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RESEARCH ARTICLE

Sine-kernel determinant on two large intervals

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Abstract

We consider the probability of two large gaps (intervals without eigenvalues) in the bulk scaling limit of the Gaussian Unitary Ensemble of random matrices. We determine the multiplicative constant in the asymptotics. We also provide the full explicit asymptotics (up to decreasing terms) for the transition between one and two large gaps.

1 | INTRODUCTION

Let K_s be the (trace class) operator on $L^2(A)$, where $A \subset \mathbb{R}$ is a finite union of intervals (gaps), with kernel $K_s(x,y) = \frac{\sin s(x-y)}{\pi(x-y)}$. Consider the Fredholm determinant

$$P_s(A) = \det(I - K_s)_A. \tag{1}$$

The determinant (1), called the sine-kernel determinant, is the probability that the set $\frac{s}{\pi}A = \{\frac{s}{\pi}x : x \in A\}$ contains no eigenvalues of the Gaussian Unitary Ensemble (GUE) of random matrices in the bulk scaling limit where the average distance between eigenvalues is 1. Similar statements hold in other contexts: the sine-process with kernel $K_s(x,y)$ is the simplest, and one of the most common and well-studied determinantal point processes appearing in random matrix theory, random partitions, and so on. Two other most common ones are the Airy and Bessel processes which appear, in particular, as the scaling limits at the edge of the spectrum of the GUE and at the origin of the Laguerre Unitary Ensemble (LUE), respectively. The corresponding Fredholm determinants on a finite union of intervals may be described in terms of solutions to integrable systems of partial differential equations (see [30, 37, 38], and [39] for an overview). If A is a single interval, Painlevé equations appear: It was discovered by Jimbo et al. [30] that $s\frac{d}{ds}\log P_s([0,\pi])$

satisfies the σ form of Painlevé V introduced by Jimbo et al. in [29, 35]. Subsequently, analogous observations were made for the edge scaling limits by Tracy and Widom, namely, the distribution of the largest eigenvalue of the GUE (the Airy-kernel determinant, widely known as the Tracy-Widom distribution [37]) and the smallest eigenvalue of the LUE (a Bessel-kernel determinant [38]) are described in terms of solutions to Painlevé II and Painlevé V, respectively.

In the present paper, we are interested in the asymptotics of $P_s(A)$ as $s \to \infty$. Consider first $P_s(A)$ when A is a single interval. We can assume without loss[1] that $A = (-1, 1)$. The asymptotics of the logarithm of (1) have the form:

$$\log P_s((-1,1)) = -\frac{s^2}{2} - \frac{1}{4}\log s + c_0 + \mathcal{O}(s^{-1}), \qquad s \to \infty, \tag{2}$$

where

$$c_0 = \frac{1}{12}\log 2 + 3\zeta'(-1). \tag{3}$$

Here $\zeta'(z)$ is the derivative of Riemann's zeta function.

The leading term $-\frac{s^2}{2}$ was found by Dyson in 1962 in one of his fundamental papers on random matrix theory [20]. Dyson used Coulomb gas arguments. The terms $-\frac{s^2}{2} - \frac{1}{4}\log s$ were computed by des Cloizeaux and Mehta [13] in 1973 who used the fact that the eigenfunctions of K_s are spheroidal functions. The constant (3), known as the Widom-Dyson constant, was identified by Dyson [21] in 1976 who used the inverse scattering techniques and the earlier work of Widom [41] on Toeplitz determinants. The works [13, 20], and [21] are not fully rigorous. The first rigorous confirmation of the main term, that is, the fact that $\log P_s((-1,1)) = -\frac{s^2}{2}(1 + o(1))$, was given by Widom [42] in 1994. The full asymptotic expansion (2), apart from the expression (3) for c_0, was proved by Deift et al. in a landmark work [18] in 1997, where the multi-interval case was also addressed. The authors of [18] used Riemann-Hilbert techniques to determine asymptotics of the logarithmic derivative $\frac{d}{ds}\log P_s(A)$, where A is one (or a union of several) interval(s). The asymptotics for $P_s(A)$ were then obtained in [18] by integrating the logarithmic derivative with respect to s. The reason the expression for c_0 was not established in [18] is that there is no initial integration point $s = s_0$ where $P_s(A)$ would be known explicitly. In [31], the author was able to justify the value of c_0 in (3) by using a different differential identity for associated Toeplitz determinants and again the result of Widom [41]. An alternative proof of (3) was given in [17], which was based on another differential identity for Toeplitz determinants. In [17], the result of [41] was also rederived this way. Both [31] and [17] relied on Riemann-Hilbert techniques. Yet another proof of (3) was given by Ehrhardt [22] who used a very different approach of operator theory. (Analogous results on the probability of a large gap were obtained for the Airy-kernel determinant in [1, 16, 37], and for the Bessel-kernel determinant in [19, 23], see [33] for an overview. For further related results on gap probabilities see [5, 9–12, 26] and references therein.)

If A is a union of several intervals, it was shown by Widom in [43] that

$$\frac{d}{ds}\log P_s(A) = -C_1 s + C_2(s) + o(1), \qquad s \to \infty, \tag{4}$$

[1] $P_s(A)$ is invariant under translations of A, and rescaling results only in the appearance of a prefactor of s.

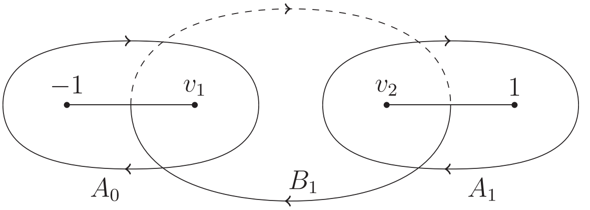

FIGURE 1 Cycles on the Riemann surface Σ.

where $C_1 > 0$ and $C_2(s)$ is a bounded oscillatory function. The constant C_1 can be computed explicitly, but $C_2(s)$ is an implicit solution of a Jacobi inversion problem. This result was extended and made more explicit by Deift et al. in [18]. We will now present the solution of [18] in the case when A is the union of two intervals, which is relevant for the present work.

As above, we assume without loss that

$$A = (-1, v_1) \cup (v_2, 1), \qquad -1 < v_1 < v_2 < 1.$$

Let $p(z) = (z^2 - 1)(z - v_1)(z - v_2)$, and consider the two-sheeted Riemann surface Σ of the function $p(z)^{1/2}$. On the first sheet $p(z)^{1/2}/z^2 \to 1$ as $z \to \infty$, while on the second, $p(z)^{1/2}/z^2 \to -1$ as $z \to \infty$. The sheets are glued at the cuts $(-1, v_1)$, $(v_2, 1)$. Each point $z \in \overline{\mathbb{C}} \setminus ((-1, v_1) \cup (v_2, 1))$ (including infinity) has two images on Σ. The Riemann surface Σ is topologically a torus.

Let the elliptic integrals $I_j = I_j(v_1, v_2) > 0$, $J_j = J_j(v_1, v_2) > 0$ be given by

$$I_j = \int_{v_2}^{1} \frac{x^j dx}{\sqrt{|p(x)|}} = \frac{i}{2} \int_{A_1} \frac{x^j dx}{p(x)^{1/2}}, \qquad J_j = \int_{v_1}^{v_2} \frac{x^j dx}{\sqrt{|p(x)|}} = \frac{1}{2} \int_{B_1} \frac{x^j dx}{p(x)^{1/2}}, \qquad j = 0, 1, 2, \tag{5}$$

where the loops (cycles) A_1, B_1 are shown in Figure 1. The loops A_0, A_1 lie on the first sheet, and the loop B_1 passes from one to the other: the part of it denoted by a solid line is on the first sheet, the other is on the second.

Let

$$\psi(z) = \frac{q(z)}{p(z)^{1/2}}, \qquad q(z) = (z - x_1)(z - x_2), \tag{6}$$

where the constants $x_1 \in (-1, v_1)$ and $x_2 \in (v_2, 1)$ are defined by the conditions

$$\int_{A_j} \psi(z) dz = 0, \qquad j = 0, 1. \tag{7}$$

It follows that

$$x_1 + x_2 = \frac{v_1 + v_2}{2}, \tag{8}$$

$$x_1 x_2 = \left(-I_2 + \frac{v_1 + v_2}{2} I_1\right) \frac{1}{I_0}, \tag{9}$$

which gives an explicit expression for $q(z)$ in terms of elliptic integrals.

Note that (7) implies that $\psi(z)$ has no residue at infinity. More precisely, we obtain as $z \to \infty$ on the first sheet

$$\psi(z) = 1 + \frac{G_0}{z^2} + \mathcal{O}(z^{-3}), \qquad G_0 = -\frac{I_2 - \frac{v_1+v_2}{2} I_1}{I_0} + \frac{1}{2} + \frac{(v_2 - v_1)^2}{8}. \tag{10}$$

As shown in [18], $G_0 > 0$.

Denote the holomorphic differential

$$\omega = i \frac{dz}{2I_0 p(z)^{1/2}}. \tag{11}$$

Clearly, it is normalized:

$$\int_{A_1} \omega = -\int_{A_0} \omega = 1. \tag{12}$$

Let

$$\tau = \int_{B_1} \omega = i\frac{J_0}{I_0}, \qquad \Omega = -\frac{1}{2\pi}\int_{B_1} \psi(x)dx = \frac{1}{\pi}\int_{v_1}^{v_2} \psi(x)dx = \frac{1}{I_0}, \tag{13}$$

where the integration $\int_{v_1}^{v_2} \psi(x)dx$ is taken on the first sheet, and where the last equation for Ω follows by Riemann's period relations (Lemma 3.45 in [18] for $n = 1$). Recall the definition (A.1) in Appendix A of the third Jacobian θ-function $\theta_3(z;\tau)$. Deift et al. found in [18] that

$$\log P_s((-1, v_1) \cup (v_2, 1)) = -s^2 G_0 + \widehat{G}_1 \log s + \log \theta_3(s\Omega; \tau) + c_1 + \mathcal{O}(s^{-1}), \qquad s \to \infty, \tag{14}$$

with G_0 as in (10), and τ, Ω as in (13). Constants $\widehat{G}_1$, c_1 are independent of s. The constant $\widehat{G}_1$ is written in [18] in terms of a limit of an integral of a combination of θ-functions. The constant term c_1 remained undetermined (for the same reason as given above in the case of one interval).

The main result of the present paper is the expression for the constant term c_1, which completes the description of the asymptotics (14). We also find that the original expression for $\widehat{G}_1$ in [18] can be simplified, and we obtain that $\widehat{G}_1 = -1/2$ (see Appendix B). We also determine this coefficient $-1/2$ of $\log s$ in a different way, as a direct result of our computation of (14) which also produced c_1. We describe this computation in more detail below in the introduction.

Thus, we obtain

Theorem 1. *The asymptotics (14) hold with*

$$\widehat{G}_1 = -\frac{1}{2}, \qquad c_1 = -\frac{1}{2}\log\frac{I_0}{\pi} - \frac{1}{8}\sum_{y\in\{-1,v_1,v_2,1\}} \log|q(y)| + 2c_0, \qquad c_0 = \frac{1}{12}\log 2 + 3\zeta'(-1). \tag{15}$$

Remark 2. Using a connection between the elliptic integral I_0 and $\theta_3(0)$, equation (101) below, and substituting $\widehat{G}_1, c_1$ into (14), we can write[2]

$$\log P_s((-1,v_1)\cup(v_2,1)) = -s^2G_0 - \frac{1}{2}\log s + \log\frac{\theta_3(s\Omega;\tau)}{\theta_3(0;\tau)} + \frac{1}{4}\log(1-v_1)(1+v_2) - \frac{1}{8}\sum_{y\in\{-1,v_1,v_2,1\}}\log|q(y)| + 2c_0 + \mathcal{O}(s^{-1}), \qquad s\to\infty. \quad (17)$$

Remark 3. The elliptic integrals I_j, J_j can be reduced to the complete ones. In particular, in the symmetric case of $-v_1 = v_2 = v$, (14) becomes (by a straightforward use of (A.37) in Appendix A)

$$\log P_s((-1,-v)\cup(v,1)) = -s^2\left(\frac{1+v^2}{2} - \frac{E(v')}{K(v')}\right) - \frac{1}{2}\log\frac{s}{\pi} + \log\theta_3\left(\frac{s}{K(v')}; 2i\frac{K(v)}{K(v')}\right) - \frac{1}{4}\log[(K(v')-E(v'))(E(v')-v^2K(v'))] + 2c_0 + \mathcal{O}(s^{-1}), \quad (18)$$

where $v' = \sqrt{1-v^2}$, and $K(z)$, $E(z)$ are the complete elliptic integrals of first and second kind, respectively, see (A.32).

Analogous results to (14), although up to an undetermined constant term, were recently obtained for the Airy and Bessel kernel determinants by Blackstone et al. in [6, 7] and [8], respectively. The latter paper dealt not only with determinants supported on two intervals, but any fixed number of intervals. Another related recent study is [32], whose authors drew inspiration from techniques of the present paper to obtain the full asymptotics (including the constant term) for the Airy kernel determinant supported on two intervals.

The asymptotics (14) with the coefficients given by (10), (13), (15) can be extended (with a worse error term) to various double scaling regimes where v_1, v_2 are allowed to approach each other or the endpoints ± 1 at a sufficiently slow rate as $s\to\infty$: Theorems 4, 10 below. In Section 10, we prove

Theorem 4 (Extension to slowly merging gaps). *For a fixed $\epsilon > 0$, let $-1+\epsilon \le v_1 < v_2 \le 1-\epsilon$ be such that $2\nu \equiv v_2 - v_1 > s^{-5/4}$. Then the asymptotics (14) hold with the error term $\mathcal{O}(s^{-1/9})$. In particular, if $s\nu \to 0$ as $s\to\infty$, the expansion of the terms in (14) gives*

$$\log P_s((-1,v_1)\cup(v_2,1)) = s^2\left(-\frac{1}{2} + \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}}\right) - \frac{1}{2}\log s + \frac{1}{4}\log\log(\gamma\nu)^{-1} - \langle\omega_0\rangle^2\log(\gamma\nu)^{-1} + \log\left(1+(\gamma\nu)^{1-2|\langle\omega_0\rangle|}\right) - \frac{1}{8}\log|\alpha\beta| + 2c_0 + o(1), \quad (19)$$

[2] Perhaps, the corresponding formula for the logarithm of the probability of $n+1$ gaps $A = \cup_{j=0}^n (a_j, b_j)$ is

$$\log\det(I-K_s)_A = -\alpha s^2 - \frac{n+1}{4}\log s + \log\frac{\theta(sV)}{\theta(0)} + \frac{1}{4}\sum_{0\le j<k\le n}\log(b_k-b_j)(a_k-a_j) - \frac{1}{8}\sum_{j=0}^{n}\log|q(a_j)q(b_j)| + (n+1)c_0 + \mathcal{O}(s^{-1}), \qquad s\to\infty. \quad (16)$$

The coefficient α here was determined in [18], and V, q, and the multivariable θ-function are in the notation of [18]. Certainly (16) behaves the way we would expect when all the gaps separate, cf. Theorem 11 below.

where $-\alpha = 1 + \frac{v_2+v_1}{2} > 0$, $\beta = 1 - \frac{v_2+v_1}{2} > 0$, $\gamma = \frac{1}{8}(\beta^{-1} + |\alpha|^{-1})$,

$$\omega_0 = \frac{s\sqrt{|\alpha\beta|}}{\log(\gamma\nu)^{-1}} > 0, \tag{20}$$

and $\langle x\rangle \in (-1/2, 1/2]$ *denotes the difference between* x *and the integer nearest to it.*

Remark 5. In Theorem 4, the rate $-5/4$ which appears in the condition $2\nu > s^{-5/4}$ can be somewhat decreased with an appropriate change of the error term $\mathcal{O}(s^{-1/9})$.

Remark 6. Using the translational invariance of $\det(I - K_s)$, we see by the shift of variable $x \to x - \frac{v_1+v_2}{2}$ that

$$P_s((-1, v_1) \cup (v_2, 1)) = P_s((\alpha, -\nu) \cup (\nu, \beta)).$$

Thus Theorem 4 provides the asymptotics for $P_s((-1, v_1) \cup (v_2, 1))$ in the case when $|v_1 - v_2| > s^{-5/4}$. In recent work [24], we obtained the asymptotics of $P_s((-1, v_1) \cup (v_2, 1)) = P_s((\alpha, -\nu) \cup (\nu, \beta))$ in the case of two gaps merging into one, that is, where v_1, v_2 are scaled with s in such a way that $|v_1 - v_2| \le 1/(s\log^2 s)$ while being bounded away from ± 1. We also showed implicitly that the asymptotics we obtained in that case uniformly connect to those of fixed $v_1 < v_2$. Theorem 4 provides an explicit matching: More precisely, we showed in [24] that[3]

Theorem 7 Splitting of the gap $(-1, 1)$ [24]. *As* $s \to \infty$, *uniformly for* $\nu = \frac{v_2-v_1}{2} \in (0, \nu_0)$, *where* $s\nu_0 \log \nu_0^{-1} \to 0$,

$$\begin{aligned}\log P_s((-1, v_1) \cup (v_2, 1)) = &-\frac{s^2}{2} + s\sqrt{|\alpha\beta|}\left(\omega_0 - \frac{\langle\omega_0\rangle^2}{\omega_0}\right) - \frac{1}{4}\log s + c_0 + \log\left(\frac{2^{2k^2-k}}{\pi^k}\frac{G(k+1)^4}{G(2k+1)}\right)\\ &+ \log\left(1 + 2\pi\kappa_{k-1}^2(\gamma\nu)^{1+2\langle\omega_0\rangle}\right) + \log\left(1 + (2\pi\kappa_k^2)^{-1}(\gamma\nu)^{1-2\langle\omega_0\rangle}\right)\\ &+ \mathcal{O}\left(\max\left\{s\nu_0\log\nu_0^{-1}, \frac{1}{\log\nu_0^{-1}}, \frac{1}{s}\right\}\right), \qquad k = \omega_0 - \langle\omega_0\rangle,\end{aligned} \tag{21}$$

where G *is the Barnes G-function, and where* κ_j *is the leading coefficient of the Legendre polynomial of degree* j *orthonormal on the interval* $[-2, 2]$, *given by*

$$\kappa_j = 4^{-j-1/2}\sqrt{2j+1}\frac{(2j)!}{j!^2}, \quad j = 1, 2, \dots, \qquad \kappa_0 = 1/2, \qquad \kappa_{-1} = 0. \tag{22}$$

The rest of notation in (21) is from Theorem 4.

[3] In [24], $\beta - \alpha$ was arbitrary, but by a rescaling argument we can assume without loss that $\beta - \alpha = 2$, which is the assumption in the present work.

As $s \to \infty$, uniformly for $\nu \in (\nu_1, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \to 0$, $\frac{s}{\log \nu_1^{-1}} \to \infty$ (i.e., $k \to \infty$), formula (21) reduces to

$$\log P_s((-1, v_1) \cup (v_2, 1)) = s^2\left(-\frac{1}{2} + \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}}\right) - \frac{1}{2}\log s + \frac{1}{4}\log\log(\gamma\nu)^{-1} - \langle\omega_0\rangle^2 \log(\gamma\nu)^{-1}$$
$$+ \log\left(1 + (\gamma\nu)^{1-2|\langle\omega_0\rangle|}\right) - \frac{1}{8}\log|\alpha\beta| + 2c_0 + \mathcal{O}\left(\max\left\{s\nu_0\log\nu_0^{-1}, \frac{1}{\log\nu_0^{-1}}, \frac{\log\nu_1^{-1}}{s}\right\}\right). \tag{23}$$

Thus we see that the asymptotic regime of Theorem 4 overlaps with that of Theorem 7 (for example, $\nu = s^{-6/5}$ belongs to both regimes), and comparing (19) with (23) we see an explicit matching. Taken together, these theorems describe the asymptotics for two large gaps and one large gap (note that (21) reduces to (2) when $\nu \to 0$ sufficiently rapidly) as well as the transition between them.

Our strategy to prove Theorem 1 relies on connecting the asymptotics for fixed $v_1 < v_2$ with another double-scaling regime, namely the one where v_1 approaches -1, and v_2 approaches 1. In this regime the scaled gaps, $s(-1, v_1)$, $s(v_2, 1)$, although still growing with s, become small in comparison with the separation between them, and we show that in that case $P_s((-1, v_1) \cup (v_2, 1))$ splits to the main orders into the product of $P_s(-1, v_1)$ and $P_s(v_2, 1)$. The advantage is that for each of the separate gaps we can use an appropriately rescaled asymptotics (2) which contains the constant c_0. More precisely, we prove in Section 2 by elementary arguments the following

Lemma 8 (Separation of gaps). *Let*

$$A_s = \left(-1, -1 + \frac{2t}{s}\right) \cup \left(1 - \frac{2t}{s}, 1\right), \qquad t = \frac{1}{2}(\log s)^{1/4}.$$

Then

$$\log\det(I - K_s)_{A_s} = -t^2 - \frac{1}{2}\log t + 2c_0 + \mathcal{O}(1/t), \qquad t \to \infty. \tag{24}$$

Remark 9. The rate of increase of t, $t = \frac{1}{2}(\log s)^{1/4}$, can be replaced with a slower rate of growth with s, and the statement will still hold.

Now we describe the steps of the proof of Theorem 1. First, we obtain in Section 3 an identity (equation (42) of Lemma 14) for the derivative $\frac{\partial}{\partial v_2}\log P_s((-1, v_1) \cup (v_2, 1))$ in terms of a certain Riemann-Hilbert (RH) problem, the Φ-RH problem. The fact that we use a differential identity with respect to one of the edges (v_2) of the gaps is crucial in allowing us to determine the constant c_1.

We then give in Section 4.4 an asymptotic solution of the Φ-RH problem as $s \to \infty$ with v_1, v_2 fixed. This problem is very similar to that solved in [18], and its solution involves the Jacobian θ-functions (we give a collection of various useful properties of θ-functions in the Appendix A below). In Section 4.5, we show that the solution of the Φ-RH problem can be extended to the double-scaling range where v_2 is allowed to approach 1 at such a rate that $(1 - v_2)s \to \infty$

(by symmetry, also v_1 is allowed to approach -1 so that $(1+v_1)s \to \infty$). It is this extension which eventually provides a connection with Lemma 8.

In Section 5, we then substitute the solution into our differential identity (see (164), (170)). In Proposition 17, we characterize the main asymptotic terms (equation (171)) in the differential identity using averaging with respect to fast oscillations.

A large part of our work, Sections 7, 8, 9, is to bring the expression (171) to an explicit form. This relies, apart from the use of standard formulae, on (specific to our setting) identities for θ-functions obtained in Lemma 16 of Section 4.2. As a result, we obtain an explicit form (194) for the non-small part (171) of the right-hand side of the differential identity (42).

We then, by Proposition 17, integrate the resulting identity with respect to v_2 from the point when $v_2 = -v_1$ is close to 1 to a fixed $v_2 = -v_1$, and then, with v_1 fixed, over v_2, so that at one of the integration limits we can use the result of Lemma 8. This proves Theorem 1. Thus the part $2c_0$ of the constant c_1 in (15) comes from Lemma 8, while the rest of c_1 comes from the integration.

As a byproduct of our proof we also obtain the following extension of the asymptotics (14).

Theorem 10 (Extension to separation of gaps). *For a fixed $\epsilon > 0$, let $-1 < v_1 < v_2 < 1$ be such that $v_2 - v_1 \geq \epsilon$, $(1-v_2)s \to \infty$, $(1+v_1)s \to \infty$. Then the asymptotics (14) hold with the error term* $\mathcal{O}(\max\{\frac{1}{(1-v_2)s}, \frac{1}{(1+v_1)s}\})$.

The independence of separated gaps established in Lemma 8 for the gaps contracting to -1 and 1, respectively, with the rate $(\log s)^{1/4}/s$ can now be extended to a slower rate of contraction. Namely, relying on Theorem 10 and evaluating the terms G_0, c_1, and τ in the limit $v_2 \to 1$ and $v_1 \to -1$, we obtain the following result in Section 6.

Theorem 11 (Independence of separated gaps). *Let $v_1 = -1 + s^{-\rho_1}$ and $v_2 = 1 - s^{-\rho_2}$, where $\rho_1, \rho_2 \in (1/2, 1)$. Then as $s \to \infty$,*

$$\frac{P_s([-1, v_1] \cup [v_2, 1])}{P_s([-1, v_1])P_s([v_2, 1])} \to 1. \tag{25}$$

More generally, the limit (25) holds in any scaling limit where

$$\min\left\{ s(1-v_2),\, s(1+v_1),\, \frac{1}{s(1-v_2)^2},\, \frac{1}{s(1+v_1)^2} \right\} \to \infty. \tag{26}$$

Note that the use of Toeplitz determinants in [17, 31] was essential to determine the constant c_0 in the asymptotics for one gap. In this paper, however, we use Lemma 8 which, in turn, relies on the already known constant c_0.

2 | SEPARATION OF GAPS: PROOF OF LEMMA 8

For $w > 2$ let

$$A^{(w)} = A_1^{(w)} \cup A_2^{(w)}, \qquad A_1^{(w)} = (-w, -w+1), \qquad A_2^{(w)} = (w-1, w).$$

With t as in Lemma 8 and $v = s/(2t)$, we have

$$\det(I - K_s)_{A_s} = \det(I - K_{2t})_{A^{(v)}}. \tag{27}$$

By (2) and translational invariance, as $t \to \infty$,

$$\det(I - K_{2t})_{A_1^{(v)}} = \det(I - K_{2t})_{A_2^{(v)}} = \det(I - K_t)_{(-1,1)} = e^{c_0} t^{-1/4} e^{-t^2/2}(1 + \mathcal{O}(1/t)).$$

Therefore, upon setting $u = 2t$, $w = v$, we obtain Lemma 8 as a direct consequence of the following lemma we now prove.

Lemma 12. *Let $u, w > 2$. There exist absolute constants $C_3, C_4 > 0$ such that*

$$\left|\det(I - K_u)_{A^{(w)}} - \det(I - K_u)_{A_1^{(w)}} \det(I - K_u)_{A_2^{(w)}}\right| \le \frac{C_3}{w} e^{C_4 u^2}. \tag{28}$$

We start with

Proposition 13. *Let $m \in \{0, 1, \dots\}$ and B be an $m+1 \times m+1$ matrix satisfying $|B_{jk}| \le u$ for all $j, k = 1, \dots, m+1$. Let $\widehat{X}$ be a set of indices j, k such that $|B_{jk}| < 1/w$ for all $(j,k) \in \widehat{X}$ and set*

$$\widehat{B}_{jk} = \begin{cases} B_{jk} & \text{if } (j,k) \notin \widehat{X}, \\ 0 & \text{if } (j,k) \in \widehat{X}. \end{cases}$$

Then

$$|\det B - \det \widehat{B}| \le \frac{1}{w}(C_1 u)^m \sqrt{m!} \tag{29}$$

for a sufficiently large absolute constant $C_1 > 0$.

Proof. Let $B^{(0)} = B$ and

$$B^{(\ell)}_{jk} = \begin{cases} B_{jk} & \text{if } (j,k) \notin \widehat{X}, \\ 0 & \text{if } (j,k) \in \widehat{X} \text{ and } j \le \ell \end{cases} \qquad \ell = 1, \dots, m+1. \tag{30}$$

In particular, $\widehat{B} = B^{(m+1)}$.

Expanding B and $B^{(1)}$ in the first row we have

$$|\det B - \det B^{(1)}| \le \frac{1}{w} \sum_{k=1}^{m+1} |\det B^{(0)(1k)}|, \tag{31}$$

where $B^{(0)(jk)}$ is the $m \times m$ matrix obtained by removing the jth row and the kth column from $B = B^{(0)}$. Similarly, for any $\ell = 1, 2, \dots, m$, expanding in the $\ell + 1$ row, we have

$$|\det B^{(\ell)} - \det B^{(\ell+1)}| \le \frac{1}{w} \sum_{k=1}^{m+1} |\det B^{(\ell)(\ell+1\,k)}|, \tag{32}$$

Inequalities (31), (32) imply

$$|\det B - \det \widehat{B}| \le \frac{1}{w} \sum_{\ell=0}^{m} \sum_{k=1}^{m+1} |\det B^{(\ell)(\ell+1\,k)}|. \tag{33}$$

Hadamard's inequality yields

$$|\det B^{(\ell)(\ell+1\,k)}| \leq u^m m^{m/2}, \tag{34}$$

and so

$$|\det B - \det \widehat{B}| \leq \frac{1}{w}(m+1)^2 u^m m^{m/2} \leq \frac{1}{w}(C_1 u)^m \sqrt{m!} \tag{35}$$

for some $C_1 > 0$. □

Proof of Lemma 12. Let

$$\widehat{K}_u(x,y) = \begin{cases} K_u(x,y) & \text{if } x,y \in A_1^{(w)} \text{ or } x,y \in A_2^{(w)} \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

If we set

$$B = \det(K_u(x_j, y_k))_{j,k=1}^{m+1}, \qquad \widehat{B} = \det(\widehat{K}_u(x_j, y_k))_{j,k=1}^{m+1}, \tag{37}$$

with $x_j, y_k \in A^{(w)}$, then $B, \widehat{B}$ satisfy the conditions of Proposition 13 for some $\widehat{X}$. By (29) and the definition of the Fredholm determinant, we have for sufficiently large absolute constants $C_j > 0$

$$\begin{aligned}
&|\det(I - K_u)_{A^{(v)}} - \det(I - \widehat{K}_u)_{A^{(v)}}| \\
&\leq \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \int_{A^{(w)}} dx_1 \cdots \int_{A^{(w)}} dx_{m+1} \left| \det(K_u(x_i, y_j))_{i,j=1}^{m+1} - \det(\widehat{K}_u(x_i, y_j))_{i,j=1}^{m+1} \right| \\
&\leq \frac{1}{w} \sum_{m=0}^{\infty} \frac{(C_2 u)^m}{\sqrt{m!}} \leq \frac{1}{w} \sqrt{\sum_{m=0}^{\infty} \frac{(C_2 u)^{2m}(m+1)^2}{m!}} \sqrt{\sum_{m=0}^{\infty} \frac{1}{(m+1)^2}} \leq \frac{C_3}{w} e^{C_4 u^2}.
\end{aligned} \tag{38}$$

The reason for introducing $\widehat{K}$ is that the corresponding Fredholm determinant splits into the product of the determinants over $A_1^{(w)}$ and $A_2^{(w)}$. Indeed,

$$\begin{aligned}
\det(I - \widehat{K}_u)_{A^{(w)}} &= I + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \frac{(-1)^m}{(m-k)!k!} \\
&\quad \times \int_{\substack{x_1,\dots,x_k \in A_1^{(w)} \\ x_{k+1},\dots,x_m \in A_2^{(w)}}} \det K_u(x_i - x_j)_{i,j=1}^{k} \det K_u(x_i - x_j)_{i,j=k+1}^{m} dx_1 \dots dx_m \\
&= \det(I - K_u)_{A_1^{(w)}} \det(I - K_u)_{A_2^{(w)}}.
\end{aligned} \tag{39}$$

Combining this with the estimate (38) proves the lemma. □

3 | DIFFERENTIAL IDENTITY

Consider the following Riemann-Hilbert problem for a 2×2 matrix valued function $\Phi(w) = \Phi(w; s)$, where $s > 0$.

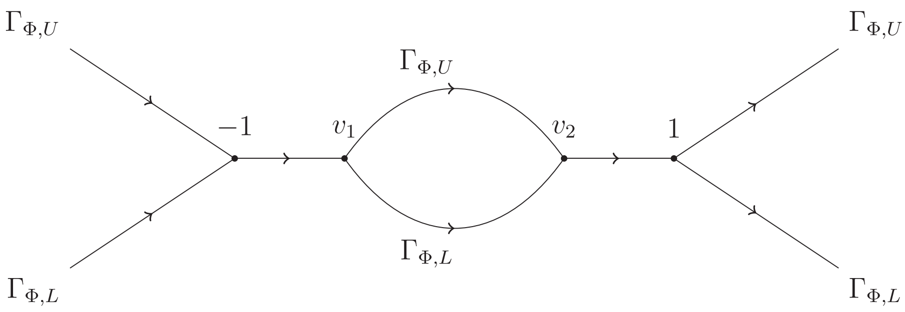

FIGURE 2 The jump contour Γ_Φ.

Let Γ_Φ be the contour shown in Figure 2, where as usual the + side of the contour is on the left w.r.t. the direction shown by the arrow, and the − side is on the right.

RH problem for Φ

(a) Φ is analytic for $w \in \mathbb{C} \setminus \Gamma_\Phi$.
(b) Φ has L^2 boundary values $\Phi_+(w)$, $\Phi_-(w)$ as the point $w \in \Gamma_\Phi$ is approached nontangentially from the + side, − side, respectively. These values are related by the jump condition $\Phi_+(w) = \Phi_-(w)J_\Phi(w)$, where

$$J_\Phi(w) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } w \in I = (-1, v_1) \cup (v_2, 1), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } w \in \Gamma_{\Phi,\mathrm{L}}, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for } w \in \Gamma_{\Phi,\mathrm{U}}. \end{cases} \tag{40}$$

(c) As $w \to \infty$,

$$\Phi(w) = \left(I + \mathcal{O}\left(\frac{1}{w}\right)\right) \begin{pmatrix} e^{isw} & 0 \\ 0 & e^{-isw} \end{pmatrix}. \tag{41}$$

Remarks.

1) As usual, we write for brevity

$$\begin{pmatrix} e^{isw} & 0 \\ 0 & e^{-isw} \end{pmatrix} = e^{isw\sigma_3}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2) By general theory, see, for example, [14], if this problem has a solution $\Phi(w)$, then the solution is unique. In Section 3.1, we show that the RH problem for Φ may be constructed explicitly in terms of the m-RH problem from [18]. It was proven in [18, Proposition 2.18] that a solution

exists to the m-RH problem, and thus there exists a solution to our RH problem for Φ for any $s > 0$.

The rest of this section will be devoted to two different proofs of the following

Lemma 14 (Differential identity). *Let $\Phi(z) = \Phi(z; s)$ solve the RH problem for Φ. The Fredholm determinant (1) satisfies:*

$$\frac{\partial}{\partial v_2} \log \det(I - K_s)_{(-1,v_1)\cup(v_2,1)} = \mathcal{F}_s(v_1, v_2) \equiv \frac{i}{2\pi} \left[\Phi_+^{-1}(v_2)\Phi'_+(v_2)\right]_{12}, \tag{42}$$

where $\Phi'(z) = \frac{d}{dz}\Phi(z)$ *and* $\Phi_+^{-1}(v_2)\Phi'_+(v_2) = \lim_{\epsilon\downarrow 0} \Phi^{-1}(v_2 + i\epsilon)\Phi'(v_2 + i\epsilon)$. *Moreover, if* $-v_1 = v_2 = v$,

$$\frac{\partial}{\partial v} \log \det(I - K_s)_{(-1,-v)\cup(v,1)} = 2\mathcal{F}_s(-v, v). \tag{43}$$

3.1 | First proof of Lemma 14

The proof of identities of type (42) using the theory of integrable operators is standard [3, 4, 18, 28]. We give an outline. First, we write the kernel of the (integrable) operator K_s in the form

$$K_s(x, y) = \frac{\vec{\lambda}^T(x)\vec{\mu}(y)}{x - y} = \frac{\sum_{j=1}^2 \lambda_j(x)\mu_j(y)}{x - y}, \qquad \vec{\lambda}(z) = \begin{pmatrix} e^{isz} \\ -e^{-isz} \end{pmatrix}, \qquad \vec{\mu} = \frac{1}{2\pi i}\begin{pmatrix} e^{-isz} \\ e^{isz} \end{pmatrix}. \tag{44}$$

Note that $\sum_{j=1}^2 \lambda_j(z)\mu_j(z) = 0$. The resolvent of the operator K_s,

$$(I - K_s)^{-1} = I + R_s,$$

has the property [18, Lemma 2.8] that the kernel of R_s is of the form

$$R_s(x, y) = \frac{\vec{\Lambda}^T(x)\vec{M}(y)}{x - y}, \qquad \Lambda_j = (I - K_s)^{-1}\lambda_j, \qquad M_j = (I - K_s^T)^{-1}\mu_j, \qquad j = 1, 2, \tag{45}$$

and moreover, $\sum_{j=1}^2 \Lambda_j(z)M_j(z) = 0$. The functions $\Lambda(z)$ and $M(z)$ for $z \in A$ can be written as [18, Lemma 2.12]

$$\vec{\Lambda}(z) = \widehat{m}_+(z)\vec{\lambda}(z), \qquad \vec{M}(z) = (\widehat{m}_+^{-1}(z))^T\vec{\mu}(z), \tag{46}$$

where $\widehat{m}(z)$ is the 2×2 matrix valued function which solves the following RHP (this is the m-RHP of [18] up to a slight modification: λ_2, μ_2 are replaced by $-\lambda_2, -\mu_2$, respectively):

RH problem for $\widehat{m}$

(a) $\widehat{m}(z)$ is analytic in $\mathbb{C} \setminus \overline{A}$.

(b) $\widehat{m}(z)$ has L^2 boundary values related by the condition $\widehat{m}_+(x) = \widehat{m}_-(x)J_m(x)$ for $x \in A$, with

$$J_m(x) = I - 2\pi i\vec{\lambda}(x)\vec{\mu}^T(x). \tag{47}$$

(c) $\widehat{m}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

This problem is reduced to a constant jump problem by the transformation

$$\widehat{\psi}(z) = \widehat{m}(z)e^{isz\sigma_3}. \tag{48}$$

Indeed so defined $\widehat{\psi}(z)$ satisfies

RH problem for $\widehat{\psi}(z)$

(a) $\widehat{\psi}(z)$ is analytic in $\mathbb{C} \setminus \overline{A}$.
(b) $\widehat{\psi}(z)$ has L^2 boundary values related by the condition $\widehat{\psi}_+(x) = \widehat{\psi}_-(x)\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ for $x \in A$.
(c) $\widehat{\psi}(z) = (I + \mathcal{O}(z^{-1}))e^{isz\sigma_3}$ as $z \to \infty$.

It is now straightforward to verify that the solution to the Φ-RH problem is written in terms of $\widehat{\psi}(z)$ as follows: $\Phi(z) = \widehat{\psi}(z)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ above $\Gamma_{\Phi,U}$ (see Figure 2); $\Phi(z) = \widehat{\psi}(z)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ below $\Gamma_{\Phi,L}$; and $\Phi(z) = \widehat{\psi}(z)$ inside the lenses in Figure 2.

Writing $\widehat{m}$ in terms of Φ in (46), we obtain

$$\vec{\Lambda}(z) = \begin{pmatrix} -\Phi_{12,+}(z) \\ -\Phi_{22,+}(z) \end{pmatrix}, \qquad \vec{M}(z) = \frac{1}{2\pi i}\begin{pmatrix} \Phi_{22,+}(z) \\ -\Phi_{12,+}(z) \end{pmatrix}, \qquad z \in A. \tag{49}$$

Now the logarithmic derivative of the determinant

$$\begin{aligned} \frac{\partial}{\partial v_2} \log\det(I - K_s)_{(-1,v_1)\cup(v_2,1)} &= -\mathrm{tr}\left((I - K_s)^{-1}\frac{\partial K_s}{\partial v_2}\right) = ((I - K_s)^{-1}K_s)(v_2, v_2) \\ &= ((I - K_s)^{-1}(K_s - I + I))(v_2, v_2) = R_s(v_2, v_2) = -(\Lambda_1(v_2)M_1'(v_2) + \Lambda_2(v_2)M_2'(v_2)). \end{aligned} \tag{50}$$

Substituting here (49), we obtain (42). The identity (43) is obtained similarly.

3.2 | Differential identity for Toeplitz determinants

For the second proof of Lemma 14, we will first represent the Fredholm determinant $\det(I - K_s)_A$ in terms of a special Toeplitz determinant and then obtain (42) as a limit of the corresponding differential identity for Toeplitz determinants. This way of proving Lemma 14 has a potential advantage of future applications to computing probabilities in the Circular Unitary Ensemble of random matrix theory, and to the theory of orthogonal polynomials.

Let $J = J_1 \cup J_2$ be the union of two disjoint arcs J_1 and J_2 on the unit circle C. We parametrize the endpoints of J_1 by $a_1 = e^{i\phi_1}, a_2 = e^{i\phi_2}$ and the endpoints of J_2 by $b = e^{i\phi_0}$, $\bar{b} = e^{-i\phi_0}$, see Figure 3. Let f be the indicator function of the set J:

$$f(z) = \begin{cases} 1 & \text{for } z \in J, \\ 0 & \text{for } z \notin J. \end{cases}$$

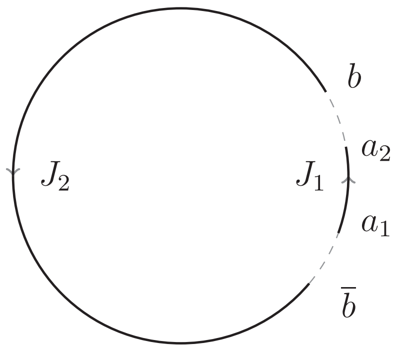

FIGURE 3 Arc J_1 on the right and J_2 on the left.

Consider the n-dimensional Toeplitz determinant with symbol f on the unit circle C:

$$D_n(f) = \det(f_{j-k})_{j,k=0}^{n-1}, \qquad f_j = \int_C f(z) z^{-j} \frac{dz}{2\pi i z} = \int_J z^{-j} \frac{dz}{2\pi i z},$$

where the integration is in the counterclockwise direction.

If the end-points of the arcs vary with n as follows, $\phi_0 = 2s/n$ and $\phi_j = 2v_j s/n$ for $j = 1, 2$, then it is easily verified that

$$\lim_{n\to\infty} D_n(f) = \det(I - K_s)_{(-1,v_1)\cup(v_2,1)}. \tag{51}$$

We will now obtain a differential identity for $D_n(f)$, and in the next subsection, by taking $n \to \infty$ and using (51), will prove Lemma 14.

Since f is nonnegative, it follows from the multiple integral representation for Toeplitz determinants that $D_j(f) > 0$ for all $j = 1, 2, \dots$. Set $D_0(f) = 1$. Define the polynomials $\psi_0 = 1/\sqrt{f_0}$, ψ_j, $j = 1, 2, \dots$ by

$$\psi_j(z) = \frac{1}{\sqrt{D_j(f) D_{j+1}(f)}} \det \begin{pmatrix} f_0 & f_{-1} & \dots & f_{-j+1} & f_{-j} \\ f_1 & f_0 & \dots & f_{-j+2} & f_{-j+1} \\ & & \ddots & & \\ f_{j-1} & f_{j-2} & \dots & f_0 & f_{-1} \\ 1 & z & \dots & z^{j-1} & z^j \end{pmatrix} = \chi_j z^j + \dots,$$

where the leading coefficient χ_j is given by

$$\chi_j = \sqrt{\frac{D_j(f)}{D_{j+1}(f)}}. \tag{52}$$

These polynomials are orthonormal on J:

$$\int_J \psi_k(z) \overline{\psi_j(z)} \frac{dz}{2\pi i z} = \delta_{jk}, \qquad j, k = 0, 1, \dots \tag{53}$$

For a given $n \geq 1$, define the matrix-valued function $Y = Y(z)$ in terms of the orthogonal polynomials:

$$Y(z) = \begin{pmatrix} \chi_n^{-1}\psi_n(z) & \chi_n^{-1}\int_J \frac{\psi_n(\zeta)}{\zeta - z}\frac{d\zeta}{2\pi i\zeta^n} \\ -\chi_{n-1}z^{n-1}\overline{\psi}_{n-1}(z^{-1}) & -\chi_{n-1}\int_J \frac{\overline{\psi}_{n-1}(\zeta^{-1})}{\zeta - z}\frac{d\zeta}{2\pi i\zeta} \end{pmatrix}. \tag{54}$$

The function Y is a unique solution to the following RH Problem:

(a) $Y : \mathbb{C} \setminus J \to \mathbb{C}^{2\times 2}$ is analytic;
(b) $Y_+(z) = Y_-(z)\begin{pmatrix} 1 & z^{-n} \\ 0 & 1 \end{pmatrix}$ for $z \in J$;
(c) $Y(z) = (I + \mathcal{O}(1/z))z^{n\sigma_3}$ as $z \to \infty$.

This fact was initially noticed in [25] for orthogonal polynomials on the real line and extended to the case of orthogonal polynomials on the unit circle in [2]. As in [15, 31], we will use the orthogonal polynomials to obtain a differential identity for $\log D_n(f)$ in terms of the solution to the RH problem for Y. Namely, we have

Proposition 15.

(a) Let $a_2 = e^{i\phi_2}$. The Toeplitz determinant $D_n(f)$ satisfies

$$\frac{\partial}{\partial\phi_2}\log D_n(f) = -\frac{1}{2\pi}F(a_2), \tag{55}$$

where F is given by

$$F(z) = -z^{-n+1}[Y^{-1}(z)Y'(z)]_{21}. \tag{56}$$

(b) Let $a_2 = \overline{a}_1 = e^{i\phi_2}$. Then

$$\frac{d}{d\phi_2}\log D_n(f) = -\frac{1}{\pi}F(a_2). \tag{57}$$

Proof. From the definition of the orthogonal polynomials it is clear that

$$D_n(f) = \prod_{j=0}^{n-1}\chi_j^{-2}. \tag{58}$$

The orthogonality conditions imply that, with $z = e^{i\theta}$,

$$\frac{1}{2\pi}\int_J \frac{\partial\psi_j(z)}{\partial\phi_2}\overline{\psi_j(z)}d\theta = \frac{1}{2\pi}\int_J \frac{\partial\chi_j}{\partial\phi_2}(z^j + \text{polynomial of degree } j-1)\overline{\psi_j(z)}d\theta = \frac{1}{\chi_j}\frac{\partial\chi_j}{\partial\phi_2}, \tag{59}$$

and similarly,

$$\frac{1}{2\pi}\int_J \psi_j(z)\frac{\partial\overline{\psi_j(z)}}{\partial\phi_2}d\theta = \frac{1}{\chi_j}\frac{\partial\chi_j}{\partial\phi_2}. \tag{60}$$

By (58)–(60) we obtain:

$$\frac{\partial}{\partial \phi_2}\log(D_n(f)) = -2\sum_{j=0}^{n-1}\frac{\partial \chi_j}{\partial \phi_2}/\chi_j = -\frac{1}{2\pi}\int_J \frac{\partial}{\partial \phi_2}\left(\sum_{j=0}^{n-1}|\psi_j(z)|^2\right)d\theta. \tag{61}$$

The Christoffel-Darboux formula for orthogonal polynomials on the unit circle (see, e.g., equation (2.8) in [15]) states that

$$-\sum_{k=0}^{n-1}|\psi_k(z)|^2 = n|\psi_n(z)|^2 - 2\mathrm{Re}\left(z\overline{\psi_n(z)}\psi_n'(z)\right) \quad \text{for } z \in C. \tag{62}$$

On the other hand, using the following identity (equation (2.4) in [15])

$$\chi_n\overline{\psi_n(z)} = \chi_{n-1}z^{-1}\overline{\psi_{n-1}(z)} + \overline{\psi_n(0)}z^{-n}\psi_n(z), \tag{63}$$

and (54), we easily verify that

$$F(z) = -z^{-n+1}[Y^{-1}(z)Y'(z)]_{21} = n|\psi_n(z)|^2 - 2\mathrm{Re}\left(z\overline{\psi_n(z)}\psi_n'(z)\right) \quad \text{for } z \in C. \tag{64}$$

Substitution of (62), (64) into (61) gives

$$\frac{\partial}{\partial \phi_2}\log D_n(f) = \frac{1}{2\pi}\int_J \frac{\partial}{\partial \phi_2}(F(z))d\theta. \tag{65}$$

Since by orthogonality

$$\int_J F(z)\frac{d\theta}{2\pi} = -\int_J \sum_{k=0}^{n-1}|\psi_k(z)|^2\frac{d\theta}{2\pi} = -n,$$

we obtain

$$0 = \frac{\partial}{\partial \phi_2}\left(\int_J F(z)d\theta\right) = F(a_2) + \int_J \frac{\partial}{\partial \phi_2}F(z)d\theta, \tag{66}$$

and proposition 15 (a) follows from (65). Part (b) is proved similarly. □

3.3 | Limit $n \to \infty$: Second proof of Lemma 14

As we are eventually interested in the limit $n \to \infty$, we first reduce the Y RH problem to an approximate problem for Φ which does not contain the parameter n, and the dependence on n is in the error of approximation.

Let

$$T(z) = \begin{cases} Y(z) & |z| < 1, \\ Y(z)z^{-n\sigma_3} & |z| > 1. \end{cases} \tag{67}$$

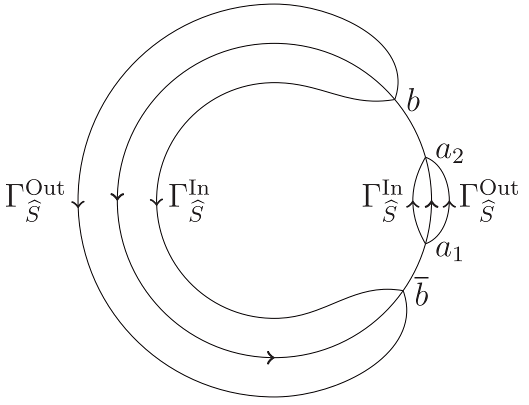

FIGURE 4 Contour $\Gamma_{\widehat{S}}$.

We open the lenses around J_1 and J_2, see Figure 4. Denote the edges of the lenses inside the unit disc by $\Gamma^{\mathrm{In}}_{\widehat{S}}$, the edges of the lenses outside the unit disc by $\Gamma^{\mathrm{Out}}_{\widehat{S}}$, and let

$$\widehat{S}(z) = \begin{cases} T(z) & \text{outside the lenses,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -z^n & 1 \end{pmatrix} & \text{inside the lenses, for } |z| < 1, \\ T(z)\begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix} & \text{inside the lenses, for } |z| > 1. \end{cases} \tag{68}$$

Then $\widehat{S}$ satisfies the following RH problem:

(a) $\widehat{S}$ is analytic on $\mathbb{C} \setminus (C \cup \Gamma^{\mathrm{In}}_{\widehat{S}} \cup \Gamma^{\mathrm{Out}}_{\widehat{S}})$.
(b) The jumps of $\widehat{S}$ are given by $\widehat{S}_+(z) = \widehat{S}_-(z) J_{\widehat{S}}(z)$, where

$$J_{\widehat{S}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in J, \\ \begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix} & \text{for } z \in \Gamma^{\mathrm{Out}}_{\widehat{S}} \\ \begin{pmatrix} 1 & 0 \\ z^{n} & 1 \end{pmatrix} & \text{for } z \in \Gamma^{\mathrm{In}}_{\widehat{S}}, \\ z^{n\sigma_3} & \text{for } z \in C \setminus J. \end{cases}$$

(c) As $z \to \infty$,

$$\widehat{S}(z) = I + \mathcal{O}\big(z^{-1}\big). \tag{69}$$

We assume that the lenses around J_1 and the contour part $C \setminus J$ are contained within the set $|z-1| < 1/2$. The following function $\mathcal{M}$ will approximate $\widehat{S}$ for $|z-1| > 1/2$:

$$\mathcal{M}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & |z| < 1, \\ I & |z| > 1. \end{cases} \tag{70}$$

For $|z-1| < 1/2$, we construct the following function Q. Let

$$w(z) = -i\frac{n}{2s}\log z, \tag{71}$$

so that $w(e^{2it\frac{s}{n}}) = t$ for any t, and define

$$Q(z) = \begin{cases} \Phi(w(z);s)z^{-\frac{n}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & |z| < 1, \\ \Phi(w(z);s)z^{-\frac{n}{2}\sigma_3} & |z| > 1, \end{cases} \tag{72}$$

where Φ is the solution of the Φ RH problem at the beginning of the section.

Let

$$\widehat{R}(z) = \begin{cases} \widehat{S}\mathcal{M}^{-1} & \text{for } |z-1| > 1/2, \\ \widehat{S}Q^{-1} & \text{for } |z-1| < 1/2. \end{cases} \tag{73}$$

Then $\widehat{R}$ is analytic for $\mathbb{C} \setminus \Gamma_{\widehat{R}}$, where

$$\widehat{\Gamma}_R = \{\text{the edge of the lens for } |z-1| > 1/2\} \cup \{z : |z-1| = 1/2\}. \tag{74}$$

We have using (41), (72),

$$\widehat{R}_+^{-1}(z)\widehat{R}_-(z) = Q(z)\mathcal{M}^{-1}(z) = I + \mathcal{O}(1/n)$$

uniformly on the circle $|z-1| = 1/2$ oriented counterclockwise. Furthermore, $\widehat{R}_+^{-1}\widehat{R}_- - I = \mathcal{O}(e^{-n\epsilon})$, $\epsilon > 0$, uniformly on the edges of the lenses. Thus, by standard small norm analysis (see, e.g., [14]),

$$\widehat{R}(z) = I + \mathcal{O}(1/n), \qquad \widehat{R}'(z) = \mathcal{O}(1/n), \tag{75}$$

uniformly for $z \in \mathbb{C}$.

We now express $F(a_2)$ from Proposition 15 in terms of elements of Φ. Tracing back the transformations, we see that as z approaches a_2 from the *inside* of the unit circle and being outside the lens,

$$Y(z) = T(z) = \widehat{S}(z) = \widehat{R}(z)Q(z) = \widehat{R}(z)\Phi(w(z))z^{-(n/2)\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using this, we obtain

$$-z^{n+1}(Y(z)^{-1}Y'(z))_{21} = z\left(\Phi(w(z))^{-1}\frac{d}{dz}\Phi(w(z))\right)_{12} + z\big(\Phi^{-1}\mathcal{O}(1/n)\Phi\big)_{12}$$
$$= z\left(\Phi(w)^{-1}\frac{d}{dw}\Phi(w)\right)_{12}\frac{dw}{dz} + z\big(\Phi^{-1}\mathcal{O}(1/n)\Phi\big)_{12}$$
$$= -\frac{in}{2s}\left(\Phi(w)^{-1}\frac{d}{dw}\Phi(w)\right)_{12} + z\big(\Phi^{-1}\mathcal{O}(1/n)\Phi\big)_{12}.$$

Taking the limit $z \to a_2 = \exp(i\phi_2) = \exp(i2v_2 s/n)$ along this trajectory, we obtain

$$F(a_2) = -\frac{in}{2s}\left[\Phi_+^{-1}(v_2)\Phi_+'(v_2)\right]_{12} + \mathcal{O}(1/n), \tag{76}$$

as $n \to \infty$. Substituting this into (55), recalling (51), and noting that $dv_2/d\phi_2 = n/(2s)$, proves (42). The symmetric case identity (43) follows from (57). Thus we finished the proof of Lemma 14.

We now solve the RH problem for Φ, compute the asymptotics of the r.h.s. of (42), integrate it, and use Lemma 8 at one of the integration limits to obtain Theorem 1.

4 | SOLUTION OF THE RH PROBLEM FOR Φ

In this section, the main objective is to provide asymptotics for $\Phi(z) = \Phi(z; s)$ as $s \to \infty$. We construct an outside parameterix in Section 4.1, local parametrices in Section 4.3, and solve a small norm problem in Section 4.4. In Section 4.4 we consider v_1 and v_2 to be fixed as $s \to \infty$, and in Section 4.5 we extend the solution to the regime where $v_2 \to 1$ such that $s(1-v_2) \to \infty$, and also to the regime where $v_1 \to -1$ such that $s(1+v_1) \to \infty$. Additionally, in Section 4.2, we provide some identities for θ functions which we will rely on later in the paper.

Recall the definition of $\psi(z)$ in (6), and for $z \in \mathbb{C} \setminus \overline{(-1, v_1) \cup (v_2, 1)}$ on the first sheet of the Riemann surface Σ, let

$$\phi(z) = \int_1^z \psi(\xi)d\xi. \tag{77}$$

We see by (7) that $\phi(z)$ is a well defined function, analytic on $\mathbb{C} \setminus \overline{(-1, v_1) \cup (v_2, 1)}$. Since $\psi_+ = -\psi_-$ on $(-1, v_1) \cup (v_2, 1)$, we have

$$\phi_+(z) + \phi_-(z) = \begin{cases} 0 & \text{for } z \in (v_2, 1), \\ -2\pi\Omega & \text{for } z \in (-1, v_1), \end{cases} \qquad \Omega = \frac{1}{\pi}\int_{v_1}^{v_2}\psi(x)dx > 0. \tag{78}$$

Since by (7) $\psi(z)$ has zero residue at infinity, $\psi(z) = 1 + \mathcal{O}(1/z^2)$ as $z \to \infty$, and we have

$$\phi(z) = z + \mathcal{O}(1), \qquad z \to \infty. \tag{79}$$

Let

$$S(z) = e^{is\ell\sigma_3}\Phi(z)e^{-is\phi(z)\sigma_3}, \qquad \ell = \int_1^\infty (\psi(x) - 1)dx - 1, \tag{80}$$

then S satisfies the following RH problem.

RH Problem for $\mathcal{S}$

(a) $\mathcal{S}$ is analytic for $z \in \mathbb{C} \setminus \Gamma_\Phi$,
(b) $\mathcal{S}$ has jumps given by $\mathcal{S}_+(z) = \mathcal{S}_-(z) J_{\mathcal{S}}(z)$, where

$$J_{\mathcal{S}}(z) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } z \in (v_2, 1), \\ \begin{pmatrix} 0 & -e^{-2\pi i s\Omega} \\ e^{2\pi i s\Omega} & 0 \end{pmatrix} & \text{for } z \in (-1, v_1), \\ \begin{pmatrix} 1 & 0 \\ e^{-2is\phi(z)} & 1 \end{pmatrix} & \text{for } z \in \Gamma_{\Phi,\mathrm{L}}, \\ \begin{pmatrix} 1 & -e^{2is\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z \in \Gamma_{\Phi,\mathrm{U}}. \end{cases} \tag{81}$$

(c) As $z \to \infty$,

$$\mathcal{S}(z) = I + \mathcal{O}\left(\frac{1}{z}\right). \tag{82}$$

We need the conditions $\operatorname{Im} \phi(z) < 0$, $\operatorname{Im} \phi(z) > 0$, to hold uniformly on $\Gamma_{\Phi,\mathrm{L}}$, $\Gamma_{\Phi,\mathrm{U}}$, respectively, away from some fixed ϵ neighborhoods of the end-points for the corresponding jumps to be exponentially close to the identity. Since (79) is uniform as $|z| \to \infty$, the conditions hold for $|z| > W$ for some sufficiently large but fixed $W > 0$. Since $\frac{d}{dx}\phi(x) = \psi(x) > 0$ for $x \in \mathbb{R} \setminus \overline{(-1, v_1) \cup (v_2, 1)}$, the conditions hold on the contour as stated assuming (and we do this) that the angle between the parts of $\Gamma_{\Phi,\mathrm{L}}$, $\Gamma_{\Phi,\mathrm{U}}$ emanating from ± 1 and the real axis was chosen to be sufficiently small and the lens around (v_1, v_2) was sufficiently narrow. Therefore

$$J_{\mathcal{S}}(z) = I + \mathcal{O}\left(e^{-cs(1+|z|)}\right), \tag{83}$$

as $s \to \infty$, for some constant $c > 0$, uniformly on $\Gamma_{\Phi,\mathrm{L}}$, $\Gamma_{\Phi,\mathrm{U}}$ away from fixed ϵ-neighborhoods of $\pm 1, v_1, v_2$.

4.1 | Outside parametrix and θ-functions

Consider the following RH problem for the 2×2-matrix valued function $\mathcal{N}(z; \omega)$ with a real parameter ω, which will give an approximate solution to the Φ RH problem away from the edge points $\pm 1, v_1, v_2$, when $\omega = s\Omega$. Later on we also construct approximate solutions (local parametrices) on a neighborhood of each edge point, and match them to the leading order with $\mathcal{N}(z; \omega)$ on the boundaries of the neighborhoods.

RH problem for $\mathcal{N}$

(a) $\mathcal{N}(z)$ is analytic on $\mathbb{C} \setminus \overline{(-1, v_1) \cup (v_2, 1)}$.
(b) On $(-1, v_1) \cup (v_2, 1)$, $\mathcal{N}$ has L^2 boundary values related by the jump conditions:

$$\mathcal{N}_+(z) = \mathcal{N}_-(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } z \in (v_2, 1),$$

$$\mathcal{N}_+(z) = \mathcal{N}_-(z) \begin{pmatrix} 0 & -e^{-2\pi i \omega} \\ e^{2\pi i \omega} & 0 \end{pmatrix} \qquad \text{for } z \in (-1, v_1).$$

(c) As $z \to \infty$,

$$\mathcal{N}(z) = I + \mathcal{O}(z^{-1}). \tag{84}$$

A more general problem with jumps on m intervals was solved in [18] in terms of multidimensional θ-functions. We now present the solution in our case of two intervals: $(-1, v_1)$, $(v_2, 1)$. Let

$$\gamma(z) = \left(\frac{(z-1)(z-v_1)}{(z-v_2)(z+1)} \right)^{1/4}, \tag{85}$$

also with branch cuts on $(-1, v_1) \cup (v_2, 1)$, such that $\gamma(z) \to 1$ as $z \to \infty$ on the first sheet of the Riemann surface Σ.

Recall the definition of the holomorphic differential (11). Let u be the following analytic function on $\mathbb{C} \setminus \{(-\infty, v_1] \cup [v_2, +\infty)\}$:

$$u(z) = -\int_{v_2}^{z} \omega, \tag{86}$$

with integration taken on the first sheet. Note that, $\mod \mathbb{Z}$,

$$u(-1) = -\frac{\tau}{2} - \frac{1}{2}, \qquad u(v_1) = -\frac{\tau}{2}, \qquad u(v_2) = 0, \qquad u(1) = -\frac{1}{2}. \tag{87}$$

The function $u(z)$ extends to the Riemann surface Σ and is then called the Abel map. It maps the Riemann surface onto the torus where θ-functions are defined.

A simple calculation (see [18]) shows that the function $\gamma(z) - \gamma(z)^{-1}$ has a single zero on (v_1, v_2) on the first sheet, denote it by $\hat{z}$, and no zeros on the second sheet. We have

$$\hat{z} = \frac{v_1 + v_2}{2 + v_1 - v_2}. \tag{88}$$

Similarly, the function $\gamma(z) + \gamma(z)^{-1}$ has no zeros on the first sheet and one zero on the second.

Let

$$d = -\frac{1-\tau}{2} - \int_{v_2}^{\hat{z}} \omega, \tag{89}$$

with integration taken on the first sheet.

Consider the third Jacobian θ-function $\theta(z) = \theta_3(z;\tau)$ (see Appendix A). Since $\theta((1-\tau)/2) = 0$, we have $\theta(u(\hat{z}) - d) = 0$. The function $\theta(u(z) - d) = 0$ has no other zeros on the Riemann surface. The function $\theta(u(z) + d) = 0$ has only one zero on the Riemann surface located on the second sheet which coincides with the only zero of $\gamma(z) + \gamma(z)^{-1}$.

By an argument in [18] we have

$$u(\infty) + d = m\tau \mod \mathbb{Z},$$

for some integer m. Consider the integral of $\boldsymbol{\omega}$ along the closed contour composed of a large interval along the real axis and a semicircle in the upper half-plane. Then using (12) and the definition of τ in (13) we obtain in the case $v_1 = -v_2$ that $u(\infty) + d = 0 \mod \mathbb{Z}$ with $u(z)$ considered on the first sheet. Therefore also in the general case of v_1, v_2, by continuity,

$$u(\infty) + d = 0 \mod \mathbb{Z}. \tag{90}$$

The solution to the RH problem for $\mathcal{N}$ is given by

$$\mathcal{N}(z;\omega) = \begin{pmatrix} \frac{\gamma+\gamma^{-1}}{2} m_{11} & -\frac{\gamma-\gamma^{-1}}{2i} m_{12} \\ \frac{\gamma-\gamma^{-1}}{2i} m_{21} & \frac{\gamma+\gamma^{-1}}{2} m_{22} \end{pmatrix},$$

$$m(z) = \frac{\theta(0)}{\theta(\omega)} \times \begin{pmatrix} \dfrac{\theta(u(z)+\omega+d)}{\theta(u(z)+d)} & \dfrac{\theta(u(z)-\omega-d)}{\theta(u(z)-d)} \\ \dfrac{\theta(u(z)+\omega-d)}{\theta(u(z)-d)} & \dfrac{\theta(u(z)-\omega+d)}{\theta(u(z)+d)} \end{pmatrix} \tag{91}$$

with z on the first sheet. To see that $\mathcal{N}$ solves the RH problem for $\mathcal{N}$, one makes several observations. First note that $\gamma(z)$ is analytic on $\mathbb{C} \setminus \overline{(-1, v_1) \cup (v_2, 1)}$ and

$$\gamma_+(z) = i\gamma_-(z), \quad z \in (-1, v_1) \cup (v_2, 1).$$

Hence for $w \in (-1, v_1) \cup (v_2, 1)$

$$\begin{aligned} \left(\frac{\gamma+\gamma^{-1}}{2}\right)_+ &= -\left(\frac{\gamma-\gamma^{-1}}{2i}\right)_-; \\ \left(\frac{\gamma-\gamma^{-1}}{2i}\right)_+ &= \left(\frac{\gamma+\gamma^{-1}}{2}\right)_-. \end{aligned} \tag{92}$$

Secondly, as follows from (A.2) and the relations

$$u_+(z) = \begin{cases} -u_-(z) \mod \mathbb{Z} & z \in (v_2, 1), \\ -u_-(z) - \tau \mod \mathbb{Z} & z \in (-1, v_1), \end{cases} \tag{93}$$

the matrix m has the jumps:

$$\begin{aligned} m_+(z) &= m_-(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } z \in (v_2, 1), \\ m_+(z) &= m_-(z) \begin{pmatrix} 0 & e^{-2\pi i\omega} \\ e^{2\pi i\omega} & 0 \end{pmatrix} \quad \text{for } z \in (-1, v_1). \end{aligned} \tag{94}$$

The singularities of m cancel with the zeros of $\gamma \pm \gamma^{-1}$. Furthermore,

$$\mathcal{N}(z) = I + \mathcal{O}(z^{-1})$$

as $z \to \infty$.

4.2 | Identities for θ-functions

Our proof of Theorem 1 will use several identities satisfied by θ-functions. We present these identities now. Standard definitions and properties of theta-functions that we need are summarized in Appendix A.

Lemma 16. *With the coefficients of the expansion γ_0, u_0, γ_1, u_1, given in (157) below we have:*

(a) For any[4] $\omega \in \mathbb{R}$,

$$\frac{\theta_3(0)^2\theta_3(d+\omega)\theta_3(d-\omega)}{\theta_3(d)^2\theta_3(\omega)^2}\left(1-\frac{\gamma_0^2 u_0}{2}\left[\frac{\theta_3'(d+\omega)}{\theta_3(d+\omega)}+\frac{\theta_3'(d-\omega)}{\theta_3(d-\omega)}-2\frac{\theta_3'(d)}{\theta_3(d)}\right]\right)=1. \tag{95}$$

(b)

$$\frac{\theta_1'(d)}{\theta_1(d)}-\frac{\theta_3'(d)}{\theta_3(d)}=\frac{1}{\gamma_0^2 u_0}=-iI_0(1+v_2). \tag{96}$$

(c)

$$\left(\frac{\theta_1(d)}{\theta_3(d)}\right)'''=\frac{3}{\gamma_0^2 u_0}\left(\frac{\theta_1(d)}{\theta_3(d)}\right)''-\frac{6(2\gamma_1+u_1)}{\gamma_0^2 u_0^3}\left(\frac{\theta_1(d)}{\theta_3(d)}\right). \tag{97}$$

(d) For $z_0 \in \{-1, v_1, v_2, 1\}$,

$$\frac{\theta_1^2(u(z_0)+d)}{\theta_3^2(u(z_0)+d)}\left(\frac{\theta_3}{\theta_1'}\right)^2 h(z_0)=-\frac{1}{I_0^2},\qquad h(z)=(z-1)(z-v_1)+(z-v_2)(z+1). \tag{98}$$

(e)

$$\theta_4(0;\tau)^4=\theta_4^4=\frac{I_0^2}{\pi^2}2(v_2-v_1). \tag{99}$$

(f)

$$\theta_2(0;\tau)^4=\theta_2^4=\frac{I_0^2}{\pi^2}(1+v_1)(1-v_2). \tag{100}$$

[4] If $d \pm \omega$ is a zero of θ_3, we multiply through in (95) before evaluating. We adopt the same convention in other formulae below. Furthermore, it is easily seen that $\theta_3(d) \neq 0$ and $\theta_3(\omega) \neq 0$ for any $\omega \in \mathbb{R}$.

(g)

$$\theta_3(0;\tau)^4 = \theta_3^4 = \frac{I_0^2}{\pi^2}(1-v_1)(1+v_2). \tag{101}$$

Proof. We begin by proving (a). Consider $\eta_1(z)$ defined by

$$\begin{aligned}\eta_1(z) = \left(\frac{\gamma(z)+\gamma^{-1}(z)}{2}\right)^2 \frac{\theta_3^2\theta_3(u(z)+\omega+d)\theta_3(u(z)-\omega+d)}{\theta_3(\omega)^2\theta_3(u(z)+d)^2} \\ - \left(\frac{\gamma(z)-\gamma^{-1}(z)}{2}\right)^2 \frac{\theta_3^2\theta_3(-u(z)+\omega+d)\theta_3(-u(z)-\omega+d)}{\theta_3(\omega)^2\theta_3(-u(z)+d)^2}\end{aligned} \tag{102}$$

Of course, we know that $\eta_1(z) = \det \mathcal{N}(z) = 1$ for all z from the Riemann-Hilbert problem. However, it is easy to provide a direct proof: By (92) and (94), and the fact that $\frac{\theta_3(\xi+\omega+d)\theta_3(\xi-\omega+d)}{\theta_3(\xi+d)^2}$ is an elliptic function of ξ, the function $\eta_1(z)$ no jumps on A and is thus meromorphic. By the fact that $\theta_3(\pm u(z)+d)$ has the same zeros as $\gamma(z)\pm\gamma(z)^{-1}$, respectively, it follows that $\eta_1(z)$ has no singularities and is an entire function. By (90), $\eta_1(z)\to 1$ as $z\to\infty$, and thus $\eta_1(z)=1$ for all $z\in\mathbb{C}$ by Liouville's theorem.

The expansion of $\eta_1(z)$ as $z\to v_2$ (using (157) below) shows that

$$\eta_1(z) \to \frac{\theta_3^2\theta_3(d+\omega)\theta_3(d-\omega)}{\theta_3(d)^2\theta_3(\omega)^2}\left(1-\frac{\gamma_0^2u_0}{2}\left[\frac{\theta_3'(d+\omega)}{\theta_3(d+\omega)}+\frac{\theta_3'(d-\omega)}{\theta_3(d-\omega)}-2\frac{\theta_3'(d)}{\theta_3(d)}\right]\right), \tag{103}$$

and since $\eta_1(v_2)=1$, we obtain Part (a) of the proposition.

Now consider

$$\eta_2(z) = \left(\frac{\gamma(z)+\gamma^{-1}(z)}{2}\right)^2 \frac{\theta_1^2(u(z)+d)}{\theta_3^2(u(z)+d)} - \left(\frac{\gamma(z)-\gamma^{-1}(z)}{2}\right)^2 \frac{\theta_1^2(-u(z)+d)}{\theta_3^2(-u(z)+d)}. \tag{104}$$

By the fact that $\frac{\theta_1^2(\xi)}{\theta_3^2(\xi)}$ is an elliptic function of ξ, and by (92) and (93), it follows that $\eta_2(z)$ is a meromorphic function, and again by cancelation of the poles from $\theta_3(\pm u(z)+d)$ by the zeros of $\gamma(z)\pm\gamma(z)^{-1}$, it follows that $\eta_2(z)$ in fact is entire. As $z\to\infty$, $\eta_2(z)\to 0$ by (90) since $\theta_1(0)=0$, and thus, $\eta_2(z)\equiv 0$ by Liouville's theorem. We see from the expansion of $\eta_2(z)$ in powers of $z-v_2$ as $z\to v_2$ that

$$\eta_2(z) \to \frac{\theta_1(d)}{\theta_3(d)}\left[\frac{\theta_1(d)}{\theta_3(d)} - \gamma_0^2u_0\left(\frac{\theta_1(d)}{\theta_3(d)}\right)'\right], \qquad z\to v_2. \tag{105}$$

Since this limit is zero, we obtain that

$$\left(\frac{\theta_1(d)}{\theta_3(d)}\right)' = \frac{1}{\gamma_0^2u_0}\frac{\theta_1(d)}{\theta_3(d)}, \tag{106}$$

which gives Part (b) of the proposition.

To prove part (c), we consider the coefficient of the first power $z - v_2$ in the expansion of $\eta_2(z)$ as $z \to v_2$. Denote here $g(z) = \frac{\theta_1(z)}{\theta_3(z)}$, then as $z \to v_2$,

$$0 = \eta_2(z) - \eta_2(v_2) = 4(z - v_2)\Bigg[-\gamma_0^2 \frac{u_0^3}{6}\big(g'''(d)g(d) + 3g''(d)g'(d)\big)$$
$$- u_0 g'(d)g(d)\big(u_1\gamma_0^2 + \gamma_0^{-2} + 2\gamma_1\gamma_0^2\big) + u_0^2\big(g''(d)g(d) + g'(d)^2\big)\Bigg] + \mathcal{O}((z - v_2)^2). \quad (107)$$

By substituting the identity for $g'(d)$ from Part (b) of the proposition into the right hand side of (107) and setting the resulting coefficient of $z - v_2$ equal to zero, we obtain Part (c).

Finally, to prove Part (d), we consider

$$\eta_3(z) = R(z)\left[\left(\frac{\gamma(z) + \gamma^{-1}(z)}{2}\right)^2 \frac{\theta_1^2(u(z) + d)}{\theta_3^2(u(z) + d)} + \left(\frac{\gamma(z) - \gamma^{-1}(z)}{2}\right)^2 \frac{\theta_1^2(-u(z) + d)}{\theta_3^2(-u(z) + d)}\right]. \quad (108)$$

By the same arguments as for η_1 and η_2 (and in addition by the fact that $R_+ = -R_-$ on A), it follows that η_3 is entire. By recalling the definition of u in (86), by (90), and by the definition of γ in (85), we obtain

$$\lim_{z\to\infty} \eta_3(z) = -\frac{1}{4I_0^2}\left(\frac{\theta_1'}{\theta_3}\right)^2 + \frac{(2 + v_1 - v_2)^2}{16}\left(\frac{\theta_1(2d)}{\theta_3(2d)}\right)^2, \quad (109)$$

so that $\eta_3(z)$ is identically equal to this constant. Now consider the asymptotics of $\eta_2(z)$ as $z \to \infty$. We have

$$0 \equiv \eta_2(z) = -z^{-2}\left[\frac{1}{4I_0^2}\left(\frac{\theta_1'}{\theta_3}\right)^2 + \frac{(2 + v_1 - v_2)^2}{16}\left(\frac{\theta_1(2d)}{\theta_3(2d)}\right)^2\right] + \mathcal{O}(z^{-3}), \quad (110)$$

from which we conclude that

$$-\frac{1}{4I_0^2}\left(\frac{\theta_1'}{\theta_3}\right)^2 = \frac{(2 + v_1 - v_2)^2}{16}\left(\frac{\theta_1(2d)}{\theta_3(2d)}\right)^2. \quad (111)$$

By substituting this into (109), we obtain

$$\eta_3(z) = -\frac{1}{2I_0^2}\left(\frac{\theta_1'}{\theta_3}\right)^2 \quad (112)$$

for all $z \in \mathbb{C}$. On the other hand, for $z_0 \in \{-1, v_1, v_2, 1\}$, from (108) by (87) and ellipticity,

$$\eta_3(z_0) = \frac{1}{2}\frac{\theta_1^2(u(z_0) + d)}{\theta_3^2(u(z_0) + d)}h(z_0). \quad (113)$$

Equating this to (112) we obtain Part (d). To show Parts (e), (f), (g), we consider the function (as usual, theta functions written without argument stand for their values with argument zero)

$$\frac{\theta_3(u(z))^2}{\theta_1(u(z))^2}+I_0^2(v_2-v_1)(v_2^2-1)\frac{\theta_3^2}{\theta_1'^2}\frac{1}{z-v_2}. \tag{114}$$

As before, we see that this function is identically constant. By evaluating at infinity, it is equal to $\frac{\theta_3(d)^2}{\theta_1(d)^2}$. On the other hand, part (d) at $z_0=v_2$ gives

$$\frac{\theta_3(d)^2}{\theta_1(d)^2}=-I_0^2(v_2-v_1)(v_2-1)\frac{\theta_3^2}{\theta_1'^2}. \tag{115}$$

Equating this constant to (114) we obtain the identity for all z:

$$\frac{\theta_3(u(z))^2}{\theta_1(u(z))^2}=-I_0^2(v_2-v_1)(v_2-1)\frac{\theta_3^2}{\theta_1'^2}\frac{z+1}{z-v_2}. \tag{116}$$

Evaluating it at $z=1$ (recall from (87) that $u(1)=1/2 \mod \mathbb{Z}$ and recall the definition of $\theta_j(z)$ from Appendix A), and using the identity $\theta_1'=\pi\theta_2\theta_3\theta_4$, we obtain Part (e). We similarly obtain Part (f) by evaluating (116) at $z=v_1$. Finally, we obtain Part (g) by making use of the identity $\theta_3^4=\theta_2^4+\theta_4^4$. □

4.3 | Local parametrices

Our goal in this section is to construct a function P on a neighborhood of each point of the set $\mathcal{T}=\{-1,v_1,v_2,1\}$, with the same jump conditions as $\mathcal{S}$ on these neighborhoods, and with an asymptotic behavior matching that of $\mathcal{N}$ to the main order on the boundaries of these neighborhoods. The first step is to recall the following model RH problem from [34] with an explicit solution in terms of Bessel functions.

RH problem for Ψ

(a) $\Psi:\mathbb{C}\setminus\Gamma_\Psi\to\mathbb{C}^{2\times 2}$ is analytic, where $\Gamma_\Psi=\mathbb{R}^-\cup\Gamma_\Psi^\pm$, with $\Gamma_\Psi^\pm=\{xe^{\pm\frac{2\pi}{3}i}: x\in\mathbb{R}^+\}$, and with orientation of $\mathbb{R}^-$, $\Gamma_\Psi^\pm$ towards zero.

(b) Ψ satisfies the jump conditions:

$$\Psi_+(\zeta)=\Psi_-(\zeta)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad\text{for } \zeta\in\mathbb{R}^-,$$

$$\Psi_+(\zeta)=\Psi_-(\zeta)\begin{pmatrix}1&0\\1&1\end{pmatrix}\quad\text{for } \zeta\in\Gamma_\Psi^\pm.$$

(c) As $\zeta\to\infty$,

$$\Psi(\zeta)=\left(\pi\zeta^{\frac12}\right)^{-\frac{\sigma_3}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}\left(I+\frac{1}{8\sqrt{\zeta}}\begin{pmatrix}-1&-2i\\-2i&1\end{pmatrix}-\frac{3}{2^7\zeta}\begin{pmatrix}1&-4i\\4i&1\end{pmatrix}+\mathcal{O}\left(\zeta^{-\frac32}\right)\right)e^{\zeta^{\frac12}\sigma_3}.$$

(d) As $\zeta\to 0$, $\Psi(\zeta)=\mathcal{O}(\log|\zeta|)$.

For $|\arg \zeta| < 2\pi/3$, we have

$$\Psi(\zeta) = \begin{pmatrix} I_0(\zeta^{1/2}) & \frac{i}{\pi} K_0(\zeta^{1/2}) \\ \pi i \zeta^{1/2} I_0'(\zeta^{1/2}) & -\zeta^{1/2} K_0'(\zeta^{1/2}) \end{pmatrix}, \tag{117}$$

where I_0 and K_0 are Bessel functions, $I_0'(x) = \frac{d}{dx} I_0(x)$, $K_0'(x) = \frac{d}{dx} K_0(x)$. For definitions and properties of Bessel functions see, for example [27]. Here the principal branch of $\zeta^{1/2}$ with the cut along the negative real axis is chosen. For the explicit expression of the solution in the sector $|\arg \zeta| > 2/3$, see [34].

We have the following useful asymptotics as $z \to 0$ for I_0:

$$I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \mathcal{O}(z^6). \tag{118}$$

We denote by $U^{(p)}$ fixed open nonintersecting balls containing $p \in \mathcal{T} = \{-1, v_1, v_2, 1\}$. Recalling ψ in (6), we define $\zeta = \zeta^{(p)}$ on $U^{(p)}$ by

$$\zeta^{(p)}(z) = -\left(s \int_p^z \psi(\xi) d\xi \right)^2. \tag{119}$$

As $z \to p$, we have the expansion

$$\zeta^{(p)}(z) = (z - p) s^2 \widetilde{\zeta}_0 (1 + o(1)), \qquad \widetilde{\zeta}_0 = -\frac{4(p - x_1)^2 (p - x_2)^2}{\prod_{q \in \mathcal{T} \setminus \{p\}} (p - q)}. \tag{120}$$

Note that $\zeta^{(p)}(z)$ is a conformal mapping of $U^{(p)}$ onto a neighborhood of zero. Observe also that $\widetilde{\zeta}_0 > 0$ for $p = v_2, -1$, and $\widetilde{\zeta}_0 < 0$ for $p = v_1, 1$, and so the contours in $U^{(p)}$ are mapped from the z-plane to the ζ-plane accordingly. We choose the exact form of the contours in the z-plane so that their images are direct lines.

Keeping in mind our conventions for the root branches, we obtain

$$\begin{aligned} (\zeta^{(p)}(z))^{1/2} + i(\phi(z) - \phi(p)) &= 0, \qquad \operatorname{Im} z > 0 \\ (\zeta^{(p)}(z))^{1/2} - i(\phi(z) - \phi(p)) &= 0, \qquad \operatorname{Im} z < 0 \end{aligned} \tag{121}$$

By (7), (77) and the definition of Ω in (13),

$$\phi(v_2) = \phi(1) = 0, \qquad \phi(-1) = \phi(v_1) = -\pi\Omega. \tag{122}$$

Let

$$X(z) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } \operatorname{Im} z > 0, \\ I & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{123}$$

For $p = -1, v_2$, we define the local parametrix on $U^{(p)}$ by

$$P(z) = E(z)\Psi(\zeta(z))X(z)e^{-is\phi(z)\sigma_3},$$
$$E(z) = \mathcal{N}(z; s\Omega)e^{is\phi(p)\sigma_3}X(z)^{-1}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left(\pi\zeta^{\frac{1}{2}}\right)^{\frac{1}{2}\sigma_3}, \tag{124}$$

where we have suppressed the superscript in $\zeta = \zeta^{(p)}$, and the branch cut for $\zeta^{1/4}$ is the same one as for $\zeta^{1/2}$.

Using the jump conditions, it is straightforward to verify that $E(z)$ has no jumps in $U^{(p)}$, and since its singularity at p is removable, $E(z)$ is analytic in the neighborhood $U^{(p)}$, $p = -1, v_2$.

Furthermore, it is easy to verify that $P(z)$ satisfies the same jump conditions as $S(z)$ in $U^{(p)}$, $p = -1, v_2$.

Finally, using the condition (c) in the Ψ-RHP and (121), we obtain for $w \in \partial U^{(p)}$

$$P(z)\mathcal{N}(z; s\Omega)^{-1} = I + \Delta_1(z) + \mathcal{O}(1/s^2), \qquad \Delta_1(z) = \mathcal{O}(1/s), \tag{125}$$

uniformly on the boundary as $s \to \infty$, where

$$\Delta_1(z) \equiv \Delta_1(z; s\Omega);$$
$$\Delta_1(z; \omega) = \frac{\mp 1}{8\sqrt{\zeta(z)}}\mathcal{N}(z; \omega)e^{is\phi(p)\sigma_3}\begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}e^{-is\phi(p)\sigma_3}\mathcal{N}^{-1}(z; \omega), \qquad p = -1, v_2, \tag{126}$$

where $\mp$ is taken to be $-$ on $U^{(p)} \cap \mathbb{C}_+$, and $+$ on $U^{(p)} \cap \mathbb{C}_-$. Note that $\Delta_1(z)$ is meromorphic in $U^{(p)}$, $p = -1, v_2$, with the first-order pole at $z = p$.

Similarly, for $p = v_1, 1$, we define the local parametrix on $U^{(p)}$ by

$$P(z) = E(z)\sigma_3\Psi(\zeta(z))\sigma_3 X(z)e^{-is\phi(z)\sigma_3},$$
$$E(z) = \mathcal{N}(z; s\Omega)e^{is\phi(p)\sigma_3}X(z)^{-1}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(\pi\zeta^{\frac{1}{2}}\right)^{\frac{1}{2}\sigma_3}. \tag{127}$$

Here $E(z)$ is analytic on $U^{(p)}$, $P(z)$ has the same jumps as $S(z)$ in $U^{(p)}$, $p = v_1, 1$, and the same condition (125) holds with

$$\Delta_1(z) \equiv \Delta_1(z; s\Omega);$$
$$\Delta_1(z; \omega) = \frac{\mp 1}{8\sqrt{\zeta(z)}}\mathcal{N}(z; \omega)e^{is\phi(p)\sigma_3}\begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix}e^{-is\phi(p)\sigma_3}\mathcal{N}^{-1}(z; \omega), \qquad p = v_1, 1, \tag{128}$$

where $\mp$ is taken to be $-$ on $U^{(p)} \cap \mathbb{C}_+$, and $+$ on $U^{(p)} \cap \mathbb{C}_-$. As at $v_2, -1$, $\Delta_1(z)$ in (128) is meromorphic in $U^{(p)}$, $p = v_1, 1$, with the first-order pole at $z = p$.

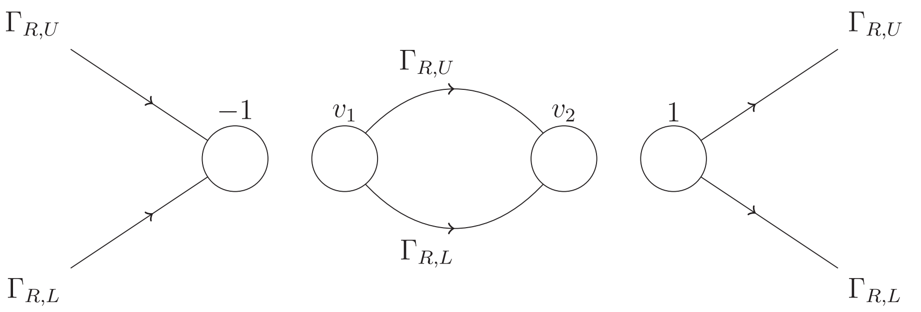

FIGURE 5 The jump contour Γ_R.

4.4 | Small norm RH problem: Solution of the Φ-RH problem for fixed v_1, v_2

Let

$$R(z) = \begin{cases} \mathcal{S}(z)\mathcal{N}^{-1}(z; s\Omega) & \text{for } z \in \mathbb{C} \setminus \left(\cup_{p\in\mathcal{T}} U^{(p)}\right), \\ \mathcal{S}(z)P^{-1}(z) & \text{for } z \in \cup_{p\in\mathcal{T}} U^{(p)}. \end{cases} \tag{129}$$

Then $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_R$, where Γ_R is as in Figure 5. We have

$$R_+(z) = R_-(z)J_R(z), \qquad J_R(z) = \begin{cases} P(z)\mathcal{N}^{-1}(z) & \text{for } z \in \cup_{p\in\mathcal{T}} \partial U^{(p)}, \\ \mathcal{N}(z)J_S(z)\mathcal{N}^{-1}(z) & \text{for } z \in \Gamma_R \setminus \left(\cup_{p\in\mathcal{T}} \partial U^{(p)}\right). \end{cases} \tag{130}$$

By (83) and (125), it follows that

$$J_R(z) = I + \mathcal{O}(s^{-1}/(|z^2| + 1))), \tag{131}$$

as $s \to \infty$, uniformly for $z \in \Gamma_R$, and by the definition of $\mathcal{S}$ and $\mathcal{N}$, we have

$$R(z) = I + \mathcal{O}(z^{-1}), \tag{132}$$

as $z \to \infty$. By standard small norm analysis, it follows that there is a solution to the RH problem for R for s sufficiently large, and that

$$R(z) = I + \mathcal{O}(1/s), \tag{133}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$ as $s \to \infty$. As usual, we expand R in the powers of the small parameter, $1/s$ in our case, to write

$$R(z) = I + R_1(z) + \mathcal{O}(1/s^2), \tag{134}$$

where R_1 solves the following RH problem. $R_1(z)$ is analytic outside the *clockwise* oriented boundaries $\partial U^{(p)}$ of the neighborhoods $U^{(p)}$,

$$R_{1+}(z) = R_{1-}(z) + \Delta_1(z), \qquad z \in \cup_{p\in\mathcal{T}} \partial U^{(p)},$$

and $R_1(z) \to 0$ as $z \to \infty$. The solution to this problem is given by

$$R_1(z) = \frac{1}{2\pi i} \int_{\cup_{p\in\mathcal{T}} \partial U^{(p)}} \frac{\Delta_1(x; s\Omega)}{x - z} dx, \qquad z \in \mathbb{C} \setminus \cup_{p\in\mathcal{T}} \partial U^{(p)}, \tag{135}$$

where the integrals are taken with *clockwise* orientation.

Taking (134) (one can obtain further terms in that expansion in a standard way) and tracing back the transformations $R \to S \to \Phi$, we obtain an asymptotic solution of the Φ-RH problem.

Additionally, we will need the main asympotitcs of $\frac{d}{dz}R(z)$, which we obtain from the standard representation

$$R(z) = I + \int_{\Gamma_R} \frac{R_-(\xi)(I - J_R(\xi))}{\xi - z} \frac{d\xi}{2\pi i}. \tag{136}$$

It follows that

$$\frac{d}{dz}R(z) = \int_{\Gamma_R} \frac{R_-(\xi)(I - J_R(\xi))}{(\xi - z)^2} \frac{d\xi}{2\pi i}, \tag{137}$$

and by (131) and (133) we obtain

$$\frac{d}{dz}R(v_2) = \mathcal{O}\big(s^{-1}\big). \tag{138}$$

4.5 | Extension of the solution to the regimes $v_2 \to 1$, $s(1 - v_2) \to \infty$; $v_1 \to -1$, $s(1 + v_1) \to \infty$

In our solution of the previous section, the end-points $-1 < v_1 < v_2 < 1$ were fixed. In this section, we show that the solution can be extended to the regime where v_2 not only can be fixed but can also approach 1 (and v_1 approach -1) sufficiently slowly as $s \to \infty$. This will be needed for the proof of Theorem 1 below.

More precisely, we fix $\epsilon > 0$ and assume

$$1 - v_2 \le 1 + v_1, \qquad v_2 - v_1 \ge \epsilon, \qquad s(1 - v_2) \to \infty. \tag{139}$$

We let $U^{(v_2)}$ and $U^{(1)}$ have radius equal to $c(1 - v_2)$, and similarly $U^{(v_1)}$ and $U^{(-1)}$ have radius equal to $c(1 + v_1)$, for some fixed and sufficiently small $c > 0$. Note that the neighborhoods can now contract with growing s.

As $v_2 \to 1$, $I_j \to \frac{\pi}{\sqrt{2(1-v_1)}}$, for $j = 0, 1, 2$, and computing an additional term in the expansion we find by (9) that

$$x_1 x_2 = \frac{v_1 - v_2}{2} - \frac{(1 - v_2)(1 + v_1)}{4} + \mathcal{O}\big((1 - v_2)^2\big), \qquad v_2 \to 1$$

uniformly in the regime (139). By (8),

$$x_1 = \frac{v_1 - 1}{2} + \mathcal{O}\big((1 - v_2)^2\big), \qquad x_2 = \frac{1 + v_2}{2} + \mathcal{O}\big((1 - v_2)^2\big). \tag{140}$$

By (140) and (119),

$$\frac{1}{\sqrt{\zeta(z)}} = \mathcal{O}\left(\frac{1}{s(1-v_2)}\right), \tag{141}$$

uniformly in the regime (139) and also uniformly for $z \in \partial U^{(p)}$, $p \in \mathcal{T} = \{-1, v_1, v_2, 1\}$.

Next we will show that $\mathcal{N}$ and $\mathcal{N}^{-1}$ are uniformly bounded on $\partial U^{(p)}$ for $p \in \mathcal{T}$. As $v_2 \to 1$ (under conditions (139)), we see from (85) that both $\gamma(z)$ and $\gamma^{-1}(z)$ are uniformly bounded also on $\partial U^{(p)}$ for $p \in \mathcal{T}$.

We now consider θ-functions, and start with τ. For J_0, we have

$$J_0 = \int_{v_1}^{v_2-\sqrt{1-v_2}} \frac{dx\left(1+\mathcal{O}\left(\sqrt{1-v_2}\right)\right)}{(1-x)\sqrt{(x+1)(x-v_1)}} + \int_{v_2-\sqrt{1-v_2}}^{v_2} \frac{dx\left(1+\mathcal{O}\left(\sqrt{1-v_2}\right)\right)}{\sqrt{2(1-v_1)(1-x)(v_2-x)}}, \tag{142}$$

as $v_2 \to 1$, and since

$$\frac{d}{dz}\log\left(\frac{\sqrt{z^2-1}+(it+\sqrt{1-t^2})z+i}{\sqrt{z^2-1}+\left(it-\sqrt{1-t^2}\right)z+i}\right) = \frac{\sqrt{1-t^2}}{(zt+1)\sqrt{z^2-1}}, \tag{143}$$

for any parameter t, it follows that

$$J_0 = \frac{1}{\sqrt{2(1-v_1)}}\left[5\log 2 + \log(1-v_2)^{-1} + \log\frac{1-v_1}{1+v_1}\right]\left(1+\mathcal{O}\left(\sqrt{1-v_2}\right)\right). \tag{144}$$

Thus, since $I_0 = \frac{\pi}{\sqrt{2(1-v_1)}}(1+\mathcal{O}(1-v_2))$,

$$\tau = i\frac{J_0}{I_0} = \frac{i}{\pi}\left[5\log 2 + \log\frac{1}{1-v_2} + \log\frac{1-v_1}{1+v_1}\right]\left(1+\mathcal{O}\left(\sqrt{1-v_2}\right)\right), \qquad v_2 \to 1 \tag{145}$$

in the regime (139), so that we have $-i\tau \to +\infty$.

As $-i\tau \to +\infty$, $\frac{\theta_3}{\theta_3(\omega)} \to 1$ for any $\omega \in \mathbb{R}$. We also observe that as $-i\tau \to +\infty$, the fraction

$$\frac{\theta(\xi+\omega;\tau)}{\theta(\xi;\tau)} \tag{146}$$

is bounded uniformly under conditions (139) and over all $\omega \in [0,1)$, for ξ bounded away from the zero of the θ-function $\frac{1+\tau}{2}$ modulo the lattice, and the same holds for derivatives of (146) with respect to ξ, ω, and τ. We now show that $\xi = u(z) \pm d$ remains bounded away from $\frac{1+\tau}{2}$ modulo the lattice for $z \in \partial U^{(p)}$, $p \in \mathcal{T}$.

We have by (90), (86), (87),

$$d = -u(\infty) = -\tau/2 + 1/2 + \frac{i}{2I_0}\int_{-\infty}^{-1}\frac{dx}{\sqrt{p(x)}} \mod \mathbb{Z}. \tag{147}$$

As $v_2 \to 1$,

$$\int_{-\infty}^{-1}\frac{dx}{\sqrt{|p(x)|}} = \int_{-\infty}^{-1}\frac{dx(1+\mathcal{O}(1-v_2))}{(1-x)\sqrt{(-1-x)(v_1-x)}}, \tag{148}$$

and by using (143)

$$\frac{1}{2I_0}\int_{-\infty}^{-1}\frac{dx}{\sqrt{p(x)}}=\frac{\sqrt{2(1-v_1)}}{(3-v_1)\pi\sqrt{1-\left(\frac{1+v_1}{3-v_1}\right)^2}}\log\left(\frac{1+\frac{1+v_1}{3-v_1}+\sqrt{1-\left(\frac{1+v_1}{3-v_1}\right)^2}}{1+\frac{1+v_1}{3-v_1}-\sqrt{1-\left(\frac{1+v_1}{3-v_1}\right)^2}}\right)(1+\mathcal{O}(1-v_2)), \tag{149}$$

as $v_2\to 1$ in the regime (139). We also have in the same regime by the definition (86) of $u(z)$,

$$u(z)=-\frac{i}{2\pi}\int_{v_2}^{z}\frac{dz}{((z-v_2)(z-1))^{1/2}}(1+\mathcal{O}(1-v_2)),\qquad z\in\partial U^{(v_2)}\cup\partial U^{(1)},$$
$$u(z)=-\frac{\tau}{2}-\frac{i\sqrt{1-v_1}}{\sqrt{2}\pi}\int_{v_1}^{z}\frac{dz}{((z+1)(z-v_1))^{1/2}(z-1)}(1+\mathcal{O}(1-v_2)),\quad z\in\partial U^{(v_1)}\cup\partial U^{(-1)}. \tag{150}$$

We note that (149) is bounded below by a fixed positive constant $c_1>0$ under conditions (139) and is uniformly to the main order $\frac{1}{2\pi}\log(1+v_1)^{-1}$, which is less or equal to $|\tau|/4$, since $\tau\sim\frac{i}{\pi}(\log(1-v_2)^{-1}+\log(1+v_1)^{-1})$. By (150), provided c is sufficiently small (where we recall that the radii of $U^{(v_2)}$ and $U^{(1)}$ are equal to $c(1-v_2)$, and the radii of $U^{(v_1)}$ and $U^{(-1)}$ are equal to $c(1+v_1)$), $c_1/2<|\mathrm{Im}\,(u(z)-d-\tau/2+1/2)|<\tau/3$ for $z\in\overline{U^{(v_2)}}$, and as a consequence $u(z)-d$ is bounded away from $\tau/2+1/2$ modulo the lattice. Similarly, it is straightforward to verify that $u(z)\pm d$ is bounded away from $\tau/2+1/2$ on $\overline{U^{(p)}}$, for $p\in\mathcal{T}$. By the boundedness of (146), it follows that $m_{ij}(z;\omega)$ and $\frac{\partial m_{ij}(z;\omega)}{\partial\omega}$ are uniformly bounded for $i,j\in\{1,2\}$ and for $z\in\overline{U^{(p)}}$, with $p\in\mathcal{T}$, and for future reference we note that by the boundedness of the derivatives of (146) with respect to ξ,ω,τ,

$$\frac{\partial}{\partial v_2}\frac{\partial m_{ij}(z;\omega)}{\partial\omega},\ \frac{\partial m_{ij}(z;\omega)}{\partial v_2}=\mathcal{O}\left(\max\left\{\left|\frac{\partial d}{\partial v_2}\right|,\left|\frac{\partial u(z)}{\partial v_2}\right|,\left|\frac{\partial\tau}{\partial v_2}\right|\right\}\right), \tag{151}$$

as $v_2\to 1$ in the regime (139), for $z\in\overline{U^{(p)}}$.

Combining the statements about boundedness of m and γ and γ^{-1}, it follows that $\mathcal{N}(z)$ and $\mathcal{N}(z)^{-1}$ are uniformly bounded for $z\in U^{(p)}$, $p\in\mathcal{T}$, and thus by (141), the jump matrix $J_R(z)$ for $R(z)$ on $\partial U^{(p)}$, $p\in\mathcal{T}$, has the form

$$P(z)\mathcal{N}(z;s\Omega)^{-1}=I+\mathcal{O}\left(\frac{1}{s(1-v_2)}\right), \tag{152}$$

as $s\to\infty$, uniformly under conditions (139) and also uniformly for $z\in\partial U^{(p)}$, $p\in\mathcal{T}$.

The analysis of $J_R(z)$ on the rest of the jump contour is similar, and we obtain uniformly for (139) and uniformly on this part of the contour

$$\mathcal{N}(z;s\Omega)J_S(z)\mathcal{N}(z;s\Omega)^{-1}=I+\mathcal{O}\left(e^{-s(1-v_2)c'(1+|z|)}\right),\qquad c'>0. \tag{153}$$

Thus we have a small norm problem for R, and as in the previous section we now obtain

$$R(z)=I+\mathcal{O}\left(\frac{1}{s(1-v_2)}\right), \tag{154}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$ under conditions (139). Therefore the solution of the Φ-Riemann-Hilbert problem for fixed v_1, v_2 extends to the regime (139). Note, however, that the error terms are different from those in the previous section.

By (136), (152), and (153)

$$R'(z)\big|_{z=v_2} = \mathcal{O}\left(\frac{1}{s(1-v_2)^2}\right). \tag{155}$$

5 | PRELIMINARY ASYMPTOTIC FORMULA FOR THE DETERMINANT

For $\nu = z - v_2$ in a neighborhood of 0, we write the expansions of $\zeta(z)$,

$$\sqrt{\zeta(\nu+v_2)} = s\zeta_0\sqrt{\nu}(1+\zeta_1\nu+\mathcal{O}(\nu^2)), \qquad \zeta_0 = \frac{2(v_2-x_1)(x_2-v_2)}{\sqrt{(1-v_2^2)(v_2-v_1)}} > 0, \tag{156}$$

where $-\pi < \arg\nu < \pi$, and the branch cut is on $(-\infty, 0]$. Similarly, we expand $\gamma(z)$, $m(z)$, and $u(z)$,

$$\begin{aligned} &\gamma(\nu+v_2) = \gamma_0\nu^{-1/4}(1+\gamma_1\nu+\mathcal{O}(\nu^2)), \qquad \gamma_0 e^{-\pi i/4} = \left(\frac{(1-v_2)(v_2-v_1)}{1+v_2}\right)^{1/4} > 0,\\ &u(\nu+v_2) = -u_0\nu^{1/2}(1+u_1\nu+\mathcal{O}(\nu^2)), \qquad u_0 = \frac{1}{I_0\sqrt{(v_2-v_1)(1-v_2^2)}} > 0, \\ &m_{jk}(\nu+v_2) = m_{jk,0} + m_{jk,1}\nu^{1/2} + m_{jk,2}\nu + \mathcal{O}(\nu^{3/2}), \end{aligned} \tag{157}$$

but with branches chosen such that $0 < \arg\nu < 2\pi$, and the branch cut on $[0, +\infty)$. Here m_{jk} are the matrix elements of m. Thus, $\arg\nu$ in (156) and (157) are the same for $\operatorname{Im}\nu > 0$, but are different for $\operatorname{Im}\nu < 0$.

Using the definition of m and the jump conditions (94), we easily obtain the relations:

$$\begin{aligned} m_{11,0} &= m_{12,0}, \qquad m_{21,0} = m_{22,0},\\ m_{11,1} &= -m_{12,1}, \quad m_{21,1} = -m_{22,1},\\ m_{11,2} &= m_{12,2}, \qquad m_{21,2} = m_{22,2}. \end{aligned} \tag{158}$$

We also find

$$\begin{aligned} m_{jj,0} &= m_{jj,0}(\omega) = \frac{\theta(0)\theta(\pm\omega+d)}{\theta(\omega)\theta(d)}\\ m_{jj,1} &= -m_{jj,0}u_0\left(\frac{\theta'(\pm\omega+d)}{\theta(\pm\omega+d)} - \frac{\theta'(d)}{\theta(d)}\right),\\ m_{jj,2} &= \frac{m_{jj,0}u_0^2}{2}\left(\frac{\theta''(\pm\omega+d)}{\theta(\pm\omega+d)} - \frac{\theta''(d)}{\theta(d)} + 2\left(\frac{\theta'(d)}{\theta(d)}\right)^2 - 2\frac{\theta'(\pm\omega+d)\theta'(d)}{\theta(\pm\omega+d)\theta(d)}\right), \end{aligned} \tag{159}$$

where $\pm$ means $+$ for $j = 1$ and $-$ for $j = 2$.

Let

$$\widehat{P}(z) = \mathcal{N}(z; s\Omega)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left(\pi\zeta^{\frac{1}{2}}\right)^{\frac{1}{2}\sigma_3}\Psi(\zeta(z)) \tag{160}$$

By the definition of $\mathcal{S}$ in (80), R in (129) and X in (123), and the fact that $\phi(v_2) = 0$,

$$\left[\Phi_+^{-1}(v_2)\Phi_+'(v_2)\right]_{12} = -\left[\widehat{P}_+^{-1}(v_2)\widehat{P}_+'(v_2) + \widehat{P}_+^{-1}(v_2)R^{-1}(v_2)R'(v_2)\widehat{P}_+(v_2)\right]_{21}. \tag{161}$$

With the notation of (156) and (157) (where the branches of $\sqrt{\nu}$ coincide for $\operatorname{Im}\nu > 0$), it is a straightforward calculation relying on the expansion of $I_0(z)$ in (118), the definition of $\mathcal{N}$ in (91), and the identities for m_{ij} in (158), to obtain

$$\begin{aligned} \widehat{P}_+(v_2) &= -\gamma_0\sqrt{\frac{\pi s\zeta_0}{2}}\begin{pmatrix} im_{11,0} & * \\ m_{22,0} & * \end{pmatrix}, \\ \widehat{P}_+'(v_2) &= -\gamma_0\sqrt{\frac{\pi s\zeta_0}{2}}\begin{pmatrix} im_{11,0}\left[\frac{m_{11,2}}{m_{11,0}} + \frac{m_{11,1}}{m_{11,0}}\left(\gamma_0^{-2} - \frac{s\zeta_0}{2}\right) + \frac{\zeta_1}{2} + \frac{s^2\zeta_0^2}{4} + \gamma_1 - \gamma_0^{-2}\frac{s\zeta_0}{2}\right] & * \\ m_{22,0}\left[\frac{m_{22,2}}{m_{22,0}} + \frac{m_{22,1}}{m_{22,0}}\left(\gamma_0^{-2} + \frac{s\zeta_0}{2}\right) + \frac{\zeta_1}{2} + \frac{s^2\zeta_0^2}{4} + \gamma_1 + \gamma_0^{-2}\frac{s\zeta_0}{2}\right] & * \end{pmatrix}, \end{aligned} \tag{162}$$

where we are uninterested in the entries $*$, and $\omega = s\Omega$ in $m_{jj,k}$.

We will now make use of the first identity (95) in Lemma 16, which, by the definitions of $m_{jk,\ell}$, we can write in the form

$$m_{11,0}m_{22,0} + \frac{\gamma_0^2}{2}(m_{11,0}m_{22,1} + m_{22,0}m_{11,1}) = 1. \tag{163}$$

Using this relation, we obtain by (161) and (162) for the r.h.s. of the differential identity of Lemma 14,

$$\begin{aligned} \mathcal{F}_s(v_1, v_2) = \frac{i}{2\pi}\left[\Phi_+^{-1}(v_2)\Phi_+'(v_2)\right]_{12} &= \frac{s^2\zeta_0^2}{4} - \frac{s\zeta_0}{4}m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2 + \Gamma_1\right) \\ &+ \frac{is\zeta_0\gamma_0^2}{4}\begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix}R^{-1}(v_2)R'(v_2)\begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix} \end{aligned} \tag{164}$$

where

$$\Gamma_j = \frac{m_{11,j}}{m_{11,0}} - \frac{m_{22,j}}{m_{22,0}}, \tag{165}$$

and we take $\omega = s\Omega$ in $m_{jj,k}$.

Now the more explicit asymptotic expression of (164) is different (in the error term) for fixed v_1, v_2 (Section 4.4) and for the double scaling regime of Section 4.5.

For fixed v_1, v_2, by (134), (135),

$$\begin{aligned} \mathcal{F}_s(v_1, v_2) = \frac{i}{2\pi}\left[\Phi_+^{-1}(v_2)\Phi_+'(v_2)\right]_{12} &= \frac{s^2\zeta_0^2}{4} - \frac{s\zeta_0}{4}m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2 + \Gamma_1\right) \\ &+ \frac{i\zeta_0\gamma_0^2}{4}W(s\Omega) + \mathcal{O}\left(s^{-1}\right), \end{aligned} \tag{166}$$

as $s \to \infty$ (uniformly for v_1, v_2 bounded away from each other and $\{-1,1\}$), where

$$W(\omega) = \begin{pmatrix} im_{22,0}(\omega) & m_{11,0}(\omega) \end{pmatrix} \sum_{p\in\{-1,v_1,v_2,1\}} \int_{\partial U^{(p)}} \frac{s\Delta_1(z;\omega)dz}{2\pi i(z-v_2)^2} \begin{pmatrix} m_{11,0}(\omega) \\ -im_{22,0}(\omega) \end{pmatrix} \tag{167}$$

with integration in the clockwise direction.

For the regime (139) of Section 4.5, by (154) and boundedness of m_{jk},

$$s \begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix} R^{-1}(v_2)R'(v_2) \begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix} = W(s\Omega) + \mathcal{O}\left(\frac{1}{s(1-v_2)^3}\right), \tag{168}$$

and since by (156) and (157), and the formulas for x_1, x_2 in (140), we have

$$\zeta_0\gamma_0^2 = \mathcal{O}(1-v_2), \qquad v_2 \to 1, \tag{169}$$

equation (164) becomes

$$\mathcal{F}_s(v_1,v_2) = \frac{s^2\zeta_0^2}{4} - \frac{\zeta_0 s}{4} m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2 + \Gamma_1\right) + \frac{i\zeta_0\gamma_0^2}{4} W(s\Omega) + \mathcal{O}\left(\frac{1}{s(1-v_2)^2}\right), \tag{170}$$

uniformly under conditions (139).

Proposition 17. *Let*

$$D(v_1,v_2) = \frac{s^2\zeta_0^2}{4} - \frac{s\zeta_0}{4} m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2 + \Gamma_1\right) + \frac{i\zeta_0\gamma_0^2}{4}\int_0^1 W(\omega)d\omega, \tag{171}$$

where ζ_0 and γ_0 are given in (156), (157), $\Gamma_j = \frac{m_{11,j}}{m_{11,0}} - \frac{m_{22,j}}{m_{22,0}}$, with $m_{jj,k} = m_{jj,k}(s\Omega)$ from (159), and where W is given in (167) (with Δ_1 defined by (126) and (128)).

(a) Let $V \in (0,1)$, and let $\widehat{A} = (-1,-V)\cup(V,1)$. Let $v_2 = -v_1$, and denote $v = v_2$. Fix $\epsilon > 0$. Then

$$\log\det(I-K_s)_{\widehat{A}} - \log\det(I-K_s)_{A_s} = 2\int_{1-\frac{2t}{s}}^{V} D(-v,v)dv + \mathcal{O}\left(\frac{1}{t}\right),$$

as $s\to\infty$, uniformly for $\epsilon \le V \le 1-\frac{2t}{s}$, where $t(s)\to\infty$, $t \le \frac{1}{2}(\log s)^{1/4}$, and $A_s = (-1,-1+2t/s)\cup(1-2t/s,1)$.

(b) Let $-1 < V_1 < 0$ and V_2 be fixed, $V_1 < V_2 < 1$, and denote $A = (-1,V_1)\cup(V_2,1)$. Then

$$\log\det(I-K_s)_A - \log\det(I-K_s)_{(-1,V_1)\cup(-V_1,1)} = \int_{-V_1}^{V_2} D(V_1,v_2)dv_2 + \mathcal{O}\left(\frac{1}{s}\right),$$

as $s\to\infty$.

(c) Let $A = (-1,V_1)\cup(V_2,1)$, and a fixed $\epsilon > 0$, and with $-1 < V_1 < \widehat{V}_2 < 1$ and

$$1-V_2 \le 1-\widehat{V}_2 \le 1+V_1, \qquad V_2 - V_1 \ge \epsilon, \qquad s(1-V_2)\to\infty. \tag{172}$$

Then

$$\log\det(I-K_s)_A - \log\det(I-K_s)_{(-1,V_1)\cup(\widehat{V}_2,1)} = \int_{\widehat{V}_2}^{V_2} D(V_1,v_2)dv_2 + \mathcal{O}\left(\frac{1}{(1-V_2)s}\right),$$

as $s\to\infty$, uniformly in the regime (172).

Remark 18. In the proof, considering the effects of averaging w.r.t. $\omega = s\Omega$, we will show that (171) gives the main contribution, and the error terms are as presented.

Remark 19. Part (a) allows us to integrate over symmetric intervals from the position of two small ones at 1 and -1 (where Lemma 8 holds) to general symmetric intervals with a fixed $0 < V < 1$. Part (b) allows then to move the V_2 edge to an arbitrary fixed position $V_1 \equiv -V < V_2 < 1$. Note that the condition $-1 < V_1 < 0$ here is not a loss of generality for $\det(I - K_s)_A$, since we can use the symmetry $x \to -x$ of the determinant.

Part (c) allows us to to integrate to reach a scaling limit where $V_2 = V_2(s)$ can approach 1 provided $s(1 - V_2) \to \infty$ and V_1 is fixed.

Finally, choose a $V_1(s) = -V_2(s)$ such that $2t = (1 + V_1)s \to \infty$ (in this case, Lemma 8 still holds by Remark 9), and then, if needed, move V_2 closer to 1 using Part (c). Then, if needed, one can use the symmetry $x \to -x$, to reach an arbitrary situation with $(1 + V_1)s \to \infty$, $(1 - V_2)s \to \infty$.

Proof. We first prove Part (b) of the proposition, then Part (c), and finally Part (a). By (166) and the differential identity (42), all we need to do for the proof of Part (b) is to show that, with $\widehat{V}_2 = -V_1$,

$$\int_{\widehat{V}_2}^{V_2} \zeta_0 \gamma_0^2 W(s\Omega) dv_2 = \int_{\widehat{V}_2}^{V_2} \zeta_0 \gamma_0^2 \int_0^1 W(\omega) d\omega dv_2 + \mathcal{O}(s^{-1}), \tag{173}$$

as $s \to \infty$. Denote $f(\omega; v_2, v_1) = \zeta_0 \gamma_0^2 W(\omega)$. This function is analytic in both ω and v_2 (v_2 is bounded away from v_1 and 1). Let f_j denote its Fourier coefficients with respect to ω, so that

$$f(\omega; v_2, v_1) = \zeta_0 \gamma_0^2 W(\omega) = \sum_{j=-\infty}^{\infty} f_j(v_2, v_1) e^{2\pi i j \omega}. \tag{174}$$

For $j \neq 0$, it follows by integration by parts that

$$\begin{aligned} \left| \int_{\widehat{V}_2}^{V_2} f_j(v_2, v_1) e^{2\pi i j s \Omega} dv_2 \right| &= \frac{1}{2\pi |j| s} \left| \left[\frac{f_j(v_2, v_1) e^{2\pi i j s \Omega}}{\frac{\partial}{\partial v_2} \Omega(v_2, v_1)} \right]_{\widehat{V}_2}^{V_2} \right. \\ &\quad \left. - \int_{\widehat{V}_2}^{V_2} \frac{\partial}{\partial v_2} \left(\frac{f_j(v_2, v_1)}{\frac{\partial}{\partial v_2} \Omega(v_2, v_1)} \right) e^{2\pi i j s \Omega} dv_2 \right|. \end{aligned} \tag{175}$$

In Proposition 24 (b) below we give an explicit formula for $\frac{\partial}{\partial v_2} \Omega(v_2, v_1)$, and in particular it is a strictly positive differentiable function bounded away from zero when v_2 is bounded away from v_1 and 1. Thus

$$\begin{aligned} \int_{\widehat{V}_2}^{V_2} f(s\Omega; v_2, v_1) dv_2 &= \sum_{j=-\infty}^{\infty} \int_{\widehat{V}_2}^{V_2} f_j(v_2, v_1) e^{2\pi i j s \Omega} dv_2 \\ &= \int_{\widehat{V}_2}^{V_2} f_0(v_2, v_1) dv_2 + \mathcal{O}\left(\frac{1}{s}\right), \qquad s \to \infty, \end{aligned} \tag{176}$$

which yields (173) since $f_0(v_2, v_1) = \zeta_0 \gamma_0^2 \int_0^1 W(\omega) d\omega$.

We now prove Part (c) of the proposition.

Substituting (140) into the expression (225) for $\frac{\partial \Omega}{\partial v_2}$ in Proposition 24 below, and also using (219), we obtain

$$\frac{\partial \Omega}{\partial v_2} = \frac{3 - v_1}{4\pi\sqrt{2(1 - v_1)}} + \mathcal{O}(1 - v_2), \qquad \frac{\partial^2 \Omega}{\partial v_2^2} = \mathcal{O}((1 - v_2)^{-1}), \tag{177}$$

and, in particular, $\frac{\partial \Omega}{\partial v_2}$ remains bounded away from 0.

We now show that

$$f_j(v_2, v_1) = \mathcal{O}\left(\frac{1}{j(1 - v_2)}\right), \qquad \frac{\partial}{\partial v_2} f_j(v_2, v_1) = \mathcal{O}\left(\frac{1}{j(1 - v_2)^2}\right), \tag{178}$$

as $v_2 \to 1$, for $j \neq 0$, uniformly under conditions (139), which proves Part (c) of the proposition by (170) and arguments similar to those we used in the proof of Part (b).

Since

$$\left|f_j(v_2, v_1)\right| = \left|\int_0^1 f(\omega; v_2, v_1)e^{-2\pi i j\omega} d\omega\right| = \left|\frac{1}{2\pi j}\int_0^1 \frac{\partial}{\partial \omega} f(\omega; v_2, v_1)e^{-2\pi i j\omega} d\omega\right|, \qquad j \neq 0, \tag{179}$$

and similarly for $\frac{\partial}{\partial v_2} f_j(v_2, v_1)$, it suffices to show that

$$\frac{\partial}{\partial \omega} f(\omega; v_2, v_1) = \mathcal{O}\left(\frac{1}{1 - v_2}\right), \qquad \frac{\partial}{\partial \omega}\frac{\partial}{\partial v_2} f(\omega; v_2, v_1) = \mathcal{O}\left(\frac{1}{(1 - v_2)^2}\right), \tag{180}$$

as $v_2 \to 1$.

It follows by the definition of W in (167), (152), (169), and the arguments of the previous section that

$$f(\omega; v_2, v_1) = \zeta_0\gamma_0^2 W(\omega) = \mathcal{O}\left(\frac{1}{1 - v_2}\right) \tag{181}$$

as $v_2 \to 1$ under conditions (139).

We recall that $\frac{\partial m_{ij}(z)}{\partial \omega}$, $i, j \in \{1, 2\}$, are uniformly bounded for $z \in \overline{U^{(p)}}$, $p \in \mathcal{T}$, and so $\frac{\partial f(\omega; v_2, v_1)}{\partial \omega}$ satisfies the same upper bound as $f(\omega; v_2, v_1)$ given in (181), proving the first bound in (180).

To obtain the second one, we observe first that by (145),

$$\frac{\partial}{\partial v_2}\tau = \mathcal{O}\left(\frac{1}{1 - v_2}\right), \qquad v_2 \to 1, \tag{182}$$

and by (150),

$$\frac{\partial u(z)}{\partial v_2} = \mathcal{O}\left(\frac{1}{1 - v_2}\right), \tag{183}$$

as $v_2 \to 1$, uniformly for $z \in \partial U^{(p)}$, $p \in \mathcal{T}$.

It follows by (147) and (149) that $\frac{\partial}{\partial v_2} d = \mathcal{O}(\frac{1}{1 - v_2})$, as $v_2 \to 1$. Thus, by (151),

$$\frac{\partial m_{ij}(z; \omega)}{\partial v_2} = \mathcal{O}\left(\frac{1}{1 - v_2}\right), \tag{184}$$

as $v_2 \to 1$, uniformly for $z \in \overline{U^{(p)}}$, $p \in \mathcal{T}$. Furthermore, by the definition (85),

$$\frac{\partial}{\partial v_2}\gamma(z),\ \frac{\partial}{\partial v_2}\gamma^{-1}(z) = \mathcal{O}\left(\frac{1}{1-v_2}\right). \tag{185}$$

By (140) and (119),

$$\frac{\partial}{\partial v_2}\left(\frac{1}{\sqrt{\zeta(z)}}\right) = \mathcal{O}\left(\frac{1}{s(1-v_2)^2}\right). \tag{186}$$

The above bounds taken together imply

$$s\frac{\partial}{\partial v_2}\Delta_1(z) = \mathcal{O}\left(\frac{1}{(1-v_2)^2}\right). \tag{187}$$

It follows by the definition of W in (167), (169), and boundedness of m_{jk} that

$$\frac{\partial}{\partial v_2}f(\omega; v_2, v_1) = \mathcal{O}\left(\frac{1}{(1-v_2)^2}\right), \tag{188}$$

as $v_2 \to 1$, uniformly under conditions (139). Since $\frac{\partial}{\partial\omega}\frac{\partial m_{ij}(z)}{\partial v_2} = \mathcal{O}(\frac{\partial m_{ij}(z)}{\partial v_2})$, it follows that $\frac{\partial}{\partial\omega}\frac{\partial}{\partial v_2}f(\omega; v_2, v_1) = \mathcal{O}(\frac{\partial}{\partial v_2}f(\omega; v_2, v_1))$, which proves the second bound in (180), completing the proof of Part (c) of the proposition.

To show Part (a), we let $v_2 = -v_1 = v$, and take the limit $s \to \infty$ such that $\epsilon < v < 1 - \frac{M}{s}$ for some $\epsilon > 0$ and a sufficiently large M. By (43),

$$\frac{\partial}{\partial v}\det(I - K_s)_{(-1,-v)\cup(v,1)} = 2\mathcal{F}_s(-v, v). \tag{189}$$

We observe that (170) is valid also for $v_2 = -v_1 = v$, and all that remains to finish the proof of Part (a) of the proposition is to consider the Fourier coefficients of f. In place of (175), we have

$$\left|\int_{\hat{V}}^{V} f_j(v,-v)e^{2\pi i j s\Omega}dv\right| = \frac{1}{2\pi|j|s}\left|\left[\frac{f_j(v,-v)e^{2\pi i j s\Omega}}{\frac{\partial}{\partial v}\Omega(v,-v)}\right]_{\hat{V}}^{V} - \int_{\hat{V}}^{V}\frac{\partial}{\partial v}\left(\frac{f_j(v,-v)}{\frac{\partial}{\partial v}\Omega(v,-v)}\right)e^{2\pi i j s\Omega}dv\right|. \tag{190}$$

By above arguments, it suffices to show that the right hand side of (190) is of order $\frac{1}{j^2 s(1-v)^2}$. To do this we need the first bound in (180), which holds also for $v_2 = -v_1 = v$, and additionally we need to prove that

$$\frac{\partial}{\partial\omega}\frac{\partial}{\partial v}f(\omega; v, -v) = \mathcal{O}\left(\frac{1}{(1-v)^2}\right), \tag{191}$$

as $v \to 1$, and that $\frac{d}{dv}\Omega(v,-v)$ remains bounded away from 0. Note that, using contour integration,

$$\Omega^{-1}(v_2, v_1) = I_0 = \int_{v_2}^{1}\frac{dx}{\sqrt{|p(x)|}} = \int_{-1}^{v_1}\frac{dx}{\sqrt{|p(x)|}},$$

and therefore

$$\Omega(v_2, v_1) = \Omega(-v_1, -v_2), \qquad \frac{\partial}{\partial v}\Omega(v, -v) = 2\frac{\partial}{\partial v_2}\Omega(v_2, -v)\Big|_{v_2=v}. \tag{192}$$

The last derivative is thus bounded away from 0 by (177).

In order to prove (191), we simply observe that the bounds obtained in (183)–(186) also hold for the derivatives with respect to v instead of v_2, which yields

$$\frac{\partial}{\partial v} f(\omega; v, -v) = \mathcal{O}\left(\frac{1}{(1-v)^2}\right), \tag{193}$$

as $v \to 1$. Since $\frac{\partial}{\partial \omega}\frac{\partial m_{ij}(z)}{\partial v} = \mathcal{O}(\frac{\partial m_{ij}(z)}{\partial v})$, it follows that $\frac{\partial}{\partial \omega}\frac{\partial}{\partial v} f(\omega; v, -v) = \mathcal{O}(\frac{\partial}{\partial v} f(\omega; v, -v))$, which proves (191) and thus Part (a) of the proposition. □

6 | PROOF OF THEOREMS 1, 10, AND 11

In the next 3 sections, we show that (171) in Proposition 17 can be written as

$$D(v_1, v_2) = \frac{\partial}{\partial v_2}\mathcal{G}(s; v_1, v_2) + \frac{\partial \tau}{\partial v_2}\int_0^1 \frac{\partial}{\partial \tau}\log\theta_3(\omega;\tau)d\omega - \frac{\partial \tau}{\partial v_2}\frac{\partial}{\partial \tau}\log\theta_3(s\Omega;\tau), \tag{194}$$

where

$$\mathcal{G} = s^2\left(\frac{I_2 - \frac{v_2+v_1}{2}I_1}{I_0} - \frac{(v_2-v_1)^2}{8}\right) + \log\theta(s\Omega;\tau) - \frac{1}{2}\log I_0 - \frac{1}{8}\sum_{y\in\{-1,v_1,v_2,1\}}\log|q(y)|. \tag{195}$$

We now use (194) to prove Theorems 1, 10, and 11. First, we show that with $\hat{V}_2$ fixed, and in all asymptotic regimes of Proposition 17,

$$\int_{\hat{V}_2}^{V_2}\left(\frac{\partial \tau}{\partial v_2}\int_0^1 \frac{\partial}{\partial \tau}\log\theta_3(\omega;\tau)d\omega - \frac{\partial \tau}{\partial v_2}\frac{\partial}{\partial \tau}\log\theta_3(s\Omega;\tau)\right)dv_2 = \mathcal{O}\left(\frac{1}{s(1-V_2)}\right), \qquad s\to\infty, \tag{196}$$

uniformly in integration regimes of Proposition 17, and so this part only contributes to the error term.

Using the differential equation (A.10) and (182), we write

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \tau}{\partial v_2}\frac{\partial}{\partial \tau}\log\theta_3(\omega;\tau)\right) = \frac{1}{4\pi i}\frac{\partial \tau}{\partial v_2}\left(\frac{\theta_3''}{\theta_3}\right)'(\omega) = \mathcal{O}\left(\frac{1}{1-v_2}\right). \tag{197}$$

Also since by (145),

$$\frac{\partial^2 \tau}{\partial v_2^2} = \mathcal{O}\left(\frac{1}{(1-v_2)^2}\right), \tag{198}$$

we similarly obtain

$$\frac{\partial}{\partial\omega}\frac{\partial}{\partial v_2}\left(\frac{\partial\tau}{\partial v_2}\frac{\partial}{\partial\tau}\log\theta_3(\omega;\tau)\right)=\mathcal{O}\left(\frac{1}{(1-v_2)^2}\right). \tag{199}$$

The estimates (197) and (199) imply, by similar arguments to (179), (180), (175), the estimate (196).

We now apply Part (a) of Proposition 17 to integrate (194) from the position of 2 symmetric small intervals $v=-v_1=v_2=1-\frac{2t}{s}$, $t=\frac{1}{2}\log(s)^{1/4}$, where Lemma 8 can be applied, to the case of $V=-v_1=v_2>0$ *fixed*. If $-v_1=v_2=v$, by symmetry under the exchange $v_2\to-v_1, v_1\to-v_2$,

$$2\frac{\partial}{\partial v_2}\mathcal{G}(s;v_1,v_2)=\frac{\partial}{\partial v}\mathcal{G}(s;-v,v).$$

Thus, applying Part (a) of Proposition 17 and using Lemma 8, we obtain

$$\log\det(I-K_s)_{\hat{A}}=\mathcal{G}(s;-V,V)-\mathcal{G}\left(s;-1+\frac{2t}{s},1-\frac{2t}{s}\right)-t^2-\frac{1}{2}\log t+2c_0+\mathcal{O}(1/t). \tag{200}$$

To finish the proof of Theorem 1 in the symmetric case, we need to estimate $\mathcal{G}(s;-1+\frac{2t}{s},1-\frac{2t}{s})$. Using formulae (A.37), (A.36), we obtain in our case $v=1-\frac{2t}{s}$ (recall that $v'^2=1-v^2$)

$$I_0(-v,v)=\frac{\pi}{2}\left(1+\frac{t}{s}+\frac{5t^2}{4s^2}+\mathcal{O}((t/s)^3)\right),\qquad \frac{I_2(-v,v)}{I_0(-v,v)}=1-\frac{2t}{s}+\frac{t^2}{s^2}+\mathcal{O}((t/s)^3), \tag{201}$$

and so the term with s^2 in $\mathcal{G}(s;-1+\frac{2t}{s},1-\frac{2t}{s})$ becomes

$$\frac{I_2(-v,v)}{I_0(-v,v)}-\frac{v^2}{2}=\frac{1}{2}-\frac{t^2}{s^2}+\mathcal{O}((t/s)^3). \tag{202}$$

The term $\log\theta$ gives a contribution only to the error term, indeed, since by (A.36)

$$J_0(-v,v)=2K(v)=\left(\log\frac{4s}{t}\right)(1+\mathcal{O}(t/s)),\qquad \tau=i\frac{J_0}{I_0}=\frac{2i}{\pi}\left(\log\frac{4s}{t}\right)(1+\mathcal{O}(t/s)),$$

we have that

$$\log\theta(s\Omega)=\log\left(1+\mathcal{O}((t/s)^2)\right)=\mathcal{O}((t/s)^2),\qquad -v_1=v_2=v=1-\frac{2t}{s}. \tag{203}$$

Finally, in this case

$$|q(1)|=|q(-1)|=1-\frac{I_2}{I_0}=\frac{2t}{s}(1+\mathcal{O}(t/s)),\qquad |q(-v)|=|q(v)|=\frac{I_2}{I_0}-v^2=\frac{2t}{s}(1+\mathcal{O}(t/s)), \tag{204}$$

and so

$$-\frac{1}{8}\sum_{y\in\{-1,v_1,v_2,1\}}\log|q(y)|=-\frac{1}{2}\log\frac{2t}{s}+\mathcal{O}(t/s). \tag{205}$$

Substituting (201), (202), (203), (205) into the expression (195) for $\mathcal{G}(s;-1+\frac{2t}{s},1-\frac{2t}{s})$, and that, in turn, into (200), we obtain asymptotics (14) with an error term $o(1)$ and with $\hat{G}_1$ and c_1 as in (15) in the case $-v_1 = v_2 = V > 0$. We then extend it to the general case of fixed $-1 < v_1 < v_2 < 1$ by now a straightforward application of Part (b) of Proposition 17. (In fact, for $v_1 < 0$, but the general case follows by a symmetry argument: see Remark 19.) Now since by [18], (14) (with the error term $\mathcal{O}(s^{-1})$) holds for *some* constants $\widehat{G}_1$, c_1, these must be equal to those in (15). This completes the proof of Theorem 1, assuming (194).

Given Theorem 1, we immediately obtain Theorem 10 by applying Part (c) of Proposition 17 and a symmetry argument as discussed in Remark 19.

Given Theorem 10, we now consider the limit where $v_2 \to 1$ and $v_1 \to -1$ in order to prove Theorem 11. We do this by evaluating G_0, τ, and c_1 in (14) as $\max\{1 - v_2,\ 1 + v_1\} \to 0$ (the regime (139)) and using Theorem 10. From (145), we know that $-i\tau \to +\infty$, and it follows that $\theta_3(s\Omega; \tau) \to 1$. Substituting the asymptotics of x_1 and x_2 from (140) into the definition of q in (6), we obtain

$$\sum_{y\in\{-1,v_1,v_2,1\}} \log|q(y)| = 2\log(1-v_2)(1+v_1) + o(1), \tag{206}$$

as $v_2 \to 1$ and $v_1 \to -1$. From Section 4.5 we recall that $I_0 \to \pi/2$, and combining this with (206) and our formula for c_1 in Theorem 1 we obtain

$$c_1 = -\frac{1}{4}\log\frac{(1-v_2)(1+v_1)}{4} + 2c_0 + o(1), \tag{207}$$

as $v_2 \to 1$ and $v_1 \to -1$. Now consider G_0. A straightforward (albeit somewhat lengthy) calculation yields

$$G_0 = -\frac{(1-v_2)^2}{8} - \frac{(1+v_1)^2}{8} + \mathcal{O}\big(\max\{(1-v_2)^4, (1+v_1)^4\}\big), \qquad v_2 \to 1 \text{ and } v_1 \to -1. \tag{208}$$

Substituting (207) and (208) into (14) with the error term of Theorem 10, we obtain

$$\begin{aligned}\log P_s(A) = &-\frac{s^2(1-v_2)^2}{8} - \frac{s^2(1+v_1)^2}{8} - \frac{1}{4}\log\frac{s^2(1-v_2)(1+v_1)}{4} + 2c_0\\ &+ o(1) + \mathcal{O}\left(\max\left\{\frac{1}{s(1-v_2)}, \frac{1}{s(1+v_1)}, s(1-v_2)^2, s(1+v_1)^2\right\}\right)\end{aligned}$$

for the scaling regime of Theorem 11, where the term $o(1)$ is independent of s. Thus, by the asymptotics for a single gap in (2), we obtain Theorem 11.

We now return to the proof of Theorem 1. All that remains is to verify (194). In Section 7 we consider the leading order term in (171), in Section 8 we consider the term involving $(\gamma_0^2\Gamma_2 + \Gamma_1)$, which yields the derivative of $\log\theta(s\Omega)$, and in Section 9 we consider the term with $\int_0^1 W(\omega)d\omega$, which yields the constant. Thus, we will prove the following three lemmata, which taken together imply (194).

Lemma 20.

$$\frac{\zeta_0^2}{4} = \frac{\partial}{\partial v_2}\left(\frac{I_2 - \frac{v_2+v_1}{2}I_1}{I_0} - \frac{(v_2-v_1)^2}{8}\right). \tag{209}$$

Lemma 21.

$$-\frac{s\zeta_0}{4}m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2+\Gamma_1\right)=\frac{\partial}{\partial v_2}\log\theta_3(s\Omega;\tau)-\frac{\partial\tau}{\partial v_2}\frac{\partial}{\partial\tau}\log\theta_3(s\Omega;\tau). \tag{210}$$

Note that the r.h.s. here equals the partial derivative $s\frac{\partial\Omega}{\partial v_2}\frac{\partial}{\partial(s\Omega)}\log\theta_3(s\Omega;\tau)$ *with* τ *fixed.*

Lemma 22.

$$\frac{i\zeta_0\gamma_0^2}{4}\int_0^1 W(\omega)d\omega=-\frac{\partial}{\partial v_2}\left(\frac{1}{2}\log I_0+\frac{1}{8}\sum_{y\in\{-1,v_1,v_2,1\}}\log|q(y)|\right)+\frac{\partial\tau}{\partial v_2}\int_0^1\frac{\partial}{\partial\tau}\log\theta_3(\omega;\tau)d\omega, \tag{211}$$

where $W(\omega)$ *is given in (167).*

7 | THE LEADING ORDER TERM: PROOF OF LEMMA 21

Recall from (5) the notation for I_j, J_j, $j=0,1,2$. We will calculate the derivatives $\frac{\partial}{\partial v_2}I_j$, $j=0,1,2$ in terms of the integrals themselves. The crucial identity here is (217) below.

First, we have

$$\frac{\partial I_j}{\partial v_2}=\frac{i}{4}\int_{A_1}\frac{z^j}{(z-v_2)\sqrt{p(z)}}dz,\qquad j=0,1,2. \tag{212}$$

Therefore,

$$\frac{\partial I_1}{\partial v_2}=\frac{i}{4}\int_{A_1}\frac{z-v_2+v_2}{(z-v_2)\sqrt{p(z)}}dz=I_0/2+v_2\frac{\partial I_0}{\partial v_2}, \tag{213}$$

and similarly,

$$\frac{\partial I_2}{\partial v_2}=I_1/2+v_2\frac{\partial I_1}{\partial v_2}. \tag{214}$$

The last two equations imply

$$\frac{\partial I_2}{\partial v_2}=v_2^2\frac{\partial I_0}{\partial v_2}+I_1/2+v_2I_0/2. \tag{215}$$

From here and (213),

$$\frac{\partial}{\partial v_2}(2I_2-(v_2+v_1)I_1)=(v_2-v_1)\left[v_2\frac{\partial I_0}{\partial v_2}+I_0/2\right]. \tag{216}$$

We will now express the derivative $\partial I_0/\partial v_2$ in terms of I_js. To this end, observe that

$$\begin{aligned}0=\frac{i}{2}\int_{A_1}\frac{d}{dz}\sqrt{\frac{(z^2-1)(z-v_1)}{z-v_2}}dz&=-i\frac{v_2-v_1}{4}\int_{A_1}\frac{z^2-1}{(z-v_2)\sqrt{p(z)}}dz+I_2-v_1I_1\\&=-(v_2-v_1)\frac{\partial}{\partial v_2}(I_2-I_0)+I_2-v_1I_1,\end{aligned} \tag{217}$$

so that

$$\frac{\partial}{\partial v_2}(I_2 - I_0) = \frac{I_2 - v_1 I_1}{v_2 - v_1}. \tag{218}$$

Using this equation and (215) we have

$$\frac{\partial I_0}{\partial v_2} = \frac{-I_2 + \frac{v_2+v_1}{2} I_1 + \frac{v_2(v_2-v_1)}{2} I_0}{(1-v_2^2)(v_2-v_1)}. \tag{219}$$

This and (216) imply

$$\frac{\partial}{\partial v_2}\left(\frac{I_2 - \frac{v_2+v_1}{2} I_1}{I_0}\right) = \frac{v_2 - v_1}{4} + \frac{\left(2I_2 - (v_1+v_2)I_1 + v_2(v_1-v_2)I_0\right)^2}{4I_0^2(1-v_2^2)(v_2-v_1)}. \tag{220}$$

By the formulas for x_1 and x_2 in (8) and (9), and the formula for ζ_0 in (156), we finish the proof of Lemma 21.

Remark 23. We also observe for future reference that the arguments may be copied line for line and applied to the integrals $J_j = \int_{v_1}^{v_2} \frac{x^j\,dx}{\sqrt{|p(x)|}}$ (by instead considering an integral over a closed loop containing (v_1, v_2) and different branches of the roots), and we obtain the analogues to (219) and (216):

$$\frac{\partial J_0}{\partial v_2} = \frac{-J_2 + \frac{v_2+v_1}{2} J_1 + \frac{v_2(v_2-v_1)}{2} J_0}{(1-v_2^2)(v_2-v_1)}, \tag{221}$$

$$\frac{\partial}{\partial v_2}(2J_2 - (v_2+v_1)J_1) = (v_2 - v_1)\left[v_2 \frac{\partial J_0}{\partial v_2} + J_0/2\right]. \tag{222}$$

8 | THE FLUCTUATIONS: PROOF OF LEMMA 21

We write the first subleading term in (171) in the form, using (156), (157) for ζ_0, u_0,

$$\frac{s\zeta_0 u_0}{4} T_1(s\Omega), \qquad \frac{\zeta_0 u_0}{4} = \frac{(v_2 - x_1)(x_2 - v_1)}{2I_0(v_2-v_1)(1-v_2^2)}, \qquad T_1(\omega) = -\frac{m_{11,0} m_{22,0}}{u_0}\left(\gamma_0^2 \Gamma_2 + \Gamma_1\right). \tag{223}$$

Our goal in this section is to prove the following proposition, of which Lemma 21 is an immediate corollary.

Proposition 24. *There hold the identities:*

(a)

$$-i\frac{\partial \tau}{\partial v_2} = \frac{\partial |\tau|}{\partial v_2} = \frac{\pi}{I_0^2(1-v_2^2)(v_2-v_1)} = \pi u_0^2, \tag{224}$$

(b)

$$\frac{\partial\Omega}{\partial v_2} = \frac{(v_2 - x_1)(x_2 - v_2)}{I_0(1 - v_2^2)(v_2 - v_1)}, \tag{225}$$

(c)

$$T_1(\omega) = 2\frac{\theta_3'(\omega)}{\theta_3(\omega)}. \tag{226}$$

Proof. To show Part (a) note that in the notation of (5)

$$|\tau| = \frac{J_0}{I_0},$$

and therefore, using (219), (221), we have

$$\frac{\partial|\tau|}{\partial v_2} = \frac{I_2 J_0 - J_2 I_0 - \frac{v_1+v_2}{2}(I_1 J_0 - I_0 J_1)}{I_0^2(1 - v_2^2)(v_2 - v_1)}, \tag{227}$$

which gives Part (a) of the proposition by Riemann's bilinear relations (A.30).

Part (b) follows from (13) and (219) by using (8), (9):

$$\frac{\partial\Omega}{\partial v_2} = -\frac{1}{I_0^2}\frac{\partial I_0}{\partial v_2} = -\frac{x_1x_2 + v_2(v_2 - v_1)/2}{I_0(1 - v_2^2)(v_2 - v_1)} = \frac{(v_2 - x_1)(x_2 - v_2)}{I_0(1 - v_2^2)(v_2 - v_1)}. \tag{228}$$

We will now show Part (c). Substituting the definitions of $m_{jj,k}$ and Γ_j into T_1 in (223), and using the identity (96) of Lemma 16, we write T_1 in the form

$$T_1(\omega) = \gamma_0^2 u_0 \frac{\theta(0)^2\theta(\omega + d)\theta(\omega - d)}{\theta(d)^2\theta(\omega)^2}\left[\frac{\theta_1'(d)}{\theta_1(d)}\left(\frac{\theta'(\omega + d)}{\theta(\omega + d)} + \frac{\theta'(\omega - d)}{\theta(\omega - d)}\right)\right.$$
$$\left.-\frac{1}{2}\left(\frac{\theta''(\omega + d)}{\theta(\omega + d)} - \frac{\theta''(\omega - d)}{\theta(\omega - d)}\right)\right]. \tag{229}$$

We now show that $T_1(\omega)$ has the same behavior as $2\theta'(\omega)/\theta(\omega)$ under the shift $\omega \to \omega + \tau$, and therefore their difference is an elliptic function. We obtain using (A.5)

$$T_1(\omega + \tau) = T_1(\omega) + f(\omega),$$

where

$$f(\omega) = 2\pi i \gamma_0^2 u_0 \frac{\theta(0)^2\theta(\omega + d)\theta(\omega - d)}{\theta(d)^2\theta(\omega)^2}\left[\frac{\theta'(\omega + d)}{\theta(\omega + d)} - \frac{\theta'(\omega - d)}{\theta(\omega - d)} - 2\frac{\theta_1'(d)}{\theta_1(d)}\right]. \tag{230}$$

It is easily seen that $f(\omega) = f(\omega + \tau) = f(\omega + 1)$, so that f is elliptic. Furthermore, at the zero $(1 + \tau)/2$ of $\theta(\omega)$, by (A.6),

$$\frac{\theta'(\omega + d)}{\theta(\omega + d)} - \frac{\theta'(\omega - d)}{\theta(\omega - d)} = \frac{\theta_1'(d + \nu)}{\theta_1(d + \nu)} + \frac{\theta_1'(d - \nu)}{\theta_1(d - \nu)} = 2\frac{\theta_1'(d)}{\theta_1(d)} + \mathcal{O}(\nu), \qquad \nu = \omega - \frac{1 + \tau}{2},$$

and thus the expression in the square brackets in (230) vanishes as $\omega \to (1+\tau)/2$. So the pole of f at $(1+\tau)/2$ cannot have the order larger then 1. Thus f is an elliptic function with at most single first-order pole modulo the lattice, which means f is a constant. At $\omega = 0$,

$$f(0) = 4\pi i\gamma_0^2 u_0 \left(\frac{\theta'(d)}{\theta(d)} - \frac{\theta_1'(d)}{\theta_1(d)} \right) = -4\pi i$$

by (96) of Lemma 16. Thus

$$f(\omega) \equiv -4\pi i.$$

This immediately implies that the function

$$T_1(\omega) - 2\frac{\theta'(\omega)}{\theta(\omega)}$$

is elliptic. To analyze its behavior at the pole, it is convenient to write T_1 in terms of θ_1 by (A.6), (A.7) with $\nu = \omega - \frac{1+\tau}{2}$:

$$\begin{aligned} T_1(\omega) = &-\gamma_0^2 u_0 \frac{\theta(0)^2\theta_1(d+\nu)\theta_1(d-\nu)}{\theta(d)^2\theta_1(\nu)^2} \left[\frac{\theta_1'(d)}{\theta_1(d)} \left(\frac{\theta_1'(d+\nu)}{\theta_1(d+\nu)} - \frac{\theta_1'(d-\nu)}{\theta_1(d-\nu)} \right) \right. \\ &\left. -\frac{1}{2}\left(\frac{\theta_1''(d+\nu)}{\theta_1(d+\nu)} - \frac{\theta_1''(d-\nu)}{\theta_1(d-\nu)} \right) - 2\pi i \frac{\theta_1'(d)}{\theta_1(d)} + \pi i \left(\frac{\theta_1'(d+\nu)}{\theta_1(d+\nu)} + \frac{\theta_1'(d-\nu)}{\theta_1(d-\nu)} \right) \right]. \end{aligned} \tag{231}$$

It is obvious from this representation that the expression in the square brackets vanishes at $\nu = 0$, and therefore the order of the pole of T_1 at $\nu = 0$ is no larger than 1. Since the same is true for $\theta'(\omega)/\theta(\omega) = \theta_1'(\nu)/\theta_1(\nu) - i\pi$,

$$T_1(\omega) - 2\frac{\theta'(\omega)}{\theta(\omega)} \equiv \text{const}.$$

The value of this constant is easy to obtain by setting $\omega = 0$: since both $T_1(0) = 0$ (see (229)) and $\theta'(0) = 0$, this value is 0, which proves Part (c). ☐

9 | THE CONSTANT: PROOF OF LEMMA 22

Recalling (167), we write the term with W in (171)

$$\frac{i\zeta_0\gamma_0^2}{4}\int_0^1 W(\omega)d\omega = \frac{i\zeta_0\gamma_0^2}{4}\int_0^1 (T_2(\omega) + T_3(\omega))d\omega, \tag{232}$$

where

$$\begin{aligned} T_2 &= \begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix} \sum_{p\in\{-1,v_1,1\}} \int_{\partial U^{(p)}} \frac{s\Delta_1(z)dz}{2\pi i(z-v_2)^2} \begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix}, \\ T_3 &= \begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix} \int_{\partial U^{(v_2)}} \frac{s\Delta_1(z)dz}{2\pi i(z-v_2)^2} \begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix}, \end{aligned} \tag{233}$$

with the integrals traversed *clockwise*.

In this section we show (in subsection 9.1) that

$$\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_2(\omega)d\omega = \frac{1}{8}\sum_{y\in\{-1,v_1,1\}}\frac{\partial}{\partial v_2}\left[-\log|q(y)| + \frac{1}{2}\log|(y-v_2)|\right], \tag{234}$$

and (in subsection 9.2) that

$$\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_3(\omega)d\omega - \frac{\partial\tau}{\partial v_2}\int_0^1 \frac{\partial}{\partial\tau}\log\theta(\omega;\tau)d\omega = -\left(\frac{1}{16}\frac{\partial}{\partial v_2}\log\left[(1-v_2^2)(v_2-v_1)\right]\right.$$
$$\left. + \frac{1}{2}\frac{\partial}{\partial v_2}\log I_0 + \frac{1}{8}\frac{\partial}{\partial v_2}\log|q(v_2)|\right). \tag{235}$$

Substituting the last two equations into (232), we prove Lemma 22.

9.1 | Evaluation of T_2

Our goal in this section is to obtain (234). We first compute $T_2(\omega)$. By the definition of $\mathcal{N}$ in (91) and by (87) and (A.3), with $\omega = \pi\Omega$,

$$\mathcal{N}(z)e^{-i\pi\omega\sigma_3} = \frac{\gamma(z)\theta_3}{2\theta_3(\omega)}\begin{pmatrix}\frac{\theta_1(\omega+d)}{\theta_1(d)} & i\frac{\theta_1(\omega+d)}{\theta_1(d)}\\ -i\frac{\theta_1(d-\omega)}{\theta_1(d)} & \frac{\theta_1(d-\omega)}{\theta_1(d)}\end{pmatrix} + o(1), \qquad z\to -1$$

$$\mathcal{N}(z)e^{-i\pi\omega\sigma_3} = \frac{\gamma(z)^{-1}\theta_3}{2\theta_3(\omega)}\begin{pmatrix}\frac{\theta_2(\omega+d)}{\theta_2(d)} & -i\frac{\theta_2(\omega+d)}{\theta_2(d)}\\ i\frac{\theta_2(d-\omega)}{\theta_2(d)} & \frac{\theta_2(d-\omega)}{\theta_2(d)}\end{pmatrix} + o(1), \qquad z\to v_1 \tag{236}$$

$$\mathcal{N}(z) = \frac{\gamma(z)^{-1}\theta_3}{2\theta_3(\omega)}\begin{pmatrix}\frac{\theta_4(\omega+d)}{\theta_4(d)} & -i\frac{\theta_4(\omega+d)}{\theta_4(d)}\\ i\frac{\theta_4(d-\omega)}{\theta_4(d)} & \frac{\theta_4(d-\omega)}{\theta_4(d)}\end{pmatrix} + o(1), \qquad z\to 1$$

(Note that $\theta_j(d)\neq 0$, $j=1,2,3,4$, by the argument following (89). Moreover, $\theta_3(\omega)\neq 0$ for $\omega\in\mathbb{R}$.) Thus, by (122), (126), (128), and the definition of $m_{jj,0}$ in (159), a straightforward calculation yields

$$\begin{pmatrix} im_{22,0} & m_{11,0}\end{pmatrix} s\Delta_1(z)\begin{pmatrix} m_{11,0}\\ -im_{22,0}\end{pmatrix} = \begin{cases}\pm\frac{is\gamma(z)^2}{16\sqrt{\zeta(z)}}F_1(\omega)+o(1), & \text{as } z\to -1,\\ \mp\frac{is\gamma(z)^{-2}}{16\sqrt{\zeta(z)}}F_2(\omega)+o(1), & \text{as } z\to v_1,\\ \mp\frac{is\gamma(z)^{-2}}{16\sqrt{\zeta(z)}}F_4(\omega)+o(1), & \text{as } z\to 1,\end{cases} \tag{237}$$

where the upper sign is taken if $\operatorname{Im} z > 0$, the lower if $\operatorname{Im} z < 0$, and F_j is given by

$$F_j(\omega) = \frac{\theta_3^4\left[\theta_j(\omega+d)\theta_3(\omega-d)+\theta_j(\omega-d)\theta_3(\omega+d)\right]^2}{\theta_3(\omega)^4\theta_3(d)^2\theta_j(d)^2}, \qquad j=1,2,4. \tag{238}$$

To compute the residue of (237) at -1, we need to analyze $\pm\frac{\gamma(z)^2}{\sqrt{\zeta(z)}}$ at -1. It is meromorphic, and we need to determine the sign of its residue (the absolute value follows straightforwardly from the expansions of $\gamma(z)$ and $\zeta(z)$). Let $x \in U^{(-1)}$, with $x=-1+\epsilon$, $\epsilon>0$, and x lying on the positive side of the cut. For such x, $\gamma(x)^2 = i|\gamma(x)^2|$ by (85), and by the expansion (120), and in particular the fact that $\widetilde{\zeta}_0$ is positive, we have that $\sqrt{\zeta(x)}$ is positive. Thus $\frac{\gamma(x)^2}{\sqrt{\zeta(x)}} = i|\frac{\gamma(x)^2}{\sqrt{\zeta(x)}}|$, and by (85) and (120),

$$\pm\frac{is\gamma(z)^2}{\sqrt{\zeta(z)}} = -\frac{1}{z+1}\frac{1+v_1}{|q(-1)|}(1+\varepsilon_1(z)) \tag{239}$$

in a neighborhood of -1, where $\varepsilon_1(z)$ is an analytic function uniformly $o(1)$ as $z\to-1$.

Similar analysis in the neighborhoods $U^{(v_1)}$, $U^{(1)}$ yields

$$\mp\frac{is\gamma(z)^{-2}}{\sqrt{\zeta(z)}} = \begin{cases} \frac{1}{z-v_1}\frac{(v_2-v_1)(1+v_1)}{2|q(v_1)|}(1+\varepsilon_2(z)) & \text{for } z\in U^{(v_1)},\\ \frac{1}{z-1}\frac{1-v_2}{|q(1)|}(1+\varepsilon_4(z)) & \text{for } z\in U^{(1)}, \end{cases} \tag{240}$$

where $\varepsilon_2(z)$, $\varepsilon_4(z)$ are analytic function uniformly $o(1)$ as $z\to v_1, 1$, respectively.

Thus by the definition of T_2 in (233), computing the residue by (237) (note the negative orientation of the contours), we obtain

$$T_2 = \frac{1+v_1}{16(1+v_2)^2|q(-1)|}F_1(\omega) - \frac{1+v_1}{32(v_2-v_1)|q(v_1)|}F_2(\omega) - \frac{1}{16(1-v_2)|q(1)|}F_4(\omega). \tag{241}$$

We now evaluate $\int_0^1 F_j(\omega)d\omega$. It is easily seen that $F_j(\omega)$, $j=1,2,4$, are elliptic functions. We start with F_1. Note first that since θ_3 is even and θ_1 is odd, we have $F_1(0)=0$. By the definition of θ_1 and θ_3, we have

$$\frac{\left(\theta_3(\omega-d)\theta_1(\omega+d)+\theta_3(\omega+d)\theta_1(\omega-d)\right)^2}{\theta_3(\omega)^4} = -\frac{\left(\theta_1(\nu-d)\theta_3(\nu+d)+\theta_1(\nu+d)\theta_3(\nu-d)\right)^2}{\theta_1(\nu)^4}, \tag{242}$$

where $\nu = \omega - \frac{1+\tau}{2}$. Thus, as $\nu\to0$, the r.h.s. of this equation becomes

$$-4\frac{\left(\theta_1'(d)\theta_3(d)-\theta_1(d)\theta_3'(d)\right)^2}{\left(\theta_1'\right)^4\nu^2} + \mathcal{O}\left(\nu^{-1}\right). \tag{243}$$

Thus we can apply Lemma A.1 in Appendix A to F_1, which gives

$$\int_0^1 F_1(\omega)d\omega = -4\left(\frac{\theta_3}{\theta_1'}\right)^4\frac{\theta_3''}{\theta_3}\left(\frac{\theta_1'(d)}{\theta_1(d)} - \frac{\theta_3'(d)}{\theta_3(d)}\right)^2. \tag{244}$$

Using here the identity (96) of Lemma 16, and then the equation $\theta_1' = \pi\theta_2\theta_3\theta_4$, we finally obtain

$$\int_0^1 F_1(\omega)d\omega = 4\left(\frac{\theta_3}{\theta_1'}\right)^4 \frac{\theta_3''}{\theta_3} I_0^2(1+v_2)^2 = 4\left(\frac{1}{\pi\theta_2\theta_4}\right)^4 \frac{\theta_3''}{\theta_3} I_0^2(1+v_2)^2. \tag{245}$$

We now evaluate the integrals of F_2 and F_4. Applying the summation formulae (A.8) and (A.9) to the definition of F_2 and F_4, respectively, in (238), we obtain

$$F_2(\omega) = \frac{4\theta_3^2}{\theta_2^2}\frac{\theta_2(\omega)^2}{\theta_3(\omega)^2}, \qquad F_4(\omega) = \frac{4\theta_3^2}{\theta_4^2}\frac{\theta_4(\omega)^2}{\theta_3(\omega)^2}. \tag{246}$$

By the definitions of θ_j for $j = 1,2,3,4$, we have with $\nu = \omega - \frac{1+\tau}{2}$

$$\begin{aligned} \frac{\theta_2(\omega)^2}{\theta_3(\omega)^2} &= \frac{\theta_4(\nu)^2}{\theta_1(\nu)^2} = \frac{\theta_4^2}{\left(\theta_1'\right)^2}\nu^{-2} + \mathcal{O}\left(\nu^{-1}\right), \qquad \nu \to 0, \\ \frac{\theta_4(\omega)^2}{\theta_3(\omega)^2} &= -\frac{\theta_2(\nu)^2}{\theta_1(\nu)^2} = -\frac{\theta_2^2}{\left(\theta_1'\right)^2}\nu^{-2} + \mathcal{O}\left(\nu^{-1}\right), \qquad \nu \to 0; \end{aligned} \tag{247}$$

and applying Lemma A.1, we obtain

$$\int_0^1 \frac{\theta_2(\omega)^2}{\theta_3(\omega)^2}d\omega = \frac{\theta_4^2}{\left(\theta_1'\right)^2}\frac{\theta_3''}{\theta_3} + \frac{\theta_2^2}{\theta_3^2}, \qquad \int_0^1 \frac{\theta_4(\omega)^2}{\theta_3(\omega)^2}d\omega = -\frac{\theta_2^2}{\left(\theta_1'\right)^2}\frac{\theta_3''}{\theta_3} + \frac{\theta_4^2}{\theta_3^2}. \tag{248}$$

From here, by (246) and the equation $\theta_1' = \pi\theta_2\theta_3\theta_4$,

$$\int_0^1 F_2(\omega)d\omega = 4\left(\frac{1}{\pi^2\theta_2^4}\frac{\theta_3''}{\theta_3} + 1\right), \qquad \int_0^1 F_4(\omega)d\omega = 4\left(-\frac{1}{\pi^2\theta_4^4}\frac{\theta_3''}{\theta_3} + 1\right). \tag{249}$$

Integrating (241) by (245), (249), we obtain

$$\begin{aligned} \int_0^1 T_2(\omega)d\omega = {} & \frac{(1+v_1)I_0^2}{4|q(-1)|}\left(\frac{1}{\pi\theta_2\theta_4}\right)^4\frac{\theta_3''}{\theta_3} - \frac{1+v_1}{8(v_2-v_1)|q(v_1)|}\left(\frac{1}{\pi^2\theta_2^4}\frac{\theta_3''}{\theta_3} + 1\right) \\ & - \frac{1}{4(1-v_2)|q(1)|}\left(-\frac{1}{\pi^2\theta_4^4}\frac{\theta_3''}{\theta_3} + 1\right). \end{aligned} \tag{250}$$

We now express all the θ-constants here in terms of elliptic integrals. For θ_2^4, θ_2^4, this was already done in (100), (99) of Lemma 16. To obtain an expression for $\frac{\theta_3''}{\theta_3}$, we first note that by the differential equation (A.10) satisfied by θ-functions, and then by (224) and (101),

$$\frac{\theta_3''}{\theta_3} = 4\pi i\frac{\partial}{\partial\tau}\log\theta_3 = \pi i\frac{1}{\partial\tau/\partial v_2}\frac{\partial}{\partial v_2}\log\theta_3^4 = I_0^2(1-v_2)^2(v_2-v_1)\frac{\partial}{\partial v_2}\log(I_0^2(1+v_2)). \tag{251}$$

We now use (219), (9), and then the expression $|q(v_2)| = (v_2 - x_1)(x_2 - v_2) = -x_1x_2 - v_2(v_2 - v_1)/2$, to obtain from here

$$\frac{\theta_3''}{\theta_3} = 2I_0^2\left(x_1x_2 + \frac{v_2 - v_1}{2}\right) = 2I_0^2\left(-|q(v_2)| + \frac{(1-v_2)(v_2-v_1)}{2}\right). \tag{252}$$

Substituting this expression as well as (100), (99) into (250), and using the fact that by (157), (156),

$$\frac{i\gamma_0^2\zeta_0}{4} = -\frac{|q(v_2)|}{2(1+v_2)},$$

we obtain

$$\begin{aligned}\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_2(\omega)d\omega &= \frac{1}{8}\frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)}\left(\frac{1}{|q(-1)|} - \frac{1}{|q(v_1)|} + \frac{1}{|q(1)|}\right)\\ &+ \frac{1}{16}\frac{|q(v_2)|}{(1-v_2^2)(v_2-v_1)}\left(-\frac{(1-v_2)(v_2-v_1)}{|q(-1)|} + \frac{1-v_2^2}{|q(v_1)|} + \frac{(1+v_2)(v_2-v_1)}{|q(1)|}\right).\end{aligned} \tag{253}$$

In the last three terms here, we express $|q(v_2)|$ by $|q(-1)|$, $|q(v_1)|$, $|q(1)|$, respectively, for example, for the last term we write (recall (8))

$$|q(v_2)| = -x_1x_2 - v_2(v_2-v_1)/2 = -|q(1)| + 1 - (v_1+v_2)/2 - v_2(v_2-v_1)/2. \tag{254}$$

This allows us to write (253) in the form

$$\begin{aligned}\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_2(\omega)d\omega &= \frac{1}{8}\frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)}\left(\frac{1}{|q(-1)|} - \frac{1}{|q(v_1)|} + \frac{1}{|q(1)|}\right)\\ &+ \frac{1}{16}\left(\frac{1}{1+v_2} + \frac{1}{v_2-v_1} - \frac{1}{1-v_2} - \frac{2-(v_2-v_1)}{2|q(-1)|} - \frac{v_2+v_1}{2|q(v_1)|} + \frac{2+v_2-v_1}{2|q(1)|}\right).\end{aligned} \tag{255}$$

On the other hand, by (9) and (220),

$$\frac{\partial}{\partial v_2}|q(-1)| = \frac{\partial}{\partial v_2}\left(1 + x_1x_2 + \frac{v_1+v_2}{2}\right) = -\frac{v_2-v_1}{4} - \frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)} + \frac{1}{2}, \tag{256}$$

and

$$\begin{aligned}\frac{\partial}{\partial v_2}|q(v_1)| &= \frac{v_2-v_1}{4} + \frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)} + \frac{v_1}{2},\\ \frac{\partial}{\partial v_2}|q(1)| &= -\frac{v_2-v_1}{4} - \frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)} - \frac{1}{2}.\end{aligned} \tag{257}$$

We therefore easily obtain the expression for $\frac{\partial}{\partial v_2}\log|q(-1)q(v_1)q(1)|$. Comparing it with (255) shows (234).

9.2 | Evaluation of T_3

Now consider T_3. Our aim in this section is to prove (235). We write $\mathcal{N}$ in (91) in the form

$$\mathcal{N}(z) = A(z;s\Omega) + B(z;s\Omega), \qquad A = \frac{1}{2}\begin{pmatrix} A_1 & iA_1 \\ -iA_2 & A_2 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} B_1 & -iB_1 \\ iB_2 & B_2 \end{pmatrix},$$

$$A_j(z;\omega) = \frac{\theta_3}{2\theta_3(\omega)}\left[\left(\gamma(z)+\gamma(z)^{-1}\right)\frac{\theta_3(u(z)\pm\omega+d)}{\theta_3(u(z)+d)} + \left(\gamma(z)-\gamma(z)^{-1}\right)\frac{\theta_3(-u(z)\pm\omega+d)}{\theta_3(-u(z)+d)}\right],$$

$$B_j(z;\omega) = \frac{\theta_3}{2\theta_3(\omega)}\left[\left(\gamma(z)+\gamma(z)^{-1}\right)\frac{\theta_3(u(z)\pm\omega+d)}{\theta_3(u(z)+d)} - \left(\gamma(z)-\gamma(z)^{-1}\right)\frac{\theta_3(-u(z)\pm\omega+d)}{\theta_3(-u(z)+d)}\right], \tag{258}$$

where $\pm$ means + for $j = 1$ and − for $j = 2$. Using the jump conditions (92), (93), we observe that $(z-v_2)^{1/4}A(z)$ and $(z-v_2)^{-1/4}B(z)$ are analytic on $U^{(v_2)}$.

Since $\Delta_1(z)$ in (126) for $p = v_2$ is meromorphic on $U^{(v_2)}$, all odd powers of roots $(z-v_2)^{1/2}$ in the expansion of (126) cancel, and it follows that for $z \in U^{(v_2)}$ and $\operatorname{Im} z > 0$,

$$\Delta_1(z) = -\frac{1}{32\sqrt{\zeta(z)}}\left[\begin{pmatrix} A_1 & iA_1 \\ -iA_2 & A_2 \end{pmatrix}\begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}\begin{pmatrix} A_2 & -iA_1 \\ iA_2 & A_1 \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} B_1 & -iB_1 \\ iB_2 & B_2 \end{pmatrix}\begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}\begin{pmatrix} B_2 & iB_1 \\ -iB_2 & B_1 \end{pmatrix}\right]. \tag{259}$$

Therefore

$$\begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix}\Delta_1(z)\begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix} = \frac{i}{16\sqrt{\zeta(z)}}\left[(m_{22,0}A_1 - m_{11,0}A_2)^2 + 3(m_{22,0}B_1 + m_{11,0}B_2)^2\right].$$

Expanding $A_1(z)$ and $A_2(z)$ as $z \to v_2$, we obtain using (156), (157), and (158),

$$m_{22,0}A_1(z) - m_{11,0}A_2(z) = -\gamma_0^{-1}u_0T_1(\omega)(z-v_2)^{3/4} + \mathcal{O}\left((z-v_2)^{5/4}\right) \tag{260}$$

with $T_1(\omega)$ as defined in (223). By (226) in Proposition 24, this equals

$$-\frac{2u_0}{\gamma_0}\frac{\theta_3'(\omega)}{\theta_3(\omega)}(z-v_2)^{3/4} + \mathcal{O}\left((z-v_2)^{5/4}\right). \tag{261}$$

So that by the definition of T_3 in (233), we obtain computing the residue for the first term,

$$T_3(\omega) = -\frac{iu_0^2}{4\gamma_0^2\zeta_0}\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2 + \int_{\partial U^{(v_2)}}\frac{3i[m_{22,0}B_1(z) + m_{11,0}B_2(z)]^2}{16(z-v_2)^2\sqrt{\zeta(z)}}\frac{dz}{2\pi i}, \tag{262}$$

where the integration is in the negative direction around v_2, and where $\sqrt{\zeta}$ and B_1, B_2 are understood to be the analytic continuation from $\operatorname{Im} z > 0$.

We now write the average

$$\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_3(\omega)d\omega = \frac{u_0^2}{16}\int_0^1\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2 d\omega + \frac{i\gamma_0^2\zeta_0 Q}{4}, \tag{263}$$

where

$$Q = \int_0^1 d\omega \int_{\partial U^{(v_2)}} \frac{3is[m_{22,0}B_1(z;\omega) + m_{11,0}B_2(z;\omega)]^2}{16(z-v_2)^2\sqrt{\zeta(z)}}\frac{dz}{2\pi i}. \tag{264}$$

To compare with Lemma 21, we will now single out a contribution from

$$\delta = \frac{\partial\tau}{\partial v_2}\int_0^1 \frac{\partial}{\partial\tau}\log\theta_3(\omega)d\omega. \tag{265}$$

Using the differential equation (A.10) and the fact that

$$0 = \int_0^1\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)' d\omega = \int_0^1\left[\frac{\theta_3''(\omega)}{\theta_3(\omega)} - \left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2\right]d\omega,$$

we can write

$$\delta = i\frac{\partial|\tau|}{\partial v_2}\int_0^1 \frac{\theta_3''(\omega)}{\theta_3(\omega)}\frac{d\omega}{4\pi i} = \frac{\partial|\tau|}{\partial v_2}\int_0^1\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2\frac{d\omega}{4\pi}. \tag{266}$$

Since, by (224), $\pi u_0^2 = \frac{\partial|\tau|}{\partial v_2}$, we can rewrite (263) in the form

$$\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_3(\omega)d\omega = -\frac{3}{16\pi}\frac{\partial|\tau|}{\partial v_2}\int_0^1\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2 d\omega + \frac{i\gamma_0^2\zeta_0 Q}{4} + \delta. \tag{267}$$

Now by (A.19),

$$\int_0^1\left(\frac{\theta_3'(\omega)}{\theta_3(\omega)}\right)^2 d\omega = \frac{\pi^2}{3} + \frac{\theta_1'''}{3\theta_1'}.$$

Using the identity $\theta_1' = \pi\theta_2\theta_3\theta_4$, and the identities (99)–(101) of Lemma 16, we write here

$$\begin{aligned}\frac{\theta_1'''}{\theta_1'} &= 4\pi i\frac{\partial}{\partial\tau}\log\left(\theta_1'\right) = \frac{\pi i}{\frac{\partial\tau}{\partial v_2}}\frac{\partial}{\partial v_2}\log\left(\theta_1'\right)^4 = \frac{\pi}{\frac{\partial|\tau|}{\partial v_2}}\frac{\partial}{\partial v_2}\log\left(\theta_2\theta_3\theta_4\right)^4\\ &= \frac{\pi}{\frac{\partial|\tau|}{\partial v_2}}\frac{\partial}{\partial v_2}\log\left[I_0^6(1-v_2^2)(v_2-v_1)\right],\end{aligned} \tag{268}$$

so that we can rewrite (267) in the form

$$\frac{i\gamma_0^2\zeta_0}{4}\int_0^1 T_3(\omega)d\omega = -\frac{1}{16}\left(\pi\frac{\partial|\tau|}{\partial v_2} + \frac{\partial}{\partial v_2}\log\left[I_0^6(1-v_2^2)(v_2-v_1)\right]\right) + \frac{i\gamma_0^2\zeta_0 Q}{4} + \delta. \tag{269}$$

It remains to evaluate Q defined in (264). To simplify the computations, we first do the averaging over ω and only then compute the residue in this case.

As above for $A(z)$, we expand $B(z)$ to obtain

$$\gamma(z)[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)]-2=\mathcal{O}(z-v_2),\qquad z\to v_2, \tag{270}$$

and therefore

$$\gamma(z)^2[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)]^2=-4+4\gamma(z)[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)]+\mathcal{O}\big((z-v_2)^2\big), \tag{271}$$

as $z\to v_2$. Thus, upon changing the order of integration,

$$Q=\int_{\partial U^{(v_2)}}\frac{dz}{2\pi i}\frac{3is}{4\gamma^2(z)(z-v_2)^2\sqrt{\zeta(z)}}\left[-1+\gamma(z)\int_0^1 d\omega\big[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)\big]\right]. \tag{272}$$

By the definition of B_1 and B_2 in (258) and the formula for $m_{11,0}$ and $m_{22,0}$ in (159), we have

$$\int_0^1 d\omega\big[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)\big]=\int_0^1(\widetilde{q}(\omega)+\widetilde{q}(-\omega))d\omega, \tag{273}$$

where

$$\widetilde{q}(\omega)=\frac{\theta_3^2\theta(-\omega+d)}{2\theta(d)\theta(\omega)^2}\left(\big(\gamma(z)+\gamma(z)^{-1}\big)\frac{\theta(u(z)+\omega+d)}{\theta(u(z)+d)}-\big(\gamma(z)-\gamma(z)^{-1}\big)\frac{\theta(-u(z)+\omega+d)}{\theta(-u(z)+d)}\right). \tag{274}$$

Since $\widetilde{q}(-\omega)=\widetilde{q}(1-\omega)$, we have that

$$\int_0^1\widetilde{q}(-\omega)d\omega=\int_0^1\widetilde{q}(\omega)d\omega, \tag{275}$$

Applying (A.20) to evaluate $\int_0^1\widetilde{q}(\omega)d\omega$, we obtain:

$$\gamma(z)\int_0^1 d\omega\big[m_{22,0}B_1(z;\omega)+m_{11,0}B_2(z;\omega)\big]=\frac{\pi\theta_3^2 g(d)}{\left(\theta_1'\right)^2\sin(\pi u)}$$
$$\times\left\{\big(\gamma(z)^2+1\big)g(d+u)[f(d)-f(d+u)]+\big(\gamma(z)^2-1\big)g(d-u)[f(d)-f(d-u)]\right\}, \tag{276}$$

where

$$g(x)=\frac{\theta_1(x)}{\theta_3(x)},\qquad f(x)=\frac{\theta_1'(x)}{\theta_1(x)}. \tag{277}$$

Note that (A.13) gives for the derivative of $f(z)$

$$f'(x)=-\left(\frac{\theta_1'}{\theta_3}\right)^2\frac{1}{g(x)^2}+\frac{\theta_3''}{\theta_3}. \tag{278}$$

Using this, we have, in particular, as $z \to v_2$, that is, $u \to 0$,

$$f(d) - f(d \pm u) = \left(\frac{\theta_1'}{\theta_3}\right)^2 \left[\pm u \left(\frac{1}{g(d)^2} - \frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2 \right) - u^2 \frac{g'(d)}{g(d)^3} \right.$$
$$\left. \pm \frac{u^3}{3}\left(-\frac{g''(d)}{g(d)^3} + 3\frac{g'(d)^2}{g(d)^4}\right) + \frac{u^4}{12}\left(-\frac{g'''(d)}{g(d)^3} + \frac{9g''(d)g'(d)}{g(d)^4} - \frac{12g'(d)^3}{g(d)^5}\right)\right] + \mathcal{O}(u^5). \quad (279)$$

Expanding also the other terms in (276), and also expanding u by (157), we obtain that, as $z \to v_2$,

$$\gamma(z)\int_0^1 d\omega\left[m_{22,0}B_1(z;\omega) + m_{11,0}B_2(z;\omega)\right] = g(d)\left(1 + \frac{\pi^2}{6}u_0^2(z - v_2) + \mathcal{O}((z - v_2)^2)\right)$$
$$\times \left[H_0 + u_0\gamma_0^2(1 + (z - v_2)(u_1 + 2\gamma_1))H_1 + (z - v_2)(u_0^2 H_2 - u_0^3\gamma_0^2 H_3) + \mathcal{O}((z - v_2)^{3/2})\right], \quad (280)$$

where

$$H_0 = 2g(d)\left(\frac{1}{g(d)^2} - \frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2\right), \qquad H_1 = 2g'(d)\frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2,$$
$$H_2 = g''(d)\left(\frac{1}{3g(d)^2} - \frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2\right), \quad H_3 = \frac{g'''(d)}{6}\left(\frac{1}{g(d)^2} - 2\frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2\right) - \frac{1}{6}\frac{g''(d)g'(d)}{g(d)^3}. \quad (281)$$

By applying (106) and (97) of Proposition 16, we simplify the combinations of the H_j as follows:

$$H_0 + u_0\gamma_0^2 H_1 = \frac{2}{g(d)}, \qquad u_0^2 H_2 - u_0^3\gamma_0^2 H_3 = \frac{2\gamma_1 + u_1}{g(d)}\left(1 - 2g(d)^2\frac{\theta_3''}{\theta_3}\left(\frac{\theta_3}{\theta_1'}\right)^2\right), \quad (282)$$

which allows us to write (280) in the form

$$\gamma(z)\int_0^1 d\omega\left[m_{22,0}B_1(z;\omega) + m_{11,0}B_2(z;\omega)\right] = 2 + \left(\frac{\pi^2}{3}u_0^2 + (2\gamma_1 + u_1)\right)(z - v_2) + \mathcal{O}((z - v_2)^{3/2}). \quad (283)$$

Substituting this into (272) and calculating the residue, we obtain

$$Q = \frac{3i}{4\gamma_0^2\zeta_0}\left(\zeta_1 - u_1 - \frac{\pi^2}{3}u_0^2\right) = \frac{3i}{4\gamma_0^2\zeta_0}\left(\zeta_1 - u_1 - \frac{\pi}{3}\frac{\partial|\tau|}{\partial v_2}\right). \quad (284)$$

For the coefficients ζ_1, u_1 in expansions (156) and (157) we easily obtain:

$$\zeta_1 = \frac{1}{3}\frac{d}{dx}\log q(x)|_{x=v_2} - \frac{1}{6}\frac{\partial}{\partial v_2}\log(v_2^2 - 1)(v_2 - v_1),$$
$$u_1 = -\frac{1}{6}\frac{\partial}{\partial v_2}\log(v_2^2 - 1)(v_2 - v_1), \quad (285)$$

so that

$$\zeta_1 - u_1 = \frac{1}{3}\frac{d}{dx}\log q(x)|_{x=v_2} = \frac{2v_2 - (v_1+v_2)/2}{-3|q(v_2)|}.$$

On the other hand, by (254) and (257),

$$\frac{\partial}{\partial v_2}|q(v_2)| = -\frac{3}{4}v_2 + \frac{v_1}{4} + \frac{q(v_2)^2}{(1-v_2^2)(v_2-v_1)}, \tag{286}$$

and by (219), (9),

$$\frac{\partial}{\partial v_2}\log I_0 = -\frac{|q(v_2)|}{(1-v_2^2)(v_2-v_1)}. \tag{287}$$

These equations imply

$$\zeta_1 - u_1 = \frac{2}{3}\frac{\partial}{\partial v_2}\log(|q(v_2)|I_0). \tag{288}$$

Substituting this into (284) for Q, and that, in turn, into (269), we obtain (235).

10 | SLOW MERGING OF GAPS: PROOF OF THEOREM 4

10.1 | Solution of the Φ-RH problem as $v_2 - v_1 \to 0$.

We consider the asymptotics of the Φ-RH problem in the double-scaling regime where $\nu = \frac{v_2-v_1}{2}$ can approach zero with $s \to \infty$ at a rate such that $2\nu > \frac{1}{s^{2-\varepsilon}}$, for any fixed $\varepsilon > 0$.

Let

$$-\alpha = 1 + \frac{v_2+v_1}{2} > 0, \qquad \beta = 1 - \frac{v_2+v_1}{2} > 0, \qquad \gamma = \frac{\beta^{-1}+|\alpha|^{-1}}{8}. \tag{289}$$

We need to evaluate the integrals I_j in the limit $\nu \to 0$. To do this (and to make a comparison with [24] easier), we first change integration variable $x = t + \frac{v_1+v_2}{2}$, which maps $(v_2, 1)$ to (ν, β); we then split this interval into $(\nu, \sqrt{\nu}) \cup [\sqrt{\nu}, \beta)$ and use a change of variable $y = t/\sqrt{\nu}$ for integration over the first one. We then obtain:[5]

$$I_2 - \frac{v_2+v_1}{2}I_1 = \sqrt{|\alpha\beta|} + \mathcal{O}\left(\nu^2\log\nu^{-1}\right), \tag{290}$$

$$I_0 = \frac{\log(\gamma\nu)^{-1}}{\sqrt{|\alpha\beta|}} + \mathcal{O}\left(\nu^2\log\nu^{-1}\right). \tag{291}$$

Hence, by (9),

$$x_1x_2 = \left(-I_2 + \frac{v_1+v_2}{2}I_1\right)\frac{1}{I_0} = -\frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}} + \mathcal{O}\left(\nu^2\right). \tag{292}$$

[5] Cf. equations (278)– (280) in [24].

Let the neighborhoods $U^{(v_1)}, U^{(v_2)}$ have radius $\nu/3$; they will be therefore contracting as $\nu \to 0$. We now evaluate the jumps $J_S(z)$ of S on the edges of the lenses $\Gamma_{\Phi,L} \cup \Gamma_{\Phi,U}$. Recall from (83) that these jumps were exponentially close to the identity, in the case where v_1 and v_2 were fixed. For $z \in \Gamma_{\Phi,L} \cup \Gamma_{\Phi,U}$ and z bounded away from the points v_1, v_2, it is clear that the jumps are still exponentially close to the identity so that (83) holds, and we consider the case where $z \to \frac{v_1+v_2}{2}$ along the edges of the lenses. We substitute (292) into the definition of ϕ in (77) and obtain (taking $u = \frac{z-v_2}{v_2-v_1}$)

$$\phi(z) = \frac{\pm i\sqrt{|\alpha\beta|}}{\log{(\gamma\nu)}^{-1}} \int_0^{\frac{z-v_2}{2\nu}} \frac{du}{\sqrt{u(u+1)}} \left(1 + \mathcal{O}\left(z - \frac{v_1+v_2}{2}\right)\right), \tag{293}$$

as $\nu \to 0$, and $z \to \frac{v_1+v_2}{2}$. Here $'+'$ sign is taken on $\Gamma_{\Phi,U}$, and $'-'$ sign is taken on $\Gamma_{\Phi,L}$, and thus $\operatorname{Im}\phi(z) < 0$, $\operatorname{Im}\phi(z) > 0$ on $\Gamma_{\Phi,L}$ and $\Gamma_{\Phi,U,}$ respectively. Worsening somewhat the error term, we have that

$$J_S(z) = I + \mathcal{O}\left(e^{-c\sqrt{s}(|z|+1)}\right), \qquad c > 0, \tag{294}$$

as $s \to \infty$, uniformly for $2\nu > s^{-2+\varepsilon}$ and $z \in \Gamma_{\Phi,L} \cup \Gamma_{\Phi,U}$.

Next we consider the jumps of R on the boundary $\partial U^{(p)}$ for $p \in \mathcal{T} = \{-1, v_1, v_2, 1\}$. Estimating $\phi(z)$ as above but now in the definition of ζ in (119), we obtain that as $s \to \infty$, uniformly for $2\nu > s^{-2+\varepsilon}$,

$$\frac{1}{\zeta(z)^{1/2}} = \begin{cases} \mathcal{O}\left(\frac{\log \nu^{-1}}{s}\right), & \text{uniformly on } \partial U^{(v_1)} \text{ and } \partial U^{(v_2)}, \\ \mathcal{O}\left(\frac{1}{s}\right), & \text{uniformly on } \partial U^{(1)} \text{ and } \partial U^{(-1)}. \end{cases} \tag{295}$$

To estimate $\Delta(z)$, we need to consider $\mathcal{N}$. We first observe that by the definition (85), $\gamma(z), \gamma(z)^{-1} = \mathcal{O}(1)$ uniformly on $\partial U^{(p)}$ for $p \in \mathcal{T}$ as $\nu \to 0$. Using (291) and a simpler expansion for J_0, we obtain

$$\tau = i\frac{J_0}{I_0} = \frac{\pi i}{\log{(\gamma\nu)}^{-1}}\left(1 + \mathcal{O}\left(\nu^2\right)\right), \qquad \nu \to 0, \tag{296}$$

and define

$$\kappa = e^{-\pi i/\tau} = [\gamma\nu]^{1+\mathcal{O}(\nu^2)}. \tag{297}$$

By the inversion formula (A.11) for θ-functions,

$$\theta(z) = \frac{1}{\sqrt{-i\tau}} \sum_k e^{-\frac{i\pi}{\tau}(k-z)^2} = \frac{\kappa^{\langle z\rangle^2}}{\sqrt{-i\tau}}\left(1 + \kappa^{1-2\langle z\rangle} + \kappa^{1+2\langle z\rangle}\right) + \mathcal{O}\left(\frac{\kappa^{9/4}}{\sqrt{|\tau|}}\right), \tag{298}$$

where

$$z = j + \langle z\rangle, \qquad -1/2 < \langle \operatorname{Re} z\rangle \le 1/2, \quad j \in \mathbb{Z}. \tag{299}$$

We now show, in (301) below, that $\Delta(z)$, which enters the jump matrix for R, may be too large for certain parameter sets, which makes it necessary to modify the solution of the RH problem. First, a simple analysis of (89) shows that $d \to -1/2$ as $\nu \to 0$. On the boundary of $U^{(v_1)}, U^{(v_2)}$, we have $|u(z)| \to 0$, uniformly in z. Therefore, using the boundedness of γ, γ^{-1} on $\partial U^{(p)}$ for $p \in \mathcal{T}$, and applying (298), we have for the 11 element of $\mathcal{N}$ on $\partial U^{(v_1)}$ if $\langle\omega\rangle > 0$ (and thus $u(z) + d + \langle\omega\rangle = \langle u(z) + d + \langle\omega\rangle\rangle$ for ν sufficiently small):

$$|\mathcal{N}_{11}| \le C\left|\frac{\theta(0)}{\theta(\omega)}\frac{\theta(u(z)+d+\omega)}{\theta(u(z)+d)}\right| \le C_1\frac{1}{\kappa^{\langle\omega\rangle^2}}\frac{\kappa^{\langle\omega\rangle^2-\langle\omega\rangle+1/4}}{\kappa^{1/4}} = C_1\kappa^{-\langle\omega\rangle} \le C_2\nu^{-\langle\omega\rangle}, \qquad \omega = s\Omega, \tag{300}$$

for some constants $C, C_1, C_2 > 0$. Similarly, we analyze the behavior of $\mathcal{N}_{11}$ for $\langle\omega\rangle < 0$, the behavior of other matrix elements of $\mathcal{N}$ on $\partial U^{(v_1)}$, as well as the behavior of $\mathcal{N}$ on $\partial U^{(v_2)}$ and $\partial U^{(\pm 1)}$. We find that the estimate (300) is the worst (note that, in fact, the estimates for $\mathcal{N}$ on $\partial U^{(\pm 1)}$ are much better), and thus recalling (295), we have

$$\begin{aligned}\Delta(z) &= \mathcal{N}(z)\mathcal{O}\left(\frac{\log\nu^{-1}}{s}\right)\mathcal{N}(z)^{-1}\\ &= \mathcal{O}\left(\frac{\log\nu^{-1}}{s}\nu^{-2|\langle s\Omega\rangle|}\right),\end{aligned} \tag{301}$$

as $s \to \infty$ and $\nu \to 0$, for $z \in \partial U^{(v_p)}$. Thus if, for example, $\nu = \frac{1}{s}$ and $|\langle s\Omega\rangle| = 1/2$ (which is a case we need to deal with since the splitting of the gap regime described in [24] breaks down in this limit), we cannot say that Δ is small, and so the corresponding jump of R is not guaranteed to be close to the identity, and so we cannot claim solvability of the R-RH problem. However, it was shown in [24] for the case of the RH problem of [18] that we can modify the solution to ensure solvability for the range $2\nu > s^{-5/4}$. We now provide more details of that construction in the present case, and apply it for all values of $\langle s\Omega\rangle$.

Let

$$t = \langle s\Omega\rangle + k/2, \tag{302}$$

where $k = \pm 1$ is chosen such that $-1/2 < t \le 1/2$. Consider the following function:

$$\begin{aligned}\widetilde{\mathcal{N}}(z) &= \begin{pmatrix} \frac{\delta+\delta^{-1}}{2}\widetilde{m}_{11} & \frac{\delta-\delta^{-1}}{2i}\widetilde{m}_{12}\\ -\frac{\delta-\delta^{-1}}{2i}\widetilde{m}_{21} & \frac{\delta+\delta^{-1}}{2}\widetilde{m}_{22}\end{pmatrix},\\ \widetilde{m}(z) &= \begin{pmatrix} \frac{\theta(u(z_-)+d')}{\theta(u(z_-)+t+d')} & 0\\ 0 & \frac{\theta(u(z_-)+d')}{\theta(u(z_-)-t+d')}\end{pmatrix}\begin{pmatrix} \frac{\theta(u(z)+t+d')}{\theta(u(z)+d')} & \frac{\theta(u(z)-t-d')}{\theta(u(z)-d')}\\ \frac{\theta(u(z)+t-d')}{\theta(u(z)-d')} & \frac{\theta(u(z)-t+d')}{\theta(u(z)+d')}\end{pmatrix},\end{aligned} \tag{303}$$

where the constant d' will be fixed later on, and we now take

$$\delta(z) = \nu^{-1/4}\left(\frac{(z-v_1)(z-v_2)}{z^2-1}\right)^{1/4}, \tag{304}$$

with branch cuts on $(-1, v_1) \cup (v_2, 1)$, and positive as $z \to \infty$ on the first sheet of the Riemann surface Σ. We have

$$\delta(z)_+ = \begin{cases} i\delta(z)_- & \text{on} \quad (-1, v_1) \\ -i\delta(z)_- & \text{on} \quad (v_2, 1) \end{cases}$$

It is easy to verify that $\widetilde{\mathcal{N}}(z)$ satisfies the same jump conditions as $\mathcal{N}$:

$$\begin{aligned} \widetilde{\mathcal{N}}_+(z) &= \widetilde{\mathcal{N}}_-(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} && \text{for } z \in (v_2, 1), \\ \widetilde{\mathcal{N}}_+(z) &= \widetilde{\mathcal{N}}_-(z) \begin{pmatrix} 0 & e^{-2\pi i(s\Omega + k/2)} \\ -e^{2\pi i(s\Omega+k/2)} & 0 \end{pmatrix} = \widetilde{\mathcal{N}}_-(z) \begin{pmatrix} 0 & -e^{-2\pi i s\Omega} \\ e^{2\pi i s\Omega} & 0 \end{pmatrix} && \text{for } z \in (-1, v_1). \end{aligned} \tag{305}$$

Furthermore, one verifies that $\delta(z) - \delta(z)^{-1}$ has two zeros at z_+, z_- located on the first sheet and such that $\delta(z_+) = \delta(z_-) = 1$ and

$$z_\pm = \frac{v_1 + v_2}{2} \pm i\sqrt{\nu|\alpha\beta|} + \mathcal{O}(\nu), \qquad \nu \to 0. \tag{306}$$

Set

$$d' = u(z_+) + 1/2 + \tau/2,$$

then it follows by the properties of the Abel map $u(z)$ (86) that $\theta(u(z) - d')$ has a single zero at z_+, and $\theta(u(z) + d')$ has no zeros on the first sheet $\mathbb{C} \setminus A$. Thus $\widetilde{\mathcal{N}}(z_-) = I$, and since $\det \widetilde{\mathcal{N}}$ extends to an entire function, $\det \widetilde{\mathcal{N}}(z) = 1$ for $z \in \mathbb{C}$. Considering the zeros and poles of the meromorphic function $\delta^{-2} - 1$ on Σ, and using the Abel theorem, we have

$$u(v_1) + u(v_2) - u(z_-) - u(z_+) \equiv 0, \tag{307}$$

modulo the lattice. Since $u(v_1) + u(v_2) \equiv u(v_1) \equiv \frac{\tau}{2}$,

$$u(z_+) + u(z_-) \equiv -\tau/2, \tag{308}$$

$$u(z_-) + d' \equiv 1/2. \tag{309}$$

Using the change of integration variable $x = t + \frac{v_1+v_2}{2}$ as above, we obtain (from now on always on the first sheet, so modulo $\mathbb{Z}$)

$$\begin{aligned} u(z_+) = -\frac{i}{2I_0}\int_{v_2}^{z_+} \frac{dx}{p(x)^{1/2}} &= -\frac{1}{2I_0\sqrt{|\alpha\beta|}}\int_{\nu}^{i\sqrt{|\alpha\beta|\nu}} \frac{dt}{(t^2-\nu^2)^{1/2}}\left(1 + \mathcal{O}(\sqrt{\nu})\right) \\ &= -\frac{1+\tau}{4} - \frac{\hat{\varepsilon}}{2} + \mathcal{O}(\sqrt{\nu}), \end{aligned} \tag{310}$$

as $\nu \to 0$, where $\hat{\varepsilon}$ is real, satisfying $\hat{\varepsilon} \to 0$ as $\nu \to 0$. Similarly,

$$u(z_-) = \frac{1-\tau}{4} + \frac{\hat{\varepsilon}}{2} + \mathcal{O}(\sqrt{\nu}). \tag{311}$$

Therefore by the definition of d',

$$d' = \frac{1+\tau}{4} - \frac{\hat{\epsilon}}{2} + \mathcal{O}(\sqrt{\nu}), \tag{312}$$

and

$$u(z_-) + d' = 1/2.$$

Thus, since θ is an even function,

$$\widetilde{m}(z) = \frac{\theta(1/2)}{\theta(t+1/2)} \begin{pmatrix} \dfrac{\theta(u(z)+t+d')}{\theta(u(z)+d')} & \dfrac{\theta(-u(z)+t+d')}{\theta(-u(z)+d')} \\ \dfrac{\theta(u(z)+t-d')}{\theta(u(z)-d')} & \dfrac{\theta(-u(z)+t-d')}{\theta(-u(z)-d')} \end{pmatrix}. \tag{313}$$

By (297), $\nu \to 0$ corresponds to $\kappa \to 0$. By (298),

$$\frac{\theta(1/2)}{\theta(1/2+t)} = \frac{2\kappa^{|t|-|t|^2}}{1+\kappa^{2|t|}}(1+\mathcal{O}(\kappa)) = \mathcal{O}\left(\nu^{|t|-|t|^2}\right), \qquad \nu \to 0. \tag{314}$$

As $\nu \to 0$, we have $u(z) \to 0$ uniformly for z in the closure of $U^{(v_1)} \cup U^{(v_2)}$, and by (312),

$$d' \pm u(z) = \langle d' \pm u(z) \rangle \to 1/4. \tag{315}$$

Consider first the case $0 < t \le 1/4$. Pick $0 < \epsilon < \varepsilon/8$. Then, uniformly on the closure of $U^{(v_1)} \cup U^{(v_2)}$,

$$\frac{\theta(1/2)\theta(d' \pm u(z)+t)}{\theta(1/2+t)\theta(d' \pm u(z))} = \mathcal{O}\left(\kappa^{t-t^2}\kappa^{t^2+2t(\pm u(z)+d')}\right) = \mathcal{O}\left(\nu^{3t/2-\epsilon}\right),$$

which is the asymptotics of $\widetilde{m}(z)_{11}, \widetilde{m}(z)_{12}$. Moreover,

$$\frac{\theta(1/2)\theta(-d' \pm u(z)+t)}{\theta(1/2+t)\theta(-d' \pm u(z))} = \mathcal{O}\left(\kappa^{t-t^2}\kappa^{t^2+2t(\pm u(z)-d')}\right) = \mathcal{O}\left(\nu^{t/2-\epsilon}\right),$$

which is the asymptotics of $\widetilde{m}(z)_{21}, \widetilde{m}(z)_{22}$.

For $1/4 < t \le 1/2$, we have $\langle \pm u(z) + t + d' \rangle = \pm u(z) + t + d' - 1$ so that

$$\frac{\theta(1/2)\theta(d' \pm u(z)+t)}{\theta(1/2+t)\theta(d' \pm u(z))} = \mathcal{O}\left(\kappa^{t-t^2}\kappa^{t^2+2t(\pm u(z)+d'-1)+(3/4)^2-(1/4)^2-\epsilon}\right) = \mathcal{O}\left(\nu^{(1-t)/2-\epsilon}\right),$$

which is the asymptotics of $\widetilde{m}(z)_{11}, \widetilde{m}(z)_{12}$, and finally

$$\frac{\theta(1/2)\theta(d' \pm u(z)+t)}{\theta(1/2+t)\theta(d' \pm u(z))} = \mathcal{O}\left(\nu^{t/2-\epsilon}\right),$$

which is the asymptotics of $\widetilde{m}(z)_{21}, \widetilde{m}(z)_{22}$.

Similarly, we analyze the case of $-1/2 < t \le 0$. Collecting the results together, we obtain

$$\widetilde{m}(z) = \mathcal{O}\left(\nu^{|t|/2-\epsilon}\right) + \mathcal{O}\left(\nu^{(1-|t|)/2-\epsilon}\right) = \mathcal{O}(\nu^{-\epsilon}), \qquad \nu \to 0, \tag{316}$$

uniformly on the closure of $U^{(v_1)} \cup U^{(v_2)}$. By similar arguments, we obtain the same estimate also on the closure of $U^{(1)} \cup U^{(-1)}$ (in this case, $|u(z) + 1/2| \le \epsilon'$, $\epsilon' > 0$.)

On the other hand, the definition of δ gives

$$\delta(z)+\delta(z)^{-1},\ \delta(z)-\delta(z)^{-1}=\mathcal{O}\big(\nu^{-1/4}\big), \tag{317}$$

uniformly for $z\in\partial U^{(p)}$ as $\nu\to 0$, for $p\in\mathcal{T}=\{-1,v_1,v_2,1\}$.

Thus,

$$\widetilde{\mathcal{N}}(z),\widetilde{\mathcal{N}}(z)^{-1}=\mathcal{O}\left(\frac{1}{\nu^{1/4+\epsilon}}\right), \tag{318}$$

as $\nu\to 0$, uniformly on $\partial U^{(p)}$ for $p\in\mathcal{T}$.

Since the solution to the RH problem for $\mathcal{N}$ is unique, we have

$$\mathcal{N}(z)=\widetilde{\mathcal{N}}(\infty)^{-1}\widetilde{\mathcal{N}}(z). \tag{319}$$

Define the new local parametrices by

$$\widetilde{P}(z)=\widetilde{\mathcal{N}}(\infty)P(z), \tag{320}$$

and let

$$\widetilde{R}(z)=\begin{cases}\widetilde{\mathcal{N}}(\infty)S(z)\widetilde{\mathcal{N}}(z)^{-1} & z\in\mathbb{C}\setminus\cup_{p\in\mathcal{T}}U^{(p)},\\ \widetilde{\mathcal{N}}(\infty)S(z)\widetilde{P}(z)^{-1} & z\in\cup_{p\in\mathcal{T}}U^{(p)},\end{cases} \tag{321}$$

Then $\widetilde{R}(z)\to 1$ as $z\to\infty$; and $\widetilde{R}(z)$ has jumps on Γ_R, see Figure 5. By (124) and the expansion of ζ in (295), the jumps of $\widetilde{R}(z)$ on $\partial U^{(p)}$ have the form

$$\widetilde{P}(z)\widetilde{\mathcal{N}}^{-1}(z)=I+\widetilde{\Delta}(z),\qquad \widetilde{\Delta}(z)=\mathcal{O}\left(\frac{\log\nu^{-1}}{s\nu^{1/2+2\epsilon}}\right), \tag{322}$$

uniformly for $z\in U^{(p)}$ as $s\to\infty$ for $2\nu>s^{-2+\epsilon}$.

For the proof of Lemma 25 below, we will also require the finer estimate

$$\widetilde{\Delta}(z)=\widetilde{\Delta}_1(z)+\widetilde{\mathcal{N}}(z)\mathcal{O}\left(\frac{\left(\log\nu^{-1}\right)^2}{s^2}\right)\widetilde{\mathcal{N}}(z)^{-1}, \tag{323}$$

where

$$\begin{aligned}\widetilde{\Delta}_1(z)&=\frac{\mp 1}{8\sqrt{\zeta(z)}}\widetilde{\mathcal{N}}(z)e^{is\phi(p)\sigma_3}\begin{pmatrix}-1 & -2i\\ -2i & 1\end{pmatrix}e^{-is\phi(p)\sigma_3}\widetilde{\mathcal{N}}^{-1}(z),\quad p=-1,v_2,\\ \widetilde{\Delta}_1(z)&=\frac{\mp 1}{8\sqrt{\zeta(z)}}\widetilde{\mathcal{N}}(z)e^{is\phi(p)\sigma_3}\begin{pmatrix}-1 & 2i\\ 2i & 1\end{pmatrix}e^{-is\phi(p)\sigma_3}\widetilde{\mathcal{N}}^{-1}(z),\quad p=v_1,1,\end{aligned} \tag{324}$$

where $\mp$ means $+$ for $\operatorname{Im} z<0$ and $-$ for $\operatorname{Im} z>0$.

By (294), the jumps of $\widetilde{R}(z)$ on the rest of the contour are estimated as follows (we decrease $c>0$ somewhat)

$$\widetilde{\mathcal{N}}(z)J_S(z)\widetilde{\mathcal{N}}(z)^{-1}=I+\mathcal{O}\left(e^{-c\sqrt{s}(|z|+1)}\right),\qquad c>0, \tag{325}$$

as $s \to \infty$, uniformly for $2\nu > s^{-2+\varepsilon}$ and for $z \in \Gamma_{R,L} \cup \Gamma_{R,U}$. Thus $\widetilde{R}$ satisfies a small-norm problem and therefore has a solution for s sufficiently large and $2\nu > s^{-2+\varepsilon}$, and

$$\widetilde{R}(z) = I + \mathcal{O}\left(\frac{\log \nu^{-1}}{s\nu^{1/2+2\varepsilon}}\right), \tag{326}$$

as $s \to \infty$, uniformly for $2\nu > s^{-2+\varepsilon}$, and uniformly for $z \in \mathbb{C} \setminus \Gamma_R$.

Since the RH problem for $\widetilde{R}$ has a unique solution, the RH problem for S (and hence for Φ) has a unique solution obtained by tracing back the transformations.

10.2 | Integration of the differential identity

We now prove

Lemma 25. *Let $-1 < V_1 < \widehat{V}_2 < 1$ be fixed, and $V_1 < V_2 < \widehat{V}_2$ be such that $|V_2 - V_1| > s^{-5/4}$. Then, uniformly for such V_2 as $s \to \infty$,*

$$\log\det(I - K_s)_A - \log\det(I - K_s)_{(-1,V_1)\cup(\widehat{V}_2,1)} = \int_{\widehat{V}_2}^{V_2} D(V_1, v_2)dv_2 + \mathcal{O}(s^{-1/9}), \tag{327}$$

where D is defined in (171) of Proposition 17.

Proof. In this proof, ϵ stands for a sufficiently small positive constant whose value may vary from line to line.

In the previous section, we obtained the asymptotic solution of the S-RH problem in the regime $s \to \infty$, $2\nu > s^{-2+\epsilon}$. By (129), R is also well defined in this regime,

$$R(z) = \widetilde{N}(\infty)^{-1}\widetilde{R}(z)\widetilde{N}(\infty), \tag{328}$$

and thus (164) holds. We now aim to prove the analogue of (166), namely

$$\mathcal{F}_s(v_1, v_2) = \frac{s^2\zeta_0^2}{4} - \frac{\zeta_0 s}{4} m_{11,0}m_{22,0}\left(\gamma_0^2\Gamma_2 + \Gamma_1\right) + \frac{i\zeta_0\gamma_0^2}{4}W(s\Omega) + \mathcal{O}\left(\frac{1}{s\nu^{3/2+\epsilon}} + \frac{1}{s^2\nu^{5/2+\epsilon}}\right), \tag{329}$$

as $s \to \infty$, uniformly for $2\nu > s^{-5/4}$, with the same notation as in (164), (166).

By (91) and (303), using (158) and similar identities for $\widetilde{m}_{jk}$, we obtain

$$\widetilde{\mathcal{N}}(z) = \frac{\delta^{-1}(z)}{2}\begin{pmatrix} \widetilde{m}_{11}(v_2) & i\widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) & \widetilde{m}_{22}(v_2)\end{pmatrix} + \mathcal{O}\left((z - v_2)^{1/4}\right), \tag{330}$$

$$\mathcal{N}(z) = \frac{\gamma(z)}{2}\begin{pmatrix} m_{11}(v_2) & im_{11}(v_2) \\ -im_{22}(v_2) & m_{22}(v_2)\end{pmatrix} + \mathcal{O}\left((z - v_2)^{1/4}\right), \tag{331}$$

as $z \to v_2$.

Thus, substituting (330) and (331) into (319) and taking the limit $z \to v_2$, we obtain

$$\begin{aligned} \begin{pmatrix} m_{11}(v_2) \\ -im_{22}(v_2) \end{pmatrix} &= \left(\lim_{z\to v_2} \frac{1}{\gamma(z)\delta(z)} \right) \widetilde{\mathcal{N}}(\infty)^{-1} \begin{pmatrix} \widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) \end{pmatrix}, \\ \begin{pmatrix} im_{22}(v_2) & m_{11}(v_2) \end{pmatrix} &= \left(\lim_{z\to v_2} \frac{1}{\gamma(z)\delta(z)} \right) \begin{pmatrix} i\widetilde{m}_{22}(v_2) & \widetilde{m}_{11}(v_2) \end{pmatrix} \widetilde{N}(\infty). \end{aligned} \tag{332}$$

By the definition of γ and δ in (85) and (304), $\lim_{z\to v_2} \gamma(z)\delta(z) = \sqrt{2}\nu^{1/4}/\sqrt{(v_2+1)}$. Thus, by (328), the third term on the right hand side of (164) is given by

$$\begin{aligned} &\frac{is\zeta_0\gamma_0^2}{4} \begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix} R^{-1}(v_2)R'(v_2) \begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix} \\ &\qquad = \frac{is\zeta_0\gamma_0^2(1+v_2)}{8\nu^{1/2}} \begin{pmatrix} i\widetilde{m}_{22}(v_2) & \widetilde{m}_{11}(v_2) \end{pmatrix} \widetilde{R}^{-1}(v_2)\widetilde{R}'(v_2) \begin{pmatrix} \widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) \end{pmatrix}, \end{aligned} \tag{333}$$

which we now evaluate. By (292), (156), (157),

$$\zeta_0\gamma_0^2 = \mathcal{O}\left(\frac{1}{\log \nu^{-1}} \right), \tag{334}$$

as $\nu \to 0$.

By the definition of Δ_1, $\widetilde{\Delta}_1$, and by (319),

$$\widetilde{\Delta}_1(z) = \widetilde{N}(\infty)\Delta_1(z)\widetilde{N}(\infty)^{-1}, \tag{335}$$

and thus, by (332), and (167),

$$W(\omega) = \frac{(1+v_2)}{2\sqrt{\nu}} \begin{pmatrix} i\widetilde{m}_{22}(v_2;\omega) & \widetilde{m}_{11}(v_2;\omega) \end{pmatrix} \sum_{p\in\mathcal{T}} \int_{\partial U^{(p)}} \frac{s\widetilde{\Delta}_1(z;\omega)}{(z-v_2)^2} \frac{dz}{2\pi i} \begin{pmatrix} \widetilde{m}_{11}(v_2;\omega) \\ -i\widetilde{m}_{22}(v_2;\omega) \end{pmatrix}. \tag{336}$$

Note that $\widetilde{R}$ satisfies (we denote the jump of $\widetilde{R}$ on Γ_R by $I + \widetilde{\Delta}(z)$)

$$\widetilde{R}(z) = I + \int_{\Gamma_R} \frac{\widetilde{R}_-(\xi)\widetilde{\Delta}(\xi)}{\xi - z} \frac{d\xi}{2\pi i}. \tag{337}$$

By (337), (322), (325), (326), and the fact that $U^{(v_1)}$ and $U^{(v_2)}$ have radius $\nu/3$,

$$\begin{aligned} \widetilde{R}'(v_2) &= \int_{\Gamma_R} \left(I + \int_{\Gamma_R} \frac{\widetilde{R}_-(u)\widetilde{\Delta}(u)}{u-\xi_-} \frac{du}{2\pi i} \right) \frac{\widetilde{\Delta}(\xi)}{(\xi-v_2)^2} \frac{d\xi}{2\pi i} \\ &= \int_{\partial U^{(v_1)}\cup\partial U^{(v_2)}} \left(I + \int_{\Gamma_R} \frac{\widetilde{\Delta}(u)}{u-\xi_-} du + \mathcal{O}\left(\frac{1}{s^2\nu^{1+4\epsilon}} \right) \right) \frac{\widetilde{\Delta}(\xi)}{(\xi-v_2)^2} \frac{d\xi}{2\pi i} \\ &\quad + \int_{\partial U^{(1)}\cup\partial U^{(-1)}} \frac{\widetilde{\Delta}(\xi)}{(\xi-v_2)^2} \frac{d\xi}{2\pi i} + \mathcal{O}\left(\frac{1}{s^2\nu^{1+4\epsilon}} \right), \\ \widetilde{R}(v_2)^{-1} &= I - \int_{\Gamma_R} \frac{\widetilde{\Delta}(u)}{u-v_2} \frac{du}{2\pi i} + \mathcal{O}\left(\frac{1}{s^2\nu^{1+4\epsilon}} \right), \end{aligned} \tag{338}$$

as $s \to \infty$, uniformly for $z \in \mathbb{C} \setminus \Gamma_R$ and for $2\nu > s^{-5/4}$. Thus,

$$\widetilde{R}(v_2)^{-1}\widetilde{R}'(v_2) = \int_{\partial U^{(v_1)} \cup \partial U^{(v_2)}} \left(I + \int_{\Gamma_R} \widetilde{\Delta}(u) \left(\frac{1}{u - \xi_-} - \frac{1}{u - v_2} \right) du + \mathcal{O}\left(\frac{1}{s^2 \nu^{1+\epsilon}} \right) \right)$$
$$\times \frac{\widetilde{\Delta}(\xi)}{(\xi - v_2)^2} \frac{d\xi}{2\pi i} + \int_{\partial U^{(1)} \cup \partial U^{(-1)}} \frac{\widetilde{\Delta}(\xi)}{(\xi - v_2)^2} \frac{d\xi}{2\pi i} + \mathcal{O}\left(\frac{1}{s^2 \nu^{1+\epsilon}} \right),$$

in the same limit. Since $\frac{1}{u-\xi_-} - \frac{1}{u-v_2} = \mathcal{O}(\nu)$ when $u \in \partial U^{(1)} \cup \partial U^{(-1)}$ and $\xi_- \in \partial U^{(v_1)} \cup \partial U^{(v_2)}$, we obtain

$$\widetilde{R}(v_2)^{-1}\widetilde{R}'(v_2) = \int_{\partial U^{(v_1)} \cup \partial U^{(v_2)}} \left(I + \int_{\partial U^{(v_1)} \cup \partial U^{(v_2)}} \widetilde{\Delta}(u) \left(\frac{1}{u - \xi_-} - \frac{1}{u - v_2} \right) du \right.$$
$$\left. + \mathcal{O}\left(\frac{1}{s^2 \nu^{1+\epsilon}} + \frac{\nu^{1/2-\epsilon}}{s} \right) \right) \frac{\widetilde{\Delta}(\xi)}{(\xi - v_2)^2} \frac{d\xi}{2\pi i} + \int_{\partial U^{(1)} \cup \partial U^{(-1)}} \frac{\widetilde{\Delta}(\xi)}{(\xi - v_2)^2} \frac{d\xi}{2\pi i} + \mathcal{O}\left(\frac{1}{s^2 \nu^{1+\epsilon}} \right). \tag{339}$$

We will now estimate (333). For estimates on $\partial U^{(-1)} \cup \partial U^{(1)}$, recall that by (316), $\widetilde{m}(v_2)$ is of order $\nu^{-\epsilon}$. For estimates on $\partial U^{(v_1)} \cup \partial U^{(v_2)}$ we need more precise information: note that by (330),

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \widetilde{\mathcal{N}}(z)^{-1} \widetilde{\mathcal{N}}(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\delta^{-1}(z)}{2} \widetilde{\mathcal{N}}(z)^{-1} \begin{pmatrix} \widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) \end{pmatrix} + \mathcal{O}\left(\nu^{-1/4-\epsilon}\delta(z)\right), \tag{340}$$

on $\partial U^{(v_1)} \cup \partial U^{(v_2)}$, and therefore

$$\widetilde{\mathcal{N}}(z)^{-1} \begin{pmatrix} \widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) \end{pmatrix} = \mathcal{O}\left(\nu^{1/4-\epsilon}\right), \tag{341}$$

as $\nu \to 0$ for $z \in \partial U^{(v_1)} \cup \partial U^{(v_2)}$. Similarly,

$$\begin{pmatrix} i\widetilde{m}_{22}(v_2) & \widetilde{m}_{11}(v_2) \end{pmatrix} \widetilde{N}(z) = \mathcal{O}\left(\nu^{1/4-\epsilon}\right). \tag{342}$$

Estimates (339), and (341), (342) on $\partial U^{(v_1)} \cup \partial U^{(v_2)}$, and $\widetilde{m}(v_2) = \mathcal{O}(\nu^{-\epsilon})$, $\mathcal{N}(z) = \mathcal{O}(\nu^{-1/4-\epsilon})$ on $\partial U^{(-1)} \cup \partial U^{(1)}$ imply that (333) can be written as

$$\frac{is\zeta_0\gamma_0^2}{4} \begin{pmatrix} im_{22,0} & m_{11,0} \end{pmatrix} R^{-1}(v_2)R'(v_2) \begin{pmatrix} m_{11,0} \\ -im_{22,0} \end{pmatrix}$$
$$= \frac{i\zeta_0\gamma_0^2}{4} \frac{s(1+v_2)}{2\nu^{1/2}} \begin{pmatrix} i\widetilde{m}_{22}(v_2) & \widetilde{m}_{11}(v_2) \end{pmatrix} \widetilde{R}^{-1}(v_2)\widetilde{R}'(v_2) \begin{pmatrix} \widetilde{m}_{11}(v_2) \\ -i\widetilde{m}_{22}(v_2) \end{pmatrix}$$
$$= \frac{i\zeta_0\gamma_0^2}{4} W(s\Omega) + \mathcal{O}\left(\frac{1}{s\nu^{3/2+\epsilon}} + \frac{1}{s^2\nu^{5/2+\epsilon}} \right). \tag{343}$$

Thus we obtained (329). After integration, the error term here yields the one not larger than that of the statement of the lemma, $\mathcal{O}(s^{-1/9})$.

To finish the proof of the lemma we need to estimate the error of replacing W with its average value. From the definition (336) and the estimates above, we deduce

$$f(\omega) = \zeta_0\gamma_0^2 W(\omega) = \mathcal{O}\left(\frac{1}{\nu^{1+\epsilon}}\right), \qquad \nu \to 0. \tag{344}$$

By (291),

$$\Omega = \frac{1}{I_0} = \frac{\sqrt{|\alpha\beta|}}{\log(\gamma\nu)^{-1}}\left(1+\mathcal{O}(\nu^2)\right), \qquad \frac{\partial\Omega}{\partial v_2} = \mathcal{O}\left(\frac{1}{\nu(\log\nu^{-1})^2}\right), \qquad \frac{\partial^2\Omega}{\partial v_2^2} = \mathcal{O}\left(\frac{1}{\nu^2(\log\nu^{-1})^2}\right), \tag{345}$$

as $\nu \to 0$.

First, we have $f = \mathcal{O}(\nu^{-1-\epsilon})$ and $\frac{\partial}{\partial v_2} f = \mathcal{O}(\nu^{-2-\epsilon})$. By the analysis leading to (316), $\frac{\partial}{\partial\omega}\widetilde{m}(v_2) = \mathcal{O}(\nu^{-\epsilon}\log\nu)$, $\omega = s\Omega$, and therefore, adjusting ϵ, we also have $\frac{\partial}{\partial\omega} f = \mathcal{O}(\nu^{-1-\epsilon})$ and $\frac{\partial}{\partial\omega}\frac{\partial}{\partial v_2} f = \mathcal{O}(\nu^{-2-\epsilon})$. Thus, by (179) and a similar expression for $\frac{\partial}{\partial v_2} f_j$, the right hand side of (175) is of order $\frac{1}{j^2 s\nu^\epsilon}$, and we obtain

$$\int_{\widehat{V}_2}^{V_2} f(s\Omega; v_2, v_1)dv_2 = \sum_{j=-\infty}^{\infty}\int_{\widehat{V}_2}^{V_2} f_j(v_2, v_1)e^{2\pi i j s\Omega}dv_2 = \int_{\widehat{V}_2}^{V_2} f_0(v_2, v_1)dv_2 + \mathcal{O}\left(\frac{1}{s\nu^\epsilon}\right), \tag{346}$$

as $s \to \infty$, uniformly for $2\nu > s^{-5/4}$. The error term here is better than the one of the statement of the lemma. Thus the lemma is proved. □

10.3 | Proof of Theorem 4

By (194) and Lemma 25, we see that to show that the expansion (14) holds in the asymptotic regime of Theorem 4 (with the error term $\mathcal{O}(s^{-1/9})$) it remains to prove that

$$\int_{\widehat{V}_2}^{V_2}\left(\frac{\partial\tau}{\partial v_2}\int_0^1 \frac{\partial}{\partial\tau}\log\theta_3(\omega;\tau)d\omega\right) - \left(\frac{\partial\tau}{\partial v_2}\frac{\partial}{\partial\tau}\log\theta_3(s\Omega;\tau)\right)dv_2 = \mathcal{O}\left(\frac{1}{s\nu^\epsilon}\right). \tag{347}$$

Since by (224), (291),

$$\frac{\partial\tau}{\partial v_2} = \frac{i\pi}{I_0^2(1-v_2^2)(v_2-v_1)} = \mathcal{O}\left(\frac{1}{\nu\log^2(\gamma\nu)^{-1}}\right), \tag{348}$$

and by (298), (297),

$$\frac{1}{\theta(\omega)}\frac{d^k}{d\omega^k}\theta(\omega) = \mathcal{O}\left(\log^k(\gamma\nu)^{-1}\right),$$

we obtain

$$\frac{\partial}{\partial \omega}\left(\frac{\partial \tau}{\partial v_2}\frac{\partial}{\partial \tau}\log \theta_3(\omega;\tau)\right) = \frac{1}{4\pi i}\frac{\partial \tau}{\partial v_2}\left(\frac{\theta_3''}{\theta_3}\right)'(\omega) = \mathcal{O}\left(\frac{\log(\gamma\nu)^{-1}}{\nu}\right). \tag{349}$$

Also since by (219),

$$\frac{\partial^2 \tau}{\partial v_2^2} = \mathcal{O}\left(\frac{1}{\nu^2 \log^2(\gamma\nu)^{-1}}\right), \tag{350}$$

we similarly obtain

$$\frac{\partial}{\partial \omega}\frac{\partial}{\partial v_2}\left(\frac{\partial \tau}{\partial v_2}\frac{\partial}{\partial \tau}\log \theta_3(\omega;\tau)\right) = \mathcal{O}\left(\frac{\log(\gamma\nu)^{-1}}{\nu^2}\right). \tag{351}$$

The estimates (349) and (351) imply, as in the proof of (346), the estimate (347). Thus, we have proven the first statement of Theorem 4.

Since we have proven the uniformity of Theorem 1 for $2\nu > s^{-5/4}$, all that remains to show (19) is to expand G_0, $\log\theta_3(s\Omega;\tau)$, and c_1 as $\nu \to 0$.

By (10) and (292),

$$G_0 = \frac{1}{2} - \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}} + \mathcal{O}(\nu^2), \tag{352}$$

as $\nu \to 0$.

By the formula for Ω in (345), θ in (298), $\varkappa$ in (297), τ in (296),

$$\log\theta_3(s\Omega;\tau) = \frac{1}{2}\log\log(\gamma\nu)^{-1} - \langle\omega_0\rangle^2\log(\gamma\nu)^{-1} + \log\left(1 + (\gamma\nu)^{1-2|\langle\omega_0\rangle|}\right) - \frac{1}{2}\log\pi + o(1), \tag{353}$$

as $s\nu \to 0$, where

$$s\Omega = \omega_0 + o(1),$$

with ω_0 given by (20).

By the asymptotics for I_0 in (291) and x_1x_2 in (292), and by (8),

$$c_1 = -\frac{1}{4}\log\log(\gamma\nu)^{-1} - \frac{1}{8}\log|\alpha\beta| + \frac{1}{2}\log\pi + 2c_0 + o(1), \tag{354}$$

as $\nu \to 0$. Thus we obtain (19) if $s\nu \to 0$.

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REFERENCES

1. J. Baik, R. Buckingham, and J. DiFranco, *Asymptotics of Tracy–Widom distributions and the total integral of a Painlevé II function*, Comm. Math. Phys. **280** (2008), 463–497.
2. J. Baik, P. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), 1119–1178.

3. P. Bleher, A. Bolibruch, A. Its, and A. Kapaev, *Linearization of the P34 equation of Painlevé-Gambier*, unpublished.
4. A. Borodin and P. Deift, *Fredholm determinants, Jimbo-Miwa-Ueno tau-functions and representation theory*, Comm. Pure Appl. Math. **55** (2002), 1160–1230.
5. T. Bothner, P. Deift, A. Its, and I. Krasovsky, *The sine process under the influence of a varying potential*, J. Math. Phys. **59** (2018), no. 9, 091414, pp. 6.
6. E. Blackstone, C. Charlier, and J. Lenells, *Oscillatory asymptotics for Airy kernel determinants on two intervals*, Int. Math. Res. Not. **4** (2022), 2636–2687.
7. E. Blackstone, C. Charlier, and J. Lenells, *Gap probabilities in the bulk of the Airy process*, Random Matrices: Theory Appl. **11** (2022), no. 2, 2250022.
8. E. Blackstone, C. Charlier, and J. Lenells, *The Bessel kernel determinant on large intervals and Birkhoff's ergodic theorem*, arXiv:2101.09216, pp. 33.
9. C. Charlier and T. Claeys, *Thinning and conditioning of the circular unitary ensemble*, Random Matrices Theory Appl. **6** (2017), no. 2, 1750007, pp. 51.
10. C. Charlier and T. Claeys, *Large gap asymptotics for Airy kernel determinant with discontinuities*, Commun. Math. Phys. **375** (2020), 1299–1339.
11. C. Christophe, J. Lenells, and J. Mauersberger, *Higher order large gap asymptotics at the hard edge for Muttalib-Borodin ensembles*, Comm. Math. Phys. **384** (2021), 829–907.
12. C. Christophe, J. Lenells, and J. Mauersberger, *The multiplicative constant for the Meijer-G kernel determinant*, Nonlinearity **34** (2021), 2837–2877.
13. J. des Cloizeaux and M. L. Mehta, *Asymptotic behaviour of spacing distributions for the eigenvalues of random matrices*, J. Math. Phys. **14** (1973), 1648–1650.
14. P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, 1998.
15. P. Deift, A. Its, and I. Krasovsky, *Toeplitz, Hankel and Toeplitz+Hankel determinants with Fisher-Hartwig singularities*, Ann. Math. **174** (2011), 1243–1299.
16. P. Deift, A. Its, and I. Krasovsky, *Asymptotics for the Airy-kernel determinant*, Comm. Math. Phys. **278** (2008), 643–678.
17. P. Deift, A. Its, I. Krasovsky, and X. Zhou, *The Widom-Dyson constant for the gap probability in random matrix theory*, J. Comput. Appl. Math. **202** (2007), 26–47.
18. P. Deift, A. Its, and X. Zhou, *A Riemann-Hilbert problem approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics*, Ann. Math. **146** (1997), 149–235.
19. P. Deift, I. Krasovsky, and J. Vasilevska, *Asymptotics for a determinant with a confluent hypergeometric kernel*, Int. Math. Res. Not. **9** (2011), 2117–2160.
20. F. Dyson, *Statistical theory of the energy levels of complex systems. II*, J. Math. Phys. **3** (1962), 157–165.
21. F. Dyson, *Fredholm determinants and inverse scattering problems*, Comm. Math. Phys. **47** (1976), 171–183.
22. T. Ehrhardt, *Dyson's constants in the asymptotics of the determinants of Wiener-Hopf-Hankel operators with the sine kernel*, Commun. Math. Phys. **272** (2007), no. 3, 683–698.
23. T. Ehrhardt, *The asymptotics of a Bessel-kernel determinant which arises in random matrix theory*, Adv. Math. **225** (2010), 3088–3133.
24. B. Fahs and I. Krasovsky, *Splitting of a gap in the bulk of the spectrum of random matrices*, Duke Math. J. **168** (2019), 3529–3590.
25. A. S. Fokas, A. R. Its, and A. V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), 395–430.
26. P. J. Forrester, *Asymptotics of spacing distributions 50 years later*, MSRI Publications **65** (2014), 199–222.
27. I. S. Gradstein and J. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, Cambridge, 2015.
28. A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov, *Differential equations for quantum correlation functions*, Int. J. Mod. Phys. B **4** (1990), 1003–1037.
29. M. Jimbo and T. Miwa, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II*, Physica **2D** (1981), 407–448.
30. M. Jimbo, T. Miwa, Y. Môri, and M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, Phys D: Nonlin. Phen. **1** (1980), no. 1, 80–158.

31. I. Krasovsky, *Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle*, Int. Math. Res. Notices IMRN **2004** (2004), 1249–1272.
32. I. Krasovsky and T.-H. Maroudas, *Airy-kernel determinant on two large intervals*, arXiv:2108.04495.
33. I. Krasovsky, *Aspects of Toeplitz determinants*, D. Lenz, F. Sobieczky, and W. Wöss (eds.), Boundaries and Spectra of Random Walks, Proceedings, Graz - St. Kathrein 2009, Progress in Probability, Birkhaeuser, Basel, 2011.
34. A. B. J. Kuijlaars, K. T-R McLaughlin, W. Van Assche, and M. Vanlessen, *The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on* $[-1,1]$, Adv. Math. **188** (2004), 337–398.
35. K. Okamoto, *Studies on the Painlevé equations II. Fifth Painlevé equation PV*, Japan. J. Math. **13** (1987), 47–76.
36. G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, Massachusetts, 1957.
37. C. A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Commun.Math. Phys. **159** (1994), 151–174.
38. C. A. Tracy and H. Widom, *Level-spacing distributions and the Bessel kernel*, Commun. Math. Phys. **161** (1994), 289–309.
39. C. A. Tracy and H. Widom, *Fredholm determinants, differential equations and matrix models*, Commun. Math. Phys. **163** (1994), 33–72.
40. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1996.
41. H. Widom, *The strong Szegő limit theorem for circular arcs*, Indiana Univ. Math. J. **21** (1971), 277–283.
42. H. Widom, *The asymptotics of a continuous analogue of orthogonal polynomials*, J. Approx. Theory **77** (1994), 51–64.
43. H. Widom, *Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals*, Comm. Math. Phys. **171** (1995), 159–180.

APPENDIX A: θ-FUNCTIONS AND ELLIPTIC INTEGRALS

Here we collect the properties of Jacobian θ-functions and elliptic integrals we need in the main text. For more information on the topic, see [27, 36, 40].

The third Jacobian θ-function is defined by a series[6]:

$$\theta_3(z;\tau) \equiv \theta_3(z) \equiv \theta(z) = \sum_{m\in\mathbb{Z}} e^{2\pi i z m + \pi i \tau m^2}, \qquad \operatorname{Im}\tau > 0. \tag{A.1}$$

The function $\theta(z)$ satisfies the periodicity properties:

$$\theta(z) = \theta(z+1), \qquad \theta(z\pm\tau) = e^{\mp 2\pi i z - \pi i \tau}\theta(z). \tag{A.2}$$

It is an entire function which is even, $\theta(z) = \theta(-z)$. Furthermore, $\theta(z)$ has a single zero modulo the lattice $(\mathbb{Z}, \tau\mathbb{Z})$ at $\frac{1+\tau}{2}$, and at the zero the derivative $\theta'(z)$ is non-zero.

The first, second, and fourth θ-functions are then defined as follows:

$$\begin{aligned} \theta_1(z) &= i e^{-\pi i z + \frac{\pi i \tau}{4}} \theta_3\left(z - \frac{\tau+1}{2}\right), \\ \theta_2(z) &= \theta_1(z+1/2) = e^{-\pi i z + \pi i \tau/4}\theta_3\left(z - \frac{\tau}{2}\right), \qquad \theta_4(z) = \theta_3(z+1/2). \end{aligned} \tag{A.3}$$

[6] θ-functions are defined in [40] with argument z/π.

The function $\theta_1(z)$ is odd, while $\theta_2(z), \theta_4(z)$ are even. The unique zeros (modulo the lattice) of θ_1, θ_2 and θ_4 are at 0,1/2 and $\tau/2$, respectively, and we have the periodicity properties:

$$\begin{aligned} &\theta_1(z+1) = -\theta_1(z), && \theta_1(z+\tau) = -e^{-2\pi iz-\pi i\tau}\theta_1(z),\\ &\theta_2(z+1) = -\theta_2(z), && \theta_2(z+\tau) = e^{-2\pi iz-\pi i\tau}\theta_2(z),\\ &\theta_4(z+1) = \theta_4(z), && \theta_4(z+\tau) = e^{-2\pi iz-\pi i\tau}\theta_4(z). \end{aligned} \tag{A.4}$$

From the periodicity properties we have

$$\begin{aligned} &\frac{\theta_j'(z+1)}{\theta_j(z+1)} = \frac{\theta_j'(z)}{\theta_j(z)}, && \frac{\theta_j'(z+\tau)}{\theta_j(z+\tau)} = \frac{\theta_j'(z)}{\theta_j(z)} - 2\pi i,\\ &\frac{\theta_j''(z+1)}{\theta_j(z+1)} = \frac{\theta_j''(z)}{\theta_j(z)}, && \frac{\theta_j''(z+\tau)}{\theta_j(z+\tau)} = \frac{\theta_j''(z)}{\theta_j(z)} - 4\pi i\frac{\theta_j'(z)}{\theta_j(z)} - 4\pi^2, \qquad j = 1,2,3,4. \end{aligned} \tag{A.5}$$

We denote $\theta_j = \theta_j(0)$, and the derivatives at zero $\theta_j' = \theta_j'(0)$, etc. In particular, we have expansions at zero: $\theta_3(z) = \theta_3 + \frac{z^2}{2}\theta_3'' + \cdots$, $\theta_1(z) = z\theta_1' + z^3\frac{\theta_1'''}{6} + \cdots$.

We will use representations of θ_3 in terms of θ_1. By (A.3),

$$\frac{\theta_3'(z)}{\theta_3(z)} = \frac{\theta_1'(\nu)}{\theta_1(\nu)} - \pi i, \qquad \nu = z - \frac{1+\tau}{2}, \tag{A.6}$$

and

$$\frac{\theta_3''(z)}{\theta_3(z)} = \frac{\theta_1''(\nu)}{\theta_1(\nu)} - 2\pi i\frac{\theta_1'(\nu)}{\theta_1(\nu)} - \pi^2, \qquad \nu = z - \frac{1+\tau}{2}. \tag{A.7}$$

θ-functions satisfy Jacobian addition relations, of which we will make use of the following two:

$$\theta_2(x+y)\theta_3(x-y) + \theta_2(x-y)\theta_3(x+y) = \frac{2}{\theta_2\theta_3}\theta_2(x)\theta_2(y)\theta_3(x)\theta_3(y), \tag{A.8}$$

$$\theta_4(x+y)\theta_3(x-y) + \theta_4(x-y)\theta_3(x+y) = \frac{2}{\theta_4\theta_3}\theta_4(x)\theta_4(y)\theta_3(x)\theta_3(y). \tag{A.9}$$

θ-functions satisfy the differential equation

$$\theta_j''(z) = 4\pi i\frac{\partial}{\partial\tau}\theta_j(z), \qquad j = 1,2,3,4, \tag{A.10}$$

some useful for us well-known identities for the values at zero:

$$\theta_1' = \pi\theta_2\theta_3\theta_4, \qquad \theta_3^4 = \theta_2^4 + \theta_4^4,$$

and the following transformation formula for $\tau \to 1/\tau$,

$$\theta_3(z) = \frac{1}{\sqrt{-i\tau}}\sum_k e^{-\frac{i\pi}{\tau}(k-z)^2}. \tag{A.11}$$

We will also need the following identity:

$$\left(\frac{\theta_3'(z)}{\theta_3(z)}\right)' = \left(\frac{\theta_1'}{\theta_3}\right)^2 \frac{\theta_1(z)^2}{\theta_3(z)^2} + \frac{\theta_3''}{\theta_3}. \tag{A.12}$$

To show it, we first observe that both sides of the equation are elliptic functions (i.e., they satisfy the periodicity relations $f(z+1) = f(z)$, $f(z+\tau) = f(z)$) with second-order pole at $z = (1+\tau)/2$. Considering the expansions of these functions at the pole, we obtain that the difference of these functions has a pole of order at most 1, and is therefore a constant. This constant is then evaluated setting $z = 0$.

Changing variable $z = \nu + \frac{1+\tau}{2}$ in (A.12), we also obtain

$$\left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)' = -\left(\frac{\theta_1'}{\theta_3}\right)^2 \frac{\theta_3(\nu)^2}{\theta_1(\nu)^2} + \frac{\theta_3''}{\theta_3}. \tag{A.13}$$

We further have

Lemma A.1. *If $g(z)$ is an elliptic function with a single pole modulo the lattice, located at $z = \frac{1+\tau}{2}$, and*

$$g\left(\nu + \frac{1+\tau}{2}\right) = c_1\nu^{-2} + \mathcal{O}(\nu^{-1}), \tag{A.14}$$

as $\nu \to 0$, then

$$g(z) = -c_1\left[\left(\frac{\theta_3'(z)}{\theta_3(z)}\right)' - \frac{\theta_3''}{\theta_3}\right] + g(0), \tag{A.15}$$

and furthermore

$$\int_0^1 g(z)dz = c_1\frac{\theta_3''}{\theta_3} + g(0). \tag{A.16}$$

Proof. The second part of the lemma, (A.16), follows directly from (A.15).

To show (A.15) note first that since $\theta_3(z)$ has a zero of order 1 at $\frac{1+\tau}{2}$,

$$\frac{\theta_3'(z)}{\theta_3(z)} = \frac{1}{z - \frac{1+\tau}{2}} + \mathcal{O}(1), \tag{A.17}$$

as $z \to \frac{1+\tau}{2}$. By the fact that $(\frac{\theta_3'(z)}{\theta_3(z)})'$ is elliptic and the hypothesis of the theorem,

$$g(z) + c_1\left(\frac{\theta_3'(z)}{\theta_3(z)}\right)' \tag{A.18}$$

is an elliptic function with a single simple pole modulo the lattice, and therefore is a constant. By (A.12), this constant is $g(0) + c_1\frac{\theta_3''}{\theta_3}$. This shows (A.15). □

Lemma A.2. *We have*

$$\int_0^1 \left(\frac{\theta_3'(z)}{\theta_3(z)}\right)^2 dz = \frac{\pi^2}{3} + \frac{\theta_1'''}{3\theta_1'}, \tag{A.19}$$

and, for any d, u,

$$\int_0^1 \frac{\theta_3(z-d)\theta_3(z+u+d)}{\theta_3(z)^2} dz = \frac{\pi\left[\theta_1'(d)\theta_1(u+d) - \theta_1(d)\theta_1'(u+d)\right]}{\left(\theta_1'\right)^2 \sin(\pi u)}. \tag{A.20}$$

Proof. Since

$$\int_0^1 \frac{\theta_3'(\omega)}{\theta_3(\omega)} d\omega = 0, \tag{A.21}$$

we have by the relation between the logarithmic derivatives of θ_1 and θ_3 in (A.6),

$$\int_0^1 \left(\frac{\theta_3'(z)}{\theta_3(z)}\right)^2 dz = \pi^2 + \int_0^1 \left(\frac{\theta_3'(z)}{\theta_3(z)} + \pi i\right)^2 dz = \pi^2 + \int_J \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^2 d\nu, \tag{A.22}$$

where

$$J = \left\{\nu = z - \frac{1+\tau}{2} \ : \ z \in (0,1)\right\}. \tag{A.23}$$

Let $\tilde{\Gamma}$ be the rectangle with corners $\pm 1/2 \pm \tau/2$, with positive orientation. Writing the integral around the contour and using the periodicity relation of θ_1'/θ_1 in (A.5), we obtain

$$\int_{\tilde{\Gamma}} \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^3 d\nu = 6\pi i \int_J \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^2 d\nu + 12\pi^2 \int_J \frac{\theta_1'(\nu)}{\theta_1(\nu)} d\nu - 8\pi^3 i. \tag{A.24}$$

By (A.6), and (A.21), $\int_J \frac{\theta_1'(\nu)}{\theta_1(\nu)} d\nu = \pi i$, and therefore

$$\int_J \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^2 d\nu = -\frac{2\pi^3}{3} + \frac{1}{6\pi i} \int_{\tilde{\Gamma}} \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^3 d\nu. \tag{A.25}$$

Since θ_1 has a single zero modulo the lattice located at 0, and since $\theta_1''(0) = 0$, we obtain

$$\int_{\tilde{\Gamma}} \left(\frac{\theta_1'(\nu)}{\theta_1(\nu)}\right)^3 d\nu = 2\pi i \frac{\theta_1'''}{\theta_1'} \tag{A.26}$$

by evaluating the residue of $(\frac{\theta_1'(\nu)}{\theta_1(\nu)})^3$ at 0. Combining (A.22), (A.25), and (A.26), we obtain (A.19).

To obtain (A.20), we first observe that by (A.3), (A.4),

$$\int_0^1 \frac{\theta(z-d)\theta(z+u+d)}{\theta(z)^2} dz = e^{-\pi i u} \int_J \frac{\theta_1(\nu-d)\theta_1(u+\nu+d)}{\theta_1(\nu)^2} d\nu, \tag{A.27}$$

where again $J = \{\nu = z - \frac{1+\tau}{2}, z \in (0,1)\}$. With $\widetilde{\Gamma}$ as above, we have by periodicity properties that

$$\int_{\widetilde{\Gamma}} \frac{\theta_1(\nu - d)\theta_1(u+\nu+d)}{\theta_1(\nu)^2} d\nu = \left(1 - e^{-2\pi i u}\right) \int_J \frac{\theta_1(\nu - d)\theta_1(u+\nu+d)}{\theta_1(\nu)^2} d\nu. \tag{A.28}$$

On the other hand, computing the residue, we obtain

$$\int_{\widetilde{\Gamma}} \frac{\theta_1(\nu - d)\theta_1(u+\nu+d)}{\theta_1(\nu)^2} d\nu = \frac{2\pi i}{(\theta_1')^2} \left(\theta_1'(d)\theta_1(u+d) - \theta_1(d)\theta_1'(u+d)\right). \tag{A.29}$$

The last 3 equations give (A.20). □

Recall the definition of the elliptic integrals $I_j = I_j(v_1, v_2)$, $J_j = J_j(v_1, v_2)$ from (5).

Lemma A.3. *There holds a Riemann's period relation:*

$$\left(I_2 - \frac{v_1+v_2}{2} I_1\right) J_0 - I_0 \left(J_2 - \frac{v_1+v_2}{2} J_1\right) = \pi. \tag{A.30}$$

Proof. We cut the Riemann surface Σ along the loops A_1, B_1, which yields a 4-gon γ with the sides $A_1, B_1, A_1^{-1}, B_1^{-1}$ (the side A_1 is identified with A_1^{-1} on the surface, the same with B_1, B_1^{-1}). The standard Riemann period relation between meromorphic differentials λ, μ on Σ is as follows:

$$\int_\gamma \Lambda\mu = \int_{A_1} \lambda \int_{B_1} \mu - \int_{A_1} \mu \int_{B_1} \lambda, \qquad \Lambda(x) = \int_{x_0}^x \lambda, \qquad x \in \Sigma, \tag{A.31}$$

where γ is traversed in the positive direction, and where x_0 is a fixed point on the surface away from the cuts.

Now taking $\lambda = \frac{x^2 - x(v_1+v_2)/2}{p(x)^{1/2}} dx$, $\mu = \frac{dx}{p(x)^{1/2}}$, we have in the local variable $\xi = 1/z$, $\lambda = \mp(1 + \mathcal{O}(\xi^2))\frac{d\xi}{\xi^2}$, $\mu = \mp(1 + \mathcal{O}(\xi))d\xi$, as $\xi \to 0$. Here the upper sign is taken on the first sheet, and the lower one on the second. Computing the residue at z-infinity (at two points on Σ corresponding to it) of $\Lambda\mu$, we obtain (A.30). □

The complete elliptic integrals of first and second kind, respectively, are defined as follows:

$$K(v) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-v^2t^2)}}, \qquad E(v) = \int_0^1 \sqrt{\frac{1-v^2t^2}{1-t^2}} dt. \tag{A.32}$$

Moreover, let

$$K'(v) = \int_1^{1/v} \frac{dt}{\sqrt{(t^2-1)(1-v^2t^2)}}, \qquad \widehat{E}(v) = \int_1^{1/v} \sqrt{\frac{1-v^2t^2}{t^2-1}} dt. \tag{A.33}$$

It is well-known that

$$K'(v) = K(v'), \qquad v' = \sqrt{1-v^2}. \tag{A.34}$$

By integrating the derivative of $t\sqrt{\frac{1-t^2}{1-v'^2t^2}}$, we also obtain that

$$\hat{E}(v) = K(v') - E(v'). \tag{A.35}$$

As $v \to 1$ (and therefore $v' \to 0$), we have the expansions:

$$\begin{aligned} K(v) &= \left(\frac{1}{2}\log\frac{1}{2-2v} + 2\log 2\right)(1 + \mathcal{O}(1-v)), \\ K(v') &= \frac{\pi}{2}\left(1 + \frac{v'^2}{4} + \frac{9v'^4}{64} + \mathcal{O}(v'^6)\right), \qquad E(v') = \frac{\pi}{2}\left(1 - \frac{v'^2}{4} - \frac{3v'^4}{64} + \mathcal{O}(v'^6)\right). \end{aligned} \tag{A.36}$$

Now consider the case symmetric intervals $-v_1 = v_2 \equiv v$. By the change of variable $x = vy$ and by using (A.35), we see that

$$I_0(-v,v) = K(v'), \qquad \frac{I_2(-v,v)}{I_0(-v,v)} = 1 - \frac{\hat{E}(v)}{K(v')} = \frac{E(v')}{K(v')}, \qquad J_0(-v,v) = 2K(v). \tag{A.37}$$

APPENDIX B: PREFACTOR OF log *s*

Here we show that the constant $\hat{G}_1$ in (14) obtained in [18] is equal to $-1/2$. Let

$$u(z) = -\frac{i}{2I_0}\int_{v_2}^{z}\frac{d\xi}{p(\xi)^{1/2}}, \tag{B.1}$$

and define

$$\rho(z,\omega) = \frac{\theta^2(0)\theta(u(z)+\omega-u(\infty))\theta(u(z)-\omega-u(\infty))}{\theta^2(\omega)\theta^2(u(z)-u(\infty))}, \qquad d = -u(\infty).$$

It is easily verified that ρ as a function of ω is elliptic: $\rho(\omega) = \rho(\omega+1) = \rho(\omega+\tau)$. Here we use our definitions of $u(z)$ (86) and d (which has the property (90)) from Section 4. However, it is straightforward to verify that ρ is exactly the function (1.30) in [18] for $n = 1$ with $x = \omega/\Omega, V = \Omega$.

Let

$$h(z) = (z-1)(z-v_1) + (z-v_2)(z+1), \tag{B.2}$$

and consider the function G_1 given by (1.33) in [18], which in our case of $n = 1$ becomes

$$G_1(t) = -\frac{1}{16}\sum_{y=\{-1,v_1,v_2,1\}}\rho(y,t\Omega)\frac{h(y)}{q(y)}.$$

It was shown in [18] that the coefficient $\hat{G}_1$ in (14) is given by

$$\hat{G}_1 = \lim_{x\to\infty}\frac{1}{x}\int_{x_0}^{x}G_1(t)dt,$$

for some fixed large x_0.

By ellipticity of ρ, this can be written in the form

$$\widehat{G}_1 = -\frac{1}{16} \sum_{y=\{-1,v_1,v_2,1\}} \frac{h(y)}{q(y)} \int_0^1 \rho(y,\omega)d\omega. \tag{B.3}$$

To compute the integral, note first that by (A.3)

$$\begin{aligned} \rho\left(y, \nu + \frac{1+\tau}{2}\right) &= \frac{\theta_3^2}{\theta_3^2(u(z)+d)} \frac{\theta_1(u(z)+d+\nu)\theta_1(-u(z)-d+\nu)}{\theta_1^2(\nu)} \\ &= -\frac{\theta_3^2}{\theta_3^2(u(z)+d)} \frac{\theta_1^2(u(z)+d)}{(\theta_1')^2\nu^2} + \mathcal{O}(\nu^{-1}), \qquad \nu \to 0. \end{aligned} \tag{B.4}$$

Using Lemma A.1 in Appendix A, we compute the integral $\int_0^1 \rho(y,\omega)d\omega$ and obtain

$$\widehat{G}_1 = -\frac{1}{16} \sum_{y\in\{-1,v_1,v_2,1\}} \frac{h(y)}{q(y)} \left(1 - \frac{\theta_3\theta_3''}{(\theta_1')^2} \frac{\theta_1^2(u(y)+d)}{\theta_3^2(u(y)+d)}\right). \tag{B.5}$$

By applying the identities (98) of Proposition 16 (d),

$$\widehat{G}_1 = -\frac{1}{16} \sum_{y\in\{-1,v_1,v_2,1\}} \frac{1}{q(y)} \left(h(y) + \frac{\theta_3''}{\theta_3 I_0^2}\right). \tag{B.6}$$

By (252),

$$\frac{\theta_3''}{\theta_3 I_0^2} = 2q(v_2) - h(v_2), \tag{B.7}$$

and therefore the term with $y = v_2$ in (B.6) is

$$\frac{1}{q(v_2)} \left(h(v_2) + \frac{\theta_3''}{\theta_3 I_0^2}\right) = 2.$$

Now note (recall (8)) that

$$2q(v_2) - h(v_2) = 2q(v_1) - h(v_1) = 2q(1) - h(1) = 2q(-1) - h(-1) = v_2 - v_1 + 2x_1x_2, \tag{B.8}$$

so that all the other terms in the sum in (B.6) are also equal 2. Therefore

$$\widehat{G}_1 = -\frac{1}{16}(2+2+2+2) = -\frac{1}{2}. \tag{B.9}$$