent ones in different ways. This corresponds to background gauge transformations and a non-invariance under them signals a 't Hooft anomaly for discrete symmetries. In d dimensions a 't Hooft anomaly is classified by a class $\alpha \in H^{d+1}(BG, U(1))$.[33]

Consider now a theory T with quenched disorder, obtained by deforming a pure theory T_0, and denote by T_h the member of the ensemble with coupling $h(x)$. Suppose T has a discrete disordered symmetry G. As we have seen this is implemented by the operators

$$\widetilde{U}_g[\Sigma^{(d-1)}] = U_g[\Sigma^{(d-1)}]\Big\langle U_g[\Sigma^{(d-1)}]\Big\rangle^{-1}\,. \tag{3.2.70}$$

We introduce a fine-enough mesh of topological operators $\widetilde{U}_{g_i}[\Sigma_i^{(d-1)}]$ satisfying (on average) the cocycle condition (3.2.69) in the three-valent junctions. Since $\widetilde{U}_{g_i}[\Sigma_i^{(d-1)}]$ is not topological in T_h, the junctions (as well as the operators $\widetilde{U}$ themselves) are not really well-defined because of UV divergences. However we can employ an arbitrary regularization scheme for these divergences, without the need of specifying a renormalization scheme to try to define the junctions and the operators $\widetilde{U}$ (recall the second comment after (3.2.41)). This is because we know that the operators become topological after the average and hence such divergences are expected to be washed away from the integration over $h(x)$. We define

$$Z_{T_h}\big[\,\{g_i\}\,, h\big] = \int \mathcal{D}\mu\, e^{-S[\phi]-\int h(x)\mathcal{O}_0(x)} \prod_i \widetilde{U}_{g_i}[\Sigma_i^{(d-1)}] = \Big\langle \prod_i \widetilde{U}_{g_i}[\Sigma_i^{(d-1)}]\Big\rangle \tag{3.2.71}$$

which, contrary to the pure case, *does* depend on the specific location of the planes $\Sigma_i^{(d-1)}$. At this point there is no notion of background gauge fields. However, as a consequence of the Ward identity discussed in section 3.2.2,

$$Z_T\big[\,\{g_i\}\,\big] = \int \mathcal{D}h\, P[h]\, Z_{T_h}\big[\,\{g_i\}\,, h\big] \tag{3.2.72}$$

[33]Strictly speaking, this is the case for bosonic theories in $d < 3$ dimensions. More in general, anomalies are classified by a cobordism group [215].

is independent of the choice of location for Σ_i and hence the set of operators $\widetilde{U}_{g_i}$ inserted in (3.2.72) corresponds to a well-defined discrete gauge field $A \in H^1(X^{(d)}, G)$. It is important to emphasize here that the gauge field A arises only after the average over $h(x)$ is performed. Differently said, if a pure system has a symmetry G, perturbing it with quenched disorder and coupling it to a background are non-commutative operations. In what follows we denote the above partition function by $Z_T[A]$.

Local modifications of the three-valent junctions change the gauge field by an exact 1-cocycle $A \to A^\lambda = A + \delta\lambda$. This can change the partition function $Z_T[A]$ by a phase, which represents a class $\alpha \in H^{d+1}(BG, U(1))$: this is the diagnostic for an 't Hooft anomaly for a discrete disordered symmetry. Since the topological operator $\widetilde{U}_g[\Sigma^{(d+1)}]$ is different from the one in the pure theory by the stacking of an $h(x)-$dependent functional, it is not a priori obvious that the contact terms arising in the local moves are the same as those in the pure theory, precisely as it occurred in the continuous case discussed in section 3.2.2. However, the fact that anomalies are classified by classes in $H^{d+1}(BG, U(1))$, which are discrete, immediately proves that they cannot depend on the strength of the disorder and must be equal to those of the pure theory. As a result, a system with a disordered symmetry with a 't Hooft anomaly cannot be trivially gapped. This is in agreement with previous works in condensed matter where – mostly in the context of topological insulators [192–196, 198] but not only (see e.g. [197]) – SPT phases of matter where the symmetry is disordered were found. We see that in general disordered symmetries can lead to protected non-trivial topological phases (see [199] for a recent analysis).[34]

Now suppose that the 't Hooft anomaly vanishes. Then $Z_T[A]$ is well defined and is possible to gauge the symmetry by summing over all consistent insertions of symmetry operators, or equivalently over cohomology classes $A \in H^1(X^{(d)}, G)$. We

[34]In [199] it is considered a Lorentizan theory with a disorder coupling depending on space but not in time. In this set-up it is found that purely disordered symmetries, i.e. in absence of pure symmetries, necessarily have a trivial t' Hooft anomaly. This is not in contradiction with our findings, based on Euclidean theories.

denote the resulting theory by T/G, whose partition function is[35]

$$Z_{T/G} = \sum_{A \in H^1(X^{(d)},G)} Z_T[A] \ . \tag{3.2.73}$$

At this point everything is essentially the same as in the pure case (see e.g. [4, 13]). The operators of T with a counterpart in T/G are the gauge-invariant ones, while we also add the $(d-2)$ dimensional operators in the twisted sector of G. Indeed the topological operators $\widetilde{U}_g[\Sigma^{(d-1)}]$ become trivial in T/G, and their boundary operators turn into genuine operators (on average). Finally, since $A \in H^1(X^{(d)}, G)$ is dynamical, T/G has a dual symmetry generated by the Wilson lines of the G gauge field. This is a $(d-2)$-form symmetry whose charged objects are the operators coming from the twisted sectors of G. For G abelian the symmetry is the Pontryagin dual $G^\vee$, while it is a non-invertible symmetry in the non-abelian case [13].

3.2.3 Disordered symmetries and the replica trick

Disordered systems are often treated by means of the *replica trick*, which expresses the averaged correlation functions as certain limits of correlation functions of a standard QFT, the replica theory. In this section we interpret the disordered symmetries from the point of view of the replica theory. In addition to provide a sanity check of the results found in section 3.2.2, the method of replicas allows us to consider emergent symmetries in the disordered theory for which the results in the previous section do not apply. We will discuss emergent symmetries in section 3.2.4. For the rest of this section and the next section we assume a Gaussian probability distribution like (3.2.18) (and its generalization for complex h) with variance v.[36]

The replica trick

To fix our notation we briefly review the *replica trick*. This is a useful tool that allows to compute connected and full (i.e. both its connected and disconnected

[35] In the pure case it is possible to modify this sum weighting the terms with phases. Consistency conditions related with associativity constraint these phases to be of the form $\int_{X^{(d)}} A^*\nu$, where $\nu \in H^d(BG, U(1))$ is a *discrete torsion class* and we think A as a homotopy class of maps $X^{(d)} \to BG$, so that $A^*\nu \in H^d(X^{(d)}, U(1))$. Since the same kind of constraints are valid also in the disordered theories, we expect the very same modification of the gauging procedure to be possible also in this context.

[36] Normalization factors of $P[h]$, which ensure that probabilities add to one, will not play a role in our considerations and are then left implicit.

parts) correlators of the disordered theory as limits of correlators of a pure theory. The starting point of the replica trick is the identity

$$W = \log Z = \lim_{n\to 0}\left(\frac{\partial Z^n}{\partial n}\right) . \tag{3.2.74}$$

The idea is to replicate the pure system n times, indexing each copy with a label a

$$Z^n[h, K_i] = \int \prod_{a=1}^{n} \mathcal{D}\mu_a \, \exp\left(-\sum_a S_{0,a} - \sum_a \int h(x)\mathcal{O}_{0,a}(x) + \sum_{i,a}\int K_i(x)\mathcal{O}_{i,a}(x)\right) , \tag{3.2.75}$$

with the same random field coupling h and external sources K_i for all replicas. When $P[h]$ is Gaussian the average over $h(x)$ can be performed explicitly and we get

$$W_n[K_i] := \int \mathcal{D}h \, P[h] Z^n[h, K^i] = \int \prod_{a=1}^{n} \mathcal{D}\mu_a \, e^{-S_{\text{rep}} + \sum_{i,a}\int K_i \mathcal{O}_{i,a}} , \tag{3.2.76}$$

where

$$S_{\text{rep}} = \sum_a S_{0,a} - \frac{v}{2}\sum_{a,b}\int d^d x \, \mathcal{O}_{0,a}(x)\mathcal{O}_{0,b}(x) \tag{3.2.77}$$

is the replica action. We see how a coupling between the replica theories has been generated after the average. Renormalization will possibly induce other couplings in the replica theory, all compatible with the symmetries of the system. Among these, importantly the replica theory enjoys an S_n replica symmetry that permutes the various copies of the pure theory. We now assume that W_n can be analytically continued for arbitrary values of n including the origin in the complex n-plane.[37] Using (3.2.74) we find

$$W_D = \lim_{n\to 0}\left(\frac{\partial W_n}{\partial n}\right) , \tag{3.2.78}$$

where W_D is defined in (3.2.23), and thus

$$\overline{\left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\right\rangle_c} = \lim_{n\to 0}\partial_n \left(\left\langle \sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\dots\right\rangle^{\text{rep}}\right) , \tag{3.2.79}$$

where we used the fact that

$$\lim_{n\to 0} W_n[K_i] = 1 . \tag{3.2.80}$$

[37] This is a notoriously subtle limit. In particular we can have the phenomenon of spontaneous replica symmetry breaking (see [216] and references therein). We assume in what follows that the replica symmetry is not spontaneously broken.

Note that the in the left hand side of (3.2.79) we have the connected part of the correlator (indicated with the subscript c) which is computed in the replica theory by a suitable limit of a full correlator. Moreover, we see from (3.2.79) that a local operator $\mathcal{O}$ inside connected correlators of the disordered theory is mapped in the replica theory to its S_n-singlet component $\sum_a \mathcal{O}_a$.

The replica trick is also useful to compute general correlation functions in the disordered theory. Denoting by

$$S_a[h] = S_{0,a} + \int h(x)\mathcal{O}_{0,a}(x)\,, \tag{3.2.81}$$

we have

$$\begin{aligned}\overline{\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\rangle} &= \int \mathcal{D}h\, P[h] \frac{\int \mathcal{D}\mu\, e^{-S[h]}\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)}{Z[h]} \\ &= \int \mathcal{D}h\, P[h] \frac{\int \prod_a \mathcal{D}\mu_a\, e^{-\sum_a S_a[h]}\mathcal{O}_{1,1}(x_1)\dots\mathcal{O}_{k,1}(x_k)}{Z[h]^n}\,,\end{aligned} \tag{3.2.82}$$

which is an identity for any positive integer n. Assuming again that it can be analytically continued for $n \to 0$ we get[38]

$$\overline{\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\rangle} = \lim_{n\to 0}\langle \mathcal{O}_{1,1}(x_1)\dots\mathcal{O}_{k,1}(x_k)\rangle^{\text{rep}}\,. \tag{3.2.83}$$

In general correlators, in contrast to connected correlators, local operators are mapped to a specific copy (the same for all operators in the correlation function) in the replica theory. Equation (3.2.83) can easily be generalized to averages of products of general correlation functions. For example, omitting for simplicity the x-dependence of the local operators, we have

$$\begin{aligned}\overline{\Big\langle \prod_{i=1}^k \mathcal{O}_i^{(1)}\Big\rangle\Big\langle \prod_{j=1}^l \mathcal{O}_j^{(2)}\Big\rangle} &= \lim_{n\to 0}\int \mathcal{D}h\, P[h] \frac{\int \prod_a\ \mathcal{D}\mu_a\, e^{-\sum_a S_a[h]} \prod_{i=1}^k \mathcal{O}_{i,1}^{(1)} \prod_{j=1}^l \mathcal{O}_{j,2}^{(2)}}{Z^n[h]} \\ &= \lim_{n\to 0}\Big\langle \prod_{i=1}^k \mathcal{O}_{i,1}^{(1)} \prod_{j=1}^l \mathcal{O}_{j,2}^{(2)}\Big\rangle^{\text{rep}}\,,\end{aligned} \tag{3.2.84}$$

and similarly for more than two products. The last observables which we need to evaluate are averages of products of N connected correlators. Before averaging, these correlators are obtained by taking functional derivatives of the product

[38]Note that we have actually taken the limit $n \to 0$ in the denominator of (3.2.82) ($Z^n[h] \to 1$) before integrating over h, while in the numerator it is kept after the integration over h.

$W[K_i^{(1)}]\cdots W[K_i^{(N)}]$. For each of them we can use the replica trick to express this product as a unique path integral. We then have

$$\overline{\prod_{l=1}^{N}\Big\langle\prod_{j_l=1}^{k_l}\mathcal{O}_{j_l}^{(l)}\Big\rangle_c}=\Big(\prod_{k=1}^{N}\lim_{n_k\to 0}\frac{\partial}{\partial n_k}\Big)\Big\langle\prod_{l=1}^{N}\prod_{j_l=1}^{k_l}\sum_{a_{j_l}^{(l)}=1}^{n_l}\mathcal{O}_{j_l,a_{j_l}^{(l)}}^{(l)}\Big\rangle^{\text{rep}}, \tag{3.2.85}$$

where S_{rep} is the replica theory for $n=\sum_{i=1}^{N}n_i$ replicas. Note that averages of products of general or connected correlators in the disordered theory are always expressed in the replica theory as suitable limits of a *single general* correlator. Since any correlator can be expanded in its connected components, (3.2.85) is actually sufficient to compute generic correlation functions of the disordered theory. Any operator of the disordered theory gives rise to a multiplet transforming in the n-dimensional (natural) representation of S_n. Averages of connected correlators of operators of the disordered theory are given by the S_n singlet operators inside the natural representation in the replica theory. More general correlation functions of the disoredered theory are instead given by considering operators singlets under subgroups $S_{n_i}\subset S_n$ induced by the natural representation in the replica theory.

Disordered symmetries from replica theory

Our first task is to understand how disordered symmetries manifest themselves in the replica theory. For concreteness we consider again the case of a $G=U(1)$ symmetry, the replica action reads

$$S_{\text{rep}}=\sum_a S_{0,a}-\frac{v}{2}\sum_{a,b}\int d^dx\,\mathcal{O}_{0,a}(x)\overline{\mathcal{O}}_{0,b}(x)\,. \tag{3.2.86}$$

The $U(1)^n$ symmetry of the replicated pure part is broken by the disorder coupling to its diagonal $U(1)$ subgroup, which is then a symmetry of the replica theory. In particular there is a conserved current

$$J_D^\mu=\sum_a J_a^\mu \tag{3.2.87}$$

constructed as the S_n singlet out of the multiplet induced by the current J^μ of the disordered theory.

We can recover the Ward identities of the disordered symmetry from those produced by J_D^μ in the replica theory by using (3.2.85) for averages of products of connected correlators. The general key idea is to write a sum of averages of products

of connected correlators with current insertions that, once mapped to correlators of the replica theory, reconstruct the complete diagonal current J_D^μ. Then we can use the Ward identity in the replica theory and finally we rewrite the results back in terms of the disordered theory.

Determining the Ward identities for averages of single connected correlators is simple, because the diagonal current J_D appears directly in the replica theory and we can immediately use the ordinary Ward identities there. We have

$$\begin{aligned}\overline{\left\langle \partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\right\rangle}_c &= \lim_{n\to 0}\partial_n\Big(\Big\langle J_D^\mu(x)\sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\dots\Big\rangle^{\text{rep}}\Big)\\ &= \sum_i q_i\delta^{(d)}(x-x_i)\lim_{n\to 0}\partial_n\big\langle\sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\dots\big\rangle^{\text{rep}}\\ &= \sum_i q_i\delta^{(d)}(x-x_i)\overline{\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\rangle}_c\,, \end{aligned} \tag{3.2.88}$$

which reproduces the connected version of (3.2.41). Averages of products of connected correlators are also easy to treat, because it is enough to consider a sum of correlators where the current is inserted in each term to reconstruct J_D in the replica theory and then use the Ward identities there. Skipping obvious steps, we get

$$\sum_{j=1}^N \overline{\langle\partial_\mu J^\mu(x)\mathcal{O}_1^{(j)}\dots\mathcal{O}_{k_j}^{(j)}\rangle_c\Big(\prod_{l\neq j}\langle\mathcal{O}_1^{(l)}\dots\mathcal{O}_{k_l}^{(l)}\Big\rangle_c\Big)} = \sum_{j=1}^N\sum_{i_j=1}^{k_j}\delta_{i_j^{(j)}}q_{i_j}^{(j)}\overline{\prod_{l=1}^N\langle\mathcal{O}_1^{(l)}\dots\mathcal{O}_{k_l}^{(l)}\rangle_c}\,, \tag{3.2.89}$$

which is similar to (3.2.45), but expressed in terms of connected correlators and the unshifted current.

Due to the different way the replica trick handles connected and general correlators, determining the Ward identities for the latter will produce the improved current $\widetilde{J}_\mu$. We use (3.2.83) to write

$$\begin{aligned}&\overline{\langle\partial^\mu J_\mu\mathcal{O}_1\cdots\mathcal{O}_n\rangle} = \lim_{n\to 0}\langle\partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\\ &= \lim_{n\to 0}\langle\partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}} - \lim_{n\to 0}\frac{1}{n-1}\langle\sum_{a=2}^n\partial^\mu J_{\mu,a}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\\ &+\lim_{n\to 0}\langle\partial^\mu J_{\mu,2}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\,.\end{aligned} \tag{3.2.90}$$

In the last step, the last two terms add to zero due to the S_n symmetry enjoyed by

the replica theory. In the limit $n \to 0$ we have

$$\begin{aligned}&\lim_{n\to 0}\langle \partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\rm rep} - \lim_{n\to 0}\frac{1}{n-1}\langle\sum_{a=2}^{n}\partial^\mu J_{\mu,a}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\rm rep}\\ &=\lim_{n\to 0}\langle \partial^\mu J_{\mu,D}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\rm rep}\end{aligned} \tag{3.2.91}$$

and

$$\lim_{n\to 0}\langle \partial^\mu J_{\mu,2}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\rm rep} = \overline{\langle \partial^\mu J_\mu\rangle\langle\mathcal{O}_1\cdots\mathcal{O}_k\rangle}\,. \tag{3.2.92}$$

Therefore, by using the standard Ward identities of the replica theory, from (3.2.90) we get (3.2.41), as expected. The Ward identities (3.2.45) for products of generic correlators can be derived using a similar treatment:

$$\begin{aligned}&\sum_{j=1}^{N}\overline{\langle\partial_\mu J^\mu\mathcal{O}_1^{(j)}\cdots\mathcal{O}_{k_j}^{(j)}\rangle\left(\prod_{l\neq j}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle\right)} = \lim_{n\to 0}\sum_{j=1}^{N}\langle\partial^\mu J_{\mu,j}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\rm rep}\\ &=\lim_{n\to 0}\sum_{j=1}^{N}\langle\partial^\mu J_{\mu,j}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\rm rep} - \lim_{n\to 0}\frac{N}{n-N}\langle\sum_{a=N+1}^{n}\partial^\mu J_{\mu,a}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\rm rep}\\ &+\lim_{n\to 0}N\langle\partial^\mu J_{\mu,N+1}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\rm rep}\\ &=\sum_{j=1}^{N}\sum_{i_j=1}^{k_j}q_{i_j}^{(j)}\delta^{(d)}(x-x_{i_j}^{(j)})\overline{\prod_{l=1}^{N}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle} + N\overline{\langle\partial^\mu J_\mu\rangle\prod_i\langle\mathcal{O}_1^{(i)}\cdots O_{k_i}^{(i)}\rangle}\,.\end{aligned} \tag{3.2.93}$$

The last term in the right-hand-side in the third row of (3.2.93) precisely combines with the left-hand-side to reproduce the shifted current $\widetilde{J}_\mu$ and hence the Ward identities (3.2.45).

The above analysis shows that the replica counterpart of the disordered symmetry is an ordinary symmetry generated by the diagonal current J_D^μ and all the Ward identities of the disordered theory reduce to Ward identities involving J_D^μ in the replica theory. The exotic selection rules (see discussion around (3.2.46)) of the disordered symmetry are a consequence of the non-trivial map between the observables of the replica theory and those in the theory with quenched disorder.

3.2.4 Disordered emergent symmetries and LogCFTs

Our analysis of Ward identities in section 3.2.2 applies for disordered symmetries, namely symmetries which are present in the underlying UV theory, are broken by

the disorder, and get restored after disorder average. On the other hand, as in pure theories, we can have genuinely emergent symmetries in the IR, namely symmetries which are not present in the UV theory even before adding the disorder coupling. If the symmetry emerges for each theory in the ensemble, then we expect that it gives rise to approximate selection rules of the same kind as in pure theories with emergent symmetries in the IR. However, we could also have symmetries that emerge in the IR only *after* disorder average. By definition, this implies the existence of additional selection rules which are valid on average in the IR of the theory. For non-emergent, actual disordered symmetries such selection rules arise from a conserved current which is a shifted version of the current operator J^μ of the UV theory $\widetilde{J}^\mu = J^\mu - \langle J^\mu \rangle$. For emergent symmetries we cannot determine its explicit form, as the description in terms of the UV action is useless, and the analysis in section 3.2.2 does not hold. However, as we will see, we can deduce which are the selection rules that the emergent disordered symmetry imposes on averaged correlation functions using the replica theory.

From a symmetry point of view, the key qualitative feature of the replica theory (for any finite n) is the presence of a S_n global permutation symmetry not present in the original theory with disorder. In the analysis in section 3.2.3 the internal symmetry G generated by the current J_D^μ commutes with S_n, namely the infinitesimal transformations $\delta\mathcal{O}_{j,a}$ of the fields do not mix different replicas. This is guaranteed by the fact that G in the replica theory is the diagonal subgroup of the G^n global symmetry of the replica theories when $v = 0$. On the other hand, in the case of an emergent symmetry this is not necessarily the case: each irreducible representation of S_n can sit in a different G-representation, or even more generally, the local operators could sit in representations of the semi-direct product $G \rtimes S_n$. We expect that emergent symmetries in the replica theory of this kind correspond to *disordered emergent symmetries* in the theory with disorder. As we will see below, even in the deep IR the resulting selection rules will be modified with respect to those coming from (3.2.41) and its generalizations. As an application we will show how these modified Ward Identities allow for logarithmic conformal field theories (LogCFTs) as IR fixed points of disordered systems.

Emergent disordered symmetries

Let us analyze in some detail the Ward Identities for emergent symmetries in the replica theory. We study theories in which the total symmetry is a direct product

$G \times S_n$, since this particular case already exhibits interesting features. For further simplification, we consider $G = U(1)$ and correlators where only the singlet and the standard representations of S_n are involved. Generalizations to other representations of S_n or more general groups G should be straightforward.

Consider the average of a single correlation function of k local operators in the disordered theory. We consider both the general and the connected part of the correlator. Using (3.2.83) and (3.2.79), they are mapped in the replica to the $n \to 0$ limit of respectively $\langle \mathcal{O}_{1,1} \dots \mathcal{O}_{k,1} \rangle^{\text{rep}}$ and $\partial_n \langle \sum_{a_1} \mathcal{O}_{1,a_1} \dots \sum_{a_k} \mathcal{O}_{k,a_k} \rangle^{\text{rep}}$, omitting the space dependence of the operators in the correlators for simplicity. The replica theory is an ordinary pure theory and the emergent symmetry should manifest with the existence of a vector local operator J_D^μ, which becomes conserved in the IR. The operator J_D^μ is necessarily a singlet of S_n, since $U(1)$ commutes with S_n by definition. Note that we do not need to assume the knowledge of the full multiplet J_a^μ for which $J_D^\mu = \sum_{a=1}^n J_a^\mu$. Indeed, while in the UV, for weak disorder, the existence of vector operators in the natural representation of S_n is guaranteed, we do not need to keep track of the IR fate of the non-singlet components. Assuming that J_D^μ is conserved in the IR also at *finite* n, the following standard selection rules on k-point correlators apply:

$$\sum_{j=1}^{k} \langle \sum_{a_j=1}^{n} \delta \mathcal{O}_{j,a_j} \prod_{j \neq i=1}^{k} \sum_{a_i=1}^{n} \mathcal{O}_{i,a_i} \rangle^{\text{rep}} = 0 \,, \tag{3.2.94}$$

$$\sum_{j=1}^{k} \langle \delta \mathcal{O}_{j,1} \prod_{j \neq i=1}^{k} \mathcal{O}_{i,1} \rangle^{\text{rep}} = 0 \,. \tag{3.2.95}$$

The key point is now to look more closely to the variations $\delta \mathcal{O}_{j,a_j}$. Indeed, the natural representation of S_n is reducible and the $\mathcal{O}_i$'s split in

$$\mathcal{O}_i^{(S)} = \sum_{a=1}^{n} \mathcal{O}_{i,a} \ , \qquad \mathcal{O}_{i,a}^{(F)} = \mathcal{O}_{i,a} - \frac{1}{n} \mathcal{O}_i^{(S)} \,, \tag{3.2.96}$$

which transform in the singlet and in the standard, or fundamental, representation respectively.[39] The $U(1)$ symmetry acts as

$$\delta \mathcal{O}_i^{(S)} = q_{S,i} \mathcal{O}_i^{(S)}, \qquad \delta \mathcal{O}_{i,a}^{(F)} = q_{F,i} \mathcal{O}_{i,a}^{(F)} \ , \tag{3.2.97}$$

[39] More general representations arise for composite operators of the disordered theory which, once replicated, correspond to multiplets of S_n transforming in a (reducible) tensor product of two or more natural representations.

where the charges are generically different, $q_{S,i} \neq q_{F,i}$, and can possibly depend on n. The variations entering the Ward identities of the replica theory are then

$$\delta\mathcal{O}_{i,a} = \delta\mathcal{O}_{i,a}^{(F)} + \frac{1}{n}\delta\mathcal{O}_{i}^{(S)} = q_{F,i}\mathcal{O}_{i,a} + \frac{\Delta q_i}{n}\sum_{a=1}^{n}\mathcal{O}_{i,a}\,, \tag{3.2.98}$$

where

$$\Delta q_i := q_{i,S} - q_{i,F}\,. \tag{3.2.99}$$

Since in connected correlators we only have singlet components, plugging (3.2.98) in (3.2.94) gives simply

$$\sum_{j=1}^{k} q_{S,j}\langle\prod_{i=1}^{k}\sum_{a_i=1}^{n}\mathcal{O}_{i,a_i}\rangle^{\text{rep}} = 0\,. \tag{3.2.100}$$

On the other hand, plugging (3.2.98) in (3.2.95) equals

$$\begin{aligned}
0 &= \sum_{j=1}^{k} q_{F,j}\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} + \sum_{j=1}^{k}\frac{\Delta q_j}{n}\langle\sum_{b=1}^{n}\mathcal{O}_{j,b}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} \\
&= \sum_{j=1}^{k}\Big(q_{F,j} + \frac{\Delta q_j}{n}\Big)\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} + \sum_{j=1}^{k}\frac{\Delta q_j}{n}\langle\sum_{b=2}^{n}\mathcal{O}_{j,b}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} \\
&= \sum_{j=1}^{k} q_{F,j}\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} + \sum_{j=1}^{k}\Delta q_j\langle\mathcal{O}_{j,2}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} \\
&+ \frac{1}{n}\sum_{j=1}^{k}\Delta q_j\Big(\langle\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}} - \langle\mathcal{O}_{j,2}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\text{rep}}\Big)\,.
\end{aligned} \tag{3.2.101}$$

The existence of the limit $n \to 0$ requires that

$$\Delta q_j(n) = nK_j + O(n^2)\,, \qquad \text{as} \quad n \to 0\,, \tag{3.2.102}$$

where

$$K_j = \frac{\partial\Delta q_j}{\partial n}\bigg|_{n=0}\,. \tag{3.2.103}$$

We can use (3.2.102) to go back to the averaged correlators of the disordered theory and obtain the desired selection rules

$$\sum_{j=1}^{k} q_j\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle} + \sum_{j=1}^{k} K_j\Big(\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle} - \overline{\langle\mathcal{O}_j\rangle\langle\prod_{j\neq i=1}^{k}\mathcal{O}_i\rangle}\Big) = 0\,, \tag{3.2.104}$$

$$\sum_{j=1}^{k} q_j\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle_c} = 0\,, \tag{3.2.105}$$

where

$$q_j = q_{F,j}|_{n=0} = q_{S,j}|_{n=0}\,, \quad j = 1, \ldots, k\,. \tag{3.2.106}$$

A similar analysis can be repeated for averages of products of correlation functions of the kind (3.2.43). We report here only the final result:

$$\sum_{m=1}^{N}\sum_{j=1}^{k_m}\Bigg[\left(q_j^{(m)} + K_j^{(m)}\right)\overline{\prod_{l=1}^{N}\langle\Upsilon^{(l)}\rangle} + K_j^{(m)}\left(\sum_{a\neq m}\overline{\langle\Upsilon_j^{(m)}\rangle\langle\mathcal{O}_j^{(m)}\Upsilon^{(a)}\rangle\prod_{l\neq m,a}\langle\Upsilon^{(l)}\rangle} - N\overline{\langle\mathcal{O}_j^{(m)}\rangle\langle\Upsilon_j^{(m)}\rangle\prod_{l\neq m}\langle\Upsilon^{(l)}\rangle}\right)\Bigg] = 0 \tag{3.2.107}$$

where we introduced the notations

$$\Upsilon^{(l)} = \prod_{i=1}^{k_l}\mathcal{O}_i^{(l)}\,, \quad \Upsilon_j^{(l)} = \prod_{i=1, i\neq j}^{k_l}\mathcal{O}_i^{(l)}\,. \tag{3.2.108}$$

When $K_j = 0$, the selection rules (3.2.104) are the standard ones associated to a $U(1)$ conserved symmetry, while for $K_j \neq 0$ we get additional terms which affect the disconnected component of the correlator only, given that the connected part satisfies the ordinary selection rule (3.2.105). The fact that (3.2.105) holds implies that in the disordered theory we have a notion of operators $\mathcal{O}_i$ carrying a definite $U(1)$ charge q_i, yet in disconnected correlators some effect is responsible for the appearance of the extra terms proportional to K_j. It would be interesting to understand the origin of these extra factors directly from the disordered theory.

For $k = 2$, (3.2.104) and (3.2.105) simplify and can be rewritten as

$$\begin{aligned}(q_1 + q_2)\overline{\langle\mathcal{O}_1\rangle_c\langle\mathcal{O}_2\rangle_c} + (K_1 + K_2)\overline{\langle\mathcal{O}_1\mathcal{O}_2\rangle_c} &= 0\,,\\ (q_1 + q_2)\overline{\langle\mathcal{O}_1\mathcal{O}_2\rangle_c} &= 0\,.\end{aligned} \tag{3.2.109}$$

If $K_1 + K_2 \neq 0$, independently of the value of $q_1 + q_2$, the connected part of the 2-point function has to vanish and only a disconnected component is allowed. We are not aware of disordered theories with $K_j \neq 0$ for an internal global symmetry. On the other hand, we will show in the next section that the exotic selection rules derived above, applied to the case of emergent conformal symmetry, are at the origin of the possible appearance of logarithmic CFTs in the IR of disordered theories.

LogCFTs

Infrared fixed points of theories with quenched disorder can be described by non-unitary LogCFTs, first discussed in 2d [201, 202]. See e.g. [217] for a review of 2d LogCFTs or [203] for an introduction to LogCFTs in d dimensions from an axiomatic point of view. It was recognized in [202] that LogCFTs are intrinsically associated in having primary operators that are highest weight of indecomposable but not irreducible representations of the conformal group. A derivation of how LogCFTs can arise as random fixed points was given in [204] and more recently in [209] by means of (suitable generalizations of) Callan-Symanzik equations, in both cases using replica methods. We provide here an alternative derivation, working out the generalization of (3.2.109) when the emergent group is assumed to be the conformal one.

In the IR fixed point of the replica theory we have a dilatation current J_d^μ which yields the topological dilatation operator

$$D\left[\Sigma^{(d-1)}\right] = \int_{\Sigma^{(d-1)}} J_d^\mu n_\mu \ . \tag{3.2.110}$$

The conformal Ward identities applied to a primary operator $\mathcal{O}$ imply

$$D\left[\Sigma_x^{(d-1)}\right]\mathcal{O}(x) = \delta_D\mathcal{O}(x) + \mathcal{O}(x)D\left[\Sigma_{\text{no}\,x}^{(d-1)}\right] \ , \tag{3.2.111}$$

where

$$\delta_D\mathcal{O} = \left(\Delta + x^\mu\partial_\mu\right)\mathcal{O}(x)\,, \tag{3.2.112}$$

$(\Sigma_{\text{no}\,x}^{(d-1)})$ $\Sigma_x^{(d-1)}$ is a closed codimension 1 surface (not) encircling x. The dilatation operator acts diagonally only on the irreducible representations (3.2.96):

$$\delta_D\mathcal{O}_i^{(S)}(x) = \left(\Delta_{S,i}(n) + x^\mu\partial_\mu\right)\mathcal{O}_i^{(S)}(x) \ , \quad \delta_D\mathcal{O}_{i,a}^{(F)}(x) = \left(\Delta_{F,i}(n) + x^\mu\partial_\mu\right)\mathcal{O}_{i,a}^{(F)}(x) \ . \tag{3.2.113}$$

Thus on $\mathcal{O}_{i,a}(x)$ we have

$$\delta_D\mathcal{O}_{i,a}(x) = \left(\Delta_{F,i} + x^\mu\partial_\mu\right)\mathcal{O}_{i,a}(x) + \frac{\Delta_{S,i} - \Delta_{F,i}}{n}\sum_{\alpha=1}^{n}\mathcal{O}_{i,\alpha}(x)\,, \tag{3.2.114}$$

where in general $\Delta_{S,i}(n) \neq \Delta_{F,i}(n)$ for finite n. We plug the above transformations in (3.2.95) with $k = 2$ and equal operators. In this way we find the analogues of (3.2.109) for scaling transformations:

$$\begin{aligned}\left(x^\mu\partial_\mu + 2\Delta\right)\overline{\langle\mathcal{O}(x)\rangle_c\langle\mathcal{O}(0)\rangle_c} + 2K\overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_c} &= 0\,,\\ \left(x^\mu\partial_\mu + 2\Delta\right)\overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_c} &= 0\,,\end{aligned} \tag{3.2.115}$$

where

$$\Delta := \Delta_F|_{n=0} = \Delta_S|_{n=0}\,, \qquad\qquad K = \partial_n(\Delta_S - \Delta_F)|_{n=0}\,. \tag{3.2.116}$$

The general solution of (3.2.115) reads

$$\begin{aligned} \overline{\langle \mathcal{O}(x)\mathcal{O}(0)\rangle}_c &= \frac{c_1}{|x|^{2\Delta}} \\ \overline{\langle \mathcal{O}(x)\mathcal{O}(0)\rangle} &= \frac{c_2}{|x|^{2\Delta}} - \frac{c_1 \log(\mu|x|)}{|x|^{2\Delta}}\,, \end{aligned} \tag{3.2.117}$$

where $c_{1,2}$ are two integration constants with mass dimension -2Δ and μ is an arbitrary mass scale. Note that in a LogCFT, due to the peculiar way dilatations act on operators, the presence of a mass scale is actually compatible with conformal symmetry (see e.g. [203] for a more detailed explanation). We see that the log term arises when $K \neq 0$, which acts as a source term in the second equation in (3.2.115).

Whenever the LogCFT has some internal global symmetry G which is not emergent in the IR but is an exact symmetry present along the whole RG flow (i.e. present for each member of the ensemble and not broken by the disorder), the derivation above shows that logarithms can only appear in two-point functions of operators singlets under G. Indeed, in the replica theory the symmetry G gets replicated in n (unbroken) copies G_a, while the conformal symmetry generally is not, being only emergent at the fixed point. A representation ρ of G acting on a primary operator $\mathcal{O}$ is then replicated into n copies ρ_a, each acting only on $\mathcal{O}_a$. Let $g \in G_a$, by simple manipulations we get

$$\begin{aligned} \rho_a(g)\cdot\mathcal{O}^{(S)} &= \rho_a(g)\cdot\mathcal{O}_a - \mathcal{O}_a + \mathcal{O}^{(S)} \\ &= (\rho_a(g) - \mathbb{1})\cdot\mathcal{O}_a^{(F)} + \frac{1}{n}\left(\rho_a(g) + (n-1)\mathbb{1}\right)\cdot\mathcal{O}^{(S)}\,. \end{aligned} \tag{3.2.118}$$

Since G_a are internal symmetries, which necessarily commute with the dilatation operator D, we have

$$0 = [D, \rho_a(g)]\cdot\mathcal{O}^{(S)} = (\Delta_F - \Delta_S)\left(\rho_a(g) - \mathbb{1}\right)\cdot\mathcal{O}_a^{(F)}\,. \tag{3.2.119}$$

Unless ρ is in the trivial representation, the only solution of (3.2.119) is

$$\Delta_S(n) = \Delta_F(n)\,, \tag{3.2.120}$$

which implies that the factor K defined in (3.2.116) vanishes, and thus logharithms cannot appear in the two-point function of $\mathcal{O}$ at the IR fixed point.

3.2.5 Symmetries in ensemble average

We discuss in this section the case in which the random coupling is taken to be constant:

$$h(x) \to h\,. \tag{3.2.121}$$

Such set-up, which does not physically describe impurities as in quenched disorder, is particularly interesting in the light of the recent understanding of the role of average QFTs in the AdS/CFT correspondence [178]. As in the case of quenched disorder, we are interested in the situation where a symmetry is explicitly broken in any element of the ensemble and we want to see when and under which conditions it can emerge after the average. To distinguish them from the case of disordered systems, we will call these symmetries *averaged symmetries*. A notable example of this kind is the $O(N)$ symmetry in the SYK model [175–177] which rotates the N Majorana fermions, broken by the random fermion coupling, and restored after average (provided the average is taken with an $O(N)$-invariant distribution, as is often the case).

We will see that the simple replacement (3.2.121) leads to crucial differences with respect to the quenched disorder case. We discuss the importance of connectedness of the full space in section 3.2.5, we derive the Ward identities and the topological operators emerging after ensemble average in section 3.2.5, and finally in section 3.2.5 we comment on the implications of our results in the context of the AdS/CFT correspondence where the ensemble average is supposed to be the dual theory of a bulk theory of gravity in $d+1$ dimensions.

Selection rules in disconnected spaces

The presence of a constant random coupling h over the entire space $X^{(d)}$ leads to a new effect, not present in the quenched disorder, which is the lack of factorization of correlation functions in *disconnected* spaces. For definiteness, consider a theory deformed by a random coupling h in a space $X^{(d)}$ which is the union of two spaces $X^{(d)} = X_1^{(d)} \sqcup X_2^{(d)}$, with $X_1^{(d)} \cap X_2^{(d)} = \emptyset$. At this stage we are not specifying whether the coupling is a constant or not, we only assume that it breaks a global 0-form symmetry G of the pure theory. For each element of the ensemble we can define a generating functional introducing sources K_i for the local operators $\mathcal{O}_i$. Since the space manifold is disconnected, for each local operator $\mathcal{O}$ we effectively need two sources, K_1 and K_2, defined in $X_1^{(d)}$ and $X_2^{(d)}$. For any h, constant or not,

the total functional factorizes[40]

$$Z[X^{(d)}, K, h] = Z[X_1^{(d)}, K_1, h]\; Z[X_2^{(d)}, K_2, h]\,, \tag{3.2.122}$$

and so will do arbitrary correlation functions of local operators Φ:

$$\langle \Phi \rangle_X = \langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle_{X_2}\,, \tag{3.2.123}$$

with obvious notation. When h is space dependent (quenched disorder), its support and its probability measure splits into X_1 and X_2. Hence quenched averaged correlators factorize in the two distinct components:[41]

$$\overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle_{X_2}} = \overline{\langle \Phi_1 \rangle}_{X_1}\, \overline{\langle \Phi_2 \rangle}_{X_2}\,. \tag{3.2.124}$$

Thanks to this factorization, the selection rules of the disordered theory are realized independently on each connected component:

$$\overline{\langle \Phi_i \rangle}_{X_i} = R_i\, \overline{\langle \Phi_i \rangle}_{X_i}\,, \qquad i = 1, 2, \qquad \text{(quenched disorder)}\,, \tag{3.2.125}$$

where R_i are the direct products of the representations of the local operators in $X_i^{(d)}$, which should each contain a singlet to get a non-vanishing correlator.

Crucially, in the ensemble average case (3.2.124) cannot hold, because a constant h does not split on the connected components and the average *correlates* the operators across $X_1^{(d)}$ and $X_2^{(d)}$. In particular, we now get the selection rules

$$\overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle}_{X_2} = R_1 \cdot R_2\, \overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle}_{X_2}\,, \qquad \text{(ensemble average)}\,. \tag{3.2.126}$$

In contrast to the quenched disorder case, *averages of single correlators in the ensemble average effectively turn into averages of products of correlators when the space is disconnected.* The constraint (3.2.126) is weaker than (3.2.125), obtained in the quenched average theory. In (3.2.126) we need the singlet to appear only in the product $R_1 \cdot R_2$, in (3.2.125) separately for R_1 and R_2. For symmetries that emerge after ensemble average, which we dub *average* symmetries, the charge is then *not* conserved on a single connected component of the manifold, but can "escape" to the

[40] This follows from the observation that any map whose domain is disconnected can be written uniquely as a sum of maps each supported in a connected component.

[41] It should not be confused this factorization of correlators in disconnected space with the non-factorization of products of averaged correlators due to quenched disorder considered in section 3.2.2 and present in any space $X^{(d)}$, connected or not.

other connected components (see the end of appendix 3.3.3 for an explicit computation in a free scalar model). We will see how this relates to the violation of global symmetries by Euclidean wormholes in section 3.2.5. The above analysis is trivally generalized to a space with an arbitrary number of disconnected components and to arbitrary products of correlation functions of local operators.

Ensemble average and Ward identities

The analysis presented in section 3.2.2 can be repeated in the case of constant h. For concreteness we consider again the case in which the pure theory has a $U(1)$ global symmetry under which $\mathcal{O}_0$ has charge q_0. We have one complex parameter h and the average generating functional is

$$\overline{Z[K_i]} = \int dh d\bar{h}\, P[\bar{h}h]\, \frac{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0+c.c.)+\int K_i\mathcal{O}_i}}{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0+c.c.)}}\,. \tag{3.2.127}$$

We derive identities between correlators by changing variables inside the various integrals in (3.2.127). By changing variable in the numerator with an infinitesimal space-dependent symmetry transformation of parameter $\epsilon(x)$, we get

$$\langle \partial_\mu J^\mu(x)\Phi\rangle = \sum_i \delta^{(d)}(x-x_i) q_i \langle\Phi\rangle + q_0\langle \mathcal{D}(x)\Phi\rangle\,, \tag{3.2.128}$$

where the sum runs over all the local operators defining Φ and we have defined

$$\mathcal{D}(x) := -h\mathcal{O}_0(x) + \bar{h}\,\overline{\mathcal{O}}_0(x)\,. \tag{3.2.129}$$

Note that (3.2.128) holds *before* taking the average. Indeed, this is nothing else than the Ward identities one obtains in a pure theory for an explicitly broken symmetry. We are now not allowed to do a change of variable in the h integral to possibly prove the vanishing on average of the last term in (3.2.128). However, we can perform a *global* transformation $h \to e^{-iq_0\epsilon}h$, with ϵ constant, inside (3.2.127). In this way, we get

$$\int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\Phi\rangle} = \int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\rangle\langle\Phi\rangle}\,, \tag{3.2.130}$$

where $X^{(d)}$ is the *full* space manifold. Finally we can perform a space dependent $U(1)$ transformation only in the path integral in the denominator of (3.2.127), getting

$$\langle \partial_\mu J^\mu\rangle = q_0\langle\mathcal{D}\rangle\,, \tag{3.2.131}$$

valid before ensemble average. From now on we will assume that $\mathcal{O}_0$ is a scalar under spatial rotations,[42] so that every element of the ensemble is $\mathfrak{so}(d)$ invariant. We then have $\langle J_\mu \rangle = 0$ and thanks to (3.2.131) the relation (3.2.130) simplifies to

$$\int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\Phi \rangle} = 0\,. \tag{3.2.132}$$

See appendix 3.3.3 for an explicit derivation of (3.2.132) for a two-point function in a simple solvable model. The combination $\partial^\mu J_\mu - q_0 \mathcal{D}(x)$ satisfies the condition

$$\int_{X^{(d)}} d^d x\, \overline{\left\langle \left(\partial^\mu J_\mu(x) - q_0 \mathcal{D}(x)\right) \Phi \right\rangle} = 0\,, \tag{3.2.133}$$

which ensures that the Ward identities (3.2.128), when integrated over the full space and after ensemble average, imply charge conservation. As expected from a spurionic argument, the symmetry is restored after average.[43]

Let us now see if we can define more general operators $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$, topological after ensemble average. The natural choice from (3.2.133) is

$$\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}] = Q[\Sigma^{(d-1)}] - q_0 \int_{D^{(d)}} d^d x\, \mathcal{D}(x)\,, \qquad Q[\Sigma^{(d-1)}] := \int_{\Sigma^{(d-1)}} n_\mu J^\mu(x)\,, \tag{3.2.134}$$

where $D^{(d)}$ is an arbitrary region such that $\partial D^{(d)} = \Sigma^{(d-1)}$. Note that this requires $\Sigma^{(d-1)}$ to be homologically trivial otherwise, by definition, the surface $D^{(d)}$ does not exist. In the terminology of [62], the operator (3.2.134) is a *non-genuine* co-dimension one operator, since it requires a topological surface attached to it.[44]

We can discuss the dependence of $\widehat{Q}$ in (3.2.134) on the choice of the filling region $D^{(d)}$. Given another such manifold $D'^{(d)}$ we can glue it along $\Sigma^{(d-1)}$ with the orientation reversal of $D^{(d)}$ to form a closed manifold $Y^{(d)} = D'^{(d)} \sqcup \overline{D^{(d)}}$, and $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$ is independent on $D^{(d)}$ if and only if

$$\int_{Y^{(d)}} \overline{\langle \mathcal{D}(x)\Phi \rangle} = 0\,. \tag{3.2.135}$$

[42]This assumption is not crucial. For non-scalar deformations, rotational invariance is broken before the average and we need to keep track of all the vacuum expectation values induced by the random variable, as done in the quenched disorder case. This can be repeated in the ensemble average case, but makes the analysis more involved.

[43]In a pure theory the identities (3.2.128) apply but $\mathcal{D}(x)$ does not integrate to zero when inserted in arbitrary correlators. As a consequence no selection rules are implied, as expected for an explicitly broken symmetry!

[44]The requirement is however of different nature. In [62] (and subsequent works) the surface is required to have a well-defined gauge-invariant operator, here the surface is required to make the operator topological (on average).

We see that (3.2.135) is not satisfied unless the space-time $X^{(d)}$ is connected, and we will generically refer to $\widehat{Q}$ as a non-genuine operator. On the other hand, if $X^{(d)}$ is connected any homologically trivial co-dimension one submanifold $\Sigma^{(d-1)}$ of $X^{(d)}$ divides $X^{(d)} - \Sigma^{(d-1)}$ in two disjoint connected components glued along $\Sigma^{(d-1)}$, hence necessarily $Y^{(d)} = X^{(d)}$ and (3.2.135) reduces to (3.2.132), showing the independence of $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$ on the filling region. $\widehat{Q}$ is still expressed with an integral over $D^{(d)}$, but the dependence of the non-genuine symmetry operator on the filling region is only apparent, and for all practical purposes this can be regarded as independent on the filling region. We refer to this situation as a quasi-genuine co-dimension one operator.

If $X^{(d)}$ has several connected components, $Y^{(d)}$ can be a proper sub-region, since adding or removing from it an entire connected component which does not intersect $\Sigma^{(d-1)}$ preserves the property that $Y^{(d)}$ is the union of regions glued along $\Sigma^{(d-1)}$. For instance if $X^{(d)}$ has two connected components $X_1^{(d)}$ and $X_2^{(d)}$, and suppose $\Sigma^{(d-1)}$ is entirely contained in $X_1^{(d)}$, the latter is divided by $\Sigma^{(d-1)}$ into two regions $D^{(d)}$ and $D'^{(d)}$, and choosing one or the other leads to different operators $\widehat{Q}[\Sigma^{(d-1)}]$, since (3.2.132) holds only in the entire space and not to each connected component:

$$\overline{\Big\langle \int_{D^{(d)}} \mathcal{D}(x)\Phi \Big\rangle} = \overline{\Big\langle \Big(\int_{D'^{(d)}} + \int_{X_2^{(d)}} \Big) \mathcal{D}(x)\Phi \Big\rangle} \neq \overline{\Big\langle \int_{D'^{(d)}} \mathcal{D}(x)\Phi \Big\rangle}\,. \tag{3.2.136}$$

In this case we cannot define a quasi-genuine co-dimension one topological operator and therefore, even if the total charge is conserved thanks to (3.2.133), we cannot measure it locally in a subregion of the entire (disconnected) space.

In order to measure the charge of operators in the whole space, we can consider $\widehat{Q}$ on a codimension 1 closed surface $\Sigma^{(d-1)} = \Sigma_1^{(d-1)} \sqcup \Sigma_2^{(d-1)}$, with $\Sigma_i^{(d-1)} \subset X_i^{(d)}$ $(i = 1, 2)$, and two regions $D_i^{(d)}$ such that $\partial D_i^{(d)} = \Sigma_i^{(d-1)}$. In each given connected component, the charge cannot be conserved, as we have seen, but if we simultaneously consider the two regions, then the Ward identities still apply. In the schematic notation of section 3.2.5 we have

$$\begin{aligned}\overline{\langle \widehat{Q}[\Sigma, D]\Phi\rangle}_X &= \overline{\langle \widehat{Q}[\Sigma_1, D_1]\Phi_1\rangle_{X_1} \langle \widehat{\Phi}_2\rangle}_{X_2} + \overline{\langle \Phi_1\rangle_{X_1} \langle \widehat{Q}[\Sigma_2, D_2]\Phi_2\rangle}_{X_2} \\ &= \Big(\chi_1(\Sigma_1) + \chi_2(\Sigma_2)\Big) \overline{\langle \widehat{\Phi}_1\rangle_{X_1} \langle \widehat{\Phi}_2\rangle}_{X_2} = \Big(\chi_1(\Sigma_1) + \chi_2(\Sigma_2)\Big) \overline{\langle \widehat{\Phi}\rangle}_X\,,\end{aligned} \tag{3.2.137}$$

where $\chi_{1,2}(\Sigma_{1,2})$ denotes the sum of the charges of the local operators $\Phi_{1,2}$ which are inside the surface $\Sigma_{1,2}^{(d-1)}$. Since $\Sigma^{(d-1)}$ depends now on $D^{(d)}$, it is crucial to

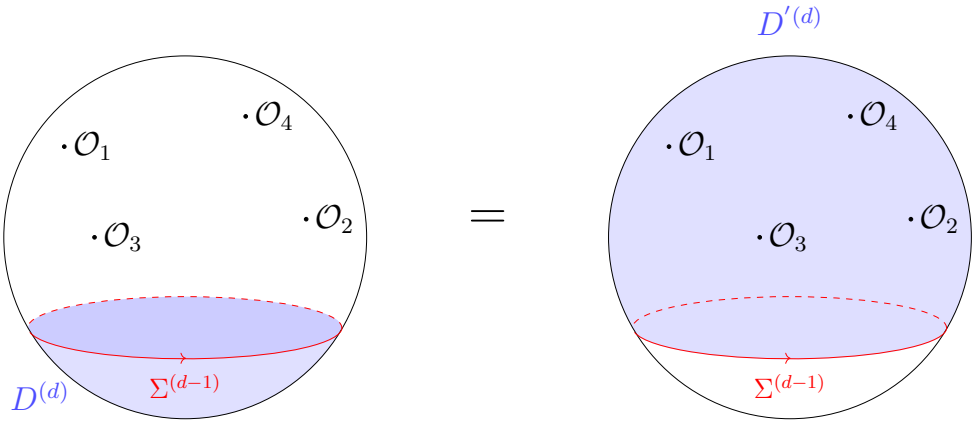


Figure 3.2: Selection rules (3.2.138) for correlators when $X^{(d)}$ is connected. The integral over the region $D^{(d)}$ in the left panel equals the integral over the region $D'^{(d)}$ in the right panel thanks to (3.2.132). When $D^{(d)}$ is shrunk to a point the region $D'^{(d)}$ extends to the whole $X^{(d)}$.

consider the complement space in both connected spaces at the same time. The generalization to spaces $X^{(d)}$ with more than two connected components is obvious.

We refer the reader to appendix 3.3.4 for a proof of the existence of the operator $\widehat{U}_g$ which implements the action of the group rather than the action of the corresponding Lie algebra. By definition, the operator $\widehat{U}_g$, given in (3.3.60), satisfies

$$\overline{\langle \widehat{U}_g[\Sigma^{(d-1)}, D^{(d)}]\mathcal{O}_1 \cdots \mathcal{O}_n \rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})}\overline{\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle}\,. \tag{3.2.138}$$

Since $\widehat{U}_g[\emptyset, X^{(d)}] = 1$, (3.2.138) implies the selection rules we derived from the spurion argument (see figure 3.2). The equivalent of (3.2.137) for a finite group action precisely reproduces the selection rule (3.2.126). With $\Sigma^{(d-1)}$ as in figure 3.3, we have

$$\overline{\langle \Phi \rangle}_X = \overline{\langle \widehat{U}_g[\Sigma, D]\Phi \rangle}_X = \overline{\langle \widehat{U}_g[\Sigma_1, D_1]\Phi_1 \rangle_{X_1} \langle \widehat{U}_g[\Sigma_2, D_2]\Phi_2 \rangle}_{X_2} = e^{i\alpha(\chi_1(\Sigma_1)+\chi_2(\Sigma_2)}\overline{\langle \Phi \rangle}_X \tag{3.2.139}$$

while, say,

$$\overline{\langle \Phi \rangle}_X = \overline{\langle \widehat{U}_g[\Sigma_1, D_1]\Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle}_{X_2} \neq e^{i\alpha\chi_1(\Sigma_1)}\overline{\langle \Phi \rangle}_X\,. \tag{3.2.140}$$

We have then found an instance of a theory with a global zero-form symmetry in the sense of giving rise to selection rules for correlation functions of local operators, but with *no* genuine co-dimension one topological operator. Aside of being topological

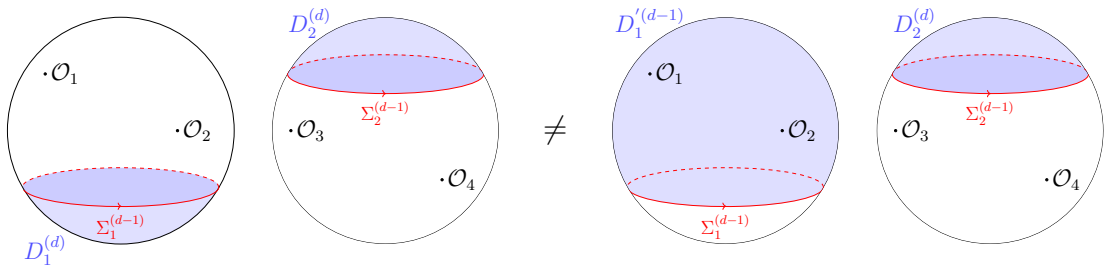


Figure 3.3: Violation of the selection rules (3.2.140) when $X^{(d)}$ is disconnected. The integral over the region $D_1^{(d)}$ in $X_1^{(d)}$ (left) is not equal to the integral over the region $D_1^{'(d)}$ in $X_1^{(d)}$ (right) because of the presence of the component $X_2^{(d)}$. An equality sign would require to reverse the region of integration also in $X_2^{(d)}$ (right) from $D_2^{(d)}$ to its complement.

only on average, the operator $\widehat{U}_g[\Sigma, D]$ is not genuine and it can be defined only on homologically trivial cycles.

The local charge violation (3.2.140) in a single connected component of space when $X^{(d)}$ is an union of several connected components indicate the presence of non-local interactions in the theory. Their presence is manifest by using the replica trick. Consider a Gaussian random distribution $P[\bar{h}h] \propto \exp(-\bar{h}h/v)$ (e.g. as in the SYK model). Repeating the steps described in section 3.2.3 we find *non-local* interactions among replicas

$$S_{\text{rep}} = \sum_{a=1}^{n} S_{0,a} - v \int d^d x \int d^d y \sum_{a,b=1}^{n} \overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y)\,. \tag{3.2.141}$$

The replica theory enjoys a diagonal $U(1)_D$ global symmetry, but the naive diagonal current $J_D^\mu = \sum_a J_a^\mu$ does not satisfy standard Ward identities. By performing an infinitesimal $U(1)$ transformation with a local parameter $\alpha(x)$ we get

$$\begin{aligned}\delta S_{\text{rep}} &= \int dx\, \alpha(x)\partial_\mu J_D^\mu(x) - q_0 v \sum_{a,b} \int dx\, dy \Big(\alpha(y) - \alpha(x)\Big)\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) \\ &= \int_{X^{(d)}} dx\, \alpha(x)\Big(\partial_\mu J_D^\mu(x) + q_0 v \sum_{a,b} \int_{X^{(d)}} dy \Big(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) - \mathcal{O}_{0,a}(x)\overline{\mathcal{O}}_{0,b}(y)\Big)\Big)\,.\end{aligned} \tag{3.2.142}$$

Thus the Ward identities for the diagonal symmetry are modified by a non-local

term and read

$$\begin{aligned}&\Big\langle\Big(\partial_\mu J^\mu_D(x)+q_0 v\sum_{a,b}\int_{X^{(d)}}d^dy\left(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y)-\overline{\mathcal{O}}_{0,a}(y)\mathcal{O}_{0,b}(x)\right)\Big)\Phi\Big\rangle^{\text{rep}}\\&\quad=\sum_i\delta^{(d)}(x-x_i)q_i\big\langle\Phi\big\rangle^{\text{rep}}\,.\end{aligned}\tag{3.2.143}$$

In the replica theory the operator

$$\partial_\mu J^\mu_D(x)+q_0 v\sum_{a,b}\int_{X^{(d)}}d^dy\,\left(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y)-\overline{\mathcal{O}}_{0,a}(y)\mathcal{O}_{0,b}(x)\right)\tag{3.2.144}$$

satisfies the Ward identities and its integral over the full space evidently vanishes (inside arbitrary correlators), implying the $U(1)_D$ selection rules. This is how the properties of the averaged symmetry show up in the replica theory, where the non-local nature of the symmetry is manifest for Gaussian distributions. The property (3.2.132) of the operator $\mathcal{D}(x)$ defined in (3.2.129) is mapped to the property of the extra term in (3.2.144) of integrating to zero exactly as an operator equation. This is consistent with the dictionary between correlators of the averaged theory and the replica one.

A gravity discussion

We have found that averaged global symmetries are intrinsically different from ordinary global symmetries. They imply selection rules as dictated by the global symmetry but, in contrast to ordinary global symmetries, they do not admit genuine co-dimension one operators, topological after average. Even in a connected space such operators cannot be defined in homologically non-trivial cycles. As a result, these symmetries cannot consistently be coupled to an external background field, at least not in a natural way.[45] Note that this is different from the concept of 't Hooft anomalies. In the latter the obstruction is in making the gauge fields dynamical but there is a well defined notion of coupling the theory to backgrounds gauge fields. The difficulty of coupling the symmetry to an external background is clear in the replica theory from the presence of the second term in (3.2.144), which is non-local and not manifestly the divergence of a current.

The results have interesting consequences when applied to averaged theories which are assumed to have an holographic dual bulk gravitational theory in asymptotically AdS space-times.

[45]For discrete symmetries, for example, coupling to an external background field corresponds to insert a mesh of symmetry defects on homologically non-trivial cycles of space.

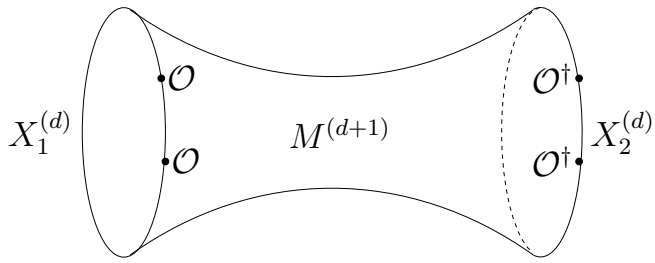


Figure 3.4: Example of a wormhole bulk geometry contributing to the average correlator $\overline{\langle \mathcal{O}\mathcal{O}\rangle\langle \mathcal{O}^\dagger\mathcal{O}^\dagger\rangle}$, with $\mathcal{O}$ a charged boundary operator.

In the ordinary AdS/CFT correspondence a given theory of gravity in asymptotically AdS space-time is dual to a given CFT. Ordinary global symmetries of the CFT become gauge symmetries in the bulk. This correspondence fits nicely with the widely accepted common lore that in quantum gravity unbroken global symmetries cannot exist [218–221]. A natural question then arises: when the dual theory is given by an ensemble average, what is the bulk interpretation of the symmetries emerging after average? In [205] (see also [206, 207]) it has been conjectured that boundary emergent symmetries correspond in the bulk to global, and not gauge, symmetries which are broken non-perturbatively by Euclidean wormhole configurations, which allow the global symmetry charge to flow from one connected component to another one, see figure 3.4. From the boundary point of view, this charge violation induced by bulk wormholes correspond to the lack of selection rules in the average theory that we have discussed before, when the space is not connected, in agreement with the findings in [205–207]. Since averaged symmetries simply cannot be gauged, our results clarify why they cannot be interpreted as gauge symmetries in the bulk, at least in the case where the average is of the form (3.2.14).[46]

Note that boundary emergent symmetries are compatible with recent works where, motivated by the connection with the lore of spectrum completeness in gravitational theories [222], "absence of global symmetries in gravitational theories" is replaced by "absence of topological operators", including those related to non-invertible symmetries [99, 223].

[46]In particular, our results do not straightforwardly apply when the average is over OPE coefficients, as e.g. discussed in [188, 189].

3.2.6 Conclusions

In this section we have studied disordered QFTs where an ordinary symmetry of a pure QFT is explicitly broken by a random coupling, but the symmetry re-emerges after quenched average. We focused our attention to understand if and under what conditions we can have operators, topological on average, in analogy to ordinary QFTs [4]. We considered *quenched disorder theories*, where the pure theory is deformed with a space dependent coupling, and *ensemble average theories*, where the latter is kept constant.

In the quenched disordered case, we can write Ward identities for averages of products of correlators and construct the symmetry operator implementing the finite group action, topological after average. Such disordered symmetries can be coupled to external background, can be gauged, and can have 't Hooft anomalies (i.e. can exclude a trivially gapped phase at long distances), precisely like ordinary symmetries. Using the replica trick, we also discussed genuinely emergent symmetries in the IR after average, namely symmetries which are not present in the UV theory even before adding the disorder coupling. We pointed out that whenever a symmetry G is emergent in the IR, exotic selection rules can explain the origin of LogCFTs.

In ensemble average theories the analogy to pure QFTs is more loose. We still have selection rules for averages of correlators and we can construct operators implementing the finite group action, but the charge operator is not purely codimension-1 and cannot be defined if $\Sigma^{(d-1)}$ is homologically non-trivial. When the space is disconnected, the selection rules apply only globally and in each connected component charge violation can occur. Such averaged symmetries cannot be coupled to background gauge fields in ordinary ways. The difficulty (impossibility) of gauging emergent boundary symmetries clarify why such symmetries cannot be identified with bulk gauge symmetries when the average theory admits a gravitational bulk dual.

It would be interesting to analyze spontaneous breaking of disordered symmetries in more detail.[47] There are essentially two ways in which the disordered symmetry could spontaneously break: i) the symmetry is spontaneously broken in the pure theory before adding the random interaction, ii) the symmetry is unbroken in the pure theory and the random interaction induces a spontaneous breaking of the disordered symmetry. Let us consider the case of continuous symmetries. From the

[47] We thank O. Aharony for a question that prompted the paragraph that follows.

replica theory point of view, i) and ii) are distinguished by which components of the replica currents J_a^μ are subject to spontaneous breaking, all components in case i) and only the singlet $\sum_a J_a^\mu$ in case ii). Assuming the existence of the analytic continuation in n and of a smooth $n \to 0$ limit, we expect for $d > 2$ gapless excitations (Goldstone boosons) in the replica theory, giving rise to power-like correlators. From the disordered theory point of view, in case i) there is a Goldstone mode in the pure theory which acquires a mass in each specific realization of the ensemble, turning into a pseudo Goldstone boson. In contrast, no Goldstone boson is present in the pure theory in the more exotic case ii). In both cases it would be nice to identify which correlators (if any) exhibit power like-behavior on average as a result of the spontaneous breaking of the disordered theory.

It would be also interesting to generalize our findings to quantum disorder, namely to Lorentzian theories where the random coupling depends only on space. The natural extension of our analysis beyond 0-form symmetries does not seem straightforward. Higher-form symmetries can be broken only by non-local deformations, which should be also taken random. It is possibly easier to consider a set-up in $d = 2$ where non-invertible symmetries can be obtained by 0-form symmetries only, and see if and in what sense we can have a non-invertible symmetry re-emerging after average.

An important remark about the ensemble average case is that, in comparing our findings with the existing literature on the factorization problem in AdS/CFT, one should keep in mind that we only considered averaging over couplings. There are other setups, like averaging over OPE coefficients [188, 189] or over different modular invariants [18], where global symmetries could behave differently from our findings. In particular [18] discusses the gauging of a 1-form global symmetry in certain gravitational toy models, but this is not in contrast with our result about the impossibility of gauging average 0-form symmetries. It is a very interesting problem for the future to discuss the status of global symmetries in these other contexts, possibly finding a unified picture.

3.3 Appendices

3.3.1 Appendix A: The Permutation Group S_N

In this appendix we describe some representations of S_N used in the main text. The simplest possible action is the natural representation, which consists of the

usual permutations of N variables. We collect the variables $\mathcal{F}_i$ in a vector $\boldsymbol{\mathcal{F}}$ and denote the action of $\sigma \in S_N$ as $\sigma \cdot \boldsymbol{\mathcal{F}} = (\mathcal{F}_{\sigma(i)}, .., \mathcal{F}_{\sigma(N)})$. Clearly this representation is reducible. The vectors with all equal entries are fixed by all permutations and span the one-dimensional trivial representation. The orthogonal complement of this subspace is given by those vectors whose components sum to zero. Thus we may construct an $N-1$ dimensional irreducible representation imposing the S_N-invariant constraint

$$\sum_{i=1}^{N} \mathcal{F}_i = 0. \tag{3.3.1}$$

This defines the standard representation, of dimension $N-1$, which we denote as $\mathfrak{S}$. We construct the dual $N-1$ dimensional representation as follows. Let us introduce another set of N variables u_i, collected in a vector $\boldsymbol{u}$, and consider the scalar product $\boldsymbol{u} \cdot \boldsymbol{\mathcal{F}} = \sum_{i=1}^{N} u_i \mathcal{F}_i$. We define the representation dual to the one carried by $\boldsymbol{\mathcal{F}}$ as the representation on $\boldsymbol{u}$ which preserves the scalar product. This means, for a pair of dual representations $R^\vee$ and R acting on vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively

$$R^\vee(\boldsymbol{a}) \cdot R(\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} \ . \tag{3.3.2}$$

Clearly acting with a permutation on the u_i or the $\mathcal{F}_i$ is equivalent, in this sense the natural representation is self-dual. For the standard representation, solving the constraint for $\mathcal{F}_N$, we get

$$\sum_{i=1}^{N} u_i \mathcal{F}_i = \sum_{i=1}^{N-1} (u_i - u_N) \mathcal{F}_i = \sum_{i=1}^{N-1} \alpha_i \mathcal{F}_i \ . \tag{3.3.3}$$

The $N-1$ coefficients $\alpha_i = u_i - u_N$ carry the representation $\mathfrak{S}^\vee$, dual to $\mathfrak{S}$, defined as

$$\sum_{i,k} \alpha_i (\mathfrak{S}_\sigma)_{ik} \mathcal{F}_k = \sum_{i,j} (\mathfrak{S}^\vee_\sigma)^{-1}_{ij} \alpha_j \mathcal{F}_i \to \mathfrak{S}^\vee_\sigma = (\mathfrak{S}_{\sigma^{-1}})^T \ . \tag{3.3.4}$$

Thus the explicit action of $\sigma \in S_N$ is

$$\mathfrak{S}^\vee_\sigma(\alpha_i) = u_{\sigma(i)} - u_{\sigma(N)} \ . \tag{3.3.5}$$

This representation respects the composition covariantly:

$$\mathfrak{S}^\vee_{\sigma_1} \circ \mathfrak{S}^\vee_{\sigma_2} = \mathfrak{S}^\vee_{\sigma_1 \circ \sigma_2} \ . \tag{3.3.6}$$

In the symmetric basis used in the text the action on the gauge fields is given by the standard representation. Consequently the action on both the charges of the Wilson lines and the continuous parameters of the GW operators is given by the dual representation $\mathfrak{S}^\vee$.

3.3.2 Appendix B: Some detail on pure Rep(G) gauge theories

As we discussed in the main text, the fusion coefficients of the condensation defects $\mathcal{C}_{\mathrm{Rep}(S_N)}$ involve the partition function of the pure $\mathrm{Rep}(S_N)$ gauge theory. Here we will provide some details on the characterization of the pure $\mathrm{Rep}(G)$ gauge theories for any finite group G, in terms commutative Frobenius algebras.

TQFTs in 2d are particularly simple because every space-like slice is a disjoint union of circles, and any compact surface Σ can be constructed by gluing pair of pants. Because of the first fact the only Hilbert space we need to assign is $\mathcal{H}_{S^1}$, while the second implies the well known result that 2d TQFTs are fully determined by commutative Frobenius algebra structure on this Hilbert space [224]. Formally this is obtained by specifying an associative and commutative multiplication $\mu : \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}$ and a linear map $\theta : \mathcal{H}_{S^1} \to \mathbb{C}$. More concretely, the Hilbert space inherits an algebra structure from the one of local operators, by using operator/state correspondence, while θ acts by taking the scalar product with the Hartle-Hawking state corresponding to the identity operator.

As a simple example, which is then easy to generalize to the case of $\mathrm{Rep}(G)$, consider the pure $\mathbb{Z}_N$ gauge theory in 2d. A clear construction is by starting from the trivial theory with $\mathbb{Z}_N$ symmetry, namely a theory of N line operators fusing according to $\mathbb{Z}_N$, and no local operator. Thus the Hilbert space is trivial, and the symmetry does not act on anything. Each line, however, has a non-empty twisted sector containing one operator. After gauging, these twist operators become local, are labeled by elements of $\mathbb{Z}_N$ and they fuse accordingly. There are also new line operators generating the dual $\widehat{\mathbb{Z}}_N = \mathrm{Hom}(\mathbb{Z}_N, U(1))$ symmetry. They are labeled by irreducible representations and act on local operators. By operators/state correspondence the Hilbert space is N dimensional, and it is in a (reducible) representation of the dual $\widehat{\mathbb{Z}}_N$ symmetry, given by the direct sum of all the irreducible representations, namely the regular representation. This has a clear interpretation in the context of gauging in fusion categories, which can be easily generalized. One can think the gauged $\mathbb{Z}_N$ symmetry as the category of representations of $\widehat{\mathbb{Z}}_N$, and the gauging is understood as the insertion of a mesh of the Frobenius algebra objects corresponding to the regular representation of $\widehat{\mathbb{Z}}_N$ [13]. The Hilbert space after gauging is the twisted sector of this algebra object, which therefore forms the regular representation of the dual symmetry. The commutative Frobenius algebra structure of the Hilbert space is then given by the Frobenius algebra structure of the regular representation of $\widehat{\mathbb{Z}}_N$ and the Hartle-Hawking state corresponds to the singlet representation.

The generalization to the pure Rep(G) gauge theory is straightforward. We start from the trivial theory enriched with topological lines forming the category Rep(G). Then we insert a fine mesh of the algebra object in the regular representation of G, and the Hilbert space of the gauged theory will be organized in such representation. This naturally has a commutative Frobenius algebra structure, which can be used to define axiomatically the theory. The dual symmetry is now generated by lines fusing according to the G group law, so that it is an invertible symmetry, possibly non-abelian.

3.3.3 Appendix C: Toy model examples

In this appendix we test some formulas of the main text in simple solvable examples. We first discuss linear random couplings in free scalar theories and establish the validity of the generalized Ward identity (3.2.41) for 2-point functions both for the case of $h(x)$ (quenched disorder) and constant h (ensemble average). Subsequently we test the 't Hooft anomaly matching condition discussed in section 3.2.2 by working out a specific example.

Free scalar theories

We consider the toy example of a complex free scalar perturbed by a linear random coupling. The action is

$$S = \int d^d x \Big(|\partial\phi|^2 + m^2|\phi|^2 + h\phi(x) + \bar{h}\bar{\phi}(x) \Big) \,. \tag{3.3.7}$$

The coupling to h explicitly breaks the $U(1)$ symmetry rotating ϕ. Here h can have or not a space dependence. In both cases we can write

$$Z[K, \bar{K}, h] = \exp\Big(\int d^d x d^d y (\bar{h} + \bar{K}(x)) G(x-y)(h + K(y)) \Big) \,, \tag{3.3.8}$$

where $G(x-y)$ is the massive scalar propagator in flat space and K, $\bar{K}$ are the external sources for ϕ and $\bar{\phi}$, respectively. We consider a Gaussian distribution with variance v and zero mean in order to simplify the expressions. In what follows we shall be sloppy with normalizations and overall constants which do not affect the main points we want to show.

Quenched disorder It is convenient to introduce a compact notation

$$
\begin{aligned}
(hG)_x &:= \int d^d w\, h(w) G(w-x)\,, \qquad & (G\bar{h})_y &:= \int d^d w\, G(y-w)\bar{h}(w)\,, \\
G_{xy} &:= G(x-y)\,, & (GG)_{xy} &:= \int d^d w\, G(x-w)G(w-y)\,,
\end{aligned}
\tag{3.3.9}
$$

so that, from (3.3.8), we get the one-point function

$$
\langle \phi(x) \rangle = Z^{-1} \frac{\delta Z}{\delta K(x)}\bigg|_{K=0} = (G\bar{h})_x\,. \tag{3.3.10}
$$

Since translation invariance is broken, this is not a constant. Similarly, for two point functions,

$$
\begin{aligned}
\langle \phi(x)\phi(y) \rangle &= Z^{-1} \frac{\delta^2 Z}{\delta K(x)\delta K(y)}\bigg|_{K=0} = (G\bar{h})_x (G\bar{h})_y\,, \\
\langle \bar{\phi}(x)\phi(y) \rangle &= Z^{-1} \frac{\delta^2 Z}{\delta \bar{K}(x)\delta K(y)}\bigg|_{K=0} = G_{xy} + (hG)_x (G\bar{h})_y\,.
\end{aligned}
\tag{3.3.11}
$$

To take the average we simply Wick contract h and $\bar{h}$ with

$$
\overline{h(x)\bar{h}(y)} = v\delta^{(d)}(x-y)\,. \tag{3.3.12}
$$

Then

$$
\overline{\langle \phi(x) \rangle} = \overline{\langle \phi(x)\phi(y) \rangle} = 0\,, \tag{3.3.13}
$$

consistently with the $U(1)$ symmetry being recovered on average. The non vanishing two-point function is

$$
\overline{\langle \bar{\phi}(x)\phi(y) \rangle} = G_{xy} + v(GG)_{xy}\,. \tag{3.3.14}
$$

The explicitly broken Ward identities for a $U(1)$ transformation read

$$
\begin{aligned}
\langle \partial_\mu J^\mu(x)\phi(y)\bar{\phi}(z) \rangle =& \delta^{(d)}(x-y)\langle \phi(y)\bar{\phi}(z) \rangle - \delta^{(d)}(x-z)\langle \phi(y)\bar{\phi}(z) \rangle \\
&- h(x)\langle \phi(x)\phi(y)\bar{\phi}(z) \rangle + \bar{h}(x)\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z) \rangle\,.
\end{aligned}
\tag{3.3.15}
$$

The last two correlators equal

$$
\begin{aligned}
\langle \phi(x)\phi(y)\bar{\phi}(z) \rangle &= G_{xz}(G\bar{h})_y + G_{yz}(G\bar{h})_x + (hG)_z (G\bar{h})_y (G\bar{h})_x\,, \\
\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z) \rangle &= G_{xy}(hG)_z + G_{yz}(hG)_x + (hG)_z (G\bar{h})_y (hG)_x\,,
\end{aligned}
\tag{3.3.16}
$$

so that

$$
\begin{aligned}
\overline{h(x)\langle \phi(x)\phi(y)\bar{\phi}(z) \rangle} &= vG_{xy}G_{xz} + vG_{yz}G(0) + v^2(GG)_{yz}G(0) + v^2(GG)_{xz}G_{xy}\,, \\
\overline{\bar{h}(x)\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z) \rangle} &= vG_{xy}G_{xz} + vG_{yz}G(0) + v^2(GG)_{yz}G(0) + v^2(GG)_{xy}G_{xz}\,.
\end{aligned}
\tag{3.3.17}
$$

The average of (3.3.15) reads then

$$\begin{aligned}\overline{\langle\partial_\mu J^\mu(x)\phi(y)\bar{\phi}(z)\rangle} &= \delta^{(d)}(x-y)\overline{\langle\phi(y)\bar{\phi}(z)\rangle} - \delta^{(d)}(x-z)\overline{\langle\phi(y)\bar{\phi}(z)\rangle} \\ &\quad - v^2((GG)_{xz}G_{xy} - (GG)_{xy}G_{xz})\,.\end{aligned} \tag{3.3.18}$$

It is straightforward to check (3.3.18) by using the explicit form of $J_\mu = \bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}$ and performing the Wick contractions. We can now explicitly check the disordered Ward identity (3.2.41). Using the equations of motion we have $\partial_\mu J^\mu = \big(\bar{h}(x)\bar{\phi}(x) - h(x)\phi(x)\big)$, so that

$$\langle\partial^\mu J_\mu(x)\rangle = \int d^d z\,\big(h(z)\bar{h}(x) - h(x)\bar{h}(z)\big)\,G_{xz}\,. \tag{3.3.19}$$

Equivalently we can directly compute

$$\langle J_\mu(x)\rangle = \int d^d w d^d z h(z)\bar{h}(w)\,\big(\partial_\mu^{(x)}G_{xz}G_{xw} - G_{xz}\partial_\mu^{(x)}G_{xw}\big) \tag{3.3.20}$$

and take a derivative. As expected from the recovery of translation invariance after the average we find $\overline{\langle\partial^\mu J_\mu\rangle} = 0$. However, due to the presence of h, inserting $\langle\partial^\mu J_\mu\rangle$ under the average modifies the correlators, in particular

$$\begin{aligned}\overline{\langle\partial^\mu J_\mu(x)\rangle\langle\phi(y)\bar{\phi}(z)\rangle} &= \int d^d w G_{xw}\overline{\big(h(w)\bar{h}(x) - h(x)\bar{h}(w)\big)\,\langle\phi(y)\bar{\phi}(z)\rangle} \\ &= -v^2\,((GG)_{xz}G_{xy} - (GG)_{xy}G_{xz})\ .\end{aligned} \tag{3.3.21}$$

This precisely corresponds to the last term in the right hand side of (3.3.18). Therefore, using the improved current $\widetilde{J}_\mu := J_\mu - \langle J_\mu\rangle$, the Ward identity (3.3.18) becomes

$$\overline{\langle\partial_\mu\widetilde{J}^\mu(x;h(x))\phi(y)\bar{\phi}(z)\rangle} = \delta^{(d)}(x-y)\overline{\langle\phi(y)\bar{\phi}(z)\rangle} - \delta^{(d)}(x-z)\overline{\langle\phi(y)\bar{\phi}(z)\rangle}\,, \tag{3.3.22}$$

in agreement with (3.2.41) with $k = 2$ operators. From here one can reproduce the exponentiation procedure and determine the presence of a topological operator in the disordered theory.

Ensemble Average When h is a constant every member of the ensemble is translation invariant. Indeed the one point function of the scalar field is now a constant:

$$\langle\phi(x)\rangle = \bar{h}\int d^d y\,G_{xy} = \frac{\bar{h}}{m^2}\,. \tag{3.3.23}$$

Note that the mass acts as a IR regulator. The two point functions are

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= \bar{h}^2\int d^dz d^dw\, G_{xz}G_{yw} = \frac{\bar{h}^2}{m^4}\,,\\ \langle\bar{\phi}(x)\phi(y)\rangle &= G_{xy} + |h|^2\int d^dz d^dw\, G_{xz}G_{yw} = G_{xy} + \frac{|h|^2}{m^4}\,.\end{aligned} \tag{3.3.24}$$

In agreement with the $U(1)$ average symmetry, the only non-vanishing average two point function is

$$\overline{\langle\bar{\phi}(x)\phi(y)\rangle} = G_{xy} + \frac{v}{m^4}\,. \tag{3.3.25}$$

The explicitly broken Ward identities are

$$\begin{aligned}\langle\partial_\mu J^\mu(x)\phi(y)\bar{\phi}(z)\rangle =& \delta^{(d)}(x-y)\langle\phi(y)\bar{\phi}(z)\rangle - \delta^{(d)}(x-z)\langle\phi(y)\bar{\phi}(z)\rangle\\ &- h\langle\phi(x)\phi(y)\bar{\phi}(z)\rangle + \bar{h}\langle\bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle\,.\end{aligned} \tag{3.3.26}$$

The operator

$$\partial^\mu J_\mu(x) + h\phi(x) - \bar{h}\bar{\phi}(x) \tag{3.3.27}$$

generates the Ward identities, and we can now explicitly check that it integrates to zero on the whole space. The left hand side of (3.3.26) vanishes when integrating x over the whole space. For the last two terms in the right hand side we get

$$\begin{aligned}\langle\phi(x)\phi(y)\bar{\phi}(z)\rangle &= \frac{\bar{h}}{m^2}\left(G_{xz}+G_{yz}\right) + \frac{h\bar{h}^2}{m^6}\,,\\ \langle\bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle &= \frac{h}{m^2}\left(G_{xy}+G_{yz}\right) + \frac{\bar{h}h^2}{m^6}\,,\end{aligned} \tag{3.3.28}$$

so that

$$\overline{h\langle\phi(x)\phi(y)\bar{\phi}(z)\rangle} - \overline{\bar{h}\langle\bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle} = \frac{v}{m^2}\left(G_{xz}-G_{xy}\right)\,. \tag{3.3.29}$$

Then, by translation invariance, we have

$$\int d^dx\left(\overline{h\langle\phi(x)\phi(y)\bar{\phi}(z)\rangle} - \overline{\bar{h}\langle\bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle}\right) = \frac{v}{m^2}\int d^dx\left(G_{xz}-G_{xy}\right) = 0\,, \tag{3.3.30}$$

where the support of the integral needs to be the entire space. In this simple example we have chosen a scalar deformation so that Poincaré invariance remains always unbroken, no tensor operator can get a vev, and all complications arising from non-vanishing vevs disappear. For example, specifying (3.3.19) to the case of constant h immediately gives $\langle\partial_\mu J^\mu\rangle = 0$.

We can also compute $\langle\bar{\phi}(x_1)\phi(x_2)\rangle$ when X is a disconnected space. For example, if $X^{(d)} = X_1^{(d)} \sqcup X_2^{(d)}$, $x_1 \in X_1^{(d)}$ an $x_2 \in X_2^{(d)}$, (3.3.8) reads

$$Z[K_{1,2}, \bar{K}_{1,2}, h] = \exp\Big(\sum_{i=1,2}\int_{X_i^{(d)}} d^dx_i d^dy_i(\bar{h} + \bar{K}_i(x_i))G(x_i - y_i)(h + K_i(y_i))\Big)\,, \tag{3.3.31}$$

and we get

$$\langle\bar{\phi}(x_1)\phi(x_2)\rangle_X = Z^{-1}\frac{\delta^2 Z}{\delta\bar{K}_1(x_1)\delta K_2(x_2)}\bigg|_{K=0} = \langle\bar{\phi}(x_1)\rangle_{X_1}\langle\phi(x_2)\rangle_{X_2} = \frac{|h|^2}{m^4}\,, \tag{3.3.32}$$

namely only the disconnected part of the correlator contributes. Averaging on h we have

$$\overline{\langle\bar{\phi}(x_1)\phi(x_2)\rangle}_X = \overline{\langle\bar{\phi}(x_1)\rangle_{X_1}\langle\phi(x_2)\rangle}_{X_2} = \frac{v}{m^4}\,. \tag{3.3.33}$$

We explicitly see that in both X_1 and X_2 the $U(1)$ symmetry is explicitly broken and conserved only globally over the entire space X.

't Hooft anomalies from replicas

We check the matching of t'Hooft anomalies between the pure and disordered theory in the simple example of the $U(1)$ chiral anomaly in $4d$. As well-known, a free massless Weyl fermion ψ in $4d$ suffers from a cubic 't Hooft anomaly, which in momentum space reads

$$p_1^\mu\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle = i\frac{k}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}\,p_2^\alpha\,p_3^\beta\,, \tag{3.3.34}$$

where $k = 1$. We deform the theory with a space dependent complex mass term $m(x)$, which explicitly breaks the $U(1)$ symmetry down to fermion parity. However, if we sample $m(x)$ from a Gaussian distribution proportional to $\bar{m}(x)m(x)$, then the disordered theory recovers the $U(1)$ symmetry via the conserved current $\widetilde{J}_\mu$. Since $\langle\widetilde{J}_\mu\rangle = 0$ before averaging, we have

$$\overline{\langle\widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle} = \overline{\langle\widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle_c} = \overline{\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle_c}\,. \tag{3.3.35}$$

The last three-point function is most easily evaluated using the replica trick. The replicated theory has n Weyl fermions with a quartic deformation (spinor indices omitted)

$$S_{\text{rep}} = \sum_{a=1}^{n} S_{0,a} + v^2\sum_{a,b}\psi_a\psi_a\overline{\psi}_b\overline{\psi}_b\,, \tag{3.3.36}$$

which is invariant under the diagonal $U(1)_D$ symmetry, with conserved current

$$J_D^\mu = \sum_a J_a^\mu \,. \tag{3.3.37}$$

According to (3.2.79), we have

$$\overline{\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle_c} = \lim_{n\to 0}\frac{\partial}{\partial n}\langle J_{D,\mu}(p_1)J_{D,\nu}(p_2)J_{D,\rho}(p_3)\rangle^{\text{rep}} \,. \tag{3.3.38}$$

The $U(1)_D$ in the replica theory also suffers from a a cubic 't Hooft anomaly

$$p_1^\mu\langle J_{D,\mu}(p_1)J_{D,\nu}(p_2)J_{D,\rho}(p_3)\rangle_{\text{rep}} = \frac{ik}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta \,, \tag{3.3.39}$$

where $k = n$, since all n fermions rotate (with the same charge) under $U(1)_D$. We then get

$$p_1^\mu\overline{\langle \widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle} = \lim_{n\to 0}\frac{\partial}{\partial n}\left(\frac{in}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta\right) = \frac{i}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta \,, \tag{3.3.40}$$

which shows that the anomaly of the pure theory persists after the quenched average and also affects the disordered symmetry, in agreement with the results in the main text.

3.3.4 Appendix D: Symmetry operators for averaged symmetries

In this appendix we prove the existence, and explicitly construct, an operator $\widehat{U}_g$ which implements the action of the group rather than the action of the corresponding Lie algebra for average symmetries. To this purpose we need to find an infinite set of operators $\widehat{Q}_n$ which have the same properties of $\widehat{Q}$ defined in (3.2.134) and which satisfy the identities

$$\langle \widehat{Q}_n\mathcal{O}_1\cdots\mathcal{O}_k\rangle = \chi^n(\Sigma^{(d-1)})\langle\mathcal{O}_1\cdots\mathcal{O}_k\rangle \,, \qquad \forall n\in\mathbb{N} \,, \tag{3.3.41}$$

where we recall that $\chi(\Sigma^{(d-1)})$ denotes the sum of the charges of the local operators which are inside the surface $\Sigma^{(d-1)}$. Note that (3.3.41) applies before ensemble averaging. We define $\widehat{Q}_0 = 1$ and $\widehat{Q}_1 = \widehat{Q}$. We find $\widehat{Q}_n[\Sigma^{(d-1)}, D^{(d)}; h]$ for $n > 1$ iteratively. Suppose that there exists an operator $\widehat{Q}_{n-1}$ such that

$$\langle \widehat{Q}_{n-1}\Phi\rangle = \chi^{n-1}(\Sigma^{(d-1)})\langle\Phi\rangle \tag{3.3.42}$$

for any product of local operators Φ. We then compute

$$\langle \widehat{Q}_{n-1}\widehat{Q}_1\Phi\rangle = \chi^{n-1}\langle Q\Phi\rangle + q_0\left(\chi+q_0\right)^{n-1}\langle h\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle - q_0\left(\chi-q_0\right)^{n-1}\langle \overline{h}\int_{D^{(d)}}\overline{\mathcal{O}_0}(x)\Phi\rangle$$
$$= \chi^n\overline{\langle\Phi\rangle} + q_0\sum_{k=0}^{n-2}\binom{n-1}{k}\chi^k q_0^{n-1-k}\left(\langle h\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle - (-1)^{n-k-1}\langle\overline{h}\int_{D^{(d)}}\overline{\mathcal{O}_0}(x)\Phi\rangle\right). \tag{3.3.43}$$

Next we introduce operators Γ_l defined in such a way that

$$\langle\Gamma_l\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle = \chi^l\langle\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle\,. \tag{3.3.44}$$

Their existence follows from the (by now assumed) existence of the operators $\widehat{Q}_n$. In fact, it is easy to see that the Γ_l's satisfy the relation

$$\widehat{Q}_l = \sum_{s=0}^{l}\binom{l}{s}q_0^{l-s}\Gamma_s\,, \tag{3.3.45}$$

valid when inserted in (vacuum to vacuum) correlators of the form $\langle\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle$. Now consider the vectors $\widehat{\mathbf{Q}} = (\widehat{Q}_0,\widehat{Q}_1,\cdots,\widehat{Q}_N)$ and $\mathbf{\Gamma} = (\Gamma_0,\Gamma_1,\cdots,\Gamma_N)$, with $\Gamma_0 = 1$. These are related as $\widehat{\mathbf{Q}} = A\cdot\mathbf{\Gamma}$ where $A = \mathbb{1}+T$ and T is a strictly lower triangular matrix with non-vanishing entries

$$T_{l,s} = \binom{l}{s}q_0^{l-s}\,. \tag{3.3.46}$$

We can invert (3.3.45) as

$$\Gamma_l = \sum_{s=0}^{l}A^{-1}_{l,s}\widehat{Q}_s\,, \tag{3.3.47}$$

where we used that

$$A^{-1} = \mathbb{1} + \sum_{i=1}^{N}(-1)^i T^i \tag{3.3.48}$$

is again a lower triangular matrix. An analogous analysis can be carried out for the operators $\overline{\Gamma}_l$ defined by

$$\langle\overline{\Gamma}_l\overline{h}\int_{D^{(d)}}\overline{\mathcal{O}_0}(x)\Phi\rangle = \chi^l\langle\overline{h}\int_{D^{(d)}}\overline{\mathcal{O}}_0(x)\Phi\rangle\,, \tag{3.3.49}$$

by simply replacing q_0 with $-q_0$, and we define $\overline{A}$ as $\widehat{\mathbf{Q}} = \overline{A} \cdot \overline{\mathbf{\Gamma}}$. We rewrite (3.3.43) as

$$
\begin{aligned}
&\langle \widehat{Q}_{n-1}\widehat{Q}_1\Phi\rangle = \langle \widehat{Q}_n\Phi\rangle \\
&+ q_0 \sum_{k=0}^{n-2} \binom{n-1}{k} q_0^{n-1-k} \left(\langle \Gamma_k h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - (-1)^{n-1-k} \langle \overline{\Gamma}_k \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right) \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \Bigg[\langle \widehat{Q}_{n-1} \left(h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x) \right) \Phi\rangle + \\
&+ \langle \Gamma_{n-1} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{\Gamma}_{n-1} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \Bigg] \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \Bigg[\langle \widehat{Q}_{n-1} \left(h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x) \right) \Phi\rangle + \\
&+ \sum_{k=0}^{n-1} \widehat{Q}_k \left(\langle A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right) \Bigg] \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \sum_{k=0}^{n-2} \widehat{Q}_k \left[\langle A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right],
\end{aligned}
\tag{3.3.50}
$$

where we used that $A^{-1}_{k,k} = \overline{A}^{-1}_{k,k} = 1$. We then find the recursion relation

$$
\widehat{Q}_n = \widehat{Q}_{n-1}\widehat{Q}_1 - q_0 \sum_{k=0}^{n-2} \widehat{Q}_k \left(A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right), \tag{3.3.51}
$$

which proves the existence of $\widehat{Q}_n$ for every values of $n \in \mathbb{N}$.
As an example consider $N = 3$. We have

$$
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_0 & 1 & 0 & 0 \\ q_0^2 & 2q_0 & 1 & 0 \\ q_0^3 & 3q_0^2 & 3q_0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q_0 & 1 & 0 & 0 \\ q_0^2 & -2q_0 & 1 & 0 \\ -q_0^3 & 3q_0^2 & -3q_0 & 1 \end{pmatrix}, \tag{3.3.52}
$$

and

$$
\begin{aligned}
\widehat{Q}_2 &= \widehat{Q}_1^2 + q_0^2 \left(h \int_{D^{(d)}} \mathcal{O}_0(x) + \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right), \\
\widehat{Q}_3 &= \widehat{Q}_2\widehat{Q}_1 - q_0^3 \int_{D^{(d)}} \mathcal{D}(x) - 2q_0^2 \widehat{Q}_1 \left(h \int_{D^{(d)}} \mathcal{O}_0 + \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right).
\end{aligned}
\tag{3.3.53}
$$

We now crucially verify that the charges $\widehat{Q}_n$ vanish when $D^{(d)} = X^{(d)}$ *after* ensemble average in arbitrary local correlators. For this purpose we derive a further constraint on correlators involving arbitrary functions of h and $\overline{h}$. Consider

$$\int dh d\bar{h}\, P[\bar{h}h] f(h,\overline{h}) \frac{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0+c.c.)+\int K_i\mathcal{O}_i}}{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0+c.c.)}}\,, \tag{3.3.54}$$

where f is an arbitrary smooth function of h and $\bar{h}$. We shift $h \to h + \epsilon\delta h$, where $\delta h = -iq_0 h$. Using that $\delta h \mathcal{O}_0 = -h\delta\mathcal{O}_0$ and expanding to linear order in ϵ we get[48]

$$iq_0 \overline{f(h,\overline{h})\langle \int_{X^{(d)}} \mathcal{D}(x;h)\mathcal{O}_1\cdots\mathcal{O}_n\rangle} = -\overline{\delta f(h,\overline{h})\langle \mathcal{O}_1\cdots\mathcal{O}_n\rangle}\,, \tag{3.3.55}$$

where

$$\delta f(h,\overline{h}) = \partial f \delta h + \overline{\partial} f \delta\overline{h}\,. \tag{3.3.56}$$

Thanks to (3.3.55) we can now show that

$$\overline{\langle \widehat{Q}_n[\emptyset, X^{(d)}; h]\Phi\rangle} = 0\,, \qquad n > 0\,. \tag{3.3.57}$$

Let us explicitly work out the $n = 2, 3$ cases. For $n = 2$ it is enough to use (3.3.55) with $f = h$ and $f = \bar{h}$ to get the identity

$$\overline{\langle \left(\int_{X^{(d)}} \mathcal{D}(x;h)\right)^2 \Phi\rangle} + \overline{\langle \left(h\int_{X^{(d)}} \mathcal{O}_0(x) + \overline{h}\int_{X^{(d)}} \overline{\mathcal{O}_0}\right)\Phi\rangle} = 0\,. \tag{3.3.58}$$

We can plug this relation into $\widehat{Q}_2$ in (3.3.53) to immediately get (3.3.57) for $n = 2$. For $n = 3$ we use (3.3.55) with the functions $h^2, \overline{h}^2$ and $h\overline{h}$. In this way we get the relations

$$\overline{\langle \left(\int_{X^{(d)}} \mathcal{D}(x)\right)^3 \Phi\rangle} = 2\overline{\langle \left(\int_{X^{(d)}} \mathcal{D}(x;h)\right)\left(h\int_{X^{(d)}} \mathcal{O}_0 + \overline{h}\int_{X^{(d)}} \overline{\mathcal{O}_0}\right)\Phi\rangle} = 0\,, \tag{3.3.59}$$

which, pluggged in $\widehat{Q}_3$ in (3.3.53) allows us to get (3.3.57) for $n = 3$. We can then construct the non-genuine symmetry operator

$$\widehat{U}_g[\Sigma^{(d-1)}, D^{(d)}; h] = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \widehat{Q}_n[\Sigma^{(d-1)}, D^{(d)}; h]\,, \qquad g = e^{i\alpha}\,, \tag{3.3.60}$$

which, similarly to $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}; h]$, becomes quasi-genuine when $D^{(d)} = X^{(d)}$.

We have then shown the existence, and explicitly constructed, the operator $\widehat{U}_g$ which implements the selection rules imposed by the emergent symmetries.

[48] An extra term coming from the denominator of (3.3.54) vanishes because of (3.2.131).

Chapter 4

Non-invertible symmetries and the AdS/CFT correspondence

In this chapter we discuss the holographic interpretation of some generalized symmetries. In particular we focus on the more exotic case of non-invertible higher-categorical symmetries arising in supersymmetric gauge theories enjoying self-dualities. Section 4.1 is based on [43] and it deals with the holographic interpretation of the non-invartible duality symmetry enjoyed by $\mathcal{N} = 4$ super Yang-Mills theory. Section 4.2 generalizes the above result by studying theories of class $\mathcal{S}$, where the duality symmetry is based on a larger duality group.

4.1 The holography of non-invertible self-duality symmetries

4.1.1 Introduction and summary of the results

Our interest in this section is in understanding how categorical symmetries appear in holography. The common lore is that a global symmetry of the boundary theory appears as a gauge symmetry, accompanied by a gauge field, in the bulk. This is confirmed and well understood in the case of invertible symmetries — both ordinary 0-form, continuous and discrete, as well as higher form. What happens for a non-invertible symmetry? What plays the role of a "non-invertible" gauge field?

We investigate this question in the specific case of *self-duality defects*.[1] The idea

[1] The holographic description of non-invertible defects of the KOZ and orbifold type has recently been investigated in [35, 39, 40].

is simple.The conformal manifold $\mathfrak{M}$ of the boundary theory is dual to a moduli space of bulk solutions in the gravitational description, while the choice of a global structure on the boundary corresponds to a certain boundary condition in gravity. The duality group Γ is a discrete gauge symmetry of string theory, which however is completely Higgsed at generic points x at which Γ acts faithfully on $\mathfrak{M}$. At a special point x which is stabilized by $G \subset \Gamma$, the duality symmetry Γ is only Higgsed to G, and in the low-energy description appears an emergent G gauge field that acts on the supergravity fields. In particular, it also acts on a low-energy topological sector of string theory whose topological (or conformal) boundary conditions encode the possible global structures of the boundary theory. It is this structure that plays the role of a "non-invertible gauge field", at least in this class of examples. The derivation and explanation of how the supergravity theory with extra gauge field gives rise to the non-invertible fusion rules of duality defects is the subject of this section. We focus on the example of 4d $\mathcal{N} = 4$ SYM and its dual type IIB string theory description. The formalism we develop is however quite general and should allow for prompt generalizations, for instance to theories of class S [225].

Let us summarize the main points of the construction. A key role is played by a topological sector of type IIB string theory compactified on S^5, which dominates at long distances: it is a 5d Chern-Simons-like (CS) theory [226–228][2]

$$S_{\mathrm{CS}} = \frac{N}{2\pi} \int b \, dc \equiv \frac{N}{4\pi} \int \mathcal{B}^{\mathsf{T}} \epsilon \, d\mathcal{B} \, . \tag{4.1.1}$$

Here b and c, that we package into $\mathcal{B} = (b, c)$, are 2-form gauge fields coming from the NS-NS and R-R 2-form potentials, respectively, while $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This theory can also be interpreted as a $\mathbb{Z}_N$ 2-form gauge theory [218], and its boundary conditions encode the global structure of the boundary theory [227]. The gauge fields b, c are dual to the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry of the set of boundary theories (taking into account the possible global structures). The 5d TQFT (4.1.1) has a global symmetry[3] $\Gamma = SL(2, \mathbb{Z}_N)$ that acts linearly on $\mathcal{B}$ and is generically spontaneously broken by the axiodilaton (on which it acts by fractional linear transformations), as well as a $\mathbb{Z}_N \times \mathbb{Z}_N$ global 2-form symmetry that shifts b, c. The symmetry defects $V_{M \in \Gamma}$ for Γ turn out to be higher gaugings [23] of the 2-form symmetry (or a subgroup

[2]In our notation, we multiply differential forms leaving all wedge products implicit.

[3]The symmetry is $SL(2, \mathbb{Z}_N)$ on spin manifolds, and a subgroup thereof on non-spin manifolds [227]. Since we are dealing with supersymmetric theories, we restrict to spin manifolds in this section.

thereof) on 4d submanifolds, with suitable choices of discrete torsion $\mathcal{T}(M)$ that we determine.

Quite interesting are the Γ-twisted sectors D_M that live at the boundary of the symmetry defects V_M. For most $M \in \Gamma$, V_M turns out to be an invertible TQFT that produces anomaly inflow and constrains its twisted sector. A minimal representative for D_M with the correct anomaly is a certain 3d TQFT $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ coupled to $\mathcal{B}$, introduced in [67]. The fusion of twisted sectors takes the schematic form

$$\begin{aligned} D_{M_2} \times D_{M_1} &= \mathcal{A}^{N,-(\mathcal{T}_2+\mathcal{T}_1)}\, D_{M_2 M_1} \ , \\ D_M \times \overline{D}_M &= \mathcal{C}^{\mathbb{Z}_N \times \mathbb{Z}_N} \ , \end{aligned} \tag{4.1.2}$$

where $\mathcal{C}^{\mathbb{Z}_N \times \mathbb{Z}_N}$ is a 3d condensate of the 2-form symmetry. Interestingly, to compute this fusion rules one uses a modified version $\mathcal{A}^{N,-\mathcal{T}_1}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_2}(\mathcal{B})$ of the stacking of TQFTs, in which the lines in the first factor acquire nontrivial braiding

$$B_{n_1 n_2} = \exp\left(\frac{2\pi i}{N}\, 2^{-1}\, n_1^{\mathsf{T}}\, \epsilon\, n_2\right) \tag{4.1.3}$$

(here $n_1, n_2 \in \mathbb{Z}_N \times \mathbb{Z}_N$ parametrize the lines of the two factors, respectively, and N is odd) with the lines in the second factor, due to the interactions with the bulk 5d theory (4.1.1).

In order to complete the holographic setup and make contact with the self-duality defects of the boundary $\mathcal{N} = 4$ SYM theory, two more steps are necessary. First, one should choose topological boundary conditions $\rho(\mathcal{L})$ for (4.1.1) on AdS_5, which are labelled by Lagrangian subgroups $\mathcal{L}$ of the $\mathbb{Z}_N \times \mathbb{Z}_N$ global 2-form symmetry and correspond to a choice of global structure ρ on the boundary. Imposing the boundary condition is equivalent to gauging $\mathcal{L}$ in the bulk [134, 229, 230], which is necessary in order to remove global symmetries from the bulk gravitational theory [18]. The topological self-duality defects of the boundary theory eventually are bulk operators placed on top of the boundary. We thus determine the pull-backs $D_{M,\mathcal{L}}$ of D_M on the boundary and their fusion rules. The answer turns out to depend on whether $\mathcal{L}$ is invariant under M or not.

Second, recall that for generic values of the axiodilaton τ, the 0-form symmetry Γ is spontaneously broken in the full supergravity theory. At the special values $\tau = i$, $e^{2\pi i/3}$ an Abelian subgroup $G = \mathbb{Z}_4$, $\mathbb{Z}_6$ generated by S, ST, respectively, is preserved, but crucially this symmetry is gauged. We study in detail the 5d theory obtained by gauging G in (4.1.1). The defects $V_{g\in G}$ become transparent, because they implement the gauging, and hence their boundaries $D_{g\in G}$ become genuine 3d

topological operators. They are ill-defined in isolation because of the anomaly, but one can form well-defined 3d topological operators $\mathfrak{D}_g$ by stacking D_g with a 3d Gukov-Witten [130], or twist, operator GW_g for the pure G gauge theory. This mechanism is similar to KOZ [119], but the role of the anomaly is played here by the torsion. The effect of gauging has also an effect on the boundary conditions, and we indicate by ρ^* the gauged boundary conditions.

The final fusion rules we find in the boundary theory take the schematic form:

$$\begin{aligned} \mathfrak{D}_{g_2,\mathcal{L}} \times \mathfrak{D}_{g_1,\mathcal{L}} &= \mathcal{N}_{g_2,g_1}\, \mathfrak{D}_{g_2g_1,\mathcal{L}} && g_2g_1 \notin \mathrm{Stab}(\rho^*)\ , \\ \mathfrak{D}_{g_2,\mathcal{L}} \times \mathfrak{D}_{g_1,\mathcal{L}} &= \mathcal{C}^{\mathbb{Z}_N}\ \mathrm{GW}_{g_2g_1} && g_2g_1 \in \mathrm{Stab}(\rho^*)\ , \\ \mathrm{GW}_{g_2} \times \mathrm{GW}_{g_1} &= \mathrm{GW}_{g_2g_2} && g_1, g_2 \in \mathrm{Stab}(\rho^*)\ . \end{aligned} \tag{4.1.4}$$

The explicit form of the TQFT coefficients $\mathcal{N}$ is given in Section 4.1.4. The third line describes a subcategory of invertible symmetries (that always includes charge conjugation). Our results reproduce the known duality and triality defects of 4d $\mathcal{N} = 4$ SYM [25, 26, 120].

This section is organized as follows. In Section 4.1.2 we recall a few facts about symmetries in holography. In Section 4.1.3 we study the 5d Chern-Simons theory, its symmetries, and its gapped boundaries. In Section 4.1.4 we describe the twist operators for the $SL(2,\mathbb{Z}_N)$ 0-form symmetry, using both a Lagrangian as well as a more formal approach based on minimal TQFTs, we compute their fusion, and the pull-back to gapped boundaries. Finally in Section 4.1.5 we address how to gauge a discrete Abelian symmetry G. That is used to present the final composition laws. We conclude in Section 4.1.6. Detailed computations are collected in several appendices.

4.1.2 Symmetries and global structures in holography

The four-dimensional $\mathcal{N} = 4$ SYM theory with gauge algebra $\mathfrak{su}(N)$ is holographically dual to type IIB string theory on asymptotically $\mathrm{AdS}_5 \times S^5$ spaces [170]. The boundary theory, however, is characterized by a specific gauge *group* with the given algebra, and thus this piece information must be encoded in the bulk theory. As explained by Witten [227], kinematical properties of the boundary theory, such as the global form of the gauge group, are captured by the long-distance behavior of the gravitational theory (or equivalently, by the behavior close to the boundary), which is encoded in the terms in the Lagrangian with the lowest number of derivatives, namely in the topological terms. One can more conveniently work with the effective

5d theory in AdS_5 obtained by reducing on the internal manifold. The 10d type IIB supergravity action contains the topological term

$$S_{\mathrm{IIB}} \supset \int_{X_{10}} B_2 \, dC_2 \, F_5 \, , \tag{4.1.5}$$

where B_2 is the NS 2-form potential, C_2 the RR 2-form potential, and F_5 is the field strength of the RR 4-form potential. In compactification on $\mathcal{M}_5 \times S^5$ with N units of 5-form flux on S^5, one obtains at low energies the 5d Chern-Simons action (4.1.1) [227, 228].

The continuous 2-form gauge fields b, c are dual to a $U(1) \times U(1)$ global 1-form symmetry of the boundary theory, whose two factors act on 't Hooft and Wilson line operators, respectively. This symmetry does not have to act faithfully on the boundary theory: it only acts faithfully on the full set of boundary theories with all possible global structures. This follows from the necessity of choosing boundary conditions. If we choose topological boundary conditions, the action (4.1.1) restricts b, c to be $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge fields [218] and accordingly restricts the 1-form symmetry. Boundary conditions $\rho(\mathcal{L})$ further set to zero a linear combination of b, c along a Lagrangian subgroup $\mathcal{L} \subset \mathbb{Z}_N \times \mathbb{Z}_N$, only leaving a 1-form symmetry of order N. Thus, the choice of boundary conditions specifies the global structure of the SYM theory [226, 227] and the spectrum of extended (here line) operators [62]. For instance, if we set $b = 0$ at the boundary, the boundary theory is $SU(N)$. Fundamental strings (that couple to b) can end on the boundary producing Wilson line operators in generic representations [231, 232], their $\mathbb{Z}_N$ charge being measured by the topological operators $e^{i\int c}$, while 't Hooft lines only exist with vanishing $\mathbb{Z}_N$ charge. On the contrary, if we set $c = 0$ we obtain the $PSU(N)_0$ theory.[4] D1-branes (that couple to c) can end on the boundary producing 't Hooft line operators with generic $\mathbb{Z}_N$ charge, the latter being measured by $e^{i\int b}$, while Wilson lines only exist in representations with trivial N-ality. One can also choose conformal boundary conditions $b = *\, c$: they give rise to an extra singleton sector [228, 233] and describe the theory $U(N)$, for which the 1-form symmetry is indeed $U(1) \times U(1)$.

Type IIB string theory also enjoys an $SL(2, \mathbb{Z})$ symmetry. As in any theory of quantum gravity, this must be a *gauge* symmetry. It acts on the axiodilaton field $\tau = C_0 + ie^{-\phi}$ by standard fractional linear transformations

$$\tau \to \frac{\underline{a}\,\tau + \underline{b}}{\underline{c}\,\tau + \underline{d}} \, , \qquad \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{4.1.6}$$

[4] See [62] and Section 4.1.3 for the meaning of this notation.

(only $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ acts on τ) and on b, c as on a doublet $\mathcal{B} = (b, c)$ in the fundamental representation,

$$\begin{pmatrix} b \\ c \end{pmatrix} \to \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} . \tag{4.1.7}$$

At generic points τ in the moduli space, $SL(2,\mathbb{Z})$ is spontaneously broken to its $\mathbb{Z}_2$ center, and thus the corresponding gauge field does not appear in the low-energy supergravity description.[5] However, special values of τ are left invariant by a larger subgroup $G \subset SL(2,\mathbb{Z})$ which therefore remains unbroken. The corresponding gauge field should then be included in the supergravity description, where it appears as an emergent gauge field for the low-energy observer. Specifically, $G = \mathbb{Z}_4$ at $\tau = i$ and $G = \mathbb{Z}_6$ at $\tau = e^{2\pi i/3}$. After compactification on S^5, we obtain a discrete G gauge field a in five dimensions, coupled to a G subgroup of the $SL(2,\mathbb{Z}_N)$ symmetry of S_{CS} (4.1.1). This is an interesting subsector of the full theory on its own. Our aim is to show that a is the gauge field corresponding to the non-invertible symmetries of the boundary theory.[6]

We conclude this section recalling that the way in which a bulk TQFT affects the symmetry on its boundary is made very clear in the recently introduced formalism of symmetry TFTs [234, 235] (see also [125, 158, 236]). Independently of holography, the symmetry TFT approach separates the local dynamics of a QFT from its global structure, by viewing a physical (or absolute) d-dimensional theory as the compound $(d+1)$-dimensional system of a slab of topological theory with two parallel boundaries. One boundary supports the relative [237] version of the theory. Roughly speaking, a relative theory is a vector of theories encoding all possible global structures. The opposite is a gapped boundary determined by some topological boundary conditions for the TFT. Their choice corresponds to picking a particular state, *i.e.*, selecting one particular absolute theory. We collect some details about the construction in Appendix 4.3.1.

[5]The $\mathbb{Z}_2$ center of $SL(2,\mathbb{Z})$, that maps $(b, c) \mapsto (-b, -c)$ but does not act on τ, is however always preserved and then the corresponding $\mathbb{Z}_2$ gauge field should be included.

[6]The non-invertibility of duality and triality defects is *only* up to condensates. It is perhaps then not surprising that the corresponding bulk gauge field is a standard discrete connection for G, though coupled to a nontrivial topological sector S_{CS}. The gauge field holographically dual to symmetries which remain non-invertible also up to condensates would presumably be a more complicated object.

4.1.3 The 5d Chern-Simons theory and its symmetries

Consider the five-dimensional Chern-Simons action [227] (see also [4, 228, 233, 238, 239])[7]

$$S[Q] = \frac{1}{4\pi} \int Q(\mathcal{B},\, d\mathcal{B}) \ , \tag{4.1.8}$$

where $\mathcal{B}$ is a vector of $2n$ 2-form gauge fields, Q is an integer-valued $2n \times 2n$ non-degenerate antisymmetric matrix (or symplectic form), and we used the notation $Q(x, y) = x^{\mathsf{T}} Q y$. We study this theory on spin manifolds. The theory has topological surface operators

$$U_m = e^{i m^{\mathsf{T}} \int \mathcal{B}} \ , \tag{4.1.9}$$

where m is an integer-valued vector in $\mathbb{Z}^{2n}$, and the integral is over a 2-dimensional surface. These operators generate an (anomalous) 2-form symmetry. The operators U_m have nontrivial linking (see Figure 4.1 left) given by the antisymmetric braiding matrix

$$B_{mm'} = e^{2\pi i\, Q^{-1}(m,m')} \ . \tag{4.1.10}$$

Any operator for which $m = Q\, k$ with $k \in \mathbb{Z}^{2n}$ is completely transparent and thus trivial. Those operators generate a lattice Λ_Q, and the 2-form symmetry defect operators are labelled by the elements of the discriminant group

$$\mathcal{D}_Q = \mathbb{Z}^{2n}/\Lambda_Q \ . \tag{4.1.11}$$

This is the 2-form symmetry of the theory. Notice that $|\mathcal{D}_Q| = |\det Q|$.

The case relevant to type IIB string theory compactified on S^5 is $n = 1$ and $Q = N\epsilon$ with $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We denote by b and c the two components of $\mathcal{B}$. The action reads:[8]

$$S = \frac{N}{4\pi} \int \mathcal{B}^{\mathsf{T}} \epsilon\, d\mathcal{B} = \frac{N}{4\pi} \int \langle \mathcal{B},\, d\mathcal{B} \rangle = \frac{N}{4\pi} \int \big(b\, dc - c\, db \big) \ . \tag{4.1.12}$$

[7] This action, as written, is not well defined [227]. When the spacetime manifold M_5 is the boundary of a six-manifold Z, one can define $S[Q] = \frac{1}{4\pi} \int_Z Q(d\mathcal{B}, d\mathcal{B})$. However, the bordism group in five dimensions is non-trivial and thus this cannot be done in general. One could instead use the formalism of Cheeger-Simons differential characters [240].

[8] We work with an antisymmetric 5d Lagrangian, which is manifestly invariant under $SL(2,\mathbb{Z})$ symmetry. One should however keep in mind that, as written, the action is not well defined (see footnote 7), and thus conclusions drawn from it should be taken with care. It turns out [227] that for N odd, the theory is $SL(2,\mathbb{Z})$ invariant only on spin manifolds, while on non-spin manifolds it is invariant under the subgroup $\Gamma(2)$ generated by S and T^2.

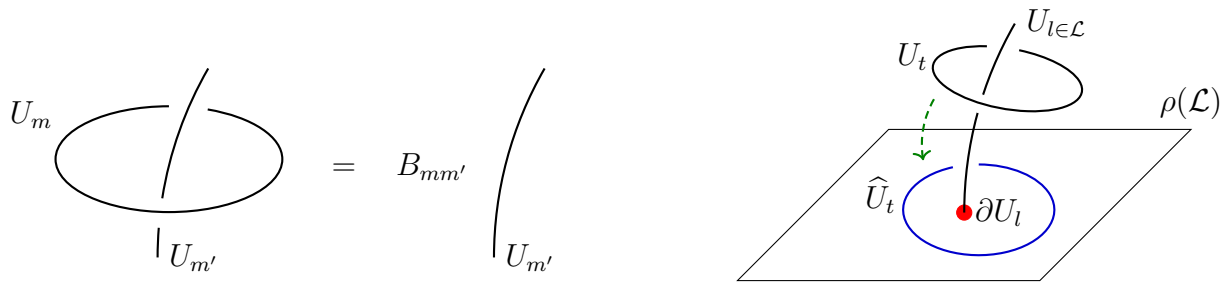


Figure 4.1: Left: Antisymmetric braiding $B_{mm'}$ between 2-dimensional defects U_m in 5d Chern-Simons theory. Right: Induced braiding between 2-dimensional defects $\widehat{U}_t$ and line defects $\partial U_{l\in\mathcal{L}}$ on gapped boundaries $\rho(\mathcal{L})$.

We introduced the antisymmetric Dirac pairing $\langle x, y\rangle = x_{\mathrm{e}} y_{\mathrm{m}} - x_{\mathrm{m}} y_{\mathrm{e}}$, where $x = (x_{\mathrm{e}}, x_{\mathrm{m}})$ is the expression of a vector in components. When describing the surface operators U_m, it might be convenient to package the information about m and the geometric 2-cycle wrapped by U_m into $\gamma \in H_2(M_5, \mathbb{Z}_N \times \mathbb{Z}_N)$, or its Poincaré-dual cocycle $\mathrm{PD}(\gamma) \in H^3(M_5, \mathbb{Z}_N \times \mathbb{Z}_N)$. In this case $U(\gamma)$ is described by the insertion of

$$U(\gamma) = \exp\left(i \int \mathcal{B}^{\mathsf{T}}\, \mathrm{PD}(\gamma)\right) \tag{4.1.13}$$

in the path integral.

In the general case, gapped boundary conditions $\rho(\mathcal{L})$ are in bijection with Lagrangian subgroups $\mathcal{L}$ of $\mathcal{D}_Q$. A subgroup is called Lagrangian if all its elements are mutually transparent, *i.e.*, if $B_{ll'} = 1$ for all $l, l' \in \mathcal{L}$, and if any element outside $\mathcal{L}$ braids non-trivially with at least one $l \in \mathcal{L}$ (*i.e.*, $\mathcal{L}$ is maximal). Defining a gapped boundary $\rho(\mathcal{L})$ is equivalent to gauging the Lagrangian subgroup $\mathcal{L}$ of the 2-form symmetry [134, 229, 230].[9]

Only dyons U_l with $l \in \mathcal{L}$ may terminate on the gapped boundary, defining in this way topological line operators ∂U_l there. Besides, dyons in $\mathcal{L}$ are absorbed by the gapped boundary if they are moved to lie within it, in other words the dyons $U_{l\in\mathcal{L}}$ are completely transparent (they do not contribute to correlation functions)

[9]More precisely, gauging the discrete symmetry $\mathcal{L}$ is equivalent to inserting a network of symmetry defects for $\mathcal{L}$ in the spacetime manifold. This is also equivalent to removing a tubular neighborhood of the network from the spacetime manifold, and placing the topological boundary condition $\rho(\mathcal{L})$ there. Thus, $\rho(\mathcal{L})$ is a topological interface between the ungauged theory and the trivial theory obtained by gauging $\mathcal{L}$ (such a theory is trivial because $\mathcal{L}$ is Lagrangian).

when placed on the gapped boundary. The boundary has non-trivial topological surface operators corresponding to $m \notin \mathcal{L}$, obtained by moving $U_{m\notin\mathcal{L}}$ to lie within the boundary, however, because of the property just mentioned, those operators $\widehat{U}_t$ are labeled by conjugacy classes $t \in \mathcal{D}_Q/\mathcal{L} \equiv \mathcal{S}$. The operators $\widehat{U}_t$ are stuck to the gapped boundary (because t would be an ambiguous label in the bulk), and generate a 1-form symmetry $\mathcal{S}$ there. The charges under that symmetry are carried by the lines $\partial U_{l\in\mathcal{L}}$, as follows from the 5d braiding (see Figure 4.1 right):

$$\widehat{U}_t(\partial U_l) = e^{2\pi i\, Q^{-1}(t,l)}\, \partial U_l \ , \tag{4.1.14}$$

where, with some abuse of notation, we indicated by t any representative of its class in $\mathcal{D}_Q$.

Some properties become clear in the Lagragian description (4.1.8): a gapped boundary on X is defined by Dirichelet boundary conditions

$$l^{\mathsf{T}}\mathcal{B}\Big|_X = 0 \quad \text{(up to gauge transformations)} \qquad \text{for all } l \in \mathcal{L} \ . \tag{4.1.15}$$

Introducing a rectangular matrix L whose columns are the generators of $\mathcal{L}$ in $\mathbb{Z}^{2n}$, so that $L^{\mathsf{T}}QL = 0$, the boundary condition is $L^{\mathsf{T}}\mathcal{B}\big|_X = 0$ (up to gauge transformations). This can be imposed by a boundary TQFT:

$$S_{\substack{\text{gapped}\\ \text{boundary}}}[L] = \frac{1}{2\pi}\int Q\Big(\eta,\, L^{\mathsf{T}}(\mathcal{B} - d\xi)\Big) \ + \ \text{counterterms} \ , \tag{4.1.16}$$

where η is a 2-form gauge field in $\mathbb{R}^{2n}/\langle\mathcal{L}\rangle$, ξ is a 1-form gauge field in $\langle\mathcal{L}\rangle$, and $\langle\mathcal{L}\rangle$ is the real span of $\mathcal{L}$. The counterterms only involve $\mathcal{B}$, and are fixed by overall gauge invariance. To give an example, consider the type IIB case $Q = \left(\begin{smallmatrix} 0 & N \\ -N & 0 \end{smallmatrix}\right)$ and take the *electric boundary* $\rho(\mathcal{L})$ where $\mathcal{L}$ is generated by $l = (1,0)$, corresponding to the boundary condition $b\big|_X = 0$ (up to gauge transformations). The boundary action is

$$S_{\substack{\text{electric}\\ \text{boundary}}} = \frac{N}{2\pi}\int\left[\eta\,(b - d\xi) - \frac{1}{2}\,bc\right] . \tag{4.1.17}$$

If we introduce a coordinate r transverse to the boundary, place the boundary at $r = 0$ and the bulk in the region $r < 0$, the full bulk plus boundary system has action

$$S_{\substack{\text{bulk plus}\\ \text{boundary}}} = \frac{N}{4\pi}\int_{r<0}\big(b\,dc - c\,db\big) + \frac{N}{2\pi}\int_{r=0}\left[\eta\,(b - d\xi) - \frac{1}{2}\,bc\right] . \tag{4.1.18}$$

The equations of motion fix the following conditions on the boundary:

$$b = d\xi \ , \qquad c = \eta \ , \qquad \eta \in H^2(M_5, \mathbb{Z}_N) \ . \tag{4.1.19}$$

Thus, b is set to be pure gauge, while η is the pull-back of c to the boundary and c remains unconstrained ($c \in H^2(M_5, \mathbb{Z}_N)$ is already imposed by the bulk EOMs). The system is invariant under the following gauge transformations:

$$b \to b + d\alpha_{\rm e} \,, \qquad c \to c + d\alpha_{\rm m} \,, \qquad \eta \to \eta + d\alpha_{\rm m} \,, \qquad \xi \to \xi + \alpha_{\rm e} \,. \quad (4.1.20)$$

Interpreting instead $\mathcal{L}$ as a subgroup of $\mathcal{D}_Q$ that is gauged, the dyons $U_{l\in\mathcal{L}}$ become trivial in the bulk because they are pure gauge and can be absorbed by the network of defects. On the contrary, the operators with $m \notin \mathcal{L}$ are projected out in the bulk (using the fact that $\mathcal{L}$ is Lagrangian) and can only exist on the boundary.

In the holographic setup, the 2-form symmetry $\mathcal{L}$ that we gauge in the bulk dictates what is the spectrum of physical lines in the holographic boundary [18,227]. Thus, the surfaces U_l with $l \in \mathcal{L}$ become trivial in the bulk, but if they are attached to the holographic boundary, their end-lines $\partial U_l \equiv W_l$ are the physical line operators of the boundary theory (notice that these are no longer topological, due to the holographic boundary conditions).[10] The 1-form symmetry of the boundary theory under which the lines W_l are charged is generated by the surface operators $\widehat{U}_t$, that can only live on the boundary.

Coming back to type IIB string theory, where $Q = N\epsilon$, the simplest case to discuss is when N is prime. We label the bulk surfaces U_m by $m = (m_{\rm e}, m_{\rm m})$, where $m_{\rm e}$ and $m_{\rm m}$ are the electric and magnetic charges, respectively. The topological sector has $N+1$ gapped boundary conditions:[11]

- An electric gapped boundary $\rho(e)$, for which $\mathcal{L}$ is generated by $l = (1, 0)$. As a gauging, this is obtained by condensing the electric surfaces $(m_{\rm e}, 0) \in \mathcal{L}$ (while in terms of a gapped boundary, this is implemented by setting $b = 0$ there). It corresponds to the global variant $SU(N)$ of the boundary theory. The Wilson lines $W_{l\in\mathcal{L}}$ are endpoints of bulk surfaces U_l. For instance, the Wilson line in the fundamental representation is the endpoint of a fundamental string [231, 232], which couples as $e^{i\int b}$ to the NS B-field b. The boundary 1-form symmetry is generated by the surfaces $\widehat{U}_t$, and we can take for $\mathcal{S} \cong \mathbb{Z}_N$ the representatives $t = (0, t_{\rm m})$.

[10] In the picture in which the bulk with gauged $\mathcal{L}$ is substituted by a slab of bulk between the holographic boundary and a gapped boundary $\rho(\mathcal{L})$, the operators U_l can be stretched between a copy of W_l in the holographic boundary and a copy of W_l in the gapped boundary.

[11] See [241] for a recent in depth study of gapped boundary conditions in the 5d Chern-Simons theory.

- N magnetic gapped boundaries $\rho(m)_r$ with $r = 0, \ldots, N-1$, for which $\mathcal{L}$ is generated by $l = (r, 1)$. They are obtained by condensing the dyonic surfaces $(rm_{\rm m}, m_{\rm m}) \in \mathcal{L}$ (or by setting $rb + c = 0$ on a gapped boundary). They correspond to the global variants $PSU(N)_r$ of the boundary theory [62]. The 't Hooft or dyonic lines are endpoint of bulk surfaces $U_{l\in\mathcal{L}}$, for instance for $r = 0$ the basic 't Hooft line is the endpoint of a D1-brane, which couples as $e^{i\int c}$ to the Ramond field c. The boundary 1-form symmetry is generated by surfaces $\widehat{U}_t$, represented for instance by $t = (t_{\rm e}, 0)$.

If N is not prime, then there is a larger number $\sigma_1(N) = \sum_{k|N} k$ of Lagrangian subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$, corresponding to global variants of the boundary theory of the form $\big(SU(N)/\mathbb{Z}_k\big)_r$.

Global 0-form symmetries

The theories (4.1.8) can have 0-form symmetries as well. On spin manifolds, a (unitary) 0-form symmetry ω is an automorphism of the discriminant group $\mathcal{D}_Q$ that preserves the quadratic form:

$$\omega^{\mathsf{T}} Q^{-1} \omega = Q^{-1} \mod 1 \,. \tag{4.1.21}$$

Since ω is invertible, it maps Lagrangian subgroups to Lagrangian subgroups. We say that a gapped boundary $\rho(\mathcal{L})$ is ω-invariant if the corresponding Lagrangian subgroup is:

$$\omega\, \mathcal{L} = \mathcal{L} \,. \tag{4.1.22}$$

In the type IIB example, the 0-form symmetry group Γ is $SL(2, \mathbb{Z}_N)$, whose generators act on electric and magnetic charges as follows:

$$S : (e, m) \mapsto (-m, e) \,, \qquad T : (e, m) \mapsto (e + m, m) \,, \qquad C : (e, m) \mapsto (-e, -m) \,. \tag{4.1.23}$$

They satisfy $S^2 = C$, $T^N = \mathbb{1}$, and $(ST)^3 = C$. If M is the matrix acting on charges, then M^{T} gives the action on the gauge fields $\mathcal{B}$, as it follows from (4.1.9). This means that in our conventions

$$S : (b, c) \mapsto (c, -b) \,, \qquad T : (b, c) \mapsto (b, c + b) \,, \qquad C : (b, c) \mapsto (-b, -c) \,. \tag{4.1.24}$$

All subgroups of $\mathcal{D}_Q$ are invariant under C. For N prime we also have:

$$\rho(m)_r \xrightarrow{T} \rho(m)_{r+1} \,, \qquad \rho(e) \xleftrightarrow{S} \rho(m)_0 \,, \qquad \rho(m)_r \xleftrightarrow{S} \rho(m)_{r_S} \quad \text{for } r \neq 0 \tag{4.1.25}$$

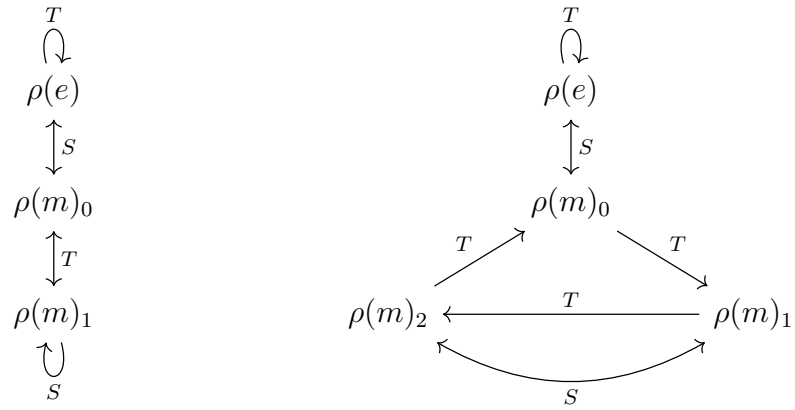


Figure 4.2: Action of $PSL(2,\mathbb{Z}_2) \cong S_3$ (left) and $PSL(2,\mathbb{Z}_3) \cong A_4$ (right) on Lagrangian subgroups (gapped boundaries).

where $r_S = -r^{-1}$ in $\mathbb{Z}_N$, while $\rho(e)$ is invariant under T (see Figure 4.2 for two examples). Lagrangian subgroups form two-terms orbits under S, except for $\rho(m)_r$ with $r^2 = -1$ mod N which are invariant. Similarly, they form three-terms orbits under ST, except for $\rho(m)_r$ with $r(r+1) = -1$ mod N which are invariant. Gapped boundaries corresponding to ω-invariant subgroups $\mathcal{L}$ allow for a 0-form symmetry action of the subgroup $G \subset \Gamma$ which stabilizes them.[12] This is clear from the Lagrangian description of the gapped boundaries, $S[L]$ in (4.1.16). The action of the 0-form symmetry does not leave the coupling to η invariant, but it can be reabsorbed in a redefinition of the generators L of $\mathcal{L}$.

Symmetry defects from higher gauging

In unitary TQFTs without local operators, all 0-form symmetries are expected to be generated by codimension-1 symmetry defects that are condensations of higher-form symmetries. This statement can be proven in the context of three-dimensional modular tensor categories (MTCs) [23, 242], while it seems plausible for higher dimensional TQFTs [23]. In this section we construct the $SL(2,\mathbb{Z}_N)$ symmetry generators of the 5d CS theory (4.1.12), in terms of condensations of the 2-form symmetry on 4d submanifolds. The $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry generated by the topological surface operators U_m in the 5d bulk becomes a 1-form symmetry on a 4d submanifold Σ on which we perform the condensation.

[12]This is true if G is a normal subgroup of Γ. This will always be so in the cases of interest to us.

We assume that the fusion algebra of surface operators is strictly associative, and since surfaces cannot braid in 4d, we can condense any subgroup $\mathcal{A} \subset \mathbb{Z}_N \times \mathbb{Z}_N$ of the 2-form symmetry. While condensing on a (spin) 4-manifold Σ, we have the possibility to add discrete torsion in the form of Dijkgraaf-Witten terms [243]. When we gauge the full group $\mathbb{Z}_N \times \mathbb{Z}_N$, the torsion is classified by $\mathbb{Z}_N^3$ and we label it by $x, y, z \in \mathbb{Z}_N$. In terms of the background $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ that we decompose into $\varphi_{\mathrm{e}}, \varphi_{\mathrm{m}} \in H^2(\Sigma, \mathbb{Z}_N)$, the phase of discrete torsion is given by

$$\Theta_{x,y,z} = \exp\left[\frac{2\pi i}{2N}\int_\Sigma \Big(y\,\mathfrak{P}(\varphi_{\mathrm{e}}) + z\,\mathfrak{P}(\varphi_{\mathrm{m}}) + 2x\,\varphi_{\mathrm{e}} \cup \varphi_{\mathrm{m}}\Big)\right] . \tag{4.1.26}$$

Here $\mathfrak{P} : H^2(\Sigma, \mathbb{Z}_N) \to H^4(\Sigma, \mathbb{Z}_{N\,\gcd(N,2)})$ is the Pontryagin square operation [244]. For N even, $\mathfrak{P}(\varphi)$ takes values in $\mathbb{Z}_{2N}$ and on spin manifolds it is an even class, therefore $y, z \in \mathbb{Z}_N$. For N odd, $\mathfrak{P}(\varphi)$ takes values in $\mathbb{Z}_N$ and we interpret the exponent as $\frac{2\pi i}{N}\,2^{-1} y \int \mathfrak{P}(\varphi_{\mathrm{e}})$ where $2^{-1} = \frac{N+1}{2} \bmod N$, and similarly for $z\,\mathfrak{P}(\varphi_{\mathrm{m}})$, therefore $y, z \in \mathbb{Z}_N$ once again. On the other hand, when we gauge a $\mathbb{Z}_N$ subgroup, the torsion is classified by $\mathbb{Z}_N$ and then only a combination of x, y, z appears. For simplicity, we will only consider the case that N is a prime number, because then $\mathbb{Z}_N$ does not contain non-trivial proper subgroups, and all its non-zero elements are invertible.

We want to compute the action of the 0-form condensation defects V on the 2-form defects U_l. To that purpose, we place U_l along $\mathbb{R}^2$ and wrap V around them, namely we place V on $\mathbb{R}^2 \times S^2$ with S^2 surrounding U_l. It turns out that it is more clear to perform condensation on compact submanifolds, therefore we substitute $\mathbb{R}^2$ with T^2. Eventually, we place U_l on T^2 and V on $\Sigma \equiv T^2 \times S^2$ around U_l (as in Figure 4.3 center).

To condense on Σ, we decompose the 2-form symmetry background $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ into a pair of backgrounds $\{\phi^{T^2}, \phi^{S^2}\}$ on the two factors of Σ, and we denote by $n = (n_{\mathrm{e}}, n_{\mathrm{m}})$ the holonomy of ϕ^{S^2} on S^2 (representing defects on T^2) and by $m = (m_{\mathrm{e}}, m_{\mathrm{m}})$ the holonomy of ϕ^{T^2} on T^2 (representing defects on S^2). Given a class $(x, y, z) \in \mathbb{Z}_N^3$ representing the choice of discrete torsion (4.1.26), its contribution to the path integral is

$$\Theta_{x,y,z}(n, m) = \exp\left[\frac{2\pi i}{N}\Big(x\left(n_{\mathrm{e}} m_{\mathrm{m}} + n_{\mathrm{m}} m_{\mathrm{e}}\right) + y\, n_{\mathrm{e}} m_{\mathrm{e}} + z\, n_{\mathrm{m}} m_{\mathrm{m}}\Big)\right] = \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}} \mathcal{T} n\right] \tag{4.1.27}$$

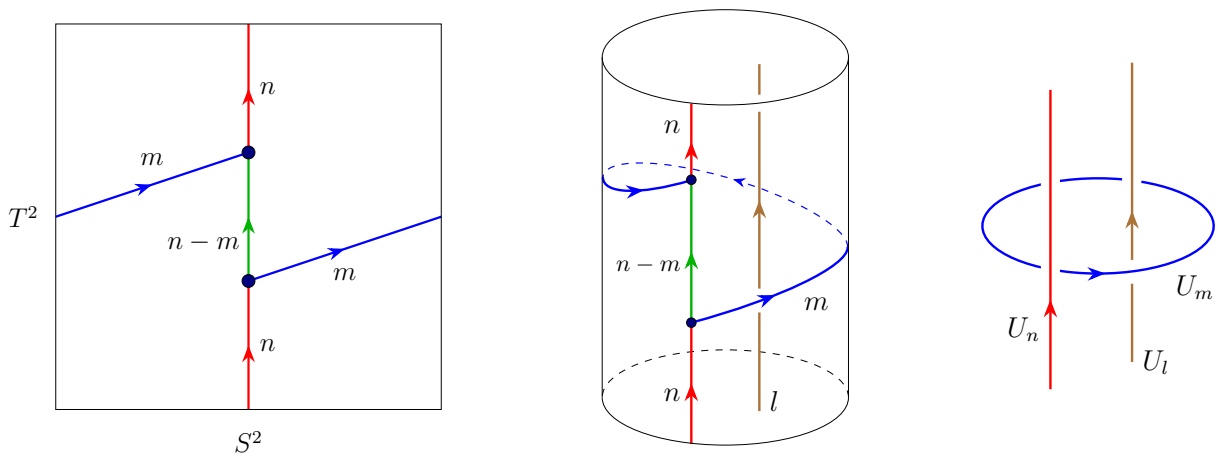


Figure 4.3: The condensation defect on $\Sigma = T^2 \times S^2$ with its network of 2d defects (left) surrounds a topological defect U_l placed on $T^2 \times \{0\}$ (center), where $0 \in B^3$ is the center of the 3-ball whose boundary is S^2. Up to a phase (4.1.29), the network can be resolved into a collection of closed surfaces with no junctions (right).

where we introduced the symmetric matrix of discrete torsions

$$\mathcal{T} = \begin{pmatrix} y & x \\ x & z \end{pmatrix} , \tag{4.1.28}$$

whose entries are in $\mathbb{Z}_N$. We can label the condensation defects of the 5d Chern-Simons theory as $V[\mathcal{A}, \mathcal{T}]$, where $\mathcal{A}$ is the condensed subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ and $\mathcal{T}$ is the matrix of discrete torsions. When $\mathcal{A} = \mathbb{Z}_N \times \mathbb{Z}_N$ we omit it, while when $\mathcal{A}$ is one-dimensional we denote it by one of its generators (p, q).

The condensation on Σ involves a network of 2-dimensional defects, as in Figure 4.3 left. Instead of working with a network (that requires to understand the trivalent junctions), we can resolve it into a pair of 2-dimensional defects: one U_n along T^2 on an outer copy of Σ, and one U_m along S^2 on an inner copy of Σ (Fig. 4.3 right). This operation involves a phase, and is equivalent to a normal ordering prescription. More generally, for N odd we can write

$$U(\gamma_1 + \gamma_2) = \exp\left[-\frac{2\pi i}{N}\, 2^{-1} \langle \gamma_1, \gamma_2 \rangle\right] U(\gamma_1)\, U(\gamma_2) \,. \tag{4.1.29}$$

(The case of N even is discussed below.) Here $\gamma_i \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ represent two defects on Σ, while $\langle\, ,\, \rangle$ is the product of the (symmetric) cup product on Σ and the

(antisymmetric) Dirac pairing in $\mathbb{Z}_N \times \mathbb{Z}_N$. On the right-hand-side, $U(\gamma_1)$ is outer while $U(\gamma_2)$ is inner. This is essentially a square root of the braiding matrix

$$B_{mn} = e^{\frac{2\pi i}{N}\langle m,n\rangle} \tag{4.1.30}$$

as in Figure 4.1 left. We obtain:

$$\begin{aligned} V[\mathcal{A},\mathcal{T}]\,U_l &= \frac{1}{\left|H^2(\Sigma,\mathcal{A})\right|^{1/2}} \sum_{n,m\in\mathcal{A}} \Theta_{x,y,z}(n,m)\; e^{-\frac{2\pi i}{N}2^{-1}\langle n,m\rangle}\, B_{ml}\, U_{l+n} \\ &= \frac{1}{|\mathcal{A}|} \sum_{n,m\in\mathcal{A}} \exp\Bigg[\frac{2\pi i}{N}\Bigg(m_{\mathrm{e}}\Big(y\,n_{\mathrm{e}} + x\,n_{\mathrm{m}} + \tfrac{1}{2}n_{\mathrm{m}} + l_{\mathrm{m}}\Big) \\ &\qquad\qquad + m_{\mathrm{m}}\Big(x\,n_{\mathrm{e}} + z\,n_{\mathrm{m}} - \tfrac{1}{2}n_{\mathrm{e}} - l_{\mathrm{e}}\Big)\Bigg)\Bigg]\, U_{l+n}\ . \end{aligned} \tag{4.1.31}$$

The sum over m produces a delta function for n. When this has exactly one solution, the sum over n selects a defect U_{Ml} where $M \in SL(2,\mathbb{Z}_N)$ is the group element corresponding to the condensation defect $V[\mathcal{A},\mathcal{T}] \equiv V_M$. The cases in which there are multiple or no solutions, even though they are not relevant to our purposes, will be discussed at the end.

If $\mathcal{A} \cong \mathbb{Z}_N$ is generated by (p,q), we can write

$$m_{\mathrm{e}} = \mu\, p\ , \qquad m_{\mathrm{m}} = \mu\, q\ , \qquad n_{\mathrm{e}} = \nu\, p\ , \qquad n_{\mathrm{m}} = \nu\, q\ . \tag{4.1.32}$$

Notice that the phase (4.1.29) trivializes. The sum over μ produces a delta function that fixes $p\,l_{\mathrm{m}} - q\,l_{\mathrm{e}} + \xi\,\nu = 0$ and selects one value for ν (as long as $\xi \neq 0$), where $\xi = 2pqx + yp^2 + zq^2$. This reproduces the action of[13]

$$M = T^k_{\mathcal{H}} \equiv \mathcal{H} T^k \mathcal{H}^{-1} \tag{4.1.33}$$

for $k = -\xi^{-1}$ and

$$\mathcal{H} = \begin{pmatrix} p & * \\ q & * \end{pmatrix} \in SL(2,\mathbb{Z}_N)\ . \tag{4.1.34}$$

The three parameters x, y, z enter only in the combination ξ, as expected since the discrete torsion is classified by $\mathbb{Z}_N$. Since T^k leaves invariant the vector $v = (1,0)$, then $T^k_{\mathcal{H}}$ leaves invariant the vector $\mathcal{H}v = (p,q)$, and we obtain the defect implementing $T^k_{\mathcal{H}}$ by condensing the algebra generated by (p,q) (with a non-vanishing torsion

[13] One has $T^k_{\mathcal{H}} = \begin{pmatrix} 1-kpq & kp^2 \\ -kq^2 & 1+kpq \end{pmatrix}$, $\nu = kpl_{\mathrm{m}} - kql_{\mathrm{e}}$ and so $T^k_{\mathcal{H}}\begin{pmatrix} l_{\mathrm{e}} \\ l_{\mathrm{m}} \end{pmatrix} = \begin{pmatrix} l_{\mathrm{e}} + \nu p \\ l_{\mathrm{m}} + \nu q \end{pmatrix}$.

determined by k). For instance, T^k is obtained by condensing the electric surfaces $(n_{\rm e}, 0)$, while its electromagnetic dual ST^kS^{-1} is realized by condensing the magnetic surfaces $(0, n_{\rm m})$. An element of $SL(2,\mathbb{Z}_N)$ (with N prime) can be written as $\mathcal{H}T^k\mathcal{H}^{-1}$ if and only if its trace is 2 mod N. There are N^2 such elements, including the identity.[14] Indeed condensation produces $N-1$ defects (as we change the torsion ξ) for each of the $N+1$ $\mathbb{Z}_N$ subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$, besides the identity (which is formally obtained by condensing the trivial subgroup $(0,0)$). We will comment on the case with vanishing torsion below.

The elements of $SL(2,\mathbb{Z}_N)$ (N prime) with trace different from 2 are obtained by condensing the full $\mathbb{Z}_N \times \mathbb{Z}_N$. The sum over m produces a delta function that fixes[15] $\left(\mathcal{T}+\frac{\epsilon}{2}\right)n+\epsilon l = 0$ and selects one value of n (as long as $\left(\mathcal{T}+\frac{\epsilon}{2}\right)$ is invertible). This reproduces the action of

$$M = \left(\mathcal{T}+\frac{\epsilon}{2}\right)^{-1}\left(\mathcal{T}-\frac{\epsilon}{2}\right) . \tag{4.1.35}$$

Note that $\det\left(\mathcal{T}\pm\frac{\epsilon}{2}\right) = \left(2-\operatorname{Tr} M\right)^{-1}$, therefore all elements $M \in SL(2,\mathbb{Z}_N)$ with $\operatorname{Tr} M \neq 2$ can be obtained this way. The relation can be inverted:

$$\mathcal{T} = \frac{\epsilon}{2}\left(1+M\right)\left(1-M\right)^{-1} . \tag{4.1.36}$$

Notice that the two factors on the right-hand-side commute. Moreover, in $SL(2,\mathbb{Z}_N)$ we have $\det(1\pm M) = 2 \pm \operatorname{Tr} M$. The following relation is also useful:

$$\mathcal{T}+\frac{\epsilon}{2} = \epsilon\left(1-M\right)^{-1} . \tag{4.1.37}$$

Explicitly, the discrete torsion that produces the symmetry defect for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}_N)$ with $\operatorname{Tr} M \neq 2$ is $x = \frac{d-a}{2(2-a-d)}$, $y = \frac{c}{2-a-d}$, $z = -\frac{b}{2-a-d}$. Finally, assuming that $\left(\mathcal{T}+\frac{\epsilon}{2}\right)$ is invertible, $\det\mathcal{T} = \operatorname{Tr}(1+M)\left[4\operatorname{Tr}(1-M)\right]^{-1}$ therefore $\mathcal{T}$ is invertible if and only if $\operatorname{Tr} M \neq -2$ mod N. The case $\mathcal{T}=0$ corresponds to $M=-\mathbb{1}\equiv C$ which is charge conjugation. The case that $\mathcal{T}$ has rank 1 corresponds (for N prime) to $M = C\mathcal{H}T^k\mathcal{H}^{-1}$ where

$$\mathcal{T} = \frac{k}{4}\begin{pmatrix} q^2 & -pq \\ -pq & p^2 \end{pmatrix} = \frac{k}{4}\left(\epsilon v\right)\cdot\left(\epsilon v\right)^{\mathsf{T}} , \qquad v = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \text{and} \qquad \mathcal{H} = \begin{pmatrix} p & * \\ q & * \end{pmatrix} . \tag{4.1.38}$$

[14] All matrices $M \in SL(2,\mathbb{Z}_N)$ with $\operatorname{Tr} M = 2$ can be written as $M = \left(\begin{smallmatrix} 1-\alpha & \beta \\ \gamma & 1+\alpha \end{smallmatrix}\right)$ with $\alpha^2 = \beta\gamma$ mod N. This equation, for N prime, has N^2-1 solutions with at least one of α, β, γ not zero. One can also easily show that, for N prime, any such matrix M can be written as in footnote 13. The total number of elements in $SL(2,\mathbb{Z}_N)$ (N prime) is instead N^3-N.

[15] When working in $\mathbb{Z}_N$ with N prime, by fractions we always mean the inverse element mod N.

$M \in SL(2,\mathbb{Z}_N)$	$M \cdot (l_1, l_2)$	$\mathcal{A}$	(x,y,z)
$C = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	$(-l_1,\, -l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$(0,0,0)$
$CT^k = \left(\begin{smallmatrix} -1 & -k \\ 0 & -1 \end{smallmatrix}\right)$	$(-l_1 - kl_2,\, -l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$(0,0,\frac{1}{4}k)$
$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$(-l_2,\, l_1)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\left(0,\frac{1}{2},\frac{1}{2}\right)$
$ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$	$(-l_2,\, l_1 + l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\left(\frac{1}{2},1,1\right)$
$(ST)^2 = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$(-l_1 - l_2,\, l_1)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\left(\frac{1}{6},\frac{1}{3},\frac{1}{3}\right)$
$T^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$	$(l_1 + kl_2,\, l_2)$	$\langle (1,0) \rangle$	$\xi = -k^{-1}$

Table 4.1: Examples of $SL(2,\mathbb{Z}_N)$ condensation defects, obtained by condensing $\mathcal{A} \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ with torsion (x,y,z), for $N > 3$ prime (for $N = 2,3$ some of those formulas are different).

In Table 4.1 we summarize a few examples.

Small values of N. Some of the previous formulas are ill-defined for small N. For $N = 2$, and more generally for N even, we cannot use the normal ordering prescription in (4.1.29) because 2^{-1} is ill-defined. However, notice that the phase that enters in the definition (4.1.31) of the operator $V[\mathcal{A},\mathcal{T}]$ is the product of the torsion and the normal ordering phases:

$$\exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}} \begin{pmatrix} y & \tilde{x} \\ \tilde{x}-1 & z \end{pmatrix} n\right] \equiv \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}} \widetilde{\mathcal{T}} n\right], \tag{4.1.39}$$

where $\tilde{x} = x + \frac{1}{2}$ and $\widetilde{\mathcal{T}} = \mathcal{T} + \frac{\epsilon}{2}$. The quantities $\tilde{x} \in \mathbb{Z}_N$ and $\widetilde{\mathcal{T}}$ are well defined, even for N even, and we can use them to classify the torsion. The group $SL(2,\mathbb{Z}_2) \cong PSL(2,\mathbb{Z}_2) \cong S_3$ (note that $C \cong \mathbb{1}$) has 6 elements, 4 of which have trace equal to 2 mod 2:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad STS^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad S = (TS)T(TS)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.1.40}$$

besides the identity. The corresponding defect operators are obtained by condensing the $\mathbb{Z}_2$ subgroup generated by $(1,0)$, $(0,1)$ and $(1,1)$, respectively, with non-

vanishing torsion $\xi = pq + yp^2 + zq^2 = 1$. The remaining elements,

$$ST = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad (ST)^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} , \tag{4.1.41}$$

have trace equal to 1 and are described by gauging the full $\mathbb{Z}_2 \times \mathbb{Z}_2$. The relation between torsion and symmetry action M is as in (4.1.35) and (4.1.37), as long as one parametrizes the torsion using $\widetilde{\mathcal{T}}$, therefore $\widetilde{\mathcal{T}} = \epsilon\,(1-M)^{-1}$. One finds that ST is obtained from torsion $(\tilde{x}, y, z) = (1, 1, 1)$, while $(ST)^2$ is obtained from $(\tilde{x}, y, z) = (0, 1, 1)$. These two values of the torsion are the only possible ones providing a matrix $\widetilde{\mathcal{T}}$ invertible in $\mathbb{Z}_2$.

For $N = 3$, the element $(ST)^2$ in Table 4.1 has trace equal to 2 mod 3. Indeed we can write $(ST)^2 = \mathcal{H}T^2\mathcal{H}^{-1}$ with $\mathcal{H} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and thus the corresponding defect operator is obtained by condensing the $\mathbb{Z}_3$ subgroup generated by $(1, 1)$ with torsion $\xi = 1$.

Fusion. The fusion rules of (invertible) condensation defects correctly satisfy the product of $SL(2, \mathbb{Z}_N)$. The method we describe below is general, however for brevity we only exhibit the product of defects obtained by condensing the full group $\mathbb{Z}_N \times \mathbb{Z}_N$. The defect operators $V[\mathcal{T}]$ on Σ defined in (4.1.31) can be rewritten as

$$V[\mathcal{T}] = \frac{1}{N^2} \sum_{n,m\,\in\,\mathbb{Z}_N\times\mathbb{Z}_N} \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}}\left(\mathcal{T} + \frac{\epsilon}{2}\right) n\right] U_n[T^2]\, U_m[S^2] \tag{4.1.42}$$

where we indicated whether the two-dimensional defects U are placed on T^2 or S^2, and rightmost operators are inner. Using the braiding matrix (4.1.30), we obtain

$$\begin{aligned} &V[\mathcal{T}_2]\, V[\mathcal{T}_1] = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.1.43) \\ &\quad = \frac{1}{N^4} \sum_{\substack{n,m \\ n',m'}} \exp\left[\frac{2\pi i}{N}\left(m^{\mathsf{T}}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right) n + m'^{\mathsf{T}}\left(\mathcal{T}_1 + \frac{\epsilon}{2}\right) n' + m^{\mathsf{T}}\epsilon n'\right)\right] U_{n+n'}[T^2]\, U_{m+m'}[S^2] \,. \end{aligned}$$

Setting $n = l - n'$, $m = k - m'$ and performing the sum over m' produces a delta function on $(\mathcal{T}_1 + \mathcal{T}_2)n' = \left(\mathcal{T}_2 + \frac{\epsilon}{2}\right)l$. When $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible, one eliminates n' obtaining

$$V[\mathcal{T}_2] \times V[\mathcal{T}_1] = V[\mathcal{T}_{21}] \tag{4.1.44}$$

with

$$\mathcal{T}_{21} = \mathcal{T}_2 - \left(\mathcal{T}_2 - \frac{\epsilon}{2}\right)\left(\mathcal{T}_1 + \mathcal{T}_2\right)^{-1}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right) . \tag{4.1.45}$$

The relation (4.1.45) can be rewritten as

$$\mathcal{T}_{21} + \frac{\epsilon}{2} = \left(\mathcal{T}_1 + \frac{\epsilon}{2}\right)(\mathcal{T}_1 + \mathcal{T}_2)^{-1}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right) . \tag{4.1.46}$$

Together with (4.1.37), with a little bit of algebra, it implies $M_{21} = M_2 M_1$ as expected.

When $\mathcal{T}_1 + \mathcal{T}_2 = 0$, the sum over m' and n' sets $l = k = 0$. We conclude that

$$V[\mathcal{T}] \times V[-\mathcal{T}] = \mathbb{1} , \tag{4.1.47}$$

in agreement with the fact that $M(-\mathcal{T}) = M(\mathcal{T})^{-1}$. The case in which $\mathcal{T}_1 + \mathcal{T}_2$ has rank 1 can be treated in a similar way. In particular, from (4.1.37) it follows that

$$(1 - M_2)\,\epsilon^{-1}(\mathcal{T}_1 + \mathcal{T}_2)(1 - M_1)\,\epsilon^{-1} = (1 - M_2 M_1)\,\epsilon^{-1} . \tag{4.1.48}$$

Taking the determinant on both sides and using that $\det(1 - M) = \mathrm{Tr}(1 - M)$ for M in $SL(2, \mathbb{Z})$, we conclude that $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible if and only if $M_2 M_1$ has trace different from 2 mod N, whilst $(\mathcal{T}_1 + \mathcal{T}_2)$ has rank 1 if and only if $M_2 M_1$ has trace equal to 2 mod N but is not the identity, and thus the corresponding symmetry operator is described by the condensation of a subgroup $\mathcal{A} \cong \mathbb{Z}_N$.

Degenerate torsion and non-invertible surfaces. Besides the $SL(2, \mathbb{Z}_N)$ symmetry defects, higher gauging can produce projectors when the symmetry we condense on a submanifold could also be condensed in the bulk [23]. This is the case when the condensed group is non-anomalous and the discrete torsion would be allowed in 5d. One example is the condensation of $\mathcal{A} = \langle (p, q) \rangle \cong \mathbb{Z}_N$ with vanishing torsion. From the analysis that follows (4.1.31) we see that the delta function introduced by the sum over μ either has no solution, or has $|\mathcal{A}| = N$ solutions, and the operator $V[\mathcal{A}, 0]$ acts on the surfaces U_l as

$$V[\mathcal{A}, 0]\, U_l = \begin{cases} 0 & \text{if } l \notin \mathcal{A} , \\ \sum_{n \in \mathcal{A}} U_n & \text{if } l \in \mathcal{A} . \end{cases} \tag{4.1.49}$$

This is consistent with the following non-invertible composition law [23]:

$$V[\mathcal{A}, 0] \times V[\mathcal{A}, 0] = \left|H^2(\Sigma, \mathcal{A})\right|^{1/2} V[\mathcal{A}, 0] , \tag{4.1.50}$$

where the coefficient on the right-hand-side is the partition function of a TQFT.

Besides, we obtain a non-invertible surface when we condense the full $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry (which is anomalous in the bulk) with a torsion matrix $\mathcal{T}$ such that $\left(\mathcal{T} + \frac{\epsilon}{2}\right)$ is not invertible. Notice that, since $\mathcal{T}$ is symmetric and ϵ antisymmetric, if $\left(\mathcal{T} + \frac{\epsilon}{2}\right)$ is non-invertible then it has rank 1.[16] In that case, there exist two integer vectors $v_{1,2} \in \mathbb{Z}_N \times \mathbb{Z}_N$ such that

$$\mathcal{T} + \frac{\epsilon}{2} = (\epsilon v_1) \cdot (\epsilon v_2)^{\mathsf{T}} \qquad \text{and} \qquad v_1^{\mathsf{T}} \epsilon\, v_2 = 1 \ . \tag{4.1.51}$$

The second condition comes from the antisymmetric part of the matrix. The sum over m in (4.1.31) gives a delta function on the solutions to $\left(\mathcal{T} + \frac{\epsilon}{2}\right) n = -\epsilon l$, that takes the form

$$\left(v_2^{\mathsf{T}} \epsilon\, n\right) v_1 = l \ . \tag{4.1.52}$$

Let $\mathcal{A}_{1,2} \cong \mathbb{Z}_N$ be the two subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ generated by $v_{1,2}$, respectively. If $l \notin \mathcal{A}_1$ then (4.1.52) has no solution for n. On the contrary, if $l \in \mathcal{A}_1$ then the solutions are $n = -l + \nu v_2$ with any $\nu \in \mathbb{Z}_N$. We obtain:

$$V[\mathcal{T}]\, U_l = \begin{cases} 0 & \text{if } l \notin \mathcal{A}_1 \ , \\ \sum_{n \in \mathcal{A}_2} U_n & \text{if } l \in \mathcal{A}_1 \ . \end{cases} \tag{4.1.53}$$

In fact, given two different $\mathbb{Z}_N$ subgroups $\mathcal{A}_1 \neq \mathcal{A}_2$ (then, for N prime, $\mathcal{A}_1 \cap \mathcal{A}_2 = (0,0)$ necessarily), one easily checks the composition law

$$V[\mathcal{A}_2, 0]\, V[\mathcal{A}_1, 0] = V[\mathcal{T}] \tag{4.1.54}$$

where, on the right-hand-side, $\mathcal{T}$ is given by (4.1.51).

Continuum description of symmetry defects

In view of describing the twisted sectors of the $SL(2, \mathbb{Z}_N)$ symmetry, it is useful to reformulate the previous discussion in terms of continuum Lagrangians. We take N odd. When the condensed group is $\mathcal{A} = \mathbb{Z}_N \times \mathbb{Z}_N$, the defect $V[\mathcal{T}]$ is described by a 4d TQFT with two dynamical 2-forms $\Phi = (\varphi_{\mathrm{e}}, \varphi_{\mathrm{m}})$ and four 1-forms $\Psi = (\psi_{\mathrm{e}}, \psi_{\mathrm{m}})$, $\Gamma = (\gamma_{\mathrm{e}}, \gamma_{\mathrm{m}})$ with action [157]:

$$S[\mathcal{T}] = \frac{N}{2\pi} \int_\Sigma \left[\mathcal{B}^{\mathsf{T}} \left(\Phi + d\Gamma\right) + \Phi^{\mathsf{T}} d\Psi + \frac{1}{2}\, \Phi^{\mathsf{T}} \mathcal{T} \Phi \right] . \tag{4.1.55}$$

The torsion is parametrized by the symmetric matrix $\mathcal{T}$ with entries in $\mathbb{Z}_N$ and such that $\mathcal{T} + \frac{\epsilon}{2}$ is invertible.[17] On the other hand, when $\mathcal{A} \cong \mathbb{Z}_N$ is generated by (p, q)

[16] This is also true for $N = 2$, because the matrix $\widetilde{\mathcal{T}} = \left(\begin{smallmatrix} y & \tilde{x} \\ \tilde{x}-1 & z \end{smallmatrix}\right)$ cannot be zero.

[17] After integrating over Ψ, the periods of Φ are multiples of $\frac{2\pi}{N}$. Thus on spin manifolds Σ, shifts of the entries of $\mathcal{T}$ by N leave e^{iS} invariant [67].

we only keep one 2-form φ and two 1-forms ψ, γ with action:

$$S\big[\langle (p,q)\rangle,\, \xi\big] = \frac{N}{2\pi}\int_\Sigma \Big[\big(pb+qc\big)\,\big(\varphi + d\gamma\big) + \varphi\, d\psi + \frac{\xi}{2}\,\varphi\varphi\Big]\,. \tag{4.1.56}$$

The torsion is parametrized by a non-vanishing $\xi \in \mathbb{Z}_N$.

Integrating over Ψ and Γ in (4.1.55) forces Φ and the pull-back of $\mathcal{B}$ to be in $H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$. Then Φ can be identified with the Poincaré dual to a 2-cycle $\sigma \in H_2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$. Since Φ couples to $\mathcal{B}$, $\Phi = \mathrm{PD}(\sigma)$ represents a two-dimensional defect $U[\sigma]$ wrapped on σ, and the theory (4.1.55) reproduces higher gauging of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry on Σ with torsion $\mathcal{T}$. A similar discussion applies to (4.1.56). The action (4.1.55) is invariant under the following gauge transformations:

$$\mathcal{B} \to \mathcal{B} + d\alpha\,, \qquad \Phi \to \Phi + d\lambda\,, \qquad \Psi \to \Psi - \mathcal{T}\lambda - \alpha + d\mu\,, \qquad \Gamma \to \Gamma - \lambda + d\nu\,. \tag{4.1.57}$$

Considering $\mathcal{B}$ as a background field, the theory (4.1.55) is of a different type depending on whether $\mathcal{T}$ is an invertible matrix over $\mathbb{Z}_N$ or not.

If $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, then (4.1.55) is an invertible TQFT. Indeed, adapting the discussion in [157] to our case, all closed surfaces $\exp\big(im^{\mathsf{T}}\oint\Phi\big)$ are gauge invariant and implement a $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry, however, because of the equation of motion $\mathcal{T}\Phi = -d\Psi$, when m is in the image of the map $\mathcal{T} : \mathbb{Z}_N^2 \to \mathbb{Z}_N^2$, the surface acts trivially. Therefore only $(\mathbb{Z}_N \times \mathbb{Z}_N)/\,\mathbb{I}\mathrm{m}\,\mathcal{T}$ acts faithfully, and if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then there is no faithful action at all. On the other hand, the line integrals of Ψ might not be gauge invariant by themselves and need to be the boundary of an open surface $\mathcal{D}$: $\exp\big(ik^{\mathsf{T}}\oint_\ell\Psi + ik^{\mathsf{T}}\mathcal{T}\int_{\mathcal{D}}\Phi\big)$ with $\ell = \partial\mathcal{D}$. They become pure line operators when the surface is transparent, *i.e.*, when $\mathcal{T}k = 0 \bmod N$. Hence the 2-form symmetry of the theory is $\ker\mathcal{T} \subset \mathbb{Z}_N \times \mathbb{Z}_N$, which is trivial if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$. Summarizing, if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then the theory (4.1.55) has no topological operators, and is thus an invertible TQFT. This implies that we could integrate out the fields Φ and Ψ. Their equations of motion say that $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ and $\mathcal{T}^{-1}(\mathcal{B} + d\Psi) + \Phi = \check{\Phi}$, where $\check{\Phi} \in H^2(\Sigma, \mathbb{Z}_1 \times \mathbb{Z}_1)$ is a gauge field with integer periods, while $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$ in $\mathbb{Z}_N$. Substituting into the action, one obtains

$$S_{\text{invertible}}[\mathcal{T}] = \frac{N}{2\pi}\int_\Sigma \Big[\mathcal{B}^{\mathsf{T}} d\widetilde{\Gamma} - \frac{1}{2}\,\mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B}\Big]\,, \tag{4.1.58}$$

up to total derivatives and multiples of 2π, where $\widetilde{\Gamma} = \Gamma - \mathcal{T}^{-1}\Psi$ transforms as $\widetilde{\Gamma} \to \widetilde{\Gamma} + \mathcal{T}^{-1}\alpha$ under gauge transformations.

If, on the contrary, $\mathcal{T}$ is a non-invertible matrix, then the 4d theory is a non-trivial TQFT with surface and line operators labeled by $(\mathbb{Z}_N \times \mathbb{Z}_N)/\,\mathbb{I}\mathrm{m}\,\mathcal{T}$ and $\ker \mathcal{T}$, respectively. Recall that this case corresponds to $SL(2,\mathbb{Z}_N)$ matrices M with $\mathrm{Tr}\, M = -2 \bmod N$, which are of the form $M = C\mathcal{H}T^k\mathcal{H}^{-1}$. In the special case $\mathcal{T} = 0$ (that corresponds to $M = C$) the 4d theory (4.1.55) is a pure $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge theory, whose 1-form symmetry is coupled to the background field $\mathcal{B}$.

We can verify that $S[\mathcal{T}]$ in (4.1.55) implements the correct transformation of 2d defects U_l. We introduce a coordinate r transverse to the 4d defect, such that $\Sigma = \{r = 0\}$, and consider the bulk-plus-defect action

$$S_{\substack{\text{bulk plus}\\ \text{defect}}} = \frac{N}{4\pi}\int \mathcal{B}^{\mathsf{T}}\epsilon\, d\mathcal{B} + \frac{N}{2\pi}\int_{r=0}\left[\mathcal{B}^{\mathsf{T}}\big(\Phi + d\Gamma\big) + \Phi^{\mathsf{T}} d\Psi + \frac{1}{2}\,\Phi^{\mathsf{T}}\mathcal{T}\Phi\right] . \tag{4.1.59}$$

Integrating out the gauge field Φ we obtain an effective description of the interface, which induces a discontinuity

$$\mathcal{B}_L = M^{\mathsf{T}}\mathcal{B}_R \tag{4.1.60}$$

in the gauge field $\mathcal{B}$ (L, R stand for left/right at $r < 0$ and $r > 0$, respectively). Here M is transposed because the $SL(2,\mathbb{Z})$ action on fields is dual to the one on charges, that we previously denoted by M. Indeed, imagine placing a 2d defect operator U_l in the region $r < 0$ (see Figure 4.4) which, compared with our previous setup in Figure 4.3 center, would be the interior region. The expectation value of the operator is $\exp\big(il^{\mathsf{T}}\int\mathcal{B}_L\big) = \exp\big(il^{\mathsf{T}}M^{\mathsf{T}}\int\mathcal{B}_R\big)$. Thus, for an external observer, the compound system of the 4d defect on Σ wrapping the 2d operator U_l appears as a 2d operator U_{Ml}. Let us determine M from (4.1.59). After choosing a gauge (λ, α) in which Γ and Ψ are zero, the equations of motion for $\mathcal{B}$ and Φ read

$$0 = \big(\mathcal{B} + \mathcal{T}\Phi\big)\,\delta(r)dr \tag{4.1.61}$$

$$\epsilon\, d\mathcal{B} = -\Phi\,\delta(r)dr\ . \tag{4.1.62}$$

The gauge field Φ acts as a source for $\mathcal{B}$. Working in a gauge in which $\mathcal{B}_{ri} = 0$, we have $\partial_r\mathcal{B}(r) = \epsilon\,\Phi\,\delta(r)$. This differential equation can be solved: $\mathcal{B}(r) = \mathcal{B}_L + \epsilon\,\Phi\,\theta(r)$, where $\mathcal{B}_L$ is the value of $\mathcal{B}$ for $r < 0$. Multiplying by $\delta(r)$, integrating in a neighbourhood of $r = 0$ and using $\delta(r) = \partial_r\theta(r)$, we obtain $\mathcal{B}(0) = \mathcal{B}_L + \frac{\epsilon}{2}\,\Phi = -\mathcal{T}\Phi$. The second equality follows from (4.1.61). Finally, evaluating at $r > 0$ we find $\mathcal{B}_R = \mathcal{B}_L + \epsilon\,\Phi$ which implies

$$\mathcal{B}_R = \left[1 - \epsilon\Big(\mathcal{T} + \frac{\epsilon}{2}\Big)^{-1}\right]\mathcal{B}_L\ . \tag{4.1.63}$$

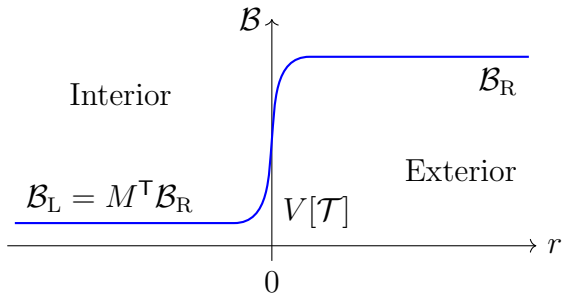


Figure 4.4: The 4d symmetry defect $V[\mathcal{T}]$ induces a discontinuity in the gauge field $\mathcal{B}$ across its surface. Compared with the setup of Figure 4.3 center, the region $r < 0$ is the interior of the cylinder while $r > 0$ is the exterior.

This discontinuity, when written in terms of M using (4.1.37), is exactly (4.1.60). If $\mathcal{T}$ is invertible, one can repeat the computation using (4.1.58) obtaining the same result.

When $\mathcal{A} \cong \mathbb{Z}_N$ one should use the defect Lagrangian (4.1.56) with only one gauge field φ. For instance, when the defect action is coupled to b (*i.e.*, $(p,q) = (1,0)$) and with torsion $\xi \neq 0$, the equation of motion from c simply sets $db = 0$ implying $b_L = b_R$, while the equation of motion from b, after substituting for the solution $\varphi = -\xi^{-1} b(0)$, gives

$$c_L = c_R - \xi^{-1} b_R \,. \tag{4.1.64}$$

This corresponds to the action of T^k with $k = -\xi^{-1}$.

Fusion of defects. We can derive the fusion of defects — that we already analyzed around (4.1.44) in terms of the discrete formalism — using continuum Lagrangians. We place two defects, with action as in (4.1.55), along two codimension-1 surfaces $\Sigma_{1,2}$ at positions $r_{1,2}$ with $r_1 < r_2$. They act as sources for the bulk gauge fields $\mathcal{B}$:

$$\epsilon\, d\mathcal{B} = -\big(\Phi_1 + d\Gamma_1\big)\,\delta(r - r_1)dr - \big(\Phi_2 + d\Gamma_2\big)\,\delta(r - r_2)dr \,. \tag{4.1.65}$$

Since $d\Phi_{1,2} = 0$ from the equations of motion, we can solve the equation as

$$\mathcal{B} = \mathcal{B}_0 + \epsilon\,\big(\Phi_1 + d\Gamma_1\big)\,\theta(r - r_1) + \epsilon\,\big(\Phi_2 + d\Gamma_2\big)\,\theta(r - r_2) \,. \tag{4.1.66}$$

Here $\mathcal{B}_0$ is a background value for $\mathcal{B}$, before adding the effect of the defects. It turns out that a crucial role in computing the fusion is played by the slab of bulk

theory in between the two defects, which produces a phase factor. There are two contributions. One comes from substituting (4.1.66) in the bulk action:

$$\frac{N}{4\pi}\int_{r_2}(\Phi_1+d\Gamma_1)^{\mathsf{T}}\epsilon\,(\Phi_2+d\Gamma_2)\,\theta(r_2-r_1)+\frac{N}{4\pi}\int_{r_1}(\Phi_2+d\Gamma_2)^{\mathsf{T}}\epsilon\,(\Phi_1+d\Gamma_1)\,\theta(r_1-r_2)$$
$$=\frac{N}{4\pi}\int_{r_2}\big(\Phi_1+d\Gamma_1\big)^{\mathsf{T}}\epsilon\,\big(\Phi_2+d\Gamma_2\big)\ . \tag{4.1.67}$$

Another one comes from substituting (4.1.66) in the two defect actions. The defect at $r=r_2$ produces $-\frac{N}{2\pi}\int_{r_2}(\Phi_1+d\Gamma_1)^{\mathsf{T}}\epsilon\,(\Phi_2+d\Gamma_2)$, while the one at $r=r_1$ does not give any contribution. In those substitutions we did not include the background $\mathcal{B}_0$, that we will couple to the final effective action. Collecting the contributions, we obtain the following action for the product of defects:

$$S[\mathcal{T}_{21}]=S[\mathcal{T}_1]+S[\mathcal{T}_2]-\frac{N}{4\pi}\int\big(\Phi_1+d\Gamma_1\big)^{\mathsf{T}}\epsilon\,\big(\Phi_2+d\Gamma_2\big)\ . \tag{4.1.68}$$

We can interpret the effect of the last term in the path integral as a phase due to the braiding between 2-dimensional defects U_m. To write out the effective action, we identify $r_1=r_2=0$ and simply write $\mathcal{B}_0\to\mathcal{B}$ for the background field. We also change variables to $\Phi=\Phi_1+\Phi_2$, $\widetilde{\Psi}=\Psi_1-\Psi_2$ and $\Gamma=\Gamma_1+\Gamma_2$. We obtain:

$$S[\mathcal{T}_{21}]=\frac{N}{2\pi}\int_\Sigma\Big[\mathcal{B}^{\mathsf{T}}(\Phi+d\Gamma)+\Phi^{\mathsf{T}}\Big(d\Psi_2+\frac{\epsilon}{2}\,d\Gamma_1\Big)+\frac{1}{2}\,\Phi^{\mathsf{T}}\mathcal{T}_2\Phi\Big]+S_{\text{int}}(\Phi,\Phi_1)$$
$$S_{\text{int}}=\frac{N}{2\pi}\int_\Sigma\Big[\Phi_1^{\mathsf{T}}\Big(d\widetilde{\Psi}-\Big(\mathcal{T}_2+\frac{\epsilon}{2}\Big)\Phi-\frac{\epsilon}{2}\,d\Gamma\Big)+\frac{1}{2}\,\Phi_1^{\mathsf{T}}\big(\mathcal{T}_1+\mathcal{T}_2\big)\Phi_1-\frac{1}{2}\,d\Gamma_1^{\mathsf{T}}\epsilon\,d\Gamma\Big]\ . \tag{4.1.69}$$

The field Φ_1, which is forced to be a cochain in $H^2(\Sigma,\mathbb{Z}_N\times\mathbb{Z}_N)$ by the equations of motion, does not directly couple to the bulk. The last term is a total derivative that vanishes on closed manifolds.

When $(\mathcal{T}_1+\mathcal{T}_2)$ is invertible in $\mathbb{Z}_N$, then Φ_1 appears quadratically and can be integrated out, obtaining:

$$S[\mathcal{T}_{21}]=\frac{N}{2\pi}\int_\Sigma\Big[\mathcal{B}^{\mathsf{T}}(\Phi+d\Gamma)+\Phi^{\mathsf{T}}d\Psi+\frac{1}{2}\,\Phi^{\mathsf{T}}\mathcal{T}_{21}\Phi$$
$$-\frac{1}{2}\,d\Big(\Big(\widetilde{\Psi}-\frac{\epsilon}{2}\,\Gamma\Big)^{\mathsf{T}}\big(\mathcal{T}_1+\mathcal{T}_2\big)^{-1}d\Big(\widetilde{\Psi}-\frac{\epsilon}{2}\,\Gamma\Big)\Big)\Big]\ , \tag{4.1.70}$$

where $\mathcal{T}_{21}$ is the matrix (4.1.45), we defined $\Psi=\Psi_2+\frac{\epsilon}{2}\Gamma_1+\big(\mathcal{T}_2-\frac{\epsilon}{2}\big)\big(\mathcal{T}_1+\mathcal{T}_2\big)^{-1}\big(\widetilde{\Psi}-\frac{\epsilon}{2}\Gamma\big)$, and $(\mathcal{T}_1+\mathcal{T}_2)^{-1}$ is the inverse in $\mathbb{Z}_N$. The last term is a total derivative and

can be ignored on closed manifolds. We reproduce the action of a single defect with discrete torsion $\mathcal{T}_{21}$, which corresponds to $M_{21} = M_2 M_1$.

When $\mathcal{T}_2 = -\mathcal{T}_1 \equiv \mathcal{T}$, then Φ_1 is a Lagrange multiplier imposing $\Phi = \big(\mathcal{T} + \frac{\epsilon}{2}\big)^{-1} d\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$ and the defect Lagrangian, up to total derivatives, simply becomes

$$S = \frac{N}{2\pi} \int_\Sigma \mathcal{B}^\intercal d\widehat{\Gamma} \,, \tag{4.1.71}$$

where $\widehat{\Gamma} = \Gamma + \big(\mathcal{T} + \frac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$. On closed manifolds, this reproduces the result $V[\mathcal{T}] \times V[-\mathcal{T}] = \mathbb{1}$. Indeed the action (4.1.71) simply imposes that the pullback of $\mathcal{B}$ be in $H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ without any discontinuity between the L and R regions.

The other cases can be dealt with in a similar way. When $\mathcal{T}_1 + \mathcal{T}_2$ has rank one, the component of Φ_1 living in the kernel of $\mathcal{T}_1 + \mathcal{T}_2$ acts as a Lagrange multiplier, setting to zero one component of Φ, while the component in the cokernel produces the torsion term for the remaining component of Φ. Fusions involving defects from the condensation of $\mathcal{A} \cong \mathbb{Z}_N$ can be studied similarly.

4.1.4 Twisted sectors and non-invertible defects

Whenever a theory has a discrete 0-form symmetry Γ, one can consider its twisted sectors. In particular, there exist codimension-2 operators that live at the boundary of the codimension-1 defect operators implementing Γ. We call them the codimension-2 operators in the twisted sector. Gauging a (non-anomalous) subgroup $G \subset \Gamma$, the corresponding defects become transparent and the codimension-2 operators at their boundary get promoted to genuine operators of the gauged theory.[18] For instance, in 2d CFTs the twisted sectors are described by local operators at the end of defect (or twist) lines, and their inclusion in the gauged theory is required by modular invariance. In 3d TQFT the twisted sectors are described by line operators at the end of defect surfaces, and the modular tensor category (MTC) of lines gets promoted to a G-crossed MTC [138], also in order to assure modularity.

The situation in higher dimensions is less well understood. In this section we study the twisted sectors of the 5d Chern-Simons theory, exploiting the Lagrangian description of codimension-1 symmetry defects that implement $SL(2, \mathbb{Z}_N)$. In particular, we describe the 3d twist defects $D[\mathcal{T}]$ and $D[\mathcal{A}, \xi]$ (or more compactly D_M) at the boundary of 4d symmetry defects $V[\mathcal{T}]$ and $V[\mathcal{A}, \xi]$ (or V_M), respectively.

[18] When G is Abelian, these are the codimension-2 operators charged under the $(d-2)$-form symmetry $\widehat{G}$ dual to G ($\widehat{G}$ is the Pontryagin dual) and implemented by the Wilson lines of G.

Lagrangian description of $D[\mathcal{T}]$

We can obtain a Lagrangian description of the 3d twisted-sector operators — that we dub $D[\mathcal{T}]$ — at the boundary of 4d $SL(2,\mathbb{Z}_N)$ symmetry defect operators $V[\mathcal{T}]$ from the Lagrangian description (4.1.55) of the latter.[19] As we will see in a moment, it is convenient to perform an integration by parts of the couplings $\mathcal{B}^\mathsf{T} d\Gamma$ and $\Phi^\mathsf{T} d\Psi$ and use the following equivalent Lagrangian for the 4d defect operators $V[\mathcal{T}]$:

$$S[\mathcal{T}] = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^\mathsf{T}\Phi + \Gamma^\mathsf{T} d\mathcal{B} + \Psi^\mathsf{T} d\Phi + \frac{1}{2}\,\Phi^\mathsf{T}\mathcal{T}\Phi\right] . \tag{4.1.72}$$

In the presence of a boundary $Y = \partial\Sigma$, this action is not invariant under the gauge transformations (4.1.57), rather, it shifts by a boundary term (up to integer multiples of 2π):

$$S \;\to\; S + \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}(\lambda - d\nu) + \Phi^\mathsf{T}(\alpha + \mathcal{T}\lambda - d\mu) + \alpha^\mathsf{T} d\lambda + \frac{1}{2}\,\lambda^\mathsf{T}\mathcal{T} d\lambda\right] . \tag{4.1.73}$$

This can by canceled by the following boundary action:

$$S_{\text{twist}}[\mathcal{T}] = \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}\Gamma + \Phi^\mathsf{T}\Psi + \Gamma^\mathsf{T} d\Psi - \frac{1}{2}\,\Gamma^\mathsf{T}\mathcal{T} d\Gamma\right] . \tag{4.1.74}$$

The reason why we wrote the 4d action as in (4.1.72) is that the 4d fields Γ and Ψ only appear as Lagrange multipliers with no derivatives, and thus their path-integrals at different spacetime points are independent. On the contrary, they appear dynamically (with derivatives) in the 3d action (4.1.74) and therefore their restrictions to Y can be treated as independent 3d fields, or edge modes. From the 3d point of view, the fields $\mathcal{B}$ and Φ appear as background fields (that can be integrated afterwards in 5 and 4 dimensions, respectively).[20] The coupled 4d-3d system is gauge invariant. We call the 3d defect defined by $S_{\text{twist}}[\mathcal{T}]$ a twist defect $D[\mathcal{T}]$ associated to the $SL(2,\mathbb{Z}_N)$ element $M(\mathcal{T})$ (4.1.35).

The actions (4.1.72) and (4.1.74) are invariant under all elements $M' \in SL(2,\mathbb{Z}_N)$ that commute with M, if we supplement the transformation $\mathcal{B} \to M'^\mathsf{T}\mathcal{B}$ with[21]

$$\Phi \;\to\; M'^{-1}\Phi\ , \qquad \Gamma \;\to\; M'^{-1}\Gamma\ , \qquad \Psi \;\to\; M'^\mathsf{T}\Psi\ . \tag{4.1.75}$$

[19] A similar discussion would apply to the defects $D[\mathcal{A},\xi]$ at the boundary of $V[\mathcal{A},\xi]$, derived from (4.1.56).

[20] Using the equivalent action (4.1.55) one obtains the boundary action $S'_{\text{twist}} = \frac{N}{2\pi}\int_Y \big[\Gamma^\mathsf{T} d\Psi - \frac{1}{2}\Gamma^\mathsf{T}\mathcal{T} d\Gamma\big]$ in which the couplings to $\mathcal{B}$ and Φ are not manifest.

[21] Invariance of the last term follows from the fact that M' commutes with M if and only if $M'^\mathsf{T}\mathcal{T}M' = \mathcal{T}$.

Such an invariance is expected since, in general, acting with a 0-form symmetry h on a twisted sector D_g gives an element of $D_{hgh^{-1}}$. This will be important when gauging a subgroup of $SL(2,\mathbb{Z}_N)$.

Let us analyze the content of the three-dimensional theory $D[\mathcal{T}]$. For simplicity, we only consider the cases in which $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, or $\mathcal{T}=0$. We start with the former. Setting $\mathcal{B}=\Phi=0$, (4.1.74) is the action of an Abelian Chern-Simons theory with four gauge fields, whose level matrix K and its inverse are

$$K = N\begin{pmatrix} -\mathcal{T} & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,, \qquad\qquad K^{-1} = N^{-1}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & \mathcal{T} \end{pmatrix}\,. \tag{4.1.76}$$

There are $|\det K| = N^4$ line operators, given by $e^{i\int(n^{\mathsf{T}}\Gamma+m^{\mathsf{T}}\Psi)}$ with $n,m\in\mathbb{Z}_N\times\mathbb{Z}_N$. Not all of them, however, are genuine 3d line operators in the coupled 4d-3d system (keeping the 5d bulk as a background), rather some of them live at the end of a bulk surface $e^{in^{\mathsf{T}}\int\Phi}$. This follows from the gauge transformations (4.1.57). A basis of genuine line operators is given by

$$W_n = \exp\left[in^{\mathsf{T}}\int\Big(\mathcal{T}\Gamma-\Psi\Big)\right]\,. \tag{4.1.77}$$

We have chosen the parametrization such that W_n has charge $n=(n_{\rm e},n_{\rm m})$ under the $\mathbb{Z}_N\times\mathbb{Z}_N$ 1-form symmetry that couples to $\mathcal{B}$.[22] These lines have spin

$$\theta[W_n] = \exp\left(-\frac{\pi i}{N}\,n^{\mathsf{T}}\mathcal{T}n\right)\,, \tag{4.1.78}$$

and give a $\mathbb{Z}_N\times\mathbb{Z}_N$ generalization of the $\mathcal{A}^{N,p}$ minimal TQFTs introduced in [67] (see Appendix F there and Appendix 4.3.2 here). Indeed, these lines have braiding $B_{ab}=\frac{\theta_{a+b}}{\theta_a\theta_b}=\exp\big[-\frac{2\pi i}{N}a^{\mathsf{T}}\mathcal{T}b\big]$ and, taken in isolation, give rise to a consistent MTC with unitary S-matrix $S_{ab}=\frac{1}{N}B_{ab}$. We will use the notation $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ to denote the theory of these lines:

$$\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})\ \subset\ D[\mathcal{T}]\,. \tag{4.1.79}$$

There is some redundancy in the nomenclature of the theories $\mathcal{A}^{N,-\mathcal{T}}$: for all matrices $\mathcal{Q}$ invertible in $\mathbb{Z}_N$, the theory $\mathcal{A}^{N,-\mathcal{Q}^{\mathsf{T}}\mathcal{T}\mathcal{Q}}$ (where the product of matrices is in $\mathbb{Z}_N$) is equivalent to $\mathcal{A}^{N,-\mathcal{T}}$ up to a relabelling of the lines $n\to\mathcal{Q}n$. They are distinguished, however, by how they couple to $\mathcal{B}$. We will refer to the theory (4.1.78) in which W_n has charge n as $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$. Notice that this theory is not coupled to the 4d field Φ.

[22] Indeed, under the transformation $\mathcal{B}\to\mathcal{B}+d\alpha$, $\Psi\to\Psi-\alpha$, the operator gets a phase $W_n\to e^{in^{\mathsf{T}}\int\alpha}W_n$.

The remaining lines are not genuine in the coupled 4d-3d system, and are generated by

$$L_m = \exp\left[-im^{\mathsf{T}}\left(\int_{\partial X}\Psi + \mathcal{T}\int_X \Phi\right)\right] \tag{4.1.80}$$

in addition to W_n, where X is a two-dimensional open surface ending on $D[\mathcal{T}]$. The twisted sector, as an isolated 3d theory, is formed by *both* genuine and non-genuine line operators. We chose the generators L_m such that in 3d (*i.e.*, switching the background Φ off) they have trivial braiding with W_n. Indeed, the twisted sector can be decomposed as

$$D[\mathcal{T}] = \mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \times \mathcal{A}^{N,\mathcal{T}}(\mathcal{B}+\mathcal{T}\Phi)\ , \tag{4.1.81}$$

where the two factors are the MTCs of W_n and L_m, respectively.[23] However, as we will see in Section 4.1.5, once a subgroup of the $SL(2,\mathbb{Z}_N)$ 0-form symmetry is gauged in the bulk, some of the 4d operators become transparent and only the subcategory $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ of genuine operators survives.

The $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry of $\mathcal{A}^{N,-\mathcal{T}}$ is anomalous, since the lines W_n that generate it have non-trivial braiding. Turning on the background field $\mathcal{B}$ coupled to the 1-form symmetry, the anomaly is canceled [67] by the following four-dimensional inflow action:[24]

$$I_{\mathcal{T}}(\mathcal{B}) = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^{\mathsf{T}} d\widetilde{\Gamma} - \frac{1}{2}\mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B}\right]\ , \tag{4.1.82}$$

where the dynamical field $\widetilde{\Gamma}$ imposes $\mathcal{B} \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ on shell, and $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$ in $\mathbb{Z}_N$. This implies that $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ is not invariant under the gauge transformation $\mathcal{B} \to \mathcal{B} + d\alpha$, $\widetilde{\Gamma} \to \widetilde{\Gamma} + \mathcal{T}^{-1}\alpha$ but rather its path integral picks up a phase:

$$\exp\left[-\frac{iN}{2\pi}\int_Y \left(\alpha^{\mathsf{T}} d\widetilde{\Gamma} + \frac{1}{2}\alpha^{\mathsf{T}}\mathcal{T}^{-1} d\alpha\right)\right]\ . \tag{4.1.83}$$

Indeed one can check that the anomaly inflow action (4.1.82) for $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \times \mathcal{A}^{N,\mathcal{T}}(\mathcal{B}+\mathcal{T}\Phi)$, if supplemented by the condition that $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$, coincides with the 4d action (4.1.55) for the defect $V[\mathcal{T}]$. Alternatively, one can start with the action (4.1.72) for $V[\mathcal{T}]$ and integrate out Φ. This is possible because, as stressed

[23] One could also consider the non-genuine operators $\ell_m = \exp\left[im^{\mathsf{T}}\left(\int_{\partial X}\Gamma + \int_X \Phi\right)\right]$ which do not couple to $\mathcal{B}$, however they have vanishing spin and do not form a MTC by themselves.

[24] In the conventions of [67], the 1-form symmetry is generated by the lines $\widetilde{W}_n \equiv W_{-\mathcal{T}^{-1}n}$ which have charge $-\mathcal{T}^{-1}n$ and spin $\exp\left(-\frac{\pi i}{N}\, n^{\mathsf{T}}\mathcal{T}^{-1}n\right)$. This theory, that [67] would call $\mathcal{A}^{N,-\mathcal{T}^{-1}}$, has an anomaly that is canceled by (4.1.82).

after (4.1.57), the theory is trivial as long as $\mathcal{T}$ is invertible in $\mathbb{Z}_N$. We already did this computation in (4.1.58): one is left with the invertible TQFT (4.1.82) in the 4d bulk and $\mathcal{A}^{N,-\mathcal{T}}$ on the 3d boundary. Either way, the coupled 4d-3d system is anomaly free.

The case of $\mathcal{T} = 0$, which describes the charge conjugation operator V_C, needs a separate discussion. Contrary to the previous case, there is no consistent MTC that describes the lines W_n decoupled from Φ. Those lines have trivial spin and braiding among themselves. This phenomenon was already observed in [67] and is a consequence of the non-invertibility of the 4d 2-form gauge theory for Φ. The action for the twisted sector $D[\mathcal{T} = 0] \equiv D_C$ is

$$S_{\text{twist}}[\mathcal{T} = 0] = \frac{N}{2\pi} \int_Y \Big[\mathcal{B}^{\mathsf{T}} \Gamma + \Phi^{\mathsf{T}} \Psi + \Gamma^{\mathsf{T}} d\Psi \Big] \,. \tag{4.1.84}$$

This is a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge theory (described by the 3d fields Γ, Ψ) coupled to the backgrounds fields $\mathcal{B}$ and Φ for the two copies of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry, and we denote it by $(\mathbb{Z}_N \times \mathbb{Z}_N)_0(\mathcal{B}, \Phi)$.

Degeneracies. We ask what is the degeneracy of the twisted sectors, *i.e.*, how many boundaries an $SL(2, \mathbb{Z}_N)$ symmetry defect V can have. In three-dimensional TQFTs with a 0-form symmetry Γ, the number of simple lines in a twisted sector labeled by $g \in \Gamma$ is equal to the number of g-invariant simple lines in the untwisted sector [138]. In our case, the 5d CS theory has no genuine codimension-2 operators (besides the trivial one), therefore we expect every twisted sector to be unique. One could argue that we should also consider the operators obtained by fusing $D[\mathcal{T}]$ with codimension-2 condensation defects obtained from the bulk 2-form symmetry.

We can show that for defects $V[\mathcal{T}]$ obtained by condensing the full $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry, the boundary $D[\mathcal{T}]$ is left invariant by every such fusion, up to stacking with a decoupled TQFT. Indeed, fusing $D[\mathcal{T}]$ with a 2d symmetry defect $U(\gamma)$ with $\gamma \in H_2(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ is equivalent to adding the following coupling to the action (4.1.74) of $D[\mathcal{T}]$:

$$\delta S_{\text{twist}}[\mathcal{T}] = \int_Y \mathcal{B}^{\mathsf{T}} \Gamma_\gamma \,, \qquad \Gamma_\gamma = \text{PD}(\gamma) \tag{4.1.85}$$

where $\text{PD}(\gamma) \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ is the Poincaré dual to γ on Y. Given a continuum description of the class Γ_γ, for instance through a delta 1-form, the extra coupling can be reabsorbed by the field redefinition $\Gamma \to \Gamma - \frac{2\pi}{N}\Gamma_\gamma$, $\Phi \to \Phi + \frac{2\pi}{N} d\Gamma_\gamma$, $\mathcal{B} \to$

$\mathcal{B} - \frac{2\pi}{N}\mathcal{T} d\Gamma_\gamma$, which however produces a phase

$$\exp\left(-\frac{2\pi i}{N}\int \frac{1}{2}\,\Gamma_\gamma^{\mathsf{T}}\mathcal{T} d\Gamma_\gamma\right) = \exp\left(-\frac{2\pi i}{N}\int \frac{1}{2}\,\Gamma_\gamma^{\mathsf{T}}\, N\mathcal{T}\,\beta(\Gamma_\gamma)\right) \equiv Q_{N\mathcal{T}}(\Gamma_\gamma)\ . \quad (4.1.86)$$

Notice that, in the continuum description on the left-hand-side, $d\Gamma_\gamma$ is a class with values in N times $\mathbb{Z}\times\mathbb{Z}$ rather than identically zero. On the right-hand-side we wrote the phase in a more precise way in terms of $\Gamma_\gamma \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ and the Bockstein map associated to the short exact sequence $0 \to \mathbb{Z}_N \xrightarrow{N} \mathbb{Z}_{N^2} \xrightarrow{\text{mod } N} \mathbb{Z}_N \to 0$ so that $\beta(\Gamma_\gamma) \in H^2(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$. The integrals in (4.1.86) are well defined on generic manifolds if $N\mathcal{T}$ is an even matrix, and on spin manifolds if $N\mathcal{T}$ is a more general integer matrix. Hence

$$U(\gamma) \times D[\mathcal{T}] = e^{iQ_{N\mathcal{T}}(\Gamma_\gamma)}\, D[\mathcal{T}]\ . \quad (4.1.87)$$

A similar effect has already been appreciated in dealing with N-ality defects in [25, 119].

Now, a 3d condensation defect for the $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry can be thought of as a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ Dijkgraaf-Witten (DW) theory, possibly with torsion $\mathcal{P}$, coupled to the dynamical field $\mathcal{B}$. The coupling is precisely (4.1.85) with Γ_γ substituted by the dynamical gauge field of the DW theory. We denote the 3d condensation defect as $\mathcal{C}_{\mathcal{P}}^{\mathbb{Z}_N\times\mathbb{Z}_N}$, and omit the subscript when there is no torsion. Stacking the condensation defect on $D[\mathcal{T}]$ replaces the coupling to $\mathcal{B}$ with the torsion term $Q_{N\mathcal{T}}(\Gamma_\gamma)$: this produces a shift $\delta\mathcal{P} = -N\mathcal{T}$ of the torsion of the DW theory. It turns out (see below) that if N is odd and the theory is spin, then shifts of the torsion components by multiples of N give equivalent theories, and so in our case the shift is immaterial. We conclude that

$$\mathcal{C}_{\mathcal{P}}^{\mathbb{Z}_N\times\mathbb{Z}_N} \times D[\mathcal{T}] = (\mathbb{Z}_N \times \mathbb{Z}_N)_{\mathcal{P}}\, D[\mathcal{T}]\ . \quad (4.1.88)$$

The factor on the right-hand-side is a decoupled Dijkgraaf-Witten TQFT. A similar argument applies to any other 3d condensate in which only a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ is condensed (possibly with torsion): they can all be absorbed by $D[\mathcal{T}]$. We conclude that there is no degeneracy in these twisted sectors.

When, on the other hand, the defect $V[\mathcal{A}, \xi]$ is obtained by condensing a subgroup $\mathcal{A}$ of $\mathbb{Z}_N \times \mathbb{Z}_N$, then only condensates of surfaces in $\mathcal{A}$ can similarly be absorbed by D, while more general surfaces in $\mathbb{Z}_N \times \mathbb{Z}_N$ cannot and give rise to a genuine degeneracy of the twisted sector. Since surfaces in $\mathcal{A}$ are absorbed, the degeneracy is given by all condensates (with torsion) of the quotient group $(\mathbb{Z}_N \times \mathbb{Z}_N)/\mathcal{A}$ (or its subgroups).

The last case is the 4d indentity interface $V_{\mathbb{1}}$, on which we do not gauge any symmetry. Its sector, which is the untwisted sector, consists of all possible 3d condensates in $\mathbb{Z}_N \times \mathbb{Z}_N$.

Dijkgraaf-Witten theories The 3d $\mathbb{Z}_N^k$ Dijkgraaf-Witten theories can be described by the following Abelian Chern-Simons action:

$$S_{\mathrm{DW}}[\mathcal{T}] = \int_Y \left[\frac{N}{2\pi}\, x^{\mathsf{T}} dy + \frac{1}{4\pi}\, y^{\mathsf{T}} \mathcal{P} dy \right] \tag{4.1.89}$$

where x, y are k-dimensional vectors of Abelian vector fields and $\mathcal{P}$ is a $k \times k$ symmetric integer matrix. The theory is bosonic if $\mathcal{P}$ is even (*i.e.*, if its diagonal entries are even), otherwise it is spin. The level matrix is $K = \left(\begin{smallmatrix} 0 & N\mathbb{1} \\ N\mathbb{1} & \mathcal{P} \end{smallmatrix}\right)$. The theory has N^{2k} lines labelled by $n \in \mathbb{Z}_N^{2k}$ with spin

$$\theta[n] = \exp\!\big(\pi i\, n^{\mathsf{T}} K^{-1} n\big) \qquad \text{where} \qquad K^{-1} = \frac{1}{N^2} \begin{pmatrix} -\mathcal{P} & N\,\mathbb{1} \\ N\,\mathbb{1} & 0 \end{pmatrix} . \tag{4.1.90}$$

In all cases, a shift of $\mathcal{P}$ by N times an even integer matrix gives an equivalent theory, *i.e.*, the diagonal entries of $\mathcal{P}$ are defined modulo $2N$ while the off-diagonal entries modulo N. This follows from the field redefinition $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} \mathbb{1} & \mathcal{Q} \\ 0 & \mathbb{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ where $\mathcal{Q}$ is an integer matrix, or equivalently, from the relabelling $n \to \left(\begin{smallmatrix} \mathbb{1} & 0 \\ \mathcal{Q} & \mathbb{1} \end{smallmatrix}\right) n$ of the lines. If N is odd, in addition, theories in which the entries of $\mathcal{P}$ differ by multiples of N are equivalent as spin theories.[25] This follows from the fact that the relabelling $n \to \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 2^{-1}\mathcal{Q} & \mathbb{1} \end{smallmatrix}\right) n$ (where 2^{-1} is the inverse in $\mathbb{Z}_N$) preserves the spin modulo a sign, which can be cancelled by fusing with the transparent fermion.

The coupling of the electric 1-form symmetry to a $\mathbb{Z}_N^k$ background field $\mathcal{B}$ is described by

$$S_{\mathrm{DW}}[\mathcal{T}](\mathcal{B}) = \int_Y \left[\frac{N}{2\pi} \left(\mathcal{B}^{\mathsf{T}} y + x^{\mathsf{T}} dy \right) + \frac{1}{4\pi}\, y^{\mathsf{T}} \mathcal{P} dy \right] , \tag{4.1.91}$$

invariant under $\mathcal{B} \to \mathcal{B} + d\alpha$, $x \to x - \alpha$. The lines labelled by $n = (n_{\mathrm{e}}, n_{\mathrm{m}})$ have charge $-n_{\mathrm{e}}$ under the electric 1-form symmetry. This statement persists under shifts of the components of $\mathcal{P}$ by multiples of N.

[25] This is not true, in general, if N is even. A counterexample for $k = 1$ is the family of four $\mathbb{Z}_2$ theories.

Fusion of twist defects

We now study the fusion of two twist defects $D[\mathcal{T}_1]$ and $D[\mathcal{T}_2]$. As expected, the fusion is compatible with the group product rule $M_{21} = M_2 M_1$ of 4d $SL(2,\mathbb{Z}_N)$ defect operators $V[\mathcal{T}]$, *i.e.*, of twisted sectors, however we would like to understand which condensates and decoupled TQFTs can be generated.

As already discussed in Section 4.1.3 for the fusion of 4d defects, the 5d bulk provides a crucial contribution to the fusion of 3d twist defects as well. The bulk contribution was computed in (4.1.68), thus the total action for the system of two 4d defects with boundary located on the same 3d (spin) manifold Y is

$$S = S[\mathcal{T}_1] + S[\mathcal{T}_2] - \frac{N}{4\pi}\int (\Phi_1 + d\Gamma_1)^{\mathsf{T}} \epsilon\, (\Phi_2 + d\Gamma_2) + S_{\text{twist}}[\mathcal{T}_1] + S_{\text{twist}}[\mathcal{T}_2]\ , \quad (4.1.92)$$

where, this times, we use the 4d action (4.1.72) for the symmetry defects.

The computation in the 4d bulk is similar to the one we did in Section 4.1.3. One introduces $\Phi = \Phi_1 + \Phi_2$, $\Gamma = \Gamma_1 + \Gamma_2$, $\widetilde{\Psi} = \Psi_1 - \Psi_2$, and eliminates Φ_2, Γ_2, Ψ_1. If $(\mathcal{T}_1 + \mathcal{T}_2)$ is an invertible matrix in $\mathbb{Z}_N$, the field Φ_1 can be integrated out leaving the bulk theory

$$S_{\text{bulk}} = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^{\mathsf{T}}\Phi + \Gamma^{\mathsf{T}} d\mathcal{B} + \Psi^{\mathsf{T}} d\Phi + \frac{1}{2}\Phi^{\mathsf{T}}\mathcal{T}_{21}\Phi\right]\ , \quad (4.1.93)$$

where $\mathcal{T}_{21}$ is given in (4.1.45) and $\Psi = \Psi_2 + \frac{\epsilon}{2}\Gamma_1 + \big(\mathcal{T}_2 - \frac{\epsilon}{2}\big)(\mathcal{T}_1 + \mathcal{T}_2)^{-1}\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$. This is the theory $S[\mathcal{T}_{21}]$. There are leftover boundary terms, that together with $S_{\text{twist}}[\mathcal{T}_1] + S_{\text{twist}}[\mathcal{T}_2]$ give

$$\begin{aligned} S_{\text{boundary}} = \frac{N}{2\pi}\int_Y \Big[&\mathcal{B}^{\mathsf{T}}\Gamma + \Phi^{\mathsf{T}}\Psi + \Gamma^{\mathsf{T}} d\Psi_2 - \frac{1}{2}\Gamma^{\mathsf{T}}\mathcal{T}_2 d\Gamma + \Gamma_1^{\mathsf{T}} d\Big(\widetilde{\Psi} + \big(\mathcal{T}_2 - \tfrac{\epsilon}{2}\big)\Gamma\Big) \\ &- \frac{1}{2}\Gamma_1^{\mathsf{T}}(\mathcal{T}_1 + \mathcal{T}_2) d\Gamma_1 - \frac{1}{2}\big(\widetilde{\Psi} - \tfrac{\epsilon}{2}\Gamma\big)^{\mathsf{T}}(\mathcal{T}_1 + \mathcal{T}_2)^{-1} d\big(\widetilde{\Psi} - \tfrac{\epsilon}{2}\Gamma\big)\Big]\ . \end{aligned} \quad (4.1.94)$$

The gauge transformations of the new fields are

$$\mathcal{B} \to \mathcal{B} + d\alpha\ , \quad \Phi \to \Phi + d\lambda\ , \quad \Psi \to \Psi - \mathcal{T}_{21}\lambda - \alpha + d\mu\ , \quad \Gamma \to \Gamma - \lambda + d\nu \quad (4.1.95)$$

where $\lambda = \lambda_1 + \lambda_2$. The theory (4.1.94) is not trivial and we cannot integrate other fields out. We perform a more rigorous analysis of it below, but for now, in order to understand the physics, let us perform an approximate computation. We introduce a new 1-form field

$$H = \Psi_1 - \mathcal{T}_1\Gamma_1 - \Psi_2 + \mathcal{T}_2\Gamma_2 = \widetilde{\Psi} + \mathcal{T}_2\Gamma - (\mathcal{T}_1 + \mathcal{T}_2)\,\Gamma_1\ . \quad (4.1.96)$$

This combination is special because it is invariant under the gauge transformations (4.1.57) parametrized by λ_1, λ_2, α. We eliminate Γ_1 in favor of H: this is not a legit operation since $(\mathcal{T}_1+\mathcal{T}_2)$ is not a unimodular integer matrix, but let us proceed anyway and treat $(\mathcal{T}_1+\mathcal{T}_2)^{-1}$ as the inverse in $\mathbb{Q}$. Up to total derivatives, we obtain

$$S_{\text{boundary}} \sim \frac{N}{2\pi}\int_Y \left[\mathcal{B}^{\mathsf{T}}\Gamma + \Phi^{\mathsf{T}}\Psi + \Gamma^{\mathsf{T}} d\Psi - \frac{1}{2}\Gamma^{\mathsf{T}}\mathcal{T}_{21} d\Gamma\right] - \frac{N}{4\pi}\int_Y H^{\mathsf{T}}(\mathcal{T}_1+\mathcal{T}_2)^{-1} dH \ . \tag{4.1.97}$$

The first term is the expected action $S_{\text{twist}}[\mathcal{T}_{21}]$ of the twisted sector $D[\mathcal{T}_{21}]$. The second term is a decoupled TQFT, described by a Chern-Simons action with fractional level-matrix. Perturbatively, it behaves as the theory $\mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}$ (while it is not well defined at the non-perturbative level).

If $(\mathcal{T}_1+\mathcal{T}_2)$ is not invertible in $\mathbb{Z}_N$ then the procedure has to be slightly changed. Let us discuss the case $\mathcal{T}_2 = -\mathcal{T}_1 \equiv \mathcal{T}$, corresponding to the fusion of a defect with its "inverse". This case is interesting because the fusion of two defects in inverse twisted sectors must produce an operator in the untwisted sector, which however contains all three-dimensional condensation defects. Starting with (4.1.92) and performing the field redefinitions to Φ, Γ, $\widetilde{\Psi}$, in the 4d bulk one finds Φ_1 to be a Lagrange multiplier imposing $\Phi = d\big(\mathcal{T}+\frac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\frac{\epsilon}{2}\Gamma\big)$. It is convenient to define

$$\begin{aligned}\widehat{\Gamma} &= \Gamma + \big(\mathcal{T}+\tfrac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\tfrac{\epsilon}{2}\Gamma\big)\\ \widehat{\Psi} &= \Psi_2 + \mathcal{T}\Gamma_1 + \mathcal{T}\big(\mathcal{T}+\tfrac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\tfrac{\epsilon}{2}\Gamma\big)\ .\end{aligned} \tag{4.1.98}$$

Then the bulk action simply reduces to the completely trivial theory

$$S_{\text{bulk}} = \frac{N}{2\pi}\int_\Sigma \widehat{\Gamma}^{\mathsf{T}} d\mathcal{B} \tag{4.1.99}$$

that describes the identity operator $V_{\mathbb{1}}$. The boundary terms instead give

$$S_{\text{boundary}} = \frac{N}{2\pi}\int_Y \left[\mathcal{B}^{\mathsf{T}}\widehat{\Gamma} + \widehat{\Gamma}^{\mathsf{T}} d\widehat{\Psi} - \frac{1}{2}\widehat{\Gamma}^{\mathsf{T}}\mathcal{T} d\widehat{\Gamma}\right] . \tag{4.1.100}$$

The fields $\widehat{\Gamma}$, $\widehat{\Psi}$ are invariant under the gauge transformations λ_1, λ_2, indeed this 3d theory does not need to be attached to any 4d theory. On the other hand, $\widehat{\Psi} \to \widehat{\Psi} - \alpha$ under gauge transformations of $\mathcal{B}$ (while $\widehat{\Gamma}$ is invariant). The action (4.1.100) describes a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ Dijkgraaf-Witten theory with torsion equal to $-N\mathcal{T}$, in which a $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry is coupled to $\mathcal{B}$ — as in (4.1.91). Alternatively, this can be though of as a 3d condensation defect for the $\mathbb{Z}_N \times \mathbb{Z}_N$

global 2-form symmetry of the 5d bulk theory: $\widehat{\Psi}$ forces $\widehat{\Gamma} \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$, then $e^{i\frac{N}{2\pi}\int \mathcal{B}^{\mathsf{T}}\widehat{\Gamma}}$ is a two-dimensional operator of the 5d theory placed on the Poincaré dual to $\widehat{\Gamma}$ within Y, and the last term in (4.1.100) produces a phase weighing the sum over surfaces. We dubbed such a 3d condensation defect $\mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N}_{-N\mathcal{T}} \equiv \mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N}$, since we are considering N odd. Therefore, the fusion of a twist defect with its "inverse" is given by

$$D[\mathcal{T}] \times D[-\mathcal{T}] = D[\mathcal{T}] \times \overline{D}[\mathcal{T}] = \mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N} \ . \tag{4.1.101}$$

A more rigorous analysis of twisted sectors. The analysis of the fusion of twist defects we performed in (4.1.97) using the Lagrangian formulation, while suggesting the correct result, was imprecise. We can obtain a more rigorous and precise derivation by studying the algebra of topological operators.

As discussed in Section 4.1.4, if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then the twist operator $D[\mathcal{T}]$ hosts a MTC of local line operators (which are not coupled to the 4d defect) forming the minimal TQFT $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$. When we fuse two twist operators $D[\mathcal{T}_1]$ and $D[\mathcal{T}_2]$, the set of local line operators is not simply the stacking of the two TQFTs because of the bulk contribution. Taken separately, the two minimal TQFTs have lines $W^{(1)}_{n_1}$ and $W^{(2)}_{n_2}$, respectively. The 5d dynamical bulk field $\mathcal{B}$, however, generates a non-trivial braiding between the two sets of lines:

$$B_{W^{(1)},W^{(2)}} = \exp\left(\frac{2\pi i}{N}\, n_1^{\mathsf{T}} \frac{\epsilon}{2} n_2\right) \tag{4.1.102}$$

where we are taking N odd. This follows from the boundary term $-\frac{N}{2\pi}\int_Y d\Gamma_1^{\mathsf{T}} \frac{\epsilon}{2} d\Gamma_2$ in (4.1.92) and the expression (4.1.77) for the local lines. It can also be understood as follows. In canonical quantization, the braiding matrix appears as a non-trivial commutator

$$W^{(1)}\, W^{(2)} = B_{W^{(1)},W^{(2)}}\, W^{(2)}\, W^{(1)} \ , \tag{4.1.103}$$

where the operators are time ordered. If $W^{(i)}$ were local lines in the full theory, this would be trivial because the lines would live on separate defects. However, in the full theory $\mathcal{B}$ is dynamical and thus both $W^{(1)}$ and $W^{(2)}$, which are coupled to $\mathcal{B}$, must be the end-lines of suitable bulk surfaces $U(\gamma) = e^{i\int_\gamma \mathcal{B}}$. Likewise, also the product $W^{(1)}W^{(2)}$ must be attached to a bulk surface with the correct charge (see Figure 4.5). Commuting the order in which the end-lines are fused has the effect of half-braiding the attached bulk surfaces, which is captured by the normal ordering phase $\exp\big(\frac{2\pi i}{N}\, 2^{-1}\langle n_1, n_2\rangle\big)$ we already introduced in (4.1.29). This is precisely the braiding (4.1.102).

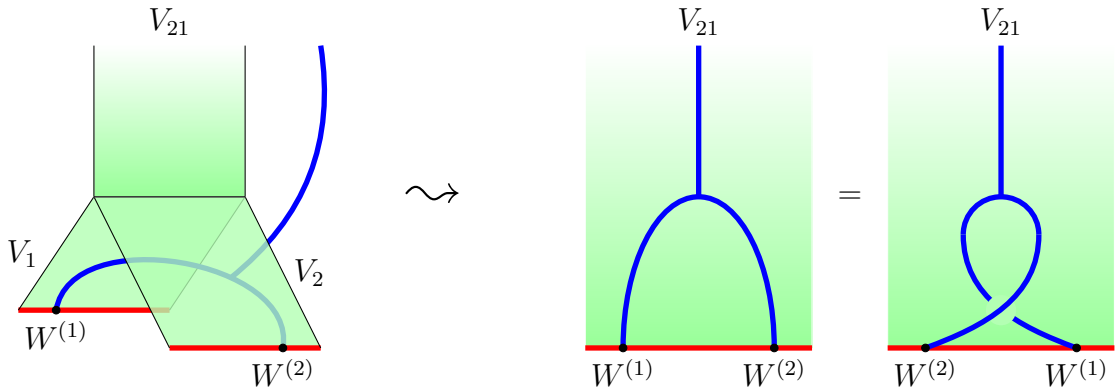


Figure 4.5: Braiding between lines $W^{(1)}$ and $W^{(2)}$ from bulk ordering. We represented the lines $W^{(i)}$ by black points, the 3d twist sectors $D[\mathcal{T}_i]$ by red lines, the surfaces $U(\gamma)$ by blue lines, and the 4d condensation defects V_i by green surfaces. Left: bulk definition of fusion. Right: two different ordering procedures, related by the half-braiding phase of the bulk 5d theory. In canonical quantization, time runs horizontally.

We indicate the product of the two sectors $\mathcal{A}^{N,-\mathcal{T}_i}$ deformed by the extra braiding (4.1.102) as $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$, in order to distinguish it from the standard decoupled tensor product. We label the lines of this theory by $\mathcal{N} = (n_1, n_2)$. The spin of the lines of $W^{(1)}$ and $W^{(2)}$ is undeformed, while the spin of product of lines can be computed using $\theta_{a+b} = \theta_a \theta_b B_{ab}$. We obtain

$$\theta[W_{\mathcal{N}}] = \exp\Big(\pi i \, \mathcal{N}^{\mathsf{T}} \mathcal{K}_{21} \mathcal{N}\Big) \qquad\qquad \mathcal{K}_{21} = \frac{1}{N} \begin{pmatrix} -\mathcal{T}_1 & \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & -\mathcal{T}_2 \end{pmatrix} . \tag{4.1.104}$$

The line $W_{\mathcal{N}}$ has charge $n_1 + n_2$ under the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry coupled to $\mathcal{B}$. We can identify a subset of lines that are decoupled from $\mathcal{B}$ and, under certain conditions, form a consistent, independent, and local 3d MTC. These are the lines with $\mathcal{N} = (l, -l)$: they exist without an attached bulk surface, and can be thought of as sitting at opposite ends of a $\mathcal{B}$ surface before fusion, see Figure 4.6. The spin of these lines is $\exp\big(-\frac{\pi i}{N} \, l^{\mathsf{T}} (\mathcal{T}_1 + \mathcal{T}_2) l\big)$ and thus, as long as $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible in $\mathbb{Z}_N$, they form the consistent MTC $\mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}$. The remaining lines are coupled to $\mathcal{B}$. We can identify a subset that has trivial braiding with the lines of $\mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}$. They are given by $\mathcal{N}_\eta = (\xi, -\xi + \eta)$ with $\xi = (\mathcal{T}_1 + \mathcal{T}_2)^{-1}\big(\mathcal{T}_2 + \frac{\epsilon}{2}\big)\eta$ and their spin is $\exp\big(-\frac{\pi i}{N} \, \eta^{\mathsf{T}} \mathcal{T}_{21} \eta\big)$ where the matrix $\mathcal{T}_{21}$ is the one in (4.1.45). Since the line $\mathcal{N}_\eta$ has charge η under the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry, they form the MTC $\mathcal{A}^{N,-\mathcal{T}_{21}}(\mathcal{B})$.

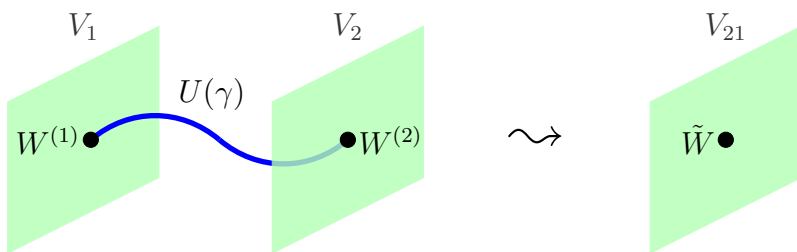


Figure 4.6: The lines $\tilde{W}$ of V_{21} that are decoupled from $\mathcal{B}$ can be seen as products of end-lines that are attached to a surface $U(\gamma)$ stretched between the two defects V_1 and V_2. Once the defects are fused, the lines $\tilde{W}$ become local in V_{21}.

Hence we arrive to the result

$$\mathcal{A}^{N,-\mathcal{T}_2}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}(\mathcal{B}) = \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} \times \mathcal{A}^{N,-\mathcal{T}_{21}}(\mathcal{B}) \ . \tag{4.1.105}$$

The product on the right-hand-side is the standard tensor product. The result is in accord with the factorization theorem of [67]. We have thus shown that:

$$D[\mathcal{T}_2] \times D[\mathcal{T}_1] = \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} \ D[\mathcal{T}_{21}] \ , \tag{4.1.106}$$

as long as as both $(\mathcal{T}_1 + \mathcal{T}_2)$ and $\mathcal{T}_{21}$ are invertible in $\mathbb{Z}_N$, as suggested by (4.1.97).

The result could be confronted with the known composition of minimal TQFTs $\mathcal{A}^{N,p}$ [25], namely $\mathcal{A}^{N,p} \times \mathcal{A}^{N,q} = \mathcal{A}^{N,\,p+q} \times \mathcal{A}^{N,\,(p^{-1}+q^{-1})^{-1}}$ valid when $\gcd(p+q, N) = 1$. While we found an equivalent expression for the decoupled lines on the right-hand-side, the lines coupled to $\mathcal{B}$ fuse differently because of the bulk dynamics.

Let us mention two cases in which the decomposition (4.1.105) fails. One case is when $\mathcal{T}_1 + \mathcal{T}_2 = 0$, namely when we consider the fusion $D[\mathcal{T}] \times D[-\mathcal{T}]$ in the untwisted sector. Set $\mathcal{T}_2 = -\mathcal{T}_1 = \mathcal{T}$. The lines decoupled from $\mathcal{B}$ have vanishing spin and form a Lagrangian subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, signaling that $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$ must be a Dijkgraaf-Witten theory. Indeed, exploiting (4.1.104), we can exhibit the set of lines $E_n = \left(\begin{smallmatrix} \mathbb{1} \\ -\mathbb{1} \end{smallmatrix}\right)\left(\mathcal{T} - \frac{\epsilon}{2}\right)^{-1} n$ decoupled from $\mathcal{B}$ and with vanishing spin, a set of lines $M_m = (m, 0)$ with charge m under $\mathcal{B}$ and with spin $\exp\left(\frac{\pi i}{N} m^{\mathsf{T}} \mathcal{T} m\right)$, and show that the two sets have canonical braiding $\exp\left(\frac{2\pi i}{N} n^{\mathsf{T}} m\right)$. This is precisely the content of the theory (4.1.100). We thus reproduce the result (4.1.101).

Another special case is when $\mathcal{T}_{21} = 0$, namely when we consider two defects $V[\mathcal{T}_2]$, $V[\mathcal{T}_1]$ that fuse into the charge-conjugation defect $V_C \equiv V[\mathcal{T} = 0]$. The two torsion matrices must be related by $\mathcal{T}_1 = -\frac{\epsilon}{2}\mathcal{T}_2^{-1}\frac{\epsilon}{2}$. When this happens, the product $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$ is not a MTC because it contains a subcategory of transparent

lines — the lines in the C twisted sector which couple only to $\mathcal{B}$ and not to Φ. In this case, we can study the fusion of the full twisted sectors, including the lines coupled to Φ (see Appendix 4.3.3). The final result is:

$$D[\mathcal{T}_1] \times D[\mathcal{T}_2] = \big(\mathbb{Z}_N \times \mathbb{Z}_N\big)_0(\Phi, \Phi_1)\; D[0] \; . \tag{4.1.107}$$

We can compare this result with our standard computation by using the factorization (also proven in the same appendix):[26]

$$\big(\mathbb{Z}_N \times \mathbb{Z}_N\big)_0(\Phi, \Phi_1) = \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} \times \mathcal{A}^{N,\mathcal{T}_1+\mathcal{T}_2}(\Phi, \Phi_1) \; . \tag{4.1.108}$$

Thus the result coincides with the remaining cases if we discard the term coupling to Φ_1 (which is integrated out in the 4d bulk computation).

Summarizing, we have obtained the following bulk fusion rules:

$$\begin{aligned} U(\gamma) \times D[\mathcal{T}] &= e^{iQ_N \mathcal{T}(\Gamma_\gamma)}\; D[\mathcal{T}] \; , \\ D[\mathcal{T}_2] \times D[\mathcal{T}_1] &= \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}\; D[\mathcal{T}_{21}] \; , \\ D[\mathcal{T}] \times \overline{D}[\mathcal{T}] &= \mathcal{C}^{\mathbb{Z}_N \times \mathbb{Z}_N} \; . \end{aligned} \tag{4.1.109}$$

We now extend our analysis to the physically relevant case of fusion on a gapped boundary.

Fusion on gapped boundaries

In Section 4.1.3 we discussed gapped boundaries $\rho(\mathcal{L})$ of the bulk 5d theory. These are defined by choosing a Lagrangian subgroup $\mathcal{L} \subset \mathcal{D}_Q \equiv \mathbb{Z}_N \times \mathbb{Z}_N$. The boundary condition sets to 1 the surface operators U_n with $n \in \mathcal{L}$, which are then screened on the boundary:

$$U_n\big|_X = 1 \qquad \text{if } n \in \mathcal{L} \; . \tag{4.1.110}$$

In terms of fields, one imposes Dirichelet boundary conditions $l^{\mathsf{T}}\mathcal{B}\big|_X = 0$ (up to gauge transformations) for all $l \in \mathcal{L}$. For N odd prime, the $N+1$ Lagrangian subgroups of $\mathcal{D}_Q$ are all isomorphic to $\mathbb{Z}_N$ and are generated by a single vector l. Thus the gapped boundaries are implemented by

$$l^{\mathsf{T}}\mathcal{B}\big|_X \equiv b_l = 0 \qquad \text{(up to gauge transformations)} \; . \tag{4.1.111}$$

[26]Note that $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 - \frac{\epsilon}{2}\mathcal{T}_1^{-1}\frac{\epsilon}{2} = (\mathcal{T}_1 + \frac{\epsilon}{2})\mathcal{T}_1^{-1}(\mathcal{T}_1 - \frac{\epsilon}{2})$ which is invertible under our assumptions.

In Section 4.1.3 we introduced the 1-form symmetry group $\mathcal{S} = \mathcal{D}_Q/\mathcal{L}$ of the gapped boundary. Here we also introduce the lattice $\mathcal{L}_\perp$ dual to $\mathcal{L}$ with respect to the Dirac pairing, and the vector $l_\perp = \epsilon\, l$ that generates $\mathcal{L}_\perp$. It satisfies $l_\perp^\mathsf{T} l = 0 \bmod N$.[27] Using this vector we can solve the boundary conditions by setting:

$$\mathcal{B}\big|_X = \tilde{b}_\perp\, l_\perp \ . \tag{4.1.112}$$

Notice that this is a condition on the field and not on the charges.

In this section we want to understand the fate of various types of defects once they are placed on the gapped boundary, or when they terminate on it. We already discussed the case of the 2d surfaces U_p with $p \in \mathcal{L}$: they can terminate on the gapped boundary, and become trivial if they are placed on top of it. On the other hand, if we fuse a 4d defect V_M (implementing the action of $M \in SL(2,\mathbb{Z}_N)$ on the gauge field $\mathcal{B}$) with a gapped boundary $\rho(\mathcal{L})$ we obtain a new gapped boundary $\rho(M\mathcal{L})$.

Let us now discuss the properties of the twist defects $D[\mathcal{T}]$ on a gapped boundary. Focusing on the case that $V[\mathcal{T}]$ comes from the condensation of $\mathbb{Z}_N \times \mathbb{Z}_N$ and that $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, in Section 4.1.4 we discussed the 3d sector $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ of lines W_n on $D[\mathcal{T}]$ that are decoupled from $V[\mathcal{T}]$ but that cancel its anomaly (4.1.82). Those lines are charged under $\mathcal{B}$, and thus are the end-lines of surfaces U_n in the fully dynamical theory. The subsector of lines $W_{n=sl}$ ($s \in \mathbb{Z}_N$) with charge proportional to l are attached to 2d surfaces of b_l, and form a consistent MTC $\mathcal{A}^{N,-t_l}$ for a 1-form symmetry $\mathcal{L} \cong \mathbb{Z}_N$, where $t_l \in \mathbb{Z}_N$ is

$$t_l = l^\mathsf{T} \mathcal{T} l \ , \tag{4.1.113}$$

provided that $t_l \neq 0$, namely, that the boundary $\rho(\mathcal{L})$ is *not* invariant under $V[\mathcal{T}]$.[28] (The case that $\mathcal{L}$ is invariant under $V[\mathcal{T}]$ will be dealt with in Section 4.1.4.) On the gapped boundary we set $b_l = 0$ (up to gauge transformations), therefore this

[27] The lattice $\mathcal{L}$ could be self-dual, in which case $l_\perp = s\, l$ for some $s \in \mathbb{Z}$. In particular, for N prime, the self-dual lattices are in one-to-one correspondence with the roots $s^2 = -1$ and are generated by $l = (1, s)$.

[28] If $\mathcal{L}$ is invariant under M, then $Ml = sl$ for some $s \in \mathbb{Z}_N$. Note that $s \neq 0$, and since we are considering here defects V_M such that $\mathrm{Tr}\, M \neq 2 \bmod N$, then $s \neq 1$. From (4.1.36) one finds $l^\mathsf{T}\mathcal{T} l = \frac{1+s}{1-s}\, l^\mathsf{T} \frac{\epsilon}{2} l = 0$. On the contrary, if $l^\mathsf{T}\mathcal{T} l = 0$ then $\mathcal{T} l = r\, \frac{\epsilon}{2} l$ for some $r \in \mathbb{Z}_N$ and here $r \neq -1$. From (4.1.35) one finds $Ml = \frac{r-1}{r+1}\, l$. This shows that $\mathcal{L}$ is invariant under M if and only if $t_l = 0$. Besides, when $\mathcal{T}$ is invertible and thus $\mathrm{Tr}\, M \neq -2 \bmod N$, then a similar argument also shows an if and only if $t_\perp = 0$.

sector becomes a decoupled TQFT. This allows us to define a minimal boundary twist defect $D_{\mathcal{L}}[\mathcal{T}]$, obtained by discarding the decoupled TQFT $\mathcal{A}^{N,-t_l}$.[29] The lines that braid trivially with $\mathcal{A}^{N,-t_l}$ can be generated by $n = \mathcal{T}^{-1} l_\perp$ and form a MTC $\mathcal{A}^{N,-t_\perp}$ with $t_\perp \in \mathbb{Z}_N$ defined as

$$t_\perp = l_\perp^{\mathsf{T}} \mathcal{T}^{-1} l_\perp \ . \tag{4.1.114}$$

These lines are coupled to the gauge field $b_\perp \equiv l_\perp^{\mathsf{T}} \mathcal{T}^{-1} \mathcal{B}$.[30] (We omit the dependence of $t_\perp$ and $b_\perp$ on $\mathcal{T}$ in order not to clutter.) We have then proved the factorization:

$$\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) = \mathcal{A}^{N,-t_l}(b_l) \times \mathcal{A}^{N,-t_\perp}(b_\perp) \ . \tag{4.1.115}$$

When we move the twist defect $D[\mathcal{T}]$ on top of a gap boundary, the first factor on the decouples yielding $D[\mathcal{T}]\big|_{\text{boundary}} = \mathcal{A}^{N,-t_l} \times D_{\mathcal{L}}[\mathcal{T}]$. We obtain:

$$D_{\mathcal{L}}[\mathcal{T}] = \mathcal{A}^{N,-t_\perp}(b_\perp) \qquad\qquad \text{for } M\mathcal{L} \neq \mathcal{L} \ . \tag{4.1.116}$$

Notice that $M\mathcal{L} = \mathcal{L}$ if and only if $\mathcal{T}\mathcal{L} = \mathcal{L}_\perp$ (see footnote 28). As we will see, this definition of $D_{\mathcal{L}}[\mathcal{T}]$ is consistent under fusion. Notice also that the twist defect $D_{\mathcal{L}}[\mathcal{T}]$, as opposed to $D[\mathcal{T}]$, is stuck on the gapped boundary.

As a check of (4.1.115), one can take the anomaly inflow action (4.1.82) and impose the boundary condition $b_l = 0$. This can be done by parametrizing a gauge field in the quotient group as $\mathcal{B} = t_\perp^{-1} b_\perp l_\perp$, which yields:

$$I(b_\perp) = \frac{N}{2\pi} \int_{4\text{d}} \left[b_\perp d\widetilde{\gamma} - \frac{1}{2}\, b_\perp t_\perp^{-1} b_\perp \right] \tag{4.1.117}$$

as expected (here $\widetilde{\gamma} = t_\perp^{-1} l_\perp^{\mathsf{T}} \widetilde{\Gamma}$). Thus, the theory $D_{\mathcal{L}}[\mathcal{T}]$ is the minimal one required to cancel the anomaly on the gapped boundary.

In order to compute the fusion $D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1]$ on a gapped boundary, we need to understand how to impose the boundary condition on the product theory $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$. Following our previous reasoning, the lines $W^{(1)}_{s_1 l}$ and $W^{(2)}_{s_2 l}$ with charges in $\mathcal{L}$ are the end-lines of surfaces of b_l but decouple from $\mathcal{B}$ on the boundary. Since they are all in the same Lagrangian subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, the two groups maintain trivial mutual braiding even after the deformation by $\mathcal{B}$. We have thus identified a

[29] The operation of discarding $\mathcal{A}^{N,-t_l}$ can be implemented as $D_{\mathcal{L}} = \left[D\big|_{\text{boundary}} \times \mathcal{A}^{N,t_l} \right] / \mathbb{Z}_N$ [67].

[30] The splitting of $\mathcal{B}$ into b_l and $b_\perp$ is well defined as long as the boundary is not invariant under $V[\mathcal{T}]$. Otherwise, $\mathcal{T} l \propto l_\perp$ and so $b_\perp \propto b_l$ which vanishes on the gapped boundary.

subset of lines that couple to b_l and form the MTC $\mathcal{A}^{N,-t_{2,l}}(b_l) \times \mathcal{A}^{N,-t_{1,l}}(b_l)$, where $t_{j,l} = l^\intercal \mathcal{T}_j l$. The lines $W_\mathcal{N}$ that braid trivially with that subset, as we will see, form a MTC $\mathcal{A}^{N,-\mathcal{R}_{21}}$ coupled to $\mathcal{B}$ for some matrix $\mathcal{R}_{21}$:

$$\mathcal{A}^{N,-\mathcal{T}_2}(\mathcal{B}) \times_\mathcal{B} \mathcal{A}^{N,-\mathcal{T}_1}(\mathcal{B}) = \mathcal{A}^{N,-t_{2,l}}(b_l) \times \mathcal{A}^{N,-t_{1,l}}(b_l) \times \mathcal{A}^{N,-\mathcal{R}_{21}}(\mathcal{B}) \ . \tag{4.1.118}$$

On the gapped boundary, the first two factors on the right-hand side decouple and moreover are precisely the two factors that are discarded in the definition of $D_\mathcal{L}[\mathcal{T}_1]$ and $D_\mathcal{L}[\mathcal{T}_2]$. After imposing $b_l\big|_X = 0$, the third factor only couples to a projection of $\mathcal{B}$. As we will see, such a projection is the very one predicted by fusion, namely to $b_\perp = l_\perp^\intercal \mathcal{T}_{21}^{-1}\mathcal{B}$. Besides, we expect the MTC $\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp)$ to be the product of a MTC $\mathcal{N}_{21}$ that does not couple to $b_\perp$, and the MTC $\mathcal{A}^{N,-t_{21}^\perp}(b_\perp)$ (where $t_{21}^\perp = l_\perp^\intercal \mathcal{T}_{21}^{-1} l_\perp$) that lives on the twisted sector $D_\mathcal{L}[\mathcal{T}_{21}]$. We will verify this expectation, and show that

$$\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp) = \mathcal{N}_{21} \times \mathcal{A}^{N,-t_{21}^\perp}(b_\perp) \ . \tag{4.1.119}$$

These relations imply the fusion rules

$$D_\mathcal{L}[\mathcal{T}_2] \times D_\mathcal{L}[\mathcal{T}_1] = \mathcal{N}_{21}\ D_\mathcal{L}[\mathcal{T}_{21}] \ , \tag{4.1.120}$$

where the decoupled TQFT $\mathcal{N}_{21}$ plays the role of a fusion coefficient.

Let us compute $\mathcal{N}_{21}$. The N^2 lines $W_\mathcal{N}$ of $\mathcal{A}^{N,-\mathcal{R}_{21}}$, that braid trivially with the first two factors on the of (4.1.118), have charges $\mathcal{N} = (\xi_1, \xi_2)$ determined by solving the equations

$$\begin{aligned} \mathcal{T}_1\, \xi_1 - \frac{\epsilon}{2}\, \xi_2 &= a_1\, l_\perp \\ \mathcal{T}_2\, \xi_2 + \frac{\epsilon}{2}\, \xi_1 &= a_2\, l_\perp \end{aligned} \tag{4.1.121}$$

for some coefficients $a_{1,2} \in \mathbb{Z}_N$ that depend on the line. In fact, one can use a_1, a_2 to parametrize the solutions. We first consider the simple case $\mathcal{T}_1 = \mathcal{T}_2$, then the generic case, and finally the exceptional case $\mathcal{T}_1 = -\mathcal{T}_2$.

Case $\boldsymbol{\mathcal{T}_1 = \mathcal{T}_2 \equiv \mathcal{T}}$. This case computes the square of a defect $D_\mathcal{L}[\mathcal{T}]$. Noticing from (4.1.45) that $\mathcal{T}_{21} = \frac{1}{2}\big(\mathcal{T} + \frac{\epsilon}{2}\mathcal{T}^{-1}\frac{\epsilon}{2}\big)$, we find:

$$\xi_1 = \frac{1}{2}\,\mathcal{T}_{21}^{-1}\Big(a_1 + a_2\,\frac{\epsilon}{2}\,\mathcal{T}^{-1}\Big) l_\perp \ , \qquad \xi_2 = \frac{1}{2}\,\mathcal{T}_{21}^{-1}\Big(a_2 - a_1\,\frac{\epsilon}{2}\,\mathcal{T}^{-1}\Big) l_\perp \ . \tag{4.1.122}$$

The charge of a line under $\mathcal{B}$ is $\xi_1 + \xi_2$. One can check that the lines with $a_1 = a_2$ have charge proportional to $\mathcal{T}_{21}^{-1} l_\perp$, and so they couple to $b_\perp$. With some algebra[31] and

[31] One should use that $\mathcal{T}_{21}\epsilon\mathcal{T} = \mathcal{T}\epsilon\mathcal{T}_{21}$. It also implies that such a matrix is antisymmetric.

(4.1.104), one can check that those lines braid trivially with the lines with $a_1 = -a_2$. This suggests to label the lines in terms of $a, c \in \mathbb{Z}_N$ and set $a_1 = a - c$, $a_2 = a + c$. The spin of a line labelled by (a, c) is found to be

$$\theta\big[W_{(a,c)}\big] = \exp\bigg(-\frac{\pi i}{N}\, t_{21}^{\perp}\,\big(a^2 + c^2\big)\bigg)\ , \tag{4.1.123}$$

where $t_{21}^{\perp} = l_{\perp}^{\mathsf{T}} \mathcal{T}_{21}^{-1} l_{\perp}$. As long as $t_{21}^{\perp} \neq 0$, such lines form the theory $\mathcal{A}^{N,-\mathcal{R}_{21}}$ with

$$\mathcal{R}_{21} = \begin{pmatrix} t_{21}^{\perp} & 0 \\ 0 & t_{21}^{\perp} \end{pmatrix}\ . \tag{4.1.124}$$

The subset of lines $(a, 0)$ form the MTC $\mathcal{A}^{N,-t_{21}^{\perp}}(b_{\perp})$, as expected. The lines $(0, c)$ have charges under $\mathcal{B}$ proportional to $\mathcal{T}_{21}^{-1}\epsilon\mathcal{T}^{-1}l_{\perp}$, which has vanishing contraction with $l_{\perp}^{\mathsf{T}}$ and thus is proportional to l. On the gapped boundary $b_l = 0$ and hence these lines form a decoupled MTC

$$\mathcal{N}_{21} = \mathcal{A}^{N,-t_{21}^{\perp}}\ . \tag{4.1.125}$$

We have obtained the fusion rule

$$D_{\mathcal{L}}[\mathcal{T}] \times D_{\mathcal{L}}[\mathcal{T}] = \mathcal{A}^{N,-t_{21}^{\perp}}\ D_{\mathcal{L}}[\mathcal{T}_{21}]\ . \tag{4.1.126}$$

Notice that this fusion rule is the same (with the same $\mathcal{N}_{21}$) on all gapped boundaries $\rho(\mathcal{L})$ belonging to the same orbit under $V[\mathcal{T}]$. This follows from footnote 21.

Generic case. In order to treat the general case it is convenient to parametrize the lines $(\xi_1, \xi_2) = (v, \eta - v)$ in terms of two vectors v, η, so that the charge of a line under $\mathcal{B}$ is η, and redefine the numbers $a_1 = p + q$, $a_2 = q$. The equations (4.1.121) become

$$\begin{aligned} \big(\mathcal{T}_1 + \mathcal{T}_2\big)v - \big(\mathcal{T}_2 + \tfrac{\epsilon}{2}\big)\eta &= p\, l_{\perp} \\ \mathcal{T}_2\, \eta - \big(\mathcal{T}_2 - \tfrac{\epsilon}{2}\big)v &= q\, l_{\perp}\ . \end{aligned} \tag{4.1.127}$$

Defining $\Gamma = \big(\mathcal{T}_2 - \frac{\epsilon}{2}\big)\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1}$, the solutions are

$$v = q\, \Gamma^{\mathsf{T}}\mathcal{T}_{21}^{-1}l_{\perp} + p\,\Big[\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1} + \Gamma^{\mathsf{T}}\mathcal{T}_{21}^{-1}\Gamma\Big]l_{\perp}\ , \qquad \eta = q\, \mathcal{T}_{21}^{-1}l_{\perp} + p\, \mathcal{T}_{21}^{-1}\Gamma l_{\perp} \tag{4.1.128}$$

and can be labelled by $q, p \in \mathbb{Z}_N$. Substituting in (4.1.104), the spins of the lines are

$$\theta\big[W_{(q,p)}\big] = \exp\bigg(-\frac{\pi i}{N}\,(q,p)\mathcal{R}_{21}\binom{q}{p}\bigg) \qquad \text{with} \qquad \mathcal{R}_{21} = \begin{pmatrix} t_{21}^{\perp} & c_{\mathrm{o}} \\ c_{\mathrm{o}} & c_{\mathrm{d}} \end{pmatrix}\ , \tag{4.1.129}$$

where

$$c_{\rm o} = l_\perp^{\mathsf{T}}\, \mathcal{T}_{21}^{-1}\Gamma\, l_\perp\ , \qquad c_{\rm d} = l_\perp^{\mathsf{T}}\left[\left(\mathcal{T}_1+\mathcal{T}_2\right)^{-1} + \Gamma^{\mathsf{T}}\mathcal{T}_{21}^{-1}\Gamma\right] l_\perp = l_\perp^{\mathsf{T}}\left(\mathcal{T}_1 + \tfrac{\epsilon}{2}\mathcal{T}_2^{-1}\tfrac{\epsilon}{2}\right)^{-1} l_\perp\ . \tag{4.1.130}$$

The subset of lines $(q,0)$ have charges η proportional to $\mathcal{T}_{21}^{-1}l_\perp$ and thus couple to $b_\perp$. Their spins show that they form the MTC $\mathcal{A}^{N,-t_{21}^\perp}(b_\perp)$. On the other hand, the subset of lines (q,p) with $q=-(t_{21}^\perp)^{-1}c_{\rm o}\,p$ braid trivially with the former subset and constitute the theory $\mathcal{N}_{21}$. Their charges η are such that $l_\perp^{\mathsf{T}}\eta=0$, therefore they are decoupled from $\mathcal{B}$ on the gapped boundary. Their spins show that

$$\mathcal{N}_{21} = \mathcal{A}^{N,-n_{21}} \qquad \text{with} \qquad n_{21} = c_{\rm d} - (t_{21}^\perp)^{-1}c_{\rm o}^2 = (t_{21}^\perp)^{-1}\det\mathcal{R}_{21}\ . \tag{4.1.131}$$

One should recall that, in the absence of a coupling to $\mathcal{B}$, the theories $\mathcal{A}^{N,-p}$ and $\mathcal{A}^{N,-pr^2}$ are equivalent for any invertible $r\in\mathbb{Z}_N$, and thus for N odd prime the only physical information in $n_{21}\neq 0$ is whether it is a quadratic residue or not. This is detected by the Legendre symbol $n_{21}^{(N-1)/2}$ mod $N\in\{1,-1\}$.[32]

Case $\mathcal{T}_2=-\mathcal{T}_1\equiv\mathcal{T}$. This is the case leading to condensation. The equations for lines in $\mathcal{A}^{N,-\mathcal{R}_{21}}$ are just $\mathcal{T}\xi_1+\frac{\epsilon}{2}\xi_2=-a_1 l_\perp$ and $\mathcal{T}\xi_2+\frac{\epsilon}{2}\xi_1=a_2 l_\perp$. The general solution is

$$\xi_1 = \left[a\left(\mathcal{T}+\frac{\epsilon}{2}\right)^{-1} - c\left(\mathcal{T}-\frac{\epsilon}{2}\right)^{-1}\right] l_\perp\ , \qquad \xi_2 = \left[a\left(\mathcal{T}+\frac{\epsilon}{2}\right)^{-1} + c\left(\mathcal{T}-\frac{\epsilon}{2}\right)^{-1}\right] l_\perp \tag{4.1.132}$$

where we redefined $a_1=c-a$ and $a_2=c+a$. For these lines:

$$\theta\big[W_{(a,c)}\big] = \exp\!\left(-\frac{2\pi i}{N}\,ac\;2l_\perp^{\mathsf{T}}\left(\mathcal{T}+\frac{\epsilon}{2}\right)^{-1} l_\perp\right)\ . \tag{4.1.133}$$

Lines with either a or $c=0$ have vanishing spin, which indicates that we are dealing with a DW type theory. The lines with $a=0$ (electric) do not couple to $\mathcal{B}$ since they have $\xi_1+\xi_2=0$. Redefining $a\to\left[2l_\perp^{\mathsf{T}}\left(\mathcal{T}+\frac{\epsilon}{2}\right)^{-1}l_\perp\right]^{-1}$ gives the canonical braiding $B_{ac}=e^{\frac{2\pi i}{N}ac}$. Thus

$$\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp) = (\mathbb{Z}_N)_0(b_\perp) = \mathcal{C}^{\mathbb{Z}_N}\ . \tag{4.1.134}$$

[32] In the higher-rank case the situation is similar. For N odd prime, one can always bring a symmetric matrix $\mathcal{T}$ with values in $\mathbb{Z}_N$ to a diagonal form $U^{\mathsf{T}}\mathcal{T}U=\mathrm{diag}(t_1,\dots,t_r)$ using an invertible matrix U (see, *e.g.*, [245]). The TQFT is then characterized by the number of $+1$ and -1 Legendre symbols of the t_i's.

We conclude that:

$$D_{\mathcal{L}}[\mathcal{T}] \times \overline{D}_{\mathcal{L}}[\mathcal{T}] = \mathcal{C}^{\mathbb{Z}_N} \ . \tag{4.1.135}$$

The condensate $\mathcal{C}^{\mathbb{Z}_N}$ is for the 1-form symmetry $\mathcal{S} = (\mathbb{Z}_N \times \mathbb{Z}_N)/\mathcal{L} \cong \mathbb{Z}_N$ that exists on the gapped boundary.

Examples. We can now apply our formalism to the known cases of duality and triality defects. We consider a generic boundary $\rho(\mathcal{L})$, but assume that it is not invariant under any symmetry defect appearing below (apart from C, which leaves every boundary invariant). For the application to self-duality defects, we must compute the fusion $D_{\mathcal{L}}[S] \times D_{\mathcal{L}}[S]$. This is a special case, since the involves charge conjugation. The explicit computation is done in Appendix 4.3.3 (see also the comments in Section 4.1.4). The complete fusion gives a coefficient which is a product of DW theories, these all admit a universal boundary condition which allows us to set them to one. This corresponds to the Dirichlet boundary of the DW theory. After this we find:

$$\begin{aligned} D_{\mathcal{L}}[S] \times D_{\mathcal{L}}[S] &= (\mathbb{Z}_N)(\tilde{b}_\perp, \phi_\perp)\ D^{\text{triv}}_{\mathcal{L}}[0] \\ D_{\mathcal{L}}[S] \times \overline{D}_{\mathcal{L}}[S] &= \mathcal{C}^{\mathbb{Z}_N} \ . \end{aligned} \tag{4.1.136}$$

For triality defects we compute:[33]

$$\begin{aligned} D_{\mathcal{L}}[ST] \times D_{\mathcal{L}}[ST] &= \mathcal{A}^{N,-p_{ST}}\ D_{\mathcal{L}}\big[(ST)^2\big] \\ D_{\mathcal{L}}[CST] \times D_{\mathcal{L}}[CST] &= \mathcal{A}^{N,-p_{ST}}\ D_{\mathcal{L}}\big[(ST)^2\big] \\ D_{\mathcal{L}}\big[(ST)^2\big] \times D_{\mathcal{L}}\big[(ST)^2\big] &= \mathcal{A}^{N,\,p_{ST}}\ D_{\mathcal{L}}[CST] \end{aligned} \tag{4.1.137}$$

where

$$p_{ST} = \begin{cases} 1 & \text{if } \mathcal{L} = \mathcal{L}(e)\ , \\ r^2 + r + 1 & \text{if } \mathcal{L} = \mathcal{L}(m)_r\ . \end{cases} \tag{4.1.138}$$

These fusions agree with those computed in [25] on the electric boundary.[34] Notice that $p_{ST} \neq 0 \mod N$ as long as the boundary Lagrangian subgroup $\mathcal{L}$ is not invariant under ST.

[33] To get to the result we use the property $\mathcal{A}^{N,pr^2} = \mathcal{A}^{N,p}$ for $\gcd(r, N) = 1$.

[34] One uses that $\mathcal{A}^{N,1} = U(1)_N$ [67]. Notice that the conventions of [25] defined in their eqns. (6.7)–(6.9) differ from ours, and their defects are the orientation reversal of ours, leading to a sign change in the level.

Other defects. Another interesting case is when the 4d defects $V_M = V[\mathcal{A}, \xi]$ are obtained by condensing a $\mathbb{Z}_N$ subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, corresponding to the elements $M \in SL(2, \mathbb{Z}_N)$ that are conjugate to T^k for some k. For simplicity let us consider the case $M = T^k$ with $k = 1, \ldots, N-1$. The twisted sectors $D_{T^k, \mathcal{L}}$ are described by the minimal theories $\mathcal{A}^{N,k^{-1}}(b)$ for $\mathbb{Z}_N$ coupled to the bulk field b. For the lines in these theories there is no extra contribution to the braiding when we stack the theories, and thus they fuse in the standard way:

$$D_{T^k, \mathcal{L}} \times D_{T^{k'}, \mathcal{L}} = \mathcal{A}^{N, k^{-1}+k'^{-1}} \; D_{T^{k+k'}, \mathcal{L}} \tag{4.1.139}$$

as long as $k + k' \neq 0 \bmod N$. This formula is in agreement with the fusion law of N-ality defects found in [25, 119]. Notice that these twist sectors are not unique since they can be fused to 3d condensates for the magnetic symmetry. However, on the magnetic boundaries $\mathcal{L}(m)$ (on which the twisted sector D_{T^k} hosts a minimal theory) we can take the condensates to be generated by the magnetic symmetry $l(m) \in \mathcal{L}(m)$.[35]. On the magnetic boundary these condensates however become all decoupled DW theories since $l(m)^{\mathsf{T}} \mathcal{B}\big|_X = 0$.

Twist defects and boundary-changing operators Consider starting with a twist defect D_M (attached to a 4d symmetry defect V_M) in the bulk and moving it on top of a gapped boundary $\rho(\mathcal{L})$. We are here interested in the case that $\rho(\mathcal{L})$ is not invariant under M. As discussed before (4.1.116), the defect D_M on the boundary decomposes into $D_{M,\mathcal{L}}$ and a decoupled TQFT. We conclude that $D_{M,\mathcal{L}}$ is an interface between two copies of $\rho(\mathcal{L})$, or using categorical terms, it defines a morphism $D_{M,\mathcal{L}} : M \times \rho(\mathcal{L}) \to \rho(\mathcal{L})$. This is depicted in Figure 4.7 left. On the other hand, if we bring the symmetry defect V_M on top of the boundary we obtain an action $V_M \times \rho(\mathcal{L}) = \rho(M\mathcal{L})$. Thus, we can construct an interface between $\rho(\mathcal{L})$ and $\rho(M\mathcal{L})$ by fusing the boundary with V_M only on a half-space and then letting V_M escape in the bulk, as in Figure 4.7 center. This defines a morphism $\mathcal{U}_M : M \times \rho(\mathcal{L}) \to \rho(M\mathcal{L})$. Since both interfaces sit at the end of a symmetry defect V_M, it is possible to define a local boundary-changing operator as the morphism $\varphi_M = \mathcal{U}_M \circ D^{\dagger}_{M,\mathcal{L}} : \rho(\mathcal{L}) \to \rho(M\mathcal{L})$, as in Figure 4.7 right.

Recall that, in the ungauged theory, one can expect to define only duality *interfaces*. The interface is a composite object given by a discrete gauging operation

[35] We can always arrive at this choice since any two magnetic lattices differ by electric ones, which can be absorbed by D_{T^k}

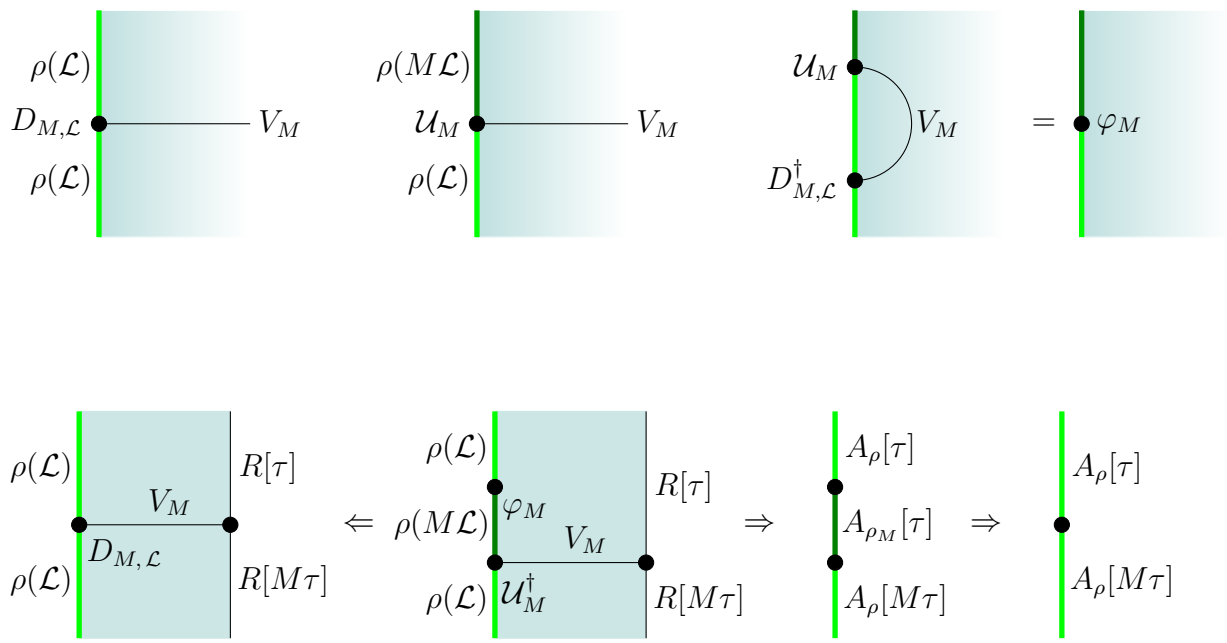


Figure 4.7: The way in which the symmetry TFT implements the construction of [26, 120]. Above: definition of various morphisms. Below: construction of the duality interface $D_{M,\mathcal{L}}$.

composed with an invertible duality transformation. On the TQFT side this is described by acting with $\mathcal{U}_M^\dagger$ to map the boundary to $\rho(M\mathcal{L})$ and then using φ_M to go back to $\rho(\mathcal{L})$. After compactifying the slab of symmetry TFT this gives an interface : $A_\rho[\tau] \to A_{\rho_M}[\tau] \to A_\rho[M\tau]$ between absolute theories. Shrinking the middle part of the drawing gives the duality interface. On the other hand the fusion $\varphi_M \times \mathcal{U}_M^\dagger = D_{M,\mathcal{L}}$ holds, since $D_{M,\mathcal{L}}$ is unique as a twist defect on the gapped boundary. We can thus identify the defect $D_{M,\mathcal{L}}$ on the bottom-left of Figure 4.7 with the duality interface in the absolute theory $A_\rho[\tau]$.

Boundaries with a stabilizer

Let us also discuss the properties of a twist defect $D_{\mathcal{L}}[\mathcal{T}]$ on a gapped boundary $\rho(\mathcal{L})$ that is invariant under the corresponding symmetry defect $V[\mathcal{T}]$. This means that M is in the stabilizer H of $\mathcal{L}$ in $SL(2,\mathbb{Z}_N)$. We can gather information on the degrees of freedom living on $D_{\mathcal{L}}[\mathcal{T}]$ by computing the anomaly inflow. The invertible TQFT living on $V[\mathcal{T}]$ is (4.1.82). On the gapped boundary we parametrize[36] $\mathcal{B} = \tilde{b}_\perp l_\perp$ and obtain

$$I_{\mathcal{T}}\big|_{\rho(\mathcal{L})} = \frac{N}{2\pi}\int\left[\tilde{\gamma}_\perp\, d\tilde{b}_\perp - \frac{1}{2}\, t_\perp\, \tilde{b}_\perp \tilde{b}_\perp\right] \tag{4.1.140}$$

(where $\tilde{\gamma}_\perp = l_\perp^{\mathsf{T}}\widetilde{\Gamma}$). Since now $t_\perp = 0$ (see footnote 28), the anomaly is trivialized.

What happens to the lines in the twisted sector can be understood using the minimal theory description. We start with the twist defect $D[\mathcal{T}]$ in the bulk and push it onto an invariant boundary $\mathcal{L}$. Normally we would now separate the degrees of freedom which decouple on the boundary, which form a $\mathcal{A}^{N,-t_l}(b_l)$ factor. This is generated by lines $L_s \equiv W_{n=s\,l}$. If the boundary is invariant then $t_l = 0$ and this procedure is ill defined as the L_s all have vanishing spin. They thus form a Lagrangian subalgebra. This means that $\mathcal{A}^{N,-\mathcal{T}}$ should rather be thought of as a DW theory coupled to $\tilde{b}_\perp$. Since the lines with trivial spin are also uncharged under $\tilde{b}_\perp$ this can be thought of as a condensate:

$$\mathcal{A}^{N,\mathcal{T}}(\mathcal{B})\big|_X = \mathcal{C}^{\mathbb{Z}_N}\ . \tag{4.1.141}$$

To be more precise we can choose a generator u of $\mathcal{S}$. Since by definition $u^{\mathsf{T}} l_\perp \neq 0$ lines $\tilde{L}_r \equiv W_{n=ru}$ are charged under $\tilde{b}_\perp$. These lines have spin:

$$\theta\left[\tilde{L}_r\right] = \exp\left(-\frac{\pi i}{N}\, r^2\, u^{\mathsf{T}}\mathcal{T} u\right) \tag{4.1.142}$$

[36]Here the normalization is different than before (4.1.117), because $t_\perp = 0$ when $\mathcal{L}$ is invariant under M.

and braid with the electric lines L_s:

$$B_{r,s} = \exp\left(-\frac{2\pi i}{N}\, rs\, u^{\mathsf{T}}\mathcal{T}l\right) \neq 1\,, \tag{4.1.143}$$

since on the invariant boundary $\mathcal{T}\mathcal{L} = \mathcal{L}_\perp$. Properly redefining L_s we can make this braiding into canonical one. As we have already commented there is no canonical choice for u, since we are free to shift it by vectors in $\mathcal{L}$. The shift $u \to u + l$ does not affect the braiding with L_r but it does affect the spin of $\tilde{L}_s$:

$$\theta\left[\tilde{L}_r\right] \to \theta\left[\tilde{L}_r\right] \exp\left(-\frac{2\pi i}{N}\, r^2\, u^{\mathsf{T}}\mathcal{T}l\right) \tag{4.1.144}$$

For N odd and on spin manifolds we can use this to set $\theta[L_r]$ to one.

Since the defect $V[\mathcal{T}]$ has trivial anomaly on $\rho(\mathcal{L})$, it can end there without adding new degrees of freedom. Therefore the twist defect $D_{\mathcal{L}}[\mathcal{T}]$ is trivial (invertible) on an invariant boundary:

$$D[\mathcal{T}]\big|_X = \mathcal{C}^{\mathbb{Z}_N}\, D^{\text{triv}}_{\mathcal{L}}[\mathcal{T}]\,, \tag{4.1.145}$$

where the superscript is useful to remember this fact.

The same phenomenon appears if we consider a fusion $D_{\mathcal{L}}[\mathcal{T}_1] \times D_{\mathcal{L}}[\mathcal{T}_2]$ in which $V[\mathcal{T}_{21}]$ leaves the boundary $\rho(\mathcal{L})$ invariant, but neither $V[\mathcal{T}_1]$ nor $V[\mathcal{T}_2]$ do. We proceed as in the usual case by separating out the lines coupling to b_l from both terms in the fusion. This is a well defined procedure since t_1^l, $t_2^l \neq 0$ (due to $\mathcal{L}$ not being invariant under neither $\mathcal{T}_1$ nor $\mathcal{T}_2$). Based on the previous remarks we expect $\mathcal{A}^{N,-\mathcal{R}_{2,1}}$ to also be a condensate. It is clear that the theory contains a Lagrangian algebra generated by $W_{(q,0)}$ in (4.1.129). In the generic discussion these lines were coupled to $b_\perp$, however if the boundary is invariant they are not.[37] These form the set of "electric" lines. The magnetic lines $W_{(0,p)}$ instead couple to $\tilde{b}_\perp$, but have nontrivial spin:

$$\theta\left[W_{(0,p)}\right] = \exp\left(-\frac{\pi i}{N} c_{\text{d}}\, p^2\right)\,, \tag{4.1.146}$$

As before, we can redefine the magnetic lines by summing a multiple of the electric ones to set this to zero. Notice that the discussion here is also consistent with the example of $\mathcal{T}_2 = -\mathcal{T}_1$ discussed before, when the final result is a condensate and the identity defect leaves all boundaries invariant.

[37] The charge under the gauge symmetry for $\tilde{b}_\perp$ is $q l_\perp^{\mathsf{T}} \mathcal{T}_{2,1}^{-1} l_\perp$, which vanishes when the boundary is invariant.

We are now in a position to write down the full result of the boundary fusion for $D_{\mathcal{L}}[\mathcal{T}]$:

$$\begin{aligned} D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1] &= \mathcal{N}_{21}\, D_{\mathcal{L}}[\mathcal{T}_{21}] \ , \qquad && \text{if } V[\mathcal{T}_{2,1}]|\rho(\mathcal{L})\rangle \neq |\rho(\mathcal{L})\rangle \ , \\ D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1] &= \mathcal{C}^{\mathbb{Z}_N}\, D_{\mathcal{L}}^{\text{triv}}[\mathcal{T}] \ , \qquad && \text{if } V[\mathcal{T}_{2,1}]|\rho(\mathcal{L})\rangle = |\rho(\mathcal{L})\rangle \ , \\ D_{\mathcal{L}}[\mathcal{T}] \times \overline{D_{\mathcal{L}}[\mathcal{T}]} &= \mathcal{C}^{\mathbb{Z}_N} \ . \end{aligned} \tag{4.1.147}$$

This will have a more natural interpretation in the gauged theory. In that case we will see that anomaly cancellation forces the Gukov Witten operator GW$[\mathcal{T}]$ to exist only as a bound state with the twist defect $D[\mathcal{T}]$ for $V[\mathcal{T}]$. When the boundary $\mathcal{L}$ is $\mathcal{T}$-invariant there is no anomaly to cancel and GW$[\mathcal{T}]$ can exist as a genuine defect on the gapped boundary. The fusion rule above tell us that, when two bound operator fuse onto an invariant one, such fusion is always accompanied by the appearence of a condensation defect. This is consistent with the fact that defects $D_{\mathcal{L}}[\mathcal{T}]$ absorb surface defects $e^{i\int b_\perp}$, which survive on the gapped boundary. In the absence of the condensation defect the cannot absorb such lines and fusion would be inconsistent.

4.1.5 The gauged theory

Finally, we discuss the effect of gauging a discrete subgroup $G \subset SL(2,\mathbb{Z}_N)$ in the bulk TQFT. In the application to $\mathcal{N}=4$ SYM, the only relevant groups (including the action of charge conjugation) are $\mathbb{Z}_4$ and $\mathbb{Z}_6$ generated by S and ST, respectively. Notice that they are both Abelian. The construction we present below applies to a generic Abelian G, while the non-Abelian case requires modifications that might be important in discussing theories of class S (we comment on that in the conclusions).

We will first describe abstractly the spectrum of operators in the gauged theory. We follow the rules for discrete gauging described for 3d MTCs in [138] and recently extended to higher dimensions in [24]. Particular care will be needed in describing the Gukov-Witten operators of G gauge theory, as they get dressed by the corresponding twist defects $D[\mathcal{T}]$. We will present the construction of these operators, that we dub $\mathfrak{D}[\mathcal{T}]$. Finally, we will study gapped boundaries $|\rho^*\rangle$ in the gauged theory in terms of orbits of boundaries $|\rho\rangle$ in the ungauged theory. This allows for a simple derivation of the fusion rules. We will also comment on the differences arising when the boundary has a nontrivial stabilizer.

In the following we will restrict to the study of twisted sectors $D[\mathcal{T}]$ for which $M(\mathcal{T})$ is an element of G. Together with the assumption that G is Abelian, this

ensures that different twisted sectors do not mix among each other and that the genuine codimension-2 operators $\mathfrak{D}[\mathcal{T}]$ of the gauged theory are still labelled by group elements.[38]

Spectrum of bulk operators The spectrum of topological operators in the gauged theory can be obtained, at least at a formal level, by applying standard rules for gauging a discrete 0-form symmetry to the ungauged theory. These are nicely summarized in [24]. Let us start with the surface defects U_n that implement the 2-form symmetry. These operators are in general not gauge invariant, as G acts on them nontrivially. We can build gauge-invariant combinations by considering orbits under G:

$$U^*_{[n]} = \frac{1}{\big|\mathrm{Stab}(n)\big|} \sum_{g\in G} U(g\,n)\ , \tag{4.1.148}$$

where $\mathrm{Stab}(n)$ is the stabilizer group for n as an element of $\mathbb{Z}_N \times \mathbb{Z}_N$. When n admits a nontrivial stabilizer, the surface $U^*_{[n]}$ supports nontrivial line defects labelled by representations of $\mathrm{Stab}(n)$. In the cases considered here, that is N prime and $G = \mathbb{Z}_4$ or $\mathbb{Z}_6$, the only surface with a nontrivial stabilizer is the identity, while all others ones do not host any line.

As an example, in the case of the $\mathbb{Z}_4$ subgroup of $SL(2,\mathbb{Z}_N)$ generated by S, a dyon (e,m) is mapped to an orbit

$$[e,m] = (e,m) + (m,-e) + (-e,-m) + (-m,e)\ . \tag{4.1.149}$$

These objects are non-invertible and their fusion is

$$[e,m]\times[e',m'] = [e+e',m+m'] + [e+m',m-e'] + [e-e',m-m'] + [e-m',m+e']\ . \tag{4.1.150}$$

More interesting is the situation for codimension-2 operators. We have already discussed that in the ungauged theory, genuine 3d operators are necessarily condensation defects. After the discrete gauging the situation is different. The twist defects $D[\mathcal{T}]$ for surfaces $V[\mathcal{T}]$ generating G become "liberated" — in the sense that they become genuine 3d operators — since the surfaces $V[\mathcal{T}]$ are transparent in the gauged theory. One could think of the liberated defects as arising from the"lassoed" configuration shown in Figure 4.8 after summing over G. Since G is Abelian, each twist sector is left fixed by the action of the lassos and it gives rise to a single genuine

[38] In the general case they are labelled by conjugacy classes under the adjoint action of G.

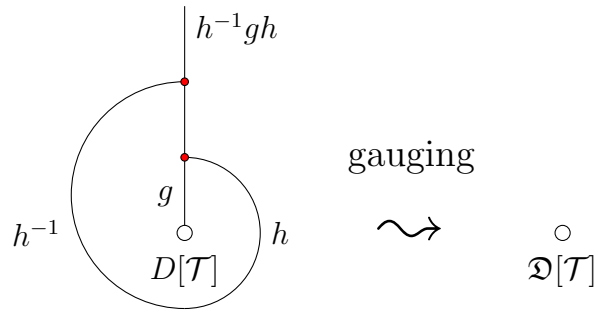


Figure 4.8: In the gauged theory, a twist defect $D[\mathcal{T}]$ (with $M(\mathcal{T}) \equiv g \in G$) is dressed by codimension-1 surfaces of G labelled by h.

operator $\mathfrak{D}[\mathcal{T}]$.[39] The action of a lasso $V_{M'}$ reduces to a 0-form symmetry action on $D[\mathcal{T}]$, which maps $W_n \mapsto W_{M'n}$. This is indeed a symmetry of the theory, since

$$M'^{\mathsf{T}}\, \mathcal{T}\, M' = \mathcal{T} \tag{4.1.151}$$

and thus it preserves the braiding. Summing over such action means that the 0-form symmetry on the defect is gauged, so we would like to conclude that $\mathfrak{D}[\mathcal{T}]$ is $D[\mathcal{T}]/G$.

This description is slightly imprecise, because $D[\mathcal{T}]$ lives at the boundary of $V[\mathcal{T}]$. Indeed, the gauging process can be thought of as coupling the original system to a discrete G gauge theory. Its gauge field $a \in H^1(M_5, G)$ couples minimally to the 0-form symmetry defects $V_{M\in G}$ of the original theory (more details in Section 4.1.5). In this setup, inserting a twist defect $D[\mathcal{T}]$ is only consistent at locations where a is not closed: it must instead satisfy $\delta a = g$ schematically. Another way of saying this is that a exhibits a nontrivial holonomy g around the 3-cycle Y on which $D[\mathcal{T}]$ lies. This is the description of Gukov-Witten defect operators in G gauge theory, that we indicate as GW_g. We infer that a more precise definition of the new operators is:

$$\mathfrak{D}[\mathcal{T}] = \mathrm{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]/G\ . \tag{4.1.152}$$

The appearance of this "bound state" has a simple explanation: In the original theory, the defect $D[\mathcal{T}]$ was not gauge invariant due to anomaly inflow from $V[\mathcal{T}]$. The GW operator is not gauge invariant either, as it carries the anomaly of $V[\mathcal{T}]$. Their combination is a well defined operator in the gauged theory. This is a close

[39] When instead G is non-Abelian, twist defects also combine into orbits and the situation is more subtle.

cousin of the mechanism described in [119]. We also learn that $\mathfrak{D}[\mathcal{T}]$ is charged under the dual $\widehat{G}$ 3-form symmetry.

The exception is the twist defect $D[\mathcal{T}=0] \equiv D_C$ for charge conjugation. In this case there is no anomaly inflow and therefore the GW operator for C defines a genuine, group-like object in the gauged theory. This suggests that we should interpret the contributions from D_C arising upon fusion as decoupled condensates after gauging.

The following table summarizes the properties of some objects in the gauged theory:

Original object	Gauged object	Emergent lines	Grouplike?
$(0,0)$	$[0,0]$	$\mathrm{Rep}(G)$	YES
(e,m)	$[e,m] = \oplus_{g\in G}\, g(e,m)$	none	NO
$D[\mathcal{T}]$	$\mathfrak{D}[\mathcal{T}] = \mathrm{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]/G$	$\mathrm{Rep}(G)$	NO
D_C	GW_C	none	YES

Hybrid formulation of the gauged theory

In order to give a Lagrangian description of the gauging of the subgroup $G \subset SL(2,\mathbb{Z}_N)$ in the 5d Chern-Simons theory, we employ a sort of hybrid formulation in which the Chern-Simons theory is described by continuum gauge fields, while the gauge field for G is described using singular cochains (see, *e.g.*, [60, 139] or the appendix in [140]).

First of all, on the spacetime manifold M_5 one chooses a simplicial triangulation. This is made of vertices or 0-simplices p_i with an arbitrary ordering for the index i, edges or 1-simplices p_{ij} (with $i<j$) connecting the vertices p_i and p_j, 2-simplices p_{ijk} (with $i<j<k$) bounded by p_{ij}, p_{jk} and p_{ik}, and so on. All simplices are contractible, and M_5 is the union of all 5-simplices. A gauge field a for the discrete gauge group G is a 1-cochain $a \in C^1(M_5, G)$ that assigns an element $a_{ij} \in G$ to each 1-simplex p_{ij} (with $i<j$), with the constraint that $da = \mathbb{1}$. We use multiplicative notation and define the differential as $(da)_{ijk} = a_{jk}a_{ik}^{-1}a_{ij}$ (with $i<j<k$). We will only consider the case that G is Abelian. Gauge transformations then map $a_{ij} \mapsto (d\lambda)_{ij}a_{ij}$ where $d\lambda_{ij} = \lambda_j\lambda_i^{-1}$ and $\lambda \in C^0(M_5, G)$ in a 0-cochain. The gauging of G is described by a sum over $a \in H^1(M_5, G)$ in cohomology.

Then we construct a covering of M_5 by closed patches that is dual to the triangulation, as follows. Each patch U_i is a 5d contractible manifold with boundary that contains the 0-simplex p_i. Then each non-empty intersection $U_{i_1\ldots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$

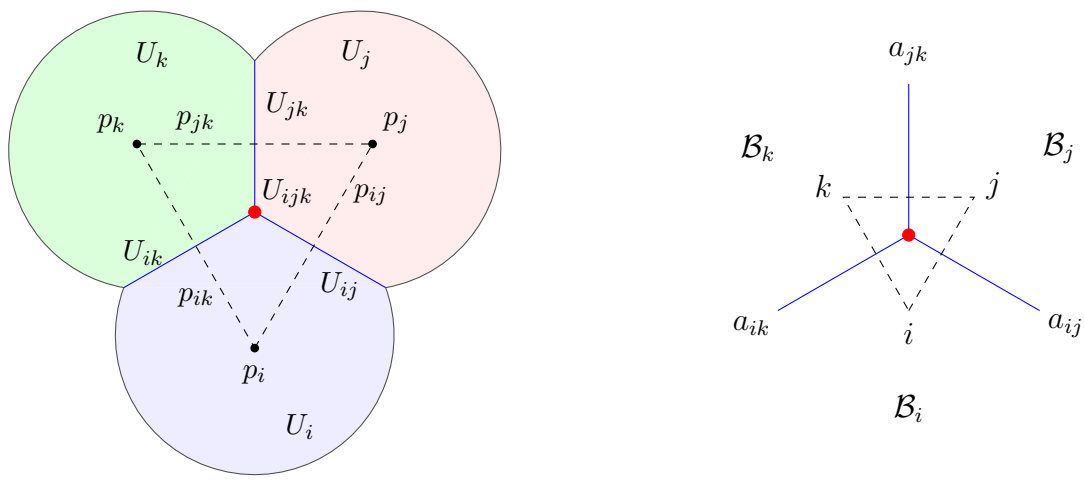


Figure 4.9: Left: representation of the simplicial triangulation and the covering by closed sets U_i, U_j, U_k near a triple intersection. Right: assignment of fields $\mathcal{B}_i$ and a_{ij}.

(with $i_1 < \ldots < i_k$ and $k = 2, \ldots, 6$) is a $(6-k)$-dimensional contractible manifold with boundary that intersects the $(k-1)$-simplex $p_{i_1 \ldots i_k}$ at one point. We give a graphical representation of this covering in Figure 4.9.

On every patch U_i we define gauge fields $\mathcal{B}_i$ with values in an Abelian group $\mathcal{A}$ (either continuous or discrete), and along the intersections U_{ij} we glue them using a group homomorphism $\theta : G \to (\mathcal{A})$ and the gauge field a:[40]

$$\mathcal{B}_i = \theta(a_{ij})\, \mathcal{B}_j \qquad\qquad \text{across } U_{ij}\ . \tag{4.1.153}$$

The gauge field $\mathcal{B}$ is thus a piecewise-smooth field with $\mathcal{B}\big|_{U_i} = \mathcal{B}_i$. Closeness of a guarantees that each $\mathcal{B}_i$ can be smooth and have a well-defined limit at triple intersections U_{ijk}. In particular, we can always find a gauge in which $a_{ij} = a_{jk} = a_{ik} = 1$ around a given triple intersection U_{ijk}, and in that gauge $\mathcal{B}$ can be smooth at the intersection.

The construction is quite general. In our case $\mathcal{B}_i$ are continuous 2-form gauge fields valued in $\mathcal{A} = U(1)^2$, while $SL(2,\mathbb{Z})$ has the natural action on $\mathcal{A}$ and $G \subset SL(2,\mathbb{Z})$ is an Abelian subgroup. We should now understand how to construct the action. Integrating $S = \frac{N}{4\pi}\int \langle \mathcal{B}, d\mathcal{B}\rangle$ with the discontinuous gluing conditions (4.1.153) leads to singularities, in particular the derivative $d\mathcal{B}$ has delta-function

[40]Besides, one could also have gauge transformations of $\mathcal{B}_i$, but we keep them implicit here.

singularities along the surfaces U_{ij}. To remedy, we introduce a covariant derivative d_a that removes those singularities:

$$d_a\mathcal{B} = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(\mathcal{B}_j - \mathcal{B}_i\big) = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\,\sigma(a_{ij})\,\mathcal{B}_j \;\equiv\; \Big(d - \delta_a^{(1)}\sigma(a)\Big)\mathcal{B}\;, \tag{4.1.154}$$

where $\delta^{(1)}(U_{ij})$ is a delta-1-form, $\sigma(a) \equiv \mathbb{1} - \theta(a)$, and in the last expression we used a more compact notation. In this way, $d\mathcal{B}$ is a piecewise-smooth field such that $d\mathcal{B}\big|_{\mathring{U}_i} = d\mathcal{B}_i$ with discontinuities across U_{ij} but no delta-function singularities. We can then construct the action

$$S = \frac{N}{4\pi}\int \langle \mathcal{B}, d_a\mathcal{B}\rangle = \sum_{U_i} \frac{N}{4\pi}\int_{U_i} \langle \mathcal{B}_i, d\mathcal{B}_i\rangle\;. \tag{4.1.155}$$

The covariant derivative d_a can be integrated by parts, and the action is invariant under gauge transformations of a.

In order to discuss 1-form gauge transformations, we need to compute the square d_a^2 of the covariant derivative. It turns out that, to do that, we ought to be more careful and write $d_a\mathcal{B} = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(\mathcal{B}_j^{(ij)} - \mathcal{B}_i^{(ij)}\big)$ where the label (ij) reminds us that we are taking the limit of $\mathcal{B}_i$ or $\mathcal{B}_j$ towards U_{ij}. Then $d_a d\mathcal{B} = -\sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(d\mathcal{B}_j^{(ij)} - d\mathcal{B}_i^{(ij)}\big)$, and finally

$$\begin{aligned} d_a^2\,\mathcal{B} &= -\sum_{U_{ijk}} \delta^{(2)}(U_{ijk})\Big[\big(\mathcal{B}_j^{(ij)} - \mathcal{B}_i^{(ij)}\big) + \big(\mathcal{B}_k^{(jk)} - \mathcal{B}_j^{(jk)}\big) - \big(\mathcal{B}_k^{(ik)} - \mathcal{B}_i^{(ik)}\big)\Big] \\ &\equiv -\sum_{U_{ijk}} \delta^{(2)}(U_{ijk})\,\sigma(da_{ijk})\,\mathcal{B}\;. \end{aligned} \tag{4.1.156}$$

In the first equality we used that $d\big(\delta^{(1)}(U_{ij})\big) = \delta^{(2)}(\partial U_{ij})$ and that the boundary of a double intersection is a collection of triple intersections (with suitable signs due to orientations). In the second line we introduced a compact notation. Indeed, if a is closed ($da = \mathbb{1}$) then each $\mathcal{B}_i$ can be smooth and taking the limit towards U_{ijk} in each patch, the first line of (4.1.156) equals zero. If, instead, a is not closed, then the G bundle can have non-trivial holonomies around the triple intersections and the $\mathcal{B}_i$'s cannot be smooth there. Given $(da)_{ijk} = g$, consider a gauge in which $a_{ij} = a_{jk} = 1$, $a_{ik} = g^{-1}$ (see Figure 4.10 right). Then the contribution to (4.1.156) from U_{ijk} becomes $-\delta^{(2)}(U_{ijk})\,\sigma(da_{ijk})\,\mathcal{B}_i^{(ik)}$. We thus write the compact formula

$$d_a^2 = -\delta_a^{(2)}\,\sigma(da)\;. \tag{4.1.157}$$

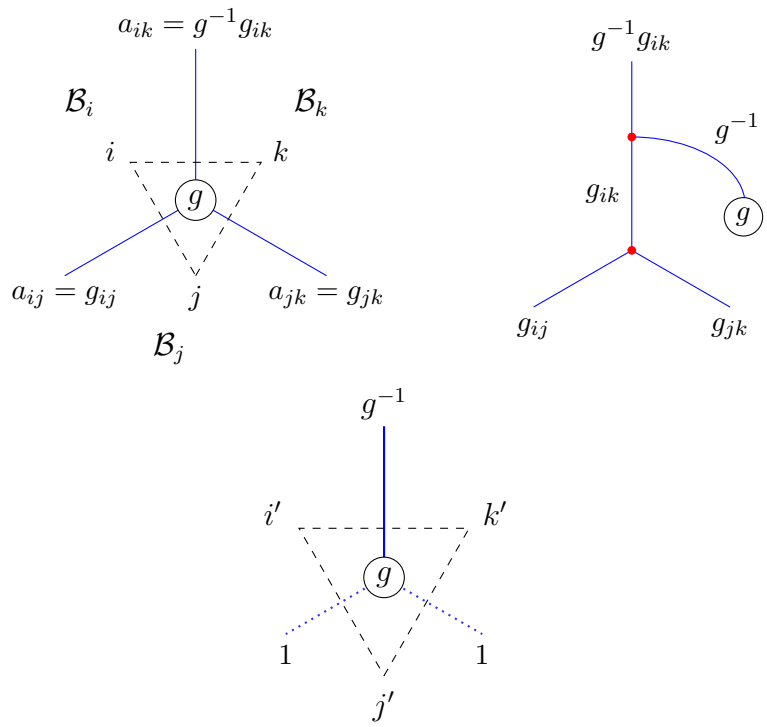


Figure 4.10: Left: A triple intersection U_{ijk} hosts a GW operator for $g \in G$. We parametrized the gauge field a in terms of $g_{ij}g_{jk} = g_{ik}$ and $da_{ijk} = g$. Center: A refinement of the triangulation such that the GW operator is pulled away from the junctions. Right: A zoom on the gauge field configuration around the isolated GW operator.

In the presence of a background for a, 1-form gauge transformations of $\mathcal{B}$ become

$$\mathcal{B} \to \mathcal{B} + d_a\alpha \ , \tag{4.1.158}$$

and the action (4.1.155) remains gauge invariant as long as a is flat.

The theory in which G is gauged involves a sum over choices of a on double intersections U_{ij} that satisfy the closeness condition $da = \mathbb{1}$. A single symmetry defect $U(\gamma)$ in the ungauged theory is mapped to a sum over its G-orbit in the gauged theory. These are precisely the $[e, m]$ defect operators we introduced before.

On the other hand, we can introduce Gukow-Witten operators in the gauged theory [130]. These are codimension-2 disorder operators defined by a nontrivial holonomy $g \in G$ for a around a 3d submanifold γ'. In the hybrid formulation, such a GW operator displaced along a collection of triple intersections U_{ijk} is defined by

a sum in the path integral over cochains a such that

$$da_{ijk} = g \qquad \text{whenever } U_{ijk} \subset \gamma' \ , \tag{4.1.159}$$

as in Figure 4.10. More generally, a collection of GW operators is described by an exact cochain $h \in C^2(M_5, G)$, and it prescribes to sum over cochains a with $da = h$ in the gauged theory. As mentioned above, requiring the $\mathcal{B}_i$'s to be smooth in their own patches in a neighborhood of a triple intersection, forces them to be invariant under g there.[41] This is a boundary condition naturally implemented on the GW operators, consistent with the fact that g-twisted sectors absorb the surfaces of $\mathcal{B}$ not stabilized by g.

Indeed, we can identify a double intersection U_{ij} with gauge field a_{ij} as an alternative description for the 4d symmetry defect V_M with $M = \theta(a_{ij})^{\mathsf{T}}$. This is already apparent if we compare the relation $\mathcal{B}_i = \theta(a_{ij})\,\mathcal{B}_j$ between the fields on the two sides of the intersection and (4.1.60), but it also follows from the action. Let us rewrite (4.1.155) as

$$S = \frac{N}{4\pi}\int \langle \mathcal{B}, d\mathcal{B}\rangle - \frac{N}{4\pi}\sum_{U_{ij}}\int_{U_{ij}} \langle \mathcal{B}, \mathcal{B}_j - \mathcal{B}_i\rangle \ . \tag{4.1.160}$$

The first term imposes that $\mathcal{B}$ is a $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge field. When $\mathcal{T}(M)$ is invertible, we can identify the second term with the reduced defect action (4.1.58). Recall from the discussion in Section 4.1.3 that the relation between M and the torsion matrix $\mathcal{T}$ follows from determining the field on the defect $\mathcal{B}(0) = \frac{1}{2}(\mathcal{B}_{\mathrm{L}} + \mathcal{B}_{\mathrm{R}}) = -\mathcal{T}\Phi$, where $\epsilon\,\Phi = \mathcal{B}_{\mathrm{R}} - \mathcal{B}_{\mathrm{L}}$, in terms of the left/right fields $\mathcal{B}_{\mathrm{L/R}}$. Substituting $\mathcal{B}_{\mathrm{R}} - \mathcal{B}_{\mathrm{L}} = -\epsilon\,\mathcal{T}^{-1}\mathcal{B}(0)$ into (4.1.160), the second term becomes

$$-\frac{N}{4\pi}\sum_{U_{ij}}\int_{U_{ij}} \mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B} \tag{4.1.161}$$

that reproduces (4.1.58).

To compute how a GW operator transforms under gauge transformations (4.1.158) we simply evaluate the variation of the action (4.1.155) on a non-closed gauge configuration as in (4.1.159):

$$\delta_\alpha S = -\sum_{U_{ijk}} \frac{N}{4\pi}\int_{U_{ijk}} \big\langle 2\mathcal{B} + d_a\alpha,\, \sigma(da_{ijk})\,\alpha\big\rangle \ . \tag{4.1.162}$$

[41] If $\mathcal{B}_i$ is smooth at U_{ijk}, then it has a well-define limit there. The limits in the three patches U_i, U_j, U_k are related by $\mathcal{B}_i = \theta(a_{ij})\,\mathcal{B}_j = \theta(a_{ik})\,\mathcal{B}_k$ and $\mathcal{B}_j = \theta(a_{jk})\,\mathcal{B}_k$. Recalling that G is Abelian, this implies $\theta(da_{ijk})\,\mathcal{B}_i = \mathcal{B}_i$ and similarly for $\mathcal{B}_j$ and $\mathcal{B}_k$.

In the gauge of Figure 4.10 right, as above, $\sigma(da_{ijk})\,\alpha = \alpha_i^{(ik)} - \alpha_k^{(ik)} = \epsilon\,\mathcal{T}^{-1}\alpha(0)$ in terms of the gauge transformation parameter on the defect. Substituting in (4.1.162) and using that the boundary conditions fix $\mathcal{B} = 0$ on the GW operator, we obtain

$$\delta_\alpha S = \sum_{U_{ijk}} \frac{N}{4\pi} \int_{U_{ijk}} \alpha^{\mathsf{T}} \mathcal{T}^{-1} d\alpha \ . \tag{4.1.163}$$

As in the description of Section 4.1.4 in terms of symmetry defects V_M, also in the hybrid formulation we find that pure GW operators are not gauge invariant in this theory. We can construct gauge-invariant operators by dressing the GW operators with the twisted sectors $D[\mathcal{T}]$, whose variation (4.1.83) is opposite to (4.1.163).

In the case of the symmetry defect V_C, the field on the defect is simply $\mathcal{B}(0) = 0$ and thus its gauge transformation parameter $\alpha(0)$ vanishes as well. This means that the gauge variation (4.1.162) vanishes and the GW operator for C is a well-defined gauge-invariant (invertible) topological operator in the gauged theory.

Gapped boundaries and non-invertible fusion rules

We consider now gapped boundaries in the gauged theory. We can use to our advantage the study and classification we already did in the ungauged theory. In order to construct a gauged boundary $|\rho^*\rangle$, we proceed in two steps. First we take a boundary $|\rho\rangle$ in the ungauged theory and make it invariant under the G action:

$$|\rho\rangle \quad \to \quad \frac{1}{\big|\mathrm{Stab}(\rho)\big|} \sum_{g\in G} |\rho_g\rangle \ . \tag{4.1.164}$$

As long as G is Abelian, we can associate a stabilizer $H \subset G$ in a consistent way also to the gauged boundary $|\rho^*\rangle$, since $\mathrm{Stab}(\rho_g) = \mathrm{Stab}(\rho)$. This does not specify a boundary condition completely, since it does not prescribe boundary conditions for neither the $\mathrm{Rep}(G)$ dual symmetry lines, nor the codimension-2 defects $\mathfrak{D}[\mathcal{T}]$. They form a canonically-conjugated pair of variables, since they braid nontrivially. Therefore, the second step is to choose boundary conditions for them. We choose to impose Dirichlet boundary conditions on a:[42]

$$|\mathrm{Dir}\rangle : \quad a = 0 \ . \tag{4.1.165}$$

Then the operators $\mathfrak{D}[\mathcal{T}]$ still exist on the gapped boundary as confined excitations.

[42]This is the same choice made in the holographic setup of [18].

These are not the only meaningful boundary conditions one could consider. Indeed it would be interesting to understand the effect of Dirichlet boundary conditions on the $\mathfrak{D}[\mathcal{T}]$'s, or of mixed ones. That they might be useful to describe theories in which either charge conjugation C (this has been studied, *e.g.*, in [24, 246, 247]) or the full categorical symmetry, are gauged. We hope to come back to these questions in the future.[43]

With Dirichlet boundary conditions on a, we define:

$$|\rho^*\rangle = \frac{1}{\left|\mathrm{Stab}(\rho)\right|}\left(\sum_{g\in G}|\rho_g\rangle\right)\times|\mathrm{Dir}\rangle\ . \tag{4.1.167}$$

The Dirichlet boundary condition on a greatly simplifies the discussion. The operators $\mathfrak{D}[\mathcal{T}]$, which away from the boundary host a $\mathrm{Rep}(G)$ worth of lines constructed with the gauge field a, on the gapped boundary reduce to a direct product $\mathrm{GW}_{M(\mathcal{T})}\times D[\mathcal{T}]$. The Gukov-Witten operators still exist on $|\rho^*\rangle$ and have group-like fusion. We will now show that the fusion of the twist operators $D_{\mathcal{L}}[\mathcal{T}]$ is the same on each gapped boundary $|\rho_g\rangle$ in the $|\rho^*\rangle$ orbit. This allows to use the results already derived for the boundary fusion.

We need to show that the various minimal theories we constructed in Section 4.1.4 in order to study the fusion of twist defects, are isomorphic for boundaries in the same G-orbit. Let $M_g\in G\subset SL(2,\mathbb{Z}_N)$ be a generator of G, and $M(\mathcal{T})$ be the element of G associated to the twist defect $D[\mathcal{T}]$ we want to study. Let $\left|\rho(\mathcal{L})\right\rangle$ be a gapped boundary defined by the Lagrangian subgroup $\mathcal{L}$ with generator l. The Lagrangian subgroup of $|\rho_g\rangle$ is $M_g\mathcal{L}$, and since $\mathcal{L}^{\mathsf{T}}\mathcal{L}_\perp=0$, we have

$$\mathcal{L}_\perp^g = M_g^{-1\mathsf{T}}\,\mathcal{L}_\perp\ . \tag{4.1.168}$$

The generators l and $l_\perp$ transform in a similar way. Since G is Abelian and $\epsilon M_g = M_g^{-1\mathsf{T}}\epsilon$, then $\mathcal{T}=M_g^{\mathsf{T}}\mathcal{T}M_g$ and so both t_l and $t_\perp$ are invariant along the orbit.

[43]In the same spirit, we could consider boundaries twisted by the dual $\check{G}$ symmetry. This amounts to choosing a representation α of G and define, for a boundary with a trivial stabilizer,

$$|\rho_\alpha^*\rangle = \sum\nolimits_{g\in G}\chi_\alpha(g)\,|\rho_g\rangle\times|\mathrm{Dir}\rangle\ . \tag{4.1.166}$$

These boundaries have vanishing overlap with the relative theory if we assume absolute theories in the same orbit to have the same partition function. When a stabilizer is present we can only twist by characters of $G/\mathrm{Stab}(\rho)$, while boundaries split into copies labelled by representations of $\mathrm{Stab}(\rho)$. We do not know how to interpret these splitted boundaries from the point of view of the 4d QFT, thus we only consider the ones labelled by the trivial representation.

Besides, $\Gamma M_g = M_g \Gamma$ and thus the theory $\mathcal{R}_{21}$ is invariant as well. Since all relevant building blocks are isomorphic on boundaries that sit inside the same G-orbit, we conclude that fusion only depends on the orbit $|\rho^*\rangle$.

A new ingredient appears when fusion produces a defect $\mathcal{D}[\mathcal{T}_{21}]$ such that $M(\mathcal{T}_{21})$ stabilizes $|\rho\rangle$. As we discussed, in these cases the minimal theory is replaced by a condensate. After gauging G, we are left with the GW operator $\mathrm{GW}_{M(\mathcal{T}_{21})}$.

Using all of the above, we finally obtain the categorical fusion rules in the boundary theory specified by the gapped boundary $|\rho^*\rangle$:

$$\begin{aligned} \mathfrak{D}_{\rho^*}[\mathcal{T}_2] \times \mathfrak{D}_{\rho^*}[\mathcal{T}_1] &= \mathcal{N}_{21}\, \mathfrak{D}_{\rho^*}[\mathcal{T}_{21}] && M_{21} \notin \mathrm{Stab}(\rho^*)\ , \\ \mathfrak{D}_{\rho^*}[\mathcal{T}_2] \times \mathfrak{D}_{\rho^*}[\mathcal{T}_1] &= \mathcal{C}^{\mathbb{Z}_N}\ \mathrm{GW}_{M(\mathcal{T}_{21})} && M_{21} \in \mathrm{Stab}(\rho^*)\ , \\ \mathrm{GW}_{M(\mathcal{T}_2)} \times \mathrm{GW}_{M(\mathcal{T}_1)} &= \mathrm{GW}_{M(\mathcal{T}_{21})} && M_1, M_2 \in \mathrm{Stab}(\rho^*)\ . \end{aligned} \tag{4.1.169}$$

In the second line, the condensate is for the 1-form symmetry $\mathcal{S}$ on the gapped boundary, and the DW description couples to $\tilde{b}_\perp$. For defects in which only one $\mathbb{Z}_N$ factor is gauged, on the other hand, the fusions are as follows:[44]

$$\begin{aligned} \mathfrak{D}_{T^k,\rho^*} \times \mathfrak{D}_{T^{k'},\rho^*} &= \mathcal{A}^{N,\,k^{-1}+k'^{-1}}\ \mathfrak{D}_{T^{k+k'},\rho^*} && T \notin \mathrm{Stab}(\rho^*)\ , \\ \mathrm{GW}_{T^k} \times \mathrm{GW}_{T^k} &= \mathrm{GW}_{T^{k+k'}} && T \in \mathrm{Stab}(\rho^*)\ . \end{aligned} \tag{4.1.170}$$

The same can be said for conjugacy classes $g = \mathcal{H}^{-1} T \mathcal{H}$.

4.1.6 Conclusions and future directions

In this section we have studied how non-invertible self-duality symmetries arise in holography, through the presence in the gravitational bulk of emergent discrete gauge fields at self-dual points on the moduli space. Although we have focused on the specific example of the 4d $\mathcal{N} = 4$ super-Yang-Mills theory with gauge algebra $\mathfrak{su}(N)$, our methods are rather general and should be applicable to a wide range of other theories, for instance to $\mathcal{N} = 2$ theories of class S. This will be the focus of the next section.

The key role is played by a topological low-energy sector of type IIB string theory on S^5: a 5d Chern-Simons-like topological field theory of 2-form gauge fields — equivalent to a $\mathbb{Z}_N$ discrete 2-form gauge theory — further orbifolded by a discrete Abelian symmetry G. It is essentially the symmetry TFT for $SU(N)$ $\mathcal{N} = 4$ SYM. This theory appears to be quite interesting and rich in its own right, both before and

[44] These can be thought of as the case of T^k modulo conjugation.

after gauging G. We have studied various aspects of the theory. Before orbifolding, we have analyzed the 4d symmetry defects associated to an $SL(2,\mathbb{Z}_N)$ 0-form global symmetry of the theory, the associated twisted sectors that live at the boundary of the defects, and their fusion. We have also investigated topological (gapped) boundaries, and how the various defects reduce when they are brought there. Then, we have studied the effect of gauging a subgroup $G \subset SL(2,\mathbb{Z}_N)$. In particular, the "liberated" twist defects $\mathfrak{D}_{g,\rho^*}$ that live on gapped boundaries turn out to be the self-duality defects of $\mathcal{N} = 4$ SYM. We derived their fusion rules using our formalism, confirming the results previously obtained in field theory.

We conclude listing a few open questions for future research.

The case of N not prime. For the sake of simplicity, we have restricted our analyses to the case of N prime throughout our paper. This technical assumption allowed us to exploit the multiplicative group structure of $\mathbb{Z}_N^*$, simplifying many formulas. When N is not prime, the situation is technically more complicated, both because the number of subgroups and global structures grows with the number of prime factors in N, and because the fusion relations for minimal theories become more involved.

General formulation of the symmetry TFT. In the last part of our work, we have resorted to a hybrid formulation of the gauged TQFT that uses both discrete and continuous gauge fields. It would be pleasant to give a completely general description in terms of the correct cohomology theory. A promising route could be to employ Deligne-Beilinson twisted cocycles.

Anomalies for N-ality symmetries. In spite of the many recent developments, a clear understanding of 't Hooft anomalies for non-invertible symmetries in $d > 2$ is still lackluster. The main obstacle is to give a concrete implementation of the associativity conditions for n-categories. The higher-dimensional TQFT approach might help to give an alternative concrete route to such questions: instead of choosing Dirichlet boundary conditions for the discrete 1-form gauge fields a, one might try to define Neumann boundaries instead. On these, the non-invertible defects $\mathfrak{D}[\mathcal{T}]$ are effectively gauged and they define an absolute theory which is obtained from the ones we have studied here by gauging the non-invertible symmetry. The failure to find such a boundary would signal an 't Hooft anomaly. We implement this idea in [126].

4.2 Non-invertible duality defects in class S theories

4.2.1 Introduction

Another very natural setting in which non-invertible defects may appear are $4d$ $\mathcal{N}=2$ theories of class $\mathcal{S}$ [225], as noted in [30]. These theories are obtained as the dimensional reduction of a 6d $\mathcal{N}=(2,0)$ SCFT on a Riemann surface Σ_g[45], and have a conformal manifold (*i.e.* a space of exactly marginal deformations) equal to the moduli space of complex structures of the Riemann surface, whose point we denote generically by Ω. Moreover, the theories have a large one-form symmetry group, and their global forms (*i.e.* the set of their genuine line operators) are classified by Lagrangian lattices $\mathcal{L}$ inside $H_1(\Sigma_g, \mathbb{Z}_N)$ [97, 248, 249] and also an extended duality group $\mathrm{MCG}(\Sigma_g)$ given by the group of large diffeomorphisms of the underlying Riemann surface [225]. The classification of non-invertible duality defects for these theories can be done in three steps:

1. Find Riemann surfaces Σ_g with a nontrivial automorphism group $G(\Omega) \subset \mathrm{MCG}(\Sigma_g)$. These will be the self-dual loci for the class $\mathcal{S}$ theory. This problem has been solved for high enough genus in the mathematical literature [250].

2. Understand the action of the duality group $G(\Omega)$ on global variants, which are Lagrangian lattices in $H_1(\Sigma_g, \mathbb{Z}_N)$.

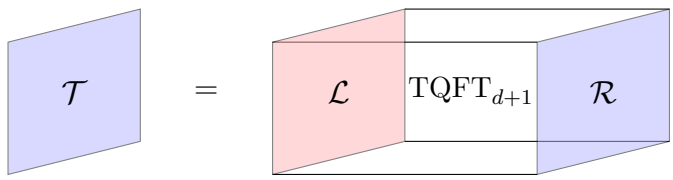
3. Study the action of discrete gauging operations Φ on half-space. These turn out to form a central extension of $\mathrm{Sp}(2g, \mathbb{Z}_N)$, which we dub $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$. These operations have already been studied in detail in [25, 26] for the case of $\mathbb{Z}_N$ one-form symmetry. We generalize and streamline their construction.

We construct a duality defect $\mathfrak{D}^M_{\mathcal{L}}$ composing the duality action $M \in \mathrm{Sp}(2g, \mathbb{Z}_N)$ with an appropriate topological manipulation $\Phi^M_{\mathcal{L}}$ which restores the initial duality frame choice. We can then compute the full set of fusion rules and the action on the line operators of the theory. Interestingly we find that from the action of $\mathfrak{D}^M_{\mathcal{L}}$ on lines we can define a property called *rank* which almost fixes the structure of the fusion algebra. The rank can be understood physically as the fact that while

[45] In this work we only focus on theories of type A_{N-1} in the absence of punctures. We expect many of our results to extend to the case of regular punctures, while we have nothing definite to say about the irregular ones. Furthermore for technical reasons we assume N to be a prime number.

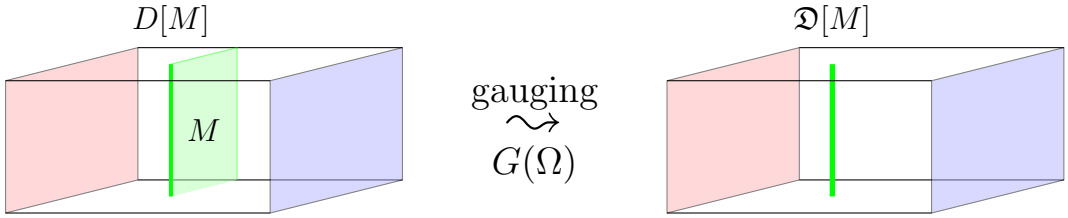
the (non-invertible) duality defects acts on the genuine lines by mapping them to non-genuine ones, there can be a subset of them on which the symmetry acts as a standard automorphism.

A second, complementary approach is to study a 5d Symmetry TFT for the topological operators in our theory. In this approach the SCFT $\mathcal{T}$ is expanded into a topological $d+1$-dimensional slab



(4.2.1)

with topological boundary conditions $\mathcal{L}$ on the farther end specifying the global structure of $\mathcal{T}$. The bulk TQFT can be constructed explicitly when an holographic dual of $\mathcal{T}$ is known. The symmetry TFT for duality defects in $\mathcal{N}=4$ SYM has been recently derived in [43, 251] and it was discussed in the previous section. The self-duality symmetry has a simple interpretation in terms of topological twist defects $D[M]$ for a bulk zero-form symmetry $G(\Omega)$. At special points in the gravitational moduli space, which correspond to self-dual SCFTs, the Symmetry TFT must be modified by gauging a subgroup of the zero-form symmetry. The twisted sectors then become liberated codimension-2 operators $\mathfrak{D}[M]$ which, when placed at the boundary, give rise to the codimension-1 duality defects of the SCFT:



(4.2.2)

For class $\mathcal{S}$ theories the gauged zero-form symmetry corresponds the a subgroup $G(\Omega)$ of the large diffeomorphisms of Σ_g which is un-Higgsed at low energies. Thanks to this description it is possible to compute the fusion algebra for the non-invertible duality defects by carefully examining the composition laws for $D[M]$. We explicitly construct the Symmetry TFT and use this approach to confirm our previous results, thus also providing a highly non-trivial check for the holographic proposal of [43].

The structure of the fusion rules is rather simple to describe. The duality defects compose in a group-like manner (*i.e.* the fusion is graded by $G(\Omega)$) and the categorical structure shows up as either decoupled TQFT "coefficients" $\mathcal{N}$ or condensation

defects $\mathcal{C}$:

$$\mathfrak{D}_{\mathcal{L}}^{M_1} \times \mathfrak{D}_{\mathcal{L}}^{M_2} = \mathcal{N}^{(1,2)}\, \mathcal{C}^{\mathcal{A}_{1,2}}\, \mathfrak{D}_{\mathcal{L}}^{M_1\, M_2}. \tag{4.2.3}$$

The TQFT coefficient can be chosen as decoupled minimal $\mathcal{A}^{N,\mathcal{N}^{(1,2)}}$ TQFTs [67] modulo congruence. For a given prime N there are omly two possible inequivalent $\mathcal{N}^{(1,2)}$. The condensation instead refers to the higher gauging of a subgroup $\mathcal{A}$ of the $(\mathbb{Z}_N)^g$ one-form symmetry. The appearance of both of this structures follows from some rather simple observations regarding the rank of the defects participating in the fusion process. Interestingly, when the group is non-abelian, the categorical data is also non-commutative, in the sense that $\mathfrak{D}_{\mathcal{L}}^{M_1} \times \mathfrak{D}_{\mathcal{L}}^{M_2}$ and $\mathfrak{D}_{\mathcal{L}}^{M_2} \times \mathfrak{D}_{\mathcal{L}}^{M_1}$ can display different categorical structures. They are however consistent with associativity, albeit in a nontrivial way.

This section is organized as follows: in Section 4.2.2 we discuss in detail the classification of different global forms for the class $\mathcal{S}$ theories we will study. In Section 4.2.3 we give a precise definition of the non-invertible duality defects and describe in detail the algebra of discrete topological manipulations Φ, its action on global variants and the way in which it can be used to extract the fusion rules. In Section 4.2.4 we introduce the concept of *rank* of a non-invertible duality defect, we describe how to compute it and how it can be used to (almost) fix the form of the fusion algebra. In Section 4.2.5 we give an alternative method to extract such data from a 5d TQFT description, following the analysis of [43]. In Section 4.2.6 we give some explicit applications of our methods to low genus cases. We conclude in Section 4.2.7 with open questions and prospects for future investigations. Various technical details and tables of some fusion rules can be found in the Appendices.

4.2.2 Global variants and Lagrangian lattices

We consider 4d gauge theories with semi-simple gauge algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the Langlands dual algebra, which is isomorphic to $\mathfrak{g}$ in the simply-laced cases. If we denote by $\widetilde{G}$ and $\widetilde{G}^*$ the simply connected groups with algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively, in absence of charged matter the full set of line operators is labelled by a lattice $\Gamma = Z(\widetilde{G}) \times Z(\widetilde{G}^*)$, which comes with a natural non-degenerate antisymmetric pairing $\langle\, ,\, \rangle$. $Z(\widetilde{G})$ and $Z(\widetilde{G}^*)$ are isomorphic, and label respectively electric and magnetic charges. Γ includes mutually non-local operators and therefore, as pointed out in [59, 62], the set of genuine line operators $W_{l\in\mathcal{L}}$ of the theory is specified by

the choice of a maximal isotropic (i.e. Lagrangian) sublattice $\mathcal{L} \subset \Gamma$. The one-form symmetry instead is identified with

$$\mathcal{S} = \Gamma / \mathcal{L} \ . \tag{4.2.4}$$

$\mathcal{S}$ can also be understood as the set labelling the non-genuine line operators $T_{s \in \mathcal{S}}$, which live in the twisted sectors of the one-form symmetry. The reason why $\mathcal{S}$ is a quotient is that by adding a genuine line to a non-genuine one, the resulting line remains in the same twisted sector. We will focus on the case in which $Z(\widetilde{G})$ does not have non-trivial proper subgroups. Then the exact sequence $1 \to \mathcal{L} \to \Gamma \to \mathcal{S} \to 1$ splits, so it is always possible to choose a representative of $\mathcal{S}$ (by abuse of notation we call it $\mathcal{S}$ itself) which is also Lagrangian, and such that

$$\Gamma = \mathcal{L} \oplus \mathcal{S} \ . \tag{4.2.5}$$

Since we will study class $\mathcal{S}$ theories, we will be mostly interested in theories with charged matter which can partially screen the line operators. In this case $Z(\widetilde{G}) \cong Z(\widetilde{G}^*)$ is replaced by a quotient $Z(\widetilde{G})/\Lambda$, where $\Lambda \subset Z(\widetilde{G})$ is the subgroup of charges screened by matter. Consider a class $\mathcal{S}$ theory of type $\mathfrak{g}$, obtained by compactifying the 6d $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$ on a genus g Riemann surface Σ_g without punctures. The weakly coupled corners of the conformal manifold where we have Lagrangian descriptions correspond to giving a pair of pants decomposition in terms of $2g-2$ three-punctured spheres glued by $3g-3$ very long tubes. The Lagrangian is written in terms of $3g-3$ $\mathcal{N}=2$ vector multiples of $\mathfrak{g}$ coupled to $2g-2$ copies of the $T_{\mathfrak{g}}$ theory, namely tri-fundamental hypermultiplets, corresponding to the three-punctured spheres [225]. Denote by $\widetilde{G}_0$ the simply connected group with algebra $\mathfrak{g}$, so that $\widetilde{G} = \widetilde{G}_0^{3g-3}$. Each tri-fundamental is charged with respect to the diagonal $Z(\widetilde{G}_0)$ of the three vector multiplets coupled to it. Since each vector multiplet is coupled to exactly two $T_{\mathfrak{g}}$ and Σ_g has no puncture, the diagonal of all the $Z(\widetilde{G}_0)$ charges of the hypermultiplets is not acted upon by the center symmetry. This means that Λ has co-dimension one in $Z(\widetilde{G}_0)^{2g-2}$, and the set of unscreened electric charges is $Z(\widetilde{G}_0)^g$.

The bottom line is that the classifying lattice for global variants of the class $\mathcal{S}$ theory is $\Gamma = Z(\widetilde{G}_0)^{2g}$, which coincides with $H^1(\Sigma_g, Z(\widetilde{G}_0))$ [97, 248].

In the case of $\mathfrak{g} = A_{N-1}$, we have $Z(\widetilde{G}_0) = \mathbb{Z}_N$ and $\Gamma = (\mathbb{Z}_N)^{2g}$. The pairing on Γ is given by

$$\langle v, u \rangle = v^{\mathsf{T}} \mathcal{J} u \quad , \qquad \mathcal{J} = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix} \ . \tag{4.2.6}$$

In this paper we will restrict to the case where N is a prime number. Then a Lagrangian lattice $\mathcal{L}$ specifying a global variant corresponds to the choice of g linearly independent vectors $v_1, ..., v_g \in (\mathbb{Z}_N)^{2g}$ such that

$$v_i^{\mathsf{T}} \mathcal{J} v_j = 0 \quad , \quad \forall i, j = 1, ..., g \ . \tag{4.2.7}$$

We will label these lattices by $2g \times g$ matrices $\mathcal{L} = (v_1, ..., v_g)$ of rank g, which satisfy $\mathcal{L}^{\mathsf{T}} \mathcal{J} \mathcal{L} = 0$. One such matrix is

$$\mathcal{E} = \begin{pmatrix} \mathbb{1}_g \\ 0 \end{pmatrix} \tag{4.2.8}$$

and we will call the corresponding theory the *electric variant*. All the others are obtained by acting on $\mathcal{E}$ with matrices $M \in \mathrm{Sp}(2g, \mathbb{Z}_N)$. The action on $\mathrm{Sp}(2g, \mathbb{Z}_N)$ on Lagrangian lattices $\mathcal{L}$ is transitive, and the stabilizer is isomorphic to the group of symplectic matrices leaving $\mathcal{E}$ invariant up to a change of basis. These matrices are of the form

$$\begin{pmatrix} u & us \\ 0 & u^{\mathsf{T}^{-1}} \end{pmatrix} \ , \qquad u \in \mathrm{GL}(g, \mathbb{Z}_N) \ , \quad s^{\mathsf{T}} = s \ , \tag{4.2.9}$$

and generate the parabolic subgroup $\mathcal{P}(2g, \mathbb{Z}_N) \subset \mathrm{Sp}(2g, \mathbb{Z}_N)$. We conclude that the global variants are labelled by the right coset $\mathrm{Sp}(2g, \mathbb{Z}_N)/\mathcal{P}(2g, \mathbb{Z}_N)$, and therefore their number is[46]

$$N_{\text{global variants}} = \prod_{k=0}^{g-1} (N^{k+1} + 1) \ . \tag{4.2.10}$$

Note that for $g = 1$ we obtain $N + 1$, which is indeed the number of global variants of $\mathfrak{su}(N)$ YM theories for N prime, including the electric variant $\mathrm{SU}(N)$ and the N magnetic variants $(N)_r$, $r = 0, ..., N - 1$. In this case all other variants can be reached from the electric $\mathrm{SU}(N)$ variant by gauging the $\mathbb{Z}_N$ one-form symmetry with an appropriate discrete torsion [4, 157]. In order to extend this idea to generic g we rewrite (4.2.10) using the q-binomial theorem as

$$N_{\text{global variants}} = \sum_{k=0}^{g} \begin{pmatrix} g \\ k \end{pmatrix}_N N^{\frac{k(k+1)}{2}} \ . \tag{4.2.11}$$

Here we introduced the Gaussian binomial coefficient

$$\begin{pmatrix} g \\ k \end{pmatrix}_N = \frac{(1 - N^g)(1 - N^{g-1}) \cdots (1 - N^{g-k+1})}{(1 - N)(1 - N^2) \cdots (1 - N^k)} \tag{4.2.12}$$

[46] We use that $|\mathrm{Sp}(2g, \mathbb{Z}_N)| = N^{g^2} \prod_{k=1}^{g} (N^{2k} - 1)$ and $|\mathcal{P}(2g, \mathbb{Z}_N)| = N^{\frac{g(g+1)}{2}} \prod_{k=0}^{g-1} (N^g - N^k)$.

which also counts the number of $(\mathbb{Z}_N)^k$ subgroups of $(\mathbb{Z}_N)^g$. After this manipulation, equation (4.2.11) has a clear interpretation as the number inequivalent ways to gauge a $(\mathbb{Z}_N)^k$ subgroup of the $(\mathbb{Z}_N)^g$ one-form symmetry with possible discrete torsion, which for N prime is encoded in a $k \times k$ symmetric matrix.

Another convenient way to label the global variants, also used in [47], is to use $2g \times 2g$ symplectic matrices M instead, subject to the identification $M \sim MP$, $P \in \mathcal{P}(2g, \mathbb{Z}_N)$. Writing the symplectic matrices in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad A^{\mathsf{T}}C - C^{\mathsf{T}}A = B^{\mathsf{T}}D - D^{\mathsf{T}}B = 0 \, , \quad A^{\mathsf{T}}D - C^{\mathsf{T}}B = \mathbb{1}_g \tag{4.2.13}$$

the Lagrangian lattice labelling the global variant is

$$\mathcal{L} = \begin{pmatrix} A \\ C \end{pmatrix} \tag{4.2.14}$$

with the identification $A \sim Au$, $C \sim Cu$ where $u \in \mathrm{GL}(2g, \mathbb{Z}_N)$. By abuse of notation we will denote this right action on the $2g \times g$ matrix $\mathcal{L}$ as $\mathcal{L} \to \mathcal{L}P$. The condition $A^{\mathsf{T}}C = C^{\mathsf{T}}A$ is precisely the requirement $\mathcal{L}^{\mathsf{T}}\mathcal{J}\mathcal{L} = 0$ that the genuine lines are mutually local. This way of labelling the global variants also makes explicit the choice of representative for $\mathcal{S} = \Gamma/\mathcal{L}$. Indeed this is just

$$\mathcal{S} = \begin{pmatrix} B \\ D \end{pmatrix} . \tag{4.2.15}$$

The equation $B^{\mathsf{T}}D = D^{\mathsf{T}}B$ implies that $\mathcal{S}$ is also Lagrangian and the condition $A^{\mathsf{T}}D - C^{\mathsf{T}}B = \mathbb{1}_g$ is nothing but $\mathcal{L}^{\mathsf{T}}\mathcal{J}\mathcal{S} = \mathbb{1}_g$, namely the fact that the two have canonical pairing.

We will see shortly that there is a natural mapping between this parametrization and the inequivalent ways of gauging the one-form symmetry with a choice of discrete torsion. Roughly speaking C will encode the information about the choice of gauged subgroup, while the choice of discrete torsion is encoded in A.

4.2.3 Duality defects and discrete topological manipulations

In this Section we define generic duality defects in theories with a $(\mathbb{Z}_N)^g$ one-form symmetry and describe their composition properties. In theories of class $\mathcal{S}$ the duality group has a natural $\mathrm{Sp}(2g, \mathbb{Z}_N)$ action[47] on $H_1(\Sigma_g, \mathbb{Z}_N)$ which sends a Lagrangian

[47] Geometrically this follows from the short exact sequence

$$1 \to \mathrm{Tor} \to \mathrm{MCG}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1 \tag{4.2.16}$$

lattice $\mathcal{L}$ to

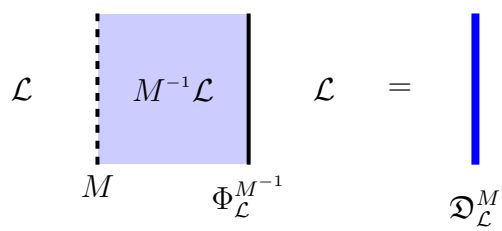
$$\mathcal{L} \to M\mathcal{L}\,, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}_N) \tag{4.2.17}$$

while acting on the complex structure matrix as $\Omega \to M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$.

As remarked in [26, 120] in the case of $g = 1$, the same action on global variants can also be realized by appropriately choosing a topological manipulation $\Phi_{\mathcal{L}}^{M}$. This corresponds to gauging a subgroup $\mathcal{A}$ of the one-form symmetry, possibly with discrete torsion. The space of topological manipulations forms a central extension of $\mathrm{Sp}(2g, \mathbb{Z}_N)$, which we denote $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$. Let us clarify that, despite our notation, the transformation $\Phi_{\mathcal{L}}^{M}$ is *not* associated to the element M in $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$. The group of topological manipulations instead acts on the right on matrices $\mathcal{L}$ of $\mathrm{Sp}(2g, \mathbb{Z}_N)/\mathcal{P}(2g, \mathbb{Z}_N)$ parametrizing the possible global structures. This assures that the duality action and the topological manipulations commute.

Given a point Ω on the conformal manifold $\mathcal{M}$ stabilized by a subgroup $G(\Omega) \subset \mathrm{MCG}(\Sigma_g)$ and a choice of global variant $\mathcal{L}$, we can define the duality defect $\mathfrak{D}_M^{\mathcal{L}}$, $M \in G(\Omega)$ by composition[48]

$$\mathfrak{D}_M^{\mathcal{L}} = M \circ \Phi_{\mathcal{L}}^{M^{-1}} \tag{4.2.18}$$

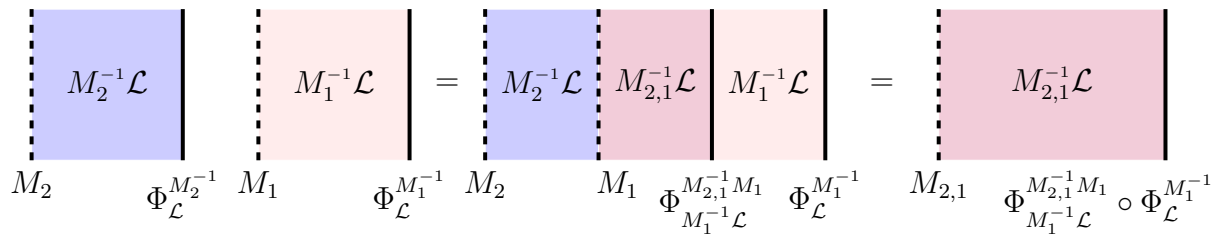


The fusion $\mathfrak{D}_{\mathcal{L}}^{M_2} \times \mathfrak{D}_{\mathcal{L}}^{M_1}$ between duality defects can be understood from the compositions laws for the topological manipulations $\Phi_{\mathcal{L}}^{M}$ on half space. After expanding both defects into slabs we slide the duality transformation M_1 across $\Phi_{\mathcal{L}}^{M_2^{-1}}$ as shown

where $\mathrm{MCG}(\Sigma_g)$ is mapping class group and Tor is the Torelli group.

[48] Our conventions are that defects act on operators on their right.

below:[49]



(4.2.19)

Since the left duality action and the right topological actions commute we have

$$\Phi_{M_1^{-1}\mathcal{L}}^{M_{2,1}^{-1}M_1} = \Phi_{\mathcal{L}}^{M_2^{-1}} . \tag{4.2.20}$$

We will now discuss the structure of $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$, starting with the example of $g = 1$ to then move onto the most general case.

Topological manipulations for $\mathbb{Z}_N$: $\mathrm{SL}(2, \mathbb{Z}_N)_T$

Let us review the structure of $\mathrm{SL}(2, \mathbb{Z}_N)_T$ [4,25,87], the space of topological manipulations for a $\mathbb{Z}_N$ symmetry. The generators are:

$$\begin{aligned}
\sigma : \quad & [\sigma Z](B) = \frac{1}{|H^2(X, \mathbb{Z}_N)|^{1/2}} \sum_{b \in H^2(X, \mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N} \int b \cup B\right) Z(b) , \\
\tau(k) : \quad & [\tau(k) Z](B) = \exp\left(\frac{2\pi i k}{2N} \int \mathfrak{P}(B)\right) Z(B) , \\
\nu(u) : \quad & [\nu(u) Z](B) = Z(uB) \quad u \in \mathbb{Z}_N^\times ,
\end{aligned} \tag{4.2.21}$$

$\mathfrak{P}$ being the Pontryagin square operation $\mathfrak{P} : H^2(X, \mathbb{Z}_N) \to H^4(X, \mathbb{Z}_{\gcd(2,N)N})$. Strictly speaking the discrete gauging σ maps the $\mathbb{Z}_N$ 1-form symmetry to its Pontryagin dual $(\mathbb{Z}_N)^* \cong \mathbb{Z}_N$. Since we want to perform successive gauging procedures we will always implicitly use this isomorphism. Throughout the rest of the paper we will often omit the overall normalization factor for the discrete gauging. These operations correspond to matrices $M \in \mathrm{SL}(2, \mathbb{Z}_N)_T$

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \tau(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} , \ k \in \mathbb{Z}_N \quad \nu(u) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} , \ u \in \mathbb{Z}_N^\times \tag{4.2.22}$$

[49] To avoid clutter we leave implicit the labelling $\mathcal{L}$ of the original theory. We also use the shorthand $M_{2,1} = M_2 M_1$.

acting on the matrix $(\mathcal{L} \mid \mathcal{S})$ on the right. The transformations $\tau(k)$ and $\nu(u)$ do not alter the global structure, *i.e.* they leave $\mathcal{L}$ invariant. $\nu(u)$ corresponds to a different choice of basis in the space of lines, while $\tau(k)$ amounts to a background discrete theta angle. Together they form the parabolic subgroup $\mathcal{P}(2, \mathbb{Z}_N)$ of $\mathrm{SL}(2, \mathbb{Z}_N)$. The algebra of these transformations can be computed straightforwardly. The most interesting relation, which we call the *"K-formula"* (see appendix 4.3.5 for a derivation), follows from considering a two-fold gauging process and reads

$$\sigma\ \tau(k)\ \sigma = Y_k\ \nu(-k^{-1})\ \tau(-k)\ \sigma\ \tau(-k^{-1})\,. \tag{4.2.23}$$

where

$$Y_k = \sum_{b \in H^2(X, \mathbb{Z}_N)} \exp\left(\frac{2\pi i k}{2N} \mathfrak{P}(b)\right) \tag{4.2.24}$$

is an invertible $4d$ two-form gauge theory [67]. If instead we have no intermediate torsion

$$\sigma\ \sigma = \nu(-1)\,. \tag{4.2.25}$$

Thus, whenever we have subsequent σ insertions, we can use the K-formula to reduce their number. Repeating this process we can bring every element of $\Phi \in \mathrm{SL}(2, \mathbb{Z}_N)_T$ into the *Standard form*:

$$\Phi = P(u, s)\ \sigma\ \tau(k)\,, \quad P(u, s) = \nu(u)\ \tau(s)\,. \tag{4.2.26}$$

Henceforth topological manipulations are always assumed to be in the standard form.

Action on global variants Let us briefly discuss the action of $\mathrm{SL}(2, \mathbb{Z}_N)_T$ on global variants. This will be used to set up a precise dictionary between the matrix $\mathcal{L}$ and the discrete operations Φ. Let

$$\Phi_k = \sigma\ \tau(k) \tag{4.2.27}$$

and $\mathcal{L}$ be the chosen variant for our 4d gauge theory. This has both genuine lines $W_l\,, l \in \mathcal{L}$ and twisted sector lines $T_s\,, s \in \mathcal{S}$ which are attached to open one-form symmetry surfaces U_s. Our aim is to understand the spectrum of genuine lines after applying Φ_k. This amounts to classify which lines of the ungauged theory are invariant under background $\mathbb{Z}_N$ gauge transformations

$$B \to B + \delta\lambda\,, \tag{4.2.28}$$

in the presence of a background discrete theta angle $e^{\frac{2\pi i k}{2N}\mathfrak{P}(B)}$. Genuine lines W_l are charged under the one-form symmetry and they will pick up a phase $e^{\frac{2\pi i}{N} l \int \lambda} = e^{-\frac{2\pi i}{N} l \int \lambda \cup \mathrm{PD}(\gamma)}$. Due to the presence of the discrete theta angle non-genuine lines T_s are not invariant either. Their insertion on a curve γ corresponds to a background B which fulfills $\delta B = s\mathrm{PD}(\gamma)$. The discrete torsion term then fails to be gauge invariant by a phase $e^{\frac{2\pi i k}{N} s \int \lambda \cup \mathrm{PD}(\gamma)}$. Thus gauge invariant operators are generated by the dyonic line $D_{k,1} \equiv W_k\, T_1$: the generator of the new Lagrangian lattice $\Phi_k \mathcal{L}$ is $k\mathcal{L} + \mathcal{S}$.

Fusion rules To derive the fusion rules we analyze two subtleties involving the half-space composition in $\mathrm{SL}(2,\, \mathbb{Z}_N)_T$. The first comes from the invertible theories Y_k. While on a closed manifold their partition function is just a phase, on a manifold X^+ with boundary ∂X^+, the TQFT Y_k becomes the anomaly-inflow theory for a 3d TQFT [67]. This theory is not completely determined by the anomaly alone, however a minimal choice always exists and is given by the minimal TQFT $\mathcal{A}^{N,k}$.[50] The composite system $Y_k(X^+) \times \mathcal{A}^{N,k}(\partial X^+)$ is anomaly-free and well defined. Thus on half-space we should interpret the appearance of Y_k as an indicator of the presence of a decoupled 3d TQFT $\mathcal{A}^{N,k}$. This gives our first rule:

$$\boxed{Y_k \text{ on half space } X^+ = \text{decoupled TQFT } \mathcal{A}^{N,k} \text{ on } \partial X^+} \tag{4.2.29}$$

The second subtlety has to do with the appearance of condensates $\mathcal{C}^{\mathbb{Z}_N}$ for the one-form symmetry [23]. As explained in [25] condensates appear when two gauging operations compensate each other in half-space. For instance consider $\sigma\,\sigma\,\tau(k)$ on X^+

$$[\sigma\,\sigma\,\tau(k) Z]\,(B) = \sum_{b,\, c \in H^2(X^+, \mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N} \int b \cup (c + B) + \frac{k}{2}\mathfrak{P}(c)\right)\, Z(c)\,. \tag{4.2.30}$$

Naively we could integrate out b to enforce $c = -B$. However on half space this is no longer true: as in the previous case, the theory for the b field is inconsistent on its own in the presence of a boundary. To recover gauge invariance the minimal choice is to add a 3d $\mathbb{Z}_N$ gauge theory with DW twist $\alpha = -kN$. Its gauge field a will couple to the bulk one-form symmetry. Now the integral on half space can be performed, leaving behind a boundary term:

$$\mathcal{C}^{\mathbb{Z}_N} = \sum_{\gamma \in H_1(\partial X^+, \mathbb{Z}_N)} U(\gamma)\, \exp\left(\frac{2\pi i k}{2} \int \mathrm{PD}(\gamma)\, \beta\left(\mathrm{PD}(\gamma)\right)\right)\,, \quad \mathrm{PD}(\gamma) = a \tag{4.2.31}$$

[50] The sign of k depends on conventions for the orientation.

with β the Bockstein map: $H^1(Y,\mathbb{Z}_N) \to H^2(Y,\mathbb{Z}_N)$.[51] This is exactly the condensation defect. Thus the second rule is:

$$\boxed{\text{Un-gauging } \mathbb{Z}_N \text{ on half space } X^+ = \text{condensation defect } \mathcal{C}^{\mathbb{Z}_N} \text{ on } \partial X^+} \tag{4.2.32}$$

Example: Triality defects We give a sample computation for the fusion of defects $D_{\mathcal{L}}^{M_1} \times D_{\mathcal{L}}^{M_2}$. We start by determining the topological manipulations $\Phi_{\mathcal{L}}^{M^{-1}}$ mapping $\mathcal{L}$ to $M^{-1}\mathcal{L}$. This can be done by parametrizing $\mathcal{L} = \sigma\ \tau(k_{\mathcal{L}})\ \mathcal{E}$, for some known $k_{\mathcal{L}}$. We first compose with $\tau(-k_{\mathcal{L}})\ \sigma^{-1}$ to reach $\mathcal{E}$ and perform a second discrete gauging to reach $M^{-1}\mathcal{L}$. The K-formula gives

$$\Phi_{\mathcal{L}}^{M^{-1}} \overset{\text{Standard}}{=} \nu(q^{-1})\ \tau(-q)\ \sigma\ \tau(-q^{-1})\,, \quad q = k_{M^{-1}\mathcal{L}} - k_{\mathcal{L}}\,. \tag{4.2.33}$$

If instead $q = 0$ then $\Phi_{\mathcal{L}}^{M^{-1}}$ is a parabolic element and the global structure is left invariant $M^{-1}\mathcal{L} = \mathcal{L}$. Having computed both $\Phi_{\mathcal{L}}^{M_1^{-1}}$ and $\Phi_{M_1^{-1}\mathcal{L}}^{M_{2,1}^{-1}\,M_1}$ the fusion rules are obtained by applying the rules (4.2.29), (4.2.32).

To illustrate this we consider the triality defect $\mathfrak{D}_{\mathcal{L}}^{ST}$ which appears in $\mathcal{N}=4$ $\mathfrak{su}(N)$ SYM at $\tau = e^{\frac{2\pi i}{3}}$. The transformations are

$$\Phi_{\mathcal{L}}^{(ST)^{-1}} = \begin{cases} \sigma\,, & \mathcal{L} = \mathcal{E} \\ \nu(-a^{-1})\ \tau(-a)\ \sigma\ \tau(-a^{-1})\,,\ a = -r^{-1}(1+r+r^2)\,, & \mathcal{L} = (r,1)^{\mathsf{T}} \end{cases}$$

$$\Phi_{(ST)^{-1}\mathcal{L}}^{(ST)^{-1}} = \begin{cases} \nu(-1)\ \tau(-1)\ \sigma\ \tau(-1)\,, & \mathcal{L} = \mathcal{E} \\ \nu(-b^{-1})\ \tau(-b)\ \sigma\ \tau(-b^{-1})\,,\ b = (r+1)^{-1}r^{-1}(r^2+r+1)\,, & \mathcal{L} = (r,1)^{\mathsf{T}} \end{cases} \tag{4.2.34}$$

which lead to the fusion rules[52]

$$\mathfrak{D}_{\mathcal{L}}^{ST} \times \mathfrak{D}_{\mathcal{L}}^{ST} = \begin{cases} \mathcal{A}^{N,-(r^2+r+1)} \times \mathfrak{D}_{\mathcal{L}}^{(ST)^2}\,, & \text{if } r^2+r+1 \neq 0 \mod N \\ \mathcal{C}^{\mathbb{Z}_N} \times \mathcal{U}_{\mathcal{L}}^{(ST)^2}\,, & \text{if } r^2+r+1 = 0 \mod N \end{cases} \tag{4.2.35}$$

Topological manipulations for $(\mathbb{Z}_N)^g$: $\mathrm{Sp}(2g,\ \mathbb{Z}_N)_T$

We now generalize our analysis to the group $\mathrm{Sp}(2g,\mathbb{Z}_N)_T$ describing topological manipulations for 4d theories with $(\mathbb{Z}_N)^g$ one-form symmetry. These are the relevant

[51] Since we work with N odd and on spin manifold, such DW term is always trivial and we can safely forget about this label in condensates.

[52] The TQFT coefficient comes from $-a - a^2b^{-1} = (r^2+r+1)r^{-1}(1-r-1) = -(r^2+r+1)$. This matches previous results using 5d techniques [43].

ones for the description of self-duality defects in A_{N-1} theories of class $\mathcal{S}$. The general strategy is logically similar to the $g = 1$ case but more technical. In this section we streamline the key features and the results, relegating long computations to Appendix 4.3.5

Let us start by discussing and setting the notations for the gauging of subgroups $\mathcal{A}$ of the one-form symmetry $(\mathbb{Z}_N)^g$. We describe $\mathcal{A}$ as a sub-lattice of $(\mathbb{Z}_N)^g$. Since N is a prime number, $\mathcal{A}$ is isomorphic to $(\mathbb{Z}_N)^r$ for some r. We will call r the rank of this sub-lattice. $\mathcal{A}$ can be specified by choosing r generators $\mathcal{A}_1$ ... $\mathcal{A}_r$ which we package into a rectangular matrix

$$C_{\mathcal{A}} = \begin{pmatrix} \mathcal{A}_1 & \dots & \mathcal{A}_r \end{pmatrix} . \tag{4.2.36}$$

This description is redundant since $C_{\mathcal{A}} u$, $u \in \mathrm{GL}(r, \mathbb{Z}_N)$ describes the same lattice (this is just a different choice of basis inside $\mathcal{A}$) and the number of distinct lattices of rank r is given by the q-binomial coefficient $\binom{g}{r}_N$. Given a lattice $\mathcal{A}$ we denote by $\tilde{\mathcal{A}}$ the quotient $(\mathbb{Z}_N)^g/\mathcal{A}$. Since for N prime the sequence $1 \to \mathcal{A} \to (\mathbb{Z}_N)^g \to \tilde{\mathcal{A}} \to 1$ splits, we get $(\mathbb{Z}_N)^g = \mathcal{A} \times \tilde{\mathcal{A}}$. We will make a choice for the splitting by specifying a second matrix $C_{\tilde{\mathcal{A}}}$, labeling the generators of $\widetilde{\mathcal{A}}$. We define duals $C^*_{\mathcal{A}}$ and $C^*_{\tilde{\mathcal{A}}}$ by the equations

$$C^*_{\mathcal{A}}\, C_{\mathcal{A}} = \mathbb{1}_r\,, \quad C^*_{\tilde{\mathcal{A}}}\, C_{\tilde{\mathcal{A}}} = \mathbb{1}_{g-r}\,, \quad C^*_{\mathcal{A}}\, C_{\tilde{\mathcal{A}}} = C^*_{\tilde{\mathcal{A}}}\, C_{\mathcal{A}} = 0\,. \tag{4.2.37}$$

The completeness relation reads $\mathbb{1}_g = C_{\mathcal{A}}\, C^*_{\mathcal{A}} + C_{\tilde{\mathcal{A}}}\, C^*_{\tilde{\mathcal{A}}}$. We also denote the "join" and "meet" operations on lattices by $\mathcal{A} \vee \mathcal{B}$ and $\mathcal{A} \wedge \mathcal{B}$ respectively.

Given a linear map $M : (\mathbb{Z}_N)^g \to (\mathbb{Z}_N)^g$ we define its restriction into $\mathcal{A}$ by

$$M_{\mathcal{A}} = C^{\mathsf{T}}_{\mathcal{A}}\, M C_{\mathcal{A}} \tag{4.2.38}$$

and a lift back to the original space by

$$M_{\mathcal{A}}{}^{\mathcal{A}} = (C^*_{\mathcal{A}})^{\mathsf{T}}\, M_{\mathcal{A}}\, C^*_{\mathcal{A}}\,, \tag{4.2.39}$$

such that $((M_{\mathcal{A}})^{\mathcal{A}})_{\mathcal{A}} = M_{\mathcal{A}}$. Given two sub-lattices $\mathcal{A}$ and $\mathcal{B}$ with trivial meet we also define double restrictions

$$M_{\mathcal{A}\mathcal{B}} = C^{\mathsf{T}}_{\mathcal{A}} M C_{\mathcal{B}} \equiv (M_{\mathcal{B}\mathcal{A}})^{\mathsf{T}}\,, \tag{4.2.40}$$

and double lifts

$$(M_{\mathcal{A}\mathcal{B}})^{\mathcal{A}\mathcal{B}} = (C^*_{\mathcal{A}})^{\mathsf{T}}\, M_{\mathcal{A}\mathcal{B}}\, C^*_{\mathcal{B}}\,. \tag{4.2.41}$$

Given these definitions, an (overcomplete) set of generators for $\mathrm{Sp}(2g,\,\mathbb{Z}_N)_T$ is given by

$$\begin{aligned}
\sigma(C_{\mathcal{A}}):\quad & [\sigma(C_{\mathcal{A}})Z]\,(B) = \frac{1}{|H^2(X,\mathcal{A})|^{\frac{1}{2}}}\sum_{b_{\mathcal{A}}\in H^2(X,\mathcal{A})} e^{\frac{2\pi i}{N}\int b_{\mathcal{A}}\cup B_{\mathcal{A}}}\; Z(C_{\mathcal{A}}b_{\mathcal{A}}+C_{\tilde{\mathcal{A}}}B_{\tilde{\mathcal{A}}})\,,\\
\tau(S):\quad & [\tau(S)Z]\,(B) = \exp\left(\frac{2\pi i}{2N}\int \mathfrak{P}^S(B)\right)\; Z(B)\,,\\
\nu(U):\quad & [\nu(U)Z]\,(B) = Z(UB)\,,\quad U\in \mathrm{GL}(g,\mathbb{Z}_N)
\end{aligned} \tag{4.2.42}$$

where the (generalized) Pontryagin square $\mathfrak{P}^S$ is defined as

$$\mathfrak{P}^S(B) = \sum_i S_{ii}\;\mathfrak{P}(B_i) + 2\sum_{i>j} S_{ij}\;B_i\cup B_j\,. \tag{4.2.43}$$

The minimal coupling $b_{\mathcal{A}}\cup B_{\mathcal{A}}$ follows from considering the cup product on $(\mathbb{Z}_N)^g$: $b\cup B\equiv B^*\cup b$. We restrict $b=C_{\mathcal{A}}b_{\mathcal{A}}$ and expand $B^*=B^*_{\mathcal{A}}C^*_{\mathcal{A}}+B^*_{\tilde{\mathcal{A}}}C^*_{\tilde{\mathcal{A}}}$. We then declare $B^*_{\mathcal{A}}$ to be the new $\mathcal{A}$ background $B_{\mathcal{A}}$. With these conventions gauging $\mathcal{A}$ twice leads back to the original theory up to charge conjugation on $C_{\mathcal{A}}$.[53]

We would now like to prove a generalized the *"K-formula"* in this setting. We consider a double gauging

$$\sigma(C_{\mathcal{A}})\;\tau(S)\;\sigma(C_{\mathcal{B}})\,. \tag{4.2.44}$$

When we gauge the full group $(\mathbb{Z}_N)^g$ with an invertible torsion S the K-formula mimics the one found in the case $g=1$ and we get

$$\sigma(\mathbb{1})\;\tau(S)\;\sigma(\mathbb{1}) = Y_S\;\nu(-S^{-1})\;\tau(-S)\;\sigma(\mathbb{1})\;\tau(-S^{-1})\,, \tag{4.2.45}$$

where

$$Y_S = \sum_{\alpha_{\mathcal{A}}\in H^2(X,\mathcal{A})} \exp\left(\frac{2\pi i}{2N}\mathfrak{P}^S(\alpha_{\mathcal{A}})\right) \tag{4.2.46}$$

In the case when we gauge a subgroup $\mathcal{A}$ with also a non-invertible torsion matrix $\mathcal{S}$ the situation is technically more involved but conceptually clear: there will be a subgroup of $\mathcal{A}$ (related to the kernel of $\mathcal{S}$) which is actually ungauged while the quotient will get the standard decoupled TQFT as in the more simple case of (4.2.45). The full derivation of the K-formula for the general case in shown in Appendix 4.3.5.

[53] On the other hand the standard isomorphism $(V^*)^*=V$ gives a gauging operation such that $\sigma(C_{\mathcal{A}})\sigma({C^*_{\mathcal{A}}}^{\mathsf{T}})$ leads back to the original theory. The two differ by left composition with a $\nu(U)$ transformation.

Action on global variants As in the case of $g = 1$, each global variant $\mathcal{L}$ can be reached from a reference boundary $\mathcal{E}$ by a gauging operation $\sigma(C_{\mathcal{A}})$ with some discrete torsion $\tau({S_{\mathcal{A}}}^{\mathcal{A}})$. To find the explicit map we consider applying

$$\Phi_{\mathcal{A},\, S_{\mathcal{A}}} = \sigma(C_{\mathcal{A}})\ \tau({S_{\mathcal{A}}}^{\mathcal{A}}) \tag{4.2.47}$$

to the electric variant $\mathcal{E} = \mathbb{1}_{2g}$. Again the genuine lines after the gauging will be the gauge invariant lines in the presence of the background torsion. In our conventions Wilson lines W_l are labelled by an element l of the dual space. Their charge under $\mathcal{A}$ gauge transformations $B \to B + C_{\mathcal{A}}\, \delta\lambda_{\mathcal{A}}$ is

$$l^{\mathsf{T}}\, C_{\mathcal{A}} \int \mathrm{PD}(\gamma) \cup \lambda\,. \tag{4.2.48}$$

From this it follows that lines labelled by dual generators $C^*_{\tilde{\mathcal{A}}}$ are still genuine after the gauging. This fixes the first $r(\mathcal{A})$ columns of $\mathcal{L}$ to be:

$$\begin{pmatrix} {C^*_{\tilde{\mathcal{A}}}}^{\mathsf{T}} \\ 0 \end{pmatrix}. \tag{4.2.49}$$

As in the $g = 1$ case, 't Hooft lines are also charged due to the discrete theta angle. The charge of T_s is

$$q_{T_s} = -s^{\mathsf{T}}\, {C^*_{\mathcal{A}}}^{\mathsf{T}} (S_{\mathcal{A}})^{\mathcal{A}} \int \mathrm{PD}(\gamma) \cup \lambda\,. \tag{4.2.50}$$

Therefore neutral dyons must fulfill

$$l^{\mathsf{T}}\, C_{\mathcal{A}} - s^{\mathsf{T}}\, {C^*_{\mathcal{A}}}^{\mathsf{T}} {S_{\mathcal{A}}}^{\mathcal{A}} = 0 \tag{4.2.51}$$

which is solved by $s = C_{\mathcal{A}}$ and $l = {C^*_{\mathcal{A}}}^{\mathsf{T}}\, S_{\mathcal{A}}$. These give the last $g - r(\mathcal{A})$ columns of $\mathcal{L}$:

$$\begin{pmatrix} {C^*_{\mathcal{A}}}^{\mathsf{T}}\, S_{\mathcal{A}} \\ C_{\mathcal{A}} \end{pmatrix}. \tag{4.2.52}$$

It can be shown that, if $\mathcal{L} = \begin{pmatrix} A \\ C \end{pmatrix}$, then $A^{\mathsf{T}}\, C$ is symmetric and has rank $r(\mathcal{A})$.

Fusion rules In a similar way as before we can also state the two basic rules for the composition in half-space:

$$\boxed{Y_{S_{\mathcal{C}}} \text{ on half space } X^+ \sim \text{ decoupled TQFT-coefficient } \mathcal{A}^{N,S_{\mathcal{C}}} \text{ on } \partial X^+} \tag{4.2.53}$$

and

$$\boxed{\text{Un-gauging } \mathcal{A} \subset (\mathbb{Z}_N)^g \text{ on half space } X^+ \sim \text{condensation defect } \mathcal{C}^{\mathcal{A}} \text{ on } \partial X^+} \tag{4.2.54}$$

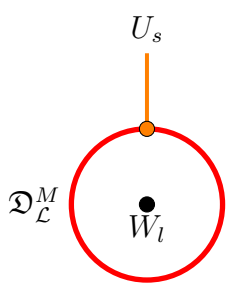
Above we have defined a generalized minimal TQFT $\mathcal{A}^{N,S_\mathcal{C}}$ which has line operators isomorphic to $\mathcal{C}$ and spins $\theta_v = \exp\left(\frac{2\pi i}{2N} v^\mathsf{T} S_\mathcal{C} v\right)$. Since the theory is decoupled the only relevant information is contained in $S_\mathcal{C}$ modulo congruence by $\mathrm{GL}(r(\mathcal{C}), \mathbb{Z}_N)$. When N is prime one can show that there are only two inequivalent choices for each rank $r(\mathcal{C})$ (see appendix 4.3.6 for a proof of this statement), which we denote:

$$\mathcal{N}^{r,+}, \quad \mathcal{N}^{r,-}. \tag{4.2.55}$$

As an example, if the rank is $r = 1$ there are two classes with representatives $\mathcal{A}^{N,1}$ and $\mathcal{A}^{N,q'}$, where q' is not a perfect square in $\mathbb{Z}_N$. It is always possible to choose the representatives to be $\left(\mathcal{A}^{N,1}\right)^r$ for $\mathcal{N}^{r,+}$ and $\left(\mathcal{A}^{N,1}\right)^{r-1} \times \mathcal{A}^{N,q'}$ for $\mathcal{N}^{r,-}$.

4.2.4 Action on line operators: the rank

We now describe how duality defects act on line operators and introduce a new concept: the rank of a non-invertible symmetry. This can be used to almost entirely fix the fusion rules of duality defects, apart from the choice of quadratic form in the decoupled TQFT. The action of duality defects on generic line operators for $\mathbb{Z}_N$ one-form symmetry has been introduced in [120]. The first key feature, already noted in [14], is that non-invertible symmetries can lead to nontrivial maps $\mathfrak{D}^M_\mathcal{L}$: $\mathcal{H}_0 \to \mathcal{H}_s$ between the untwisted ($\mathcal{H}_0$) and twisted ($\mathcal{H}_s$) Hilbert space. In our case the twisting is by the one-form symmetry defect U_s. These follow from the existence of nontrivial junctions between $\mathfrak{D}$ and U_s. In radial quantization we represent them in the following way



(4.2.56)

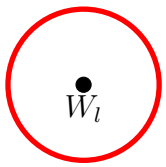
We first discuss the untwisted action. Let us suppose that $\Phi^{M^{-1}}_\mathcal{L}$ is obtained by gauging $\mathcal{A}$ with discrete torsion $S_\mathcal{A}$. If W_l is charged under $\mathcal{A}$ then the operator

is killed by the gauging interface. If it is uncharged, it will be mapped by M onto another genuine operator W_{Ml}. Consistency implies that genuine line operators uncharged under $\mathcal{A}$ corresponds to the sublattice $\mathcal{K} \subset \mathcal{L}$ such that $M\mathcal{K} \subset \mathcal{L}$.

$\mathcal{K}$ can be explicitly computed as

$$\mathcal{K} = \mathcal{L} \wedge M^{-1}\, \mathcal{L}\,. \tag{4.2.57}$$

Thus we conclude that

$\mathfrak{D}^M_{\mathcal{L}}$ 

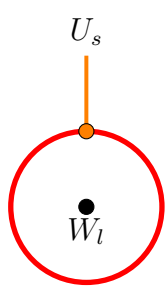
$= \delta_{l\in\mathcal{K}}\, \langle \mathfrak{D}^M_{\mathcal{L}}(\Sigma)\rangle$ 

(4.2.58)

The gauged group $\mathcal{A}$, which is a subgroup $\mathcal{S}_{\mathcal{A}} \subset \mathcal{S}$ must satisfy:

$$\langle \mathcal{S}_{\mathcal{A}},\, \mathcal{K}\rangle = 0\,. \tag{4.2.59}$$

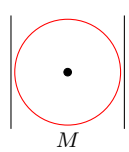
It is clear that the ranks of the lattices satisfy $r(\mathcal{A}) = g - r(\mathcal{K})$. We call $r(\mathcal{A})$ the *rank* of the non-invertible defect $\mathfrak{D}^M_{\mathcal{L}}$.[54] Duality defects with $r(\mathcal{A}) = 0$ are invertible. The maps on the twisted sector $\mathcal{H}_s$ can be understood in a similar way, but now the line W_l can be charged under the gauged symmetry. Consistency with the duality transformation requires that $Ml = s \mod \mathcal{L}$ so:

$\mathfrak{D}^M_{\mathcal{L}}$ 

$= \delta_{[Ml],\, s}\, \langle \mathfrak{D}^M_{\mathcal{L}}(\Sigma)\rangle$ 

(4.2.60)

The characterization of the rank of the symmetry using $\mathcal{K}$ can be used to understand the structure of the fusion algebra without performing the direct computation. First notice that the rank can be written as

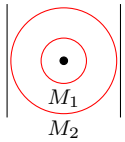
$r(\mathfrak{D}^M_{\mathcal{L}}) = g -$ 

(4.2.61)

[54] Sometimes we use $r(\mathfrak{D})$ or $r(M)$ instead.

where $|\cdot|$ is the dimension of the image of the map inside. After computing the fusion we define

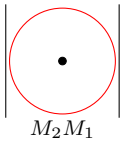
$$r\left(\mathfrak{D}_{\mathcal{L}}^{M_2} \times \mathfrak{D}_{\mathcal{L}}^{M_1}\right) = g - \Big|$$



$$\Big| \tag{4.2.62}$$

Notice that this is different from

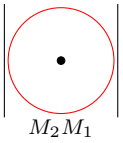
$$r(\mathfrak{D}_{\mathcal{L}}^{M_2 M_1}) = \; g - \Big|$$



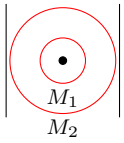
$$\Big| \tag{4.2.63}$$

because the image of (4.2.62) is spanned by $\mathcal{K}_{M_1} \wedge M_1^{-1}\mathcal{K}_{M_2}$ while for (4.2.63) by $\mathcal{K}_{M_{2,1}}$ which is a larger vector space. When the two do not agree a further object is needed to make the fusion consistent. This is a condensate $\mathcal{C}_{2,1}$. It's rank is computed by:

$$r\left(\mathcal{C}_{2,1}\right) = \Big|$$



$$\Big| \; - \; \Big|$$



$$\Big| \tag{4.2.64}$$

This information is readily obtainable as soon as we know $\mathcal{K}$ for the various defects.

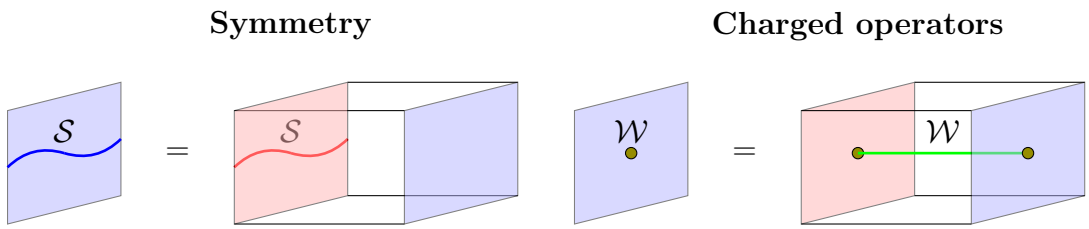
4.2.5 The $5d$ Symmetry TFT

Another viable path to compute the fusion algebra of the duality symmetries is using the bulk Symmetry TFT description [43]. The symmetry TFT is a $d+1$-dimensional topological theory which encodes the discrete symmetries of a given d-dimensional QFT. More specifically a d-dimensional *absolute* QFT is isomorphic to a *relative* QFT living at the boundary of a $(d+1)$-dimensional slab where the symmetry TFT lives. At the other boundary one should impose gapped (topological) boundary conditions $\mathcal{L}$ which select the given global structure of the absolute theory (see figure 4.2.1).

In our applications the boundary conditions are of Dirichlet type for the generators of the Lagrangian algebra $\mathcal{L}$[55]. Defects in the quotient $\mathcal{S} = \Gamma/\mathcal{L}$, when pushed

[55]In this sense ρ is equivalent to perform a gauging of a Lagrangian algebra on the bulk TFT [18, 134].

to the boundary, are the topological operators of the d-dimensional absolute theory. Operators in $\mathcal{L}$ can instead end the topological boundary, and their endpoints describe operators charged under $\mathcal{S}$.



(4.2.65)

Recently various authors have shown in detail how to derive the Symmetry TFT for theories with an holographic dual starting from from string/M-theory [30, 35, 43, 47, 125, 236]. The symmetry TFT for the duality defects in $\mathcal{N} = 4$ SYM was first introduced in [251], while in [43] it was embedded in a holographic framework and the fusion rules for such defects are derived from the bulk formalism. In this section we extend this analysis to the case $g > 1$ and explain how features introduced in Section 4.2.3 emerge form the bulk.

The symmetry TFT for 6d $\mathcal{N} = (2, 0)$ theories of type A_{N-1} has a simple holographic derivation, since the theory can be realized by a stack of N M5 branes in flat space-time. These induce N units of $G_4 = d\mathcal{C}$ flux on a round S^4, where $\mathcal{C}$ is the 3-form potential. The symmetry TFT can be derived from the reduction of the topological terms in the eleven dimensional supergravity action. The resulting 7-dimensional topological theory is a Chern-Simons theory [252]

$$S_{7d} = \frac{N}{4\pi} \int_{Y_7} \mathcal{C}\, d\mathcal{C}\,. \tag{4.2.66}$$

The operators of the theory are

$$C_m = e^{im \int \mathcal{C}} \quad , \qquad m = 0, ..., N-1 \tag{4.2.67}$$

generating a $\mathbb{Z}_N^{[3]}$ 3-form symmetry. The TFT (4.2.66) generically does not have gapped boundary conditions, indicating the fact that 6d $\mathcal{N} = (2, 0)$ SCFTs are intrinsically relative [253].

The symmetry TFT for class $\mathcal{S}$ theories is obtained by considering a seven-dimensional space-time of the from $X_7 = X_5 \times \Sigma_g$ and reducing on Σ_g. The 5d

symmetry TFT takes the form[56]

$$S_{5d} = \frac{N}{4\pi} \int \sum_{i,j=1}^{2g} \mathcal{B}_i \, \mathcal{J}_{ij} \, d\mathcal{B}_j \ , \qquad \mathcal{J} = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix} . \tag{4.2.68}$$

However this is too naive. In the eleven-dimensional theory all the global symmetries are gauged since gravity is not decoupled. Whenever we choose a vacuum in such a gravity theories part of these gauge symmetries are Higgsed and appear as global symmetries of the IR effective theory. Among these are large diffeomorphisms of Σ_g, for which the order parameter can be taken as the complex structure matrix Ω. However, if our choice of Σ_g has discrete isometries, we will get a residual gauge symmetry in the compactified theory. Thus, whenever the Riemann surface has a nontrivial automorphism group $G(\Omega) \subset \mathrm{Sp}(2g, \mathbb{Z})$, we get an emergent gauge symmetry in the symmetry TFT. This acts on charge labels of the 2-form gauge fields, transforming in the fundamental representation of $\mathrm{Sp}(2g,\ \mathbb{Z}_N)$. We will argue that this emergent gauge symmetry is responsible for the non-invertible duality defects.

An analogous scenario is enjoyed by $\mathcal{N} = 4$ SYM, viewed as the theory on D3 branes in type IIB, at $\tau = i, e^{2\pi i/3}$. From an holographic point of view, the axio-dilaton VEV of type IIB string theory Higgses the $\mathrm{SL}(2, \mathbb{Z})$ gauge symmetry which therefore appears as a global symmetry of the supergravity theory. However at $\tau = i, e^{2\pi i/3}$ the VEV of such a field is invariant under the $\mathbb{Z}_{4,6}$ subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and then an emergent gauge field is still present in the infrared. In [43] it was shown that this gauge field is responsible for the topological duality (and triality) defects of the dual gauge theory. The same conclusion is reached if we regard $\mathcal{N} = 4$ SYM as the theory obtained by compactifying the 6d $\mathcal{N} = (2, 0)$ theory on a torus with appropriate modular parameter $\tau = i, e^{2\pi i/3}$. Indeed M-theory compactified on a small torus is equivalent to type IIB string theory [254], in which S-duality is realized as modular transformation of the torus. Thus we see that from the M-theory perspective, the maximally supersymmetric case analyzed in [43] is not special, and the un-Higgsed subgroup can be embedded in the group of large diffeomorphisms in string theory.

[56] We assume that $H_2(X_5, \mathbb{Z})$ is trivial.

Duality defects from the Symmetry TFT

Let us describe the construction of duality defects in the Symmetry TFT and how to compute fusion rules. Most results are a straightforward generalization of [43], to which we refer for a thorough analysis. The symmetries of (4.2.68) are a 2-form symmetry $(\mathbb{Z}_N^{[2]})^{2g}$ generated by the topological surface operators

$$U_m = e^{im^{\mathsf{T}} \int \mathcal{B}} \tag{4.2.69}$$

and a zero-form symmetry $\mathrm{Sp}(2g, \mathbb{Z}_N)$ acting on the gauge fields as

$$\mathcal{B} \to M^{\mathsf{T}} \mathcal{B} \ , \qquad M \in \mathrm{Sp}(2g, \mathbb{Z}_N) \ . \tag{4.2.70}$$

All the topological defects implementing this zero-form symmetry are condensation defects [23] constructed by higher-gauging a subgroup $\mathcal{A}$ of the 2-form symmetry on a codimension one manifold with an appropriate choice of discrete torsion.

Given a subgroup $\mathcal{A} \subset (\mathbb{Z}_N)^{2g}$ and a symmetric torsion matrix $\mathcal{T}_\mathcal{A}$ we can define a condensation defect on a compact four manifold Σ with $H_2(\Sigma, \mathbb{Z}) = \mathbb{Z}^2$ as a sum

$$V[\mathcal{T}_\mathcal{A}] = \sum_{m,m' \in \mathcal{A}} \exp\left(-\frac{2\pi i}{N} \left(m^{\mathsf{T}} \left(\mathcal{T}_\mathcal{A} + 2^{-1} \mathcal{J}_\mathcal{A}\right) m'\right)\right) U_{C_\mathcal{A} m'}(\gamma)\, U_{C_\mathcal{A} m}(\gamma') \,, \tag{4.2.71}$$

with γ and γ' being the generators of $H_2(\Sigma, \mathbb{Z})$.[57] To construct a topological defect which implements $M \in \mathrm{Sp}(2g, \mathbb{Z}_N)$ we need to impose the correct group action on surface operators U_k. That is

$$\begin{aligned} V[\mathcal{T}_\mathcal{A}] U_k(\gamma) = & \sum_{m,m' \in \mathcal{A}} \exp\left(-\frac{2\pi i}{N} \left(m^{\mathsf{T}} \left(\mathcal{T}_\mathcal{A} + 2^{-1} \mathcal{J}_\mathcal{A}\right) m' + m^{\mathsf{T}} C_\mathcal{A}^* \mathcal{J} k\right)\right) U_{k + C_\mathcal{A} m'}(\gamma) \\ = & \, U_{M \cdot k}(\gamma) \,, \end{aligned} \tag{4.2.72}$$

which implies

$$M = \left(\mathbb{1}_{2g} - C_\mathcal{A} \left(\mathcal{T}_\mathcal{A} + 2^{-1} \mathcal{J}_\mathcal{A}\right)^{-1} C_\mathcal{A}^* \mathcal{J}\right) \ . \tag{4.2.73}$$

Notice that since the image of $(M - \mathbb{1}_{2g})$ is isomorphic to $\mathcal{A}$. Inverting (4.2.73) we

[57] We will follow the notation of 4.2.3 for lattice operations.

find[58]

$$\mathcal{T}_{\mathcal{A}} = -2^{-1}\mathcal{J}_{\mathcal{A}} + \left[\left((M - \mathbb{1}_{2g})\mathcal{J}\right)_{\mathcal{A}}\right]^{-1} \tag{4.2.75}$$

This determines univocally the zero-form symmetry defects. For a full gauging $C_{\mathcal{A}} = \mathbb{1}_{2g}$ and we get

$$\mathcal{T} = 2^{-1}\mathcal{J}\left(\mathbb{1}_{2g} + M\right)\left(\mathbb{1}_{2g} - M\right)^{-1} , \tag{4.2.76}$$

which generalizes [43].

The fusion of two condensation defects can be computed from (4.2.71). After some algebra we find that

$$V[\mathcal{T}^{(1)}_{\mathcal{A}}] \times V[\mathcal{T}^{(2)}_{\mathcal{B}}] = V[\mathcal{T}^{(2,1)}_{\mathcal{A},\mathcal{B}}] , \tag{4.2.77}$$

where

$$\mathcal{T}^{(2,1)}_{\mathcal{A},\mathcal{B}} = \mathcal{T}^{(2)}_{\mathcal{A}\vee\mathcal{B}} - \left(\mathcal{T}^{(2)}_{\mathcal{A}\vee\mathcal{B},\mathcal{A}} - 2^{-1}\mathcal{J}_{\mathcal{A}\vee\mathcal{B},\mathcal{A}}\right)\left(\mathcal{T}^{(1)}_{\mathcal{A}} + \mathcal{T}^{(2)}_{\mathcal{A}}\right)^{-1}\left(\mathcal{T}^{(2)}_{\mathcal{A}\vee\mathcal{B},\mathcal{A}} + 2^{-1}\mathcal{J}_{\mathcal{A}\vee\mathcal{B},\mathcal{A}}\right) \tag{4.2.78}$$

is the torsion matrix corresponding to the $\mathrm{Sp}(2g, \mathbb{Z}_N)$ element $M_2 M_1$. Following [43] we can give a Lagrangian description of the defect

$$S_V = \frac{N}{4\pi}\int_{\Sigma_4} \mathcal{B}_{\mathcal{A}}\Phi_{\mathcal{A}} + \Gamma^{\mathsf{T}}_{\mathcal{A}} d\mathcal{B}_{\mathcal{A}} + \Psi^{\mathsf{T}}_{\mathcal{A}} d\Phi_{\mathcal{A}} + \frac{1}{2}\Phi^{\mathsf{T}}_{\mathcal{A}}\mathcal{T}_{\mathcal{A}}\Phi_{\mathcal{A}} \tag{4.2.79}$$

where $\Phi_{\mathcal{A}}, \Gamma_{\mathcal{A}}, \Psi_{\mathcal{A}}$ are auxiliary fields.[59] On a closed Σ_4 this is gauge invariant under

$$\begin{aligned} &\mathcal{B}_{\mathcal{A}} \to \mathcal{B}_{\mathcal{A}} + d\alpha_{\mathcal{A}} , \quad \Phi_{\mathcal{A}} \to \Phi_{\mathcal{A}} + d\lambda_{\mathcal{A}} , \\ &\Psi_{\mathcal{A}} \to \Psi_{\mathcal{A}} - \mathcal{T}_{\mathcal{A}}\lambda_{\mathcal{A}} - \alpha_{\mathcal{A}} + d\mu_{\mathcal{A}} , \quad \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{A}} - \lambda_{\mathcal{A}} + d\nu_{\mathcal{A}} . \end{aligned} \tag{4.2.81}$$

[58] Let us consider for example

$$M = \begin{pmatrix} \mathbb{1}_g & B \\ 0 & \mathbb{1}_g \end{pmatrix} \tag{4.2.74}$$

with B non-singular. The image of $M - \mathbb{1}_{2g}$ is contained in the "electric" $(\mathbb{Z}_N)^g$. To implement this symmetry it suffices to gauge the aforementioned electric $(\mathbb{Z}_N)^g$ only.

[59] Alternatively, in discrete notation it can be written as

$$\frac{2\pi}{N}\int \mathcal{B}_{\mathcal{A}} \cup \Phi_{\mathcal{A}} + \frac{1}{2}\mathfrak{P}^{T_{\mathcal{A}}}(\Phi_{\mathcal{A}}) , \tag{4.2.80}$$

with $\Phi_{\mathcal{A}} \in H^2(\Sigma_4, \mathcal{A})$ and $B_{\mathcal{A}} \in H^2(\Sigma_4, \mathcal{A}^*)$.

Twisted sectors and fusion rules We can now define the twist defects associated to zero-form symmetry operators $V[\mathcal{T}_\mathcal{A}]$. Given a p-dimensional topological operator V, we can consider its twisted Hilbert space $\mathcal{H}_V$, which is spanned by non-genuine $p-1$ dimensional topological defects on which V can end. If V implements an anomaly-free symmetry, gauging V liberates the twist defects, which become genuine $p-1$ dimensional operators in the gauged theory. To construct the twist defects $D[\mathcal{T}_\mathcal{A}]$ for $V[\mathcal{T}_\mathcal{A}]$ we can impose Dirichlet boundary conditions on $\Phi_\mathcal{A}$.[60] The minimal description of $D[\mathcal{T}_\mathcal{A}]$ can be found by compensating for the lack of gauge invariance on an open Σ_4 by adding additional degrees of freedom of $Y = \partial\Sigma_4$. We find[61]

$$S_{twist} = \frac{N}{4\pi}\int_Y \mathcal{B}_\mathcal{A}^\mathsf{T}\Gamma_\mathcal{A} + \Phi_\mathcal{A}^\mathsf{T}\Psi_\mathcal{A} + \Gamma_\mathcal{A} d\Psi_\mathcal{A} - \frac{1}{2}\Gamma_\mathcal{A}^\mathsf{T}\mathcal{T}_\mathcal{A} d\Gamma_\mathcal{A}\ . \tag{4.2.83}$$

where $\Gamma_\mathcal{A}$ and $\Psi_\mathcal{A}$ are now regarded as auxiliary 3d fields.

If $\mathcal{T}_\mathcal{A}$ is full rank we can obtain a simpler description of $D[\mathcal{T}_\mathcal{A}]$ by integrating out $\Phi_\mathcal{A}$. The 4d defect $V[\mathcal{T}_\mathcal{A}]$ becomes

$$S_V = \frac{N}{2\pi}\int_{\Sigma_4} \mathcal{B}_\mathcal{A}^\mathsf{T} d\Gamma_\mathcal{A} - \frac{1}{2}\mathcal{B}_\mathcal{A}^\mathsf{T}\mathcal{T}_\mathcal{A}^{-1}\mathcal{B}_\mathcal{A} = -\frac{2\pi i}{2N}\int \mathfrak{P}^{\mathcal{T}_\mathcal{A}^{-1}}(\mathcal{B}_\mathcal{A})\,. \tag{4.2.84}$$

The corresponding twist defect is described by a minimal $\mathcal{A}^{N,-\mathcal{T}_\mathcal{A}}$ TQFT [67]. This is a 3d TQFT hosting $N^{r(\mathcal{A})}$ lines W_n, which fuse according to $\mathcal{A}$ and have spins

$$\theta(W_n) = \exp\left(\frac{\pi i}{N} n^\mathsf{T}\mathcal{T}_\mathcal{A} n\right) \tag{4.2.85}$$

A simple way to derive this fact is to interpret (4.2.84) as an anomaly-inflow action for a $\mathcal{A}$ one-form symmetry in 3d. The minimal TQFT $\mathcal{A}^{N,-\mathcal{T}_\mathcal{A}}$ is the "smallest"[62] possible MTC saturating the anomaly [67]. If $\mathcal{T}_\mathcal{A}$ is not full-rank this reasoning fails as the 4d theory is not an invertible TQFT and to use such a description one should

[60] This is only true if $T_\mathcal{A}$ is an invertible matrix, in which case the defect theory is an invertible TQFT which has only one allowed boundary condition. Studying twist defects for which $T_\mathcal{A}$ is not full rank is challenging and we do not consider them in this work.

[61] In discrete notation this reads

$$-\frac{2\pi}{2}\int \gamma_\mathcal{A} \cup \mathcal{T}_\mathcal{A}\, \beta(\gamma_\mathcal{A})\,, \quad \gamma_\mathcal{A} \in H^1(Y,\mathcal{A})\,. \tag{4.2.82}$$

with β the Bockstein map.

[62] By smallest we mean that each theory $\mathcal{T}$ with the same anomaly can be written as $\mathcal{T} = \mathcal{A}^{N,\mathcal{T}_\mathcal{A}} \times \mathcal{T}'$ for some $\mathcal{T}'$ decoupled from $\mathcal{A}$.

presumably make a choice of boundary condition on $\partial\Sigma_4$ first. We do not discuss such cases here. Notice that if $\mathcal{A} \neq (\mathbb{Z}_N)^{2g}$ the twist defect is not unique as we can always fuse it to a 3d condensate of $(\mathbb{Z}_N)^{2g}/\mathcal{A}$.

Fusion rules for twist defects $D[\mathcal{T}^{(1)}_{\mathcal{A}}]$ and $D[\mathcal{T}^{(2)}_{\mathcal{B}}]$ can be computed by noticing that lines $W^{(1)}$ and $W^{(2)}$ are not mutually local, as they are attached to bulk U_m surfaces [43]. Taking this into account lines of the composite defect $D[\mathcal{T}^{(1)}_{\mathcal{A}}] \times_{\mathcal{B}} D[\mathcal{T}^{(2)}_{\mathcal{B}}]$, where $\times_{\mathcal{B}}$ reinforces that it is not a naive tensor product, braid through

$$\mathcal{K}_{21} = \begin{pmatrix} \mathcal{T}^{(1)}_{\mathcal{A}} & 2^{-1}\,\mathcal{J}_{\mathcal{A},\mathcal{B}} \\ -2^{-1}\,\mathcal{J}_{\mathcal{B},\mathcal{A}} & \mathcal{T}^{(2)}_{\mathcal{B}} \end{pmatrix}. \tag{4.2.86}$$

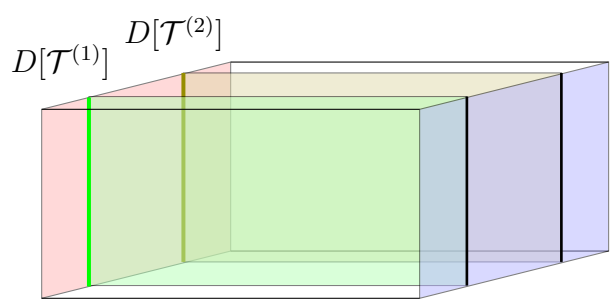
To compute the fusion product one must isolate the group of uncharged lines under the bulk 1-form symmetry from the above. The remaining coupled theory is precisely the twist defect $D[\mathcal{T}^{(1,2}]$. For instance, when $C_{\mathcal{A}} = C_{\mathcal{B}} = \mathbb{1}$, we find

$$D[\mathcal{T}^{(1)}] \;\times\; D[\mathcal{T}^{(2)}] = \mathcal{A}^{N,-\mathcal{T}^{(1)}-\mathcal{T}^{(2)}} \times D[\mathcal{T}^{(2,1)}]\,, \tag{4.2.87}$$

as long as $\mathcal{T}^{(2,1)}$ also has full rank. In the opposite case, when $\mathcal{T}^{(2)} = -\mathcal{T}^{(1)}$ and the final zero-form symmetry defect is the identity, we find

$$D[\mathcal{T}^{(1)}] \;\times\; D[\mathcal{T}^{(2)}] = \mathcal{C}^{(\mathbb{Z}_N)^{2g}}\,. \tag{4.2.88}$$

Fusion on gapped boundaries What we are actually interested in is to discuss the composition laws for twist defects once they are brought onto a gapped boundary $\mathcal{L}$. Before the gauging of the zero-form symmetry $G(\Omega)$ these describe fusion rules for duality *interfaces*



(4.2.89)

Here we will mostly discuss the case in which $\mathcal{A} = \mathcal{B} = (\mathbb{Z}_N)^{2g}$. However we give a general algorithm at the end of the Section. In section 4.2.2 we have learned that global variants correspond to Lagrangian lattices $\mathcal{L}$ of $(\mathbb{Z}_N)^{2g}$. From the symmetry TFT perspective this is encoded in a boundary condition which sets:

$$U_m = \mathbb{1}\,, \quad \text{if } m \in \mathcal{L} \tag{4.2.90}$$

or, alternatively on the fields

$$\mathcal{B}_{\mathcal{L}} \equiv \mathcal{L}^{\mathsf{T}} \mathcal{B} = 0 \qquad \text{(up to gauge transformations)}. \tag{4.2.91}$$

There are two cases to discuss, depending on whether the rank of $\mathcal{T}_{\mathcal{L}}$ is maximal or not. Let us start by assuming that $\mathcal{T}_{\mathcal{L}}$ is an invertible $g \times g$ matrix. We will treat the other cases in 4.2.5.

Since on the gapped boundary a subgroup $\mathcal{L}$ of the $(\mathbb{Z}_N)^{2g}$ two-form symmetry acts trivially, lines W_l with $l \in \mathcal{L}$ of a twist defect $D[\mathcal{T}]$ which are only charged under $\mathcal{L}$ completely decouple. These form a $\mathcal{A}^{N,-\mathcal{T}_{\mathcal{L}}}$ minimal theory, which we consider screened on the boundary.[63] The coupled twist defect is described by:

$$D_{\mathcal{L}}[\mathcal{T}] = \frac{D[\mathcal{T}] \times \mathcal{A}^{N,T_{\mathcal{L}}}}{\mathcal{L}} \,. \tag{4.2.92}$$

If $\mathcal{T}$ is full rank, we can faithfully parametrize lines surviving the quotient by $\mathcal{S} = \mathcal{T}^{-1}\mathcal{L}_{\perp}$, with $\mathcal{L}_{\perp} = \mathcal{J}\ \mathcal{L}$ the dual lattice of $\mathcal{L}$. These lines form a minimal theory

$$\mathcal{A}^{N,-\mathcal{T}_{\perp}} \,, \quad \text{with } \mathcal{T}_{\perp} = \mathcal{T}_{\mathcal{L}_{\perp}}^{-1} \,. \tag{4.2.93}$$

A complementary procedure, which leads to the same answer, is to impose the boundary conditions $\mathcal{L}^{\mathsf{T}}\mathcal{B} = 0$ directly on the "anomaly inflow" action (4.2.84). The leftover anomaly is precisely captured by the minimal theory $\mathcal{A}^{N,-\mathcal{T}_{\perp}}$.

Boundary fusion rules can be computed by applying the screening process to the $\mathcal{A}^{N,\mathcal{T}^{(1)}} \times_{\mathcal{B}} \mathcal{A}^{N,\mathcal{T}^{(2)}}$ theory instead. We define:

$$\mathcal{A}^{N,\mathcal{R}_{2,1}} = \frac{\mathcal{A}^{N,\mathcal{T}^{(1)}} \times_{\mathcal{B}} \mathcal{A}^{N,\mathcal{T}^{(2)}} \times \mathcal{A}^{N,\mathcal{T}_{\mathcal{L}}^{(1)}} \times \mathcal{A}^{N,\mathcal{T}_{\mathcal{L}}^{(2)}}}{\mathcal{L} \times \mathcal{L}} \,. \tag{4.2.94}$$

When $\mathcal{T}^{(2,1)}$ and its restriction to $\mathcal{L}$ are also full rank we can parametrize

$$\mathcal{R}_{21} = \begin{pmatrix} \mathcal{T}_{\perp}^{(2,1)} & c_0 \\ c_0 & c_d \end{pmatrix} \qquad \begin{aligned} c_0 &= \mathcal{L}_{\perp}^{\mathsf{T}}\ \mathcal{T}^{(2,1)^{-1}}\ \left(\mathcal{T}^{(2)} - 2^{-1}\ \mathcal{J}\right)\left(\mathcal{T}^{(1)} + \mathcal{T}^{(2)}\right)^{-1}\ \mathcal{L}_{\perp} \\ c_d &= \mathcal{L}_{\perp}^{\mathsf{T}}\ \left(\mathcal{T}^{(1)} + 4^{-1}\mathcal{J}\ \mathcal{T}^{(2)^{-1}}\ \mathcal{J}\right)\ \mathcal{L}_{\perp}\ . \end{aligned} \tag{4.2.95}$$

This theory splits into the outgoing defect $D_{\mathcal{L}}[\mathcal{T}^{(2,1)}]$, described by the upper-left corner of the matrix, and a decoupled TQFT coefficient which can be computed on a case by case basis. The fusion rules then read

$$D_{\mathcal{L}}[\mathcal{T}^{(1)}]\ \times\ D_{\mathcal{L}}[\mathcal{T}^{(2)}] = \mathcal{N}_{21}\ D_{\mathcal{L}}[\mathcal{T}^{(2,1}] \,. \tag{4.2.96}$$

If the rank of $\mathcal{T}_{\mathcal{L}}^{(2,1)}$ decreases instead, the fusion is accompanied by a condensation. The next part of this Section will clarify how to treat this case.

[63] Notice that this is a well defined MTC only if $\mathcal{T}_{\mathcal{L}}$ is invertible.

Lower rank defects: rank from the bulk

We now discuss the case in which the matrix $\mathcal{T}_{\mathcal{L}}$ has a kernel. It is a matter of simple algebra to show that $\mathcal{T}_{\mathcal{L}}$ has a kernel if and only if the sublattice $\mathcal{K} = \mathcal{L} \wedge M^{-1}\mathcal{L}$ which is mapped inside $\mathcal{L}$ by M is non empty. The kernel is then spanned by vectors

$$\mathcal{K}_{\text{Ker}} = (\mathbb{1}_{2g} - M)\, \mathcal{K}\,. \tag{4.2.97}$$

Let's now discuss the implications of a nontrivial $\mathcal{K}_{\text{Ker}}$ on the boundary defect $D_{\mathcal{L}}(\mathcal{T})$. It is clear that now our previous algorithm to screen decoupled lines fails, as the $\mathcal{A}^{N,-\mathcal{T}_{\mathcal{L}}}$ theory is ill defined. Instead lines in $\mathcal{K}_{\text{Ker}}$ form a condensable subgroup of $\mathcal{A}^{N,\mathcal{T}}$ and are gauged by the boundary conditions. Defining $\mathcal{L}_{\mathcal{K}} = \mathcal{L}/\mathcal{K}_{Ker}$ the correct prescription is

$$D_{\mathcal{L}}[\mathcal{T}] = \frac{\mathcal{A}^{N,-\mathcal{T}} \times \mathcal{A}^{N,\mathcal{T}_{\mathcal{L}_{\mathcal{K}}}}}{\mathcal{L}} \tag{4.2.98}$$

where we have used that $\mathcal{L}_{\mathcal{K}} \times \mathcal{K}_{\text{Ker}} = \mathcal{L}$. The same conclusion can be reached also from the anomaly theory (4.2.84): when $\mathcal{K}$ is non-empty the anomaly theory is of lower rank, with a kernel spanned by $\mathcal{J}\,(M + \mathbb{1}_{2g})\,\mathcal{K}$. In both cases the rank of the boundary defect decreases by $\dim(\mathcal{K})$, which reproduces the result found in Section 4.2.4.

We can now state an algorithm to compute fusion rules from the bulk Symmetry TFT.

1. Find the groups $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_{2,1}$ of the ingoing and outgoing defects.

2. Construct the boundary theory for the fusion

$$\mathcal{A}^{N,\mathcal{R}_{2,1}} = \frac{\mathcal{A}^{N,\mathcal{T}^{(1)}} \times_{\mathcal{B}} \mathcal{A}^{N,\mathcal{T}^{(2)}} \times \mathcal{A}^{N,\mathcal{T}^{(1)}_{\mathcal{L}_{\mathcal{K}}}} \times \mathcal{A}^{N,\mathcal{T}^{(2)}_{\mathcal{L}_{\mathcal{K}}}}}{\mathcal{L} \times \mathcal{L}}\,. \tag{4.2.99}$$

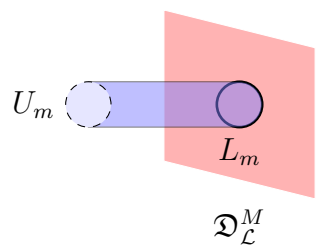
3. Find a splitting:

$$\mathcal{A}^{N,\mathcal{R}_{2,1}} = \mathcal{N}_{2,1} \times \mathcal{C}^{\mathcal{A}_{\text{Cond}}} \times D_{\mathcal{L}}[\mathcal{T}^{(2,1)}]\,. \tag{4.2.100}$$

   This is tedious but doable, since the condensed lines are neutral under the symmetry of $D_{\mathcal{L}}[\mathcal{T}^{(2,1)}]$.

Rank and Fusion Having explained how the rank of a duality defect can be understood both from a 4d and 5d perspective, we conclude by giving an explicit application of this concept.

It is rather simple, given defects $\mathfrak{D}_{\mathcal{L}}^{M_1}$, $\mathfrak{D}_{\mathcal{L}}^{M_2}$ and $\mathfrak{D}_{\mathcal{L}}^{M_1 M_2}$, to compute the associated groups $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_{1,2}$. We now argue that this information almost unequivocally fixes the fusion algebra $\mathfrak{D}_{\mathcal{L}}^{M_1} \times \mathfrak{D}_{\mathcal{L}}^{M_2}$. First notice that the composite defect $\mathfrak{D}_{\mathcal{L}}^{M_1} \times \mathfrak{D}_{\mathcal{L}}^{M_2}$ host $N^{r(\mathfrak{D}_{\mathcal{L}}^{M_1})+r(\mathfrak{D}_{\mathcal{L}}^{M_2})}$ non-genuine line excitations, on which the gauged one-form symmetry surfaces can end:



(4.2.101)

These can be thought of as two-morphisms $L_m : \mathbb{1}_{\mathfrak{D}_{\mathcal{L}}^{M}} \times U_m \to \mathbb{1}_{\mathfrak{D}_{\mathcal{L}}^{M}}$ and their category is explicitly described in the bulk symmetry TFT by a minimal TQFT (4.2.98).

The number of these lines cannot change upon performing fusion, so it must be matched on the other side. We have $N^{r(\mathfrak{D}_{\mathcal{L}}^{M_2 M_1})}$ lines from the outgoing defect and $N^{2r(\mathcal{C}_{2,1})}$ lines from the condensation defect. The factor of 2 comes from regarding the condensate as a DW theory coupled to a dynamical two-form field. The subgroup of the one-form symmetry which is condensed can be computed as a quotient $\mathcal{A}_{\text{Cond}} = (\mathcal{A}_1 \vee \mathcal{A}_2)/\mathcal{A}_{1,2}$. The remaining lines will necessarily form a decoupled TQFT. This determines the rank of the TQFT coefficient. The only undetermined datum is its class modulo congruence.

The gauged theory

The correct 5d bulk theory describing the special loci of the conformal manifold where we get the duality defects is obtained from (4.2.68) by gauging the automorphism group $G(\Omega) \subset \text{Sp}(2g, \mathbb{Z}_N)$ of the Riemann surface on which we compactify the 6d $\mathcal{N} = (2,0)$ theory and the 7d TQFT. We will henceforth drop the suffix Ω.

The gauging consists in coupling (4.2.68) to a pure $G(\Omega)$ gauge theory, which renders the four-dimensional defects labelled by $M \in G(\Omega)$ transparent. Their twist defects become genuine three-dimensional operators, provided we dress them by the naive Gukov-Witten operators (GW) of the pure $G(\Omega)$ gauge theory, as explained in [43]. A convenient way to describe the gauged theory is through the

hybrid formulation introduced in [43], in which the bulk 2-form fields $\mathcal{B}_i$ are kept continuous, while the 1-form G gauge field A is a singular (or Cech) cochain. The general treatment goes in parallel to the case of $\mathcal{N} = 4$ SYM (namely $g = 1$) studied in [43]. Here we focus on the main differences when $G(\Omega)$ is non-abelian.

The first issue is defining non-abelian discrete gauge fields, which are expected to be described by $H^1(X_5, G)$. However this object is not a group and its definition is somewhat subtle. Let us sketch how this is done.

We choose a good cover $X_5 = \bigcup_i \mathcal{U}_i$ of the manifold, and we assume an ordering for the indices. The covering is dual to a simplicial decomposition: patches are associated with vertices (or 0-simplices) $v_i \in \mathcal{U}_i$, double intersections $\mathcal{U}_{ij}$ to 1-simplices v_{ij}, $i < j$, crossing a co-dimension one plane and oriented from v_i to v_j, and triple intersections $\mathcal{U}_{ijk}$ are associated with co-dimension two planes orthogonal to 2-simplices v_{ijk}, with $i < j < k$. A zero cochain $\lambda \in C^0(X, G)$ associates an element $\lambda_i \in G$ to each vertex v_i, while a 1-cochain $A \in C^1(X, G)$ is an assignment of $A_{ij} \in G$ for each 1-simplex v_{ij}, and so on. Given $A, B \in C^p(X, G)$, the group structure of G is used to construct $AB \in C^p(X, G)$

$$(AB)_{i_0,\ldots,i_p} = A_{i_0,\ldots,i_p} B_{i_0,\ldots,i_p} \in G \ . \tag{4.2.102}$$

making $C^p(X, G)$ a group. We would like to define differentials $\delta_p : C^p(X, G) \to C^{p+1}(X, G)$ such that $\delta_{p+1}\delta_p = 0$. For generic p this is not possible, but fortunately we only need the cases $p = 0, 1$, for which we introduce

$$(\delta_0 \lambda)_{ij} = \lambda_i \lambda_j^{-1} \qquad (\delta_1 A)_{ijk} = A_{jk} A_{ik}^{-1} A_{ij} \ , \tag{4.2.103}$$

satisfying $\delta_1\delta_0 = 0$. These maps are *not* homomorphisms and thus $\mathrm{Ker}(\delta_1)$ and $\mathrm{Im}(\delta_0)$ are not groups. However on $\mathrm{Ker}(\delta_1)$ we can introduce the equivalence relation $\sim$ as

$$A \sim B \quad \Longleftrightarrow \quad A_{ij} = \lambda_i B_{ij} \lambda_j^{-1} \ , \tag{4.2.104}$$

which is well defined since

$$\delta_1 A_{ijk} = \lambda_j (\delta_1 B_{ijk}) \lambda_j^{-1} \ . \tag{4.2.105}$$

Quotienting by $\sim$ defines $H^1(X, G)$. In physical terms the equivalence relation above is a gauge transformation. $H^1(X, G)$ is not a group, but this does not affect the formalism of [43] in any way.

The second issue is that while the twisted sectors of the four-dimensional defects (and the duality defects) are labelled by elements of G, the GW operators are

labelled by conjugacy classes. Indeed a GW operator for the discrete gauge theory is defined by a singular connection A such that

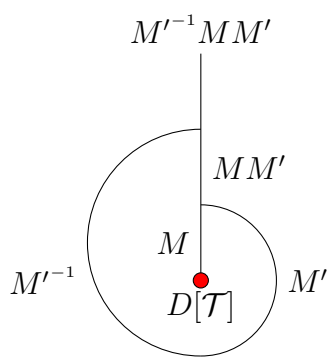
$$\delta_1 A_{ijk} = g\,, \tag{4.2.106}$$

around a patch U_{ijk}. Since a gauge transformation (4.2.104) by $\lambda \in C^0(X, G)$ on A maps $\delta_1 A_{ijk}$ to $\lambda_j \delta_1 A_{ijk} \lambda_j^{-1}$ we have to declare that GW operators labelled by elements in the same conjugacy classes are equivalent: thus a GW operator is labelled by a conjugacy class $[g]$ rather than an element of the group.

This fact is reflected on twist defects as follows. For G an abelian group (as in [43]) the 3d action (4.2.83) defining the twist defect $D[\mathcal{T}(M)]$ has a 0-form symmetry G, which acts as $\mathcal{B} \to {M'^{-1}}^{\mathsf{T}} \mathcal{B}$, $\Phi \to M'\Phi$, $\Gamma \to M'\Gamma$, $\Psi \to {M'^{-1}}^{\mathsf{T}} \Psi$, where $M' \in G$. Indeed

$$\mathcal{T}(M) \to {M'}^{\mathsf{T}}\, \mathcal{T}(M)\, M' = \mathcal{T}(M'^{-1} M M') \tag{4.2.107}$$

which for G abelian is $\mathcal{T}(M)$. For a non-abelian G, while the GW operators are labelled by conjugacy classes, the 3d twist defects are labelled by elements $M \in G$. However because of (4.2.107) the 3d action does not have the G symmetry in the non-abelian case, but the action of $M' \in G$ on $D[\mathcal{T}(M)]$ produces a different defect:



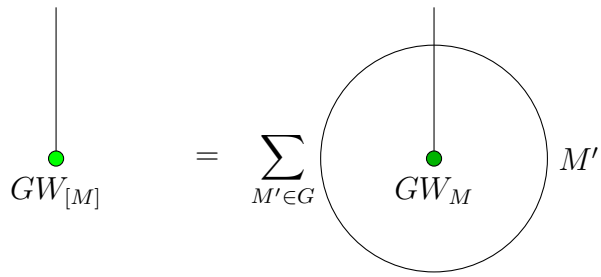
(4.2.108)

Thus we cannot simply covariantize the action to generate a good operator in the gauged theory. What we have to do, instead, is to sum over all defects which are in the same orbits for the adjoint action of G on itself:

$$D[\mathcal{T}(M)] \to D[\mathcal{T}(M)]/G = \bigoplus_{\widetilde{M} \in [M]} D[\mathcal{T}(\widetilde{M})]\,. \tag{4.2.109}$$

We conclude that in the gauged theory also the twist defects are labelled by the conjugacy classes of G, and they form a compound with the GW operators defined by the equation (4.2.106). A convenient way to represent these operators is to

start from the *naked*, non gauge-invariant three-dimensional operators labelled by elements $M \in G$, then acting on it with gauge transformations and summing over them:



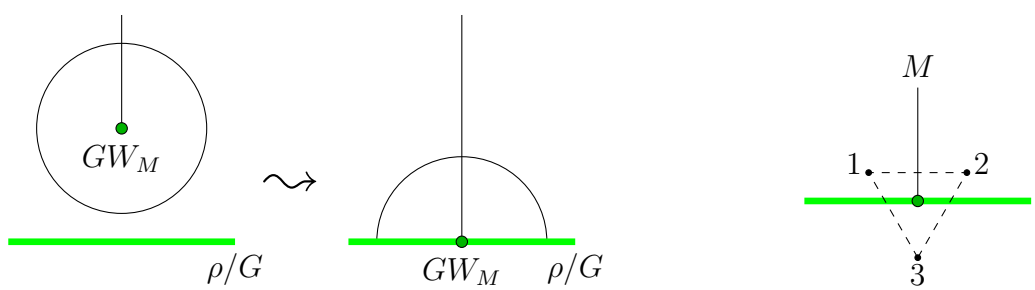
(4.2.110)

The four-dimensional defects implementing M, M' are precisely the location of the co-dimension one plane orthogonal to the 1-simplices associated with the gauge field $A \in H^1(X_5, G)$, and therefore are transparent in the gauged theory. Notice that in the bulk, the 3d operators of the gauged theory have two sources of non-invertibility. The first one, which we discussed here and in [43], has to do with the appearance of TQFT coefficients and condensates, while the second one comes from the fact that these defects are sum of several defects of the ungauged theory leading to a non-invertibility of orbifold type as in [29, 33].

A gapped boundary of the gauged theory can be described as a non-simple but G-invariant boundary in the ungauged theory, tensored with Dirichlet boundary conditions for the G gauge field:

$$|\rho/G\rangle = \frac{1}{|\mathrm{Stab}(\rho)|} \sum_{M \in G} |\rho_M\rangle \times |A = 0\rangle \tag{4.2.111}$$

Bringing the GW operator onto the gapped boundary liberates the naked GW since four-dimensional surfaces implementing G are absorbed by $|\rho/G\rangle$:



(4.2.112)

Alternatively, we can just think of the gauge transformations also being frozen on the boundary, so that $(\delta_1 A)_{ijk} = M \in G$ is a well defined boundary condition. This is depicted on the right side of (4.2.112), with $A_{13} = A_{23} = 1$ and $A_{12} = M$. The last

specification should be thought of as a boundary condition for the insertion of the boundary defect. Thus we conclude that upon bringing twist defect onto a gapped boundary the non-invertibility of the orbifold type disappears and these operators are labelled by element $M \in G$ instead of conjugacy classes, as we expect from the $4d$ construction.

4.2.6 Applications and Examples

We conclude with some explicit computations for $g = 2$. The cases of higher genus can be treated similarly, however the number of global variants soon becomes prohibitive and we do not expect any new features to emerge. The list of discrete automorphism groups for $g = 2$ Riemann surfaces without puntures is the following [250]:

$G(\Omega)$	# Moduli
$\mathbb{Z}_{10}$	0
$(\mathbb{Z}_2 \times \mathbb{Z}_6) \rtimes \mathbb{Z}_2$	0
$\mathbb{Z}_{12} \times \mathbb{Z}_2$	0
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	0
$\mathrm{GL}(2,3)$	0
$\mathbb{Z}_3 \times (\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	0
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1
D_8	1
$\mathbb{Z}_2 \times \mathbb{Z}_6$	1
D_{12}	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2

We focus on two representative cases: the largest cyclic group $\mathbb{Z}_{4g+2} = \mathbb{Z}_2^C \times \mathbb{Z}_{2g+1}$ and the symmetry enhancement from D_{4g+4} to $(\mathbb{Z}_{2g+2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. Both cases are present for every genus. To compute the fusion rules using the discrete gauging description we first determine the matrix $\Phi_{\mathcal{L}}^{M^{-1}}$ which sends $\mathcal{L}$ to $M^{-1}\mathcal{L}$ (seen as a member of the quotient $\mathrm{Sp}(2g, \mathbb{Z}_N)/\mathcal{P}(2g, \mathbb{Z}_N)$) via its right action. This is just

$$\Phi_{\mathcal{L}}^{M^{-1}} = \mathcal{L}^{-1}\, M^{-1}\, \mathcal{L}\,. \tag{4.2.113}$$

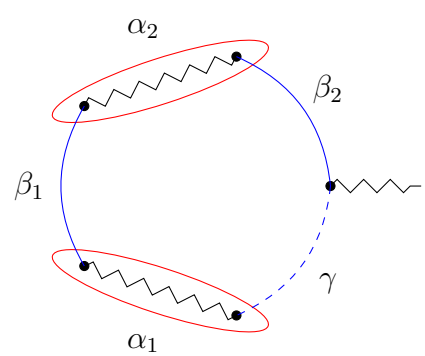
We then decompose it in the standard form as explained in Appendix 4.3.4.[64] All the remaining categorical data are extracted by applying the K-formula.

[64]Notice that the electric boundary is just $\mathcal{L} = \mathbb{1}_{2g}$, to topological manipulations there can be extracted by putting M^{-1} in standard form.

Cyclic group $\mathbb{Z}_{4g+2}$ The Riemann surface with such an automorphism group is rather simple to describe, it corresponds to the hyperelliptic curve:

$$y^2 = x^{2g+1} - 1\,. \tag{4.2.114}$$

The symmetry is comprised by the hyperelliptic involution $\mathbb{Z}_2^C \ : Cy = -y$ and a discrete rotation $\mathbb{Z}_{2g+1} \ : \rho x = e^{-\frac{2\pi i}{2g+1}}\, x$. The action on homology cycles is best seen by representing the curve as a branched cover of the complex plane (here for $g = 2$):



(4.2.115)

The cycles relate to the standard basis through $\alpha_i = A_i$, $\beta_1 = B_1 - B_2$, $\beta_2 = B_2$. C interchanges the two sheets, reversing the orientation of cycles, while ρ corresponds to a discrete clockwise rotation. They correspond to matrices $M \in \mathrm{Sp}(4, \mathbb{Z})$

$$M_C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad M_\rho = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix}. \tag{4.2.116}$$

In this example the only interesting defect is M_ρ, since C leaves all the Lagrangian lattices $\mathcal{L}$ invariant and is thus invertible. It can be checked that, for all $\mathcal{L}$:

$$\mathfrak{D}_{\mathcal{L}}^{M_\rho} \times \mathfrak{D}_{\mathcal{L}}^{M_\rho{}^\dagger} = \mathcal{C}^{\mathcal{A}}\,, \tag{4.2.117}$$

where $\mathcal{A}$ is subgroup of the one-form symmetry being gauged by $\Phi_{\mathcal{L}}^{M_\rho^{-1}}$. For $g = 2$ this can either be the full $\mathbb{Z}_N^2$, a one dimensional subgroup $\mathcal{A}$ or nothing at all, in which case the defect is invertible.

The fusion rules for M_ρ with itself are more interesting. For $g = 2$ there can be several patterns, depending on $r(M_\rho)$ and $r(M_\rho^2)$. The allowed patterns are given in the Table below:

$r(M_\rho)$	$r(M_\rho^2)$	$\mathfrak{D}_{\mathcal{L}}^{M_\rho} \times \mathfrak{D}_{\mathcal{L}}^{M_\rho}$
2	2	$\mathcal{N}^{2,\pm}\, \mathfrak{D}_{\mathcal{L}}^{M_\rho^2}$
2	1	$\mathcal{C}^{\mathbb{Z}_N}\, \mathcal{N}^{1,\pm}\, \mathfrak{D}_{\mathcal{L}}^{M_\rho^2}$
2	0	$\mathcal{C}^{\mathbb{Z}_N^2}\, \mathcal{U}_{\mathcal{L}}^{M_\rho^2}$
1	2	$\mathfrak{D}_{\mathcal{L}}^{M_\rho^2}$
1	1	$\mathcal{N}^{1,\pm}\, \mathfrak{D}_{\mathcal{L}}^{M_\rho^2}$
1	0	$\mathcal{C}^{\mathbb{Z}_N}\, \mathcal{U}_{\mathcal{L}}^{M_\rho^2}$
0	0	$\mathcal{U}_{\mathcal{L}}^{M_\rho^2}$

As explained in appendix 4.3.6 we only have two choices $\mathcal{N}^{r,\pm}$ for the TQFT coefficients at a given rank. In our conventions $\mathcal{N}^{r,+}$ is represented by $\left(\mathcal{A}^{N,1}\right)^r$ while $\mathcal{N}^{r,-}$ by $\left(\mathcal{A}^{N,1}\right)^{r-1} \times \mathcal{A}^{N,q'}$, q' not being a perfect square.

Notice that, even if all three defects are non-invertible, they can fuse as an invertible symmetry sometimes.

Dihedral group D_{4g+4} and symmetry enhancement The second example is the simplest non-abelian group D_{4g+4}. This example also enjoys two other features of non-invertible symmetries in class $\mathcal{S}$ theories, namely the presence of moduli spaces on which the symmetry is realized and its enhancement. As before we can represent the surface hosting such automorphism group by an hyperelliptic curve[65]

$$y^2 = \left(x^{g+1} - \lambda\right)\left(x^{g+1} - 1/\lambda\right) . \tag{4.2.119}$$

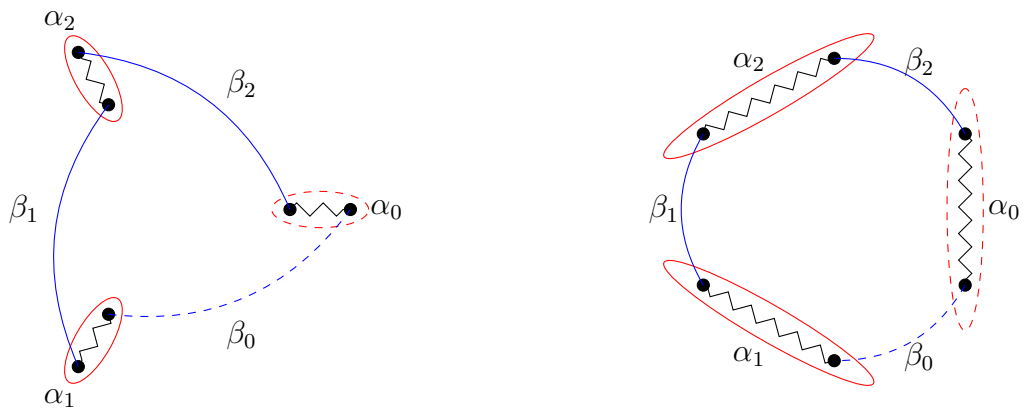
The symmetry is generated by the hyperelliptic involution $\mathbb{Z}_2^C$, a rotation $\mathbb{Z}_{g+1}^t$: $tx = e^{-\frac{2\pi i}{g+1}} x$ and a reflection $\mathbb{Z}_2^r$: $r(y,\ x) = (y\ x^{-(g+1)},\ x^{-1})$. It is simple to show that they combine into a dihedral group D_{4g+4}. The symmetry is enhanced at the special point $\lambda = i$ by a further $\mathbb{Z}_2$ symmetry $\mathbb{Z}_2^\sigma$: $\sigma x = -x$ to the group $(\mathbb{Z}_{2g+2} \times \mathbb{Z}_2^C) \rtimes \mathbb{Z}_2^r$. σ should be thought as an emergent S-duality. Again the action of the symmetry group on homology is easier to visualize by employing a branched

[65]This is true for even genus, if g is odd instead

$$y^2 = x\left(x^g - \lambda\right)\left(x^g - 1/\lambda\right) \tag{4.2.118}$$

and the dihedral group is D_{4g}.

cover



(4.2.120)

The left picture is the curve for $\lambda \in \mathbb{R}^+$, while the right one represents the special point $\lambda = i$. The actions of t and σ are just rotations, while r flips the picture around the real axis while also interchanging the two ends of each cut. For $g = 2$ their matrix representation is

$$M_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \quad M_r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \quad M_\sigma = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \tag{4.2.121}$$

In this case the duality group is non-abelian, and one might wonder whether the categorical structure of its fusion rules is also non-commutative.

The answer is positive. For example consider $N = 3$ and

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mathfrak{D}_{\mathcal{L}}^{M_r} \times \mathfrak{D}_{\mathcal{L}}^{M_\sigma} = \mathcal{N}^{2,+}\, \mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma} \ , \ \mathfrak{D}_{\mathcal{L}}^{M_\sigma} \times \mathfrak{D}_{\mathcal{L}}^{M_r} = \mathcal{N}^{2,-}\, \mathfrak{D}_{\mathcal{L}}^{M_\sigma\, r} \, . \tag{4.2.122}$$

The non-commutativity may also involve condensation defects, for example

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathfrak{D}_{\mathcal{L}}^{M_r} \times \mathfrak{D}_{\mathcal{L}}^{M_\sigma} = \mathcal{N}^{2,-}\, \mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma} \ , \ \mathfrak{D}_{\mathcal{L}}^{M_\sigma} \times \mathfrak{D}_{\mathcal{L}}^{M_r} = \mathcal{C}^{\mathbb{Z}_N^2}\, \mathcal{U}_{\mathcal{L}}^{M_\sigma\, r} \, . \tag{4.2.123}$$

While this happens, associativity must still hold. Let us again consider the second example above and compute $\mathfrak{D}_{\mathcal{L}}^{M_r} \times \mathfrak{D}_{\mathcal{L}}^{M_\sigma} \times \mathfrak{D}_{\mathcal{L}}^{M_r} = \mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma\,r}$, in the two fusion channels we get

$$\mathfrak{D}_{\mathcal{L}}^{M_r} \times \mathfrak{D}_{\mathcal{L}}^{M_\sigma} \times \mathfrak{D}_{\mathcal{L}}^{M_r} = \begin{cases} (\mathcal{N}^{2,-})^2 \;\; \mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma\,r} \\ \mathcal{C}^{\mathbb{Z}_N^2} \;\; \mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma\,r} \end{cases} \tag{4.2.124}$$

To compare the expressions notice that, for $N = 3$ $(\mathcal{N}^{2,-})^2 = \mathcal{N}^{4,+} = (\mathbb{Z}_N)^2$ while the condensation can be absorbed by the $\mathfrak{D}_{\mathcal{L}}^{M_r\,\sigma\,r}$ defect as it is full rank, leaving behind a $(\mathbb{Z}_N)^2$ partition function.

4.2.7 Conclusions

The main goal of this section was understanding non-invertible duality defects in theories with an extended 1-form symmetry $(\mathbb{Z}_N)^g$. These are naturally realized in 4d SCFT of class $\mathcal{S}$ whose Riemann surface Σ_g has a nontrivial automorphism group $G(\Omega)$ which acts as self-dualities. Due to the large 1-form symmetry group the structure of the non-invertible defects is more intricate than *e.g.* the case of $\mathcal{N} = 4$ SYM. Given two elements M_1 and M_2 in $G(\Omega)$ and a choice $\mathcal{L}$ of global variant the generic fusion between duality defects takes the form

$$\mathfrak{D}_{\mathcal{L}}^{M_1} \times \mathfrak{D}_{\mathcal{L}}^{M_2} = \mathcal{N}^{r,\pm}\, \mathcal{C}^{\mathcal{A}}\, \mathfrak{D}_{\mathcal{L}}^{M_{1,2}}\,, \tag{4.2.125}$$

where $\mathcal{N}^{r,\pm}$ are decoupled TQFTs and $\mathcal{C}^{\mathcal{A}}$ condensation defects for $\mathcal{A} \subset (\mathbb{Z}_N)^g$. We have given two ways to understand this structure: either from a purely QFT perspective or from 5d TQFT description. In the former it can be understood by considering the algebra of the group $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$ of discrete topological manipulations Φ on half space, while in the latter it descends from the description of twist defects in a certain 5d Chern-Simons theory, which become liberated as special points of the gravitational moduli space where $G(\Omega)$ remains un-Higgsed. This generalizes the previous analysis of [43] discussed in section 4.1. The validity of our approach is shown in various concrete examples for $N = 3$ and $g = 2$, pointing out some interesting new properties such as the non-commutativity of the non-invertible symmetry algebra.

Finally, we have given slick way to derive the fusion rules by analyzing the action of the non-invertible defects on genuine line operators. This has led us to introduce the concept of "rank" of a non-invertible symmetry.

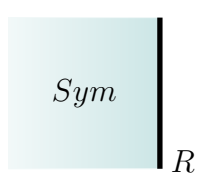
Let us close by commenting on open questions and natural generalizations of our results. Some natural extensions are the addition of punctures on the Riemann

surface — both regular and irregular — and the study of class $\mathcal{S}$ theories of D and E-type. The one-form symmetry structure in these cases has been studied *e.g.* in [97, 249]. We expect that, as long as only regular punctures are involved, our methods should extend without great novelties. A second question concerns 't Hooft anomalies and the possibility of gauging these duality symmetries. This should have an interpretation in both the 4d SCFT and the 5d Symmetry TFT. Finally, we are left wandering whether preserving these symmetries also gives rise to new exotic RG flows. It is simple to prove, at least under some mild assumptions[66], that these symmetries cannot generically be realized by an SPT in the IR. It would be interesting to study the existence of such RG flows further.

4.3 Appendices

4.3.1 Appendix A: Basic manipulations with symmetry TFT

We review here some basic facts about the symmetry TFT approach to global variants of gauge theories. The idea is simple: in order to describe an n-dimensional gauge theory $\mathcal{T}$, we introduce an auxiliary system, comprised of a n-dimensional *relative* theory R together with an $(n+1)$-dimensional non-invertible TQFT Sym. The relative theory contains all the information about $\mathcal{T}$ which is insensitive to the global structure, such as correlators of local operators, possibly charged under flavor symmetries. Other properties of the theory, such as its 1-form symmetry, depend on the choice of the global structure and thus both the topological defects generating them and the charged objects are not part of R. The geometric setup is as follows:

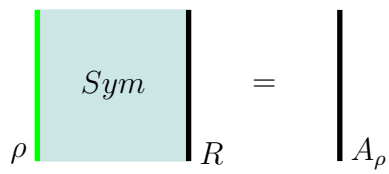


From the point of view of the $(n+1)$-dimensional theory, the output of the n-dimensional boundary manifold is not a complex number but a vector in a finite-dimensional vector space, namely the Hilbert space of Sym. From the point of

[66]The proof mimics the arguments of [120].

view of R, this Hilbert space is the vector space of partition functions. This does not define an absolute theory, as the bulk Hilbert space of Sym is in general not one-dimensional (*i.e.*, Sym in not invertible). This can be fixed by the choice of a topological boundary ρ. In general there will be multiple independent such ρ's, each one specifying an absolute theory.

Since ρ is topological, we take Sym in a slab with R and ρ boundary conditions on the two sides, and collapse the picture onto X, thus obtaining a local (absolute) theory A_ρ:



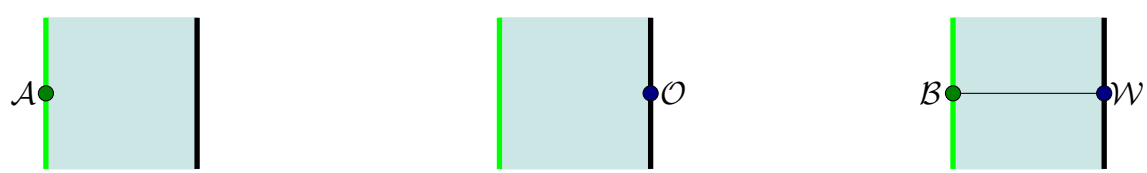
Using standard arguments, one can view the choice of ρ as the gauging of a "maximal" non-anomalous generalized symmetry $\mathcal{A}_\rho$ inside Sym. On the other hand, one can expand the state on the r.h.s. as $|\mathcal{R}\rangle = \sum_\gamma \mathcal{Z}[\gamma]\,|\gamma\rangle$, with $|\gamma\rangle$ an orthonormal basis for the TQFT Hilbert space. Computing the overlap gives the partition function of the absolute theory:

$$\mathcal{Z}[A_\rho] = \langle\rho|\mathcal{R}\rangle = \sum\nolimits_\gamma \langle\rho|\gamma\rangle\ \mathcal{Z}[\gamma]\ . \tag{4.3.1}$$

In our case $\gamma \in H^2(X, \mathbb{Z}_N \times \mathbb{Z}_N)$ is a surface in the TQFT and

$$\mathcal{Z}[A_\rho] = \sum\nolimits_{\gamma\in\mathcal{L}(\rho)} \mathcal{Z}[\gamma]\ . \tag{4.3.2}$$

Using the symmetry TFT construction we can define various objects in the absolute theory on the slab geometry:



A topological object $\mathcal{A}$ inside A_ρ | An $\mathcal{A}$-neutral operator $\mathcal{O}$ inside A_ρ | An $\mathcal{A}$-charged object $\mathcal{W}_\mathcal{B}$ inside A_ρ

The fact that $\mathcal{O}$ is neutral while $\mathcal{W}_\mathcal{B}$ is charged under $\mathcal{A}$ follows from sliding the $\mathcal{A}$ operator in the pictures above before squashing the setup into the absolute theory.

4.3.2 Appendix B: Properties of minimal TQFTs

Three-dimensional TQFTs with discrete 1-form symmetry group $\mathbb{Z}_N$ (or, more generally, products of the form $\prod_I \mathbb{Z}_{N_I}$) and fixed anomaly for said 1-form symmetry admit powerful classification results in terms of "minimal" TQFTs $\mathcal{A}^{\{N_I\},\{p_{IJ}\}}$, as pioneered in [67]. Here we review some important consequences of the classification results for $\mathbb{Z}_N$ (theories $\mathcal{A}^{N,p}$) and $\mathbb{Z}_N^r$ (theories $\mathcal{A}^{N,\mathcal{T}}$). In the main text we only use $r = 1$ and $r = 2$.

The possible anomalies for a $\mathbb{Z}_N$ 1-form symmetry in 3d are labelled by an integer p defined modulo $2N$ (or modulo N on spin manifolds) and can be represented by the following 4d inflow action [67]:[67]

$$I_{\mathcal{T}} = \frac{2\pi p}{2N} \int_X \mathfrak{P}(B) \ , \qquad\qquad B \in H^2(X, \mathbb{Z}_N) \ . \tag{4.3.3}$$

If we assume that there are no non-anomalous subgroups, that is $\gcd(N, p) = 1$, to such anomaly we associate a minimal TQFT $\mathcal{A}^{N,p}$. This theory has N line operators W_l, $l = 0, \ldots, N-1$ that form a $\mathbb{Z}_N$ fusion algebra, with spins

$$\theta_l = \exp\left(\frac{2\pi i\, p}{2N}\, l^2\right) \ . \tag{4.3.4}$$

If the theory is bosonic then $\theta_N = 1$ and $Np \in 2\mathbb{Z}$. In this thesis we deal with spin theories, in which case $W_N = \psi$ can be a transparent fermion. In bosonic theories p is identified with $p + 2N$, while in the spin case $p + N$ gives rise to the same spin theory. We stress that $\mathcal{A}^{N,p}$ is a well defined 3d TQFT if and only if $\gcd(N, p) = 1$, as otherwise the theory has transparent bosonic lines, which give a non-unitary S matrix. Otherwise, the S-matrix is given by

$$S_{l\,l'} = \frac{1}{N^{1/2}} \exp\left(\frac{2\pi i\, p}{N}\, l\, l'\right) \ . \tag{4.3.5}$$

An important result of [67] is that a 3d TQFT T with a $\mathbb{Z}_N$ 1-form symmetry with anomaly p admits an expansion in the $\mathcal{A}^{N,p}$ (hence the name "minimal"):

$$T = \mathcal{A}^{N,p} \times T' \ , \qquad\qquad T' = \frac{\mathcal{A}^{N,-p} \times T}{\mathbb{Z}_N} \ , \tag{4.3.6}$$

which can be derived using the identity

$$\left(\mathbb{Z}_N\right)_{-Np} = \mathcal{A}^{N,p} \times \mathcal{A}^{N,-p} \ . \tag{4.3.7}$$

[67] The anomaly is generated by $-I_{\mathcal{T}}$ as customary.

The product (or stacking) of two minimal theories $\mathcal{A}^{N,p} \times \mathcal{A}^{N,q}$ is also simple to compute, as long as $\gcd(p+q, N) = 1$:

$$\mathcal{A}^{N,p} \times \mathcal{A}^{N,q} = \mathcal{A}^{N,(p^{-1}+q^{-1})^{-1}} \times \mathcal{A}^{N,p+q} \,, \tag{4.3.8}$$

where inverses are taken in $\mathbb{Z}_N$. Minimal theories have a large degree of redundance, indeed let r be coprime with N, then

$$\mathcal{A}^{N,p} \sim \mathcal{A}^{N,r^2 p} \tag{4.3.9}$$

as MTCs. The transformation is equivalent to choosing a different generator for $\mathbb{Z}_N$. Note that this implies that $\mathcal{A}^{N,p} \sim \mathcal{A}^{N,p^{-1}}$. Using these conventions, we can set the line $W_{l=1}$ as the generator of $\mathbb{Z}_N$. It then follows from the S-matrix that the generator has charge p under $\mathbb{Z}_N$. Alternatively, we could use the line $W_{l=p^{-1}}$ as the fundamental line. This line has unit charge under $\mathbb{Z}_N$. The change of variables affects the inflow action, which is then labelled by p^{-1} instead. In the main text we choose to work under this choice of generator.

The construction can be generalized to multiple $\mathbb{Z}_N$ factors. For simplicity we treat the case in which N is prime, as in the main text. A theory $\mathcal{A}^{N,\mathcal{T}}$ is then described by a symmetric $r \times r$ matrix $\mathcal{T}$, whose lines have spins

$$\theta_l = \exp\left(\frac{2\pi i}{2N}\, l^{\mathsf{T}}\, \mathcal{T}\, l\right) = \exp\left(\frac{2\pi i}{2N}\left[\sum_i \mathcal{T}_{ii} l_i^2 + 2\sum_{i>j} \mathcal{T}_{ij} l_i l_j\right]\right) \,. \tag{4.3.10}$$

A bosonic theory requires $N\mathcal{T}_{ii} \in 2\mathbb{Z}$ and $N\mathcal{T}_{ij} \in \mathbb{Z}$, while a spin theory can have $\mathcal{T}_{ii} \in \mathbb{Z}$ and $\mathcal{T}_{ij} \in \mathbb{Z}$. The condition of having a well defined S-matrix requires that $\mathcal{T}$ is an invertible matrix over $\mathbb{Z}_N^r$, this is the natural generalization of the gcd condition for $r = 1$. To this TQFT we can associate an anomaly theory as in the previous case:

$$I_{\mathcal{T}} = \frac{2\pi}{2N}\mathfrak{P}^{\mathcal{T}}(B)\,, \quad \mathfrak{P}^{\mathcal{T}}(B) = \sum_i^r \mathcal{T}_{ii}\mathfrak{P}(B_i) + \sum_{i>j}^r 2\mathcal{T}_{ij} B_i \cup B_j\,, \quad B \in H^2(X, \mathbb{Z}_N^r)\,. \tag{4.3.11}$$

As in the previous case there is a large degree of redundancy in these theories. Let $\mathcal{N}$ be an invertible matrix over $\mathbb{Z}_N$, then:

$$\mathcal{A}^{N,\,\mathcal{N}^{\mathsf{T}}\mathcal{T}\mathcal{N}} = \mathcal{A}^{N,\,\mathcal{T}}\,, \tag{4.3.12}$$

as $\mathcal{N}$ just implements a redefinition of the generators. Since $\mathcal{T}$ is a nondegenerate symmetric quadratic form it can be diagonalized with coefficients in $\mathbb{Z}_N$ by a suitable

$\mathcal{N}$: $\mathcal{T} = \mathrm{diag}(t_i,\ ...,\ t_r)$ with $\gcd(t_i, N) = 1$. The only relevant information about the theory (without specifying the coupling to the two-form gauge field B) are thus the quadratic residue classes for the t_i. As for the one-dimensional case, in the main text we use a slightly different convention in which the fundamental lines have charge one. To go back to the standard convention one has to substitute $\mathcal{T}$ by $\mathcal{T}^{-1}$ in the formulas.

An important novelty with respect to the one-dimensional case is that generalized minimal theories can have anomaly free subgroups. For spin theories these are generated by vectors l such that:

$$l^{\mathsf{T}}\mathcal{T}l = 0 \mod N\,. \tag{4.3.13}$$

Let us take $r = 2$ and N prime. For $r = 2$ $\mathcal{A}^{N,\mathcal{T}}$ theories also contain twisted $\mathbb{Z}_N$ DW theories DW(α) but with a twist matrix which is a multiple of N. If N is prime every l will generate a Lagrangian subgroup, so a solution to (4.3.13) implies that the theory is DW for a certain choice of torsion. This is important in the main text, as it implements the correct fusion laws for twisted sectors on invariant boundaries.

Notice that the factorization theorem for $r = 1$ still applies. This means that, given a $\mathbb{Z}_N$ subgroup with nontrivial anomaly p, we can write:

$$\mathcal{A}^{N,\mathcal{T}} = \mathcal{A}^{N,p} \times \mathcal{A}^{N,\mathcal{T}'}\,, \tag{4.3.14}$$

And $\mathcal{T}'$ is an $r-1 \times r-1$ matrix. We use this decomposition property multiple times throughout our work.

4.3.3 Appendix C: The case of charge conjugation

Here we expand on the case of charge conjugation C, which is the only 0-form symmetry defect with vanishing torsion. Since the 4d defect theory (4.1.55) or (4.1.72) with $\mathcal{T} = 0$ is a non-invertible TQFT, the twisted sector of the charge-conjugation defect does not host a well defined MTC of line operators [67]. Consider the case of two defects $V[\mathcal{T}_1]$ and $V[\mathcal{T}_2]$ whose fusion is C. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are such that $\mathcal{T}_{21} = 0$ (they fuse onto C), then

$$\mathcal{T}_2 = -\frac{\epsilon}{2}\,\mathcal{T}_1^{-1}\,\frac{\epsilon}{2}\,. \tag{4.3.15}$$

Using our formalism, we indeed find that the braiding in $\mathcal{A}^{N,-\mathcal{T}_1} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_2}$ is degenerate, because the braiding matrix $\mathcal{K}_{21}$ in (4.1.104) has $\det \mathcal{K}_{21} = 0$.[68]

[68] One uses that if $K = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and A is invertible, then $\det(K) = \det(A)\det(D - CA^{-1}B)$. Another simple way to see this is to perform the redefinition $n_1 \to \mathcal{T}_1^{-1}\frac{\epsilon}{2}n_1$, then the matrix

Bulk fusion. Lines in the kernel of K couple to $\mathcal{B}$, but have vanishing spin and do not braid with anything else and thus do not form a well defined MTC. They are the naive restriction of the lines Ψ of the C-twisted sector when we decouple the lines charged under Φ.

To understand the fusion we must also take into account nonlocal lines. The $D[\mathcal{T}_1] \times D[\mathcal{T}_2]$ system is described by a braiding matrix:

$$K^{-1} = \frac{1}{N} \begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & \mathcal{T}_1 & 0 & \frac{\epsilon}{2} \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & -\frac{\epsilon}{2} & \mathbb{1} & \mathcal{T}_2 \end{pmatrix}, \tag{4.3.16}$$

with a basis made up of $(\Gamma_1, \Psi_1, \Gamma_2, \Psi_2)$. Thus we label a line by its charges (n_1, m_1, n_2, m_2) under the above generators. The vectors $(\mathcal{T}_1 m_1, -m_1, 0, 0)$ and $(0, 0, \mathcal{T}_2 m_2, -m_2)$ are charged only under $\mathcal{B}$ transformations, while the vectors $(n_1, 0, 0, 0)$ and $(0, 0, n_2, 0)$ are charged under Φ_1 and Φ_2 respectively. In the variables Φ, Φ_1 the lines charged only under Φ are $(n_1, 0, n_1, 0)$.

We want to decompose this system. First, lines of the form:

$$\widetilde{L} = \begin{pmatrix} \mathcal{T}_1 n \\ -n \\ -\mathcal{T}_2 n \\ n \end{pmatrix}, \tag{4.3.17}$$

are neutral w.r.t. all gauge transformations of the 4d bulk and form an $\mathcal{A}^{N,-(\mathcal{T}_1+\mathcal{T}_2)}$ decoupled theory. Lines which do not braid with them must satisfy the condition

$$n_1 - n_2 + \frac{\epsilon}{2}(m_1 + m_2) = 0. \tag{4.3.18}$$

We choose the basis:

$$L_{n_1} = \begin{pmatrix} n_1 \\ 0 \\ n_1 \\ 0 \end{pmatrix}, \quad L_{n_2} = \begin{pmatrix} \frac{\epsilon}{2} n_2 \\ n_2 \\ \frac{\epsilon}{2} n_2 \\ -n_2 \end{pmatrix}, \quad L_{n_3} = \begin{pmatrix} \frac{\epsilon}{2}(\Gamma - 1) n_3 \\ \Gamma n_3 \\ \frac{\epsilon}{2} \Gamma n_3 \\ (1 - \Gamma) n_3 \end{pmatrix} \tag{4.3.19}$$

with $\Gamma = (\mathcal{T}_1 + \mathcal{T}_2)^{-1} (\mathcal{T}_2 + \frac{\epsilon}{2})$. Notice that the relevant definitions can be read off from our Lagrangian computations in Section 4.1.4. The line L_1 is charged only

becomes: $\mathcal{K}_{2,1} = \begin{pmatrix} \mathcal{T}_2 & \mathcal{T}_2 \\ \mathcal{T}_2 & \mathcal{T}_2 \end{pmatrix}$, which indeed has half rank.

under Φ, L_{n_2} under Φ and Φ_1 while L_{n_3} only under $\mathcal{B}$ and Φ. Between these lines, L_{n_2} have nontrivial spin :

$$\theta_{n_2} = \exp\left(\frac{\pi i}{N} n_2^{\mathsf{T}} (\mathcal{T}_1 + \mathcal{T}_2) n_2\right) , \tag{4.3.20}$$

and does not braid with both L_1 and L_3. Therefore it forms a decoupled $\mathcal{A}^{N,\mathcal{T}_1+\mathcal{T}_2}(\widetilde{\Phi})$ factor where $\widetilde{\Phi} = (T_1 + T_2)(\Phi_1 + \Gamma\Phi)$. Since $\theta_{n_1} = 1$, $\theta_{n_3} = \exp\left(\frac{\pi i}{N} n_3^{\mathsf{T}} \mathcal{T}_{2,1} n_3\right)$, $B_{n_1 n_3} = \exp\left(\frac{2\pi i}{N} n_1^{\mathsf{T}} n_3\right)$ they form a $(\mathbb{Z}_{N\times N})_{N\mathcal{T}_{2,1}}(\mathcal{B}, \Phi)$ theory, which is the twisted sector for $V[\mathcal{T}_{2,1}]$:

$$D[\mathcal{T}_1] \times D[\mathcal{T}_2] = \left[\mathcal{A}^{N,\mathcal{T}_1+\mathcal{T}_2}(\widetilde{\Phi}) \times \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}\right] D[\mathcal{T}_{21}] . \tag{4.3.21}$$

Notice that:

$$\mathcal{A}^{N,\mathcal{T}_1+\mathcal{T}_2} \times \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} = (\mathbb{Z}_N \times \mathbb{Z}_N)_{N(\mathcal{T}_1+\mathcal{T}_2)} \tag{4.3.22}$$

and also that the decoupled coefficient $\mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}$ is the same decoupled TQFT as in the normal bulk fusion, which is the correct leftover coefficient once Φ_1 is integrated out. This formula generalizes smoothly to the case of charge conjugation for which $\mathcal{T}_{2,1} = 0$.

Boundary fusion Now we can understand fusions on a gapped boundary $\mathcal{L}$. All gapped boundaries are C-invariant, thus the twisted sector will host a genuine GW operator in the gauged theory, plus a condensate coming from the fusion.

First we must discuss what happens to the full defect $D[\mathcal{T}]$ when it approaches the gapped boundary. In the 4d zero form symmetry defect $V[\mathcal{T}]$ we have a coupling $\mathcal{B}^{\mathsf{T}}\,\Phi$. As we move to the boundary this becomes $\mathcal{B}^{\mathsf{T}}\,\Phi \to \tilde{b}_\perp\, l_\perp^{\mathsf{T}}\Phi$. We expand

$$\begin{aligned} \Phi &= l\phi_l + \frac{u\phi_\perp}{l_\perp^{\mathsf{T}} u} \;\Rightarrow\; l_\perp^{\mathsf{T}}\Phi = \Phi_\perp , \\ \mathcal{B} &= l_\perp \tilde{b}_\perp + \frac{u_\perp b_l}{u_\perp^{\mathsf{T}} l} \;\Rightarrow\; l^{\mathsf{T}}\mathcal{B} = b_l , \end{aligned} \tag{4.3.23}$$

where u is the generator of $\mathcal{S}$ defined in Section 4.1.3 and $u_\perp = \langle\, , u\rangle$ is such that $l_\perp^{\mathsf{T}} u = l^{\mathsf{T}} u_\perp \neq 0$ [69]. Labelling a line $L_{n,m}$ in the twisted sector by its charges $(n,\, m)$ under Γ and Ψ we find that the lines

$$\begin{pmatrix} u_\perp \\ 0 \end{pmatrix} , \begin{pmatrix} -\mathcal{T} l \\ l \end{pmatrix} \tag{4.3.24}$$

[69] When $\mathcal{T} \neq 0$ we can choose $u = \mathcal{T}^{-1} l_\perp$ and we find $b_\perp \equiv l_\perp^{\mathsf{T}} \mathcal{T}^{-1} \mathcal{B} = t_\perp \tilde{b}_\perp$.

are charged only under ϕ_l and b_l respectively. They form a DW theory with braiding matrix

$$K^{-1} = \begin{pmatrix} 0 & l^{\mathsf{T}} u_\perp \\ l^{\mathsf{T}} u_\perp & -t_l \end{pmatrix} . \tag{4.3.25}$$

After a rescaling of the electric generator this becomes a $(\mathbb{Z}_N)_{Nt_l}(\phi_l, b_l)$ DW theory. Remaining lines need to have trivial braiding with these generators. They have a basis given by

$$\begin{pmatrix} 0 \\ u \end{pmatrix} , \quad \begin{pmatrix} l_\perp \\ 0 \end{pmatrix} , \tag{4.3.26}$$

with braiding matrix

$$\tilde{K}^{-1} = \begin{pmatrix} 0 & l_\perp^{\mathsf{T}} u \\ l_\perp^{\mathsf{T}} u & u^{\mathsf{T}} \mathcal{T} u \end{pmatrix} . \tag{4.3.27}$$

To get a more familiar result notice that $u = \mathcal{T}^{-1} l_\perp$ is a good choice for u as long as the boundary is not invariant. In these variables the lines

$$\begin{pmatrix} l_\perp \\ -\mathcal{T}^{-1} l_\perp \end{pmatrix} , \tag{4.3.28}$$

have spin $\exp\left(-\frac{\pi i}{N} t_\perp\right)$ and couple only to $b_\perp$ with unit charge. They thus correctly reproduce the sub-theory $\mathcal{A}^{N,-t_\perp}(b_\perp)$. We conclude that this procedure is consistent with the one used in section 4.1.4 where the field Φ was integrated out. However this procedure is more general and in particular it can be extended to the case of charge conjugation. We have shown that in general

$$D[\mathcal{T}] = (\mathbb{Z}_N)_{-Nt_l}(b_l, \phi_l) \times (\mathbb{Z}_N)_{Nu^{\mathsf{T}}\mathcal{T}u}(\tilde{b}_\perp, \phi_\perp) . \tag{4.3.29}$$

We want now to discuss the fusions of two twist defects on the gapped boundary. A simple way to derive the boundary fusion is to start from the formula (4.3.21), impose boundary conditions which set the decoupled DW theories to one on the boundary (which is a consistent boundary condition) and divide

$$(\mathbb{Z}_N \times \mathbb{Z}_N)_0(\mathcal{B}, \Phi) = (\mathbb{Z}_N)_0(\phi_l, b_l) \times (\mathbb{Z}_N)_0(\tilde{b}_\perp, \phi_\perp) . \tag{4.3.30}$$

The first term is generated by lines:

$$\begin{pmatrix} u_\perp \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 0 \\ l \end{pmatrix} , \tag{4.3.31}$$

While the second by lines:

$$\begin{pmatrix} 0 \\ u \end{pmatrix} , \quad \begin{pmatrix} l_\perp \\ 0 \end{pmatrix} . \tag{4.3.32}$$

Notice that the braiding are non-degenerate owning to $l_\perp^\mathsf{T} u = l^\mathsf{T} u_\perp \neq 0$. The first term is also a decoupled DW theory which can be set to one on the boundary, while the second term is a condensate for the $\mathbb{Z}_N$ surviving there. One would then conclude

$$D_\mathcal{L}[\mathcal{T}_1] \times D_\mathcal{L}[\mathcal{T}_2] = (\mathbb{Z}_N)_0(\tilde{b}_\perp, \phi_\perp)\, D_\mathcal{L}[0] \,, \tag{4.3.33}$$

for a trivial $D_\mathcal{L}[0]$.

4.3.4 Appendix D: Matrix representation of $\mathrm{Sp}(2g,\, \mathbb{Z}_N)_T$

In this appendix we describe the matrix representation of generic topological manipulations $\Phi \in \mathrm{Sp}(2g,\, \mathbb{Z}_N)$. The matrices generating the parabolic group are

$$\tau(S) = \begin{pmatrix} \mathbb{1} & S \\ 0 & \mathbb{1} \end{pmatrix} , \quad \nu(U) = \begin{pmatrix} U^{-1\mathsf{T}} & 0 \\ 0 & U \end{pmatrix} . \tag{4.3.34}$$

Notice that, under right composition[70]

$$\tau(S)\, \nu(U) = \begin{pmatrix} U^{-1\mathsf{T}} & U^{-1\mathsf{T}}\, S \\ 0 & U \end{pmatrix} = \nu(U)\, \tau(U^{-1\mathsf{T}}\, S\, U^{-1}) \tag{4.3.35}$$

as it should be. A generic gauging matrix $\sigma(C_\mathcal{A})$, satisfying

$$\sigma(C_\mathcal{A})\, \sigma(C_\mathcal{A}) = \nu(\mathbf{C}_\mathcal{A}) \,, \quad \mathbf{C}_\mathcal{A} = -C_\mathcal{A}\, C_\mathcal{A}^* + C_{\tilde{\mathcal{A}}}\, C_{\tilde{\mathcal{A}}}^* \tag{4.3.36}$$

can be constructed in the following way: start with the rank r projector $P = \sum_{j=1}^r E_j$ and its complementary $P^\perp = \mathbb{1} - P$. Define the matrix:

$$\sigma(P) = \begin{pmatrix} P^\perp & -P \\ P & P^\perp \end{pmatrix} \tag{4.3.37}$$

[70] To follow the notation in the main paper, topological manipulations act multiplicatively on the right, however if taken as a matrix representation they act on the left. All the compositions are written in the former conventions.

which describes the gauging of $E_1 \vee E_2 \vee ... \vee E_r$. It is an $\mathrm{Sp}(2g, \mathbb{Z}_N)_T$ matrix and $\sigma(P)^2 = \begin{pmatrix} P^\perp - P & 0 \\ 0 & P^\perp - P \end{pmatrix}$. Consider

$$\sigma(C_\mathcal{A}) \equiv \nu(u^{-1})\, \sigma(P)\, \nu(u) = \begin{pmatrix} u^{-1\mathsf{T}}\, P^\perp\, u^\mathsf{T} & -u^{-1\mathsf{T}}\, P\, u^{-1} \\ u\, P\, u^\mathsf{T} & u\, P^\perp\, u^{-1} \end{pmatrix}, \tag{4.3.38}$$

where $u = (C_\mathcal{A} \mid C_{\tilde{\mathcal{A}}})$. Then $uP = (C_\mathcal{A} \mid 0)$, $uP^\perp = (0 \mid C_{\tilde{\mathcal{A}}})$, while

$$Pu^{-1} = \begin{pmatrix} C^*_\mathcal{A} \\ 0 \end{pmatrix} \quad P^\perp u^{-1} = \begin{pmatrix} 0 \\ C^*_{\tilde{\mathcal{A}}} \end{pmatrix}. \tag{4.3.39}$$

Then

$$\sigma(C_\mathcal{A})^2 = \begin{pmatrix} u^{-1\mathsf{T}}\, (P^\perp - P)\, u^\mathsf{T} & 0 \\ 0 & u(P^\perp - P)u^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_\mathcal{A}^{-1\mathsf{T}} & 0 \\ 0 & \mathbf{C}_\mathcal{A} \end{pmatrix}. \tag{4.3.40}$$

We can also write down the matrix corresponding to $\sigma(C_\mathcal{A})\, \tau({S_\mathcal{A}}^\mathcal{A}) \equiv \Phi_{\mathcal{A}, S_\mathcal{A}}$:[71]

$$\Phi_{\mathcal{A}, S_\mathcal{A}} = \begin{pmatrix} u^{-1\mathsf{T}}(P^\perp + S_\mathcal{A})u^\mathsf{T} & -u^{-1\mathsf{T}}\, P\, u^{-1} \\ u\, P\, u^\mathsf{T} & u\, P^\perp\, u^{-1} \end{pmatrix}. \tag{4.3.41}$$

From these definitions it is possible to reconstruct the full algebra of $\mathrm{Sp}(2g\,, \mathbb{Z}_N)_T$ barring the central extension. We also want to prove the standard-form decomposition for elements $\Phi \in \mathrm{Sp}(2g, \mathbb{Z}_N)_T$. That is, we want to write [72]:

$$\Phi = \nu(V)\, \tau(S')\, \sigma(C_\mathcal{A})\, \tau(S)\,, \quad S = {S_\mathcal{A}}^\mathcal{A} = (u^{-1})^\mathsf{T}\, P\, s\, Pu^{-1}\,. \tag{4.3.42}$$

We parametrize $\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. A short computation shows that $C = u\, P\, u^\mathsf{T}\, V^{-1\mathsf{T}}$, so the matrices u and V can be extracted by computing the Smith Normal Form of C. Having done this we find a matrix $Y_\mathcal{A} = \sigma(C_\mathcal{A})\, \tau(S)$ such that $\Phi\, Y_\mathcal{A}^{-1}$ is parabolic. This can be done if the equation[73]

$$P\, s\, Pu^\mathsf{T} - P\, u^\mathsf{T}\, A\, V^\mathsf{T} = 0 \mod N \tag{4.3.43}$$

has solutions. Imposing that Φ is symplectic we get that $AV^\mathsf{T} = Q\, u\, P\, u^\mathsf{T}$ for some matrix Q. One finally sets $s = u^\mathsf{T}\, Q\, u$ to solve the equation. With this procedure it is possible to put any discrete manipulation Φ into the standard form.

[71] We think of $S_\mathcal{A}$ as a matrix with non-zero entries only in the upper-left $r \times r$ corner.

[72] Here matrices are written on the left, but composition should be understood on the right.

[73] This comes from setting the bottom left block of $\Phi\, Y_\mathcal{A}^{-1}$ to zero.

4.3.5 Appendix E: Composition laws for topological manipulations

K-formula for $g = 1$. Let us first derive the K-formula for the $g = 1$ case. Such computation is relatively easy but it is also helpful in order to understand the more complicated computation done for generic g.

We start by recalling the generic definition of the topological manipulation $\sigma\ \tau(k)\ \sigma$

$$[\sigma\ \tau(k)\ \sigma Z]\,(B) = \sum_{b,\,c\,\in\,H^2(X,\mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N}\int b\cup(c+B) + \frac{k}{2}\mathfrak{P}(b)\right)\ Z(c)\,. \quad (4.3.44)$$

By changing variables $b \to b - k^{-1}(c+B)$, we cancel the minimal coupling between b and c fields, obtaining

$$\begin{aligned} &= Y_k\ \exp\left(-\frac{2\pi i k^{-1}}{2N}\mathfrak{P}(B)\right) \sum_{c\in H^2(X,\mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N}\int c\cup(-k^{-1}B) - \frac{k^{-1}}{2}\mathfrak{P}(c)\right)\ Z(c) \\ &= Y_k\left[\nu(-k^{-1})\ \tau(-k)\ \sigma\ \tau(-k^{-1})Z\right](B) \end{aligned} \quad (4.3.45)$$

where the central extension is

$$Y_k = \sum_{b\in H^2(X,\mathbb{Z}_N)} \exp\left(\frac{2\pi i k}{2N}\mathfrak{P}(b)\right)\,. \quad (4.3.46)$$

K-formula for generic g Let us now treat the general case of $\sigma(C_{\mathcal{A}})\ \tau(S)\ \sigma(C_{\mathcal{B}})$. First, we want to restrict to the case in which $S = (S_{\mathcal{A}})^{\mathcal{A}}$. This can be done straightforwardly by expanding the quadratic form $\mathfrak{P}^S$ leading to the identity

$$\begin{aligned} &\sigma(C_{\mathcal{A}})\tau(S) = \tau(S_{\tilde{\mathcal{A}}}{}^{\tilde{\mathcal{A}}})\ \nu(V_{\mathcal{A}})\ \sigma(C_{\mathcal{A}})\ \tau(S_{\mathcal{A}}{}^{\mathcal{A}}) \\ &V_{\mathcal{A}} = C_{\mathcal{A}}\ C^*_{\mathcal{A}} + C_{\tilde{\mathcal{A}}}\ C^*_{\tilde{\mathcal{A}}} + C_{\mathcal{A}}\ S_{\mathcal{A}\tilde{\mathcal{A}}}\ C^*_{\tilde{\mathcal{A}}}\,. \end{aligned} \quad (4.3.47)$$

We then assume safely that $S = S_{\mathcal{A}}{}^{\mathcal{A}}$. Secondly, we want to change our basis of generators so that $C_{\mathcal{A}} = (C_{\mathcal{A}'} \mid C_{\mathcal{C}})$, $C_{\mathcal{B}} = (C_{\mathcal{B}'} \mid C_{\mathcal{C}})$ with $\mathcal{C} = \mathcal{A} \wedge \mathcal{B}$. This can be implemented through right multiplication by a $\mathrm{GL}(r(\mathcal{A}),\mathbb{Z}_N)$ matrix $u_{\mathcal{A}}$ and similarly for $\mathcal{B}$. Using the definition of σ it is simple to see that the two are related by ν transformations. With this in mind we can assume that all the matrices are

already given in their "split" form. The double gauging reads, explicitly

$$\begin{aligned}
&\left[\sigma(C_{\mathcal{A}})\,\tau({S_{\mathcal{A}}}^{\mathcal{A}})\,\sigma(C_{\mathcal{B}})Z\right](B) = \\
&= \sum_{\substack{\alpha_{\mathcal{A}'},\alpha(\mathcal{C})\\ \beta_{\mathcal{B}'},\beta_{\mathcal{C}}}} \exp\left(\frac{2\pi i}{N}\int \alpha_{\mathcal{A}'}\cup B_{\mathcal{A}'} + \beta_{\mathcal{B}'}\cup B_{\mathcal{B}'} + \alpha_{\mathcal{C}}\cup(\beta_{\mathcal{C}}+B_{\mathcal{C}}) + \frac{1}{2}\mathfrak{P}^{S}(C_{\mathcal{A}'}\alpha_{\mathcal{A}} + C_{\mathcal{C}}\alpha_{\mathcal{C}})\right) \\
&\qquad\qquad \times Z\left(C_{\mathcal{A}'}\alpha_{\mathcal{A}'} + C_{\mathcal{B}'}\beta_{\mathcal{B}'} + C_{\mathcal{C}}\beta_{\mathcal{C}} + \tilde{C}\tilde{B}\right).
\end{aligned} \tag{4.3.48}$$

The quadratic function $\mathfrak{P}^S$ expands as $\mathfrak{P}^{S_{\mathcal{C}}}(\alpha_{\mathcal{C}}) + \mathfrak{P}^{S_{\mathcal{A}'}}(\alpha_{\mathcal{A}'}) + 2\alpha_{\mathcal{C}}\cup S_{\mathcal{C}\,\mathcal{A}'}\alpha_{\mathcal{A}'}$. If $S_{\mathcal{C}}$ is invertible we redefine:

$$\alpha_{\mathcal{C}} \to \alpha_{\mathcal{C}} - S_{\mathcal{C}}^{-1}(\beta_{\mathcal{C}} + B_{\mathcal{C}} + S_{\mathcal{C}\,\mathcal{A}'}\,\alpha_{\mathcal{A}'})\,, \tag{4.3.49}$$

eliminating the linear couplings for $\alpha_{\mathcal{C}}$. Expanding:

$$\begin{aligned}
(4.3.48) = &\left(\sum_{\alpha_{\mathcal{C}}} \exp\left(\frac{2\pi i}{2N}\mathfrak{P}^{S_{\mathcal{C}}}(\alpha_{\mathcal{C}})\right)\right) \exp\left(-\frac{2\pi i}{2N}\int \mathfrak{P}^{{S_{\mathcal{C}}}^{-1}}(B_{\mathcal{C}})\right)\times \\
&\times \sum_{\substack{\alpha_{\mathcal{A}'}\\ \beta_{\mathcal{B}'},\beta_{\mathcal{C}}}} \exp\left(\frac{2\pi i}{N}\int \alpha_{\mathcal{A}'}\cup(B_{\mathcal{A}'} - S_{\mathcal{A}'\,\mathcal{C}}\,S_{\mathcal{C}}^{-1}B_{\mathcal{C}}) - \beta_{\mathcal{C}}\cup {S_{\mathcal{C}}}^{-1}\,B_{\mathcal{C}} + \beta_{\mathcal{B}'}\cup B_{\mathcal{B}'}\right)\times \\
&\times \exp\left(\frac{2\pi i}{2N}\int \mathfrak{P}^{S_{\mathcal{A}'} - S_{\mathcal{A}'\,\mathcal{C}}\,S_{\mathcal{C}}^{-1}\,S_{\mathcal{C}\,\mathcal{A}'}}(\alpha_{\mathcal{A}'}) - \mathfrak{P}^{{S_{\mathcal{C}}}^{-1}}(\beta_{\mathcal{C}}) - \alpha_{\mathcal{A}'}\cup S_{\mathcal{A}'\,\mathcal{C}}\,{S_{\mathcal{C}}}^{-1}\,\beta_{\mathcal{C}}\right)\times \\
&\times Z\left(C_{\mathcal{A}'}\,\alpha_{\mathcal{A}'} + C_{\mathcal{B}'}\,\beta_{\mathcal{B}'} + C_{\mathcal{C}}\,\beta_{\mathcal{C}} + \tilde{C}\,\tilde{B}\right).
\end{aligned} \tag{4.3.50}$$

The new torsion is thus given by the matrix

$$\begin{aligned}
-X_{\mathcal{A}'\,\mathcal{B}'\,\mathcal{C}} =& {S_{\mathcal{A}'}}^{\mathcal{A}'} - \left(S_{\mathcal{A}'\,\mathcal{C}}\,S_{\mathcal{C}}^{-1}\,S_{\mathcal{C}\,\mathcal{A}'}\right)^{\mathcal{A}'} - \\
&\left(S_{\mathcal{A}'\,\mathcal{C}}\,{S_{\mathcal{C}}}^{-1}\right)^{\mathcal{A}'\,\mathcal{C}} - \left({S_{\mathcal{C}}}^{-1}S_{\mathcal{C}\,\mathcal{A}'}\right)^{\mathcal{C}\,\mathcal{A}'} - {S_{\mathcal{C}}}^{-1\mathcal{C}}
\end{aligned} \tag{4.3.51}$$

while to get the correct couplings we must perform a redefinition on $B = C_{\mathcal{A}'}\,B_{\mathcal{A}'} + C_{\mathcal{B}'}\,B_{\mathcal{B}'} + C_{\mathcal{C}}\,B_{\mathcal{C}} + \tilde{\mathcal{C}}\,\tilde{B}$ by a matrix:

$$U_{\mathcal{A}'\,\mathcal{B}'\,\mathcal{C}} = C_{\mathcal{A}'}\left(C^*_{\mathcal{A}'} - S_{\mathcal{A}'\,\mathcal{C}}\,S_{\mathcal{C}}^{-1}C^*_{\mathcal{C}}\right) - C_{\mathcal{C}}\,S_{\mathcal{C}}^{-1}\,C^*_{\mathcal{C}} + C_{\mathcal{B}'}\,C^*_{\mathcal{B}'} + \tilde{C}\,\tilde{C}^*\,. \tag{4.3.52}$$

This shows that:

$$\begin{aligned}
&\sigma(C_{\mathcal{A}})\,\tau((S_{\mathcal{A}})^{\mathcal{A}})\,\sigma(C_{\mathcal{B}}) = \\
&\qquad Y_{S_{\mathcal{C}}}\,\tau(-{S_{\mathcal{C}}}^{-1\mathcal{C}})\,\nu(U_{\mathcal{A}'\,\mathcal{B}'\,\mathcal{C}})\,\sigma(C_{\mathcal{A}'}\mid C_{\mathcal{B}'}\mid C_{\mathcal{C}})\,\tau(-X_{\mathcal{A}'\,\mathcal{B}'\,\mathcal{C}})\,.
\end{aligned} \tag{4.3.53}$$

Now we consider the case in which $S_{\mathcal{C}}$ has a kernel. Since we work at $g = 2$ we only consider the case in which (I) $S_{\mathcal{C}} = 0$ or (II) $C_{\mathcal{A}} = C_{\mathcal{B}} = \mathbb{1}$ and $S_{\mathcal{C}}$ has a kernel $\mathcal{K}$.

(I) Since $S_{\mathcal{C}} = 0$ $\alpha_{\mathcal{C}}$ enters the equation linearly and imposes a constraint:

$$\beta_{\mathcal{C}} = -(B_{\mathcal{C}} + S_{\mathcal{C}\mathcal{A}'}\ \alpha_{\mathcal{A}'})\,. \tag{4.3.54}$$

Letting $C_{\mathcal{D}} = (C_{\mathcal{A}'} - C_{\mathcal{C}}\ S_{\mathcal{C}\mathcal{A}'})$ we find:

$$\begin{aligned}(4.3.48) = \sum_{\alpha_{\mathcal{A}'},\beta_{\mathcal{B}'}} \exp\left(\frac{2\pi i}{N}\int \alpha_{\mathcal{A}'}\cup B_{\mathcal{A}'} + \beta_{\mathcal{B}'}\cup B_{\mathcal{B}'} + \frac{1}{2}\mathfrak{P}^{S_{\mathcal{A}'}}(\alpha_{\mathcal{A}'})\right)\times \\ \times Z\left(C_{\mathcal{D}}\ \alpha_{\mathcal{A}'} + C_{\mathcal{B}'}\ \beta_{\mathcal{B}'} - C_{\mathcal{C}}\ B_{\mathcal{C}} + \tilde{C}\ \tilde{B}\right)\,.\end{aligned} \tag{4.3.55}$$

This is the partition function of a theory in which $C_{\mathcal{D}} \mid C_{\mathcal{B}'}$ is gauged with torsion ${S_{\mathcal{A}'}}^{\mathcal{D}}$. The background is:

$$B' = C_{\mathcal{D}}\ B_{\mathcal{A}'} + C_{\mathcal{B}'}\ B_{\mathcal{B}'} - C_{\mathcal{C}}\ B_{\mathcal{C}} + \tilde{C}\ \tilde{B} \tag{4.3.56}$$

and is obtained for the original one by a transformation

$$U_{\mathcal{A}\,\mathcal{B}} = C_{\mathcal{D}}\ C^*_{\mathcal{A}'} + C_{\mathcal{B}'}\ C^*_{\mathcal{B}'} - C_{\mathcal{C}}\ C^*_{\mathcal{C}} + \tilde{\mathcal{C}}\ \tilde{\mathcal{C}}^*\,. \tag{4.3.57}$$

Therefore in case **(I)** we have

$$\sigma(C_{\mathcal{A}})\ \tau((S_{\mathcal{A}})^{\mathcal{A}})\ \sigma(C_{\mathcal{B}}) = \nu(U_{\mathcal{A}\,\mathcal{B}})\ \sigma(C_{\mathcal{D}} \mid C_{\mathcal{B}'})\ \tau((S_{\mathcal{A}'})^{\mathcal{D}})\,. \tag{4.3.58}$$

(II) Let $\mathcal{K}$ be the kernel of S. The decompose the dynamical field $\beta = C_{\mathcal{K}}\beta_{\mathcal{K}} + C_{\tilde{\mathcal{K}}}\ \beta_{\tilde{\mathcal{K}}}$. The discrete gauging is:

$$\begin{aligned}&[\sigma(\mathbb{1})\ \tau(S)\ \sigma(\mathbb{1})\ Z]\,(B) = \\ &\sum_{\substack{\alpha_{\mathcal{K}},\alpha_{\tilde{\mathcal{K}}}\\ \beta_{\mathcal{K}},\beta_{\tilde{\mathcal{K}}}}} \exp\left(\frac{2\pi i}{N}\int \alpha_{\mathcal{K}}\cup(\beta_{cK} + B_{\mathcal{K}}) + \alpha_{\tilde{\mathcal{K}}}\cup(\beta_{\tilde{\mathcal{K}}} + B_{\tilde{\mathcal{K}}}) + \frac{1}{2}\mathfrak{P}^{S_{\tilde{\mathcal{K}}}}(\alpha_{\tilde{\mathcal{K}}})\right)\times \\ &\times Z\left(C_{\mathcal{K}}\ \beta_{\mathcal{K}} + C_{\tilde{\mathcal{K}}}\ \beta_{\tilde{\mathcal{K}}}\right)\,.\end{aligned} \tag{4.3.59}$$

Since $\alpha_{\mathcal{K}}$ appears linearly we integrate it out, fixing $\beta_{\mathcal{K}} = -B_{\mathcal{K}}$. Thus finding

$$\sigma(\mathbb{1})\ \tau(S)\ \sigma(\mathbb{1}) = \nu\left(-C_{\mathcal{K}}\ C^*_{\mathcal{K}} + C_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}\right)\ \sigma(C_{\tilde{\mathcal{K}}})\ \tau({S_{\tilde{\mathcal{K}}}}^{\tilde{\mathcal{K}}})\ \sigma(C_{\tilde{\mathcal{K}}})\,. \tag{4.3.60}$$

Applying the K-formula the second term becomes

$$\sigma(C_{\tilde{\mathcal{K}}})\ \tau({S_{\tilde{\mathcal{K}}}}^{\tilde{\mathcal{K}}})\ \sigma(C_{\tilde{\mathcal{K}}}) = Y_{S_{\tilde{\mathcal{K}}}}\ \tau\left(-S^{-1\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}}\right)\ \nu\left(C_{\mathcal{K}}\ C^*_{\mathcal{K}} - C_{\tilde{\mathcal{K}}}\ S^{-1}_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}\right)\ \sigma(C_{\tilde{\mathcal{K}}})\ \tau\left(-S^{-1\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}}\right)\,. \tag{4.3.61}$$

Defining $U = -C_{\mathcal{K}}\ C^*_{\mathcal{K}} + C_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}$ and $V = C_{\mathcal{K}}\ C^*_{\mathcal{K}} - C_{\tilde{\mathcal{K}}}\ S^{-1}_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}$ we have $V^{-1} = C_{\mathcal{K}}\ C^*_{\mathcal{K}} - C_{\tilde{\mathcal{K}}}\ S_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}$, $V\ U = W = -\left(C_{\mathcal{K}}\ C^*_{\mathcal{K}} + C_{\tilde{\mathcal{K}}}\ S^{-1}_{\tilde{\mathcal{K}}}\ C^*_{\tilde{\mathcal{K}}}\right)$ and $(V^{-1})^{\mathsf{T}}\ S^{-1\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}}\ V^{-1} = {S_{\tilde{\mathcal{K}}}}^{\tilde{\mathcal{K}}}$. The final formula is

$$\sigma(\mathbb{1})\ \tau(S)\ \sigma(\mathbb{1}) = Y_{S_{\tilde{\mathcal{K}}}}\ \nu(W)\ \tau(-{S_{\tilde{\mathcal{K}}}}^{\tilde{\mathcal{K}}})\ \sigma(C_{\tilde{\mathcal{K}}})\ \tau(-S^{-1\tilde{\mathcal{K}}}_{\tilde{\mathcal{K}}})\,. \tag{4.3.62}$$

Composing $\sigma(C_{\mathcal{A}})$ and $\nu(U)$ Lastly we need to understand the composition

$$[\sigma(C_{\mathcal{A}})\ \nu(U)\ Z]\,(B) = \sum_{\alpha_{\mathcal{A}}} \exp\left(\frac{2\pi i}{N}\int \alpha_{\mathcal{A}} \cup B_{\mathcal{A}}\right)\ Z\left(U\left(C_{\mathcal{A}}\ \alpha_{\mathcal{A}} + C_{\tilde{\mathcal{A}}}\ B_{\tilde{\mathcal{A}}}\right)\right)\ . \tag{4.3.63}$$

Defining $C_{\mathcal{A}_U} = UC_{\mathcal{A}}$ and $C_{\tilde{\mathcal{A}}_U} = UC_{\tilde{\mathcal{A}}}$[74] we can write this as:

$$(4.3.63) = [\sigma(C_{\mathcal{A}_U})\ Z]\,(B' = C_{\mathcal{A}_U}\ B_{\mathcal{A}} + C_{\tilde{\mathcal{A}}_U}\ B_{\tilde{\mathcal{A}}}) \tag{4.3.64}$$

The original couplings were instead $B = C_{\mathcal{A}}\ B_{\mathcal{A}} + C_{\tilde{\mathcal{A}}}\ B_{\tilde{\mathcal{A}}}$, so we need to compose with $\nu(U)$

$$\sigma(C_{\mathcal{A}})\ \nu(U) = \nu(U)\ \sigma(C_{\mathcal{A}_U})\ . \tag{4.3.65}$$

4.3.6 Appendix F: Quadratic forms over $\mathbb{Z}_N$

We want to classify all the quadratic forms $q : \mathbb{Z}_N^g \to \mathbb{Z}_N$ for N prime, namely symmetric $g \times g$ matrices $\mathcal{T}$ with $\mathbb{Z}_N$ entries, up to the equivalence relation

$$\mathcal{T} \sim R^T \mathcal{T} R\ , \qquad R \in \mathrm{GL}(g, \mathbb{Z}_N)\ . \tag{4.3.66}$$

In the case $g = 1$ it is easy to see that there are three classes: 0, the perfect squares, and the non perfect squares. Consider now $g = 2$. First we put $\mathcal{T}$ in diagonal form with eigenvalues p_1, p_2, which can be swapped by a congruence transformation. If $\mathcal{T}$ is non singular there are in principle three cases: p_1, p_2 are both perfect squares, one of the two is a perfect square but the other is not, and both are not perfect squares. A congruence transformation with $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \neq 0$ preserves the diagonal form if and only if $abp_1 + cdp_2 = 0$, and the new eigenvalues are $p_1' = a^2 p_1 + c^2 p_2$, $p_2' = b^2 p_1 + d^2 p_2$. If p_1, p_2 are both perfect squares we can set $p_1 = p_2 = 1$, and to preserve the diagonal form we must have $c = sa$, $b = -sd$ with $s \neq 0$, implying

$$p_1' = a^2(s^2+1)\ \ , \qquad p_2' = d^2(s^2+1)\ . \tag{4.3.67}$$

We conclude that the case in which p_1, p_2 are both perfect squares is equivalent to that in which they are both not perfect square, while the other case form a distinct class. By taking into account the 3 classes at non-maximal rank we get 5 classes.

For generic g, if $\mathcal{T}$ of maximal rank we put it in diagonal form with eigenvalues $p_1, ..., p_g$. In principle there are $g+1$ cases, depending on the number $k = 0, ..., g$ of

[74] The duals instead transform with the inverse matrix, e.g. $C^*_{\mathcal{A}_U} = C_{\mathcal{A}} U^{-1}$.

eigenvalues which are perfect squares. However, using the result at $g = 2$ the cases k and $k+2$ are equivalent, while k and $k+1$ are distinct. Thus we get 2 *new* classes at rank g which we did not have at rank $g-1$. Thus the number of classes for $g \times g$ matrices is

$$n_{\text{classes}}(g) = 1 + 2g \ . \tag{4.3.68}$$

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