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MODULI SPACES OF $\mathbb{Z}/k\mathbb{Z}$ -CONSTELLATIONS OVER \mathbb{A}^2

MICHELE GRAFFEO

ABSTRACT. Let $\rho : \mathbb{Z}/k\mathbb{Z} \rightarrow \mathrm{SL}(2, \mathbb{C})$ be a representation of a finite abelian group and let $\Theta^{\mathrm{gen}} \subset \mathrm{Hom}_{\mathbb{Z}}(R(\mathbb{Z}/k\mathbb{Z}), \mathbb{Q})$ be the space of generic stability conditions on the set of G -constellations. We provide a combinatorial description of all the chambers $C \subset \Theta^{\mathrm{gen}}$ and prove that there are $k!$ of them. Moreover, we introduce the notion of simple chamber and we show that, in order to know all toric G -constellations, it is enough to build all simple chambers. We also prove that there are $k \cdot 2^{k-2}$ simple chambers. Finally, we provide an explicit formula for the tautological bundles \mathcal{R}_C over the moduli spaces \mathcal{M}_C for all chambers $C \subset \Theta^{\mathrm{gen}}$ which only depends upon the chamber stair which is a combinatorial object attached to the chamber C .

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0. INTRODUCTION

Given a Gorenstein singular variety X , a crepant resolution is a proper birational morphism

$$Y \xrightarrow{\varepsilon} X$$

where Y is smooth and the canonical bundle is preserved, i.e. $\omega_Y = \varepsilon^* \omega_X$.

It was proven by Watanabe in [29] that the singularities of the form \mathbb{A}^n/G , where $G \subset \mathrm{SL}(n, \mathbb{C})$ is a finite subgroup, are Gorenstein. Their crepant resolutions appear in several fields of Algebraic Geometry and Mathematical Physics, for example see [4, 17, 25] and the references therein.

Even though, in general, crepant resolutions may not exist, their existence is guaranteed in dimension 2 and 3: see [9] for dimension 2, and see Roan [26, 27], Ito [16] and Markushevich [20] for dimension 3. In particular, the 3-dimensional case was solved by a case by case analysis, taking advantage of the fact that the conjugacy classes of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ were listed, for example in [31].

More recently, in [3], Bridgeland, King and Reid proved in one shot that a resolution always exists in dimension 3. The resolution that they proposed is made in terms of G -clusters, i.e. G -equivariant zero-dimensional subschemes Z of \mathbb{A}^n such that $H^0(Z, \mathcal{O}_Z) \cong \mathbb{C}[G]$ as G -modules

(Definition 1.1). In particular, in [3] it was proved that there exists a crepant resolution

$$G\text{-Hilb}(\mathbb{A}^3) \rightarrow \mathbb{A}^3/G$$

where $G\text{-Hilb}(\mathbb{A}^3)$ is the fine moduli space of G -clusters. Notice that this result had already been obtained for abelian actions by Nakamura in [22].

In [7] Craw and Ishii generalized the notion of G -cluster to that of G -constellation, i.e. a coherent G -sheaf \mathcal{F} such that $H^0(\mathbb{A}^n, \mathcal{F}) \cong \mathbb{C}[G]$ as G -modules (Definition 1.4). Moreover, in the case of G abelian the authors in [7] introduced a notion of θ -stability for G -constellations (Definition 1.8), following the ideas in King [18]. They proved that, for any abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$ and for any crepant resolution $Y \xrightarrow{\varepsilon} \mathbb{A}^3/G$ there exists at least a generic stability condition θ and an isomorphism $\mathcal{M}_\theta \xrightarrow{\varphi} Y$ such that the composition $\varepsilon \circ \varphi$ agrees with the restriction

$$\mathcal{M}_\theta \rightarrow \mathbb{A}^3/G$$

of the Hilbert–Chow morphism, to the irreducible component \mathcal{M}_θ of the fine moduli space of θ -stable G -constellations containing free orbits. Moreover, they conjectured that the same is true for any finite subgroup of $\mathrm{SL}(3, \mathbb{C})$. Recently, this conjecture has been affirmatively solved by Yamagishi in [30].

It turns out that the space of generic stability conditions $\Theta^{\mathrm{gen}} \subset \Theta$ is a disjoint union of connected components called chambers. Moreover, in each chamber C , the notion of stability is constant, i.e. for any $\theta, \theta' \in C$, a G -constellation is θ -stable if and only if it is θ' -stable.

In this paper I will focus on the 2-dimensional abelian case, i.e. the case when $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite abelian, and hence cyclic, subgroup. In the literature the singularity \mathbb{A}^2/G is sometimes called the $A_{|G|-1}$ singularity. This case is particularly simple from the point of view of the resolution because we know, from classical surface theory, that there is a unique minimal crepant resolution. Therefore, all the moduli spaces \mathcal{M}_θ are isomorphic as quasi-projective varieties. As a consequence, in order to distinguish two chambers it is enough to study their universal families $\mathcal{U}_C \in \mathrm{Ob} \mathrm{Coh}(\mathcal{M}_C \times \mathbb{A}^2)$. The first main result in the paper is the following.

THEOREM 4.17. If $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} is the disjoint union of $k!$ chambers.

The result in Theorem 4.17 can be also recovered, via different arguments, from the theory developed by Kronheimer in [19] (See also [5, Chapter 3-§3] for the algebraic interpretation), but the approach to the abelian case here is different and it helps to prove the other results.

In order to prove Theorem 4.17, I will give an exhaustive combinatorial description of the toric points of the spaces \mathcal{M}_θ in terms of very classical combinatorial objects, namely skew Ferrers diagrams. Such diagrams are standard tools in many branches of mathematics, e.g. enumerative geometry, group theory, commutative algebra etc (for example [2, 12, 21]).

Next, I will introduce the notion of simple chamber (Definition 5.8) and I will show that, for any indecomposable G -constellation \mathcal{F} , there exists at least a simple chamber C such that \mathcal{F} is θ -stable for all $\theta \in C$. This property makes simple chambers useful, because knowing them is the same as knowing all the G -constellations. In order to define simple chambers, I will need to construct chamber stairs (Definition 5.2), combinatorial objects that I will use to encode all the data of a chamber C .

The second theorem I prove is the following.

THEOREM 5.15. If $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} contains $k \cdot 2^{k-2}$ simple chambers.

Finally, in Theorem 6.5 I will give a commutative algebra construction that allows one to write an explicit formula for the tautological bundle

$$\mathcal{R}_\theta \in \mathrm{Ob} \mathrm{Coh}(\mathcal{M}_\theta),$$

i.e. the pushforward of the universal family $\mathcal{U}_\theta \in \mathrm{Ob} \mathrm{Coh}(\mathcal{M}_\theta \times \mathbb{A}^2)$ via the first projection. This construction can be easily implemented using some software such as Macaulay2 [14]. Moreover, it provides a realization of all the moduli spaces \mathcal{M}_θ as a G -invariant subvariety of $\mathrm{Quot}_{\mathcal{K}_C}^{[G]}(\mathbb{A}^2)$ where $\mathcal{K}_C \in \mathrm{Ob} \mathrm{Coh}(\mathbb{A}^2)$ is an ideal sheaf dependent only upon the chamber C such that $\theta \in C$ (see Corollary 6.8). This solves, in 2-dimensions, a problem related to the one raised by Nakamura in [22, Problem 6.5.] and it also implies that to give a chamber is equivalent to give its chamber stair (Definition 5.2).

This paper gives some contributions to the solution of several open problems regarding the subject, and provides some techniques that seem to be applicable to more general situations, such as some non-abelian, even 3-dimensional, case for example following the ideas in [23, 24].

After providing, in the first section, some technical preliminaries and some known facts, I will devote the second section to a brief description of the singularity \mathbb{A}^2/G and to its minimal resolution.

In the third section I will prove that the toric G -constellations are completely described in terms of G -stairs, which are certain diagrams whose definition I will give in Definition 3.19.

The following sections (4 and 5), are devoted to the proofs of the two main theorems, while in the last section I will give the above mentioned commutative algebra construction.

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1. PRELIMINARIES

Given a finite group G and a representation $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$, we have an action of G on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, given by

$$\begin{aligned} G \times \mathbb{C}[x_1, \dots, x_n] &\longrightarrow \mathbb{C}[x_1, \dots, x_n] \\ (g, p) &\longmapsto p \circ \rho(g)^{-1} \end{aligned}$$

where p and $\rho(g)^{-1}$ are thought respectively as a polynomial and a linear function. Out of this, we can build the quotient singularity

$$\mathbb{A}^n/G = \mathrm{Spec} \mathbb{C}[x_1, \dots, x_n]^G$$

whose points parametrize the set-theoretic orbits of the action of G on \mathbb{A}^n induced by ρ .

Given a representation $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$, a ρ -equivariant sheaf (or a ρ -sheaf in the sense of [3]) is a coherent sheaf $\mathcal{F} \in \mathrm{Ob} \mathrm{Coh}(\mathbb{A}^n)$ together with a lift to \mathcal{F} of the G -action on \mathbb{A}^n induced by ρ , i.e. for all $g \in G$ there are morphisms $\lambda_g^{\mathcal{F}} : \mathcal{F} \rightarrow \rho(g)^* \mathcal{F}$ such that:

- $\lambda_{1_G}^{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$,
- $\lambda_{hg}^{\mathcal{F}} = \rho(g)^*(\lambda_h^{\mathcal{F}}) \circ \lambda_g^{\mathcal{F}}$,

where 1_G is the unit of G . In particular, this induces a structure of representation on the vector space $H^0(\mathbb{A}^n, \mathcal{F})$ as above

$$\begin{aligned} G \times H^0(\mathbb{A}^n, \mathcal{F}) &\longrightarrow H^0(\mathbb{A}^n, \mathcal{F}) \\ (g, s) &\longmapsto (\lambda_g^{\mathcal{F}})^{-1} \circ \rho(g)^*(s). \end{aligned}$$

Whenever the representation is an inclusion $G \subset \mathrm{GL}(n, \mathbb{C})$ we will omit the representation and we will talk about G -equivariant sheaf (or G -sheaf).

Definition 1.1. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup. A G -cluster is a zero-dimensional subscheme Z of \mathbb{A}^n such that:

- the structure sheaf \mathcal{O}_Z is G -equivariant, i.e. the ideal I_Z is invariant with respect to the action of G on $\mathbb{C}[x_1, \dots, x_n]$, and
- if $\rho_{\mathrm{reg}} : G \rightarrow \mathrm{GL}(\mathbb{C}[G])$ is the regular representation, then there is an isomorphism of representations

$$\varphi : H^0(Z, \mathcal{O}_Z) \rightarrow \mathbb{C}[G],$$

i.e. φ is an isomorphism of vector spaces such that the following diagram:

$$\begin{array}{ccc} G \times H^0(Z, \mathcal{O}_Z) & \longrightarrow & H^0(Z, \mathcal{O}_Z) \\ \mathrm{id}_G \times \varphi \downarrow & & \downarrow \varphi \\ G \times \mathbb{C}[G] & \longrightarrow & \mathbb{C}[G] \end{array}$$

where the horizontal arrows are the G -actions, commutes.

We will denote by $\mathrm{Hilb}^G(\mathbb{A}^n)$ the fine moduli space of G -clusters and, by $G\text{-Hilb}(\mathbb{A}^n)$ the irreducible component of $\mathrm{Hilb}^G(\mathbb{A}^n)$ containing the free G -orbits.

Recall that, for all $n \geq 1$ and for all $G \subset \mathrm{SL}(n, \mathbb{C})$ finite subgroup, the singularities of the form \mathbb{A}^n/G are Gorenstein (cf. [29]).

Theorem 1.2 ([3, Theorem 1.2]). *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup where $n = 2, 3$. Then, the Hilbert-Chow morphism*

$$Y := G\text{-Hilb}(\mathbb{A}^n) \xrightarrow{\varepsilon} \mathbb{A}^n/G =: X$$

is a crepant resolution of singularities, i.e. $\omega_Y \cong \varepsilon^ \omega_X$.*

Remark 1.3. The Hilbert-Chow morphism ε mentioned in Theorem 1.2 is a G -equivariant version of the usual Hilbert-Chow morphism

$$\bar{\varepsilon} : \mathrm{Hilb}^{|G|}(\mathbb{A}^n) \rightarrow \mathrm{Sym}^{|G|}(\mathbb{A}^n).$$

In particular ε can be thought of as the restriction of $\bar{\varepsilon}$ to the G -invariant subvariety $G\text{-Hilb}(\mathbb{A}^n) \subset \mathrm{Hilb}^{|G|}(\mathbb{A}^n)$.

A natural generalization of the concept of a G -cluster is given in [7], and it is achieved by consider coherent $\mathcal{O}_{\mathbb{A}^n}$ -modules which are not necessarily the structure sheaves of zerodimensional subschemes of \mathbb{A}^n .

Definition 1.4 ([7, Definition 2.1]). Let $G \subset GL(n, \mathbb{C})$ be a finite subgroup. A G -constellation is a coherent $\mathcal{O}_{\mathbb{A}^n}$ -module \mathcal{F} on \mathbb{A}^n such that:

- \mathcal{F} is G -equivariant, i.e. there is a fixed lift on \mathcal{F} of the G -action on \mathbb{A}^n , and
- there is an isomorphism of representations

$$\varphi : H^0(\mathbb{A}^n, \mathcal{F}) \rightarrow \mathbb{C}[G].$$

Remark 1.5. Since a G -constellation \mathcal{F} is a coherent sheaf on the affine variety \mathbb{A}^n , sometimes, by abuse of notations, we address the name G -constellation to the space of global sections $H^0(\mathbb{A}^n, \mathcal{F})$ as well as \mathcal{F} and, sometimes, we treat a G -constellation as if it were a $\mathbb{C}[x_1, \dots, x_n]$ -module, meaning that we are working with the space of its global sections.

Remark 1.6. The G -equivariance hypothesis implies that the support of a G -constellation is a union of G -orbits. Moreover, for dimensional reasons, the only constellations supported on a free orbit Z are isomorphic to the structure sheaf \mathcal{O}_Z .

Remark 1.7. Recall that (see, for example, [12, chapters 1 and 2]), given a finite group G and the set of isomorphism classes of its irreducible representations

$$\text{Irr}(G) = \{\text{Irreducible representations}\}/\text{iso},$$

there is a ring isomorphism

$$\Psi : R(G) \xrightarrow{\sim} \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}\rho,$$

where $(R(G), \oplus)$ is the Grothendieck group of isomorphism classes of representations of G , and the ring structure (on both sides) is induced by tensor product \otimes of representations. Moreover $\text{Irr}(G) = \{\rho_1, \dots, \rho_s\}$ is finite, and we have the correspondence:

$$R(G) \xrightarrow{\Psi} \bigoplus_{i=1}^s \mathbb{Z}\rho_i$$

$$\mathbb{C}[G] \longmapsto (\dim \rho_1, \dots, \dim \rho_s).$$

Following the ideas in [18], the above mentioned properties allow one to introduce a notion of stability on the set of G -constellations. Given a finite subgroup $G \subset SL(n, \mathbb{C})$ (where $n = 2, 3$), the *space of stability conditions* for G -constellations is

$$\Theta = \{ \theta \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \}$$

Definition 1.8. Let $\theta \in \Theta$ be a stability condition. A G -constellation \mathcal{F} is said to be θ -*(semi)stable* if, for any proper G -equivariant subsheaf $0 \subsetneq \mathcal{E} \subsetneq \mathcal{F}$, we have

$$\theta(H^0(\mathbb{A}^n, \mathcal{E})) \underset{(\geq)}{>} 0.$$

A stability condition θ is *generic* if the notion of θ -semistability is equivalent to the notion of θ -stability. Finally, we denote by $\Theta^{\text{gen}} \subset \Theta$ the subset of generic stability conditions.

Definition 1.9. A G -constellation \mathcal{F} is *indecomposable* if it cannot be written as a direct sum

$$\mathcal{F} = \mathcal{E}_1 \oplus \mathcal{E}_2,$$

where $\mathcal{E}_1, \mathcal{E}_2$ are proper G -subsheaves, and it is *decomposable* otherwise.

Remark 1.10. If we think of a G -constellation as its space of global sections, a G -constellation $F = H^0(\mathcal{F}, \mathbb{A}^n)$ is indecomposable if it cannot be written as a direct sum

$$F = E_1 \oplus E_2,$$

where E_1, E_2 are proper G -equivariant $\mathbb{C}[x_1, \dots, x_n]$ -submodules.

Remark 1.11. If \mathcal{F} is decomposable, then it is not θ -stable for any stability condition $\theta \in \Theta$.

Since, for our purpose, we are interested in indecomposable G -constellations, whenever not specified a G -constellation will always be indecomposable.

Remark 1.12. If $Z \subset \mathbb{A}^n$ is a free orbit, then \mathcal{O}_Z does not admit any proper G -subsheaf. Therefore, it is θ -stable for all $\theta \in \Theta$.

Definition 1.13. Let $\theta \in \Theta^{\text{gen}}$ be a generic stability condition. We denote by \mathcal{M}_θ the (fine) moduli space of θ -stable G -constellations.

The theorem below brings together results from [7, 3, 30].

Theorem 1.14. *The following results are true for $n = 2, 3$.*

- *The subset $\Theta^{\text{gen}} \subset \Theta$ of generic parameters is open and dense. It is the disjoint union of finitely many open convex polyhedral cones in Θ called chambers.*
- *For generic $\theta \in \Theta^{\text{gen}}$, the moduli space \mathcal{M}_θ exists and it depends only upon the chamber $C \subset \Theta^{\text{gen}}$ containing θ , so we write \mathcal{M}_C in place of \mathcal{M}_θ for any $\theta \in C$. Moreover, the Hilbert–Chow morphism, which associates to each G -constellation \mathcal{F} its support $\text{Supp}(\mathcal{F})$, $\varepsilon: \mathcal{M}_C \rightarrow \mathbb{A}^n/G$, is a crepant resolution.*
- *(Craw–Ishii Theorem [7]) Given a finite abelian subgroup $G \subset \text{SL}(n, \mathbb{C})$, suppose $Y \xrightarrow{\varepsilon} \mathbb{A}^n/G$ is a projective crepant resolution. Then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$ and $\varepsilon = \varepsilon_C$ is the Hilbert–Chow morphism.*
- *(Yamagishi Theorem [30]) Given a finite subgroup $G \subset \text{SL}(n, \mathbb{C})$, suppose $Y \xrightarrow{\varepsilon} \mathbb{A}^n/G$ is a projective crepant resolution. Then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$.*
- *There exists a chamber $C_G \subset \Theta^{\text{gen}}$ such that $\mathcal{M}_{C_G} = G\text{-Hilb}(\mathbb{A}^n)$.*

We will adopt the same notation as [7] for the universal family of C -stable G -constellations, namely $\mathcal{U}_C \in \text{Ob Coh}(\mathcal{M}_C \times \mathbb{A}^n)$, and for the tautological bundle $\mathcal{R}_C := (\pi_{\mathcal{M}_C})_* \mathcal{U}_C$.

Remark 1.15. The hypothesis of Theorem 1.14, Remarks 1.6 and 1.12 imply, together with the third point of Theorem 1.14, that if we denote by $U_C = \mathcal{M}_C \setminus \text{Exc}(\varepsilon_C)$ the complement of the exceptional locus of the Hilbert–Chow morphism then, for any two chambers $C, C' \subset \Theta^{\text{gen}}$, then there is a canonical isomorphism of families over \mathbb{A}^n/G -schemes

$$\begin{array}{ccc} \mathcal{U}_C|_{U_C \times \mathbb{A}^n} & \cong & \mathcal{U}_{C'}|_{U_{C'} \times \mathbb{A}^n} \\ \downarrow & & \downarrow \\ U_C \times \mathbb{A}^n & & U_{C'} \times \mathbb{A}^n, \end{array}$$

i.e. there exists a unique isomorphism $\varphi_C: U_C \rightarrow U_{C'}$ such that the diagram

$$\begin{array}{ccc} U_C & \xrightarrow{\varphi} & U_{C'} \\ & \searrow \varepsilon_C & \swarrow \varepsilon_{C'} \\ & \mathbb{A}^n/G & \end{array}$$

commutes and $\mathcal{U}_C|_{U_C \times \mathbb{A}^n} \cong (\varphi \times \text{id}_{\mathbb{A}^n})^* \mathcal{U}_{C'}|_{U_{C'} \times \mathbb{A}^n}$.

In particular, any U_C parametrizes the free orbits of the G -action as the complement of the singular locus of \mathbb{A}^n/G does.

2. THE TWO-DIMENSIONAL ABELIAN CASE

In this section we introduce some notation that we will use throughout the rest of the paper. Moreover, we give a very brief description of the singularities $A_{|G|-1}$ and of their respective resolutions.

Throughout all the section, we fix a finite abelian subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$.

2.1. The action of G . Whenever $G \subset \mathrm{SL}(n, \mathbb{C})$ is a finite abelian subgroup, it is well known that its irreducible representations are 1-dimensional and that the group G and the set $\mathrm{Irr}(G)$ are in bijection. Moreover, the map Ψ in Remark 1.7 is such that

$$\begin{aligned} R(G) &\xrightarrow{\Psi} \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}\rho \\ \mathbb{C}[G] &\longmapsto (1, \dots, 1). \end{aligned}$$

In particular, in dimension 2, it is well known that all finite abelian subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ are cyclic. Moreover, for any $k \geq 1$, there is only one conjugacy class of abelian subgroups of $\mathrm{SL}(2, \mathbb{C})$ isomorphic to $\mathbb{Z}/k\mathbb{Z}$. In what follows we will choose, as representative of such conjugacy class,

$$(2.1) \quad \mathbb{Z}/k\mathbb{Z} \cong G = \left\langle g_k = \begin{pmatrix} \xi_k^{-1} & 0 \\ 0 & \xi_k \end{pmatrix} \right\rangle \subset \mathrm{SL}(2, \mathbb{C}),$$

where ξ_k is a (fixed) primitive k -th root of unity.

We adopt the following notation for the irreducible representations of G :

$$\mathrm{Irr}(G) = \left\{ \begin{array}{l} \rho_i : \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathbb{C}^* \\ g_k \longmapsto \xi_k^i \end{array} \middle| i = 0, \dots, k-1 \right\}.$$

Sometimes, we will identify $\mathrm{Irr}(G)$ with the set $\{0, \dots, k-1\}$ according to the bijection $\rho_j \mapsto j$. Notice that, one may also identify $(\mathrm{Irr}(G), \otimes)$ with the abelian group $(\mathbb{Z}/k\mathbb{Z}, +)$, but in what follows we will mostly deal with $\mathrm{Irr}(G)$ as a set of indices, hence we will ignore the natural group structure on it.

2.2. The quotient singularity \mathbb{A}^2/G and its resolution. The singularity obtained in this case is the so-called A_{k-1} singularity, i.e.

$$A_{k-1} := \mathbb{A}^2/G.$$

This is a rational double point. It is well known that it has a unique minimal, in fact crepant, resolution $Y \xrightarrow{\varepsilon} A_{k-1}$ whose exceptional divisor is a chain of $k-1$ smooth (-2) -rational projective curves.

As a consequence of Theorem 1.14 and of the uniqueness of the minimal model of a surface, for any chamber C , there is an isomorphism of varieties $\varphi_C : \mathcal{M}_C \xrightarrow{\sim} Y$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}_C & \xrightarrow{\varphi_C} & Y \\ & \searrow \varepsilon_C & \swarrow \varepsilon \\ & & A_{k-1} \end{array}$$

commutes. What changes between two different chambers C, C' is that they have different universal families $\mathcal{U}_C, \mathcal{U}_{C'} \in \text{Ob Coh}(Y \times \mathbb{A}^2)$.

3. TORIC G -CONSTELLATIONS

This section is devoted to the study of toric G -constellations, i.e. those G -constellations which, in addition to being G -sheaves, are also \mathbb{T}^2 -sheaves. As it usually happens when dealing with \mathbb{T}^2 -modules, we will see that the $\mathbb{C}[x, y]$ -module structure of a toric G -constellation is fully described in terms of combinatorial objects, namely the skew Ferrers diagrams.

This way of proceeding in the description of a \mathbb{T}^2 -module is not new, and it is actually adopted very often in the literature; for example in the study of monomial ideals (see [2]) or, more generally, in the study of \mathbb{T}^2 -modules of finite length (see [21]).

Although many statements can be generalized to higher dimension, from now on we will focus on the 2-dimensional case.

3.1. The torus action. Recall that \mathbb{A}^2 is a toric variety via the standard torus action:

$$(3.1) \quad \begin{array}{ccc} \mathbb{T}^2 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ ((\sigma_1, \sigma_2), (x, y)) & \longmapsto & (\sigma_1 \cdot x, \sigma_2 \cdot y). \end{array}$$

Notice that, under our assumptions, G is a finite subgroup of the torus \mathbb{T}^2 . Hence, the action of \mathbb{T}^2 commutes with the action of the finite abelian (diagonal) subgroup $G \subset \mathbb{T}^2$.

This implies that, given a θ -stable G -constellation \mathcal{F} and an element $\sigma \in \mathbb{T}^2$, the pullback $\sigma^* \mathcal{F}$ is a θ -stable G -constellation. Indeed, σ^* induces an isomorphism between the global sections of $\sigma^* \mathcal{F}$ and \mathcal{F} and hence, $\dim H^0(\mathbb{A}^2, \sigma^* \mathcal{F}) = k$. Moreover, $\sigma^* \mathcal{F}$ is still a G -sheaf if we define, for all $g \in G$, the morphisms $\lambda_g^{\sigma^* \mathcal{F}} : \sigma^* \mathcal{F} \rightarrow g^* \sigma^* \mathcal{F}$ as

$$\lambda_g^{\sigma^* \mathcal{F}} = \sigma^* \lambda_g^{\mathcal{F}}.$$

Such morphisms are well defined because σ^* and g^* commute, i.e. $g^* \sigma^* \mathcal{F} \cong \sigma^* g^* \mathcal{F}$ for all $(g, \sigma) \in G \times \mathbb{T}^2$. Finally, we have to check that $\sigma^* \mathcal{F}$ is θ -stable. This follows from the fact that both the groups $G \subset \mathbb{T}^2$ act diagonally and, as a consequence, if $\mathcal{E} \subset \mathcal{F}$ is a proper G -subsheaf and

$$H^0(\mathbb{A}^2, \mathcal{E}) = \bigoplus_{j=1}^r \rho_{i_j}$$

as representations, then $\sigma^* \mathcal{E} \subset \sigma^* \mathcal{F}$ is a proper G -subsheaf and

$$H^0(\mathbb{A}^2, \sigma^* \mathcal{E}) = \bigoplus_{j=1}^r \rho_{i_j}$$

as representations.

Definition 3.1. As explained above, the torus \mathbb{T}^2 acts on \mathcal{M}_C , for any chamber C . We say that a (indecomposable) G -constellation \mathcal{F} is *toric* if it corresponds to a torus fixed point.

Remark 3.2. A G -constellation \mathcal{F} is toric if and only if it admits a structure of \mathbb{T}^2 -sheaf. Indeed, if \mathcal{F} is a torus fixed point one possible \mathbb{T}^2 -structure is obtained from the following associations

$$\begin{array}{ccc} \mathbb{T}^2 \times H^0(\mathbb{A}^2, \mathcal{F}) & \longrightarrow & H^0(\mathbb{A}^2, \mathcal{F}) \\ (\sigma, p) & \longmapsto & p \circ \sigma^{-1}, \end{array}$$

for $\sigma \in \mathbb{T}^2$ acting on \mathbb{A}^2 as in (3.1). We stress that the \mathbb{T}^2 -equivariant structure on \mathcal{F} is not unique. Indeed any such structure can be twisted by characters of \mathbb{T}^2 .

Definition 3.3. We say that a G -constellation \mathcal{F} is *nilpotent* if the endomorphisms $x \cdot$ and $y \cdot$ of the $\mathbb{C}[x, y]$ -module $H^0(\mathbb{A}^2, \mathcal{F})$ are nilpotent.

Remark 3.4. A G -constellation \mathcal{F} is supported at the origin $0 \in \mathbb{A}^2$ if and only if it is nilpotent. This follows from the relation between the annihilator of a $\mathbb{C}[x, y]$ -module and the support of the sheaf associated to it (see [10, Section 2.2]). Moreover, Theorem 1.14 implies that nilpotent C -stable G -constellations correspond to points of the exceptional locus of the crepant resolution \mathcal{M}_C .

Remark 3.5. Given a G -constellation $F = H^0(\mathbb{A}^2, \mathcal{F})$, we can compare its structures of G -representation and of $\mathbb{C}[x, y]$ -module. Looking at the induced action of G on $\mathbb{C}[x, y]$, it turns out that, if $s \in \rho_i$ via the isomorphism $F \cong \mathbb{C}[G]$ then:

$$x \cdot s \in \rho_{i+1},$$

and,

$$y \cdot s \in \rho_{i-1}.$$

Proposition 3.6. *If $F = H^0(\mathbb{A}^2, \mathcal{F})$ is a nilpotent G -constellation then the endomorphism xy is the zero endomorphism.*

Proof. The G -constellation F is a k -dimensional \mathbb{C} -vector space. Let us pick a basis

$$\{v_0, \dots, v_{k-1}\}$$

of F such that, for all $i = 0, \dots, k-1$, $v_i \in \rho_i$ under the isomorphism $F \cong \mathbb{C}[G]$. As in Remark 3.5, for all $i = 0, \dots, k-1$, we have:

$$x \cdot v_i \in \rho_{i+1},$$

and,

$$y \cdot v_i \in \rho_{i-1}$$

where the indices are thought modulo k . In other words,

$$x \cdot v_i \in \text{Span}(v_{i+1}) \text{ and } y \cdot v_i \in \text{Span}(v_{i-1}).$$

Therefore, we get:

$$xy \cdot v_i \in \text{Span}(v_i), \quad \forall i = 0, \dots, k-1$$

i.e.

$$xy \cdot v_i = \alpha_i v_i, \text{ with } \alpha_i \in \mathbb{C}, \quad \forall i = 0, \dots, k-1.$$

Now, the nilpotency hypothesis implies that $\alpha_i = 0$ for all $i = 0, \dots, k-1$. \square

Remark 3.7. If a G -constellation $F = H^0(\mathbb{A}^2, \mathcal{F})$ is toric, then it is also nilpotent. Indeed, following the same logic as in the proof of Proposition 3.6 we have

$$x^k \cdot v_i = \alpha_i v_i, \text{ with } \alpha_i \in \mathbb{C}, \quad \forall i = 0, \dots, k-1,$$

but torus equivariancy implies $\alpha_i = 0$ for all $i = 0, \dots, k-1$.

3.2. Skew Ferrers diagrams and G -stairs. The advantage of working with toric G -constellations is that their spaces of global sections can be described in terms of monomial ideals whose data are described by means of combinatorial objects.

We can associate, to each element of the natural plane \mathbb{N}^2 , two labels: namely a monomial and an irreducible representation. We achieve this by saying that *a polynomial* $p \in \mathbb{C}[x, y]$ belongs to an irreducible representation ρ_i if

$$\forall g \in G, \quad g \cdot p = \rho_i(g)p$$

i.e. p is an eigenfunction for the linear map $g \cdot$ with the complex number $\rho_i(g)$ as eigenvector. In particular, with the notations in Section 2.1, the monomial $x^i y^j$ belongs to the irreducible representation ρ_{i-j} of the abelian group G , where the index is thought modulo k . According to this association, we can define the *representation tableau* \mathcal{T}_G as

$$\mathcal{T}_G = \{ (i, j, t) \in \mathbb{N}^2 \times \text{Irr}(G) \mid i - j \equiv t \pmod{k} \} \subset \mathbb{N}^2 \times \text{Irr}(G).$$

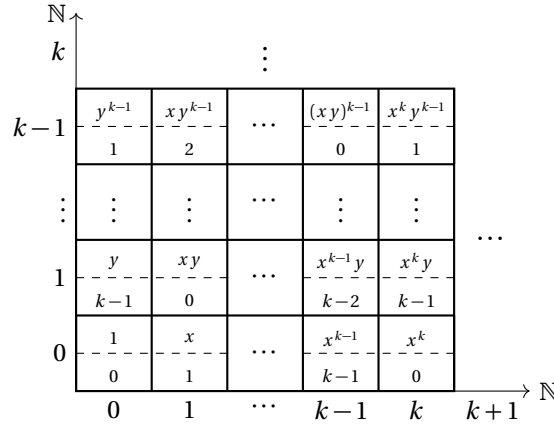


FIGURE 1. The representation tableau \mathcal{T}_G .

Notice that the labeling with the representation is superfluous because the first projection

$$\pi_{\mathbb{N}^2} : \mathcal{T}_G \rightarrow \mathbb{N}^2$$

is a bijection. In any case, this notation is useful to keep in mind that we are dealing with the representation structure as well as with the module structure.

In summary, the representation tableau has the property that

- moving to the right "increases" the irreducible representation by 1 (mod k)*
- moving up "decreases" the irreducible representation by 1 (mod k).*

Definition 3.8. A *Ferrers diagram* (Fd) is a subset A of the natural plane \mathbb{N}^2 such that

$$(\mathbb{N}^2 \setminus A) + \mathbb{N}^2 \subset (\mathbb{N}^2 \setminus A)$$

i.e. there exist $s \geq 0$ and $t_0 \geq \dots \geq t_s \geq 0$ such that

$$A = \{ (i, j) \mid i = 0, \dots, s \text{ and } j = 0, \dots, t_i \}.$$

Remark 3.9. In the literature there is some ambiguity about the name to be given to such diagrams. Indeed, sometimes, they are also called Young tableaux and, by Ferrers diagrams, something else is meant (for some different notations, see for example [12, 1]). In any case, we will adopt the notation in [8].

Pictorially, we see s consecutive columns of weakly decreasing heights. An example is depicted in Figure 2.

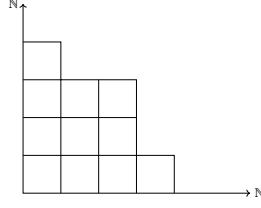


FIGURE 2. An example of Fd where $s = 3$, $t_0 = 3$, $t_1 = 2$, $t_2 = 2$, $t_3 = 0$.

Remark 3.10. We briefly recall that, starting from a Ferrers diagram A , we can build a torus-invariant zero-dimensional subscheme Z of \mathbb{A}^2 . Indeed, if $B = \mathbb{N}^2 \setminus A$ is the complement of A , then

$$I_Z = \{ x^{b_1} y^{b_2} \mid (b_1, b_2) \in B \}$$

is the ideal of the above mentioned subscheme $Z \subset \mathbb{A}^2$. In particular, the $\mathbb{C}[x, y]$ -module structure of $H^0(\mathbb{A}^2, \mathcal{O}_Z) = \mathbb{C}[x, y]/I_Z$ is encoded in the Fd, by saying that a box, labeled by the monomial $m \in \mathbb{C}[x, y]$, corresponds to the one-dimensional vector subspace of $H^0(\mathbb{A}^2, \mathcal{O}_Z)$ generated by m , and

moving to the right in the Fd is the multiplication by x
moving up in the Fd is the multiplication by y .

Definition 3.11. Let $\Gamma \subset \mathbb{N}^2$ be a subset of the natural plane. We will say that Γ is a *skew Ferrers diagram (sFd)* if there exist two Ferrers diagrams $\Gamma_1, \Gamma_2 \subset \mathbb{N}^2$ such that $\Gamma = \Gamma_1 \setminus \Gamma_2$.

Moreover, we will say that a sFd Γ is *connected* if, for any decomposition

$$\Gamma = \Gamma_1 \cup \Gamma_2$$

as disjoint union, there are at least a box in Γ_1 and a box in Γ_2 which share an edge.

Lemma 3.12. Let $A_1, A_2 \subset \mathbb{N}^2$ be two Ferrers diagrams and let $\Gamma \subset \mathbb{N}^2$ be the skew Ferrers diagram $\Gamma = A_1 \setminus A_2$. Consider, for $i = 1, 2$, the ideals

$$I_{A_i} = (\{ x^{b_1} y^{b_2} \in \mathbb{C}[x, y] \mid (b_1, b_2) \in \mathbb{N}^2 \setminus A_i \}).$$

Then, the isomorphism class of the torus equivariant $\mathbb{C}[x, y]$ -module

$$M_\Gamma = I_{A_2}/I_{A_2} \cap I_{A_1} = I_{A_2}/I_{A_2 \cup A_1},$$

is independent of the choice of A_1, A_2 . Equivalently, for any other choice of $A'_1, A'_2 \subset \mathbb{N}^2$ such that $\Gamma = A'_1 \setminus A'_2$, the torus equivariant $\mathbb{C}[x, y]$ -modules M_Γ and $I_{A'_2}/I_{A'_2 \cup A'_1}$ are isomorphic.

Proof. The fact that M_Γ does not depend on the decomposition $\Gamma = A_1 \setminus A_2$ follows noticing that, if we pick another decomposition $\Gamma = A'_1 \setminus A'_2$, then the isomorphism of \mathbb{C} -vector spaces

$$I_{A_2}/I_{A_2} \cap I_{A_1} \rightarrow I_{A'_2}/I_{A'_2} \cap I_{A'_1},$$

which associates the class $x^\alpha y^\beta + I_{A_2} \cap I_{A_1}$ to the class $x^\alpha y^\beta + I_{A'_2} \cap I_{A'_1}$, is an isomorphism of $\mathbb{C}[x, y]$ -modules. \square

Now, instead of focusing just on subsets of the natural plane \mathbb{N}^2 , we introduce more structure by looking at subsets of the representation tableau.

In some instances, we will need to work with abstract sFd's obtained forgetting about the monomials.

Definition 3.13. A G -sFd is a subset $A \subset \mathcal{T}_G$ of the representation tableau whose image $\pi_{\mathbb{N}^2}(A)$, under the first projection

$$\pi_{\mathbb{N}^2} : \mathcal{T}_G \rightarrow \mathbb{N}^2,$$

is a sFd.

An *abstract G -sFd* is a diagram Γ made of boxes labeled by the irreducible representations of G that can be embedded into the representation tableau as a G -sFd.

Example 3.14. Consider the $\mathbb{Z}/3\mathbb{Z}$ -action on \mathbb{A}^2 defined in (2.1). In Figure 3 are shown an abstract G -sFd and two of its possible realizations as G -sFd.

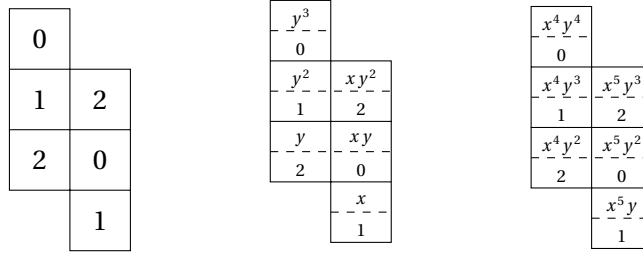


FIGURE 3. An abstract $\mathbb{Z}/3\mathbb{Z}$ -sFd and two of its possible realizations as $\mathbb{Z}/3\mathbb{Z}$ -sFd.

On the other hand, the diagram in Figure 4 is not an abstract G -sFd.



FIGURE 4.

Remark 3.15. Given any subset Ξ of the representation tableau and any monomial $x^\alpha y^\beta$ we will denote by $x^\alpha y^\beta \cdot \Xi$ the subset of the representation tableau obtained by translating Ξ α steps to the right and β steps up. Notice that this is compatible with the association $\mathbb{N}^2 \leftrightarrow \{\text{monomials in two variables}\}$ as explained in Remark 3.10.

Lemma 3.16. *If \mathcal{F} is a torus equivariant G -constellation then there exists a basis $\{v_0, \dots, v_{k-1}\}$ of $F = H^0(\mathbb{A}^2, \mathcal{F})$ such that*

- (1) for all $i = 0, \dots, k-1$, we have $v_i \in \rho_i$,
- (2) for all $i = 0, \dots, k-1$, the sections v_i are semi-invariant functions with respect some character χ_i of \mathbb{T}^2 , i.e. $(a, b) \cdot v_i = \chi_i(a, b)v_i$ for all $(a, b) \in \mathbb{T}^2$,
- (3) for all $i = 0, \dots, k-1$,

$$\begin{cases} x \cdot v_i \in \{v_{i+1}, 0\}, \\ y \cdot v_i \in \{v_{i-1}, 0\}. \end{cases}$$

Proof. We can always pick a basis $\{\tilde{v}_0, \dots, \tilde{v}_{k-1}\}$ which satisfies (1) and (2). Moreover, it follows from Remark 3.5 that:

$$\begin{cases} x \cdot \tilde{v}_i \in \text{Span}(\tilde{v}_{i+1}), \\ y \cdot \tilde{v}_i \in \text{Span}(\tilde{v}_{i-1}), \end{cases}$$

where the indices are thought modulo k . The fact that \mathcal{F} is toric implies that there are no "cycles", i.e. there are no $1 < s < k$ and

$$\left\{ (i_j, k_j, h_j, \sigma_j) \in \text{Irr}(G) \times \mathbb{N}^2 \times \mathbb{C}^* \left| \begin{array}{l} j = 1, \dots, s, \\ i_j \neq i_{j'} \text{ for } j \neq j', \\ k_j + h_{j+1} > 0 \end{array} \right. \right\}$$

where the indices are thought modulo s , such that

$$(3.2) \quad \begin{cases} (x \cdot)^{k_1} \tilde{v}_{i_1} &= \sigma_1 (y \cdot)^{h_2} \tilde{v}_{i_2}, \\ (x \cdot)^{k_2} \tilde{v}_{i_2} &= \sigma_2 (y \cdot)^{h_3} \tilde{v}_{i_3}, \\ &\vdots \\ (x \cdot)^{k_{s-1}} \tilde{v}_{i_{s-1}} &= \sigma_{s-1} (y \cdot)^{h_s} \tilde{v}_{i_s}, \\ (x \cdot)^{k_s} \tilde{v}_{i_s} &= \sigma_s (y \cdot)^{h_1} \tilde{v}_{i_1}. \end{cases}$$

Indeed, x and y are semi-invariant functions with respect to the characters

$$\mathbb{T}^2 \xrightarrow{\lambda_x} \mathbb{C}^*$$

$$(a, b) \longmapsto a$$

and

$$\mathbb{T}^2 \xrightarrow{\lambda_y} \mathbb{C}^*$$

$$(a, b) \longmapsto b$$

of the torus \mathbb{T}^2 . Then, if we act on both sides of the Equations (3.2) with some $(a, b) \in \mathbb{T}^2$, we get:

$$(3.3) \quad \begin{cases} \lambda_x(a, b)^{k_1} \chi_{i_1}(a, b) (x \cdot)^{k_1} \tilde{v}_{i_1} = \sigma_1 \lambda_y(a, b)^{h_2} \chi_{i_2}(a, b) (y \cdot)^{h_2} \tilde{v}_{i_2}, \\ \lambda_x(a, b)^{k_2} \chi_{i_2}(a, b) (x \cdot)^{k_2} \tilde{v}_{i_2} = \sigma_2 \lambda_y(a, b)^{h_3} \chi_{i_3}(a, b) (y \cdot)^{h_3} \tilde{v}_{i_3}, \\ &\vdots \\ \lambda_x(a, b)^{k_{s-1}} \chi_{i_{s-1}}(a, b) (x \cdot)^{k_{s-1}} \tilde{v}_{i_{s-1}} = \sigma_{s-1} \lambda_y(a, b)^{h_s} \chi_{i_s}(a, b) (y \cdot)^{h_s} \tilde{v}_{i_s}, \\ \lambda_x(a, b)^{k_s} \chi_{i_s}(a, b) (x \cdot)^{k_s} \tilde{v}_{i_s} = \sigma_s \lambda_y(a, b)^{h_1} \chi_{i_1}(a, b) (y \cdot)^{h_1} \tilde{v}_{i_1}, \end{cases}$$

Now, the System (3.3) is equivalent to:

$$\begin{cases} a^{k_1} \chi_{i_1}(a, b) = b^{h_2} \chi_{i_2}(a, b), \\ a^{k_2} \chi_{i_2}(a, b) = b^{h_3} \chi_{i_3}(a, b), \\ &\vdots \\ a^{k_{s-1}} \chi_{i_{s-1}}(a, b) = b^{h_s} \chi_{i_s}(a, b), \\ a^{k_s} \chi_{i_s}(a, b) = b^{h_1} \chi_{i_1}(a, b), \end{cases}$$

which is equivalent to

$$(3.4) \quad a^{k_1 + \dots + k_s} = b^{h_1 + \dots + h_s} \quad \forall (a, b) \in \mathbb{T}^2.$$

Finally, the only solution of Equation (3.4) is

$$k_1 = \dots = k_s = h_1 = \dots = h_s = 0,$$

which contradicts the hypothesis $k_i + h_{i+1} > 0$ for all $i = 1, \dots, s$.

We are now ready to build the requested basis. Let $\{w_1, \dots, w_\ell\} \subset \{\tilde{v}_0, \dots, \tilde{v}_{k-1}\}$ be a minimal set of generators of the $\mathbb{C}[x, y]$ -module F , i.e. the set

$$\{w_j + \mathfrak{m} \cdot F \in F/\mathfrak{m} \cdot F \mid j = 1, \dots, \ell\}$$

is a basis of the \mathbb{C} -vector space $F/\mathfrak{m} \cdot F$. Let us also denote by F_j , for $j = 1, \dots, \ell$, the submodule generated by w_j . We start by taking, for all $j = 1, \dots, \ell$, as basis of F_j the set

$$B_j = \{x^\alpha y^\beta w_j \mid \alpha \cdot \beta = 0\}.$$

The problem is that in general the union of all B_j 's is not a basis of F because there can be some relations $x^\alpha w_i = \mu y^\beta w_j$ for $i \neq j$ and $\mu \in \mathbb{C}^* \setminus 1$. The fact that there are no cycles implies that we can re-scale all the elements in each B_j obtaining new \bar{B}_j so that $\bigcup_j \bar{B}_j$ is a basis of F that verifies properties (1), (2), (3). \square

Proposition 3.17. *Given a, possibly decomposable, torus equivariant G -constellation $F = H^0(\mathbb{A}^2, \mathcal{F})$, there is (at least) one G -sFd whose associated $\mathbb{C}[x, y]$ -module is a G -constellation isomorphic to F .*

Remark 3.18. If we find one G -sFd with the required property, then there are infinitely many of them. Indeed, a special property of the representation tableau is that translations enjoy some periodicity properties.

Let Γ be a G -sFd, then:

- (1) multiplication by x has period k , i.e there is an isomorphism of $\mathbb{C}[x, y]$ -modules

$$M_\Gamma \xrightarrow{\sim} M_{x^k \cdot \Gamma}$$

which induces an isomorphism of representations between M_Γ and $M_{x^k \cdot \Gamma}$;

- (2) multiplication by y has period k , i.e there is an isomorphism of $\mathbb{C}[x, y]$ -modules

$$M_\Gamma \xrightarrow{\sim} M_{y^k \cdot \Gamma}$$

which induces an isomorphism of representations between M_Γ and $M_{y^k \cdot \Gamma}$;

- (3) multiplication by xy is an isomorphism, i.e there is an isomorphism of $\mathbb{C}[x, y]$ -modules

$$M_\Gamma \xrightarrow{\sim} M_{xy \cdot \Gamma}$$

which induces an isomorphism of representations between M_Γ and $M_{xy \cdot \Gamma}$.

In particular, all these G -sFd's correspond to the same abstract G -sFd.

Proof. (of Proposition 3.17). Let $\{v_0, \dots, v_{k-1}\}$ be a \mathbb{C} -basis of F with the properties listed in Lemma 3.16, and let $\{w_j = v_{i_j} \mid j = 1, \dots, s\}$ be a minimal set of generators of F as a $\mathbb{C}[x, y]$ -module (see the proof of Lemma 3.16). Denote by F_j , for $j = 1, \dots, s$, the $\mathbb{C}[x, y]$ -submodule of F generated by w_j . We can represent each F_j by using diagrams of the form shown in Figure 5, where the integers k_j and h_j are defined by

$$k_j = \max \{ \alpha \mid (x \cdot)^\alpha w_j \neq 0 \}$$

and

$$h_j = \max \{ \alpha \mid (y \cdot)^\alpha w_j \neq 0 \},$$

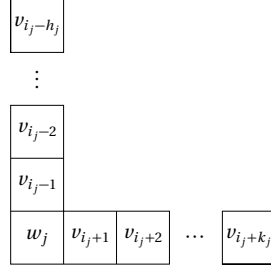


FIGURE 5.

and they are well defined because any toric G -constellation is nilpotent by Remark 3.7.

The $\mathbb{C}[x, y]$ -module structure of F_j is encoded in the fact that the multiplication by x (resp. y) sends the generator of a box (i.e., the generator of the corresponding vector space) to the generator of the box on the right (resp. above). If there is no box on the right (resp. above) this means that the multiplication by x (resp. y) is zero.

Now, we have to glue these diagrams to form the required G -sFd. We glue them along boxes with the same labels. First, notice that, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(x \cdot)^r w_j = (x \cdot)^t w_{j'}$, i.e. $i_j + r = i_{j'} + t$ modulo k , then

$$(x \cdot)^r w_j = (x \cdot)^t w_{j'} = 0.$$

Indeed, if $r < t$ (the case $r \geq t$ is analogous) then, a representation argument (see Proposition 3.6) tells us that $w_j = (x \cdot)^{t-r} w_{j'}$, which, whenever $(x \cdot)^r w_i \neq 0$, contradicts the minimality of the generating set $\{w_1, \dots, w_s\}$. Analogously, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(y \cdot)^r w_j = (y \cdot)^t w_{j'}$, then $(y \cdot)^r w_j = 0$.

Now we show that, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(x \cdot)^r w_j = (y \cdot)^t w_{j'}$, then $r = k_j$ and $t = h_{j'}$. Suppose, by contradiction, that there exists $1 \leq r < k_j$ such that $(x \cdot)^r w_j = (y \cdot)^t w_{j'}$ (the case $1 \leq t < h_{j'}$ is similar). In particular, the minimality assumption implies $t \geq 1$. Since $r < k_j$, by definition of k_j , we have $(x \cdot)^{r+1} w_j \neq 0$. Therefore, we get

$$0 \neq (x \cdot)^{r+1} w_j = x \cdot ((x \cdot)^r w_j) = x \cdot y^t \cdot w_{j'} = (x y) \cdot y^{t-1} \cdot w_{j'} = 0$$

which gives a contradiction.

We show now that there are no “cycles”. Explicitly, suppose that, up to reordering the v_i 's, and consequently the w_i 's, we have already glued ℓ diagrams of the form depicted in Fig. 5 to a diagram of the form shown in Figure 6. Then, we want to show that there is no gluing $(x \cdot)^{k_\ell} w_\ell = \sigma (y \cdot)^{h_1} w_1$ for some $\sigma \in \mathbb{C}^*$, i.e. no gluing of the first and the last boxes of the above diagram. The presence of this cycle would translate into the following system of equalities

$$\begin{cases} (x \cdot)^{k_1} w_1 & = (y \cdot)^{h_2} w_2, \\ (x \cdot)^{k_2} w_2 & = (y \cdot)^{h_3} w_3, \\ & \vdots \\ (x \cdot)^{k_{\ell-1}} w_{\ell-1} & = (y \cdot)^{h_\ell} w_\ell, \\ (x \cdot)^{k_\ell} w_\ell & = \sigma (y \cdot)^{h_1} w_1, \end{cases}$$

which cannot be verified by any toric G -constellation as explained in the proof of Lemma 3.16.

So far we have proven that each connected component of the required G -sFd have the shape depicted in Figure 7.

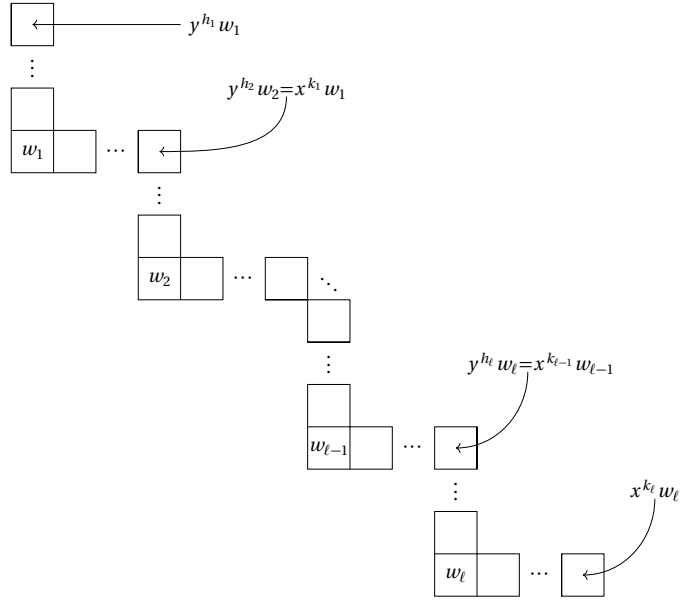


FIGURE 6.

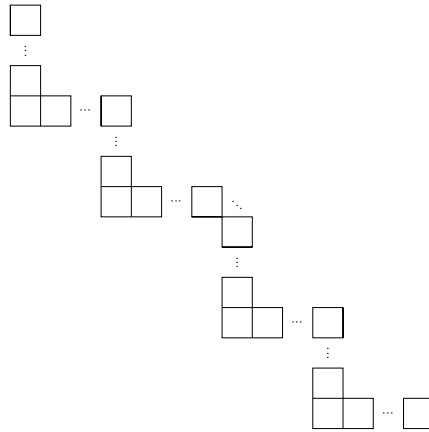


FIGURE 7.

Moreover, if we forget about the reordering, each box contains a label v_i whose index increases by one when moving to the right or downward in the diagram. Since we have chosen $v_i \in \rho_i$ for $i = 0, \dots, k - 1$, this diagram fits in the representation tableau (see Section 3.2), i.e. it is an abstract G -sFd. After performing all possible gluings, we obtain a number of abstract G -sFd's A_1, \dots, A_m whose shape is drawn in Figure 7.

The last thing to do is to show that we can realize A_1, \dots, A_m as subsets $\Gamma_1, \dots, \Gamma_m$ of the representation tableau to get a G -sFd, i.e. in such a way that

$$\pi_{\mathbb{N}^2} \left(\bigcup_{i=1}^m \Gamma_i \right)$$

is a sFd. This can be done in many ways and we explain one possible way to proceed.

We start by realizing A_1, \dots, A_m as disjoint G -sFd's $\Gamma_1, \dots, \Gamma_m$. This can always be done because, as we observed, A_1, \dots, A_m are abstract G -sFd's and, from any choice of realizations

$\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_m$ of them as non-necessarily disjoint G -sFd's, we can obtain disjoint $\Gamma_1, \dots, \Gamma_m$ by performing the translations described in Remark 3.18.

At this point, we have m disjoint G -sFd's as described in Figure 8, where just the labels of

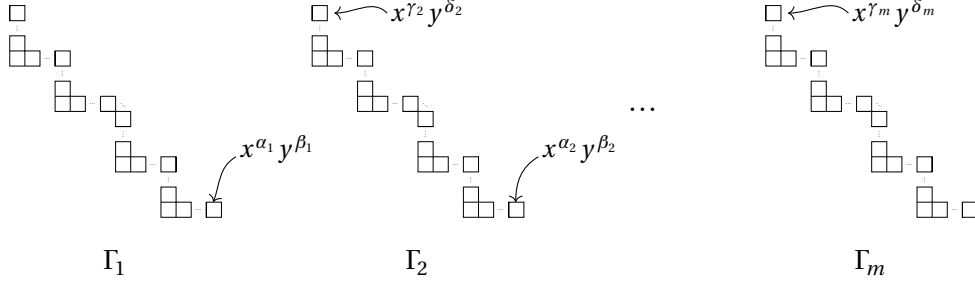


FIGURE 8.

the boxes we are interested in are shown. The problem is that, in general, the union $\bigcup_{i=1}^m \Gamma_i$ is not a G -sFd, i.e. $\pi_{\mathbb{N}^2} \left(\bigcup_{i=1}^m \Gamma_i \right)$ is not a sFd. In order to solve this problem, we have to perform some translations, and a possible choice of G -sFd is

$$\Gamma = \bigcup_{i=1}^m \bar{\Gamma}_i,$$

where

$$\bar{\Gamma}_i = x^{k \sum_{j=1}^{i-1} \alpha_j} y^{k \sum_{j=1+i}^m \delta_j} \cdot \Gamma_i \quad \text{for } i = 1, \dots, m.$$

The proof that Γ is a G -sFd is now an easy check. \square

As a byproduct of the proof, we also get that any G -sFd associated to a toric G -constellation has a particular shape.

Definition 3.19. We say that a connected G -sFd Γ is a *stair* if

$$(m, n) \in \pi_{\mathbb{N}^2}(\Gamma) \Rightarrow (m+1, n+1), (m-1, n-1) \notin \pi_{\mathbb{N}^2}(\Gamma).$$

Moreover,

- a G -stair is a stair made of k boxes,
- an *abstract (G -)stair* is an abstract G -sFd whose realization in the representation tableau is a (G -)stair,
- given a stair Γ , the (*anti*)generators of Γ are the boxes positioned in the (top) lower corners of Γ (see Figure 9),
- a substair is any (possibly not connected) subset of a stair.

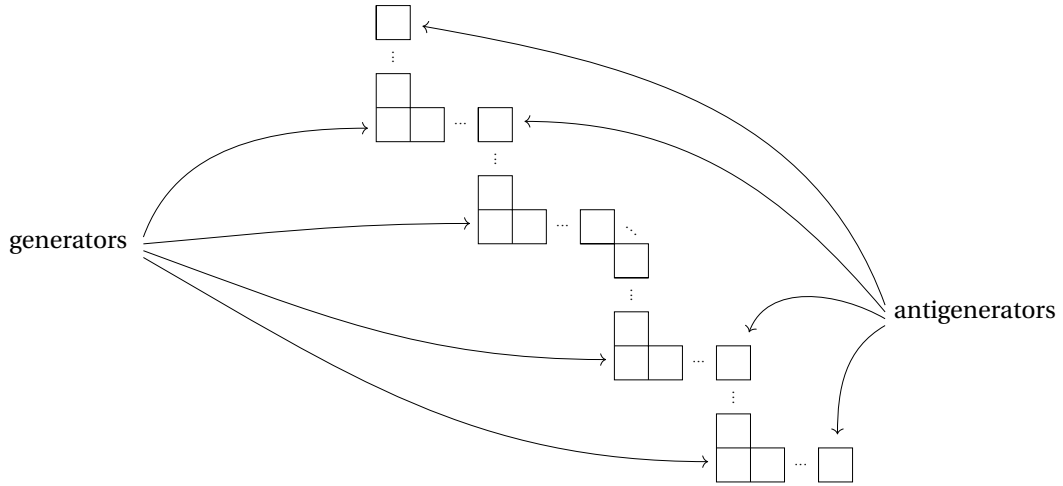


FIGURE 9. Generators and antigenerators of a stair.

Remark 3.20. If \mathcal{F} is any torus equivariant G -constellation, and $\Gamma_{\mathcal{F}}$ is any G -sFd associated to \mathcal{F} , then $\Gamma_{\mathcal{F}}$ is connected, i.e. it is a G -stair, if and only if \mathcal{F} is indecomposable, i.e. if it is toric.

In this case we will refer to the upper left box as the first box and we will refer to the lower right box as the last box. In this way, we provide a total order to the boxes of a G -stair and, consequently, we provide a total order also to the irreducible representations of G .

Remark 3.21. The set of generators of a stair Γ corresponds to a minimal set of generators of the $\mathbb{C}[x, y]$ -module M_{Γ} associated to Γ , i.e. $m_1, \dots, m_s \in M_{\Gamma}$ such that

$$\{ m_i + \mathfrak{m} \cdot M_{\Gamma} \in M_{\Gamma} / \mathfrak{m} \cdot M_{\Gamma} \mid i = 1, \dots, s \}$$

is a \mathbb{C} -basis of $M_{\Gamma} / \mathfrak{m} \cdot M_{\Gamma}$. Antigenerators correspond to one-dimensional $\mathbb{C}[x, y]$ -submodules of M_{Γ} , i.e. they form a \mathbb{C} -basis of the so-called socle

$$(0 :_{M_{\Gamma}} \mathfrak{m}) = \{ m \in M_{\Gamma} \mid \mathfrak{m} \cdot m = 0 \in M_{\Gamma} \}.$$

Since each irreducible representation of G appears once in a G -stair L , sometimes, with abuse of notation, we will say that an irreducible representation is a (anti)generator for L .

Definition 3.22. Given a connected G -sFd Γ , we denote respectively by $h(\Gamma)$ and $w(\Gamma)$ the *height* and the *width* of Γ , i.e. the height and the width of the smallest rectangle in \mathbb{N}^2 containing $\pi_{\mathbb{N}^2}(\Gamma)$.

Moreover, the *height* and the *width*, $h(\mathcal{F})$ and $w(\mathcal{F})$, of a toric G -constellation \mathcal{F} are respectively the height and the width of any G -stair which represents \mathcal{F} .

4. THE CHAMBER DECOMPOSITION OF Θ AND THE MODULI SPACES \mathcal{M}_C

This section is devoted to the proof of the first main result (Theorem 4.17). In the first part of the section we analyze the toric points of \mathcal{M}_C and the corresponding G -constellations. Then, we show how to construct 1-dimensional families of nilpotent G -constellations. Finally, in the last part, we give the proof of the first main result.

4.1. **The crepant resolution \mathcal{M}_C and its toric points.** As noticed in Section 2.2, the crepant resolution $\mathcal{M}_C \xrightarrow{\varepsilon_C} \mathbb{A}^2/G$ does not depend on the chamber C , i.e. for all $C, C' \in \Theta^{\text{gen}}$ different chambers, there exists a canonical isomorphism $\varphi: \mathcal{M}_C \xrightarrow{\sim} \mathcal{M}_{C'}$ such that the diagram

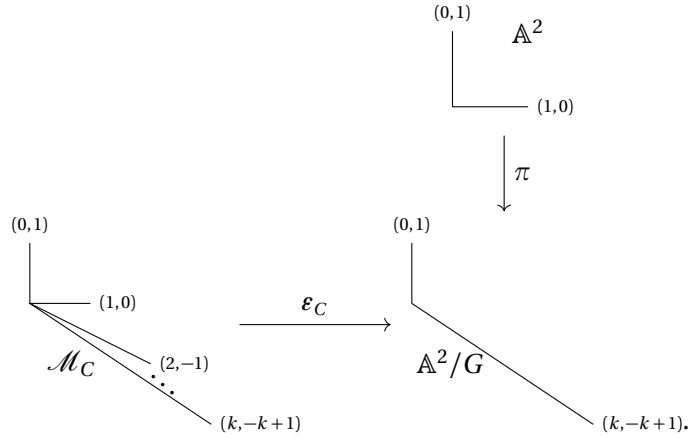
$$\begin{array}{ccc} \mathcal{M}_C & \xrightarrow{\varphi} & \mathcal{M}_{C'} \\ & \searrow \varepsilon_C & \swarrow \varepsilon_{C'} \\ & \mathbb{A}^2/G & \end{array}$$

commutes.

The varieties \mathbb{A}^2 , \mathbb{A}^2/G and \mathcal{M}_C are toric (see for example [6, Chapter 10] or [11, Chapter 2]) and we can rewrite the diagram

$$\begin{array}{ccc} & \mathbb{A}^2 & \\ & \downarrow \pi & \\ \mathcal{M}_C & \xrightarrow{\varepsilon_C} & \mathbb{A}^2/G \end{array}$$

in terms of fans as follows:



In particular, \mathcal{M}_C is covered by the k toric charts $U_j \cong \mathbb{A}^2$, for $j = 1, \dots, k$, associated to the maximal cones of the fan for \mathcal{M}_C showed above.

Let us identify \mathbb{A}^2/G with the subvariety of \mathbb{A}^3

$$\mathbb{A}^2/G = \{(\alpha, \beta, \gamma) \in \mathbb{A}^3 \mid \alpha\beta - \gamma^k = 0\},$$

and let us put (toric) coordinates a_j, c_j on each U_j for $j = 1, \dots, k$. Then, we can encode the diagram above into the following k diagrams

$$\begin{array}{ccc} & \mathbb{A}^2 & (x, y) \\ & \downarrow \pi & \downarrow \\ U_j & \xrightarrow{\varepsilon_j} & \mathbb{A}^2/G & (x^k, y^k, xy) \\ (a_j, c_j) & \longmapsto & (a_j^{k-j+1} c_j^{k-j}, a_j^{j-1} c_j^j, a_j c_j) \end{array}$$

for $j = 1, \dots, k$. In this way, we obtain some relations between the coordinates x, y on \mathbb{A}^2 and the coordinates a_j, c_j on U_j , namely

$$(4.1) \quad \begin{aligned} a_j &= x^j y^{j-k}, \\ c_j &= x^{1-j} y^{k-j+1}. \end{aligned}$$

Formally, these are relations between regular functions $x, y, a_j, c_j \in \mathbb{C}[a_j, c_j] \otimes_{\mathbb{C}[x, y]^G} \mathbb{C}[x, y]$ defined on $U_j \times_{\mathbb{A}^2/G} \mathbb{A}^2 = \text{Spec} \left(\left(\mathbb{C}[a_j, c_j] \otimes_{\mathbb{C}[x, y]^G} \mathbb{C}[x, y] \right)_{\text{red}} \right)$.

Remark 4.1. The toric points of \mathcal{M}_C are the origins of the charts U_j and they correspond to the toric C -stable G -constellations. Indeed, the torus \mathbb{T}^2/G acts on \mathcal{M}_C making it into a toric variety, as described at the beginning of this section, and this toric action coincides with the action

$$\begin{aligned} \mathbb{T}^2 \times \mathcal{M}_C &\longrightarrow \mathcal{M}_C \\ (\sigma, [\mathcal{F}]) &\longmapsto [\sigma^* \mathcal{F}]. \end{aligned}$$

This is a consequence of the universal property of \mathcal{M}_C . Notice that, outside the exceptional locus of \mathcal{M}_C , i.e. on the open subset of free orbits, a direct computation is enough to show that the two actions agree.

Hence we have a total order on the toric G -constellations over \mathcal{M}_C , in the sense that the first toric G -constellation is the G -constellation over the origin of U_1 , the second one is the G -constellation over the origin of U_2 , and so on.

Remark 4.2. Let Γ be a G -stair. Then there exists a unique $\sigma \in \text{Irr}(G)$ such that

$$y \cdot \sigma = 0 \text{ and } x \cdot \sigma \otimes \rho_{-1} = 0$$

in Γ . In particular, the representation σ corresponds to the first box of Γ . This representation is important because, if we want to deform in a non-trivial way the G -constellation \mathcal{F}_Γ associated to Γ keeping the property of being nilpotent, there are only two ways to do it, namely to modify the $\mathbb{C}[x, y]$ -module structure of \mathcal{F}_Γ by imposing

$$y \cdot \sigma = \lambda \cdot \sigma \otimes \rho_{-1}, \quad \lambda \in \mathbb{C}^*$$

or

$$x \cdot \sigma \otimes \rho_{-1} = \mu \cdot \sigma, \quad \mu \in \mathbb{C}^*.$$

Indeed, if $y \cdot \sigma = \lambda \cdot \sigma \otimes \rho_{-1}$ is not zero, then the nilpotency hypothesis implies

$$x \cdot \sigma \otimes \rho_{-1} = \frac{1}{\lambda} x y \cdot \sigma = 0,$$

and the other case is similar. Comparing this with the proof of Lemma 3.16 one can show that letting λ (resp. μ) varying in \mathbb{C}^* all the G -constellations so obtained are not isomorphic to each other (as G -constellations). In particular λ, μ are coordinates on a chart of \mathcal{M}_C around \mathcal{F}_Γ .

As a consequence of the above remark, we obtain the following lemma.

Lemma 4.3. *If \mathcal{F}_j is the toric G -constellation over the origin of the j -th chart of some \mathcal{M}_C , then we have*

$$\mathfrak{h}(\mathcal{F}_j) = k - j + 1$$

or, equivalently

$$\mathfrak{w}(\mathcal{F}_j) = j.$$

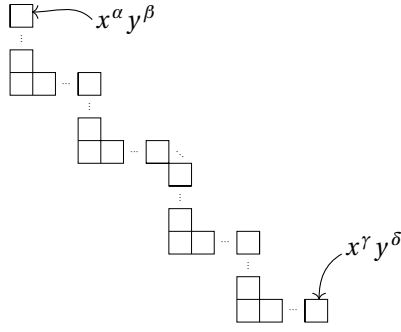


FIGURE 10.

Proof. Let $\Gamma_j \subset \mathcal{T}_G$ be a G -stair for \mathcal{F}_j . In particular, it has the form in Figure 10 where just the labels of the boxes we are interested in are shown. Recall, from Section 3.2, that, if we write the skew Ferrers diagram $\pi_{\mathbb{N}^2}(\Gamma_j) = A \setminus B$ as the difference of two Ferrers diagrams A and B , then $\mathcal{F}_j \cong M_{\Gamma_j}$, where

$$M_{\Gamma_j} \cong \frac{I_A}{I_A \cap I_B},$$

and I_A, I_B are as in the proof of Lemma 3.12. Now, if we deform \mathcal{F}_j as in Remark 4.2, by using the parameters $a_j, c_j \in \mathbb{C}$, we get relations:

$$x \cdot x^\gamma y^\delta = a_j x^\alpha y^\beta$$

$$y \cdot x^\alpha y^\beta = c_j x^\gamma y^\delta$$

and, the relations (4.1) tell us that

$$(\gamma - \alpha + 1, \delta - \beta) = (\mathfrak{w}(\mathcal{F}), -\mathfrak{h}(\mathcal{F}) + 1) = (j, j - k) \in \mathbb{N}^2$$

$$(\alpha - \gamma, \beta - \delta + 1) = (-\mathfrak{w}(\mathcal{F}) + 1, \mathfrak{h}(\mathcal{F})) = (1 - j, k - j + 1) \in \mathbb{N}^2$$

which completes the proof. \square

Remark 4.4. Lemma 4.3 implies that any two toric G -constellations of the same height (or equivalently width) cannot belong to the same chamber, i.e. they cannot be θ -stable for the same generic parameter $\theta \in \Theta^{\text{gen}}$ simultaneously.

4.2. One dimensional families.

Definition 4.5. Given a toric G -constellation \mathcal{F} and its abstract G -stair $\Gamma_{\mathcal{F}}$, its *favorite condition* is the stability condition $\theta_{\mathcal{F}} \in \Theta$ defined by:

$$(\theta_{\mathcal{F}})_i = \begin{cases} -2 & \text{if } \rho_i \text{ is a generator and it is neither the first nor the last box of } \Gamma_{\mathcal{F}}, \\ -1 & \text{if } \rho_i \text{ is a generator and it is either the first or the last box of } \Gamma_{\mathcal{F}}, \\ 2 & \text{if } \rho_i \text{ is an antigenerator and it is neither the first nor the last box of } \Gamma_{\mathcal{F}}, \\ 1 & \text{if } \rho_i \text{ is an antigenerator and it is either the first or the last box of } \Gamma_{\mathcal{F}}, \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the *cone of good conditions* for \mathcal{F} , is the cone:

$$\Theta_{\mathcal{F}} = \{ \theta \in \Theta^{\text{gen}} \mid \mathcal{F} \text{ is } \theta\text{-stable} \}.$$

Remark 4.6. It is worth mentioning that the favorite condition $\theta_{\mathcal{F}}$ of a toric G -constellation \mathcal{F} can be understood as the stability condition determined by an appropriate flow on a certain quiver as explained in [28, §6].

Definition 4.7. Let Γ be a stair and let $\Gamma' \subset \Gamma$ be a substair. We say that an element $v \in \Gamma'$ is

- a *left internal endpoint* of Γ' if there exists $w \in \Gamma \setminus \Gamma'$ such that $x \cdot w = v$ or if $y \cdot v \in \Gamma \setminus \Gamma'$;
- a *right internal endpoint* of Γ' if there exists $w \in \Gamma \setminus \Gamma'$ such that $y \cdot w = v$ or if $x \cdot v \in \Gamma \setminus \Gamma'$.

Moreover, we say that

- a left (resp. right) internal endpoint is a *horizontal left (resp. right) cut* if $y \cdot v \in \Gamma \setminus \Gamma'$ (resp. there exists $w \in \Gamma \setminus \Gamma'$ such that $y \cdot w = v$);
- a left (resp. right) internal endpoint is a *vertical left (resp. right) cut* if there exists $w \in \Gamma \setminus \Gamma'$ such that $x \cdot w = v$ (resp. $x \cdot v \in \Gamma \setminus \Gamma'$);

Remark 4.8. If \mathcal{F} is a G -constellation and $\Gamma_{\mathcal{F}}$ is a G -stair for \mathcal{F} , then a substair $\Gamma \subset \Gamma_{\mathcal{F}}$ corresponds to a G -equivariant $\mathbb{C}[x, y]$ -submodule \mathcal{E}_{Γ} of \mathcal{F} if and only if it has only vertical left cuts and horizontal right cuts. Moreover, if Γ is connected and $\theta_{\mathcal{F}}$ is the favorite condition of \mathcal{F} , then,

$$\theta_{\mathcal{F}}(\mathcal{E}_{\Gamma}) = \begin{cases} 1 & \text{if } \Gamma \text{ has one internal endpoint,} \\ 2 & \text{if } \Gamma \text{ has two internal endpoints.} \end{cases}$$

Remark 4.9. Let \mathcal{F} be a toric G -constellation with abstract G -stair $\Gamma_{\mathcal{F}}$ and let $\mathcal{E} < \mathcal{F}$ be a subrepresentation, i.e. a G -invariant linear subspace, whose substair $\Gamma_{\mathcal{E}} \subset \Gamma_{\mathcal{F}}$ is connected. Then, if $\Gamma_{\mathcal{E}}$ has two horizontal cuts or two vertical cuts and $\theta_{\mathcal{F}}$ is the favorite condition of \mathcal{F} , we have

$$\theta_{\mathcal{F}}(\mathcal{E}) = 0.$$

Remark 4.10. The following properties are easy to check for a toric G -constellation \mathcal{F} :

- favorite conditions are never generic,
- the G -constellation \mathcal{F} is $\theta_{\mathcal{F}}$ -stable,
- there exist generic conditions $\theta \in \Theta^{\text{gen}}$ such that \mathcal{F} is θ -stable, i.e. the cone of good conditions $\Theta_{\mathcal{F}}$ is not empty.

Moreover, given a chamber C , we have:

$$C = \bigcap_{[\mathcal{F}] \in \mathcal{M}_C} \Theta_{\mathcal{F}}.$$

For example, one can prove the third property using the openness of the nonempty set $\{\theta \in \Theta \mid \mathcal{F} \text{ is strictly } \theta\text{-stable}\}$ and the denseness of Θ^{gen} . However, we give here an alternative proof of this fact as in what follows we shall need a similar argument.

Let ρ_i be any irreducible representation, we denote by \mathcal{F}_{ρ_i} the G -equivariant $\mathbb{C}[x, y]$ -submodule of \mathcal{F} generated by ρ_i and, we denote by $\Gamma_{\rho_i} \subset \Gamma_{\mathcal{F}}$ the abstract substair and G -stair corresponding to \mathcal{F}_{ρ_i} and \mathcal{F} respectively.

Consider an $\varepsilon \in \Theta$ with the following properties:

$$\left\{ \begin{array}{ll} \varepsilon_i = 0 & \text{if } \rho_i \text{ is an antigenerator,} \\ \varepsilon_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antigenerator,} \\ \varepsilon_i = - \sum_{\rho_j \in (\Gamma_{\rho_i} \setminus \rho_i)} \varepsilon_j & \text{if } \rho_i \text{ is a generator,} \\ \sum_{\rho_i \text{ generator}} \varepsilon_i < 1. & \end{array} \right.$$

Then, for any subrepresentation $\mathcal{E} < \mathcal{F}$, we have

$$\varepsilon(\mathcal{E}) > - \sum_{\rho_i \text{ generator}} \varepsilon_i > -1.$$

Hence, the G -constellation \mathcal{F} is $(\theta_{\mathcal{F}} + \varepsilon)$ -stable. Indeed, Remark 4.8 implies that, given an indecomposable proper G -equivariant $\mathbb{C}[x, y]$ -submodule we have

$$(\theta_{\mathcal{F}} + \varepsilon)(\mathcal{E}) > 0.$$

On the contrary, if \mathcal{E} is not indecomposable then it is a direct sum of indecomposable components and $(\theta_{\mathcal{F}} + \varepsilon)(\mathcal{E}) > 0$ follows by the additivity of $\theta_{\mathcal{F}} + \varepsilon$ on direct sums.

We conclude by noticing that $\Theta \setminus \Theta^{\text{gen}}$ is a union of hyperplanes and so, there is at least a choice $\varepsilon \in \Theta$ such that $\theta_{\mathcal{F}} + \varepsilon$ is generic.

We will see in the proof of Theorem 4.17 that there is an easier way, which does not involve any ε , to prove that $\Theta_{\mathcal{F}}$ is not empty.

Definition 4.11. An *abstract linking stair* is an abstract stair made of $2k$ boxes obtained from an abstract G -stair Γ in either of the following ways:

- (1) (*decreasing linking stair* of Γ) take two copies of Γ and make a new abstract stair by gluing the right edge of the last box of one copy to the left edge of the first box of the other copy;
- (2) (*increasing linking stair* of Γ) take two copies of Γ and make a new abstract stair by gluing the lower edge of the last box of one copy to the upper edge of the first box of the other copy.

A *linking stair* is a realization of an abstract linking stair as a subset of the representation tableau.

Remark 4.12. An abstract linking stair contains exactly k different abstract G -stairs.

Proposition 4.13. *Let Γ be the abstract G -stair of a G -constellation \mathcal{F} and let L be its abstract decreasing linking stair. Consider any G -stair $\Gamma' \subset L$ and its associated G -constellation \mathcal{F}' . Then, the following are equivalent:*

- (1) *there exists at least a chamber C such that both \mathcal{F} and \mathcal{F}' belong to C , i.e. $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$,*
- (2) $\mathfrak{h}(\mathcal{F}') = \mathfrak{h}(\mathcal{F}) - 1$,
- (3) *the substair $\Gamma' \subset L$ has a horizontal left cut.*

In particular, \mathcal{F}' is the G -constellation next to \mathcal{F} in \mathcal{M}_C as per Remark 4.1.

Example 4.14. Figure 11 describes the situation via an example. Here, we are considering the $\mathbb{Z}/9\mathbb{Z}$ -action on \mathbb{A}^2 given in (2.1).

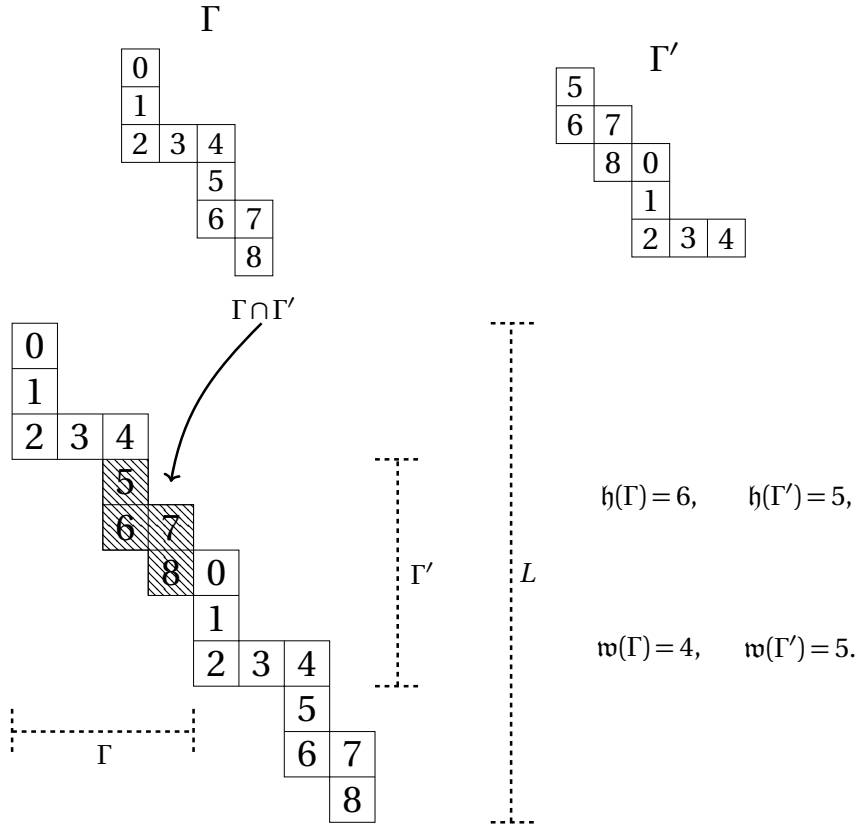


FIGURE 11. The abstract linking stair L of an abstract G -stair Γ and a substair Γ' of L which satisfies the hypotheses of Proposition 4.13.

Proof. (of Proposition 4.13). We start by introducing some notation.

Let $\mathcal{F}, \mathcal{F}'$ be two G -constellations. Given a proper subrepresentation $\mathcal{E} < \mathcal{F}$ (resp. $\mathcal{E}' < \mathcal{F}'$), we denote by \mathcal{E}' (resp. \mathcal{E}) the corresponding subrepresentation $\mathcal{E}' < \mathcal{F}'$ (resp. $\mathcal{E} < \mathcal{F}$). Here, by “corresponding” we mean that, since \mathcal{E} is a subrepresentation of the regular representation $\mathbb{C}[G]$ of an abelian group, it decomposes as a direct sum of distinct indecomposable representations $\mathcal{E} \cong \bigoplus_j \rho_{i_j}$. Then, we denote by \mathcal{E}' the subrepresentation of $\mathcal{F}' \cong \mathbb{C}[G]$ given by the same summands:

$$\mathcal{E}' \cong \bigoplus_j \rho_{i_j}.$$

In particular, for all $\theta \in \Theta$, the two rational numbers

$$\theta(\mathcal{E}) \text{ and } \theta(\mathcal{E}')$$

are the same. Moreover, we denote by $\Gamma_{\mathcal{E}} \subset \Gamma$ (resp. $\Gamma_{\mathcal{E}'} \subset \Gamma'$) the substair associated to \mathcal{E} (resp. \mathcal{E}').

Notice that, given a proper G -equivariant $\mathbb{C}[x, y]$ -submodule $\mathcal{E} < \mathcal{F}$, the subrepresentation \mathcal{E}' is not necessarily a $\mathbb{C}[x, y]$ -submodule of \mathcal{F}' . We are now ready to proceed with the proof.

(2) \Leftrightarrow (3) We omit the easy proof.

(1) \Rightarrow (3) Suppose by contradiction that $\Gamma' \subset L$ has a vertical left cut. Then, by Remark 4.8, the subrepresentation $\mathcal{E}_{\Gamma \cap \Gamma'} < \mathcal{F}$ is a $\mathbb{C}[x, y]$ -submodule because, in Γ , the substair $\Gamma \cap \Gamma'$ has a vertical left cut by hypothesis and its last box is not internal. At the same time,

again by Remark 4.8, $\mathcal{E}'_{\Gamma \cap \Gamma'} < \mathcal{F}'$ is the complement of a $\mathbb{C}[x, y]$ -submodule, because its first box is not internal and it has a vertical right cut. Hence,

$$C \subset \Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \subset \{\theta(\mathcal{E}_{\Gamma \cap \Gamma'}) > 0\} \cap \{-\theta(\mathcal{E}'_{\Gamma \cap \Gamma'}) > 0\} = \emptyset,$$

which contradicts (I).

(3) \Rightarrow (I) In order to prove statement (I), we need to show that

$$\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset.$$

We start by identifying the proper indecomposable G -equivariant subsheaves $\mathcal{E} < \mathcal{F}$ (resp. $\mathcal{E}' < \mathcal{F}'$) such that also \mathcal{E}' (resp. \mathcal{E}) is a proper G -equivariant subsheaf of \mathcal{F} (resp. \mathcal{F}').

Let $\mathcal{E}' < \mathcal{F}'$ be a proper indecomposable G -equivariant submodule of \mathcal{F}' ; we consider three different cases.

Case 1. Both the first and the last box of the substair $\Gamma_{\mathcal{E}'} \subset \Gamma'$ are internal endpoints. Then, the same happens for $\Gamma_{\mathcal{E}} \subset \Gamma$. This is true because Γ has a vertical right cut in L , by the construction of a decreasing linking stair (see Definition 4.11), and hence, the right internal endpoint of $\Gamma_{\mathcal{E}'}$ in Γ' , which is a horizontal cut by Remark 4.8, is different from the right internal endpoint of Γ in L . Therefore, both internal endpoints of $\Gamma_{\mathcal{E}'}$ correspond to internal endpoints of $\Gamma_{\mathcal{E}}$ of the same respective nature. As a consequence, the subrepresentation \mathcal{E} is a proper, non necessarily indecomposable, G -equivariant submodule of \mathcal{F} .

Case 2. The substair $\Gamma_{\mathcal{E}'}$ has only the vertical left cut in Γ' , and hence, its last box coincides with the last box of Γ' . In particular, this box is not the right internal endpoint of Γ in L . We have to study the nature of the internal endpoints of $\Gamma_{\mathcal{E}}$. Notice first that it is enough to study the right internal endpoint of $\Gamma_{\mathcal{E}}$ because, if $\Gamma_{\mathcal{E}}$ has still left internal endpoint, then it is a vertical left cut. Let ρ_i be the label on the last box of Γ' , then, the label on the horizontal left cut of Γ' (i.e. its first box) is ρ_{i+1} . Now, since, by hypothesis (3), the box labeled by ρ_{i+1} is a horizontal left cut of $\Gamma' \subset L$, the box labeled by ρ_i in Γ has to be a horizontal right cut for the substair $\Gamma_{\mathcal{E}}$. Therefore, $\Gamma_{\mathcal{E}}$ has only vertical left cuts and horizontal right cuts, and so, by Remark 4.8, \mathcal{E} is a proper, non necessarily indecomposable, G -equivariant submodule.

Case 3. The substair $\Gamma_{\mathcal{E}'}$ has only the horizontal right cut, i.e. its first box coincides with the first box of Γ' . First of all notice that, as for the first analyzed case, the right internal endpoint of $\Gamma_{\mathcal{E}'}$ in Γ' , which is a horizontal cut by hypothesis, is different from the right internal endpoint of Γ in L , which is vertical by definition of decreasing linking stair. Therefore, the box of Γ with the same label as the horizontal right cut of $\Gamma_{\mathcal{E}'}$ is an internal endpoint of $\Gamma_{\mathcal{E}}$ and it is a horizontal right cut. Finally, the first box of Γ' in L is a left internal endpoint for $\Gamma_{\mathcal{E}}$, and so it is a horizontal left cut by point (3) of the statement. As a consequence, $\Gamma_{\mathcal{E}}$ has two horizontal cuts.

In summary, if $\mathcal{E}' < \mathcal{F}'$ is a proper indecomposable G -equivariant submodule of \mathcal{F}' such that $\Gamma_{\mathcal{E}'}$ has a vertical left cut, then also $\mathcal{E} < \mathcal{F}$ is a proper, non necessarily indecomposable, G -equivariant submodule. While, if $\Gamma_{\mathcal{E}'} < \Gamma'$ has only the right horizontal cut, then $\Gamma_{\mathcal{E}}$ has two horizontal cuts.

Following the same logic, if $\mathcal{E} < \mathcal{F}$ is a proper indecomposable G -equivariant submodule of \mathcal{F} such that $\Gamma_{\mathcal{E}}$ has a horizontal right cut, then also $\mathcal{E}' < \mathcal{F}'$ is a proper, non

necessarily indecomposable, G -equivariant submodule. While, if $\Gamma_{\mathcal{E}} < \Gamma$ has only the left vertical cut, then $\Gamma_{\mathcal{E}'}$ has two vertical cuts.

We are now ready to exhibit a $\theta \in \Theta^{\text{gen}}$ such \mathcal{F} and \mathcal{F}' are θ -stable. Let $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{F}'}$ be the respective favorite conditions for \mathcal{F} and \mathcal{F}' and let $\theta = \theta_{\mathcal{F}} + \theta_{\mathcal{F}'}$ be their sum. Then, both \mathcal{F} and \mathcal{F}' are θ -stable. Indeed,

- if $\mathcal{E} < \mathcal{F}$ is a proper indecomposable G -equivariant $\mathbb{C}[x, y]$ -submodule of \mathcal{F} such that also \mathcal{E}' is a $\mathbb{C}[x, y]$ -submodule of \mathcal{F}' , then

$$\theta(\mathcal{E}) = \theta_{\mathcal{F}}(\mathcal{E}) + \theta_{\mathcal{F}'}(\mathcal{E}) = \theta_{\mathcal{F}}(\mathcal{E}) + \theta_{\mathcal{F}'}(\mathcal{E}') > 0$$

follows from the fact that \mathcal{F} is $\theta_{\mathcal{F}}$ -stable and \mathcal{F}' is $\theta_{\mathcal{F}'}$ -stable (see Remark 4.10);

- if $\mathcal{E}' < \mathcal{F}'$ is a proper indecomposable G -equivariant $\mathbb{C}[x, y]$ -submodule of \mathcal{F}' such that $\Gamma_{\mathcal{E}}$ has two horizontal cuts, then

$$\theta(\mathcal{E}') = \theta_{\mathcal{F}}(\mathcal{E}') + \theta_{\mathcal{F}'}(\mathcal{E}') = \theta_{\mathcal{F}}(\mathcal{E}) + \theta_{\mathcal{F}'}(\mathcal{E}') = \theta_{\mathcal{F}'}(\mathcal{E}') = 1 > 0$$

follows from the fact that \mathcal{F}' is $\theta_{\mathcal{F}'}$ -stable (see Remark 4.10) and from Remarks 4.8 and 4.9;

- if $\mathcal{E} < \mathcal{F}$ is a proper indecomposable G -equivariant $\mathbb{C}[x, y]$ -submodule of \mathcal{F} such that $\Gamma_{\mathcal{E}'}$ has two vertical cuts, then

$$\theta(\mathcal{E}) = \theta_{\mathcal{F}}(\mathcal{E}) + \theta_{\mathcal{F}'}(\mathcal{E}) = \theta_{\mathcal{F}}(\mathcal{E}) + \theta_{\mathcal{F}'}(\mathcal{E}') = \theta_{\mathcal{F}}(\mathcal{E}) = 1 > 0$$

follows from the fact that \mathcal{F} is $\theta_{\mathcal{F}}$ -stable (see Remark 4.10) and from Remarks 4.8 and 4.9;

- if $\mathcal{E} < \mathcal{F}$ (resp. $\mathcal{E}' < \mathcal{F}'$) is a proper decomposable G -equivariant $\mathbb{C}[x, y]$ -submodule, then

$$\theta(\mathcal{E}) > 0$$

follows by applying the previous points to the indecomposable components of \mathcal{E} and from the additivity of θ .

The last issue here is that, in general, such θ is not generic, i.e.

$$\theta \in \overline{\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'}} \setminus \Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'}$$

In order to solve this problem, we can perturb $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{F}'}$ the same way as we did in Remark 4.10 thus obtaining a generic $\tilde{\theta} \in \Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'}$. Consider the stability conditions $\varepsilon, \varepsilon' \in \Theta$ defined as follows:

$$\left\{ \begin{array}{ll} \varepsilon_i = 0 & \text{if } \rho_i \text{ is an antgenerator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i = 0 & \text{if } \rho_i \text{ is an antgenerator of } \Gamma_{\mathcal{F}'}, \\ \varepsilon_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antgenerator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antgenerator of } \Gamma_{\mathcal{F}'}, \\ \varepsilon_i = - \sum_{\rho_j \in (\Gamma_{\rho_i} \setminus \rho_i)} \varepsilon_j & \text{if } \rho_i \text{ is a generator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i = - \sum_{\rho_j \in (\Gamma'_{\rho_i} \setminus \rho_i)} \varepsilon'_j & \text{if } \rho_i \text{ is a generator of } \Gamma_{\mathcal{F}'}, \\ \sum_{\rho_i \text{ generator of } \Gamma_{\mathcal{F}}} \varepsilon_i + \sum_{\rho_i \text{ generator of } \Gamma_{\mathcal{F}'}} \varepsilon'_i < 1, \end{array} \right.$$

where, as in Remark 4.10, $\Gamma_{\rho_i} \subset \Gamma$ (resp. $\Gamma'_{\rho_i} \subset \Gamma'$) is the substair associated to the $\mathbb{C}[x, y]$ -submodule of \mathcal{F} (resp. \mathcal{F}') generated by the irreducible subrepresentation ρ_i .

Now, if

$$\tilde{\theta} = (\theta_{\mathcal{F}} + \varepsilon) + (\theta_{\mathcal{F}'} + \varepsilon')$$

then \mathcal{F} and \mathcal{F}' are $\tilde{\theta}$ -stable, and ε and ε' can be chosen in such a way that $\tilde{\theta}$ is generic. As a consequence $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$. □

We will see, in the proof of Theorem 4.17, that there is an easier way to prove that $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'}$ is not empty. By following the same logic, one can prove a similar statement for the increasing linking stairs.

Proposition 4.15. *Let Γ be the abstract G -stair of a G -constellation \mathcal{F} and let L be its abstract increasing linking stair. Consider any G -stair $\Gamma' \subset L$ and its associated G -constellation \mathcal{F}' . Then, the following are equivalent:*

- (1) *there exists at least a chamber C such that both \mathcal{F} and \mathcal{F}' belong to C , i.e. $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$,*
- (2) *$\mathfrak{h}(\mathcal{F}') = \mathfrak{h}(\mathcal{F}) + 1$,*
- (3) *the substair $\Gamma' \subset L$ has a vertical right cut.*

In particular, \mathcal{F} is the G -constellation next to \mathcal{F}' in \mathcal{M}_C in the sense of Remark 4.1.

4.3. Counting the chambers.

Remark 4.16. Propositions 4.13 and 4.15 provide a way to build 1-dimensional families of nilpotent G -constellations. In particular, each of this families corresponds to some exceptional line in some \mathcal{M}_C . Moreover, the two gluings described in the definition of linking stair are nothing but the two possible ways of deforming a toric G -constellation keeping the property of being nilpotent described in Remark 4.2. This implies that the families coming from Proposition 4.13 and Proposition 4.15 are exactly the 1-dimensional families of nilpotent G -constellations appearing in the moduli spaces \mathcal{M}_C .

An easy combinatorial computation tells us that the maximum number of chambers is $k!$. Indeed, if we start by a G -constellation \mathcal{F}_1 of maximum height $\mathfrak{h}(\mathcal{F}) = k$, i.e. \mathcal{F}_1 has one of the k abstract G -stairs shown in Figure 12, we can construct toric G -constellations $\mathcal{F}_2, \dots, \mathcal{F}_k$

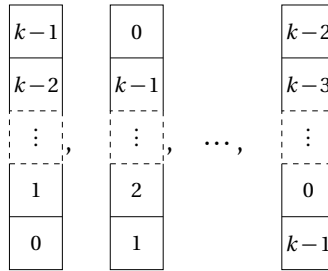


FIGURE 12. The abstract G -stairs of maximum height.

with respective abstract G -stairs Γ_j for $j = 2, \dots, k$ by recursively applying the prescriptions in Proposition 4.13. Precisely, for any $j > 1$, each Γ_j is a connected substair, with horizontal left cut, of the decreasing linking stair of Γ_{j-1} .

To conclude that the maximum number of chambers is $k!$, we notice that the j -th time that we apply Proposition 4.13 there are $k - j$ possible G -stairs with horizontal left cut in the decreasing linking stair of the abstract G -stair of \mathcal{F}_j .

Theorem 4.17. *If $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} is the disjoint union of $k!$ chambers.*

Proof. It is enough to show that, if $\mathcal{F}_1, \dots, \mathcal{F}_k$ are as in Remark 4.16, then there exists a chamber

$$C = \Theta_{\mathcal{F}_1} \cap \Theta_{\mathcal{F}_2} \cap \dots \cap \Theta_{\mathcal{F}_k} \neq \emptyset,$$

such that \mathcal{F}_j is C -stable for all $j = 1, \dots, k$. We claim that, if, for all $j = 1, \dots, k$, the favorite condition of \mathcal{F}_j is $\theta_{\mathcal{F}_j}$, then

$$\theta = \sum_{j=1}^k \theta_{\mathcal{F}_j} \in C.$$

A priori, in order to prove the claim, we need to show both that θ is generic and that every \mathcal{F}_j is θ -stable. In fact, it is enough to show just that every \mathcal{F}_j is θ -stable, because this implies that \mathcal{M}_θ has k torus fixed-points and, as a consequence, that θ is generic.

Let $\mathcal{E}_j < \mathcal{F}_j$ be a proper G -equivariant indecomposable $\mathbb{C}[x, y]$ -submodule of \mathcal{F}_j with substair $\Gamma_{\mathcal{E}_j} \subset \Gamma_{\mathcal{F}_j}$. Suppose also that $\mathcal{E}_j = \bigoplus_{s=m}^n \rho_s$, where $0 \leq m \leq n \leq k-1$. We denote by \mathcal{E}_i , for $i = 1, \dots, j-1, j+1, \dots, k$, the subrepresentation of \mathcal{F}_i corresponding to \mathcal{E}_j , i.e.

$$\mathcal{E}_i = \bigoplus_{s=m}^n \rho_s, \quad \forall i = 1, \dots, j-1, j+1, \dots, k.$$

Notice that

- if $\Gamma_{\mathcal{E}_{j+1}}$ has two vertical cuts, then $\Gamma_{\mathcal{E}_i}$ has two vertical cuts for every $i > j+1$;
- if $\Gamma_{\mathcal{E}_{j-1}}$ has two horizontal cuts, then $\Gamma_{\mathcal{E}_i}$ has two horizontal cuts for every $i < j-1$.

This is true because every time we increase (resp. decrease) the index i , we perform a horizontal left (resp. vertical right) cut in the decreasing (resp. increasing) linking stair which does not affect the vertical left (resp. horizontal right) cut of $\Gamma_{\mathcal{E}_{j+1}}$ (resp. $\Gamma_{\mathcal{E}_{j-1}}$).

Hence, for all $i = 1, \dots, j-1, j+1, \dots, k$, we have $\theta_{\mathcal{F}_i}(\mathcal{E}_j) \geq 0$ and, as a consequence

$$\theta(\mathcal{E}_j) = \left(\theta_{\mathcal{F}_j} + \sum_{i \neq j} \theta_{\mathcal{F}_i} \right) (\mathcal{E}_j) > 0.$$

□

Remark 4.18. The proof of Theorem 4.17 provides an alternative way to prove that

$$\Theta_{\mathcal{F}} \neq \emptyset$$

in Remark 4.10 and, that

$$\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$$

in the last part of the third point of the proof of Proposition 4.13.

For example, let \mathcal{F} be a toric G -constellation with abstract G -stair of height $\mathfrak{h}(\mathcal{F}) = j$. We construct $\mathcal{F}_1, \dots, \mathcal{F}_{j-1}, \mathcal{F}_{j+1}, \dots, \mathcal{F}_k$ by recursively applying Propositions 4.13 and 4.15, i.e.

- if $i > j$, then \mathcal{F}_i has, as G -stair, a G -substair, with a horizontal left cut, of the decreasing linking stair of \mathcal{F}_{i-1} ,
- if $i < j$, then \mathcal{F}_i has, as G -stair, a G -substair, with a vertical right cut, of the increasing linking stair of \mathcal{F}_{i+1} .

Then, if $\theta = \theta_{\mathcal{F}} + \sum \theta_{\mathcal{F}_i}$ is the sum of all favorite conditions, we have $\theta \in \Theta_{\mathcal{F}}$.

5. SIMPLE CHAMBERS

In this section we firstly introduce the notion of chamber stair. Roughly speaking, it is a stair that encodes all the data needed to reconstruct a chamber. Then, we define simple chambers, which are a particular kind of chambers with the property that any toric G -constellation belongs to at least one of them. Finally, we prove that there are exactly $k \cdot 2^{k-2}$ simple chambers.

Remark 5.1. Given a chamber $C \subset \Theta^{\text{gen}}$ we can make a stair Γ_C out of it and we say that Γ_C is the chamber stair of C .

Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be the toric G -constellations in \mathcal{M}_C . As explained in Proposition 4.13 (resp. Proposition 4.15), the abstract G -stairs Γ_j, Γ_{j+1} of two consecutive G -constellations $\mathcal{F}_j, \mathcal{F}_{j+1}$ are substairs of the same stair L , namely the decreasing linking stair of Γ_j (resp. the increasing linking stair of Γ_{j+1}). Moreover they have non-empty intersection in L .

Now, if $\Gamma_1, \dots, \Gamma_k$ are the respective abstract G -stairs of $\mathcal{F}_1, \dots, \mathcal{F}_k$, we can construct a new abstract stair Γ_C by gluing consecutive abstract G -stairs along their common parts.

Definition 5.2. The *abstract chamber stair of C* or the *abstract C -stair* is the abstract stair Γ_C obtained as described above.

Example 5.3. Consider the case $G \cong \mathbb{Z}/5\mathbb{Z}$. Figure 13 explains how to build an abstract C -stair starting from the abstract G -stairs of the G -constellations in some chamber C . In particular,

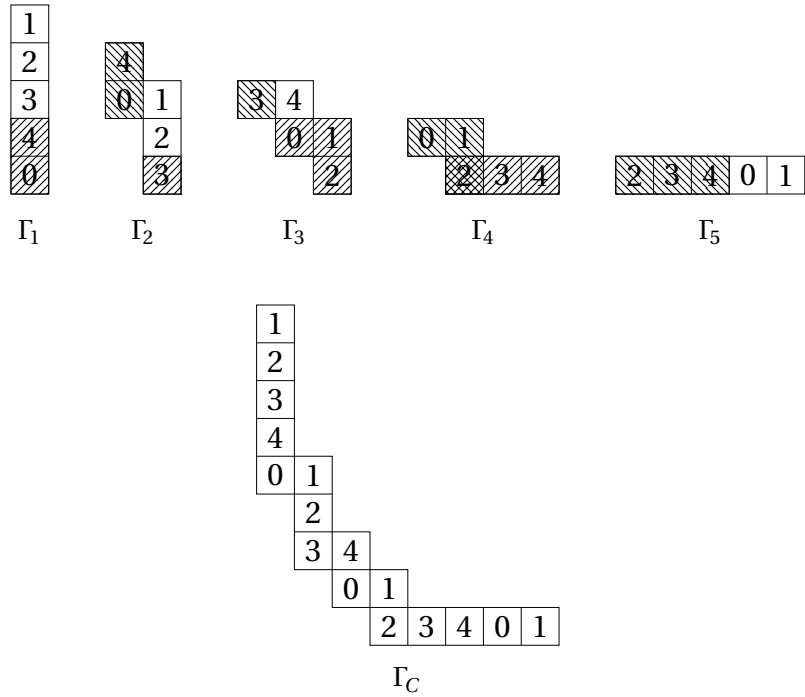


FIGURE 13. The abstract C -stair Γ_C is obtained by gluing, along their common part, the abstract $\mathbb{Z}/5\mathbb{Z}$ -stairs Γ_i and Γ_{i+1} for $i = 1, \dots, 4$.

we have glued the boxes of an abstract G -stair with the boxes of the next abstract G -stair.

Definition 5.4. A *chamber stair associated to C* or a *C -stair* is any realization $\tilde{\Gamma}_C$ of the abstract chamber stair Γ_C associated to C as a subset of the representation tableau.

Remark 5.5. Let $C \in \Theta^{\text{gen}}$ be a chamber and let $\Gamma_C \in \mathcal{T}_G$ be a C -stair. Consider a G -stair $\Gamma \subset \Gamma_C$ of width $\mathfrak{w}(\Gamma) = j$ and the associated G -constellation \mathcal{F}_Γ . Let us also denote by $b, b' \in \Gamma$ the first and the last box of Γ . Suppose that \mathcal{F}_Γ is not C -stable. Then, there are two consecutive C -stable G -constellations \mathcal{F} and \mathcal{F}' with associated respective G -stairs $\Gamma_{\mathcal{F}}, \Gamma_{\mathcal{F}'} \subset \Gamma_C$ such that $b \in \Gamma_{\mathcal{F}}$ and $b' \in \Gamma_{\mathcal{F}'}$.

Therefore, Γ is a substair of both the decreasing linking stair L of $\Gamma_{\mathcal{F}}$ and the increasing linking stair L' of $\Gamma_{\mathcal{F}'}$. In particular, as a consequence of Propositions 4.13 and Proposition 4.15, one and only one between the following two possibilities must occur, namely:

$$(5.1) \quad \begin{aligned} & \mathfrak{w}(\mathcal{F}) = j - 1, \mathfrak{w}(\mathcal{F}') = j, \text{ and } b \text{ (resp. } b') \text{ is a left (resp. right) horizontal cut of } \Gamma \text{ in } L, \\ & \mathfrak{w}(\mathcal{F}) = j, \mathfrak{w}(\mathcal{F}') = j + 1, \text{ and } b \text{ (resp. } b') \text{ is a right (resp. left) vertical cut of } \Gamma \text{ in } L. \end{aligned}$$

On the other hand, again as a consequence of Proposition 4.13 and Proposition 4.15, if \mathcal{F}_Γ is C -stable, none of the conditions in (5.1) can hold true, and in this case Γ has horizontal left cut and vertical right cut in Γ_C .

Summing up, if $\Gamma \subset \Gamma_C$ is a connected G -substair associated to a toric G -constellation \mathcal{F}_Γ then only the following two cases can occur:

- the G -constellation \mathcal{F}_Γ is C -stable and Γ has a horizontal left cut and a vertical right cut, or
- the G -constellation \mathcal{F}_Γ is not C -stable and Γ has two horizontal cuts or two vertical cuts.

Remark 5.6. Different chambers have different abstract chamber stairs.

First, recall from Remark 5.5 that, as per Proposition 4.13, the G -stair of any toric C -stable G -constellation has a vertical right cut in the C -stair and a horizontal right cut in the decreasing linking stair of the previous G -constellation.

Suppose that two chambers C and C' have the same abstract chamber stair Γ . In particular, from the construction of abstract chamber stairs, it follows that C and C' have the same first (in the sense of Remark 4.1) toric G -constellation. Suppose that C and C' differ for the j -th toric G -constellation. This translates into the fact that, if \mathcal{F}_j and \mathcal{F}'_j are the respective j -th G -constellations of C and C' and Γ_j, Γ'_j are their abstract G -stairs, then

$$\Gamma_j \neq \Gamma'_j.$$

Let us denote by \mathcal{F}_{j-1} the $(j-1)$ -th toric G -constellation of C (and C') and by Γ_{j-1} its abstract G -stair. Then, both Γ_j and Γ'_j are substairs of the decreasing linking stair L_{j-1} of Γ_{j-1} and they have horizontal right cut in L_{j-1} as noticed above. Since, Γ_{j-1}, Γ_j and Γ'_j are connected and $\Gamma_{j-1} \cap \Gamma_j, \Gamma_{j-1} \cap \Gamma'_j \neq \emptyset$ in L_{j-1} , it follows that:

$$\Gamma_{j-1} \cup \Gamma_j \subsetneq \Gamma_{j-1} \cup \Gamma'_j \text{ or } \Gamma_{j-1} \cup \Gamma_j \supsetneq \Gamma_{j-1} \cup \Gamma'_j.$$

Finally, if without loss of generality we suppose

$$\Gamma_{j-1} \cup \Gamma_j \subsetneq \Gamma_{j-1} \cup \Gamma'_j \subset \Gamma,$$

then we get a contradiction. Indeed, as noticed at the beginning, Γ_j has a vertical right cut in Γ , but it has to have a horizontal right cut in $\Gamma_{j-1} \cup \Gamma'_j$ because it is a connected substair of L_{j-1} which strictly contains Γ_j .

Remark 5.7. Since the abstract chamber stair Γ_C of a chamber C contains a copy of the abstract G -stairs of the toric C -stable G -constellations, we will think of such abstract G -stairs as substairs of Γ_C .

Similarly, given a C -stair $\tilde{\Gamma}_C \subset \mathcal{T}_G$ which realize Γ_C , we will realize the abstract G -stairs associated to the G -constellations in C as substairs of $\tilde{\Gamma}_C$.

Definition 5.8. Given a chamber C , we say that a toric C -stable G -constellation is C -characteristic if its abstract G -stair has the same generators as the abstract C -stair, see Definition 3.19.

We say that a chamber C is *simple* if there is a toric C -stable G -constellation whose abstract G -stair has the same generators of the abstract C -stair, i.e. if there exists at least one C -characteristic G -constellation.

Example 5.9. An example of a simple chamber is given by the chamber C_G in Theorem 1.14, i.e. the chamber whose associated moduli space is G -Hilb(\mathbb{A}^2). In particular, the abstract C_G -stair has only one generator, namely ρ_0 .

Definition 5.10. Let Γ be a G -stair and let ρ_i and ρ_j be its first and its last generators.

- The *left tail* of Γ is the substair of Γ given by

$$\text{lt}(\Gamma) = \{ y^s \cdot \rho_i \mid s > 0 \}.$$

- The *right tail* of Γ is the substair of Γ given by

$$\text{rt}(\Gamma) = \{ x^s \cdot \rho_j \mid s > 0 \}.$$

- The *tail* of Γ is the substair of Γ given by

$$\text{t}(\Gamma) = \text{lt}(\Gamma) \cup \text{rt}(\Gamma).$$

Similarly one can define left/right tails for abstract G -stairs.

Remark 5.11. If two G -stairs Γ and Γ' have the same generators, then they differ by their tails, i.e. the following equality of subsets of the representation tableau holds true:

$$\Gamma \setminus \text{t}(\Gamma) = \Gamma' \setminus \text{t}(\Gamma')$$

In particular, if a G -stair Γ has a tail of cardinality m , then there are $m + 1$ G -stairs with the same generators as Γ .

In simple words, the other G -stairs are obtained by moving some boxes from the left tail to the right tail (and viceversa) of Γ .

Proposition 5.12. *The following properties are true.*

- (1) *Any toric G -constellation is C -stable for some simple chamber C .*
- (2) *Given a simple chamber C , and a C -characteristic G -constellation \mathcal{F} , there is an algorithm to produce all the toric C -stable constellations.*
- (3) *If C is a simple chamber, all the toric G -constellations that admit a G -stair with the same generators as the C -stair belong to C , i.e. they are C -stable. In particular, they are C -characteristic.*

Proof. Let Γ_C be the abstract C -stair. We prove the first two points in a constructive way. In order to do so, we show that, given a toric G -constellation \mathcal{F} , there is a unique simple chamber C such that \mathcal{F} is C -characteristic.

Let \mathcal{F} be a toric G -constellation with associated abstract G -stair $\Gamma_{\mathcal{F}}$ of height $\mathfrak{h}(\mathcal{F}) = j$. In order to build a chamber starting from \mathcal{F} , we have to first recursively apply Propositions 4.13 and 4.15 $j - 1$ times and $k - j$ times respectively, to obtain k toric constellations

$$\mathcal{F}_1, \dots, \mathcal{F}_{j-1}, \mathcal{F}, \mathcal{F}_{j+1}, \dots, \mathcal{F}_k$$

and, finally, apply Theorem 4.17 to conclude that there exists a chamber C such that the constellations $\mathcal{F}_1, \dots, \mathcal{F}_{j-1}, \mathcal{F}, \mathcal{F}_{j+1}, \dots, \mathcal{F}_k$ correspond to the toric points of \mathcal{M}_C .

The condition that the chamber must be simple translates into the fact that, at every step, no new generators appear. This may be only achieved by performing, every time that we apply Proposition 4.13 (resp. Proposition 4.15), the first (resp. last) possible horizontal (resp. vertical) cut in the decreasing (resp. increasing) linking stair.

In order to prove the last point, we start by considering a G -constellation \mathcal{F} whose abstract G -stair $\Gamma_{\mathcal{F}}$ has the same generators as the C -stair and such that it has empty right tail, i.e. $\mathfrak{t}(\Gamma_{\mathcal{F}}) = \mathfrak{t}(\Gamma_C)$.

Let $m = \#\mathfrak{t}(\Gamma_{\mathcal{F}})$ be the cardinality of the left tail of $\Gamma_{\mathcal{F}}$. The first m times we apply Proposition 4.13 by performing the first possible horizontal cut we increase the cardinality of $\mathfrak{r}(\Gamma_{\mathcal{F}})$ by 1 and, consequently, we decrease the cardinality of $\mathfrak{t}(\Gamma_{\mathcal{F}})$ by 1. In this way we find, as explained in Remark 5.11, all the toric G -constellations which admit a G -stair with the same generators as the C -stair and all of them are C -stable by Theorem 4.17. \square

Lemma 5.13. *Let Γ be a G -stair. Then Γ has at most*

$$\left\lfloor \frac{k+1}{2} \right\rfloor$$

generators.

Proof. The statement follows from the following observation. If a stair has r generators, then it has at least $2r - 1$ boxes, as shown in Figure 14.

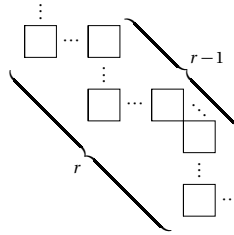


FIGURE 14.

Now, a G -stair has exactly k boxes. Hence,

$$r \leq \left\lfloor \frac{k+1}{2} \right\rfloor.$$

\square

Example 5.14. Non-simple chambers exist.

As already mentioned in Theorem 1.14, there is a chamber C_G such that $G\text{-Hilb}(\mathbb{A}^2) \cong \mathcal{M}_{C_G}$ as moduli spaces. In particular,

$$C_G \subset \{ \theta \in \Theta \mid \theta_0 < 0, \theta_i > 0 \forall i = 1, \dots, k-1 \},$$

and the abstract G -stairs of its toric constellations are shown in Figure 15.

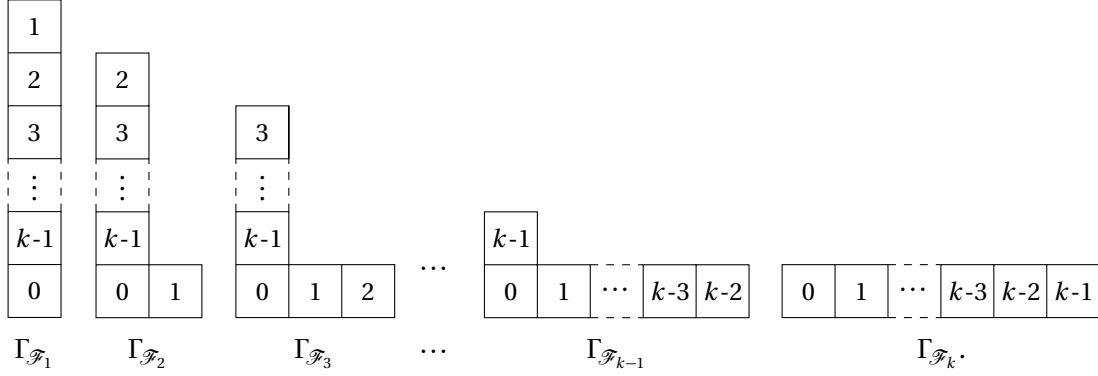


FIGURE 15. The abstract G -stairs of the C_G -stable toric G -constellations.

Notice that, for $i = 1, \dots, k$ and $j = 0, \dots, k-1$, the favorite conditions $\theta_{\mathcal{F}_i}$ are defined by

$$(\theta_{\mathcal{F}_i})_j = \begin{cases} -2 & \text{if } j = 0 \text{ \& } i \neq 1, k, \\ -1 & \text{if } j = 0 \text{ \& } (i = 1 \text{ or } i = k), \\ 1 & \text{if } j = i-1 \neq 0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

and that the condition

$$\theta = \sum_{i=1}^k \theta_{\mathcal{F}_i} = (-2k+2, \underbrace{2, \dots, 2}_{k-1})$$

belongs to C_G . More precisely, the moduli space $G\text{-Hilb}(\mathbb{A}^2)$ parametrises all the toric G -constellations generated by the trivial representation. As a consequence, the abstract G -stairs $\Gamma_{\mathcal{F}_i}$, for $i = 1, \dots, k$, have as only generator the trivial representation.

Let us reverse this property by asking the presence of just one antigeenerator, for example, the trivial representation. It is easy to see that there is a chamber C_G^{OP} whose toric G -constellations, as requested, have the abstract G -stairs in Figure 16. In particular,

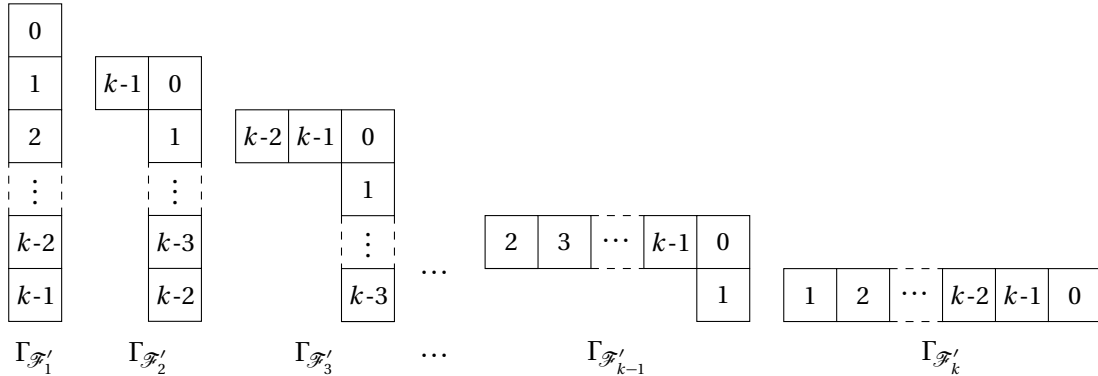


FIGURE 16. The abstract G -stairs of the C_G^{OP} -stable toric G -constellations.

$$C_G^{\text{OP}} \subset \{\theta \in \Theta \mid \theta_0 > 0, \theta_i < 0 \forall i = 1, \dots, k-1\}.$$

We denote the associated moduli space by

$$G\text{-Hilb}^{\text{OP}}(\mathbb{A}^2) := \mathcal{M}_{C_G^{\text{OP}}}.$$

Notice that, while $C_{\mathbb{Z}/3\mathbb{Z}}^{\text{OP}}$ is simple, $C_{\mathbb{Z}/k\mathbb{Z}}^{\text{OP}}$ is not simple for $k \geq 4$ because the number of generators of the $C_{\mathbb{Z}/k\mathbb{Z}}^{\text{OP}}$ -stair is

$$k-1 > \left\lfloor \frac{k+1}{2} \right\rfloor \quad \forall k \geq 4.$$

Therefore, as a consequence of Lemma 5.13, there is no $C_{\mathbb{Z}/k\mathbb{Z}}^{\text{OP}}$ -characteristic G -constellation.

We show, as an example, the abstract chamber stairs of C_G and C_G^{OP} in the case $k = 5$.

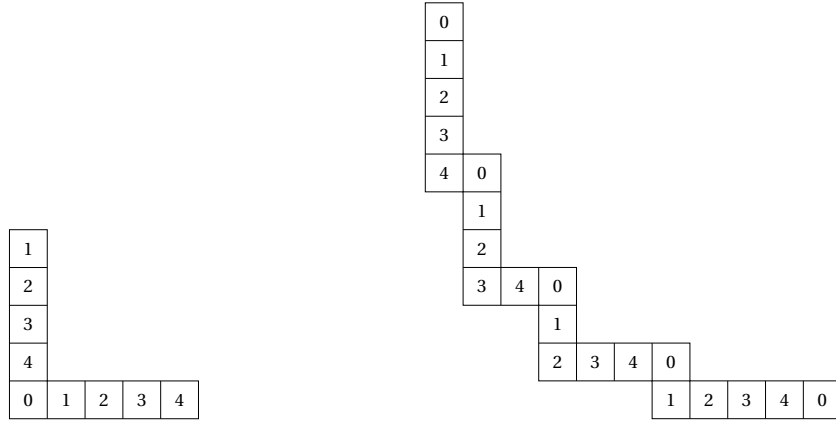


FIGURE 17. The abstract $C_{\mathbb{Z}/5\mathbb{Z}}$ -stair and the abstract $C_{\mathbb{Z}/5\mathbb{Z}}^{\text{OP}}$ -stair.

Theorem 5.15. *If $G \subset \text{SL}(2, \mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} contains $k \cdot 2^{k-2}$ simple chambers.*

Proof. Let \mathcal{B} be the set of possible sets of generators for a G -stair, i.e.

$$\mathcal{B} = \{ A \subset \mathcal{T}_G \mid \text{there exists a } G\text{-stair whose generators are the elements in } A \},$$

and let \mathcal{G} be the set of all G -stairs

$$\mathcal{G} = \{ \Gamma \subset \mathcal{T}_G \mid \Gamma \text{ is a } G\text{-stair} \}.$$

Consider the subsemigroup Z of \mathcal{T}_G

$$Z = \{ (\alpha k + \gamma, \beta k + \gamma, \rho_0) \in \mathcal{T}_G \mid \alpha, \beta, \gamma \geq 0 \}.$$

We denote by $\overline{\mathcal{B}}$ and $\overline{\mathcal{G}}$ the set of equivalence classes

$$\overline{\mathcal{B}} = \mathcal{B} / \sim_Z, \text{ and } \overline{\mathcal{G}} = \mathcal{G} / \sim_Z$$

where, if $A_1, A_2 \in \mathcal{B}$ (resp. $\Gamma_1, \Gamma_2 \in \mathcal{G}$), then $A_1 \sim_Z A_2$ (resp. $\Gamma_1 \sim_Z \Gamma_2$) if there exist $z \in Z$ such that

$$A_1 = A_2 + z \text{ or } A_2 = A_1 + z \quad (\text{resp. } \Gamma_1 = \Gamma_2 + z \text{ or } \Gamma_2 = \Gamma_1 + z).$$

Notice that, if two G -stairs are \sim_Z -equivalent also their sets of generators are \sim_Z -equivalent. However, the contrary is not true.

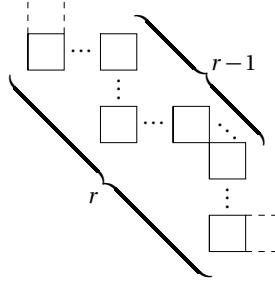


FIGURE 18.

Now, the number of simple chambers equals the cardinality of $\overline{\mathcal{B}}$. Indeed, Proposition 5.12 implies that the chamber C is uniquely determined by a constellation \mathcal{F} whose G -stair is C -characteristic. More precisely, C is uniquely determined by the generators of any characteristic C -stair $\Gamma_{\mathcal{F}}$. Although there are infinitely many G -stairs corresponding to \mathcal{F} , Remark 3.18 tells us that two G -stairs correspond to the same G -constellation if and only if they differ by an element in Z , i.e. they are \sim_Z -equivalent.

Let \mathcal{G}_r be the set of G -stairs with r generators and let $\overline{\mathcal{G}}_r = \mathcal{G}_r / \sim_Z$ be the induced quotient. We have a surjective map

$$\Psi : \mathcal{G} \rightarrow \mathcal{B}$$

which associates to each G -stair its set of generators, and this map descends to the sets of equivalence classes

$$\overline{\Psi} : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{B}},$$

because \sim_Z -equivalent G -stairs correspond to \sim_Z -equivalent sets of generators.

Now, $\overline{\mathcal{B}}$ decomposes as a disjoint union (see Lemma 5.13) as follows:

$$\overline{\mathcal{B}} = \bigsqcup_{r=1}^{\lfloor \frac{k+1}{2} \rfloor} \overline{\Psi}(\overline{\mathcal{G}}_r).$$

Our strategy is to compute $\overline{\Psi}(\overline{\mathcal{G}}_r)$ for every $1 \leq r \leq \lfloor \frac{k+1}{2} \rfloor$ and then sum over all r . For $r = 1$ we have $|\overline{\Psi}(\overline{\mathcal{G}}_1)| = k$. If we impose the presence of $r \geq 2$ generators and of a tail of cardinality j then there are

$$k \cdot \binom{k-2-j}{2r-3}$$

elements in $\overline{\Psi}(\overline{\mathcal{G}}_r)$ which comes from G -stairs with a tail of cardinality j . Indeed, as shown in Figure 18, we have $2r - 1$ fixed boxes (generators and anti-generators), j boxes contained in the tails (dashed areas) and $k - 2r + 1 - j$ boxes left to arrange in $2r - 2$ places (dotted areas). The number of possible ways to arrange the boxes is computed via the stars and bars method¹. In particular, there are

$$\binom{(2r-2) + (k-2r+1-j) - 1}{k-2r+1-j} = \binom{k-2-j}{2r-3}$$

of them.

¹In a more suggestive way, one can say “combinations with repetition of $2r - 2$ elements of class $k - 2r + 1 - j$ ”.

Finally, if we sum over all possible r and j , we get

$$k \cdot \left[1 + \sum_{r=2}^{\lfloor \frac{k+1}{2} \rfloor} \sum_{j=0}^{k-2r+1} \binom{k-2-j}{2r-3} \right] = k \cdot 2^{k-2}.$$

□

Remark 5.16. An easy combinatorial computation shows that the set $\overline{\mathcal{G}}$ in the proof of Theorem 5.15 has cardinality $k \cdot 2^{k-1}$, i.e. that there are exactly $k \cdot 2^{k-1}$ isomorphism classes of toric G -constellations.

We conclude this section with two examples which help to understand the notions just introduced.

Example 5.17. In this example we treat the case $G \cong \mathbb{Z}/5\mathbb{Z}$.

The following picture contains a list of the possible shapes of the abstract chamber stairs of simple chambers and, in each case, the shapes of the G -stairs associated to the toric G -constellations belonging to the respective simple chamber.

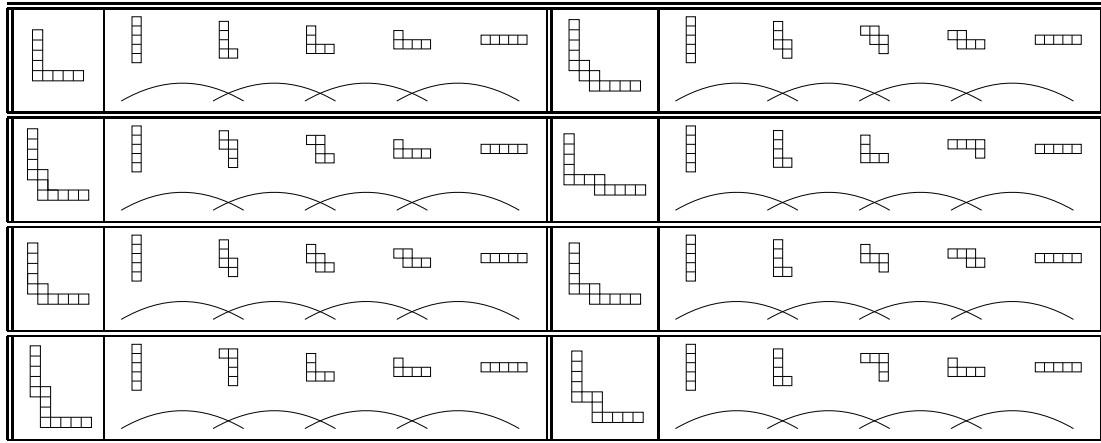
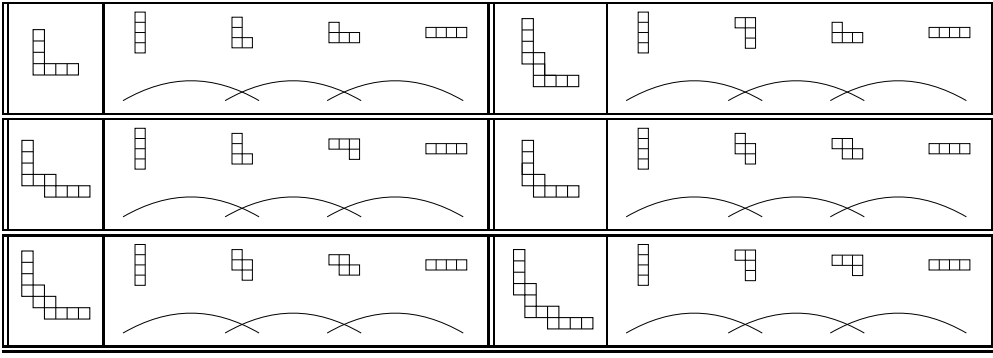


FIGURE 19. Description of the simple chambers for the action of $\mathbb{Z}/5\mathbb{Z}$.

As predicted by Theorem 5.15, the possible shapes for the chamber stairs of simple chambers are $8 = 2^{5-2}$, and there are 5 different ways to label each chamber stair.

Example 5.18. In this example we treat the case $G \cong \mathbb{Z}/4\mathbb{Z}$.

The following picture contains a list of the possible shapes of the abstract chamber stairs and, in each case, the shapes of the G -stairs associated to the toric G -constellations belonging to the respective chamber.


 FIGURE 20. Description of the chambers for the action of $\mathbb{Z}/4\mathbb{Z}$.

Notice that the first $4 = 2^{4-2}$ pictures correspond to simple chambers. Moreover, as predicted by Theorem 4.17, the possible shapes for the chamber stairs are $6 = (4-1)!$, and there are 4 different ways to label each chamber stair.

Note also that, after having labeled each box appropriately, the first and last chambers in Figure 20 correspond to C_G and C_G^{OP} respectively (see Example 5.14).

6. THE COSTRUCTION OF THE TAUTOLOGICAL BUNDLES \mathcal{R}_C

The quasi projective variety \mathcal{M}_C is a fine moduli space obtained by GIT as described in [18] by King. In particular, there exists a universal family $\mathcal{U}_C \in \text{Ob Coh}(\mathcal{M}_C \times \mathbb{A}^2)$. The tautological bundle is the pushforward

$$\mathcal{R}_C = (\pi_{\mathcal{M}_C})_*(\mathcal{U}_C).$$

It is a vector bundle of rank $k = |G|$ whose fibers are G -constellations and, more precisely, over each point $[\mathcal{F}] \in \mathcal{M}_C$ the fiber $(\mathcal{R}_C)_{[\mathcal{F}]}$ is canonically isomorphic to the space of global sections $H^0(\mathbb{A}^2, \mathcal{F})$.

In this section we give an explicit construction of the tautological bundles \mathcal{R}_C for all chambers $C \subset \Theta^{\text{gen}}$ in terms of their chamber stairs. We will adopt the same notation as in Section 4.1.

Notation 6.1. From now on, given a coherent monomial ideal sheaf $\mathcal{K} \subset \mathcal{O}_{\mathbb{A}^2}$, we denote by $\widetilde{\mathcal{K}}$ the \mathcal{O}_Y -module defined by

$$\widetilde{\mathcal{K}} = \varepsilon^* \pi_* \mathcal{K} / \text{Tor}_{\mathcal{O}_Y} \varepsilon^* \pi_* \mathcal{K}.$$

Lemma 6.2. Suppose that \mathcal{K} is generated by the monomials $x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s}$. Then, on each toric chart $U_j \subset Y$ with coordinates (a_j, c_j) , the sheaf $\widetilde{\mathcal{K}}$ agrees with the sheaf \mathcal{H}_j associated to the $\mathbb{C}[a_j, c_j]$ -module:

$$H_j = \frac{K_j}{K_j \cap I_j} \subset \frac{\mathbb{C}[a_j, c_j, x, y]}{K_j \cap I_j},$$

where K_j and I_j are the ideals of $\mathbb{C}[a_j, c_j, x, y]$ given by

$$K_j = (x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s})$$

and

$$I_j = (a_j y^{k-j} - x^j, c_j x^{j-1} - y^{k-j+1}, a_j c_j - x y),$$

and the gluings on the intersections $U_i \cap U_j$, for $1 \leq i, j \leq k$, are given by:

$$\begin{aligned} \Gamma(U_i \cap U_j, \mathcal{H}_i) &\xrightarrow{\varphi_{ij}} \Gamma(U_i \cap U_j, \mathcal{H}_j) \\ x &\longmapsto x, \\ y &\longmapsto y, \\ a_i &\longmapsto a_j^{i-j+1} c_j^{i-j}, \\ c_i &\longmapsto a_j^{j-i} c_j^{j-i+1}. \end{aligned}$$

Proof. The proof is achieved by direct computation, after noticing that the gluings on the intersections are deduced from the toric description of the toric quasiprojective variety \mathcal{M}_C given at the beginning of Section 4.1 and, in particular, from Equations (4.1). \square

Remark 6.3. Using the the relations given in (4.1), the modules H_j , for $j = 1, \dots, k$, can be regarded as $\mathbb{C}[a_j, c_j]$ -submodules of the rational function field $\mathbb{C}(x, y)$.

Remark 6.4. If $x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s}$ are the generators of some C -stair Γ_C and \mathcal{K} is defined as in Lemma 6.2, all the G -sFd associated to the toric fibers of $\widetilde{\mathcal{K}}$ are substairs of Γ_C . This is a consequence of Nakayama's Lemma together with the following three facts:

$$(6.1) \quad \begin{aligned} \forall j = 1, \dots, k, \forall i = 1, \dots, s \quad &x^{\alpha_i+1} y^{\beta_i+1} \in (K_j \cap I_j) + (a_j, c_j), \\ \forall j = 1, \dots, k, \quad &x^{\alpha_1} y^{\beta_1+k} \in (K_j \cap I_j) + (a_j, c_j), \\ \forall j = 1, \dots, k, \quad &x^{\alpha_s+k} y^{\beta_s} \in (K_j \cap I_j) + (a_j, c_j). \end{aligned}$$

The relations (6.1) follow from the easy observations that

$$\begin{aligned} x^{\alpha_i} y^{\beta_i} \cdot (a_j c_j - x y) &= a_j c_j x^{\alpha_i} y^{\beta_i} - x^{\alpha_i+1} y^{\beta_i+1} \in K_j \cap I_j, \\ y^{j-1} \cdot x^{\alpha_1} y^{\beta_1} \cdot (c_j x^{j-1} - y^{k-j+1}) &= c_j x^{\alpha_1+j-1} y^{\beta_1+j-1} - x^{\alpha_1} y^{\beta_1+k} \in K_j \cap I_j, \\ x^{k-j} \cdot x^{\alpha_s} y^{\beta_s} \cdot (a_j y^{k-j} - x^j) &= a_j x^{\alpha_s+k-j} y^{\beta_s+k-j} - x^{\alpha_s+k} y^{\beta_s} \in K_j \cap I_j. \end{aligned}$$

In this last part of the paper we state and prove the last main theorem. Before to give the proof, we also state and prove some corollaries and results needed in the proof.

Theorem 6.5. *Let $C \subset \Theta^{\text{gen}}$ be a chamber and let $\Gamma_C \subset \mathcal{T}_G$ be a C -stair. Suppose that Γ_C has $s \geq 1$ ordered (see Remark 3.20) generators v_1, \dots, v_s with associated monomials*

$$x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s} \in \mathbb{C}[x, y].$$

Consider the ideal sheaf $\mathcal{K} = (x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s}) \mathcal{O}_{\mathbb{A}^2}$, then

$$\mathcal{R}_C \cong \varepsilon^* \pi_* \mathcal{K} / \text{Tor}_{\mathcal{O}_{\mathcal{M}_C}}(\varepsilon^* \pi_* \mathcal{K}).$$

The following corollaries are direct consequences of Theorem 6.5.

Corollary 6.6. *On each toric chart $U_j \subset \mathcal{M}_C$ with coordinates (a_j, c_j) , the tautological bundle $\mathcal{R}_C|_{U_j}$ agrees with the sheaf \mathcal{H}_j associated to the $\mathbb{C}[a_j, c_j]$ -module H_j in Lemma 6.2.*

Corollary 6.7. *In the hypotheses of Theorem 6.5, the \mathcal{O}_Y -module $\widetilde{\mathcal{K}}$ is locally free of rank $|G|$.*

Corollary 6.8. *Let C and \mathcal{K} be as in Theorem 6.5. Then, \mathcal{M}_C can be identified with a closed G -invariant subvariety of $\text{Quot}_{\mathcal{K}}^{[G]}(\mathbb{A}^2)$.*

Remark 6.9. Corollary 6.7 is a generalisation of [13, Proposition 2.4.] in the abelian setting.

Remark 6.10. For the trivial ideal $K = (1) = \mathbb{C}[x, y]$ Corollary 6.6 recovers Nakamura's description of the G -Hilbert scheme when G is abelian (see [22]).

Remark 6.11. Notice that, over the origin of the first and the last charts, the \mathcal{O}_{U_1} -module \mathcal{H}_1 and the \mathcal{O}_{U_k} -module \mathcal{H}_k have, as toric fibers, the expected G -constellations \mathcal{F}_1 and \mathcal{F}_k , i.e

$$\mathcal{F}_1 \cong \mathcal{H}_{10_1} \cong \frac{(x^{\alpha_1} y^{\beta_1})}{(x^{\alpha_1} y^{\beta_1+k}, x^{\alpha_1+1} y^{\beta_1})} \subset \frac{\mathbb{C}[x, y]}{(x^{\alpha_1} y^{\beta_1+k}, x^{\alpha_1+1} y^{\beta_1})}$$

and

$$\mathcal{F}_k \cong \mathcal{H}_{k0_k} \cong \frac{(x^{\alpha_s} y^{\beta_s})}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1})} \subset \frac{\mathbb{C}[x, y]}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1})},$$

where $0_i \in U_i$ is, for $i = 1, k$, the origin.

We prove this only for the origin of U_k , the other proof is similar. We start by showing that

$$x^{\alpha_i} y^{\beta_i} \in (K_k \cap I_k) + (a_k, c_k) \text{ for } i = 1, \dots, s-1.$$

Notice that, for all $i = 1, \dots, s-1$, we have

$$\alpha_i \geq 0, \quad \beta_i > \beta_{i+1} > \beta_s \geq 0, \quad \alpha_i + k - 1 \geq \alpha_{i+1}.$$

Therefore, we can write:

$$c_k x^{\alpha_i+k-1} y^{\beta_i-1} - x^{\beta_i} y^{\alpha_i} = \begin{cases} c_k x^{\alpha_i+k-1-\alpha_{i+1}} y^{\beta_i-1-\beta_{i+1}} (x^{\alpha_{i+1}} y^{\beta_{i+1}}) - x^{\alpha_i} y^{\beta_i} \\ (x^{\alpha_i} y^{\beta_i-1})(c_k x^{k-1} - y), \end{cases}$$

which implies

$$x^{\alpha_i} y^{\beta_i} \in (K_k \cap I_k) + (a_k, c_k) \forall i = 1, \dots, s-1.$$

Now, we have

$$K_k \cap I_k + (a_k, c_k) = (x^{\alpha_s} y^{\beta_s}) \cap I_k + (a_k, c_k) = (x^{\alpha_s} y^{\beta_s}) \cdot I_k + (a_k, c_k) = (x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k),$$

which gives

$$\mathcal{H}_{k0_k} \cong \frac{(x^{\alpha_s} y^{\beta_s})}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k)} \subset \frac{\mathbb{C}[x, y, a_k, c_k]}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k)}.$$

Definition 6.12. Let $K \subset \mathbb{C}[x, y]$ be the ideal generated by the (ordered) set of monomials

$$\{ x^{\alpha_i} y^{\beta_i} \mid i = 1, \dots, s \}$$

associated to the generators of some chamber stair Γ_C and let $\Gamma_K = \{(m, i) \in \mathcal{T}_C \mid m \in K\}$ be the subset of the representation tableau corresponding to K . Given a monomial $m_b \in K$ corresponding to a box $b \in \Gamma_C \subset \Gamma_K$, we say that:

- the property **(A_j)** holds for m_b (or for b) if

$$x^{-j} y^{k-j} \cdot m_b \in \Gamma_K,$$

- the property **(C_j)** holds for m_b (or for b) if

$$x^{j-1} y^{-k+j-1} \cdot m_b \in \Gamma_K.$$

Lemma 6.13. *If the property **(A_j)** (resp. **(C_j)**) holds for a box $b \in \Gamma_C$ then it holds also for the box after (resp. before) b .*

Proof. Let $m_b = x^\alpha y^\beta$ be the monomial associated to the box b . From Definition 6.12, it follows immediately that, if the property (\mathbf{A}_j) (resp. (\mathbf{C}_j)) holds for b , then it holds for all the monomials $x^\gamma y^\delta$ such that $\gamma \geq \alpha$ and $\delta \geq \beta$. This proves the Lemma in the case in which the box after (resp. before) b is on the right (resp. above) b .

We prove the remaining case for the property (\mathbf{C}_j) and we leave the similar proof for (\mathbf{A}_j) . We have to prove that, if two monomials of the form $x^\alpha y^\beta, x^{\alpha-1} y^\beta$ correspond to some successive boxes in Γ_C and the property (\mathbf{C}_j) holds for $x^\alpha y^\beta$ then it holds also for $x^{\alpha-1} y^\beta$. In other words, we suppose that

$$m_1 = x^{\alpha+j-1} y^{\beta-k+j-1} \in K,$$

and we want to prove that

$$m_2 = x^{\alpha+j-2} y^{\beta-k+j-1} \in K.$$

Let b_1, b_2 be the boxes corresponding to m_1, m_2 and let b be the box corresponding to $x^{\alpha-1} y^\beta$. If $b_1 \in \Gamma_K \setminus \Gamma_C$ it follows easily that $b_2 \in \Gamma_K$. Suppose $b_1 \in \Gamma_C$ and consider the connected substair $\Gamma \subset \Gamma_C$ whose first box is b and whose last box is b_1 . We have, by construction,

$$\mathfrak{w}(\Gamma) = j \text{ and } \mathfrak{h}(\Gamma) = k - j + 2,$$

which imply that Γ contains $k + 1$ boxes.

Let $\Gamma' = \Gamma \setminus \{b_1\}$ be the connected G -substair of Γ_C obtained by removing the last box from Γ and let $b' \in \Gamma_C$ be the last box of Γ' . Now, by construction, b is a vertical left cut for Γ' in Γ_C and, as a consequence of Remark 5.5 also b' is a vertical cut. Therefore b' must correspond to the monomial m_2 from which it directly follows

$$b' = b_2 \in \Gamma_C.$$

Which implies the thesis. \square

Proof. (of Theorem 6.5). If we endow the product $\mathcal{M}_C \times \mathbb{A}^2$ with the G -action defined by

$$\begin{aligned} G \times \mathcal{M}_C \times \mathbb{A}^2 &\longrightarrow \mathcal{M}_C \times \mathbb{A}^2 \\ (g_k^i, p, (x, y)) &\longmapsto (p, (\xi_k^{-i} x, \xi_k^i y)). \end{aligned}$$

where g_k is the (fixed) generator of the cyclic group G (see subsection 2.1), it turns out that the $\mathcal{O}_{\mathcal{M}_C \times \mathbb{A}^2}$ -module $\widetilde{\mathcal{K}}$ is G -equivariant with respect to this action.

To prove the theorem, we use the description of $\widetilde{\mathcal{K}}$ given in Corollary 6.6. We know from Remark 1.6 that the tautological bundles \mathcal{R}_C and \mathcal{R}_{C_G} agree on the complement U_C of the exceptional locus of \mathcal{M}_C . Moreover, we have, as a consequence of the construction of $\widetilde{\mathcal{K}}$ and of Remark 6.10, isomorphisms

$$\mathcal{R}_C|_{U_C} \cong \mathcal{R}_{C_G}|_{U_C} \cong \widetilde{\mathcal{K}}|_{U_C} \cong \mathcal{O}_{U_C}^{\oplus k}.$$

Now we show that the fibers of \mathcal{R}_C and $\widetilde{\mathcal{K}}$ over the toric points of \mathcal{M}_C are the same G -constellations. This will be enough to prove the statement, because each chamber is uniquely identified by its toric G -constellations. We split this part in several steps:

STEP 0 Over each point of $p \in \mathcal{M}_C$ the fiber $\widetilde{\mathcal{K}}_p$ is a G -equivariant $\mathbb{C}[x, y]$ -module and, over each origin $0_j \in U_j$ the fibre $\widetilde{\mathcal{K}}_{0_j}$ is also \mathbb{T}^2 -equivariant. This follows from the fact that the ideal K_j is generated by monomials and that the ideal I_j is generated by G -eigenbinomials (recall that the group G acts trivially on U_j) of positive degrees in the variables a_j, c_j .

STEP 1 All the G -sFd associated to the toric fibers of $\widetilde{\mathcal{H}}$ are substairs of the C -stair Γ_C . For this, see Remark 6.4.

STEP 2 For all $j = 1, \dots, k$, the j -th torus equivariant G -module $\widetilde{\mathcal{H}}_{0_j}$ is indecomposable. Let $\Gamma_j \subset \Gamma_C$ be the G -sFd associated to $\widetilde{\mathcal{H}}_{0_j}$. Then, the G -constellation $\widetilde{\mathcal{H}}_{0_j}$ is indecomposable if and only if Γ_j is connected.

First observe that, for a box $b \in \Gamma_C$ both the properties (\mathbf{A}_j) and (\mathbf{C}_j) implies that the corresponding monomial m_b belongs to $(K_j \cap I_j) + (a_j, c_j)$. This is true because, if $m_b = x^\alpha y^\beta$, then

$$(6.2) \quad \begin{aligned} (\mathbf{A}_j) &\Rightarrow a_j x^{\alpha-j} y^{\beta+k-j} - x^\alpha y^\beta \in K_j \cap I_j, \\ (\mathbf{C}_j) &\Rightarrow c_j x^{\alpha+j-1} y^{\beta-k+j-1} - x^\alpha y^\beta \in K_j \cap I_j. \end{aligned}$$

On the other hand, $b \in \Gamma_C \setminus \Gamma_j$ if and only if $m_b \in (K_j \cap I_j) + (a_j, c_j)$. In particular, by construction, at least one of the following relations is true.

- (1) $a_j x^{\alpha-j} y^{\beta+k-j} - x^\alpha y^\beta \in K_j \cap I_j$,
- (2) $c_j x^{\alpha+j-1} y^{\beta-k+j-1} - x^\alpha y^\beta \in K_j \cap I_j$,
- (3) $a_j c_j x^{\alpha-1} y^{\beta-1} - x^\alpha y^\beta \in K_j \cap I_j$.

Notice that $b \in \Gamma_C$ implies (see STEP 1) that (3) can not hold true. Therefore, given $b \in \Gamma_C$, it belongs to Γ_j if and only if one among the two properties (\mathbf{A}_j) and (\mathbf{C}_j) holds for b . Now, the connectedness of Γ_j is a consequence of Lemma 6.13.

STEP 3 Let, for all $j = 1, \dots, k$, $\mathfrak{m}_j \subset \mathcal{O}_{\mathcal{M}_C, 0_j}$ be the maximal ideal, and let

$$F_j = \widetilde{\mathcal{H}}_{0_j} / \mathfrak{m}_j = \widetilde{\mathcal{H}}_{0_j} \otimes_{\mathcal{O}_{\mathcal{M}_C, 0_j}} (\mathcal{O}_{\mathcal{M}_C, 0_j} / \mathfrak{m}_j)$$

be the fibre of the sheaf $\widetilde{\mathcal{H}}$ over the point 0_j . We show now that

$$\dim_{\mathbb{C}} F_j = k.$$

This implies, together with the previous step that, for all $j = 1, \dots, k$, the G -module $\widetilde{\mathcal{H}}_{0_j}$ is a toric G -constellation. First notice that, by semicontinuity of the dimension of the fibers of a coherent sheaf (cf. [15, Example 12.7.2]), we have

$$(6.3) \quad \dim_{\mathbb{C}} F_j \geq k.$$

Let us suppose $j \geq 2$, the case $j = 1$ was shown in Remark 6.11. Let $\Gamma_j \subset \Gamma_C$ be, as in the previous step, the G -sFd associated to $\widetilde{\mathcal{H}}_{0_j}$, and let $x^\alpha y^\beta$ be the monomial in $\mathbb{C}[x, y] \subset \mathbb{C}[a_j, c_j, x, y]$ corresponding to the first box of Γ_j . Then, if the monomial $x^{\alpha+a} y^{\beta+b}$ corresponds to a box of Γ_C for some $a \geq j$ and $j-k \leq b \leq 0$, it has the property (\mathbf{A}_j) . As in STEP 2, Lemma 6.13 implies that $x^{\alpha+a} y^{\beta+b} \notin \Gamma_i$. Suppose that, $x^\gamma y^{\beta-k+j-1} \in \Gamma_i$ for some $\alpha+1 \leq \gamma \leq \alpha+j$. Then, if we have $x^{\gamma'} y^\beta \in \Gamma_i$ for some $\gamma-j+1 \leq \gamma' \leq \gamma$, the following relation

$$c_j x^{\gamma'+j-1} y^{\beta-k+j-1} - x^{\gamma'} y^\beta \in K_j \cap I_j,$$

implies that $x^{\gamma'} y^\beta \in K_j \cap I_j + (a_j, c_j)$. Now, by construction we have $\alpha-j+2 \leq \gamma' \leq \alpha+j$ and we have fixed $j \geq 2$. Thus, $\gamma' = \alpha$ gives the contradiction $x^\alpha y^\beta \notin \Gamma_i$.

As a consequence, $x^\gamma y^{\beta-k+j-1} \notin \Gamma_i$ for all $\alpha+1 \leq \gamma \leq \alpha+j$. Now, thanks to the connectedness proven in STEP 2, we have

$$\mathfrak{w}(\widetilde{\mathcal{H}}_{0_j}) \leq j \quad \text{and} \quad \mathfrak{h}(\widetilde{\mathcal{H}}_{0_j}) \leq k-j+1,$$

which together imply

$$\dim_{\mathbb{C}} F_j = \mathfrak{w}(\widetilde{\mathcal{K}}_{0_j}) + \mathfrak{h}(\widetilde{\mathcal{K}}_{0_j}) - 1 \leq k.$$

The equality $\dim_{\mathbb{C}} F_j = k$ follows now from Equation (6.3).

STEP 4 As an immediate consequence of the previous step, for all $j = 1, \dots, k$, the G -constellation $\widetilde{\mathcal{K}}_{0_j}$ has width $\mathfrak{w}(\widetilde{\mathcal{K}}_{0_j}) = j$, see Lemma 4.3. Hence, for $j = 1, \dots, k$, they are different to each other.

Now, the above listed properties imply that $\widetilde{\mathcal{K}}$ is the tautological bundle $\mathcal{R}_{C'}$ of some chamber $C' \subset \Theta^{\text{gen}}$ which admits Γ_C as C' -stair and this, by Remark 5.6, implies $C' = C$. \square

Remark 6.14. As expected, in dimension 3 Theorem 6.5 is in general false. For instance, given the $(\mathbb{Z}/2\mathbb{Z})^2$ -action over \mathbb{A}^3 defined by the inclusion

$$\begin{aligned} (\mathbb{Z}/2\mathbb{Z})^2 &\longrightarrow \text{SL}(3, \mathbb{C}) \\ (1, 0) &\longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ (0, 1) &\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

the quotient singularity $X = \mathbb{A}^3/(\mathbb{Z}/2\mathbb{Z})^2$ admits four different crepant resolutions $\varepsilon_i : Y_i \rightarrow X$, for $i = 1, \dots, 4$. All of them are toric and they are described by the planar graphs in Figure 21.

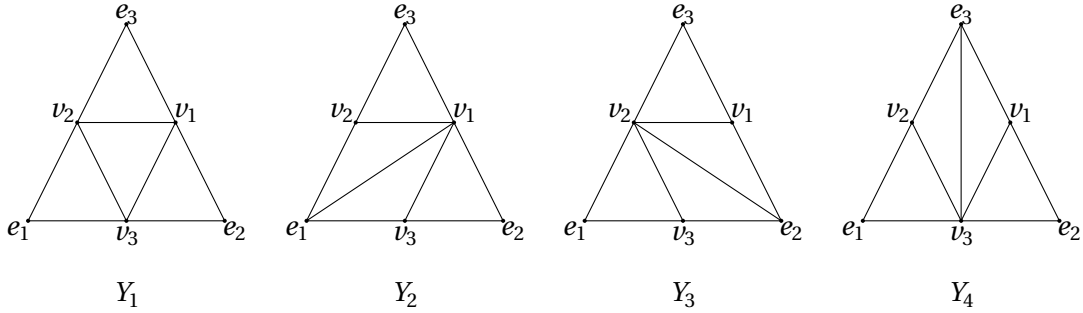


FIGURE 21. Toric description of the crepant resolutions of $\mathbb{A}^3/(\mathbb{Z}/2\mathbb{Z})^2$.

These diagrams are obtained by considering a fan Σ_i for each resolution Y_i then, each simplex in the planar graph is the intersection of a cone in Σ_i , with the plane containing the heads of the rays that generate Σ_i . Notice that Y_1 differs from the other resolutions by just one flop $Y_1 \xleftarrow{\sigma_i} Y_i$ for $i = 2, 3, 4$.

Now, let $\tilde{\mathcal{O}}_i$, for $i = 1, \dots, 4$, be the torsion free \mathcal{O}_{Y_i} -module defined by

$$\tilde{\mathcal{O}}_i = \varepsilon_i^* \pi_* \mathcal{O}_{\mathbb{A}^3} / \text{Tor}_{\mathcal{O}_{Y_i}} \varepsilon_i^* \pi_* \mathcal{O}_{\mathbb{A}^3},$$

where $\pi : \mathbb{A}^3 \rightarrow X$ is the canonical projection. A direct computation shows that only $\tilde{\mathcal{O}}_1$ is locally free, and, for $i = 2, 3, 4$, the locus where $\tilde{\mathcal{O}}_i$ fails to be locally free coincides with the line

flopped by σ_i . In this setting, it can be shown that the pair $(Y_1, \tilde{\mathcal{O}}_i)$ is canonically isomorphic to the pair $((\mathbb{Z}/2\mathbb{Z})^2 - \text{Hilb}(\mathbb{A}^3), \mathcal{R})$ where \mathcal{R} is the tautological bundle.

My future project is to work out conditions on an ideal sheaf $\mathcal{K} \subset \mathcal{O}_{\mathbb{A}^3}$ and a crepant resolution Y of \mathbb{A}^3/G , for $G \subset \text{SL}(3, \mathbb{C})$ finite subgroup, in order to have $\tilde{\mathcal{K}}$ locally free and isomorphic to $\mathcal{O}_{\mathbb{A}^3}[G]$.

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