Logarithmic soft graviton theorems from superrotation Ward identities

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ABSTRACT: Soft graviton theorems receive one-loop contributions that are logarithmic in the energy of the soft graviton, and which are closely related to tails of gravitational waveforms. We demonstrate that these logarithmic corrections are encoded in the Ward identity of superrotation symmetries, i.e. they follow from conservation of superrotation charge across spatial infinity i^0. Our proof relies on a careful analysis of the radiative phase space admitting such gravitational tails, and the determination of the fluxes through null infinity $\mathscr{I}$ that act as canonical generators of superrotations on both gravitational and matter fields. All logarithmic terms are derived from the fluxes through correlations of the supertranslation Goldstone mode, provided care is taken in manipulating gravitationally interacting (i.e. dressed) rather than free fields. In cases where massive particles take part in the scattering process, logarithmic corrections also partly arise from the superrotation charge generator at timelike infinity $i^{\pm}$.

Contents

1   Introduction

Soft graviton theorems and their relation with asymptotic symmetries play an important role in developing an understanding of the infrared structure of quantum gravity. Influential work [1–4] has indeed shown that conservation of supertranslation charge is equivalent to Weinberg's leading soft graviton theorem [5], while conservation of superrotation charge encodes the tree-level subleading soft graviton theorem [6] (the case of massive particle scattering was treated in [7]). The soft graviton theorems are therefore the most direct physical consequence of the extended BMS symmetries [8–12] in the context of scattering amplitudes.

Soft graviton theorems receive loop corrections beyond the leading order $O(\omega^{-1})$ in the soft energy ω expansion, and it is of primary interest to assess whether these also consistently follow from BMS conservation laws, i.e., whether BMS symmetries hold in the quantum theory. However the literature on loop corrections to soft graviton theorems contains seemingly different results. On the one hand a general analysis of Bern, Davies and Nohle showed that loop corrections appear at order $O(\omega^0)$ in the soft expansion [13], while more recently Sen et al. [14–17] have obtained corrections at order $O(\ln \omega)$ (see also [18, 19] for a derivation of log terms in the eikonal approach) and related them to gravitational tail effects in classical gravity [20, 21]. In previous works [22, 23], a detailed derivation of the $O(\omega^0)$ corrections as a consequence of the conservation of superrotation charge was provided (see also [24, 25]). Establishing this correspondence required significant control over the radiative gravitational phase space, and therefore informs us about foundational aspects of the theory. The goal of the present work will be to unify these earlier results with the $O(\ln \omega)$ corrections found by Sahoo and Sen [16] (see also the recent generalisation [26]).

Soft graviton theorems associated with superrotations also play a central role in flat holography. Indeed in that context the subleading soft graviton theorem maps onto the conformal Ward identity of a two-dimensional celestial CFT [27], or that of a three-dimensional Carrollian CFT [28, 29]. The stress tensor of the corresponding boundary theory is identified with the integral density of the soft superrotation flux, whose determination is thus of central importance.

The paper is organised as follows. In section 2 we review the $O(\ln \omega)$ corrections found by Sahoo and Sen [16] and observe that they coincide with the $O(\omega^0)$ corrections worked out in [13] for the case where all scattered particles are massless, provided one substitutes $\ln \omega$ by the

cutoff used to regulate infrared divergences. This strongly suggests that the methods developed in previous works [22, 23] and which accounted for the $O(\omega^0)$ corrections to the subleading soft graviton theorem can also account for the generic $O(\ln \omega)$ corrections. To demonstrate this, our main task will be to extend our previous analysis to the case where scattered particles can be massive. To this end we review in section 3 the exponentiation of infrared divergence in scattering amplitudes without soft graviton emission [5], and their description in terms of correlation functions of the supertranslation Goldstone mode [30–32]. We point out that a slight modification of the Goldstone two-point function needs to be considered in order to account for the so-called 'Coulomb' phases corresponding to classical long-range interaction between asymptotic states. In section 4 we describe the radiative phase space of asymptotically flat gravity and its organisation with respect to extended BMS symmetries. In particular, we provide a symplectic form such that these symmetries are realised canonically and identify the canonical generators with BMS fluxes [33]. This generalises the construction provided in [22] and building upon [2, 33–35] to the case where gravitational tails are allowed in the phase space. This results in a new term in the soft superrotation flux compared to previous works [22, 33], which however vanishes in a superrotation frame of reference where the Liouville field is zero or in case gravitational tails need not be considered explicitly. Finally in section 5 we derive all the logarithmic corrections found by Sahoo and Sen [16] from superrotation charge conservation. While some of these corrections directly arise from the soft superrotation flux as in [22], for massive particle scattering some corrections also arise from the generator of superrotation at timelike infinities $i^{\pm}$ when considering matter fields interacting with the gravitational field, i.e. dressed fields rather than free fields.[1]

2 Logarithmic corrections to soft graviton theorems

Soft graviton theorems relate a scattering amplitude $\mathcal{M}_n(p_1, \ldots, p_n)$ of n hard particles of momenta p_i to the amplitude $\mathcal{M}_{n+1}(q, p_1, \ldots, p_n)$ that contains an additional external soft graviton of momentum $q = \omega \hat{q}$. Assuming a power series expansion in the soft momentum, the tree-level soft graviton theorem at leading and subleading order in ω takes the form [5, 6]

$$\mathcal{M}_{n+1} \overset{\omega \to 0}{=} \left[\omega^{-1} \hat{S}_n^{(0)} + S_n^{(1)} \right] \mathcal{M}_n + O(\omega), \tag{2.1}$$

where the leading and subleading soft factors are respectively given by

$$\hat{S}_n^{(0)} = \frac{\kappa}{2} \sum_{i=1}^{n} \frac{p_i^{\mu} p_i^{\nu} \varepsilon_{\mu\nu}(\hat{q})}{p_i \cdot \hat{q}}, \quad S_n^{(1)} = -\frac{i\kappa}{2} \sum_{i=1}^{n} \frac{p_i^{\mu} \varepsilon_{\mu\nu}(\hat{q}) q_{\lambda}}{p_i \cdot q} \left(J_i^{\lambda\nu} + S_i^{\lambda\nu} \right) \equiv S_n^{(1)J} + S_n^{(1)S}, \tag{2.2}$$

where $\kappa = \sqrt{32\pi G}$, $\varepsilon_{\mu\nu}(\hat{q})$ is the polarisation tensor of the soft graviton, $J_i^{\mu\nu} = -i(p_i^{\mu} \partial_{p_i}^{\nu} - p_i^{\nu} \partial_{p_i}^{\mu})$ and $S_i^{\mu\nu}$ are the orbital and spin angular momentum of the i-th particle, respectively.

Sahoo and Sen have shown that one-loop corrections generate logarithmic corrections which actually dominate over the $O(\omega^0)$ term in the expansion [16],

$$\mathcal{M}_{n+1} \overset{\omega \to 0}{=} \left[\omega^{-1} \hat{S}_n^{(0)} - \frac{\kappa^2}{4} \ln \omega \, S_n^{(\ln)} \right] \mathcal{M}_n + O(\omega^0). \tag{2.3}$$

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[1] Loop corrections in QED were similarly discussed in [36, 37].

Note that these loop corrections necessarily introduce an infrared length scale so as to make the argument of the logarithm dimensionless. These logarithmic corrections are explicitly given by

$$
\begin{aligned}
S_n^{(\ln)} = {} & \frac{i\kappa}{8\pi} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot q} \sum_j \delta_{\eta,\eta_j} \, q \cdot p_j \\
& + \frac{i\kappa}{16\pi} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\nu q_\rho}{p_i \cdot q} \sum_j \delta_{\eta_i,\eta_j} (p_i \cdot p_j)(p_i^\mu p_j^\rho - p_j^\mu p_i^\rho) \frac{2(p_i \cdot p_j)^2 - 3p_i^2 p_j^2}{\left[(p_i \cdot p_j)^2 - p_i^2 p_j^2 \right]^{3/2}} \\
& - \frac{\kappa}{8\pi^2} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\nu p_i^\nu}{p_i \cdot q} \sum_j q \cdot p_j \ln|\hat{q} \cdot \hat{p}_j| \\
& - \frac{\kappa}{32\pi^2} \sum_i \frac{p_i^\mu \varepsilon_{\mu\nu} q_\lambda}{p_i \cdot q} \left(p_i^\lambda \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\lambda}} \right) \sum_j \frac{2(p_i \cdot p_j)^2 - p_i^2 p_j^2}{[(p_i \cdot p_j)^2 - p_i^2 p_j^2]^{1/2}} \ln\left(\frac{p_i \cdot p_j + \sqrt{(p_i \cdot p_j)^2 - p_i^2 p_j^2}}{p_i \cdot p_j - \sqrt{(p_i \cdot p_j)^2 - p_i^2 p_j^2}} \right).
\end{aligned}
\tag{2.4}
$$

Note that the formulae of Sahoo and Sen are presented with $\kappa = 2$ in our conventions. In [16] the first two lines and last two lines are referred to as the 'classical' and 'quantum' contributions, respectively, even though they all arise from one-loop diagrams. The reason for this distinction is that the 'classical' terms can be obtained from purely classical considerations. In particular they naturally map to late-time gravitational tails [15, 17]. The first term in (2.4) represents the effect of gravitational drag on the soft graviton due to the other hard particles in the final state. The second term represents the effect of late time gravitational radiation due to the late time acceleration of the particles via long range gravitational interaction.

The 'quantum terms' were obtained by explicit evaluation of one-loop diagrams in a theory of minimally coupled scalars in [16], and the generalisation to spinning fields was recently given in [26]. In particular, the first term comes from a region of loop integration where the loop momentum is small compared to ω but large compared to the infrared cut-off R^{-1}, which corresponds to the best energy resolution of a detector situated a distance R from the scattering center. The second term comes from regions where the loop momentum is large compared to ω but small compared to all other energy scales involved in the process (energies of hard particles).

Note that the expression presented in [16] did not include the terms $i = j$ in the second and fourth line of (2.4), the reason being that diagrams involving a virtual soft graviton emitted and absorbed from the same external leg were discarded. However Weinberg has argued that the infrared contributions arising from such diagrams should be kept, and are in fact crucial to ensure cancellations of infrared divergences in inclusive cross sections [38]. Here we have therefore allowed the terms with $i = j$ in the formula (2.4). We will see that these terms will naturally arise from the derivation of the logarithmic corrections based on superrotation Ward identities that we will present in this work.

There is a significantly more compact way to write (2.4), which will turn out very useful in carrying our analysis. For this we introduce the relative velocity of the particles i and j,

$$
\beta_{ij} \equiv \sqrt{1 - (\hat{p}_i \cdot \hat{p}_j)^{-2}},
\tag{2.5}
$$

where $\hat{p}_i$ is the normalised momentum satisfying $\hat{p}_i^2 = -1$, and the quantities[2]

$$\sigma_n = \frac{1}{2(8\pi)^2} \sum_{ij} \eta_i \eta_j \, m_i m_j \, \frac{1 + \beta_{ij}^2}{\beta_{ij}\sqrt{1 - \beta_{ij}^2}} \left(i\pi \delta_{\eta_i, \eta_j} - \frac{1}{2} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} \right), \qquad (2.6)$$

$$\hat{\sigma}'_{n+1}(\hat{q}) = \frac{1}{2(4\pi)^2} \sum_{i=1}^{n} (p_i \cdot \hat{q}) \ln(\hat{p}_i \cdot \hat{q}). \qquad (2.7)$$

In writing (2.6) we have assumed that all hard particles are massive. If the i-th particle is massless, then the corresponding terms in the sum are replaced by

$$\frac{1}{(8\pi)^2} \, \eta_i \eta_j \, |p_i \cdot p_j| \left(i\pi \delta_{\eta_i, \eta_j} - \ln |\hat{p}_i \cdot \hat{p}_j| \right) = \frac{1}{(8\pi)^2} \, (p_i \cdot p_j) \ln(\hat{p}_i \cdot \hat{p}_j), \qquad (2.8)$$

which is valid whether the j-th particle is massless or massive. In fact $\hat{\sigma}'_{n+1}(\hat{q})$ derives from σ_{n+1} in case where the $(n+1)$-th particle is a (soft) graviton with normalised momentum $\hat{q}$,

$$\sigma_{n+1} = \sigma_n + \hat{\sigma}'_{n+1}(\hat{q}). \qquad (2.9)$$

In terms of these quantities the logarithmic soft factor (2.4) can be simply written[3]

$$S_n^{(\ln)}(\hat{q}) = -8 \left(\hat{\sigma}'_{n+1}(\hat{q}) \, \hat{S}_n^{(0)}(\hat{q}) - S_n^{(1)J}(\hat{q}) \, \sigma_n \right), \qquad (2.10)$$

where $S_n^{(1)J}(\hat{q})$ is the orbital part of the subleading soft operator defined in (2.2), and where we have kept the dependence on the soft graviton momentum direction $\hat{q}$ explicit. The fact that the logarithmic corrections do not depend on the spin of the particles has been confirmed in [26].

At this point it is very informative to make a comparison between the expression (2.10) and the earlier result of Bern, Davies and Nohle [13] regarding infrared divergent one-loop corrections to the soft graviton theorem in purely massless scatterings, namely

$$\mathcal{M}_{n+1} \overset{\omega \to 0}{=} \left[\omega^{-1} \hat{S}_n^{(0)} + S_n^{(1)} + \kappa^2 \epsilon^{-1} \left(\hat{\sigma}'_{n+1} \hat{S}_n^{(0)} - \left(S_n^{(1)J} \sigma_n \right) \right) \right] \mathcal{M}_n + \mathcal{O}(\omega), \qquad (2.11)$$

where $\epsilon = (4 - d)/2$ is an infrared regulator in dimensional regularisation. While this method differs from the one used by Sahoo and Sen,[4] a direct comparison between the infrared divergent terms in (2.11) and the logarithmic terms derived by Sahoo and Sen and given in (2.10) allows to identify the terms that multiply ϵ^{-1} with the ones in front of $\ln \omega$. We can understand this correspondence in the following way: if instead of dimensional regularisation we used energy cutoffs as Weinberg originally discussed [5], we would replace ϵ^{-1} by

$$\epsilon^{-1} \quad \longleftrightarrow \quad \ln \frac{\Lambda}{\lambda}, \qquad (2.12)$$

where λ is the infrared regulator while Λ is a dividing energy scale below which virtual gravitons are considered soft and some approximations can be applied. The terms $\ln(\omega R)$ can be thus identified with (2.12) with the energy ω of the emitted soft graviton as the dividing energy scale Λ and the inverse of the distance of the detector R as the IR cut-off energy scale.

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[2] $\eta_i = +1 \, (-1)$ for an outgoing (incoming) particle.

[3] This formula can be found in (6.29) of [16], where $K_{gr}^{reg} = -8 \ln \omega \, \sigma_n$ and $\widehat{S}_{gr}^{(1)} = S_n^{(1)J}$.

[4] The IR divergences in [16] are dealt with by means of the Grammer-Yennie formalism which provides a prescription to extract an IR-finite S-matrix.

3 Soft factorisation and supertranslation Goldstone

An important feature of gravitational scattering is that infrared divergences factorise from the rest of the amplitude [5]. For a scattering of n hard particles, this factorisation takes the form

$$\mathcal{M}_n = \mathcal{M}_{\text{soft}}\, \mathcal{M}_{\text{finite}}\,, \tag{3.1}$$

where $\mathcal{M}_{\text{soft}}$ accounts for the infrared divergences coming from soft virtual gravitons propagating between external legs,

$$\mathcal{M}_{\text{soft}} = \exp\left[\frac{1}{\epsilon}\frac{\kappa^2}{2(8\pi)^2}\sum_{ij}\eta_i\eta_j\, m_i m_j\, \frac{1+\beta_{ij}^2}{\beta_{ij}\sqrt{1-\beta_{ij}^2}}\left(i\pi\delta_{\eta_i,\eta_j} - \frac{1}{2}\ln\frac{1+\beta_{ij}}{1-\beta_{ij}}\right)\right]\,, \tag{3.2}$$

with β_{ij} the relative velocity of particle i and j given in (2.5). Notice also the relationship $\mathcal{M}_{\text{soft}} = \exp[\kappa^2\epsilon^{-1}\sigma_n]$. The soft factor (3.2) actually vanishes as the infrared regulator is removed ($\epsilon \to 0^+$) but fortunately drops out of the experimentally observable inclusive cross-sections. Note that the imaginary part of the exponent in (3.2) only involves pairs of particles that are both incoming or outgoing, and corresponds to the effect of the gravitational potential attraction between these particles. Such pure phases are often referred to as 'Coulomb phases' by abuse of language and in analogy with quantum electrodynamics.

In the language of celestial holography, the scattering amplitude $\mathcal{M}_n$ is interpreted as the correlation function of n 'celestial' operators,[5]

$$\mathcal{M}_n = \langle\text{out}|\mathcal{S}|\text{in}\rangle = \langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle\,, \tag{3.3}$$

where each operator $\mathcal{O}_i$ represents an external particle with momentum p_i. The factorisation (3.2) of the amplitudes motivates the factorisation of the celestial operators themselves [30],

$$\mathcal{O}_1(p_1) = \mathcal{W}_1(p_1)\,\tilde{\mathcal{O}}_1(p_1)\,, \tag{3.4}$$

such that $\mathcal{M}_{\text{soft}}$ and $\mathcal{M}_{\text{finite}}$ be respectively obtained as

$$\mathcal{M}_{\text{soft}} = \langle\mathcal{W}_1\ldots\mathcal{W}_n\rangle\,, \qquad \mathcal{M}_{\text{finite}} = \langle\tilde{\mathcal{O}}_1\ldots\tilde{\mathcal{O}}_n\rangle\,. \tag{3.5}$$

The $\tilde{\mathcal{O}}(p)$'s are dressed operators and $\mathcal{M}_{\text{finite}}$ is the scattering amplitude of dressed particles.

In practice a specific parametrisation of massless momenta q in terms of energy and coordinates on the celestial sphere is used,

$$q^\mu = \omega\,\eta\,(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) \equiv \omega\,\hat{q}^\mu\,, \tag{3.6}$$

while for a massive momentum p a parametrisation in terms of coordinates $(\rho, z, \bar{z})$ on a three-dimensional hyperboloid is used,

$$p^\mu = \frac{m}{2\rho}\,\eta\,(1+\rho^2(1+z\bar{z}), \rho^2(z+\bar{z}), -i\rho^2(z-\bar{z}), -1+\rho^2(1-z\bar{z})) \equiv m\,\hat{p}^\mu\,. \tag{3.7}$$

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[5]Usually the term 'celestial' is reserved to operators recast in boost basis, while here we deal with momentum basis operators.

We will reserve the letter q for massless momenta only, while the letter p will always refer to the momentum of one of the hard external particles that can be either massive or massless.

The operators $\tilde{\mathcal{O}}_i$ are to be understood as (conformally) dressed operators and the vertex operators $\mathcal{W}_i$ as the corresponding operator dressings [30, 31, 39, 40]. For a massless particle with momentum $p = \omega\,\hat{p}$ parametrised as in (3.6), the vertex operator is given in terms of the supertranslation Goldstone mode $C^{(0)}$ by [30]

$$\mathcal{W}(p) = e^{i\omega C^{(0)}(\hat{p})}\,, \tag{3.8}$$

while for a massive particle with momentum $p = m\,\hat{p}$ parametrised as in (3.7), it is given by

$$\mathcal{W}(p) = \exp\left[\frac{im}{2}\int d^2\hat{q}\,\mathcal{G}(\hat{p};\hat{q})\,C^{(0)}(\hat{q})\right]\,, \tag{3.9}$$

where it is understood that $\hat{q} = \hat{q}(w,\bar{w})$ shares the time-orientation of $\hat{p}$, namely $\eta(\hat{q}) = \eta(\hat{p})$, and where $d^2\hat{q} = dw\,d\bar{w}$ and $\mathcal{G}(\hat{p};\hat{q})$ is the 'bulk-boundary propagator' [41]

$$\mathcal{G}(\hat{p};\hat{q}) = \frac{1}{\pi}\left(\frac{\rho}{1 + \rho^2|z - w|^2}\right)^3\,. \tag{3.10}$$

Using the vertex operators (3.8)–(3.9) together with the Goldstone two-point function,

$$\begin{aligned}
\langle C^{(0)}(\hat{q}_i)C^{(0)}(\hat{q}_j)\rangle &= -\frac{1}{\epsilon}\frac{\kappa^2}{2(4\pi)^2}\,(\hat{q}_i\cdot\hat{q}_j)\ln(\hat{q}_i\cdot\hat{q}_j) \\
&= \frac{1}{\epsilon}\frac{\kappa^2}{(4\pi)^2}\,\eta_i\eta_j\,|z_{ij}|^2\left(\ln|z_{ij}|^2 - i\pi\delta_{\eta_i,\eta_j}\right)\,,
\end{aligned} \tag{3.11}$$

it was shown that the soft factor (3.2) is indeed recovered [30]. Note that the imaginary part of the above expression had not been explicitly considered in previous work [22, 30, 32], although it is necessary here in order to account for the 'Coulomb phases' in (3.2). This new contribution will actually play a crucial role in recovering the classical logarithmic corrections to the soft theorems. Also note that the Goldstone two-point function is still a Green's function for the operator $\partial^2\bar{\partial}^2$ in agreement with the effective action derived in [32], whereas the new imaginary term corresponds to a different choice of boundary condition.

With this at hand, the insertion of $C^{(0)}$ in the $\mathcal{S}$-matrix can be computed along the same line as in [22], by considering the soft factor of a scattering with an additional massless particle of momentum $q = \omega\,\hat{q}(z,\bar{z})$ and taking suitable derivative with respect to ω,

$$\begin{aligned}
\langle C^{(0)}(\hat{q})\mathcal{W}_1\ldots\mathcal{W}_n\rangle &= -i\partial_\omega\langle\mathcal{W}(q)\mathcal{W}_1\ldots\mathcal{W}_n\rangle\big|_{\omega=0} = -i\partial_\omega\mathcal{M}_{\text{soft}}(q,p_1,\ldots,p_n)\big|_{\omega=0} \\
&= -i\frac{\kappa^2}{\epsilon}\hat{\sigma}'_{n+1}(\hat{q})\,\langle\mathcal{W}_1\ldots\mathcal{W}_n\rangle\,.
\end{aligned} \tag{3.12}$$

Since the operators $\tilde{\mathcal{O}}_i$ and $\mathcal{W}_i$ do not interact, this property is also valid at the level of the full $\mathcal{S}$-matrix itself,

$$\langle\text{out}|C^{(0)}(\hat{q})\,\mathcal{S}|\text{in}\rangle = -i\frac{\kappa^2}{\epsilon}\,\hat{\sigma}'_{n+1}(\hat{q})\,\langle\text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{3.13}$$

Note that the computation in this section was done in dimensional regularization with an eikonal approximation which produced a $\frac{1}{\epsilon}$ regulator. We will stick to this scheme in the rest

of the paper, however, the computation could have been repeated with a cut-off regularization, producing a $\ln \omega$ regulator instead. Then the rest of the results would follow in the same way, with the identification of equation (2.12). The expression given in (3.13) complements the leading soft graviton theorem, which describes the insertion of the leading soft news $\mathcal{N}_{zz}^{(0)}$ (see below (4.5)) into $\mathcal{S}$-matrix elements [2],

$$\langle \text{out}| \mathcal{N}_{zz}^{(0)}(\hat{q}) \, \mathcal{S} | \text{in} \rangle = -\frac{\kappa}{16\pi} \hat{S}_n^{(0)+}(\hat{q}) \, \langle \text{out}| \mathcal{S} | \text{in} \rangle \,. \tag{3.14}$$

4 Radiative phase space at null infinity

In this section we give a detailed description of the radiative phase space at null infinity $\mathscr{I}^+$, following notations and conventions of [22]. We extend that previous analysis by allowing for gravitational tails as they directly relate to the logarithmic corrections to the soft graviton theorems which we aim to describe in this work.

We cover $\mathscr{I}^+$ with coordinates $(u, z, \bar{z})$, where u is a retarded time and $(z, \bar{z})$ are complex stereographic coordinates on the celestial sphere. We choose the representative of the degenerate boundary metric to be $ds^2|_{\mathscr{I}^+} = 0 \, du^2 + 2 \, dz \, d\bar{z}$ and we denote $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$. The radiative data is encoded in the asymptotic shear $C_{zz}(u, z, \bar{z})$ ($C_{zz}^* = C_{\bar{z}\bar{z}}$) and the Bondi news tensor $N_{zz} = \partial_u C_{zz}$. We will assume the following falloffs in u as $u \to \pm\infty$, which are compatible with the action of BMS symmetries on the phase space [42, 43], and are sufficiently weak to encompass gravitational tails [20, 21, 44],

$$C_{zz} = (u + C_{\pm}) N_{zz}^{\text{vac}} - 2\partial^2 C_{\pm} + \frac{1}{u} C_{zz}^{L, \pm} + o(u^{-1}) \,, \quad N_{zz} = N_{zz}^{\text{vac}} - \frac{C_{zz}^{L, \pm}}{u^2} + o(u^{-2}) \,. \tag{4.1}$$

Here, $C_{\pm}(z, \bar{z})$ correspond to the values of the supertranslation field at the corners $\mathscr{I}_{\pm}^+$ of null infinity and encoding the displacement memory effect [45]. The presence of the new term $C_{zz}^{L, \pm}(z, \bar{z})$ in the expansion (4.1) is necessary in order to account for gravitational tails [14–17]. The vacuum news tensor $N_{zz}^{\text{vac}}(z)$ [42, 46], identified with the tracefree part of the Geroch tensor [47–49], is given in terms of a Liouville field $\varphi(z)$,

$$N_{zz}^{\text{vac}} = \frac{1}{2} (\partial \varphi)^2 - \partial^2 \varphi \,. \tag{4.2}$$

The latter encodes the refraction/velocity kick memory effects [42]. To make the covariance of the expressions under the superrotations manifest, it is useful to introduce the derivative operators [33, 48, 50, 51]

$$\mathscr{D} \phi_{h, \bar{h}} = [\partial - h\partial\varphi] \phi_{h, \bar{h}} \,, \qquad \bar{\mathscr{D}} \phi_{h, \bar{h}} = [\bar{\partial} - \bar{h}\bar{\partial}\bar{\varphi}] \phi_{h, \bar{h}} \,, \tag{4.3}$$

which, when acting on conformal fields $\phi_{h, \bar{h}}$, produce conformal fields of weights $(h + 1, \bar{h})$ and $(h, \bar{h} + 1)$, respectively.

The radiative data can be split into hard and soft variables [22, 35, 48]. To do so we define $C_{zz}^{(0)}$, $\tilde{C}_{zz}$ and $\tilde{N}_{zz}$ through

$$\begin{aligned}
C_{zz} &= u N_{zz}^{\text{vac}} + C_{zz}^{(0)} + \tilde{C}_{zz} \,, \quad N_{zz} = N_{zz}^{\text{vac}} + \tilde{N}_{zz} \,, \\
C_{zz}^{(0)} &= -2\mathscr{D}^2 C^{(0)} \,, \qquad\qquad\quad C^{(0)} = \frac{1}{2}(C_+ + C_-) \,.
\end{aligned} \tag{4.4}$$

Moreover, the leading soft news $\mathcal{N}_{zz}^{(0)}$ and subleading soft news $\mathcal{N}_{zz}^{(1)}$ are defined by

$$\mathcal{N}_{zz}^{(0)} \equiv \int_{-\infty}^{+\infty} du\, \tilde{N}_{zz} = -4\mathscr{D}^2 N^{(0)}, \qquad \mathcal{N}_{zz}^{(1)} \equiv \int_{-\infty}^{+\infty} du\, u \tilde{N}_{zz}. \tag{4.5}$$

Notice that the subleading soft news $\mathcal{N}_{zz}^{(1)}$ is formally divergent due to the tail terms parametrised by C_{zz}^L in (4.1). This potential issue is removed once we turn to scattering amplitudes since the quantum Hilbert space simply discards such tails at the cost of generating infrared divergences at one loop; see also the discussion after (5.7). This can be viewed as a reshuffling of the infrared divergences associated with the physics of logarithmic corrections studied in this paper. Obviously, it would be desirable to come up with a quantisation of gravity that does account for gravitational tails in physical scattered states, and which does not require the introduction of infrared cutoffs at any step. We will not attempt to achieve this here, although the phase space description provided in this section might offer the basis to carry out such a program.

Now we wish to provide a symplectic form on this radiative phase space such that extended BMS symmetries are canonical transformations, and identify the corresponding canonical generators. The hard variables are $\{\tilde{C}_{zz}, \tilde{N}_{zz}\}$ while the soft variables are $\{C^{(0)}, \varphi, \mathcal{N}_{zz}^{(0)}, \mathcal{N}_{zz}^{(1)}\}$. The appropriate symplectic form Ω on the radiative phase space at $\mathscr{I}^+$ is the sum of a hard and a soft contribution [22, 35],

$$\Omega = \Omega^{\text{hard}} + \Omega^{\text{soft}},$$
$$\Omega^{\text{hard}} = \frac{1}{32\pi G} \int_{\mathscr{I}^+} du\, d^2z \left[\delta \tilde{N}_{zz} \wedge \delta \tilde{C}_{\bar{z}\bar{z}} + c.c.\right] + \text{matter sector}, \tag{4.6}$$
$$\Omega^{\text{soft}} = \frac{1}{32\pi G} \int_{\mathcal{S}} d^2z \left[\delta \mathcal{N}_{zz}^{(0)} \wedge \delta C_{\bar{z}\bar{z}}^{(0)} + \delta \Pi_{zz} \wedge \delta N_{\bar{z}\bar{z}}^{\text{vac}} + c.c.\right],$$

with the constraint

$$\Pi_{zz} = 2\mathcal{N}_{zz}^{(1)} + C^{(0)}\mathcal{N}_{zz}^{(0)} + (\varphi + \bar{\varphi})\Delta C_{zz}^L. \tag{4.7}$$

Notice the shift[6] of Π_{zz} compared to [22, 35] which involves $\Delta C_{zz}^L \equiv C_{zz}^{L,+} - C_{zz}^{L,-}$. Under BMS transformations parametrised by supertranslations $\mathcal{T}(z, \bar{z})$ and superrotations $(\mathcal{Y}(z), \bar{\mathcal{Y}}(\bar{z}))$, we have

$$\delta_{(\mathcal{T}, \mathcal{Y}, \bar{\mathcal{Y}})} \Delta C_{zz}^L = (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \partial\mathcal{Y} - \bar{\partial}\bar{\mathcal{Y}})\Delta C_{zz}^L,$$
$$\delta_{(\mathcal{T}, \mathcal{Y}, \bar{\mathcal{Y}})} \mathcal{N}_{zz}^{(1)} = (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \partial\mathcal{Y} - \bar{\partial}\bar{\mathcal{Y}})\mathcal{N}_{zz}^{(1)} - \frac{1}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})\Delta C_{zz}^L - \mathcal{T}\mathcal{N}_{zz}^{(0)}. \tag{4.8}$$

Using (4.8), one can now show that the combination (4.7) transforms nicely under BMS transformations,

$$\delta_{(\mathcal{T}, \mathcal{Y}, \bar{\mathcal{Y}})} \Pi_{zz} = (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \partial\mathcal{Y} - \bar{\partial}\bar{\mathcal{Y}})\Pi_{zz} - \mathcal{T}\mathcal{N}_{zz}^{(0)}. \tag{4.9}$$

The transformation of the other canonical variables remain exactly the same as in [22] and will not be repeated here.

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[6]$\Pi_{zz}^{here} = \Pi_{zz}^{there} + (\varphi + \bar{\varphi})\Delta C_{zz}^L.$

We then obtain the canonical generators of BMS symmetries, also known as BMS fluxes, by contracting the symplectic structure (4.6) with the corresponding Hamiltonian vector fields,

$$i_{\delta_{(\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}})}}\Omega^{\text{soft}} = \delta F^{\text{soft}}_{(\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}})}, \qquad i_{\delta_{(\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}})}}\Omega^{\text{hard}} = \delta F^{\text{hard}}_{(\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}})}. \tag{4.10}$$

The supertranslation fluxes are given by

$$F^{\text{hard}}_{\mathcal{T}} = -\frac{1}{16\pi G}\int du d^2z\,\mathcal{T}\left[\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}} + 16\pi G\,T^{(2)}_{uu}\right],$$
$$F^{\text{soft}}_{\mathcal{T}} = \frac{1}{8\pi G}\int d^2z\,\mathcal{T}\left[\mathscr{D}^2\mathcal{N}^{(0)}_{\bar{z}\bar{z}}\right], \tag{4.11}$$

where $T^{(2)}_{\mu\nu}$ denotes the order $O(r^{-2})$ in the expansion of the $\mu\nu$-component of the matter stress tensor, while the superrotation fluxes are given by

$$F^{\text{hard}}_{\mathcal{Y}} = \frac{1}{16\pi G}\int du d^2z\,\mathcal{Y}\left[\frac{3}{2}\tilde{C}_{zz}\partial\tilde{N}_{\bar{z}\bar{z}} + \frac{1}{2}\tilde{N}_{\bar{z}\bar{z}}\partial\tilde{C}_{zz} + \frac{u}{2}\partial(\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}}) + 16\pi G\left(\frac{u}{2}\partial_z T^{(2)}_{uu} - T^{(2)}_{uz}\right)\right],$$
$$F^{\text{soft}}_{\mathcal{Y}} = \frac{1}{16\pi G}\int d^2z\,\mathcal{Y}\left[-\mathscr{D}^3\left(\mathcal{N}^{(1)}_{\bar{z}\bar{z}} + \frac{1}{2}(\varphi+\bar{\varphi})\Delta C^L_{\bar{z}\bar{z}}\right) + \frac{3}{2}C^{(0)}_{zz}\mathscr{D}\mathcal{N}^{(0)}_{\bar{z}\bar{z}} + \frac{1}{2}\mathcal{N}^{(0)}_{\bar{z}\bar{z}}\mathscr{D}C^{(0)}_{zz}\right], \tag{4.12}$$

together with the complex conjugate expressions associated with $\bar{\mathcal{Y}}$. These fluxes are identical to those presented in [22] when setting $\Delta C^L_{\bar{z}\bar{z}} = 0$. Note that $F^{\text{soft}}_{\mathcal{Y}}$ contains logarithmic infrared divergences through $\mathcal{N}^{(1)}_{\bar{z}\bar{z}}$, in agreement with earlier observations [22, 52].

For completeness we express these fluxes as a difference of charges between the corners $\mathscr{I}^+_\pm$ of future null infinity. Following the notations and conventions of [22], and introducing the mass and angular momentum aspects in terms of Newman-Penrose scalars,[7]

$$\mathcal{M} = -\frac{1}{2}(\Psi^0_2 + \bar{\Psi}^0_2), \qquad \mathcal{N} = -\Psi^0_1 + u\,\bar{\partial}\Psi^0_2, \tag{4.13}$$

the total BMS fluxes can be written

$$F(\mathscr{I}^+) \equiv F^{\text{hard}} + F^{\text{soft}} = Q(\mathscr{I}^+_+) - Q(\mathscr{I}^+_-), \tag{4.14}$$

with

$$Q(\mathscr{I}^+_\pm) = \frac{1}{32\pi G}\int d^2z\left[8\mathcal{T}\mathcal{M} + \mathcal{Y}\left[4\bar{\mathcal{N}} - \partial(N^{\text{vac}}_{zz}C^L_{\bar{z}\bar{z}}) + \mathscr{D}^3((\varphi+\bar{\varphi})C^L_{\bar{z}\bar{z}})\right] + c.c.\right]\Big|_{\mathscr{I}^+_\pm}. \tag{4.15}$$

The last two terms featuring $C^L_{\bar{z}\bar{z}}$ give new contributions compared to the expression of BMS charges given in [22].

5 Superrotation Ward identity

We finally come to the derivation of the logarithmic corrections to soft graviton theorems from the Ward identity of superrotations. First we briefly remind the general procedure

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[7]Notice that the falloffs (4.1) imply a logarithmic divergence in the angular momentum aspect ($\mathcal{N} \sim \ln u$) [52], although the mass aspect remains finite ($\mathcal{M} \sim u^0$). This is the same divergence as that appearing in the flux (4.12) through $\mathcal{N}^{(1)}_{\bar{z}\bar{z}}$.

allowing the derivation of soft graviton theorems from BMS conservation laws [1, 2, 4, 7]. The key ingredient is the conservation of the BMS charges across spatial infinity i^0,

$$Q(\mathscr{I}_-^+) = Q(\mathscr{I}_+^-) \equiv Q(i^0) \,. \tag{5.1}$$

It should be pointed out that genuine conservation laws at spatial infinity have only been established for the global BMS group containing supertranslations but no superrotations [53]. Alternatively (5.1) can be established provided gravitational fields satisfy suitable antipodal matching relations between $\mathscr{I}_-^+$ and $\mathscr{I}_+^-$ in the asymptotic solution space of interest. This in particular requires to connect gravitational solutions in asymptotic gauges adapted to $\mathscr{I}^\pm$ and i^0, respectively, which has become an active subject of investigations [52, 54–60]. For superrotations the relevant antipodal matching is that of the angular momentum aspect, which has been established in an asymptotic solution space without gravitational tails near spatial infinity, i.e., in a phase space where the physical news tensor falls-off as $\tilde{N}_{zz} = o(|u|^{-2})$ in the limit $u \to -\infty$ [60]. The logarithmic corrections to the soft theorems discussed here are however tightly connected to existence of such tails [15, 17], and to establish the validity of the antipodal matching in this context therefore constitutes an important open problem. In what follows we will simply assume the validity of (5.1) and work out its implications. We also work in a reference superrotation frame where $\varphi = \bar{\varphi} = 0$ and therefore the derivative operator (4.3) is the standard derivative $\mathscr{D} = \partial_z \equiv \partial$.

Writing the BMS fluxes at past and future null infinity as a net difference of charges,

$$\begin{aligned} F(\mathscr{I}^+) &= Q(\mathscr{I}_+^+) - Q(\mathscr{I}_-^+) \,, \\ F(\mathscr{I}^-) &= Q(\mathscr{I}_+^-) - Q(\mathscr{I}_-^-) \,, \end{aligned} \tag{5.2}$$

the conservation law (5.1) takes the form

$$Q(i^0) = Q(\mathscr{I}_+^+) - F(\mathscr{I}^+) = Q(\mathscr{I}_-^-) + F(\mathscr{I}^-) \,. \tag{5.3}$$

Invariance of the $\mathcal{S}$-matrix under BMS symmetries can be stated as

$$\left[Q(i^0), \mathcal{S} \right] = 0 \,, \tag{5.4}$$

or using (5.3),

$$\langle \text{out} | \left(F(\mathscr{I}^+) - Q(\mathscr{I}_+^+) \right) \mathcal{S} + \mathcal{S} \left(Q(\mathscr{I}_-^-) + F(\mathscr{I}^-) \right) | \text{in} \rangle = 0 \,. \tag{5.5}$$

One is left to evaluate the effect of inserting these charges and fluxes in the $\mathcal{S}$-matrix elements. We will focus on superrotations in this work, although the discussion also applied to supertranslations thus far. The hard part of the superrotation flux $F_{\mathcal{Y}}^{\text{hard}}(\mathscr{I}^+)$ generates the symmetry transformations of the massless outgoing states, while the charge $Q_{\mathcal{Y}}(\mathscr{I}_+^+)$ at the top corner of $\mathscr{I}$ generates those of the massive outgoing states. A similar statement applies to incoming states. Together these terms account for the tree-level subleading soft factor $S_n^{(1)}$ in (2.2). Indeed, insertion of the linear piece in the soft flux (4.12), namely

$$F_{\mathcal{Y}}^{\text{soft},0} = -\frac{2}{\kappa^2} \int d^2z \, \mathcal{Y} \, \partial^3 \mathcal{N}_{\bar{z}\bar{z}}^{(1)} \,, \tag{5.6}$$

adds an external soft graviton to the original amplitude since the subleading soft news can be expressed in terms of graviton operators as [4]

$$\mathcal{N}_{\bar{z}\bar{z}}^{(1)} = \frac{i\kappa}{16\pi} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \left[a_-(\omega \hat{q}) - a_+^\dagger(\omega \hat{q}) \right]. \tag{5.7}$$

In this way one recovers the subleading soft graviton theorem at tree-level [4, 7]. It should be emphasised that (5.7) is only valid in the space of rapidly decreasing functions as $u \to \pm\infty$ [61], which is the basis for the perturbative approach to quantum gravity within which the logarithmic corrections (2.4) have been established. In particular, this functional space does not account for asymptotic wavepackets featuring gravitational tails, a limitation striking back in the form of infrared divergences. In a previous publication [22] (see also [23]), it was shown that the infrared divergent one-loop corrections to the soft graviton theorem that were derived in [13] are accounted for by insertion of the remaining terms in the soft flux (4.12), which can be rewritten as

$$F_{\mathcal{Y}}^{\text{soft,new}} = \frac{2}{\kappa^2} \int d^2 z \, \mathcal{Y} \left[-\partial^3 (C^{(0)} \mathcal{N}_{\bar{z}\bar{z}}^{(0)}) + 3\bar{\partial}^2 \mathcal{N}_{zz}^{(0)} \partial C^{(0)} + C^{(0)} \partial \bar{\partial}^2 \mathcal{N}_{zz}^{(0)} \right]. \tag{5.8}$$

The focus of that previous analysis was on recovering the result of Bern et al. which concerns the IR-divergent one-loop contributions to the soft graviton theorem in case all particles are massless. Here we extend that analysis so that we recover the loop corrections found by Sahoo and Sen that also apply to the scattering of massive particles. As we discussed in section 2 these results take the exact same form when expressed in terms of the quantities σ_n, $\hat{\sigma}'_{n+1}$ and $\hat{S}_n^{(0)}$, $S_n^{(1)}$, modulo the identification between the ϵ^{-1} and $\ln \omega$ terms. Hence it is just a matter of extending the derivation presented in [22] in a way that it also applies to massive particles.

The two terms in (2.10) will arise in two distinct ways. The first term $\hat{\sigma}'_{n+1} \hat{S}_n^{(0)}$ will be obtained by insertion of the soft flux contribution $F_{\mathcal{Y}}^{\text{soft,new}}$ into the $\mathcal{S}$-matrix, while the second term $S_n^{(1)J} \sigma_n$ will arise from a contribution to the charges at timelike infinity, $Q_{\mathcal{Y}}^{i^+} \equiv Q_{\mathcal{Y}}(\mathscr{I}_+^+)$ and $Q_{\mathcal{Y}}^{i^-} \equiv Q_{\mathcal{Y}}(\mathscr{I}_-^-)$ which is associated with the gravitational dressing of the massive scalar fields. It is interesting to note that the imaginary part of these terms can be given a very clear classical interpretation: $\text{Im}\,(\hat{\sigma}'_{n+1} \hat{S}_n^{(0)})$ results from the gravitational drag exerted by the hard particles on the soft graviton, while $\text{Im}\,(S_n^{(1)J} \sigma_n)$ results from the long range 'Coulombic'/Newtonian interaction between the hard particles. Quantum mechanically, $\hat{\sigma}'_{n+1} \hat{S}_n^{(0)}$ and $S_n^{(1)J} \sigma_n$ arise from Feynman diagrams where the soft graviton attaches to a virtual soft graviton or to an external hard particle, respectively [16].

We start by inserting the soft flux contribution $F_{\mathcal{Y}}^{\text{soft,new}}$ as given in (5.8) into the $\mathcal{S}$-matrix, upon using the formulae (3.13) and (3.14) for the insertions of $C^{(0)}$ and $\mathcal{N}_{zz}^{(0)}$, respectively. This yields

$$\langle \text{out} | F_{\mathcal{Y}}^{\text{soft,new}}(\mathscr{I}^+) \mathcal{S} + \mathcal{S} F_{\mathcal{Y}}^{\text{soft,new}}(\mathscr{I}^-) | \text{in} \rangle$$

$$= -\frac{i\kappa}{8\pi\epsilon} \int d^2 z \, \mathcal{Y} \left[\partial^3 (\hat{\sigma}'_{n+1} \hat{S}_n^{(0)-}) - \hat{\sigma}'_{n+1} \partial \bar{\partial}^2 \hat{S}_n^{(0)+} - 3\partial \hat{\sigma}'_{n+1} \bar{\partial}^2 \hat{S}_n^{(0)+} \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle \tag{5.9}$$

$$= -\frac{i\kappa}{16\pi\epsilon} \int d^2 z \, \mathcal{Y} \, \partial^3 (\hat{\sigma}'_{n+1} \hat{S}_n^{(0)-}) \langle \text{out} | \mathcal{S} | \text{in} \rangle,$$

where in the second equality we used (5.18) in [22]. This accounts for the first term in (2.10).

We now move to the derivation of the remaining term, $S_n^{(1)J}\sigma_n$, from the hard part of the superrotation generators. This includes the fluxes acting on the massless particles at null infinity and the charges acting on massive particles at timelike infinity. As massive fields propagate towards future timelike infinity i^+ rather than null infinity, it is convenient to use hyperbolic coordinates (τ, x^a) within the future light cone,

$$ds^2 = -d\tau^2 + \tau^2 h_{ab}\, dx^a\, dx^b = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{\rho^2} + \rho^2\, dz\, d\bar{z}\right). \tag{5.10}$$

The three-dimensional hyperboloid $\mathcal{H}$ covered by the coordinates $x^a = (\rho, z, \bar{z})$ describes the directions of approach to i^+. Obviously a similar construction applies in the neighborhood of past timelike infinity i^-. In this coordinate system superrotations are parametrised by a vector field $\mathcal{Y}_{\mathcal{H}}^a\, \partial_a$ satisfying $D_a \mathcal{Y}_{\mathcal{H}}^a = 0$ and determined directly by the standard superrotation parameters $\mathcal{Y}^A = (\mathcal{Y}, \bar{\mathcal{Y}})$ through the integral relation [7, 41]

$$\mathcal{Y}_{\mathcal{H}}^a(\hat{p}) = \int d^2\hat{q}\, G_A^a(\hat{p}; \hat{q}) \mathcal{Y}^A(\hat{q}), \qquad \hat{p} = \hat{p}(\rho, z, \bar{z}), \quad \hat{q} = \hat{q}(w, \bar{w}), \tag{5.11}$$

where the 'bulk-boundary propagator' G_A^a admits the representation

$$G_w^a(\hat{p}; \hat{q})\partial_a = \frac{i}{4\pi}\, \partial_w^3 \left(\frac{\hat{p}^\mu\, \varepsilon_{\mu\nu}^-(\hat{q})\, \hat{q}_\lambda}{\hat{p}\cdot\hat{q}}\right) J^{\lambda\nu}(\hat{p}). \tag{5.12}$$

Asymptotically superrotations are diffeomorphisms generated by the vector field $\mathcal{Y}_{\mathcal{H}}^a \partial_a$ such that the charge generating the corresponding asymptotic transformation of the massive matter fields is simply given by [7]

$$Q_{\mathcal{Y}}^{i^+} = \lim_{\tau\to\infty} \frac{\tau^3}{8} \int_{\mathcal{H}} d^3\hat{p}\, \mathcal{Y}_{\mathcal{H}}^a(\hat{p})\, T_{\tau a}(\tau, \hat{p}), \tag{5.13}$$

where $T_{\mu\nu}$ is the matter stress tensor and $d^3\hat{p}$ denotes the volume element of the unit hyperboloid,

$$\int d^3\hat{p}\, E_{\hat{p}}\, \delta^3(\hat{p}) = 1, \qquad E_{\hat{p}} \equiv \sqrt{|\vec{p}/m|^2 + 1}. \tag{5.14}$$

Ideally one should prove that this quantity is indeed equal to $Q_{\mathcal{Y}}(\mathscr{I}_+^+)$ as defined in (4.15). Progress in this direction has been reported in [52], however here we will simply assume its validity as customarily done. For simplicity we will analyse the case of massive scalar fields which is appropriate to the scattering of massive spinless particles. Asymptotically free massive scalar fields can be written in terms of the creation/annihilation operators [7],

$$\phi^{(\text{free})}(\tau, x^a) = \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left(b(p)\, e^{-i\tau m} + b(p)^\dagger\, e^{i\tau m}\right) + O(\tau^{-5/2}), \tag{5.15}$$

with the position-momentum relation $p = m\, \hat{p}(x^a)$ given as in (3.7), following from a stationary phase approximation of the momentum mode decomposition of the field in the large-τ limit. In turn the stress tensor components of interest take the form

$$T_{\tau a}^{(\text{free})} = \frac{im^2}{4(2\pi\tau)^3} \left[b^\dagger\partial_a b - \partial_a b^\dagger b + \frac{1}{2}\partial_a\left(b^{\dagger 2} e^{2im\tau} - b^2 e^{-2im\tau}\right)\right] + O(\tau^{-4}), \tag{5.16}$$

such that using $D_a \mathcal{Y}^a_{\mathcal{H}} = 0$ the charge (5.13) becomes

$$Q^{i^+(free)}_{\mathcal{Y}} = \frac{im^2}{32(2\pi)^3} \int_{\mathcal{H}} d^3\hat{p}\, \mathcal{Y}^a_{\mathcal{H}}(\hat{p}) \left(b^\dagger \partial_a b - \partial_a b^\dagger b \right) . \tag{5.17}$$

However the charge formula (5.13) really features matter fields in gravitational interaction rather than free fields. This will be taken care of by dressing the free field operators. In particular the 'classical' part of the dressing will properly account for the long-range gravitational interaction between scattered particles. Dressing the field ϕ hence amounts to performing the replacement

$$b(\hat{p})^\dagger \quad \mapsto \quad \tilde{b}(\hat{p})^\dagger = \exp\left[-\frac{im}{2} \int d^2\hat{q}\, \mathcal{G}(\hat{p};\hat{q})\, C^{(0)}(\hat{q}) \right] b(\hat{p})^\dagger , \tag{5.18}$$

as can be directly inferred from (3.4)–(3.9). Indeed in the language of celestial holography the creation operator $b(p)^\dagger$ is represented by a local operator $\mathcal{O}(p)$. The dressed superrotation charge thus acquires an extra contribution,

$$Q^{i^+}_{\mathcal{Y}} = Q^{i^+(free)}_{\mathcal{Y}} - \Delta Q^{i^+}_{\mathcal{Y}} , \tag{5.19}$$

where

$$\Delta Q^{i^+}_{\mathcal{Y}} = \frac{m^3}{32(2\pi)^3} \int d^2\hat{q} \int d^3\hat{p}\, \mathcal{Y}^a_{\mathcal{H}}(\hat{p})\, \partial_a \mathcal{G}(\hat{p};\hat{q})\, C^{(0)}(\hat{q})\, b(\hat{p})^\dagger b(\hat{p}) . \tag{5.20}$$

The contribution of the free charge $Q^{i^+(free)}_{\mathcal{Y}}$ to the Ward identity (5.5) yields the leading soft factor $\hat{S}^{(0)}_n$ [7]. We are left to evaluate the result of inserting $\Delta Q^{i^+}_{\mathcal{Y}}$ and $\Delta Q^{i^-}_{\mathcal{Y}}$ into $\mathcal{S}$-matrix elements. We will set $\bar{\mathcal{Y}} = 0$ for ease of notation. First we recall the identities [22, 30]

$$\sigma_n = -\frac{\epsilon}{8\kappa^2} \sum_{ij=1}^n m_i m_j \int d^2\hat{q}\, d^2\hat{q}'\, \mathcal{G}(\hat{p}_i;\hat{q}) \mathcal{G}(\hat{p}_j;\hat{q}') \langle C^{(0)}(\hat{q}) C^{(0)}(\hat{q}') \rangle , \tag{5.21}$$

$$\hat{\sigma}'_{n+1}(\hat{q}') = -\frac{\epsilon}{2\kappa^2} \sum_{i=1}^n m_i \int d^2\hat{q}\, \mathcal{G}(\hat{p}_i;\hat{q}) \, \langle C^{(0)}(\hat{q}') C^{(0)}(\hat{q}) \rangle$$

$$= \frac{1}{(8\pi)^2} \sum_{i=1}^n m_i \int d^2\hat{q}\, \mathcal{G}(\hat{p}_i;\hat{q})\, (\hat{q}\cdot\hat{q}') \ln(\hat{q}\cdot\hat{q}') , \tag{5.22}$$

such that

$$\partial^3 S^{(1)J-}_n(\hat{q})\, \sigma_n = \frac{1}{2} \sum_i^n m_i \int d^2\hat{q}'\, \mathcal{G}(\hat{p}_i;\hat{q}')\, \partial^3 S^{(1)J-}_n(\hat{q})\, \hat{\sigma}'_{n+1}(\hat{q}') . \tag{5.23}$$

We then have

$$
\begin{aligned}
&\langle \text{out}|\Delta Q_{\mathcal{Y}}^{i^+}\mathcal{S} - \mathcal{S}\Delta Q_{\mathcal{Y}}^{i^-}|\text{in}\rangle \\
&= \frac{1}{16}\sum_i m_i \int d^2\hat{q}\,\mathcal{Y}_{\mathcal{H}}^a(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\langle\text{out}|C^{(0)}(\hat{q})\mathcal{S}|\text{in}\rangle \\
&= \frac{-i\kappa^2}{16\epsilon}\sum_i m_i \int d^2\hat{q}\,\mathcal{Y}_{\mathcal{H}}^a(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\hat{\sigma}'_{n+1}(\hat{q})\,\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&= \frac{i}{32}\sum_{ij} m_i m_j \int d^2\hat{q}'\,\mathcal{G}(\hat{p}_j;\hat{q}')\int d^2\hat{q}\,\mathcal{Y}_{\mathcal{H}}^a(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\langle C^{(0)}(\hat{q})C^{(0)}(\hat{q}')\rangle\,\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&= \frac{-i\kappa}{32\pi\epsilon}\sum_j m_j \int d^2\hat{q}'\,\mathcal{G}(\hat{p}_j;\hat{q}')\int d^2\hat{q}\,\mathcal{Y}(\hat{q})\,\partial^3 S_n^{(1)J-}(\hat{q})\,\hat{\sigma}'_{n+1}(\hat{q}')\,\langle\text{out}|\mathcal{S}|\text{in}\rangle \\
&= \frac{-i\kappa}{16\pi\epsilon}\int d^2\hat{q}\,\mathcal{Y}(\hat{q})\,\partial^3 S_n^{(1)J-}(\hat{q})\,\sigma_n\,\langle\text{out}|\mathcal{S}|\text{in}\rangle.
\end{aligned}
\tag{5.24}
$$

In the first equality we acted on the massive external states with the particle number operator [7]

$$
b(\hat{p})^\dagger\,b(\hat{p})\,|p'\rangle = m^{-2}(2\pi)^3\,(2E_{\hat{p}})\,\delta^3(\hat{p}-\hat{p}')\,|p'\rangle.
\tag{5.25}
$$

The second equality is obtained from (3.13), while the third equality relies on (5.22). The fourth equality holds thanks to (5.12) together with (5.22). The last equality follows directly from (5.23).

We observe that the last expression of (5.24) indeed accounts for the second term $S_n^{(1)J}\sigma_n$ in (2.10). Together with (5.9) we have therefore recovered all logarithmic corrections to the soft graviton theorems when external states are massive. The treatment of massless particles goes along similar lines. In particular it requires dressing the massless fields (including the graviton field if some of the scattered particles are hard gravitons) in the hard part of the flux $F_{\mathcal{Y}}^{\text{hard}}$ given in (4.12), which should then produce the corresponding massless contributions in $S_n^{(1)J}\sigma_n$. The massless contributions to $\hat{\sigma}'_{n+1}\,\hat{S}_n^{(0)}$ are still captured by (5.9) without modifications. Note that the result for massless particles is also directly obtained by taking the corresponding massless limits.

Acknowledgments

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