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A study of Quot schemes on smooth curves

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Abstract

We study the geometry and topology of Quot schemes on smooth projective curves. First, we give an explicit presentation of the rational cohomology ring of the Quot scheme parametrising torsion quotients on $\mathbb{P}^1$. Next, we construct a stratification of the corresponding relative Quot scheme, which recovers several known results by specialisation. We also use this stratification to prove that the integral cohomology of the Quot scheme parametrising torsion quotients is torsion-free, thereby strengthening the first result. Finally, we study the cohomology of Schur complexes associated with tautological complexes on the Quot scheme parametrising positive rank quotients on $\mathbb{P}^1$, and we construct exceptional collections in its derived category.

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मेरा नेपाली साथीहरूलाई बिर्सने त कुरै भएन। म विशेषगरी सुजन, ठूले र उज्ज्वलप्रति धेरै आभारी छु। मैले सुजनलाई नेपालमा अर्थशास्त्रको स्नातकोत्तर पढ्दाको समयमा भेटेको थिएँ। त्यसपछि हामीले विभिन्न विषयहरूमा धेरै कुराकानी गरेका छौँ, जसबाट मैले सधैं धेरै कुरा सिकेको छु। ठूले मेरो बाल्यकालदेखिको साथी हो र अहिलेसम्म पनि हामी सम्पर्कमा छौँ। हामीबीच भएका सबै कुराकानीहरूका लागि र यी वर्षहरूमा पाएका सबै सुझावहरूका लागि म अत्यन्तै आभारी छु। उज्ज्वललाई सबैभन्दा पछि चिनेको भए पनि, मलाई गणितप्रति जिज्ञासा जगाउनमा उस्को ठूलो हात छ। उस्कै कारण मेरो एलन हकलबेरीसँग चिनजान भयो, जसबाट मैले धेरै गणित सिक्ने अवसर पाएँ। तिमीहरू प्रत्येक अझै पनि मेरो साथी रहेकोमा म धेरै आभारी छु।

अन्त्यमा, म मेरी आमा, मेरा बुवा, मेरी बहिनी, नानी दिदी र मेरी हजुरआमालाई पनि धन्यवाद दिन चाहन्छु। मेरो लागि गर्नुभएको सबै कुराका लागि म सधैं आभारी रहनेछु।

Contents

Chapter 1

Introduction

In 1857, Bernhard Riemann, in [57], sought criteria that would determine a complex algebraic curve up to isomorphism. This led him to the notion of moduli, which are the independent parameters needed to specify the isomorphism class of such a curve. Although it took roughly half a century for this programme to be put on a completely rigorous footing, the idea of moduli proved extremely fruitful. In modern algebraic geometry, this philosophy is embodied in the study of moduli spaces, geometric spaces whose points represent the isomorphism classes of the objects under consideration.

In the second half of the twentieth century, algebraic geometers began to apply this point of view far beyond curves, including to fundamental objects like sheaves on schemes. In particular, Alexander Grothendieck, in [28], introduced the moduli space of quotients of a fixed coherent sheaf on a projective scheme, which he called the Quot scheme, the central object of this thesis. The Quot scheme plays a fundamental role in the construction of moduli space of sheaves on schemes [33]. Consequently, understanding its geometry provides a natural first approximation to the study of more intricate moduli spaces of sheaves; see, for instance, [60, 55, 51] for the case of moduli space of stable bundles on curves.

In this thesis, we investigate three distinct but related questions arising in the study of Quot schemes on curves, working throughout over the complex numbers $\mathbb{C}$. Let C be a smooth projective curve of genus g and let V be a vector bundle of rank n on C. The Quot scheme $\mathrm{Quot}_d(V, r, C)$ parametrises exact sequences

$$0 \to E \to V \to F \to 0 \quad \text{with rank } F = r \text{ and } \deg F = d.$$

This is a fine moduli space, and hence admits a universal exact sequence

$$0 \to \mathcal{E} \to \rho^* V \to \mathcal{F} \to 0 \tag{1.1}$$

where ρ denotes the projection morphism to C; given a point $q \in \mathrm{Quot}_d(V, r, C)$, the restriction of the sequence (1.1) to $\{q\} \times C$ recovers the exact sequence corresponding to q. It is always projective and, for d sufficiently large, is irreducible and generically smooth of the expected dimension [9, 54]. The Quot scheme on a curve can be regarded as a compactification of the space of morphisms of degree d from C to the Grassmannian $\mathrm{Gr}(r, n)$, parametrising r dimensional quotients of $\mathbb{C}^n$. This perspective has been used to relate it to other moduli spaces (see [9, 54, 43]). Moreover, its intersection theory has been studied extensively, culminating in the Vafa–Intriligator formula [8, 61, 47], which has, in particular, been applied to the study of the intersection theory of the moduli space of vector bundles on a curve [44].

Quot schemes on curves are not smooth in general. In the following two settings they are always smooth.

1. The $r = 0$ case. The Quot scheme $\mathrm{Quot}_d(V, C) := \mathrm{Quot}_d(V, 0, C)$ of rank zero quotients of V is always smooth of dimension nd. Its Betti numbers are well known [11] and depend only on the rank and degree of V [5, 56]. More recently, the additive structure of the cohomology was explored in [46], where the authors construct a canonical basis for $H^*(\mathrm{Quot}_d(V, C))$ via an action of the shifted Yangian of $\mathfrak{sl}_2$. The derived category of $\mathrm{Quot}_d(V, C)$ is also known to admit a semi-orthogonal decomposition [64, 46].

2. The $C \cong \mathbb{P}^1$ case. Let V be a balanced vector bundle (see Chapter 2 for the definition) on $\mathbb{P}^1$. Then the Quot scheme $\mathrm{Quot}_d(V, r, \mathbb{P}^1)$ parametrising rank r quotients of V is smooth of dimension $r(n-r) + nd$. In the special case where V is the trivial bundle $\mathcal{O}_{\mathbb{P}^1}^n$, this Quot scheme was studied in [63], where the Chow ring is described and the Betti numbers are computed. A closed formula for the generating function of the Poincaré polynomial was later obtained in [18] via the study of Hyperquot schemes on $\mathbb{P}^1$.

These two classes of smooth Quot schemes will be our main objects of study. We also study a relative variant of $\mathrm{Quot}_d(V, C)$, denoted $\mathrm{Quot}_d(V, \varphi)$, where $\varphi \colon X \to Y$ is a family of smooth projective curves; we refer the reader to Chapter 2 for background on these spaces. The thesis is organised into three parts.

1. A study of a presentation of the cohomology ring of the Quot scheme $\mathrm{Quot}_d(V, C)$. This is based on joint work with Alina Marian, and is the content of Chapter 3.

2. A study of a stratification of the relative Quot scheme $\mathrm{Quot}_d(V, \varphi)$. This is based on joint work with Barbara Fantechi and Elisa Vitale, and is presented in Chapter 4.

3. A study of the cohomology of Schur complexes associated to tautological complexes on the Quot scheme $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$. This is based on the preprint [27] with Feiyang Lin and Shubham Sinha, and constitutes Chapter 5.

In the remainder of this chapter, we follow the outline above to summarise the contents of this thesis and highlight our contributions.

1.1 Cohomology ring of $\mathrm{Quot}_d(V, C)$.

Our goal is to determine a presentation of the rational cohomology ring $H^*(\mathrm{Quot}_d(V, C))$. The only known result in this direction is due to [15], where a presentation of the cohomology ring of $\mathrm{Quot}_2(\mathcal{O}_{\mathbb{P}^1}^2, \mathbb{P}^1)$ is computed.

Fix a basis $\{1, \gamma_1, \ldots, \gamma_{2g}, h\}$ of $H^*(C)$. The Chern classes of the universal subbundle $\mathcal{E}$ admit a Künneth decomposition

$$c_i(\mathcal{E}) = a_i \otimes 1 + \sum_{i=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes h \text{ for } 1 \leq i \leq n, \tag{1.2}$$

where a_i, b_i^j, f_i are classes in the cohomology ring $H^*(\mathrm{Quot}_d(V, C))$ for all i and j. Observe, in particular, that $f_1 \in H^0(\mathrm{Quot}_d(V, C))$. These classes generate the cohomology ring $H^*(\mathrm{Quot}_d(V, C))$ as an algebra (see Proposition 3.0.1). Define

$$\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] := \mathbb{Q}[a_1, \ldots, a_n, f_2, \ldots, f_n] \otimes \bigwedge(b_1^1, \ldots, b_1^{2g}, \ldots, b_n^1, \ldots, b_n^{2g}). \tag{1.3}$$

Therefore, there is a surjective homomorphism from $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] \twoheadrightarrow H^*(\mathrm{Quot}_d(V, C))$. Our aim is to describe its kernel, which we denote by $\mathcal{I}_d$.

To construct elements of $\mathcal{I}_d$, we use David Mumford's idea [4] of using vanishing Chern classes of *tautological bundles*. Let L be a line bundle of degree m on C and consider the coherent sheaf $\mathcal{F} \otimes \rho^* L$ on $\mathrm{Quot}_d(V, C) \times C$. Pushing forward along the projection map π to $\mathrm{Quot}_d(V, C)$, the higher direct images vanish, hence by Grauert's theorem [29, Chapter III, Corollary 12.9], we can define a tautological bundle,

$$L^{[d]} := \pi_*(\mathcal{F} \otimes \rho^* L).$$

The rank of $L^{[d]}$ is d, and therefore $c_{d+i}(L^{[d]}) = 0$ for all $i > 0$. Using Grothendieck–Riemann–Roch formula (2.2.4) to write these vanishing Chern classes in terms of a_i, b_i^j, f_i yields elements of $\mathcal{I}_d$; we call them *Mumford relations*. Such relations have been studied, for instance, for moduli spaces of vector bundles on smooth projective curves [37] and for moduli space of sheaves on $\mathbb{P}^2$ [53].

We next introduce a mild generalisation which produces a larger collection of relations. Let Jac be the Jacobian of degree zero line bundles on C, and let $\mathcal{P}$ be a Poincaré line bundle on $\mathsf{Jac} \times C$. Consider the coherent sheaf $\mathcal{F} \otimes \mathcal{P} \otimes \rho^* L$ on $\mathrm{Quot}_d(V, C) \times \mathsf{Jac} \times C$. As before, pushing forward along the projection morphism π to $\mathrm{Quot}_d(V, C) \times \mathsf{Jac}$, the higher direct images vanish, so by Grauert's theorem, we can define a *generalised tautological bundle*

$$\mathsf{F}_m := \pi_*(\mathcal{F} \otimes \mathcal{P} \otimes L). \tag{1.4}$$

(The vector bundle F_m depends on L, but we abuse the notation as the Chern class $c(\mathsf{F}_m) \in H^*(\mathrm{Quot}_d(V, C) \times \mathsf{Jac})$ depends only on $m = \deg L$.) The rank of F_m is d, so $c_{d+i}(\mathsf{F}_m) = 0$ for all $i > 0$. Let $\delta \in H^*(\mathsf{Jac})$. Define the slant product

$$-/\delta \colon H^*(\mathrm{Quot}_d(V, C) \times \mathsf{Jac}) \to H^*(\mathrm{Quot}_d(V, C)); \ \alpha \mapsto \alpha/\delta,$$

where α is paired with the Poincaré dual of δ and pushed forward to obtain α/δ. Using Grothendieck–Riemann–Roch formula (2.2.4), we may express $c(\mathsf{F}_m)/\delta$ as an element in $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$. This procedure produces elements of $\mathcal{I}_d$, which we call *generalised Mumford relations*; Mumford relations are special cases of these relations. Analogous relations have also been defined in the setting of moduli space of vector bundles on smooth projective curves and of moduli spaces of sheaves on $\mathbb{P}^2$ [38]. In the former case, they generate the full ideal of relations [21].

Our main theorem is that when C is isomorphic to $\mathbb{P}^1$, the Mumford relations generate the ideal $\mathcal{I}_d$. Moreover, for a general C, we prove, conditional on the conjectural Lemma 3.2.6, that the generalised Mumford relations generate $\mathcal{I}_d$.

Theorem (3.0.2)**.** *The following statements hold.*

1. *The ideal of relations $\mathcal{I}_d$ of $H^*(\mathrm{Quot}_d(V, \mathbb{P}^1))$ is generated by Chern classes $c_{d+i}(L^{[d]})$, expressed as elements in $\mathbb{Q}[\mathsf{a}, \mathsf{f}]$, for all $L \in \mathrm{Pic}(\mathbb{P}^1)$ and $i > 0$.*

2. *For a general curve C, and assuming the conjectural Lemma 3.2.6, the ideal of relations $\mathcal{I}_d$ of $H^*(\mathrm{Quot}_d(V, C))$ is generated by the Künneth components of Chern classes $c_{d+i}(\mathsf{F}_m)$, expressed as elements in $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$, for all $m \in \mathbb{Z}$ and $i > 0$; equivalently,*

$$\mathcal{I}_d = \langle c_{d+i}(\mathsf{F}_m)/\delta \mid \delta \in H^*(\mathsf{Jac}), m \in \mathbb{Z}, i > 0 \rangle \subset \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}].$$

Remark 1.1.1. We state Theorem 3.0.2 for all integers m. In the proof, it is necessary to consider infinitely many values of m. Nevertheless, we expect that only finitely many values of m (depending on C, n and d) suffice, although at present we do not have an explicit bound.

Remark 1.1.2. The theorem immediately implies that the surjection $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] \twoheadrightarrow H^*(\mathrm{Quot}_d(V, C))$ is a linear isomorphism in degrees strictly less than $2(d - g + 1)$. One may view this as a stabilisation of the cohomology ring $H^*(\mathrm{Quot}_d(V, C))$ to the free algebra $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$. This conclusion is also compatible with what follows from the well known formula (3.41) for the Poincaré polynomial of $\mathrm{Quot}_d(V, C)$.

Approach of proof. Theorem 3.0.2 is proved by analysing the geometry of the nested Quot scheme $\mathrm{Quot}_{d,d+1}(V, C)$ (see Chapter 3). In particular, we exploit its relationship with Quot schemes parametrising quotients of consecutive degrees d and $d+1$ in order to setup an induction argument.

1.2 Stratification of $\mathrm{Quot}_d(V, \varphi)$.

The Quot scheme $\mathrm{Quot}_d(\mathcal{O}_C^n, C)$ carries a $\mathbb{C}^*$–action induced by a lift of the standard $(\mathbb{C}^*)^n$-action on the trivial bundle $\mathcal{O}_C^n$. In this setting, the Białynicki–Birula decomposition [10] yields a stratification of $\mathrm{Quot}_d(\mathcal{O}_C^n, C)$. This stratification is computed explicitly in [11], where it is used to determine the class of $\mathrm{Quot}_d(\mathcal{O}_C^n, C)$ in $\mathrm{K}_0(\mathrm{Var}_k)$ (see Chapter 2 for the definition) and, in particular, its Betti numbers.

Theorem ([11]). *For every $d \geq 0$, the following identity holds in $\mathrm{K}_0(\mathrm{Var}_k)$,*

$$[\mathrm{Quot}_d(\mathcal{O}_C^n, C)] = \sum_{\substack{\underline{m} \in \mathbb{N}^n \\ \sum m_i = d}} [\mathrm{Sym}^{m_1} C \times \ldots \times \mathrm{Sym}^{m_n} C]\, [\mathbb{A}^1]^{\sum_{i=1}^n m_i(n-i)}. \tag{1.5}$$

In [5, 56] it was shown that the expression (1.5) is in fact independent of the vector bundle V. A similar formula for the nested Quot scheme (see Chapter 4 for the definition) was obtained in [50], again using the Białynicki–Birula decomposition.

Theorem ([50]). *For any sequence of positive integers $d_{s+1} \geq \cdots \geq d_1 \geq 0$, the following identity holds in $\mathrm{K}_0(\mathrm{Var}_k)$,*

$$[\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V, C)] = \prod_{k=1}^{s+1} \left[\mathrm{Quot}_{d_k - d_{k-1}}(\mathcal{O}_C^n, C)\right]. \tag{1.6}$$

Our goal is to describe the Białynicki–Birula strata intrinsically, using the geometry of the Quot scheme rather than a torus action. More precisely, we prove a stratification result for the relative Quot scheme for any family $\varphi\colon X \to Y$ of smooth projective varieties, under the assumption that the rank n vector bundle V on X admits a *filtration with line bundle quotients*; that is, V has a filtration by vector subbundles,

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V. \tag{1.7}$$

with rank $V_i = i$ and V_i/V_{i-1} a line bundle for all i.

Consider the relative Quot scheme $\mathrm{Quot}_d(V, \varphi)$. On $\mathrm{Quot}_d(V, \varphi) \times_Y X$, we have the universal exact sequence

$$0 \to \mathcal{E} \to \rho^* V \to \mathcal{F} \to 0 \tag{1.8}$$

where ρ denotes the projection to X. Set $\widetilde{V} := V_{n-1}$ and let $L := V/\widetilde{V}$ be the quotient line bundle. Define

$$\widetilde{\mathcal{E}} := \mathcal{E} \cap \rho^* \widetilde{V}, \quad \widetilde{\mathcal{F}} := \rho^* \widetilde{V}/\widetilde{\mathcal{E}}.$$

Since $\widetilde{\mathcal{F}}$ is not flat in general, we consider its flattening stratification [24, Theorem 5.13],

$$\mathrm{Quot}_d(V, \varphi) = \bigsqcup_{m=0}^{d} \mathrm{Quot}_{d,m}. \tag{1.9}$$

On $\mathrm{Quot}_{d,m} \times_Y X$, the restriction of $\widetilde{\mathcal{F}}$ is flat over $\mathrm{Quot}_{d,m}$ and has degree m along the fibres of φ. This gives rise to a morphism $\eta_m\colon \mathrm{Quot}_{d,m} \to \mathrm{Quot}_m(\widetilde{V}, \varphi)$. Next, consider the restriction of the morphism $\rho^* L \to \mathcal{F}/\widetilde{\mathcal{F}}$ to $\mathrm{Quot}_{d,m} \times_Y X$. Since the restriction of $\mathcal{F}/\widetilde{\mathcal{F}}$ is flat of degree $d - m$ along the fibres of φ, we obtain a morphism $\nu_{d-m}\colon \mathrm{Quot}_{d,m} \to \mathrm{Sym}^{d-m}_Y X$. Combining these, we define

$$\Phi_{d,m} := (\eta_m, \nu_{d-m})\colon \mathrm{Quot}_{d,m} \to \mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y \mathrm{Sym}^{d-m}_Y X.$$

Let $\widetilde{\mathcal{F}}_m$ denote the universal quotient on $\mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y X$, and let $\mathcal{A}_{d-m}$ denote the universal subbundle on $\mathrm{Sym}^{d-m}_Y X \times_Y X$. On $\mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y \mathrm{Sym}^{d-m}_Y X \times_Y X$, we consider the sheaf $\mathcal{A}^\vee_{d-m} \otimes \widetilde{\mathcal{F}}_m$. Pushing this sheaf forward along the projection π to $\mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y \mathrm{Sym}^{d-m}_Y X$, the higher direct images vanish. By Grauert's theorem, we therefore obtain a vector bundle

$$\mathbb{S}_{d,m} := \pi_*(\mathcal{A}^\vee_{d-m} \otimes \widetilde{\mathcal{F}}_m).$$

The following key lemma underlies our main result.

Lemma 1.2.1 (4.0.1)**.** *The morphism $\Phi_{d,m}$ is an affine bundle, and $\mathrm{Quot}_{d,m}$ is isomorphic to the total space of the vector bundle $\mathbb{S}_{d,m}$.*

Our main theorem is as follows.

Theorem (4.0.2)**.** *Let V be a vector bundle of rank n on X filtered with line bundle quotients. The Quot scheme $\mathrm{Quot}_d(V, \varphi)$ admits a stratification by locally closed subvarieties $\mathrm{Quot}_{d,\underline{m}}$ which are affine bundles over $\mathrm{Sym}^{m_1}_Y X \times_Y \ldots \times_Y \mathrm{Sym}^{m_n}_Y X$ where $\underline{m} = (m_1, \ldots, m_n)$ ranges over n-tuples of non-negative integers with $m_1 + \ldots + m_n = d$.*

Formulae (1.5) and (1.6) immediately follow from Theorem 4.0.2.

Remark 1.2.2. The hypothesis that V admits a filtration with line bundle quotients is essential for constructing morphisms $\Phi_{d,m}$. When $Y = \mathrm{pt}$, this condition is automatic, since any vector bundle on smooth projective curves admits such a filtration. Moreover, whenever Y is irreducible, it can stratified so that, over the preimage of each stratum, such a filtration exists for any V.

When $Y = \mathrm{pt}$, we further establish that the integral cohomology $H^*(\mathrm{Quot}_d(V, C), \mathbb{Z})$ is torsion-free. In particular, Theorem 3.0.2 provides a complete description of the cohomology ring.

Proposition 1.2.3 (4.1.5)**.** *The integral cohomology of* $\mathrm{Quot}_d(V, C)$ *is torsion-free.*

1.3 Schur complexes on $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$.

Consider the Quot scheme $\mathrm{Quot}_d(V, r, C)$ parametrising quotients of V on C of rank r and degree d. Given a vector bundle M on $\mathrm{Quot}_d(V, r, C)$, we define the associated *tautological complex* on $\mathrm{Quot}_d(V, r, C)$,

$$M^{[d]} := \pi_*(\mathcal{F} \otimes \rho^* M)$$

where π denotes the projection to $\mathrm{Quot}_d(V, r, C)$. Whenever $M^{[d]}$ is a vector bundle, we refer to it as the tautological bundle associated to M, as in Section 1.1. For a line bundle L, the cohomology of the tautological bundle $L^{[d]}$ has been studied in several settings. The earliest result in this direction is due to [49], where global sections were computed in the case of symmetric products of curves $\mathrm{Sym}^d C$. In fact, a general formula holds, see [39, Proposition 2.21],

$$H^\bullet(\mathrm{Sym}^d C, L^{[d]}) \cong H^\bullet(C, L) \otimes \mathrm{Sym}^{d-1} H^\bullet(C, \mathcal{O}_C). \tag{1.10}$$

The study of Schur bundles associated to tautological bundles was propelled by the conjectural formula in [52],

$$H^\bullet\left(\mathrm{Quot}_d(V, C), \wedge^k L^{[d]}\right) \cong \wedge^k H^\bullet(C, V \otimes L) \otimes \mathrm{Sym}^{d-k} H^\bullet(C, \mathcal{O}_C) \text{ for } 1 \leq k \leq d. \tag{1.11}$$

This was proved partially in [39, 48], and a complete proof was given in [45]; formulae reminiscent of (1.10) and (1.11) also hold for the Hilbert scheme of points $\mathrm{Hilb}^d S$ on a smooth projective surface S; see [58, 40]. The cohomology of the tangent bundle on $\mathrm{Quot}_d(V, C)$ is also known due to [12].

The computation of the cohomology of Schur bundles associated to tautological bundles on $\mathbb{P}^1$ was initiated in [48], where the authors compute the cohomology of exterior and symmetric powers of tautological bundles associated to line bundles on $\mathbb{P}^1$. In particular, the following formulae are conjectured.

Conjecture 1.3.1 ([48])**.** *Let L be a line bundle on $\mathbb{P}^1$, and assume the inequality*

$$k < (nd + n)/(n - r).$$

Then,

$$\chi\left(\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1), \wedge^k L^{[d]}\right) = \binom{n\chi(\mathbb{P}^1, L)}{k},$$
$$\chi\left(\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1), \mathrm{Sym}^k L^{[d]}\right) = \binom{n\chi(\mathbb{P}^1, L) + k - 1}{k}.$$

Conjecture 1.3.1 follows immediately by specialising our main theorem to exterior and symmetric powers. Our main theorem is as follows.

Theorem (5.0.1)**.** *Let λ be a partition satisfying*

$$|\lambda| < (nd + n)/(n - r).$$

For any vector bundle M on $\mathbb{P}^1$, consider the associated Schur complex $\mathbb{S}^\lambda M^{[d]}$ (see Chapter 2 for the definition). In this case, there is a natural isomorphism,

$$H^\bullet\left(\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1), \mathbb{S}^\lambda M^{[d]}\right) \cong \mathbb{S}^\lambda H^\bullet(\mathbb{P}^1, M^{\oplus n}).$$

Remark 1.3.2. For arbitrary values of d, n and r, the bound on the size of the partition λ in Theorem 5.0.1 is sharp. However, for some values of d, n and r, it is possible to get a sharper bound. For example, when $d = 0$, Borel–Weil–Bott theorem (see Chapter 5) gives the optimal result, which is substantially stronger than ours.

To illustrate the sharpness of the general bound, consider the case $n = 2$, $r = 1$ and $d = 2$. By [27, Theorem 1.7], the vector bundle $\wedge^6\mathcal{O}_{\mathbb{P}^1}(4)^{[2]}$ has no higher cohomology. However, the natural morphism

$$\wedge^6 H^0\left(\mathcal{O}_{\mathbb{P}^1}(4)^2\right) \to H^0\left(\mathrm{Quot}_2(\mathcal{O}_{\mathbb{P}^1}^2, 1, \mathbb{P}^1), \wedge^6\mathcal{O}_{\mathbb{P}^1}(4)^{[2]}\right)$$

is a surjection but not an isomorphism; see [27, Example 5.8].

Remark 1.3.3. In [27], Theorem 5.0.1 is established in greater generality, where the trivial bundle $\mathcal{O}_{\mathbb{P}^1}^n$ is replaced by an arbitrary vector bundle V, provided that the Quot scheme $\mathrm{Quot}_d(V, r, \mathbb{P}^1)$ is irreducible. Moreover, the cohomology of duals of certain tautological bundles is also investigated; see [27, Theorem 1.10].

Similar techniques can also be used to study exceptional collections in the bounded derived category $\mathrm{D}^b(\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1))$. We first fix an ordering on the set of all partitions. Given two partitions λ and μ, we write

$$\lambda \prec \mu \text{ if } |\lambda| < |\mu|, \text{ or if } |\lambda| = |\mu| \text{ and } \lambda_i < \mu_i \text{ for some } i. \tag{1.12}$$

Theorem (5.0.2)**.** *Let λ be a partition, and let L be a line bundle of degree m on $\mathbb{P}^1$. For all $m \geq d$, the set*

$$\left\{\mathbb{S}^\lambda L^{[d]} \mid |\lambda| \leq d,\ \lambda_1 < n - r\right\},$$

forms an exceptional collection with respect to the ordering defined above.

Remark 1.3.4. The exceptional collections in Theorem 5.0.2 are not full. For example, consider the case $d = 0$, $n = 2$ and $r = 1$. Then $\mathrm{Quot}_0(\mathcal{O}_{\mathbb{P}^1}^2, 1, \mathbb{P}^1) \cong \mathbb{P}^1$, and it is well known know that $D^b(\mathbb{P}^1) \cong \langle \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\rangle$ [7]. However the exceptional collection given above only produces the subcategory $\langle \mathcal{O}_{\mathbb{P}^1}\rangle$ for all $m \geq 0$.

Approach of proof. Theorems 5.0.1 and 5.0.2 are proved using a geometric construction. We follow the approach of [48] and, for each tautological bundle, construct an embedding of the Quot scheme $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$ into a product of Grassmannians in such a way that the tautological bundle admits a Koszul resolution whose terms are expressed in terms of universal bundles on the Grassmannians. This reduces the problem to an application of the Borel–Weil–Bott theorem (see Chapter 5) together with various properties of Littlewood–Richardson coefficients (see Chapter 2).

Chapter 2

Preliminaries

2.1 Quot schemes

Let $\varphi\colon X \to Y$ be a family of smooth projective curves, and let V be a vector bundle of rank n on X. For $y \in Y$, write $X_y := \varphi^{-1}(y)$ for the fibre of φ over y.

Define a functor $\mathfrak{Quot}_d(V, \varphi)$ from the category of Noetherian schemes to the category of sets as follows. For each Noetherian scheme T, the set $\mathfrak{Quot}_d(V, r, \varphi)(T)$ consists of equivalence class of exact sequences on $X \times_Y T$,

$$0 \to \mathcal{E}_T \to p_T^* V \to \mathcal{F}_T \to 0$$

where $\mathcal{F}_T$ is flat over T, and for every $y \in Y$, the restriction of $\mathcal{F}_T$ to the fibre X_y has rank r and degree d. The equivalence relation is given by the identification of two exact sequences with the same kernel.

Theorem 2.1.1 ([28])**.** *There exists a projective Y-scheme $\mathrm{Quot}_d(V, r, \varphi)$ representing the functor $\mathfrak{Quot}_d(V, r, \varphi)$.*

Assume $Y = \mathrm{Spec}(k)$. In this setting, there exists a non-negative integer $d_0(V, r)$ such that the Quot scheme $\mathrm{Quot}_d(V, r, C)$ is irreducible for all $d \geq d_0(V, r)$ [54]. The Quot scheme is equipped with deformation-obstruction complex $\mathrm{Ext}^\bullet_\pi(\mathcal{E}, \mathcal{F})$ [24], where π denotes the projection morphism to $\mathrm{Quot}_d(V, r, C)$. In particular, when C is isomorphic to $\mathbb{P}^1$ and $V \cong \mathcal{O}_{\mathbb{P}^1}^n$, the Quot scheme is smooth and irreducible of dimension $r(n-r)+nd$. Similarly, in the case $r = 0$ and for a general curve C, the Quot scheme is smooth and irreducible of dimension nd. As we specialise in various ways throughout this thesis, we fix the following notation:

1. If $r = 0$ and for Y arbitrary, we write $\mathrm{Quot}_d(V, \varphi)$.
2. If $r = 0$ and $Y = \mathrm{Spec}(k)$, we write $\mathrm{Quot}_d(V, C)$, or simply $\mathrm{Quot}_d(V)$.
3. If $r \neq 0$ and $Y = \mathrm{Spec}(k)$, we write $\mathrm{Quot}_d(V, r, C)$.

Properties of the Quot scheme. Set $r = 0$. If we further assume $n = 1$, then $\mathrm{Quot}_d(V, \varphi)$ is isomorphic to the relative Hilbert scheme $\mathrm{Hilb}^d_Y(X)$. Since φ is a family of curves, there is an isomorphism

$$\mathrm{Hilb}^d_Y(X) \cong \mathrm{Sym}^d_Y(X). \tag{2.1}$$

The relative Quot scheme $\mathrm{Quot}_d(V, \varphi)$ is related to the relative symmetric product $\mathrm{Sym}^d_Y(X)$ by a proper morphism, called the Quot–to–Chow morphism (defined in [28, Section 6])

$$\sigma\colon \mathrm{Quot}_d(V, \varphi) \to \mathrm{Sym}^d_Y(X). \tag{2.2}$$

When $Y = \mathrm{Spec}(k)$, the morphism σ is known to be flat [26], and the fibres over points of the form $dp \in \mathrm{Sym}^d C$ are normal varieties [34].

The Quot scheme behaves well under base change. More precisely, given a morphism $\psi\colon Z \to Y$, there is an isomorphism of Z-schemes

$$\mathrm{Quot}_d(p_X^* V, \tau) \cong \mathrm{Quot}_d(V, \varphi) \times_Y Z \tag{2.3}$$

where $\tau\colon X \times_Y Z \to Z$ denotes the pullback of φ and $p_X\colon X \times_Y Z \to X$ is the projection.

2.2 Chern classes and pushforward formulae

Chern classes. Let X be a smooth projective variety, and let E be a vector bundle of rank n on X. Denote by $\pi\colon \mathbb{P}_X(E) \to X$ the projective bundle parametrising one-dimensional quotients of E. It comes equipped with the tautological bundle $\mathcal{O}(-1)$; we write $\mathcal{O}(1)$ for its dual. The first Chern class $c_1(\mathcal{O}(1)) \in H^2(X)$ is the Poincaré dual of the divisor associated to $\mathcal{O}(1)$. Set

$$c_1(\mathcal{O}(1)) = \lambda.$$

Let $c_i(E)$ denote the i-th Chern class of E, and define the homogenised total Chern class of E by

$$c(E, z) := \sum_{i=0}^{n} (-1)^i c_i(E) z^{n-i}.$$

Following Grothendieck, the total Chern class relates to the first Chern $c_1(\mathcal{O}(1))$ on $\mathbb{P}_X(E)$ via the pushforward identity (see [25, Example 3.2.1]).

Proposition 2.2.1. *We have the identity*

$$\pi_*\left(\frac{1}{z-\lambda}\right) = \frac{1}{c(E, z)}. \tag{2.4}$$

Proof. By [25, Example 3.2.1], we get that the pushforward of total Segre class of $\mathcal{O}(-1)$ is precisely the total Segre class of $E^\vee$

$$\pi_*(s(\mathcal{O}(-1))) = s(\mathrm{E}^\vee).$$

In particular, $\pi_*(\lambda^{n-1+i}) = s_i(\mathrm{E}^\vee)$ and $\pi_*(\lambda^j) = 0$ for all $j < n-1$. Thus

$$\pi_*\left(\frac{1}{z-\lambda}\right) = \sum_{i=0}^{\infty} \frac{s_i(\mathrm{E}^\vee)}{z^{n+i}} = \frac{1}{z^n} \frac{1}{1 - \frac{c_1(\mathrm{E})}{z} + \ldots + (-1)^n \frac{c_n(\mathrm{E})}{z^n}} = \frac{1}{c(\mathrm{E}, z)}. \qquad \square$$

Remark 2.2.2. We also occasionally use the more standard notation $c_t(E) = \sum_{i=0}^n c_i(E) t^i$ for the total Chern class, and sometimes write simply $c(E)$.

The pushforward identity (2.4) admits a natural generalisation. Let $P(\lambda)$ be a polynomial in λ. Then

$$\pi_*\left(\frac{P(\lambda)}{z-\lambda}\right) = \left.\frac{P(z)}{c(E,z)}\right|_{z<0}, \tag{2.5}$$

where the right hand side is understood as the Laurent expansion in z, keeping only the terms with negative degree.

Virtual projective bundle. Let E and F be two vector bundles on X, and let $\phi\colon F \to E$ be a morphism. On $\mathbb{P}_X(E)$ consider the composition

$$\pi^*F \to \pi^*E \twoheadrightarrow \mathcal{O}(1).$$

This defines a section s of the vector bundle $\pi^*F^\vee \otimes \mathcal{O}(1)$ on $\mathbb{P}_X(E)$. We denote by $\mathbb{P}_X(E-F)$ the zero locus of s, and refer to it as a *virtual projective bundle*. It comes with a natural closed embedding $\iota\colon \mathbb{P}_X(E-F) \hookrightarrow \mathbb{P}_X(E)$. Let

$$c_1(\iota^*\mathcal{O}(1)) = \mu.$$

Assuming that $\mathbb{P}_X(E-F)$ has the expected dimension $\dim X + \operatorname{rank} E - \operatorname{rank} F - 1$, we obtain the following pushforward formula.

Proposition 2.2.3. *Write $\tau = \pi \circ \iota$. Then,*

$$\tau_*\left(\frac{\mu^k}{z-\mu}\right) = \left.\frac{z^k c(F,z)}{c(E,z)}\right|_{z<0}$$

Proof. Since the expected dimension coincides with the actual dimension, the fundamental class of $\mathbb{P}_X(E-F)$ in $H^*(\mathbb{P}_X(E))$ is

$$[\mathbb{P}_X(E-F)] = c_{\text{top}}(F^\vee \otimes \mathcal{O}(1)).$$

Using the pushforward identity (2.5), we compute,

$$\tau_*\left(\frac{\mu^k}{z-\mu}\right) = \pi_*\left(\frac{\lambda^k \cdot c_{\text{top}}(F^\vee \otimes \mathcal{O}(1))}{z-\lambda}\right) = \pi_*\left(\frac{\lambda^k \cdot c(F,\lambda)}{z-\lambda}\right) = \left.\frac{z^k c(F,z)}{c(E,z)}\right|_{z<0},$$

as claimed. □

Grothendieck–Riemann–Roch Formula. Let C be a smooth projective curve of genus g, and let L a line bundle of degree d on C. The Riemann–Roch formula

$$\chi(L) = d + 1 - g$$

computes the holomorphic Euler characteristic of L. The Hirzebruch–Riemann–Roch formula generalises this to a vector bundle E on a smooth projective variety X, giving

$$\chi(E) = \int_X \operatorname{ch}(E) \cdot \operatorname{td}(X),$$

in terms of the Chern character $\operatorname{ch}(E)$ of E and the Todd class $\operatorname{td}(X)$; see [25, Chapter 3].

Grothendieck–Riemann–Roch formula further generalises Hirzebruch–Riemann–Roch formula to morphisms between varieties.

Theorem 2.2.4 (Grothendieck–Riemann–Roch). *Let $f\colon X \to Y$ be a projective morphism between smooth projective varieties with relative tangent bundle T_f. Let $\mathcal{F}$ be a coherent sheaf $\mathcal{F}$ on X such that $R^i f_*\mathcal{F} = 0$ for all $i \geq 1$. Then the Chern character of the pushforward $f_*\mathcal{F}$ satisfies*

$$\operatorname{ch}(f_*\mathcal{F}) = f_*\left(\operatorname{ch}(\mathcal{F}) \cdot \operatorname{td}(T_f)\right).$$

Hirzebruch–Riemann–Roch formula is recovered from Grothendieck–Riemann–Roch formula by taking Y to be a point; see [25, Chapter 15] for the general version of Theorem 2.2.4.

2.3 Stacks of vector bundles on smooth projective curves

Cohomology of smooth algebraic stacks. Let $\mathfrak{Y}$ be a smooth algebraic stack of finite type over $\mathbb{C}$, and let $Y \to \mathfrak{Y}$ be a smooth atlas. For each $q \geq 0$, set $\mathfrak{Y}_q := Y \times_{\mathfrak{Y}} \cdots \times_{\mathfrak{Y}} Y$, the q–fold fibre product. The rational singular cohomology of $\mathfrak{Y}$ may be defined (see [31]) via the spectral sequence,

$$E_1^{p,q} = H^p(\mathfrak{Y}_q, \mathbb{Q}) \implies H^{p+q}(\mathfrak{Y}, \mathbb{Q}).$$

One can also approximate the cohomology $\mathfrak{Y}$ by restricting to open substacks of increasing codimension.

Lemma 2.3.1 ([59]). *Let $\mathfrak{Y}$ be a smooth algebraic stack of finite type, and let $\mathfrak{U} \subset \mathfrak{Y}$ be an open substack whose complement has codimension n. Then, for all $j < 2n$, the restriction map induces an isomorphism*

$$H^j(\mathfrak{Y}) \cong H^j(\mathfrak{U}).$$

Now let $\mathfrak{X}$ be a smooth algebraic stack, locally of finite type over $\mathbb{C}$. Following [31, 59], we define the rational cohomology $H^*(\mathfrak{X}, \mathbb{Q})$ as the inverse limit of the cohomologies of all open substacks of finite type,

$$H^*(\mathfrak{X}, \mathbb{Q}) := \varprojlim_{\mathfrak{U} \subset \mathfrak{X} \text{ f.t. open}} H^*(\mathfrak{U}, \mathbb{Q}). \tag{2.6}$$

Stack of vector bundles. Let $\operatorname{Bun}_{\mathrm{GL}_n(\mathbb{C})}$ denote the stack of principal $\mathrm{GL}_n(\mathbb{C})$-bundles on a smooth projective curve C of genus g. Equivalently, it parametrises vector bundles of rank n on C, and we adopt this interpretation.

Theorem 2.3.2 ([6]). *The stack $\operatorname{Bun}_{\mathrm{GL}_n(\mathbb{C})}$ is a smooth algebraic stack of dimension $(g-1)n^2$.*

The connected components of $\operatorname{Bun}_{\mathrm{GL}_n(\mathbb{C})}$ are in bijection with the fundamental group $\pi_1(\mathrm{GL}_n(\mathbb{C}))$ [32]; these components record the degree of the corresponding vector bundles.

Proposition 2.3.3. *There is an isomorphism $\pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{Z}$. Consequently, the stack decomposes as*

$$\mathrm{Bun}_{\mathrm{GL}_n(\mathbb{C})} = \bigsqcup_d \mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})},$$

where $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}$ is the component parametrising vector bundles of rank n and degree d. Each of these connected components is locally of finite type.

Proof. By [16, Lemma 2.6], there is a fibration $\mathrm{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^{n-1} \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^n \setminus \{0\}$. Applying the long exact sequence in homotopy and using that a sphere S^k is $(k-1)$-connected, we get the required isomorphism. The decomposition then follows immediately, and the fact that $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}$ is locally of finite type is proven in [2]. $\square$

Since $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}$ is a smooth, there exists a universal bundle $\mathbb{E}$ on $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})} \times C$. Fix a basis $\{1, \gamma_1, \ldots, \gamma_{2g}, h\}$ for $H^*(C)$. Then the Chern classes of $\mathcal{E}$ admit a Künneth decomposition

$$c_i(\mathbb{E}) = a_i \otimes 1 + \sum_{j=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes h.$$

Collecting the Künneth components in $H^*(\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})})$, one defines a free graded algebra $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$ as in (1.3).

Theorem 2.3.4 ([59])**.** *The cohomology ring $H^*(\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})})$ is isomorphic to $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$.*

Stack of globally generated bundles on $\mathbb{P}^1$. Let $\mathcal{V}_d$ denote the open substack of $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}$ parametrising globally generated vector bundles of rank n and degree d on $\mathbb{P}^1$.

Theorem 2.3.5 ([13])**.** *The stack $\mathcal{V}_d$ is a smooth global quotient stack of dimension $-n^2$.*

Since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, any vector bundle $E \in \mathcal{V}_d$ admits an isomorphism

$$E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with integers $0 \leq a_1 \leq \ldots \leq a_n$. We then say that E has splitting type $\overline{a} = (a_1 \leq \ldots \leq a_n)$. This yields a stratification

$$\mathcal{V}_d = \bigsqcup_{\overline{a}} \Sigma_{\overline{a}},$$

where $\Sigma_{\overline{a}}$ is the substack of bundles of splitting type $\overline{a}$. For each d, there is a unique splitting type, called *the balanced splitting type*, characterised by $|a_1 - a_n| \leq 1$.

Proposition 2.3.6. *The stratum $\Sigma_{\overline{a}}$ corresponding to the balanced splitting type is the unique open and dense stratum in $\mathcal{V}_d$.*

Proof. For each splitting type $\overline{a}$, the stratum $\Sigma_{\overline{a}}$ is isomorphic to the classifying stack $BG_{\overline{a}}$ where

$$G_{\overline{a}} \cong \mathrm{Aut}\left(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)\right).$$

Note that there are isomorphisms

$$\operatorname{Aut}(\mathcal{O}_{\mathbb{P}^1}(a)^k) \cong \mathrm{GL}_k(\mathbb{C}) \text{ and } \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(a)^k, \mathcal{O}_{\mathbb{P}^1}(a+t)^j) \cong \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}^k, \mathcal{O}_{\mathbb{P}^1}(t)^j).$$

Suppose the splitting type $\overline{a}$ is balanced, and write $\overline{a} = (b, \ldots, b, b+1, \ldots, b+1)$ with k copies of b. Then

$$\dim \Sigma_{\overline{a}} = -\dim G_{\overline{a}} = -(k^2 + (n-k)^2 + 2k(n-k)) = -n^2 = \dim \mathcal{V}_d.$$

(Observe that b does not appear in the above computation.) We can similarly verify that every other stratum is of higher codimension. □

2.4 Grothendieck ring of varieties

Let Var_k denote the set of isomorphism classes of irreducible varieties over $k = \mathbb{C}$. Consider the free abelian group generated by Var_k, modulo the cut–and–paste relation

$$[X] = [Z] + [X \setminus Z]$$

for every closed subvariety $Z \subset X$. The resulting group is denoted $\mathrm{K}_0(\mathrm{Var}_k)$ and is called the *Grothendieck group of varieties*. It becomes a ring by setting

$$[X] \cdot [Y] := [X \times_k Y]$$

for any two varieties X and Y. The class of the affine line $\mathbb{A}^1_k$ plays a distinguished role and is denoted $\mathbb{L}$.

Given locally closed subvarieties $Z_1, \ldots, Z_k \subset X$, the cut–and–paste relation

$$[\cup_i Z_i] = [Z_1] + \ldots [Z_k]$$

holds in $\mathrm{K}_0(\mathrm{Var}_k)$. Moreover, if $\varphi \colon X \to Y$ is a Zariski locally trivial fibration with fibre F, then we have the identity (see [17])

$$[X] = [F] \cdot [Y] \,. \tag{2.7}$$

Example 2.4.1. Smooth projective varieties $\mathbb{P}^n$ and $\mathrm{Gr}(r,n)$ admit cellular decompositions, hence stratifications by affine spaces; this yields explicit expressions for their classes in $\mathrm{K}_0(\mathrm{Var}_k)$,

$$[\mathbb{P}^n_k] = \mathbb{L}^n + \mathbb{L}^{n-1} + \ldots + 1 \text{ and } [\mathrm{Gr}(r,n)] = \frac{(\mathbb{L}^n - 1) \cdot (\mathbb{L}^{n-1} - 1) \cdot \cdots \cdot (\mathbb{L}^{r+1} - 1)}{(\mathbb{L}^{n-r} - 1) \cdot \cdots \cdot (\mathbb{L} - 1)}.$$

Example 2.4.2. Consider the blow-up $\mathrm{Bl}_Z X$ of X along a subvariety Z, with exceptional divisor E. In $\mathrm{K}_0(\mathrm{Var}_k)$, one has the identity

$$[\mathrm{Bl}_Z(X)] = [X] - [Z] + [E].$$

Universal property of $\mathrm{K}_0(\mathrm{Var}_k)$ and relation to other rings. The ring $\mathrm{K}_0(\mathrm{Var}_k)$ satisfies a *universal property*: any additive invariant of varieties factors uniquely through it. More precisely, given a map $\lambda : \mathrm{Var}_k \to G$ to an abelian group G that satisfies $\lambda(\emptyset) = 0$ and

$$\lambda(X) = \lambda(Z) + \lambda(X \setminus Z)$$

whenever $Z \subset X$ is a closed subvariety of X, there exists a unique group homomorphism $\bar{\lambda} : \mathrm{K}_0(\mathrm{Var}_Y) \to G$ with $\bar{\lambda}([X]) = \lambda(X)$. In particular, the compactly supported Euler characteristic

$$\chi_c(X) := \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbb{Q}} H_c^i(X, \mathbb{Q}),$$

which is an additive invariant, factors through $\mathrm{K}_0(\mathrm{Var}_k)$. Another important example of an additive invariant is the *virtual Hodge polynomial*.

Theorem 2.4.3 ([19],[20])**.** *There is an assignment* $\mathrm{MHD} : \mathrm{Var}_{\mathbb{C}} \to \mathbb{Z}[u, v]$*, such that for X smooth and projective*

$$\mathrm{MHD}(X) := \sum_{p,q} \dim_{\mathbb{C}} H^p(X, \Omega_X^p) u^p v^q$$

is the Hodge polynomial.

Consequently, knowing the class of a smooth projective variety X in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ suffices to compute its Hodge numbers, and hence Betti numbers.

Remark 2.4.4. Every variety admits a stratification by smooth locally closed subvarieties. Therefore, for the purposes of cut–and–paste constructions, one may equivalently work with smooth varieties. Likewise, replacing varieties by schemes of finite type does not change the resulting Grothendieck ring [16, Lemma 2.12].

Białynicki-Birula decomposition. Let X be a smooth projective variety equipped with a $\mathbb{C}^*$-action. A fundamental tool for understanding both the cohomology of X and its class in $\mathrm{K}_0(\mathrm{Var}_k)$ is the following theorem.

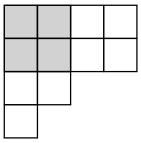
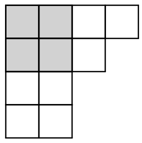
Theorem 2.4.5 ([10])**.** *Let X be a smooth projective variety with a $\mathbb{C}^*$-action, and let $F_1, \ldots, F_k$ be the connected components of the fixed locus $X^{\mathbb{C}^*}$. Then there exists a decomposition $X = \sqcup_i F_i^+$, such that each $F_i^+ \to F_i$ is an affine bundle.*

2.5 Schur bundles and complexes

Partitions and integer sequences. An *integer partition*, or simply a partition, is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers. Partitions are conveniently represented by Young diagrams. Given a partition λ, its *conjugate partition* $\lambda^\dagger$ is obtained by interchanging rows and columns.

Definition 2.5.1 (Rank of an integer partition)**.** The *rank* of a partition λ is the side length of the largest square contained in its Young diagram.

Example 2.5.2. Let $\lambda = (5, 4, 2, 1)$. Then $\lambda^\dagger = (4, 3, 2, 2, 1)$. Their Young diagrams are as follows.

λ:  $\lambda^\dagger$: 

The largest square contained each of these Young diagrams has side length 2. Hence $\operatorname{rank} \lambda = \operatorname{rank} \lambda^\dagger = 2$.

Schur bundles. Let $V := \mathbb{C}^n$ be a complex vector space. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, there is an irreducible polynomial representation $\mathbb{S}^\lambda V$ of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ. Its character is the Schur polynomial $s_\lambda(x_1, \dots, x_k)$.

Theorem 2.5.3 (Hook–content formula)**.** *The dimension of* $\mathbb{S}^\lambda V$ *is*

$$\dim \mathbb{S}^\lambda V = s_\lambda(1, \dots, 1) = \prod_c \frac{n + \mathrm{content}(c)}{\mathrm{hook}(c)},$$

where the product runs over all cells $c = (i, j)$ *in the Young diagram of* λ*; here* $\mathrm{content}(c) = j - i$*, and the hook length is* $\mathrm{hook}(c) = 1 + (\lambda_i - j) + (\lambda^\dagger_j - i)$*.*

Given two partitions λ and μ, the tensor product of the corresponding polynomial representations $\mathbb{S}^\lambda V$ and $\mathbb{S}^\mu V$ decomposes as

$$\mathbb{S}^\lambda V \otimes \mathbb{S}^\mu V = \bigoplus_\nu (\mathbb{S}^\nu V)^{\oplus c^\nu_{\lambda,\mu}}, \tag{2.8}$$

where $c^\nu_{\lambda,\mu}$ are the *Littlewood–Richardson coefficients*, computable via the *Littlewood–Richardson rule.*

Remark 2.5.4. More generally, if $\eta = (\eta_1, \dots, \eta_n)$ is any non-increasing sequence of integers, one can define an irreducible rational representation $\mathbb{S}^\eta V$ of $\mathrm{GL}_n(\mathbb{C})$. Choose an integer $k \geq -\eta_n$, so that $\eta + (k)^n := (\eta_1 + k, \dots, \eta_n + k)$ is a partition. Then

$$\mathbb{S}^\eta V = \det(V)^{-k} \otimes \mathbb{S}^{\eta + (k)^n} V.$$

The decomposition (2.8) extends to non-increasing sequences of integers. Given η, ρ, χ non-increasing integer sequences, choose k, l so that $\eta + (k)^n, \rho + (l)^n$ and $\chi + (k+l)^n$ are partitions, then the Littlewood–Richardson coefficient $c^\chi_{\eta,\rho}$ is given by the identity

$$c^\chi_{\eta,\rho} = c^{\chi + (k+l)^n}_{\eta + (k)^n, \rho + (l)^n}. \tag{2.9}$$

This invariance under twist is very useful in computations.

Proposition 2.5.5. *For non-increasing integer sequences* η, ρ *and* χ*, as in remark 2.5.4, one has* $c^\chi_{\eta,\rho} \neq 0$ *if and only if* $c^\rho_{\chi,-\eta} \neq 0$*.*

Proof. The Littlewood–Richardson coefficients $c^\chi_{\eta,\rho}$ are precisely the dimension of the vector space

$$\mathrm{Hom}(\mathbb{S}^\chi V, \mathbb{S}^\eta V \otimes \mathbb{S}^\rho V) \cong \mathrm{Hom}(\mathbb{S}^\rho V, \mathbb{S}^\chi V \otimes \mathbb{S}^{-\eta} V)^\vee.$$ ∎

Two further identities that are frequently used in computation are the following.

Proposition 2.5.6. *Let V and W be two complex vector spaces. Then the following identities hold.*

1. *The ith exterior power of $V \otimes W$ decomposes as*

$$\bigwedge^{i} V \otimes W = \bigoplus_{|\mu|=i} \mathbb{S}^{\mu} V \otimes \mathbb{S}^{\mu^{\dagger}} W, \tag{2.10}$$

   where the direct sum ranges over all partitions μ of size i. This identity is also known as Cauchy's formula*.*

2. *For a partition μ, we have*

$$\mathbb{S}^{\mu}(V \oplus W) = \bigoplus_{\alpha,\beta} \left(\mathbb{S}^{\alpha} V \otimes \mathbb{S}^{\beta} W\right)^{\oplus c^{\mu}_{\alpha,\beta}}. \tag{2.11}$$

   where the sum runs over all partitions α, β with $c^{\mu}_{\alpha,\beta} \neq 0$.

Everything above carries over verbatim to vector bundles. For a vector bundle V on a scheme X and a non-increasing integer sequence η, the bundle $\mathbb{S}^{\eta} V$ is called the *Schur bundle* of V.

Littlewood–Richardson rule. Littlewood–Richardson rule is a combinatorial rule for computing the Littlewood–Richardson coefficients. We record a few standard consequences.

Proposition 2.5.7. *Let α, β and μ be partitions such that $c^{\mu}_{\alpha,\beta} \neq 0$. Then*

1. $|\alpha| + |\beta| = |\mu|$,

2. $c^{\mu}_{\alpha,\beta} = c^{\mu^{\dagger}}_{\alpha^{\dagger},\beta^{\dagger}}$,

3. *Weyl's inequality: for all $i, j \geq 1$,*

$$\alpha_i + \beta_j \geq \mu_{i+j-1}, \tag{2.12}$$

4. *The first dominance inequality: $\mu \leq \alpha + \beta$, meaning*

$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \alpha_i + \beta_i \text{ for all } j \geq 1,$$

5. *The second dominance inequality: $\alpha \cup \beta \leq \mu$, giving*

$$\sum_{i=1}^{j} \alpha_i + \beta_i \leq \sum_{i=1}^{2j} \mu_i \text{ for all } j \geq 1.$$

Proof. See [48] for the proofs. $\square$

Schur complexes. Let λ be a partition, and let π_λ be the corresponding irreducible representation of the symmetric group $\mathfrak{S}_n$. For a perfect complex $A^\bullet$, one defines the Schur complex

$$\mathbb{S}^\lambda(A^\bullet) = (\pi_\lambda \otimes (A^\bullet)^{\otimes n})^{\mathfrak{S}_n}.$$

Tensor products of complexes respect quasi-isomorphisms, and in characteristic 0 taking $\mathfrak{S}_n$-invariants is exact, so this construction is well-defined on the derived category; see [23, Page 302 between Theorems 9.11.4 and 9.11.5].

When $A^\bullet$ has length two, there is an explicit description due to [65]. Suppose $\Phi\colon A_1 \to A_2$ is a morphism of vector bundles on a locally Noetherian scheme X. One defines a cohomological Schur complex $(\mathbb{S}^\lambda\Phi)^\bullet$, whose degree t term is indexed by partitions $\nu \subset \lambda$ of size t,

$$(\mathbb{S}^\lambda\Phi)^t = \bigoplus_{\nu\subseteq\lambda, |\nu|=t} \mathbb{S}^{\lambda/\nu}A_1 \otimes \mathbb{S}^{\nu^\dagger}A_2, \tag{2.13}$$

where $\mathbb{S}^{\lambda/\nu}A_1 = \oplus_\mu(\mathbb{S}^\mu A_1)^{\oplus c^\lambda_{\nu\mu}}$. The first few terms of the complex $(\mathbb{S}^\lambda\Phi)^\bullet$ are

$$\mathbb{S}^\lambda A_1 \to \mathbb{S}^{\lambda/(1)}A_1 \otimes A_2 \to \mathbb{S}^{\lambda/(2)}A_1 \otimes \wedge^2 A_2 \oplus \mathbb{S}^{\lambda/(1^2)}A_1 \otimes \mathrm{Sym}^2 A_2 \to \cdots$$

There is also a homological version $(\mathbb{S}^\lambda\Phi)^\bullet$, whose degree $(-t)$ term is

$$(\mathbb{S}^\lambda\Phi)_t = \bigoplus_{\nu\subseteq\lambda, |\nu|=q} \mathbb{S}^{\nu^\dagger}A_1 \otimes \mathbb{S}^{\lambda/\nu}A_2, \tag{2.14}$$

where $\mathbb{S}^{\lambda/\nu}A_2 = \oplus_\mu(\mathbb{S}^\mu A_2)^{\oplus c^\lambda_{\nu\mu}}$. The first few terms of the complex $(\mathbb{S}^\lambda\Phi)^\bullet$ are

$$\cdots \to \mathrm{Sym}^2 A_1 \otimes \mathbb{S}^{\lambda/(1^2)}A_2 \oplus \wedge^2 A_1 \otimes \mathbb{S}^{\lambda/(2)}A_2 \to A_1 \otimes \mathbb{S}^{\lambda/(1)}A_2 \to \mathbb{S}^\lambda A_2.$$

The cohomological and homological constructions are related by the identity $\mathbb{S}^\lambda(\Phi)_\bullet = \mathbb{S}^{\lambda^\dagger}(\Phi)^\bullet[\ell]$ for $\ell = |\lambda|$.

Proposition 2.5.8. *Given a short exact sequence of vector bundles*

$$0 \to A_1 \xrightarrow{\Psi} A_2 \xrightarrow{\Phi} A_3 \to 0. \tag{2.15}$$

The induced sequences $(\mathbb{S}^\lambda\Psi)_\bullet \to \mathbb{S}^\lambda A_3 \to 0$ *and* $0 \to \mathbb{S}^\lambda A_1 \to (\mathbb{S}^\lambda\Phi)^\bullet$ *are exact.*

Proof. Since A_3 is locally free, the short exact sequence (2.15) splits locally. As acyclicity can be checked locally, [1, Corollary V.1.15] implies that the Schur complex $(\mathbb{S}^\lambda\Psi)_\bullet$ is acyclic. Moreover, this shows that the cokernel agrees locally with $\mathbb{S}^\lambda A_3$. Functoriality of Schur complexes then implies that these local isomorphisms glue, and hence the cokernel is isomorphic to $\mathbb{S}^\lambda A_3$. The second statement follows by applying the same argument to the dual of (2.15) and then dualising the resulting complex. ∎

Chapter 3

Cohomology of Quot schemes of rank zero quotients on curves

Let C be a smooth projective curve. In this chapter, we study the cohomology ring of the Quot scheme $\mathrm{Quot}_d(V,C)$, which parametrises rank zero quotients of the vector bundle V of rank n on C. Our main result is an explicit presentation of the rational cohomology ring $H^*(\mathrm{Quot}_d(V,\mathbb{P}^1))$, while for a general curve C we obtain a similar presentation, conditional on a conjectural lemma. As the integral cohomology of $\mathrm{Quot}_d(V,C)$ is torsion-free (see Chapter 4), this yields a complete description of the cohomology ring. In the second part of the chapter, we focus entirely on $H^*(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1},\mathbb{P}^1))$ and explain the geometric origin of certain relations.

Recall from Section 1.1 that on $\mathrm{Quot}_d(V,C)\times C$, we have the universal sequence

$$0\to\mathcal{E}\to\rho^*V\to\mathcal{F}\to 0$$

where ρ denotes the projection to C. Consider the classes a_i, b_i^j, f_i arising as Künneth components in the decomposition (1.2) of Chern classes of $\mathcal{E}$.

Proposition 3.0.1. *The classes a_i, b_i^j, f_i for $1\le i\le n$ and $1\le j\le 2g$ multiplicatively generate the ring $H^*(\mathrm{Quot}_d(V,C))$.*

Proof. On $\mathrm{Quot}_d(V,C)\times\mathrm{Quot}_d(V,C)\times C$, consider the two universal sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \rho^*V \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}_2 \arrow[r] & \rho^*V \arrow[r] & \mathcal{F}_2 \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

The induced morphism $\mathcal{E}_1\to\rho^*V\to\mathcal{F}_2$ determines a section of the vector bundle $\pi_*(\mathcal{E}_1^\vee\otimes\mathcal{F}_2)$, where π is the projection to $\mathrm{Quot}_d(V,C)\times\mathrm{Quot}_d(V,C)$. The vanishing locus of this section is precisely the diagonal Δ. Since Δ has the expected codimension, one has the identity $[\Delta]=c_{\mathrm{top}}(\pi_*(\mathcal{E}_1^\vee\otimes\mathcal{F}_2))$ for its fundamental class in $H^*(\mathrm{Quot}_d(V,C))\otimes H^*(\mathrm{Quot}_d(V,C))$. By Grothendieck–Riemann–Roch formula (2.2.4), the top Chern class $c_{\mathrm{top}}(\pi_*(\mathcal{E}_1^\vee\otimes\mathcal{F}_2))$ can be expressed in terms of the Künneth components of Chern classes of $\mathcal{E}_1$ and $\mathcal{E}_2$. The standard argument, as in [62, Theorem 2.1], then implies that these components generate the cohomology ring. $\square$

Recall from Section 1.1 the ideal $\mathcal{I}_d \subset \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$ that arises as the kernel of surjection $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] \to H^*(\mathrm{Quot}_d(V, C))$. The Mumford relations are obtained by expressing the Chern classes $c_{d+i}(L^{[d]})$, for all $i > 0$, as polynomials in $\mathsf{a}, \mathsf{b}, \mathsf{f}$. Meanwhile, the generalised Mumford relations are obtained by taking slant products of the Chern classes $c_{d+i}(\mathsf{F}_m)$, again for all $i > 0$, with $\delta \in H^*(\mathsf{Jac})$, and expressing $c_{d+i}(\mathsf{F}_m)/\delta$ as elements of $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}]$. In this way, one obtains polynomials in $\mathsf{a}, \mathsf{b}, \mathsf{f}$ lying in $\mathcal{I}_d$.

Our main theorem asserts that Mumford relations generate $\mathcal{I}_d$ when $C \cong \mathbb{P}^1$. For a general curve, we prove, assuming the conjectural Lemma 3.2.6, that the generalised Mumford relations generate $\mathcal{I}_d$.

Theorem 3.0.2. *The following statements hold.*

1. *The ideal of relations $\mathcal{I}_d$ of $H^*(\mathrm{Quot}_d(V, \mathbb{P}^1))$ is generated by Chern classes $c_{d+i}(L^{[d]})$, expressed as polynomials in a_i, f_i, for all $L \in \mathrm{Pic}(\mathbb{P}^1)$.*

2. *For a general curve C, and assuming the conjectural Lemma 3.2.6, the ideal of relations $\mathcal{I}_d$ of $H^*(\mathrm{Quot}_d(V, C))$ is generated by the Künneth components of Chern classes $c_{d+i}(\mathsf{F}_m)$, expressed as polynomials in a_i, b_i^j, f_i for $m \in \mathbb{Z}$ and $i > 0$; equivalently,*
$$\mathcal{I}_d = \langle c_{d+i}(\mathsf{F}_m)/\delta \mid \delta \in H^*(\mathsf{Jac}), m \in \mathbb{Z}, i > 0 \rangle \subset \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}].$$

Plan of proof. The proof proceeds by induction on d. The base case $d = 1$ relies on the isomorphism $\mathrm{Quot}_1(V, C) \cong \mathbb{P}_C(V)$, while the induction step requires a detailed study of the geometry of nested Quot scheme.

The nested Quot scheme $\mathrm{Quot}_{d,d+1}(V, C)$ (see Section 3.1) fits in the following diagram, together with the natural morphisms to successive Quot schemes as well as to the curve C.

$$\begin{tikzcd}
 & \mathrm{Quot}_{d,d+1}(V,C) \arrow[dl, "p_-"'] \arrow[d, "p_C"] \arrow[dr, "p_+"] & \\
\mathrm{Quot}_d(V,C) & C & \mathrm{Quot}_{d+1}(V,C)
\end{tikzcd} \tag{3.2}$$

Assuming the theorem for $H^*(\mathrm{Quot}_d(V, C))$, let $\mathcal{I}_{d,d+1}$ denote the pullback of $\mathcal{I}_d$ along p_-. The goal is to push $\mathcal{I}_{d,d+1}$ forward to $\mathrm{Quot}_{d+1}(V, C)$ and show that the resulting ideal is generated by generalised Mumford relations and coincides with $\mathcal{I}_{d+1}$.

Since pushforward morphism is defined only at the level of cohomology, we construct a lift to the cohomology rings of suitable stacks of vector bundles in order to define a pushforward for polynomials in $\mathsf{a}, \mathsf{b}, \mathsf{f}$. Concretely, we associate to $\mathrm{Quot}_d(V, C)$ the stack $\mathcal{B}_d$ together with a surjection $H^*(\mathcal{B}_d) \to H^*(\mathrm{Quot}_d(V, C))$ whose kernel is $\mathcal{I}_d$. Similarly, we associate to $\mathrm{Quot}_{d,d+1}(V, C)$ the nested stack $\mathcal{B}_{d,d+1}$, such that $\mathcal{I}_{d,d+1}$ is the kernel of a surjection $H^*(\mathcal{B}_{d,d+1}) \to H^*(\mathrm{Quot}_{d,d+1}(V, C))$. These stacks fit together in a diagram similar to (3.2).

We then define an $H^*(\mathcal{B}_{d+1})$–linear morphism $\Phi\colon H^*(\mathcal{B}_{d,d+1}) \to H^*(\mathcal{B}_{d+1})$ which lifts the pushforward p_{+*} in cohomology and maps $\mathcal{I}_{d,d+1}$ surjectively onto $\mathcal{I}_{d+1}$. The remaining task is to show that every element of $\Phi(\mathcal{I}_{d,d+1})$ is of the required form; this is accomplished using the pushforward formula (3.23) together with conjectural Lemma 3.2.6.

3.1 Nested Quot scheme and pushforward formulae

The goal of this section is to introduce the nested Quot scheme and set up the induction argument.

Nested Quot scheme on curves. Define the nested Quot scheme $\mathrm{Quot}_{d,d+1}(V,C)$ to be the closed subscheme of $\mathrm{Quot}_d(V,C) \times \mathrm{Quot}_{d+1}(V,C)$ given by

$$\mathrm{Quot}_{d,d+1}(V,C) := \left\{ (E \hookrightarrow V, \widetilde{E} \hookrightarrow V) \mid \text{There is an injection } \widetilde{E} \hookrightarrow E \right\}. \tag{3.3}$$

Denote by p_- and p_+ the projections to $\mathrm{Quot}_d(V,C)$ and $\mathrm{Quot}_{d+1}(V,C)$, respectively. We also have a morphism $p_C \colon \mathrm{Quot}_{d,d+1}(V,C) \to C$, which sends $(E \subset V, \widetilde{E} \subset V)$ to the support of length one torsion sheaf $\widetilde{E}/E$. These maps fit into the diagram (3.2). For later use, we set

$$\widehat{p}_- := p_- \times p_C, \ \ \widehat{p}_+ := p_+ \times p_C. \tag{3.4}$$

Following [46], we may use the morphisms p_- and p_C to identify $\mathrm{Quot}_{d,d+1}(V,C)$ with the projective bundle of one-dimensional quotients of the universal subbundle $\mathcal{E}$ on $\mathrm{Quot}_d(V,C) \times C$,

$$\begin{tikzcd} \mathrm{Quot}_{d,d+1}(V,C) \arrow[r, "\cong"] \arrow[dr, "p_- \times p_C"'] & \mathbb{P}_{\mathrm{Quot}_d(V,C)\times C}(\mathcal{E}) \arrow[d] \\ & \mathrm{Quot}_d(V,C) \times C \end{tikzcd} . \tag{3.5}$$

It follows immediately that $\mathrm{Quot}_{d,d+1}(V,C)$ is a smooth projective variety of dimension $n(d+1)$. Let $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ denote the dual of the tautological line bundle on $P_{\mathrm{Quot}_d(V,C)\times C}(\mathcal{E})$, and set

$$\lambda := c_1(\mathcal{L}).$$

The morphism p_+ admits a simple description on a dense open subset of $\mathrm{Quot}_{d+1}(V,C)$. Indeed, consider those points $(\widetilde{E} \subset V)$ in $\mathrm{Quot}_{d+1}(V,C)$ for which the support of the torsion sheaf $V/\widetilde{E}$ consists of $d+1$ distinct points of C. This defines a non-empty open subscheme. Choosing any d of these $d+1$ points determines a point in the fibre of p_+, and in particular p_+ is generically finite of degree $d+1$.

On $\mathrm{Quot}_{d+1}(V,C) \times C$ we have the universal sequence

$$0 \to \widetilde{\mathcal{E}} \to \rho^* V \to \widetilde{\mathcal{F}} \to 0. \tag{3.6}$$

Combining p_+ with p_C, we obtain the following description of $\mathrm{Quot}_{d,d+1}(V,C)$.

Lemma 3.1.1 ([46])**.** *The morphism $p_+ \times p_C$ identifies $\mathrm{Quot}_{d,d+1}(V,C)$ as the virtual projective bundle*

$$\mathbb{P}_{\mathrm{Quot}_{d+1}(V,C)\times C}(\widetilde{\mathcal{E}}^\vee \otimes K_C - V^\vee \otimes K_C) \tag{3.7}$$

with respect to the dual of the first morphism in the universal sequence (3.6). Under this identification, the dual tautological line bundle corresponds to $\mathcal{L}^\vee$.

In summary, the nested Quot scheme admits two equivalent descriptions

$$\mathbb{P}_{\mathrm{Quot}_d(V)\times C}(\mathcal{E}) \cong \mathrm{Quot}_{d,d+1}(V,C) \cong \mathbb{P}_{\mathrm{Quot}_{d+1}(V,C)\times C}(\widetilde{\mathcal{E}}^\vee \otimes \mathcal{K}_C - V^\vee \otimes \mathcal{K}_C). \tag{3.8}$$

Pulling back the universal sequences on $\mathrm{Quot}_d(V,C)\times C$ and $\mathrm{Quot}_{d+1}(V,C)\times C$ to $\mathrm{Quot}_{d,d+1}(V,C)\times C$, one obtains the universal exact sequence

$$0 \to (p_+ \times \mathrm{id}_C)^*\widetilde{\mathcal{E}} \to (p_- \times \mathrm{id}_C)^*\mathcal{E} \to \mathcal{L}\otimes \mathcal{O}_\Delta \to 0 \tag{3.9}$$

where Δ denotes the pullback of the diagonal $\Delta_C \subset C\times C$ along $p_C \times \mathrm{id}_C$. The sequence (3.9) sits in a larger diagram together with the corresponding sequence for the universal quotients

$$0 \to \mathcal{L}\otimes \mathcal{O}_\Delta \to (p_+ \times \mathrm{id}_C)^*\widetilde{\mathcal{F}} \to (p_- \times \mathrm{id}_C)^*\mathcal{F} \to 0. \tag{3.10}$$

Since Δ is isomorphic to $\mathrm{Quot}_{d,d+1}(V,C)$, restricting the universal sequence (3.9) to Δ yields the following.

Lemma 3.1.2. *There is an exact sequence on* $\mathrm{Quot}_{d,d+1}(V,C)$*,*

$$0 \to \mathcal{L}\otimes p_C^* K_C \to (p_+ \times p_C)^*\widetilde{\mathcal{E}} \to (p_- \times p_C)^*\mathcal{E} \to \mathcal{L} \to 0. \tag{3.11}$$

Proof. Restricting the universal sequence (3.9) to Δ gives

$$0 \to \mathrm{Tor}_1(\mathcal{L}\otimes\mathcal{O}_\Delta, \mathcal{O}_\Delta) \to (\widehat{p}_+)^*\widetilde{\mathcal{E}} \to (\widehat{p}_-)^*\mathcal{E} \to \mathcal{L} \to 0.$$

Now pull back the diagonal sequence $0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$ on $C\times C$ along $p_C \times \mathrm{id}_C$, take a tensor product with $\mathcal{L}$ and then restrict it to Δ. This produces the exact sequence

$$0 \to \mathrm{Tor}_1(\mathcal{L}\otimes\mathcal{O}_\Delta, \mathcal{O}_\Delta) \to \mathcal{L}\otimes \mathcal{O}(-\Delta)|_\Delta \to \mathcal{L} \to \mathcal{L} \to 0.$$

The rightmost map is a surjection of line bundles, hence an isomorphism. Consequently, the first morphism is also an isomorphism, and the claim follows. □

Künneth decomposition of the universal Chern classes. The universal sequence (3.9) on $\mathrm{Quot}_{d,d+1}(V,C)\times C$ yields the following identity in $H^*(\mathrm{Quot}_{d,d+1}(V,C)\times C)$

$$c(\mathcal{E}) = c(\widetilde{\mathcal{E}})\cdot c(\mathcal{L}\otimes\mathcal{O}_\Delta) = c(\widetilde{\mathcal{E}})\cdot \frac{1+\lambda}{1+\lambda - c_1(\Delta)}. \tag{3.12}$$

As before, fix a basis $\{1, \gamma_1, \ldots, \gamma_{2g}, h\}$ for the cohomology of C, viewed as the second factor in $\mathrm{Quot}_{d,d+1}(V,C)\times C$. With respect to this basis, the Chern classes of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ admit Künneth decompositions in $H^*(\mathrm{Quot}_{d,d+1}(V,C))\otimes H^*(C)$

$$c_i(\mathcal{E}) = a_i \otimes 1 + \sum_{i=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes h \quad \text{and} \quad c_i(\widetilde{\mathcal{E}}) = \widetilde{a}_i + \sum_{i=1}^{2g} \widetilde{b}_i^j \otimes \gamma_j + \widetilde{f}_i \otimes h. \tag{3.13}$$

Next, fix a basis $\{\sigma_0, \sigma_1, \ldots, \sigma_{2g+1}\}$ for $H^*(C)$, now viewed as the cohomology of the target curve of the morphism p_C. We fix the presentation

$$H^*(C) = \mathbb{Q}[\sigma_0, \ldots, \sigma_{2g+1}]/(\mathrm{T}(\sigma))$$

where $\mathrm{T}(\sigma)$ denotes the geometric relations; we abbreviate this as $\mathbb{Q}[\sigma]/(\mathrm{T}(\sigma))$. With these conventions, the class of the diagonal decomposes as

$$c_1(\Delta) = \sigma_0 \otimes 1 + \sum_{j=1}^{g} \sigma_{j+g} \otimes \gamma_j - \sum_{j=1}^{g} \sigma_j \otimes \gamma_{j+g} + \sigma_{2g+1} \otimes h. \tag{3.14}$$

Combining the identity (3.12) with the Künneth decompositions (3.13) and (3.14), we obtain the following result describing the change of variables.

Proposition 3.1.3. *In $H^*(\mathrm{Quot}_{d,d+1}(V,C))$, the classes $\mathsf{a}, \mathsf{b}, \mathsf{f}$ can be expressed in terms of $\widetilde{\mathsf{a}}, \widetilde{\mathsf{b}}, \widetilde{\mathsf{f}}, \sigma, \lambda$.*

Relation to the nested stack of vector bundles on C. Define the stack of vector bundles of rank n and degree $-d + \deg(V)$ on C by

$$\mathcal{B}_d := \mathrm{Bun}^{-d+\deg(V)}_{\mathrm{GL}_n(\mathbb{C})}.$$

Let the nested stack $\mathcal{B}_{d,d+1}$ as a closed substack of $\mathcal{B}_d \times \mathcal{B}_{d+1}$ given by

$$\mathcal{B}_{d,d+1} := \left\{ (E, \widetilde{E}) \mid \text{There is an injection } \widetilde{E} \hookrightarrow E. \right\}.$$

Let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be the universal bundles on $\mathcal{B}_d \times C$ and $\mathcal{B}_{d+1} \times C$, respectively. In analogy with the nested Quot scheme, using an analogue of exact sequence (3.11) on $\mathcal{B}_{d,d+1}$, one obtains the following identifications with projective bundles.

$$\begin{tikzcd} \mathbb{P}_{\mathcal{B}_d \times C}(\mathcal{G}) \arrow[r, "\simeq"] \arrow[d, "\widehat{\beta}_d"'] & \mathcal{B}_{d,d+1} & \mathbb{P}_{\mathcal{B}_{d+1} \times C}(\widetilde{\mathcal{G}}^\vee \otimes K_C) \arrow[l, "\simeq"'] \arrow[d, "\widehat{\beta}_{d+1}"] \\ \mathcal{B}_d \times C & & \mathcal{B}_{d+1} \times C \end{tikzcd} \tag{3.15}$$

Let $\mathcal{M}$ be the dual tautological line bundle on $\mathbb{P}_{\mathcal{B}_d \times C}(\mathcal{G})$, and let $\mathcal{N}$ be the dual tautological bundle on $\mathbb{P}_{\mathcal{B}_{d+1} \times C}(\widetilde{\mathcal{G}})$. Under the above identifications, these line bundles satisfy $\mathcal{N} = \mathcal{M}^\vee$. To match the notation used for Quot schemes, we write

$$c_1(\mathcal{M}) = \lambda.$$

Forgetting the embeddings into V, one obtains natural morphisms

$$\alpha_d \colon \mathrm{Quot}_d(V,C) \to \mathcal{B}_d \text{ and } \alpha_{d,d+1} \colon \mathrm{Quot}_{d,d+1}(V,C) \to \mathcal{B}_{d,d+1}$$

which fit into the commutative diagram

$$\begin{tikzcd} \mathrm{Quot}_{d,d+1}(V) \arrow[r, hook, "\iota"] \arrow[rr, bend left, "\widehat{p}_+"] \arrow[dr, "\alpha_{d,d+1}"'] & \mathbb{P}_{\mathrm{Quot}_{d+1}(V,C) \times C}(\widetilde{\mathcal{E}}^\vee \otimes K_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}_{d+1}(V,C) \times C \arrow[d, "\alpha_{d+1} \times \mathrm{id}_C"] \\ & \mathcal{B}_{d,d+1} \arrow[r, "\widehat{\beta}_{d+1}"'] & \mathcal{B}_{d+1} \times C \end{tikzcd} . \tag{3.16}$$

Pushforward of the ideal of relations. Fix a basis $\{1, \gamma_1, \ldots, \gamma_{2g}, h\}$ of $H^*(C)$. To keep the notation consistent with the Quot scheme setting, we write the Künneth decompositions on $H^*(\mathcal{B}_{d,d+1}) \otimes H^*(C)$,

$$c_i(\mathcal{G}) = a_i \otimes 1 + \sum_{i=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes h \quad \text{and} \quad c_i(\widetilde{\mathcal{G}}) = \widetilde{a}_i + \sum_{i=1}^{2g} \widetilde{b}_i^j \otimes \gamma_j + \widetilde{f}_i \otimes h.$$

By Theorem 2.3.4, the cohomology rings of $\mathcal{B}_d$ and $\mathcal{B}_{d+1}$ are freely generated, as a graded algebras, by the Künneth components. In particular,

$$H^*(\mathcal{B}_d) \cong \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] \text{ and } H^*(\mathcal{B}_{d+1}) \cong \mathbb{Q}[\widetilde{\mathsf{a}}, \widetilde{\mathsf{b}}, \widetilde{\mathsf{f}}].$$

Moreover, pulling back along α_d and α_{d+1} preserves the generators, and hence induces surjections

$$\alpha_d^* \colon H^*(\mathcal{B}_d) \to H^*(\mathrm{Quot}_d(V, C)) \text{ and } \alpha_{d+1}^* \colon H^*(\mathcal{B}_{d+1}) \to H^*(\mathrm{Quot}_{d+1}(V, C)).$$

In fact, we have $\mathcal{I}_d = \ker(\alpha_d^*)$ and $\mathcal{I}_{d+1} = \ker(\alpha_{d+1}^*)$.

The morphism $\alpha_{d,d+1}$ induces a surjection on cohomology. Define

$$\mathcal{I}_{d,d+1} := \ker \alpha_{d,d+1}^*.$$

Using the Projective bundle formula one obtains the following presentations

$$H^*(\mathcal{B}_{d,d+1}) \cong \frac{\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}, \sigma, \lambda]}{(\mathrm{T}(\sigma), c_n(\mathcal{G}^\vee \otimes \mathcal{M}))} \text{ and } H^*(\mathrm{Quot}_{d,d+1}(V, C)) \cong \frac{\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}, \sigma, \lambda]}{(\mathcal{I}_d, \mathrm{T}(\sigma), c_n(\mathcal{E}^\vee \otimes \mathcal{L}))}.$$

It then follows immediately from the definitions that $p_-^* \mathcal{I}_d = \mathcal{I}_{d,d+1}$. Our aim is to push $\mathcal{I}_{d,d+1}$ forward to $\mathrm{Quot}_{d+1}(V, C)$. Since p_{+*} only exists on cohomology, we introduce its lift at the level of polynomials in $\mathsf{a}, \mathsf{b}, \mathsf{f}$. First, we define an $H^*(\mathcal{B}_{d+1})$–linear pushforward morphism

$$\widehat{\Phi} \colon H^*(\mathcal{B}_{d,d+1}) \to H^*(\mathcal{B}_{d+1} \times C); \quad \gamma \mapsto \widehat{\beta}_{d+1*}\left(\gamma \cdot c_n(V \otimes K_C^\vee \otimes \mathcal{N})\right). \tag{3.17}$$

Proposition 3.1.4. *The $H^*(\mathcal{B}_{d+1})$–linear morphism $\widehat{\Phi}$ uniquely lifts the pushforward morphism $\widehat{p}_{+*}$.*

Proof. Viewing $\widehat{p}_+$ as a virtual projective bundle morphism, Proposition 2.2.3 yields an explicit formula for $\tau \in H^*(\mathrm{Quot}_{d,d+1}(V, C))$,

$$\widehat{p}_{+*}(\tau) = \pi_*(\tau \cdot c_n(V \otimes K_C^\vee \otimes \mathcal{O}(1))) \tag{3.18}$$

where $\pi \colon \mathbb{P}(\widetilde{\mathcal{E}}^\vee \otimes K_C) \to \mathrm{Quot}_{d+1}(V, C) \times C$ is the associated projective bundle morphism, and $\mathcal{O}(1)$ the dual tautological bundle on $\mathbb{P}(\widetilde{\mathcal{E}}^\vee \otimes K_C)$. $\square$

Now set $\Phi := \pi \circ \widehat{\Phi}$, and similarly $\beta := \pi \circ \widehat{\beta}$, where $\pi \colon \mathcal{B}_{d+1} \times C \to \mathcal{B}_{d+1}$ is the projection.

Proposition 3.1.5. *The map Φ sends $\mathcal{I}_{d,d+1}$ surjectively onto $\mathcal{I}_{d+1}$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} H^*(\mathcal{B}_{d,d+1}) \arrow[r, two heads, "\alpha^*_{d,d+1}"] \arrow[d, "\Phi"'] & H^*(\mathrm{Quot}_{d,d+1}(V)) \arrow[d, "p_{+*}"] \\ H^*(\mathcal{B}_{d+1}) \arrow[r, two heads, "\alpha^*_{d+1}"'] & H^*(\mathrm{Quot}_{d+1}(V)). \end{tikzcd} \tag{3.19}$$

One has $\beta^*_{d+1}\mathcal{I}_{d+1} \subset \mathcal{I}_{d,d+1}$. Moreover, for any $\tau \in \mathcal{I}_{d+1}$,

$$\begin{aligned} \Phi(\beta^*_{d+1}(\tau)) =& \tau \cdot \beta_{d+1*}(c_n(V \otimes K_C^\vee \otimes \mathcal{N})) \\ =& \tau \cdot \pi_*(c_1(\mathcal{G}^\vee \otimes K_C) + (e-\kappa)h) \\ =& \tau \cdot \pi_*(a_1 + (d+1)h) \\ =& (d+1)\tau, \end{aligned}$$

where $e = \deg V$ and $\kappa = c_1(K_C)$. This gives the claimed surjectivity. $\square$

Pushforward of Chern classes. We use the pushforward morphism (3.17) to derive two pushforward identities that will be used in the proof of Theorem 3.0.2.

Let Jac be the Jacobian parametrising degree zero line bundles on C, and let $\mathcal{P}$ be the Poincaré line bundle on $\mathsf{Jac} \times C$. For an integer m, we consider the sheaf $\mathcal{F} \otimes \mathcal{P} \otimes \mathcal{O}(m\Delta)$ on $\mathrm{Quot}_d(V, C) \times C \times \mathsf{Jac} \times C$, where $\mathcal{F}$ and $\mathcal{P}$ are pulled back along the second copy of C. Pushing forward along the projection π to $\mathrm{Quot}_d(V, C) \times C \times \mathsf{Jac}$, we obtain a vector bundle of rank d,

$$\mathbb{F}_m := \pi_*(\mathcal{F} \otimes \mathcal{P} \otimes \mathcal{O}(m\Delta)).$$

Proposition 3.1.6. *On* $\mathrm{Quot}_d(V, C) \times C \times \mathsf{Jac}$ *there is a short exact sequence*

$$0 \to \mathbb{F}_m \to \mathbb{F}_{m+1} \to \mathcal{F} \otimes \mathcal{P} \otimes K_C^{\otimes -(m+1)} \to 0. \tag{3.20}$$

Proof. Start with the diagonal sequence $0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$ on $\mathrm{Quot}_d(V, C) \times C \times \mathsf{Jac} \times C$. Tensor it with $\mathcal{F} \otimes \mathcal{P} \otimes \mathcal{O}((m+1)\Delta)$ and push it forward along π. This produces the stated exact sequence. $\square$

Let $\widetilde{\mathbb{F}}_m$ denote the analogous bundle on $\mathrm{Quot}_{d+1}(V, C) \times C \times \mathsf{Jac}$. It relates to $\mathbb{F}_m$ as follows.

Lemma 3.1.7. *On* $\mathrm{Quot}_{d,d+1}(V, C) \times \mathsf{Jac}$ *there is a short exact sequence*

$$0 \to \mathcal{L} \otimes \mathcal{P} \otimes K_C^{\otimes -m} \to (\widehat{p}_+ \times \mathrm{id}_{\mathsf{Jac}})^*\widetilde{\mathbb{F}}_m \to (\widehat{p}_- \times \mathrm{id}_{\mathsf{Jac}})^*\mathbb{F}_m \to 0.$$

Set $\zeta := c_1(\mathcal{P})$ *and* $\kappa := c_1(p_C^* K_C)$. *In particular, we obtain the following formula*

$$c(\mathbb{F}_m, z) = \frac{c(\widetilde{\mathbb{F}}_m, z)}{(z - \lambda - \zeta + m\kappa)} \quad \textit{in } H^*(\mathrm{Quot}_{d,d+1}(V) \times \mathsf{Jac}). \tag{3.21}$$

Proof. On $\mathrm{Quot}_{d,d+1}(V,C)\times \mathsf{Jac}\times C$, take a tensor product of the universal sequence (3.10) with $(p_C\times \mathrm{id}_C)^*\mathcal{P}\otimes \mathcal{O}(m\Delta)$ to obtain

$$0\to \mathcal{L}\otimes \mathcal{O}_\Delta\otimes \mathcal{P}\otimes \mathcal{O}(m\Delta)\to \widetilde{\mathcal{F}}\otimes \mathcal{P}\otimes \mathcal{O}(m\Delta)\to \mathcal{F}\otimes \mathcal{P}\otimes \mathcal{O}(m\Delta)\to 0.$$

Pushing forward to $\mathrm{Quot}_{d,d+1}(V,C)\times \mathsf{Jac}$ and using that the normal bundle of the diagonal $\Delta_C\subset C\times C$ is $K_C^\vee$ yields the desired exact sequence. The formula (3.21) follows immediately. □

Using the Grothendieck–Riemann–Roch–formula (2.2.4) and the Künneth decomposition (3.13), we can express the total Chern class $c(\mathbb{F}_m)$ as a polynomial in the generators $\mathsf{a},\mathsf{b},\mathsf{f}$ with coefficients in $H^*(C\times \mathsf{Jac})$. By abuse of notation, we continue to denote by $c(\mathbb{F}_m)$ the unique lift of this polynomial to $H^*(\mathcal{B}_d\times C\times \mathsf{Jac})$. We further note that the identity (3.21) admits a lift to $H^*(\mathcal{B}_{d,d+1}\times \mathsf{Jac})$, which we denote using the same notation.

Proposition 3.1.8. *We have*

$$(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})c\left(\mathbb{F}_m,z\right)=c\left(\widetilde{\mathbb{F}}_m,z\right)-c\left(\widetilde{\mathbb{F}}_{m+1},z\right). \tag{3.22}$$

If we set $\mu:=-\lambda-\zeta+m\kappa$, then for all $k\geq 0$,

$$(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})\left(\mu^k\cdot c\left(\mathbb{F}_m,z\right)\right)=-z^k\cdot c\left(\widetilde{\mathbb{F}}_{m+1},z\right)+c\left(\widetilde{\mathbb{F}}_m,z\right)\cdot\sum_{i=0}^{k}(-1)^i c_i(\widetilde{\mathbb{F}}_{m+1}-\widetilde{\mathbb{F}}_m,\,z)\cdot z^{k-i}. \tag{3.23}$$

Proof. From the identity (3.21),

$$(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})c(\mathbb{F}_m,z)=c(\widetilde{\mathbb{F}}_m,z)\cdot(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})\left(\frac{1}{z-\lambda-\zeta+m\kappa}\right).$$

Setting $w=-z$, this becomes

$$(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})c(\mathbb{F}_m,z)=-c(\widetilde{\mathbb{F}}_m,z)\cdot(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})\left(\frac{1}{w-(-\lambda-\zeta+m\kappa)}\right).$$

Now view the virtual projective bundle (3.7) as being over $\mathrm{Quot}_{d+1}(V,C)\times \mathsf{Jac}\times C$, twist by $\mathcal{P}^\vee\otimes K_C^{\otimes m}$, and apply the pushforward formula (2.2.3). This yields

$$\begin{aligned}(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})c(\mathbb{F}_m,z)&=-c(\widetilde{\mathbb{F}}_m,z)\cdot c(\widetilde{\mathcal{F}}^\vee\otimes\mathcal{P}^\vee\otimes K_C^{m+1},w)\Big|_{w^{<0}}\\&=-c(\widetilde{\mathbb{F}}_m,z)\cdot c(\widetilde{\mathcal{F}}\otimes\mathcal{P}\otimes K_C^{\otimes-(m+1)},z)\Big|_{z^{<0}}.\end{aligned}$$

Finally, one uses the sequence (3.20) to simplify the right hand side

$$\begin{aligned}(\widehat{\Phi}\times \mathrm{id}_{\mathsf{Jac}*})c(\mathbb{F}_m,z)=&-c(\widetilde{\mathbb{F}}_m,z)\cdot\left(\frac{c(\widetilde{\mathbb{F}}_{m+1},z)}{c(\widetilde{\mathbb{F}}_m,z)}-1\right)\\=&c(\widetilde{\mathbb{F}}_m,z)-c(\widetilde{\mathbb{F}}_{m+1},z),\end{aligned}$$

The second identity is proved in the same way applying formula (2.2.3). □

3.2 Proof of Theorem 3.0.2

The proof proceeds by induction. We begin with the base case $d = 1$, corresponding to the Quot scheme $\mathrm{Quot}_1(V, C) \cong \mathbb{P}_C(V)$.

Base case. Let $e = \deg V$, and let $L \in \mathrm{Pic}(C)$ be a line bundle of degree $\ell := m+g-1$. Next, write the Künneth decomposition of $\mathrm{ch}(\mathcal{E})$ on $\mathrm{Quot}_1(V, C) \times C$ as

$$\mathrm{ch}_i(\mathcal{E}) = \mathrm{ch}_i(\mathcal{E}_x) \otimes 1 + \sum_{j=1}^{2g} \beta_i^j \otimes \gamma_j + \pi_*(\mathrm{ch}_{i+1} \mathcal{E}) \otimes h$$

where the coefficients β_i^j admit the explicit description

$$\beta_1^j = b_1^j \text{ and for } i > 1, (-1)^{i-1}(i-1)!\, \beta_i^j = b_i^j + \sum_{k=1}^{i-1} t_k a_{i-k} b_k^j \text{ with } t_k \in \mathbb{Q}. \tag{3.24}$$

Let $\mathcal{P}$ denote the normalised Poincaré line bundle on $\mathsf{Jac} \times C$, as described in [3]. After fixing a basis $\{\delta_1, \dots, \delta_{2g}\}$ for $\in H^1(\mathsf{Jac})$, its Chern character $\mathcal{P}$ is given by

$$\mathrm{ch}(\mathcal{P}) = 1 + \zeta - \theta h \in H^*(\mathsf{Jac} \times C) \text{ with } \zeta = \sum_{j=1}^{2g} \delta_j \otimes \gamma_j \text{ and } \theta = \sum_{j=1}^{g} \delta_j \delta_{g+j}.$$

Recall the generalised tautological bundle associated to L,

$$\mathsf{F}_\ell := \pi_*(\mathcal{F} \otimes \mathcal{P} \otimes L),$$

where $\pi : \mathrm{Quot}_1(V, C) \times \mathsf{Jac} \times C \to \mathrm{Quot}_1(V, C) \times \mathsf{Jac}$ is the projection.

Proposition 3.2.1. *The total Chern class of F_ℓ is*

$$c(\mathsf{F}_\ell) = c(L^{[1]}) \cdot e^{-n\theta} \cdot e^{\sum_{i\geq 1}(-1)^{i-1}(i-1)!\left(\mathrm{ch}_{i-1}(\mathcal{E}_x)\theta - \sum_{j=1}^{g}(-\beta_i^j \delta_{j+g} + \beta_i^{j+g}\delta_j)\right)}.$$

Proof. Apply Grothendieck–Riemann–Roch formula (2.2.4) to obtain

$$\begin{aligned}
\mathrm{ch}(\mathsf{F}_\ell) &= \pi_* \left(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(\mathcal{P}) \cdot \mathrm{ch}(L) \cdot \mathrm{td}(C)\right) \\
&= \pi_* \left(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}(L) \cdot \mathrm{td}(C)\right) + \pi_* \left(\mathrm{ch}(\mathcal{F}) \cdot (\zeta - \theta h)\right) \\
&= \mathrm{ch}(L^{[1]}) + \pi_* \left(((n + eh) - \mathrm{ch}(\mathcal{E})) \cdot (\zeta - \theta h)\right) \\
&= \mathrm{ch}(L^{[1]}) - n\theta + \mathrm{ch}(\mathcal{E}_x) \cdot \theta - \pi_* \left(\mathrm{ch}(\mathcal{E}) \cdot \zeta\right).
\end{aligned}$$

Hence

$$\mathrm{ch}(\mathsf{F}_\ell) = \mathrm{ch}(L^{[1]}) - n\theta + \mathrm{ch}(\mathcal{E}_x) \cdot \theta - \sum_{i\geq 0} \sum_{j=1}^{g} (-\beta_i^j \delta_{j+g} + \beta_i^{j+g} \delta_j).$$

Passing from the Chern character to the total Chern class gives the claimed expression. ∎

Proposition 3.2.2. *The total Chern class of $L^{[1]}$ is*

$$c(L^{[1]}) = (1 + a_1 + a_2 + \cdot + a_n)^{-m} \cdot e^{-\sum_{i\geq 1}(-1)^{i-1}\,(i-1)!\,\pi_* \mathrm{ch}_{i+1}(\mathcal{E})}.$$

Proof. By the assumption on the degree of L, we have $\mathrm{ch}(L)\cdot\mathrm{td}(C) = 1+mh$. Applying Grothendieck–Riemann–Roch formula (2.2.4) to the projection π and the bundle $L^{[1]}$ gives

$$\begin{aligned}\mathrm{ch}(L^{[1]}) &= \pi_*\left(\mathrm{ch}(\mathcal{F})\cdot\mathrm{ch}(L)\cdot\mathrm{td}(C)\right)\\ &= \pi_*\left(((n+eh)-\mathrm{ch}(\mathcal{E}))\cdot(1+mh)\right)\\ &= nm+e-m\,\mathrm{ch}(\mathcal{E}_x)-\pi_*\mathrm{ch}(\mathcal{E})\end{aligned}$$

where $\mathcal{E}_x$ denotes the pullback of $\mathcal{E}$ along $\mathrm{Quot}_1(V,C)\times\{x\}\to\mathrm{Quot}_1(V,C)\times C$. Passing from the Chern character to the total Chern class yields

$$c(L^{[1]}) = c(\mathcal{E}_x)^{-m}\cdot e^{-\sum_{i\geq 1}(-1)^{i-1}(i-1)!\,\pi_*\mathrm{ch}_{i+1}(\mathcal{E})},$$

Finally, since $c(\mathcal{E}_x)^{-m} = (1+a_1+a_2+\cdot+a_n)^{-m}$, the stated formula follows. □

Theorem 3.2.3. *The generalised Mumford relations*

$$\langle c_{1+i}(\mathsf{F}_\ell)/\delta \mid \ell\in\mathbb{Z},\ \delta\in H^*(\mathsf{Jac})\ \textit{and}\ i>0\rangle$$

imply the following identities in $H^*(\mathrm{Quot}_1(V))$.

$$b_1^j b_1^k = 0 \text{ for } |j-k|\neq g. \tag{3.25}$$
$$a_1 = -b_1^j b_1^{j+g} \text{ for } 1\leq j\leq g. \tag{3.26}$$
$$a_i = (-1)^{i-1}a_1 f_2^{i-1},\ \ 2\leq i\leq n. \tag{3.27}$$
$$b_i^j = (-1)^{i-1}b_1^j f_2^{i-1},\ \ 2\leq i\leq n. \tag{3.28}$$
$$f_{i+1} = (-1)^{i+1}\left(f_2^i-(i-1)e\,a_1 f_2^{i-1}\right),\ 2\leq i\leq n-1. \tag{3.29}$$
$$0 = f_2^n-(n-1)e\,a_1 f_2^{n-1}. \tag{3.30}$$

Proof. Since rank $\mathsf{F}_\ell = 1$, we have $c_2(\mathsf{F}_\ell)=0$. Taking the slant product with $\delta_j\delta_k$ for $|j-k|\neq g$ yields relations (3.25), while slanting with $\delta_j\delta_{j+g}$ gives relations (3.26).

To get the remaining relations, let us consider $c_i(\mathsf{F}_\ell)=0$ for $i\geq 2$. Slanting with δ_j gives

$$c_i(\mathsf{F}_\ell)/\delta_j = (i-1)\beta_i^j-\beta_{i-1}^j\cdot c_1(L^{[1]}) = 0. \tag{3.31}$$

Similarly, we also have

$$c_i(\mathsf{F}_\ell)/1 = c_i(L^{[1]}) = 0 \text{ for } i\geq 2.$$

Now we analyse vanishings $c_i(L^{[1]})=0$ for all $i\geq 2$ to recover the remaining relations. Start by defining

$$\mathsf{y}_i = (-1)^i\,(i-1)!\,\pi_*\mathrm{ch}_{i+1}(\mathcal{E}), \tag{3.32}$$

Observe that

$$c_1(L^{[1]}) = \mathsf{y}_1-ma_1 = f_2-a_1(m+g-1). \tag{3.33}$$

As rank $L^{[1]}=1$, we have $c_i(L^{[1]})=0$ for all $i\geq 2$. Observe that $c_i(L^{[1]})$ is a polynomial of degree i in m, so the coefficients of monomials m^k vanish; let $[m^k]c_i(L^{[1]})$ denote the coefficient of m^k. From $c_2(L^{[1]})=0$ one immediately obtains

$$[m^2]c_2(L^{[1]}) = a_1^2 = 0 \text{ and } [m]c_2(L^{[1]}) = a_2 = -a_1\mathsf{y}_1 = -a_1 f_2$$

and from $c_3(L^{[1]}) = 0$, one obtains

$$[m]c_3(L^{[1]}) = a_3 = a_2\mathsf{y}_1 = a_1 f_2^2.$$

Moreover, considering the relation $[1]c_i(L^{[1]}) = 0$ for each $i \geq 2$, yields

$$e^{\sum_{i\geq 1}\mathsf{y}_i}|_{\geq \deg\ 2} = 0.$$

So $\exp(\sum_{i\geq 1}\mathsf{y}_i) = \exp(1+\mathsf{y}_1)$ and, taking logarithms, one writes

$$\mathsf{y}_i = (-1)^{i+1}\frac{{\mathsf{y}_1}^i}{i}, \ \ i \geq 2. \tag{3.34}$$

Using this identity to simplify $[m]c_i(L^{[1]})$ yields relations (3.27).

To get the remaining relations, start with the vanishings $a_i b_1^j = 0$, $i \geq 1$, implied by (3.26) and (3.27). The identities (3.24) and (3.31), used recursively, yield the vanishings

$$a_i b_k^j = 0, \ k \geq 1, 1 \leq i \leq n.$$

One can then simplify the expression (3.24) to write

$$(-1)^{k-1}(k-1)!\beta_k^j = b_k^j, \ k \geq 1, \ 1 \leq j \leq 2g. \tag{3.35}$$

Finally, using the recursion (3.31) together with the expression (3.33), for $c_1(L^{[1]})$, and (3.35) gives relations (3.28).

We finally express the classes f_i in terms of f_2 and a_1, as follows. Observe first that equations from (3.25) to (3.28) imply

$$c_j(\mathcal{E})\cdot c_k(\mathcal{E})\cdot c_l(\mathcal{E}) = 0 \text{ in } H^*(\mathrm{Quot}_1(V)\times C)$$

for any triple product of Chern classes with $j, k, l \geq 1$. So the power sum polynomials p_i have simple expressions

$$p_i(\mathcal{E}) = (-1)^{i-1} i\, c_i(\mathcal{E}) + (-1)^i \sum_{j=1}^{i-1} j\, c_{i-j}(\mathcal{E})\, c_j(\mathcal{E}).$$

One has the equality $\mathsf{y}_i = (-1)^i \frac{1}{i(i+1)}\pi_*(p_{i+1})$, so using (3.34), one writes

$$\begin{aligned} {\mathsf{y}_1}^i &= (-1)^{i+1} i\, \mathsf{y}_i = -\frac{\pi_*(p_{i+1})}{i+1} \\ &= (-1)^{i+1} f_{i+1} + \frac{(-1)^i}{i+1}\sum_{j=1}^{i} j\left(a_{i+1-j}f_j + f_{i+1-j}a_j - 2\sum_{l=1}^{g} b_j^l b_{i+1-j}^{l+g}\right). \end{aligned} \tag{3.36}$$

Given the expression (3.33) for y_1, the above equation determines f_i recursively

$$f_{i+1} = (-1)^{i+1} f_2^i + a_1 \mathsf{m}_i(f_2), \tag{3.37}$$

where $\mathsf{m}_i(f_2)$ is a monomial of degree $i-1$ in f_2, which we now determine. From (3.36), we obtain

$$\begin{aligned}\mathsf{y}_1^i &= (-1)^{i+1} f_{i+1} + \frac{(-1)^i}{i+1}\sum_{j=1}^{i} j\left[(-1)^i 2a_1 f_2^{i-1} + (-1)^{i-1}2ga_1 f_2^{i-1}\right] - ea_1 f_2^{i-1}\\ &= (-1)^{i+1} f_{i+1} + i(1-g)\, a_1 f_2^{i-1} - ea_1 f_2^{i-1}.\end{aligned}$$

Using (3.33), we write

$$f_{i+1} = (-1)^{i+1}\left(f_2^i - (i-1)e\, a_1 f_2^{i-1}\right),$$

which is precisely the identity (3.29). When $i = n$, we get $f_{n+1} = 0$, hence we obtain the identity (3.30). This completes the proof. □

Since $\mathrm{Quot}_1(V) \cong \mathbb{P}_C(V)$, there is an isomorphism

$$H^*(\mathbb{P}_C(V)) \cong H^*(C)[\lambda]/\langle \lambda^n - c_1(V)\lambda^{n-1}\rangle$$

where $\lambda = c_1(\mathcal{O}_{\mathbb{P}_C(V)}(1))$. Let $\gamma = \{1, \gamma_1, \ldots, \gamma_{2g}, h\}$ be a basis of $H^*(C)$. We may then state the following corollary, which completes the proof of the base case.

Corollary 3.2.4. *Under the identifications*

$$b_1^i \mapsto -\gamma_{i+g},\ \ b_1^{i+g} \mapsto \gamma_i,\ 1 \le i \le g,\ \ f_2 \mapsto \lambda - eh,$$

we have

$$\mathbb{Q}[\mathsf{a},\mathsf{b},\mathsf{f}]/\langle c_{1+i}(\mathsf{F}_\ell)/\delta \mid \ell \in \mathbb{Z},\ \ \delta \in H^*(\mathsf{Jac}) \text{ and } i > 0\rangle \cong H^*(\mathrm{Quot}_1(V)).$$

Remark 3.2.5. One may alternatively take $d = 0$ as the base case, in which case the argument simplifies considerably. However, since the proof of Theorem 3.0.2 for general curves depends on the validity of the conjectural Lemma 3.2.6, the case $d = 1$ already provides a non-trivial example.

Induction argument. By the inductive hypothesis, we assume that Theorem 3.0.2 holds for $\mathrm{Quot}_d(V, C)$. Define

$$\mathcal{J}_d := \langle c_{d+j}(\mathsf{F}_m)/\delta \mid \delta \in H^*(\mathsf{Jac}), m \in \mathbb{Z} \text{ and } j > 0\rangle \subset \mathbb{Q}[\mathsf{a},\mathsf{b},\mathsf{f}].$$

By Proposition 3.1.5, the pushforward Φ maps $\mathcal{I}_{d,d+1}$ surjectively onto $\mathcal{I}_{d+1}$. Thus, to complete the induction step, it suffices to show that $\Phi(\mathcal{I}_{d,d+1})$ is contained in the ideal $\mathcal{J}_{d+1}$.

We now state a key technical lemma relating the Künneth components of the Chern classes $c(\mathbb{F}_{m'})$, for which we have the pushforward formula (3.23), to the Künneth components of the Chern classes $c(\mathsf{F}_m)$ appearing in the statement of Theorem 3.0.2. First, write the Künneth decomposition

$$c_{d+i}(\mathbb{F}_m) = \mathfrak{a}_{i,m} \otimes 1 + \sum_{j=1}^{2g} \mathfrak{b}_{i,m}^j \otimes \gamma_j + \mathfrak{f}_{i,m} \otimes h. \tag{3.38}$$

and note that $\mathfrak{a}_{i,m} = c_{d+i}(\mathsf{F}_m)$. Fixing the basis $\{\delta_1, \ldots, \delta_g\}$ of $H^1(\mathsf{Jac})$ fixes the basis of $H^*(\mathsf{Jac})$. Let us write the the Künneth decomposition with respect to this basis,

$$\mathfrak{a}_{i,m} = \sum_{\delta} \mathfrak{a}_{i,m,\delta} \otimes \delta, \;\; \mathfrak{b}^j_{i,m} = \sum_{\delta} \mathfrak{b}^j_{i,m,\delta} \otimes \delta \text{ and } \mathfrak{f}_{i,m} = \sum_{\delta} \mathfrak{f}_{i,m,\delta} \otimes \delta. \tag{3.39}$$

Lemma 3.2.6 (Conjecture)**.** *With the notation of* (3.38) *and* (3.39)*, for every* $\delta \in H^*(\mathsf{Jac})$*, the Künneth components* $\mathfrak{b}^j_{i,m,\delta}$ *and* $\mathfrak{f}_{i,m,\delta}$ *lie in the ideal*

$$\langle \mathfrak{a}_{l,m',\delta'} \mid l > 0, \;\; m' \in \mathbb{Z} \text{ and } \delta' \in H^*(\mathsf{Jac}) \rangle \subset \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}].$$

Proof of Lemma 3.2.6 when $C \cong \mathbb{P}^1$. Observe that

$$\mathbb{F}_m = \pi_*(\mathcal{F} \otimes \mathcal{P} \otimes \mathcal{O}(m\Delta)) \cong \pi_*(\mathcal{F} \otimes \mathcal{P} \otimes \mathcal{O}_C(m)) \otimes \mathcal{O}_C(m) = \mathsf{F}_m \otimes \mathcal{O}_C(m).$$

Hence

$$\begin{aligned} c_t(\mathbb{F}_m) =& c_t(\mathsf{F}_m \otimes \mathcal{O}_C(m)) \\ =& c_t(\mathsf{F}_m) + m\left(dtc_t(\mathsf{F}_m) - t^2 c_t'(\mathsf{F}_m)\right) \otimes h \end{aligned}$$

where $c_t'(\mathsf{F}_m)$ denotes the derivative of $c_t(\mathsf{F}_m)$ with respect to t. In particular,

$$c_{d+i}(\mathbb{F}_m) = c_{d+i}(\mathsf{F}_m) + m(dc_{d+i-1}(\mathsf{F}_m) - (d+i-1)c_{d+i-1}(\mathsf{F}_m)) \otimes h,$$

and the claim follows. $\square$

Remark 3.2.7. The argument above uses that $\mathcal{O}(\Delta_{\mathbb{P}^1})$ on $\mathbb{P}^1 \times \mathbb{P}^1$ splits as a tensor product of pullbacks from the two factors. For a general curve, such decomposition fails, as reflected in the decomposition (3.14) of the diagonal class.

Proof of Theorem 3.0.2 assuming Lemma 3.2.6. Assume inductively that $\mathcal{I}_d = \mathcal{J}_d$. By Proposition 3.1.5, $\Phi(\mathcal{I}_{d,d+1}) = \mathcal{I}_{d+1}$, so it suffices to prove $\Phi(\mathcal{I}_{d,d+1}) \subset \mathcal{J}_{d+1}$. Write

$$\phi := \widehat{\Phi} \times \mathrm{id}_{\mathsf{Jac}*}.$$

Using decompositions (3.38) and (3.39), along with the definition of ϕ, we can write down the Künneth components of $\phi(\lambda^k \cdot c_{d+i}(\mathbb{F}_m))$

$$\begin{aligned} \phi(\lambda^k \cdot c_{d+i}(\mathbb{F}_m))/(1 \otimes \delta) &= \phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(1 \otimes \delta), \\ \phi(\lambda^k \cdot c_{d+i}(\mathbb{F}_m))/(\gamma_j \otimes \delta) &= \phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(\gamma_j \otimes \delta) \\ &\quad + \phi(\lambda^k \mathfrak{b}^j_{i,m,\delta} \otimes (\gamma_j \otimes \delta))/(\gamma_j \otimes \delta) \text{ and} \\ \phi(\lambda^k \cdot c_{d+i}(\mathbb{F}_m))/(h \otimes \delta) &= \phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(h \otimes \delta) \\ &\quad + \sum_{j=1}^{2g} \phi(\lambda^k \mathfrak{b}^j_{i,m,\delta} \otimes (\gamma_j \otimes \delta))/(h \otimes \delta) \\ &\quad + \phi(\lambda^k \mathfrak{f}_{i,m,\delta} \otimes (h \otimes \delta))/(h \otimes \delta). \end{aligned}$$

We claim that each of the summands above lands in $\mathcal{J}_{d+1}$.

1. $\phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(1 \otimes \delta) \in \mathcal{J}_{d+1}$ for all $\delta \in H^*(\mathsf{Jac})$:
Since $\lambda^k \mathfrak{a}_{i,m,\delta} \in \mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}, \lambda]$, we have $\phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta)) = \widehat{\Phi}(\lambda^k \mathfrak{a}_{i,m,\delta}) \otimes \delta$. So the claim follows from the pushforward equation (3.23).

2. $\phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(\gamma_j \otimes \delta)$, $\phi(\lambda^k \mathfrak{b}^j_{i,m,\delta} \otimes (\gamma_j \otimes \delta))/(\gamma_j \otimes \delta) \in \mathcal{J}_{d+1}$ for all $1 \leq j \leq 2g$ and $\delta \in H^*(\mathsf{Jac})$:
Using Lemma 3.2.6 and Proposition 3.1.3, write
$$\mathfrak{b}^j_{i,m,\delta} = \sum_{t,m',\delta'} c_{t,m',\delta'} \mathfrak{a}_{t,m',\delta'} \text{ with } c_{t,m',\delta'} \in \mathbb{Q}[\widetilde{\mathsf{a}}, \widetilde{\mathsf{b}}, \widetilde{\mathsf{f}}, \lambda, \gamma]. \tag{3.40}$$
Then
$$\phi(\lambda^k \mathfrak{b}^j_{i,m,\delta} \otimes (\gamma_j \otimes \delta))/(\gamma_j \otimes \delta) = \sum_{t,m',\delta'} \phi\left(\lambda^k c_{t,m',\delta'} \mathfrak{a}_{t,m',\delta'} \otimes (1 \otimes \delta)\right)/(1 \otimes \delta),$$
so we conclude by part 1. Applying equation (3.23) and Lemma 3.2.6 yields $\phi(\lambda^k \cdot c_{d+i}(\mathbb{F}_m))/(\gamma_j \otimes \delta) \in \mathcal{J}_{d+1}$. Hence, $\phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(\gamma_j \otimes \delta) \in \mathcal{J}_{d+1}$ as well.

3. $\phi(\lambda^k \mathfrak{a}_{i,m,\delta} \otimes (1 \otimes \delta))/(h \otimes \delta)$, $\phi(\lambda^k \mathfrak{b}^j_{i,m,\delta} \otimes (\gamma_j \otimes \delta))/(h \otimes \delta)$ and $\phi(\lambda^k \mathfrak{f}_{i,m,\delta} \otimes (h \otimes \delta))/(h \otimes \delta) \in \mathcal{J}_{d+1}$ for all $1 \leq j \leq 2g$ and $\delta \in H^*(\mathsf{Jac})$:
Using Lemma 3.2.6 and Proposition 3.1.3, in the same manner as in (3.40), we can write $\mathfrak{f}_{i,m,\delta}$ as an element in $\mathcal{J}_d$. Then the claim follows exactly as in part 2.

Therefore $\Phi(\mathcal{I}_{d,d+1}) \subset \mathcal{J}_{d+1}$, completing the induction. □

Corollary 3.2.8. *The surjection $\mathbb{Q}[\mathsf{a}, \mathsf{b}, \mathsf{f}] \twoheadrightarrow H^*(\mathrm{Quot}_d(V, C))$ is a linear isomorphism in degrees strictly less than $2(d - g + 1)$.*

Remark 3.2.9. Theorem 3.0.2 yields, in particular, a presentation of the cohomology ring $H^*(\mathrm{Sym}^d C)$. Indeed, when $n = 1$ there is an isomorphism $\mathrm{Quot}_d(V, C) \cong \mathrm{Sym}^d C$. We expect, however, that the resulting presentation of $H^*(\mathrm{Sym}^d C)$ differs from the one given in [42], except in the case when C is isomorphic to $\mathbb{P}^1$.

Remark 3.2.10. Theorem 3.0.2 can also be used to give a presentation of the cohomology ring of the stack $\mathrm{Coh}^d_0(C)$ of rank zero coherent sheaves of degree d on C. This can be seen by realizing $\mathrm{Quot}_d(\mathcal{O}^n_C, C)$ as an open subscheme of codimension n inside a vector bundle over $\mathrm{Coh}^d_0(C)$. One can then define an inverse system $H^*(\mathrm{Quot}_d(\mathcal{O}^{n+1}_C, C)) \to H^*(\mathrm{Quot}_d(\mathcal{O}^n_C, C))$ by forgetting the Künneth generators coming from top Chern class. In this way one obtains a presentation
$$H^*(\mathrm{Coh}^d_0(C)) = \varprojlim_n H^*(\mathrm{Quot}_d(\mathcal{O}^n_C, C)),$$
using the definition (2.6). A different presentation of $H^*(\mathrm{Coh}^d_0(C))$ was computed in [30].

3.3 Specialisation to $\mathrm{Quot}_d(V, \mathbb{P}^1)$

In this section, we specialise to $\mathrm{Quot}_d(V, \mathbb{P}^1)$ and present a few results that are specific to this setting.

We begin by recalling the generating function of the Poincaré polynomials of $\mathrm{Quot}_d(V, C)$ as d varies, for a general curve C.

Theorem 3.3.1. *Let $P(\mathrm{Quot}_d(V,C),z)$ denote the Poincaré polynomial of the Quot scheme $\mathrm{Quot}_d(V,C)$. Then we have the generating series*

$$\sum_{d=0}^{\infty} P(\mathrm{Quot}_d(V,C),z)t^d = \prod_{i=0}^{n-1} \frac{(1+tz^{2i+1})^{2g}}{(1-tz^{2i})(1-tz^{2i+2})}. \tag{3.41}$$

By manipulating this generating function, one obtains the following useful corollary.

Corollary 3.3.2. *The surjection $\mathbb{Q}[\mathsf{a},\mathsf{b},\mathsf{f}] \to H^*(\mathrm{Quot}_d(V,C))$ is a linear isomorphism in degrees strictly less than $2(d-g+1)$. Moreover, if $d > 2g-2$, then there are $d-2g+2$ relations in degree $2(d-g+1)$.*

Remark 3.3.3. As observed in the previous section, Theorem 3.0.2 already establishes the first part of the corollary without any reference to the Poincaré polynomial.

Specialising Corollary 3.3.2 to the case $C \cong \mathbb{P}^1$, one observes that $H^*(\mathrm{Quot}_d(V,\mathbb{P}^1))$ has $d+2$ relations in degree $2d+2$. We shall provide a geometric interpretation of these relations when $V \cong \mathcal{O}_{\mathbb{P}^1}^n$ and $n \geq d+1$. Before doing so, we record the following conjecture, which arose from computer calculations.

Conjecture 3.3.4. *Let $n=2$ and d be odd. Then the ideal of relations in $H^*(\mathrm{Quot}_d(V,\mathbb{P}^1))$ is generated by $d+2$ relations in degree $2d+2$.*

Stack of globally generated bundles on $\mathbb{P}^1$. Recall the stack $\mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}(\mathbb{P}^1)$, which parametrises vector bundles of rank n and degree d on $\mathbb{P}^1$. Inside this stack, we define an open substack $\mathcal{V}_d$ consisting of globally generated vector bundles given by

$$\mathcal{V}_d = \left\{ E \in \mathrm{Bun}^d_{\mathrm{GL}_n(\mathbb{C})}(\mathbb{P}^1) \mid E \text{ is isomorphic to } \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \text{ with } a_i \geq 0 \right\}.$$

Theorem 3.3.5 ([13])**.** *The stack $\mathcal{V}_d$ is a smooth Artin stack of dimension $-n^2$.*

Let $\mathcal{G}$ denote the universal bundle on $\mathcal{V}_d \times \mathbb{P}^1$, and let π be the projection to $\mathcal{V}_d$. We then define a vector bundle

$$\mathcal{M}_d := \mathrm{Hom}(\mathcal{O}^n, \pi_*\mathcal{G})$$

on $\mathcal{V}_d$. The Quot scheme embeds into $\mathcal{M}_d$ via an open embedding $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1},\mathbb{P}^1) \hookrightarrow \mathcal{M}_d$, which sends an inclusion $E \subset \mathcal{O}^n_{\mathbb{P}^1}$ to its dual $\mathcal{O}^n_{\mathbb{P}^1} \to E^\vee$. Let ϕ_d denote the composition of this open embedding with the projection $\mathcal{M}_d \to \mathcal{V}_d$. By an application of the excision sequence in cohomology, this immediately yields a surjection

$$\phi_d^* \colon H^*(\mathcal{V}_d) \to H^*(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1},\mathbb{P}^1)).$$

This allows us to apply Theorem 3.0.2 to investigate $H^*(\mathcal{V}_d)$. Fix a basis $\{1,h\}$ for $H^*(\mathbb{P}^1)$. The Künneth decomposition of the Chern classes of the universal bundle $\mathcal{G}$ on $\mathcal{V}_d \times \mathbb{P}^1$ may then be written as

$$c_i(\mathcal{G}) = \mathfrak{a}_i \otimes 1 + \mathfrak{f}_i \otimes h.$$

By [41], we know that the classes $\mathfrak{a}_i, \mathfrak{f}_i$ generate the ring $H^*(\mathcal{V}_d)$. In particular, we obtain the following result.

Proposition 3.3.6. *The cohomology ring $H^*(\mathcal{V}_d)$ has no relations in degrees strictly less than $2d+2$.*

Proof. Observe that $\phi_d^*(\mathfrak{a}_i) = (-1)^i a_i$ and $\phi_d^*(\mathfrak{f}_i) = (-1)^i f_i$. Since $H^*(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{P}^1))$ has no relations in degrees strictly less than $2d+2$, it follows that the same is true for $H^*(\mathcal{V}_d)$. □

Remark 3.3.7. An alternative proof of the above proposition is given in [41], based on a detailed analysis of the geometry of $\mathcal{V}_d$. Moreover, using their methods, one can compute the Poincaré polynomial

$$P(\mathcal{V}_d, z) = \frac{\prod_{i=1}^{n-1}(1 - z^{2(d+i)})}{\prod_{i=1}^{n-1}(1 - z^{2i}) \prod_{i=1}^{n}(z - t^{2i})}, \tag{3.42}$$

which yields further information about the cohomology of $\mathcal{V}_d$. We note that the explicit formula (3.42) does not appear in [41].

Geometric interpretation of the relations. We now study the complement $\mathcal{Z}_d$ of $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{P}^1)$ in $\mathcal{M}_d$, with the aim of understanding the geometric origin of the relations in degree $2d+2$. On $\mathcal{M}_d \times \mathbb{P}^1$, consider the universal morphism

$$\xi_d \colon \mathcal{O}^n \to \mathcal{G}.$$

Taking the determinant of this morphism and pushing it forward along the projection $\pi\colon \mathcal{M}_d \times \mathbb{P}^1 \to \mathcal{M}_d$ yields a section $\mathcal{O} \to \pi_*(\det \mathcal{G})$ of the vector bundle $\pi_*(\det \mathcal{G})$. The vanishing locus of this section is precisely $\mathcal{Z}_d$. Since rank $\pi_*(\det \mathcal{G}) = d+1$, we obtain a bound on the codimension

$$\operatorname{codim} \mathcal{Z}_d \leq d+1.$$

Moreover, we have the following result.

Theorem 3.3.8. *The closed substack $\mathcal{Z}_d \subset \mathcal{M}_d$ has $d+1$ irreducible components in codimension $d+1$ whenever $n \geq d+1$.*

Proof. Fix a smooth chart of $\mathsf{V} \to \mathcal{V}_d$. For a splitting type $\overline{a} := (a_1 \leq \ldots \leq a_n)$, we consider the corresponding splitting locus $\mathsf{V}_{\overline{a}} \subset \mathsf{V}$. These data fit into a Cartesian diagram

$$\begin{tikzcd} \mathsf{M}_{\overline{a}} \arrow[r] \arrow[d] & \mathsf{M} \arrow[r] \arrow[d] & \mathcal{M}_d \arrow[d] \\ \mathsf{V}_{\overline{a}} \arrow[r] & \mathsf{V} \arrow[r] & \mathcal{V}_d \end{tikzcd} .$$

We denote by the same symbol ξ_d the restriction of the universal morphism to $\mathsf{M} \times \mathbb{P}^1$. Define locally closed subschemes

$$\mathsf{M}_i := \left\{ x \in \mathsf{M} \mid \operatorname{rank} \ \xi_d|_{\{x\}\times\mathbb{P}^1} = i \right\}.$$

It suffices to show that M_{n-1} has $(d+1)$ irreducible components in codimension $(d+1)$, since the remaining strata contribute only in higher codimensions. Set

$$\widetilde{\mathsf{M}}_{\overline{a}} := \mathsf{M}_{\overline{a}} \cap \mathsf{M}_{n-1}$$

viewed as a subvariety of M. Over $\widetilde{\mathsf{M}}_{\overline{a}} \times \mathbb{P}^1$, we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{O}^n \arrow[rr] \arrow[dr] & & \bigoplus_i \mathcal{O}(a_i) \arrow[r] & \mathcal{H} \arrow[r] & 0 \\ & & & \mathcal{S} \arrow[ur, hook] & & & \end{tikzcd} \tag{3.43}$$

where $\mathcal{L}$ and $\mathcal{H}$ are coherent sheaves of rank 1, and $\mathcal{S}$ is a vector bundle of rank $n-1$. We now consider the flattening stratification of $\mathcal{H}$,

$$\widetilde{\mathsf{M}}_{\overline{a}} = \bigsqcup_k \widetilde{\mathsf{M}}_{\overline{a},k},$$

where $\widetilde{\mathsf{M}}_{\overline{a},k}$ parametrises exact sequences of the form

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-k) \to \mathcal{O}_{\mathbb{P}^1}^n \to \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \to H \to 0.$$

The restriction of the diagram (3.43) to $\widetilde{\mathsf{M}}_{\overline{a},k} \times \mathbb{P}^1$ is determined by

1. The restriction of $0 \to \mathcal{S} \to \bigoplus_i \mathcal{O}(a_i) \to \mathcal{H} \to 0$, which can be viewed as the universal sequence on the Quot scheme $\mathrm{Quot}_{d-k}(\bigoplus_i \mathcal{O}(a_i), 1, \mathbb{P}^1)$ parametrising rank 1, degree k quotients of $\bigoplus_i \mathcal{O}(a_i)$.

2. A surjection $\mathcal{O}^n \to \mathcal{S}$, which determines an open subset of $\mathrm{Hom}(\mathcal{O}^n, S)$ for each point $0 \to S \to \bigoplus_i \mathcal{O}(a_i) \to H \to 0$ of the above Quot scheme.

When $\overline{a}$ is the balanced splitting type, the Quot scheme $\mathrm{Quot}_{d-k}(\bigoplus_i \mathcal{O}(a_i), 1, \mathbb{P}^1)$ is a smooth projective variety of dimension $n(d-k+1)-(d+1)$ whenever it is non-empty, and $\dim \mathrm{Hom}(\mathcal{O}^n, S) = n^2 + (k-1)n$. Under the assumption $n \geq d+1$, we have $n(d-k+1)-(d+1) \geq 0$, so the Quot scheme above is indeed non-empty. Consequently,

$$\operatorname{codim} \widetilde{\mathsf{M}}_{\overline{a},k} = \dim \mathsf{M}_{\overline{a}} - \dim \widetilde{\mathsf{M}}_{\overline{a},k} = n^2+nd-(n^2+(k-1)n+n(d-k+1)-(d+1)) = d+1.$$

The closures of the strata $\widetilde{\mathsf{M}}_{\overline{a},k}$ therefore give rise to exactly $(d+1)$ irreducible components in codimension $(d+1)$. $\square$

Using the excision sequence, we conclude that these $(d+1)$ irreducible components of $\mathcal{Z}_d$ contribute to $d+1$ relations in degree $2d+2$. The remaining relation arises from the unique (up to scalar) relation in $H^*(\mathcal{V}_d)$ in degree $2d+2$, as shown in [41].

Chapter 4

Stratification of Quot schemes on curves

In this chapter, we study the geometry of the relative Quot scheme $\mathrm{Quot}_d(V,\varphi)$ *associated with a family* $\varphi\colon X \to Y$ *of smooth projective curves. We provide an explicit geometric stratification of* $\mathrm{Quot}_d(V,\varphi)$ *in the case where* V *admits a filtration whose successive quotients are line bundles. This result has immediate applications to the stratification of the Quot scheme* $\mathrm{Quot}_d(V,C)$*, and we also present several further applications.*

Let $\varphi\colon X \to Y$ be a family of smooth projective curves, and let V be a vector bundle of rank n on X that admits a filtration with line bundle quotients:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V.$$

Consider the relative Quot scheme $\mathrm{Quot}_d(V,\varphi)$ as defined in Chapter 2. Recall from Section 1.2, it admits a stratification

$$\mathrm{Quot}_d(V,\varphi) = \bigsqcup_{m=0}^{d} \mathrm{Quot}_{d,m},$$

such that for each stratum there is a morphism

$$\Phi_{d,m}\colon \mathrm{Quot}_{d,m} \to \mathrm{Quot}_m(V_{n-1},\varphi) \times_Y \mathrm{Sym}^{d-m}_Y X.$$

Let $\widetilde{\mathcal{F}}_m$ denote the universal quotient on $\mathrm{Quot}_d(V_{n-1},\varphi) \times_Y X$, and let $\mathcal{A}_{d-m}$ denote the universal subbundle on $\mathrm{Sym}^{d-m}_Y X \times_Y X$. Define a vector bundle

$$\mathbb{S}_{d,m} := \pi_*(\mathcal{A}^\vee_{d-m} \otimes \widetilde{\mathcal{F}}_m)$$

on $\mathrm{Quot}_d(V_{n-1},\varphi) \times_Y \mathrm{Sym}^{d-m}_Y X$. We may now state the following lemma.

Lemma 4.0.1. *The morphism* $\Phi_{d,m}$ *is an affine bundle, and* $\mathrm{Quot}_{d,m}$ *is isomorphic to the total space of the vector bundle* $\mathbb{S}_{d,m}$*.*

Our main theorem follows from Lemma 4.0.1 by an inductive argument.

Theorem 4.0.2. *Let V be a vector bundle of rank n on X filtered with line bundle quotients. The Quot scheme $\mathrm{Quot}_d(V,\varphi)$ admits a stratification by locally closed subvarieties $\mathrm{Quot}_{d,\underline{m}}$ which are affine bundles over $\mathrm{Sym}^{m_1}_Y X \times_Y \ldots \times_Y \mathrm{Sym}^{m_n}_Y X$ where $\underline{m} = (m_1,\ldots,m_n)$ ranges over n-tuples of non-negative integers with $m_1+\ldots+m_n = d$.*

Theorem 4.0.2 has numerous corollaries and applications. We postpone the discussion of these results to sections 4.1 and 4.2, after completing the proof of the theorem.

4.1 Stratification of $\mathrm{Quot}_d(V,\varphi)$ for V filtered with line bundle quotients

In this section, we prove Lemma 4.0.1 and Theorem 4.0.2. We also derive several corollaries.

Lifting Theorem. Recall that we have a morphism $\Phi_{d,m}\colon \mathrm{Quot}_{d,m} \to \mathrm{Quot}_d(\widetilde{V},\varphi) \times_Y \mathrm{Sym}^{d-m}_Y X$. On $\mathrm{Quot}_m(\widetilde{V},\varphi) \times_Y \mathrm{Sym}^{d-m}_Y X \times_Y X$, we have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \widetilde{\mathcal{E}}_m \arrow[r] & \rho^*\widetilde{V} \arrow[r] \arrow[d] & \widetilde{\mathcal{F}}_m \arrow[r] & 0 \\
 & & \rho^*V \arrow[d] & & \\
0 \arrow[r] & \mathcal{A}_{d-m} \arrow[r] & \rho^*L \arrow[r] \arrow[d] & \mathcal{B}_{d-m} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd} \tag{4.1}$$

where the top and bottom rows are universal sequences on $\mathrm{Quot}_m(\widetilde{V},\varphi) \times_Y X$ and $\mathrm{Sym}^{d-m}_Y X \times_Y X$, respectively. Our goal is to complete this diagram by constructing a lift on $\mathrm{Quot}_{d,m} \times_Y X$. Define the morphisms

$$\alpha_{d,m}\colon \widetilde{\mathcal{E}}_m \to \rho^*V \text{ and } \beta_{d,m}\colon \rho^*V \to \mathcal{B}_{d-m}$$

obtained by compositions in the diagram above. We then have the following; a closely related statement appears in [24, Theorem 6.4.5].

Theorem 4.1.1. *The following statements hold.*

1. *Define $\mathcal{K}_{d,m} := \ker(\beta_{d,m})/\mathrm{im}(\alpha_{d,m})$. Then the diagram (4.1) induces a short exact sequence in $\mathrm{Ext}^1(\mathcal{A}_{d-m},\widetilde{\mathcal{F}}_m)$,*

$$0 \to \widetilde{\mathcal{F}}_m \to \mathcal{K}_{d,m} \to \mathcal{A}_{d-m} \to 0. \tag{4.2}$$

2. *Splittings of sequence (4.2) are in bijection with inclusions $\mathcal{E} \subset \rho^*V$, up to automorphisms, for which the diagram (4.1) can be completed.*

Remark 4.1.2. For a split sequence $0 \to A \to B \to C \to 0$, the set of all possible splittings is in bijection with $\mathrm{Hom}(C,A)$.

Proofs of Lemma 4.0.1 and Theorem 4.0.2 With the machinery developed above, we now proceed directly to the proofs.

Proof of Lemma 4.0.1. Let $\mathbb{S}_{d,m}$ denote the total space of the vector bundle $\pi_*(\mathcal{A}_{d-m}^\vee \otimes \widetilde{\mathcal{F}}_m)$. On $\mathbb{S}_{d,m} \times_Y X$, there is the universal morphism (see [24, Theorem 5.8])

$$\phi_{d,m} \colon \mathcal{A}_{d-m} \to \widetilde{\mathcal{F}}_m.$$

The vanishing $R^1\pi_*(\mathcal{A}_{d-m}^\vee \otimes \widetilde{\mathcal{F}}_m) = 0$ implies that the Leray spectral sequence degenerates, yielding a natural isomorphism

$$H^1(\mathcal{A}_{d-m}^\vee \otimes \widetilde{\mathcal{F}}_m) \cong H^1(\pi_*(\mathcal{A}_{d-m}^\vee \otimes \widetilde{\mathcal{F}}_m)). \tag{4.3}$$

Let $U \subset \mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y \mathrm{Sym}_Y^{d-m} X$ be an affine open subset, and set $U_X := U \times_Y X$. Then $H^1(U_X, (\mathcal{A}_{d-m}^\vee \otimes \widetilde{\mathcal{F}}_m)|_{U_X}) = 0$. Let W_X denote the preimage of U_X in $\mathbb{S}_d \times_Y X$. We therefore obtain the vanishing

$$\mathrm{Ext}^1\left(\mathcal{A}_{d-m}|_{W_X}, \widetilde{\mathcal{F}}_m|_{W_X}\right) = 0.$$

Applying Theorem 4.1.1 and Remark 4.1.2 to the pullback of diagram (4.1) to W_X, we see that liftings are in bijection with elements of $\mathrm{Hom}(\mathcal{A}_{d-m}|_{W_X}, \widetilde{\mathcal{F}}_m|_{W_X})$. Choosing the restriction of $\phi_{d,m}$ to W_X therefore yields a lifting, which defines a morphism $\chi \colon W_X \to \mathrm{Quot}_{d,m}$. By construction, χ is an isomorphism onto its image and is compatible with the natural projection to $U \subset \mathrm{Quot}_m(\widetilde{V}, \varphi) \times_Y \mathrm{Sym}_Y^{d-m} X$. $\square$

We can immediately prove the main result.

Proof of Theorem 4.0.2. We proceed by induction on the rank n of V. When $n = 1$, the statement follows directly from the isomorphism (2.1).

Now assume $n > 1$. Since V is filtered with line bundle quotients, we obtain the stratification (1.9). In particular, for each $m = 0, \ldots, d$, we have morphisms $\Phi_{d,m} \colon \mathrm{Quot}_{d,m} \to \mathrm{Quot}_m(V_{n-1}) \times_Y \mathrm{Sym}_Y^{d-m} X$. By the inductive hypothesis, the Quot scheme $\mathrm{Quot}_m(V_{n-1}, \varphi)$ admits a stratification by locally closed subvarieties $\mathrm{Quot}_{m,\underline{\alpha}}$, each of which is an affine bundle over $\mathrm{Sym}_Y^{\alpha_1} X \times_Y \ldots \times_Y \mathrm{Sym}_Y^{\alpha_{n-1}} X$ where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ is a $(n-1)$-tuple of non-negative integers satisfying $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_{n-1} = m$. To conclude the proof, set $\underline{m} := (\alpha_1, \ldots, \alpha_{n-1}, d-m)$ and define $\mathrm{Quot}_{d,\underline{m}}$ to be the pullback of $\mathrm{Quot}_{m,\underline{\alpha}} \times_Y \mathrm{Sym}_Y^{d-m} X$ along $\Phi_{d,m}$, as shown in the following Cartesian diagram

$$\begin{tikzcd}
\mathrm{Quot}_{d,\underline{m}} \arrow[r] \arrow[d] & \mathrm{Quot}_{d,m} \arrow[d, "\Phi_{d,m}"] \\
\mathrm{Quot}_{m,\underline{\alpha}} \times_Y \mathrm{Sym}_Y^{d-m} X \arrow[r] \arrow[d] & \mathrm{Quot}_m(V_{n-1}, \varphi) \times_Y \mathrm{Sym}_Y^{d-m} X \arrow[d] \\
\mathrm{Sym}_Y^{\alpha_1} X \times_Y \ldots \times_Y \mathrm{Sym}_Y^{\alpha_{n-1}} X \times_Y \mathrm{Sym}_Y^{d-m} X \arrow[r] & \mathrm{Sym}_Y^{m} X \times_Y \mathrm{Sym}_Y^{d-m} X
\end{tikzcd}$$

where the morphism $\mathrm{Quot}_m(V_{n-1}, \varphi) \to \mathrm{Sym}_Y^m X$ in the bottom right is the Quot–to–Chow morphism (2.2). $\square$

Let $\mathrm{Quot}_d(V, \varphi)$ be as in Theorem 4.0.2. We then obtain the following formula for its class in $\mathrm{K}_0(\mathrm{Var}_k)$.

Corollary 4.1.3. *For every $d \geq 0$, the following identity holds in $\mathrm{K}_0(\mathrm{Var}_k)$,*

$$[\mathrm{Quot}_d(V, \varphi)] = \sum_{\substack{\underline{m} \in \mathbb{N}^n \\ \sum m_i = d}} [\mathrm{Sym}_Y^{m_1} X \times_Y \ldots \times_Y \mathrm{Sym}_Y^{m_n} X] \, \mathbb{L}^{\boldsymbol{a}(\underline{m})}, \tag{4.4}$$

where $\boldsymbol{a}(\underline{m}) := \sum_{i=1}^n m_i(n-i)$.

Remark 4.1.4. It follows immediately from Corollary 4.1.3 that the class $[\mathrm{Quot}_d(V, \varphi)]$ depends only on d, n, and the morphism $\varphi \colon X \to Y$. When $Y = \mathrm{pt}$, the scheme X is a smooth projective curve. Since every vector bundle on a smooth projective curve admits a filtration with line bundle quotients, formula (4.4) recovers the result for $[\mathrm{Quot}_d(\mathcal{O}_C^n, C)]$ in [11], as well as its extension to $[\mathrm{Quot}_d(V, C)]$ for an arbitrary vector bundle V on a smooth projective curve, as obtained in [5]. We note that the previous approaches rely heavily on Białynicki–Birula decomposition.

Torsion-freeness of $H^*(\mathrm{Quot}_d(V, C))$ We use Theorem 4.0.2 to prove that the integral cohomology $H^*(\mathrm{Quot}_d(V, C))$ is torsion-free.

Proposition 4.1.5. *The integral cohomology of $\mathrm{Quot}_d(V, C)$ is torsion-free.*

Proof. As V is filtered with line bundle quotients, we may choose an exact sequence

$$0 \to L \to V \to V_{n-1} \to 0.$$

On $\mathrm{Ext}^1(V_{n-1}, L) \times C$, there is a universal extension $\mathcal{V}$ of L by V_{n-1}. Write the projection $\pi \colon \mathrm{Ext}^1(V_{n-1}, L) \times C \to \mathrm{Ext}^1(V_{n-1}, L)$, and consider the relative Quot scheme $f \colon \mathrm{Quot}_d(\mathcal{V}, \pi) \to \mathrm{Ext}^1(V_{n-1}, L)$; this morphism f is smooth and projective. Hence, by Ehresmann's theorem [22], all fibres of f are diffeomorphic. In particular, the fibres $\mathrm{Quot}_d(V, C)$ and $\mathrm{Quot}_d(L \oplus V_{n-1}, C)$ have isomorphic integral cohomology. Repeating this argument, we may reduce to the case in which V splits as a direct sum of line bundles. In the split setting, $\mathrm{Quot}_d(V, C)$ admits a $\mathbb{C}^*$-action induced by a $(\mathbb{C}^*)^n$-action on V. The associated Białynicki-Birula decomposition yields the stratification [10] of Theorem 4.0.2. Moreover, by [36, Formula 0.1 and Section 1.], one obtains an additive decomposition

$$H^i\left(\mathrm{Quot}_d(V, C), \mathbb{Z}\right) \cong \bigoplus_{\underline{m}} H_c^{i - 2\mathrm{codim}(\mathrm{Quot}_{d,\underline{m}})}\left(\mathrm{Quot}_{d,\underline{m}}, \mathbb{Z}\right).$$

Finally, each stratum $\mathrm{Quot}_{d,\underline{m}}$ is an affine bundle over $\mathrm{Sym}^{m_1} C \times \ldots \times \mathrm{Sym}^{m_n} C$, whose integral cohomology is torsion-free [42]. It theorefore follows that $H^*(\mathrm{Quot}_d(V, C), \mathbb{Z})$ is torsion-free, as claimed. □

4.2 Non-filtered bundles and applications

In this section, we consider a vector bundle V on X that does not necessarily admit a filtration with line bundle quotients. For any family $\varphi \colon X \to Y$ of smooth projective curves, we prove that the base scheme Y admits a stratification such that, on the preimage of each stratum, V admits a filtration with line bundle quotients. We then provide two applications of this result in conjunction with the results of the previous section.

Non-filtered bundles on trivial families

Lemma 4.2.1. *Assume Y is irreducible. Then there exists a non-empty open subset $U \subset Y$ such that V is filtered with line bundle quotients on $\varphi^{-1}(U)$.*

Proof. If V is a line bundle, there is nothing to prove. We therefore assume that rank $V > 1$. Twist V by an ample line bundle $\mathcal{O}(1)$ on X so that $V(m) := V \otimes \mathcal{O}(m)$ is globally generated. Let $\xi \in Y$ be the generic point of Y, and let X_ξ denote the base change of X to ξ. The restriction $V(m)|_{X_\xi}$ is globally generated, and since X_ξ is a curve and rank $V > 1$, we may choose a section $s \in \Gamma(V(m))$ whose restriction $s|_{X_\xi}$ does not vanish anywhere along X_ξ. In particular, the zero locus $Z(s)$ does not contain X_ξ. Define

$$U := \varphi(X \setminus Z(s)) \subset Y,$$

and set $X_U := \varphi^{-1}(U)$. Then $\xi \in U$ and $X_\xi \subset X_U$. On X_U, we therefore obtain a nowhere vanishing morphism

$$\mathcal{O}_{X_U}(-m) \to V|_{X_U}.$$

Its cokernel is a vector bundle of strictly smaller rank, to which the induction hypothesis applies. The result then follows by induction. □

Theorem 4.2.2. *Let $\varphi\colon X = Y \times C \to Y$ be a trivial family of curves, and let V be a vector bundle of rank n on X. Then the following identity holds in $\mathrm{K}_0(\mathrm{Var}_k)$*

$$[\mathrm{Quot}_d(V,\varphi)] = [Y]\,[\mathrm{Quot}_d(\mathcal{O}_C^n, C)]\,. \tag{4.5}$$

Proof. We divide the proof into two steps.

1. *V is filtered with line bundle quotients.*
   In this case, Corollary (4.1.3) applies. Since $X = Y \times C$, the relative symmetric product $\mathrm{Sym}_Y^m(Y \times C)$ is isomorphic to $\mathrm{Sym}^m(C) \times Y$. We therefore obtain

   $$[\mathrm{Sym}_Y^{m_1}(Y \times C) \times_Y \cdots \times_Y \mathrm{Sym}_Y^{m_n}(Y \times C)] = [\mathrm{Sym}^{m_1} C] \cdot \ldots \cdot [\mathrm{Sym}^{m_n} C]\,[Y]\,.$$

   The claim follows immediately.

2. *V is not filtered with line bundle quotients.*
   If $\dim Y = 0$, the statement is clear. Assume therefore that the statement holds whenever $\dim Y < k$. Let $Y_1, \ldots, Y_l$ be the irreducible components of Y of dimension k, and set

   $$W_i := Y_i \setminus \bigcup_{j \neq i} Y_j.$$

   Each W_i is irreducible, so by Lemma 4.2.1 there exists an open subset $U_i \subset W_i$ on which V is filtered with line bundle quotients. On each U_i, the result follows from previous step. Set $U := \cup_i U_i$, and let $Z := Y \setminus U$. Since $\dim Z < k$, The induction hypothesis applies to Z. Using isomorphism (2.3), we obtain

   $$\begin{aligned}[\mathrm{Quot}_d(V,\varphi)] &= [\mathrm{Quot}_d(V,\varphi) \times_Y U] + [\mathrm{Quot}_d(V,\varphi) \times_Y Z] \\ &= \sum_{i=1}^{l} [\mathrm{Quot}_d(V|_{U_i \times C}, \varphi|_{U_i \times C})] + [\mathrm{Quot}_d(V|_{Z \times C}, \varphi|_{Z \times C})] \\ &= [Y]\,[\mathrm{Quot}_d(\mathcal{O}_C^n, C)]\,.\end{aligned}$$

   This completes the proof. □

Nested relative Quot scheme. We now prove an analogue of Theorem 4.2.2 for relative nested Quot scheme. Throughout, we work with the trivial family $\varphi\colon Y \times C \to Y$ of smooth projective curves.

Let $0 = d_0 \leq d_1 \leq \cdots \leq d_{s+1}$ be integers. The relative nested Quot scheme $\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V,\varphi)$ is a Y-scheme whose fibre over $y \in Y$ is the nested Quot scheme,

$$\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V,\varphi)_y = \left\{ \left(E_{s+1} \subset \ldots \subset E_1 \subset V|_{\varphi}^{-1}(y)\right) \mid \deg E_i = d_i \text{ for all } i. \right\}.$$

Let $\tau_s\colon \mathrm{Quot}_{d_1,\ldots,d_s}(V,\varphi) \times C \to \mathrm{Quot}_{d_1,\ldots,d_s}(V,\varphi)$ be the projection. The relative nested Quot scheme admits a recursive description via the isomorphism

$$\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V,\varphi) \cong \mathrm{Quot}_{d_{s+1}-d_s}(\mathcal{E}_s,\tau_s), \tag{4.6}$$

where $\mathcal{E}_s$ denotes the pullback of universal subbundle on $\mathrm{Quot}_{d_s}(V,\varphi) \times C$ to the product $\mathrm{Quot}_{d_1,\ldots,d_s}(V,\varphi) \times C$ along the natural forgetful morphism.

Theorem 4.2.3. *The following identity holds in* $\mathrm{K}_0(\mathrm{Var}_k)$*,*

$$\left[\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V,\varphi)\right] = [Y] \prod_{k=1}^{s+1} \left[\mathrm{Quot}_{d_k-d_{k-1}}(\mathcal{O}_C^n, C)\right].$$

Proof. We proceed by induction on s. The case $s = 0$ follows immediately from Corollary 4.2.2. For general s, we combine the isomorphism (4.6) with Corollary 4.2.2 to obtain,

$$\begin{aligned}\left[\mathrm{Quot}_{d_1,\ldots,d_{s+1}}(V,\varphi)\right] &= \left[\mathrm{Quot}_{d_{s+1}-d_s}(\mathcal{E}_s,\tau_s)\right] \\ &= \left[\mathrm{Quot}_{d_1,\ldots,d_s}(V,\varphi)\right]\left[\mathrm{Quot}_{d_{s+1}-d_s}(\mathcal{O}_C^n, C)\right] \\ &= [Y] \prod_{k=1}^{s+1} \left[\mathrm{Quot}_{d_k-d_{k-1}}(\mathcal{O}_C^n, C)\right],\end{aligned}$$

as required. □

Remark 4.2.4. When $Y = \mathrm{pt}$, Theorem 4.2.3 together with Corollary 4.1.3 recovers the formula for the class of nested Quot scheme in $\mathrm{K}_0(\mathrm{Var}_k)$ proved in [50], albeit in a different form. We note that the proof in [50] relies on Białynicki–Birula decomposition.

The Quot scheme of positive rank quotients. We now consider the Quot scheme $\mathrm{Quot}_d(\mathcal{O}_C^n, r, C)$ of rank r and degree d quotients of $\mathcal{O}_C^n$ on a smooth projective curve C. The goal of this section is to give a geometric stratification of $\mathrm{Quot}_d(\mathcal{O}_C^n, r, C)$ using Theorem 4.0.2. In particular, we prove the following result.

Theorem 4.2.5. *The class* $[\mathrm{Quot}_d(\mathcal{O}_C^n, r, C)]$ *is divisible by the class* $[\mathrm{Gr}(r,n)]$ *in* $\mathrm{K}_0(\mathrm{Var}_k)$*.*

Remark 4.2.6. If there existed a $\mathrm{GL}_n(\mathbb{C})$-equivariant surjective morphism

$$\mathrm{Quot}_d(\mathcal{O}_C^n, r, C) \to \mathrm{Gr}(r,n),$$

then it would automatically be a Zariski locally trivial fibration by Lemma 4.2.7 below, and Theorem 4.2.5 would follow immediately. However, such a morphism does not exist in general. For instance, take $d = 1$, $r = 1$, $n = 2$, and $C \cong \mathbb{P}^1$. Then the Quot scheme $\mathrm{Quot}_1(\mathcal{O}^2, 1, \mathbb{P}^1)$ is isomorphic to $\mathbb{P}^3$, while $\mathrm{Gr}(1,2) \cong \mathbb{P}^1$; clearly, there are no surjective morphisms $\mathbb{P}^3 \to \mathbb{P}^1$.

Consider the universal sequence on $\mathrm{Quot}_d(\mathcal{O}_C^n, r, C) \times C$,

$$0 \to \mathcal{E} \to \rho^* \mathcal{O}_C^n \to \mathcal{F} \to 0. \tag{4.7}$$

As in [8], we stratify $\mathrm{Quot}_d(\mathcal{O}_C^n, r, C)$ according to the degree of torsion in the quotient

$$\mathrm{Quot}_d(\mathcal{O}_C^n, r, C) = \bigsqcup_{m=0}^{d} \mathrm{Quot}_m.$$

The locally closed subvarieties Quot_m parametrise surjections $\mathcal{O}_C^n \to F$ fitting into exact sequences

$$0 \to T \to F \to B \to 0,$$

where B is a vector bundle of rank r and degree $d-m$, and T is a torsion sheaf of degree m. On $\mathrm{Quot}_m \times C$, define the sheaf

$$\mathcal{T} := \mathcal{E}xt^1 \left(\mathcal{E}xt^1 \left(\mathcal{F}|_{\mathrm{Quot}_m \times C}, \mathcal{O}_{\mathrm{Quot}_m \times C} \right), \mathcal{O}_{\mathrm{Quot}_m \times C} \right).$$

We use the same notation for the restriction of sequence (4.7) to $\mathrm{Quot}_m \times C$. This yields the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \rho^* \mathcal{O}^n \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{B} \arrow[r, equal] \arrow[d] & \mathcal{B} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}.$$

The middle column parametrises vector bundle quotients of $\mathcal{O}_C^n$ of rank r and degree $d-m$, while the top row parametrises torsion quotients of degree m. The middle column is therefore identified with the space of morphisms $C \to \mathrm{Gr}(r,n)$ of degree $d-m$, which we denote by $\mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n))$. We may describe a point of Quot_m as follows: first choose an element $\widetilde{E} \subset \mathcal{O}_C^n$ in $\mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n))$, and then choose a quotient of $\widetilde{E}$ of degree m. This description allows us to realise Quot_m as a relative Quot scheme over the morphism $\tau_{d-m} \colon \mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n)) \times C \to \mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n))$; explicitly we have an isomorphism

$$\mathrm{Quot}_m \cong \mathrm{Quot}_m(\widetilde{\mathcal{E}}, \tau_{d-m}). \tag{4.8}$$

To prove Theorem 4.2.5, let us examine $\mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n))$ more closely. Fix a point $p \in C$, and consider the evaluation morphism

$$ev_p \colon \mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n)) \to \mathrm{Gr}(r,n), \quad \phi \mapsto \phi(p).$$

This morphism is $\mathrm{GL}_n(\mathbb{C})$-equivariant, and hence its fibres are all isomorphic; we denote the fibre by F_{d-m}.

Lemma 4.2.7. *The morphism ev_p is a Zariski locally trivial fibration.*

Proof. The morphism $\pi\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Gr}(r,n)$ is a Zariski locally trivial fibration by [14, Theorem 4.13]. Let $U \subset \mathrm{Gr}(r,n)$ be an open subset over which π trivialises, and choose a local section $s_U : U \to \pi^{-1}(U)$. Define a morphism

$$\varphi_U : U \times F_{d-m} \to ev_p^{-1}(U) \text{ given by } (u,t) \mapsto s_U(u)t$$

whose inverse is given by $x \mapsto (ev_p(x), s_U(ev_p(x))^{-1}x)$. □

We can now prove Theorem 4.2.5.

Proof of Theorem 4.2.5. Applying Theorem 4.2.2 to the isomorphism (4.8), we obtain

$$[\mathrm{Quot}_m] = [\mathrm{Mor}_{d-m}(C, \mathrm{Gr}(r,n))] \cdot \left[\mathrm{Quot}_m(\mathcal{O}_C^{n-r}, C)\right].$$

Combining this with Lemma 4.2.7, we conclude that

$$[\mathrm{Quot}_d(\mathcal{O}_C^n, r, C)] = [\mathrm{Gr}(r,n)] \sum_{m=0}^{d} [F_{d-m}] \cdot \left[\mathrm{Quot}_m(\mathcal{O}_C^{n-r}, C)\right]. \qquad \square$$

Remark 4.2.8. Theorem 4.2.5 immediately implies that the Poincaré polynomial of $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$ is divisible by the Poincaré polynomial of $\mathrm{Gr}(r,n)$. This observation already appears in [18], where it is obtained using the Białynicki–Birula decomposition.

Chapter 5

Cohomology of Schur functors of tautological bundles

In this chapter, we study the cohomology of tautological complexes *on the Quot scheme* $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$ *parametrising rank* r *and degree* d *quotients of* $\mathcal{O}^n_{\mathbb{P}^1}$ *on* $\mathbb{P}^1$*. The cohomology of the Schur complexes associated to these tautological complexes admits particularly clean formulae, which we describe explicitly. These descriptions also allow us to construct explicit exceptional collections in the bounded derived category* $\mathrm{D}^b(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1))$ *of coherent sheaves on* $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$*.*

Recall from Section 1.3 that, given a vector bundle M on $\mathbb{P}^1$, one can associate to it a tautological complex

$$M^{[d]} := \pi_*(\mathcal{F} \otimes \rho^* M)$$

on $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$. The main result of this chapter is as follows.

Theorem 5.0.1. *Let* λ *be a partition satisfying the inequality* $|\lambda| < (nd + n)/(n - r)$*. Then there is a natural isomorphism,*

$$H^\bullet(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1), \mathbb{S}^\lambda M^{[d]}) \cong \mathbb{S}^\lambda H^\bullet(\mathbb{P}^1, M^{\oplus n}).$$

Again, recalling from Section 1.3 the ordering (1.12) on the set of all partitions, we state the following result.

Theorem 5.0.2. *Let* λ *be a partition, and let* L *be line bundle of degree* m *on* $\mathbb{P}^1$*. For each* $m \geq d$*, the collection*

$$\left\{ \mathbb{S}^\lambda L^{[d]} \mid |\lambda| \leq d,\ \lambda_1 < n - r \right\},$$

forms an exceptional collection in $\mathrm{D}^b(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1))$*.*

Plan of proof. For each $m \geq d$, there is an embedding $\iota_m \colon \mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1) \hookrightarrow \mathrm{Gr}_{m-1} \times \mathrm{Gr}_m$ whose image is cut out by a section of a vector bundle that can be expressed in terms of universal bundles on Gr_{m-1} and Gr_m; see [63]. This description allows us to resolve tautological vector bundles $\mathcal{O}_{\mathbb{P}^1}(m-1)^{[d]}$ and $\mathcal{O}_{\mathbb{P}^1}(m)^{[d]}$ on $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$ via Koszul resolutions whose terms are written entirely in terms of universal bundles on $\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m$. At this point, we may apply the classical Borel–Weil–Bott theorem together with the existence of full exceptional collection on the derived category of Grassmannians [35] to prove Theorem 5.0.2 completely. The same approach also establishes Theorem 5.0.1 for two-fold tensor products of tautological bundles associated to sufficiently positive line bundles on $\mathbb{P}^1$. To prove Theorem 5.0.1 in full generality, we employ an induction argument based on an explicit resolution of tautological complexes by vector bundles to which the previous result applies.

Notation. When working with several line bundles simultaneously, we adopt the notation $\mathsf{F}_m := L^{[d]}$, where m denotes the degree of the line bundle L.

5.1 Borel–Weil–Bott Theorem and indices

Borel–Weil–Bott theorem. Let $\mathrm{Gr}(r,n)$ be the Grassmannian of $n-r$ dimensional subspaces of the complex vector space $\mathbb{C}^n$. On $\mathrm{Gr}(r,n)$, we have the universal exact sequence

$$0 \to \mathcal{A} \to \mathcal{O}^n_{\mathrm{Gr}(r,n)} \to \mathcal{B} \to 0. \tag{5.1}$$

The cohomology of the universal bundles is governed by the Borel–Weil–Bott theorem. A convenient formulation of this theorem may be found in [65].

Theorem 5.1.1 (Borel–Weil–Bott). *Let $\mu = (\mu_1, \ldots, \mu_{n-r})$ and $\nu = (\nu_1, \ldots, \nu_r)$ be two non-increasing sequences of integers. Define*

$$\omega := (\mu, \nu) + (n-1, n-2, \ldots, 0).$$

Then the cohomology of the vector bundle $\mathbb{S}^\mu \mathcal{A}^\vee \otimes \mathbb{S}^\nu \mathcal{B}^\vee$ is described completely as follows.

1. *All cohomology groups vanish if the sequence ω contains repetitions,*
2. *Otherwise, let $\sigma \in \mathfrak{S}_n$ be the permutation that rearranges ω into a strictly decreasing sequence. Writing $\gamma := \sigma \circ \omega - (n-1, n-2, \ldots, 0)$, we have*

$$H^\ell\left(\mathrm{Gr}(r,n),\ \mathbb{S}^\mu \mathcal{A}^\vee \otimes \mathbb{S}^\nu \mathcal{B}^\vee\right) = \begin{cases} \mathbb{S}^\gamma(\mathbb{C}^n) & \text{if } \ell = \mathrm{length}(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 5.1.1, to show that all the cohomology groups of the bundle $\mathbb{S}^\mu \mathcal{A}^\vee \otimes \mathbb{S}^\nu \mathcal{B}^\vee$ vanish, it suffices to verify that ω has repetitions. If ω has no repetitions, we say that the bundle has *non-vanishing cohomology*. We now introduce two indices that capture this non-vanishing behaviour.

t-index of a non-increasing sequence of integers. t-index of a partition was introduced in [48] to study the cohomology of the vector bundle $\mathbb{S}^\nu \mathcal{B}^\vee$ for ν a partition. We give here a definition that applies to any non-increasing sequence of integers.

Definition 5.1.2 (t–index)**.** Fix a positive integer t. We say that a non-increasing sequence of integers ν has a *well-defined t–index j* if there exists a non-negative integer j such that the inequalities

$$\nu_j \geq j + t \text{ and } \nu_{j+1} \leq j \tag{5.2}$$

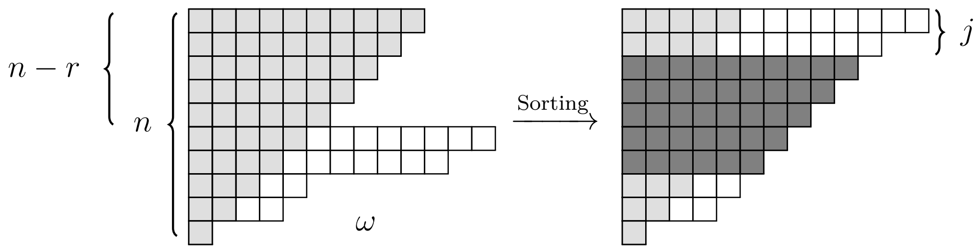
are satisfied.

Lemma 5.1.3. *Let $\nu = (\nu_1, \ldots, \nu_r)$ be a non-increasing sequence of integers. Then the Schur bundle $\mathbb{S}^\nu(\mathcal{B}^\vee)$ has non-vanishing cohomology if and only if ν has a well-defined $(n-r)$–index j.*

Proof. By Theorem 5.1.1, the bundle $\mathbb{S}^\nu(\mathcal{B}^\vee)$ has non-vanishing cohomology if and only if

$$\omega = (f_{n-r}, \ldots, f_1, g_1, \ldots g_r) := (0, \ldots, 0, \nu_1, \ldots, \nu_r) + (n-1, n-2, \ldots, r, r-1, \ldots, 0)$$

has no repetitions. The sequence ω can be represented diagrammatically as shown below, where the shaded part corresponds to the partition $(n-1, \ldots, 0)$ and the unshaded part corresponds to the partition ν.



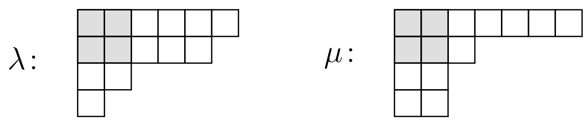
Let ℓ denote the length of the permutation that sorts ω into a strictly decreasing sequence. The integers $f_{n-r} > \cdots > f_1$ form a block of consecutive integers, represented by black boxes in the diagram on the right. Hence there exists an integer j such that the inequalities

$$g_1 > \cdots g_j > f_{n-r} > \cdots > f_1 > g_{j+1} > \cdots > g_r$$

hold. The inequalities in (5.2) follow immediately from $g_j > f_{n-r}$ and $f_1 > g_{j+1}$. $\square$

Remark 5.1.4. The 0–index is well-defined for any integer partition and equals the rank of the partition. If an integer partition λ has a well-defined t–index j, then j is also the 0–index of λ. In particular, whenever the t–index is well-defined, it is independent of t and equals the rank of the partition. Moreover, if j is the 0–index of λ, then λ has a well-defined t–index if and only if $t \leq \lambda_j - j$; see example 5.1.5 below.

Example 5.1.5. We give two examples of partitions, one of which has a well-defined 3–index and one which does not. Consider the partitions $\lambda = (6, 5, 2, 1)$ and $\mu = (7, 3, 2, 2)$.



Both partitions have rank 2, so the t-index, whenever well-defined, must be 2. However, λ has a well-defined t–index for $t = 0, 1, 2, 3$, whereas μ has a well-defined t–index only for $t = 0, 1$.

(t,η)–index of a partition. We define a relative version of t–index, which provides a criterion for detecting non-vanishing cohomology of vector bundles of the form $\mathbb{S}^{\mu}\mathcal{A}\otimes\mathbb{S}^{\eta}\mathcal{B}$, where μ is a partition and η is a non-increasing sequence of integers.

Definition 5.1.6. Fix a positive integer t and a non-increasing sequence of integers $\eta=(\eta_1,\dots,\eta_r)=(\gamma,-\delta)$, where γ and δ are partitions. We say that a partition $\mu=(\mu_1,\dots,\mu_{n-r})$ has a *well-defined* (t,η)*–index* if there exists a non-negative integer i such that, for all $k\geq 1$, the inequalities

$$\mu_{i+1-k}\geq i+t-\gamma_k^{\dagger} \text{ and } \mu_{i+k}\leq i+\delta_k^{\dagger} \tag{5.3}$$

hold.

Lemma 5.1.7. *Let $\mu=(\mu_1,\dots,\mu_{n-r})$ be a partition, and let $\eta=(\eta_1,\dots,\eta_r)=(\gamma,-\delta)$ be a non-increasing sequence of integers, where γ and δ are partitions. If the Schur bundle $\mathbb{S}^{\mu}\mathcal{A}\otimes\mathbb{S}^{\eta}\mathcal{B}$ has non-vanishing cohomology, then μ has a well-defined (r,η)-index i.*

Proof. Write $\gamma=(\gamma_1,\dots,\gamma_p)$ and $-\delta=(-\delta_q,\dots,-\delta_1)$. Consider the sequence,

$$(f_{n-r},\dots,f_1,g_1,\dots,g_p,h_q,\dots,h_1):=(-\mu_{n-r},\dots,\gamma_1,\dots,-\delta_q,\dots)+(n-1,\dots,0),$$

and set $\omega=(f_{n-r},\dots,f_1,g_1,\dots,g_p,h_q,\dots,h_1)$. By assumption, ω has no repetitions. We show that μ has a well-defined (r,η)–index. Define

$$i:=\#\{j:q>f_j \text{ and } 1\leq j\leq n-r\}, \tag{5.4}$$

and set $i=0$ whenever $f_1\geq q$. We first verify the inequalities in (5.3) involving the initial i parts of μ. Suppose, for a contradiction, that there exists an integer k with $1\leq k\leq i$ such that $\mu_{i+1-k}<i+r-\delta_k^{\dagger}$. We obtain a contradiction by constructing a sequence of integers

$$\phi:=(f_{i-(k-1)},f_{i-(k-2)}\dots,f_i,h_{\delta_k^{\dagger}+1},h_{\delta_k^{\dagger}+2},\dots,h_q)$$

containing $q-\delta_k^{\dagger}+k$ elements lying strictly between $\delta_k^{\dagger}-k$ and $q=\delta_1^{\dagger}$. This forces a repetition in ω. Indeed, the definitions of i and q imply $q>f_{i+1-e}$ and $q>h_{e'}$ for all $e,e'\geq 1$. For $1\leq e\leq k$, we therefore have

$$f_{i+1-e}\geq f_{i+1-k}=-\mu_{i+1-k}+r+i-k>\delta_k^{\dagger}-k.$$

Moreover, whenever $e'>\delta_k^{\dagger}$, we have $\delta_{e'}<k$, and hence

$$h_{e'}=-\delta_{e'}+(e'-1)>-k+\delta_k^{\dagger}.$$

Thus all elements of ϕ lie in the claimed interval, contradicting the assumption that ω has no repetitions.

The second family of inequalities in (5.3) is proved similarly. Suppose, for a contradiction, that there exists $k\geq 1$ such that $\mu_{i+k}>i+\gamma_k^{\dagger}$. We then construct a sequence of integers

$$\psi:=(f_{i+1},f_{i+2},\dots,f_{i+k},g_{\gamma_k^{\dagger}+1},g_{\gamma_k^{\dagger}+2},\dots,g_p)$$

which contains $k+p-\gamma_k^\dagger$ elements lying strictly between $q-1$ and $p+q-\gamma_k^\dagger+k-1$, again forcing a repetition. Indeed, the definitions of i and q imply the inequalities $f_{i+e} > q-1$ and $g_{e'} > q-1$ for all $e, e' \geq 1$. For $1 \leq e \leq k$, we have

$$f_{i+e} \leq -\mu_{i+k} + r + i + k - 1 < p + q - \gamma_k^\dagger + k - 1.$$

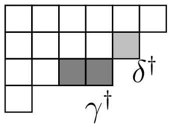
Moreover, whenever $e' > \gamma_k^\dagger$, we have $\gamma_{e'} < k$, and hence

$$g_{e'} < n - r - \gamma_k^\dagger + k - 1 = p + q - \gamma_k^\dagger + k - 1.$$

Thus all elements of ψ lie in the required interval, contradicting the assumption that ω has no repetitions. This completes the proof. $\square$

Example 5.1.8. Consider the sequence $\eta = (1, 1, -1)$. Both partitions λ and μ from Example 5.1.5 have a well-defined (t, η)-index 2 for $t = 2$.

Example 5.1.9. Let η be as in Example 5.1.8, and consider the partition $\nu = (6, 4, 4, 1)$.

$\nu =$


For $t = 2$, the partition ν has a well-defined (t, η)–index 2. In the diagram, the partition $\gamma^\dagger$, in black, is contained in ν, whereas $\delta^\dagger$, in grey, is not. It is therefore clear that $(2, \eta)$–index of ν differs from its rank.

5.2 Cohomology of tensor products of Schur bundles

In this section, we use the machinery developed in the preceding section to study the cohomology of the vector bundles $\mathbb{S}^\lambda \mathsf{F}_m \otimes \mathbb{S}^\nu \mathsf{F}_{m+1}$ on the Quot scheme $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$.

Strømme Embedding Recall that on $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1) \times \mathbb{P}^1$ we have the universal exact sequence

$$0 \to \mathcal{E} \to \rho^* \mathcal{O}_{\mathbb{P}^1}^n \to \mathcal{F} \to 0.$$

For $m \geq d-1$, let L be a line bundle of degree m. Twisting the universal sequence by L and pushing it forward along the projection π to $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$, we obtain an exact sequence of vector bundles

$$0 \to \pi_*(\mathcal{E} \otimes \rho^* L) \to H^0(L)^{\oplus n} \otimes \mathcal{O}_{\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)} \to \mathsf{F}_m \to 0.$$

This sequence induces a morphism $\phi_m \colon \mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1) \to \mathrm{Gr}(r_m, n_m)$, where $\mathrm{Gr}(r_m, n_m)$ denotes the Grassmannian of $r_m = (m+1)r + d$ dimensional quotient spaces of $n_m = (m+1)n$ dimensional vector space $H^0(L)^{\oplus n}$. Let

$$0 \to \mathcal{A}_m \to H^0(L)^{\oplus n} \otimes \mathcal{O}_{\mathrm{Gr}(r_m, n_m)} \to \mathcal{B}_m \to 0$$

be the universal exact sequence on $\mathrm{Gr}(r_m, n_m)$. We may now state the following theorem, which realises $\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}^n, r, \mathbb{P}^1)$ as a closed subvariety of the product $\mathrm{Gr}(r_{m-1}, n_{m-1}) \times \mathrm{Gr}(r_m, n_m)$.

Theorem 5.2.1 ([63])**.** *For all $m \geq d$, the morphism*

$$\iota_m := (\phi_{m-1}, \phi_m)\colon \mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1) \to \mathrm{Gr}(r_{m-1}, n_{m-1}) \times \mathrm{Gr}(r_m, n_m)$$

is a closed embedding. Moreover, the image $\iota_m(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1))$ is cut out by a regular section of the vector bundle $\mathcal{A}^\vee_{m-1} \otimes \mathcal{B}_m \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$.

For the remainder of the chapter, we use the shorthand $\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m$ to denote the product $\mathrm{Gr}(r_{m-1}, n_{m-1}) \times \mathrm{Gr}(r_m, n_m)$. We also introduce the notation

$$\mathcal{K}_m := \mathcal{A}^\vee_{m-1} \otimes \mathcal{B}_m \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$

for the vector bundle appearing in Theorem 5.2.1. An immediate corollary of Theorem 5.2.1 is the existence of a *Koszul resolution*,

$$\cdots \to \bigwedge^2 \mathcal{K}^\vee_m \to \mathcal{K}^\vee_m \to \mathcal{O}_{\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m} \to \mathcal{O}_{\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)} \to 0. \tag{5.5}$$

For later use, we record the following isomorphism.

Proposition 5.2.2. *There is a decomposition into the direct sum of Schur bundles,*

$$\bigwedge^t \mathcal{K}^\vee_m \cong \bigoplus_{|\mu|=t} \mathbb{S}^\mu \mathcal{A}_{m-1} \boxtimes \left(\bigoplus_{\alpha,\beta,\sigma} (\mathbb{S}^\sigma \mathcal{B}^\vee_m)^{\oplus c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta}} \right).$$

Proof. Using the definition of $\mathcal{K}_m$ and applying Cauchy's formula (2.10), we obtain

$$\bigwedge^t \mathcal{K}^\vee_m = \bigoplus_{|\mu|=t} \mathbb{S}^\mu \mathcal{A}_{m-1} \boxtimes \mathbb{S}^{\mu^\dagger}(\mathcal{B}^\vee_m \oplus \mathcal{B}^\vee_m).$$

Applying formula (2.11), we may write $\mathbb{S}^{\mu^\dagger}(\mathcal{B}^\vee_m \oplus \mathcal{B}^\vee_m)$ as $\bigoplus_{\alpha,\beta} (\mathbb{S}^\alpha \mathcal{B}^\vee_m \otimes \mathbb{S}^\beta \mathcal{B}^\vee_m)^{\oplus c^{\mu^\dagger}_{\alpha,\beta}}$. Finally, decomposing each tensor product $\mathbb{S}^\alpha \mathcal{B}^\vee_m \otimes \mathbb{S}^\beta \mathcal{B}^\vee_m$ using Littlewood–Richardson rule yields the stated decomposition. □

By construction of the embedding ι_m, we have natural isomorphisms,

$$\mathsf{F}_{m-1} \cong \iota^*_m \mathcal{B}_{m-1} \text{ and } \mathsf{F}_m \cong \iota^*_m \mathcal{B}_m. \tag{5.6}$$

We may therefore use the Koszul resolution (5.5) to compute the cohomology of the Schur bundles on the Quot scheme. In particular, we record the Koszul resolutions for the bundles $\mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}_m$ and $\mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}^\vee_{m-1}$ in the following proposition.

Proposition 5.2.3. *For each $t \geq 0$, define vector bundles*

$$\mathcal{V}_t := \left(\mathbb{S}^\lambda \mathcal{B}_{m-1} \boxtimes \mathbb{S}^\nu \mathcal{B}_m\right) \otimes \bigwedge^t \mathcal{K}^\vee \text{ and } \mathcal{W}_t := \left(\mathbb{S}^\lambda \mathcal{B}_{m-1} \boxtimes \mathbb{S}^\nu \mathcal{B}^\vee_{m-1}\right) \otimes \bigwedge^t \mathcal{K}^\vee.$$

Then the vector bundle $\mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}_m$ admits a resolution

$$\cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{V}_0 \to \mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}_m \to 0, \tag{5.7}$$

and the vector bundle $\mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}^\vee_{m-1}$ admits a resolution

$$\cdots \to \mathcal{W}_2 \to \mathcal{W}_1 \to \mathcal{W}_0 \to \mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}^\vee_{m-1} \to 0. \tag{5.8}$$

Cohomology of the vector bundle $\mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}_m$. Fix two line bundles L_{m-1} and L_m on $\mathbb{P}^1$ of degrees $m-1$ and m, respectively. We now prove the following key technical result, which is required for the proof of Theorem 5.0.1.

Theorem 5.2.4. *Let λ and ν be integer partitions satisfying the inequality $(n-r)(|\lambda|+|\nu|) < nd+n$. Then, for m sufficiently large, we have*

$$H^0\left(\mathrm{Quot}_d, \mathbb{S}^\lambda \mathsf{F}_{m-1} \otimes \mathbb{S}^\nu \mathsf{F}_m\right) \cong \mathbb{S}^\lambda H^0\left(\mathbb{P}^1, L_{m-1}^{\oplus n}\right) \otimes \mathbb{S}^\nu H^0\left(\mathbb{P}^1, L_m^{\oplus n}\right),$$

and all the higher cohomology groups vanish.

Proof. Recall the Koszul resolution (5.7). By Theorem 5.1.1, we have

$$H^0(\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m, \mathcal{V}_0) = \mathbb{S}^\lambda H^0\left(\mathbb{P}^1, L_{m-1}^{\oplus n}\right) \otimes \mathbb{S}^\nu H^0\left(\mathbb{P}^1, L_m^{\oplus n}\right),$$

and all the higher cohomology groups of $\mathcal{V}_0$ vanish. To conclude, we consider the spectral sequence $H^i(\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m, \mathcal{V}_j) \implies H^{i-j}(\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m, \mathcal{V}_\bullet)$, and show that all the cohomology groups of $\mathcal{V}_j$ vanish for all $j > 0$.

Using the definition of $\mathcal{V}_t$ and the decomposition in Proposition 5.2.2, we can write

$$\mathcal{V}_t = \bigoplus_{|\mu|=t} \mathbb{S}^\mu \mathcal{A}_{m-1} \otimes \mathbb{S}^\lambda \mathcal{B}_{m-1} \boxtimes \left(\bigoplus_{\alpha,\beta,\sigma} \mathbb{S}^\nu \mathcal{B}_m \otimes (\mathbb{S}^\sigma \mathcal{B}_m^\vee)^{\oplus c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta}} \right).$$

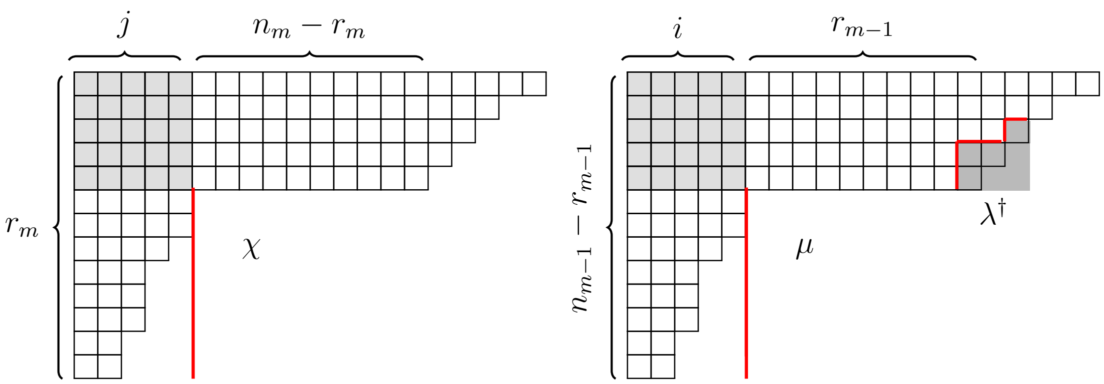
Applying Littlewood–Richardson rule, we may further decompose the direct summands further and obtain the summands of the form

$$\left(\mathbb{S}^\mu \mathcal{A}_{m-1} \otimes \mathbb{S}^{-\lambda} \mathcal{B}_{m-1}^\vee\right) \boxtimes \mathbb{S}^\chi \mathcal{B}_m^\vee \text{ with } c^{\mu^\dagger}_{\alpha,\beta} \cdot c^\sigma_{\alpha,\beta} \cdot c^\chi_{\sigma,-\nu} \neq 0.$$

Suppose, for a contradiction, that for some choice of partitions $\mu, \lambda, \nu, \alpha, \beta, \sigma$ and some non-increasing sequence of integers χ, the summand above has non-vanishing cohomology. Then, by Lemmas 5.1.3 and 5.1.7, we may define indices

$$\begin{aligned} j &:= (n_m - r_m)\text{–index of } \chi, \text{ and} \\ i &:= (r_{m-1}, -\lambda)\text{–index of } \mu. \end{aligned}$$

These indices are illustrated in the diagrams below; the red lines indicate the inequalities (5.2) for χ and (5.3) for μ.



Before proceeding, note that $c^{\chi}_{\sigma,-\nu} \neq 0$ implies that, upon twisting by $(\nu_1)^{r_m}$,

$$c^{\chi}_{\sigma,-\nu} = c^{\chi+(\nu_1)^{r_m}}_{\sigma,-\nu+(\nu_1)^{r_m}} \neq 0. \tag{5.9}$$

We now show that each possible inequality relating j and i leads to a contradiction.

1. *Suppose $j > i$.*
   Using $c^{\chi}_{\sigma,-\nu} \neq 0$ and the diagram for χ, we obtain

   $$j + n_m - r_m \leq \chi_j \leq \sigma_j - \nu_r \leq \sigma_j,$$

   where the penultimate inequality follows from Weyl's inequality (2.12) applied to the twisted Littlewood–Richardson coefficient in equation (5.9). Applying Weyl's inequality (2.12) to $c^{\sigma}_{\alpha,\beta} \neq 0$ gives $\sigma_j \leq \alpha_j + \beta_1$. Since $c^{\mu^\dagger}_{\alpha,\beta} \neq 0$, we have $\beta_1 \leq \mu^\dagger_1 \leq n_{m-1} - r_{m-1} = \operatorname{rank} \mathcal{A}_{m-1}$. Hence

   $$j + n_m - r_m \leq \sigma_j \leq \alpha_j + \beta_1 \leq \alpha_j + n_{m-1} - r_{m-1},$$

   so $j + n - r \leq \alpha_j$. Taking conjugate yields $j \leq \alpha^\dagger_{j+(n-r)}$. Using $j > i$, we obtain

   $$j \leq \alpha^\dagger_{j+(n-r)} \leq \mu_{j+(n-r)} \leq \mu_{i+(n-r)} \leq i,$$

   a contradiction.

2. *Suppose $2j < i$.*
   From $c^{\sigma}_{\alpha,\beta} \neq 0$, we have $\alpha^\dagger_{j+1} \leq \sigma^\dagger_{j+1}$. Let $\chi^\dagger_k$ denote the number of parts of χ that are at least k. By Part (2) of Lemma 5.2.5 applied to $c^{\chi}_{\sigma,-\nu} \neq 0$, we obtain $\sigma^\dagger_{j+1} \leq j + |\nu|$, so $\alpha^\dagger_{j+1} \leq j + |\nu|$. Similarly, one also has $\beta^\dagger_{j+1} \leq j + |\nu|$. Using $2j < i$ and the diagram for μ, we deduce

   $$r_{m-1} + i - \lambda^\dagger_{i-2j} \leq \mu_{2j+1} \leq \alpha^\dagger_{j+1} + \beta^\dagger_{j+1} \leq 2j + 2|\nu|.$$

   Thus $r_{m-1} - \lambda^\dagger_{i-2j} - 2|\nu| \leq 2j - i$. Since $r_{m-1} = mr + d$, taking m sufficiently large yields a contradiction.

3. *Suppose $j < i$.*
   We may assume $2j \geq i$. Using the diagrams for χ and μ, together with equation (5.9) and dominance inequalities $\sigma \leq \alpha + \beta$ and $\alpha \cup \beta \leq \mu^\dagger$, we obtain

   $$j(n_m - r_m + j) \leq \sum_{k=1}^{j} \chi_k \leq \sum_{k=1}^{j} \sigma_k \leq \sum_{k=1}^{j} \alpha_k + \beta_k \leq \sum_{k=1}^{2j} \mu^\dagger_k \leq i(n_{m-1} - r_{m-1}) + (2j - i)i. \tag{5.10}$$

   Rearranging yields

   $$j(n - r) \leq (i - j)(n_{m-1} - r_{m-1}) - (i - j)^2. \tag{5.11}$$

   Similarly, consider the diagrams for χ and μ, together with the inequalities $\mu \leq \alpha^\dagger + \beta^\dagger$ and $\alpha^\dagger \cup \beta^\dagger \leq \sigma^\dagger$. Moreover, applying [27, Lemma 4.5] to $c^{\chi}_{\sigma,-\nu} \neq 0$, we obtain

   $$i(r_{m-1} + i) \leq \sum_{k=1}^{i} \mu_i + |\lambda| \leq \sum_{k=1}^{2i} \sigma^\dagger_k + |\lambda| \leq \sum_{k=1}^{2i} \chi^\dagger_k + |\nu| + |\lambda| \leq j r_m + (2i - j)j + |\nu| + |\lambda|. \tag{5.12}$$

Rearranging gives

$$jr \geq (i-j)r_{m-1} + (i-j)^2 - |\nu| - |\lambda|. \tag{5.13}$$

Multiplying equation (5.11) by r and equation (5.13) by $(n-r)$, and using $i-j>0$, yields

$$(n-r)(|\nu|+|\lambda|) \geq nd + n,$$

contradicting the hypothesis.

4. *Suppose $i = j$.*
   Substituting $i = j$ into equation (5.11) gives $i(n-r) \leq 0$, hence $i = 0$. Substituting $i = j$ into equation (5.13) then gives $|\nu| + |\lambda| \leq 0$, so $\nu = \lambda = \emptyset$. By Lemma 5.1.7, this forces $\mu = \emptyset$, contradicting $|\mu| = t \geq 1$. □

Lemma 5.2.5. *Let $\lambda, \nu, \gamma, \delta$ be partitions and set $\eta := (\gamma, -\delta)$. If $c^{\eta}_{\nu,-\lambda} \neq 0$, then*

1. *$|\gamma| \leq |\nu|$ and $|\delta| \leq |\lambda|$,*
2. *suppose η has a well-defined t–index j. Then, $\nu^{\dagger}_{j+1} \leq j + |\lambda|$.*

Proof. By Proposition 2.5.5, we have $c^{\eta}_{\nu,-\lambda} \neq 0$ if and only if $c^{\nu}_{\eta,\lambda} \neq 0$. Define integer partitions $\rho := \eta + (\delta_1)^n$ and $\tau := \nu + (\delta_1)^n$. We prove the two statements separately.

1. Since $\rho \subseteq \tau$, we immediately obtain $|\gamma| \leq |\nu|$. Moreover, we have $|\gamma| - |\delta| = |\nu| - |\lambda|$. It therefore suffices to show $|\lambda| - |\delta| \geq 0$. Indeed,
$$|\lambda| - |\delta| \geq |\lambda| - |\nu| + |\nu| - |\delta| \geq |\lambda| - |\nu| + |\gamma| - |\delta| = 0.$$
2. Using identity (2.9), we have $\rho_{j+1} \leq j + \delta_1$. Equivalently, $\rho^{\dagger}_{j+\delta_1+1} \leq j$. Since $|\tau/\rho| = |\lambda|$, it follows that $\tau^{\dagger}_{j+\delta_1+1} \leq j + |\lambda|$. Finally, since τ is obtained from ν by adding δ_1 to each part, we have $\nu^{\dagger}_{j+1} = \tau^{\dagger}_{j+\delta_1+1}$, and the claim follows. □

Exceptional collections in $\mathrm{D}^b(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1))$. We now describe some exceptional collections in the derived category of coherent sheaves on $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$. We begin by computing the Ext groups $\mathrm{Ext}^{\bullet}(\mathbb{S}^{\nu}\mathsf{F}_{m-1}, \mathbb{S}^{\lambda}\mathsf{F}_{m-1})$.

Theorem 5.2.6. *Let λ and ν be two partitions satisfying $r|\nu| + (n-r)|\lambda| < dn + n$ and $\nu_1 < n - r$. Then, for all $m \geq d$, we have*

$$\mathrm{Hom}\left(\mathbb{S}^{\nu}\mathsf{F}_{m-1}, \mathbb{S}^{\lambda}\mathsf{F}_{m-1}\right) \cong \mathrm{Hom}\left(\mathbb{S}^{\nu}\mathcal{B}_{m-1}, \mathbb{S}^{\lambda}\mathcal{B}_{m-1}\right),$$

and the higher Ext *groups vanish.*

Proof. We closely follow the proof of Theorem 5.2.4. Recall the Koszul resolution (5.8). By definition,

$$H^0\left(\mathrm{Gr}_{m-1} \times \mathrm{Gr}_m, \mathcal{W}_0\right) = \mathrm{Hom}\left(\mathbb{S}^{\nu}\mathcal{B}_{m-1}, \mathbb{S}^{\lambda}\mathcal{B}_{m-1}\right),$$

and by Theorem 5.1.1 all the higher cohomology groups of $\mathcal{W}_0$ vanish. As in the proof of Theorem 5.2.4, it therefore suffices to show that all cohomology groups of $\mathcal{W}_j$ vanish for every $j > 0$.

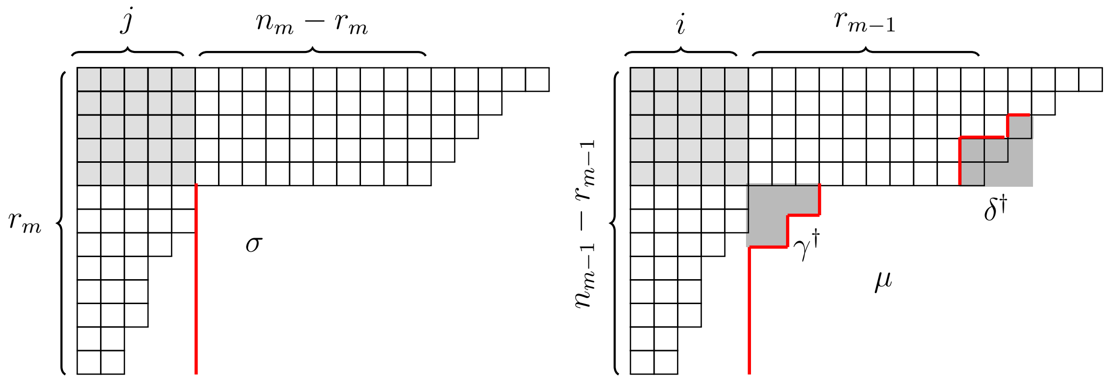
Using the definition of $\mathcal{W}_t$, Proposition 5.2.2, Littlewood-Richardson rule, we may decompose $\mathcal{W}_t$ into direct summands of the form

$$\left(\mathbb{S}^{\mu}\mathcal{A}_{m-1} \otimes \mathbb{S}^{\eta}\mathcal{B}_{m-1}^{\vee}\right) \boxtimes \mathbb{S}^{\sigma}\mathcal{B}_m^{\vee} \text{ with } c^{\mu^\dagger}_{\alpha,\beta} \cdot c^{\sigma}_{\alpha,\beta} \cdot c^{\eta}_{\nu,-\lambda} \neq 0.$$

Suppose, for a contradiction, that for some choice of partitions $\mu, \lambda, \nu, \alpha, \beta, \sigma$ and some non-increasing sequence η, the above direct summand has non-vanishing cohomology. Then, by Lemmas 5.1.3 and 5.1.7, we may define indices

$$\begin{aligned} j &:= (n_m - r_m)\text{–index of } \sigma, \text{ and} \\ i &:= (r_{m-1}, \eta)\text{–index of } \mu. \end{aligned}$$

Write $\eta = (\gamma, -\delta)$, where γ and δ are integer partitions. The indices i and j are illustrated in the diagrams below; the red lines indicate the inequalities (5.2) for σ and (5.3) for μ.



We now show that any possible inequality between i and j leads to a contradiction.

1. *Suppose $j > i$.*
   Arguing as in (1) of Theorem 5.2.4, we have $\alpha_j + \beta_1 \geq \sigma_j$ and $\beta_1 \leq n_{m-1} - r_{m-1}$. Hence
   $$j + n_m - r_m \leq \sigma_j \leq \alpha_j + \beta_1 \leq \alpha_j + n_{m-1} - r_{m-1},$$
   so $j + n - r \leq \alpha_j$. Taking conjugate gives $j \leq \alpha^\dagger_{j+(n-r)}$. Using the assumption $j > i$, we obtain
   $$j \leq \alpha^\dagger_{j+(n-r)} \leq \mu_{j+(n-r)} \leq \mu_{i+(n-r)} \leq i + \gamma^\dagger_{n-r}.$$
   Since $c^{\eta}_{\nu,-\lambda} \neq 0$, we have $\gamma_1 < \nu_1$, and by hypothesis $\nu_1 < n - r$. In particular, $\gamma^\dagger_{n-r} = 0$, so $j \leq i$, a contradiction.

2. *Suppose $2j < i$.*
   We argue as in (2) of Theorem 5.2.4. From $c^{\sigma}_{\alpha,\beta} \neq 0$, we obtain bounds $\alpha^\dagger \leq j$ and $\beta^\dagger_{j+1} \leq j$. By Weyl's inequality (2.12) we have $\mu_{2j+1} \leq \alpha^\dagger_{j+1} + \beta^\dagger_{j+1} \leq 2j$. Using $2j < i$ and the diagram, we then get
   $$i \leq i + r_{m-1} - \delta^\dagger_{i-2j} \leq \mu_{2j+1} \leq 2j,$$
   a contradiction.

3. *Suppose $j < i$.*
   Assume $2j \geq i$. As in (3) of Theorem 5.2.4, using dominance inequalities together with the diagram yields

$$j(n_m - r_m + j) \leq \sum_{k=1}^{j} \sigma_k \leq \sum_{k=1}^{j} \alpha_k + \beta_k \leq \sum_{k=1}^{2j} \mu_k^\dagger \leq i(n_{m-1} - r_{m-1}) + (2j - i)i + |\gamma|. \tag{5.14}$$

   Rearranging gives

$$j(n - r) \leq (i - j)(n_{m-1} - r_{m-1}) - (i - j)^2 + |\gamma|. \tag{5.15}$$

   Similarly, we obtain

$$i(r_{m-1} + i) - |\delta| \leq \sum_{k=1}^{i} \mu_k \leq \sum_{k=1}^{i} \alpha_k^\dagger + \beta_k^\dagger \leq \sum_{k=1}^{2i} \sigma_k^\dagger \leq j r_m + (2i - j)j, \tag{5.16}$$

   and hence

$$jr \geq (i - j)r_{m-1} + (i - j)^2 - |\delta|. \tag{5.17}$$

   Multiplying inequality (5.15) by r and inequality (5.17) by $(n - r)$, and using $i - j > 0$, we obtain

$$r|\gamma| + (n - r)|\delta| \geq nd(i - j) + n(i - j)^2 \geq nd + n.$$

   Since $c^{\eta}_{\gamma,-\lambda} \neq 0$, Part (1) of Lemma 5.2.5 gives $|\gamma| \leq |\nu|$ and $|\delta| \leq |\lambda|$. Hence $nd + n \leq r|\nu| + (n - r)|\lambda|$, contradicting the hypothesis of the lemma.

4. *Suppose $i = j$.*
   Since $c^{\eta}_{\nu,-\lambda} \neq 0$, we have $\gamma_1 < \nu_1$, and therefore

$$\mu_{i+k}^\dagger \leq i + \gamma_1 \leq i + \nu_1 < i + n - r \text{ for any } 1 \leq k \leq 2j - i.$$

   Modifying the chain of inequalities (5.14) accordingly yields

$$j(n_m - r_m + j) \leq i(n_{m-1} - r_{m-1}) + (2j - i)(i + \gamma_1) < i(n_{m-1} - r_{m-1}) + (2j - i)(n - r + i).$$

   Setting $i = j$ gives $j(n - r) < j(n - r)$, which is a contradiction. $\square$

Remark 5.2.7. The proofs of Theorems 5.2.4 and 5.2.6 are very similar. We chose to present both proofs for clarity, but they can be combined into a single argument, as we do in [27].

Before proving Theorem 5.0.2, we recall a theorem of Kapranov [35] describing the bounded derived category of a Grassmannian $\mathrm{Gr}(r, n)$ of r-dimensional quotients of $\mathbb{C}^n$. Let $\mathcal{B}$ denote the universal quotient.

Theorem 5.2.8 ([35]). *The derived category $\mathrm{D}^b(\mathrm{Gr}(r, n))$ admits a full exceptional collection*

$$\left\{ \mathbb{S}^\lambda \mathcal{B} \mid \lambda_1 \leq n - r,\ \lambda_1^\dagger \leq r \right\}.$$

The proof of Theorem 5.0.2 is now reduced to combining Theorem 5.2.6 with Kapranov's exceptional collection.

Proof of Theorem 5.0.2. By Theorem 5.2.6, it suffices to show that the set

$$\left\{\mathbb{S}^\lambda \mathcal{B}_{m-1} \mid |\lambda| \leq d,\ \lambda_1 < n-r\right\},$$

is contained in the full exceptional collection on $\mathrm{Gr}(r_{m-1}, n_{m-1})$ described in Theorem 5.2.8. If $d = 0$, then $\lambda_1 = 0 \leq n_{m-1} - r_{m-1}$ for all $m \geq d$. Assume now $d > 0$. Then for all $m \geq d$ we have

$$\lambda_1 \leq n-r-1 \leq (n-r-1)m \leq (n-r)m - d = n_{m-1} - r_{m-1}.$$

The condition $\lambda_1^\dagger \leq r_{m-1}$ is automatic, since rank $\mathcal{B}_{m-1} = r_{m-1}$. This completes the proof. $\square$

5.3 Proof of Theorem 5.0.1

Let us record the following important proposition from [63, Proposition 1.1].

Proposition 5.3.1. *Let X be a scheme of finite type and $\mathcal{F}$ a coherent sheaf on $X \times \mathbb{P}^1$ such that $R^1\pi_*\left(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\right) = 0$, where π denotes the projection to X. Then there is an exact sequence,*

$$0 \to \pi^*\left(\pi_*\left(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\right)\right) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \pi^*(\pi_*\mathcal{F}) \to \mathcal{F} \to 0.$$

Lemma 5.3.2. *For $m >> 0$, let $\lambda^\ell, \ldots, \lambda^m$ be integer partitions satisfying the inequality $\sum_{i=\ell}^m |\lambda^i| < (nd+n)/(n-r)$. Then, for line bundles $L_\ell, \ldots, L_m$ of degrees $\ell, \ldots, m$ on $\mathbb{P}^1$, we have the cohomological identity*

$$H^\bullet\left(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1), \bigotimes_{i=\ell}^m \mathbb{S}^{\lambda^i}\mathsf{F}_i\right) \cong \bigotimes_{i=\ell}^m \mathbb{S}^{\lambda^i} H^\bullet\left(\mathbb{P}^1, L_i^{\oplus n}\right). \tag{5.18}$$

Proof. For ease of notation, set

$$W_i := H^0\left(\mathbb{P}^1, L_i^{\oplus n}\right) \otimes \mathcal{O}_{\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)}.$$

We argue by downward induction on ℓ, proving the following strengthened statement: The isomorphism (5.18) holds and is induced by the natural maps

$$\bigotimes_{i=\ell}^m \mathbb{S}^{\lambda^i} H^\bullet\left(\mathbb{P}^1, L_i^{\oplus n}\right) \to H^\bullet\left(\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1), \bigotimes_{i=\ell}^m \mathbb{S}^{\lambda^i}\mathsf{F}_i\right).$$

For the base case, we set $\ell = m-1$; the statement follows from Theorem 5.2.4. Now assume $\ell < m-1$ and suppose that the claim holds for $\ell+1$. Observe that the hypotheses of Proposition 5.3.1 apply to both $L_m^{\oplus n}$ and $\mathcal{F} \otimes L_m$. We therefore obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \pi^*W_{m-1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] \arrow[d] & \pi^*W_m \arrow[r] \arrow[d] & L_m^{\oplus n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^*\mathsf{F}_{m-1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & \pi^*\mathsf{F}_m \arrow[r] & \mathcal{F} \otimes L_m \arrow[r] & 0
\end{tikzcd} \tag{5.19}$$

Take a tensor product of this diagram with $L_{\ell-m}$ and push forward to $\mathrm{Quot}_d(\mathcal{O}^n_{\mathbb{P}^1}, r, \mathbb{P}^1)$. This yields a commutative diagram of distinguished triangles

$$\begin{tikzcd} W_\ell \arrow[r] \arrow[d] & W_{m-1} \otimes H^1(L_{\ell-m-1}) \arrow[r, "\Psi"] \arrow[d] & W_m \otimes H^1(L_{\ell-m}) \arrow[r, "+1"] \arrow[d] & {} \\ \mathsf{F}_\ell \arrow[r] & \mathsf{F}_{m-1} \otimes H^1(L_{\ell-m-1}) \arrow[r, "\Phi"] & \mathsf{F}_m \otimes H^1(L_{\ell-m}) \arrow[r, "+1"] & {} \end{tikzcd} \tag{5.20}$$

Taking Schur complexes of the morphisms Ψ and Φ produces a morphism of complexes $\mathbb{S}^{\lambda^\ell}\Psi \to \mathbb{S}^{\lambda^\ell}\Phi$; this gives us a morphism $\mathbb{S}^{\lambda^\ell}W_\ell \to \mathbb{S}^{\lambda^\ell}\mathsf{F}_\ell$. Now set $\mathcal{T}_{\ell+1} := \otimes_{i=\ell+1}^m \mathbb{S}^{\lambda^i}\mathsf{F}_i$, and take a tensor product of the previous morphism of Schur complexes with $\mathcal{T}_{\ell+1}$. By construction and the induction hypothesis, each term $(\mathbb{S}^{\lambda^\ell}\Phi)_t \otimes \mathcal{T}_{\ell+1}$ only involves Schur complexes of F_i with $i \geq \ell+1$. We may therefore apply the Littlewood-Richardson rule to simplify these terms. Applying the cohomology functor $H^\bullet$ then yields, for each t, an isomorphism

$$H^\bullet\left((\mathbb{S}^{\lambda^\ell}\Psi)_t \otimes \mathcal{T}_{\ell+1}\right) \to H^\bullet\left((\mathbb{S}^{\lambda^\ell}\Phi)_t \otimes \mathcal{T}_{\ell+1}\right).$$

Finally, consider the commutative diagram of triangles

$$\begin{tikzcd} H^\bullet\left((\mathbb{S}^{\lambda^\ell}\Psi)_{t+1} \otimes \mathcal{T}_{\ell+1}\right)[-t-1] \arrow[r] \arrow[d] & H^\bullet\left(\tau_{\leq t+1}(\mathbb{S}^{\lambda^\ell}\Psi) \otimes \mathcal{T}_{\ell+1}\right) \arrow[r] \arrow[d] & H^\bullet\left(\tau_{\leq t}(\mathbb{S}^{\lambda^\ell}\Psi) \otimes \mathcal{T}_{\ell+1}\right) \arrow[r, "+1"] \arrow[d] & {} \\ H^\bullet\left((\mathbb{S}^{\lambda^\ell}\Phi)_{t+1} \otimes \mathcal{T}_{\ell+1}\right)[-t-1] \arrow[r] & H^\bullet\left(\tau_{\leq t+1}(\mathbb{S}^{\lambda^\ell}\Phi) \otimes \mathcal{T}_{\ell+1}\right) \arrow[r] & H^\bullet\left(\tau_{\leq t}(\mathbb{S}^{\lambda^\ell}\Phi) \otimes \mathcal{T}_{\ell+1}\right) \arrow[r, "+1"] & {} \end{tikzcd}$$

and argue inductively on t to conclude that $H^\bullet\left(\mathbb{S}^{\lambda^\ell}\Psi \otimes \mathcal{T}_{\ell+1}\right) \to H^\bullet\left(\mathbb{S}^{\lambda^\ell}\Phi \otimes \mathcal{T}_{\ell+1}\right)$ is an isomorphism. This completes the proof. $\square$

Proof of Theorem 5.0.1. Since M is a vector bundle on $\mathbb{P}^1$, it splits as $M \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ for integers a_i. Consequently, $M^{[d]} \cong \oplus_i \mathsf{F}_{a_i}$. Using formula (2.11), we may write $\mathbb{S}^\lambda M^{[d]}$ as a direct sum of tensor products of Schur bundles of the form $\mathbb{S}^{\lambda^i}\mathsf{F}_{a_i}$. Since $|\lambda| < (nd+n)/(n-r)$, the hypotheses of Lemma 5.3.2 apply, and therefore $H^\bullet(\mathrm{Quot}_d(\mathcal{O}_{\mathbb{P}^1}, r, \mathbb{P}^1), \mathbb{S}^\lambda M^{[d]})$ is identified with the corresponding direct sum of tensor products of $\mathbb{S}^{\lambda_i}H^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)^{\oplus n})$. Applying the decomposition formula (2.11) in reverse allows us to reconstruct $\mathbb{S}^\lambda H^\bullet(\mathbb{P}^1, M^{\oplus n})$, which proves the claimed isomorphism. $\square$

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