SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Pseudo-Kähler geometry of Hitchin representations and convex projective structures

Candidate:
Nicholas Rungi

Advisor:
Prof. Andrea Tamburelli

Academic Year 2022-23

Abstract

In this thesis we study the symplectic and pseudo-Riemannian geometry of the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component associated with a closed orientable surface, using an approach coming from the theory of symplectic reduction in an infinite-dimensional context.
In the case where the closed surface is homeomorphic to a torus, for each choice of a smooth real function with certain properties, we prove the existence of a pseudo-Kähler metric on the deformation space of properly convex projective structures. Moreover, we define a circle action and a $\mathrm{SL}(2,\mathbb{R})$-action on the aforementioned space, which turn out to be Hamiltonian with respect to our symplectic form, and we give an explicit description of the moment maps. Then, we study the symplectic geometry of the deformation space as a completely integrable Hamiltonian system, and we find a geometric global Darboux frame for the symplectic form using the theory of complete Lagrangian fibrations.
In the case of genus $g \geqslant 2$ we define a mapping class group invariant pseudo-Kähler metric on the Hitchin component, by using a general construction of Donaldson. The complex structure is exactly the one coming from the identification with the holomorphic bundle of cubic differentials over Teichmüller space. In particular, we prove that Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$ has an interpretation as moment map for the action of an infinite-dimensional Lie group.

Acknowledgements

It might be a platitude, but the first person I really have to thank for the realization of this thesis is my advisor Andrea Tamburelli. We met each other for the first time in June 2021 when he was still finishing his post-doc in Houston. Although full of commitments, he always managed to find two or three hours each week to discuss about mathematics. These meetings, albeit online, were a major boost in the completion of this project, as he always had an answer to my math related questions and advised me on how best to optimize the work to be done. I also thank him for all the academic tips he gave me during these two years, as they turned out to be very useful in the end. One of the most valuable lessons I learned from him is the remarkable difference between "studying mathematics" and "doing research in mathematics".

I am truly grateful to my future post-doc mentor Andrea Seppi for hosting me more than once at Université Grenoble Alpes. I am looking forward to start working with him, and I am confident that these two years will be very productive and at the same time enjoyable.

I also want to thank Ugo Bruzzo and Jacopo Stoppa who, during my first year as a PhD student at SISSA, helped me in making an important decision for the years to come.

A special thanks goes to the committee that decided to attend my discussion: Francesco Bonsante, Gabriele Mondello, Antonio Lerario, Jean-Marc Schlenker, Andrea Seppi and Jacopo Stoppa. In particular, I would like to thank Jean-Marc Schlenker and Andrea Seppi for taking the time to carefully referee my thesis.

It would not have been the same without all the fantastic people I have met in SISSA during these years. The environment was very stimulating especially because of the large number of seminars and lectures available, which spanned the various areas of geometry and the interplay with mathematical physics. I would especially like to thank Lorenzo Sillari, Paolo Tomasini, Giuseppe Orsatti and Giacomo Leccese for sharing some great time with me and for exchanging advice on how to deal with the various difficulties that a PhD entails.

A warm mention to my longtime friends in Colleferro: each time I came back home it felt like only one week had passed, instead of months. I consider myself lucky to have such a safe point of reference, outside of family, that you know you can count on when you need it. For this I wanted to give special thanks to my two friends Carlo Ciucci and Lorenzo Noro.

A special place in these acknowledgements is dedicated to my parents Patrizia and Matteo: without all their support, in various ways, that I received from an early age, I most likely would not be here writing these lines now. I was always encouraged to do what I liked best and they respected my choices by supporting me as best they could. I believe there is no more genuine evidence than this of the care they have for me. They probably know that I will never be able to find a way to thank them enough, maybe there is no need to find it, or maybe it doesn't even exist when it comes to unconditional love.

Last but not least, my girlfriend (and I would say half-life partner) Melania: we have known each other since we were high school kids, and we have been through so much together. So many positive experiences that have allowed us to be mutually enriched and to grow up at the same time. The three years we spent living together in Trieste were great. It wouldn't have been the same if once I got back from SISSA there wasn't her to laugh with, joke with, play with, and why not, get angry with! The love he has shown me in the 14 years that we have known each other is something extraordinary, which hardly and not everyone can find in life, but I have been lucky enough to receive it.

I would like to end these acknowledgments with a separate word about the city of Trieste. I never expected to find such a beautiful city in which to spend three years. It may be small, and with few things to offer during the winter, but, in my opinion, in the time of spring or early fall it expresses its beauty at its best. The presence of the sea and the many hiking trails in the surrounding area make it even more special. It would be great if someday I had a chance to go back there again.

Contents

Introduction

This thesis tries to enlarge the current knowledge regarding the global geometry of higher rank Teichmüller spaces. In recent years, people have been interested in the study of the geometric and dynamical properties of surface group representations into Lie groups of rank at least two, with the aim of generalizing the classical Teichmüller theory that concerns representations into $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ ([Wie18]). Various connected components in the corresponding higher rank character varieties have been found to share many similarities with the classical Teichmüller space. Some of these components are the Hitchin components, defined for semi-simple real split Lie groups ([Hit92]). In particular, any of these connected components contains a copy of Teichmüller space, to which one refers as the Fuchsian locus. The main motivation for this thesis comes from the study of the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component from a pseudo-Riemannian and symplectic point of view, with the aim of giving a natural generalization of the Weil-Petersson Kähler metric defined on Teichmüller space ([Wei58], [Ahl61a], [Ahl61b]).

Given a smooth closed surface Σ of genus $g \geqslant 2$, discrete and faithful surface group representations in $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ are known to be holonomies of hyperbolic structures, and the corresponding connected component in the character variety recovers the Teichmüller space $\mathcal{T}(\Sigma)$. Similarly, every representation in the Hitchin component $\mathrm{Hit}_3(\Sigma)$ for $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$ is discrete and faithful, and they can be viewed as holonomies of convex projective structures on the surface ([Gol90a],[CG93]). There is a natural symplectic form ω_{G} on the Hitchin component (and also defined on much more general spaces), found by Goldman using the explicit description of the Zariski tangent space to a point, and the correspondence between surface group representations and flat bundles ([Gol84]). In particular, it is shown that ω_{G} restricts to a multiple of the Weil-Petersson symplectic form on the Fuchsian locus. In the early 2000s, Labourie ([Lab07]) and Loftin ([Lof01]) proved independently, using the theory of hyperbolic affine spheres and harmonic maps in symmetric spaces, that the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component can be endowed with a mapping class group invariant complex structure $\mathbf{I}$. Such a complex structure comes from the identification of the aforementioned component with the holomorphic bundle of cubic differentials over Teichmüller space.

Conjecture. *The symplectic form ω_{G} is compatible with Labourie and Loftin's complex structure, so that they define a mapping class group invariant Kähler metric on the* $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$*-Hitchin component extending the Weil-Petersson metric on Teichmüller space.*

Goldman's symplectic form is defined using the algebraic description of the Hitchin component, but the mapping class group invariant complex structure $\mathbf{I}$ comes from the parameterization with holomorphic cubic differentials. Because of the different way in which the symplectic form and the complex structure are defined, it is still unclear whether $\omega_{\mathrm{G}}(\mathbf{I}\cdot,\cdot)$ defines a Riemannian metric.

Later on, three more Riemannian metrics on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component were defined: one by Darvishzadeh and Goldman ([DW96]), one by Li ([Li16]) and another by Bridgeman-Canary-Labourie-Sambarino ([Bri+15]) called *pressure metric* (defined also on much more general spaces). Regarding the first two it has been shown that they restrict to a multiple of the Weil-Petersson metric on Teichmüller space, which is totally geodesic in $\mathrm{Hit}_3(\Sigma)$ with respect to the metric found by Li. As far as pressure metric is concerned, very little is known and this is partly due to its complicated expression ([LW18],[Dai19]). In all three cases the relation with Labourie and Loftin's complex structure is unknown.

Recently, Kim and Zhang ([KZ17]), using various notions of positivity for holomorphic bundles on Kähler manifolds, have succeeded in showing the existence of a Kähler metric on $\mathrm{Hit}_3(\Sigma)$, which restricts to a multiple of the Weil-Petersson one on the Fuchsian locus. Even if this metric is natural, namely invariant under the action of the mapping class group, the relation of its complex structure with the one found by Labourie and Loftin is still mysterious ([Lab17, §1.2 and §1.3]).

It is therefore unknown whether there is a symplectic form (or Riemannian metric) on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component that gives rise to a Kähler metric when matched with the complex structure $\mathbf{I}$. This thesis attempts to answer this question by proving the following result:

Theorem A. *There exists a closed* 2*-form* $\boldsymbol{\omega}$ *on* $\mathrm{Hit}_3(\Sigma)$ *such that* $\mathbf{g}(\cdot,\cdot):=\boldsymbol{\omega}(\mathbf{I}\cdot,\cdot)$ *defines a pseudo-Riemannian metric of signature* $(6g-6,10g-10)$*. Moreover, the triple* $(\mathbf{g},\boldsymbol{\omega},\mathbf{I})$ *gives rise to a mapping class group invariant pseudo-Kähler structure on a neighborhood of the Fuchsian locus in the Hitchin component, and it restricts to a multiple of the Weil-Petersson Kähler metric on Teichmüller space, which embeds as a totally geodesic submanifold.*

The above statement seems to suggest that the right structure to be sought is a pseudo-Kähler one, namely the metric is no longer required to be positive-definite. The tensor $\boldsymbol{\omega}$ and $\mathbf{g}$ are explict and defined on the whole $\mathrm{Hit}_3(\Sigma)$ but, because we cannot exclude that $\boldsymbol{\omega}$ might be degenerate outside the Fuchsian locus, the triple $(\boldsymbol{\omega},\mathbf{g},\mathbf{I})$ defines a-priori a pseudo-Kähler structure only on a neighborhood of it. In particular, one can recognise the dimension of Teichmüller space with regard to the positive part of $\mathbf{g}$, and the real dimension of the space of holomorphic cubic differentials with regard to the negative one.

There is a well-defined action of the mapping class group on $\mathrm{Hit}_3(\Sigma)$, whose quotient $\mathcal{C}(\Sigma)$ results in a complex orbifold smooth at generic points ([Lof01, Proposition 4.1.2]), which fibres over the moduli space of Riemann surfaces $\mathcal{M}_g$.

Corollary B. *There exists an orbifold neighborhood of the moduli space of Riemann surfaces of genus $g \geqslant 2$ inside $\mathcal{C}(\Sigma)$ endowed with a pseudo-Kähler orbifold structure. Such a structure restricts to a multiple of the Weil-Petersson orbifold Kähler structure on $\mathcal{M}_g$.*

According to Labourie and Loftin's parameterization of $\mathrm{Hit}_3(\Sigma)$ as the holomorphic bundle of cubic differentials over $\mathcal{T}(\Sigma)$, one can induce a circle action on the Hitchin component, which corresponds to rotation of the fibres in the vector bundle description.

Theorem C. *The aforementioned circle action on* $\mathrm{Hit}_3(\Sigma)$ *is Hamiltonian with respect to* $\boldsymbol{\omega}$ *and it acts by preserving the pseudo-Riemannian metric* $\mathbf{g}$.

The proof of the above theorems relies on various techniques: symplectic reduction theory; elliptic operators on compact manifolds, and a general construction of moment maps in an infinite-dimensional context given by Donaldson ([Don03]). In what follows we will give a rough idea of the proof strategy, while also stating other fundamental and non-immediate results related to the main theorem.

The main theorem of this thesis is inspired by a similar result obtained in the case of maximal globally hyperbolic anti-de Sitter three-manifolds ([MST21]), where the authors developed part of the techniques the we used in our work. On the one hand the overall strategy follows the lines of the anti-de Sitter setting ([MST21, §1.7]), but on the other we encountered more difficulties during some steps of the proof that will be explained on a case-by-case basis (see also the last paragraph of the introduction for a more general discussion).

The genus one case:

The very first step is to prove similar statements when the smooth closed surface is a torus T^2, following the lines of [MST21, §3.2 and §3.3]. In this case, the natural space to study is the deformation space of properly convex $\mathbb{RP}^2$-structures on the torus. In this regard, let us introduce $\mathcal{J}(\mathbb{R}^2)$ to be the space of (almost) complex structures on $\mathbb{R}^2$ compatible with the standard area form $\rho_0 = \mathrm{d}x \wedge \mathrm{d}y$, namely all the endomorphisms $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that $J^2 = -\mathbb{1}$ and for which $\{v, Jv\}$ is a positive basis, whenever $v \neq 0$. For any such J, let $g_J^0(\cdot,\cdot) := \rho_0(\cdot, J\cdot)$ be the associated scalar product on $\mathbb{R}^2$. There is an identification between this space and the hyperbolic plane $\mathbb{H}^2$, so that the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$ by

Möbius transformations results in an action by conjugation on $\mathcal{J}(\mathbb{R}^2)$. Let us denote with $D^3(\mathcal{J}(\mathbb{R}^2))$ the real vector bundle over $\mathcal{J}(\mathbb{R}^2)$ whose fibre over a point J is given by all 1-forms A with values in the bundle of g_J^0-symmetric and trace-less endomorphisms of $\mathbb{R}^2$ such that $A(J\cdot) = A(\cdot)J$ and $A(X)Y = A(Y)X$ for all $X, Y \in \mathbb{R}^2$. Under the identification $\mathcal{J}(\mathbb{R}^2) \cong \mathbb{H}^2 \cong \mathcal{T}(T^2)$ the vector bundle $D^3(\mathcal{J}(\mathbb{R}^2))$ can be identified with $Q^3\big(\mathcal{T}(T^2)\big)$, namely the bundle of holomorphic cubic differentials over $\mathcal{T}(T^2)$. By using the theory of hyperbolic affine spheres in $\mathbb{R}^3$, the complement of the zero section in $Q^3\big(\mathcal{T}(T^2)\big)$ can be identified with the deformation space of properly convex $\mathbb{RP}^2$-structures on the torus, denoted with $\mathcal{B}_0(T^2)$. Then, for any choice of a smooth function $f : [0, +\infty) \to (-\infty, 0]$ such that $f(0) = 0, f'(t) < 0\ \forall t > 0$ and $\lim\limits_{t\to+\infty} f(t) = -\infty$, we prove the following:

Theorem D. *For any function f as above, there exists an* $\mathrm{SL}(2,\mathbb{R})$*-invariant pseudo-Kähler structure $(\widehat{\boldsymbol{\omega}}_f, \widehat{\mathbf{I}}, \widehat{\mathbf{g}}_f)$ on $D^3(\mathcal{J}(\mathbb{R}^2))$ which restricts to a mapping class group invariant pseudo-Kähler metric on $\mathcal{B}_0(T^2)$.*

By exploiting the isomorphism $D^3(\mathcal{J}(\mathbb{R}^2)) \cong Q^3\big(\mathcal{T}(T^2)\big)$, we can induce a circle action on $D^3(\mathcal{J}(\mathbb{R}^2))$ corresponding to a rotation of the fibre in the holomorphic bundle description. Together with the $\mathrm{SL}(2,\mathbb{R})$-action, we get two further results:

Theorem E. *For any function f as above, the circle action on $D^3(\mathcal{J}(\mathbb{R}^2))$ is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ and it preserves the psuedo-metric $\widehat{\mathbf{g}}_f$. Moreover, the Hamiltonian function can be explicitly expressed in terms of f.*

Theorem F. *The* $\mathrm{SL}(2,\mathbb{R})$*-action on $D^3(\mathcal{J}(\Sigma))$ is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ and the moment map $\widehat{\mu} : D^3(\mathcal{J}(\mathbb{R}^2)) \to \mathfrak{sl}(2,\mathbb{R})^*$ can be explicitly expressed in terms of f.*

In particular, by taking the action of the subgroup $\mathbb{R}^* < \mathrm{SL}(2,\mathbb{R})$ generated by the diagonal matrices, it is possible to explicitly compute the Hamiltonian function H_2 with respect to this restricted action. Together with the Hamiltonian function H_1 of the circular action, we get the existence of two commuting Hamiltonian vector fields $\mathbb{X}_{H_1}, \mathbb{X}_{H_2}$ on $\mathcal{B}_0(T^2)$. In other terms, the space $(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f)$ has the structure of a complete Hamiltonian integrable system. The main issue is that each fiber of the associated Lagrangian fibration $H := (H_1, H_2) : \big(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f\big) \to B \subset \mathbb{R}^2$ is diffeomorphic to $\mathbb{R} \times S^1$. However, since the base space B is contractible and the Hamiltonian vector fields $\mathbb{X}_{H_1}, \mathbb{X}_{H_2}$ are complete, we can apply the theory of complete Lagrangian fibration to obtain the following:

Theorem G. *The collection $\{\theta, H_1, s, H_2\}$ is a global Darboux frame for $\widehat{\boldsymbol{\omega}}_f$, where $(s, \theta) \in \mathbb{R} \times S^1 \cong H^{-1}(b)$ for each $b \in B$, and correspond to the angle coordinates of the completely integrable Hamiltonian system $(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f, H_1, H_2)$.*

The general case:

Now let Σ be a smooth closed connected and oriented surface of genus $g \geqslant 2$. The crucial step in moving from the genus one case to the higher genus case, consists in the following construction. Let ρ be a fixed area form on Σ, then for any (almost) complex structure J on Σ, let $g_J := \rho(\cdot, J\cdot)$ be the associated Riemannian metric. Now consider the space formed by pairs (J, A), where J is an (almost) complex structure on Σ, compatible with the given orientation, and A is a 1-form with values in the bundle of trace-less and g_J-symmetric endomorphisms of $T\Sigma$ such that $A(J\cdot) = A(\cdot)J$ and $A(X)Y = A(Y)X, \ \forall X, Y \in \Gamma(T\Sigma)$. This space, denoted by $D^3(\mathcal{J}(\Sigma))$, is of infinite dimension and it carries a pseudo-Kähler structure as its analogue $D^3(\mathcal{J}(\mathbb{R}^2))$. In fact, one can choose an area-preserving linear isomorphism from $\mathbb{R}^2$ to $T_x\Sigma$, which induces an identification between $D^3(\mathcal{J}(\mathbb{R}^2))$ and $D^3\big(\mathcal{J}(T_x\Sigma)\big)$. Since the pseudo-Kähler metric on $D^3(\mathcal{J}(\mathbb{R}^2))$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, the induced structure does not depend on the chosen area-preserving linear isomorphism. Then, one can (formally) integrate each element of the pseudo-Kähler structure on Σ, evaluated on first-order deformations $(\dot{J}, \dot{A})$. Slightly more in detail, let P be the $\mathrm{SL}(2,\mathbb{R})$-frame bundle over Σ whose fibres over a point $x \in \Sigma$ are linear maps $F : \mathbb{R}^2 \to T_x\Sigma$ such that $F^*\rho_x$ is the standard area form on $\mathbb{R}^2$. Let us define the fibre bundle

$$P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) := P \times D^3(\mathcal{J}(\mathbb{R}^2)) \Big/ \mathrm{SL}(2,\mathbb{R})$$

where $\mathrm{SL}(2,\mathbb{R})$ acts diagonally on the two factors. The space $D^3(\mathcal{J}(\Sigma))$ can be identified with the space of smooth sections of such fibre bundle. Hence, as explained above, one can introduce a symplectic form $\boldsymbol{\omega}_f$ and a pseudo-Riemannian metric $\mathbf{g}_f$ on $D^3(\mathcal{J}(\Sigma))$ by formally integrating the ones induced on each fibre of $T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$, denoted with $\widehat{\boldsymbol{\omega}}_f$ and $\widehat{\mathbf{g}}_f$. Here $T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ stands for the vertical sub-bundle of $TP\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ with respect to the projection map $P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) \to \Sigma$. Similarly, a complex structure $\mathbf{I}$ is obtained on the infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$ of smooth sections, by applying point-wise $\hat{\mathbf{I}}$, which is defined on $D^3(\mathcal{J}(\mathbb{R}^2))$. It should be noted that the symplectic form $\hat{\boldsymbol{\omega}}_f$ and the pseudo-Riemannian metric $\hat{\mathbf{g}}_f$ both depend on the choice of a smooth function f, as they arise from the construction on $D^3(\mathcal{J}(\mathbb{R}^2))$. In particular, the expression for $\boldsymbol{\omega}_f$ and $\mathbf{g}_f$ combined with $\mathbf{I}$ effectively gives us a (formal) family of pseudo-Kahler metrics on the space of smooth sections $D^3(\mathcal{J}(\Sigma))$. Instead, we are interested in inducing such structures on a certain submanifold, whose elements (J, A) will be identified with the set of embedding data of hyperbolic affine spheres in $\mathbb{R}^3$. In order to do so, a particular choice of the function f appearing in the expression of $\boldsymbol{\omega}_f$ and $\mathbf{g}_f$ has to be made. Let $F : [0, +\infty) \to \mathbb{R}$ be the unique smooth function such that $ce^{-F(t)} - 2te^{-3F(t)} + 1 = 0$, where c is a constant depending only on the topology and the area of (Σ, ρ). Let us define a new metric in the same conformal class of g_J by the formula $h := e^{F(t)}g_J$, where the function F is computed in $||A||^2_{g_J}$ (the norm of the tensor A with respect to g_J) divided by 8. Then, imposing the equations governing the embedding data of hyperbolic affine spheres on the pair (h, A), we get a $\mathrm{Ham}(\Sigma, \rho)$-invariant submanifold $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ of the space of smooth sections (J, A), whose quotient $\widetilde{\mathcal{B}}(\Sigma)$ by $\mathrm{Ham}(\Sigma, \rho)$, is a smooth manifold of dimension $16g - 16 + 2g$. This will be a consequence of a simple application of Moser's trick in symplectic geometry, of the particular choice of the function f in terms of F, and finally of the existence and uniqueness of hyperbolic affine sphere immersions in $\mathbb{R}^3$. It turns out that such a manifold is not diffeomorphic to the Hitchin component, as its dimension exceeds that of $\mathrm{Hit}_3(\Sigma)$ by $2g$. As we shall see later, the tangent space to this manifold splits as the $\mathbf{g}_f$-orthogonal direct sum of the tangent space to $\mathrm{Hit}_3(\Sigma)$ and the tangent to the orbit of harmonic vector fields. For this reason, the further (finite-dimensional) quotient of $\widetilde{\mathcal{B}}(\Sigma)$ by $\mathrm{Symp}_0(\Sigma,\rho)/\mathrm{Ham}(\Sigma,\rho)\cong H^1_{\mathrm{dR}}(\Sigma,\mathbb{R})$ gives us the desired Hitchin component.

The candidate for the tangent space to the Hitchin component:

As explained in the previous paragraph, in order to actually obtain the Hitchin component from the space $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, and thus induce a pseudo-Kähler structure $(\mathbf{g}_f,\mathbf{I},\boldsymbol{\omega}_f)$ on it, we need to perform two quotients: the first by $\mathrm{Ham}(\Sigma,\rho)$ and the second by $\mathrm{Symp}_0(\Sigma,\rho)/\mathrm{Ham}(\Sigma,\rho)$. The idea is to define a distribution $\{W_{(J,A)}\}_{(J,A)}$ of $\mathrm{Ham}(\Sigma,\rho)$-invariant subspaces inside the tangent space to $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$. Each vector space $W_{(J,A)}$ of this distribution will be defined by a system of partial differential equations and will be point-wise isomorphic to the tangent space of $\widetilde{\mathcal{B}}(\Sigma)$. In analogy with the anti-de Sitter case ([MST21, Lemma 4.18]), the first result that conceals a number of technical difficulties shows, using an argument from the theory of elliptic operators on compact manifolds, that the dimension of each $W_{(J,A)}$ is bounded below by the expected dimension of the quotient manifold.

Theorem H. *Let (J,A) be a point in the infinite-dimensional space $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$. Let $W_{(J,A)}$ be the vector space of solutions of the following system:*

$$\begin{cases} \mathrm{d}\big(\operatorname{div}\big((f-1)\dot J\big)+\mathrm{d}\dot f\circ J-\frac{f'}{6}\beta\big)=0\\ \mathrm{d}\big(\operatorname{div}\big((f-1)\dot J\big)\circ J+\mathrm{d}\dot f_0\circ J-\frac{f'}{6}\beta\circ J\big)=0\\ \mathrm{d}^{\nabla}\dot A_0(\bullet,\bullet)-J(\operatorname{div}\dot J\wedge A)(\bullet,\bullet)=0 \end{cases}$$

where $\dot A_0$ is the trace-less part of the first order variation of A, ∇ is the Levi-Civita connection with respect to g_J, $\beta(\bullet):=\langle(\nabla_\bullet A)J,\dot A_0\rangle$ is a 1-form and $\dot f_0=-\frac{f'}{4}\langle A,\dot A_0J\rangle$ is a smooth function on Σ. Then, $\dim W_{(J,A)}\geqslant 16g-16+2g$.

The second difficult statement, which will be consequence of the above theorem, also involves a large number of technical details. It allows us to identify each subspace $W_{(J,A)}$ with the tangent space to the first quotient space $\widetilde{\mathcal{B}}(\Sigma)$ at the point (J,A), as it happens in [MST21, §4.5] with the appropriate differences.

Theorem J. *For every element $(J,A)\in\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, the vector space $W_{(J,A)}$ is contained inside $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ and it is invariant by the complex structure $\mathbf{I}$. Moreover, the collection $\{W_{(J,A)}\}_{(J,A)}$ defines a $\mathrm{Ham}(\Sigma,\rho)$-invariant distribution on $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ and the natural projection $\pi:\widetilde{\mathcal{HS}}_0(\Sigma,\rho)\to\widetilde{\mathcal{B}}(\Sigma)$ induces a linear isomorphism*

$$\mathrm{d}_{(J,A)}\pi: W_{(J,A)}\longrightarrow T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma)$$

In particular, we can restrict the pseudo-Kähler structure $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ from the ambient space to the finite dimensional manifold $\widetilde{\mathcal{B}}(\Sigma)$. Since the pseudo-metric $\mathbf{g}_f$ is not positive-definite, it is not immediate that it is still non-degenerate when restricted to the subspaces $W_{(J,A)}$. This is indeed the biggest issue to be addressed, and it will be discussed in a subsequent paragraph of the following introduction. At this point, one can proceed by performing the finite-dimensional quotient $\widetilde{\mathcal{B}}(\Sigma)/H$, where $H := \mathrm{Symp}_0(\Sigma,\rho)/\,\mathrm{Ham}(\Sigma,\rho) \cong H^1_{\mathrm{dR}}(\Sigma,\mathbb{R})$. Such a quotient is isomorphic to the Hitchin component, and there is a $\mathbf{g}_f$-orthogonal decomposition $W_{(J,A)} = V_{(J,A)} \oplus S_{(J,A)}$, where $V_{(J,A)}$ is the tangent to $\mathrm{Hit}_3(\Sigma)$ and $S_{(J,A)}$ is a copy of H.

Theorem K. *The H-action on $\widetilde{\mathcal{B}}(\Sigma)$ is free and proper, with complex and symplectic H-orbits. Moreover, the pseudo-Kähler structure $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ descends to the quotient which is identified with* $\mathrm{Hit}_3(\Sigma)$*. Finally, the complex structure* $\mathbf{I}$ *induced on the* $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$*-Hitchin component coincides with the one found by Labourie and Loftin.*

The relation with moment maps and symplectic reduction

While Theorem H and Theorem J can be proven with self-contained arguments, it is not clear how to obtain the differential equations defining the subspace $W_{(J,A)}$. In fact, their origin must be sought in the context of moment maps and symplectic reductions, but in an infinite-dimensional context. For this reason, we will briefly explain how to characterize the subspaces $W_{(J,A)}$ in these terms and how the presence of isotropic vectors for $\mathbf{g}_f$ generates further difficulties. In the torus case we showed that the action of $\mathrm{SL}(2,\mathbb{R})$ on $D^3(\mathcal{J}(\mathbb{R}^2))$ is Hamiltonian with respect to the symplectic form $\hat{\boldsymbol{\omega}}_f$ and we computed explicitly the moment map $\widehat{\mu} : D^3(\mathcal{J}(\mathbb{R}^2)) \to \mathfrak{sl}(2,\mathbb{R})^*$. A general theorem of Donaldson ([Don03]), allows us to promote the previous result to a Hamiltonian action of $\mathrm{Ham}(\Sigma,\rho)$ on $D^3(\mathcal{J}(\Sigma))$, with respect to the symplectic form $\boldsymbol{\omega}_f$. In this case, the moment map $\boldsymbol{\mu}$ associates to each pair $(J,A) \in D^3(\mathcal{J}(\Sigma))$ an element in the dual Lie algebra of Hamiltonian vector fields on the surface. It turns out that to obtain an honest moment map $\widetilde{\boldsymbol{\mu}}$ for the action of the group of Hamiltonian diffeomorphisms, one has to add a scalar multiple of the area form ρ. At this point, it can be shown that the submanifold $\widetilde{\boldsymbol{\mu}}^{-1}(0)$ intersected with the set $\mathcal{M}_{\mathrm{C}} = \{(J,A) \in D^3(\mathcal{J}(\Sigma)) \mid d^\nabla A = 0\}$ is equal to $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$. Inspired by classical symplectic reduction theory, one is tempted to induce the pseudo-Riemannian metric $\mathbf{g}_f$ and the symplectic form $\boldsymbol{\omega}_f$ on the quotient $\big(\widetilde{\boldsymbol{\mu}}^{-1}(0) \cap \mathcal{M}_{\mathrm{C}}\big)/\,\mathrm{Ham}(\Sigma,\rho)$. The issue is that, in our case, the tangent space $T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ is a Krein space ([AI81]). Roughly speaking, a Krein space is a (real or complex) infinite-dimensional vector space endowed with an indefinite inner product which admits an orthogonal direct sum decomposition in positive and negative part. Moreover, the pseudo-metric restricted to both the positive and negative part induces a complete norm. The presence of the indefinite metric does not allow us, like in the Hilbert case ([Tro12, Theorem 1.3.2]), to identify the $\mathbf{g}_f$-orthogonal to the $\mathrm{Ham}(\Sigma,\rho)$-orbit inside $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ with the $\mathbf{I}$-invariant distribution tangent to the finite dimensional manifold $\widetilde{\mathcal{B}}(\Sigma)$. Despite that, by imitating the reduction in the positive-definite case, we are able to give a characterization of the subspace $W_{(J,A)}$ as follows:

Theorem L. *For any $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, the vector space $W_{(J,A)}$ is the largest subspace in $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ that is:*

- *invariant under the complex structure $\mathbf{I}$;*
- *$\mathbf{g}_f$-orthogonal to the orbit $T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big)$.*

The proof of this theorem is independent of the other results, but it serves as a motivation for defining the subspace $W_{(J,A)}$ as the solution of a system of partial differential equations.

The pseudo-metric is non-degenerate.

Theorem J together with Theorem K allows us to induce the pseudo-Kähler structure $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ from the infinite-dimensional manifold $D^3(\mathcal{J}(\Sigma))$ to the Hitchin component, but, a-priori, it may be degenerate. However, exploiting the explicit expression of $\mathbf{g}_f$, we can prove that, at least on the Fuchsian locus, there are no non-zero degenerate vectors. As for the tangent directions to points away from the Fuchsian locus, the analysis becomes very complicated. On the one hand, we know the exact expression of $\mathbf{g}_f$, but on the other hand, the model $V_{(J,A)}$ of the tangent space to the Hitchin component is described by very complicated PDEs, whose solution is far from being explicit. The idea, is to look for a subspace of $T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ (possibly of infinite dimension) whose elements have a treatable description for our purpose. This is the tangent space $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ to the set of pairs $(J,A) \in D^3(\mathcal{J}(\Sigma))$ satisfying the Codazzi-like equation $\mathrm{d}^\nabla A = 0$ for hyperbolic affine spheres. We will show that the set $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ contains the tangent space to the $\mathrm{Diff}(\Sigma)$-orbit, which in turn splits as a direct sum of three subspaces. Then, using the relation between the PDEs describing $V_{(J,A)}$ and the theory of symplectic reduction, the following $\mathbf{g}_f$-orthogonal decomposition of $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ can be obtained:

$$V_{(J,A)} \overset{\perp_{\mathbf{g}_f}}{\oplus} S_{(J,A)} \overset{\perp_{\mathbf{g}_f}}{\oplus} T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big) \overset{\perp_{\mathbf{g}_f}}{\oplus} \mathbf{I}\Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big)\Big)\ .$$

The existence of the moment map $\tilde{\boldsymbol{\mu}}$ for the action of $\mathrm{Ham}(\Sigma,\rho)$ on $D^3(\mathcal{J}(\Sigma))$ and an explicit calculation allow us to conclude that $\mathbf{g}_f$ restricted to the Hamiltonian orbit is non-degenerate. Moreover, using a highly non-trivial integration by parts we prove that $\mathbf{g}_f$ is non-degenerate even when restricted to the subspace $S_{(J,A)}$. Finally, using the relation $\mathbf{g}_f(\mathbf{I}\cdot,\mathbf{I}\cdot) = \mathbf{g}_f(\cdot,\cdot)$ one gets the following further result

Theorem M. *The pseudo-Riemannian metric $\mathbf{g}_f$ is non-degenerate on $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ if and only if it is non-degenerate on $V_{(J,A)}$, namely on the Hitchin component.*

In this regard, we introduce the notion of Krein space and some useful results that may lead to a better understanding of (possible) degenerate vectors for $\mathbf{g}_f$ away from the Fuchsian locus.

Comparison with the anti-de Sitter case

As mentioned earlier, part of the techniques we use to construct the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component as a symplectic quotient and the definition of the pseudo-Kähler metric are based on a previous work ([MST21]), where the authors defined a para-hyperkähler structure on the deformation space of maximal globally hyperbolic anti-de Sitter 3-manifolds, denoted with $\mathcal{MGH}(\Sigma)$. Such a deformation space can be identified with a maximal component in the $\mathbb{P}\mathrm{SL}(2,\mathbb{R})\times\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ character variety, which consists entirely of discrete and faithful representations. In particular, such a space is parameterized by two copies of Teichmüller space ([Mes07],[KS07]) and it is isomorphic to the cotangent bundle $T^*\mathcal{T}(\Sigma)$ ([KS07]).
As can be seen, the first major difference lies in the fact that $\mathrm{Hit}_3(\Sigma)$ cannot be isomorphic to $\mathcal{T}(\Sigma)\times\mathcal{T}(\Sigma)$, since its real dimension is equal to $16g-16$. This does not allow, unlike the anti-de Sitter case, to define a natural para-complex structure $\mathbf{J}$ that together with $\mathbf{I}$ gives rise to another para-complex structure $\mathbf{K}:=\mathbf{IJ}$. Moreover, the parameterization $\mathcal{MGH}(\Sigma)\cong T^*\mathcal{T}(\Sigma)$ as a holomorphic vector bundle, gives rise to a complex symplectic structure on $\mathcal{MGH}(\Sigma)$, which is missing for the Hitchin component. This is the reason why with our construction we only obtain a pseudo-Kähler metric.
The different descriptions as holomorphic vector bundles over $\mathcal{T}(\Sigma)$ lead to different computations along the way. In fact, in the $\mathbb{P}\mathrm{SL}(2,\mathbb{R})\times\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ setting one has to work with a pair given by a complex structure and a holomorphic quadratic differential on Σ, the real part of which corresponds to the second fundamental form of the immersion as a maximal surface in AdS 3-manifolds, namely it is an endomorphism of $T\Sigma$. In our case, the real part of a holomorphic cubic differential is, up to the contraction with the metric, an $\mathrm{End}(T\Sigma)$-valued 1-form. On the one hand, the additional 1-form part makes the analysis more difficult, but on the other we still succeed in obtaining similar results in regard to some key steps in the construction (Proposition 3.39 and Proposition 3.41).
The presence of other two moment maps in the AdS seeting ([MST21, Theorem 6.5]), allowed the authors to obtain $\mathcal{MGH}(\Sigma)$ as the quotient of an infinite-dimensional space by the group of all symplectomorphisms of the surface isotopic to the identity. In our case, not knowing whether the equation $\mathrm{d}^\nabla A=0$ can be interpreted as a moment map, we had to resort to the use of two quotients, which led to further difficulties developed in Section 3.4.1. It is also worth mentioning that since the PDE's defining the distribution tangent to the deformation space are much more complicated in our setting, it was necessary to employ a deep analysis of the associated differential operators (Section 3.2.4).
Finally, the most relevant part: the pseudo-metric is non-degenerate on the deformation space. In [MST21], the authors were able to identify the three symplectic forms they defined on $\mathcal{MGH}(\Sigma)$ with already known symplectic forms (thus non-degenerate), in terms of the

various parameterizations given above. In our setting, we do not know what the relation between our symplectic form $\boldsymbol{\omega}$ and Goldman's one is, because of the particular choice of function f that must be made and on which $\boldsymbol{\omega}$ depends. This led us to a careful analysis of the involved infinite-dimensional spaces and to obtain some partial results towards the non-existence of degenerate vectors for $\mathbf{g}$ away from the Fuchsian locus (Section 3.4.2 and 3.4.4).

Outline of the thesis

The thesis is structured as follows. Chapter 1 introduces the Hitchin component for the Lie group $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$ and it explains the classical relation with convex projective structures and hyperbolic affine sphere immersions. In Chapter 2 we study the deformation space of properly convex projective structures on the torus, and we introduce an explicit family of pseudo-Kähler metrics which are invariant by the action of the mapping class group. The material of this chapter can be found in:

[RT21] Rungi N., Tamburelli A., *Pseudo-Kähler geometry of properly convex projective structures on the torus.*

Chapter 3 deals with the construction of a pseudo-Kähler metric $(\mathbf{g},\mathbf{I},\boldsymbol{\omega})$ on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component associated with a genus $g \geqslant 2$ surface, so that $\mathbf{I}$ is exactly the complex structure found by Labourie and Loftin. We prove that such a component, and pseudo-Kähler metric, can be obtained by means of symplectic reduction theory in an infinite-dimensioanl context. In particular, we find an interpretation of Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$ as a moment map for the action of an infinite-dimensional Lie group. The material covered here has appeared in the preprint:

[RT23] Rungi N., Tamburelli A., *The* $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$*-Hitchin component as an infinite-dimensional pseudo-Kähler reduction.*

In Chapter 4 we use the explicit description of the family of pseudo-Kähler structures introduced in the torus case to study some metric and symplectic properties. In particular, using the theory of complete Lagrangian fibrations, we prove the existence of a geometric global Darboux frame for the symplectic form. In addition, we succeed in describing the explicit form of an arbitrary isometry of the space, for a particular choice of the pseudo-Kähler metric among those introduced. The material covered here is presented in:

[RT22] Rungi N., Tamburelli A., *Global Darboux coordinates for complete Lagrangian fibrations and an application to the deformation space of projective structures in genus one* (to appear in *Journal of Symplectic Geometry*, Volume 22 - Issue 2).

Chapter 1

Background materials

In the first chapter we introduce the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component of a smooth closed oriented surface Σ of genus $g \geqslant 2$, and we explain its relation with convex $\mathbb{R}\mathbb{P}^2$-structures and hyperbolic affine sphere immersion. The material covered here is classical, and the main purpose is to fix notation and recall fundamental results on the topic.

1.1 The $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component

Let Σ be a closed, connected smooth and oriented surface of genus $g \geqslant 2$ and consider the space $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{P}\mathrm{SL}(3,\mathbb{R}))$ of all representations from $\pi_1(\Sigma)$ to $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$. This set has a topology induced by the inclusion

$$\begin{aligned} \mathrm{Hom}(\pi_1(\Sigma),\, \mathbb{P}\mathrm{SL}(3,\mathbb{R})) &\hookrightarrow \mathbb{P}\mathrm{SL}(3,\mathbb{R})^{2g} \\ \rho &\longmapsto \big(\rho(a_1), \dots, \rho(b_g)\big) \end{aligned}$$

where $a_1, \dots, b_g$ are generators of $\pi_1(\Sigma)$ subject to the relation $\prod_{i=1}^{g} \big[a_i, b_i\big] = 1$. There is a natural action of $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$ on this space given by conjugation: for $\gamma \in \pi_1(\Sigma)$ and $P \in \mathbb{P}\mathrm{SL}(3,\mathbb{R})$

$$(P \cdot \rho)(\gamma) := P^{-1}\rho(\gamma)P \ . \tag{1.1.1}$$

In order to get a Hausdorff quotient space, one needs to restrict to the *completely reducible* representations, i.e. those $\rho : \pi_1(\Sigma) \to \mathbb{P}\mathrm{SL}(3,\mathbb{R})$ which split as a direct sum of irreducible representations. If we denote by $\mathrm{Hom}^+(\pi_1(\Sigma), \mathbb{P}\mathrm{SL}(3,\mathbb{R}))$ the space of such representations, the quotient space

$$\mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3,\mathbb{R})\big) := \mathrm{Hom}^+(\pi_1(\Sigma), \mathbb{P}\mathrm{SL}(3,\mathbb{R})) \Big/ \mathbb{P}\mathrm{SL}(3,\mathbb{R})$$

is called the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-*character variety*. It is a real algebraic variety (possibly singular), whose real dimension at a smooth point is equal to $-8\chi(\Sigma)$.

Theorem 1.1 (Hitchin [Hit92]). *The real algebraic variety $\mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3,\mathbb{R})\big)$ has three connected components: the one containing the class of the trivial representation, the one consisting of representations whose associated flat $\mathbb{R}^3$-bundles have non-zero second Stiefel-Whitney class and the one consisting of representations connected to those arising as uniformization. Moreover, the third one is contained in the smooth locus of $\mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3,\mathbb{R})\big)$ and it is diffeomorphic to $\mathbb{R}^{-8\chi(\Sigma)}$.*

It must be noted that the there is no topological invariant which distinguishes the first component to the third one, as they are both formed by representations whose associated flat $\mathbb{R}^3$-bundles have zero second Stiefel-Whitney class. The most interesting component in the above list is the last one, which will be denoted by $\mathrm{Hit}_3(\Sigma)$ throughout the discussion. In Hitchin's original paper ([Hit92]) it was called the "Teichmüller component" since it seemed to be a natural generalization of the Teichmüller component $\mathcal{T}^{\mathfrak{rep}}(\Sigma)$ for $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$, which is actually contained in $\mathrm{Hit}_3(\Sigma)$. Nowadays it is known as the *Hitchin component* and for our particular case (also for $\mathbb{P}\mathrm{SL}(n,\mathbb{R})$) there is a quite explicit description of its construction and of the inclusion $\mathcal{T}^{\mathfrak{rep}}(\Sigma) \hookrightarrow \mathrm{Hit}_3(\Sigma)$. Let us identify $\mathbb{R}^3$ with the space of homogeneous polynomials in two variables x, y of degree 2, i.e. $\mathbb{R}^3 \cong \mathrm{Span}_{\mathbb{R}}\{x^2, xy, y^2\}$. There is an action of $\mathrm{SL}(2,\mathbb{R})$ on such space:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^{2-i}y^i := (ax+cy)^{2-i}(bx+dy)^i, \qquad i = 0, 1, 2$$

which induces a (unique up to conjugation) representation $\tau_3 : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ given by:

$$\tau_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

It is immediate to see that one gets an induced representation $\mathbb{P}\mathrm{SL}(2,\mathbb{R}) \to \mathbb{P}\mathrm{SL}(3,\mathbb{R})$ still denoted by τ_3. For any discrete and faithful representation $j : \pi_1(\Sigma) \to \mathbb{P}\mathrm{SL}(2,\mathbb{R})$, the composition $\tau_3 \circ j : \pi_1(\Sigma) \to \mathbb{P}\mathrm{SL}(3,\mathbb{R})$ is discrete and faithful as well. The Hitchin component can be defined as the connected component of $\mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3,\mathbb{R})\big)$ containing $\tau_3 \circ j$, i.e. it is formed by all the representations obtained as deformations of the *Fuchsian* ones. In particular, the composition $\tau_3 \circ j$ induces an inclusion of $\mathcal{T}^{\mathfrak{rep}}(\Sigma)$ in $\mathrm{Hit}_3(\Sigma)$, whose image is called the *Fuchsian locus* and it will be denoted by $\mathcal{F}(\Sigma)$.

1.2 Deformation space of convex $\mathbb{RP}^2$-structures

An $\mathbb{RP}^2$-*structure* on a smooth connected surface S is a maximal $\mathbb{RP}^2$-atlas, namely an atlas in which the local charts take value in the real projective plane and the transition functions restrict to projective transformations on each connected component of the subset where defined. Once a maximal $\mathbb{RP}^2$-atlas is given, we say that S is an $\mathbb{RP}^2$-*surface*. By

unravelling the definition it is easy to see that if S is an $\mathbb{RP}^2$-surface and $p : \widetilde{S} \to S$ is its universal cover, then $\widetilde{S}$ inherits an $\mathbb{RP}^2$-structure from the one of S.
A domain (open and connected) $\Omega \subset \mathbb{RP}^2$ is said to be *convex* if there exists a projective line l disjoint from Ω such that $\Omega \subset \mathbb{RP}^2 \setminus l \cong \mathbb{A}^2$ is convex in the usual sense. By definition $\mathbb{R}^2$ is convex but $\mathbb{RP}^2$ is not. It is not difficult to show that this notion of convexity in the real projective plane does not add new convex sets with respect to those usual in affine spaces ([APS04, §1]) .

Definition 1.2. An $\mathbb{RP}^2$-surface S is *convex* if it is projectively isomorphic to a quotient $\Omega/_\Gamma$, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathrm{Proj}(\Omega) \subset \mathrm{SL}(3,\mathbb{R})$ is a discrete group of projective transformations preserving Ω acting freely and properly discontinuously on Ω. The surface S is *properly convex* if Ω is bounded in some affine space.

There is a well-known equivalent way of defining a convex $\mathbb{RP}^2$-surface in terms of the existence of a pair of maps with special properties. This is the following:

Theorem 1.3 (Development Theorem)**.** *Let S be an $\mathbb{RP}^2$-surface, then the following are equivalent:*

(1) S is convex

(2) There exists a pair (dev, h), where $\mathrm{dev} : \widetilde{S} \to \mathbb{RP}^2$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$ called the developing map and $h : \pi_1(S) \to \mathrm{SL}(3,\mathbb{R})$ is a group homomorphism called the holonomy representation, such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\mathrm{dev}"] \arrow[d, "\gamma"'] & \mathbb{RP}^2 \arrow[d, "h(\gamma)"] \\ \widetilde{S} \arrow[r, "\mathrm{dev}"] & \mathbb{RP}^2 \end{tikzcd} \tag{1.2.1}$$

Moreover, if $(\widetilde{dev}, \widetilde{h})$ is another such pair, then $\exists g \in \mathrm{SL}(3,\mathbb{R})$ such that:

$$\widetilde{\mathrm{dev}} = g \circ \mathrm{dev}, \qquad \widetilde{h}(\gamma) = g \circ h(\gamma) \circ g^{-1}, \ \forall \gamma \in \pi_1(S) \ .$$

It is clear from the statement that if S is convex, then its universal cover $\widetilde{S}$ can be identified with a convex domain $\Omega \subset \mathbb{RP}^2$ via the developing map and the discrete subgroup Γ can be identified with $\pi_1(S)$ via the holonomy homomorphism. From this point on we will focus only on the case in which the surface is closed and orientable, hence it will be denoted with Σ.

Definition 1.4. Let Σ be a smooth, closed and orientable surface. A *(properly) convex* $\mathbb{RP}^2$*-structure* on Σ is a pair (ϕ, M), where $\phi : \Sigma \to M$ is a diffeomorphism (called the *marking*) and $M \cong \Omega/_\Gamma$ is a (properly) convex $\mathbb{RP}^2$-surface.

One can define an equivalence relation on such pairs, namely $(\phi_1, M_1) \sim (\phi_2, M_2)$ if and only if there exists a projective isomorphism $\Psi : M_1 \to M_2$ such that the new marking $\Psi \circ \phi_1$ is isotopic to ϕ_2. Now we are ready to introduce the main space that we are going to study in this article: *the deformation space of (properly) convex $\mathbb{RP}^2$-structures*

$$\mathcal{B}(\Sigma) := \{(f, M) \text{ convex } \mathbb{RP}^2 - \text{structure on } \Sigma\} /_{\sim},$$

$$\mathcal{B}_0(\Sigma) := \{(f, M) \text{ properly convex } \mathbb{RP}^2 - \text{structure on } \Sigma\} /_{\sim} \ .$$

The behavior of this space depends highly on the genus of the surface and, as one can imagine, there are notable differences between the flat case (genus one) and the hyperbolic one ($g \geqslant 2$).

Proposition 1.5 ([Kui53],[Ben60])**.** *If Σ is a convex $\mathbb{RP}^2$-surface with $g \geqslant 2$, then it must be properly convex. Moreover, the boundary $\partial\Omega$ is always strictly convex and C^1, and it must be either and ellipse or a Jordan curve which is nowhere C^2. In particular, there is an identification $\mathcal{B}(\Sigma) \equiv \mathcal{B}_0(\Sigma)$.*

In the case of the torus this is no longer true, for instance there are many convex $\mathbb{RP}^2$-structures which are not properly convex: affine and Euclidean ones. They can not be properly convex since the developing map identifies the universal cover of T^2 with a copy of $\mathbb{R}^2$ inside $\mathbb{RP}^2$, which is convex but not bounded (see [Gol22, §8.5]).
To any equivalence class of convex $\mathbb{RP}^2$-structures on Σ there is an associated class of representations $[\rho]$, with $\rho : \pi_1(\Sigma) \to \mathbb{P}\mathrm{SL}(3, \mathbb{R})$, by Theorem 1.3. In particular, this association defines the so-called *monodromy map* $\mathfrak{hol} : \mathcal{B}(\Sigma) \to \mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3, \mathbb{R})\big)$ whose image is contained in the space of discrete and faithful representations.

Theorem 1.6 ([Gol90a], [CG93])**.** *The map $\mathfrak{hol} : \mathcal{B}(\Sigma) \to \mathfrak{R}\big(\Sigma, \mathbb{P}\mathrm{SL}(3, \mathbb{R})\big)$ induces an isomorphism between $\mathcal{B}(\Sigma)$ and $\mathrm{Hit}_3(\Sigma)$. In particular, any deformation of a Fuchsian representation $\tau \circ j : \pi_1(\Sigma) \to \mathbb{P}\mathrm{SL}(3, \mathbb{R})$ can be realized as the holonomy of a convex $\mathbb{RP}^2$-structure on Σ.*

1.3 Hyperbolic affine spheres

Let Σ be a closed surface of genus $g \geqslant 2$ with universal cover $\widetilde{\Sigma}$ and let $f : \widetilde{\Sigma} \to \mathbb{R}^3$ be an immersion with $\tilde{\xi} : \widetilde{\Sigma} \to \mathbb{R}^3$ a transverse vector field to $f(\widetilde{\Sigma})$. This means that for all $x \in \widetilde{\Sigma}$ we have a splitting:

$$T_{f(x)}\mathbb{R}^3 = f_* T_x \widetilde{\Sigma} + \mathbb{R}\tilde{\xi}_x \ .$$

Let D be the standard flat connection on $\mathbb{R}^3$ and suppose the structure equations of the immersed surface are given by:

$$\begin{aligned} D_X Y &= \nabla_X Y + h(X, Y)\xi \\ D_X \xi &= -S(X) \end{aligned} \tag{1.3.1}$$

where ∇ is a torsion-free connection on $\widetilde{\Sigma}$ called the *Blaschke connection*, ξ is the *affine normal* of the immersion (see [Lof01, §3.1] for example), h is a metric on $\widetilde{\Sigma}$ called the *Blaschke metric* and S is an endomorphism of $T\widetilde{\Sigma}$ called the *affine shape operator*.

Definition 1.7. Let N be an immersed hypersurface in $\mathbb{R}^3$ with structure equations given by (1.3.1). Then N is called a *hyperbolic affine sphere* if $S = -\mathrm{Id}_{TN}$.

The properties of the global geometry of hyperbolic affine spheres were conjectured by Calabi ([Cal72]) and proved by Cheng-Yau ([CY77], [CY86]) and Calabi-Nirenberg (with clarifications by Gigena ([Gig81]) and Li ([Li90], [Li92])). The most important result (stated only in $\mathbb{R}^3$ but true in arbitrary $\mathbb{R}^n$) is the following:

Theorem 1.8 (Cheng-Yau-Calabi-Nirenberg)**.** *Given a constant $\lambda < 0$ and a convex, bounded domain $\Omega \subset \mathbb{R}^2$, there is a unique properly embedded hyperbolic affine sphere $N \subset \mathbb{R}^3$ with affine shape operator $S = \lambda \cdot \mathrm{Id}_{TN}$ and center 0 asymptotic to the boundary of the cone $\mathcal{C}(\Omega) := \{(tx, t) \mid x \in \Omega, t > 0\} \subset \mathbb{R}^3$. For any immersed hyperbolic affine sphere $f : N \to \mathbb{R}^3$, properness of the immersion is equivalent to the completeness of the Blaschke metric, and any such N is a properly embedded hypersurface asymptotic to the boundary of the cone given by the convex hull of N and its center.*

We can use the above theorem to describe a $\mathrm{Diff}(\Sigma)$-equivariant one-to-one correspondence between convex $\mathbb{RP}^2$-structures and hyperbolic affine spheres. In fact, given a convex $\mathbb{RP}^2$-structure $\phi : \Sigma \to M \cong \Omega/\Gamma$, where $\Omega \subset \mathbb{R}^2$ is bounded, there exists a unique hyperbolic affine sphere $\mathcal{H} \subset \mathbb{R}^3$ asymptotic to the boundary of the cone $\mathcal{C}(\Omega) \subset \mathbb{R}^3$ (Theorem 1.8). Such a hyperbolic affine sphere $\mathcal{H}$ is invariant under automorphisms of $\mathcal{C}(\Omega)$, seen as a subgroup of $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$. The restriction of the projection $\pi : \mathcal{C}(\Omega) \to \Omega$ induces a diffeomorphism of $\mathcal{H}$ onto Ω. By equivariance, the tensor h and the connection ∇ descend to the quotient $\Omega/\Gamma \cong M$. Viceversa, given an embedding of the universal cover $\widetilde{\Sigma} \hookrightarrow \mathbb{R}^3$ as a $\widetilde{\Gamma}$-equivariant hyperbolic affine sphere, with $\widetilde{\Gamma} \cong \pi_1(\Sigma)$, one gets an identification of $\widetilde{\Sigma}$ with a domain $\Omega \subset \mathbb{RP}^2$, via the developing map. Then, Theorem 1.8 implies that $\widetilde{\Sigma}$ is asymptotic to a cone over Ω. The action of $\widetilde{\Gamma}$ on $\widetilde{\Sigma} \subset \mathbb{R}^3$ corresponds to an action of a group $\Gamma < \mathbb{P}\mathrm{SL}(3, \mathbb{R})$, isomorphic to $\pi_1(\Sigma)$, on the domain Ω so that $\Sigma \cong \Omega/\Gamma$.

Let $f : (\widetilde{\Sigma}, h, \nabla) \to \mathbb{R}^3$ be an immersed hyperbolic affine sphere, where h is the Blaschke metric and ∇ is the Blaschke connection. If ∇^h denotes the Levi-Civita connection with respect to h, then $\nabla = \nabla^h + A$, where A is a section of $T^*(\Sigma) \otimes \mathrm{End}(T\Sigma)$ called the *Pick form*. In particular, for every $X \in \Gamma(T\Sigma)$ the quantity $A(X)$ is an endomorphism of $T\Sigma$.

Definition 1.9. The *Pick tensor* is the $(0, 3)$-tensor defined by

$$C(X, Y, Z) := h(A(X)Y, Z), \quad \forall X, Y, Z \in \Gamma(T\Sigma) \ . \tag{1.3.2}$$

Corollary 1.10. *If $f : (\widetilde{\Sigma}, h, \nabla) \hookrightarrow \mathbb{R}^3$ is an immersed hyperbolic affine sphere, then the Pick tensor is totally symmetric, namely in index notation C_{ijk} we have*

$$C_{ijk} = C_{\sigma(ijk)}, \quad \forall \sigma \in \mathfrak{S}_3 \ .$$

In particular, this is equivalent to the requirement that the endomorphism $A(X)$ is h-symmetric for all $X \in \Gamma(T\Sigma)$ and

$$A(X)Y = A(Y)X, \quad \forall X, Y \in \Gamma(T\Sigma) \ . \tag{1.3.3}$$

Theorem 1.11 ([BH13, Lemma 4.8])**.** *Let Σ be a closed oriented surface of genus $g \geqslant 1$. Let h be a Riemannian metric on Σ and J be the induced (almost) complex structure. Suppose that a $(1,2)$ tensor A and a $(0,3)$ tensor C are related by $A = h^{-1}C$. Assume further that the tensor C is totally symmetric. Then, $A(X)$ is trace-free for all $X \in \Gamma(T\Sigma)$ if and only if C is the real part of a complex cubic differential, which can be expressed as $q = C(\cdot,\cdot,\cdot) - iC(J\cdot,\cdot,\cdot)$. If this holds, then the following are equivalent:*
- $\mathrm{d}^{\nabla^h} A = 0$;
- *C is the real part of a holomorphic cubic differential $q = C(\cdot,\cdot,\cdot) - iC(J\cdot,\cdot,\cdot)$;*
- $(\nabla^h_{JX} A)(\cdot) = (\nabla^h_X A)(J\cdot), \forall X \in \Gamma(T\Sigma)$.

The embedding data of hyperbolic affine spheres in $\mathbb{R}^3$ can be described in terms of the Blaschke metric h and the Pick form A satisfying the following equations:

$$\begin{cases} K_h - ||q||_h^2 = -1 \\ \mathrm{d}^{\nabla^h} A = 0 \ , \end{cases} \tag{HS}$$

where $q = C(\cdot,\cdot,\cdot) - iC(J\cdot,\cdot,\cdot)$ is the holomorphic cubic differential determined by the Pick tensor C, K_h is the Gaussian curvature of the Blaschke metric h and $A = h^{-1}C$ is the associated $\mathrm{End}_0(T\Sigma, h)$-valued 1-form. Moreover, for any tangent vector fields X, Y, Z on Σ the exterior-derivative $\mathrm{d}^{\nabla^h} A$ is the $\mathrm{End}_0(T\Sigma, h)$-valued 2-form

$$\big(\mathrm{d}^{\nabla^h} A\big)(X,Y)Z = \big(\nabla^h_X A\big)(Y)Z - \big(\nabla^h_Y A\big)(X)Z \ , \tag{1.3.4}$$

where $\mathrm{End}_0(T\Sigma, h)$ denotes the vector bundle of h-symmetric and trace-less endomorphisms of the tangent bundle.

Remark 1.12. Notice that the second equation in (HS) is invariant under conformal change of metric. In fact, it is equivalent to require that the cubic differential $q = C(\cdot,\cdot,\cdot) - iC(J\cdot,\cdot,\cdot)$ is holomorphic with respect to the complex structure defined by the conformal class of h. For this reason, in the following discussion, we will use either the tensor A or C according to which is more convenient.

Viceversa, every pair (h, C) satisfying (HS), with h a Riemannian metric and C a totally symmetric $(0,3)$-tensor equal to the real part of a h-cubic differential, i.e. a cubic differential that is holomorphic for the conformal class of h, represents the embedding data of a hyperbolic affine sphere in $\mathbb{R}^3$ ([Wan90],[Lof01],[BH13]). Considering that such a correspondence is natural by the action of $\mathrm{Diff}(\Sigma)$, we introduce the space parameterizing the embedding data of $\pi_1(\Sigma)$-equivariant hyperbolic affine spheres in $\mathbb{R}^3$ as:

$$\mathcal{HS}(\Sigma) := \left\{ (h, C) \;\middle|\; \begin{array}{l} h \text{ is a Riemannian metric} \\ C \text{ is the real part of a } h\text{-cubic differential} \\ \text{equations (HS) are satisfied} \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma) \tag{1.3.5}$$

Thus, according to the above discussion, we obtain the following result:

Proposition 1.13. *Let Σ be a closed surface of genus $g \geqslant 2$, then there exists a $MCG(\Sigma)$-invariant homeomorphism between $\mathcal{B}(\Sigma)$ and $\mathcal{HS}(\Sigma)$, given by the embedding data of the unique equivariant hyperbolic affine sphere.*

Because of this identification, for the rest of the discussion we will equivalently use one of the two pieces of notation in Proposition 1.13 to denote the deformation space of convex $\mathbb{RP}^2$-structures, hence the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component.

1.4 Wang's equation

Here we discuss the relation between the hyperbolic affine sphere immersion $f : \widetilde{\Sigma} \to \mathbb{R}^3$ and the conformal geometry of the surface. In particular, it is possible to rewrite the structure equations (1.3.1) in terms of a local holomorphic coordinate on the surface. Since we are interested in equivariant hyperbolic affine spheres, we can pick a parameterization $f : \Delta \to \mathbb{R}^3$, where Δ is a simply-connected domain in $\mathbb{C}$ biholomorphic to the open unit disk. Let $z = x + iy$ be a local conformal coordinate with respect to the Blaschke metric h, so that $h = e^{\psi}|\mathrm{d}z|^2$, where $|\mathrm{d}z|^2$ is defined as the symmetric product between $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. Since $\{e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$ is a h-orthonormal basis of the tangent space, the affine normal satisfies

$$\det\Big(e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y, \xi\Big) = 1$$

which implies

$$\det(f_x, f_y, \xi) = e^{\psi} \ .$$

By rewriting all in terms of

$$\frac{\partial f}{\partial z} = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + i f_y)$$

we get

$$\det(f_z, f_{\bar{z}}, \xi) = i e^{\psi} \ .$$

The affine structure equations are

$$\begin{aligned} D_X Y &= \nabla_X Y + h(X,Y)\xi \\ D_X \xi &= -\lambda \cdot X \ . \end{aligned} \tag{1.4.1}$$

Now consider the coordinate frame $\{e_1 := f_z := f_*(\frac{\partial}{\partial z}),\ e_{\bar{1}} := f_{\bar{z}} := f_*(\frac{\partial}{\partial \bar{z}})\}$. Hence,

$$h(f_z, f_z) = h(f_{\bar{z}}, f_{\bar{z}}) = 0, \quad h(f_z, f_{\bar{z}}) = \frac{1}{2}e^{\psi}.$$

Let θ be the matrix of connection one-forms for ∇, i.e.

$$\nabla e_i = \theta_i^j e_j, \quad i, j \in \{1, \bar{1}\}.$$

If $\hat{\theta}$ is the matrix of connection one-forms of the Levi-Civita connection, then

$$\hat{\theta}_{\bar{1}}^1 = \hat{\theta}_1^{\bar{1}} = 0, \quad \hat{\theta}_1^1 = \partial\psi, \quad \hat{\theta}_{\bar{1}}^{\bar{1}} = \bar{\partial}\psi \ .$$

The difference $\nabla - \nabla^h$ is equal to the so-called *Pick form*, namely the section of $\mathrm{End}_0(T\Sigma, h) \otimes T^*\Sigma$ satisfying (1.3.2). In local coordinates

$$\theta_i^j - \hat{\theta}_i^j = A_{ik}^j \rho^k, \quad i, j \in \{1, \bar{1}\}$$

where $\{\rho^1 = \mathrm{d}z, \rho^{\bar{1}} = \mathrm{d}\bar{z}\}$ is the dual frame of one-forms. By lowering an index we get the Pick tensor

$$C_{ijk} = h_{il} A_{jk}^l, \quad i, j, k \in \{1, \bar{1}\}$$

which is totally symmetric, as one can see from the last equation. In particular, all the components of C must vanish except for C_{111} and $\overline{C_{111}} = C_{\bar{1}\bar{1}\bar{1}}$. This discussion completely determines θ, indeed

$$\theta = \begin{pmatrix} \theta_1^1 & \theta_{\bar{1}}^1 \\ \theta_1^{\bar{1}} & \theta_{\bar{1}}^{\bar{1}} \end{pmatrix} = \begin{pmatrix} \partial\psi & e^{-\psi}\bar{Q}\mathrm{d}\bar{z} \\ e^{-\psi}Q\mathrm{d}z & \bar{\partial}\psi \end{pmatrix}$$

where $Q := 2C_{111}$ is a smooth function on the affine sphere.
Since D is the standard (flat) connection on $\mathbb{R}^3$, by using the structure equations (1.4.1) we have

$$\begin{aligned} f_{zz} &:= D_{f_z} f_z = \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \psi_z f_z + e^{-\psi} Q f_{\bar{z}} \\ f_{\bar{z}\bar{z}} &:= D_{f_{\bar{z}}} f_{\bar{z}} = \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial \bar{z}} = \psi_{\bar{z}} f_{\bar{z}} + e^{-\psi} \bar{Q} f_z \\ f_{z\bar{z}} &:= D_{f_z} f_{\bar{z}} = \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \bar{z}} + h(f_z, f_{\bar{z}})\xi = \frac{1}{2}e^{\psi}\xi \ . \end{aligned} \tag{1.4.2}$$

We can translate our affine sphere so that $\xi = f$, hence by combining (1.4.2) with this last equation we get a 1^{st}-order linear system in $\mathbf{F}^t := (f_z, f_{\bar{z}}, \xi)$, given by

$$\begin{aligned} \frac{\partial}{\partial z}\begin{pmatrix} f_z \\ f_{\bar{z}} \\ \xi \end{pmatrix} &= \begin{pmatrix} \psi_z & Qe^{-\psi} & 0 \\ 0 & 0 & \frac{1}{2}e^{\psi} \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_z \\ f_{\bar{z}} \\ \xi \end{pmatrix} \\ \frac{\partial}{\partial \bar{z}}\begin{pmatrix} f_z \\ f_{\bar{z}} \\ \xi \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\psi} \\ \bar{Q}e^{-\psi} & \psi_{\bar{z}} & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} f_z \\ f_{\bar{z}} \\ \xi \end{pmatrix} . \end{aligned} \tag{1.4.3}$$

Given an initial condition for $\mathbf{F}^t$ at $z_0 \in \Delta$, there exists a unique solution to this system as long as the following integrability conditions are satisfied

$$\begin{aligned} &\psi_{z\bar{z}} + |Q|^2 e^{-2\psi} - \frac{1}{2}e^{\psi} = 0 \\ &Q_{\bar{z}} = 0 \ . \end{aligned} \tag{1.4.4}$$

The second equation and the definition of Q implies that $q := Q\mathrm{d}z^3$ is a holomorphic cubic differential over Δ.

Remark 1.14. From now on, we rescale the cubic differential $q = Q\mathrm{d}z^3 \mapsto q' = Q'\mathrm{d}z^3 := \sqrt{2}Q\mathrm{d}z^3$, so that equations (1.4.4) become:

$$\begin{aligned} &\psi_{z\bar{z}} + \frac{1}{2}|Q'|^2 e^{-2\psi} - \frac{1}{2}e^{\psi} = 0 \\ &Q'_{\bar{z}} = 0 \ . \end{aligned} \tag{1.4.5}$$

Moreover, we will denote, by abuse of notation, the rescaled cubic differential with $q = Q\mathrm{d}z^3$. For this reason, some of the formulae that will follow in the torus case will differ by a multiplicative factor from those presented in [RT21]. This rescaling is done to be then consistent with what will be explained in the genus $g \geqslant 2$ case.

Now let (Σ, J) be a closed Riemann surface with genus $g \geqslant 2$. By the well-known Poincaré-Koebe Uniformization Theorem we can pick a Riemannian metric g_0 of constant curvature k_0 on Σ which is compatible with the initial complex structure J. Let $H^0(\Sigma, K^3)$ be the holomorphic sections of the tri-canonical bundle over (Σ, J), namely the $\mathbb{C}$-vector space of holomorphic cubic differentials. It is easy to see, using the Riemann-Roch Theorem, that this space has complex dimension equal to $5g-5$. If $z = x + iy$ is a local holomorphic coordinate on (Σ, J), then we can define a norm on $H^0(\Sigma, K^3)$, given by:

$$||q||^2_{g_0} := |Q|^2 e^{-3\phi} \ ,$$

where $q = Q\mathrm{d}z^3$ and $g_0 = e^{\phi}|\mathrm{d}z|^2$ in this local coordinate.

Theorem 1.15 ([Wan90]). *Pick the metric g_0 so that its Gaussian curvature is equal to -1. Let $h = e^u g_0$ be a Riemannian metric in the same conformal class as g_0 and $q \in H^0(\Sigma, K^3)$, then $\psi = u + \phi$ satisfies the first equation of (1.4.5) if and only if the metric h satisfies:*

$$K_h - ||q||_h^2 = -1 \ , \tag{1.4.6}$$

where K_h is the Gaussian curvature of h and $||q||_h^2 = ||q||_{g_0}^2 e^{-3u}$.

Lemma 1.16. *In the setting of the previous theorem, the metric h satisfies equation (1.4.6) if and only if the function $u : \Sigma \to \mathbb{R}$ satisfies the following semi-linear elliptic equation*

$$\Delta_{g_0} u + 2||q||_{g_0}^2 e^{-2u} - 2e^u + 2 = 0 \tag{1.4.7}$$

Proof. This is an easy application of the formula for the curvature $K_h = e^{-u}(k_0 - \frac{1}{2}\Delta_{g_0} u)$ under conformal change of metric $h = e^u g_0$. In fact, since g_0 can be chosen so that $k_0 = -1$, we get

$$K_h - ||q||_h^2 + 1 = -e^{-u} - \frac{1}{2}e^{-u}\Delta_{g_0} u - e^{-3u}||q||_{g_0}^2 + 1 \ .$$

Multiplying the right-hand side of the equation above by the factor $-2e^u$, we have the following equivalence

$$-e^{-u} - \frac{1}{2}e^{-u}\Delta_{g_0} u - e^{-3u}||q||_{g_0}^2 + 1 = 0 \iff \Delta_{g_0} u + 2||q||_{g_0}^2 e^{-2u} - 2e^u + 2 = 0 \ .$$

□

The original approach used by Wang to study existence and uniqueness of the solution to (1.4.7) (and thus to (1.4.6)) was the theory of elliptic operators between Sobolev spaces ([Wan90, §4]). About ten years later, Loftin simplified a lot the original argument by using the theory of sub and sup-solutions.

Lemma 1.17 ([SY94, Proposition V.1.1]). *Let $(M, \tilde{g})$ be a smooth compact Riemannian manifold. Consider the following differential equation:*

$$\Delta_{\tilde{g}} u + f(p, u) = 0 \ , \tag{1.4.8}$$

where f is a smooth function on $M \times \mathbb{R}$. Suppose there exist $\phi, \psi \in C^2(M)$ satisfying:

$$\Delta\phi + f(p, \phi) \geqslant 0 \ , \quad \Delta\psi + f(p, \psi) \leqslant 0 \ , \quad \phi \leqslant \psi \ .$$

Then, Equation (1.4.8) has a smooth solution u such that $\phi \leqslant u \leqslant \psi$. The functions ϕ and ψ are called respectively a sub-solution and a sup-solution for (1.4.8).

Proposition 1.18 ([Lof01]). *Let $(M, \tilde{g})$ be a smooth compact Riemannian manifold and let $\tilde{\varphi}$ be a smooth non-negative function on M. Then, the equation*

$$\Delta_{\tilde{g}} u + \tilde{\varphi}(p) e^{-2u} - 2e^u + 2 = 0 \tag{1.4.9}$$

has a unique smooth solution.

Proof. For the existence part, we need to find a sub and sup-solution to (1.4.9) and then appeal to Lemma 1.17 with $f(p,u) := \tilde{\varphi}(p)e^{-2u} - 2e^{u} + 2$. The sub-solution is given by $\phi := 0$ since $\tilde{\varphi}(p)$ is non-negative by hypothesis. To find the sup-solution ψ we look for a non-negative constant $c \equiv \psi$ such that $f(p,c) \leqslant 0$, which is equivalent, after multiplying by e^{2c}, to $\tilde{\varphi}(p) - 2e^{3c} + 2e^{2c} \leqslant 0$. In order to do so, let us first define $H := \max_{q\in M}\tilde{\varphi}(q)$, which is strictly positive since $\tilde{\varphi}$ is non-negative and non-constant, and set m to be the positive root of the equation $2x^3 - 2x^2 - H = 0$, so that $H = 2m^3 - 2m^2$. By definition, $m > 1$ and the sup-solution is given by $c \equiv \psi := \log m > 0$. In fact,

$$\tilde{\varphi}(p) - 2e^{3c} + 2e^{2c} = \tilde{\varphi}(p) - 2m^3 + 2m^2 = \tilde{\varphi}(p) - H \leqslant 0 \ .$$

The smoothness of the solution follows from standard arguments of elliptic theory.
For the uniqueness part we need to apply the maximum principle. Suppose u_1, u_2 are two solutions of (1.4.9) and let $x \in M$ be a maximum of $u_1 - u_2$, then $\Delta_{\tilde{g}}(u_1 - u_2)(x) \leqslant 0$. Since u_1, u_2 both satisfy (1.4.9), we get:

$$\tilde{\varphi}(p)e^{-2u_2(x)} - 2e^{u_2(x)} + 2 \leqslant \tilde{\varphi}(p)e^{-2u_1(x)} - 2e^{u_1(x)} + 2$$

but, the function $\tilde{\varphi}(p)e^{-2u} - 2e^{u} + 2$ is strictly decreasing in u, so it implies: $(u_1 - u_2)(x) \leqslant 0$. In particular, since x is a maximum of $u_1 - u_2$ we get

$$(u_1 - u_2)(y) \leqslant 0, \quad \forall y \in M \ .$$

Arguing with a minimum point it follows that the reverse inequality holds on the whole M, hence $u_1 \equiv u_2$. □

Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the conformal universal covering, namely $\widetilde{\Sigma}$ is biholomorphic to the open unit disk in $\mathbb{C}$. Given a holomorphic cubic differential q on Σ, we get by Proposition 1.18 a unique pair (h, q) satisfying (1.4.7) on Σ. Then, by Proposition 1.15 the pair (π^*h, π^*q) satisfies (1.4.5), where $\pi^*h = e^{\psi}|\mathrm{d}z|^2$ on $\widetilde{\Sigma}$. In particular, (π^*h, π^*q) determines a hyperbolic affine sphere $f : \widetilde{\Sigma} \to \mathbb{R}^3$ with π^*h as its Blaschke metric and it is complete since $\pi : (\widetilde{\Sigma}, \pi^*h) \to (\Sigma, h)$ is a local isometry and Σ is compact. Moreover, it can be proved that the deck transformation group of Σ can be regarded as a discrete subgroup of the unimodular affine group acting on the affine sphere $f : \widetilde{\Sigma} \to \mathbb{R}^3$. This holds because given any $\gamma \in \pi_1(\Sigma)$ we have $(\gamma^* \circ \pi^*)h = \pi^*h$ and $(\gamma^* \circ \pi^*)q = \pi^*q$, but the Blaschke metric and the Pick form completely determine the affine sphere up to unimodular affine transformations. Hence, the map which sends the point $f(p)$ to $f(\gamma(p))$, with $p \in \Sigma$, must be the restriction of an unimodular affine transformation in $\mathbb{R}^3$. By the standard theory of affine differential geometry it follows that the given construction yields all complete hyperbolic affine spheres which admit the action of a discrete subgroup of the unimodular affine group in $\mathbb{R}^3$ with compact quotient.

Corollary 1.19 ([Wan90; Lof01])**.** *A hyperbolic affine sphere in $\mathbb{R}^3$ with center 0 which admits a properly discontinuous action of a discrete group $\Gamma < \mathrm{SL}(3,\mathbb{R})$, so that the quotient* *is a closed oriented surface Σ of genus $g \geqslant 2$, is completely determined by a conformal structure on Σ and a section $q \in H^0(\Sigma, K^3)$. Moreover, all such hyperbolic affine spheres are determined in this way.*

1.5 Labourie and Loftin's parameterization

Let $\pi : Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big) \to \mathcal{T}^{\mathfrak{c}}(\Sigma)$ be the holomorphic vector bundle of cubic differentials over Teichmüller space. The fibre over an equivalence class $[J] \in \mathcal{T}^{\mathfrak{c}}(\Sigma)$ is the $\mathbb{C}$-vector space of holomorphic sections of the tri-canonical bundle. Given any pair $([J], q) \in Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$, we have an embedding $\widetilde{\Sigma} \to \mathbb{R}^3$ as a (equivariant) hyperbolic affine sphere whose Pick tensor and Blaschke metric are completely determined by $[J]$ and q (Corollary 1.19). In particular, by the argument in Section 1.2, we get a family of convex $\mathbb{RP}^2$-structures on Σ in the same $\mathrm{Diff}_0(\Sigma)$-orbit. Conversely, if we start with an equivalence class of convex $\mathbb{RP}^2$-structures, by Theorem 1.8 we get an (equivariant) embedding $\widetilde{\Sigma} \to \mathbb{R}^3$ as a hyperbolic affine sphere, which is equivalent to a pair $([J], q)$ as above. In the end, the main result is the following:

Theorem 1.20 ([Lof01],[Lab07])**.** *Let $\Phi : \mathcal{B}(\Sigma) \to Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$ be the map which associates to each equivalence class of convex $\mathbb{RP}^2$-structures the pair $([J], q)$ described above. Then, Φ is an homeomorphism.*

There is a pull-back action of $\mathrm{MCG}(\Sigma)$ on $Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$ given by:

$$[\psi] \cdot ([J], q) := ([\psi^* J], \psi^* q) \ .$$

It is well defined as it does not depend on the chosen representative in $[\psi] \in \mathrm{MCG}(\Sigma)$. Moreover, the pair $([\psi^* J], \psi^* q)$ still defines a point in $Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$ as $\psi^* q$ is holomorphic with respect to $\psi^* J$ if and only if q is J-holomorphic. In particular, the mapping class group $\mathrm{MCG}(\Sigma)$ acts on $\mathcal{B}(\Sigma)$ and $\mathrm{Hit}_3(\Sigma)$ by:

$$[\psi] \cdot [f, M] := [f \circ \psi, M], \qquad [\psi] \cdot [\rho] := [\rho \circ \psi_*] \tag{1.5.1}$$

for $[\psi] \in \mathrm{MCG}(\Sigma)$, $[f, M] \in \mathcal{B}(\Sigma)$ and $[\rho] \in \mathrm{Hit}_3(\Sigma)$, so that the monodromy map $\mathfrak{hol}$ induces a $\mathrm{MCG}(\Sigma)$-equivariant isomorphism between $\mathcal{B}(\Sigma)$ and $\mathrm{Hit}_3(\Sigma)$ (see Theorem 1.6). We get the following remarkable consequence:

Corollary 1.21 ([Lof01],[Lab07])**.** *The space* $\mathrm{Hit}_3(\Sigma)$ *carries a mapping class group invariant complex structure, denoted with* $\mathbf{I}$.

Chapter 2

The torus case

In this chapter we first study Wang's equation when Σ is a torus and we look at the associated flat hyperbolic affine sphere in $\mathbb{R}^3$. In particular, we get a correspondence between the deformation space of properly convex $\mathbb{RP}^2$-structures on T^2 and the complement of the zero section of the holomorphic bundle of cubic differentials over Teichmüller space. Using such a correspondence we define an explicit family of pseudo-Kähler structures on the aforementioned deformation space, which is invariant by the action of $\mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$. Finally, we prove that a circle action and a $\mathrm{SL}(2,\mathbb{R})$-action on the deformation space are both Hamiltonian and we compute the associated moment maps.

2.1 The parameterization in genus one

Let us consider the case when the Riemann surface $(\Sigma, J) = (T^2, J)$ has genus one. Then, we can always pick a flat metric g_0 so that $g_0 = |\mathrm{d}z|^2$ in coordinates. A holomorphic cubic differential q on T^2 is given (globally) by $q = c\mathrm{d}z^3$, with $c \in \mathbb{C}$, hence in this case equation (1.4.7) is

$$\Delta_0 u + 2|c|^2 e^{-2u} - 2e^u = 0 \ , \tag{2.1.1}$$

where u is the conformal parameter of the new metric $g = e^u g_0$ and $\Delta_0 = 4\partial_z\partial_{\bar{z}}$ is the standard Laplacian. Notice that if the holomorphic cubic differential is zero, namely if $c = 0$, then we get $\Delta_0 u = 2e^u$ and by integrating with respect to the volume form of g_0 it follows that

$$\int_{T^2} \Delta_0 u \ \mathrm{d}\mu_0 = 2\int_{T^2} e^u \ \mathrm{d}\mu_0 \ ,$$

which is not possible since the left hand side of the equation is zero and the right one is strictly positive.

Proposition 2.1. *Provided $c \neq 0$, Equation (2.1.1) has a unique constant solution given by* $u = \log\left(|c|^{\frac{2}{3}}\right)$

Proof. It is straightforward to see that $\log\left(|c|^{\frac{2}{3}}\right)$ satisfies (2.1.1). Suppose u is any other solution and let $p \in T^2$ be a maximum point for u. This implies that $(\Delta_0 u)(p) \leqslant 0$, hence

$$2e^{u(p)} \leqslant 2|c|^2 e^{-2u(p)} \quad \Longrightarrow \quad u(p) \leqslant \log\left(|c|^{\frac{2}{3}}\right)$$

Since p is a point of maximum, we get

$$u(x) \leqslant u(p) \leqslant \log\left(|c|^{\frac{2}{3}}\right), \quad \forall x \in T^2$$

Arguing in the same way with a point of minimum, we get $u(x) \geqslant \log\left(|c|^{\frac{2}{3}}\right) \forall x \in T^2$, thus the only possibility is that

$$u \equiv \log\left(|c|^{\frac{2}{3}}\right) .$$

□

We can already notice a first difference with the case genus $g \geqslant 2$, in which the solution to the semi-linear elliptic equation (1.4.7) could always be found. On the torus, if on the one hand we have to place restrictions on the possible values of the cubic holomorphic differential, on the other hand the treatment is considerably simplified. In this case, since the metric g_0 can be chosen to be flat, the function ψ of equations (1.4.5) coincides with the unique solution of (2.1.1). In particular, following the argument explained at the beginning of Section 1.4, we can rewrite the first order system of ODEs (1.4.3)[1] in the following way

$$\begin{aligned} \frac{\partial}{\partial z}\begin{pmatrix} f_z \\ f_{\bar z} \\ f \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} c e^{-\psi} & 0 \\ 0 & 0 & \frac{1}{2}e^{\psi} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_z \\ f_{\bar z} \\ f \end{pmatrix} \\ \frac{\partial}{\partial \bar z}\begin{pmatrix} f_z \\ f_{\bar z} \\ f \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\psi} \\ \frac{1}{\sqrt{2}} \bar c e^{-\psi} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_z \\ f_{\bar z} \\ f \end{pmatrix} . \end{aligned} \tag{2.1.2}$$

In a more compact form if $\mathbf{F}^t = (f_z, f_{\bar z}, f)$ and A, B are the 3×3 matrices in the first and second equation respectively, we get

$$\begin{cases} \frac{\partial}{\partial z}\mathbf{F} = A \cdot \mathbf{F} \\ \frac{\partial}{\partial \bar z}\mathbf{F} = B \cdot \mathbf{F} . \end{cases}$$

[1]The factor $\frac{1}{\sqrt{2}}$ in front of the cubic differential part appears because of the rescaling explained in Remark 1.14

As one can easily check, since A and B commute, a solution of this system is given by

$$\mathbf{F}(z,\bar{z}) = e^{Az+B\bar{z}} \cdot C \ , \tag{2.1.3}$$

where C is a constant matrix determined by the initial data. Now we are going to compute an explicit solution of (2.1.2), that is we will find a formula for the parameterization $f : \mathbb{C} \to \mathbb{R}^3$ of the hyperbolic affine 2-sphere. Then, using Theorem 1.8 we will get that this affine sphere is asymptotic to a cone over a bounded domain Ω, which will turn out to be projectively equivalent to a triangle. The main result of this section is the following:

Theorem 2.2. *Let $\Omega/_\Gamma$ be a properly convex $\mathbb{RP}^2$-structure on T^2, then Ω is projectively equivalent to a triangle in $\mathbb{R}^3$ with vertices* $\{(1,0,0);(0,1,0);(0,0,1)\}$.

Proof. Recall that the holomorphic cubic differential is given by $q = c\mathrm{d}z^3$, with $c \neq 0$ and $c = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \in \mathbb{R}$. Since to find the solution $\mathbf{F}$ of (2.1.2) we have to compute the exponential of a sum of matrices and Az and $B\bar{z}$ commute, we can find a common basis of eigenvectors that diagonalizes them simultaneously, namely we can find an invertible matrix P such that

$$Az = PD_{Az}P^{-1} \quad \text{and} \quad B\bar{z} = PD_{B\bar{z}}P^{-1}$$

with $D_{Az}, D_{B\bar{z}}$ diagonal matrices. From this it follows that

$$Az + B\bar{z} = PD_{Az+B\bar{z}}P^{-1} \quad \Longrightarrow \quad e^{Az+B\bar{z}} = Pe^{D_{Az+B\bar{z}}}P^{-1} \ ,$$

where $D_{Az+B\bar{z}} = D_{Az} + D_{B\bar{z}}$.

A common basis of eigenvectors is given by

$$\vec{v}_0 = \begin{pmatrix} e^{i\frac{2}{3}\theta} \\ 1 \\ \left(\frac{\rho}{2\sqrt{2}}\right)^{-\frac{1}{3}} e^{i\frac{\theta}{3}} \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} \zeta^2 e^{i\frac{2}{3}\theta} \\ 1 \\ \zeta\left(\frac{\rho}{2\sqrt{2}}\right)^{-\frac{1}{3}} e^{i\frac{\theta}{3}} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} \zeta e^{i\frac{2}{3}\theta} \\ 1 \\ \zeta^2\left(\frac{\rho}{2\sqrt{2}}\right)^{-\frac{1}{3}} e^{i\frac{\theta}{3}} \end{pmatrix}$$

with eigenvalues $\{\lambda_0 z, \zeta\lambda_0 z, \zeta^2\lambda_0 z\}$ for Az and eigenvalues $\{\bar{\lambda}_0\bar{z}, \zeta^2\bar{\lambda}_0\bar{z}, \zeta\bar{\lambda}_0\bar{z}\}$ for $B\bar{z}$, where $\zeta = e^{i\frac{2\pi}{3}}$ is a 3rd primitive root of unity and $\lambda_0 := \left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{i\frac{\theta}{3}}$. Hence, the matrix P is given by $(\vec{v}_0 \mid \vec{v}_1 \mid \vec{v}_2)$ and the eigenvalues of $Az + B\bar{z}$ are $\{2\mathcal{R}e(\lambda_0 z), 2\mathcal{R}e(\lambda_0 z\zeta), 2\mathcal{R}e(\lambda_0 z\zeta^2)\}$. At this point, it is easy to compute the matrix $e^{Az+B\bar{z}}$ and find the parameterization f, being it the third row of the solution of the system $\mathbf{F}$. The vector $f = (f_1, f_2, f_3)^t$ we obtain takes values in $\mathbb{C}^3$ and not in $\mathbb{R}^3$ as one might expect. This happens because we still have to make a choice of the initial data. Then, by choosing the following constant matrix C in (2.1.3)

$$C = \begin{pmatrix} \left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{i\frac{\theta}{3}} & \zeta\left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{i\frac{\theta}{3}} & \zeta^2\left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{i\frac{\theta}{3}} \\ \left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{-i\frac{\theta}{3}} & \zeta^2\left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{-i\frac{\theta}{3}} & \zeta\left(\frac{\rho}{2\sqrt{2}}\right)^{\frac{1}{3}} e^{-i\frac{\theta}{3}} \\ 1 & 1 & 1 \end{pmatrix} \tag{2.1.4}$$

we get

$$f(z,\bar{z}) = \begin{pmatrix} e^{2\mathcal{R}e(\lambda_0 z)} \\ e^{2\mathcal{R}e(\zeta\lambda_0 z)} \\ e^{2\mathcal{R}e(\zeta^2\lambda_0 z)} \end{pmatrix} \in \mathbb{R}^3 \ . \tag{2.1.5}$$

It is a straightforward computation to see that

$$\mathcal{R}e(\lambda_0 z) + \mathcal{R}e(\zeta\lambda_0 z) + \mathcal{R}e(\zeta^2\lambda_0 z) = 0,$$

hence showing that f is a parameterization of the hypersurface $\{(x,y,w) \in \mathbb{R}^3 \mid xyw = 1,\ x,y,w > 0\}$. In particular, the hyperbolic affine sphere we get is asymptotic to the three coordinate planes in the first octant, which are nothing but the boundary of the cone over the triangle T contained in the plane $\{(x,y,w) \in \mathbb{R}^3 \mid x+y+w=1\}$ and with vertices $\{(1,0,0);(0,1,0);(0,0,1)\}$. By Theorem 1.8, this triangle has to be projectively equivalent to the convex bounded domain Ω of the initial properly convex $\mathbb{RP}^2$-structure, where $p : \mathbb{R}^3\backslash\{0\} \to \mathbb{RP}^2$ is the standard projection. □

Remark 2.3. It must be noted that in the case of genus one, the problem of classifying convex bounded domains Ω, as in Definition 1.4, up to projective transformations and preserved by the action of a discrete subgroup $\Gamma < \mathrm{SL}(3,\mathbb{R})$ contained in $\mathrm{Proj}(\Omega)$ and isomorphic to $\mathbb{Z}\times\mathbb{Z}$, is equivalent to the problem of classifying flat hyperbolic affine spheres in $\mathbb{R}^3$ up to unimodular affine transformations. The latter was the problem studied in [MR90] which we now recovered in terms of properly convex $\mathbb{RP}^2$-structures over the torus.

Corollary 2.4. *There exists a bijection between $\mathcal{B}_0(T^2)$ and the complement of the zero section in $Q^3\big(\mathcal{T}(T^2)\big)$.*

Proof. The above bijection follows from Theorem 1.8. In fact, by the discussion in Section 1.3 to any $\big[\Omega/_\Gamma\big] \in \mathcal{B}_0(T^2)$ we have an associated equivariant hyperbolic affine sphere $M \hookrightarrow \mathbb{R}^3$ which is determined by its Blaschke metric and its Pick tensor (see Corollary 1.19). Hence, let $\chi : \mathcal{B}_0(T^2) \to Q^3_0(\mathcal{T}(T^2))$ be the map that associates to each $\big[\Omega/_\Gamma\big]$ the pair (J,q), where J is the complex structure induced by the Blaschke metric and $q = c\mathrm{d}z^3$ is a non-zero cubic holomorphic differential whose real part coincides with the Pick tensor of M. Since, by Lemma 1.16 and Proposition 2.1, for any such (J,q) we can find a unique (up to unimodular affine transformations) hyperbolic affine sphere in $\mathbb{R}^3$ that is invariant under a subgroup $\Gamma < \mathbb{P}\mathrm{SL}(3,\mathbb{R})$ isomorphic to $\pi_1(T^2)$, the map χ is a bijection. □

2.2 The pseudo-Kähler metric on the deformation space

2.2.1 Definition of the pseudo-Kähler structure

Let $\rho_0 := \mathrm{d}x_0 \wedge \mathrm{d}y_0$ be the standard area form on $\mathbb{R}^2$ and let us introduce the following space

Definition 2.5. The set $\mathcal{J}(\mathbb{R}^2)$ of ρ_0-compatible linear-complex structures on $\mathbb{R}^2$ is defined as

$$\mathcal{J}(\mathbb{R}^2) := \{J \in \mathrm{End}(\mathbb{R}^2) \mid J^2 = -\mathbb{1},\ \rho_0(v, Jv) > 0 \text{ for some } v \in \mathbb{R}^2 \backslash \{0\}\} \ .$$

This space is a 2-dimensional manifold and it is easy to see that $\forall J \in \mathcal{J}(\mathbb{R}^2)$, the pairing $g_J^0(\cdot,\cdot) := \rho_0(\cdot, J\cdot)$ is a scalar product on $\mathbb{R}^2$, with respect to which J is an orthogonal endomorphism. By differentiating the identity $J^2 = -\mathbb{1}$, it follows that

$$T_J\mathcal{J}(\mathbb{R}^2) = \{\dot{J} \in \mathrm{End}(\mathbb{R}^2) \mid J\dot{J} + \dot{J}J = 0\} \ .$$

Equivalently, the space $T_J\mathcal{J}(\mathbb{R}^2)$ can be identified with the trace-less and g_J^0-symmetric endomorphisms of $\mathbb{R}^2$. It carries a natural (almost) complex structure given by:

$$\begin{aligned} \widehat{\mathcal{I}} : T_J\mathcal{J}(\mathbb{R}^2) &\to T_J\mathcal{J}(\mathbb{R}^2) \\ \dot{J} &\mapsto -J\dot{J} \ . \end{aligned}$$

There is a natural scalar product defined on each tangent space

$$\langle \dot{J}, \dot{J}' \rangle_J := \frac{1}{2}\,\mathrm{tr}\Big(\dot{J}\dot{J}'\Big)$$

for each $\dot{J}, \dot{J}' \in \mathcal{J}(\mathbb{R}^2)$. It is easy to check that $\widehat{\mathcal{I}}$ preserves this scalar product.

Lemma 2.6. *There is a diffeomorphism between $\mathcal{J}(\mathbb{R}^2)$ and $\mathcal{T}(T^2)$, which is equivariant with respect to the action of* $\mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$.

Proof. A linear (almost) complex structure J can be thought of as a constant tensor on $\mathbb{R}^2$, which therefore induces an almost-complex structure on the torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. This gives a well-defined map from $\mathcal{J}(\mathbb{R}^2)$ to $\mathcal{T}(T^2)$, which is a bijection since any element in $\mathcal{T}(T^2)$, namely an isotopy class of almost-complex structures on T^2, can be represented as the conformal structure J_0 (multiplication by i) on $\mathbb{R}^2/\Lambda$, with $\Lambda \cong \mathbb{Z}^2$. In fact, one can assume, up to homothety of Λ, that the torus $\mathbb{R}^2/\Lambda$ has area 1, and such representation is unique up to conjugation in $\mathrm{SO}(2)$. Then, conjugating J_0 by the unique element in $\mathrm{SL}(2,\mathbb{R})$ that maps Λ to $\mathbb{Z}^2$ (as marked lattices), one can find the unique $J \in \mathcal{J}(\mathbb{R}^2)$ that is sent to the given class in $\mathcal{T}(T^2)$. After identifying $\mathrm{MCG}(T^2)$ with $\mathrm{SL}(2,\mathbb{Z})$, the bijection is clearly equivariant by construction. □

Proposition 2.7. *The holomorphic vector bundle $Q^3\big(\mathcal{T}(T^2)\big)$ can be identified with the following*

$$D^3(\mathcal{J}(\mathbb{R}^2)) := \{(J,C) \in \mathcal{J}(\mathbb{R}^2) \times S_3(\mathbb{R}^2) \mid C(J\cdot, J\cdot, J\cdot) = -C(J\cdot,\cdot,\cdot)\} \tag{2.2.1}$$

where $S_3(\mathbb{R}^2)$ is the space of totally-symmetric tri-linear forms on $\mathbb{R}^2$. Moreover if $(J,C) \in D^3(\mathcal{J}(\mathbb{R}^2))$, then

$$C(J\cdot,\cdot,\cdot) = C(\cdot, J\cdot,\cdot) = C(\cdot,\cdot,J\cdot) \ . \tag{2.2.2}$$

Proof. If $J \in \mathcal{J}(\mathbb{R}^2)$ and q is a cubic J-holomorphic differential, then $C = \mathcal{R}e(q)$ is a totally-symmetric tri-linear form on $\mathbb{R}^2$ by Theorem 1.11. In particular,

$$C(X, Y, Z) = g_J^0(A(X)Y, Z), \qquad \forall X, Y, Z \in \mathbb{R}^2 \ ,$$

where $A \in \mathrm{End}(\mathbb{R}^2) \otimes T^*\mathbb{R}^2$ and its endomorphism part is g_J^0-symmetric and trace-less. Hence, for all $X, Y, Z \in \mathbb{R}^2$, we have

$$\begin{aligned} C(JX, JY, JZ) &= -g_J^0(JA(JX)Y, JZ) && (A(JX) \in T_J\mathcal{J}(\mathbb{R}^2)) \\ &= -g_J^0(A(JX)Y, Z) && (J \text{ is } g_J^0 - \text{orthogonal}) \\ &= -C(JX, Y, Z) \ . \end{aligned}$$

We conclude that $(J, C = \mathcal{R}e(q)) \in D^3(\mathcal{J}(\mathbb{R}^2))$. Conversely, if $(J, C) \in D^3(\mathcal{J}(\mathbb{R}^2))$, then $q = C(\cdot, \cdot, \cdot) - iC(J\cdot, \cdot, \cdot)$ defines a cubic holomorphic differential by Theorem 1.11. Finally,

$$\begin{aligned} C(JX, Y, Z) &= g_J^0(A(JX)Y, Z) \\ &= -g_J^0(A(JX)JY, JZ) && (A(JX) \in T_J\mathcal{J}(\mathbb{R}^2)) \\ &= -g_J^0(A(JY)JX, JZ) && (\text{rel. } (1.3.3)) \\ &= g_J^0(JA(JY)X, JZ) && (A(JY) \in T_J\mathcal{J}(\mathbb{R}^2)) \\ &= g_J^0(A(X)JY, Z) \\ &= C(X, JY, Z) \end{aligned}$$

for all $X, Y, Z \in \mathbb{R}^2$. A similar computation shows that $C(\cdot, J\cdot, \cdot) = C(\cdot, \cdot, J\cdot)$. □

Remark 2.8. Notice that, thanks to Relation (1.3.2), the space $D^3(\mathcal{J}(\mathbb{R}^2))$ can be interpreted in terms of the tensor A, namely it is formed by all possible pairs (J, A) with $J \in \mathcal{J}(\mathbb{R}^2)$ and $A \in \mathrm{End}(\mathbb{R}^2) \otimes T^*\mathbb{R}^2$ such that:

- $A(X)Y = A(Y)X, \quad A(JX)Y = A(X)JY, \quad \forall X, Y \in \mathbb{R}^2$;
- the endomorphism $A(X)$ is g_J^0-symmetric and trace-less for all vectors X. In particular, $A(X) \in T_J\mathcal{J}(\mathbb{R}^2)$

We will make repetitive use of this correspondence, using the tensor C or the tensor A, whichever is more convenient.

Because of the identification $\mathcal{J}(\mathbb{R}^2) \cong \mathcal{T}(T^2)$, the space $D^3(\mathcal{J}(\mathbb{R}^2))$ has the structure of a vector bundle over $\mathcal{J}(\mathbb{R}^2)$, whose fiber at a point $J \in \mathcal{J}(\mathbb{R}^2)$ is a two dimensional real vector space, denoted with $D^3(\mathcal{J}(\mathbb{R}^2))_J$. Let $\{e_1, e_2\}$ be a g_J^0-orthonormal basis of $\mathbb{R}^2$ and $\{e_1^*, e_2^*\}$ be its dual, then any element A in $D^3(\mathcal{J}(\mathbb{R}^2))_J$ can be written as $A = A_1e_1^* + A_2e_2^*$, where $A_k := A(e_k)$ for $k = 1, 2$. Hence, we can introduce a scalar product on $D^3(\mathcal{J}(\mathbb{R}^2))_J$ by

$$\langle A, B \rangle_J := \mathrm{tr}(A \wedge *_J B)(e_1, e_2) \ , \tag{2.2.3}$$

or, more explicitly after expanding the wedge product and evaluating the 2-form,

$$\langle A, B\rangle_J = \operatorname{tr}(A_1B_1 + A_2B_2)e_1^* \wedge e_2^*(e_1, e_2) = \operatorname{tr}(A_1B_1 + A_2B_2)\ . \tag{2.2.4}$$

Remark 2.9. We will be assuming that the area of the torus for the flat metric g_J^0 is equal to 1. In fact, there is an equivalent description of $\mathcal{T}(T^2)$ as the space of isotopy classes of unit-area flat metrics on T^2. This can be seen thanks to the isomorphism $\mathcal{T}(T^2) \cong \mathcal{J}(\mathbb{R}^2)$ presented in Lemma 2.6. The set $\mathcal{J}(\mathbb{R}^2)$ can be interpreted as the space of all orientation preserving linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ up to rotation and/or dilation. This is equivalent to classify all possible marked lattices in $\mathbb{R}^2$ up to Euclidean isometries and homotheties. Since we can always, up to homotheties, choose a marked lattice of unit area, it follows we can always find a $J \in \mathcal{J}(\mathbb{R}^2)$ with the above property (see [FM11, §10.2]). Moreover, it is easy to check that Relation (2.2.4) does not depend on the choice of the basis.

By exploiting the definition in (2.2.4), the following relation can be deduced:

$$\langle AJ, BJ\rangle_J = \langle A, B\rangle_J\ , \tag{2.2.5}$$

which is equivalent to

$$\langle AJ, B\rangle_J = -\langle A, BJ\rangle_J\ . \tag{2.2.6}$$

Lemma 2.10. *Let $(J, A) \in D^3(\mathcal{J}(\mathbb{R}^2))$ and let $\dot{A} := (g_J^0)^{-1}\dot{C}$ denote the unique $(1,2)$-tensor such that $g_J^0(\dot{A}(X)Y, Z) = \dot{C}(X, Y, Z)$ for all $X, Y, Z \in \mathbb{R}^2$. Then, an element $(\dot{J}, \dot{A})$ belongs to $T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$ if and only if*

$$\dot{J} \in T_J\mathcal{J}(\mathbb{R}^2), \quad \operatorname{tr}\dot{A}(X) = \operatorname{tr}\Big(JA(X)\dot{J}\Big)\ \forall X \in \mathbb{R}^2, \quad \dot{A}_0 = \dot{\tilde{A}}_0 + T(J, A, \dot{J}) \tag{2.2.7}$$

where $\dot{A}_0$ is the full trace-free part of $\dot{A}$, while the tensor $\dot{\tilde{A}}_0$ is the trace-free part of $\dot{A}$ independent of $\dot{J}$ and $T(J, A, \dot{J}) = A_1J\dot{J}Ee_1^* + 2A_2J\dot{J}Ee_2^*$ in a local basis dual to a g_J^0-orthonormal frame $\{e_1, e_2\}$, with $E = \operatorname{diag}(1, -1)$.

Proof. First notice that

$$\dot{g}_J^0 = \rho(\cdot, \dot{J}\cdot) = -\rho(\cdot, J^2\dot{J}\cdot) = -g_J^0(\cdot, J\dot{J}\cdot)\ . \tag{2.2.8}$$

Then, since $A(X) = \big((g_J^0)^{-1}C\big)(X)$ and the endomorphisms $A(X)$ are trace-free for each vector X, we get

$$0 = \operatorname{tr}(A(X))' = \operatorname{tr}\big(\big((g_J^0)^{-1}C\big)(X)\big)' = -\operatorname{tr}\big((g_J^0)^{-1}\dot{g}_J^0(g_J^0)^{-1}C(X)\big) + \operatorname{tr}\Big(\dot{A}(X)\Big)\ .$$

In particular, using equation (2.2.8) we obtain $(g_J^0)^{-1}\dot{g}_J^0 = -J\dot{J}$ and hence

$$\operatorname{tr}\Big(\dot{A}(X)\Big) = \operatorname{tr}\big((g_J^0)^{-1}\dot{g}_J^0(g_J^0)^{-1}C(X)\big) = -\operatorname{tr}\Big(J\dot{J}A(X)\Big) = \operatorname{tr}\Big(JA(X)\dot{J}\Big)\ .$$

We defer the third decomposition in (2.2.7) to Section 2.2.2, where a computation in local coordinates is performed. □

The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^2)$ by conjugation and more generally on its tangent space by

$$(J,\dot{J}) \in T\mathcal{J}(\mathbb{R}^2), \quad P\cdot(J,\dot{J}) := (PJP^{-1}, P\dot{J}P^{-1})$$

with $P \in \mathrm{SL}(2,\mathbb{R})$.

Lemma 2.11. *There is an* $\mathrm{SL}(2,\mathbb{R})$ *action on* $D^3(\mathcal{J}(\mathbb{R}^2))$ *given by:*

$$P\cdot(J,A) := (PJP^{-1}, PA(P^{-1}\cdot)P^{-1}) \tag{2.2.9}$$

where $P \in \mathrm{SL}(2,\mathbb{R})$ *and* $A(P^{-1}\cdot)$ *has to be interpreted as the action of* P^{-1} *by pull-back on the one-form part of* A.

Proof. Let us first consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $Q^3\big(\mathcal{T}(T^2)\big)$ given by:

$$P\cdot(J,q) = (PJP^{-1}, (P^{-1})^*q)$$

with $P \in \mathrm{SL}(2,\mathbb{R})$ and $(J,q) \in Q^3\big(\mathcal{T}(T^2)\big)$. We need to understand how the above action transforms under the bijection of Proposition 2.7. The new cubic holomorphic differential $(P^{-1})^*q$ corresponds to the new tensor $\widetilde{C} = \mathcal{R}e\big((P^{-1})^*q\big) = C(P^{-1}\cdot, P^{-1}\cdot, P^{-1}\cdot)$ which is given by:

$$\begin{aligned}\widetilde{C}(X,Y,Z) &= C(P^{-1}X, P^{-1}Y, P^{-1}Z)\\ &= g_J^0(A\big(P^{-1}X\big)P^{-1}Y, P^{-1}Z)\\ &= \rho(PA\big(P^{-1}X\big)P^{-1}Y, PJP^{-1}Z) && (P \in \mathrm{SL}(2,\mathbb{R}))\\ &= g_{P\cdot J}^0(PA\big(P^{-1}X\big)P^{-1}Y, Z)\ .\end{aligned}$$

Hence, the corresponding $\widetilde{A}$ defined by

$$\widetilde{C}(X,Y,Z) = g_{P\cdot J}^0(\widetilde{A}(X)Y, Z)\ ,$$

is exactly $\widetilde{A} = PA(P^{-1}\cdot)P^{-1}$. □

Lemma 2.12. *For every* $P \in \mathrm{SL}(2,\mathbb{R})$ *and* $J \in \mathcal{J}(\mathbb{R}^2)$, *we have*

$$\begin{aligned}\langle P\cdot\dot{J}, P\cdot\dot{J}'\rangle_{P\cdot J} &= \langle\dot{J},\dot{J}'\rangle_J\\ \langle P\cdot A, P\cdot B\rangle_{P\cdot J} &= \langle A,B\rangle_J\end{aligned}$$

where $\dot{J},\dot{J}' \in T_J\mathcal{J}(\mathbb{R}^2)$ *and* $A,B \in D^3(\mathcal{J}(\mathbb{R}^2))_J$.

Proof. For the action on $T_J\mathcal{J}(\mathbb{R}^2)$, we have

$$\langle P\cdot\dot{J}, P\cdot\dot{J}'\rangle_{P\cdot J} = \frac{1}{2}\operatorname{tr}\Big(P\dot{J}\dot{J}'P^{-1}\Big)$$

$$\begin{aligned} &= \frac{1}{2}\,\mathrm{tr}\Big(\dot{J}\dot{J}'\Big) && \text{(trace symmetry)} \\ &= \langle \dot{J}, \dot{J}'\rangle_J \ . \end{aligned}$$

For the action on $D^3(\mathcal{J}(\mathbb{R}^2))_J$, we have

$$\begin{aligned} \langle P\cdot A, P\cdot B\rangle_{P\cdot J} &= \mathrm{tr}\big(PA_1B_1P^{-1} + PA_2B_2P^{-1}\big)\big((P^{-1})^*(e_1^*\wedge e_2^*)\big)(e_1,e_2) \\ &= \mathrm{tr}\big(P(A_1B_1 + A_2B_2)P^{-1}\big)(e_1^*\wedge e_2^*)(e_1,e_2) && (P\in \mathrm{SL}(2,\mathbb{R})) \\ &= \langle A, B\rangle_J \ . && \text{(trace symmetry)} \end{aligned}$$

□

The $\mathrm{SL}(2,\mathbb{R})$-action on $D^3(\mathcal{J}(\mathbb{R}^2))$ can be differentiated, hence we get a linear isomorphism between $T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$ and $T_{P\cdot(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$, which is given explicitly by

$$P\cdot(\dot{J},\dot{A}) = (P\dot{J}P^{-1}, P\dot{A}(P^{-1}\cdot)P^{-1})$$

where $(\dot{J},\dot{A})\in T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$ and $P\in\mathrm{SL}(2,\mathbb{R})$. Moreover, all the conditions in Lemma 2.10 are $\mathrm{SL}(2,\mathbb{R})$-invariant.
We can define a similar scalar product on pairs $\dot{A},\dot{B}$ by

$$\langle \dot{A},\dot{B}\rangle_J := \mathrm{tr}\Big(\dot{A}_1\dot{B}_1 + \dot{A}_2\dot{B}_2\Big) \ ,$$

which is $\mathrm{SL}(2,\mathbb{R})$-invariant as well.
In the following we will denote with $||\cdot||_J = ||\cdot||$ the norm induced by the scalar product $\langle\cdot,\cdot\rangle_J = \langle\cdot,\cdot\rangle$ and it will be clear from the context which one we are using. In order to simplify the notation we define $||A||_0^2 := \frac{1}{8}||A||_J^2$. Finally, since $\dot{A}$ is an element of $\mathrm{End}(\mathbb{R}^2)\otimes T^*\mathbb{R}^2$ whose endomorphism part is g_J^0-symmetric, let us consider its trace and trace-free part as in Lemma 2.10, namely if $\dot{A} = \dot{A}_1e_1^* + A_2e_2^*$, then:

$$\dot{A}_0 = (\dot{A}_1)_0e_1^* + (\dot{A}_2)_0e_2^*, \qquad \dot{A}_{\mathrm{tr}} = \frac{1}{2}\,\mathrm{tr}\Big(\dot{A}_1\Big)\mathbb{1}e_1^* + \frac{1}{2}\,\mathrm{tr}\Big(\dot{A}_2\Big)\mathbb{1}e_2^* \ .$$

Let $f:[0,+\infty)\to(-\infty,0]$ be a smooth function such that $f(0)=0$, $f'(t)<0$ for each $t>0$ and $\lim\limits_{t\to+\infty} f(t) = -\infty$. Then, we define the following symmetric bi-linear form on $T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$

$$\begin{aligned} (\widehat{\mathbf{g}}_f)_{(J,A)}\big((\dot{J},\dot{A});(\dot{J}',\dot{A}')\big) := \big(1 - f(||A||_0^2)\big)\langle\dot{J},\dot{J}'\rangle &+ \frac{f'(||A||_0^2)}{6}\langle\dot{A}_0,\dot{A}_0'\rangle \\ &- \frac{f'(||A||_0^2)}{12}\langle\dot{A}_{\mathrm{tr}},\dot{A}_{\mathrm{tr}}'\rangle \end{aligned} \tag{2.2.10}$$

and the endomorphism $\widehat{\mathbf{I}}$ of $T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$

$$\widehat{\mathbf{I}}_{(J,A)}(\dot{J},\dot{A}) := (-J\dot{J}, -\dot{A}J - A\dot{J}) \tag{2.2.11}$$

where the products $\dot{A}J$ and $A\dot{J}$ have to be interpreted as a matrix multiplication. Matching these two objects together we get the following 2-form:

$$\widehat{\boldsymbol{\omega}}_f(\cdot,\cdot) = \widehat{\mathbf{g}}_f(\cdot,\widehat{\mathbf{I}}\cdot)$$

which is given by:

$$\begin{aligned}(\widehat{\boldsymbol{\omega}}_f)_{(J,A)}\big((\dot{J},\dot{A});(\dot{J}',\dot{A}')\big) = \big(f(\|A\|_0^2)-1\big)\langle \dot{J}, J\dot{J}'\rangle - \frac{f'(\|A\|_0^2)}{12}\langle \dot{A}_{\mathrm{tr}}, *_J\dot{A}'_{\mathrm{tr}}\rangle \\ - \frac{f'(\|A\|_0^2)}{6}\langle \dot{A}_0, \dot{A}'_0 J\rangle \ .\end{aligned} \tag{2.2.12}$$

Remark 2.13. The symmetric tensor $\widehat{\mathbf{g}}_f$ and the form $\widehat{\boldsymbol{\omega}}_f$ are defined only in terms of the various scalar products $\langle\cdot,\cdot\rangle$ and $\|A\|_0^2$, hence by Lemma 2.12 they are both $\mathrm{SL}(2,\mathbb{R})$-invariant. In particular, the complex structure $\widehat{\mathbf{I}}$ is uniquely determined by the relation $\widehat{\boldsymbol{\omega}}_f(\cdot,\cdot) = \widehat{\mathbf{g}}_f(\cdot,\mathbf{I}\cdot)$ once the form $\widehat{\boldsymbol{\omega}}_f$ and the tensor $\widehat{\mathbf{g}}_f$ are given. In our case, this implies that $\widehat{\mathbf{I}}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant as well.

Lemma 2.14. *For every $\dot{J},\dot{J}' \in T_J\mathcal{J}(\mathbb{R}^2)$ we have*

$$\dot{J}\dot{J}' = \langle \dot{J},\dot{J}'\rangle_J \mathbb{1} - \langle J\dot{J},\dot{J}'\rangle_J J \ . \tag{2.2.13}$$

Proof. Notice that

$$J\dot{J}\dot{J}' = -\dot{J}J\dot{J}' = \dot{J}\dot{J}'J$$

Therefore, the matrix $\dot{J}\dot{J}'$ commutes with J, but it is straightforward to see that this is equivalent to $\dot{J}\dot{J}' \in \mathrm{Span}_{\mathbb{R}}\{\mathbb{1},J\}$, hence the thesis. □

Lemma 2.15. *Let $\{e_1,e_2\}$ be a g_J^0-orthonormal basis of $\mathbb{R}^2$ such that $Je_1 = e_2$ and $Je_2 = -e_1$, and let $\{e_1^*,e_2^*\}$ be its dual basis. Then, writing $A = A_1e_1^* + A_2e_2^*$ and $\dot{A} = \dot{A}_1e_1^* + \dot{A}_2e_2^*$ we get*

(1) $JA_2 = A_1$

(2) $$-\dot{A}J - A\dot{J} = \underbrace{-(\dot{A}_1)_0Je_1^* - (\dot{A}_2)_0Je_2^*}_{\textit{trace-less part}} \underbrace{-\frac{1}{2}\operatorname{tr}\Big(\dot{A}_2\Big)\mathbb{1}e_1^* + \frac{1}{2}\operatorname{tr}\Big(\dot{A}_1\Big)\mathbb{1}e_2^*}_{\textit{trace part}} \ .$$

Proof. (1) By definition $A_i = A(e_i)$ for $i = 1,2$ and the vector $A_i \cdot e_j$ can be written as a linear combination of e_1, e_2. Then, it is sufficient to prove that $A_1 \cdot e_i = JA_2 \cdot e_i$, for $i = 1,2$. Hence, if

$$A_1 \cdot e_1 = \alpha_{11}e_1 + \beta_{11}e_2, \qquad A_2 \cdot e_1 = \alpha_{21}e_1 + \beta_{21}e_2$$

we get $JA_2 \cdot e_1 = \alpha_{21}e_2 - \beta_{21}e_1$, but since the basis $\{e_1,e_2\}$ is g_J^0-orthonormal we can write

$$\beta_{11} = g_J^0(A_1\cdot e_1, e_2) = C(e_1,e_1,Je_1)$$

and

$$\begin{aligned}
\alpha_{21} &= g_J^0(JA_2 \cdot e_1, e_1) \\
&= g_J^0(A(e_2) \cdot e_1, e_1) && (J \text{ is } g_J^0 - \text{orthogonal}) \\
&= C(Je_1, e_1, e_1) \\
&= \beta_{11} && (C(\cdot,\cdot,\cdot) \text{ is totally-symmetric})
\end{aligned}$$

With the same argument one can prove that $-\beta_{21} = \alpha_{11}$ and that $A_1 \cdot e_2 = JA_2 \cdot e_2$, obtaining the claim.
(2) By using the decomposition $\dot{A} = \dot{A}_0 + (\dot{A})_{\mathrm{tr}}$, we get

$$-\dot{A}J = -(\dot{A}_1)_0 Je_1^* - (\dot{A}_2)_0 Je_2^* - \frac{1}{2}\operatorname{tr}\Big(\dot{A}_1\Big)Je_1^* - \frac{1}{2}\operatorname{tr}\Big(\dot{A}_2\Big)Je_2^* \ . \tag{2.2.14}$$

The same happens for the tensor A, hence, using Equation (2.2.13) on $A_1\dot{J}$ and $A_2\dot{J}$, we get

$$\begin{aligned}
-A\dot{J} &= -\frac{1}{2}\operatorname{tr}\Big(A_1\dot{J}\Big)\mathbb{1}e_1^* - \frac{1}{2}\operatorname{tr}\Big(A_2\dot{J}\Big)\mathbb{1}e_2^* + \frac{1}{2}\operatorname{tr}\Big(JA_1\dot{J}\Big)Je_1^* + \frac{1}{2}\operatorname{tr}\Big(JA_2\dot{J}\Big)Je_2^* \\
&= -\frac{1}{2}\operatorname{tr}\Big(\dot{A}_2\Big)\mathbb{1}e_1^* + \frac{1}{2}\operatorname{tr}\Big(\dot{A}_1\Big)\mathbb{1}e_2^* + \frac{1}{2}\operatorname{tr}\Big(\dot{A}_1\Big)Je_1^* + \frac{1}{2}\operatorname{tr}\Big(\dot{A}_2\Big)Je_2^*
\end{aligned}$$

where in the last equality we used (2.2.7) and $JA_2 = A_1$. It is now clear that adding the two terms $-\dot{A}J$ and $-A\dot{J}$ we get the desired formula in the statement. □

Theorem 2.16. *The triple $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ defines an* $\mathrm{SL}(2,\mathbb{R})$*-invariant pseudo-Kähler structure on $D^3(\mathcal{J}(\mathbb{R}^2))$.*

Proof. In order not to overload the following proof too much, the closedness of $\widehat{\boldsymbol{\omega}}_f$ and the non-degeneracy of $\widehat{\mathbf{g}}_f$ are postponed to Lemma 2.22 at the end of the chapter, as it requires a computation in local coordinates.

• $\widehat{\mathbf{I}}^2 = -\mathbb{1}$ *and it is integrable.*

The first claim is a calculation:

$$\begin{aligned}
\widehat{\mathbf{I}}^2_{(J,A)}(\dot{J},\dot{A}) &= \widehat{\mathbf{I}}_{(J,A)}\big(-J\dot{J}, -\dot{A}J - A\dot{J}\big) \\
&= (J^2\dot{J}, -(-\dot{A}J - A\dot{J})J + AJ\dot{J}) \\
&= (-\dot{J}, -\dot{A} + A\dot{J}J + AJ\dot{J}) \\
&= (-\dot{J}, -\dot{A}) \ . && (\dot{J}J = -J\dot{J})
\end{aligned}$$

For the second one, it is sufficient to prove that, under the bijection in Proposition 2.7, the almost-complex structure $\widehat{\mathbf{I}}$ on $D^3(\mathcal{J}(\mathbb{R}^2))$ corresponds to the multiplication by $-i$ on $Q^3\big(\mathcal{T}(T^2)\big)$. Since the latter is integrable, the former is integrable as well. To show this, we need to compute the tensor $\widetilde{C}$ associated with the variation $-i\dot{q}$ of the holomorphic cubic differential q in the fibre over J. Thanks to Proposition 2.7 this is given by

$$\widetilde{C}(\cdot,\cdot,\cdot) = \mathcal{R}e(-i\dot{q}) = -\dot{C}(\cdot, J\cdot,\cdot) - C(\cdot,\dot{J}\cdot,\cdot) \ .$$

If $\widetilde{A}$ denotes the corresponding associated tensor as in (1.3.2), we get:

$$\begin{aligned} g_J^0(\widetilde{A}(X)Y,Z) &= \widetilde{C}(X,Y,Z) \\ &= -\dot{C}(X,JY,Z) - C(X,\dot{J}Y,Z) \\ &= g_J^0\big((-\dot{A}(X)J - A(X)\dot{J})Y,Z\big) \end{aligned}$$

for all $X,Y,Z \in \Gamma(T\mathbb{R}^2)$, hence the claim.

• <u>*The metric $\widehat{\mathbf{g}}_f$ and the complex structure $\widehat{\mathbf{I}}$ are compatible.*</u>

We need to prove that

$$(\widehat{\mathbf{g}}_f)(J,A)\big(\widehat{\mathbf{I}}_{(J,A)}(\dot{J},\dot{A});\ \widehat{\mathbf{I}}_{(J,A)}(\dot{J}',\dot{A}')\big) = (\widehat{\mathbf{g}}_f)_{(J,A)}\big((\dot{J},\dot{A});(\dot{J}',\dot{A}')\big) \ .$$

By definition of $\widehat{\mathbf{I}}$ we have

$$\begin{aligned} (\widehat{\mathbf{g}}_f)_{(J,A)}\big(\widehat{\mathbf{I}}_{(J,A)}(\dot{J},\dot{A});\ \widehat{\mathbf{I}}_{(J,A)}(\dot{J}',\dot{A}')\big) &= \big(1-f(||A||_0^2)\big)\langle -J\dot{J}, -J\dot{J}'\rangle \\ &\quad + \frac{f'(||A||_0^2)}{6}\langle -(\dot{A}J + A\dot{J})_0, -(\dot{A}'J + A\dot{J}')_0\rangle \\ &\quad - \frac{f'(||A||_0^2)}{12}\langle -(\dot{A}J + A\dot{J})_{\mathrm{tr}}, -(\dot{A}'J + A\dot{J}')_{\mathrm{tr}}\rangle \ . \end{aligned}$$

Since the argument of the functions f, f' depends only on the norm of A (up to a constant) and remains unchanged when we apply $\widehat{\mathbf{I}}_{(J,A)}$, we can focus only on the scalar products part. The first term is

$$\begin{aligned} \langle -J\dot{J}, -J\dot{J}'\rangle &= \frac{1}{2}\operatorname{tr}\Big(J\dot{J}J\dot{J}'\Big) \\ &= \frac{1}{2}\operatorname{tr}\Big(\dot{J}\dot{J}'\Big) && (J\dot{J} = -\dot{J}J) \\ &= \langle \dot{J}, \dot{J}'\rangle \ . \end{aligned}$$

Applying part (2) of Lemma 2.15 and observing that $(\dot{A}_i)_0, (\dot{A}_i')_0 \in T_J\mathcal{J}(\mathbb{R}^2),\ i = 1,2$, the second term is

$$\begin{aligned} \langle -(\dot{A}J + A\dot{J})_0, -(\dot{A}'J + A\dot{J}')_0\rangle &= \operatorname{tr}\Big((\dot{A}_1)_0J(\dot{A}_1')_0J\Big) + \operatorname{tr}\Big((\dot{A}_2)_0J(\dot{A}_2')_0J\Big) \\ &= \operatorname{tr}\Big((\dot{A}_1)_0(\dot{A}_1')_0 + (\dot{A}_2)_0(\dot{A}_2')_0\Big) \\ &= \langle \dot{A}_0, \dot{A}_0'\rangle \ . \end{aligned}$$

Applying, again, part (2) of Lemma 2.15 the third term is:

$$\begin{aligned}\langle -(\dot{A}J + A\dot{J})_{\mathrm{tr}}, -(\dot{A}'J + A\dot{J}')_{\mathrm{tr}}\rangle &= \frac{1}{4}\operatorname{tr}\Big(\operatorname{tr}\Big(\dot{A}_1\Big)\operatorname{tr}\Big(\dot{A}_1'\Big)\mathbb{1} + \operatorname{tr}\Big(\dot{A}_2\Big)\operatorname{tr}\Big(\dot{A}_2'\Big)\mathbb{1}\Big)\\ &= \langle \dot{A}_{\mathrm{tr}}, \dot{A}'_{tr}\rangle\ .\end{aligned}$$

Hence, we have the claim. □

Remark 2.17. The complex structure $\widehat{\mathbf{I}}$ preserves the 0-section of $T_{(J,A)}D^3(\mathcal{J}(\mathbb{R}^2))$ since $\widehat{\mathbf{I}}_{(J,0)}(\dot{J},0) = (-J\dot{J},0)$. In particular, $\widehat{\mathbf{I}}$ still defines a complex structure on the complement of the 0-section on $D^3(\mathcal{J}(\mathbb{R}^2))$, which is identified with $Q^3_0(\mathcal{T}(T^2))$ by Proposition 2.7, which is further identified with $\mathcal{B}_0(T^2)$ by Corollary 2.4. Hence, we get a well-defined complex structure on $\mathcal{B}_0(T^2)$ which will be denoted with $\widehat{\mathbf{I}}$ by abuse of notation. The same argument holds for the pseudo-Riemannian metric $\widehat{\mathbf{g}}_f$ and the symplectic form $\widehat{\boldsymbol{\omega}}_f$.

Theorem D. *The deformation space $\mathcal{B}_0(T^2)$ admits a* $\mathrm{MCG}(T^2)$*-equivariant pseudo-Kähler structure* $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$.

Proof. By Theorem 2.16 and Remark 2.17 the deformation space $\mathcal{B}_0(T^2)$ has a well-defined pseudo-Kähler structure $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$. Since all the identifications are equivariant with respect to $\mathrm{SL}(2,\mathbb{Z}) \cong \mathrm{MCG}(T^2)$ and the triple $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, it follows that the induced pseudo-Kähler structure is $\mathrm{MCG}(T^2)$-invariant. □

2.2.2 The pseudo-metric and the symplectic form in coordinates

As we explained in the previous section, it only remains to prove that the symmetric tensor $\widehat{\mathbf{g}}_f$ and the 2-form $\widehat{\boldsymbol{\omega}}_f$ on $D^3(\mathcal{J}(\mathbb{R}^2))$ are non-degenerate and closed, respectively. In order to do so, we need to write their expression in local coordinates. First of all, it is necessary to find the analogue in coordinates of the two spaces, $\mathcal{J}(\mathbb{R}^2)$ and $D^3(\mathcal{J}(\mathbb{R}^2))$, which we have studied so far. Let $\widehat{G}$ and $\widehat{\Omega}$ be the restriction of $\widehat{\mathbf{g}}_f$ and $\widehat{\boldsymbol{\omega}}_f$ to the 0-section of $D^3(\mathcal{J}(\mathbb{R}^2))$, which is identified with $\mathcal{J}(\mathbb{R}^2)$. Then,

$$\widehat{G}_J(\dot{J},\dot{J}') = \langle \dot{J},\dot{J}'\rangle_J, \qquad \widehat{\Omega}_J(\dot{J},\dot{J}') = -\langle \dot{J}, J\dot{J}'\rangle_J$$

with $\dot{J},\dot{J}' \in T_J\mathcal{J}(\mathbb{R}^2)$. In this case $\widehat{G}_J$ is a scalar product for all $J \in \mathcal{J}(\mathbb{R}^2)$, hence $(\widehat{G},\widehat{\Omega})$ is an $\mathrm{SL}(2,\mathbb{R})$-invariant Kähler structure on $\mathcal{J}(\mathbb{R}^2)$. Moreover, the $\mathrm{SL}(2,\mathbb{R})$-action is transitive with stabilizer $\mathrm{SO}(2)$ at the standard linear complex structure

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

Therefore, $\mathcal{J}(\mathbb{R}^2) \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$.

Lemma 2.18 ([Tra18, Lemma 4.3.2])**.** *Let $\mathbb{H}^2$ be the hyperbolic plane with complex coordinate $z = x + iy$ and with Kähler structure*

$$g_{\mathbb{H}^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \qquad \omega_{\mathbb{H}^2} = -\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} \ .$$

Then, there exists a unique $\mathrm{SL}(2,\mathbb{R})$*-invariant Kähler isometry* $j : \mathbb{H}^2 \to \mathcal{J}(\mathbb{R}^2)$ *such that* $j(i) = J_0$*. It is given by the formula*

$$j(x+iy) := \begin{pmatrix} \frac{x}{y} & -\frac{x^2+y^2}{y} \\ \frac{1}{y} & -\frac{x}{y} \end{pmatrix} \ . \tag{2.2.15}$$

Remark 2.19. The minus sign in front of the area form on $\mathbb{H}^2$ shows up since we are considering the relation $\widehat{\boldsymbol{\omega}}_f(\cdot,\cdot) = \widehat{\mathbf{g}}_f(\cdot, \widehat{\mathbf{I}}\cdot)$ on $D^3(\mathcal{J}(\mathbb{R}^2))$, hence on $\mathcal{J}(\mathbb{R}^2)$.

In particular, thanks to this last lemma and the isomorphism $\mathcal{T}(T^2) \cong \mathcal{J}(\mathbb{R}^2)$, we can identify the Teichmüller space of the torus with $\mathbb{H}^2$. Whenever we are thinking of Teichmüller space of the torus as $\mathbb{H}^2$, we will denote the total space of $Q^3\big(\mathcal{T}(T^2)\big)$ as $Q^3(\mathbb{H}^2)$. In particular, we can identify $Q^3(\mathbb{H}^2)$ with $\mathbb{H}^2 \times \mathbb{C}$, where $\mathbb{C}$ is a copy of the fiber $Q^3(\mathbb{H}^2)_z$ over a point $z \in \mathbb{H}^2$. We can define an $\mathrm{SL}(2,\mathbb{R})$-action on $\mathbb{H}^2 \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,w) := \left(\frac{az+b}{cz+d}, (cz+d)^3 w\right), \qquad \text{with } (z,w) \in \mathbb{H}^2 \times \mathbb{C}, \quad ad-bc = 1 \ . \tag{2.2.16}$$

Moreover, the metric on the fiber is the one induced by the norm

$$|w|_z^2 = \mathrm{Im}(z)^3 |w|^2 \qquad \text{for } z \in \mathbb{H}^2, w \in Q^3(\mathbb{H}^2)_z \ .$$

Given $J \in \mathcal{J}(\mathbb{R}^2)$, let us define the space of J-complex symmetric tri-linear forms by

$$\begin{aligned} S_3(\mathbb{R}^2, J) : &= \{\gamma : \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \longrightarrow \mathbb{C} \mid \gamma \ \text{ is symmetric and } (J,i) - \text{tri-linear}\} \\ &\cong \{\tau : \mathbb{R}^2 \to \mathbb{C} \mid \text{for all } \alpha, \beta \in \mathbb{R} \text{ and } v \in \mathbb{R}^2 \text{ it holds } \tau(\alpha v + \beta J v) = (\alpha + i\beta)^3 \tau(v)\} \ . \end{aligned}$$

This space can be seen as the fiber of a complex line bundle $\mathcal{L}_3(\mathbb{R}^2) \to \mathcal{J}(\mathbb{R}^2)$ endowed with a natural $\mathrm{SL}(2,\mathbb{R})$-action given by

$$P \cdot (J, \gamma) := (PJP^{-1}, (P^{-1})^*\gamma), \qquad \text{for } P \in \mathrm{SL}(2,\mathbb{R}) \ .$$

It is not difficult to see that the line bundle $\mathcal{L}_3(\mathbb{R}^2)$ can be identified with $D^3(\mathcal{J}(\mathbb{R}^2))$. In particular, each fiber $S_3(\mathbb{R}^2, J)$ is endowed with a scalar product from the one on $D^3(\mathcal{J}(\mathbb{R}^2))_J$ defined in (2.2.3).

Lemma 2.20 ([Tra18, Lemma 5.2.1]). *Let us consider the map $\varphi : Q^3(\mathbb{H}^2) \to \mathrm{Hom}(\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2, \mathbb{C})$ given by*

$$\begin{aligned} \varphi(z,w) : \mathbb{R}^2 &\longrightarrow \mathbb{C} \\ v &\mapsto \bar{w}(v_1 - \bar{z}v_2)^3 \end{aligned}$$

and let $j : \mathbb{H}^2 \to \mathcal{J}(\mathbb{R}^2)$ be the map defined by (2.2.15). Then, the following holds:

- $\varphi(z,w) \in S_3(\mathbb{R}^2, j(z))$, *for all* $(z,w) \in Q^3(\mathbb{H}^2)$.
- *The fibre map $\varphi(z,\cdot) : Q^3(\mathbb{H}^2)_z \cong \mathbb{C} \to S_3(\mathbb{R}^2, j(z))$ is a complex anti-linear isometry for every $z \in \mathbb{H}^2$.*
- *The bundle map $(j,\varphi) : Q^3(\mathbb{H}^2) \to \mathcal{L}_3(\mathbb{R}^2)$ is a $\mathrm{SL}(2,\mathbb{R})$-equivariant bijection.*

At this point it easy to compute in coordinates the Pick tensor $C \in D^3(\mathcal{J}(\mathbb{R}^2))_J$, the Pick form $A \in \mathrm{End}(\mathbb{R}^2) \otimes T^*\mathbb{R}^2$ and their respective variations: $\dot{C}$ and $\dot{A} = g_J^{-1}\dot{C}$, by using this last two lemmas and the isomorphism $Q^3\big(\mathcal{T}(T^2)\big) \cong D^3(\mathcal{J}(\mathbb{R}^2))$. Let $z = x + iy$ and $w = u + iv$ be the complex coordinates on $\mathbb{H}^2$ and $\mathbb{C}$ respectively, then the bundle map (j,φ) in Lemma 2.20 is given by

$$\mathbb{H}^2 \times \mathbb{C} \ni (z,w) \longmapsto \big(j(z), C_{(z,w)}\big) \in D^3(\mathcal{J}(\mathbb{R}^2))$$

where $C_{(z,w)} = \mathcal{R}e(q_{(z,w)})$ with $q_{(z,w)} = \bar{w}(\mathrm{d}x_0 - \bar{z}\mathrm{d}y_0)^3$ (see Proposition 2.7). Hence, the Pick form $A_{(z,w)}$ will be recovered by (1.3.2). Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$, it is enough to compute the tensors at the point $(i,w) \equiv (0,1,u,v)$ for a generic $w \in \mathbb{C}$. The components of the Pick tensor $C_{(z,w)}$ are given by

$$\begin{aligned} &C_{111}(z,w) = u, \qquad C_{112}(z,w) = -xu + yv, \qquad C_{122}(z,w) = ux^2 - uy^2 - 2xyv, \\ &C_{222}(z,w) = -ux^3 - vy^3 + 3(uy^2x + x^2yv) \ . \end{aligned}$$

The remaining components are determined by the four above since C is totally-symmetric. Its variation $\dot{C}_{(i,w)}$ at (i,w) is

$$\begin{aligned} &\dot{C}_{111}(i,w) = \dot{u}, \qquad \dot{C}_{112}(i,w) = -u\dot{x} + \dot{v} + v\dot{y}, \qquad \dot{C}_{122}(i,w) = -\dot{u} - 2(u\dot{y} + v\dot{x}), \\ &\dot{C}_{222}(i,w) = -\dot{v} + 3(u\dot{x} - v\dot{y}) \ . \end{aligned}$$

The Pick form computed in (i,w) is then

$$A_{(i,w)} = \begin{pmatrix} u & v \\ v & -u \end{pmatrix} \mathrm{d}x_0 + \begin{pmatrix} v & -u \\ -u & -v \end{pmatrix} \mathrm{d}y_0 \ . \tag{2.2.17}$$

Its variation $\dot{A}$ will be given in terms of its trace-free and trace part at the point (i,w)

$$(\dot{A}_0)_{(i,w)} = \begin{pmatrix} \dot{u} + u\dot{y} + v\dot{x} & -u\dot{x} + \dot{v} + v\dot{y} \\ -u\dot{x} + \dot{v} + v\dot{y} & -\dot{u} - u\dot{y} - v\dot{x} \end{pmatrix} \mathrm{d}x_0 + \begin{pmatrix} \dot{v} + 2(v\dot{y} - u\dot{x}) & -\dot{u} - 2(u\dot{y} + v\dot{x}) \\ -\dot{u} - 2(u\dot{y} + v\dot{x}) & -\dot{v} + 2(u\dot{x} - v\dot{y}) \end{pmatrix} \mathrm{d}y_0$$

$$(\dot{A}_{\mathrm{tr}})_{(i,w)} = \begin{pmatrix} -u\dot{y} - v\dot{x} & 0 \\ 0 & -u\dot{y} - v\dot{x} \end{pmatrix} \mathrm{d}x_0 + \begin{pmatrix} u\dot{x} - v\dot{y} & 0 \\ 0 & u\dot{x} - v\dot{y} \end{pmatrix} \mathrm{d}y_0 \ .$$

Remark 2.21. Recall that by Remark 1.14, the cubic differential q, hence the tensor A and its first order variation, has to be rescaled by a factor $\frac{1}{\sqrt{2}}$. For this reason, all calculations made from here on will include the rescaling factor and their values may vary from the one in [RT21].

Thanks to this expression in coordinates and together with the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2 \times \mathbb{C}$, we are now able to write the metric $\widehat{\mathbf{g}}_f$ and the symplectic form $\widehat{\boldsymbol{\omega}}_f$ at the point (z,w). Let $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ be a real basis of the tangent space of $\mathbb{H}^2 \times \mathbb{C}$ with its dual basis $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}u, \mathrm{d}v\}$, then the expressions (2.2.10) and (2.2.12) become respectively

$$(\widehat{\mathbf{g}}_f)_{(z,w)} = \begin{pmatrix} \frac{1}{y^2}\big(1 - f + \frac{3}{2}(u^2+v^2)y^3 f'\big) & 0 & f'vy^2 & -f'uy^2 \\ 0 & \frac{1}{y^2}\big(1 - f + \frac{3}{2}(u^2+v^2)y^3 f'\big) & f'uy^2 & f'vy^2 \\ f'vy^2 & f'uy^2 & \frac{2}{3}f'y^3 & 0 \\ -f'uy^2 & f'vy^2 & 0 & \frac{2}{3}f'y^3 \end{pmatrix}$$

$$\begin{aligned} (\widehat{\boldsymbol{\omega}}_f)_{(z,w)} = &\Big(-1 + f - \frac{3}{2}f'y^3(u^2+v^2)\Big)\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} - \frac{2}{3}f'y^3 \mathrm{d}u \wedge \mathrm{d}v \\ &- y^2 f'\Big(u(\mathrm{d}x \wedge \mathrm{d}u + \mathrm{d}y \wedge \mathrm{d}v) + v(\mathrm{d}u \wedge \mathrm{d}y - \mathrm{d}v \wedge \mathrm{d}x)\Big) \end{aligned}$$

where the functions f, f' are evaluated in:

$$\frac{1}{4}\Big|\Big|\frac{1}{\sqrt{2}}A_{(z,w)}\Big|\Big|^2_{j(z)} = \frac{1}{8}||A_{(z,w)}||^2_{j(z)} = \frac{1}{2}||q_{(z,w)}||^2_{j(z)} = \frac{1}{2}y^3(u^2+v^2) \ .$$

The matrix associated with the complex structure $\widehat{\mathbf{I}}_{(z,w)} : T_{(z,w)}\big(\mathbb{H}^2 \times \mathbb{C}\big) \to T_{(z,w)}\big(\mathbb{H}^2 \times \mathbb{C}\big)$ in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ is

$$\widehat{\mathbf{I}}_{(i,w)} = \begin{pmatrix} J_0 & 0_{2\times 2} \\ 0_{2\times 2} & J_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

We will explain how to obtain the expressions above for $\widehat{\mathbf{g}}_f$ and $\widehat{\boldsymbol{\omega}}_f$ later in the section. We first show that these formulae define a non-degenerate pseudo-Riemannian metric and a closed 2-form on $\mathbb{H}^2 \times \mathbb{C}$, thus concluding the proof of Theorem 2.16.

Lemma 2.22. *The tensor $(\widehat{\mathbf{g}}_f)_{(z,w)}$ is non-degenerate and the form $(\widehat{\boldsymbol{\omega}}_f)_{(z,w)}$ is closed, for each $(z,w) \in \mathbb{H}^2 \times \mathbb{C}$.*

Proof. The tensor $\widehat{\mathbf{g}}_f$ can be written as:

$$(\widehat{\mathbf{g}}_f)_{(z,w)} = \begin{pmatrix} \Theta & \Xi \\ \Gamma & \Delta \end{pmatrix}$$

where $\Theta, \Xi, \Gamma, \Delta$ are 2×2 matrices with

$$\Theta = \frac{1}{y^2}\big(1 - f + \frac{3}{2}y^3(u^2+v^2)f'\big)\mathbb{1}_{2\times 2}, \qquad \Delta = \frac{2}{3}y^3 f'\mathbb{1}_{2\times 2} \ .$$

Hence, Ξ and Γ both commute with Θ and Δ. In this case there is an easy formula for the determinant of the 4×4 matrix, namely $\det\Big((\widehat{\mathbf{g}}_f)_{(z,w)}\Big) = \det(\Theta\Delta - \Xi\Gamma)$, where

$$\Theta\Delta = \frac{2}{3}y\Big(f' - ff' + \frac{3}{2}y^3(f')^2(u^2+v^2)\Big)\mathbb{1}_{2\times 2} \qquad \Xi\Gamma = y^4(f')^2(u^2+v^2)\mathbb{1}_{2\times 2}$$

which gives

$$\det\Big((\widehat{\mathbf{g}}_f)_{(z,w)}\Big) = \frac{4}{9}y^2(f')^2(1-f)^2 \ .$$

The right hand side of the last equation is always non-zero thanks to the property of the function f, hence $(\widehat{\mathbf{g}}_f)_{(z,w)}$ is non-degenerate at each point $(z,w) \in \mathbb{H}^2 \times \mathbb{C}$.
It only remains to prove that $(\mathrm{d}\widehat{\boldsymbol{\omega}}_f)_{(z,w)} = 0$ for each $(z,w) \in \mathbb{H}^2 \times \mathbb{C}$. By using directly the expression in coordinate, we get:

- Coefficient $\mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v$:

$$\begin{aligned}&-2y^2 f'\mathrm{d}y\wedge\mathrm{d}u\wedge\mathrm{d}v - y^5 f''(u^2+v^2)\mathrm{d}y\wedge\mathrm{d}u\wedge\mathrm{d}v - y^5 f'' u^2\mathrm{d}u\wedge\mathrm{d}y\wedge\mathrm{d}v \\ &- y^2 f'\mathrm{d}u\wedge\mathrm{d}y\wedge\mathrm{d}v - y^5 f'' v^2 \mathrm{d}v\wedge\mathrm{d}u\wedge\mathrm{d}y - y^2 f'\mathrm{d}v\wedge\mathrm{d}u\wedge\mathrm{d}y = 0\end{aligned}$$

- Coefficient $\mathrm{d}x \wedge \mathrm{d}u \wedge \mathrm{d}v$:

$$-y^5 f'' uv\mathrm{d}v\wedge\mathrm{d}x\wedge\mathrm{d}u + y^5 f'' uv\mathrm{d}u\wedge\mathrm{d}v\wedge\mathrm{d}x = 0$$

- Coefficient $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}v$:

$$\begin{aligned}&yf'v\mathrm{d}v\wedge\mathrm{d}x\wedge\mathrm{d}y - \frac{3}{2}y^4 f'' u^2 v\mathrm{d}v\wedge\mathrm{d}x\wedge\mathrm{d}y - 3yf'v\mathrm{d}v\wedge\mathrm{d}x\wedge\mathrm{d}y \\ &- \frac{3}{2}y^4 f'' v^3\mathrm{d}v\wedge\mathrm{d}x\wedge\mathrm{d}y + 2yf'v\mathrm{d}y\wedge\mathrm{d}v\wedge\mathrm{d}x + \frac{3}{2}y^4 f'' v(u^2+v^2)\mathrm{d}y\wedge\mathrm{d}v\wedge\mathrm{d}x = 0\end{aligned}$$

- Coefficient $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u$:

$$\begin{aligned}&yf'u\mathrm{d}u\wedge\mathrm{d}x\wedge\mathrm{d}y - 3yf'u\mathrm{d}u\wedge\mathrm{d}x\wedge\mathrm{d}y - \frac{3}{2}y^4 f'' u^3\mathrm{d}u\wedge\mathrm{d}x\wedge\mathrm{d}y \\ &- \frac{3}{2}y^4 f'' uv^2\mathrm{d}u\wedge\mathrm{d}x\wedge\mathrm{d}y - 2yf'u\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}u - \frac{3}{2}y^4 f''(u^2+v^2)\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}u = 0\end{aligned}$$

□

Thanks to the expression in coordinates, it is easy to see that $\widehat{\mathbf{g}}_f$ is indeed a pseudo-Riemannian metric on $\mathbb{H}^2\times\mathbb{C}$ (hence on $\mathcal{B}_0(T^2)$) since it is negative-definite when restricted to $\{0\}\times\mathbb{C}$ and it coincides with $g_{\mathbb{H}^2}$ on $\mathbb{H}^2\times\{0\}$.

In the following we will give an idea on how to compute $(\widehat{\mathbf{g}}_f)_{(i,w)}$ by using (2.2.10) and the expression of the tensors in coordinates. Finally, by using the $\mathrm{SL}(2,\mathbb{R})$-invariance we briefly sketch how to compute the tensor $\widehat{\mathbf{g}}_f$ at an arbitrary point of $\mathbb{H}^2\times\mathbb{C}$.
In order to simplify the computation we will give the expression of the associated quadratic form. The first part of the quadratic form in the tensor formalism is $\frac{1}{2}(1-f)\,\mathrm{tr}\Big(\dot{J}^2\Big)$, hence at the point (i,w) we have:

$$J=J_0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\qquad\text{and}\qquad \dot{J}=\mathrm{d}_i j(\dot{x},\dot{y})=\begin{pmatrix}\dot{x} & -\dot{y}\\ -\dot{y} & -\dot{x}\end{pmatrix}\ .$$

Thus, $\frac{1}{2}(1-f)\,\mathrm{tr}\Big(\dot{J}^2\Big)=(1-f)\big(\dot{x}^2+\dot{y}^2\big)$. Moreover, using the expression in coordinates of $(\dot{A}_0)_{(i,w)}$ and $(\dot{A}_{\mathrm{tr}})_{(i,w)}$ we get:

$$(*\dot{A}_0)_{(i,w)}=\begin{pmatrix}\dot{u}+u\dot{y}+v\dot{x} & -u\dot{x}+\dot{v}+v\dot{y}\\ -u\dot{x}+\dot{v}+v\dot{y} & -\dot{u}-u\dot{y}-v\dot{x}\end{pmatrix}\mathrm{d}y_0-\begin{pmatrix}\dot{v}+2(v\dot{y}-u\dot{x}) & -\dot{u}-2(u\dot{y}+v\dot{x})\\ -\dot{u}-2(u\dot{y}+v\dot{x}) & -\dot{v}+2(u\dot{x}-v\dot{y})\end{pmatrix}\mathrm{d}x_0,$$
$$(*\dot{A}_{\mathrm{tr}})_{(i,w)}=\begin{pmatrix}-u\dot{y}-v\dot{x} & 0\\ 0 & -u\dot{y}-v\dot{x}\end{pmatrix}\mathrm{d}y_0-\begin{pmatrix}u\dot{x}-v\dot{y} & 0\\ 0 & u\dot{x}-v\dot{y}\end{pmatrix}\mathrm{d}x_0\ .$$

Hence,

$$\frac{1}{6}\,\mathrm{tr}\Big(\dot{A}_0\wedge *\dot{A}_0\Big)=\frac{2}{3}(\dot{u}^2+\dot{v}^2)+\frac{5}{3}(u^2+v^2)(\dot{x}^2+\dot{y}^2)+2\big(u(\dot{u}\dot{y}-\dot{x}\dot{v})+v(\dot{y}\dot{v}+\dot{u}\dot{x})\big)$$
$$\frac{1}{12}\,\mathrm{tr}\Big(\dot{A}_{\mathrm{tr}}\wedge *\dot{A}_{\mathrm{tr}}\Big)=\frac{1}{6}(u^2+v^2)(\dot{x}^2+\dot{y}^2)\ .$$

The final expression for the quadratic form associated with $\widehat{\mathbf{g}}_f$ and computed at (i,w) is thus

$$\Big(1-f+\frac{3}{2}f'(u^2+v^2)\Big)\big(\dot{x}^2+\dot{y}^2\big)+\frac{2}{3}f'(\dot{u}^2+\dot{v}^2)+2f'\big(u(\dot{u}\dot{y}-\dot{x}\dot{v})+v(\dot{y}\dot{v}+\dot{u}\dot{x})\big)\quad(2.2.18)$$

Similarly, we can recover the coordinate expression of $(\widehat{\mathbf{g}}_f)_{(i,w)}$ from (2.2.18). In order to give the precise expression of $\widehat{\mathbf{g}}_f$ at an arbitrary point $(z,\widetilde{w})\in\mathbb{H}^2\times\mathbb{C}$ we need to use the $\mathrm{SL}(2,\mathbb{R})$-invariance of $\widehat{\mathbf{g}}_f$ and the fact that the $\mathrm{SL}(2,\mathbb{R})$-action on $\mathbb{H}^2$ is transitive. In fact, we can find a $P\in\mathrm{SL}(2,\mathbb{R})$ such that $P\cdot z=i$ for $z\in\mathbb{H}^2$, where $P\cdot z$ is the action via Möbius transformations. This matrix P is explicitly given by

$$P=\begin{pmatrix}\frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}}\\ 0 & \sqrt{y}\end{pmatrix}\ .$$

In particular, the point $\widetilde{w} = \widetilde{u} + i\widetilde{v} \in \mathbb{C}$ is determined by $P \cdot (z, \widetilde{w}) = (i, w)$. In fact,

$$u^2 + v^2 = y^3(\widetilde{u}^2 + \widetilde{v}^2) \ . \tag{2.2.19}$$

By using the $\mathrm{SL}(2,\mathbb{R})$-invariance we get

$$(\widehat{\mathbf{g}}_f)_{(z,\widetilde{w})}(\cdot,\cdot) = (P^*\widehat{\mathbf{g}}_f)_{(z,\widetilde{w})}(\cdot,\cdot) = (\widehat{\mathbf{g}}_f)_{(i,w)}(\mathrm{d}_{(z,\widetilde{w})}P\cdot, \mathrm{d}_{(z,\widetilde{w})}P\cdot),$$

where the differential of P at $(z, \widetilde{w})$ is given by

$$\begin{aligned} \mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial x}\right) &= \frac{1}{y}\frac{\partial}{\partial x} & \qquad \mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial y}\right) &= \frac{1}{y}\frac{\partial}{\partial y} \\ \mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial u}\right) &= y^{\frac{3}{2}}\frac{\partial}{\partial u} & \qquad \mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial v}\right) &= y^{\frac{3}{2}}\frac{\partial}{\partial v} \ . \end{aligned}$$

Now we have all the tools to compute $\widehat{\mathbf{g}}_f$ at a point $(z, \widetilde{w})$. For instance,

$$\begin{aligned} (\widehat{\mathbf{g}}_f)_{(z,\widetilde{w})}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= (\widehat{\mathbf{g}}_f)_{(i,w)}\left(\mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial x}\right), \mathrm{d}_{(z,\widetilde{w})}P\left(\frac{\partial}{\partial x}\right)\right) \\ &= \frac{1}{y^2}(\widehat{\mathbf{g}}_f)_{(i,w)}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \\ &= \frac{1}{y^2}\Big(1 - f + 3f'(u^2+v^2)\Big) \\ &= \frac{1}{y^2}\Big(1 - f + 3y^3 f'(\widetilde{u}^2+\widetilde{v}^2)\Big) \qquad \text{(Equation 2.2.19)} \end{aligned}$$

With a similar computation one can recover all the entries of the tensor $\widehat{\mathbf{g}}_f$ at every $(z, w) \in \mathbb{H}^2 \times \mathbb{C}$.

2.3 The circle action

In this section we study the behavior of the circle action on $\mathcal{B}_0(T^2)$ given by rotation of the fibres, according to the isomorphism presented in Corollary 2.4. The main result claims that the aforementioned action is Hamiltonian with respect to the symplectic form $\widehat{\boldsymbol{\omega}}_f$ and that it acts by isometries with respect to the pseudo-Riemannian metric $\widehat{\mathbf{g}}_f$. Finally, we explicitly compute the associated Hamiltonian function.

The first step is to understand how the circle action $q \mapsto e^{-i\theta}q$ on $Q^3(\mathcal{T}(T^2))$ changes under the bijection with $D^3(\mathcal{J}(\mathbb{R}^2))$ (see Proposition 2.7). In other words, if C is the Pick tensor associated with the J-holomorphic cubic differential q, namely $C = \mathcal{R}e(q)$, then we need to find the expression of the new Pick form $\widetilde{A}$ associated with $\widetilde{C} = \mathcal{R}e(e^{-i\theta}q)$. According to Theorem 1.11, we have $q = C(\cdot,\cdot,\cdot) - iC(\cdot, J\cdot,\cdot)$. In particular, the expression

$$e^{-i\theta}q = \cos\theta C(\cdot,\cdot,\cdot) + \sin\theta C(\cdot, J\cdot,\cdot) + i\Big(\cos\theta C(\cdot, J\cdot,\cdot) - \sin\theta C(\cdot,\cdot,\cdot)\Big)$$

implies that $\widetilde{C}(\cdot,\cdot,\cdot) = \cos\theta C(\cdot,\cdot,\cdot) + \sin\theta C(\cdot, J\cdot,\cdot)$ and the new Pick form is

$$\widetilde{A}(\cdot) = (g_J^0)^{-1}\widetilde{C} = \cos\theta A(\cdot) - \sin\theta A(\cdot)J \ .$$

The last equation gives an induced action on $D^3(\mathcal{J}(\mathbb{R}^2))$ by setting

$$\begin{aligned} \widehat{\Psi}_\theta \ : D^3(\mathcal{J}(\mathbb{R}^2)) &\longrightarrow D^3(\mathcal{J}(\mathbb{R}^2)) \\ (J, A) &\mapsto (J, \cos\theta A(\cdot) - \sin\theta A(\cdot)J) \ . \end{aligned}$$

It is clear from the definition that $\widehat{\Psi}_\theta$ preserves the 0-section in $D^3(\mathcal{J}(\mathbb{R}^2))$ (seen as a vector bundle over $\mathcal{J}(\mathbb{R}^2)$), hence it induces an S^1-action on $\mathcal{B}_0(T^2)$ which will still be denoted with $\widehat{\Psi}_\theta$ by abuse of notation. Before stating and proving the main result, we need a technical lemma regarding the derivative of the norm of the Pick form.

Lemma 2.23. *Let $(J, A) \in D^3(\mathcal{J}(\mathbb{R}^2))$, then*

$$\big(||A||_J^2\big)' = 2\langle A, \dot{A}_0\rangle \ . \tag{2.3.1}$$

Proof. During the proof of this lemma we use the notation of the previous section, namely $A = A_1e_1^* + A_2e_2^*$ and $\dot{A} = \dot{A}_1e_1^* + \dot{A}_2e_2^*$, with $\{e_1, e_2\}$ a g_J^0-orthonormal basis of $\mathbb{R}^2$ and $\{e_1^*, e_2^*\}$ its dual basis, with $A_i := A(e_i)$ and $\dot{A}_i := \dot{A}(e_i)$ for $i = 1, 2$. Recall that the relation between the Pick form A and the Pick tensor C is $A = (g_J^0)^{-1}C$, hence

$$\begin{aligned} A' &= \big((g_J^0)^{-1}C\big)' \\ &= -(g_J^0)^{-1}\dot{g}_J^0(g_J^0)^{-1}C + (g_J^0)^{-1}\dot{C} \\ &= J\dot{J}A + \dot{A} \ . && (\dot{g}_J^0(\cdot,\cdot) = -g_J^0(\cdot, J\dot{J}\cdot)) \end{aligned}$$

In particular, $(A_i)' = J\dot{J}A_i + \dot{A}_i$, for each $i = 1, 2$. Thus,

$$\begin{aligned} \big(||A||_J^2\big)' &= \operatorname{tr}\Big((A_1)^2 + (A_2)^2\Big)' \\ &= 2\operatorname{tr}\Big(A_1\dot{A}_1 + A_2\dot{A}_2\Big) + \operatorname{tr}\Big(J\dot{J}\big((A_1)^2 + (A_2)^2\big) + A_1J\dot{J}A_1 + A_2J\dot{J}A_2\Big) \\ &= 2\operatorname{tr}\Big(A_1\dot{A}_1 + A_2\dot{A}_2\Big) \ , \end{aligned}$$

where in the second line we used the fact that, since both $J\dot{J}A_iA_i$ and $A_iJ\dot{J}A_i$ anticommute with J for each $i = 1, 2$, the terms $\operatorname{tr}\Big(J\dot{J}A_iA_i\Big) = \operatorname{tr}\Big(A_iJ\dot{J}A_i\Big)$ vanish for each $i = 1, 2$. Thus, we get

$$\big(||A||_J^2\big)' = 2\langle A, \dot{A}\rangle \ .$$

Finally, by writing $\dot{A} = \dot{A}_0 + \dot{A}_{\mathrm{tr}}$ where

$$\dot{A}_{\mathrm{tr}} = \frac{1}{2}\operatorname{tr}\Big(\dot{A}_1\Big)\mathbb{1}e_1^* + \frac{1}{2}\operatorname{tr}\Big(\dot{A}_2\Big)\mathbb{1}e_2^* \ ,$$

we obtain

$$\langle A, \dot{A}_{\mathrm{tr}}\rangle = \frac{1}{2}\operatorname{tr}\Big(\operatorname{tr}\Big(\dot{A}_1\Big)A_1 + \operatorname{tr}\Big(\dot{A}_2\Big)A_2\Big)$$

and this last term is equal to zero since the A_i's are trace-less endomorphisms. □

Theorem E. *The S^1-action on $\mathcal{B}_0(T^2)$ is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ and it satisfies*

$$\widehat{\Psi}_\theta^*\widehat{\mathbf{g}}_f = \widehat{\mathbf{g}}_f \ .$$

The Hamiltonian function is given by $H(J,A) = \frac{2}{3}f\Big(\frac{||A||_J^2}{8}\Big)$.

Proof. The infinitesimal generator of the action is

$$X_{(J,A)} = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}\Psi_\theta(J,A) = (0, -AJ).$$

Hence,

$$\begin{aligned}
(\iota_X\widehat{\boldsymbol{\omega}}_f)_{(J,A)}(\dot{J},\dot{A}) &= (\widehat{\boldsymbol{\omega}}_f)_{(J,A)}\big((\dot{J},\dot{A}),(0,AJ)\big)\\
&= (\widehat{\mathbf{g}}_f)_{(J,A)}\big((\dot{J},\dot{A}),\mathbf{I}(0,AJ)\big)\\
&= (\widehat{\mathbf{g}}_f)_{(J,A)}\big((\dot{J},\dot{A}),(0,-AJ^2)\big)\\
&= \frac{f'}{6}\langle\dot{A}_0, A\rangle_J \ . && (A \text{ is } g_J^0-\text{traceless})
\end{aligned}$$

Now we compute the differential of $H(J,A) := \frac{2}{3}f\big(||A||_0^2\big)$, where $||A||_0^2$ is defined as $||A||_J^2$ divided by 8. This is given by

$$\begin{aligned}
\mathrm{d}_{(J,A)}H(\dot{J},\dot{A}) &= \frac{f'}{12}\big(||A||_J^2\big)'\\
&= \frac{f'}{6}\langle A, \dot{A}_0\rangle_J \ . && (\text{Lemma } 2.23)
\end{aligned}$$

Thus, the S^1-action is Hamiltonian. It only remains to prove that $\widehat{\Psi}_\theta$ is an isometry for $\widehat{\mathbf{g}}_f$. First of all we compute the differential of the action:

$$\mathrm{d}_{(J,A)}\widehat{\Psi}_\theta(\dot{J},\dot{A}) = \big(\dot{J}, \cos\theta\dot{A}(\cdot) - \sin\theta(\dot{A}(\cdot)J + A(\cdot)\dot{J})\big) \ .$$

Then, we notice that the circle action preserves the norm of the Pick form, namely $||\cos\theta A - \sin\theta AJ||_J^2 = ||A||_J^2$. In fact,

$$||\cos\theta A - \sin\theta AJ||_J^2 = \cos^2\theta ||A||_J^2 + \sin^2\theta \underbrace{||AJ||_J^2}_{(a)} - 2\cos\theta\sin\theta \underbrace{\langle A, AJ\rangle_J}_{(b)} \ . \tag{2.3.2}$$

The term (a) is

$$\begin{aligned} ||AJ||_J^2 &= \operatorname{tr}(A_1JA_1J + A_2JA_2J) \\ &= \operatorname{tr}(A_1A_1 + A_2A_2) && (A_i \in T_J\mathcal{J}(\mathbb{R}^2) \text{ and } J^2 = -\mathbb{1}) \\ &= ||A||_J^2 \ . \end{aligned}$$

The term (b) is

$$\langle A, AJ\rangle_J = \operatorname{tr}(A_1A_1J + A_2A_2J)$$

but $\operatorname{tr}(A_iJA_i) = \operatorname{tr}(JA_iA_i) = \operatorname{tr}(A_iA_iJ) = -\operatorname{tr}(A_iJA_i)$ for $i = 1, 2$, hence the term (b) is zero. In the first two equalities we used the trace symmetry and in the third one the fact that $A_i \in T_J\mathcal{J}(\mathbb{R}^2)$. The circle action $\widehat{\Psi}_\theta$ preserves the pseudo-Riemannian metric $\widehat{\mathbf{g}}_f$ if and only if the following holds

$$(\widehat{\mathbf{g}}_f)_{(J,A)}\big((\dot{J}, \dot{A}); (\dot{J}', \dot{A}')\big) = (\widehat{\mathbf{g}}_f)_{\widehat{\Psi}_\theta(J,A)}\big(\mathrm{d}_{(J,A)}\widehat{\Psi}_\theta(\dot{J}, \dot{A}); \mathrm{d}_{(J,A)}\widehat{\Psi}_\theta(\dot{J}', \dot{A}')\big) \ .$$

Let us define ψ_θ to be the second component of the differential of the circle action, namely

$$\psi_\theta(\dot{J}, \dot{A}) := \cos\theta\dot{A} - \sin\theta(\dot{A}J + A\dot{J}).$$

Then, in order to conclude the proof, we need to show

(1) $\langle \psi_\theta(\dot{J}, \dot{A})_0, \psi_\theta(\dot{J}', \dot{A}')_0\rangle = \langle \dot{A}_0, \dot{A}_0'\rangle$;

(2) $\langle \psi_\theta(\dot{J}, \dot{A})_{\mathrm{tr}}, \psi_\theta(\dot{J}', \dot{A}')_{\mathrm{tr}}\rangle = \langle \dot{A}_{\mathrm{tr}}, \dot{A}_{\mathrm{tr}}'\rangle$.

The left hand side term of (1) can be written as

$$\begin{aligned} &\cos^2\theta\langle \dot{A}_0, \dot{A}_0'\rangle + \sin^2\theta\langle (\dot{A}J + A\dot{J})_0, (\dot{A}'J + A\dot{J}')_0\rangle \\ &\qquad + \cos\theta\sin\theta\big(\langle \dot{A}_0, -(\dot{A}'J + A\dot{J}')_0\rangle + \langle -(\dot{A}J + A\dot{J})_0, \dot{A}_0'\rangle\big) \ . \end{aligned}$$

The coefficient of $\sin^2\theta$ has already been computed (see proof of Theorem 2.16) and it is equal to $\langle \dot{A}_0, \dot{A}_0'\rangle$. The coefficient of $\cos\theta\sin\theta$ is equal to

$$\operatorname{tr}\Big(-(\dot{A}_1)_0(\dot{A}_1')_0J - (\dot{A}_2)_0(\dot{A}_2')_0J - (\dot{A}_1)_0J(\dot{A}_1')_0 - (\dot{A}_2)_0J(\dot{A}_2')_0\Big)$$

and it vanishes since $(\dot{A}_i)_0, (\dot{A}'_i)_0 \in T_J\mathcal{J}(\mathbb{R}^2)$ for each $i = 1, 2$.
The left hand side term of (2) can be written as

$$\cos^2\theta\langle \dot{A}_{\mathrm{tr}}, \dot{A}'_{\mathrm{tr}}\rangle + \sin^2\theta\langle (\dot{A}J + A\dot{J})_{\mathrm{tr}}, (\dot{A}'J + A\dot{J}')_{\mathrm{tr}}\rangle \\ + \cos\theta\sin\theta\big(\langle \dot{A}_{\mathrm{tr}}, -(\dot{A}'J + A\dot{J}')_{\mathrm{tr}}\rangle + \langle -(\dot{A}J + A\dot{J})_{\mathrm{tr}}, \dot{A}'_{\mathrm{tr}}\rangle\big) \ .$$

The coefficient of $\sin^2\theta$ has already been calculated (see proof of Theorem 2.16) and it is equal to $\langle \dot{A}_{\mathrm{tr}}, \dot{A}'_{\mathrm{tr}}\rangle$. By using Lemma 2.15, the coefficient of $\cos\theta\sin\theta$ can be written as

$$\frac{1}{2}\operatorname{tr}\Big(-(\dot{A}_1)_{\mathrm{tr}}\operatorname{tr}\Big(\dot{A}'_2\Big)\mathbb{1} + (\dot{A}_2)_{\mathrm{tr}}\operatorname{tr}\Big(\dot{A}'_1\Big)\mathbb{1} - (\dot{A}'_1)_{\mathrm{tr}}\operatorname{tr}\Big(\dot{A}_2\Big)\mathbb{1} + (\dot{A}'_2)_{\mathrm{tr}}\operatorname{tr}\Big(\dot{A}_1\Big)\mathbb{1}\Big) \ .$$

Since $(\dot{A}_i)_{\mathrm{tr}} = \frac{1}{2}\operatorname{tr}\Big(\dot{A}_i\Big)\mathbb{1}$ and $(\dot{A}'_i)_{\mathrm{tr}} = \frac{1}{2}\operatorname{tr}\Big(\dot{A}'_i\Big)\mathbb{1}$ for each $i = 1, 2$, the first term of the above expression cancels out with the last one and the same happens for the second and third one. Finally, the term with $\cos\theta\sin\theta$ vanishes and we obtain the claim. □

2.4 The moment map for the SL(2, ℝ)-action

Now we will study the $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{B}_0(T^2)$ and its moment map. Recall that if $P \in \mathrm{SL}(2,\mathbb{R})$ and $(J, A) \in D^3(\mathcal{J}(\mathbb{R}^2))$, then

$$P \cdot (J, A) = (PJP^{-1}, PA(P^{-1}\cdot)P^{-1}) \ .$$

In particular, this action preserves the 0-section in $D^3(\mathcal{J}(\mathbb{R}^2))$ (seen as a vector bundle over $\mathcal{J}(\mathbb{R}^2)$), hence it induces an $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{B}_0(T^2)$, which will be denoted by $\Phi_P : \mathcal{B}_0(T^2) \to \mathcal{B}_0(T^2)$. Thanks to Lemma 2.12, it is clear that $\Phi_P^*\widehat{\boldsymbol{\omega}}_f = \widehat{\boldsymbol{\omega}}_f$, i.e. $\mathrm{SL}(2,\mathbb{R})$ acts by symplectomorphisms on $\mathcal{B}_0(T^2)$. Thus, it makes sense to ask whether the action is Hamiltonian and, if this is the case, to find the expression of the moment map. The Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ is given by $\mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathrm{End}(\mathbb{R}^2) \mid \operatorname{tr}(X) = 0\}$ with Lie bracket $[X, Y] = XY - YX$. In particular any $X \in \mathfrak{sl}(2,\mathbb{R})$ can be decomposed as $X = X^a + X^s$, where X^s is a trace-less g_J^0-symmetric matrix and X^a is a trace-less g_J^0-skew-symmetric matrix. In particular, $X^s \in T_J(\mathcal{J}(\mathbb{R}^2))$ and $X^a = -\frac{1}{2}\operatorname{tr}(JX)J$, since it commutes with J.

Theorem F. *The $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{B}_0(T^2)$ is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ with moment map $\widehat{\mu} : \mathcal{B}_0(T^2) \to \mathfrak{sl}(2,\mathbb{R})^*$ given by*

$$\widehat{\mu}_{(J,A)}(X) = \left(1 - f\left(\frac{||A||_J^2}{8}\right)\right)\operatorname{tr}(JX) \tag{2.4.1}$$

for all $X \in \mathfrak{sl}(2,\mathbb{R})$.

Proof. Let $X \in \mathfrak{sl}(2,\mathbb{R})$ and let

$$V_X(J,A) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}Je^{-tX}, (e^{-tX})^* C)|_{t=0}$$

be its infinitesimal generator. The first component is equal to $XJ - JX = [X, J]$. For the second component define $P_t := e^{tX}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}C((P_t)^{-1}\cdot, (P_t)^{-1}\cdot, (P_t)^{-1}\cdot)|_{t=0} = -C(X\cdot,\cdot,\cdot) - C(\cdot, X\cdot,\cdot) - C(\cdot,\cdot, X\cdot) \ .$$

If $\widetilde{C}(\cdot,\cdot,\cdot)$ is defined as the right hand side term of the equation above, then the new Pick form $\widetilde{A}$ satisfies

$$\begin{aligned} g_J^0(\widetilde{A}(Y)Z, W) &= \widetilde{C}(Y,Z,W) \\ &= -C(X\cdot Y, Z, W) - C(Y, X\cdot Z, W) - C(Y, Z, X\cdot W) \\ &= -g_J^0(A(X\cdot Y)Z, W) - g_J^0(A(Y)X\cdot Z, W) - g_J^0(A(Y)Z, X\cdot W) \\ &= -g_J^0(A(X\cdot Y)Z + A(Y)X\cdot Z + X^*\cdot A(Y)Z, W) \end{aligned}$$

for all $Y, Z, W \in \mathbb{R}^2$, where X^* denotes the adjoint of X with respect to g_J^0. Hence, we have

$$\widetilde{A}(\cdot) = -A(X\cdot) - AX - X^*A \ .$$

By using the decomposition $X = X^s + X^a$ in its symmetric and skew-symmetric part, we can write the second component of $V_X(J,A)$ as:

$$\underbrace{-A(X^a\cdot) - A(X^s\cdot) + [X^a, A]}_{\text{trace-less part}} - \underbrace{(AX^s + X^sA)}_{\text{trace part}} \ . \tag{2.4.2}$$

- <u>$\widehat{\mu}$ *is equivariant*</u>:

Let $P \in \mathrm{SL}(2,\mathbb{R})$ and $X \in \mathfrak{sl}(2,\mathbb{R})$, then

$$\begin{aligned} \widehat{\mu}_{P\cdot(J,A)}(X) &= \left(1 - f\left(\frac{1}{8}||P\cdot A||^2_{P\cdot J}\right)\right)\mathrm{tr}\big(PJP^{-1}X\big) \\ &= \left(1 - f\big(||A||_0^2\big)\right)\mathrm{tr}\big(JP^{-1}XP\big) \\ &= \widehat{\mu}_{(J,A)} \circ \mathrm{Ad}(P^{-1})(X) \\ &= \mathrm{Ad}^*(P)(\widehat{\mu}_{(J,A)})(X) \end{aligned}$$

where in the second equality we used Lemma 2.12 and the trace symmetry.

- <u>$\widehat{\mu}$ satisfies property (ii) in Definition B.3</u>:

Let us define $||A||_0^2$ as the norm squared of A divided by 8, and let $\widehat{\mu}^X : \mathcal{B}_0(T^2) \to \mathbb{R}$ be the map

$$\widehat{\mu}^X(J, A) = \left(1 - f\left(||A||_0^2\right)\right)\operatorname{tr}(JX) \ ,$$

then

$$\begin{aligned}
\mathrm{d}_{(J,A)}\widehat{\mu}^X(\dot{J}, \dot{A}) &= -\frac{1}{8}\left(||A||_J^2\right)' f'\left(||A||_0^2\right)\operatorname{tr}(JX) + \left(1 - f\left(||A||_0^2\right)\right)\operatorname{tr}\left(\dot{J}X\right)\\
&= -\frac{1}{4}\langle A, \dot{A}_0\rangle f'\left(||A||_0^2\right)\operatorname{tr}(JX) + \left(1 - f\left(||A||_0^2\right)\right)\operatorname{tr}\left(\dot{J}X\right)
\end{aligned}$$

where we used Lemma 2.23 in the second equality. Now let V_X be the infinitesimal generator of X, then

$$\begin{aligned}
\iota_{V_X}\widehat{\boldsymbol{\omega}}_f(\dot{J}, \dot{A}) &= \widehat{\mathbf{g}}_f(V_X(J, A), \widehat{\mathbf{I}}(\dot{J}, \dot{A}))\\
&= \frac{f-1}{2}\operatorname{tr}\left([X, J]J\dot{J}\right) + \frac{f'}{6}\langle [X^a, A] - A(X\cdot), (-\dot{A}J - A\dot{J})_0\rangle\\
&\quad - \frac{f'}{12}\langle AX^s + X^sA, (\dot{A}J + A\dot{J})_{\mathrm{tr}}\rangle \ ,
\end{aligned} \tag{2.4.3}$$

where we used the decomposition in (2.4.2). The first term of $\iota_{V_X}\widehat{\boldsymbol{\omega}}_f(\dot{J}, \dot{A})$ is

$$\frac{1-f}{2}\operatorname{tr}\left(\dot{J}X + JXJ\dot{J}\right) = (1-f)\operatorname{tr}\left(\dot{J}X\right)$$

by trace symmetry and $\dot{J}J + J\dot{J} = 0$. It only remains to show that the sum of the second and third term of (2.4.3) is equal to $-\frac{1}{4}f'\langle A, \dot{A}_0\rangle\operatorname{tr}(JX)$. The coefficient of $\frac{f'}{6}$ in (2.4.3) can be written as

$$\underbrace{\langle A(X^s\cdot), (\dot{A}J + A\dot{J})_0\rangle}_{(a)} + \underbrace{\langle [X^a, A] - A(X^a\cdot), (-\dot{A}J - A\dot{J})_0\rangle}_{(b)} \ .$$

Moreover, by using Lemma 2.15 the term with $[X^a, A]$ in (b) becomes

$$\frac{1}{2}\operatorname{tr}(JX)\operatorname{tr}\left(JA_1(\dot{A}_1)_0J + JA_2(\dot{A}_2)_0J - A_1J(\dot{A}_1)_0J - A_2J(\dot{A}_2)_0J\right) \ .$$

Using that $A_i, (\dot{A}_i) \in T_J\mathcal{J}(\mathbb{R}^2)$ for each $i = 1, 2$, the above term reduces to $-\operatorname{tr}(JX)\langle A, \dot{A}_0\rangle$. Notice that $-A(X^a\cdot) = \frac{1}{2}\operatorname{tr}(JX)A(\cdot)J$, since $C(J\cdot, \cdot, \cdot) = C(\cdot, J\cdot, \cdot)$. Hence, the term with $-A(X^a\cdot)$ in (b) becomes

$$-\frac{1}{2}\operatorname{tr}(JX)\operatorname{tr}\left(A_1J(\dot{A}_1)_0J + A_2J(\dot{A}_2)_0J\right) = -\frac{1}{2}\operatorname{tr}(JX)\langle A, \dot{A}_0\rangle \ .$$

Finally, the term (b) multiplied by $\frac{f'}{6}$ is equal to

$$-\frac{1}{4}f'\,\mathrm{tr}(JX)\langle A,\dot{A}_0\rangle\ .$$

Hence, it only remains to show that

$$\frac{f'}{6}\underbrace{\langle A(X^s\cdot),(\dot{A}J+A\dot{J})_0\rangle}_{(a)}-\frac{f'}{12}\underbrace{\langle AX^s+X^sA,(\dot{A}J+A\dot{J})_{\mathrm{tr}}\rangle}_{(c)}=0\ . \tag{2.4.4}$$

To do so, we will use a basis of $\mathfrak{sl}(2,\mathbb{R})$, namely we can write $\mathfrak{sl}(2,\mathbb{R})=\mathrm{Span}_{\mathbb{R}}(\xi_1,\xi_2,\xi_3\}$ where

$$\xi_1=J_0,\qquad \xi_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \xi_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}\ .$$

The only symmetric matrices of this basis are ξ_2 and ξ_3, hence it is sufficient to prove Equation (2.4.4) when $X^s=\xi_2$ and $X^s=\xi_3$, since all the elements are linear in $X\in\mathfrak{sl}(2,\mathbb{R})$. In both cases we use the description in coordinates $z=x+iy$ for $\mathbb{H}^2$ and $w=u+iv$ for $\mathbb{C}$, of the Pick form A and its variation $\dot{A}$, as we did in Section 2.2.2. In particular we can do the computation in $(z,w)=(i,w)$ by $\mathrm{SL}(2,\mathbb{R})$-invariance.

(i) $X^s=\xi_2$.
In this case if $\{\frac{\partial}{\partial x_0},\frac{\partial}{\partial y_0}\}$ is a g_{J_0}-orthonormal basis of $\mathbb{R}^2$, then $X^s\cdot\frac{\partial}{\partial x_0}=\frac{\partial}{\partial x_0}$ and $X^s\cdot\frac{\partial}{\partial y_0}=-\frac{\partial}{\partial y_0}$, hence $A(X^s\cdot)=A_1\mathrm{d}x_0-A_2\mathrm{d}y_0$. In particular,

$$\begin{aligned}\mathrm{tr}\Big(A_1(\dot{A}_1)_0J_0\Big)&=2(-|w|^2\dot{x}-v\dot{u}+u\dot{v})\\ \mathrm{tr}\Big(A_2(\dot{A}_2)_0J_0\Big)&=2(-2|w|^2\dot{x}+u\dot{v}-v\dot{u})\ .\end{aligned}$$

Hence,

$$\begin{aligned}\frac{f'}{6}\langle A(X^s\cdot),(\dot{A}J+A\dot{J})_0\rangle&=\frac{f'}{6}\,\mathrm{tr}\Big(A_1(\dot{A}_1)_0J-A_2(\dot{A}_2)_0J\Big)\\ &=\frac{f'}{3}|w|^2\dot{x}\ .\end{aligned}$$

On the other hand, since

$$\begin{aligned}\mathrm{tr}\Big(\dot{A}_1\Big)&=-2(u\dot{y}+v\dot{x})\qquad &\mathrm{tr}(A_1X^s)=2u\\ \mathrm{tr}\Big(\dot{A}_2\Big)&=2(u\dot{x}-v\dot{y})\qquad &\mathrm{tr}(A_2X^s)=2v,\end{aligned}$$

we get

$$-\frac{f'}{12}\langle AX^s+X^sA,(\dot{A}J+A\dot{J})_{\mathrm{tr}}\rangle=-\frac{f'}{12}\bigg(\mathrm{tr}(A_1X^s)\,\mathrm{tr}\Big(\dot{A}_2\Big)-\mathrm{tr}(A_2X^s)\,\mathrm{tr}\Big(\dot{A}_1\Big)\bigg)$$

$$= -\frac{f'}{3}|w|^2\dot{x}$$

and Equation (2.4.4) is clearly satisfied.

(ii) $X^s = \xi_3$.
With the same notation as in case (i) we have $X^s \cdot \frac{\partial}{\partial x_0} = \frac{\partial}{\partial y_0}$ and $X^s \cdot \frac{\partial}{\partial y_0} = \frac{\partial}{\partial x_0}$, hence $A(X^s\cdot) = A_2 \mathrm{d}x_0 + A_1 \mathrm{d}y_0$. Using that

$$\begin{aligned} (\dot{A}_1)_0 + (\dot{A}_2)_0 J &= \begin{pmatrix} -u\dot{y} - v\dot{x} & u\dot{x} - v\dot{y} \\ u\dot{x} - v\dot{y} & u\dot{y} + v\dot{x} \end{pmatrix} \\ \mathrm{tr}\Big(A_1\big((\dot{A}_1)_0 + (\dot{A}_2)_0 J\big)\Big) &= -2|w|^2\dot{y}\ , \end{aligned}$$

the term (a) multiplied by $\frac{f'}{6}$ is

$$\begin{aligned} \frac{f'}{6}\langle A(X^s\cdot), (\dot{A}J + A\dot{J})_0\rangle &= \frac{f'}{6}\,\mathrm{tr}\Big(A_1\big((\dot{A}_1)_0 + (\dot{A}_2)_0 J\big)\Big) \\ &= -\frac{f'}{3}|w|^2\dot{y}\ . \end{aligned}$$

On the other hand, since

$$\begin{aligned} \mathrm{tr}\Big(\dot{A}_1\Big) &= -2(u\dot{y} + v\dot{x}) \qquad & \mathrm{tr}(A_1 X^s) &= 2v \\ \mathrm{tr}\Big(\dot{A}_2\Big) &= 2(u\dot{x} - v\dot{y}) \qquad & \mathrm{tr}(A_2 X^s) &= -2u, \end{aligned}$$

the term (c) multiplied by $-\frac{f'}{12}$ is

$$-\frac{f'}{12}\Bigg(\mathrm{tr}(A_1X^s)\,\mathrm{tr}\Big(\dot{A}_2\Big) - \mathrm{tr}(A_2X^s)\,\mathrm{tr}\Big(\dot{A}_1\Big)\Bigg) = \frac{f'}{3}|w|^2\dot{y}\ .$$

Thus, Equation (2.4.4) is proved and the theorem as well. □

Chapter 3

The general case

In this chapter we state and prove the main result of the thesis, namely the existence and the explicit expression of the pseudo-Kähler metric on the Hitchin component. Because of the way it has been constructed, the pseudo-metric, a-priori, could be degenerate. Later on, we show that it is non-degenerate on the Fuchsian locus, and in the last part of the chapter, we present partial results suggesting that the same is true for points away from it.

3.1 The Weil-Petersson Kähler metric on Teichmüller space

In this section we briefly recall the definition of the group of (Hamiltonian) symplectomorphisms of a closed oriented surface of genus $g \geqslant 2$ and their corresponding Lie algebras. Next, we briefly describe the construction of the Weil-Petersson Kähler metric on Teichmüller space using the theory of symplectic reduction, which inspires our construction for $\mathrm{Hit}_3(\Sigma)$.

3.1.1 The Lie algebra of the group of (Hamiltonian) symplectomorphisms

Let ρ be a fixed area form on a closed surface Σ of genus $g \geqslant 2$. The group $\mathrm{Symp}_0(\Sigma, \rho)$ is given by those diffeomorphisms ϕ isotopic to the identity and such that $\phi^*\rho = \rho$. Thanks to Cartan's magic formula:

$$\mathcal{L}_X\rho = \iota_X \mathrm{d}\rho + \mathrm{d}(\iota_X\rho)$$

and the fact that $\mathrm{d}\rho = 0$, we obtain the following identification for the Lie algebra of $\mathrm{Symp}_0(\Sigma, \rho)$:

$$\mathfrak{S}(\Sigma, \rho) = \{X \in \Gamma(T\Sigma) \mid \mathrm{d}(\iota_X\rho) = 0\} \cong_\rho Z^1(\Sigma) \ ,$$

where the last isomorphism is given by the identification of $\Gamma(T\Sigma)$ with the space of 1-forms $\Omega^1(\Sigma)$, and $Z^1(\Sigma)$ denotes the space of closed 1-forms. A symplectomorphism ϕ is called *Hamiltonian* if there is an isotopy $\phi_\bullet : [0,1] \to \mathrm{Symp}_0(\Sigma,\rho)$, with $\phi_0 = \mathrm{Id}$ and $\phi_1 = \phi$, and a smooth family of functions $H_t : \Sigma \to \mathbb{R}$ such that $\iota_{X_t}\rho = \mathrm{d}H_t$, where X_t is the infinitesimal generator of the symplectomorphism ϕ_t. Let us denote by $\mathrm{Ham}(\Sigma,\rho)$ the group of Hamiltonian symplectomorphisms, which is a normal subgroup of $\mathrm{Symp}(\Sigma,\rho)$ ([MS17, §3.1]). The Lie algebra of $\mathrm{Ham}(\Sigma,\rho)$ can be characterized as:

$$\mathfrak{H}(\Sigma,\rho) = \{X \in \Gamma(T\Sigma) \mid \iota_X\rho \text{ is exact}\} \cong_\rho B^1(\Sigma) \ ,$$

where $B^1(\Sigma)$ is the space of exact 1-forms on Σ.

Lemma 3.1. *Let ρ be a fixed area form and J be a complex structure on Σ, then any $X \in \Gamma(T\Sigma)$ has a unique decomposition*

$$X = V + W + JW' \ , \tag{3.1.1}$$

where $W, W' \in \mathfrak{H}(\Sigma,\rho)$ and $\mathrm{d}(\iota_V\rho) = \mathrm{d}(\iota_{JV}\rho) = 0$.

Proof. Let ρ be a fixed area form on Σ and consider the induced isomorphism

$$\begin{aligned} \Gamma(T\Sigma) &\xrightarrow{\cong} \Omega^1(\Sigma) \\ X &\mapsto \iota_X\rho \ . \end{aligned}$$

For any (almost) complex structure J on Σ we get a Riemannian metric $g_J := \rho(\cdot, J\cdot)$. Hodge theory for compact Riemannian surfaces implies the existence of a decomposition

$$\Omega^1(\Sigma) = \mathrm{d}\big(C^\infty(\Sigma)\big) \oplus \mathrm{d}^*\big(\Omega^2(\Sigma)\big) \oplus \mathcal{H}^1(\Sigma) \ ,$$

where d^* is the L^2-adjoint of the exterior differential and $\mathcal{H}^1(\Sigma) = \{\alpha \in \Omega^1(\Sigma) \mid \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0\}$ is the space of harmonic 1-forms. In particular, for any $X \in \Gamma(T\Sigma)$ we have a unique decomposition $\iota_X\rho = \mathrm{d}f + \mathrm{d}^*\omega + \alpha$, with $f \in C^\infty(\Sigma), \omega \in \Omega^2(\Sigma)$ and $\alpha \in \mathcal{H}^1(\Sigma)$. Since each element of the decomposition is a 1-form, there must exist three vector fields $V, W, \widetilde{W}$ such that $\mathrm{d}f = \iota_W\rho, \mathrm{d}^*\omega = \iota_{\widetilde{W}}\rho$ and $\alpha = \iota_V\rho$, which implies that $X = V + W + \widetilde{W}$. Now notice that $\iota_W\rho$ is exact, hence $W \in \mathfrak{H}(\Sigma,\rho)$. Since α is harmonic we have $\mathrm{d}(\iota_V\rho) = \mathrm{d}^*(\iota_V\rho) = 0$, but the term in between can be written as $\mathrm{d}(\iota_V\rho \circ J)$, which implies that $\mathrm{d}(\iota_{JV}\rho) = 0$. Finally, in order to end the proof, we only need to show that $\widetilde{W} = JW'$ for some $W' \in \mathfrak{H}(\Sigma,\rho)$. This follows from the fact that $\iota_{\widetilde{W}}\rho = \mathrm{d}^*\omega = *_{g_J} \circ \mathrm{d} \circ *_{g_J}\omega = (\mathrm{d} \circ *_{g_J}\omega) \circ J$, where $*_{g_J}$ denotes the Hodge-star operator with respect to g_J. Since $\mathrm{d} \circ *_{g_J}\omega$ is an exact 1-form, there exists a vector field $W' \in \mathfrak{H}(\Sigma,\rho)$ such that $\iota_{W'}\rho \circ J = \iota_{\widetilde{W}}\rho$, hence $\widetilde{W} = JW'$. □

Because of the close connection with harmonic 1-forms, the vector fields V on (Σ, J) for which $\mathrm{d}(\iota_V\rho) = \mathrm{d}(\iota_{JV}\rho) = 0$ will be called *harmonic*. The space of harmonic vector fields on Σ will be denoted with $\mathfrak{h}_J$ and it is a Lie subalgebra of $\mathfrak{S}(\Sigma,\rho)$. Moreover, there is a splitting

$$\mathfrak{S}(\Sigma,\rho) = \mathfrak{H}(\Sigma,\rho) \oplus \mathfrak{h}_J \tag{3.1.2}$$

as infinite-dimensional vector spaces. Let $\psi \in \mathrm{Symp}_0(\Sigma, \rho)$, then there exists a family of symplectomorphisms $\{\psi_t\}$ with $\psi_1 = \psi$ and $\psi_0 = \mathrm{Id}$. Denote with X_t the vector field which generates the isotopy, namely $\partial_t \psi_t = X_t \circ \psi_t$. Then one has a well-defined map called the *Flux homomorphism*

$$\begin{aligned} \mathrm{Flux} : \mathrm{Symp}_0(\Sigma, \rho) &\longrightarrow H^1_{\mathrm{dR}}(\Sigma, \mathbb{R}) \\ \{\psi_t\} &\mapsto \int_0^1 [\iota_{X_t}\rho]\mathrm{d}t \end{aligned} \tag{3.1.3}$$

Lemma 3.2 ([MS17])**.** *The Flux homomorphism is surjective and it induces an isomorphism*

$$\mathrm{Symp}_0(\Sigma, \rho) \Big/ \mathrm{Ham}(\Sigma, \rho) \cong H^1_{dR}(\Sigma, \mathbb{R}) \ . \tag{3.1.4}$$

We end the discussion in this section by introducing two non-degenerate pairings:

$$\begin{aligned} \langle \cdot | \cdot \rangle_{\mathfrak{S}} : \Omega^1(\Sigma) \Big/ B^1(\Sigma) \times Z^1(\Sigma) &\longrightarrow \mathbb{R} \\ ([\alpha], \beta) \quad &\longmapsto \int_\Sigma \alpha \wedge \beta \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} \langle \cdot | \cdot \rangle_{\mathfrak{H}} : \Omega^1(\Sigma) \Big/ Z^1(\Sigma) \times B^1(\Sigma) &\longrightarrow \mathbb{R} \\ ([\alpha], \beta) \quad &\longmapsto \int_\Sigma \alpha \wedge \beta \ . \end{aligned}$$

Thanks to the identifications $Z^1(\Sigma) \cong_\rho \mathfrak{S}(\Sigma, \rho), B^1(\Sigma) \cong_\rho \mathfrak{H}(\Sigma, \rho)$ and the isomorphism between $B^2(\Sigma)$ and $\Omega^1(\Sigma)/Z^1(\Sigma)$ induce by the differential d, we get

$$\Omega^1(\Sigma) \Big/ B^1(\Sigma) \subset \mathfrak{S}(\Sigma, \rho)^*, \quad B^2(\Sigma) \cong_{\mathrm{d}} \Omega^1(\Sigma) \Big/ Z^1(\Sigma) \subset \mathfrak{H}(\Sigma, \rho)^* \ .$$

Remark 3.3. Observe that, since the above pairings are defined on infinite dimensional vector spaces $\mathcal{V} \times \mathcal{W}$, the notion of non-degeneracy we are referring to is the one that sometimes in the literature is called *weakly non-degenerate*, namely the induced map $\mathcal{V} \to \mathcal{W}^*$ is injective.

Using the standard property of the contraction operator ι with respect to the wedge product, for any vector field V and any 1-form α on Σ, one has

$$\iota_V \alpha \rho = \alpha \wedge \iota_V \rho \ . \tag{3.1.6}$$

Moreover, if V is Hamiltonian, namely $\iota_V \rho = \mathrm{d}H$ for some smooth function H, we get

$$\langle \alpha, \mathrm{d}H \rangle_{\mathfrak{H}} = \int_\Sigma \alpha \wedge \mathrm{d}H = \int_\Sigma \alpha(V) \rho \ , \tag{3.1.7}$$

where $[\alpha] \in \Omega^1(\Sigma)/Z^1(\Sigma)$.

3.1.2 Teichmüller space as a symplectic reduction

Let us briefly recall the construction of $\mathcal{J}(\mathbb{R}^2)$ and its tangent space carrying an $\mathrm{SL}(2,\mathbb{R})$-invariant Kähler structure (Section 2.2.1 and 2.2.2).

Let $\rho_0 := \mathrm{d}x \wedge \mathrm{d}y$ be the standard area form on $\mathbb{R}^2$ and consider the space

$$\mathcal{J}(\mathbb{R}^2) := \{J \in \mathrm{End}(\mathbb{R}^2) \mid J^2 = -\mathbb{1},\ \rho_0(v, Jv) > 0 \text{ for some } v \in \mathbb{R}^2\backslash\{0\}\} \ .$$

Such a space is a 2-dimensional manifold and it is easy to see that $\forall J \in \mathcal{J}(\mathbb{R}^2)$, the tensor $g_J^0(\cdot,\cdot) := \rho_0(\cdot, J\cdot)$ is a scalar product on $\mathbb{R}^2$, with respect to which J is an orthogonal endomorphism. The tangent space $T_J\mathcal{J}(\mathbb{R}^2)$ can be identified with the set of trace-less and g_J^0-symmetric endomorphisms of $\mathbb{R}^2$. It carries a natural (almost) complex structure given by

$$\begin{aligned} \widehat{\mathcal{I}} : T_J\mathcal{J}(\mathbb{R}^2) &\to T_J\mathcal{J}(\mathbb{R}^2) \\ \dot{J} &\mapsto -J\dot{J} \end{aligned}$$

Moreover, there is a natural scalar product defined on each tangent space

$$\langle \dot{J}, \dot{J}' \rangle_J := \frac{1}{2}\operatorname{tr}\left(\dot{J}\dot{J}'\right) \ ,$$

for every $\dot{J}, \dot{J}' \in \mathcal{J}(\mathbb{R}^2)$. The group $\mathrm{SL}(2,\mathbb{R})$ acts by conjugation on $\mathcal{J}(\mathbb{R}^2)$: for $P \in \mathrm{SL}(2,\mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^2)$ one defines $P \cdot J := PJP^{-1}$. The same formula can be used to define the $\mathrm{SL}(2,\mathbb{R})$-action on $T_J\mathcal{J}(\mathbb{R}^2)$ as well.

Lemma 3.4. *The pairing given by*

$$\widehat{\Omega}_J(\dot{J}, \dot{J}) := -\frac{1}{2}\operatorname{tr}\left(\dot{J}J\dot{J}'\right)$$

defines a symplectic form on $\mathcal{J}(\mathbb{R}^2)$, compatible with $\widehat{\mathcal{I}}$ and $\langle\cdot,\cdot\rangle_J$. In particular, the triple $(\langle\cdot,\cdot\rangle_J, \widehat{\mathcal{I}}, \widehat{\Omega}_J)$ is an $\mathrm{SL}(2,\mathbb{R})$-invariant Kähler structure on $\mathcal{J}(\mathbb{R}^2)$.

Now let P be the $\mathrm{SL}(2,\mathbb{R})$ frame bundle over (Σ,ρ), namely the fibre over a point $x \in \Sigma$ is given by those linear maps $F : \mathbb{R}^2 \to T_x\Sigma$ such that $F^*\rho_x = \rho_0$. The frame bundle P inherits the structure of an $\mathrm{SL}(2,\mathbb{R})$-principal bundle with the following action: $B\cdot(x,F) := (x, F\circ B^{-1})$, for $B \in \mathrm{SL}(2,\mathbb{R})$. Notice that any symplectomorphism ψ of (Σ,ρ) naturally lifts to a diffeomorphism $\hat{\psi}$ of the total space P, by setting

$$\hat{\psi}(x,F) := (\psi(x), \mathrm{d}_x\psi \circ F) \in P \ ,$$

for every $(x,F) \in P$. Let us define the bundle

$$P\big(\mathcal{J}(\mathbb{R}^2)\big) := P \times \mathcal{J}(\mathbb{R}^2) \Big/ \mathrm{SL}(2,\mathbb{R}) \, ,$$

where $\mathrm{SL}(2,\mathbb{R})$ acts diagonally on the two factors. Notice that a section of $P\big(\mathcal{J}(\mathbb{R}^2)\big)$ induces an almost complex structure J on Σ which is compatible with ρ, i.e. $g_J(\cdot,\cdot) := \rho(\cdot, J\cdot)$ defines a Riemannian metric on Σ. The induced almost complex structure on Σ is fibre-wise defined on $T_x\Sigma$ as: $F_x \circ J_x \circ F_x^{-1}$. It is easy to see that the section J is well-defined as if two pairs $\big((x,F),J_x\big)$ and $\big((x,F',J'_x)\big)$ differ by the diagonal action of $\mathrm{SL}(2,\mathbb{R})$, then they induce the same almost complex structure on $T_x\Sigma$. According to the above construction, let us introduce the space of almost complex structures on Σ:

$$\mathcal{J}(\Sigma) := \Gamma\big(\Sigma, P\big(\mathcal{J}(\mathbb{R}^2)\big)\big) \ .$$

Given any $J \in \mathcal{J}(\Sigma)$, a tangent vector $\dot{J} \in T_J\mathcal{J}(\Sigma)$ identifies with a section of the pull-back vector bundle $J^*\big(T^{\mathrm{vert}}P\big(\mathcal{J}(\mathbb{R}^2)\big)\big) \to \Sigma$, where $T^{\mathrm{vert}}P\big(\mathcal{J}(\mathbb{R}^2)\big)$ stands for the vertical sub-bundle of $TP\big(\mathcal{J}(\mathbb{R}^2)\big)$ with respect to the projection $\pi : P\big(\mathcal{J}(\mathbb{R}^2)\big) \to \Sigma$. Equivalently, $\dot{J}$ is a section of $\mathrm{End}(T\Sigma)$ that satisfies $\dot{J}J + J\dot{J} = 0$. One can formally define a symplectic form on the infinite-dimensional manifold $\mathcal{J}(\Sigma)$ by integrating fibre-wise that on $\mathcal{J}(\mathbb{R}^2)$. In other words,

$$\Omega_J(\dot{J},\dot{J}') := -\frac{1}{2}\int_\Sigma \mathrm{tr}\Big(\dot{J}J\dot{J}'\Big)\rho \ . \tag{3.1.8}$$

Furthermore, one obtains a complex structure $\mathcal{I}$ on $\mathcal{J}(\Sigma)$, by applying point-wise $\widehat{\mathcal{I}}$ which is defined on $\mathcal{J}(\mathbb{R}^2)$. At this point, the main goal is to explain that such a symplectic form and complex structure can be induced from the ambient $\mathcal{J}(\Sigma)$ to Teichmüller space, using the theory of symplectic reduction. In the end, one succeeds in doing more, namely, Ω_J and $\mathcal{I}$ will be part of a Kähler metric on $\mathcal{T}^{\mathfrak{c}}(\Sigma)$ which turns out to be a multiple of the Weil-Petersson metric. The first result in this direction was provided by Donaldson:

Theorem 3.5 ([Don03],[Tra18])**.** *Let* $c := \frac{2\pi\chi(\Sigma)}{Vol(\Sigma,\rho)}$, *then the function*

$$\begin{aligned} \mu : \mathcal{J}(\Sigma) &\longrightarrow \mathfrak{H}(\Sigma,\rho)^* \\ J &\longmapsto -2(K_J - c)\rho \end{aligned} \tag{3.1.9}$$

is a moment map for the action of $\mathrm{Ham}(\Sigma,\rho)$ *on* $(\mathcal{J}(\Sigma),\Omega)$, *where* $K_J \in \mathcal{C}^\infty(\Sigma)$ *is the Gaussian curvature of* g_J.

Observe that, by the Gauss-Bonnet Theorem, the 2-form $-2(K_J - c)\rho$ is exact, according to the inclusion $B^2(\Sigma) \subset \mathfrak{H}(\Sigma,\rho)^*$ introduced in Section 3.1.1. Because of property (i) in Definition (B.3), the subset $\mu^{-1}(0) \subset \mathcal{J}(\Sigma)$ is preserved by the action of $\mathrm{Ham}(\Sigma,\rho)$. In particular, any variation $\dot{J} = \mathcal{L}_X J$, with X an Hamiltonian vector field and $J \in \mathcal{J}(\Sigma)$, lies inside $\mathrm{Ker}\mathrm{d}_J\mu$, which is identified with $T_J\mu^{-1}(0)$. In other words, the tangent space to the $\mathrm{Ham}(\Sigma,\rho)$-orbit is entirely contained in the Kernel of $\mathrm{d}_J\mu$, for any $J \in \mathcal{J}(\Sigma)$. Furthermore, by property (ii) in Definition B.3, for any $J \in \mu^{-1}(0)$ the space $\mathrm{Ker}(\mathrm{d}_J\mu)$ is identified with the Ω_J-orthogonal to $T_J\big(\mathrm{Ham}(\Sigma,\rho)\cdot J\big)$, namely the tangent space to the $\mathrm{Ham}(\Sigma,\rho)$-orbit. By using a geometric characterization of the elements in the Ω_J-orthogonal to the orbit

([Don03]) one can induce a symplectic form on the quotient $\widetilde{\mathcal{T}}(\Sigma) := \mu^{-1}(0)/\mathrm{Ham}(\Sigma,\rho)$. However, the space $\widetilde{\mathcal{T}}(\Sigma)$ is not isomorphic to Teichmüller space of the surface as it is a manifold of real dimension $6g-6+2g$. The further quotient of the space $\widetilde{\mathcal{T}}(\Sigma)$ by the group

$$H := \mathrm{Symp}_0(\Sigma,\rho) \big/ \mathrm{Ham}(\Sigma,\rho) \cong H^1_{\mathrm{dR}}(\Sigma,\mathbb{R}) \ , \qquad \text{(see Lemma 3.2)}$$

can be identified with $\mathcal{T}^{\mathfrak{c}}(\Sigma)$ (see [Don03, §2.2]). The H-orbits in $\widetilde{\mathcal{T}}(\Sigma)$ are complex and symplectic submanifolds (see [Don03, §2.2] and [Tra18, Lemma 4.4.8]), hence one gets an induced symplectic form on $\mathcal{T}^{\mathfrak{c}}(\Sigma)$ given by:

$$\mathbf{\Omega}_{[J]}([\dot{J}],[\dot{J}']) = \Omega_J(\dot{J}_h,\dot{J}'_h) \ ,$$

where the vectors $\dot{J}_h, \dot{J}'_h \in \mathrm{Ker}(\mathrm{d}_J\mu)$ are lifts of $\dot{J},\dot{J}'$ that are Ω_J-orthogonal to the $\mathrm{Symp}_0(\Sigma,\rho)$-orbit. If one further re-normalizes the lift $\dot{J}_h$ so that it is L^2-orthogonal to the tangent space to the orbit, one recovers the classical description of the tangent space to Teichmüller space as the space of traceless Codazzi tensors ([Tro12]). In that case, the formula of Weil-Petersson metric is also recovered by choosing an area form ρ with $\mathrm{Vol}(\Sigma,\rho) = -2\pi\chi(\Sigma)$, which means $c=-1$ in Theorem 3.5.

Proposition 3.6 ([BMS15, §2.1])**.** *Let $\dot{J},\dot{J}'$ be elements in $T_{[J]}\mathcal{T}^{\mathfrak{c}}(\Sigma)$, then the Weil-Petersson symplectic form and metric are respectively given by:*

$$\big(\Omega_{WP}\big)_{[J]}(\dot{J},\dot{J}') = -\frac{1}{8}\int_\Sigma \mathrm{tr}\Big(\dot{J}J\dot{J}'\Big)\mathrm{d}V, \qquad \big(G_{WP}\big)_{[J]}(\dot{J},\dot{J}') = \frac{1}{8}\int_\Sigma \mathrm{tr}\Big(\dot{J}\dot{J}'\Big)\mathrm{d}V \ , \quad (3.1.10)$$

where $\mathrm{d}V$ *is the area form of the unique hyperbolic metric with conformal structure J.*

Remark 3.7. One of the key facts of this construction is that any choice of a supplement V of $T_J\big(\mathrm{Symp}_0(\Sigma,\rho)\cdot J\big)$ inside the Kernel of $\mathrm{d}\mu$ and Ω_J-orthogonal to $T_J\big(\mathrm{Symp}_0(\Sigma,\rho)\cdot J\big)$, provides a well-defined model for the tangent space to $\mathcal{T}^{\mathfrak{c}}(\Sigma)$, such that $(V,\Omega_J|_V)$ is symplectomorphic to $(T_{[J]}\mathcal{T}^{\mathfrak{c}}(\Sigma), 4\Omega_{\mathrm{WP}})$.

3.1.3 A formula for the differential of the curvature

Here we briefly explain how to derive a formula for the first variation of the curvature K_J, using the theory introduced in the previous section. That expression will be useful later, when we explain how the pseudo-metric is induced on the Hitchin component by a symplectic reduction argument. We will follow closely the approach in [MST21, §4.2].

Given any $B \in \mathrm{End}(T\Sigma)$ and given a Riemannian metric g on Σ, we define the *divergence* of the endomorphism B as the 1-form:

$$(\mathrm{div}_g B)(X) := \sum_i g\big((\nabla^g_{e_i}B)X, e_i\big) \ , \qquad (3.1.11)$$

where $(e_i)_i$ is a g-orthonormal frame of $T\Sigma$, ∇^g is the Levi-Civita connection with respect to g and X is a smooth vector field on the surface. We will denote, likewise, the divergence of a vector field V by $\mathrm{div}_g V$. Moreover, whenever there is a fixed almost complex structure J on the surface, the divergence will be taken with respect to $g_J = \rho(\cdot, J\cdot) \equiv g$. Given that J is ∇^g-parallel, namely $(\nabla^g_X J)Y = 0$ for all $X, Y \in \Gamma(T\Sigma)$, one can deduce the following useful formula:

$$\mathrm{div}_g(JB) = -(\mathrm{div}_g B) \circ J \tag{3.1.12}$$

for any trace-less and g-symmetric endomorphism B. Another relation we will be using later is the following:

$$\mathrm{div}_g(X) = \mathrm{d}(\iota_X \rho)(v, Jv) \tag{3.1.13}$$

for any unit vector v.

Lemma 3.8 ([MST21])**.** *Let X be a vector field on Σ, then*

$$\frac{1}{2}\,\mathrm{tr}\left(\dot{J}J\mathcal{L}_X J\right) = (\mathrm{div}_g \dot{J})(X) - \mathrm{div}_g(JX)\ , \tag{3.1.14}$$

where $\big(\mathcal{L}_X J\big)(Y) := [X, JY] - J([X,Y])$ for any $Y \in \Gamma(T\Sigma)$.

Proof. First notice that

$$\begin{aligned}
(\mathrm{div}_g \dot{J})(X) &= \sum_i g\big((\nabla^g_{e_i}\dot{J})X, e_i\big) \\
&= \sum_i g\big(\nabla^g_{e_i}(\dot{J}V) - \dot{J}\nabla^g_{e_i}V, e_i\big) \\
&= \mathrm{div}_g(\dot{J}V) - \sum_i g\big(\dot{J}\nabla^g_{e_i}V, e_i\big) \\
&= \mathrm{div}_g(\dot{J}V) - \mathrm{tr}\Big(\dot{J}M_V\Big)\ ,
\end{aligned}$$

where M_V stands for the endomorphism $M_V X := \nabla^g_X V$. The Lie derivative $\mathcal{L}_X J$ can be expressed as $JM_V - M_V J$ (proof of Lemma 3.26), hence we have

$$\begin{aligned}
\mathrm{tr}\Big(\dot{J}M_V\Big) &= -\,\mathrm{tr}\Big(\dot{J}JJM_V\Big) && (J^2 = -\mathbb{1}) \\
&= -\frac{1}{2}\Big(\mathrm{tr}\Big(\dot{J}JJM_V\Big) - \mathrm{tr}\Big(J\dot{J}JM_V\Big)\Big) && (\dot{J} \in T_J\mathcal{J}(\Sigma)) \\
&= -\frac{1}{2}\Big(\mathrm{tr}\Big(\dot{J}JJM_V\Big) - \mathrm{tr}\Big(\dot{J}JM_VJ\Big)\Big) \\
&= -\frac{1}{2}\,\mathrm{tr}\left(\dot{J}J\mathcal{L}_X J\right)\ ,
\end{aligned}$$

and relation (3.1.14) follows. ∎

Proposition 3.9 ([MST21]). *Let J be any almost complex structure on Σ and ρ a fixed area form, then*

$$\mathrm{d}K_J(\dot{J})\rho = \frac{1}{2}\mathrm{d}\big(\mathrm{div}_g\dot{J}\big) \ ,$$

where K_J is the Gaussian curvature of $g_J \equiv g$.

Proof. For any Hamiltonian vector field V, with Hamiltonian function H, we have

$$\begin{aligned}
\Omega(\dot{J}, \mathcal{L}_V J) &= -\frac{1}{2}\int_\Sigma \mathrm{tr}\Big(\dot{J}J\mathcal{L}_V J\Big)\rho \\
&= -\int_\Sigma \Big(\frac{1}{2}\mathrm{tr}\Big(\dot{J}J\mathcal{L}_V J\Big) + \mathrm{div}_g(\dot{J}V)\Big)\rho \\
&= -\int_\Sigma (\mathrm{div}_g\dot{J})(V)\rho && \text{(relation (3.1.14))} \\
&= -\int_\Sigma (\mathrm{div}_g\dot{J}) \wedge \iota_V\rho \\
&= -\int_\Sigma (\mathrm{div}_g\dot{J}) \wedge \mathrm{d}H\rho \ .
\end{aligned}$$

According to Theorem 3.5, the map μ satisfies

$$\langle \mathrm{d}\mu(\dot{J}) \mid V\rangle_{\mathfrak{H}} = -2\int_\Sigma H\mathrm{d}K_J(\dot{J})\rho \ .$$

On the other hand, μ being a moment map for the action of $\mathrm{Ham}(\Sigma, \rho)$, we get

$$\begin{aligned}
\langle \mathrm{d}\mu(\dot{J}) \mid V\rangle_{\mathfrak{H}} &= \Omega_J(\dot{J}, \mathcal{L}_V J) \\
&= -\int_\Sigma (\mathrm{div}_g\dot{J}) \wedge \mathrm{d}H \\
&= -\int_\Sigma H\mathrm{d}\Big(\mathrm{div}_g\dot{J}\Big) \ ,
\end{aligned}$$

again for any Hamiltonian vector field V, with Hamiltonian function H. Combining the relations above, we find that

$$-2\int_\Sigma H\mathrm{d}K_J(\dot{J})\rho = -\int_\Sigma H\mathrm{d}\Big(\mathrm{div}_g\dot{J}\Big) \ ,$$

and by letting the Hamiltonian function vary, we obtain the desired formula. □

3.2 The pseudo-Kähler metric on the Hitchin component

This is the core part of the thesis, where the main result will be proved (Section 3.2.3). In particular, after defining an infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$ starting from a general construction of Donaldson (Section 3.2.1), we want to realize the Hitchin component as a subset of $D^3(\mathcal{J}(\Sigma))$. This is done in Section 3.2.2 by using a "special" conformal change of metric on the surface and a standard application of Moser's trick in symplectic geometry. Then, we look for a specific distribution inside the tangent to a set $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ sitting inside $D^3(\mathcal{J}(\Sigma))$. Each vector space $W_{(J,A)}$ of this distribution is defined as the space of solutions to a system of PDEs. After studying in detail the above system of equations (Section 3.2.4), we show that the distribution $\{W_{(J,A)}\}_{(J,A)}$ is integrable, with integral manifold (up to a further finite dimensional decomposition) the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component (Section 3.2.5). Finally, in Section 3.2.6 we generalize the result on the circle action in the case where the surface is of genus $g \geqslant 2$.

3.2.1 Construction of $D^3(\mathcal{J}(\Sigma))$

Here we use the notations introduced in Section 3.1.2. Recall that $\mathcal{J}(\mathbb{R}^2)$ is the space of (almost) complex structures on $\mathbb{R}^2$ compatible with the standard orientation. We introduced a real vector bundle over $\mathcal{J}(\mathbb{R}^2)$ defined as

$$D^3(\mathcal{J}(\mathbb{R}^2)) := \{(J,C) \in \mathcal{J}(\mathbb{R}^2) \times S_3(\mathbb{R}^2) \mid C(J\cdot,J\cdot,J\cdot) = -C(J\cdot,\cdot,\cdot)\} \ , \tag{3.2.1}$$

where $S_3(\mathbb{R}^2)$ is the space of totally symmetric $(0,3)$-tensors. Any pair $(J,C) \in D^3(\mathcal{J}(\mathbb{R}^2))$ defines a unique pair (J,q), where q is a J-holomorphic cubic differential on $\mathbb{R}^2$. In particular, there is a $\mathrm{SL}(2,\mathbb{R})$-equivariant isomorphism between $D^3(\mathcal{J}(\mathbb{R}^2))$ and the holomorphic vector bundle of cubic differentials $Q^3\big(\mathcal{T}(T^2)\big)$ over Teichmüller space of the torus (Corollary 2.4). For any choice of a smooth function $f:[0,+\infty) \to (-\infty,0]$ with $f(0)=0, f'(t)<0$ for any $t>0$, and $\lim_{t\to+\infty} f(t) = -\infty$ we showed the existence of a $\mathrm{SL}(2,\mathbb{R})$-invariant pseudo-Kähler metric $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ on $D^3(\mathcal{J}(\mathbb{R}^2))$, which restricts to a $\mathrm{MCG}(T^2)$-invariant pseudo-Kähler structure on $\mathcal{B}_0(T^2)$ (Theorem 2.16 and D).

Now let Σ be a closed smooth surface of genus $g \geqslant 2$. The next step is to perform a construction similar to that done for $\mathcal{J}(\Sigma)$ in Section 3.1.2, so as to obtain an infinite-dimensional space, associated with Σ, and endowed with a (formal) pseudo-Kähler structure. Let P be the $\mathrm{SL}(2,\mathbb{R})$-frame bundle over (Σ,ρ) introduced in Section 3.1.2 and consider the bundle

$$P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) := P \times D^3(\mathcal{J}(\mathbb{R}^2)) \Big/ \mathrm{SL}(2,\mathbb{R}) \ ,$$

where $\mathrm{SL}(2,\mathbb{R})$ acts diagonally on the two factors. The fibre of $P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ over a point $x \in \Sigma$ identifies with $D^3\big(\mathcal{J}(T_x\Sigma)\big)$, namely the space of pairs (J_x, A_x) where J_x is an almost complex structure on $T_x\Sigma$ compatible with ρ_x, and A_x is an $\mathrm{End}_0(T_x\Sigma, (g_{J_x})_x)$-valued 1-form such that $A_x(J_x\cdot) = A_x(\cdot)J_x$ and $A_x(X)Y = A_x(Y)X, \ \forall X,Y \in \Gamma(T_x\Sigma)$. Since the pseudo-Kähler metric on $D^3(\mathcal{J}(\mathbb{R}^2))$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, each fibre $D^3\big(\mathcal{J}(T_x\Sigma)\big)$ is naturally endowed with a pseudo-Kähler structure, still denoted with $((\widehat{\mathbf{g}}_f)_x, \widehat{\mathbf{I}}_x, (\widehat{\boldsymbol{\omega}}_f)_x)$, obtained by identifying $T_x\Sigma$ with $\mathbb{R}^2$ using an area-preserving isomorphism $F_x : T_x\Sigma \to \mathbb{R}^2$. The space of smooth sections

$$D^3(\mathcal{J}(\Sigma)) := \Gamma\big(\Sigma, P(D^3(\mathcal{J}(\mathbb{R}^2)))\big)$$

is identified with the set of pairs (J, A), where J is a complex structure on Σ, and A is an $\mathrm{End}_0(T\Sigma, g_J)$-valued 1-form such that $A(J\cdot) = A(\cdot)J$ and $A(X)Y = A(Y)X$, $\forall X, Y \in \Gamma(T\Sigma)$. Moreover, there is an identification between the tangent space to $D^3(\mathcal{J}(\Sigma))$ at (J, A) and the space of sections of the vector bundle $(J, A)^*\big(T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)\big) \to \Sigma$, where $T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ stands for the vertical sub-bundle of $TP\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ with respect to the projection map $P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) \to \Sigma$. We can consider tangent vectors $(\dot{J}, \dot{A})$ at (J, A) as the data of (see Lemma 2.10):

- a section $\dot{J}$ of $\mathrm{End}(T\Sigma)$ such that $\dot{J}J + J\dot{J} = 0$, namely $\dot{J}$ is a g_J-symmetric and trace-less endomorphism of $T\Sigma$;
- an $\mathrm{End}(T\Sigma, g_J)$-valued 1-form $\dot{A}$ such that

$$\dot{A} = \dot{\tilde{A}}_0 + \underbrace{T(J, A, \dot{J}) + \frac{1}{2}\,\mathrm{tr}\Big(JA\dot{J}\Big)\mathbb{1}}_{\text{completely determined by } \dot{J}} \, , \tag{3.2.2}$$

  where $\mathbb{1}$ is the 2×2 identity matrix and $\dot{A}_0 = \dot{\tilde{A}}_0 + T(J, A, \dot{J})$ is the trace-less part of $\dot{A}$. Moreover, the trace-part $\dot{A}_{\mathrm{tr}}$ and the tensor $\dot{A}_0 - \dot{\tilde{A}}_0$ is uniquely determined by $\dot{J}$. If $\{e_1, e_2\}$ denotes a local g_J-orthonomal frame of $T\Sigma$ and $\{e_1^*, e_2^*\}$ is the dual frame, then $T(J, A, \dot{J}) = A_1 J\dot{J}Ee_1^* + 2A_2 J\dot{J}Ee_2^*$ with $E = \mathrm{diag}(1, -1)$.

The infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$ inherits a (formal) family of pseudo-Kähler structures, where the symplectic form is defined as

$$(\boldsymbol{\omega}_f)_{(J,A)}\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big) := \int_\Sigma \widehat{\boldsymbol{\omega}}_f\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big)\rho \tag{3.2.3}$$

and the pseudo-Riemannian metric is given by

$$(\mathbf{g}_f)_{(J,A)}\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big) := \int_\Sigma \widehat{\mathbf{g}}_f\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big)\rho \, , \tag{3.2.4}$$

where $\widehat{\boldsymbol{\omega}}_f$ and $\widehat{\mathbf{g}}_f$ denote, respectively, the symplectic form and pseudo-Riemannian metric induced on the pull-back of the vertical sub-bundle inside $TP\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ as described above. Likewise we get a linear endomorphism

$$\mathbf{I}_{(J,A)} : T_{(J,A)}D^3(\mathcal{J}(\Sigma)) \to T_{(J,A)}D^3(\mathcal{J}(\Sigma))$$

obtained by applying pointwisely the endomorphism $\widehat{\mathbf{I}}$ to a smooth section $(\dot{J}, \dot{A})$ (see [Koi90, §2]).

Remark 3.10. It is important to point out that the definition of each element of the pseudo-Kähler structure on $D^3(\mathcal{J}(\Sigma))$ is identical to that given in (2.2.10), (2.2.11) and (2.2.12), the only change is that now the elements $J, A, g_J, \dot{J}, \dot{A}$ are all tensors and $f\big(||A||_0^2\big)$ is a smooth function on Σ. Because of this similarity, in the remainder of the discussion we will make use of some relations proved in the previous work [RT21] and which we will recall when necessary. The general idea to keep in mind is that the identities for elements in $D^3(\mathcal{J}(\mathbb{R}^2))$ can be interpreted as point-wise identities at the level of smooth sections inside $D^3(\mathcal{J}(\Sigma))$. In both our setting and anti-de Sitter one the definition of the infinite-dimensional space follows the lines of a much more general construction given by Donaldson ([Don03, §2.1]), and for this reason the same phenomenon described above happens in either situation ([MST21, Remark 4.9]).

3.2.2 A conformal change of metric

The next step now is to introduce a conformal change of metrics on the surface that allows us to find an equivalent description of $\mathcal{HS}(\Sigma) \cong \mathrm{Hit}_3(\Sigma)$ which will be crucial for the symplectic reduction. In order to do this, we need to fix an area form ρ on the surface. Then, using the so-called Moser's trick in symplectic geometry we will obtain a different model of $\mathrm{Hit}_3(\Sigma)$ as the quotient of an infinite dimensional space by $\mathrm{Symp}_0(\Sigma, \rho)$.
First, we introduce the function that will allow us to make the conformal change of metric.

Lemma 3.11. *There exists a unique smooth function $F : [0, +\infty) \to \mathbb{R}$ such that*

$$ce^{-F(t)} - 2te^{-3F(t)} + 1 = 0, \qquad F(0) = \log|c|, \tag{3.2.5}$$

where $c := \frac{2\pi\chi(\Sigma)}{\mathrm{Vol}(\Sigma,\rho)}$ is a strictly negative constant depending only on the topology and the area of the surface. Moreover, if the function $f : [0, +\infty) \to (-\infty, 0]$ is defined as

$$f(t) := -\left(\int_0^t F'(s)s^{-\frac{1}{3}}\mathrm{d}s\right)t^{\frac{1}{3}} \tag{3.2.6}$$

then it is smooth and it satisfies the following properties:

(1) $f(0) = 0$;

(2) $f'(t) < 0$ *for all* $t > 0$;

(3) $\lim\limits_{t\to+\infty} f(t) = -\infty$.

Proof. The existence and uniqueness of the smooth function F follows from a standard application of the implicit function theorem to $G(t, y) := ce^{-y} - 2te^{-3y} + 1$. In particular, $G\big(0, F(0)\big) = 0$ implies that $F(0) = \log|c|$. Using the formulae for the derivative of the function $y = F(t)$ in terms of the derivatives of $G(t, y)$, we obtain that:

$$F'(t) > 0 \ \forall t \geqslant 0, \qquad F''(t) < 0 \ \forall t \geqslant 0, \qquad \lim_{t\to+\infty} F(t) = +\infty \ .$$

From the definition it is clear that f is smooth, $f(0) = 0$ and it only attains non-positive values. The fundamental theorem of calculus implies that

$$f'(t) = -\Big(F'(t) + \frac{t^{-\frac{2}{3}}}{3}\int_0^t F'(s)s^{-\frac{1}{3}}\mathrm{d}s\Big)\ ,$$

hence $f'(t) < 0$ for all $t > 0$. The behavior of f at infinity can be obtained by using the explicit expression

$$F(t) = \ln\left(\frac{(4t)^{\frac{1}{3}}}{\sqrt[3]{1+\sqrt{1+\frac{\zeta}{t}}} + \sqrt[3]{1-\sqrt{1+\frac{\zeta}{t}}}}\right), t \neq 0$$

that can be derived from the functional equation in the statement, which is a cubic equation in $e^{-F(t)}$, and where $\zeta = \frac{2}{27}|c|$. □

Lemma 3.12. *Let $f : [0, +\infty) \to (-\infty, 0]$ and $F : [0, +\infty) \to \mathbb{R}$ be the functions defined above. Then,*

(1) $f'(t) = -F'(t) + \dfrac{f(t)}{3t}$, *for all $t > 0$;*

(2) $1 - f(t) + 3tf'(t) > 0$, *for all $t \geqslant 0$;*

(3) $f'(t)$ *is monotonically increasing for any $t > 0$.*

Proof. The first identity can be obtained by computing the derivative of the function $f(t)$. In fact,

$$f'(t) = -\Big(F'(t) + \frac{t^{-\frac{2}{3}}}{3}\int_0^t F'(s)s^{-\frac{1}{3}}\mathrm{d}s\Big) = -F'(t) + \frac{f(t)}{3t}, \quad \forall t > 0 \tag{3.2.7}$$

Regarding the second identity, we need to use the explicit expression of $F(t)$ found in the proof of Lemma 3.11. Hence, for any $t > 0$, we have

$$F(t) = \ln\left(\frac{(4t)^{\frac{1}{3}}}{g(t)}\right), \qquad g(t) := \sqrt[3]{1+\sqrt{1+\frac{\zeta}{t}}} + \sqrt[3]{1-\sqrt{1+\frac{\zeta}{t}}}, \quad \zeta = \frac{2}{27}|c|\ .$$

This implies,

$$F'(t) = \frac{1}{3t} - \frac{g'(t)}{g(t)}\ .$$

In the end, combining (1) with the explicit expression for $F'(t)$, we get

$$1 - f(t) + 3tf'(t) = 1 - f(t) + 3t\Big(-F'(t) + \frac{f(t)}{3t}\Big)$$

$$
\begin{aligned}
&= 1 - f(t) + 3t\Big(-\frac{1}{3t} + \frac{g'(t)}{g(t)} + \frac{f(t)}{3t}\Big) \\
&= 3t\frac{g'(t)}{g(t)} .
\end{aligned}
$$

By using the classical theory of the study of a real function of one variable, we deduce that $g(t)$ is strictly positive and monotonically increasing for every $t > 0$. Moreover, at $t = 0$, we have $1 - f(0) + 3 \cdot 0 \cdot f'(0) = 1 > 0$.
Using equation (1) in the statement, we have

$$
f''(t) = -F''(t) + \frac{1}{3t^2}\big(f'(t)t - f(t)\big) \ . \tag{3.2.8}
$$

Using the definition of $f(t)$, the new function $G(t) := f'(t)t^{\frac{2}{3}} - f(t)t^{-\frac{1}{3}}$ is equal to zero when $t = 0$ and its derivative is given by

$$
\begin{aligned}
G'(t) &= \big(f'(t)t^{\frac{2}{3}} - f(t)t^{-\frac{1}{3}}\big)' \\
&= \Big(- t^{\frac{2}{3}}F'(t) - \frac{2}{3}f(t)t^{-\frac{1}{3}}\Big)' && \text{(Equation (3.2.7))} \\
&= -F''(t)t^{\frac{2}{3}} - \frac{2}{3}F'(t)t^{-\frac{1}{3}} - \frac{2}{3}f'(t)t^{-\frac{1}{3}} + \frac{2}{9}f(t)t^{-\frac{4}{3}} && \text{(Equation (3.2.7))} \\
&= -F''(t)t^{\frac{2}{3}} > 0, \quad \forall t > 0 \ .
\end{aligned}
$$

This implies that $G(t) \geqslant 0$ for any $t \geqslant 0$, hence, by using (3.2.8), $f'(t)$ is monotonically increasing for any $t > 0$. $\square$

Recall that, in general, given a (0,3)-tensor C and a Riemannian metric g on Σ, one can define $A := g^{-1}C$ to be the associated (1,2)-tensor, namely a 1-form with values in $\mathrm{End}(T\Sigma)$. Suppose also that C is totally symmetric, then according to Theorem 1.11 the tensor C is the real part of a cubic differential if and only if the endomorphism part of A is trace-less. Let us introduce the following space:

$$
\mathcal{HS}_0(\Sigma) := \left\{ (g,C) \ \middle| \ \begin{array}{l} g \text{ is a Riemannian metric on } \Sigma \\ C \text{ is the real part of a } g\text{-cubic differential} \\ \big(h := e^{F\left(\frac{||q||_g^2}{2}\right)}g, A := g^{-1}C\big) \text{ satisfy (HS)} \end{array} \right\} \Big/ \mathrm{Diff}_0(\Sigma)
$$

Notice that the map sending the pair (g, C) to (h, A), where $h = e^{F\left(\frac{||q||_g^2}{2}\right)}g$ and $A = g^{-1}C$, induces a $\mathrm{MCG}(\Sigma)$-equivariant map from $\mathcal{HS}_0(\Sigma)$ to $\mathcal{HS}(\Sigma)$. There exists an inverse map constructed by sending the pair (h, A) satisfying (HS) to the pair (g, C) where $g = e^{-F\left(\frac{||q||_g^2}{2}\right)}h$ and $C := gA$. Since all the process is invariant by the action of $\mathrm{Diff}(\Sigma)$, we get the following:

Lemma 3.13. *The correspondence described above induces a $MCG(\Sigma)$-equivariant isomorphism between $\mathcal{HS}_0(\Sigma)$ and $\mathcal{HS}(\Sigma)$.*

Let ρ be the area form fixed at the beginning of the discussion, then for any (almost) complex structure J the pairing $g_J := \rho(\cdot, J\cdot)$ defines a Riemannian metric on the surface. Let us introduce the space

$$\mathcal{HS}_0(\Sigma,\rho) := \left\{ (J,C) \;\middle|\; \begin{array}{l} J \text{ is an (almost) complex structure on } \Sigma \\ C \text{ is the real part of a } J\text{-cubic differential} \\ \big(h := e^{F\left(\frac{||q||^2_{g_J}}{2}\right)} g_J, A := g_J^{-1}C\big) \text{ satisfy (HS)} \end{array} \right\} \Big/ \mathrm{Symp}_0(\Sigma)$$

Proposition 3.14. *The map sending the pair (J,C) to $(h = e^{F\left(\frac{||q||^2_{g_J}}{2}\right)} g_J, A = g_J^{-1}C)$ induces a $MCG(\Sigma)$-equivariant homeomorphism between $\mathcal{HS}_0(\Sigma,\rho)$ and $\mathcal{HS}(\Sigma)$.*

Proof. The proof is based on the so-called Moser's trick in symplectic geometry. Since this argument is standard in contexts similar to ours, we will only give an idea of how it is applied (for more details see [Hod05, §3.2.3]). Moser's stability theorem ([MS17, Theorem 3.2.4]) claims that given a family of cohomologous symplectic forms ω_t on a closed manifold, there exists a family of diffeomorphisms ϕ_t such that $\phi_0 = \mathrm{Id}$ and $\phi_t^*\omega_t = \omega_0$. For a closed surface Σ of genus $g \geqslant 2$, given two area forms ρ, ρ' of the same total area, one can apply Moser's stability theorem to the family $\rho_t := (1-t)\rho + t\rho'$ and deduce that there exists $\phi \in \mathrm{Diff}_0(\Sigma)$ such that $\phi^*\rho' = \rho$. In particular, for any $\mathrm{Diff}_0(\Sigma)$-equivalence class $[g, C]$ in $\mathcal{HS}(\Sigma)$, there exists a representative of the form (g_J, C). Finally, if one has a family of diffeomorphisms ψ_t with $\psi_0 = \mathrm{Id}$ and $\psi_1^*\rho = \rho$, by applying Moser's stability again to $\rho_t := \psi_t^*\rho$ one can deform ψ_t to a family of symplectomorphisms ϕ_t such that $\phi_0 = \mathrm{Id}$ and $\phi_1 = \psi_1$. Combining it all together, it has been shown that

$$\mathrm{Symp}_0(\Sigma,\rho) = \mathrm{Diff}_0(\Sigma) \cap \mathrm{Symp}(\Sigma,\rho)\ .$$

$\square$

3.2.3 Proof of Theorem A

The aim of this section is to summarize the strategy of the proof of the main result. We will present preliminary results, proved later in the thesis, which will allow us to give a quite immediate proof of the main theorem. The same approach was used in [MST21, §4.4] with the appropriate differences.

Recall that $\mathcal{HS}_0(\Sigma,\rho)$ is the quotient of the infinite-dimensional space

$$\widetilde{\mathcal{HS}}_0(\Sigma,\rho) := \left\{ (J,C) \;\middle|\; \begin{array}{l} J \text{ is an (almost) complex structure on } \Sigma \\ C \text{ is the real part of a } J\text{-cubic differential} \\ \big(h := e^{F\left(\frac{||q||^2_{g_J}}{2}\right)} g_J, A := g_J^{-1}C\big) \text{ satisfy (HS)} \end{array} \right\}$$

by the action of $\mathrm{Symp}_0(\Sigma,\rho)$, where F is the smooth function defined in Lemma 3.11. The main idea is to define an $\mathrm{Ham}(\Sigma,\rho)$-invariant distribution $\{W_{(J,A)}\}_{(J,A)}$ inside $T\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, whose integral manifold $\widetilde{\mathcal{B}}(\Sigma)$ is the finite-dimensional quotient $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)/\,\mathrm{Ham}(\Sigma,\rho)$. Because of the very specific choice of the subspaces $W_{(J,A)}$, the further finite-dimensional quotient $\widetilde{\mathcal{B}}(\Sigma)/H$, with $H := \mathrm{Symp}(\Sigma,\rho)/\,\mathrm{Ham}(\Sigma,\rho)$, is isomorphic to $\mathrm{Hit}_3(\Sigma)$.

Definition 3.15. Given $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, define $W_{(J,A)}$ to be the subspace of $T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ formed by those elements $(\dot J,\dot A)$ satisfying the following system of equations:

$$\begin{cases} \mathrm{d}\Big(\operatorname{div}\big((f-1)\dot J\big) + \mathrm{d}\dot f\circ J - \frac{f'}{6}\beta\Big) = 0 \\ \mathrm{d}\Big(\operatorname{div}\big((f-1)\dot J\big)\circ J + \mathrm{d}\dot f_0\circ J - \frac{f'}{6}\beta\circ J\Big) = 0 \\ \mathrm{d}^\nabla \dot A_0(\bullet,\bullet) - J(\operatorname{div}\dot J\wedge A)(\bullet,\bullet) = 0 \end{cases} \tag{3.2.9}$$

where $\beta(\bullet) := \langle(\nabla_\bullet A)J, \dot A_0\rangle$ is a 1-form, $\dot f_0 = -\frac{f'}{4}\langle A, \dot A_0 J\rangle$ is a smooth function on Σ and f is the function given by (3.2.6). Moreover, all the expressions for f, f' and $\dot f$ are evaluated at $||A||_0^2 = \frac{||A||_J^2}{8}$.

Remark 3.16. The third equation in the above system can be re-written as $\mathrm{d}^\nabla\dot A_0(\bullet,\bullet)J = (\mathrm{div}_g\dot J\wedge A)(\bullet,\bullet)$, which is equivalent to the following:

$$\mathrm{d}^\nabla\dot A_0(\bullet,J\bullet) = \mathrm{div}_g\dot J(\bullet)A(J\bullet) - \mathrm{div}_g\dot J(J\bullet)A(\bullet)\ . \tag{3.2.10}$$

In fact, by $C^\infty(\Sigma)$-linearity, it is sufficient to perform the computation on a pair $\{X, JX\}$, for $X \in \Gamma(T\Sigma)$. Therefore, we have

$$\begin{aligned} \mathrm{d}^\nabla\dot A_0(X,JX)J &= (\mathrm{div}_g\dot J\wedge A)(X,JX) \\ &= (\mathrm{div}_g\dot J)(X)\cdot A(JX) - (\mathrm{div}_g\dot J)(JX)\cdot A(X) \end{aligned}$$

which is exactly the right-hand side of (3.2.10) computed on X (as an $\mathrm{End}_0(T\Sigma,g)$-valued 1-form). This new form of the equation will be crucial to some key steps in our argument.

Theorem H. *Let (J,A) be a point in $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, then*

$$\mathrm{dim}W_{(J,A)} \geqslant 16g - 16 + 2g\ .$$

The latter result will be a consequence of a detailed study of the system of equations defining $W_{(J,A)}$. The difficult part lies in computing the principal symbols of the matrix operator associated with the three equations in (3.2.9). It is then possible to conclude, using standard results from the theory of elliptic operators on compact manifolds.

Theorem J. *For every element* $(J, A) \in \widetilde{\mathcal{HS}}_0(\Sigma, \rho)$*, the vector space* $W_{(J,A)}$ *is contained inside* $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ *and it is invariant by the complex structure* $\mathbf{I}$*. Moreover, the collection* $\{W_{(J,A)}\}_{(J,A)}$ *defines a* $\mathrm{Ham}(\Sigma, \rho)$*-invariant distribution on* $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ *and the natural projection* $\pi : \widetilde{\mathcal{HS}}_0(\Sigma, \rho) \to \widetilde{\mathcal{B}}(\Sigma)$ *induces a linear isomorphism*

$$\mathrm{d}_{(J,A)}\pi : W_{(J,A)} \longrightarrow T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma)$$

Combining together Theorem H and Theorem J, we observe that the integral manifold $\widetilde{\mathcal{B}}(\Sigma)$ has dimension equal to $16g - 16 + 2g$ and, for this reason, cannot be isomorphic to the $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$-Hitchin component. In fact, it is necessary to perform an additional (finite-dimensional) quotient of $\widetilde{\mathcal{B}}(\Sigma)$ by the group $H := \mathrm{Symp}_0(\Sigma, \rho)/\mathrm{Ham}(\Sigma, \rho)$ isomorphic to $H^1_{\mathrm{dR}}(\Sigma, \mathbb{R})$ (see Lemma 3.2).

Theorem K. *The* H*-action on* $\widetilde{\mathcal{B}}(\Sigma)$ *is free and proper, with complex and symplectic* H*-orbits. Moreover, the pseudo-Kähler structure* $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ *descend to the quotient which is identified with* $\mathrm{Hit}_3(\Sigma)$*. Finally, the complex structure* $\mathbf{I}$ *induced on the* $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$*-Hitchin component coincides with the one found by Labourie and Loftin.*

Remark 3.17. The tangent space to the integral manifold $\widetilde{\mathcal{B}}(\Sigma)$, i.e. the subspace $W_{(J,A)}$, decomposes as a direct sum $V_{(J,A)} \oplus S_{(J,A)}$, where $V_{(J,A)}$ is the tangent space to the Hitchin component and $S_{(J,A)} := \{\big(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C\big) \mid X \in \Gamma(T\Sigma),\ \mathrm{d}(\iota_X\rho) = \mathrm{d}(\iota_{JX}\rho) = 0\}$, namely the tangent space to the harmonic orbit (see Section 3.1.1). Using the definition of $W_{(J,A)}$ in terms of the system of equations (3.2.9), we get a similar description of the tangent space to the Hitchin component. In particular, $V_{(J,A)}$ can be characterized as the subspace of $W_{(J,A)}$ (see Section 3.4.1) defined by the following system:

$$\begin{cases} \mathrm{div}\big((f-1)\dot{J}\big) + \mathrm{d}\dot{f} \circ J - \frac{f'}{6}\beta = \mathrm{d}\gamma_1 \\ \mathrm{div}\big((f-1)\dot{J}\big) \circ J + \mathrm{d}\dot{f}_0 \circ J - \frac{f'}{6}\beta \circ J = \mathrm{d}\gamma_2 \\ \mathrm{d}^\nabla \dot{A}_0(\bullet, \bullet) - J(\mathrm{div}\,\dot{J} \wedge A)(\bullet, \bullet) = 0 \end{cases} \tag{3.2.11}$$

for some $\gamma_1, \gamma_2 \in C^\infty(\Sigma)$. In a more concise form:

$$V_{(J,A)} = \left\{ (\dot{J}, \dot{A}) \in T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho) \;\middle|\; \begin{array}{l} \alpha_1 + i\alpha_2 \text{ is exact} \\ \mathrm{d}^\nabla \dot{A}_0(\bullet, \bullet) - J(\mathrm{div}\,\dot{J} \wedge A)(\bullet, \bullet) = 0 \end{array} \right\} \tag{3.2.12}$$

where α_1 and α_2 are the 1-forms in (3.2.11) defined by the LHS of the first two equations.

At this point, we have all the ingredients to present a concise proof of the main result of the thesis.

Theorem A. *Let* Σ *be a closed oriented surface of genus* $g \geqslant 2$*. Then, there exists a neighborhood* $\mathcal{N}_{\mathcal{F}(\Sigma)}$ *of the Fuchsian locus in* $\mathrm{Hit}_3(\Sigma)$*, which admits a mapping class group invariant pseudo-Kähler metric* $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$*. Moreover, the Fuchsian locus embeds as a totally geodesic submanifold and the triple* $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ *restricts to a (multiple of) the Weil-Petersson Kähler metric of Teichmüller space.*

Proof. The tangent space $T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma)$ can be identified with $W_{(J,A)}$ (Theorem J), hence we can define a complex structure $\mathbf{I}$, and a pseudo-Riemannian metric $\mathbf{g}_f$ by restriction from the infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$. This definition does not depend on the representative in the $\mathrm{Ham}(\Sigma,\rho)$-orbit by the invariance statement in Theorem J and the $\mathrm{Symp}(\Sigma,\rho)$-invariance of $\mathbf{I}$ and $\mathbf{g}_f$. It is immediate that $\mathbf{I}$ is still compatible with $\mathbf{g}_f$ and that the pairing $\mathbf{g}_f(\cdot,\mathbf{I}\cdot)$ coincides with the 2-form $\boldsymbol{\omega}_f$ restricted to $W_{(J,A)}$. Moreover, Theorem K allows us to induce the triple $(\mathbf{g}_f,\mathbf{I},\boldsymbol{\omega}_f)$ on the quotient $\widetilde{\mathcal{B}}(\Sigma)/H \cong \mathrm{Hit}_3(\Sigma)$, in such a way that the induced complex structure is equivalent to the one found by Labourie and Loftin. Thanks to the $\mathrm{Symp}(\Sigma,\rho)$-invariance of $\mathbf{g}_f$ and $\mathbf{I}$, it follows that the induced structure on $\mathrm{Hit}_3(\Sigma)$ is invariant under the action of the mapping class group, since it is isomorphic to $\mathrm{Symp}(\Sigma,\rho)/\mathrm{Symp}_0(\Sigma,\rho)$.

Notice that the Fuchsian locus $\mathcal{F}(\Sigma)$ (see Section 1.1 and 1.3) inside $\mathrm{Hit}_3(\Sigma) \cong \mathcal{HS}(\Sigma)$ corresponds to pairs (J,A) with $A=0$. According to Remark 3.17, the tangent space to $\mathcal{HS}(\Sigma)$ along the Fuchsian locus is isomorphic to $V_{(J,0)}$, and thus consists of pairs $(\dot{J},\dot{A})$ such that $\mathrm{div}_g\dot{J}=0$ and $\mathrm{d}^\nabla\dot{A}_0(\cdot,\cdot)=0$. The pseudo-metric restricted to $V_{(J,0)}$ is equal to

$$(\mathbf{g}_f)_{(J,0)}\big((\dot{J},\dot{A});(\dot{J}',\dot{A}')\big)=\int_\Sigma\langle\dot{J},\dot{J}'\rangle\rho+\int_\Sigma\frac{f'}{6}\langle\dot{A}_0,\dot{A}_0'\rangle\rho$$

since the trace-part of $\dot{A}$ is equal to zero according to relation (2.2.7). Notice that $\mathbf{g}_f$ on $V_{(J,0)}$ coincides with $4G_{\mathrm{WP}}$ along horizontal directions ($\dot{A}=0$) and it is negative-definite along vertical directions ($\dot{J}=0$). Because of the explicit description of $\mathbf{g}_f$, this is sufficient to conclude that the pseudo-metric is non-degenerate on arbitrary directions inside $V_{(J,0)}$ as well. In particular, there must exist an open neighborhood $\mathcal{N}_{\mathcal{F}(\Sigma)}$ of $\mathcal{F}(\Sigma)$ inside $\mathrm{Hit}_3(\Sigma)$ in which $\mathbf{g}_f$ is non-degenerate.

Finally, the Fuchsian locus is the set of fixed points of the circle action, that consists of isometries for $\mathbf{g}_f$ by Theorem C (which is proved in Section 3.2.6). Using a standard argument in (pseudo)-Riemannian geometry, this implies that the Fuchsian locus is totally geodesic. □

Remark 3.18. It is important to emphasize again that the triple $(\mathbf{g}_f,\mathbf{I},\boldsymbol{\omega}_f)$ is defined over the entire Hitchin component $\mathrm{Hit}_3(\Sigma)$, but it may be degenerate away from the Fuchsian locus. The main problem lies in the fact that the restriction of an indefinite metric on a subspace is not necessarily non-degenerate (as in the positive-definite case). Partial results have been obtained concerning the non-existence of degenerate vectors outside $\mathcal{N}_{\mathcal{F}(\Sigma)}$, which will be explained in detail in Section 3.4.2 and 3.4.4.

3.2.4 The system of equations

This section is devoted to the study of the system of equations defined by (3.2.9) and to the proof of Theorem H. More precisely, in Lemma 3.19 and Lemma 3.20 we study the induced connection on the endomorphism bundle and the associated exterior covariant derivative. Then, Lemma 3.21 allows us to compute the terms involving derivatives of order two of $(\dot{J}, \dot{A})$ in the first and second equation appearing in the system defining $W_{(J,A)}$. Further on, we explain how the space $W_{(J,A)}$ can be seen as the kernel of a matrix of mixed-order smooth differential operators, which is proven to be elliptic (Lemma 3.22). Finally, using the homotopy invariance of the Fredholm index for elliptic operators, we deduce the lower bound on the dimension of $W_{(J,A)}$.

Lemma 3.19. *Let ∇ be the Levi-Civita connection with respect to g_J, then the induced connection*

$$\bar{\nabla} : \Omega^0(\Sigma, \mathrm{End}_0(T\Sigma, g_J)) \longrightarrow \Omega^1(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$$

does not admit any non-zero parallel section, where $\mathrm{End}_0(T\Sigma, g_J)$ denotes the real vector bundle of g_J-symmetric and trace-less endomorphisms of $T\Sigma$.

Proof. Let $B \in \Omega^0(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$ such that $\bar{\nabla} B = 0$. Let $x_0 \in \Sigma$ be a fixed point and $x \in \Sigma$ be arbitrary. Consider a path $\gamma : [0,1] \to \Sigma$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. Let $\{e_1, e_2\}$ be a basis of $T_{x_0}\Sigma$ and denote with $\{e_1(t), e_2(t)\}$ the basis of $T_{\gamma(t)}\Sigma$ obtained by parallel transport $\{e_1, e_2\}$ along the path γ. Then, if $b_{ij}(t)$ denotes the (i,j)-th entry of $B_{\gamma(t)}$ for $i, j = 1, 2$, we have $b_{ij}(t) = g_J\big(B_{\gamma(t)}(e_j(t)), e_i(t)\big)$. By differentiating the last identity with respect to the parameter t, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{ij}(t) = g_J\Big(\underbrace{\big(\bar{\nabla}_{\dot{\gamma}} B\big)}_{=0} \big(e_j(t)\big) + B_{\gamma(t)}\big(\nabla_{\dot{\gamma}} e_j\big), e_i(t)\Big) + g_J\big(B_{\gamma(t)}\big(e_j(t)\big), \nabla_{\dot{\gamma}} e_i\big) \ .$$

Since the basis $\{e_1(t), e_2(t)\}$ has been obtained by parallel transport, we have $\nabla_{\dot{\gamma}} e_j = 0$ for any $j = 1, 2$. In particular, we deduce that each entry of B is constant along γ, hence $B_x = B_{\gamma(1)} = B_{\gamma(0)} = B_{x_0}$. Since $x \in \Sigma$ was arbitrary, it follows that the endomorphism B is actually constant on the whole surface. At this point, it is enough to show that every section of $E := \mathrm{End}_0(T\Sigma, g_J)$ admits at least one zero to conclude the proof. Since the real rank of E is equal to the real dimension of the surface, any section B is nowhere zero if and only if the Euler class $e\big(E\big)$ is trivial in $H^2(\Sigma, \mathbb{R})$. In our case, it can be shown (see for example [Tro12, §2.4]) that E is the realization of the holomorphic line bundle $K \otimes K$ defined on (Σ, J). In particular, $e\big(E\big) = c_1\big(K \otimes K\big)$, where c_1 denotes the first Chern class of a complex vector bundle. Therefore,

$$\int_\Sigma e\big(E\big) = \int_\Sigma c_1\big(K \otimes K\big) = \deg(K \otimes K) = 2(2g-2) \neq 0 \ .$$

The last chain of equalities implies that $e\big(E\big)$ is not trivial in cohomology by Poincaré duality, and thus any such section B admits at least one zero. $\square$

Lemma 3.20. *Let ∇ be the Levi-Civita connection with respect to g_J, then the exterior covariant derivative*

$$\mathrm{d}^\nabla : \Omega^1(\Sigma, \mathrm{End}_0(T\Sigma, g_J)) \longrightarrow \Omega^2(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$$

is surjective and its kernel has real dimension equal to $10g - 10$.

Proof. Recall that for $A \in \Omega^1\big(\Sigma, \mathrm{End}_0(T\Sigma, g_J)\big)$ and for any $X, Y, Z \in \Gamma(T\Sigma)$ we have

$$\big(\mathrm{d}^\nabla A\big)(X,Y)Z = \big(\nabla_X A\big)(Y)Z - \big(\nabla_Y A\big)(X)Z \ .$$

In particular, if we define the $(0,3)$-tensor $C(X,Y,Z) := g_J(A(X)Y, Z)$, then $A \in \mathrm{Ker}(\mathrm{d}^\nabla)$ if and only if C is the real part of a g_J-holomorphic cubic differential (Theorem 1.11). The space of holomorphic cubic differentials on (Σ, g_J) coincides with the space $H^0(\Sigma, K^{\otimes^3})$ of holomorphic sections of the tri-canonical bundle, which is isomorphic (as a real vector space) to $\mathbb{R}^{10g-10}$ by an easy application of Riemann-Roch Theorem for curves.
Concerning the surjectivity of d^∇, we will prove that its Co-kernel is trivial. Let us denote with $*_J$ the Hodge-star operator defined on differential forms with respect to g_J, which can be extended to an isomorphism $*_J : \Omega^k(\Sigma, \mathrm{End}_0(T\Sigma, g_J)) \xrightarrow{\cong} \Omega^{2-k}(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$. Let $\big(\mathrm{d}^\nabla\big)^*$ be the formal adjoint of the exterior covariant derivative with respect to the L^2-inner product on $\Omega^2(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$ induced by $*_J$ and integration over Σ. A standard computation shows that

$$\big(\mathrm{d}^\nabla\big)^* = - *_J \circ \mathrm{d}^\nabla \circ *_J : \Omega^2(\Sigma, \mathrm{End}_0(T\Sigma, g_J)) \longrightarrow \Omega^1(\Sigma, \mathrm{End}_0(T\Sigma, g_J)) \ .$$

Since $\mathrm{Range}(\mathrm{d}^\nabla)$ is a closed subspace of $\Omega^2(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$, we get that $\mathrm{CoKer}(\mathrm{d}^\nabla) = \mathrm{Ker}\Big(\big(\mathrm{d}^\nabla\big)^*\Big)$. In particular, if $\alpha \in \Omega^2(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$ then

$$\begin{aligned} \big(\mathrm{d}^\nabla\big)^*\alpha = 0 \iff & \qquad\qquad (*_J \text{ is an isomorphism}) \\ \mathrm{d}^\nabla(*_J\alpha) = 0 \iff & \qquad\qquad (\mathrm{d}^\nabla \equiv \bar{\nabla} \text{ on } \Omega^0(\Sigma, \mathrm{End}_0(T\Sigma, g_J))) \\ \bar{\nabla}(*_J\alpha) = 0 \ , & \end{aligned}$$

where $\bar{\nabla}$ in the last equation is the induced connection on $\Omega^0(\Sigma, \mathrm{End}_0(T\Sigma, g_J))$. According to Lemma 3.19, the induced connection $\bar{\nabla}$ does not admit any non-zero parallel section, hence $*_J\alpha = 0$, which implies $\alpha = 0$. □

Lemma 3.21. *Let $(J, A) \in D^3(\mathcal{J}(\Sigma))$ and consider the following 2-forms on the surface*

$$\begin{aligned} \eta_1 &:= \mathrm{d}\bigg(\mathrm{div}_g\big((f-1)\dot{J}\big) + \mathrm{d}\dot{f} \circ J - \frac{f'}{6}\langle(\nabla_\bullet A)J, \dot{A}_0\rangle\bigg) \ , \\ \eta_2 &:= \mathrm{d}\bigg(\mathrm{div}_g\big((f-1)\dot{J}\big) \circ J + \mathrm{d}\dot{f}_0 \circ J + \frac{f'}{6}\langle\nabla_\bullet A, \dot{A}_0\rangle\bigg) \end{aligned}$$

where $(\dot{J}, \dot{A}_0) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma))$*, the function* f *is the one defined by (3.2.6),* $\dot{f}_0 = -\frac{f'}{4}\langle A, \dot{A}_0 J\rangle$*, and* $f, f', \dot{f}, \dot{f}_0$ *are computed in* $||A||_0^2 = \frac{||A||_J^2}{8}$*. Then, the part involving second order derivatives of* $(\dot{J}, \dot{A}_0)$ *in* η_1 *and* η_2 *is, respectively*

$$(f-1)\mathrm{d}\big(\mathrm{div}_g \dot{J}\big) + \frac{f'}{4}\mathrm{d}\langle A, \nabla_{J\bullet}\dot{A}_0\rangle, \qquad (f-1)\mathrm{d}\big(\mathrm{div}_g \dot{J}\circ J\big) - \frac{f'}{4}\mathrm{d}\langle A, \big(\nabla_{J\bullet}\dot{A}_0\big)J\rangle \ .$$

Proof. By using (3.1.11), we get the following equation involving the divergence of a smooth section of $\mathrm{End}_0(T\Sigma, g_J)$ multiplied by a smooth function φ

$$\mathrm{div}_g\big(\varphi\dot{J}\big)(X) = \mathrm{d}\varphi(\dot{J}X) + \varphi\big(\mathrm{div}_g\dot{J}\big)(X), \quad \forall X \in \Gamma(T\Sigma) \ . \tag{3.2.13}$$

Therefore,

$$\mathrm{d}\Big(\mathrm{div}_g\big((f-1)\dot{J}\big)\Big) = \mathrm{d}\big(\mathrm{d}f\circ\dot{J}\big) + \mathrm{d}f \wedge \mathrm{div}_g\dot{J} + (f-1)\mathrm{d}\big(\mathrm{div}_g\dot{J}\big) \ ,$$

and it is clear that $(f-1)\mathrm{d}\big(\mathrm{div}_g\dot{J}\big)$ is the only part involving second order derivatives of $\dot{J}$ in the expression above. Regarding the other two terms in η_1, let us first define $\tau_1 := \mathrm{d}\dot{f}\circ J$ and $\tau_2 := -\frac{f'}{6}\beta$, where $\beta = \langle(\nabla_\bullet A)J, \dot{A}_0\rangle$.

<u>*The differential of* τ_1</u>

Notice that the first order variation of $f\big(||A||_0^2\big)$ is

$$\begin{aligned} \dot{f} &= \frac{f'}{8}(\langle\dot{A}, A\rangle + \langle A, \dot{A}\rangle) && \text{(Lemma 2.23)} \\ &= \frac{f'}{4}\langle A, \dot{A}_0\rangle \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{d}\tau_1 &= \mathrm{d}\Big(\mathrm{d}\big(\frac{f'}{4}\langle A, \dot{A}_0\rangle\big)\circ J\Big) \\ &= \frac{1}{4}\mathrm{d}\Big(\langle A, \dot{A}_0\rangle \mathrm{d}f'\circ J + f'\mathrm{d}\langle A, \dot{A}_0\rangle\circ J\Big) \\ &= \frac{1}{4}\mathrm{d}\Big(\frac{f''}{4}\langle A, \nabla_{J\bullet}A\rangle\langle A, \dot{A}_0\rangle + f'\big(\langle\nabla_{J\bullet}A, \dot{A}_0\rangle + \langle A, \nabla_{J\bullet}\dot{A}_0\rangle\big)\Big) \ , \end{aligned}$$

where in the last step we used that $\mathrm{d}f' = \frac{f''}{4}\langle A, \nabla_\bullet A\rangle$. The only interesting part, for our purpose, is the term containing $f'\langle A, \nabla_{J\bullet}\dot{A}_0\rangle$. In particular,

$$\frac{1}{4}\mathrm{d}\Big(f'\langle A, \nabla_{J\bullet}\dot{A}_0\rangle\Big) = \frac{f''}{16}\langle A, \nabla_\bullet A\rangle \wedge \langle A, \nabla_{J\bullet}\dot{A}_0\rangle + \frac{f'}{4}\mathrm{d}\langle A, \nabla_{J\bullet}\dot{A}_0\rangle \ .$$

Again, the only part involving second order derivatives is the term $\frac{f'}{4}\mathrm{d}\langle A, \nabla_{J\bullet}\dot{A}_0\rangle$.

<u>*The differential of* τ_2</u>

To conclude the proof it must be proven that $\mathrm{d}\tau_2$ does not involve second derivatives of $\dot{J}$ or $\dot{A}_0$. In fact, by carrying out calculations similar to those made above

$$\begin{aligned}\mathrm{d}\gamma_2 &= \mathrm{d}\bigg(-\frac{f'}{6}\langle(\nabla_\bullet A)J, \dot{A}_0\rangle\bigg) \\ &= -\frac{1}{6}\bigg(\frac{f''}{4}\langle A, \nabla_\bullet A\rangle\langle(\nabla_\bullet A)J, \dot{A}_0\rangle + f'\big(\langle(\nabla_\bullet\nabla_\bullet A)J, \dot{A}_0\rangle + \langle(\nabla_\bullet A)J, \nabla_\bullet \dot{A}_0\rangle\big)\bigg)\ .\end{aligned}$$

Because of the very similar expression of the 2-forms η_1, η_2 it is easy to see, by going over the calculations already done, that the part involving second derivatives of $(\dot{J}, \dot{A}_0)$ in η_2 is exactly

$$(f-1)\mathrm{d}\big(\mathrm{div}_g\dot{J}\circ J\big) - \frac{f'}{4}\mathrm{d}\langle A, \big(\nabla_{J\bullet}\dot{A}_0\big)J\rangle\ .$$

□

The next step is to write down in coordinates the expressions found in Lemma 3.21, so that, later, we will be able to explicitly deduce the principal symbol of the matrix of operators associated with the PDEs defining the subspace $W_{(J,A)}$. In order to do this, we need to recall the construction in coordinates for $D^3(\mathcal{J}(\mathbb{R}^2))$ (see Section 2.2.2), and then use the particular definition of $D^3(\mathcal{J}(\Sigma))$ to infer that the same can be done, point-wise, in the genus $g \geqslant 2$ case (see Remark 3.10). Let $\rho_0 := \mathrm{d}x_0 \wedge \mathrm{d}y_0$ be the standard area form on $\mathbb{R}^2$ and $g_J^0 := \rho_0(\cdot, J\cdot)$ be the associated scalar product, for some $J \in \mathcal{J}(\mathbb{R}^2)$. According to Proposition 2.7, the space $D^3(\mathcal{J}(\mathbb{R}^2))$ is $\mathrm{SL}(2,\mathbb{R})$-equivariantly isomorphic to the holomorphic vector bundle of cubic differentials over Teichmüller space of the torus, denoted with $Q^3\big(\mathcal{T}(T^2)\big)$. The latter can be identified with $\mathbb{H}^2 \times \mathbb{C}$, where $\mathbb{H}^2$ is a copy of $\mathcal{T}(T^2)$ and $\mathbb{C}$ is isomorphic to the fibre over an oriented (almost) complex structure $J : \mathbb{R}^2 \to \mathbb{R}^2$. Let $z = x + iy$ and $w = u + iv$ be the complex coordinates on $\mathbb{H}^2$ and $\mathbb{C}$, respectively. Then, we have the following correspondence

$$\mathbb{H}^2 \times \mathbb{C} \ni (z,w) \longmapsto \big(j(z), C_{(z,w)}\big) \in D^3(\mathcal{J}(\mathbb{R}^2))$$

where $C_{(z,w)} = \mathcal{R}e(q_{(z,w)})$ with $q_{(z,w)} = \bar{w}(\mathrm{d}x_0 - \bar{z}\mathrm{d}y_0)^3$ (see Lemma 2.20). Because of the $\mathrm{SL}(2,\mathbb{R})$-invariance, one can compute the pair $(j(z), C_{(z,w)})$ at points $(i,w) \equiv (0,1,u,v)$, for some $w \in \mathbb{C}$. Using the relation $A = (g_J^0)^{-1}C$, one can deduce:

$$\dot{J} = \mathrm{d}_i j(\dot{x},\dot{y}) = \begin{pmatrix} \dot{x} & -\dot{y} \\ -\dot{y} & -\dot{x}\end{pmatrix}, \qquad A_{(i,w)} = \begin{pmatrix} u & v \\ v & -u\end{pmatrix}\mathrm{d}x_0 + \begin{pmatrix} v & -u \\ -u & -v\end{pmatrix}\mathrm{d}y_0\ .$$

$$(\dot{A}_0)_{(i,w)} = \begin{pmatrix} \dot{u} + u\dot{y} + v\dot{x} & -u\dot{x} + \dot{v} + v\dot{y} \\ -u\dot{x} + \dot{v} + v\dot{y} & -\dot{u} - u\dot{y} - v\dot{x}\end{pmatrix}\mathrm{d}x_0 + \begin{pmatrix} \dot{v} + 2(v\dot{y} - u\dot{x}) & -\dot{u} - 2(u\dot{y} + v\dot{x}) \\ -\dot{u} - 2(u\dot{y} + v\dot{x}) & -\dot{v} + 2(u\dot{x} - v\dot{y})\end{pmatrix}\mathrm{d}y_0$$

$$(\dot{A}_{\mathrm{tr}})_{(i,w)} = \begin{pmatrix} -u\dot{y} - v\dot{x} & 0 \\ 0 & -u\dot{y} - v\dot{x} \end{pmatrix} \mathrm{d}x_0 + \begin{pmatrix} u\dot{x} - v\dot{y} & 0 \\ 0 & u\dot{x} - v\dot{y} \end{pmatrix} \mathrm{d}y_0 \ .$$

Now, let us define the following matrix of smooth differential operators

$$\begin{aligned} \Lambda : T_{(J,A)} D^3(\mathcal{J}(\Sigma)) &\longrightarrow \Omega^2_{\mathbb{C}}(\Sigma) \oplus \Omega^2\big(\Sigma, \mathrm{End}_0(T\Sigma, g_J)\big) \\ (\dot{J}, \dot{A}_0) &\longmapsto \big((L_1 + iL_2)(\dot{J}, \dot{A}_0), S(\dot{J}, \dot{A}_0)\big) \end{aligned} \tag{3.2.14}$$

where

$$\begin{aligned} L_1(\dot{J}, \dot{A}_0) &:= \mathrm{d}\bigg(\mathrm{div}_g\big((f-1)\dot{J}\big) + \mathrm{d}\dot{f} \circ J - \frac{f'}{6}\langle (\nabla_{\bullet} A)J, \dot{A}_0 \rangle\bigg) \in \Omega^2(\Sigma) \ , \\ L_2(\dot{J}, \dot{A}_0) &:= \mathrm{d}\bigg(\mathrm{div}_g\big((f-1)\dot{J}\big) \circ J + \mathrm{d}\dot{f}_0 \circ J + \frac{f'}{6}\langle \nabla_{\bullet} A, \dot{A}_0 \rangle\bigg) \in \Omega^2(\Sigma) \ , \\ S(\dot{J}, \dot{A}_0) &= \mathrm{d}^{\nabla} \dot{A}_0(\cdot,\cdot) - J(\mathrm{div}\, \dot{J} \wedge A)(\cdot,\cdot) \in \Omega^2\big(\Sigma, \mathrm{End}_0(T\Sigma, g_J)\big) \ . \end{aligned}$$

It is possible to define the principal symbol of a matrix of mixed-order differential operators as the matrix obtained by taking the principal symbols of each differential operator. The corresponding system of PDEs is called *elliptic*, if the symbol matrix has non-zero determinant (see [ADN64] and [Gru77] for more details).

Lemma 3.22. *Let (J, A) be an arbitrary point in $D^3(\mathcal{J}(\Sigma))$. Then, for any $p \in \Sigma$ and for any $0 \neq \xi \in T^*\Sigma$, the symbol matrix $\sigma\big(\Lambda\big)_p(\xi)$ has non-zero determinant.*

Proof. Let $\{e_1, e_2\}$ be a g_J-orthonormal basis and let $\{e_1^*, e_2^*\}$ be the dual basis, so that $\xi = \xi_1 e_1^* + \xi_2 e_2^*$. We first note that $\sigma(\Lambda)_p(\xi)$ is a 4×4 matrix as any term in Λ, involving the tensors $(\dot{J}, \dot{A}_0)$, can be written in the coordinates $(\dot{x}, \dot{y}, \dot{u}, \dot{v})$, for what explained above. Moreover, we have the following decomposition:

$$\sigma\big(\Lambda\big)_p(\xi) = \begin{pmatrix} \Theta & \Xi \\ \Gamma & \Delta \end{pmatrix} \ , \tag{3.2.15}$$

where each block is a 2×2 matrix, and each entry in the first and second block-row is a homogeneous polynomial in ξ_1, ξ_2 of degree two and one, respectively. After a fairly long computation in coordinates the final expression for $\sigma(\Lambda)_p(\xi)$ is

$$\begin{pmatrix} -2(f-1)\xi_1\xi_2 & (f-1)(\xi_1^2 - \xi_2^2) + \frac{3}{2}|w|^2 f'|\xi|^2 & -f'u(\xi_1^2\xi_2^2) & -f'v|\xi|^2 \\ (f-1)(\xi_2^2 - \xi_1^2) - \frac{3}{2}|w|^2 f'|\xi|^2 & -2(f-1)\xi_1\xi_2 & -f'v|\xi|^2 & f'u|\xi|^2 \\ -3u\xi_1 & -3v\xi_1 & -\xi_2 & \xi_1 \\ -3v\xi_1 & 3u\xi_1 & -\xi_1 & -\xi_2 \end{pmatrix} ,$$

where $|\xi|^2 := \xi_1^2 + \xi_2^2$ and the second column corresponds to the coefficient of $-\dot{y}$. We only show how to get block Θ, as with a similar calculation one can obtain the remaining ones. To write down explicitly each entry of Θ, we need to compute the principal symbol of L_1 and L_2, along directions $(\dot{x}, -\dot{y}, \dot{u}, \dot{v})$ with $\dot{u} = \dot{v} = 0$. According to Lemma 3.21, $\sigma_2(L_1)$ and $\sigma_2(L_2)$ depend, respectively, on $(f-1)\mathrm{d}(\mathrm{div}_g\dot{J}) + \frac{f'}{4}\mathrm{d}\langle A, \nabla_{J\bullet}\dot{A}_0\rangle$ and on $(f-1)\mathrm{d}(\mathrm{div}_g\dot{J}\circ J) - \frac{f'}{4}\mathrm{d}\langle A, (\nabla_{J\bullet}\dot{A}_0)J\rangle$. In particular,

$$\mathrm{d}(\mathrm{div}_g\dot{J})(\xi,\xi) = \dot{x}(-2\xi_1\xi_2) - \dot{y}(\xi_1^2-\xi_2^2)\ , \qquad \mathrm{d}\big(\langle A, \nabla_{J\bullet}\dot{A}_0\rangle\big)(\xi,\xi) = -6|w|^2(\xi_1^2+\xi_2^2)\dot{y}\ ,$$
$$\mathrm{d}(\mathrm{div}_g\dot{J}\circ J)(\xi,\xi) = \dot{x}(\xi_2^2-\xi_1^2) - \dot{y}(-2\xi_1\xi_2)\ , \qquad \mathrm{d}\big(\langle A, (\nabla_{J\bullet}\dot{A}_0)J\rangle\big)(\xi,\xi) = 6|w|^2(\xi_1^2+\xi_2^2)\dot{x}\ ,$$

where all the above equality are to be intended up to lower order terms in ξ. In the end, the upper left block in $\sigma(\Lambda)_p(\xi)$ is given by:

$$\Theta = \begin{pmatrix} -2(f-1)\xi_1\xi_2 & (f-1)(\xi_1^2-\xi_2^2)+\frac{3}{2}|w|^2f'|\xi|^2 \\ (f-1)(\xi_2^2-\xi_1^2)-\frac{3}{2}|w|^2f'|\xi|^2 & -2(f-1)\xi_1\xi_2 \end{pmatrix}\ .$$

If $\xi \neq 0$, then the matrix Δ is invertible as its determinant is equal to $|\xi|^2$. This allows us to use the determinant formula of block matrices to obtain

$$\begin{aligned}\det\big(\sigma(\Lambda)_p(\xi)\big) &= |\xi|^2\det\big(\Theta - \Xi\Delta^{-1}\Gamma\big) \\ &= |\xi|^2\Big(4\xi_1^2\xi_2^2\big(1-f+\frac{3}{2}f'|w|^2\big)^2 + (\xi_1^2-\xi_2^2)^2\big(1-f+\frac{3}{2}f'|w|^2\big)^2\Big)\ .\end{aligned}$$

Since $1-f+\frac{3}{2}f'|w|^2$ is strictly positive (Lemma 3.12), requiring that last expression vanishes is equivalent to the conditions $\xi_1\xi_2 = 0$ and $\xi_1 = \xi_2$, which clearly is possible if and only if $\xi_1 = \xi_2 = 0$. ☐

Theorem H. *Let (J,A) be a point in $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, then*

$$\mathrm{dim}W_{(J,A)} \geqslant 16g - 16 + 2g\ .$$

Proof. Notice that, the space $W_{(J,A)}$ can be seen as the kernel of Λ, namely the matrix of smooth differential operators defined in (3.2.14). Let us consider the deformation tA, for some $t \in [0,1]$, and look at the corresponding smooth 1-parameter family of matrices of differential operators:

$$\begin{aligned}\big\{\Lambda_t\big\}_{t\in[0,1]} : T_{(J,A)}D^3(\mathcal{J}(\Sigma)) &\to \Omega^2_{\mathbb{C}}(\Sigma)\oplus\Omega^2(\Sigma,\mathrm{End}_0(T\Sigma,g_J)) \\ (\dot{J},\dot{A}_0) &\longmapsto (D_t(\dot{J},\dot{A}_0), S_t(\dot{J},\dot{A}_0))\ ,\end{aligned}$$

$$\begin{aligned}D_t(\dot{J},\dot{A}_0) :=& \mathrm{d}\bigg(\mathrm{div}_g\big((f_t-1)\dot{J}\big) + \mathrm{d}\dot{f}_t\circ J - \frac{f_t'}{6}\langle(\nabla_\bullet tA)J, \dot{A}_0\rangle\bigg) \\ &+ i\mathrm{d}\bigg(\mathrm{div}_g\big((f_t-1)\dot{J}\circ J\big) + \mathrm{d}(\dot{f}_0)_t\circ J + \frac{f_t'}{6}\langle\nabla_\bullet tA, \dot{A}_0\rangle\bigg)\ , \\ S_t(\dot{J},\dot{A}_0) :=& \ \mathrm{d}^\nabla\dot{A}_0(\cdot,\cdot) - tJ(\mathrm{div}_g\dot{J}\wedge A)(\cdot,\cdot)\ ,\end{aligned}$$

where $f_t := f\big(t^2||A||_0^2\big)$, $\dot{f}_t = t\frac{f'_t}{4}\langle \dot{A}_0, A\rangle$, and $(\dot{f}_0)_t = -t\frac{f'_t}{4}\langle \dot{A}_0 J, A\rangle$. Observe that the matrix Λ_t is elliptic for any $t \in [0,1]$ (in the sense explained above) as Lemma 3.22 holds for any $(J, A) \in D^3(\mathcal{J}(\Sigma))$. In particular, since Σ is closed the operator matrix Λ_t has a well-defined index for any $t \in [0,1]$ (Lemma A.12). By definition, Λ_0 associates (up to a sign), to each $(\dot{J}, \dot{A}_0)$, the element

$$\big(\mathrm{d}\big(\mathrm{div}_g \dot{J}\big) + i\mathrm{d}\big(\mathrm{div}_g \dot{J} \circ J\big), \mathrm{d}^\nabla \dot{A}_0\big) \ .$$

The homotopy invariance of the Fredholm index (see Theorem A.13) implies the following chain of equalities:

$$\mathrm{ind}\big(\Lambda\big) = \mathrm{ind}\big(\Lambda_1\big) = \mathrm{ind}\big(\Lambda_t\big) = \mathrm{ind}\big(\Lambda_0\big) \ .$$

Since the differential equations obtained from the kernel of the matrix Λ_0 are decoupled in $\dot{J}$ and $\dot{A}_0$, we have the following index decomposition:

$$\begin{aligned}\mathrm{ind}\big(\Lambda_0\big) &= \mathrm{ind}\big(\mathrm{d}(\mathrm{div}_g \cdot) + i\mathrm{d}(\mathrm{div}_g \circ J)\big) + \mathrm{ind}\big(\mathrm{d}^\nabla\big) \\ &= \mathrm{ind}\big(\mathrm{d}(\mathrm{div}_g \cdot) + i\mathrm{d}(\mathrm{div}_g \circ J)\big) + 10g - 10 \ .\end{aligned}$$

where in the last step we used Lemma 3.20. It is well-known (see [Tro12] for example) that the divergence operator $\mathrm{div}_g : T_J\mathcal{J}(\Sigma) \to \Omega^1(\Sigma)$ is surjective and its kernel has real dimension equal to $6g - 6$. In particular, for any $\alpha \in \Omega^1(\Sigma)$ there exists $\dot{J} \in T_J\mathcal{J}(\Sigma)$ such that $\mathrm{div}_g \dot{J} = \alpha$. Any such real 1-form has a decomposition $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with $\overline{\alpha^{0,1}} = \alpha^{1,0}$. Thus,

$$\alpha + i\alpha \circ J = \alpha^{1,0} + \alpha^{0,1} + i\big(i\alpha^{1,0} - i\alpha^{0,1}\big) = 2\alpha^{0,1} \ .$$

According to this last identity and the surjectivity of the divergence operator, it follows that the cokernel of $\mathrm{d}\big((\mathrm{div}_g \cdot) + i(\mathrm{div}_g \circ J)\big)$ is isomorphic to

$$\mathrm{Coker}\big(\partial : \Omega^{0,1}(\Sigma) \longrightarrow \Omega^{1,1}(\Sigma)\big) \cong \Omega^{1,1}(\Sigma) \Big/ \mathrm{Im}(\partial) = H^{1,1}_{\partial}(\Sigma) \cong \mathbb{R}^2 \ ,$$

as there are no $(0,2)$-forms on (Σ, J). In addition, the kernel of $\mathrm{d}\big((\mathrm{div}_g \cdot) + i(\mathrm{div}_g \circ J)\big)$ is given by

$$\{\dot{J} \in T_J\mathcal{J}(\Sigma) \mid \partial\big((\mathrm{div}_g \cdot) + i(\mathrm{div}_g \circ J)\big) = 0\} \cong H^{0,1}_{\partial}(\Sigma) \times \mathrm{Ker}(\mathrm{div}_g \cdot) \cong \mathbb{R}^{6g-6} \times \mathbb{R}^{2g}$$

using again the surjectivity of the divergence operator. Therefore, we have

$$\begin{aligned}\mathrm{ind}(\Lambda_0) &= \mathrm{ind}\big(\mathrm{d}(\mathrm{div}_g \cdot) + i\mathrm{d}(\mathrm{div}_g \circ J)\big) + 10g - 10 \\ &= 6g - 6 + 2g - 2 + 10g - 10 = 16g - 16 + 2g - 2.\end{aligned}$$

To conclude, we notice that all operators D_t take value into the subspace of complex exact 2-forms, hence the dimension of the cokernel of Λ_t is at least equal to the dimension of $H^2_{\mathbb{C}}(\Sigma) \cong \mathbb{R}^2$. Thus

$$\mathrm{dim}W_{(J,A)} = \mathrm{dim}(\mathrm{Ker}(\Lambda_1))$$

$$
\begin{aligned}
&= \mathrm{ind}(\Lambda_1) + \dim(\mathrm{Coker}(\Lambda_1)) \\
&\geqslant \mathrm{ind}(\Lambda_0) + 2 \\
&= 16g - 16 + 2g - 2 + 2 = 16g - 16 + 2g \ .
\end{aligned}
$$

□

3.2.5 The preferred subspace inside the tangent to $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$

In this section we prove Theorem J by using the theory developed so far. In particular, in Lemma 3.23 and Lemma 3.25 we prove the $\mathrm{Symp}(\Sigma, \rho)$ and $\mathbf{I}$ invariance of $W_{(J,A)}$, respectively. Then, we find a formula for the action of the almost-complex structure $\mathbf{I}$ on tangent vectors to the $\mathrm{Symp}(\Sigma, \rho)$-orbit (Lemma 3.26) and we study the operator associated with the first equation in (HS) (Lemma 3.27). Finally, if $\pi : \widetilde{\mathcal{HS}}_0(\Sigma, \rho) \to \widetilde{\mathcal{B}}(\Sigma)$ denotes the quotient projection, where $\widetilde{\mathcal{B}}(\Sigma)$ is the quotient of $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ by $\mathrm{Ham}(\Sigma, \rho)$, the injectivity of the map $\mathrm{d}_{(J,A)}\pi : W_{(J,A)} \to T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma)$ is proven in Lemma 3.29 by using all the previous results. The only part of Theorem J that is left is the inclusion $W_{(J,A)} \subset T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$, as it is necessary to explain first the connection between the system of differential equations (3.2.9) and the theory of symplectic reduction. For this reason its discussion is postponed to Section 3.3.3. The results presented in this section follow closely the ones given for the anti-de Sitter case ([MST21, §4.5]), even though one of the two tensors we work with is of a different type.

Lemma 3.23. *For every symplectomorphism ϕ of (Σ, ρ) and for every $(\dot{J}, \dot{A}) \in W_{(J,A)}$, we have $(\phi^*\dot{J}, \phi^*\dot{A}) \in W_{(\phi^*J, \phi^*A)}$. In other words, the distribution $\{W_{(J,A)}\}_{(J,A)\in\widetilde{\mathcal{HS}}_0(\Sigma,\rho)}$ is invariant under the action of* $\mathrm{Symp}(\Sigma, \rho)$.

Proof. The assumption that ϕ is a symplectomorphism ($\phi^*\rho = \rho$) is crucial in order to prove that g_{ϕ^*J}, the metric associated with the area form ρ and complex structure ϕ^*J, is equal to the pull-back metric $\phi^*g_J = \phi^*\big(\rho(\cdot, J\cdot)\big)$. In other words, we are saying that $\phi : (\Sigma, g_{\phi^*J}) \to (\Sigma, g_J)$, is an isometry. In particular, for any endomorphism of the tangent bundle B we get

$$\phi^*\Big(\mathrm{div}_g B\Big) = \mathrm{div}_{\phi^*g}(\phi^*B) \ .$$

Moreover, the parts involving the scalar product between tangent vectors $\dot{J}, \dot{J}' \in T_J\mathcal{J}(\Sigma)$ and $\dot{A}, \dot{A}'$ are preserved by ϕ (see Section 2.2.1 and Section 3.2.1). As for the rest of the terms in the equations defining $W_{(J,A)}$, we see that they are preserved by ϕ using the naturality of the action and the functoriality of the involved operators, such as the induced connection ∇ and the exterior covariant derivative d^∇ on $\mathrm{End}_0(\Sigma, g)$-valued 1-form. □

Remark 3.24. Notice that the above lemma holds for any symplectomorphism ϕ not necessarily Hamiltonian. This is a stronger result than what we need in Theorem J. In particular, with the same argument it is possible to prove the $\mathrm{Symp}(\Sigma,\rho)$-invariance of $\mathbf{I}$ and $\mathbf{g}_f$.

Lemma 3.25. *For any $(J,A)\in\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, the subspace $W_{(J,A)}$ is preserved by $\mathbf{I}$.*

Proof. Recall that by definition $\mathbf{I}(\dot{J},\dot{A})=(-J\dot{J},-\dot{A}J-A\dot{J})=:(\dot{J}',\dot{A}')$. We only need to show that the pair $(\dot{J}',\dot{A}')$ still satisfies the equations defining $W_{(J,A)}$. In fact,

$$\begin{aligned}(f-1)\mathrm{div}_g(\dot{J}')&=(f-1)\mathrm{div}_g\big(-J\dot{J}\big) && \text{(rel. (3.1.12))}\\ &=(f-1)\mathrm{div}_g\dot{J}\circ J\ .\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{d}\Big(\frac{f'}{4}\langle\dot{A}_0',A\rangle\Big)\circ J=\mathrm{d}\Big(\frac{f'}{4}\langle-\dot{A}_0J,A\rangle\Big)\circ J\ ,\\ &-\frac{f'}{6}\langle(\nabla_\bullet A)J,\dot{A}_0'\rangle=-\frac{f'}{6}\langle(\nabla_\bullet A)J,-\dot{A}_0J\rangle=\frac{f'}{6}\langle\nabla_\bullet A,\dot{A}_0\rangle\ ,\end{aligned}$$

where in the last step we used relation (2.2.5). By using $\mathbf{I}^2=-\mathbb{1}$, it follows that the element $\mathbf{I}(\dot{J},\dot{A})$ satisfies the second equation in (3.2.9) as well. Regarding the last equation, notice that, according to Remark 3.16, it is equivalent to $\mathrm{d}^\nabla\dot{A}_0(\bullet,J\bullet)=\mathrm{div}_g\dot{J}(\bullet)A(J\bullet)-\mathrm{div}_g\dot{J}(J\bullet)A(\bullet)$. Therefore, for any $X\in\Gamma(T\Sigma)$, we get

$$\begin{aligned}\mathrm{d}^\nabla\big(\dot{A}_0'\big)(X,JX)&=-\mathrm{d}^\nabla\big(\dot{A}_0J\big)(X,JX)\\ &=-\big(\mathrm{d}^\nabla\dot{A}_0\big)(X,JX)J && (\nabla_\bullet J=0)\\ &=-(\mathrm{div}_g\dot{J})(X)A(JX)J+(\mathrm{div}_g\dot{J})(JX)A(X)J && (A(J\cdot)=A(\cdot)J)\\ &=(\mathrm{div}_g\dot{J})(X)A(X)+\mathrm{div}_g\dot{J}(JX)A(X)J\ .\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{div}_g\dot{J}')(X)A(JX)-(\mathrm{div}_g\dot{J}')(JX)A(X)&=-(\mathrm{div}_gJ\dot{J})(X)A(JX)+(\mathrm{div}_gJ\dot{J})(JX)A(X)\\ &=(\mathrm{div}_g\dot{J})(JX)A(JX)+(\mathrm{div}_g\dot{J})(X)A(X)\\ &=(\mathrm{div}_g\dot{J})(JX)A(X)J+(\mathrm{div}_g\dot{J})(X)A(X)\ ,\end{aligned}$$

where we used relation (3.1.12) on the first step and $A(J\cdot)=A(\cdot)J$ on the second one. □

Lemma 3.26. *For every symplectic vector field X on (Σ,ρ) and for every $(J,A)\in D^3(\mathcal{J}(\Sigma))$, with $C(\cdot,\cdot,\cdot)=g_J\big(A(\cdot)\cdot,\cdot\big)$ equal to the real part of a holomorphic cubic differential on (Σ,J), we have $\mathbf{I}\big(\mathcal{L}_XJ,g_J^{-1}\mathcal{L}_XC\big)=\big(-\mathcal{L}_{JX}J,-g_J^{-1}\mathcal{L}_{JX}C\big)$.*

Proof. For any vector field V on the surface, let use define the operator $M_V : \Gamma(T\Sigma) \to \Gamma(T\Sigma)$ as $M_V(Y) := \nabla^g_Y V$, where ∇^g is the Levi-Civita connection with respect to $g \equiv g_J = \rho(\cdot, J\cdot)$. Then, for any $Y \in \Gamma(T\Sigma)$, we have

$$\begin{aligned}(\mathcal{L}_V J)Y &= [V, JY] - J([V,Y]) \\ &= \nabla^g_V(JY) - \nabla^g_{JY}V - J(\nabla^g_V Y) + J(\nabla^g_Y V) && (\nabla^g \text{ is torsion-free}) \\ &= J(\nabla^g_V Y) - M_V(JY) - J(\nabla^g_V Y) + JM_V(Y) && (\nabla^g_\bullet J = 0) \\ &= (JM_V - M_V J)(Y) \ .\end{aligned}$$

The above computation implies that

$$\mathcal{L}_V J = JM_V - M_V J \ . \tag{3.2.16}$$

Now since $C(\cdot,\cdot,\cdot)$ is a $(0,3)$-tensor, for any $Y, Z, U \in \Gamma(T\Sigma)$, its Lie derivative can be computed as follows

$$(\mathcal{L}_V C)(Y,Z,U) = V \cdot C(Y,Z,U) - C([V,Y],Z,U) - C(Y,[V,Z],U) - C(Y,Z,[V,U]) \ .$$

Moreover, using the relation

$$V \cdot C(Y,Z,U) = (\nabla^g_V C)(Y,Z,U) + C(\nabla^g_V Y, Z, U) + C(Y, \nabla^g_V Z, U) + C(Y, Z\nabla^g_V U) \ ,$$

we obtain that

$$(\mathcal{L}_V C)(\cdot,\cdot,\cdot) = (\nabla^g_V C)(\cdot,\cdot,\cdot) + C(M_V\cdot,\cdot,\cdot) + C(\cdot,M_V\cdot,\cdot) + C(\cdot,\cdot,M_V\cdot) \ . \tag{3.2.17}$$

In particular, by re-writing the last relation using the associated $(1,2)$-tensor defined as $A = g^{-1}C$ and using the compatibility between ∇^g and the metric g, we get

$$(g^{-1}\mathcal{L}_V C)(\cdot) = (\nabla^g_V A)(\cdot) + A(M_V\cdot) + A(\cdot)M_V + M_V^* A(\cdot) \ , \tag{3.2.18}$$

where M_V^* denotes the g-adjoint operator of M_V. Now let us apply the almost-complex structure $\mathbf{I}$ to the pair $(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C)$ with X a ρ-symplectic vector field. Therefore,

$$\mathbf{I}\big(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C\big) = \big(-J\mathcal{L}_X J, -(g^{-1}\mathcal{L}_X C)(\cdot)J - A(\cdot)\mathcal{L}_X J\big) \ .$$

Since J is ∇^g-parallel then $M_{JX} = JM_X$, so that the first component of $\mathbf{I}\big(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C\big)$ is given by

$$\begin{aligned}-J\mathcal{L}_X J &= -J(JM_X - M_X J) && (\text{rel. } (3.2.16) \text{ for } V = X) \\ &= -(JM_{JX} - M_{JX}J) \\ &= -\mathcal{L}_{JX} J \ . && (\text{rel. } (3.2.16) \text{ for } V = JX)\end{aligned}$$

Regarding the second component of $\mathbf{I}\big(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C\big)$, using relation (3.2.16) and (3.2.18) for $V = X$, we have

$$\begin{aligned}
-(g^{-1}\mathcal{L}_X C)(\cdot)J - A(\cdot)\mathcal{L}_X J &= -(\nabla^g_X A)(\cdot)J - A(M_X\cdot)J - M_X^* A(\cdot)J - A(\cdot)M_X J \\
&\quad - A(\cdot)JM_X + A(\cdot)M_X J \\
&= -(\nabla^g_X A)(\cdot)J - A(M_X\cdot)J - A(\cdot)JM_X + M_X^* JA(\cdot) \ ,
\end{aligned}$$

where in the last equality we used $A(\cdot)J = -JA(\cdot)$. On the other hand, using relation (3.2.18) with $V = JX$, we get

$$\begin{aligned}
-(g^{-1}\mathcal{L}_{JX} C)(\cdot) &= -(\nabla^g_{JX} A)(\cdot) - A(M_{JX}\cdot) - A(\cdot)M_{JX} - M_{JX}^* A(\cdot) \\
&= -(\nabla^g_{JX} A)(\cdot) - A(JM_X\cdot) - A(\cdot)JM_X - (JM_X)^* A(\cdot) \quad (M_{JX} = JM_X) \\
&= -(\nabla^g_X A)(J\cdot) - A(JM_X\cdot) - A(\cdot)JM_X - (JM_X)^* A(\cdot) \quad (\text{Theorem 1.11}) \\
&= -(\nabla^g_X A)(J\cdot) - A(M_X\cdot)J - A(\cdot)JM_X + M_X^* JA(\cdot) \ ,
\end{aligned}$$

where in the last step we used $A(J\cdot) = A(\cdot)J$ and $J^* = -J$. Finally, we conclude by observing that

$$\begin{aligned}
(\nabla^g_X A)(JY)Z &= \nabla^g_X\big(A(JY)Z\big) - A\big(\nabla^g_X(JY)\big)Z - A(JY)\nabla^g_X Z \\
&= \nabla^g_X\big(A(Y)JZ\big) - A\big(J\nabla^g_X Y\big)Z - A(Y)J\nabla^g_X Z \\
&= \nabla^g_X\big(A(Y)JZ\big) - A\big(\nabla^g_X Y\big)JZ - A(Y)\nabla^g_X(JZ) \\
&= (\nabla^g_X A)(Y)JZ, \qquad \forall X, Y, Z \in \Gamma(T\Sigma) \ .
\end{aligned}$$

□

Lemma 3.27. *Let $G : D^3(\mathcal{J}(\Sigma)) \to C^\infty(\Sigma)$ be the operator defined as $G(J, A) := K_h + 1 - ||q||_h^2$, where h is the metric in the conformal class of g_J with conformal factor e^F (see (3.2.5)), and q is a cubic differential whose real part is equal to $C = g_J A$. Suppose that (J, A) satisfies equations* (HS) *and let U be a vector field on Σ. Then,*

$$\mathrm{d}_{(J,A)}G\big(\mathcal{L}_U J, g_J^{-1}\mathcal{L}_U C\big) = -\frac{1}{2}\Delta_h\lambda + (1 + 2||q||_h^2)\lambda \ ,$$

where $\lambda := \mathrm{div}_{g_J}U\left(\frac{3}{2}||q||_{g_J}^2 F'\left(\frac{||q||_{g_J}^2}{2}\right) - 1\right)$. In particular, if the element $\big(\mathcal{L}_U J, g_J^{-1}\mathcal{L}_U C\big)$ belongs to the kernel of $\mathrm{d}_{(J,A)}G$, then U is symplectic.

Proof. Let us denote with $\{\psi_t\}_{t\in[0,1]}$ the flow of U, and let (J, C) be a point in $D^3(\mathcal{J}(\Sigma))$. Consider the path $\{(J_t, C_t)\}_{t\in[0,1]} \subset D^3(\mathcal{J}(\Sigma))$ given by $(J_t, C_t) = (\psi_t^* J, \psi_t^* C)$ so that $(J_0, C_0) = (J, C)$. In particular,

$$\mathcal{L}_U J = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^* J\Big|_{t=0}, \quad g^{-1}\mathcal{L}_U C = g^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^* C\Big|_{t=0}, \quad g \equiv g_J \ .$$

The final goal will be to compute $\frac{\mathrm{d}}{\mathrm{d}t}G(J_t, C_t)|_{t=0}$. Let us first determine the Riemannian metric $g_t := \rho(\cdot, J_t\cdot)$, where $J_t = \mathrm{d}\psi_t^{-1} \circ J \circ \mathrm{d}\psi_t$.

$$\begin{aligned} g_t &= \rho\big(\cdot, (\mathrm{d}\psi_t^{-1} \circ J \circ \mathrm{d}\psi_t)\cdot\big) = \rho\big((\mathrm{d}\psi_t^{-1} \circ \mathrm{d}\psi_t)\cdot, (\mathrm{d}\psi_t^{-1} \circ J \circ \mathrm{d}\psi_t)\cdot\big) \\ &= \big(\det(\mathrm{d}\psi_t^{-1}) \circ \psi_t\big)\rho\big(\mathrm{d}\psi_t\cdot, (J \circ \mathrm{d}\psi_t)\cdot\big) = \big(\det(\mathrm{d}\psi_t^{-1}) \circ \psi_t\big)g\big(\mathrm{d}\psi_t\cdot, \mathrm{d}\psi_t\cdot\big) \\ &= \big(\det(\mathrm{d}\psi_t^{-1}) \circ \psi_t\big)\psi_t^* g \ . \end{aligned}$$

In particular, g_t is conformal to $\psi_t^* g$ with conformal factor given by $u_t := \big(\det(\mathrm{d}\psi_t^{-1}) \circ \psi_t\big)$. Now let $F : [0, +\infty) \to \mathbb{R}$ be the function defined in Lemma 3.11 and consider the conformal change of metric $h = e^F g$, where F is evaluated at $||q||_g^2$ divided by 2. The next step is to determine the Riemannian metric

$$h_t := e^{F\left(\frac{||q_t||_{g_t}^2}{2}\right)} g_t = e^{F\left(\frac{||q_t||_{g_t}^2}{2}\right)} u_t \cdot \psi_t^* g \ ,$$

where q_t is the J_t-holomorphic cubic differential whose real part is equal to C_t. Therefore,

$$\begin{aligned} h_t &= e^{F\left(\frac{||q_t||_{g_t}^2}{2}\right)} u_t \cdot \psi_t^* g = e^{F\left(\frac{||q_t||_{g_t}^2}{2}\right)} u_t \cdot e^{-F\left(\frac{||q||_g^2 \circ \psi_t}{2}\right)} \psi_t^* h \\ &= e^{F\left(\frac{||q_t||_{g_t}^2}{2}\right) - F\left(\frac{||q||_g^2 \circ \psi_t}{2}\right)} u_t \cdot \psi_t^* h = v_t \cdot \psi_t^* h \ . \end{aligned}$$

Again, the metric h_t is conformal to $\psi_t^* h$ with conformal factor v_t (notice that $v_0 = u_0 \equiv 1$). Using the formula of curvature by conformal change of metric, we get

$$\begin{aligned} K_{h_t} &= K_{v_t \psi_t^* h} \\ &= v_t^{-1}\Big(K_{\psi_t^* h} - \frac{1}{2}\Delta_{\psi_t^* h} \ln v_t\Big) \\ &= v_t^{-1}\Big(K_h \circ \psi_t - \frac{1}{2}\big(\Delta_h \ln\big(v_t \circ \psi_t^{-1}\big)\big) \circ \psi_t\Big) \ , \end{aligned}$$

where in the last equality we used the functoriality of the Gaussian curvature and of the Laplacian, namely

$$K_{\psi_t^* h} = \psi_t^*(K_h), \quad \Delta_{\psi_t^* h} \ln v_t = \psi_t^*\Big(\Delta_h \ln\big(v_t \circ \psi_t^{-1}\big)\Big) \ .$$

The last term we need to determine in $G(J_t, C_t)$ is the one involving the norm of the cubic differential q_t.

$$||q_t||_{h_t}^2 = ||q_t||_{v_t \psi_t^* h}^2 = v_t^{-3}||\psi_t^* q||_{\psi_t^* h}^2 = v_t^{-3}||q||_h^2 \circ \psi_t \ . \tag{3.2.19}$$

We can finally deduce an expression for the term

$$K_{h_t} - ||q_t||_{h_t}^2 = v_t^{-1}\Big(K_h \circ \psi_t - \frac{1}{2}\big(\Delta_h \ln\big(v_t \circ \psi_t^{-1}\big)\big) \circ \psi_t - v_t^{-2}||q||_h^2 \circ \psi_t\Big) \ ,$$

and compute the first order variation of operator G along the path $t \mapsto (J_t, C_t)$, obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(K_{h_t} - ||q||^2_{h_t} + 1\big)\Big|_{t=0} &= -\dot{v}\big(K_h - ||q||^2_h\big) + U(K_h) - \frac{1}{2}\Delta_h \dot{v} + 2\dot{v}||q||^2_h - U\big(||q||^2_h\big) \\ &= \dot{v}\big(1 + 2||q||^2_h\big) - \frac{1}{2}\Delta_h \dot{v} \ ,\end{aligned}$$

where in the last line we used that (J, C) satisfies $G(J, C) = 0$. At this point, it only remains to compute $\dot{v}$, i.e. the first order variation of v_t

$$\begin{aligned}\dot{v} = \frac{\mathrm{d}v_t}{\mathrm{d}t}\Big|_{t=0} &= \frac{\mathrm{d}(v_t \circ \psi_t^{-1})}{\mathrm{d}t}\Big|_{t=0} \qquad (v_0 \equiv 1) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} e^{F\left(\frac{||q_t||^2_{g_t} \circ \psi_t^{-1}}{2}\right) - F\left(\frac{||q||^2_g}{2}\right)} u_t \circ \psi_t^{-1}\Big|_{t=0} \\ &= \dot{u} + \frac{1}{2}F'\Big(\frac{||q||^2_g}{2}\Big)\frac{\mathrm{d}\big(||q_t||^2_{g_t} \circ \psi_t^{-1}\big)}{\mathrm{d}t}\Big|_{t=0} \ .\end{aligned}$$

By imitating the steps performed for relation (3.2.19), we deduce that

$$||q_t||^2_{g_t} = u_t^{-3}||q||^2_g \circ \psi_t \ .$$

Since (ψ_t) represents the flow of U, the first order variation of the conformal factor u_t is given by

$$\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\det\big(\mathrm{d}\psi_t^{-1}\big) \circ \psi_t\Big)\Big|_{t=0} = -\mathrm{div}_g U \ .$$

To conclude, we have

$$\begin{aligned}\dot{v} &= \dot{u} - \frac{3}{2}F'\Big(\frac{||q||^2_g}{2}\Big)\dot{u}||q||^2_g \\ &= \mathrm{div}_g U\Big(\frac{3}{2}F'\Big(\frac{||q||^2_g}{2}\Big)||q||^2_g - 1\Big) \ ,\end{aligned}$$

hence the first order variation of operator G along the path $t \mapsto (J_t, C_t)$ is

$$\mathrm{d}_{(J,A)}G\big(\mathcal{L}_U J, g^{-1}\mathcal{L}_U C\big) = -\frac{1}{2}\Delta_h \lambda + (1 + 2||q||^2_h)\lambda, \quad \lambda := \mathrm{div}_g U\Big(\frac{3}{2}F'\Big(\frac{||q||^2_g}{2}\Big)||q||^2_g - 1\Big) \ .$$

Regarding the second part of the statement, observe that the following inequality holds

$$T(\lambda) := -\frac{1}{2}\Delta_h \lambda + (1 + 2||q||^2_h)\lambda \geqslant -\frac{1}{2}\Delta_h \lambda + \lambda =: S(\lambda) \ .$$

Since the linear operator S is known to be self-adjoint and positive, hence injective, over $L^2(\Sigma, \mathrm{d}\mathfrak{a}_h)$, so is the linear operator T. Therefore, if $(\mathcal{L}_U J, g^{-1}\mathcal{L}_U C)$ lies inside the kernel of

$\mathrm{d}_{(J,A)}G$, then the function $\lambda = \mathrm{div}_g U\Big(\frac{3}{2}F'\Big(\frac{||q||_g^2}{2}\Big)||q||_g^2 - 1\Big)$ is send to 0 by the operator T. At this point, we would conclude by saying that the divergence of the vector field U is zero (see (3.1.13)), which is obviously true if $A = 0$, i.e. $q = 0$. As for the first order variation of the operator G at points where $q \neq 0$, using Lemma 3.12 with $t = \frac{||q||_J^2}{2}$, we get that the function $\frac{3}{2}F'\Big(\frac{||q||_g^2}{2}\Big)||q||_g^2 - 1$ is strictly negative. In particular, $\mathrm{div}_g U\Big(\frac{3}{2}F'\Big(\frac{||q||_g^2}{2}\Big)||q||_g^2 - 1\Big)$ is zero if and only if $\mathrm{div}_g U = 0$. □

Remark 3.28. In order to conclude the proof of the Theorem J, one of the remaining results to show is the inclusion of the subspace $W_{(J,A)}$ inside the tangent space to the infinite-dimensional space $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$. In order to show this inclusion, it is necessary to explain how the differential equations defining $W_{(J,A)}$ are related to the process of infinite-dimensional symplectic reduction. In view not to overextending the discussion too much, during the proof of the last lemma that follows, we will use a result presented and proved in Section 3.3.3.

Lemma 3.29. *For every $(J, A) \in \widetilde{\mathcal{HS}}_0(\Sigma, \rho)$, we have*

$$W_{(J,A)} \cap T_{(J,A)}\Big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\Big) = \{0\} \ .$$

In particular, the natural quotient projection $\pi : \widetilde{\mathcal{HS}}_0(\Sigma, \rho) \to \widetilde{\mathcal{B}}(\Sigma)$ induces a linear isomorphism

$$\mathrm{d}_{(J,A)}\pi : W_{(J,A)} \xrightarrow{\cong} T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma) \ .$$

Proof. Let X be a Hamiltonian vector field on Σ with Hamiltonian function H, and suppose that $(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C)$ belongs to $W_{(J,A)}$. Thus, according to Lemma 3.26 and the $\mathbf{I}$-invariance of $W_{(J,A)}$, the same has to hold for $\mathbf{I}(\mathcal{L}_X J, g^{-1}\mathcal{L}_X C) = (-\mathcal{L}_{JX} J, -g^{-1}\mathcal{L}_{JX} C)$. Since $W_{(J,A)}$ is contained in $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ (see Proposition 3.44), the differential of operator G considered in Lemma 3.27 has to send the pair $(-\mathcal{L}_{JX} J, -g^{-1}\mathcal{L}_{JX} C)$ to zero. By the second part of Lemma 3.27, we deduce that JX is ρ-symplectic, namely $\mathrm{d}(\iota_{JX}) = 0$. This implies that the 1-form $-\mathrm{d}H \circ J = -(\iota_X \rho) \circ J = \iota_{JX}\rho$ is closed, and therefore the function H is g-harmonic (since $\mathrm{d}\big(\mathrm{d}H \circ J\big) = -\Delta_g H \rho$). The only harmonic functions on a closed manifold are the constants, hence we deduce that the vector field X is equal to zero, which proves the first part of the statement. Regarding the second one, let $\widetilde{\mathcal{B}}(\Sigma)$ be the quotient of the infinite-dimensional space $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ by the group $\mathrm{Ham}(\Sigma, \rho)$ and consider the quotient projection $\pi : \widetilde{\mathcal{HS}}_0(\Sigma, \rho) \to \widetilde{\mathcal{B}}(\Sigma)$. By definition, the kernel of $\mathrm{d}_{(J,A)}\pi$ coincides with $T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big)$. Hence, by the first part of the statement, the map $\mathrm{d}_{(J,A)}\pi$ is injective. Moreover, since $\dim W_{(J,A)} \geqslant 16g - 16 + 2g$ (Theorem H) and $\dim \widetilde{\mathcal{B}}(\Sigma) = 16g - 16 + 2g$, this is actually an isomorphism. □

Remark 3.30. The above lemma shows a major difference with the $\mathbb{P}\mathrm{SL}(2, \mathbb{R}) \times \mathbb{P}\mathrm{SL}(2, \mathbb{R})$ case ([MST21, Lemma 4.21]), where the authors were able to obtain a similar result for the group of symplectomorphisms of the surface not necessarily Hamiltonian. This forces us to perform an additional (finite-dimensional) quotient to obtain the Hitchin component, and thus produces additional analytical difficulties carried out in Section 3.4.1.

3.2.6 The circle action on $\mathcal{HS}(\Sigma)$

Recall that the space $D^3(\mathcal{J}(\mathbb{R}^2))$ consists of pairs (J, A), where J is an almost-complex structure on $\mathbb{R}^2$ and A is a 1-form with values in the trace-less and g_J^0-symmetric endomorphisms bundle of $\mathbb{R}^2$ such that $A(J\cdot) = A(\cdot)J$ and $A(X)Y = A(Y)X$, $\forall X, Y \in T\mathbb{R}^2$. In particular, there is $\mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$-equivariant isomorphism between $D^3(\mathcal{J}(\mathbb{R}^2))$and the holomorphic vector bundle $Q^3\big(\mathcal{T}(T^2)\big)$ of cubic differentials over Teichmüller space of the torus (Proposition 2.7). In fact, if $(J, A) \in D^3(\mathcal{J}(\mathbb{R}^2))$ then the $(0,3)$-tensor $C(\cdot,\cdot,\cdot) = g_J^0\big(A(\cdot)\cdot,\cdot\big)$ is the real part of a J-holomorphic cubic differential q on (T^2, J). The natural S^1-action on $Q^3\big(\mathcal{T}(T^2)\big)$ given by $(J, q) \mapsto (J, e^{-i\theta}q)$, can be induced on $D^3(\mathcal{J}(\mathbb{R}^2))$ and results in the following formula

$$\begin{aligned}\widehat{\Psi}_\theta \ &: D^3(\mathcal{J}(\mathbb{R}^2)) \longrightarrow D^3(\mathcal{J}(\mathbb{R}^2)) \\ &(J, A) \mapsto (J, \cos\theta A(\cdot) - \sin\theta A(\cdot)J) \ .\end{aligned}$$

It is clear from the definition that $\widehat{\Psi}_\theta$ preserves the 0-section in $D^3(\mathcal{J}(\mathbb{R}^2))$ (seen as a vector bundle over $\mathcal{J}(\mathbb{R}^2) \cong \mathcal{T}(T^2)$), hence it induces an S^1-action on $\mathcal{B}_0(T^2)$ which will still be denoted by $\widehat{\Psi}_\theta$ by abuse of notation. In particular, we proved that $\widehat{\Psi}_\theta$ preserves $\widehat{\boldsymbol{\omega}}_f$ and it acts by isomotries on $\mathcal{B}_0(T^2)$ with respect to $\widehat{\mathbf{g}}_f$. Moreover, such action is Hamiltonian and we computed explicitly the Hamiltonian function (Theorem E).
Moving on to the case of genus $g \geqslant 2$, we still have an S^1-action on $Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$ given by $([J], q) \mapsto ([J], e^{-i\theta}q)$, which can be induced on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component using the parameterization

$$\Phi : \mathrm{Hit}_3(\Sigma) \xrightarrow{\cong} Q^3\big(\mathcal{T}^{\mathfrak{c}}(\Sigma)\big)$$

found by Labourie and Loftin (see Section 1.5). Thanks to Proposition 1.13 and to the construction explained in Section 3.2.2, we know that $\mathrm{Hit}_3(\Sigma)$ is diffeomorphic to the following space

$$\mathcal{HS}_0(\Sigma,\rho) := \left\{ (J, C) \ \middle| \begin{array}{l} J \text{ is an (almost) complex structure on } \Sigma \\ C \text{ is the real part of a } J\text{-cubic differential} \\ \big(h := e^{F\left(\frac{\|q\|^2_{g_J}}{2}\right)} g_J, A := g_J^{-1}C\big) \text{ satisfy (HS)} \end{array} \right\} \Big/ \mathrm{Symp}_0(\Sigma) \ ,$$

where $F : [0, +\infty) \to \mathbb{R}$ is the smooth function defined in Lemma 3.11. In particular, we can then describe the induced S^1-action by the following formula:

$$\begin{aligned}\Psi_\theta \ &: \mathrm{Hit}_3(\Sigma) \longrightarrow \mathrm{Hit}_3(\Sigma) \\ &(J, A) \mapsto (J, \cos\theta A(\cdot) - \sin\theta A(\cdot)J) \ .\end{aligned}$$

Theorem C. *Let ρ be a fixed area form on Σ, then the circle action on $\mathrm{Hit}_3(\Sigma)$ is Hamiltonian with respect to $\boldsymbol{\omega}_f$ and it satisfies:*

$$\Psi_\theta^* \mathbf{g}_f = \mathbf{g}_f, \quad \forall \theta \in \mathbb{R} .$$

The Hamiltonian function is given by:

$$H(J, q) := \frac{2}{3} \int_\Sigma f\Big(\frac{||q||^2_{g_J}}{2}\Big)\rho,$$

where $f : [0, +\infty) \to (-\infty, 0]$ is the smooth function defined by (3.2.6).

The proof of the above theorem is simply an adaptation of the proof made in the torus case. In fact, as already explained in Remark 3.10, identities valid for elements in $D^3(\mathcal{J}(\mathbb{R}^2))$ can be interpreted as point-wise identities for smooth sections in $D^3(\mathcal{J}(\Sigma))$, and, according to the construction explained in Section 3.2.2, the Hitchin component $\mathrm{Hit}_3(\Sigma)$ can be seen a subset of $D^3(\mathcal{J}(\Sigma))$.

3.3 The infinite dimensional symplectic reduction

In this section we present the process that led us to the definition of the pseudo-Kähler structure on the $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$-Hitchin component and the characterization of its tangent space as described in Remark 3.17. The main tool is a general theorem proved by Donaldson, which will be adapted to our case of interest. In particular, it allows us to give an interpretaton of Wang's equation for hyperbolic affine sphere in $\mathbb{R}^3$ (1.4.6) as a moment map for a Hamiltonian action in an infinite-dimensional context.

3.3.1 Donaldson's construction

Since we will be using a lot of notation from Section 3.2.1, let us briefly recall the construction of the infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$. It has been defined as the space of smooth sections of the bundle

$$P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) := P \times D^3(\mathcal{J}(\mathbb{R}^2)) \Big/ \mathrm{SL}(2, \mathbb{R}) \longrightarrow \Sigma ,$$

where $\mathrm{SL}(2, \mathbb{R})$ acts diagonally on two factors. In particular, each element in $D^3(\mathcal{J}(\Sigma))$ can be described as a pair (J, A), with J an almost-complex structure on Σ, and A a 1-form with values in the trace-less and g_J-symmetric endomorphisms of $T\Sigma$ such that $A(J\cdot) = A(\cdot)J$ and $A(X)Y = A(Y)X, \ \forall X, Y \in \Gamma(T\Sigma)$. Moreover, a tangent vector $(\dot{J}, \dot{A})$, where $\dot{A} := g_J^{-1}\dot{C}$, at (J, A) can be considered as the data of:

- a section $\dot{J}$ of $\mathrm{End}(T\Sigma)$ such that $\dot{J}J + J\dot{J} = 0$, namely $\dot{J}$ is a g_J-symmetric and trace-less endomorphism of $T\Sigma$;

- an $\mathrm{End}(T\Sigma, g_J)$-valued 1-form $\dot{A}$ such that

  $$\dot{A} = \dot{A}_0 + \frac{1}{2}\,\mathrm{tr}\Big(JA\dot{J}\Big)\mathbb{1}\ ,$$

  where $\mathbb{1}$ is the 2×2 identity matrix and $\dot{A}_0$ is the trace-less part of $\dot{A}$. In particular, the trace-part $\dot{A}_{\mathrm{tr}}$ of $\dot{A}$ is uniquely determined by $\dot{J}$.

Let us denote with $s = (J, A)$ an element in $D^3(\mathcal{J}(\Sigma))$, and with $\dot{s}$ the corresponding tangent vector. Suppose there is a $\mathrm{SL}(2,\mathbb{R})$-action on $D^3(\mathcal{J}(\mathbb{R}^2))$. Given an $\mathrm{SL}(2,\mathbb{R})$-invariant symplectic form $\widehat{\omega}$ on $D^3(\mathcal{J}(\mathbb{R}^2))$, there is an induced symplectic structure on each vertical subspace of $P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$, denoted with $\widehat{\omega}_{s(x)}$ for $x \in \Sigma$. In particular, given two tangent vectors $\dot{s}, \dot{s}' \in T_s D^3(\mathcal{J}(\Sigma))$, we can define

$$\boldsymbol{\omega}_s(\dot{s}, \dot{s}') := \int_\Sigma \widehat{\omega}_s(\dot{s}, \dot{s}')\rho\ . \tag{3.3.1}$$

This gives rise to a formal symplectic structure on $D^3(\mathcal{J}(\Sigma))$ which is invariant by the action of $\mathrm{Symp}_0(\Sigma, \rho)$. Now, if the $\mathrm{SL}(2,\mathbb{R})$-action on $D^3(\mathcal{J}(\mathbb{R}^2))$ is Hamiltonian with respect to the symplectic form $\widehat{\omega}$, and with moment map $\widehat{\mu} : D^3(\mathcal{J}(\mathbb{R}^2)) \to \mathfrak{sl}(2,\mathbb{R})^*$, given any section $s \in D^3(\mathcal{J}(\Sigma))$, we get an induced section $\widehat{\mu}_s$ of the bundle $\mathrm{End}_0(T\Sigma)^*$. Then, the following result holds

Theorem 3.31 ([Don03, Theorem 9])**.** *Let ρ be an area form on Σ and let ∇ be any torsion-free connection on $T\Sigma$ satisfying $\nabla\rho = 0$. Define the map $\boldsymbol{\mu} : D^3(\mathcal{J}(\Sigma)) \to \Omega^2(\Sigma)$ as follows:*

$$\boldsymbol{\mu}(s) := \widehat{\omega}\big(\nabla_\bullet s, \nabla_\bullet s\big) + \langle \widehat{\mu}_s \mid R^\nabla \rangle - \mathrm{d}\big(c(\nabla_\bullet \widehat{\mu}_s)\big)\ .$$

Then,

(1) $\boldsymbol{\mu}(s)$ is a closed 2-form for any $s \in D^3(\mathcal{J}(\Sigma))$;

(2) $\boldsymbol{\mu}$ is equivariant with respect to the action of $\mathrm{Ham}(\Sigma, \rho)$;

(3) Given a vector field $V \in \mathfrak{H}(\Sigma, \rho)$, and γ_V a primitive of $\iota_V\rho$, the differential of the map

$$\begin{aligned} D^3(\mathcal{J}(\Sigma)) &\longrightarrow \mathbb{R} \\ s &\mapsto \int_\Sigma \gamma_V \cdot \boldsymbol{\mu}(s) \end{aligned}$$

equals

$$\boldsymbol{\omega}_s(\dot{s}, \mathcal{L}_V s) = \int_\Sigma \widehat{\omega}_s(\dot{s}, \mathcal{L}_V s)\rho\ .$$

Before moving on, the meaning of each term in the definition of the moment map must be explained. First notice that $\nabla_\bullet s$ is a section of $T^*\Sigma \otimes s^*\big(T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)\big)$, hence we set $\widehat{\omega}\big(\nabla_\bullet s, \nabla_\bullet s\big)$ to be the 2-form on Σ given by:

$$\widehat{\omega}\big(\nabla_\bullet s, \nabla_\bullet s\big)(U,V) := \widehat{\omega}\big(\nabla_U s, \nabla_V s\big), \text{ for } U, V \in \Gamma(T\Sigma) \ ,$$

where in the RHS of last equality we apply the symplectic form $\widehat{\omega}$ on $T^{\mathrm{vert}}P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ and the wedge product on the 1-form part. Moreover, the covariant derivative $\nabla_\bullet \widehat{\mu}_s$ is a section of $T^*\Sigma \otimes \mathrm{End}_0(T\Sigma)^*$ and we get a 1-form by performing the following contraction:

$$c\big(\nabla_\bullet \widehat{\mu}_s\big)(v) := \sum_{j=1}^{2} \langle \nabla_{e_j} \widehat{\mu}_s \mid (v \otimes e_j^*)_0 \rangle, \text{ for } v \in \Gamma(T\Sigma) \ ,$$

where $\{e_1, e_2\}$ is a local orthonormal frame of $T\Sigma$ and $\{e_1^*, e_2^*\}$ is the associated orthonormal dual frame. Finally, the curvature tensor R^∇ of the torsion-free connection ∇ is defined as:

$$R^\nabla\big(U,V\big)W := \nabla_V \nabla_U W - \nabla_U \nabla_V W - \nabla_{[V,U]} W \ ,$$

for any $U, V, W \in \Gamma(T\Sigma)$. Because of the anti-symmetry in the first two entries of R^∇, it can be considered as a section of $\Omega^2(\Sigma) \otimes \mathrm{End}_0(T\Sigma)$. For this reason we can contract the endomorphism part of R^∇ with $\widehat{\mu}_s$ and obtain the 2-form on Σ denoted with $\langle \widehat{\mu}_s \mid R^\nabla \rangle$. Let us recall the following technical result that will be useful later in the construction of our moment map.

Lemma 3.32 ([Don03, Lemma 13])**.** *There exists a closed 2-form $\widehat{\omega}_{P(D^3(\mathcal{J}(\mathbb{R}^2)))}$ on $P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big)$ such that, for any section $s \in D^3(\mathcal{J}(\Sigma))$, the following holds:*

$$s^* \widehat{\omega}_{P(D^3(\mathcal{J}(\mathbb{R}^2)))} = \widehat{\omega}\big(\nabla_\bullet s, \nabla_\bullet s\big) + \langle \widehat{\mu}_s \mid R^\nabla \rangle \ .$$

In particular, since $D^3(\mathcal{J}(\mathbb{R}^2))$ is contractible, the de-Rham cohomology class of $\boldsymbol{\mu}(s)$ in $H^2_{dR}(\Sigma, \mathbb{R})$ does not depend on the chosen section.

3.3.2 The moment map on $D^3(\mathcal{J}(\Sigma))$

In Section 2.2.1 we introduced an $\mathrm{SL}(2,\mathbb{R})$-action on elements $(J, A) \in D^3(\mathcal{J}(\mathbb{R}^2))$, given by

$$P \cdot (J, A) = (PJP^{-1}, PA(P^{-1}\cdot)P^{-1}) \ ,$$

where $A(P^{-1}\cdot)$ has to be interpreted as the action of $P \in \mathrm{SL}(2,\mathbb{R})$ via pull-back on the 1-form part of A. Moreover, we proved that for any choice of a smooth function $f : [0, +\infty) \to (-\infty, 0]$ such that: $f(0) = 0, f'(t) < 0$ for any $t > 0$ and $\lim_{t \to +\infty} f(t) = -\infty$, the $\mathrm{SL}(2,\mathbb{R})$-action is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ and with moment map

$$\widehat{\mu}_{(J,A)}(X) = \left(1 - f\Big(\frac{||q||_J^2}{2}\Big)\right) \mathrm{tr}(JX) \ .$$

On the infinite-dimensional space $D^3(\mathcal{J}(\Sigma))$ we defined a (formal) family of pseudo-Kähler structures $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$, depending on the choice of a smooth function f as above (Section 3.2.1). We still denote by $\widehat{\boldsymbol{\omega}}_f$ the symplectic form induced on each fibre by an area-preserving isomorphism between $T_x\Sigma$ and $\mathbb{R}^2$, then

$$(\boldsymbol{\omega}_f)_{(J,A)}\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big) = \int_\Sigma \widehat{\boldsymbol{\omega}}_f\big((\dot{J}, \dot{A}), (\dot{J}', \dot{A}')\big)\rho$$

is obtained from relation (3.3.1) by integrating fibre-wise the family of symplectic forms introduced in the torus case. Moreover, according to Donaldson's construction of Section 3.3.1, the group $\mathrm{Symp}_0(\Sigma, \rho)$ acts on $D^3(\mathcal{J}(\Sigma))$ preserving $\boldsymbol{\omega}_f$ and the action of $\mathrm{Ham}(\Sigma, \rho)$ is Hamiltonian.

Theorem 3.33. *The moment map found by Donaldson for the action of* $\mathrm{Ham}(\Sigma, \rho)$ *on* $\big(D^3(\mathcal{J}(\Sigma)), \boldsymbol{\omega}_f\big)$ *can be expressed as:*

$$\boldsymbol{\mu}(J, A) = -\frac{2}{3} f'\Big(\frac{||\tau||^2}{2}\Big)\big(||\bar{\partial}\tau||^2 - ||\partial\tau||^2\big)\rho + 2K_J\bigg(f\Big(\frac{||\tau||_J^2}{2}\Big) - 1\bigg)\rho + 2i\bar{\partial}\partial f\Big(\frac{||\tau||^2}{2}\Big) ,$$

where τ is the complex cubic differential whose real part is equal to $C = g_J A$ and where $\bar{\partial} = \bar{\partial}_J, \partial = \partial_J$.

Proof. We will determine the expression for $\boldsymbol{\mu}$ using Theorem 3.31, hence starting from the explicit description of $\widehat{\mu}$ given in Theorem F. As a torsion-free connection ∇ we can choose the Levi-Civita connection with respect to $g_J = \rho(\cdot, J\cdot)$, which clearly satisfies $\nabla\rho = 0$. Similar computations can be found in [Tra18], where the functions f and F are chosen to have different properties.

The term $\widehat{\omega}(\nabla_\bullet s, \nabla_\bullet s)$:

Since ∇ is the Levi-Civita connection for g_J, we have $\nabla_V J = 0$, and the element $\nabla_V A$ is still an $\mathrm{End}_0(T\Sigma, g_J)$-valued 1-form for any $V \in \Gamma(T\Sigma)$. Now let $\{e_1, e_2\}$ be a local g_J-orthonormal frame of $T\Sigma$ and let $\{e_1^*, e_2^*\}$ be the dual frame. Then, we get

$$(\widehat{\omega}_f)_{(J,A)}\big((0, \nabla_{e_1}A), (0, \nabla_{e_2}A)\big) = -\frac{1}{6} f'\Big(\frac{||\tau||^2}{2}\Big)\langle \nabla_{e_1}A, (\nabla_{e_2}A)J\rangle .$$

According to the above observation, the tensors $\nabla_{e_1}A$ and $\nabla_{e_2}A$ can be written as

$$\nabla_{e_1}A = (A_1)^1 e_1^* + (A_2)^1 e_2^*, \qquad \nabla_{e_2}A = (A_1)^2 e_1^* + (A_2)^2 e_2^* ,$$

where $A_j := A(e_j)$ for $j = 1, 2$ and

$$(A_1)^k := \begin{pmatrix} a_{11}^{k1} & a_{12}^{k1} \\ a_{12}^{k1} & -a_{11}^{k1} \end{pmatrix}, \qquad (A_2)^k := \begin{pmatrix} a_{11}^{k2} & a_{12}^{k2} \\ a_{12}^{k2} & -a_{11}^{k2} \end{pmatrix}, \text{ for } k = 1, 2 .$$

Using the relation $A_2 = A_1 J$ and $\nabla_\bullet J = 0$, we get

$$(A_2)^k := \begin{pmatrix} a^{k1}_{12} & -a^{k1}_{11} \\ -a^{k1}_{11} & -a^{k1}_{12} \end{pmatrix} .$$

Moreover, by (2.2.4), we have

$$\begin{aligned} \langle \nabla_{e_1} A, (\nabla_{e_2} A) J \rangle &= \operatorname{tr}\Big((A_1)^1 (A_1)^2 J + (A_2)^1 (A_2)^2 J \Big) \\ &= 4(a^{11}_{11} a^{21}_{12} - a^{11}_{12} a^{21}_{11}) . \end{aligned}$$

Recalling that $A = g_J^{-1} C$, last formula can be written in terms of $C(\cdot,\cdot,\cdot)$ by using the following relation:

$$a^{kj}_{lm} = g_J\big((\nabla_{e_k} A)(e_j) \cdot e_l, e_m \big) = (\nabla_{e_k} C)(e_j, e_l, e_m) =: (\nabla_{e_k} C)_{jlm} .$$

In the end, we obtain

$$\begin{aligned} (\widehat{\omega}_f)_{(J,A)}\big((0,\nabla_{e_1} A),(0,\nabla_{e_2} A)\big) &= -\frac{2}{3} f'\Big(\frac{||\tau||^2_J}{2}\Big)\Big((\nabla_{e_1} C)_{111} (\nabla_{e_2} C)_{112} - (\nabla_{e_1} C)_{112} (\nabla_{e_2} C)_{111} \Big) \\ &= -\frac{2}{3} f'\Big(\frac{||\tau||^2_J}{2}\Big)\Big((\nabla_{e_1} C)_{222} (\nabla_{e_2} C)_{111} - (\nabla_{e_1} C)_{111} (\nabla_{e_2} C)_{222} \Big) , \end{aligned}$$

where in the last step we used $C(J\cdot, J\cdot, J\cdot) = -C(J\cdot,\cdot,\cdot) = -C(\cdot, J\cdot, \cdot) = -C(\cdot,\cdot,J\cdot)$. The action of the operators ∂ and $\bar{\partial}$ on τ are defined as follows:

$$(\partial\tau)(v,\cdot,\cdot,\cdot) = \frac{1}{2}\Big(\nabla_v \tau - i \nabla_{Jv}\tau\Big), \qquad (\bar{\partial}\tau)(v,\cdot,\cdot,\cdot) = \frac{1}{2}\Big(\nabla_v \tau + i \nabla_{Jv}\tau\Big) .$$

With a fairly long calculation in local coordinates we deduce

$$||(\bar{\partial}\tau)(e_1,\cdot,\cdot,\cdot)||^2 - ||(\partial\tau)(e_1,\cdot,\cdot,\cdot)||^2 = (\nabla_{e_1} C)_{222} (\nabla_{e_2} C)_{111} - (\nabla_{e_1} C)_{111} (\nabla_{e_2} C)_{222} .$$

Finally, we get

$$\widehat{\omega}(\nabla_\bullet (J,A), \nabla_\bullet (J,A)) = (\widehat{\omega}_f)_{(J,A)}\big((0,\nabla_{e_1} A),(0,\nabla_{e_2} A)\big) = -\frac{2}{3} f'\Big(\frac{||\tau||^2_J}{2}\Big)\big(||\bar{\partial}\tau||^2 - ||\partial\tau||^2\big)\rho ,$$

where $||\partial\tau||^2 = ||\partial\tau(v,\cdot,\cdot,\cdot)||^2$ and $||\bar{\partial}\tau||^2 = ||\bar{\partial}\tau(v,\cdot,\cdot,\cdot)||^2$ for some unit vector v (the norm is independent of such vector).

The term $\langle \widehat{\mu}_s \mid R^\nabla \rangle$:

Since ∇ is the Levi-Civita connection for g_J, the tensor R^∇ coincides with the Riemann tensor of g_J. A classical computation using a local orthonormal frame shows that $R^\nabla =$

$K_J\rho J$, where K_J is the Gaussian curvature of g_J. From Theorem F we have $\widehat{\mu}_{(J,A)}(\cdot) = \left(1 - f\left(\frac{||\tau||_J^2}{2}\right)\right)\operatorname{tr}(J\cdot)$, therefore

$$\langle \widehat{\mu}_s \mid R^{\nabla}\rangle = 2K_J\left(f\left(\frac{||\tau||_J^2}{2}\right) - 1\right)\rho \ .$$

The term $\mathrm{d}\big(c(\nabla_\bullet \widehat{\mu}_s)\big)$:

Notice that, for any $B \in \mathfrak{sl}(T\Sigma, \rho) \equiv \operatorname{End}_0(T\Sigma)$ and for any $v \in \Gamma(T\Sigma)$, we have

$$\nabla_v \widehat{\mu}_{(J,A)}\big(B\big) = -\mathrm{d}f\left(\frac{||\tau||_J^2}{2}\right)(v)\operatorname{tr}(JB) \ ,$$

where we used again $\nabla_\bullet J = 0$. If $\{e_1, e_2\}$ denotes a local g_J-orthonormal frame for $T\Sigma$ and $\{e_1^*, e_2^*\}$ denotes its dual frame, we obtain

$$\begin{aligned}
c(\nabla_\bullet \widehat{\mu}_{(J,A)})(v) &= \langle \nabla_{e_1}\widehat{\mu}_{(J,A)} \mid (v\otimes e_1)_0^*\rangle + \langle \nabla_{e_2}\widehat{\mu}_{(J,A)} \mid (v\otimes e_2)_0^*\rangle \\
&= -\mathrm{d}f\left(\frac{||\tau||^2}{2}\right)(e_1)\operatorname{tr}\left(J(v\otimes e_1)_0^*\right) - \mathrm{d}f\left(\frac{||\tau||^2}{2}\right)(e_2)\operatorname{tr}\left(J(v\otimes e_2)_0^*\right) \\
&= -\mathrm{d}f\left(\frac{||\tau||^2}{2}\right)(e_1)e_1^*\big(Jv\big) - \mathrm{d}f\left(\frac{||\tau||^2}{2}\right)(e_2)e_2^*\big(Jv\big) \\
&= -\left(\mathrm{d}\left(f\left(\frac{||\tau||^2}{2}\right)\right)\circ J\right)(v) \ .
\end{aligned}$$

In other words $c(\nabla_\bullet \widehat{\mu}_{(J,A)}) = -\mathrm{d}\left(f\left(\frac{||\tau||^2}{2}\right)\right)\circ J$. It is not difficult to show that, for any $\psi \in C^\infty(\Sigma)$, the following relation holds:

$$\mathrm{d}\big(\mathrm{d}\psi\circ J\big) = -\Delta_{g_J}\psi = -2i\partial_J\bar{\partial}_J\psi = 2i\bar{\partial}_J\partial_J\psi \ .$$

In the end, we get

$$\mathrm{d}\big(c(\nabla_\bullet \widehat{\mu}_{(J,A)})\big) = -\mathrm{d}\left(\mathrm{d}\left(f\left(\frac{||\tau||^2}{2}\right)\right)\circ J\right) = -2i\bar{\partial}\partial f\left(\frac{||\tau||^2}{2}\right) \ .$$

□

Corollary 3.34. *Let ρ be a fixed area form on Σ, and let $c := \frac{2\pi\chi(\Sigma)}{\operatorname{Area}(\Sigma,\rho)}$. Then, the map*

$$\begin{aligned}
\widetilde{\boldsymbol{\mu}} :& D^3(\mathcal{J}(\Sigma)) \longrightarrow B^2(\Sigma) \subset \mathfrak{H}(\Sigma,\rho)^* \\
&(J, A) \longmapsto \boldsymbol{\mu}(J,A) + 2c\rho
\end{aligned}$$

is a moment map for the action of $\operatorname{Ham}(\Sigma,\rho)$ *on* $\big(D^3(\mathcal{J}(\Sigma)), \boldsymbol{\omega}_f\big)$.

Proof. According to Lemma 3.32, the de-Rham cohomology class of the closed 2-form $\boldsymbol{\mu}(J, A)$ does not depend on the choice of the section, and the same is true for its integral over the surface. Hence, if $A = 0$, by Gauss-Bonnet Theorem we get

$$\int_\Sigma \boldsymbol{\mu}(J, 0) = -2\int_\Sigma K_J\rho = -4\pi\chi(\Sigma) \ .$$

In particular, the integral of the 2-form $\boldsymbol{\mu}(J, A) + 2c\rho$ is equal to zero. This implies that $\widetilde{\boldsymbol{\mu}}$ takes values in the space of exact 2-forms $B^2(\Sigma)$, which is contained in $\mathfrak{H}(\Sigma, \rho)^*$ (see Section 3.1.1). Finally, the properties (i) and (ii) in Definition B.3 continue to hold for $\widetilde{\boldsymbol{\mu}}$ since the additional term $2c\rho$ does not depend on the chosen section. □

In the remaining part of this section we show how, if we assume the additional hypothesis $\bar\partial_J\tau = 0$, the moment map $\widetilde{\boldsymbol{\mu}}$ we found is directly related to Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$ (see Section 1.4). The idea of proof of the following result is similar to that used in [Tra18] for a slightly different moment map.

Theorem 3.35. *Let $(J, A) \in D^3(\mathcal{J}(\Sigma))$ and suppose that $A = g_J^{-1}\mathcal{R}e(\tau)$ with $\bar\partial_J\tau = 0$, then*

$$\widetilde{\boldsymbol{\mu}}(J, A) = -2e^{F\left(\frac{||\tau||_J^2}{2}\right)}\Big(K_h - ||\tau||_h^2 + 1\Big)\rho \ , \qquad \textit{where} \ \ h := e^{F\left(\frac{||\tau||_J^2}{2}\right)}g_J \ .$$

Proof. If $A = 0$, the statement is immediate. Suppose that $A \neq 0$ and define λ to be the function $\frac{||\tau||_J^2}{2}$. Then, outside the zeroes of A it is easy to show that:

$$-\frac{i}{\lambda}\bar\partial\lambda \wedge \partial\lambda = ||\partial\tau||_J^2\rho \ , \qquad K_J\rho = -\frac{i}{3}\bar\partial\partial\log(\lambda) \ .$$

Let us assume for a moment that the following identity holds:

$$2i\bar\partial\bigg(\Big(f'(\lambda) - \frac{f(\lambda)}{3\lambda}\Big)\partial\lambda\bigg) = -\frac{2i}{3\lambda}f'(\lambda)\bar\partial\lambda \wedge \partial\lambda + 2f(\lambda)K_J\rho + 2i\bar\partial\partial f(\lambda) \ . \tag{3.3.2}$$

According to Theorem 3.33 we can write $\widetilde{\boldsymbol{\mu}}(J, A)$ as follows:

$$\widetilde{\boldsymbol{\mu}}(J, A) = -\frac{2i}{3\lambda}f'(\lambda)\bar\partial\lambda \wedge \partial\lambda + \frac{2i}{3}\big(1 - f(\lambda)\big)\bar\partial\partial\log(\lambda) + 2i\bar\partial\partial f(\lambda) + 2c\rho$$

In particular, we obtain the following sequence of identities:

$$\begin{aligned}
\widetilde{\boldsymbol{\mu}}(J, A) &= -\frac{2i}{3\lambda}f'(\lambda)\bar\partial\lambda \wedge \partial\lambda + \frac{2i}{3}\big(1 - f(\lambda)\big)\bar\partial\partial\log(\lambda) + 2i\bar\partial\partial f(\lambda) + 2c\rho \\
&= -\frac{2i}{3\lambda}f'(\lambda)\bar\partial\eta \wedge \partial\lambda + \frac{2i}{3}\big(1 - f(\lambda)\big)\big(-\frac{3}{i}K_J\rho\big) + 2i\bar\partial\partial f(\lambda) + 2c\rho \quad &&\text{(rel. (3.3.2))} \\
&= 2i\bar\partial\bigg(\Big(\frac{f(\lambda)}{3\lambda} - f'(\lambda)\Big)\partial\lambda\bigg) - 2(K_J - c)\rho &&\text{(Lemma 3.12)}
\end{aligned}$$

$$
\begin{aligned}
&= -2i\bar{\partial}F'(\lambda) \wedge \partial\lambda - 2(K_J - c)\rho \\
&= -2i\bar{\partial}\partial F(\lambda) - 2(K_J - c)\rho \\
&= -2\Big(K_J - \frac{1}{2}\Delta_{g_J}F(\lambda) - c\Big)\rho \ .
\end{aligned}
$$

Now, if h denotes the Riemannian metric on Σ conformal to g_J with conformal factor equal to $e^{F(\lambda)}$, we get

$$
K_h = e^{-F(\lambda)}\Big(K_J - \frac{1}{2}\Delta_{g_J}F(\lambda)\Big) \ .
$$

On the other hand, using the functional equation (3.2.5) satisfied by F, we have

$$
\begin{aligned}
\widetilde{\boldsymbol{\mu}}(J,A) &= -2\Big(K_J - \frac{1}{2}\Delta_{g_J}F(\lambda) - c\Big)\rho \\
&= -2\Big(e^{F(\lambda)}K_h - c\Big)\rho \\
&= -2e^{F(\lambda)}\big(K_h - ||\tau||_J^2 e^{-3F(\lambda)} + 1\big) \\
&= -2e^{F(\lambda)}\big(K_h - ||\tau||_h^2 + 1\big)\rho \ .
\end{aligned}
$$

In order to finish the proof, it only remains to show that relation (3.3.2) holds, which stems from the following identities:

$$
2fK_J\rho = \frac{2i}{3\lambda}f\Big(\frac{1}{\lambda}\bar{\partial} \wedge \partial\lambda - \bar{\partial}\partial\lambda\Big), \quad 2i\bar{\partial}\partial f = 2i\Big(f''\bar{\partial}\lambda \wedge \partial\lambda + f'\bar{\partial}\partial\lambda\Big) \ .
$$

This ends the proof outside the zeroes of A, which is a finite set in Σ. Thus, the statement follows by continuity of the expression. □

Corollary 3.36. *Let $(J,A) \in D^3(\mathcal{J}(\Sigma))$. Then $\widetilde{\boldsymbol{\mu}}(J,A) = 0$ and $\mathrm{d}^{\nabla}A = 0$ if and only if $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$.*

Proof. Recall from Section 3.2.2 that $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ is the space of pairs (J,A) such that $\big(h = e^F g_J, A\big)$ satisfies (HS) (see also Remark 1.12). By Theorem 1.11 we know that, up to contraction with the metric, A is the real part of a J-holomorphic cubic differential τ. Finally, the above theorem implies that $\widetilde{\boldsymbol{\mu}}(J,A) = 0$ if and only if $K_h - ||\tau||_h^2 = -1$. □

3.3.3 The symplectic quotient

Here we explain how the use of symplectic reduction allows us to determine (in part) the system of differential equations (3.2.9) defining $W_{(J,A)}$, hence those characterizing the tangent space to the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component (see Remark 3.17). Following in parallel the construction done for Teichmüller space in Section 3.1.2, the idea is to induce our symplectic form $\boldsymbol{\omega}_f$ from the ambient space $D^3(\mathcal{J}(\Sigma))$ to the quotient of $\widetilde{\boldsymbol{\mu}}^{-1}(0)$ by the group $\mathrm{Ham}(\Sigma,\rho)$. On the other hand, there are two major differences with the case of

$\mathcal{T}(\Sigma)$: the first is that the infinite-dimensional space $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ is cut by two equations (see Corollary 3.36) and, only one of them, has an interpretation as a moment map. In particular, we have to look at the space $\widetilde{\boldsymbol{\mu}}^{-1}(0) \cap \mathcal{M}_{\mathbb{C}}$ modulo $\mathrm{Ham}(\Sigma,\rho)$; the second is that once we induce the symplectic form on the quotient, the pairing $\boldsymbol{\omega}_f(\mathbf{I}\cdot,\cdot) = \mathbf{g}_f$ gives rise to a pseudo-Riemannian metric, and this generates additional difficulties since one is intent to identify the space $W_{(J,A)}$ with the $\mathbf{g}_f$-orthogonal to the $\mathrm{Ham}(\Sigma,\rho)$-orbit.
Our moment map $\widetilde{\boldsymbol{\mu}}$ has values in the space of exact 2-forms on the surface, which is contained in the dual Lie algebra of the Hamiltonian group (see Corollary 3.34). After recalling two technical lemmas, in Proposition 3.39 we compute a primitive of the differential of the moment map (still with values in the exact 2-forms). This will allow us, in Proposition 3.41, to perform a highly non-trivial integration by parts, which will be useful later in discussing the (possible) presence of degenerate vectors for the pseudo-metric away from the Fuchsian locus. Then, with Proposition 3.44 we prove the inclusion of $W_{(J,A)}$ inside the tangent to $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, the discussion of which had been left hanging by Section 3.2.5. Finally, inspired by the Kähler reduction of Teichmüller space, we are able to characterize $W_{(J,A)}$ as the largest subspace in $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ that is both $\mathbf{g}_f$-orthogonal to the orbit and invariant under the action of the complex structure $\mathbf{I}$.
The statements and the proofs of Proposition 3.39 and Proposition 3.41 are inspired by the analogous counterparts in the anti-de Sitter case ([MST21, Proposition 6.10 and 6.12]). Despite that, the presence of the 1-form part in the tensor A created additional problems during the development of the proofs, which will be highlighted throughout. We first recall two technical lemmas that will be useful further on.

Lemma 3.37 ([MST21, Lemma 4.16])**.** *Let B be a trace-less endomorphism of $T\Sigma$, then*

$$(\nabla_X B)Y - (\nabla_Y B)X = (\mathrm{div}_g B)(Y)X - (\mathrm{div}_g B)(X)Y \ .$$

Lemma 3.38 ([MST21, Lemma 4.15])**.** *Let $\dot{J} \in T_J\mathcal{J}(\Sigma)$ be an infinitesimal variation of a complex structure on Σ. If $\dot{\nabla}$ denotes the first order variation of the Levi-Civita connection of $g_J = \rho(\cdot, J\cdot)$ along $\dot{J}$, then the following holds:*

$$\dot{\nabla}_X Y = -\frac{1}{2}\big((\mathrm{div}\dot{J})(X)JY + J(\nabla_X \dot{J})Y\big) \ , \tag{3.3.3}$$

for every tangent vector fields X, Y on Σ.

Proposition 3.39. *For every $(J, A) \in D^3(\mathcal{J}(\Sigma))$ such that $A = g_J^{-1}\mathcal{R}e(\tau)$ with $\bar{\partial}_J\tau = 0$, and for every tangent vector $(\dot{J}, \dot{A}) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ we have*

$$\mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A}) = \mathrm{d}\Big((f-1)\mathrm{div}_g\dot{J} + \mathrm{d}f \circ \dot{J} + \mathrm{d}\dot{f} \circ J - \frac{f'}{6}\beta\Big) \ , \tag{3.3.4}$$

where $f, f', \dot{f}$ are evaluated at $\frac{||\tau||_J^2}{2}$ and β is the 1-form defined as $\beta(V) := \langle \dot{A}_0, (\nabla_V A)J\rangle$.

Proof. The most intricate part of the proof is encompassed in showing the following identity:

$$\Big(-\frac{2}{3}f'\big(||\bar\partial\tau||^2-||\partial\tau||^2\big)\rho\Big)' = -\mathrm{d}\Big(\frac{f'}{6}\beta\Big) - 2\dot f K_J\rho + \mathrm{d}(f-1)\wedge \mathrm{div}_g\dot J\ , \tag{3.3.5}$$

where the derivative is taken with respect to (J,A) along tangent directions $(\dot J,\dot A)$. Let us assume for a moment that (3.3.5) holds and let us prove the formula stated in the theorem. In fact, the other terms in $\widetilde{\boldsymbol{\mu}}$ (see Theorem 3.33) are easier to handle

$$\begin{aligned}\big(2(f-1)K_J\rho\big)' &= 2\dot f K_J\rho + 2(f-1)\mathrm{d}K_J(\dot J)\rho && \text{(Proposition 3.9)}\\ &= 2\dot f K_J\rho + (f-1)\mathrm{d}\big(\mathrm{div}_g\dot J\big)\ .\end{aligned}$$

Moreover,

$$\big(2i\bar\partial\partial f\big)' = -\big(\Delta_{g_J}f\big)'\rho = \mathrm{d}\big((\mathrm{d}f\circ J)'\big) = \mathrm{d}\big(\mathrm{d}\dot f\circ J + \mathrm{d}f\circ\dot J\big)\ .$$

Combining these formulae, we get the desired expression for the moment map

$$\begin{aligned}\mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot J,\dot A) &= -\mathrm{d}\Big(\frac{f'}{6}\beta\Big) - 2\dot f K_J\rho + \mathrm{d}f\wedge\mathrm{div}_g\dot J + \mathrm{d}\big(\mathrm{d}\dot f\circ J + \mathrm{d}f\circ\dot J\big) + 2\dot f K_J\rho + (f-1)\mathrm{d}\big(\mathrm{div}_g\dot J\big)\\ &= \mathrm{d}\Big((f-1)\mathrm{div}_g\dot J + \mathrm{d}f\circ\dot J + \mathrm{d}\dot f\circ J - \frac{f'}{6}\beta\Big)\ .\end{aligned}$$

Now let us focus on proving relation (3.3.5). As was shown in Theorem 3.33, we know that:

$$-\frac{2}{3}f'\big(||\bar\partial\tau||^2-||\partial\tau||^2\big) = (\widehat\omega_f)_{(J,A)}\big((0,\nabla_{e_1}A),(0,\nabla_{e_2}A)\big)\ , \tag{3.3.6}$$

for any choice of a local frame $\{e_1,e_2\}$ such that $\rho(e_1,e_2)=1$. Therefore, we can compute the following derivative:

$$\begin{aligned}\Big(-\frac{2}{3}f'\big(||\bar\partial\tau||^2-||\partial\tau||^2\big)\Big)' &= \Big((\widehat\omega_f)_{(J,A)}\big((0,\nabla_{e_1}A),(0,\nabla_{e_2}A)\big)\Big)'\\ &= \Big(-\frac{f'}{6}\langle\nabla_{e_1}A,(\nabla_{e_2}A)J\rangle\Big)'\ ,\end{aligned}$$

where in the second step we used that, for any $i=1,2$, the endomorphism part of $\nabla_{e_i}A$ is trace-less and g_J-symmetric. In order to simplify the computation of the derivative, let us make some preliminary observations. Since equation (3.3.6) is true for any unit volume local frame $\{e_1,e_2\}$, we can further assume that it is g_J-orthonormal and does not change as J varies along tangent directions. Moreover, the terms corresponding to variations $\dot J$ make no contributions as $\mathrm{tr}\Big(\nabla_{e_1}A\nabla_{e_2}A\dot J\Big)=0$ (see (2.2.4)). This allows us to reduce the study of the derivative to only the following terms:

$$\begin{aligned}\Big(-\frac{2}{3}f'\big(||\bar\partial\tau||^2-||\partial\tau||^2\big)\Big)' = &-\frac{f''}{24}\langle A,\dot A_0\rangle\langle\nabla_{e_1}A,(\nabla_{e_2}A)J\rangle+\\ &-\frac{f'}{6}\langle(\nabla_{e_1}A)',(\nabla_{e_2}A)J\rangle - \frac{f'}{6}\langle\nabla_{e_1}A,(\nabla_{e_2}A)'J\rangle\ ,\end{aligned} \tag{3.3.7}$$

and we expressed the first order variation of f' as $\frac{f''}{4}\langle A, \dot{A}_0\rangle$ (Lemma 2.23). At this point, using Lemma 3.38, we can obtain an expression for $(\nabla_X A)'$. In fact,

$$(\nabla_X A)' = \dot{\nabla}_X A + \nabla_X A'$$

and we can compute

$$\begin{aligned}
(\dot{\nabla}_X A)(Y)Z &= \dot{\nabla}_X\big(A(Y)Z\big) - A(\dot{\nabla}_X Y)Z - A(Y)\dot{\nabla}_X Z \\
&= \frac{1}{2}\Big(-(\mathrm{div}\dot{J})(X)JA(Y)Z - J(\nabla_X\dot{J})A(Y)Z + (\mathrm{div}\dot{J})(X)A(JY)Z + \\
&\quad + A\big(J(\nabla_X\dot{J})Y\big)Z + (\mathrm{div}\dot{J})(X)A(Y)JZ + A(Y)J(\nabla_X\dot{J})Z\Big) \\
&= \frac{1}{2}\Big(3(\mathrm{div}\dot{J})(X)A(Y)JZ + A\big(J(\nabla_X\dot{J})Y\big)Z + A(Y)J(\nabla_X\dot{J})Z - J(\nabla_X\dot{J})A(Y)Z\Big)
\end{aligned}$$

where we used $A(JY)Z = A(Y)JZ$ and $A(Y)JZ = -JA(Y)Z$. As for the term involving the derivative of A, we first notice that $A' = J\dot{J}A + \dot{A}$, hence

$$\begin{aligned}
(\nabla_X A') &= J\nabla_X\dot{J}A + J\dot{J}\nabla_X A + \nabla_X\dot{A}_0 + \nabla_X\dot{A}_{\mathrm{tr}} \\
&= J\nabla_X\dot{J}A + J\dot{J}\nabla_X A + \nabla_X\dot{A}_0 + \frac{1}{2}\Big(\mathrm{tr}\Big(\nabla_X\dot{J}JA\Big) + \mathrm{tr}\Big(\dot{J}J\nabla_X A\Big)\Big)\ .
\end{aligned}$$

Now, choosing $X = e_1$ and observing that the two trace terms in $(\nabla_{e_1}A')$ and the four elements $AJ\nabla_{e_1}\dot{J}, J\nabla_{e_1}\dot{J}A, J\nabla_{e_1}\dot{J}A, J\dot{J}\nabla_{e_1}A$ are zero once they pair with $(\nabla_{e_2}A)J$ using the scalar product (2.2.4), we get

$$\begin{aligned}
-\frac{f'}{6}\langle(\nabla_{e_1}A)', (\nabla_{e_2}A)J\rangle &= -\frac{f'}{4}(\mathrm{div}\dot{J})(e_1)\langle AJ, (\nabla_{e_2}A)J\rangle - \frac{f'}{6}\langle\nabla_{e_1}\dot{A}_0, (\nabla_{e_2}A)J\rangle \\
&\quad - \frac{f'}{12}\langle A\big(\nabla_{e_1}\dot{J}\,\cdot\,\big)J, \nabla_{e_2}AJ\rangle\ .
\end{aligned}$$

Moreover, since $\langle\nabla_{e_1}A, (\nabla_{e_2}A)'J\rangle = -\langle(\nabla_{e_1}A)J, (\nabla_{e_2}A)'\rangle$, performing a similar computation as above, we obtain

$$\begin{aligned}
\frac{f'}{6}\langle(\nabla_{e_2}A)', (\nabla_{e_1}A)J\rangle &= \frac{f'}{4}(\mathrm{div}\dot{J})(e_2)\langle AJ, (\nabla_{e_1}A)J\rangle + \frac{f'}{6}\langle\nabla_{e_2}\dot{A}_0, (\nabla_{e_1}A)J\rangle \\
&\quad + \frac{f'}{12}\langle A\big(\nabla_{e_2}\dot{J}\,\cdot\,\big)J, \nabla_{e_1}AJ\rangle\ .
\end{aligned}$$

Combining everything together in (3.3.7), we have

$$\begin{aligned}\Big(-\frac{2}{3}f'\big(||\bar\partial\tau||^2-||\partial\tau||^2\big)\Big)' = &-\frac{f''}{24}\langle A,\dot A_0\rangle\langle\nabla_{e_1}A,(\nabla_{e_2}A)J\rangle+\frac{f'}{4}\Big((\mathrm{div}\dot J)(e_2)\langle\nabla_{e_1}A,A\rangle\\ &-(\mathrm{div}\dot J)(e_1)\langle AJ,(\nabla_{e_2}A)J\rangle\Big)-\frac{f'}{6}\Big(\langle\nabla_{e_1}\dot A_0,(\nabla_{e_2}A)J\rangle\\ &+\langle\nabla_{e_1}A,(\nabla_{e_2}\dot A_0)J)\rangle\Big)+\frac{f'}{12}\Big(\langle\nabla_{e_1}A,A\big(\nabla_{e_2}\dot J\,\cdot\,)\rangle\\ &-\langle A\big(\nabla_{e_1}\dot J\,\cdot\,),\nabla_{e_2}A\rangle\Big)\ ,\end{aligned} \tag{3.3.8}$$

where we used, again, the symmetry and the compatibility of the scalar product with J (see (2.2.5)). Regarding the divergence term found in (3.3.8), it can be elaborated as follows:

$$\begin{aligned}\frac{f'}{4}\Big((\mathrm{div}\dot J)(e_2)\langle\nabla_{e_1}A,A\rangle-(\mathrm{div}\dot J)(e_1)\langle A,\nabla_{e_2}A\rangle\Big) &= -\big(\mathrm{div}\dot J\wedge\mathrm{d}f\big)(e_1,e_2)\\ &= \big(\mathrm{d}(f-1)\wedge\mathrm{div}\dot J\big)(e_1,e_2)\ ,\end{aligned}$$

where we used $\mathrm{d}f=\frac{f'}{4}\langle A,\nabla_\bullet A\rangle$. Comparing relation (3.3.5) with (3.3.8), the proof is complete if we show that

$$\begin{aligned}&(\boldsymbol{i})\ \ -\mathrm{d}\Big(\frac{f'}{6}\beta\Big)(e_1,e_2) = -\frac{f''}{24}\langle A,\dot A_0\rangle\langle\nabla_{e_1}A,(\nabla_{e_2}A)J\rangle+2\dot f K_J+\\ &\qquad\qquad -\frac{f'}{6}\Big(\langle\nabla_{e_1}\dot A_0,(\nabla_{e_2}A)J\rangle+\langle\nabla_{e_1}A,(\nabla_{e_2}\dot A_0)J)\rangle\Big)\ ,\\ &(\boldsymbol{ii})\ \ \langle\nabla_{e_1}A,A\big(\nabla_{e_2}\dot J\,\cdot\,)\rangle-\langle A\big(\nabla_{e_1}\dot J\,\cdot\,),\nabla_{e_2}A\rangle=0\ .\end{aligned}$$

Proof of relation ($\boldsymbol{i}$)

First notice that if $A=0$ then the relation is clearly satisfied. Suppose A is not identically zero, then

$$-\mathrm{d}\Big(\frac{f'}{6}\beta\Big)=-\frac{1}{6}\mathrm{d}f'\wedge\beta-\frac{f'}{6}\mathrm{d}\beta=-\frac{f''}{24}\langle A,\nabla_\bullet A\rangle\wedge\beta-\frac{f'}{6}\mathrm{d}\beta\ .$$

Regarding the differential of $\beta(\bullet)=\langle\dot A_0,(\nabla_\bullet A)J\rangle$ we get

$$\begin{aligned}\mathrm{d}\beta(e_1,e_2) &= e_1\cdot\big(\langle\dot A_0,(\nabla_{e_2}A)J\rangle\big)-e_2\cdot\big(\langle\dot A_0,(\nabla_{e_1}A)J\rangle\big)-\langle\dot A_0,(\nabla_{[e_1,e_2]}A)J\rangle\\ &= \langle\nabla_{e_1}\dot A_0,(\nabla_{e_2}A)J\rangle-\langle\nabla_{e_2}\dot A_0,(\nabla_{e_1}A)J\rangle+\langle\dot A_0,\big(\nabla_{e_1}\nabla_{e_2}A-\nabla_{e_2}\nabla_{e_1}A-\nabla_{[e_1,e_2]}A\big)J\rangle\\ &= \langle\nabla_{e_1}\dot A_0,(\nabla_{e_2}A)J\rangle-\langle\nabla_{e_2}\dot A_0,(\nabla_{e_1}A)J\rangle-3K_J\langle A,\dot A_0\rangle\ ,\end{aligned}$$

where the last equality follows from $R^\nabla(e_1,e_2)A=3K_JAJ$ since $R^\nabla(e_1,e_2)=\nabla_{e_1}\nabla_{e_2}-\nabla_{e_2}\nabla_{e_1}-\nabla_{[e_1,e_2]}$. Thus,

$$-\frac{f'}{6}\mathrm{d}\beta(e_1,e_2)=-\frac{f'}{6}\Big(\langle\nabla_{e_1}\dot A_0,(\nabla_{e_2}A)J\rangle-\langle\nabla_{e_2}\dot A_0,(\nabla_{e_1}A)J\rangle\Big)+2\dot f K_J\ . \tag{3.3.9}$$

Concerning the other therm, we need to prove that

$$\big(\langle A, \nabla_{\bullet} A\rangle \wedge \beta\big)(e_1, e_2) = \langle A, \dot{A}_0\rangle\langle \nabla_{e_1} A, (\nabla_{e_2} A)J\rangle \ .$$

Notice that, for any $p \in \Sigma$ outside the zeroes of A, the elements $(A_1)_p := \big(A(e_1)\big)_p$ and $(A_1 J)_p := \big(A(e_1)J\big)$ form a basis for the space of g_J-symmetric and trace-less endomorphisms of $T_p\Sigma$. In particular, using the scalar product $\langle \cdot, \cdot \rangle$ we can write

$$\dot{A}_0 = \frac{1}{||A||^2}\Big(\langle A, \dot{A}_0\rangle A + \langle AJ, \dot{A}_0\rangle AJ\Big), \quad \nabla_{e_1} A = \frac{1}{||A||^2}\Big(\langle A, \nabla_{e_1} A\rangle A + \langle AJ, \nabla_{e_1} A\rangle AJ\Big) \ .$$

Replacing these identities in the previous equation, we obtain

$$\begin{aligned}
\big(\langle A, \nabla_{\bullet} A\rangle \wedge \beta\big)(e_1, e_2) &= \langle A, \nabla_{e_1} A\rangle\langle \dot{A}_0, (\nabla_{e_2} A)J\rangle - \langle A, \nabla_{e_2} A\rangle\langle \dot{A}_0, (\nabla_{e_1} A)J\rangle \\
&= \frac{\langle A, \nabla_{e_1} A\rangle}{||A||^2}\Big(\langle \dot{A}_0, A\rangle\langle A, (\nabla_{e_2} A)J\rangle + \langle \dot{A}_0, AJ\rangle\langle AJ, (\nabla_{e_2} A)J\rangle\Big) \\
&\quad - \frac{\langle A, \nabla_{e_2} A\rangle}{||A||^2}\Big(\langle \dot{A}_0, A\rangle\langle A, (\nabla_{e_1} A)J\rangle + \langle \dot{A}_0, AJ\rangle\langle AJ, (\nabla_{e_1} A)J\rangle\Big) \\
&= \frac{\langle \dot{A}_0, A\rangle}{||A||^2}\Big(\langle A, \nabla_{e_1} A\rangle\langle A, (\nabla_{e_2} A)J\rangle - \langle A, \nabla_{e_2} A\rangle\langle A, (\nabla_{e_1} A)J\rangle\Big) \\
&= \frac{\langle \dot{A}_0, A\rangle}{||A||^2} \cdot \Big\langle \langle A, \nabla_{e_1} A\rangle A + \langle AJ, \nabla_{e_1} A\rangle AJ; (\nabla_{e_2} A)J\Big\rangle \\
&= \langle A, \dot{A}_0\rangle\langle \nabla_{e_1} A, (\nabla_{e_2} A)J\rangle \ .
\end{aligned}$$

Since the relation is true on the complement of a finite set in Σ (the zeroes of A), it extends on the whole surface by continuity of the expression.

Proof of relation (ii)

As explained at the beginning of the section, the presence of the 1-form part in the tensor A generates further difficulties. In fact, one has to deal with terms of the form $A\big(\nabla_{e_i} \dot{J} \cdot\big)$ which do not appear in the anti-de Sitter case. First of all notice that if A is identically zero, then the relation is clearly satisfied. Hence, let us assume that this is not the case. In the following, we will use the notations introduced in the proof of Theorem 3.33. Namely,

$$\nabla_{e_1} A = (A_1)^1 e_1^* + (A_2)^1 e_2^*, \qquad \nabla_{e_2} A = (A_1)^2 e_1^* + (A_2)^2 e_2^* \ ,$$

where $A_j := A(e_j)$ for $j = 1, 2$ and since $A = g_J^{-1} C$, we have

$$A_1 = \begin{pmatrix} C_{111} & C_{112} \\ C_{112} & -C_{111} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} C_{112} & -C_{111} \\ -C_{111} & -C_{112} \end{pmatrix},$$

$$(A_1)^k := \begin{pmatrix} (\nabla_k C)_{111} & (\nabla_k C)_{112} \\ (\nabla_k C)_{112} & -(\nabla_k C)_{111} \end{pmatrix}, \quad (A_2)^k := \begin{pmatrix} (\nabla_k C)_{112} & -(\nabla_k C)_{111} \\ -(\nabla_k C)_{111} & -(\nabla_k C)_{112} \end{pmatrix}, \ k = 1, 2$$

$$(\nabla_k C)_{jlm} := (\nabla_{e_k} C)(e_j, e_l, e_m) = g_J\big((\nabla_{e_k} A)(e_j) \cdot e_l, e_m\big) \ .$$

By assumption, C is the real part of a holomorphic cubic differential and, this is equivalent (see Theorem 1.11), to require that $(\nabla_{JX} A)(\cdot) = (\nabla_X A)(J\cdot)$ for any vector field X on the surface. In particular, we obtain the following additional relations

$$(\nabla_2 C)_{111} = (\nabla_1 C)_{112}, \qquad (\nabla_2 C)_{112} = -(\nabla_1 C)_{111} \ . \tag{3.3.10}$$

The next step is to write explicitly, in a similar way, the tensors

$$A\big(\nabla_{e_1} \dot{J} \cdot\big) = (\widetilde{A}_1)^1 e_1^* + (\widetilde{A}_2)^1 e_2^*, \qquad A\big(\nabla_{e_2} \dot{J} \cdot\big) = (\widetilde{A}_1)^2 e_1^* + (\widetilde{A}_2)^2 e_2^* \ .$$

For any $p \in \Sigma$ outside the zeroes of A, the elements A_1 and $A_2 = A_1 J$ form a basis for the space of g_J-symmetric and trace-less endomorphisms of $T_p\Sigma$. In particular, both $\nabla_{e_1}\dot{J}$ and $\nabla_{e_2}\dot{J}$ can be written in this basis as

$$\begin{aligned}
\nabla_{e_1}\dot{J} &= \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_1}\dot{J}A_1\Big)A_1 + \operatorname{tr}\Big(A_1 J \nabla_{e_1}\dot{J}\Big)A_2\Big) \\
\nabla_{e_2}\dot{J} &= \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_2}\dot{J}A_1\Big)A_1 + \operatorname{tr}\Big(A_1 J \nabla_{e_2}\dot{J}\Big)A_2\Big) \ .
\end{aligned}$$

This new form of the endomorphisms allows us to compute their values on the g_J-orthonormal basis of the tangent to the surface

$$\begin{aligned}
\nabla_{e_1}\dot{J}\cdot e_1 &= \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_1}\dot{J}A_1\Big)C_{111} + \operatorname{tr}\Big(A_1 J \nabla_{e_1}\dot{J}\Big)C_{112}\Big)e_1 \\
&\quad + \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_1}\dot{J}A_1\Big)C_{112} - \operatorname{tr}\Big(A_1 J \nabla_{e_1}\dot{J}\Big)C_{111}\Big)e_2
\end{aligned}$$

$$\begin{aligned}
\nabla_{e_1}\dot{J}\cdot e_2 &= \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_1}\dot{J}A_1\Big)C_{112} - \operatorname{tr}\Big(A_1 J \nabla_{e_1}\dot{J}\Big)C_{111}\Big)e_1 \\
&\quad - \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_1}\dot{J}A_1\Big)C_{111} + \operatorname{tr}\Big(A_1 J \nabla_{e_1}\dot{J}\Big)C_{112}\Big)e_2
\end{aligned}$$

and the same calculation can be done for $\nabla_{e_2}\dot{J}$. In particular, we obtain

$$(\widetilde{A}_1)^k = \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_k}\dot{J}A_1\Big)\big(C_{111}A_1 + C_{112}A_2\big) + \operatorname{tr}\Big(A_1 J \nabla_{e_k}\dot{J}\Big)\big(C_{112}A_1 - C_{111}A_2\big)\Big),$$

$$(\widetilde{A}_2)^k = \frac{1}{\operatorname{tr}\big(A_1^2\big)}\Big(\operatorname{tr}\Big(\nabla_{e_k}\dot{J}A_1\Big)\big(C_{112}A_1 - C_{111}A_2\big) - \operatorname{tr}\Big(A_1 J \nabla_{e_k}\dot{J}\Big)\big(C_{111}A_1 + C_{112}A_2\big)\Big) \ .$$

To conclude, we notice that $\mathrm{tr}\big(A_1^2\big) = 2\big(C_{111}^2 + C_{112}^2\big)$, hence

$$\begin{aligned}\langle \nabla_{e_1} A, A\big(\nabla_{e_2}\dot{J}\,\cdot\,\big)\rangle - \langle A\big(\nabla_{e_1}\dot{J}\,\cdot\,\big), \nabla_{e_2} A\rangle &= \mathrm{tr}\left((A_1)^1(\widetilde{A}_1)^2 + (A_2)^1(\widetilde{A}_2)^2\right) \\ &\quad - \mathrm{tr}\left((A_1)^2(\widetilde{A}_1)^1 - (A_2)^2(\widetilde{A}_2)^1\right) \\ &= 0\ .\end{aligned}$$

Since the relation is true on the complement of a finite set in Σ (the zeroes of A), it extends on the whole surface by continuity of the expression. □

Remark 3.40. In analogy with what happens for the $\mathbb{PSL}(2,\mathbb{R}) \times \mathbb{PSL}(2,\mathbb{R})$ case (see [MST21, Remark 6.11]), we fix a primitive of $\mathrm{d}\widetilde{\boldsymbol{\mu}}$ found in Proposition 3.39 and we consider the linear map $L_{(J,A)} : T_{(J,A)}D^3(\mathcal{J}(\Sigma)) \to \Omega^1(\Sigma)/B^1(\Sigma) \subset \mathfrak{S}(\Sigma,\rho)^*$ which associates to each tangent vector $(\dot{J},\dot{A})$ the above primitive (modulo exact 1-forms). With an abuse of notation we will denote this primitive by $\mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A}) \equiv L_{(J,A)}(\dot{J},\dot{A})$.

Proposition 3.41. *Let $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$, then for every $(\dot{J},\dot{A}) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ and for every symplectic vector field V, we have*

$$\boldsymbol{\omega}_f\big((\mathcal{L}_V J, g_J^{-1}\mathcal{L}_V C); (\dot{J},\dot{A})\big) = -\langle \mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A}) \mid V\rangle_{\mathfrak{S}} \tag{3.3.11}$$

Proof. Before we begin the proof of the formula stated in the proposition, let us make some preliminary remarks. For any vector field X on the surface, let use define the operator $M_X : \Gamma(T\Sigma) \to \Gamma(T\Sigma)$ as $M_X(Y) := \nabla^g_Y X$, where ∇^g is the Levi-Civita connection with respect to $g \equiv g_J = \rho(\cdot, J\cdot)$. The endomorphism M_X can be decomposed as

$$M_X = \frac{\mathrm{tr}(M_X)}{2}\mathbb{1} - \frac{\mathrm{tr}(JM_X)}{2}J + M_X^{\mathrm{s}}\ ,$$

where the first term is the trace part, the second one is the g-skew-symmetric part, and the third one is the g-symmetric and trace-less part. If $X = V$ is a ρ-symplectic vector field, then the trace part of M_V vanishes. Since J is ∇^g-parallel, we have $M_{JV} = JM_V$ and its decomposition is given by

$$M_{JV} = JM_V = \frac{\mathrm{tr}(JM_V)}{2}\mathbb{1} + 0 + JM_V^{\mathrm{s}}\ . \tag{3.3.12}$$

In particular, the g-skew-symmetric part of M_{JV} vanishes and $JM_V^{\mathrm{s}} = M_{JV}^{\mathrm{s}}$. Recall that (see (3.2.18)) we found the following formula for the Lie derivative of C expressed in terms of the tensor A

$$(g^{-1}\mathcal{L}_V C)(\cdot) = (\nabla_V A)(\cdot) + A(M_V\cdot) + A(\cdot)M_V + M_V^* A(\cdot)\ ,$$

which can be re-written using the decomposition of M_V found above

$$(g^{-1}\mathcal{L}_V C)(\cdot) = \underbrace{(\nabla_V A)(\cdot) - \frac{3}{2}\operatorname{tr}(JM_V)AJ + A\big(M_V^{\mathrm{s}}\cdot\big)}_{\text{trace-less}} + \underbrace{AM_V^{\mathrm{s}} + M_V^{\mathrm{s}}A}_{\text{trace part}} \ . \tag{3.3.13}$$

At this point, we can compute the symplectic form

$$\begin{aligned}
(\boldsymbol{\omega}_f)\big((\mathcal{L}_V J, g^{-1}\mathcal{L}_V C),(\dot{J},\dot{A})\big) &= \int_\Sigma \Big((f-1)\langle \mathcal{L}_V J, J\dot{J}\rangle - \frac{f'}{6}\langle (g^{-1}\mathcal{L}_V C)_0, (\dot{A}J + A\dot{J})_0\rangle \\
&\quad + \frac{f'}{12}\langle (g^{-1}\mathcal{L}_V C)_{\mathrm{tr}}, (\dot{A}J + A\dot{J})_{\mathrm{tr}}\rangle\Big)\rho \\
&= \int_\Sigma \Big((f-1)\langle \mathcal{L}_V J, J\dot{J}\rangle - \frac{f'}{6}\langle \nabla_V A - \frac{3}{2}\operatorname{tr}(JM_V)AJ, \dot{A}_0 J\rangle \\
&\quad - \frac{f'}{6}\big(\langle A\big(M_V^{\mathrm{s}}\cdot\big), \dot{A}_0 J\rangle - \frac{1}{2}\langle AM_V^{\mathrm{s}} + M_V^{\mathrm{s}}A, (\dot{A}J + A\dot{J})_{\mathrm{tr}}\rangle\big)\Big)\rho
\end{aligned}$$

In order to simplify the third and fourth term in the integral, we make us of the following identity which will be proven at the end

$$\langle A\big(M_V^{\mathrm{s}}\cdot\big), \dot{A}_0 J\rangle - \frac{1}{2}\langle AM_V^{\mathrm{s}} + M_V^{\mathrm{s}}A, (\dot{A}J + A\dot{J})_{\mathrm{tr}}\rangle = 0 \ . \tag{3.3.14}$$

Regarding the first term in the symplectic form, we use Lemma 3.8 and we obtain

$$\begin{aligned}
\int_\Sigma (f-1)\langle \mathcal{L}_V J, J\dot{J}\rangle\rho &= \int_\Sigma \Big((1-f)(\operatorname{div}_g \dot{J})(V) + (f-1)\operatorname{div}_g(\dot{J}V)\Big)\rho \\
&= \int_\Sigma \Big((1-f)(\operatorname{div}_g \dot{J})(V) - \mathrm{d}f(\dot{J}V) + \operatorname{div}_g\big((f-1)\dot{J}V\big)\Big)\rho \\
&= -\int_\Sigma \Big((f-1)(\operatorname{div}_g \dot{J})(V) + \mathrm{d}f(\dot{J}V)\Big)\rho \ .
\end{aligned}$$

Moving on to the second term in the symplectic form

$$\begin{aligned}
-\int_\Sigma \frac{f'}{6}\langle \nabla_V A - \frac{3}{2}\operatorname{tr}(JM_V)AJ, \dot{A}_0 J\rangle\rho &= -\int_\Sigma \Big(-\frac{f'}{6}\beta(V) - \dot{f}\operatorname{div}_g(JV)\Big)\rho \\
&= -\int_\Sigma \Big(-\frac{f'}{6}\beta(V) + \mathrm{d}\dot{f}(JV) - \operatorname{div}_g(\dot{f}JV)\Big)\rho \\
&= -\int_\Sigma \Big(-\frac{f'}{6}\beta(V) + \mathrm{d}\dot{f}(JV)\Big)\rho \ .
\end{aligned}$$

In the end, combining the above two relations with (3.3.14), we obtain

$$(\boldsymbol{\omega}_f)\big((\mathcal{L}_V J, g^{-1}\mathcal{L}_V C),(\dot{J},\dot{A})\big) = -\int_\Sigma \Big((f-1)(\operatorname{div}_g \dot{J})(V) + \mathrm{d}f(\dot{J}V) - \frac{f'}{6}\beta(V) + \mathrm{d}\dot{f}(JV)\Big)\rho$$

$$
\begin{aligned}
&= \int_\Sigma \iota_V \Big((f-1)\mathrm{div}_g \dot J + \mathrm{d}f \circ \dot J - \frac{f'}{6}\beta + \mathrm{d}\dot f \circ J \Big)\rho \\
&= \int_\Sigma \Big((f-1)\mathrm{div}_g \dot J + \mathrm{d}f \circ \dot J - \frac{f'}{6}\beta + \mathrm{d}\dot f \circ J \Big) \wedge \iota_V \rho \\
&= -\langle \mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot J, \dot A) \mid V \rangle_{\mathfrak{G}} \ ,
\end{aligned}
$$

where in the third step we used equation (3.1.6).

Proof of relation (3.3.14)

Once again, the presence of the 1-form part in A makes the analysis more difficult. In fact, there is an additional term which does not appear in the anti-de Sitter case. If $A = 0$ the identity is clearly satisfied. Suppose $A \neq 0$, then for any $p \in \Sigma$ outside the zeroes of A the elements A_1 and $A_2 = A_1 J$ form a basis for the space of g_J-symmetric and trace-less endomorphisms of $T_p\Sigma$. Let $\{e_1, e_2\}$ be a g_J-orthonormal basis and let $\{e_1^*, e_2^*\}$ be its dual. Following the approach used to prove Proposition 3.39, we have

$$
\begin{aligned}
A\big(M_V^{\mathrm{s}} \cdot\big) = {}& \frac{1}{\mathrm{tr}\big(A_1^2\big)} \Big(\mathrm{tr}(M_V^{\mathrm{s}} A_1)\big(C_{111}A_1 + C_{112}A_2\big) + \mathrm{tr}(A_1 J M_V^{\mathrm{s}})\big(C_{112}A_1 - C_{111}A_2\big)\Big) e_1^* \\
&+ \frac{1}{\mathrm{tr}\big(A_1^2\big)} \Big(\mathrm{tr}(M_V^{\mathrm{s}} A_1)\big(C_{112}A_1 - C_{111}A_2\big) + \mathrm{tr}(A_1 J M_V^{\mathrm{s}})\big(C_{111}A_1 + C_{112}A_2\big)\Big) e_2^* \ ,
\end{aligned}
$$

$$
A M_V^{\mathrm{s}} + M_V^{\mathrm{s}} A = \mathrm{tr}(M_V^{\mathrm{s}} A_1)\mathbb{1} e_1^* + \mathrm{tr}(A_1 J M_V^{\mathrm{s}})\mathbb{1} e_2^*,
$$

$$
(\dot A J + A \dot J)_{\mathrm{tr}} = \frac{1}{2}\,\mathrm{tr}\Big(\dot A_2\Big)\mathbb{1} e_1^* - \frac{1}{2}\,\mathrm{tr}\Big(\dot A_1\Big)\mathbb{1} e_2^*, \qquad \dot A_0 J = (\dot A_1)_0 J e_1^* + (\dot A_2)_0 J e_2^*
$$

In particular, we can write the two terms in (3.3.14) as follows:

$$
-\frac{1}{2}\langle A M_V^{\mathrm{s}} + M_V^{\mathrm{s}} A, (\dot A J + A\dot J)_{\mathrm{tr}} \rangle = -\frac{1}{2}\Big(\mathrm{tr}\Big(\dot A_2\Big)\,\mathrm{tr}(A_1 M_V^{\mathrm{s}}) - \mathrm{tr}\Big(\dot A_1\Big)\,\mathrm{tr}(A_1 M_{JV}^{\mathrm{s}})\Big),
$$

$$
\begin{aligned}
\langle A\big(M_V^{\mathrm{s}} \cdot\big), \dot A_0 J \rangle = {}& \frac{\mathrm{tr}(A_1 M_V^{\mathrm{s}})}{\mathrm{tr}\big(A_1^2\big)} \Big(C_{111}\,\mathrm{tr}\Big(A_1(\dot A_1)_0 J\Big) + C_{112}\,\mathrm{tr}\Big(A_2(\dot A_1)_0 J\Big)\Big) \\
&+ \frac{\mathrm{tr}(A_1 M_V^{\mathrm{s}})}{\mathrm{tr}\big(A_1^2\big)} \Big(C_{112}\,\mathrm{tr}\Big(A_1(\dot A_2)_0 J\Big) - C_{111}\,\mathrm{tr}\Big(A_2(\dot A_2)_0 J\Big)\Big) \\
&+ \frac{\mathrm{tr}(A_1 M_{JV}^{\mathrm{s}})}{\mathrm{tr}\big(A_1^2\big)} \Big(C_{112}\,\mathrm{tr}\Big(A_1(\dot A_1)_0 J\Big) - C_{111}\,\mathrm{tr}\Big(A_2(\dot A_1)_0 J\Big)\Big) \\
&- \frac{\mathrm{tr}(A_1 M_{JV}^{\mathrm{s}})}{\mathrm{tr}\big(A_1^2\big)} \Big(C_{111}\,\mathrm{tr}\Big(A_1(\dot A_2)_0 J\Big) + C_{112}\,\mathrm{tr}\Big(A_2(\dot A_2)_0 J\Big)\Big) \ .
\end{aligned}
$$

Finally, writing $\dot{A}$ in term of the variations of the tensor C, namely

$$\dot{A}_1 = \begin{pmatrix} \dot{C}_{111} & \dot{C}_{112} \\ \dot{C}_{112} & \dot{C}_{122} \end{pmatrix}, \qquad \dot{A}_2 = \begin{pmatrix} \dot{C}_{112} & \dot{C}_{122} \\ \dot{C}_{122} & \dot{C}_{222} \end{pmatrix}$$

and using that $\operatorname{tr}\Big(A_i(\dot{A}_k)_0 J\Big) = \operatorname{tr}\Big(A_i\dot{A}_k J\Big)$ for any $i, j = 1, 2$, a direct computation shows the desired equality. Since (3.3.14) holds on the complement of a finite set in Σ, it holds everywhere by continuity of the expression. □

Remark 3.42. It is crucial to emphasize the importance of the result just proved. From the general theory of moment maps (see Definition B.3) we know that (3.3.11) follows from Corollary 3.34 if $\mathrm{d}\tilde{\boldsymbol{\mu}}$ is paired with Hamiltonian vector fields. The point is that $\tilde{\boldsymbol{\mu}}$ can not be promoted to a moment map for the action of $\mathrm{Symp}(\Sigma, \rho)$, which still preserves $\boldsymbol{\omega}_f$. In particular, the formula showed above is far from being obvious when computed for a symplectic vector field, which decomposes as the sum of a harmonic and a Hamiltonian vector field (see (3.1.2)).

Lemma 3.43. *Let $(J, A) \in D^3(\mathcal{J}(\Sigma))$, then the kernel of the linearized Codazzi-like equation $\mathrm{d}^\nabla A = 0$ is given by*

$$\left\{(\dot{J}, \dot{A}) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma)) \mid \mathrm{d}^\nabla \dot{A}_0(\bullet, \bullet) - J(\operatorname{div}_g \dot{J} \wedge A)(\bullet, \bullet) = 0\right\} .$$

Proof. Recall that, for any vector fields $X, Y, Z \in \Gamma(T\Sigma)$, we have

$$(\mathrm{d}^\nabla A)(X, Y)Z = (\nabla_X A)(Y)Z - (\nabla_Y A)(X)Z . \tag{3.3.15}$$

Therefore, we need to compute the derivative of (3.3.15) with respect to variations of (J, A). For instance,

$$\Big((\nabla_X A)(Y)Z - (\nabla_Y A)(X)Z\Big)' = (\dot{\nabla}_X A)(Y)Z - (\dot{\nabla}_Y A)(X)Z + (\mathrm{d}^\nabla A')(X, Y)Z ,$$

where $A' = J\dot{J}A + \dot{A}$. The part involving the variation of the connection has already been computed in the proof of Proposition 3.39

$$(\dot{\nabla}_X A)(Y)Z = \frac{1}{2}\Big(3(\operatorname{div}\dot{J})(X)A(Y)JZ + A\big(J(\nabla_X\dot{J})Y\big)Z + A(Y)J(\nabla_X\dot{J})Z - J(\nabla_X\dot{J})A(Y)Z\Big).$$

Subtracting the term $(\dot{\nabla}_Y A)(X)Z$ from the last expression and using Lemma 3.37 on $A\big((\nabla_X\dot{J})Y - (\nabla_Y\dot{J})X\big)$, we get

$$\begin{aligned}(\dot{\nabla}_X A)(Y)Z - (\dot{\nabla}_Y A)(X)Z = {} & J\Big((\operatorname{div}\dot{J})(Y)A(X) - (\operatorname{div}\dot{J})(X)A(Y)\Big)Z + \frac{1}{2}JA(X)(\nabla_Y\dot{J})Z \\ & + \frac{1}{2}J\Big((\nabla_Y\dot{J})A(X) - A(Y)(\nabla_X\dot{J}) - (\nabla_X\dot{J})A(Y)\Big)Z\end{aligned}$$

$$= -J(\mathrm{div}\dot{J} \wedge A)(X,Y)Z + \frac{1}{2}JA(X)(\nabla_Y \dot{J})Z$$
$$+ \frac{1}{2}J\Big((\nabla_Y \dot{J})A(X) - A(Y)(\nabla_X \dot{J}) - (\nabla_X \dot{J})A(Y)\Big)Z \ .$$

Regarding the term with the exterior covariant derivative of A', we have

$$(\mathrm{d}^\nabla A')(X,Y)Z = \underbrace{(\mathrm{d}^\nabla (J\dot{J}A))(X,Y)Z}_{\text{term } (a)} + \underbrace{(\mathrm{d}^\nabla \dot{A}_{\mathrm{tr}})(X,Y)Z}_{\text{term } (b)} + (\mathrm{d}^\nabla \dot{A}_0)(X,Y)Z \ . \tag{3.3.16}$$

The term (a) is easy to handle since $\nabla_\bullet J = 0$ and $\mathrm{d}^\nabla A = 0$,

$$(\mathrm{d}^\nabla (J\dot{J}A))(X,Y)Z = \nabla_X(J\dot{J}A)(Y)Z - \nabla_Y(J\dot{J}A)(X)Z$$
$$= J\Big((\nabla_X \dot{J})A(Y) - (\nabla_Y \dot{J})A(X)\Big)Z \ .$$

As for the term (b), recall that $\dot{A}_{\mathrm{tr}} = \frac{1}{2}\,\mathrm{tr}\Big(\dot{J}JA\Big)\mathbb{1}$, hence

$$(\mathrm{d}^\nabla \dot{A}_{\mathrm{tr}})(X,Y)Z = (\nabla_X \dot{A}_{\mathrm{tr}})(Y)Z - (\nabla_Y \dot{A}_{\mathrm{tr}})(X)Z$$
$$= \frac{1}{2}\,\mathrm{tr}\Big(\nabla_X\big(\dot{J}JA(Y)\big)\Big)Z - \frac{1}{2}\,\mathrm{tr}\Big(\nabla_Y\big(\dot{J}JA(X)\big)\Big)Z$$
$$= \frac{1}{2}\,\mathrm{tr}\Big((\nabla_X \dot{J})JA(Y) - (\nabla_Y \dot{J})JA(X)\Big)Z \ .$$

We conclude if we show that

$$\frac{1}{2}\,\mathrm{tr}\Big((\nabla_X \dot{J})JA(Y) - (\nabla_Y \dot{J})JA(X)\Big)Z = -J(\nabla_X \dot{J})A(Y)Z + J(\nabla_Y \dot{J})A(X)Z$$
$$- \frac{1}{2}JA(X)(\nabla_Y \dot{J})Z - \frac{1}{2}J(\nabla_Y \dot{J})A(X)Z$$
$$+ \frac{1}{2}JA(Y)(\nabla_X \dot{J})Z + \frac{1}{2}J(\nabla_X \dot{J})A(Y)Z \ ,$$

which follows from the fact that the elements $JA(X)\nabla_Y \dot{J} - J(\nabla_Y \dot{J})A(X)$ and $J(\nabla_X \dot{J})A(Y) - JA(Y)\nabla_X \dot{J}$ are both trace-term, and they can be written as

$$JA(X)\nabla_Y \dot{J} - J(\nabla_Y \dot{J})A(X) = -\,\mathrm{tr}\Big(J(\nabla_Y \dot{J})A(X)\Big)\mathbb{1},$$

$$J(\nabla_X \dot{J})A(Y) - JA(Y)\nabla_X \dot{J} = -\,\mathrm{tr}\Big(JA(Y)\nabla_X \dot{J}\Big)\mathbb{1} \ .$$

□

Proposition 3.44. *Let* $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ *and consider the space* $W_{(J,A)}$ *defined by the system of equations (3.2.9). Then,*

$$W_{(J,A)} \subset T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)\ .$$

Proof. According to Corollary 3.36, the infinite-dimensional space $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ can be seen as the intersection of $\tilde{\boldsymbol{\mu}}^{-1}(0)$ with $\mathcal{M}_{\mathrm{C}} := \{(J,A) \in D^3(\mathcal{J}(\Sigma)) \mid \mathrm{d}^\nabla A = 0\}$. In particular, the tangent space to the pre-image of the zero locus of the moment map is identified with $\operatorname{Ker}\big(\mathrm{d}\tilde{\boldsymbol{\mu}}\big)$. On the other hand, Proposition 3.39 and Lemma 3.43 together implies that

$$(\dot J,\dot A) \in T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho) \iff \begin{cases} \mathrm{d}\Big((f-1)\mathrm{div}_g\dot J + \mathrm{d}f\circ\dot J + \mathrm{d}\dot f\circ J - \frac{f'}{6}\langle \dot A_0, (\nabla_\bullet A)J\rangle\Big) = 0 \\ \mathrm{d}^\nabla\dot A_0(\bullet,\bullet) - J(\mathrm{div}_g\dot J\wedge A)(\bullet,\bullet) = 0 \end{cases}$$

Looking again at the equations (3.2.9) defining the space $W_{(J,A)}$, it is clear that

$$W_{(J,A)} \subset T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)\ .$$

□

At this point, it must be noted that the subspace we are interested in can be described as

$$W_{(J,A)} = \left\{(\dot J,\dot A) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma)) \;\middle|\; \begin{array}{l} (\dot J,\dot A), \mathbf{I}(\dot J,\dot A) \in \operatorname{Ker}(\mathrm{d}\tilde{\boldsymbol{\mu}}) \\ \mathrm{d}^\nabla\dot A_0(\bullet,\bullet) - J(\mathrm{div}_g\dot J\wedge A)(\bullet,\bullet) = 0 \end{array}\right\} \tag{3.3.17}$$

which clarifies the connection of the first two equations in (3.2.9) with symplectic reduction theory.

Theorem G. *For any* $(J,A) \in \widetilde{\mathcal{HS}}_0(\Sigma,\rho)$*, the vector space* $W_{(J,A)}$ *is the largest subspace in* $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ *that is:*

- *invariant under the complex structure* $\mathbf{I}$*;*
- $\mathbf{g}_f$*-orthogonal to the orbit* $T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big)$.

Proof. Recall from Corollary 3.36 that the space $\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ can be identified with $\tilde{\boldsymbol{\mu}}^{-1}(0) \cap \mathcal{M}_{\mathrm{C}}$, where $\mathcal{M}_{\mathrm{C}} := \{(J,A) \in D^3(\mathcal{J}(\Sigma)) \mid \mathrm{d}^\nabla A = 0\}$. Let us denote with $\widetilde{W}$ the largest subspace in $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma,\rho)$ that is $\mathbf{g}_f$-orthogonal to $T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big)$ and $\mathbf{I}$-invariant. Suppose that $(\dot J,\dot A) \in \widetilde{W}$, hence the same is true for $\mathbf{I}(\dot J,\dot A)$ by $\mathbf{I}$-invariance. In particular, both $(\dot J,\dot A)$ and $\mathbf{I}(\dot J,\dot A)$ lie in $\operatorname{Ker}(\mathrm{d}\tilde{\boldsymbol{\mu}})$. We now note that $(\dot J,\dot A)$ is $\mathbf{g}_f$-orthogonal to the $\mathrm{Ham}(\Sigma,\rho)$-orbit if and only if $\mathbf{I}(\dot J,\dot A)$ lies in $\operatorname{Ker}(\mathrm{d}\tilde{\boldsymbol{\mu}})$ thanks to the following computation

$$\mathbf{g}_f\big((\mathcal{L}_V J, g_J^{-1}\mathcal{L}_V C),(\dot J,\dot A)\big) = -\mathbf{g}_f\big((\mathcal{L}_V J, g_J^{-1}\mathcal{L}_V C),\mathbf{I}^2(\dot J,\dot A)\big)$$

$$
\begin{aligned}
&= -\boldsymbol{\omega}_f\big((\mathcal{L}_V J, g_J^{-1}\mathcal{L}_V C), \mathbf{I}(\dot{J}, \dot{A})\big) \\
&= \langle \mathrm{d}\widetilde{\boldsymbol{\mu}}\big(\mathbf{I}(\dot{J}, \dot{A})\big) \mid V \rangle_{\mathfrak{H}}, \quad \forall\, V \in \mathfrak{H}(\Sigma, \rho)\ .
\end{aligned}
$$

by definition of moment map. This implies that an element $(\dot{J}, \dot{A})$ belongs to $\widetilde{W}$ if and only if

$$
\begin{cases}
\mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J}, \dot{A}) = 0 \\
\mathrm{d}\widetilde{\boldsymbol{\mu}}\big(\mathbf{I}(\dot{J}, \dot{A})\big) = 0 \\
\mathrm{d}^{\nabla}\dot{A}_0(\bullet, \bullet) - J(\mathrm{div}_g \dot{J} \wedge A)(\bullet, \bullet) = 0\ ,
\end{cases}
$$

which is equivalent to the system of partial differential equations (3.2.9) defining the subspace $W_{(J,A)}$ (see Proposition 3.39). □

3.4 The pseudo-metric is non-degenerate

Here we discuss the possible presence of degenerate vectors for $\mathbf{g}_f$ away from the Fuchsian locus and we present the results obtained suggesting the non-degeneracy of the pseudo-metric over the entire $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$-Hitchin component.

3.4.1 The finite-dimensional quotient

Although the main part of the results have been shown, it still remains to prove Theorem K, namely the identification of $\mathrm{Hit}_3(\Sigma)$ with the finite dimensional quotient $\widetilde{\mathcal{B}}(\Sigma)/H$, where $\widetilde{\mathcal{B}}(\Sigma)$ is the smooth manifold of real dimension $16g - 16 + 2g$ isomorphic to the quotient of the space $\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ by the group $\mathrm{Ham}(\Sigma, \rho)$ (see Theorem J), and $H := \mathrm{Symp}_0(\Sigma, \rho)/\mathrm{Ham}(\Sigma, \rho)$ is isomorphic to $H^1_{\mathrm{dR}}(\Sigma, \mathbb{R})$ (see Lemma 3.2). The tangent space $T_{[J,A]}\widetilde{\mathcal{B}}(\Sigma)$ is identified with the vector space $W_{(J,A)}$ which is defined as the space of solutions to the following system of differential equations

$$
\begin{cases}
\mathrm{d}\Big(\mathrm{div}\,\big((f-1)\dot{J}\big) + \mathrm{d}\dot{f} \circ J - \frac{f'}{6}\beta\Big) = 0 \\
\mathrm{d}\Big(\mathrm{div}\,\big((f-1)\dot{J}\big) \circ J + \mathrm{d}\dot{f}_0 \circ J - \frac{f'}{6}\beta \circ J\Big) = 0 \\
\mathrm{d}^{\nabla}\dot{A}_0(\bullet, \bullet) - J(\mathrm{div}\,\dot{J} \wedge A)(\bullet, \bullet) = 0
\end{cases}
$$

Let us denote with α_1 and α_2 the 1-forms in the above system whose differential is zero and let us introduce the vector space

$$
V_{(J,A)} := \left\{ (\dot{J}, \dot{A}) \in T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho) \;\middle|\; \begin{array}{l} \alpha_1 + i\alpha_2 \text{ is exact} \\ \mathrm{d}^{\nabla}\dot{A}_0(\bullet, \bullet) - J(\mathrm{div}\,\dot{J} \wedge A)(\bullet, \bullet) = 0 \end{array} \right\} \tag{3.4.1}
$$

It is not difficult to see, following the lines of the proof of Lemma 3.23 and Lemma 3.25, that $V_{(J,A)}$ is invariant under the action of $\mathrm{Symp}(\Sigma, \rho)$ and the complex structure $\mathbf{I}$. In what follows, although we will use the term "symplectic form" to denote $\boldsymbol{\omega}_f$, we do not yet know whether on the spaces we are considering $\boldsymbol{\omega}_f$ is actually non-degenerate. In any case, with abuse of terminology, the results we are about to present still apply.

Proposition 3.45. *There is a $\boldsymbol{\omega}_f$-orthogonal decomposition*

$$W_{(J,A)} = V_{(J,A)} \overset{\perp_{\boldsymbol{\omega}_f}}{\oplus} S_{(J,A)} \ ,$$

where $S_{(J,A)} := \{\big(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C\big) \mid X \in \Gamma(T\Sigma),\ \mathrm{d}(\iota_X\rho) = \mathrm{d}(\iota_{JX}\rho) = 0\} \cong T_{(J,A)}\big(H\cdot(J,A)\big)$ *is the tangent space to the harmonic orbit.*

Proof. Recall that, according to (3.3.11), for any symplectic vector field X on the surface and for any $(\dot{J},\dot{A}) \in T_{(J,A)}D^3(\mathcal{J}(\Sigma))$, we have

$$\boldsymbol{\omega}_f\big((\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C);(\dot{J},\dot{A})\big) = -\langle \mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A}) \mid X\rangle_{\mathfrak{S}} \ ,$$

where $\mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A})$ denotes the primitive found in Proposition 3.39 (see also Remark 3.40). In particular, if $(\dot{J},\dot{A}) \in V_{(J,A)}$ such a primitive equals the 1-form α_1 considered in (3.4.1), hence it is exact. Using the non-degenerate symplectic pairing (3.1.5), we get

$$\boldsymbol{\omega}_f\big((\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C);(\dot{J},\dot{A})\big) = -\langle \mathrm{d}\widetilde{\boldsymbol{\mu}}(\dot{J},\dot{A}), X\rangle_{\mathfrak{S}} = 0 \ ,$$

for any symplectic vector field X and for any $(\dot{J},\dot{A}) \in V_{(J,A)}$. In other words, $V_{(J,A)}$ is $\boldsymbol{\omega}_f$-orthogonal to the symplectic orbit and it coincides with the $\boldsymbol{\omega}_f$-orthogonal to $S_{(J,A)}$ inside $W_{(J,A)}$. For this reason, we can conclude if we show that

$$V_{(J,A)} \cap S_{(J,A)} = \{0\} \ .$$

Suppose there exists a harmonic vector field X such that $(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C) \in V_{(J,A)}$. By definition of $V_{(J,A)}$, the 1-form

$$\widetilde{\alpha}_1 := \mathrm{div}\Big((f-1)\mathcal{L}_X J\Big) + \mathrm{d}\dot{f}\circ J - \frac{f'}{6}\langle \big(g_J^{-1}\mathcal{L}_X C\big)_0, (\nabla_{\bullet}A)J\rangle$$

is exact. Therefore,

$$\begin{aligned}\int_\Sigma \widetilde{\alpha}_1 \wedge \iota_U\rho &= -\langle \mathrm{d}\widetilde{\boldsymbol{\mu}}\big(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C\big), U\rangle_{\mathfrak{S}} && \text{(rel. (3.1.6))}\\ &= 0, \quad \forall U \in \mathfrak{S}(\Sigma,\rho) \ .\end{aligned}$$

Since X is harmonic, we can choose $U = JX$ and obtain

$$\begin{aligned}0 &= \int_\Sigma \Big(\mathrm{div}\Big((f-1)\mathcal{L}_X J\Big) + \mathrm{d}\dot{f}\circ J - \frac{f'}{6}\beta\Big) \wedge \iota_{JX}\rho && \text{(rel. (3.1.6))}\\ &= \int_\Sigma \Big(\mathrm{div}\Big((f-1)\mathcal{L}_X J\Big) + \mathrm{d}\dot{f}\circ J - \frac{f'}{6}\beta\Big)(JX)\rho \\ &= \int_\Sigma \Big(\mathrm{div}\Big((f-1)\mathcal{L}_X J\Big) - \frac{f'}{6}\beta\Big)(JX)\rho - \int_\Sigma (\mathrm{d}\dot{f})(X)\rho\end{aligned}$$

$$= \int_\Sigma \Big(\operatorname{div}\Big((f-1)\mathcal{L}_X J\Big) - \frac{f'}{6}\beta\Big)(JX)\rho - \int_\Sigma \big(\operatorname{div}(\dot{f}X) - \dot{f}\operatorname{div}(X)\big)\rho$$
$$= \int_\Sigma \Big(\operatorname{div}\Big((f-1)\mathcal{L}_X J\Big) - \frac{f'}{6}\beta\Big)(JX)\rho \ . \qquad (X \text{ is harmonic})$$

Now let us compute the term

$$\begin{aligned}
\beta(JX) &= \langle \big(g_J^{-1}\mathcal{L}_X C\big)_0, (\nabla_{JX}A)J\rangle && \text{(Theorem 1.11)}\\
&= -\langle \big(g_J^{-1}\mathcal{L}_X C\big)_0, \nabla_X A\rangle && \text{(rel. (3.3.13))}\\
&= -\langle \nabla_X A - \frac{3}{2}\operatorname{tr}(JM_X)AJ + A\big(M_X^{\mathrm{s}}\cdot\big), \nabla_X A\rangle && (JX \text{ is symplectic})\\
&= -\langle \nabla_X A + A\big(M_X^{\mathrm{s}}\cdot\big), \nabla_X A\rangle\\
&= -||\nabla_X A||^2 - \langle A\big(M_X^{\mathrm{s}}\cdot\big), \nabla_X A\rangle && \text{(Theorem 1.11)}\\
&= -||\nabla_X A||^2 + \langle A\big(M_X^{\mathrm{s}}\cdot\big), (\nabla_{JX}A)J\rangle && (\nabla_\bullet J = 0)\\
&= -||\nabla_X A||^2 + \langle A\big(M_X^{\mathrm{s}}\cdot\big), (\nabla_{JX}A)J + A\nabla_{JX}J\rangle \ .
\end{aligned}$$

Applying equation (3.3.14) to the last term with $\dot{A}_0 = \nabla_{JX}A$ and $\dot{J} = \nabla_{JX}J$, we get

$$\begin{aligned}
\beta(JX) &= -||\nabla_X A||^2 + \langle A\big(M_X^{\mathrm{s}}\cdot\big), (\nabla_{JX}A)J + A\nabla_{JX}J\rangle\\
&= -||\nabla_X A||^2 + \frac{1}{2}\langle AM_X^{\mathrm{s}} + M_X^{\mathrm{s}}A, \Big((\nabla_{JX}A)J + A\nabla_{JX}J\Big)_{\mathrm{tr}}\rangle\\
&= -||\nabla_X A||^2 \ ,
\end{aligned}$$

where we used that the endomorphism part of $(\nabla_{JX}A)J$ is trace-less. In order to study the divergence term, let us first make some preliminary observations. Let $L : \Gamma(T\Sigma) \to \operatorname{End}_0(T\Sigma, g_J)$ be the Lie derivative operator. It can be shown that its L^2-adjoint is $L^*(\dot{J}) = -J(\operatorname{div}_{g_J}\dot{J})^\#$ ([Tro12]), where $\# : \Omega^1(\Sigma) \to \Gamma(T\Sigma)$ is the musical isomorphism induced by the metric g_J. Therefore,

$$\begin{aligned}
\int_\Sigma \Big(\operatorname{div}\Big((f-1)\mathcal{L}_X J\Big)\Big)(JX)\rho &= \int_\Sigma \langle \operatorname{div}\Big((f-1)\mathcal{L}_X J\Big)^\#, JX\rangle\rho\\
&= -\int_\Sigma \langle J\Big(\operatorname{div}\Big((f-1)\mathcal{L}_X J\Big)\Big)^\#, X\rangle\rho\\
&= \int_\Sigma \langle (f-1)\mathcal{L}_X J, \mathcal{L}_X J\rho\rangle\\
&= \int_\Sigma (f-1)||\mathcal{L}_X J||^2\rho \ .
\end{aligned}$$

Referring back to the term we are interested in, we conclude

$$\int_\Sigma (f-1)||\mathcal{L}_X J||^2\rho + \frac{1}{6}\int_\Sigma f'||\nabla_X A||^2\rho = 0$$

and, since f, f' are both strictly negative, this is possible if and only if $\mathcal{L}_X J = \nabla_X A = 0$. Given that on a Riemann surface (Σ, J) of genus $g \geqslant 2$ there are no non-zero biholomorphism isotopic to the identity, it follows that $X = 0$. $\square$

Lemma 3.46. *The vector space $S_{(J,A)}$ is a complex-symplectic subspace of $\big(W_{(J,A)}, \mathbf{I}, \boldsymbol{\omega}_f\big)$ isomorphic to $H^1_{dR}(\Sigma, \mathbb{R})$.*

Proof. Requiring $S_{(J,A)}$ to be a complex subspace of $\big(W_{(J,A)}, \mathbf{I}\big)$ is equivalent to say that it is preserved by the action of the complex structure. For instance, if $(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C) \in S_{(J,A)}$ then $\mathbf{I}\big(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C\big) = \big(-\mathcal{L}_{JX} J, -g_J^{-1}\mathcal{L}_{JX}\big)$ (see Lemma 3.26). Since X is harmonic, i.e. X and JX are symplectic vector field, the element $\big(-\mathcal{L}_{JX} J, -g_J^{-1}\mathcal{L}_{JX}\big)$ belongs to $S_{(J,A)}$ as $\mathrm{d}\big(\iota_{J^2X}\rho\big) = -\mathrm{d}\big(\iota_X\rho\big) = 0$. Moreover, according to Proposition 3.45, we have

$$S_{(J,A)} \cap \big(S_{(J,A)}\big)^{\perp_{\boldsymbol{\omega}_f}} = \{0\} \ ,$$

which implies that $S_{(J,A)}$ is a symplectic subspace of $\big(W_{(J,A)}, \boldsymbol{\omega}_f\big)$ endowed with the restricted symplectic form.
Now if $(\mathcal{L}_X J, g_J^{-1}\mathcal{L}_X C) \in S_{(J,A)}$, then $\mathrm{d}(\iota_X\rho) = \mathrm{d}(\iota_{JX}\rho) = 0$. In particular,

$$0 = \mathrm{d}(\iota_{JX}\rho) = -\mathrm{d}(\iota_X\rho \circ J)$$

and since $\iota_X\rho \circ J = *_J(\iota_X\rho)$, we conclude that $\iota_X\rho$ is a harmonic 1-form. This gives a well-defined map from $S_{(J,A)}$ to the space of harmonic 1-forms on the surface, which is isomorphic to $H^1_{\mathrm{dR}}(\Sigma, \mathbb{R})$ by Hodge theory. The map is an isomorphism since for any cohomology class $[\gamma] \in H^1_{\mathrm{dR}}(\Sigma, \mathbb{R})$ there exists a unique harmonic representative, which is of the form $\iota_X\rho$, for some harmonic vector field X on the surface (see Lemma 3.1). $\square$

Remark 3.47. It should be noted that the decomposition of Proposition 3.45 is also orthogonal with respect to $\mathbf{g}_f$. In fact, $\mathbf{g}_f(\cdot,\cdot) = \boldsymbol{\omega}_f(\mathbf{I}\cdot,\cdot)$ and using the $\mathbf{I}$-invariance of $S_{(J,A)}$ it follows that

$$V_{(J,A)} = \big(S_{(J,A)}\big)^{\perp_{\boldsymbol{\omega}_f}} = \big(S_{(J,A)}\big)^{\perp_{\mathbf{g}_f}} \subset W_{(J,A)} \ .$$

In Section 3.1.2, we discussed how to obtain Teichmüller space by means of symplectic reduction theory and we argued how the symplectic form is actually part of a Kähler metric. If μ denotes the moment map of Theorem 3.5, the quotient space $\widetilde{\mathcal{T}}(\Sigma) = \mu^{-1}(0)/\operatorname{Ham}(\Sigma,\rho)$ is a smooth manifold of dimension $6g-6+2g$ with a natural H-action. In particular, since the action is free and proper, the quotient map $p : \widetilde{\mathcal{T}}(\Sigma) \to \mathcal{T}(\Sigma)$ is an H-principal bundle. On the other hand, there is a $\mathrm{MCG}(\Sigma)$-equivariant projection map $\widetilde{\pi} : \widetilde{\mathcal{B}}(\Sigma) \to \widetilde{\mathcal{T}}(\Sigma)$ which allows us to lift the H-action from $\widetilde{\mathcal{T}}(\Sigma)$ to $\widetilde{\mathcal{B}}(\Sigma)$. By a standard argument, the H-action on $\widetilde{\mathcal{B}}(\Sigma)$ is free and proper as well (see [Lab08, Proposition 6.3.3]). In the end, the quotient

$\widetilde{\mathcal{B}}(\Sigma)/H$ results in an identification with $\mathcal{B}(\Sigma)$ so that the following diagram commutes

$$\begin{tikzcd} \widetilde{\mathcal{B}}(\Sigma) \arrow[r, "\tilde{\pi}"] \arrow[d, "p'"'] & \widetilde{\mathcal{T}}(\Sigma) \arrow[d, "p"] \\ \mathcal{B}(\Sigma) \arrow[r, "\pi"] & \mathcal{T}(\Sigma) \end{tikzcd}$$

where $\pi : \mathcal{B}(\Sigma) \to \mathcal{T}(\Sigma)$ is the MCG(Σ)-equivariant holomorphic vector bundle map given by Theorem 1.20, and $p' : \widetilde{\mathcal{B}}(\Sigma) \to \mathcal{B}(\Sigma)$ is the quotient projection. According to Proposition 3.45 and Lemma 3.46, the H-orbits in $\widetilde{\mathcal{B}}(\Sigma)$ are complex-symplectic submanifolds, therefore there is a well-defined complex structure $\mathbf{I}$ and symplectic form $\boldsymbol{\omega}_f$ on the quotient (see [Tra18, Lemma 4.4.9]), giving rise to a pseudo-Kähler metric on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component. In other words, we proved the following

Theorem F. *The H-action on $\widetilde{\mathcal{B}}(\Sigma)$ is free and proper, with complex and symplectic H-orbits. Moreover, the pseudo-Kähler structure $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ descend to the quotient which is identified with* $\mathrm{Hit}_3(\Sigma)$. *Finally, the complex structure* $\mathbf{I}$ *induced on the* $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$*-Hitchin component coincides with the one found by Labourie and Loftin.*

3.4.2 The pseudo-metric is non-degenerate on the orbit

Here we want to study the set $\mathcal{M}_{\mathrm{C}} = \{(J, A) \in D^3(\mathcal{J}(\Sigma)) \mid \mathrm{d}^\nabla A = 0\}$, namely the subspace of $D^3(\mathcal{J}(\Sigma))$ where the Codazzi-like equation for hyperbolic affine spheres (see (HS)) is satisfied.

Lemma 3.48. *Let (J, A) be a point in $\mathcal{M}_{\mathrm{C}}$, then*

$$T_{(J,A)}\big(\mathrm{Diff}(\Sigma)\cdot(J,A)\big) \subset T_{(J,A)}\mathcal{M}_{\mathrm{C}} \ .$$

Moreover, the tangent space $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ admits the following decomposition:

$$V_{(J,A)} \overset{\perp_{\mathbf{g}_f}}{\oplus} S_{(J,A)} \overset{\perp_{\mathbf{g}_f}}{\oplus} T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big) \overset{\perp_{\mathbf{g}_f}}{\oplus} \mathbf{I}\Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma,\rho)\cdot(J,A)\big)\Big) \ .$$

Proof. If $(J, A) \in \mathcal{M}_{\mathrm{C}}$, then $\mathrm{d}^\nabla A = 0$ where ∇ is the Levi-Civita connection with respect to $g_J = \rho(\cdot, J\cdot)$. In particular, $A = g_J^{-1}C = g_J^{-1}\mathcal{R}e(q)$ where q is a J-complex cubic differential on (Σ, J) so that equation $\mathrm{d}^\nabla A = 0$ is equivalent to $\bar\partial_J q = 0$ (see Theorem 1.11). Now let $X \in \Gamma(T\Sigma)$ and consider its flow $\{\phi_t\} \subset \mathrm{Diff}(\Sigma)$, namely $X = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t|_{t=0}$ and $\phi_0 = \mathrm{Id}$. Let us define

$$J_t := \mathrm{d}\phi_t^{-1} \circ J \circ \mathrm{d}\phi_t, \qquad C_t := C(\mathrm{d}\phi_t\cdot, \mathrm{d}\phi_t\cdot, \mathrm{d}\phi_t\cdot), \qquad q_t := \phi_t^* q \ .$$

It is not difficult to show that q is holomorphic with respect to J if and only if q_t is holomorphic with respect to J_t. Therefore, to conclude the proof of the first part of the statement, we only need to show that $\mathcal{R}e(C_t) = q_t$. This last identity can be proven with a computation in coordinates. In fact, let $\{x, y\}$ be isothermal coordinates on the surface, so that $g_J = e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$ and $q = (P + iQ)\mathrm{d}z^3$, with $P + iQ$ a J-holomorphic function. Then, we get

$$C = P\mathrm{d}x^3 - 3P\mathrm{d}x \odot \mathrm{d}y^2 - 3Q\mathrm{d}x^2 \odot \mathrm{d}y + Q\mathrm{d}y^3 \ ,$$

where $\odot$ denotes the symmetric product. Plugging in the action of the flow (ϕ_t) on the expressions above for q and C gives the claim. Regarding the decomposition, we already know by Lemma 3.1 that $T_{(J,A)}\big(\mathrm{Diff}(\Sigma) \cdot (J, A)\big)$ splits as a direct sum

$$T_{(J,A)}\big(H \cdot (J, A)\big) \oplus T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big) \oplus \mathbf{I}\Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big)\Big) \ ,$$

where $H := \mathrm{Symp}_0(\Sigma\, \rho)/\mathrm{Ham}(\Sigma, \rho)$. In particular, by Lemma 3.46 the tangent to the harmonic orbit is identified with $S_{(J,A)}$. Let U be a Hamiltonian vector field on the surface. The $\mathbf{g}_f$-orthogonality follows from the following computation:

$$\mathbf{g}_f\big((\mathcal{L}_U J, g^{-1}\mathcal{L}_U C); \mathbf{I}(\mathcal{L}_U J, g^{-1}\mathcal{L}_U C)\big) = \boldsymbol{\omega}_f\big((\mathcal{L}_U J, g^{-1}\mathcal{L}_U C); (\mathcal{L}_U J, g^{-1}\mathcal{L}_U C)\big) = 0 \ ,$$

and by $\mathbf{I}$-invariance of $S_{(J,A)}$, which is contained in the largest subspace in $T_{(J,A)}\widetilde{\mathcal{HS}}_0(\Sigma, \rho)$ that is $\mathbf{g}_f$-orthogonal to the Hamiltonian orbit (see Theorem G). Finally, $V_{(J,A)}$ is $\mathbf{g}_f$-orthogonal to the symplectic orbit by Proposition 3.41 and to the space $\mathbf{I}\Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho)\cdot (J, A)\big)\Big)$ by $\mathbf{I}$-invariance. □

Proposition 3.49. *Let (J, A) be a point in $\mathcal{M}_{\mathrm{C}}$. Then, the pseudo-metric $\mathbf{g}_f$ is non-degenerate when restricted to the following subspaces:*

$$S_{(J,A)}, \quad T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big), \quad \mathbf{I}\Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big)\Big).$$

Proof. The pseudo-metric $\mathbf{g}_f$ is non-degenerate on the Hamiltonian orbit as a consequence of Lemma 3.29 and Theorem G, indeed they imply together the following condition

$$T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big) \cap \Big(T_{(J,A)}\big(\mathrm{Ham}(\Sigma, \rho) \cdot (J, A)\big)\Big)^{\perp_{\mathbf{g}_f}} = \{0\} \ .$$

Moreover, the same is true on the Hamiltonian orbit after applying the complex structure $\mathbf{I}$ since $\mathbf{g}_f(\mathbf{I}\cdot, \mathbf{I}\cdot) = \mathbf{g}_f(\cdot, \cdot)$. Regarding the subspace $S_{(J,A)}$, we get the thesis directly from the proof of Lemma 3.46 (see also Remark 3.47). □

Theorem M. *Let (J, A) be a point in $\mathcal{M}_{\mathrm{C}}$. Then, the following are equivalent:*
- $\mathbf{g}_f$ *is non-degenerate on $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$;*
- $\mathbf{g}_f$ *is non-degenerate when restricted to $V_{(J,A)}$, hence on the Hitchin component.*

Proof. The tangent space $T_{(J,A)}\mathcal{M}_{\mathrm{C}}$ decomposes in the $\mathbf{g}_f$-ortoghonal direct sum of four subspaces (Lemma 3.48). Thanks to Proposition 3.49 we know that the metric $\mathbf{g}_f$ is non-degenerate on three out of four spaces, and the one not counted is exactly $V_{(J,A)}$. Using that the decomposition is $\mathbf{g}_f$-orthogonal, the thesis follows directly. □

3.4.3 Krein spaces

Using the construction of Section 3.2.1 we can define a (formal) pseudo-Kähler structure $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ on the infinite-dimensional manifold $D^3(\mathcal{J}(\Sigma))$. In particular, the pseudo-metric, and hence the symplectic form, is known to be non-degenerate. Unlike the positive definite case, the tangent space $T_{(J,A)}D^3(\mathcal{J}(\Sigma))$ will no longer have a Hilbert space structure, but rather will be a so-called Krein space. The aim of this section is to introduce such spaces by following the theory developed in [AI81]. Then, in Section 3.4.4, we explain how this approach can actually lead to the proof of the absence of degenerate vectors for $\mathbf{g}_f$ when restricted on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component.

In what follows we are going to consider a real vector space $\mathcal{F}$, possibly of infinite dimension, endowed with a symmetric bi-linear form $\langle\cdot|\cdot\rangle$.

Remark 3.50. Given a space $\big(\mathcal{F}, \langle\cdot|\cdot\rangle\big)$ as above, we do not require the value $\langle v|v\rangle$ to always be strictly positive whenever v is non-zero, but rather consider more general situations in which $\langle v|v\rangle$ can be positive, negative or null. In particular, if not specified, there could be degenerate vectors for the form $\langle\cdot|\cdot\rangle$ among those that are isotropic, i.e. $\langle v|v\rangle = 0$.

Definition 3.51. Let $\mathcal{F}$ be a real vector space with a symmetric bi-linear form $\langle\cdot|\cdot\rangle$, then

(i) $\big(\mathcal{F}, \langle\cdot|\cdot\rangle\big)$ is *non-degenerate* if there are no non-zero vectors orthogonal to the whole space $\mathcal{F}$;

(ii) let $\mathcal{L} \subset \mathcal{F}$ be a linear subspace, then it is *non-degenerate* with respect to the restricted symmetric bi-linear form if
$$\mathcal{L} \cap \big(\mathcal{L}\big)^{\perp_{\langle\cdot|\cdot\rangle}} = \{0\} \ .$$

In the following, let us denote with $\mathcal{F}_+$ and $\mathcal{F}_-$ the set of vectors $v \in \mathcal{F}$ such that $\langle v|v\rangle > 0$ and $\langle v|v\rangle < 0$, respectively.

Lemma 3.52. *Let $\mathcal{L} \subset \mathcal{F}$ be a linear subspace. If there exists a decomposition $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ in positive and negative part, then the linear subspace $\mathcal{L}$ endowed with the restricted symmetric bi-linear form is non-degenerate.*

Proof. Suppose, by contrary, there exists a non-zero vector $v \in \mathcal{L}$ such that $\langle v|w\rangle = 0$ for any $w \in \mathcal{L}$. By hypothesis, we can decompose $v = v_+ + v_-$ in its positive and negative part to get
$$\begin{aligned} 0 &= \langle v|v_+\rangle = \langle v_+|v_+\rangle + \langle v_-|v_+\rangle \\ 0 &= \langle v|v_-\rangle = \langle v_+|v_-\rangle + \langle v_-|v_-\rangle \ . \end{aligned}$$

Therefore, using the symmetry of the bi-linear form, we have the following contradiction
$$0 < -\langle v_-|v_-\rangle = \langle v_+|v_-\rangle = -\langle v_+|v_+\rangle < 0 \ .$$
∎

Remark 3.53. It must be pointed out that the decomposition of Lemma 3.52 is not required to be orthogonal and the symmetric bi-linear form $\langle\cdot|\cdot\rangle$ is not required to be non-degenerate on the ambient space. Moreover, the converse of the above statement is not true (see [AI81, Example 1.33])

Definition 3.54. Let $\mathcal{F}$ be a real vector space endowed with a symmetric bi-linear form $\langle\cdot|\cdot\rangle$ and a decomposition

$$\mathcal{F} = \mathcal{F}_+ \overset{\perp}{\oplus} \mathcal{F}_- \ , \tag{3.4.2}$$

where the symbol $\perp$ denotes the orthogonal with respect to $\langle\cdot|\cdot\rangle$. Suppose also that the linear subspace $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) endowed with $\langle\cdot|\cdot\rangle$ (resp. $-\langle\cdot|\cdot\rangle$) is a Hilbert space, then $\big(\mathcal{F}, \langle\cdot|\cdot\rangle\big)$ is called a *Krein space.*

Notice that if $\big(\mathcal{F}, \langle\cdot|\cdot\rangle\big)$ is a Krein space, the symmetric bi-linear form $\langle\cdot|\cdot\rangle$ is non-degenerate by Lemma 3.52. In particular, we can define an inner product by using the decomposition (3.4.2). In fact, if $v = v_+ + v_-$ and $w = w_+ + w_-$ is such a decomposition for some $v, w \in \mathcal{F}$, then

$$(v, w) := \langle v_+|w_+\rangle - \langle v_-|w_-\rangle \ , \tag{3.4.3}$$

is positive-definite. In particular, the subspaces $\mathcal{F}_+$ and $\mathcal{F}_-$ are orthogonal with respect to $(\cdot,\cdot)$ as well. In other words, we can think of $\mathcal{F}$, endowed with $(\cdot,\cdot)$, as a Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \overset{\perp}{\oplus} \mathcal{H}_- \ .$$

At first glance, the definition of the scalar product $(\cdot,\cdot)$ might give the impression that it depends on the chosen orthogonal decomposition (3.4.2) (which is not unique if $\mathcal{F}$ admits isotropic vectors). However, it can be shown (see [AI81, Remark 2.5 and Theorem 7.19]) that the norms induced by different orthogonal decompositions are equivalent, and thus they induce the same topology on $\mathcal{F}$. This allows us to consider continuous operators on Krein spaces and to state a result that is fundamental for our purposes.

From now on, we will denote a Krein space as a Hilbert space $\mathcal{H} = \mathcal{H}_+ \overset{\perp}{\oplus} \mathcal{H}_-$ with respect to the scalar product $(\cdot,\cdot)$, since as explained above, there is no issue with the choice of a decomposition. Such a splitting, generates two mutually complementary projectors P_+ and P_- mapping $\mathcal{H}$ on to $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. In particular, $P_+ + P_- = \mathrm{Id}_{\mathcal{H}}$ and $\big(P_\pm\big)^2 = P_\pm$. The projectors $P_\pm$ are called *canonical projectors* and they are orthogonal (self-adjoint) with respect to the scalar product $(\cdot,\cdot)$:

$$\mathcal{H} = \mathcal{H}_+ \overset{\perp}{\oplus} \mathcal{H}_- = P_+\mathcal{H} \overset{\perp}{\oplus} P_-\mathcal{H}_- \ .$$

We can also define another linear operator $\mathfrak{J} : \mathcal{H} \to \mathcal{H}$ given by $\mathfrak{J} := P_+ - P_-$, which is called the *canonical symmetry* of the Krein space $\mathcal{H}$.

Lemma 3.55 ([AI81])**.** *The canonical symmetry $\mathfrak{J}$ is a bounded linear operator and it has the following properties:*

(i) $\mathfrak{J}^* = \mathfrak{J}$;

(ii) $\mathfrak{J}^2 = \mathrm{Id}_{\mathcal{H}}$;

(iii) $\mathfrak{J}^{-1} = \mathfrak{J}^*$;

where the adjoint $\mathfrak{J}^*$ is taken with respect to $(\cdot,\cdot)$. Moreover, $\mathcal{H}_\pm$ is an eigen-subspace of $\mathfrak{J}$ with eigenvalue $\lambda = \pm 1$.

The introduction of the canonical symmetry allows us to find a close relationship between the indefinite symmetric bi-linear form $\langle\cdot|\cdot\rangle$ and the scalar product $(\cdot,\cdot)$, indeed we see from their definition that:

$$\langle v|w\rangle = (\mathfrak{J}v, w), \qquad \forall v, w \in \mathcal{H} \ . \tag{3.4.4}$$

Lemma 3.56. *Let $\mathcal{L}$ be a linear subspace of a Krein space $\big(\mathcal{F}, \langle\cdot|\cdot\rangle\big)$, then*

$$\Big(\mathcal{L}\Big)^{\perp_{(\cdot,\cdot)}} = \Big(\mathfrak{J}\mathcal{L}\Big)^{\perp_{\langle\cdot|\cdot\rangle}} \ .$$

Proof. This is simply a consequence of relation (3.4.4) and Lemma 3.55, since given $v \in \mathcal{F}$ we have

$$(v, w) = 0 \iff \langle v|\mathfrak{J}w\rangle = 0, \qquad \forall w \in \mathcal{L} \ .$$

□

3.4.4 Conclusion

In this final paragraph of the chapter we want to explain how, the approach of Krein spaces, can actually shed some light on the non-existence of degenerate vectors for the pseudo-metric when restricted to $V_{(J,A)}$. We will also discuss the reason why, in our case, knowing that $\mathbf{g}_f$ is non-degenerate on the symplectic orbit is not sufficient to conclude.

In what follows, we will recall the construction of the set $D^3(\mathcal{J}(\Sigma))$ that was explained in Section 3.2.1. The final goal is to study its tangent space as an infinite-dimensional vector space and thus its structure as a Banach space with respect to a norm that we will introduce shortly. We recall the construction made for smooth sections, but the same holds for sections of L^2 regularity with respect to a fixed area form, so that the corresponding space will be denoted with $D^3(\mathcal{J}(\Sigma))_{L^2}$. In particular, any tangent vector to $D^3(\mathcal{J}(\Sigma))_{L^2}$ will be a L^2-section as well.

The infinite dimensional manifold $D^3(\mathcal{J}(\Sigma))$ has been defined as the space of smooth sections of the bundle

$$P\big(D^3(\mathcal{J}(\mathbb{R}^2))\big) := P \times D^3(\mathcal{J}(\mathbb{R}^2)) \Big/ \mathrm{SL}(2,\mathbb{R}) \longrightarrow \Sigma \ ,$$

where $\mathrm{SL}(2,\mathbb{R})$ acts diagonally on two factors. In particular, each element in $D^3(\mathcal{J}(\Sigma))$ can be described as a pair (J, A), with J an almost-complex structure on Σ, and A a 1-form with values in the trace-less and g_J-symmetric endomorphisms of $T\Sigma$. Moreover, a tangent vector $(\dot{J}, \dot{A})$, where $\dot{A} := g_J^{-1}\dot{C}$, at (J, A) can be considered as the data of:

- a section $\dot{J}$ of $\mathrm{End}(T\Sigma)$ such that $\dot{J}J + J\dot{J} = 0$, namely $\dot{J}$ is a g_J-symmetric and trace-less endomorphism of $T\Sigma$;
- an $\mathrm{End}(T\Sigma, g_J)$-valued 1-form $\dot{A}$ such that

$$\dot{A} = \dot{\tilde{A}}_0 + \underbrace{T(J, A, \dot{J}) + \frac{1}{2}\operatorname{tr}\Big(JA\dot{J}\Big)\mathbb{1}}_{\text{completely determined by } \dot{J}} \ , \tag{3.4.5}$$

  where $\mathbb{1}$ is the 2×2 identity matrix and $\dot{A}_0 = \dot{\tilde{A}}_0 + T(J, A, \dot{J})$ is the trace-less part of $\dot{A}$. Moreover, the trace-part $\dot{A}_{\mathrm{tr}}$ and the tensor $\dot{A}_0 - \dot{\tilde{A}}_0$ is uniquely determined by $\dot{J}$ (see Lemma 2.10).

In Section 3.2.1 we also defined a (formal) pseudo-Kähler metric $(\mathbf{g}_f, \mathbf{I}, \boldsymbol{\omega}_f)$ on the infinite-dimensional manifold $D^3(\mathcal{J}(\Sigma))$. In particular, for any $(J, A) \in D^3(\mathcal{J}(\Sigma))$, the tensor

$$(\mathbf{g}_f)_{(J,A)}\big((\dot{J}, \dot{A}); (\dot{J}', \dot{A}')\big) = \int_\Sigma (1-f)\langle \dot{J}, \dot{J}'\rangle\rho + \int_\Sigma \frac{f'}{6}\Big(\langle \dot{A}_0, \dot{A}_0'\rangle - \frac{1}{2}\langle \dot{A}_{\mathrm{tr}}, \dot{A}_{\mathrm{tr}}'\rangle\Big)\rho$$

defines a symmetric bi-linear form on each tangent space $T_{(J,A)}D^3(\mathcal{J}(\Sigma))$, which is known to be non-degenerate.

Theorem 3.57. *For any $(J, A) \in D^3(\mathcal{J}(\Sigma))_{L^2}$, the tangent space $T_{(J,A)}D^3(\mathcal{J}(\Sigma))_{L^2}$ endowed with $(\mathbf{g}_f)_{(J,A)}$ is a Krein space.*

Proof. During the proof of the theorem we will denote by $\mathcal{F}_{L^2}$ the tangent space $T_{(J,A)}D^3(\mathcal{J}(\Sigma))_{L^2}$. Let $(J, A) \in D^3(\mathcal{J}(\Sigma))_{L^2}$, then for any $(\dot{J}, \dot{A}) \in \mathcal{F}_{L^2}$ there is a $\mathbf{g}_f$-orthogonal decomposition in positive and negative part, given by:

$$\mathcal{F}_{L^2} = \big(\mathcal{F}_{L^2}\big)_+ \overset{\perp_{\mathbf{g}_f}}{\oplus} \big(\mathcal{F}_{L^2}\big)_- \ , \tag{3.4.6}$$

where $\big(\mathcal{F}_{L^2}\big)_+ := \{(\dot{J}, \dot{A}) \in \mathcal{F}_{L^2} \mid \dot{\tilde{A}}_0 = 0\}$ and $\big(\mathcal{F}_{L^2}\big)_- := \{(\dot{J}, \dot{A}) \in \mathcal{F}_{L^2} \mid \dot{J} = 0\}$. In fact, using relation (3.2.2), we have

$$(\mathbf{g}_f)_{(J,A)}\big|_{(\mathcal{F}_{L^2})_-} = \int_\Sigma \frac{f'}{6}\langle \dot{\tilde{A}}_0, \dot{\tilde{A}}_0\rangle\rho < 0 \tag{Lemma 3.11}$$

$$(\mathbf{g}_f)_{(J,A)}\big|_{(\mathcal{F}_{L^2})_+} = \int_\Sigma \Big(1 - f + \frac{3}{2}||q||_J^2 f'\Big)\langle \dot{J}, \dot{J}\rangle\rho > 0 \qquad \text{(Lemma 3.12 with } t = \tfrac{||q||_J^2}{2}\text{)}$$

In particular, $(\mathbf{g}_f)_{(J,A)}$ and $-(\mathbf{g}_f)_{(J,A)}$ are positive-definite scalar products on $\big(\mathcal{F}_{L^2}\big)_+$ and $\big(\mathcal{F}_{L^2}\big)_-$, respectively. Let us denote with $||\cdot||_{\mathbf{g}_f}$ the norm induced by $\mathbf{g}_f$ on $\big(\mathcal{F}_{L^2}\big)_+$, then $-||\cdot||_{\mathbf{g}_f}$ induces a norm on $\big(\mathcal{F}_{L^2}\big)_-$. To conclude the proof of the theorem, we show that the above norms are equivalent to the standard L^2-norms:

$$||\dot{J}||_{L^2}^2 = \frac{1}{2}\int_\Sigma \mathrm{tr}\Big(\dot{J}^2\Big)\rho = \int_\Sigma ||\dot{J}||_J^2\rho, \qquad ||\dot{A}||_{L^2}^2 = \int_\Sigma \mathrm{tr}\left(\dot{A}\wedge *\dot{A}\right) = \int_\Sigma ||\dot{A}||_J^2\rho$$

defined for $(1,1)$-tensors $\dot{J}$ and $(1,2)$-tensors $\dot{A}$ (see (2.2.3) and (2.2.4)). Since the latter are complete norms, the former are complete as well. The functions f and f' appearing in the definition of $(\mathbf{g}_f)_{(J,A)}$ are computed in $\frac{||q||_J^2}{2}$, which is a smooth function on Σ. Let us denote with $m_q = 0$ (resp. M_q) the minimum (resp. the maximum) of $||q||_J^2$. Then, we have

$$\begin{aligned} -||\dot{\tilde{A}}_0||_{\mathbf{g}_f}^2 &= -\int_\Sigma \frac{f'}{6}||\dot{\tilde{A}}_0||_J^2\rho \\ &\geqslant -\frac{1}{6}\int_\Sigma f'\Big(\frac{M_q}{2}\Big)||\dot{\tilde{A}}_0||_J^2\rho \\ &= \tilde{c}(J,q)||\dot{\tilde{A}}_0||_{L^2}^2, \qquad \tilde{c}(J,q)\in\mathbb{R}^+ \end{aligned}$$

as the function $-f'$ is strictly decreasing. Moreover,

$$\begin{aligned} -||\dot{\tilde{A}}_0||_{\mathbf{g}_f}^2 &= -\int_\Sigma \frac{f'}{6}||\dot{\tilde{A}}_0||_J^2\rho \\ &\leqslant -\frac{1}{6}\int_\Sigma f'(0)||\dot{\tilde{A}}_0||_J^2\rho \\ &\leqslant \tilde{c}'(J,q)||\dot{\tilde{A}}_0||_{L^2}^2, \qquad \tilde{c}'(J,q)\in\mathbb{R}^+ \ . \end{aligned}$$

Regarding the norm induced on the positive part, first recall that $1-f(t)+3tf'(t) = 3t\frac{g'(t)}{g(t)}$ with $g(t), g'(t)$ both strictly positive for any $t>0$, and it is equal to 1 when $t=0$ (Lemma 3.12). Then, notice that $1-f\Big(\frac{||q||_J^2}{2}\Big)+\frac{3}{2}||q||_J^2 f'\Big(\frac{||q||_J^2}{2}\Big)$ is a smooth function on Σ with non negative values, hence its minimum m is positive and different from zero according to what we observed above. Therefore, we have

$$||\dot{J}||_{\mathbf{g}_f}^2 = \int_\Sigma \Big(1 - f + \frac{3}{2}||q||_J^2 f'\Big)||\dot{J}||_J^2\rho$$

$$\begin{aligned}
&\geqslant \int_\Sigma m||\dot{J}||_J^2\rho \\
&= c(J,q)||\dot{J}||_{L^2}^2, \qquad c(J,q) \in \mathbb{R}^+ \ .
\end{aligned}$$

Moreover,

$$\begin{aligned}
||\dot{J}||_{\mathbf{g}_f}^2 &= \int_\Sigma \Big(1 - f + \frac{3}{2}||q||_J^2 f'\Big)||\dot{J}||_J^2\rho \\
&\leqslant \int_\Sigma \Big(1 - f\Big)||\dot{J}||_J^2\rho && (f' < 0) \\
&\leqslant \int_\Sigma \Big(1 - f\Big(\frac{M_q}{2}\Big)\Big)||\dot{J}||_J^2\rho && (f \text{ decreasing}) \\
&= c'(J,q)||\dot{J}||_{L^2}^2, \qquad c'(J,q) \in \mathbb{R}^+ \ .
\end{aligned}$$

□

In what follows, we keep using the notation introduced in the proof of Theorem 3.57, namely $\mathcal{F}_{L^2}$ denotes the space of tangent vectors with L^2-regularity and $\mathcal{F}$ denotes the space of smooth tangent vectors. According to (3.4.6), let $(\dot{J},\dot{A}) = (\dot{J}_+ + \dot{J}_-, \dot{A}_+ + \dot{A}_-)$ be the decomposition for a vector in $\mathcal{F}_{L^2}$, then we can introduce the canonical symmetry of the Krein space:

$$\mathfrak{J}(\dot{J},\dot{A}) := (\dot{J}_+,\dot{A}_+) - (\dot{J}_-,\dot{A}_-) = (\dot{J}, T(J,A,\dot{J}) + \dot{A}_{\mathrm{tr}}) - (0,\dot{\mathring{A}}_0) \ ,$$

where $T(J,A,\dot{J})$ is the tensor defined in Lemma 2.10. In particular,

$$\begin{aligned}
(\widetilde{\mathbf{g}}_f)_{(J,A)}\big((\dot{J},\dot{A});(\dot{J}',\dot{A}')\big) &:= (\mathbf{g}_f)_{(J,A)}\big(\mathfrak{J}(\dot{J},\dot{A});(\dot{J}',\dot{A}')\big) && \text{(rel. (3.4.4))} \\
&= \int_\Sigma \Big(1 - f + \frac{3}{2}||q||_J^2 f'\Big)\langle \dot{J},\dot{J}'\rangle\rho - \int_\Sigma \frac{f'}{6}\langle \dot{\mathring{A}}_0,\dot{\mathring{A}}_0'\rangle\rho \\
&\quad + \int_\Sigma \frac{f'}{6}\Big(\langle T(J,A,\dot{J}),\dot{\mathring{A}}_0'\rangle - \langle \dot{\mathring{A}}_0, T'(J,A,\dot{J}')\rangle\Big)\rho
\end{aligned}$$

defines a positive-definite scalar product on $\mathcal{F}_{L^2}$ (see Lemma 3.12). Such a $\widetilde{\mathbf{g}}_f$ induces the norm $||(\dot{J},\dot{A})||_{\widetilde{\mathbf{g}}_f}^2 := ||\dot{J}||_{\mathbf{g}_f} - ||\dot{\mathring{A}}_0||_{\mathbf{g}_f}^2$ on $\mathcal{F}_{L^2}$ which is complete by using the same argument as in the proof of Theorem 3.57. In the end, the pair $\big(\mathcal{F}_{L^2},\widetilde{\mathbf{g}}_f\big)$ defines a Hilbert space and the decomposition (3.4.6) is orthogonal with respect to $\widetilde{\mathbf{g}}_f$ as well. Now let us consider the linear subspace $\mathcal{L}_{L^2} \subset \mathcal{F}_{L^2}$ given by $\mathcal{L}_{L^2} := T_{(J,A)}\mathcal{M}_{\mathrm{C}}^{L^2}$, namely it is formed by the L^2-tensors tangent to the space of pairs (J,A) such that $\mathrm{d}^\nabla A = 0$. By using the scalar product $\widetilde{\mathbf{g}}_f$, we get an Hilbert space decomposition

$$\mathcal{F}_{L^2} = \mathcal{L}_{L^2} \overset{\perp_{\widetilde{\mathbf{g}}_f}}{\oplus} \big(\mathcal{L}_{L^2}\big)^{\perp_{\widetilde{\mathbf{g}}_f}},$$

where the $\widetilde{\mathbf{g}}_f$-orthogonal to $\mathcal{L}_{L^2}$ can be identified with the range of the linearized $\widetilde{\mathbf{g}}_f$-adjoint of d^∇, denoted with $(\mathrm{d}^\nabla)^{*_{\widetilde{\mathbf{g}}_f}}$.

Lemma 3.58. *In the setting explained above, the following properties hold:*

- $\big(\mathrm{d}^\nabla\big)^{*_{\mathbf{g}_f}} = \mathfrak{J}\big(\mathrm{d}^\nabla\big)^{*_{\widetilde{\mathbf{g}}_f}}\mathfrak{J}$*;*

- *there are two further Hilbert space decompositions for $\mathcal{F}_{L^2}$ as:*

$$\mathcal{F}_{L^2} = \mathfrak{J}\big(\mathcal{L}_{L^2}\big) \overset{\perp_{\widetilde{\mathbf{g}}_f}}{\oplus} \mathrm{Range}\big((\mathrm{d}^\nabla)^{*_{\mathbf{g}_f}}\mathfrak{J}\big)$$

$$\mathcal{F}_{L^2} = \overline{\big(\mathcal{L}_{L^2} + \mathcal{L}_{L^2}^{\perp_{\mathbf{g}_f}}\big)} \overset{\perp_{\widetilde{\mathbf{g}}_f}}{\oplus} \mathfrak{J}\big(\mathcal{L}_0\big) \ ,$$

where $\mathcal{L}_0$ denotes the space of degenerate vectors in $\mathcal{L}_{L^2}$ with respect to $\mathbf{g}_f$, namely $\mathcal{L}_0 := \mathcal{L}_{L^2} \cap \mathcal{L}_{L^2}^{\perp_{\mathbf{g}_f}}$.

Proof. The relation between the $\mathbf{g}_f$-adjoint of d^∇ and its $\widetilde{\mathbf{g}}_f$-adjoint follows directly from $\widetilde{\mathbf{g}}_f(\cdot,\cdot) = \mathbf{g}_f(\mathfrak{J}\cdot,\cdot)$ and the properties of the canonical symmetry stated in Lemma 3.55. Regarding the Hilbert space decompositions of $\mathcal{F}_{L^2}$, the first one is obtained by applying $\mathfrak{J}$ on (3.4.6) and using that $\mathfrak{J}$ is orthogonal with respect to the scalar product $\widetilde{\mathbf{g}}_f$. The second one follows from a general argument on Krein spaces ([AI81, §7]). ☐

Remark 3.59. In Theorem M we proved that the pseudo-metric is non-degenerate on $V_{(J,A)}$ if and only if it is so on $\mathcal{L} \equiv T_{(J,A)}\mathcal{M}_{\mathrm{C}}$. The same correspondence holds for tensors with L^2-regularity, but because of a standard elliptic argument applied on $V_{(J,A)}$, one gets the following further equivalence: $\mathbf{g}_f$ is non-degenerate on $V_{(J,A)}$ (smooth sections) if and only if it is non-degenerate on $\mathcal{L}_{L^2}$.

The bottom line is that $V_{(J,A)}$ is described by a system of PDEs, whose solution is far from being explicit. On the other hand, in light of Lemma 3.48, every element inside $\mathcal{L}_{L^2}$ can be written explicitly as

$$(\dot{J},\dot{A}) = (\mathcal{L}_X J + g_J^{-1}\mathcal{R}e(q_2),\ g_J^{-1}\mathcal{L}_X C + g_J^{-1}\mathcal{R}e(q_3)) \ , \tag{3.4.7}$$

where X is a vector field on the surface and q_2, q_3 are J-holomorphic quadratic and cubic differentials, respectively.

Conjecture 3.60. *The pseudo-metric $\mathbf{g}_f$ restricted on $\mathcal{L}_{L^2}$ is non-degenerate, which is equivalent to $\mathfrak{J}\big(\mathcal{L}_0\big) = \{0\}$ or, in other words, $\mathcal{L}_{L^2} + \mathcal{L}_{L^2}^{\perp_{\mathbf{g}_f}}$ is dense in $\mathcal{F}_{L^2}$.*

We conclude the discussion by explaining why the results presented in Section 3.4.2, namely the absence of degenerate vectors on the orbit, are not sufficient to conclude in our case. The $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component can be described (see Proposition 1.13 and Proposition 3.14) as the following quotient space:

$$\mathcal{HS}_0(\Sigma,\rho) := \left\{ (J,C) \;\middle|\; \begin{array}{l} J \text{ is an (almost) complex structure on } \Sigma \\ C \text{ is the real part of a } J\text{-cubic differential } q \\ \big(h := e^{F\left(\frac{||q||^2_{g_J}}{2}\right)} g_J, A := g_J^{-1}C\big) \text{ satisfy (HS)} \end{array} \right\} \Big/ \mathrm{Symp}_0(\Sigma) \ ,$$

where (HS) is the system formed by: $K_h - ||q||^2_h = -1$, i.e. Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$ and $\mathrm{d}^\nabla A = 0$. Throughout the paper, we explained that the first equation in the above system has an interpretation as a moment map with respect to the action of the group of Hamiltonian symplectomorphisms of the surface. This allowed us, in part, to find the PDEs with which we were able to describe the space $V_{(J,A)}$. It is not clear to us whether equation $\mathrm{d}^\nabla A = 0$, which we have seen to be equivalent to requiring q to be J-holomorphic, also has an interpretation as a moment map. There are contexts in which this happens: the first one is that of self-duality equations for Higgs bundles over Riemann surfaces, provided that the complex structure on Σ is fixed at the beginning ([Hit87]); the second one is that of almost-Fuchsian hyperbolic 3-manifolds ([Don03],[Tra18]), and the last one is that of maximal globally hyperbolic Anti-de Sitter 3-manifolds ([MST21]). In the former case, the cubic differential is replaced by the Higgs field of the holomorphic bundle and the corresponding moduli space results in a *hyperKähler reduction*; in the remaining two cases q is replaced by a quadratic differential on the surface and the corresponding deformation space is constructed as a *hyperKähler reduction* for almost-Fuchsian manifolds and a *para-hyperKähler reduction* (also called *hypersymplectic reduction*) for Anti-de Sitter manifolds. The context studied by Hitchin ([Hit87]) and Donaldson-Trautwein ([Don03],[Tra18]) is quite different from ours, as in their case the metric is positive-definite on an open subset of the quotient, and the non-degeneracy on this subspace follows by a standard argument. Instead, in the case of hypersymplectic reduction ([DS08]), if the pseudo-metric is non-degenerate on the orbit then it is so on the quotient. Unfortunately, even though we know $\mathbf{g}_f$ to be non-degenerate on the symplectic orbit (Proposition 3.49), the absence of an interpretation of $\mathrm{d}^\nabla A = 0$ as a moment map for does not allow us to conclude.

Chapter 4

Symplectic and metric properties

In this chapter we return to the study of the deformation space of properly convex $\mathbb{RP}^2$-structures on the torus. In fact, we introduced a family of pseudo-Kähler metrics $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ on $\mathcal{B}_0(T^2)$ invariant by the action of $\mathrm{SL}(2,\mathbb{R})$. Any element of the aforementioned structure can be written in coordinates according to the isomorphism $\mathcal{B}_0(T^2) \cong \mathbb{H}^2 \times \mathbb{C}^*$, therefore it comes naturally to ask what might be some metric and symplectic properties of $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ and how they might depend on the choice of the smooth function f. After briefly recalling the Arnold-Liouville theorem in Hamiltonian mechanics, we introduce the theory of complete Lagrangian fibrations, and we show how they are connected with a large class of completely integrable Hamiltonian systems into which $\mathcal{B}_0(T^2)$ falls.

4.1 The Arnold-Liouville Theorem

Definition 4.1. A *Hamiltonian system* is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M, \mathbb{R})$ is a function, called the *Hamiltonian* function.

If (M, ω) is a symplectic manifold and $f \in C^\infty(M, \mathbb{R})$, then the *Hamiltonian vector field* $\mathbb{X}_f \in \Gamma(TM)$ associated with f is defined by the following property

$$\omega(\mathbb{X}_f, Y) = \mathrm{d}f(Y), \quad \forall Y \in \Gamma(TM) \ . \tag{4.1.1}$$

Definition 4.2. Let (M, ω, H) be a Hamiltonian system. A function $f \in C^\infty(M, \mathbb{R})$ is called an *integral of motion* if

$$\omega(\mathbb{X}_f, \mathbb{X}_H) = 0 \ .$$

In other words, any integral of motion f is constant along the integral curves of $\mathbb{X}_H$.

Definition 4.3. A Hamiltonian system (M, ω, H) is *completely integrable* if it possesses $n = \frac{1}{2}\dim(M)$ integrals of motion $f_1 = H, f_2, \dots, f_n$ such that:

- The differentials $(\mathrm{d}f_1)_p, \ldots, (\mathrm{d}f_n)_p$ are linearly independent for each $p \in M$;
- they are pairwise in involution, i.e. $\omega(\mathbb{X}_{f_i}, \mathbb{X}_{f_j}) = 0$ for each $i, j = 1, \ldots, n$.

The first condition in the previous definition is called *independence* and the second one is called *involutivity.* Notice that one of the integral of motion can always be taken to be the Hamiltonian function of the system. Furthermore, at each $p \in M$, the Hamiltonian vector fields associated with the integrals of motion span an isotropic subspace of T_pM. One of the most relevant results of this theory is the following

Theorem 4.4 (Arnold-Liouville, [Arn13])**.** *Let (M, ω, H) be a completely integrable Hamiltonian system of dimension $2n$ and with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of the map $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$. Then,*

- *The level set $f^{-1}(c)$ is a Lagrangian submanifold of M;*
- *if the Hamiltonian vector fields $\mathbb{X}_{f_1}, \ldots \mathbb{X}_{f_n}$ are complete on the level set $f^{-1}(c)$, then each connected component of $f^{-1}(c)$ is diffeomorphic to $\mathbb{R}^k \times T^{n-k}$, for some $0 \leqslant k \leqslant n$. Moreover, that component has coordinates $\theta_1, \ldots, \theta_n$ called angle coordinates, in which the flows of $\mathbb{X}_{f_1}, \ldots \mathbb{X}_{f_n}$ are linear;*
- *there are coordinates $\psi_1, \ldots, \psi_n$, called action coordinates such that the manifold (M, ω) is symplectomorphic to $(\mathbb{R}^{n+k} \times T^{n-k}, \omega_0)$, where $\omega_0 = \sum_{i=1}^{n} \theta_i \wedge \psi_i$.*

Remark 4.5. From a geometric point of view, regular level sets $f^{-1}(c)$ being Lagrangian submanifolds implies that, in a neighborhood of a regular value, the map $f : M \to \mathbb{R}^n$ is a *Lagrangian fibration*, i.e. it is locally trivial and its fibers are Lagrangian submanifolds ([DD08]).

On the other hand, one of the main limitation of this result is that the action coordinates $\psi_1, \ldots, \psi_n$ are, in general, not the given integrals of motion, since $\theta_1, f_1, \ldots, \theta_n, f_n$ may not form a global Darboux chart for ω.

It is worth mentioning that a first general strategy to overcome this problem was presented in [Dui80], in the case in which the Lagrangian fibration $\pi : (M, \omega) \to B \subset \mathbb{R}^n$ has fiber diffeomorphic to an n-dimensional torus. The crucial point is the existence of a global Lagrangian section $\sigma : B \to (M, \omega)$, which is guaranteed as long as $H^2(B, \mathbb{R}) \cong \{0\}$. Recently, Choi-Jung-Kim have presented an adapted version of this result, in the case where each fibre of $\pi : (M, \omega) \to B$ is diffeomorphic to $\mathbb{R}^n$ (see Theorem 3.4.5 in [CJK20]). Their main application was the existence of a global Darboux frame for Goldman symplectic form ω_{G} on the $\mathbb{P}\mathrm{SL}(3, \mathbb{R})$-Hitchin component of a surface of genus at least two.

As for our case, we are going to study Lagrangian fibrations, associated with Hamiltonian systems, whose fibre is neither compact nor simply connected. We will give a criterion on the fibration that allows the action variables to be used as part of the global Darboux frame, and we show that $\mathcal{B}_0(T^2)$ is among the Hamiltonian systems that give rise to this type of Lagrangian fibrations.

4.2 Complete Lagrangian fibrations

Here we briefly review and recall the fundamental results of the theory of complete Lagrangian fibrations, and we show how they are directly related to a large class of completely integrable Hamiltonian systems. Since we could not find references with all the necessary details, we also include the proofs of the most important steps.

Definition 4.6. A *Lagrangian fibration* is a triple (π, M, B), where (M, ω) is a symplectic manifold, B is an open subset of $\mathbb{R}^n$ contained in the set of regular values of π, the map $\pi : (M, \omega) \to B$ is a smooth surjective submersion and for each $b \in B$ the submanifold $\pi^{-1}(b)$ is Lagrangian in (M, ω).

Let $\pi : (M, \omega) \to B$ be a Lagrangian fibration and let $\alpha : B \to T^*B$ be a 1-form. Define a vector field $X_{\pi^*\alpha} \in \Gamma(TM)$ by setting

$$\omega(X_{\pi^*\alpha}, \cdot) = \pi^*\alpha \ . \tag{4.2.1}$$

Proposition 4.7. *For all $\alpha, \beta \in \Gamma(T^*B)$ and for all $f \in C^\infty(B)$ we have:*

(i) $X_{\pi^*(\alpha+\beta)} = X_{\pi^*\alpha} + X_{\pi^*\beta}$

(ii) $X_{\pi^*(f\alpha)} = (\pi^*f)X_{\pi^*\alpha}$

(iii) $X_{\pi^*\alpha} \in \mathrm{Ker}\pi_*$

(iv) $\big[X_{\pi^*\alpha}, X_{\pi^*\beta}\big] = 0$

Proof. Properties (i) and (ii) follow directly form the defining equation (4.2.1). Let $q^1, \dots, q^n$ be local coordinates on $V \subset B$ such that $\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}q^i$ for some functions $\alpha_i : V \to \mathbb{R}, i = 1, \dots, n$. Then, by properties (i) and (ii)

$$X_{\pi^*\alpha} = \sum_{i=1}^{n} (\pi^*\alpha_i) X_{\pi^*\mathrm{d}q^i} \ .$$

Since condition (iii) is pointwise, it suffices to prove that for all functions $f \in C^\infty(V)$ we get $X_{\pi^*\mathrm{d}f} \in \mathrm{Ker}\pi_*$. Let f be such a function and $Y \in \mathrm{Ker}\pi_*$. Since π^*f is constant along the fibre of $\pi : M \to B$, it follows $Y(\pi^*f) = 0$. On the other hand,

$$0 = Y(\pi^*f) = \big(\pi^*(\mathrm{d}f)\big)(Y) = \omega(X_{\pi^*\mathrm{d}f}, Y)$$

by definition of $X_{\pi^*\mathrm{d}f}$. Since the last equality holds for all vertical fields Y, we have $X_{\pi^*\mathrm{d}f} \in (\mathrm{Ker}\pi_*)^{\perp^\omega}$. The map π defines a Lagrangian fibration, which implies that $(\mathrm{Ker}\pi_*)^{\perp^\omega} = \mathrm{Ker}\pi_*$ and property (iii) follows.
For the last property, let $\alpha, \beta \in \Gamma(T^*B)$ and locally write

$$\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}q^i, \qquad \beta = \sum_{j=1}^{n} \beta_i \mathrm{d}q^i$$

for smooth functions α_i, β_j. Then

$$\begin{aligned}
[X_{\pi^*\alpha}, X_{\pi^*\beta}] = \sum_{i,j=1}^{n} [(\pi^*\alpha_i) X_{\pi^*\mathrm{d}q^i}, (\pi^*\beta_j) X_{\pi^*\mathrm{d}q^j}] = \sum_{i,j=1}^{n} \Big((\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^*\mathrm{d}q^i}, X_{\pi^*\mathrm{d}q^j}] \\
+ (\pi^*\alpha_i)\big(X_{\pi^*\mathrm{d}q^i}(\pi^*\beta_j)\big) X_{\pi^*\mathrm{d}q^j} - (\pi^*\beta_j)\big(X_{\pi^*\mathrm{d}q^j}(\pi^*\alpha_i)\big) X_{\pi^*\mathrm{d}q^i} \Big) \\
= \sum_{i,j=1}^{n} (\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^*\mathrm{d}q^i}, X_{\pi^*\mathrm{d}q^j}]
\end{aligned}$$

where the first equality follows from properties (i)-(ii) and the the third one from property (iii). Thus, it suffices to show that for any $f, g \in C^\infty(B)$ one has $[X_{\pi^*\mathrm{d}f}, X_{\pi^*\mathrm{d}g}] = 0$. Notice that the homomorphism

$$\begin{aligned}
C^\infty(M) &\to \Gamma(TM) \\
f &\mapsto X_{\mathrm{d}f}
\end{aligned}$$

is a Lie algebra homomorphism with respect to the Poisson bracket $\{\cdot,\cdot\}_\omega$ and the Lie bracket $[\cdot,\cdot]$, where $\{f, g\}_\omega := \omega(X_{\mathrm{d}f}, X_{\mathrm{d}g})$. In particular, for each $f, g \in C^\infty(B)$

$$[X_{\pi^*\mathrm{d}f}, X_{\pi^*\mathrm{d}g}] = X_{\mathrm{d}\{\pi^*f, \pi^*g\}_\omega} = 0 \ .$$

The second equality follows from the fact that $\pi : (M, \{\cdot,\cdot\}_\omega) \to (B, 0)$ is a Poisson morphism (see [Vai94]). □

Definition 4.8. A Lagrangian fibration $\pi : (M, \omega) \to B$ is *complete* if for each compactly supported 1-form α on B, the vector field $X_{\pi^*\alpha}$ defined by (4.2.1) is complete.

Recall that a Lagrangian fibration is naturally associated with a completely integrable Hamiltonian system (see Remark 4.5). The next Proposition explains why the previous hypothesis of completeness on a Lagrangian fibration is on the one hand interesting from the point of view of geometry and on the other not too restrictive.

Proposition 4.9. *Let $\pi : (M, \omega) \to B$ be a Lagrangian fibration associated with a completely integrable Hamiltonian system (M, H, ω) with integrals of motion given by $f_1 = H, f_2, \ldots, f_n$. If the Hamiltonian vector fields $\mathbb{X}_{f_1}, \ldots, \mathbb{X}_{f_n}$ are complete on $\pi^{-1}(b)$ for each $b \in B$, then the Lagrangian fibration $\pi : (M, \omega) \to B$ is complete.*

Proof. Since the Hamiltonian vector fields are vertical and linearly independent, they point-wise generate the tangent space to each fibre $\pi^{-1}(b)$. Moreover, there exist 1-forms $\alpha_i : B \to T^*B$ such that

$$\omega(\mathbb{X}_{f_i}, \cdot) = \pi^*\alpha_i, \qquad i = 1, \ldots, n$$

where $\omega(\mathbb{X}_{f_i}, \cdot) = \mathrm{d}f_i$ by (4.1.1). The 1-forms α_i are point-wise linearly independent on B, indeed if

$$a_1\alpha_1 + \cdots + a_n\alpha_n = 0, \text{ for some } a_1, \ldots, a_n \in C^\infty(B)$$

then the above sum is still equal to zero after taking the pullback via π. By using the defining property of the α_i's and the independence property of $f_1, \ldots, f_n$ we get $(\pi^*a_1)(m) = \cdots = (\pi^*a_n)(m) = 0$, $\forall m \in M$, i.e. the functions $a_1, \ldots, a_n$ are zero on the whole set B. Let $\alpha : U \subset B \to T^*U$ be a locally defined compactly supported 1-form. We need to prove that the vector field $X_{\pi^*\alpha}$, defined by (4.2.1), is complete. By the above argument, there exist n functions $g_1, \ldots, g_n$ on U such that

$$\alpha = \sum_{i=1}^{n} g_i\alpha_i \ .$$

Since α has compact support on U, the functions g_i have compact support on the same set as well. In particular, they are bounded on U. By properties (i) an (ii) of Proposition 4.7 it follows that

$$X_{\pi^*\alpha} = \sum_{i=1}^{n}(\pi^*g_i)X_{\pi^*\alpha_i} = \sum_{i=1}^{n}(\pi^*g_i)\mathbb{X}_{f_i} \ .$$

The vector field $X_{\pi^*(g_j\alpha_j)} = (\pi^*g_j)\mathbb{X}_{f_j}$ is complete for all $j = 1, \ldots, n$ by an application of the so-called "Escape Lemma" (see [Lee13], Lemma 9.19), indeed the function π^*g_j is bounded and $\mathbb{X}_{f_j}$ is complete. Moreover, since $[X_{\pi^*(g_i\alpha_i)}, X_{\pi^*(g_j\alpha_j)}] = 0$ for all $i, j = 1, \ldots, n$ by property (iv) of Proposition 4.7, it follows that $X_{\pi^*(g_i\alpha_i)} + X_{\pi^*(g_j\alpha_j)}$ defines a new complete vector field as it is the sum of two commuting complete vector fields. Applying in an iterative way the previous observation, we deduce that $X_{\pi^*\alpha}$ is complete, as well. $\square$

From now on, all Lagrangian fibrations will be complete. Let $\alpha : U \to T^*U$ be a compactly supported locally defined 1-form on the base and let $\phi^t_\alpha : \pi^{-1}(U) \to \pi^{-1}(U)$ be the flow of the vector field $X_{\pi^*\alpha}$ defined for all $t \in \mathbb{R}$. Since $X_{\pi^*\alpha}$ is vertical, its flow ϕ^t_α lies along the fibres of $\pi : (M, \omega) \to B$ for all $t \in \mathbb{R}$. Furthermore, for each $\alpha_b \in T^*B$ there exists a compactly supported locally defined 1-form $\alpha : U \to T^*U$ such that $\alpha(b) = \alpha_b$ and the value of $X_{\pi^*\alpha}$ at a point $m \in M$ only depends on α_b and not on the choice of α. Therefore, for each $\alpha_b \in T^*B$ there is a well-defined diffeomorphism

$$\phi^1_{\alpha_b} := \phi^1_\alpha|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(b)$$

where $\alpha \in \Gamma(T^*U)$ is a compactly supported form such that $\alpha(b) = \alpha_b$. In particular, for each $b \in B$ the map

$$\begin{aligned} T_b^*B &\to \mathrm{Diff}(\pi^{-1}(b)) \\ \alpha_b &\mapsto \phi^1_{\alpha_b} \end{aligned} \tag{4.2.2}$$

is a Lie group homomorphism, where T_b^*B has the structure of an abelian Lie group with respect to the sum of covectors. In other words, for each $b \in B$, the map in (4.2.2) defines a transitive action of T_b^*B on $\pi^{-1}(b)$. In general, this action is not free and for instance one can consider the associated isotropy group, namely

$$\Lambda_b := \{\alpha_b \in T_b^*B \mid \phi^1_{\alpha_b}(m) = m,\ \forall m \in \pi^{-1}(b)\}$$

known as the *period lattice*. It can be proved that it is a discrete subgroup of T_b^*B isomorphic to $\mathbb{Z}^k$, with $k = 1, \ldots, n$ (see [Dui80] for the case $k = n$ or [FGS03] in general).

Remark 4.10. In the case $M = T^*B$ and $\omega = \Omega_{\mathrm{can}}$, the transitive action is simply given by the sum of covectors and $\Lambda_b = 0$ for each $b \in B$.

Definition 4.11 ([Vai94])**.** The subset

$$\Lambda := \bigcup_{b \in B} \Lambda_b \subset T^*B$$

is called the *period net* associated with the complete Lagrangian fibration $\pi : (M, \omega) \to B$.

Lemma 4.12. *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration and let $\alpha : U \to T^*U$ be a locally defined 1-form. Then,*

$$\left(\phi^1_\alpha\right)^*\omega - \omega = \pi^*\mathrm{d}\alpha \tag{4.2.3}$$

Proof. The proof relies on the following computation

$$\begin{aligned} \left(\phi^1_\alpha\right)^*\omega - \omega &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^t_\alpha\right)^*\omega \mathrm{d}t \\ &= \int_0^1 \left(\phi^t_\alpha\right)^*\left(\mathcal{L}_{X_{\pi^*\alpha}}\omega\right)\mathrm{d}t \\ &= \int_0^1 \left(\phi^t_\alpha\right)^*\mathrm{d}\big(\omega(X_{\pi^*\alpha}, \cdot)\big)\mathrm{d}t && \text{(Cartan's magic formula)} \\ &= \int_0^1 \left(\pi \circ \phi^t_\alpha\right)^*\mathrm{d}\alpha \mathrm{d}t && \text{(Equation (4.2.1))} \\ &= \int_0^1 \pi^*\mathrm{d}\alpha \mathrm{d}t = \pi^*\mathrm{d}\alpha\ . && (\pi \circ \phi^t_\alpha = \pi, \text{ for all } t) \end{aligned}$$

□

Theorem 4.13. *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration and let Λ be the associated period net. Then,*

- *Λ is a closed Lagrangian submanifold of T^*B;*
- *the quotient $T^*B/_\Lambda$ is a smooth manifold.*

Proof. Since $\pi : (M, \omega) \to B$ is a surjective submersion, then for each $b \in B$ there exists a local section $\sigma : U \to \pi^{-1}(U)$ defined in an open neighborhood U containing b ([GM+97], Proposition 1.2.4). Fix such a section and consider the map

$$\begin{aligned} \psi_\sigma : T^*U &\to \pi^{-1}(U) \\ \alpha &\mapsto \phi^1_\alpha\big(\sigma \circ p(\alpha)\big) \end{aligned} \tag{4.2.4}$$

where $p : (T^*B, \Omega_{\text{can}}) \to B$. We first want to prove that ψ_σ is a local diffeomorphism. Since $\dim T^*U = \dim \pi^{-1}(U)$ it suffices to prove that $\mathrm{Ker}\, \mathrm{d}_\alpha \psi_\sigma = \{0\}$ for all $\alpha \in T^*U$. Fix an element $\alpha_0 \in T^*U$ and notice that if $X \in T_{\alpha_0}T^*U$ is tangent to the fibres of p, then $\mathrm{d}_{\alpha_0}\psi_\sigma(X) = 0$ if and only if $X = 0$. Therefore, if $\mathrm{d}_{\alpha_0}\psi_\sigma(Y) = 0$ and $Y \neq 0$, then $\mathrm{d}_{\alpha_0}p(Y) \neq 0$. Any such vector $Y \in T_{\alpha_0}T^*U$ is mapped to a non-zero vector $\widetilde{Y} \in T_{\psi_\sigma(\alpha_0)}\pi^{-1}(U)$ such that $\mathrm{d}\pi_{\psi_\sigma(\alpha_0)}(\widetilde{Y}) \neq 0$, since the action in (4.2.2) preserves the fibre of $\pi : (M, \omega) \to B$ and σ is an immersion. This is not possible as the vector field $\widetilde{Y}$ is vertical with respect to π. Hence, ψ_σ is a local diffeomorphism. Now let $b_0 \in U$ and $\alpha_0 \in \Lambda_{b_0}$. By definition $\psi_\sigma(\alpha_0) = (\sigma \circ p)(\alpha_0)$. The map ψ_σ is a local diffeomorphism, hence there exists an inverse ψ_σ^{-1} defined on an open neighbourhood $V \subset \pi^{-1}(U)$ of $(\sigma \circ p)(\alpha_0)$. By shrinking U if needed, we may assume that $U = \pi(V)$. The composition

$$\alpha_\sigma := \psi_\sigma^{-1} \circ \sigma : U \to T^*U$$

is a locally defined 1-form, since $p = \pi \circ \psi_\sigma$. In particular, for all $b \in U$ we get

$$\sigma(b) = \psi_\sigma \circ \alpha_\sigma(b) = \phi^1_{\alpha_\sigma(b)}\big(\sigma(b)\big)$$

which means that for all $b \in U$, $\alpha_\sigma(b) \in \Lambda_b|_U$. Define $W := \psi_\sigma^{-1}(V)$ and since ψ_σ^{-1} is an open map, W is an open neighbourhood (diffeomorphic to V) of $\alpha_\sigma(b)$. In the end, the above argument shows that $\alpha_\sigma(U) \subset W \cap \Lambda$. In order to show that Λ is a smooth submanifold of T^*B it suffices to prove that $W \cap \Lambda \subset \alpha_\sigma(U)$, since that would mean that Λ is locally given by the graph of the 1-form α_σ. Let $\beta \in W \cap \Lambda$, then there exists $m \in V = \psi_\sigma(W)$ such that

$$m = \psi_\sigma(\beta) = \phi^1_\beta\big(\sigma \circ p(\beta)\big) \ .$$

On the other hand, $\beta \in \Lambda_{p(\beta)}$ implies that for all $\widetilde{m} \in \pi^{-1}(p(\beta))$, $\phi^1_\beta(\widetilde{m}) = \widetilde{m}$. Therefore,

$$\phi^1_\beta(\sigma \circ p(\beta)) = \sigma \circ p(\beta) \ .$$

Putting all together we get

$$\psi_\sigma(\beta) = \sigma \circ p(\beta)$$

and applying ψ_σ^{-1} to both sides of the equality

$$\beta = \psi_\sigma^{-1} \circ \sigma \circ p(\beta) = \alpha_\sigma \circ p(\beta) \ .$$

Thus proving that $\beta \in \alpha_\sigma(U)$. This completes the proof that Λ is a smooth submanifold of T^*B. In order to show Λ is also closed, let $\{\beta_n\} \subset \Lambda$ be a sequence converging to $\beta \in T^*B$. By taking a small enough neighbourhood $\widetilde{W}$ of β in T^*B, it is possible to ensure that all but finitely many β_n lie in $\widetilde{W}$ and that there exists a local section $\sigma : \widetilde{U} := p(\widetilde{W}) \subset B \to M$. Again, for all but finitely many n, we have

$$\psi_\sigma(\beta_n) = \sigma \circ p(\beta_n)$$

since $\beta_n \in \Lambda_{p(\beta_n)}$ for all $n \in \mathbb{N}$. By continuity of ψ_σ the left hand side of the above equation converges to $\psi_\sigma(\beta)$ and by continuity of $\sigma \circ p$ the right hand side to $\sigma \circ p(\beta)$. Therefore, $\psi_\sigma(\beta) = \sigma \circ p(\beta)$ which means that $\beta \in \Lambda$. It only remains to show that Λ is Lagrangian in $(T^*B, \Omega_{\text{can}})$. Notice that any locally defined section $\alpha : U \to \Lambda|_U$ of $p|_\Lambda : \Lambda \subset T^*B \to B$ is a closed 1-form. In fact, for any such α we get $\phi_\alpha^1 = \text{Id}$, which implies $\left(\phi_\alpha^1\right)^* \omega = \omega$. By Lemma 4.12 it follows that $\pi^* \mathrm{d}\alpha = 0$. Since π is a submersion, we get $\mathrm{d}\alpha = 0$ as required. In the end, the closed submanifold Λ is locally given by the image of closed 1-forms, hence it is Lagrangian in $(T^*B, \Omega_{\text{can}})$. The proof of the first claim is completed.
The proof of the second one relies on the following standard result ([Vai94]): if N is a smooth manifold and $\mathcal{R}$ is an equivalence relation on N whose graph in $N \times N$ is a closed submanifold, then the quotient $N/_{\mathcal{R}}$ is a smooth manifold.
In our case, two elements $\alpha, \beta \in T^*B$ are equivalent if and only if $\alpha - \beta \in \Lambda$. The proof that

$$Q := \left\{(\alpha, \beta) \in T^*B \times T^*B \mid \alpha - \beta \in \Lambda\right\}$$

is a closed submanifold of $T^*B \times T^*B$ can be done in the same way as before. Indeed by repeating the construction above it follows that $Q \cap (W_1 \times W_2) = \alpha_{\sigma_1}(U) \times \beta_{\sigma_2}(U)$, where $\sigma_1, \sigma_2 : U \subset B \to \pi^{-1}(U) \subset M$ are local sections of π and $W_i := \psi_{\sigma_i}^{-1}(V)$ are the corresponding open neighbourhood of $\alpha_{\sigma_1}(b)$ and $\beta_{\sigma_2}(b)$, for some $b \in U = \pi(V)$. □

Corollary 4.14. *A choice of a local section $\sigma : U \subset B \to \pi^{-1}(U) \subset M$ induces a diffeomorphism*

$$\widetilde{\psi}_\sigma : T^*U/_{\Lambda|_U} \to \pi^{-1}(U)$$

which commutes with the projections onto U.

Remark 4.15. The diffeomorphism $\widetilde{\psi}_\sigma$ can be thought of as a local trivialization for the Lagrangian fibration $\pi : (M, \omega) \to B$. In particular, it sends the zero section of $T^*U \to U$ to the image of σ. The main issue of this construction is that a complete Lagrangian fibration $\pi : (M, \omega) \to B$ may not admit a globally defined section and, therefore, there is no natural choice of locally defined sections $\sigma : U \to \pi^{-1}(U)$ to construct the "trivialization" $\widetilde{\psi}_\sigma$.

By Theorem 4.13, the vertical action of Λ (sum of covectors) on the fibres of T^*B induced by a section $\alpha : U \to \Lambda|_U$ is by symplectomorphisms with respect to Ω_{can}. In particular, this implies that the quotient space $T^*B/_\Lambda$ inherits a symplectic form $\widetilde{\omega}$ which makes the induced projection

$$\widetilde{p} : \big(T^*B/_\Lambda, \widetilde{\omega}\big) \to B$$

a complete Lagrangian fibration.

Definition 4.16. Given a complete Lagrangian fibration $\pi : (M, \omega) \to B$ with period net $\Lambda \subset T^*B$, the complete Lagrangian fibration given by

$$\widetilde{p} : \big(T^*B/_\Lambda, \widetilde{\omega}\big) \to B$$

is called the *symplectic reference* fibration associated to $\pi : (M, \omega) \to B$.

Remark 4.17. Any symplectic reference Lagrangian fibration admits a globally defined Lagrangian section, obtained as the image of the zero section $0 : B \hookrightarrow T^*B$ inside $T^*B/_\Lambda$. In fact, if $q : (T^*B, \Omega_{\mathrm{can}}) \to (T^*B/_\Lambda, \widetilde{\omega})$ is the quotient projection such that $q^*\widetilde{\omega} = \Omega_{\mathrm{can}}$ and $s := q \circ 0$, then

$$s^*\widetilde{\omega} = 0^*(q^*\widetilde{\omega}) = 0^*\Omega_{\mathrm{can}} = 0$$

hence $s : B \to (T^*B/_\Lambda, \widetilde{\omega})$ is a globally defined Lagrangian section.

4.2.1 The existence of global Lagrangian sections

Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration as in the previous sections and let $U_i, U_j \subset B$ be open subsets such that $U_i \cap U_j \neq \varnothing$. Pick sections $\sigma_i : U_i \to \pi^{-1}(U_i), \sigma_j : U_j \to \pi^{-1}(U_j)$ and construct local trivializations $\widetilde{\psi}_{\sigma_i}, \widetilde{\psi}_{\sigma_j}$ as in Corollary 4.14. Consider the diffeomorphism

$$\widetilde{\psi}_{\sigma_j}^{-1} \circ \widetilde{\psi}_{\sigma_i} : T^*\big(U_i \cap U_j\big)/_\Lambda|_{U_i \cap U_j} \to T^*\big(U_i \cap U_j\big)/_\Lambda|_{U_i \cap U_j}$$

which leaves the projection onto B invariant and it sends the zero section to $\widetilde{\psi}_{\sigma_j}^{-1}(\sigma_i)$ (see Remark 4.15). It can be proved ([Dui80],[DD87]) that $\widetilde{\psi}_{\sigma_j}^{-1}(\sigma_i)$ is the unique section s_{ji} of $T^*\big(U_i \cap U_j\big)/_\Lambda|_{U_i \cap U_j} \to U_i \cap U_j$ satisfying $\psi_{s_{ji}}^{-1}(\sigma_j) = \sigma_i$. Fixing a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$ in the sense of Leary, i.e. all subsets U_i and all finite intersections of these subsets are contractible, the above construction yields locally defined smooth sections s_{ji} for each pair i, j whose respective open sets in $\mathcal{U}$ intersect non-trivially. By definition, the family s_{ji} defines a Čech 1-cocyle for the cohomology of B with coefficients in the sheaf $C^\infty\big(T^*B/_\Lambda\big)$ of smooth sections of $T^*B/_\Lambda \to B$ and, therefore, a cohomology class $\eta \in H^1\big(B, C^\infty\big(T^*B/_\Lambda\big)\big)$. Let $\mathcal{F}_\Lambda$ be the sheaf of smooth sections of $p|_\Lambda : \Lambda \to B$. There is a short exact sequence of sheaves ([Dui80],[DD87])

$$0 \to \mathcal{F}_\Lambda \to C^\infty(T^*B) \to C^\infty\big(T^*B/_\Lambda\big) \to 0 \tag{4.2.5}$$

where the first map is induced by the inclusion $\Lambda \hookrightarrow T^*B$ and $C^\infty(T^*B)$ is the sheaf of 1-forms of B. It is a standard result that the sheaf $C^\infty(T^*B)$ is *fine*, in particular it is acyclic since B is a paracompact Hausdorff space ([Gun15]). Then, for all $k \geqslant 1$ we have

$$H^k\big(B, C^\infty(T^*B)\big) \cong \{0\} \ .$$

The long exact sequence in cohomology induced by the short exact sequence in (4.2.5) induces an isomorphism

$$\Phi : H^1\big(B, C^\infty(T^*B/_\Lambda)\big) \xrightarrow{\cong} H^2(B, \mathcal{F}_\Lambda)$$

Theorem 4.18 ([Dui80],[DD87]). *The image $\Phi(\eta) =: c_\Lambda \in H^2(B, \mathcal{F}_\Lambda)$ is called the Chern class associated with the Lagrangian fibration $\pi : (M, \omega) \to B$ and $c_\Lambda = 0$ if and only if there exists a globally defined section $\sigma : B \to M$.*

Remark 4.19. The topological (indeed smooth) structure of a complete Lagrangian fibration $\pi : (M, \omega) \to B$ is completely determined by its *period net* Λ and its Chern class $c_\Lambda \in H^2(B, \mathcal{F}_\Lambda)$. More precisely, two complete Lagrangian fibrations are fiber-wise diffeomorphic if and only if they have diffeomorphic period nets and equal (up to diffeomorphism relating the period nets) Chern classes.

In light of the results of the previous section it makes sense to ask for a symplectic classification of complete Lagrangian fibrations. In particular, one might be interested in understanding when the diffeomorphism $\widetilde{\psi}_\sigma$ of Corollary 4.14 can be chosen to be a symplectomorphism between $\big(T^*U/_\Lambda|_U, \widetilde{\omega}\big)$ and $\big(\pi^{-1}(U), \omega\big)$. The first step in this direction is the existence of local Lagrangian sections.

Theorem 4.20 ([FGS03]). *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration. Then, for each $b \in B$ there exists a neighborhood $U \subset B$ of b and a local Lagrangian section $\sigma : U \to \pi^{-1}(U)$.*

Corollary 4.21. *The diffeomorphism $\widetilde{\psi}_\sigma$ is a symplectomorphism from $\big(T^*U/_\Lambda|_U, \widetilde{\omega}\big)$ to $\big(\pi^{-1}(U), \omega\big)$ if and only if the local section σ is Lagrangian.*

Proof. Let $\alpha \in \Gamma(T^*U)$ be a locally defined 1-form on B and let $q : T^*U \to T^*U/_\Lambda|_U$ be the restricted quotient map. Then, $q \circ \alpha : U \to T^*U/_\Lambda|_U$ is a local section of the symplectic reference Lagrangian fibration associated with $\pi : (M, \omega) \to B$ (see Definition 4.16). Applying Lemma 4.12 we get

$$\begin{aligned}
(\phi^1_\alpha)^*\omega &= \omega + \pi^*\mathrm{d}\alpha \\
&= \omega + \pi^*\alpha^*\Omega_{\mathrm{can}} && (\mathrm{d}\alpha = \alpha^*\Omega_{\mathrm{can}}) \\
&= \omega + \pi^*(q \circ \alpha)^*\widetilde{\omega} \ . && (q^*\widetilde{\omega} = \Omega_{\mathrm{can}})
\end{aligned}$$

Applying σ^* to both sides of the equation and using $\pi \circ \sigma = \mathrm{Id}_U$ we have

$$(\phi^1_\alpha \circ \sigma)^*\omega = \sigma^*\omega + (q \circ \alpha)^*\widetilde{\omega}\ .$$

Moreover, since by definition $\phi^1_\alpha \circ \sigma = \widetilde{\psi}_\sigma \circ q \circ \alpha$ and $\widetilde{p} \circ q \circ \alpha = \mathrm{Id}_U$ (see (4.2.4) and Definition 4.16), the last equality can be written as

$$(q \circ \alpha)^*\big((\widetilde{\psi}_\sigma)^*\omega + \widetilde{p}^*\sigma^*\omega - \widetilde{\omega}\big) = 0\ .$$

<u>*Claim*</u>: There exists a locally defined 2-form β on $U \subset B$ such that $(\widetilde{\psi}_\sigma)^*\omega - \widetilde{\omega} = \widetilde{p}^*\beta$. Assuming the claim we can conclude the proof of the theorem. In fact, if such β exists we get

$$(q \circ \alpha)^*\big(\widetilde{p}^*\beta + \widetilde{p}^*\sigma^*\omega\big) = 0\ .$$

Using again that $\widetilde{p} \circ q \circ \alpha = \mathrm{Id}_U$ we obtain $\beta = \sigma^*\omega$, hence

$$(\widetilde{\psi}_\sigma)^*\omega - \widetilde{\omega} = \widetilde{p}^*\sigma^*\omega\ .$$

At this point it is clear that σ is Lagrangian (i.e. $\sigma^*\omega = 0$) if and only if $\widetilde{\psi}_\sigma$ is a symplectomorphism. Finally, the proof of the claim above can be found in [Gro01, Proposition 2.3]. ☐

Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration and choose a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B such that there exists a local Lagrangian section $\sigma_i : U_i \to \pi^{-1}(U)$ for each $i \in I$ (Theorem 4.20). Using Corollary 4.21 we can apply verbatim the construction made at the beginning of the section replacing "diffeomorphism" with "symplectomorphism". In particular, we get the existence of local Lagrangian sections s_{ji} for $\widetilde{p} : \big(T^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j}, \widetilde{\omega}\big) \to U_i \cap U_j$. Let us denote this sheaf of Lagrangian sections by $\mathcal{Z}^1(T^*B/\Lambda)$. As before, the family $\{s_{ji}\}_{i,j \in I}$ defines a Čech cohomology class $\xi \in H^1(B, \mathcal{Z}^1(T^*B/_\Lambda))$, called the *Lagrangian Chern class* associated with the complete Lagrangian fibration $\pi : (M, \omega) \to B$.

Proposition 4.22. *The map $p|_\Lambda : \Lambda \to B$ is a covering space.*

Proof. Notice that the smooth submanifold $\iota : \Lambda \hookrightarrow T^*B$ intersects T^*_bB, for each $b \in B$, at the period lattice $\Lambda_b \cong \mathbb{Z}^k, 1 \leqslant k \leqslant n$ (see Definiton 4.11). Hence, the fibre $(p|_\Lambda)^{-1}(b) \cong \Lambda \cap T^*_bB \cong \Lambda_b \cong \mathbb{Z}^k$ is discrete. Since $p : T^*B \to B$ is a vector bundle, for each $b \in B$ there exists an open neighborhood U_b such that $p^{-1}(U_b) \cong U_b \times T^*_bB$. In particular,

$$\begin{aligned}(p|_\Lambda)^{-1}(U_b) &= \iota^{-1}\big(p^{-1}(U_b)\big)\\ &\cong p^{-1}(U_b) \cap \Lambda\\ &\cong U_b \times (T^*_bB \cap \Lambda)\\ &\cong U_b \times \mathbb{Z}^k\ .\end{aligned}$$

☐

Theorem 4.23 ([Dui80],[DD87]). *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration with period net Λ and vanishing Chern class $c_\Lambda = 0$. Then, it admits a global Lagrangian section if and only if $\xi = 0$.*

Corollary 4.24. *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration over a contractible open connected subset B in $\mathbb{R}^n$. Then, it admits a global Lagrangian section $\sigma : B \to (M, \omega)$.*

Proof. Let Λ be the period net of the complete Lagrangian fibration. There exists a short exact sequence of sheaves

$$0 \to \mathcal{F}_\Lambda \to \mathcal{Z}^1(T^*B) \to \mathcal{Z}^1(T^*B/_\Lambda) \to 0,$$

where $\mathcal{Z}^1(T^*B)$ denotes the sheaf of closed 1-forms on B and $\mathcal{F}_\Lambda$ is the sheaf of sections of the covering $p|_\Lambda : \Lambda \to B$. The sheaf $\mathcal{Z}^1(T^*B)$ can be equivalently described as the sheaf of Lagrangian sections of $p : (T^*B, \Omega_{\text{can}}) \to B$. The long exact sequence induced in cohomology gives

$$\begin{aligned} \ldots \to H^1(B, \mathcal{Z}^1(T^*B)) \to H^1(B, \mathcal{Z}^1(T^*B/_\Lambda)) \xrightarrow{\delta} H^2(B, \mathcal{F}_\Lambda) \to \\ \to H^2(B, \mathcal{Z}^1(T^*B)) \to H^2(B, \mathcal{Z}^1(T^*B/_\Lambda)) \to \ldots . \end{aligned}$$

Using the following isomorphism

$$H^k(B, \mathcal{Z}^1(T^*B)) \cong H^{k+1}_{\text{dR}}(B, \mathbb{R}), \quad \text{if } k \geqslant 1 \qquad \text{(see [Gun15] and [BT+82])}$$

and the hypothesis that B is contractible we get

$$H^1(B, \mathcal{Z}^1(T^*B/_\Lambda)) \cong H^2(B, \mathcal{F}_\Lambda) \; . \tag{4.2.6}$$

On the other hand, the sheaf $\mathcal{F}_\Lambda$ is the sheaf of sections of a covering space over B (Proposition 4.22), hence it is locally constant on B[1]. It is a standard result that over a smooth manifold X, locally constant sheaves of abelian groups $\mathcal{F}_G$ (also known as *local systems*) correspond to representations $\rho : \pi(X, x_0) \to \text{Aut}(G)$ (see [Dim04] Proposition 2.5.1). In our case $X = B$ is contractible, thus any representation as above is trivial and the local system $\mathcal{F}_\Lambda$ is actually isomorphic to the constant sheaf $\underline{\mathbb{Z}}^k$, for some $1 \leqslant k \leqslant n$ (see Definition 4.11). In the end,

$$\begin{aligned} H^1(B, \mathcal{Z}^1(T^*B/_\Lambda)) &\cong H^2(B, \mathcal{F}_\Lambda) \\ &\cong H^2(B, \underline{\mathbb{Z}}^k) && \text{(sheaf cohomology)} \\ &\cong H^2(B, \mathbb{Z}^k) && \text{(singular cohomology)} \\ &\cong \Big(H^2(B, \mathbb{Z})\Big)^k \cong \{0\} \end{aligned}$$

In particular, both the Chern class and the Lagrangian Chern class of the fibration vanish. As a consequence of Theorem 4.18 and Theorem 4.23 we get the existence of a global Lagrangian section $\sigma : B \to (M, \omega)$. $\square$

[1] A sheaf of abelian groups $\mathcal{F}$ on X is locally constant if for each $x \in X$ there exists a neighborhood U containing x such that $\mathcal{F}|_U$ is a constant sheaf on U.

4.3 The Hamiltonian actions

Here we recall the existence of the circle and $\mathrm{SL}(2,\mathbb{R})$ actions we defined on $\mathcal{B}_0(T^2)$. Then, we compute the Hamiltonian function of the associated restricted $\mathbb{R}^* < \mathrm{SL}(2,\mathbb{R})$ action.

Lemma 4.25. *Let (M,ω) be a symplectic manifold endowed with a Hamiltonian G-action and moment map $\mu_G : M \to \mathfrak{g}^*$. If $H \leqslant G$ is any closed subgroup, then the restricted H-action is Hamiltonian with moment map $\mu_H : M \to \mathfrak{h}^*$ given by $\mu_H := |_{\mathfrak{h}} \circ \mu_G$, where $|_{\mathfrak{h}} : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map which associates to each functional on $\mathfrak{g}$ its restriction on $\mathfrak{h}$.*

From Lemma 2.20 and Corollary 2.4 the deformation space $\mathcal{B}_0(T^2)$ is diffeomorphic to $\mathbb{H}^2 \times \mathbb{C}^*$. Let us denote with (z,w) the coordinates on $\mathbb{H}^2 \times \mathbb{C}$. Since the circle action

$$(z,w) \mapsto (z, e^{-i\theta}w), \quad \theta \in \mathbb{R}$$

preserves $\mathbb{H}^2 \times \{0\}$, we can consider $\widehat{\Psi}_\theta$ to be the induced S^1-action on $\mathcal{B}_0(T^2)$. We proved that such an action is Hamiltonian with respect to $\widehat{\boldsymbol{\omega}}_f$ and $\widehat{\Psi}_\theta^*\widehat{\mathbf{g}}_f = \widehat{\mathbf{g}}_f$. In particular the Hamiltonian function is given by

$$H_1(z,w) = \frac{2}{3} f\Big(\frac{\mathrm{Im}(z)^3|w|^2}{2}\Big) \ .$$

The $\mathrm{SL}(2,\mathbb{R})$-action defined in (2.2.16) preserves $\mathbb{H}^2 \times \{0\}$ as well, and we proved that it is Hamiltonian on $\mathcal{B}_0(T^2) \cong \mathbb{H}^2 \times \mathbb{C}$ with associated moment map

$$\widehat{\mu}^X(z,w) = \bigg(1 - f\Big(\frac{\mathrm{Im}(z)^3|w|^2}{2}\Big)\bigg) \operatorname{tr}(j(z)X), \quad X \in \mathfrak{sl}(2,\mathbb{R}) \ .$$

Notice that inside $\mathrm{SL}(2,\mathbb{R})$ there is the subgroup of diagonal matrices with determinant equal to one, namely

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \ \middle|\ \lambda \in \mathbb{R}^* \right\} < \mathrm{SL}(2,\mathbb{R}) \ . \tag{4.3.1}$$

In particular, such a subgroup can be identified with a copy of $\mathbb{R}^*$ which still acts in a Hamiltonian fashion (Lemma 4.25) on the space $\mathcal{B}_0(T^2)$.

Lemma 4.26. *Let $\mathbb{R}^*$ be a copy of the subgroup of diagonal matrices in $\mathrm{SL}(2,\mathbb{R})$ and consider its restricted Hamiltonian action on $\mathcal{B}_0(T^2)$, then the Hamiltonian function is given by*

$$H_2(z,w) = 2\frac{x}{y}\bigg(1 - f\Big(\frac{y^3|w|^2}{2}\Big)\bigg) \ .$$

Proof. The Lie algebra of $\mathbb{R}^*$ can be identified with

$$\mathfrak{h} := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \ \middle|\ \alpha \in \mathbb{R} \right\} .$$

By Lemma 4.25, the the associated moment map for the restricted $\mathbb{R}^*$-action $\widehat{\mu}_{\mathfrak{h}} : \mathcal{B}_0(T^2) \to \mathfrak{h}^*$ is

$$\widehat{\mu}_{\mathfrak{h}}^X(z,w) = \left(1 - f\Big(\frac{y^3|w|^2}{2}\Big)\right) \operatorname{tr}(j(z)X),$$

where $X \in \mathfrak{h}$. Let $\xi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}$, then the Hamiltonian function $H_2 : \mathcal{B}_0(T^2) \to \mathbb{R}$ is $H_2(z,w) := \mu_{\mathfrak{h}}^{\xi}(z,w)$, given that $\mathrm{d}\mu_{\mathfrak{h}}^{\xi} = \widehat{\boldsymbol{\omega}}_f(V_\xi, \cdot)$, where

$$V_\xi = 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 3\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)$$

is the infinitesimal generator of the action. Finally, since

$$\operatorname{tr}(j(z)\xi) = \operatorname{tr}\left(\begin{pmatrix} \frac{x}{y} & -\frac{x^2+y^2}{y} \\ \frac{1}{y} & -\frac{x}{y} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = 2\frac{x}{y}$$

we get $H_2(z,w) = 2\dfrac{x}{y}\left(1 - f\Big(\dfrac{y^3|w|^2}{2}\Big)\right)$. □

4.3.1 Global Darboux coordinates

In this section we prove the main result regarding the symplectic geometry of $\big(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f\big)$.

Proposition 4.27. *The Hamiltonian system $(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f, H_1)$ is completely integrable. The integrals of motion are given by*

$$H_1(z,w) = \frac{2}{3} f\Big(\frac{y^3|w|^2}{2}\Big), \quad H_2(z,w) = 2\frac{x}{y}\left(1 - f\Big(\frac{y^3|w|^2}{2}\Big)\right) .$$

Proof. Let $\mathbb{X}_{H_1}, \mathbb{X}_{H_2}$ be the Hamiltonian vector fields associated with H_1, H_2. An explicit expression is given by

$$\mathbb{X}_{H_1} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}, \quad \mathbb{X}_{H_2} = 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 3\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) .$$

It is clear that they are point-wise linearly independent on $\mathcal{B}_0(T^2)$, hence to end the proof we only need to show that they are involutive. The symplectic form is

$$\begin{aligned} \widehat{\boldsymbol{\omega}}_f = &\left(-1 + f - \frac{3}{2} f' y^3 (u^2+v^2)\right) \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} - \frac{2}{3} f' y^3 \mathrm{d}u \wedge \mathrm{d}v \\ &- y^2 f' \Big(u(\mathrm{d}x \wedge \mathrm{d}u + \mathrm{d}y \wedge \mathrm{d}v) + v(\mathrm{d}u \wedge \mathrm{d}y - \mathrm{d}v \wedge \mathrm{d}x)\Big) . \end{aligned}$$

Moreover,

$$\widehat{\boldsymbol{\omega}}_f(\mathbb{X}_{H_1},\mathbb{X}_{H_2}) = 2\left(ux\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial v},\frac{\partial}{\partial x}\right) + uy\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial v},\frac{\partial}{\partial y}\right) - vx\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial u},\frac{\partial}{\partial x}\right) - vy\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial u},\frac{\partial}{\partial y}\right)\right) - 3\left(u^2\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial v},\frac{\partial}{\partial u}\right) - v^2\widehat{\boldsymbol{\omega}}_f\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)\right).$$

It is easy to see that the term at the right hand side of the last equality is equal to zero, hence we get the claim. □

Let $H := (H_1, H_2) : (\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f) \to B$ be the Lagrangian fibration associated with the above completely integrable Hamiltonian system (see Remark 4.5), where $B := H\big(\mathcal{B}_0(T^2)\big) \subset \mathbb{R}^2$. Using the explicit expression of the integrals of motion and the properties of the function f, it is clear that B is homeomorphic to $U := \big\{(u_1,u_2) \in \mathbb{R}^2 \mid u_1 < 0\big\}$, hence it is contractible. Moreover, any $b = (b_1, b_2) \in B$ is a regular value for H and each fiber

$$H^{-1}(b) = \left\{(z,w) \in \mathcal{B}_0(T^2) \;\middle|\; \frac{2}{3}f\left(\frac{y^3|w|^2}{2}\right) = b_1,\ 2\frac{x}{y}\left(1 - f\left(\frac{y^3|w|^2}{2}\right)\right) = b_2\right\}$$

is diffeomorphic to $\mathbb{R} \times S^1$.

Remark 4.28. The fact that each fiber is diffeomorphic to $\mathbb{R} \times S^1$ can be seen directly from Theorem 4.4, since the vector fields $\mathbb{X}_{H_1}, \mathbb{X}_{H_2}$ are complete on $H^{-1}(b)$, for each $b \in B$. Indeed $\mathbb{X}_{H_1}$ is the generator of the counter clock-wise rotation in the plane and the integral curve of $\mathbb{X}_{H_2}$ passing through the point (z,w) is $\gamma_{(z,w)}(t) = (e^{2t}z, e^{-3t}w)$, which is defined for all $t \in \mathbb{R}$.

Theorem G. *Let $(s,\theta) \in \mathbb{R} \times S^1$ be the angle coordinates of $(\mathcal{B}_0(T^2), H_1, \widehat{\boldsymbol{\omega}}_f)$ given by the Arnold-Liouville theorem. Then, $\big\{\theta, H_1, s, H_2\big\}$ is a global Darboux frame for $\widehat{\boldsymbol{\omega}}_f$.*

Proof. The Lagrangian fibration $H : (\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f) \to B$ is the one arising from the completely integrable Hamiltonian system $(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f, H_1)$. Since the vector fields $\mathbb{X}_{H_1}, \mathbb{X}_{H_2}$ are complete on each fiber $H^{-1}(b)$, by Proposition 4.9 the associated Lagrangian fibration $H : (\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f) \to B$ is complete (see Definition 4.8). Moreover, the base B is a contractible open subset of $\mathbb{R}^2$. Using Corollary 4.24 we get the existence of a global Lagrangian section $\sigma : B \to \mathcal{B}_0(T^2)$. In particular, $\sigma(B)$ is a Lagrangian submanifold of $(\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f)$, $\sigma(b) \in H^{-1}(b)$ for each $b \in B$ and $H \circ \sigma = \mathrm{Id}_B$. Let $b = (b_1, b_2) \in B$, then the vector fields

$$\frac{\partial}{\partial H_i} = \mathrm{d}\sigma\left(\frac{\partial}{\partial b_i}\right), \quad i = 1,2$$

are tangent to $\sigma(B)$ and they generate a Lagrangian subspace of $T_{\sigma(b)}\mathcal{B}_0(T^2)$. In fact,

$$(\widehat{\boldsymbol{\omega}}_f)_{\sigma(b)}\left(\frac{\partial}{\partial H_1},\frac{\partial}{\partial H_2}\right) = (\widehat{\boldsymbol{\omega}}_f)_{\sigma(b)}\left(\mathrm{d}\sigma\left(\frac{\partial}{\partial b_1}\right), \mathrm{d}\sigma\left(\frac{\partial}{\partial b_2}\right)\right)$$

$$= (\sigma^* \widehat{\boldsymbol{\omega}}_f)_b \left(\frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2} \right)$$
$$= 0 \ . \qquad \text{(The section } \sigma \text{ is Lagrangian)}$$

Let $(s, \theta) \in \mathbb{R} \times S^1$ be the angle coordinates given by the Arnlod-Liouville Theorem, then the vector fields $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial s}$ are point-wise tangent to the fiber of $H : (\mathcal{B}_0(T^2), \widehat{\boldsymbol{\omega}}_f) \to B$. In particular, they correspond to $\mathbb{X}_{H_1}$ and $\mathbb{X}_{H_2}$ respectively. Hence,

$$\widehat{\boldsymbol{\omega}}_f \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial s} \right) = \widehat{\boldsymbol{\omega}}_f \big(\mathbb{X}_{H_1}, \mathbb{X}_{H_2} \big) = 0 \ . \qquad \text{(Involution)}$$

Let us denote the coordinate s with g_1 and θ with g_2. In order to conclude the proof of the theorem, we need to show that

$$(\widehat{\boldsymbol{\omega}}_f)_x \left(\frac{\partial}{\partial g_i}, \frac{\partial}{\partial H_j} \right) = \delta^i_j, \quad \forall x \in \mathcal{B}_0(T^2) \ . \tag{4.3.2}$$

Suppose first $x \in \sigma(B)$, hence $(x_1, x_2) = \sigma(b_1, b_2)$ for some $(b_1, b_2) \in B$. Then,

$$\begin{aligned}
(\widehat{\boldsymbol{\omega}}_f)_x \left(\frac{\partial}{\partial g_i}, \frac{\partial}{\partial H_j} \right) &= (\widehat{\boldsymbol{\omega}}_f)_x \left(\mathbb{X}_{H_i}, \mathrm{d}_b \sigma \left(\frac{\partial}{\partial b_j} \right) \right) \\
&= \mathrm{d}_x H_i \left(\mathrm{d}_b \sigma \left(\frac{\partial}{\partial b_j} \right) \right) && (4.1.1) \\
&= \mathrm{d}_b \big(H_i \circ \sigma \big) \left(\frac{\partial}{\partial b_j} \right) && \text{(Chain rule)} \\
&= \delta^i_j \ . && (H_i \circ \sigma = b_i)
\end{aligned}$$

Now let x be an arbitrary point of $\mathcal{B}_0(T^2)$ and let Ψ^t_i be the Hamiltonian flow associated with H_i. Since the flow action on the fiber $H^{-1}(b)$ is transitive, we can always assume that $x = \Psi^t_i\big(\sigma(H(x))\big)$, where $b = H(x)$. In particular, we have that the vector field $\frac{\partial}{\partial H_j}$ computed at $x = \Psi^t_i\big(\sigma(H(x))\big)$ is equal to $\mathrm{d}\Psi^t_i \left(\frac{\partial}{\partial H_j} \right)$, where now the vector field inside the differential of Ψ^t_i is computed at $\sigma(H(x))$. Hence,

$$\begin{aligned}
(\widehat{\boldsymbol{\omega}}_f)_x \left(\frac{\partial}{\partial g_i}, \frac{\partial}{\partial H_j} \right) &= (\widehat{\boldsymbol{\omega}}_f)_x \left(\mathbb{X}_{H_i}, \mathrm{d}\Psi^t_i \left(\frac{\partial}{\partial H_j} \right) \right) \\
&= \left(\big((\Psi^t_i)^{-1} \big)^* \widehat{\boldsymbol{\omega}}_f \right)_x \left(\mathbb{X}_{H_i}, \mathrm{d}\Psi^t_i \left(\frac{\partial}{\partial H_j} \right) \right) && (\Psi^t_i \text{ preserves } \widehat{\boldsymbol{\omega}}_f) \\
&= (\widehat{\boldsymbol{\omega}}_f)_{\sigma(H(x))} \left(\big(\mathrm{d}\Psi^t_i \big)^{-1} \Big(\mathbb{X}_{H_i} \Big), \frac{\partial}{\partial H_j} \right)
\end{aligned}$$

$$
\begin{aligned}
&= (\widehat{\boldsymbol{\omega}}_f)_{\sigma(H(x))}\left(\mathbb{X}_{H_i}, \frac{\partial}{\partial H_j}\right) && (\Psi_i^t \text{ is the flow associated with } H_i) \\
&= \delta_i^j && (\sigma(H(x)) \in \sigma(B))
\end{aligned}
$$

□

4.4 The Ricci tensor and the scalar curvature

In this section we show that the copy of the hyperbolic plane $\mathbb{H}^2 \times \{0\} \subset \mathbb{H}^2 \times \mathbb{C} \cong D^3(\mathcal{J}(\mathbb{R}^2))$ is the only embedded submanifold with scalar curvature equal to 2, whenever $f(t) = -kt$ for $k > 0$. The formulae given for the scalar curvature of $\widehat{\mathbf{g}}_f$ differ by a rescaling factor with respect the ones in [RT22] (see Remark 1.14 and Remark 2.21).

A *Kähler manifold* is a particular case of a pseudo-Kähler one, namely when the pseudo-Riemannian metric has index equal to zero. For this very reason it is natural to ask whether some properties of Kähler manifolds still holds in this more general setting. Now we briefly recall the definition of some curvature tensors defined on Kähler manifolds and we will explain how their formulae still hold in the pseudo-Riemannian setting as long as the pseudo-metric is of neutral signature.

Let (M, g, I) be a Kähler manifold of complex dimension n. The tensor I can be extended by $\mathbb{C}$-linearity on the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$. Since $I^2 = -\mathbb{1}$ there is an eigenbundle decomposition $T_{\mathbb{C}}M = T^{1,0} \oplus T^{0.1}M$, where

$$T^{1,0}M := \{X \in T_{\mathbb{C}}M \mid I(X) = i \cdot X\}, \qquad T^{0,1}M := \{X \in T_{\mathbb{C}}M \mid I(X) = -i \cdot X\}\ .$$

The bundle $T^{1,0}M$ is called the *holomorphic tangent bundle* and $T^{0,1}M$ the *anti-holomorphic tangent bundle* of M, in particular they are the conjugate of each other. If $(z_1, \dots, z_n)$ are local holomorphic coordinates on M, the n-dimensional complex vector space $T^{1,0}M$ is generated by $\{\frac{\partial}{\partial z_1}, \dots \frac{\partial}{\partial z_n}\}$. Since $z_k = x_k + iy_k$ for each $k = 1, \dots, n$ we have

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \qquad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right), \quad \forall k = 1, \dots n\ .$$

If we denote with $g^{\mathbb{C}}$ the $\mathbb{C}$-linear extension of g to $T_{\mathbb{C}}M$, then locally it can be written as

$$g^{\mathbb{C}} := \sum_{j,k} g^{\mathbb{C}}_{j\bar{k}}\big(\mathrm{d}z^j \otimes \mathrm{d}\bar{z}^k + \mathrm{d}\bar{z}^k \otimes \mathrm{d}z^j\big)$$

where $g^{\mathbb{C}}_{j\bar{k}} := g^{\mathbb{C}}\big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\big) = \frac{1}{4}\Big(g\big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\big) + g\big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\big) - ig\big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\big) + ig\big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\big)\Big)$, since the Hermitian condition implies that $g^{\mathbb{C}}_{jk} = g^{\mathbb{C}}_{\bar{j}\bar{k}} = 0$ and the symmetry that $\overline{g^{\mathbb{C}}_{j\bar{k}}} = g^{\mathbb{C}}_{k\bar{j}}$ for each $j, k = 1, \ldots, n$.

In the following, by abuse of notation, we will denote with g the metric extended by $\mathbb{C}$-linearity on $T_{\mathbb{C}}M$. If ∇ denotes the Levi-Civita connection of g, then the only non vanishing Christoffel symbols are

$$\Gamma^i_{jk} := g^{i\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial z_j}, \qquad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} := \overline{\Gamma^i_{jk}}$$

where $g^{j\bar{k}}$ denotes the inverse metric computed on $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}$. The Riemann curvature tensor $R \in \Gamma\big(T^*_{\mathbb{C}}M \otimes T_{\mathbb{C}}M \otimes \mathrm{End}(T_{\mathbb{C}}M)\big)$ of ∇ is given by

$$R_i{}^j{}_{k\bar{l}} = -\frac{\Gamma^j_{ki}}{\partial \bar{z}_l}, \qquad R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_\alpha{}^j{}_{\gamma\bar{\delta}} g_{j\bar{\beta}} \ .$$

As a consequence of the Bianchi identity, the Riemann tensor enjoys the following symmetries

$$R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}} \ .$$

Finally, the Ricci tensor and the scalar curvature are given, respectively, by:

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \qquad \mathrm{scal}(g) = g^{i\bar{j}} R_{i\bar{j}} \ .$$

Remark 4.29. All the properties listed so far hold in the case of pseudo-Kähler manifolds, indeed they are only a consequence of the fact that the metric is non-degenerate and that $\nabla g = \nabla I = 0$ (see [Zhe01]).

Lemma 4.30. *Let (M, g, I) be a pseudo-Kähler manifold of real dimension $4n$ and of neutral signature $(2n, 2n)$, then*

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\big(\det(g)\big) \ .$$

Proof. First, notice that $\log\big(\det(g)\big)$ is well-defined since the pseudo-metric g is of neutral signature, hence $\det(g) > 0$. Then, by using the formulae above, we get

$$\begin{aligned}
R_{i\bar{j}} &= g^{k\bar{l}} R_{k\bar{l}i\bar{j}} && (\text{symmetry of } R_{i\bar{j}k\bar{l}}) \\
&= R_p{}^p{}_{i\bar{j}} && (R_{k\bar{l}i\bar{j}} = g_{p\bar{l}} R_k{}^p{}_{i\bar{j}}) \\
&= -\frac{\partial \Gamma^p_{ip}}{\partial \bar{z}_j} \\
&= -\frac{\partial}{\partial \bar{z}_j}\Big(g^{p\bar{q}} \frac{\partial g_{p\bar{q}}}{\partial z_i}\Big) \\
&= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\big(\det(g)\big) \ . && (\text{Jacobi's formula})
\end{aligned}$$

□

Before computing the Ricci tensor and the scalar curvature of the new metrics, it should be noted that it is sufficient to do the computation at points $(i, u) \in \mathbb{H}^2 \times \mathbb{C}$. In fact, the $\mathrm{SL}(2, \mathbb{R})$-action introduced in Section 2.2.1 allows us to reduce to the points (i, w) and the natural S^1-action on $\mathbb{C}$ introduced in Section 2.3, to the points (i, u), since both actions are isometric (Theorem 2.16 and Theorem E). Furthermore, we need to write $\det(\widehat{\mathbf{g}}_f)$ and the inverse of the metric $\widehat{\mathbf{g}}_f$, extended by $\mathbb{C}$-linearity on $T_{\mathbb{C}}(\mathbb{H}^2 \times \mathbb{C})$, in terms of the coordinates (z, w). We have:

$$
\begin{aligned}
(\widehat{\mathbf{g}}_f^{z\bar{z}})_{(i,u)} &= \frac{1}{1-f}, \quad (\widehat{\mathbf{g}}_f^{w\bar{w}})_{(i,u)} = \frac{3(1-f+3f'u^2)}{4f'(1-f)} \\
(\widehat{\mathbf{g}}_f^{z\bar{w}})_{(i,u)} &= i\frac{3u}{2(1-f)}, \quad \det(\widehat{\mathbf{g}}_f)_{(z,w)} = \frac{4}{9}\,\mathrm{Im}(z)^2(f')^2(1-f)^2 \ .
\end{aligned}
$$

Proposition 4.31. *The Ricci tensor and the scalar curvature of the pseudo-Kähler metrics* $(\widehat{\mathbf{g}}_f, \widehat{\mathbf{I}}, \widehat{\boldsymbol{\omega}}_f)$ *are given by:*

$$
\begin{aligned}
(R_{z\bar{z}})_{(i,u)} &= \frac{1}{2} + 3u^2\left(\frac{f'}{1-f} - \frac{f''}{f'}\right) + \frac{9}{2}u^4 G_f \\
(R_{w\bar{w}})_{(i,u)} &= -2\left(\frac{f''}{f'} - \frac{f'}{1-f}\right) + 2u^2 G_f \\
(R_{z\bar{w}})_{(i,u)} &= i\left(3u\left(\frac{f''}{f'} - \frac{f'}{1-f}\right) - 3u^3 G_f\right) \\
\mathrm{scal}(\widehat{\mathbf{g}}_f)_{(i,u)} &= \frac{2}{1-f} - \frac{3}{2}\frac{f''}{(f')^2} + \frac{3u^2}{1-f}\left(6u^2 G_f + \frac{11}{2}\left(\frac{f'}{1-f} - \frac{f''}{f'}\right) + \frac{G_f(1-f)}{2f'}\right)
\end{aligned}
$$

where $G_f := \dfrac{f''(1-f) + (f')^2}{(1-f)^2} - \dfrac{f''' \cdot f - (f'')^2}{(f')^2}$.

Proof. Using the formulae above and the symmetries $\overline{R_{i\bar{j}}} = R_{j\bar{i}}$, the Ricci tensor is given by

$$
\mathrm{Ric}_{\widehat{\mathbf{g}}_f} = R_{z\bar{z}}\mathrm{d}z \otimes \mathrm{d}\bar{z} + R_{w\bar{w}}\mathrm{d}w \otimes \mathrm{d}\bar{w} + 2\mathcal{R}e(R_{z\bar{w}})\mathrm{d}z \otimes \mathrm{d}\bar{w} \ .
$$

According to Lemma 4.30, the components can be computed as

$$
R_{z\bar{z}} = -\frac{\partial^2}{\partial z \partial \bar{z}}\log\big(\det(\widehat{\mathbf{g}}_f)\big), \quad R_{w\bar{w}} = -\frac{\partial^2}{\partial w \partial \bar{w}}\log\big(\det(\widehat{\mathbf{g}}_f)\big), \quad R_{z\bar{w}} = -\frac{\partial^2}{\partial z \partial \bar{w}}\log\big(\det(\widehat{\mathbf{g}}_f)\big) \ .
$$

Using the expression of $\det(\widehat{\mathbf{g}}_f)$ found above we get

$$
\log\big(\det(\widehat{\mathbf{g}}_f)\big) = \log\left(\frac{4}{9}\right) + 2\log\big(\mathrm{Im}(z)\big) + \log\big((f')^2\big) + 2\log\big(1-f\big) \ .
$$

Finally, recalling that the functions f, f', f'', f''' are all evaluated at $\mathrm{Im}(z)^3|w|^2$ and using the formula $\dfrac{\partial}{\partial z}\mathrm{Im}(z)^l = \dfrac{l}{(2i)^l}\mathrm{Im}(z)^{l-1}$ we obtain the desired expression for the components

of the Ricci tensor.
The scalar curvature is given, by definition, by

$$\mathrm{scal}(\widehat{\mathbf{g}}_f) = \widehat{\mathbf{g}}_f^{z\bar{z}} R_{z\bar{z}} + \widehat{\mathbf{g}}_f^{w\bar{w}} R_{w\bar{w}} + \widehat{\mathbf{g}}_f^{z\bar{w}} R_{z\bar{w}} + \widehat{\mathbf{g}}_f^{w\bar{z}} R_{w\bar{z}} \ .$$

Since $\widehat{\mathbf{g}}_f^{w\bar{z}} R_{w\bar{z}} = \overline{\widehat{\mathbf{g}}_f^{z\bar{w}} R_{z\bar{w}}}$, the final expression is

$$\mathrm{scal}(\widehat{\mathbf{g}}_f) = \widehat{\mathbf{g}}_f^{z\bar{z}} R_{z\bar{z}} + \widehat{\mathbf{g}}_f^{w\bar{w}} R_{w\bar{w}} + 2\mathcal{R}e(\widehat{\mathbf{g}}_f^{z\bar{w}} R_{z\bar{w}}) \ .$$

Now, we can directly compute the scalar curvature at the points (i,u). By a simple, but long enough, direct calculation, one gets the desired formula. □

As one can see, these expressions are too complicated to be able to make any estimates on the scalar curvature. On the other hand, on $\mathbb{H}^2 \times \{0\} \subset \mathbb{H}^2 \times \mathbb{C}$ the expression is considerably simplified, indeed given that $f(0) = 0$, it follows that

$$\mathrm{scal}(\widehat{\mathbf{g}}_f)_{(i,0)} = 2 - \frac{3}{2}\frac{f''(0)}{f'(0)^2} \ . \tag{4.4.1}$$

In particular, if we pick the function f to be of the form $f(t) = -kt$, with $k > 0$, it becomes clear that the scalar curvature on $\mathbb{H}^2 \times \{0\}$ is constant and equal to 2.

Corollary 4.32. *For any $(i,u) \in \mathbb{H}^2 \times \mathbb{C}^*$ and for $f(t) = -kt$, with $k > 0$, the scalar curvature* $\mathrm{scal}(\widehat{\mathbf{g}}_f)_{(i,u)}$ is strictly less then 2.

Proof. For this choice of f, at the point (i,u) we have

$$f'(t) = -k, \quad f'' = f''' = 0, \quad G_f(t) = \frac{k^2}{(1+kt^2)}, \qquad t = u^2 \neq 0 \ .$$

Thanks to Proposition 4.31 it follows that

$$\mathrm{scal}(\widehat{\mathbf{g}}_f)_{(i,u)} = \frac{2}{1+kt}\left(1 + 3t\left(\frac{6tk^2}{(1+kt)^2} - \frac{6k}{1+kt}\right)\right) \ .$$

Using that $\dfrac{1}{1+kt} < 1$ for all $k > 0$ and $t > 0$, we obtain

$$\mathrm{scal}(\widehat{\mathbf{g}}_f)_{(i,u)} < 2 + \frac{36tk}{1+kt}\left(\frac{tk}{1+kt} - 1\right) \ .$$

The last quantity is strictly less then 2 since

$$\frac{36tk}{1+kt} > 0, \quad \frac{tk}{1+kt} - 1 < 0, \quad \forall t,k > 0 \ .$$

□

4.4.1 The isometry group

It is clear from Theorem 2.16 and Theorem E, that any matrix in $\mathbb{PSL}(2,\mathbb{R})$ and any rotation of the fiber generated by S^1, is an isometry of $\mathbb{H}^2 \times \mathbb{C}$ with respect to $\widehat{\mathbf{g}}_f$. Plus, the two actions commute. We will show that, whenever $f(t) = -kt$, with $k > 0$, any other isometry h, isotopic to the identity, can be written as composition $h = P \circ e^{i\theta}$ for some $(P, e^{i\theta}) \in \mathbb{PSL}(2,\mathbb{R}) \times S^1$. Finally, we deduce the expression for an arbitrary isometry of the space.

Lemma 4.33. *Let $h_1, h_2 : (M_1, g_1) \to (M_2, g_2)$ be two isometries between smooth connected pseudo-Riemannian manifolds. If there is a point $p \in M_1$ such that $h_1(p) = h_2(p)$ and $\mathrm{d}_p h_1 \equiv \mathrm{d}_p h_2$, then $h_1 \equiv h_2$.*

Proof. Let $\mathcal{C} := \{q \in M_1 \mid \mathrm{d}_q h_1 = \mathrm{d}_q h_2\}$. Then, by continuity $\mathcal{C}$ is a closed subset in M_1. Since $p \in \mathcal{C}$ by hypothesis, it is non-empty. Therefore, it only remains to show that $\mathcal{C}$ is open in M_1. We infer that if $q \in \mathcal{C}$, then any normal neighborhood $\mathcal{U}$ of q is contained in $\mathcal{C}$. In fact, if $r \in \mathcal{U}$ there exists a $v \in T_q M_1$ such that $\gamma_v(1) = \exp_q(v) = r$. Thus,

$$h_1(r) = h_1\big(\gamma_v(1)\big) = \gamma_{\mathrm{d}h_1(v)}(1) = \gamma_{\mathrm{d}h_2(v)}(1) = h_2\big(\gamma_v(1)\big) = h_2(r) \ .$$

In other words, the functions h_1 and h_2 coincide when restricted on $\mathcal{U}$, hence $\mathrm{d}_r h_1 = \mathrm{d}_r h_2$ for all $r \in \mathcal{U}$, which implies $\mathcal{U} \subset \mathcal{C}$. □

Theorem 4.34. *Let $\mathrm{Isom}_0\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$ be the connected component of the identity of the isometry group $\mathrm{Isom}\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$. If $f(t) = -kt$, with $k > 0$, then $\mathrm{Isom}_0\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big) \cong \mathbb{PSL}(2,\mathbb{R}) \times S^1$.*

Proof. First notice that each isometry $h \in \mathrm{Isom}_0\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$ preserves the copy of the hyperbolic plane $\mathbb{H}^2 \times \{0\}$. In fact, if there was an isometry $\widetilde{h}$ with $\widetilde{h}(z,0) = (z', w)$ for some $(z', w) \in \mathbb{H}^2 \times \mathbb{C}^*$, then we would get the following contradiction

$$\begin{aligned} 2 &= \mathrm{scal}(\widehat{\mathbf{g}}_f)_{(z,0)} \\ &= \big(\widetilde{h}^* \mathrm{scal}(\widehat{\mathbf{g}}_f)\big)_{(z,0)} && (\widetilde{h} \text{ is an isometry}) \\ &= \mathrm{scal}(\widehat{\mathbf{g}}_f)_{(z',w)} \\ &< 2 \ . && \text{(Corollary 4.32)} \end{aligned}$$

Pick any $h \in \mathrm{Isom}_0\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$ such that $h(z,0) = (z',0)$ for some $z, z' \in \mathbb{H}^2$. We can always assume that $h(i,0) = (i,0)$, indeed there exist two matrices $P, P' \in \mathrm{SL}(2,\mathbb{R})$ such that $(z,0) = P \cdot (i,0)$, $(z',0) = P' \cdot (i,0)$, hence the isometry $(P')^{-1} \circ h \circ P$ would fix the point $(i,0)$. If we consider the linear map $\mathrm{d}_{(i,0)} h : T_i\mathbb{H}^2 \times T_0\mathbb{C} \to T_i\mathbb{H}^2 \times T_0\mathbb{C}$ restricted to horizontal directions, we can again assume, up to pre- and post-composition with elements in $\mathbb{PSL}(2,\mathbb{R})$ as before, that $\mathrm{d}_{(i,0)} h(Z,0) = (Z,0)$, for all $Z \in T_i\mathbb{H}^2$. This implies that

$$\mathrm{d}_{(i,0)} h_{|_{T_i\mathbb{H}^2}} = \mathrm{Id}_{T_i\mathbb{H}^2} \ .$$

Now if $(0,U) \in T_i\mathbb{H}^2 \times T_0\mathbb{C}^*$ is a real vertical direction, then $\mathrm{d}_{(i,0)}h(0,U) = (0,W)$ for some $W \in T_0\mathbb{C}^*$. In particular, since h is an isometry we get

$$||U||^2_{\widehat{\mathbf{g}}_f} = ||W||^2_{\widehat{\mathbf{g}}_f} \ ,$$

which implies that $W = e^{i\theta}U$ for some $\theta \in \mathbb{R}$. Furthermore, since the circular action is an isometry for $\widehat{\mathbf{g}}_f$ that is trivial on the base $\mathbb{H}^2$, up to pre- and post-composing with rotations in the complex plane we have

$$\mathrm{d}_{(i,0)}h(Z,U) = (Z,U)$$

for all $Z \in T_i\mathbb{H}^2$. Since h is orientation preserving, we deduce that $d_{(i,0)}h = \mathrm{Id}$, since h should also fix an imaginary vertical tangent vector. In the end, using Lemma 4.33, we obtain that $h = \mathrm{Id}$ on the whole $\mathbb{H}^2 \times \mathbb{C}$ after possibly pre- and post-composing h by elements of $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ and rotations. Therefore, h was of the form $h = P \circ e^{i\theta}$ for some $(P, e^{i\theta}) \in \mathbb{P}\mathrm{SL}(2,\mathbb{R}) \times S^1$. □

During the proof of the theorem we used that each isometry isotopic to the identity preserves the orientations on both $\mathbb{H}^2$ and $\mathbb{C}$. There are other three possibilities for an arbitrary isometry $h \in \mathrm{Isom}\big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$:

- h reverses the orientation on $\mathbb{H}^2$ and preserves the orientation on $\mathbb{C}$. Then, by composing with $h_1(z,w) := (-\bar{z}, w)$ we get an isometry preserving both orientations. Hence, the proof of Theorem 4.34 holds for $h \circ h_1$.
- h preserves the orientation on $\mathbb{H}^2$ and reverses the orientation on $\mathbb{C}$. Then, by composing with $h_2(z,w) = (z, \bar{w})$ we get an isometry preserving both orientations. Hence, we have the same conclusion as above for $h \circ h_2$.
- Finally, h reverses both the orientations. Then, the same argument applies to $h \circ h_1 \circ h_2$.

In the end, we proved the following

Corollary 4.35. *If $f(t) = -kt$, with $k > 0$, then any isometry $h : \big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big) \to \big(\mathbb{H}^2 \times \mathbb{C}, \widehat{\mathbf{g}}_f\big)$ can be written as*

$$h = P \circ e^{i\theta}, \quad h = P \circ e^{i\theta} \circ h_1, \quad h = P \circ e^{i\theta} \circ h_2, \quad h = P \circ e^{i\theta} \circ h_1 \circ h_2$$

for some $P \in \mathbb{P}\mathrm{SL}(2,\mathbb{R})$ and $e^{i\theta} \in S^1$.

Chapter 5

Further developments

While this thesis answers, at least partially, some questions that have long remained unanswered, it also introduces new ones that deserve to be analyzed and hopefully answered in the near future.

5.1 Relation with Goldman's symplectic form

In Chapter 3 we proved the existence of a (possibly new) symplectic form $\boldsymbol{\omega}_f$ on the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component (Theorem A), which is known to be non-degenerate in a neighborhood of the Fuchsian locus. It is thus natural to ask about the relation between $\boldsymbol{\omega}_f$ and Goldman's symplectic form ω_{G}. According to our construction (Section 3.3.1) we need first to understand what happens in the torus case, where we have a family of symplectic forms parameterized by smooth functions.

Question 5.1. *Does there exist a smooth function $f : [0,+\infty) \to (-\infty,0]$ with $f(0)=0$, $f'(t)<0$ $\forall t>0$ and $\lim\limits_{t\to+\infty} f(t) = -\infty$ such that $\widehat{\boldsymbol{\omega}}_f = k\omega_{\mathrm{G}}$, for some $k\in\mathbb{R}$?*

If this were true for the torus, then one could try to prove something similar in the genus $g \geqslant 2$ case using Donaldson's construction. In particular, an affirmative answer would imply that our symplectic form $\boldsymbol{\omega}_f$ is non-degenerate on the entire Hitchin component and would show that Goldman's symplectic ω_{G} form is compatible with Labourie and Loftin's complex structure, giving rise to a pseudo-Kähler metric.

Another possible approach comes from an equivalent expression for ω_{G} found by Goldman. He showed that the space of projective equivalence classes of affine connections on Σ can be realized as a symplectic quotient using the theory of infinite-dimensional symplectic reduction ([Gol90b]). Because of the equivalence with the deformation space of convex

$\mathbb{RP}^2$-structures on Σ, he showed that Goldman's symplectic form can be expressed as:

$$\omega_{\mathrm{G}}(\dot{\sigma}_1, \dot{\sigma}_2) = \frac{1}{3}\int_{\Sigma} \operatorname{tr}\dot{\sigma}_1 \wedge \operatorname{tr}\dot{\sigma}_2 - \int_{\Sigma} \operatorname{tr}(\dot{\sigma}_1 \wedge \dot{\sigma}_2),$$

where $\dot{\sigma}_1, \dot{\sigma}_2$ are tangent vectors to the space of connections, namely $\mathrm{End}(T\Sigma)$-valued 1-forms. In particular, they are deformations of the canonical projectively flat connection (the Blaschke connection) coming from the hyperbolic affine sphere formulation. Given that we were able to write Labourie and Loftin's complex structure $\mathbf{I}$ in terms of pairs $(\dot{J}, \dot{A})$, with $\dot{J}$ a variation of an almost complex structure on Σ and $\dot{A}$ a variation of the Pick form of the corresponding hyperbolic affine sphere, it would be interesting to understand the action of $\mathbf{I}$ on tangent vectors $\dot{\sigma}_1, \dot{\sigma}_2$.

Question 5.2. *Using the above expression for ω_{G}, is it true that*

$$\omega_{\mathrm{G}}\big(\mathbf{I}(\dot{\sigma}_1), \dot{\sigma}_2\big) = -\boldsymbol{\omega}_{\mathrm{G}}\big(\dot{\sigma}_1, \mathbf{I}(\dot{\sigma}_2)\big),$$

or, in other words, that ω_{G} is compatible with $\mathbf{I}$?

5.2 A new geometric transition

In this thesis we proved that

$$K_h - ||q||_h^2 = -1,$$

namely Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$, has an interpretation as a moment map for the action of an infinite-dimensional Lie group. It is interesting to note that ([LM16]), by changing the sign in front of the cubic differential part, one obtains the equation governing minimal Lagrangian immersions in the complex hyperbolic plane $\mathbb{CH}^2$. In particular, using Trautwein's result ([Tra18]), it can be shown that an open subset of the moduli space of such minimal Lagrangian immersions inherits a mapping class group invariant Kähler metric, and there is an analogous moment map interpretation for the corresponding equation. The aforementioned open subset correspond to an open subset in the $\mathrm{SU}(2,1)$-character variety of the surface Σ ([HLL13],[LM13],[LM19]), so that the Kähler metric is defined on a neighborhood of the "Fuchsian locus". Moreover, by letting q tends to zero both equations degenerate to the constant curvature equation defining the Teichmüller space. At the level of Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{su}(2,1)$ are the 8-dimensional real split and real quasi-split forms of $\mathfrak{sl}(3,\mathbb{C})$ of rank 2 and 1, respectively.

Question 5.3. *Is there a geometric transition from a hyperbolic affine sphere in $\mathbb{R}^3$ to a minimal Lagrangian in $\mathbb{CH}^2$? If so, what is the intermediate geometry and the corresponding immersion?*

Question 5.4. *How is the pseudo-Kähler metric we introduced on $\mathrm{Hit}_3(\Sigma)$ related to the Kähler one introduced by Trautwein? How do they interact under the above geometric transition?*

5.3 What about other rank 2 split Lie groups?

One of the key results we used for our construction was the natural isomorphism between the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component and a holomorphic bundle over $\mathcal{T}(\Sigma)$. In fact, according to this parameterization, $\mathrm{Hit}_3(\Sigma)$ inherits a complex structure invariant under the action of the mapping class group. Recently, Labourie ([Lab17]) has shown a similar result regarding the Hitchin component for a general real simple split Lie group of rank 2, which up to isomorphism is $\mathbb{P}\mathrm{SL}(3,\mathbb{R}), \mathbb{P}\mathrm{Sp}(4,\mathbb{R})$ or the real split form $\mathrm{G}_2^{\mathbb{R}}$ of the exceptional G_2. In the $\mathbb{P}\mathrm{Sp}(4,\mathbb{R})$ case, the Hitchin component is parameterized by the bundle of holomorphic quartic differentials, and in the case of $\mathrm{G}_2^{\mathbb{R}}$ by the bundle of holomorphic sextic differentials. In particular, both connected components inherit a natural complex structure.

Question 5.5. *Does there exist a symplectic form $\boldsymbol{\omega}$ on the $\mathbb{P}\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{G}_2^{\mathbb{R}}$ Hitchin component compatible with the aforementioned complex structure? Do they give rise to a natural pseudo-Kähler structure which restrics to (a multiple of) the Weil-Petersson metric on Teichmüller space?*

The first major difference in these two cases, is that Hitchin equations of the associated cyclic Higgs bundle over $(\Sigma, J) \in \mathcal{T}(\Sigma)$ (the equivalent of Wang's equation for $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$), form a coupled system of PDE's. Assuming that Donaldson's construction can also be applied in these cases, there remains the problem of understanding how the two coupled equations can be interpreted as the intersection of the zero locus of two moment maps. In fact, as explained several times throughout Section 3.3.1, Donaldson's theorem provides us with only one. In particular, such a moment map depends on a particular choice of smooth function f. So it is natural to ask the following:

Question 5.6. *Is it possible to find two smooth functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying some functional equation, so that Donaldson's theorem provides two moment maps whose zero locus can be identified with the above system of PDE's?*

One possible approach, when $G = \mathbb{P}\mathrm{Sp}(4,\mathbb{R})$, comes from the exceptional isomorphism $\mathbb{P}\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{SO}_0(2,3)$. In fact, in this case, thanks to the work in [CTT19], for any Hitchin representation (actually maximal) $\rho : \pi_1(\Sigma) \to \mathrm{SO}_0(2,3)$ there exists a unique ρ-equivariant embedding $\widetilde{\Sigma} \hookrightarrow \mathbb{H}^{2,2}$ as a *maximal space-like surface*. The equations governing the embedding data of such surfaces are quite similar to the one governing hyperbolic affine spheres in $\mathbb{R}^3$. It turns out that, at least when $\Sigma = T^2$, there is an explicit (actually constant) solution to the self-duality equations for the corresponding $\mathrm{SO}_0(2,3)$-Higgs bundle, and this allows us to write the induced metric on the normal tangent bundle of the embedded surface in terms of the metric induced on the tangent bundle. Then, combining the work in [LT23] and the very recent one in [Nie22], we expect to define a pseudo-Kähler structure on the $\mathrm{SO}_0(2,3)$-Hitchin component of the torus using the same approach as the one presented in this thesis.

Appendix A

Elliptic operators on compact manifolds

In this appendix we recall the fundamental results about elliptic differential operators defined on smooth compact manifolds that we used during the proof of the main result of the thesis. The material covered here is classical ([Dem97],[War83],[Gil18],[Nic20]).

A.1 Sobolev space of sections

Let M be a smooth compact n-manifold equipped with a Riemannian metric g and let $\mathrm{d}V_g$ be its volume form, normalized to unit volume, i.e.

$$\int_M \mathrm{d}V_g = 1 \ .$$

Definition A.1. Let $p \geqslant 1$, define the *p-Lebesgue space* as

$$L^p(M) := \left\{ f : M \longrightarrow \mathbb{R} \ \middle| \ \left(\int_M |f|^p \mathrm{d}V_g \right)^{\frac{1}{p}} < \infty \right\} \ .$$

This is a Banach space for $p \geqslant 1$ with norm given by $\|f\|_p := \left(\int_M |f|^p \mathrm{d}V_g \right)^{\frac{1}{p}}$, and it is a Hilbert space when $p = 2$ with the following scalar product

$$(f, g) := \int_M f \cdot g \ \mathrm{d}V_g \ . \tag{A.1.1}$$

Definition A.2. Let $p \geqslant 1$ and k be a non negative integer. Define the *(k,p)-Sobolev space* $W^{k,p}(M)$ to be the set of $f \in L^p(M)$ such that f is k-times weakly differentiable and

$|D^\alpha f| \in L^p(M)$ for $|\alpha| \leqslant k$, where in a local coordinate $x = (x_1, \dots, x_n)$ we have

$$D^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}, \qquad \alpha = (\alpha_1, \dots, \alpha_n) \ .$$

Then $W^{k,p}(M)$ is a Banach space with the Sobolev norm

$$\|f\|_{W^{k,p}} = \Big(\sum_{|\alpha|\leqslant k} \int_M |D^\alpha f|^p \mathrm{d}V_g\Big)^{\frac{1}{p}} \tag{A.1.2}$$

and $W^{k,2}(M)$ is a Hilbert space (with scalar product similar to (A.1.1)).
Let $\pi : E \to M$ be a real (or complex) vector bundle and let $\langle \cdot, \cdot \rangle_E$ be a scalar product on E. We define, for $p \geqslant 1$, the space $L^p(M, E)$ to be the set of locally integrable sections of E over M, namely it is formed by the smooth sections $s \in \Gamma(M, E)$ for which the norm

$$\|s\|_p = \Big(\int_M |s|^p \mathrm{d}V_g\Big)^{\frac{1}{p}}$$

is finite, where $|s|^2 = \langle s, s \rangle_E$. Let ∇ be a connection on E compatible with $\langle \cdot, \cdot \rangle$, we define the space $W^{k,p}(M, E)$ as the completion of $\Gamma(M, E)$ with respect to the norm

$$\|s\|_{W^{k,p}} := \Big(\sum_{j\leqslant k} \int_M \|\nabla^j s\|^p \mathrm{d}V_g\Big)^{\frac{1}{p}} \ .$$

In particular when $p = 2$ we obtain that $W^{k,2}(M, E)$ is the completion of $\Gamma(M, E)$ with respect to

$$\|s\|^2_{W^{k,2}} = \sum_{j\leqslant k} \int_M \|\nabla^j s\|^2 \mathrm{d}V_g \ , \tag{A.1.3}$$

where

$$\begin{aligned} \|\nabla^0 s\|^2 &= \langle s, s \rangle_E \\ \|\nabla s\|^2 &= \langle \nabla s, \nabla s \rangle_{T^*(M)\otimes E} \\ \|\nabla^2 s\|^2 &= \langle \nabla^2 s, \nabla^2 s \rangle_{T^*(M)\otimes T^*(M)\otimes E} \ . \end{aligned}$$

and $W^{0,2}(M, E) = L^2(M, E)$. The scalar product on $W^{k,2}(M, E)$ is simply given by $(s, s)_k := \big(\|s\|_{W^{k,2}}\big)^2$ and it can be proved that

$$\bigcap_{k=1}^{\infty} W^{k,2}(M, E) = \Gamma(M, E) \ .$$

Proposition A.3. Let M be a compact smooth n-dimensional manifold, then the Sobolev norms (A.1.3) associated with two different choices of Riemannian metrics on M, scalar products on E and connections, are equivalent. Furthermore each of these norms is in turn equivalent to the norm associated with the scalar product

$$\langle s, s' \rangle_k := \sum_j \left\langle (\psi_j s) \circ \phi_j^{-1}; (\psi_j s') \circ \phi_j^{-1} \right\rangle_k ,$$

where $\big(U_j, \phi_j\big)$ is an atlas for M which is trivializing for E and with the property that the image of ϕ_j is all contained in a foundamental domain for $T^n = \mathbb{R}^n/\mathbb{Z}^n$, and $\{\psi_j\}$ is a partition of unity subordinate to $\{U_j\}$.

A.2 Differential operators over compact manifolds

Definition A.4. Let E and F be real (or complex) vector bundles over a smooth compact n-manifold M. Let $L : \Gamma(M; E) \longrightarrow \Gamma(M; F)$ be a $\mathbb{K}$-linear map, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say that L is a *differential operator* if for any trivializing chart U for E and F we have

$$\begin{aligned} &\chi : U \longrightarrow A \subset \mathbb{R}^n \\ &\varphi : E\big|_U \longrightarrow A \times \mathbb{K}^l \\ &\psi : F\big|_U \longrightarrow A \times \mathbb{K}^m \end{aligned}$$

and the following diagram is commutative:

$$\begin{tikzcd} \Gamma_0(U, E\big|_U) \arrow[r, "r_U \circ L \circ i_U"] \arrow[d] & \Gamma(U, F\big|_U) \\ \Gamma_0(A, A \times \mathbb{K}^l) \arrow[r, "L_U"] & \Gamma(A, A \times \mathbb{K}^m) \arrow[u] \end{tikzcd}$$

where $i_U : \Gamma_0(U, E\big|_U) \hookrightarrow \Gamma(M, E)$ is the immersion of compact support functions, r_U : $\Gamma(M, F) \longrightarrow \Gamma(U, F\big|_U)$ is the restriction and L_U is a matrix of differential operators.

The operator L is of *order* k if in any trivialization L_U involves no derivatives of order bigger than k, hence locally $(L_U)_{ij} = \sum\limits_{|\alpha| \leqslant k} a_\alpha D^\alpha$.
We denote the space of differential operators of order k over M as $\mathrm{Diff}^k(M; E; F)$.

Theorem A.5. Let $L : \Gamma(M; E) \longrightarrow \Gamma(M; F)$ be a differential operator of order k. Then there is a $\mathbb{K}$-linear extension $\tilde{L} : W^{k+l,2}(M; E) \longrightarrow W^{l,2}(M; F)$, for every non negative integer $l \geqslant 0$, such that $\tilde{L}\big|_{C^\infty(M;E)} = L$.

Let $L \in$Diff$^k(M;E;F)$, it is possible to define the *principal symbol* of L

$$\sigma_k(L) \in \Gamma(T^*(M); \mathrm{Hom}(\pi^*E, \pi^*F))$$

as follows. Let $(x,\xi) \in T^*(M)$ and $e_x \in E_x$ be given; find $f \in C^\infty(M)$ and $e \in \Gamma(M;E)$ such that $\mathrm{d}f|_x = \xi$ and $e(x) = e_x$. Define $\sigma_k(L)_x(\xi) \in \mathrm{Hom}(E_x, F_x)$ ([Dem97]) to be:

$$\sigma_k(L)_x(\xi)(e_x) := L\big((f - f(x))^k e\big)(x) \tag{A.2.1}$$

It can be checked that $\sigma_k(L)_x(\xi)$ does not depend on the choices made and that it is a linear map from E_x to F_x. In a local trivialization if $\xi = \xi_1 \mathrm{d}x^1 + \cdots + \xi_n \mathrm{d}x^n$, using the construction above, one has

$$\sigma_k\Big(\sum_{|\alpha|\leqslant k} a_\alpha(ij) D^\alpha\Big)_x(\xi) = \sum_{|\alpha|=k} a_\alpha(ij)(\xi_1)^{\alpha_1} \dots (\xi_n)^{\alpha_n} \ .$$

Definition A.6. A differential operator $L : \Gamma(M;E) \longrightarrow \Gamma(M;F)$ of order k is *elliptic* if for all $(x,\xi) \in T^*(M)\backslash\{0\}$ the principal symbol of L evaluated on (x,ξ) is an isomorphism.

Lemma A.7 (Weyl Lemma)**.** *Let $L : \Gamma(M;E) \longrightarrow \Gamma(M;F)$ be an elliptic operator of order k and let $s \in W^{k,2}(M,E)$, then if $Ls = 0$ holds weakly in $L^2(M,E)$ it follows that $s \in \Gamma(M,E)$.*

Example A.8. Let $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be the *Euclidean Laplacian*, namely

$$\Delta(f) := -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f, \quad f \in C^\infty(\mathbb{R}^n) \ .$$

Let $0 \neq \xi \in T_x^*\mathbb{R}^n$ be such that $\mathrm{d}_x f = \xi$. Then,

$$\begin{aligned}
\sigma_2(\Delta)_x(\xi) &= -\sigma_2\Big(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\Big)_x(\xi) \\
&= -\sum_{i=1}^n \xi_i^2 \\
&= -\sum_{i=1}^n \Big(\frac{\partial}{\partial x_i} f|_x\Big)^2 \\
&= -|\mathrm{d}_x f|^2 \neq 0 \ .
\end{aligned}$$

Thus the Euclidean Laplacian Δ on $\mathbb{R}^n$ is an elliptic operator of order 2. The same proof applies to the *Laplace-Beltrami* operator $\Delta_L : C^\infty(M) \to C^\infty(M)$ defined on a smooth compact Riemannian manifold (M,g) of dimension n as the divergence of the gradient. In fact, in a local coordinate $x = (x_1, \dots, x_n)$ we have

$$\Delta_L = \sum_{i,j} \Big(g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_l g^{ij} \Gamma_{ij}^l \frac{\partial}{\partial x_l}\Big),$$

where $(g^{ij})_{ij}$ is the matrix associated with the inverse metric and the Γ_{ij}^{l}'s denote the Christoffel symbols.

Given (M, g) a smooth compact Riemannian n-manifold and given E, F two smooth real (or complex) vector bundles over M, let us consider $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$ scalar products on E and F, respectively. Define the following inner product on the space of smooth sections of E

$$(s,t)_E := \int_M \langle s,t\rangle_E \mathrm{d}V_g \quad \forall s,t \in \Gamma(M;E) \tag{A.2.2}$$

and similarly on $\Gamma(M,F)$.

Definition A.9. Let $L : \Gamma(M,E) \to \Gamma(M,F)$ be a smooth differential operator of order k, then its *adjoint* $L^* : \Gamma(M,F) \to \Gamma(M,E)$ is the smooth differential operator of order k uniquely determined by the following property:

$$\big(L(s),t\big)_F = \big(s, L^*(t)\big)_E, \qquad s \in \Gamma(M;E), \quad t \in \Gamma(M;F)\ .$$

Theorem A.10. *If $L : \Gamma(M,E) \to \Gamma(M,F)$ is an elliptic differential operator between the space of smooth sections of two real (or complex) vector bundles of the same rank, then one has the following decomposition:*

$$\Gamma(M,E) = \mathrm{Ker}(L) \oplus \mathrm{Range}(L^*)$$

and the space $\mathrm{Ker}(L)$ is finite dimensional. Moreover, the direct sum decomposition is orthogonal with respect to $(\cdot,\cdot)_E$.

Definition A.11. Given a bounded linear operator $T : X \to Y$ between Banach spaces, we say it is *Fredholm* if $\mathrm{Ker}(T)$ and $\mathrm{Coker}(T) := Y/\mathrm{Range}(T)$ are both finite dimensional and $\mathrm{Range}(T)$ is closed. The number:

$$\mathrm{ind}(T) := \dim \mathrm{Ker}(T) - \dim \mathrm{Coker}(T) \in \mathbb{Z}$$

is called the *Fredholm index* of T.

Lemma A.12. *Any elliptic operator $L : \Gamma(M,E) \to \Gamma(M,F)$ between the space of smooth sections of $\mathbb{K}$-vector bundles over a compact smooth manifold M is Fredholm.*

Theorem A.13 (Homotopy invariance of the index)**.** *Suppose that we have a continuous path*

$$L_t : \Gamma(M,E) \longrightarrow \Gamma(M,F), \quad t \in [0,1]$$

of elliptic differential operators of order k between the space of smooth sections of two $\mathbb{K}$-vector bundles over a smooth compact manifold M. Then,

$$\mathrm{ind}(L_t) = \mathrm{ind}(L_0), \quad \forall t \in [0,1]\ .$$

Appendix B

Symplectic reduction theory

In this appendix we recall the notion of Hamiltonian action on a symplectic manifold with associated moment map. We briefly explain the construction of symplectic reduction when the manifold is of finite dimension, as it inspires the infinite-dimensional case. We will not go into details since the material covered here is classical and can be found in many books and papers ([MS17; DD08]). Finally, after recalling the definition of pseudo-Kähler manifold, we state and prove the Marsden-Meyer-Weinstein theorem in the pseudo-Kähler setting, since it is not easily found in the literature.

B.1 The Marsden-Meyer-Weinstein theorem

Definition B.1. A *symplectic manifold* is a pair (M, ω), where M is a real smooth manifold and ω is a closed non-degenerate 2-form on M, called the *symplectic form*.

One can think of the closed 2-form ω as a family of skew-symmetric non-degenerate bilinear forms $\omega_p : T_pM \times T_pM \to \mathbb{R}$, for any $p \in M$. In particular, the non-degeneracy of ω implies that $\dim M$ is even and there is an induced isomorphism $T_p^*M \cong T_pM$ by ω_p, for any $p \in M$.

Definition B.2. Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M. Let $\psi_g : M \to M$ be the map $\psi_g(p) := g \cdot p$, then we say the group G acts by *symplectomorphisms* on (M, ω) if $\psi_g^*\omega = \omega$ for all $g \in G$.

Definition B.3. Let G be a Lie group, with Lie algebra $\mathfrak{g}$, acting on a symplectic manifold (M, ω) by symplectomorphisms. We say the action is *Hamiltonian* if there exists a smooth function $\mu : M \to \mathfrak{g}^*$ satisfying the following properties:

(i) The function μ is equivariant with respect to the G-action on M and the co-adjoint action on $\mathfrak{g}^*$, namely

$$\mu_{g \cdot p} = \mathrm{Ad}^*(g)(\mu_p) := \mu_p \circ \mathrm{Ad}(g^{-1}) \in \mathfrak{g}^* \ . \tag{B.1.1}$$

(ii) Given $\xi \in \mathfrak{g}$, let X_ξ be the vector field on M generating the action of the 1-parameter subgroup generated by ξ, i.e. $X_\xi = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) \cdot p|_{t=0}$. Then, for every $\xi \in \mathfrak{g}$ we have

$$\mathrm{d}\mu^\xi = \iota_{X_\xi}\omega = \omega(X_\xi, \cdot) \ , \tag{B.1.2}$$

where $\mu^\xi : M \to \mathbb{R}$ is the function $\mu^\xi(p) := \mu_p(\xi)$.

A map μ satisfying the two properties above is called a *moment map* for the Hamiltonian action.

Given a symplectic action of a Lie group G on a symplectic manifold (M, ω), one can ask whether a quotient exists in the category of symplectic manifolds. It is clear that the topological quotient always exists, but it is not necessarily a smooth manifold, for example when the G-action is not proper or free. Although the action is required to be free and proper, the resulting quotient manifold may have odd dimension and so it will not admit a symplectic form. All in all, the topological quotient M/G does not in general provide a suitable quotient in symplectic geometry. Nevertheless, the existence of a moment map for a Hamiltonian action allows us to induce a symplectic structure on the quotient of a level set of the moment map. In fact, since $0 \in \mathfrak{g}^*$ is fixed by the co-adjoint action, equivariance of μ implies that the preimage $\mu^{-1}(0) \subset M$ is preserved by the action of G.

Theorem B.4 (Marsden-Weinstein-Meyer [MW74; Wei80])**.** *Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms. Suppose the action is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map, and suppose the restricted G-action on $\mu^{-1}(0)$ is free and proper. Then, the following holds:*
- *The topological quotient $\mu^{-1}(0)/G$ is a smooth manifold of dimension* $\dim M - 2\dim G$ *and the quotient map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle;*
- *there exists a unique symplectic form ω_{red} on $\mu^{-1}(0)/G$ such that $\pi^*\omega_{red} = \iota^*\omega$.*

The pair $(\mu^{-1}(0), \omega_{\mathrm{red}})$ is called the *symplectic quotient* or *Marsden-Weinstein-Meyer quotient* of (M, ω). The main steps of the proof can be summarized as follows:

Step 1:

If $\mathfrak{g}_p$ denotes the Lie algebra of the stabilizer of $p \in M$, then $\mathrm{d}_p\mu : T_pM \to \mathfrak{g}^*$ satisfies:

$$\mathrm{Ker}(\mathrm{d}_p\mu) = \Big(T_p\big(G \cdot p\big)\Big)^{\perp_{\omega_p}}, \qquad \mathrm{Im}(\mathrm{d}_p\mu) = \mathfrak{g}_p^0 := \{\xi \in \mathfrak{g} \mid \langle \xi, X \rangle = 0, \ \forall X \in \mathfrak{g}_p\} \ .$$

In other words, the kernel of the differential of the moment map can be identified with the symplectic orthogonal of the tangent space to the G-orbit, and the image of the differential is identified with the annihilator of $\mathfrak{g}_p$. In particular, it can be proven that $T_p\big(G \cdot p\big)$ is an isotropic subspace of (T_pM, ω_p), namely $(\omega)_p|_{T_p(G\cdot p)} \equiv 0$.

Step 2:

The G-action on $\mu^{-1}(0)$ is free, then 0 is a regular value of μ. In particular, $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to the dimension of G. Finally, using an "equivariant" version of tubular neighborhood theorem (sometimes called *slice theorem*), one gets that the quotient projection $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle.

Step 3:

The symplectic form ω is induced on the quotient $\mu^{-1}(0)/G$ by using a standard argument in symplectic geometry, namely if (V, ω) denotes a symplectic vector space which admits an isotropic subspace U (i.e. $\omega|_U \equiv 0$), then there is a natural induced symplectic form on the quotient $U^{\perp_\omega}/U$. In our case, we can pick $V = T_pM$ and $U = T_p\big(G \cdot p\big)$ which is an isotropic subspace of (T_pM, ω_p) by Step 1. In the end, at the level of tangent spaces, we get the following identifications:

$$U^{\perp_\omega} \big/ U \cong \mathrm{Ker}(\mathrm{d}_p\mu) \Big/ T_p\big(G \cdot p\big) \cong T_{[p]}\Big(\mu^{-1}(0) \big/ G\Big) .$$

In other words, the tangent space to the quotient is identified with the symplectic orthogonal to $T_p\big(G \cdot p\big)$ inside $\mathrm{Ker}(\mathrm{d}_p\mu)$.

B.2 Pseudo-Kähler reduction

A *pseudo-Riemannian* metric g on a smooth n-manifold M is an everywhere non-degenerate, smooth, symmetric $(0, 2)$-tensor. The *index* of g is the maximal rank k of the smooth distribution where it is negative-definite. For instance, if $k = 0$ then g is a Riemannian metric. Now let J be a complex structure on M, then (g, J) is a *pseudo-Hermitian structure* if

$$g(JX, JY) = g(X, Y), \qquad \forall X, Y \in T_pM, p \in M \ . \tag{B.2.1}$$

Notice that, due to this last condition, the index of g in this case is always even $k = 2s$, where s is called the *complex index* and it satisfies $1 \leqslant s \leqslant m = \dim_{\mathbb{C}} M$. The *fundamental 2-form* ω of a pseudo-Hermitian manifold (M, g, J) is defined by:

$$\omega(X, Y) := g(X, JY), \qquad \forall X, Y \in T_pM, p \in M \ . \tag{B.2.2}$$

Definition B.5. A pseudo-Hermitian manifold (M, g, J, ω) is called *pseudo-Kähler* if the fundamental 2-form is closed, namely if $\mathrm{d}\omega = 0$. In this case the corresponding metric is called *pseudo-Kähler*. Moreover, if g is positive-definite then (M, g, J, ω) is called a *Kähler* manifold.

Let (M, g, J, ω) be a pseudo-Kähler manifold and suppose there is an action of a Lie group G on M which preserves the symplectic form ω and the pseudo-Riemannian metric g. Let us also assume that the action is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Then, one is tempted to mimic the symplectic reduction case and try to induce the pseudo-Kähler structure on the quotient $\mu^{-1}(0)/G$. Indeed, the same can be done with the appropriate adjustements:

Theorem B.6 (Pseudo-Kähler reduction). *Let G be a Lie group acting on a pseudo-Kähler manifold (M, ω, J, g) by isometries and by symplectomorphisms. Suppose the action is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Suppose that the restricted G-action on $\mu^{-1}(0)$ is free and proper and that the pseudo-metric g restricted to the orbit $T_p\big(G \cdot p\big) \subset T_p\mu^{-1}(0)$ is non-degenerate. Then, the following holds:*
- *The topological quotient $\mu^{-1}(0)/G$ is a smooth manifold of dimension $\dim M - 2\dim G$ and the quotient map $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle;*
- *there exists a unique pseudo-Riemannian metric g_{red} and complex structure J_{red} on $\mu^{-1}(0)/G$ such that*

$$\pi^* g_{red} = \iota^* g, \quad \pi^* J_{red} = \iota^* J$$

and the pairing $\omega_{red} = g_{red}(\cdot, J_{red}\cdot)$ is a symplectic form on the quotient.

Proof. During the proof we will assume the first claim of the theorem to be true, namely the existence of the G-principal bundle $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$, and we will explain, step by step, how to induce the pseudo-Riemannian metric and complex structure on the quotient. Let V be the vertical bundle of the above G-principal bundle, namely $V_p = \mathrm{Ker}(\mathrm{d}_p\pi)$ for all $p \in \mu^{-1}(0)$, and let N denotes the normal bundle of the inclusion $\iota : \mu^{-1}(0) \hookrightarrow M$.

Step 1: The pseudo-metric is non-degenerate when restricted to V_p and N_p.

Let $\xi \in \mathfrak{g}$ and let X_ξ be its infinitesimal generator, then we have

$$g(\mathrm{grad}\mu^\xi, Y) = \mathrm{d}\mu^\xi(Y) = \omega(X_\xi, Y) = -g(JX_\xi, Y) \ , \quad \forall Y \in \Gamma(TM)$$

which implies that $\mathrm{grad}\mu^\xi = -JX_\xi$. Now, let $\xi_1, \dots, \xi_k$ be a basis for $\mathfrak{g}$ and $\eta^1, \dots, \eta^k \in \mathfrak{g}^*$ its dual basis. Then, the moment map μ can be seen as a smooth map from M to $\mathbb{R}^k \cong \mathfrak{g}^*$, as follows

$$\mu(p) = \mu^{\xi_1}(p)\eta^1 + \cdots + \mu^{\xi_k}(p)\eta^k \ , \forall p \in M \ ,$$

where μ^{ξ_j} is a C^∞ function from M to $\mathbb{R}$, for any $j = 1, \dots, k$. A standard argument shows that a global frame for the normal bundle N is given by

$$\{\mathrm{grad}\mu^{\xi_1}, \dots, \mathrm{grad}\mu^{\xi_k}\} = \{JX_{\xi_1}, \dots, JX_{\xi_k}\} \ .$$

Moreover, since the restricted action of G on $\mu^{-1}(0)$ is free, for all $p \in \mu^{-1}(0)$ each stabilizer G_p is trivial, and the differential of the orbit map $\Phi_p(h) = h \cdot p \in G \cdot p$, for $h \in G$, induces a linear isomorphism

$$T_p(G \cdot p) \cong \mathfrak{g} \ .$$

In particular, from the above discussion, we deduce that $\{X_{\xi_1}, \dots, X_{\xi_k}\}$ is a global frame for the vertical bundle V. In other words, for any $p \in \mu^{-1}(0)$, we showed that the set

$$\{X_{\xi_1}, \dots, X_{\xi_k}, JX_{\xi_1}, \dots, JX_{\xi_k}\}$$

is a basis for

$$V_p \oplus N_p \cong T_p(G \cdot p) \oplus J\Big(T_p(G \cdot p)\Big) \ .$$

By hypothesis, the pseudo-metric g is non-degenerate when restricted to $T_p(G \cdot p) \cong V_p$, and together with the pseudo-hermitian condition (B.2.1), we deduce that g is non-degenerate when restricted to $J\Big(T_p(G \cdot p)\Big) \cong N_p$ as well. Finally, by using (B.2.2), we get a g-orthogonal decomposition

$$V_p \overset{\perp}{\oplus} N_p \cong T_p(G \cdot p) \overset{\perp}{\oplus} J\Big(T_p(G \cdot p)\Big) \ ,$$

which implies that the restricted pseudo-metric is non-degenerate on the direct sum $V_p \overset{\perp}{\oplus} N_p$ for any $p \in \mu^{-1}(0)$.

<u>*Step 2:*</u> The space N_p is the g-orthogonal to $\mathrm{Ker}(\mathrm{d}_p\mu)$.

Notice that for any $\xi \in \mathfrak{g}$ and for any $w \in \mathrm{Ker}(\mathrm{d}_p\mu) = T_p\mu^{-1}(0)$, we get

$$g(w, JX_\xi) = \omega(w, X_\xi) = \mathrm{d}_p\mu^\xi(w) = 0 \ .$$

According to what has been shown in Step 1, we know that $N_p = J\Big(T_p(G \cdot p)\Big)$, and we can deduce the following g-orthogonal decomposition:

$$T_pM = T_p\mu^{-1}(0) \overset{\perp}{\oplus} N_p, \quad \forall p \in \mu^{-1}(0) \ ,$$

which implies that the pseudo-metric is non-degenerate when restricted to $T_p\mu^{-1}(0) = \mathrm{Ker}(\mathrm{d}_p\mu)$.

<u>*Step 3:*</u> The choice of the supplement to the orbit.

Let us define the following space:

$$H_p := \{v \in \mathrm{Ker}(\mathrm{d_p}\mu) \mid J(v) \in \mathrm{Ker}(\mathrm{d}_p\mu)\}, \quad \forall p \in \mu^{-1}(0) \ .$$

Notice that H_p is J-invariant by definition. We want to prove that there is a g-orthogonal decomposition

$$T_p\mu^{-1}(0) = H_p \overset{\perp}{\oplus} V_p, \quad \forall p \in \mu^{-1}(0) \ .$$

Regarding the direct sum decomposition, suppose by contrary there exists $0 \neq v \in H_p \cap T_p\big(G \cdot p\big)$, then by definition of H_p the element $J(v)$ still belongs to $\mathrm{Ker}(\mathrm{d}_p\mu) = T_p\mu^{-1}(0)$.

Moreover, since by hypothesis $v \in T_p(G \cdot p) = V_p$ we deduce that $J(v) \in N_p$ by Step 1. In particular, the element $J(v)$ is g-orthogonal to $\mathrm{Ker}(\mathrm{d}_p\mu)$ by Step 2. This is possible if and only if $J(v) = 0$ as the pseudo-metric is non-degenerate on $\mathrm{Ker}(\mathrm{d}_p\mu)$. Since J is an isomorphism and we assumed v to be non-zero, we get a contradiction. Finally, notice that an element v is g-orthogonal to $T_p(G \cdot p)$ if and only if $J(v) \in \mathrm{Ker}(\mathrm{d}_p\mu)$ as shown by the following computation:

$$g(X_\xi, v) = -g(X_\xi, J^2(v)) = -\omega(X_\xi, J(v)) = \mathrm{d}_p\mu^\xi\big(J(v)\big), \quad \forall \xi \in \mathfrak{g}.$$

Hence, by definition of H_p we can conclude that $H_p \perp V_p$.

Before going on with the proof, let us give a brief summary of what has been deduced so far. For any $p \in \mu^{-1}(0)$ we proved the existence of the following g-orthogonal decomposition:

$$T_pM = H_p \overset{\perp}{\oplus} T_p(G \cdot p) \overset{\perp}{\oplus} J\Big(T_p(G \cdot p)\Big) \ ,$$

where H_p can be identified with the tangent to the quotient space $\mu^{-1}(0)/G$. Moreover, the pseudo-metric g is non-degenerate whenever is restricted to one of the above spaces. In particular, we will denote with g_{red} the pseudo-metric induced on the quotient, so that $\pi^* g_{\mathrm{red}} = \iota^* g$.

<u>*Step 4:*</u> The induced almost complex structure on $\mu^{-1}(0)/G$.

We first observe that, if Y is a vector field on $\mu^{-1}(0)/G$, its horizontal lift $\widetilde{Y}$ is not a vector field on M, but only on $\mu^{-1}(0)$. Thus, it's not clear a-priori how to apply the complex structure to such lifts. Nevertheless, the map

$$\begin{aligned} \mu^{-1}(0) &\longrightarrow H \\ p &\mapsto J_p(\widetilde{Y}_p) \end{aligned}$$

defines a smooth G-invariant section of H, since the complex structure preserves H, $\widetilde{Y}$ is G-invariant and G preserves J. We will denote such a section by $J\widetilde{Y}$. Since $J\widetilde{Y}$ is a G-invariant horizontal section, it is the lift of a unique smooth vector field $\mathrm{d}\pi(J\widetilde{Y})$ on $\mu^{-1}(0)/G$. In other words, we have a way of applying the induced complex structure J_{red} to vector fields on $\mu^{-1}(0)/G$, by the following formula:

$$J_{\mathrm{red}}(Y) := \mathrm{d}\pi(J\widetilde{Y}) \ , \quad \forall Y \in \Gamma\Big(T(\mu^{-1}(0)/G)\Big) \ .$$

It is easy to see that J_{red} is a $(1,1)$-tensor and C^∞ linear. In fact,

$$J_{\mathrm{red}}(fY) = \mathrm{d}\pi\big(J(\widetilde{fY})\big) = \mathrm{d}\pi\big((f \circ \pi)J\widetilde{Y}\big) = f\mathrm{d}\pi(J\widetilde{Y}) = fJ_{\mathrm{red}}(Y) \ ,$$

for any C^∞ function f on $\mu^{-1}(0)/G$. Finally, to show that $J_{\mathrm{red}}^2 = -\mathbb{1}$ we only need to notice that the horizontal lift of $\mathrm{d}\pi(J\widetilde{Y})$ coincides with $J\widetilde{Y}$. Hence,

$$J_{\mathrm{red}}\big(J_{\mathrm{red}}(Y)\big) = J_{\mathrm{red}}\big(\mathrm{d}\pi(J\widetilde{Y})\big) = \mathrm{d}\pi\Big(J\widetilde{\mathrm{d}\pi(J\widetilde{Y})}\Big) = \mathrm{d}\pi\big(J^2(\widetilde{Y})\big) = -\mathrm{d}\pi(\widetilde{Y}) = -Y \ .$$

Step 5: $\hat{\nabla} J_{\mathrm{red}} = 0$, where $\hat{\nabla}$ is the Levi-Civita connection with respect to g_{red}.

Let ∇ be the Levi-Civita connection of g and let $P_H : TM \to H$ be the orthogonal projection. We claim that for any Y, Z smooth vector fields on $\mu^{-1}(0)/G$, we have

$$\hat{\nabla}_Z Y = \mathrm{d}\pi\Big(P_H\big(\nabla_{Z^*} Y^*\big)\Big) \ ,$$

where Y^*, Z^* are arbitrary smooth extension to a neighborhood of $\mu^{-1}(0)$. In fact, if $\widetilde{\nabla}$ denotes the Levi-Civita connection of $\iota^* g$ on $\mu^{-1}(0) \subset M$, then it is standard to prove that

$$\widetilde{\nabla}_Z Y = \Big(\nabla_{Z^*} Y^*\Big)^{\perp} \ ,$$

where $\perp$ is the orthogonal projection onto $T\mu^{-1}(0)$ with respect to the pseudo-metric g on M. The Levi-Civita connection $\hat{\nabla}$ is obtained by first projecting $\nabla_{Z^*} Y^*$ onto $T\mu^{-1}(0)$, and then projecting onto H and using the correspondence between G-invariant sections of H and vector fields on $\mu^{-1}(0)/G$. This procedure gives exactly the formula written above. Finally, we need to prove that for any smooth vector fields Y, Z on $\mu^{-1}(0)/G$ we have $(\hat{\nabla}_Z J_{\mathrm{red}})Y = 0$, which is equivalent to $\big(\hat{\nabla}_Z J_{\mathrm{red}}\big)Y = J_{\mathrm{red}}\big(\hat{\nabla}_Z Y\big)$. By taking the horizontal lift of the terms we are interested in, we get

$$\begin{aligned}
\Big(\hat{\nabla}_Z J_{\mathrm{red}} Y\Big)^{\mathrm{hor}} &= P_H\Big(\nabla_{Z^*}\big(J_{\mathrm{red}} Y\big)^*\Big) \\
&= P_H\big(\nabla_{Z^*} J Y^*\big) && (\nabla J = 0 \text{ on } M) \\
&= P_H\big(J \nabla_{Z^*} Y^*\big) && (J \text{ preserves } H) \\
&= J P_H\big(\nabla_{Z^*} Y^*\big) \\
&= \Big(J_{\mathrm{red}}\big(\hat{\nabla}_Z Y\big)\Big)^{\mathrm{hor}} \ .
\end{aligned}$$

Step 6: The pair $(g_{\mathrm{red}}, J_{\mathrm{red}})$ defines a pseudo-Kähler metric on $\mu^{-1}(0)/G$.

We first observe that the pseudo-metric g_{red} satisfies Equation (B.2.1), indeed for any smooth vector fields Y, Z on $\mu^{-1}(0)/G$ we have

$$\begin{aligned}
g_{\mathrm{red}}(J_{\mathrm{red}} Y, J_{\mathrm{red}} Z) &= g_{\mathrm{red}}\big(\mathrm{d}\pi(J\widetilde{Y}), \mathrm{d}\pi(J\widetilde{Z})\big) = (\pi^* g_{\mathrm{red}})\big(J\widetilde{Y}, J\widetilde{Z}\big) = (\iota^* g)\big(J\widetilde{Y}, J\widetilde{Z}\big) \\
&= (\iota^* g)(\widetilde{Y}, \widetilde{Z}) = g_{\mathrm{red}}(Y, Z) \ .
\end{aligned}$$

Then, let us denote with ω_{red} the tensor obtained as follows:

$$\omega_{\text{red}}(Y,Z) = g_{\text{red}}(Y, J_{\text{red}}Z) \ , \quad \forall Y,Z \in \Gamma\big(T(\mu^{-1}(0)/G)\big) \ .$$

It follows easily, by using (B.2.1), that ω_{red} defines a 2-form on $\mu^{-1}(0)/G$. Finally, in the setting of Kähler geometry, the integrability of the almost complex structure and the closedness of the fundamental 2-form are equivalent to the requirement that the almost complex structure is parallel with respect to the Levi-Civita connection ([Voi02]). It turns out that the same proof can be adapted to the pseudo-Kähler case, hence $\widehat{\nabla} J_{\text{red}} = 0$ if and only if J_{red} is integrable and $\mathrm{d}\omega_{\text{red}} = 0$. This directly implies that $(g_{\text{red}}, J_{\text{red}}, \omega_{\text{red}})$ defines a pseudo-Kähler structure on $\mu^{-1}(0)/G$. $\square$

Bibliography

[ADN64] Shmuel Agmon, Avron Douglis, and Louis Nirenberg. "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II". in: *Communications on pure and applied mathematics* 17.1 (1964), pp. 35–92.

[Ahl61a] Lars V Ahlfors. "Some remarks on Teichmuller's space of Riemann surfaces". In: *Annals of Mathematics* (1961), pp. 171–191.

[Ahl61b] Lars Valerian Ahlfors. *Curvature properties of Teichmuller's space.* Mathematical Sciences Directorate, Air Force Office of Scientific Research, 1961.

[AI81] T Ya Azizov and IS Iokhvidov. *Linear operators in spaces with indefinite metric and their applications.* Vol. 15. Springer, 1981, pp. 438–490.

[APS04] Mats Andersson, Mikael Passare, Ragnar Sigurdsson, et al. *Complex convexity and analytic functionals.* Vol. 225. Springer Science & Business Media, 2004.

[Arn13] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics.* Vol. 60. Springer Science & Business Media, 2013.

[Ben60] Jean-Paul Benzécri. "Sur les variétés localement affines et localement projectives". In: *Bulletin de la Société Mathématique de France* 88 (1960), pp. 229–332.

[BH13] Yves Benoist and Dominique Hulin. "Cubic differentials and finite volume convex projective surfaces". In: *Geometry & Topology* 17.1 (2013), pp. 595–620.

[BMS15] Francesco Bonsante, Gabriele Mondello, and Jean-Marc Schlenker. "A cyclic extension of the earthquake flow II". in: *Annales Scientifiques de l'ENS* 48.4 (2015), pp. 811–859.

[Bri+15] Martin Bridgeman et al. "The pressure metric for Anosov representations". In: *Geometric and Functional Analysis* 25.4 (2015), pp. 1089–1179.

[BT+82] Raoul Bott, Loring W Tu, et al. *Differential forms in algebraic topology.* Vol. 82. Springer, 1982.

[Cal72] Eugenio Calabi. "Complete affine hyperspheres I". in: *Instituto Nazionale di Alta Matematica Symposia Mathematica.* Vol. 10. 1972, pp. 19–38.

[CG93] Suhyoung Choi and William M Goldman. "Convex real projective structures on closed surfaces are closed". In: *Proceedings of the American Mathematical Society* 118.2 (1993), pp. 657–661.

[CJK20] Suhyoung Choi, Hongtaek Jung, and Hong Chan Kim. "Symplectic coordinates on $\mathbb{P}\mathrm{SL}_3(\mathbb{R})$-Hitchin components". In: *Pure and applied mathematics quarterly* 16.5 (2020), pp. 1321–1386.

[CTT19] Brian Collier, Nicolas Tholozan, and Jérémy Toulisse. "The Geometry of Maximal Representations of Surface Groups into $\mathrm{SO}_0(2,n)$". In: *Duke Mathematical Journal* 168.15 (2019), pp. 2873–2949.

[CY77] Shiu-Yuen Cheng and Shing-Tung Yau. "On the regularity of the Monge-Ampère equation det $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = f(x,u)$". In: *Communications on Pure and Applied Mathematics* 30.1 (1977), pp. 41–68.

[CY86] Shiu-Yuen Cheng and Shing-Tung Yau. "Complete affine hypersurfaces. Part I. The completeness of affine metrics". In: *Communications on Pure and Applied Mathematics* 39.6 (1986), pp. 839–866.

[Dai19] Xian Dai. "Geodesic Coordinates for the Pressure Metric at the Fuchsian Locus". In: *arXiv:1910.00792* (2019).

[DD08] Ana Cannas Da Silva and A Cannas Da Salva. *Lectures on symplectic geometry.* Vol. 3575. Springer, 2008.

[DD87] Pierre Dazord and Thomas Delzant. "Le probleme général des variables actions-angles". In: *Journal of differential geometry* 26.2 (1987), pp. 223–251.

[Dem97] Jean-Pierre Demailly. *Complex analytic and differential geometry.* Grenoble: Université de Grenoble, 1997.

[Dim04] Alexandru Dimca. *Sheaves in topology.* Springer Science & Business Media, 2004.

[Don03] Simon K Donaldson. "Moment maps in differential geometry". In: *Surveys in differential geometry* 8.1 (2003), pp. 171–189.

[DS08] Andrew Dancer and Andrew Swann. "Hypersymplectic manifolds". In: *Recent developments in pseudo-Riemannian geometry* (2008), pp. 97–148.

[Dui80] Johannes J Duistermaat. "On global action-angle coordinates". In: *Communications on pure and applied mathematics* 33.6 (1980), pp. 687–706.

[DW96] Mehdi-Reza Darvishzadeh and M Goldman William. "Deformation spaces of convex real-projective structures and hyperbolic affine structures". In: *Journal of the Korean Mathematical Society* 33.3 (1996), pp. 625–639.

[FGS03] Emanuele Fiorani, Giovanni Giachetta, and Gennadi Sardanashvily. "The Liouville-Arnold-Nekhoroshev theorem for non-compact invariant manifolds". In: *Journal of Physics A: Mathematical and General* 36.7 (2003), pp. L101–L107.

[FM11] Benson Farb and Dan Margalit. *A primer on mapping class groups (pms-49).* Princeton university press, 2011.

[Gig81] Salvador Gigena. "On a conjecture by E. Calabi". In: *Geometriae Dedicata* 11.4 (1981), pp. 387–396.

[Gil18] Peter B Gilkey. *Invariance theory: the heat equation and the Atiyah-Singer index theorem.* Vol. 16. CRC press, 2018.

[GM+97] Giovanni Giachetta, Luigi Mangiarotti, et al. *New Lagrangian and Hamiltonian methods in field theory.* World Scientific, 1997.

[Gol22] William M Goldman. *Geometric structures on manifolds.* Vol. 227. American Mathematical Society, 2022.

[Gol84] William M Goldman. "The symplectic nature of fundamental groups of surfaces". In: *Advances in Mathematics* 54.2 (1984), pp. 200–225.

[Gol90a] William M Goldman. "Convex real projective structures on compact surfaces". In: *Journal of Differential Geometry* 31.3 (1990), pp. 791–845.

[Gol90b] William M Goldman. "The symplectic geometry of affine connenctions on surfaces." In: *Journal für die reine und angewandte Mathematik* 407 (1990), pp. 126–159.

[Gro01] Mark Gross. "Special Lagrangian fibrations II: geometry". In: *AMS/IP Studies in Advanced Mathematics* 23 (2001), pp. 95–150.

[Gru77] Gerd Grubb. "Boundary problems for systems of partial differential operators of mixed order". In: *Journal of Functional Analysis* 26.2 (1977), pp. 131–165.

[Gun15] Robert C Gunning. "Lectures on Riemann surfaces". In: *Lectures on Riemann Surfaces.* Princeton University Press, 2015.

[Hit87] Nigel J Hitchin. "The self-duality equations on a Riemann surface". In: *Proceedings of the London Mathematical Society* 3.1 (1987), pp. 59–126.

[Hit92] Nigel J Hitchin. "Lie groups and Teichmüller space". In: *Topology* 31.3 (1992), pp. 449–473.

[HLL13] Zheng Huang, John Loftin, and Marcello Lucia. "Holomorphic cubic differentials and minimal lagrangian surfaces in $\mathbb{CH}^2$". In: *Mathematical Research Letters* 20.3 (2013), pp. 501–520.

[Hod05] Thomas Wolf Stephen Hodge. "Hyperkähler geometry and Teichmüller space". PhD thesis. Imperial College London, 2005.

[Koi90] Norihito Koiso. "On the complex structure of a manifold of sections". In: *Osaka Journal of Mathematics* 27 (1990), pp. 175–183.

[KS07] Kirill Krasnov and Jean-Marc Schlenker. "Minimal surfaces and particles in 3-manifolds". In: *Geometriae Dedicata* 126 (2007), pp. 187–254.

[Kui53] Nicolaas Kuiper. "On convex locally projective spaces". In: *Convegno Internazionale di Geometria Differenziale, Italia* (1953), pp. 200–213.

[KZ17] Inkang Kim and Genkai Zhang. "Kähler metric on the space of convex real projective structures on surface". In: *Journal of Differential Geometry* 106.1 (2017), pp. 127–137.

[Lab07] François Labourie. "Flat projective structures on surfaces and cubic holomorphic differentials". In: *Pure Appl. Math. Q.* 3.4, Special Issue: In honor of Grigory Margulis. Part 1 (2007), pp. 1057–1099. ISSN: 1558-8599.

[Lab08] François Labourie. "Cross ratios, Anosov representations and the energy functional on Teichmüller space". In: *Annales scientifiques de l'Ecole normale supérieure.* Vol. 41. 3. 2008, pp. 439–471.

[Lab17] François Labourie. "Cyclic surfaces and Hitchin components in rank 2". In: *Annals of Mathematics* 185.1 (2017), pp. 1–58.

[Lee13] John M Lee. "Smooth manifolds". In: *Introduction to Smooth Manifolds.* Springer, 2013, pp. 1–31.

[Li16] Qiongling Li. "Teichmüller space is totally geodesic in Goldman space". In: *Asian Journal of Mathematics* 20.1 (2016), pp. 21–46.

[Li90] An-Min Li. "Calabi conjecture on hyperbolic affine hyperspheres". In: *Mathematische Zeitschrift* 203.1 (1990), pp. 483–491.

[Li92] An-Min Li. "Calabi conjecture on hyperbolic affine hyperspheres (2)". In: *Mathematische Annalen* 293.1 (1992), pp. 485–493.

[LM13] John Loftin and Ian McIntosh. "Minimal Lagrangian surfaces in $\mathbb{CH}^2$ and representations of surface groups into $\mathrm{SU}(2,1)$". In: *Geometriae Dedicata* 162.1 (2013), pp. 67–93.

[LM16] John Loftin and Ian McIntosh. "Cubic differentials in the differential geometry of surfaces". In: *Handbook of Teichmüller Theory* 6 (2016), pp. 231–274.

[LM19] John Loftin and Ian McIntosh. "Equivariant minimal surfaces in $\mathbb{CH}^2$ and their Higgs bundles". In: *Asian Journal of Mathematics* 23.1 (2019), pp. 71–106.

[Lof01] John C Loftin. "Affine spheres and convex $\mathbb{RP}^n$-manifolds". In: *American Journal of Mathematics* (2001), pp. 255–274.

[LT23] François Labourie and Jérémy Toulisse. "Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces". In: *Inventiones mathematicae* (2023), pp. 1–88.

[LW18] François Labourie and Richard Wentworth. "Variations along the Fuchsian locus". In: *Ann. Sci. Ec. Norm. Sup ér.* 51.2 (2018), pp. 487–547.

[Mes07] Geoffrey Mess. "Lorentz spacetimes of constant curvature". In: *Geometriae Dedicata* 126 (2007), pp. 3–45.

[MR90] Martin A Magid and Patrick J Ryan. "Flat affine spheres in R 3". In: *Geometriae Dedicata* 33.3 (1990), pp. 277–288.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Vol. 27. Oxford University Press, 2017.

[MST21] Filippo Mazzoli, Andrea Seppi, and Andrea Tamburelli. "Para-hyperKähler geometry of the deformation space of maximal globally hyperbolic anti-de Sitter three-manifolds". In: *arXiv:2107.10363* (2021).

[MW74] Jerrold Marsden and Alan Weinstein. "Reduction of symplectic manifolds with symmetry". In: *Reports on mathematical physics* 5.1 (1974), pp. 121–130.

[Nic20] Liviu I Nicolaescu. *Lectures on the Geometry of Manifolds*. World Scientific, 2020.

[Nie22] Xin Nie. "Cyclic Higgs bundles and minimal surfaces in pseudo-hyperbolic spaces". In: *arXiv:2206.13357* (2022).

[RT21] Nicholas Rungi and Andrea Tamburelli. "Pseudo-Kähler geometry of properly convex projective structures on the torus". In: *arXiv:2112.08979* (2021).

[RT22] Nicholas Rungi and Andrea Tamburelli. "Global Darboux coordinates for complete Lagrangian fibrations and an application to the deformation space of $\mathbb{RP}^2$-structures in genus one". In: *arXiv:2208.05336 (to appear in Journal of Symplectic Geometry)* (2022).

[RT23] Nicholas Rungi and Andrea Tamburelli. "The $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component as an infinite-dimensional pseudo-Kähler reduction". In: *arXiv:2306.02699* (2023).

[SY94] Richard M Schoen and Shing-Tung Yau. *Lectures on differential geometry*. Vol. 2. International press Cambridge, MA, 1994.

[Tra18] Samuel Trautwein. "Infinite dimensional GIT and moment maps in differential geometry". PhD thesis. ETH Zurich, 2018.

[Tro12] Anthony Tromba. *Teichmüller theory in Riemannian geometry*. Birkhäuser, 2012.

[Vai94] Izu Vaisman. *Lectures on the geometry of Poisson manifolds*. Vol. 118. Springer Science & Business Media, 1994.

[Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I: Volume 1*. Vol. 76. Cambridge University Press, 2002.

[Wan90] Changping Wang. "Some Examples of Complete Hyperbolic Affine 2-Spheres in $\mathbb{R}^3$". In: *Global differential geometry and global analysis* 15 (1990), pp. 271–280.

[War83] Frank W Warner. *Foundations of differentiable manifolds and Lie groups*. Vol. 94. Springer Science & Business Media, 1983.

[Wei58] André Weil. "Modules des surfaces de Riemann". In: *Seminare N. Bourbaki* 168 (1958), pp. 413–419.

[Wei80] Alan Weinstein. "The symplectic "category"". In: *Differential Geometric Methods in Mathematical Physics: Clausthal 1980 Proceedings of an International Conference Held at the Technical University of Clausthal* (1980), pp. 45–51.

[Wie18] Anna Wienhard. "An invitation to higher Teichmüller theory". In: *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*. World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039.

[Zhe01] Fangyang Zheng. *Complex differential geometry*. 18. American Mathematical Soc., 2001.