

SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

PhD course in Theoretical Particle Physics

A tale of mirror symmetry and 3d $\mathcal{N} \leq 2$ CS theories

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Academic Year 2025/26



“The unseen harmony is stronger than the visible one.”

— HERACLITUS

Abstract

This thesis investigates new directions in three-dimensional supersymmetric (SUSY) Chern–Simons (CS) theories, with a focus on the rich interplay between dualities, moduli spaces, and infrared (IR) physics.

In the first part, we develop a systematic algorithmic framework for deriving the mirror duals of 3d $\mathcal{N} = 2$ SQCD-CS theories with matter. This approach not only streamlines the construction of mirror pairs under 3d mirror symmetry but also yields non-trivial predictions for the IR dynamics of the theories under study. We also analyze the renormalization group (RG) flows resulting from mass deformations of the newly found dualities and further explore the vast landscape of $\mathcal{N} = 2$ SQCD-CS matter theories.

In the second part, we venture beyond supersymmetry and propose a family of new bosonization dualities for non-supersymmetric 3d CS-QCD theories with both bosonic and fermionic matter. We support these conjectures with a broad range of explicit examples and independent consistency checks.

The third part is devoted to the moduli space geometry of *improved bifundamentals*, a distinguished class of 3d $\mathcal{N} = 2$ theories that arise naturally in the context of mirror symmetry with four supercharges. We perform a detailed analysis of these moduli spaces using the Hilbert series (HS) and the Superconformal Index (SCI) as primary computational tools.

A mamma e papà

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following works:

- S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota and A. Shri, *Planar Abelian Mirror Duals of $\mathcal{N} = 2$ SQCD₃*, Phys.Rev.D 112 (2025) 10, [arXiv:2411.05620] [1]
- S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota and A. Shri, *A chiral-planar dualization algorithm for 3d $\mathcal{N} = 2$ Chern–Simons–matter theories*, JHEP 10 (2025) 211, [arXiv:2505.02913] [2]
- S. Benvenuti, R. Comi, G. Pedde Ungureanu, S. Rota and A. Shri, *Universal Planar Abelian Duals of 3d $\mathcal{N} = 2$ Unitary CS-SQCD*, [arXiv:2603.08842] [3]
- S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota and A. Shri, *Planar Abelian Duals of Chern–Simons QCD*, JHEP 07 (2026) 209, [arXiv:2506.05465] [4]
- S. Benvenuti, G. Pedde Ungureanu, *Hilbert Series and Superconformal Indices of the Improved Bifundamentals*, [arXiv:2505.07952] [5]

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Chapter 1

Introduction

If it turned out that some physical system could not be described by a quantum field theory, it would be a sensation; if it turned out that the system did not obey the rules of quantum mechanics and relativity, it would be a cataclysm.

— Steven Weinberg, *The Quantum Theory of Fields*, Volume I [6]

Quantum Field Theory (QFT) is the most comprehensive framework developed to understand fundamental physics, successfully unifying quantum mechanics with special relativity and providing the mathematical foundation for the Standard Model. Nearly a century after its first formulation [7–10], QFT continues to find applications across an extraordinary range of physical phenomena, from nuclear and condensed matter physics to cosmology. Yet despite these successes, generic interacting QFTs remain analytically intractable beyond perturbation theory, and it is precisely in this non-perturbative regime that the richest physics often lies.

The obstacles are well-known. Non-perturbative dynamics, such as confinement, chiral symmetry breaking, and the structure of infrared (IR) fixed points, resist analytic treatment in generic gauge theories. Scalar masses suffer from quadratic ultraviolet (UV) divergences, rendering low scales unnaturally fine-tuned [11]. Determining whether a symmetry is broken dynamically, counting vacua, or identifying phases requires tools that generic QFTs simply do not provide.

Restricting to theories with extra spacetime symmetry, namely supersymmetry (SUSY) [12–15], transforms this landscape dramatically. Beyond its phenomenological appeal, SUSY provides an analytically controlled setting in which non-perturbative questions become tractable, making it an ideal laboratory for understanding strongly coupled dynamics. The key advantages are the following. First, in a supersymmetric quantum field theory, bosonic and fermionic loop contributions cancel in the UV, eliminating quadratic divergences in scalar masses and leaving only a milder logarithmic sensitivity to high scales [16]. Second, SUSY combined with holomorphy leads to powerful non-renormalization theorems: superpotential couplings are protected beyond specific orders, and certain BPS-protected quantities are exactly computable [17–19]. Third, SUSY allows the definition of the Witten index, a topological invariant that counts supersymmetric vacua and constrains the patterns of dynamical symmetry breaking [20–22]. It is in this spirit that the present thesis is written.

Three-dimensional supersymmetric theories. In this thesis we focus specifically on QFTs in three spacetime dimensions. While lower-dimensional theories might seem less directly relevant to phenomenology, three-dimensional supersymmetric gauge theories have proven to be an extraordinarily rich and tractable setting for exploring non-perturbative physics, for several reasons [23–25].

To begin with, three-dimensional gauge theories possess a topological symmetry, also called magnetic symmetry $U(1)_J$, which is exact at the quantum level for Abelian gauge groups and has no four-dimensional analog. This symmetry is generated by the conserved current $J = *F$, and its charged operators, the monopole operators, are disorder operators that create units of magnetic flux around a point in spacetime [26–30]. These operators play a central role throughout this thesis, both as generators of the moduli space of supersymmetric vacua and as the key objects exchanged under duality.

Furthermore, three-dimensional gauge theories admit Chern–Simons (CS) interactions [31]: topological terms for the gauge field that give the photon a mass, lift the Coulomb branch, and modify the statistics of charged particles through flux attachment. CS terms are constrained by quantization conditions and are generated at one loop when massive fermions are integrated out [32–34], creating a rich interplay between matter content, CS levels, and the quantum structure of the moduli space. Theories with both CS interactions and matter, called CS-matter theories, exhibit a landscape of IR dualities that is significantly richer than their four-dimensional counterparts, and remains only partially explored [24, 35].

Finally, and perhaps most importantly for this thesis, three-dimensional $\mathcal{N} = 4$ and $\mathcal{N} = 2$ gauge theories exhibit **mirror symmetry**: a duality that exchanges the Coulomb branch and Higgs branch of the moduli space, swaps mass parameters with Fayet–Iliopoulos terms, and maps perturbative degrees of freedom on one side to non-perturbative monopole operators on the other [36–39]. Mirror symmetry has its origins in string theory, where it can be derived through $SL(2, \mathbb{Z})$ transformations of Type IIB brane configurations [40], but its consequences for field theory are profound and far from fully understood, especially once CS interactions are included and supersymmetry is reduced from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

State of the art and open problems. For $\mathcal{N} = 4$ theories, mirror symmetry is by now well-established. For linear quivers with unitary gauge groups, the mirror dual can be derived systematically via Hanany–Witten brane moves [40, 41], and this has been extended to circular quivers, orthosymplectic quivers, and quivers involving orientifold planes [37, 42–44]. A systematic *algorithmic* approach to $\mathcal{N} = 4$ mirror symmetry, based on the $SL(2, \mathbb{Z})$ action on Abelian theories and a set of local duality moves—the so-called dualization algorithm—was developed in recent years [45, 46] and provides a powerful machine for generating mirror pairs.

However, extending this technology to $\mathcal{N} = 2$ theories with Chern–Simons interactions presents significant new challenges. Reducing from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ by turning on real mass deformations generates CS terms and breaks the symmetry between the two branches, making the structure of the mirror dual much harder to control [23]. Prior to the work underlying this thesis, the mirror duals of general $\mathcal{N} = 2$ SQCD-CS theories with both fundamental and anti-fundamental matter were not known systematically: results existed for special cases [47–55], but no algorithmic framework was available to treat general CS levels and matter content in a unified way. The first paper underlying this thesis [1] established the existence of *planar Abelian* mirror duals for a broad class

of $\mathcal{N} = 2$ SQCD-CS theories, providing the foundation for the algorithmic treatment developed subsequently in [2].

A related open problem concerns a distinguished class of $\mathcal{N} = 2$ theories known as **improved bifundamentals**. The families of these theories, denoted in this thesis by FT_N , FM_N , FE_N , FC_N , FH_N and their variants, arise naturally as building blocks in mirror symmetry with four supercharges [56, 57], playing a role analogous to that of the $T[SU(N)]$ theory in $\mathcal{N} = 4$ contexts. Despite their importance, a systematic understanding of their moduli spaces was lacking: the geometry of the Coulomb and Higgs branches, possible symmetry enhancements at the conformal fixed point, and the operator content of these theories had not been worked out in detail.

Finally, a third open direction concerns the extension of duality ideas *beyond* supersymmetry. In recent years, bosonization dualities in 2+1 dimensions, relating CS theories coupled to bosonic matter to CS theories coupled to fermionic matter, have attracted enormous attention [29, 58–63], connecting to condensed matter phenomena such as the quantum Hall effect, composite fermions, and dualities of topological insulators [64, 65]. For QED₃-type Abelian theories, a rich web of such dualities is now established [61, 66–68]. For non-Abelian CS-QCD theories with both bosonic and fermionic matter *simultaneously* present, however, the structure of bosonization dualities was only partially understood [66, 69–71], and explicit proposals with broad ranges of gauge groups, CS levels, and flavor content were largely missing.

Summary of results. This thesis addresses all three open problems described above. The work is based on five publications [1–5] and is organized as follows.

Chapter 3 develops a systematic, algorithmic framework for deriving the mirror duals of $\mathcal{N} = 2$ SQCD-CS theories with matter. The starting point is the known $\mathcal{N} = 4$ mirror pair for SQCD with F flavors. By turning on carefully chosen real mass deformations, one generates CS terms and reduces supersymmetry to $\mathcal{N} = 2$, obtaining a new class of mirror pairs we call *chiral-planar* duals. The key technical advance is the identification of the correct $\mathcal{N} = 2$ analog of the $SL(2, \mathbb{Z})$ S-wall, together with a set of fusion and gluing rules that allow the dualization algorithm to be extended to CS-matter theories. The resulting algorithm is both systematic and predictive: it applies uniformly to theories with a *specific* CS level, with *both* fundamental and antifundamental fields, and with *arbitrary* gauge group ranks, while yielding non-trivial statements about the IR dynamics of theories that would otherwise be inaccessible. We illustrate the algorithm through a broad set of examples, including theories with unitary and special unitary gauge groups, as well as non-constant rank quivers. Moreover, we further investigate the landscape of $\mathcal{N} = 2$ CS-SQCD by turning on mass deformations and following the corresponding flows, and we propose new mirror duals for $U(N)_k$ SQCD with F fundamentals, for arbitrary values of the parameters N , k , and F .

Chapter 4 ventures beyond supersymmetry and proposes a family of new bosonization dualities for non-supersymmetric 3d CS-QCD theories with both bosonic and fermionic matter simultaneously. The motivation comes partly from the SUSY results of the earlier chapter: in many cases, non-supersymmetric dualities can be obtained by taking appropriate decoupling limits of supersymmetric ones. The proposals are supported by a broad range of consistency checks, including matching of gauge-invariant operators, compatibility with mass deformations and RG flows, consistency with known Abelian dualities in special limits, particle-vortex duality, and derivation of TQFT dualities from mass deformations. We also make progress towards the purely fermionic case, which presents

new challenges not present in the purely bosonic one.

Chapter 5 is devoted to the moduli space geometry of improved bifundamentals. Using the Hilbert series (HS) and the Superconformal Index (SCI) as primary computational tools, we carry out a detailed analysis of the Higgs and Coulomb branches of the theories FT_N , FM_N , FE_N , FC_N and FH_N , at generic N . The Hilbert series allows us to read off the generators and relations of the chiral ring, identify the geometry of the moduli space as an algebraic variety, and detect unexpected symmetry enhancements. For the $N = 2$ cases we find a rich pattern of such enhancements, including an exceptional model $\widetilde{FE}_2$ whose moduli space geometry is governed by an exceptional Lie algebra. These results provide new data on the operator spectrum of these important theories and clarify their role in the broader network of $\mathcal{N} = 2$ mirror symmetry.

Taken together, these three directions share a common theme: the use of duality as a precision tool for extracting exact non-perturbative information about strongly coupled quantum field theories. Mirror symmetry, moduli space geometry, and bosonization dualities are all facets of the same underlying insight: apparently different QFTs can describe identical physics in the IR, and the techniques developed in each part of this thesis reinforce and inform the others. The remainder of the thesis begins with Chapter 2, which reviews the theoretical background necessary to follow the subsequent developments, including $\mathcal{N} = 2$ SUSY in three dimensions, quiver gauge theories, the main IR dualities, and the localization techniques used to compute partition functions exactly.

Chapter 2

Basics of supersymmetry and quiver gauge theories

In this chapter, we briefly review several topics in supersymmetry that will be used frequently throughout the rest of the thesis. Since the main results of the PhD concern QFTs in three spacetime dimensions, we mainly focus on 2+1-dimensional SQFTs.

The chapter is organized as follows. In section 2.1 we review the basic building blocks of 3d $\mathcal{N} = 2$ gauge theories: the field content, the possible lagrangian interactions, the structure of the moduli space of vacua, and the role of monopole operators. Section 2.2 introduces the quiver diagram notation that will be used to compactly describe the gauge theories studied in this thesis. In section 2.3 we review the infrared dualities that form the backbone of our analysis: Seiberg duality in 4d, Aharony duality and its Chern–Simons deformations (Giveon-Kutasov duality) in 3d, and mirror symmetry. Finally, section 2.4 presents the technique of supersymmetric localization, which provides the exact partition functions used to test the dualities in the subsequent chapters.

2.1 Quick review of (2+1)D $\mathcal{N} = 2$ theories

In this section we review the matter content and properties of 3d theories with four supercharges, following mainly [23–25, 72]. This class of theories has the same number of supercharges as 4d $\mathcal{N} = 1$. The reason we are not interested in 3d $\mathcal{N} = 1$ theories is that $\mathcal{N} = 1$ in 3d carries only two real supercharges, which is too few to exploit holomorphy in the superpotential. The connection between supersymmetry and holomorphic non-renormalization theorems requires a complex structure on superspace, which in turn requires at least four real supercharges. Without this, we cannot control the non-perturbative dynamics. Instead, similarly to the 4d case, theories with four supercharges possess holomorphic objects and non-renormalization theorems, which enable us to compute some properties of these theories exactly.

The SUSY algebra of these theories follows simply from reducing $d = 4$ $\mathcal{N} = 1$ supersymmetry down to three dimensions; for a review of the 4d algebra, see for example Chaps. 2–3 of [73]:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma_{\alpha\beta}^a P_a + 2i\varepsilon_{\alpha\beta} Z, \quad (2.1)$$

where the σ^a are chosen to be real and symmetric. The quantity Z is a real central term, which in the dimensional reduction viewpoint corresponds to the momentum P_3 in the

reduced direction. The spinors Q and $\bar{Q}$ are complex, and thus include twice the minimal number of charges in three dimensions. The notation Q and $\bar{Q}$ is not strictly necessary in 3d: the 3d Lorentz group $SO(1, 2)$ has only real spinor representations, so there is no notion of chirality and Q_α and $\bar{Q}_\alpha$ are related by complex conjugation rather than being independent spinors. Equivalently, there is no γ^5 matrix in 3d. We nonetheless keep this notation to maintain consistency with the 4d formalism and to make the $U(1)_R$ charge assignments manifest. If we define two complex spinor coordinates $\theta^\alpha, \bar{\theta}^\alpha$, a standard choice for the supercharges is

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\bar{\theta}^\beta (\gamma^a)_{\alpha\beta} \partial_a, \quad \bar{Q}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} + i\theta^\beta (\gamma^a)_{\alpha\beta} \partial_a, \quad (2.2)$$

where Einstein summation convention is understood. Also, γ^a for $a = 0, 1, 2$ are the 3d gamma matrices that satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ in Minkowski signature $\eta^{ab} = \text{diag}(-, +, +)$.

The automorphism of this algebra is $U(1)_R$, rotating the supercharges: under $U(1)_R$, the supercharges Q_α and $\bar{Q}_\alpha$ transform with charges $+1$ and -1 , respectively.

2.1.1 SUSY multiplets

Similarly to 4d, we have chiral superfields

$$X = (\phi, \psi, F) \quad (2.3)$$

with ϕ a complex scalar, ψ a complex fermion, and F an auxiliary scalar field, satisfying $\bar{D}_\alpha X = 0$; we also have anti-chiral superfields

$$\bar{X} = (\bar{\phi}, \bar{\psi}, \bar{F}), \quad (2.4)$$

satisfying $D_\alpha \bar{X} = 0$, where a standard choice for the covariant derivatives $\bar{D}$ and D is

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\bar{\theta}^\beta (\gamma^a)_{\alpha\beta} \partial_a, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta^\beta (\gamma^a)_{\alpha\beta} \partial_a, \quad (2.5)$$

which satisfy the algebra

$$\{D_\alpha, \bar{D}_\beta\} = -2i(\gamma^a)_{\alpha\beta} \partial_a, \quad \{D_\alpha, D_\beta\} = \{\bar{D}_\alpha, \bar{D}_\beta\} = 0, \quad (2.6)$$

and the D commute with the Q in all combinations.

There are also vector superfields satisfying the reality condition $V = V^\dagger$, whose matter content is

$$V = (\sigma, A_i, \lambda, \bar{\lambda}, D) \quad (2.7)$$

whose lowest component σ is a real scalar field, coming from the compactification of the A_3 component of the 4d gauge field, while A_i is the 3d gauge field, λ and $\bar{\lambda}$ are the 2-component Majorana fermions (which may equivalently be written as a single Dirac fermion $\Lambda = \lambda + i\bar{\lambda}$) and D is the auxiliary scalar field. In addition, we can have real superfields Σ satisfying the following conditions

$$\varepsilon^{\alpha\beta} D_\alpha D_\beta \Sigma = \varepsilon^{\alpha\beta} \bar{D}_\alpha \bar{D}_\beta \Sigma = 0, \quad (2.8)$$

where $\varepsilon^{\alpha\beta}$ is the 2×2 antisymmetric tensor with $\varepsilon^{01} = +1$, whose lowest component is a real scalar field. Such superfields are called *linear* and have the matter content

$$\Sigma = (\sigma_\Sigma, j_i, \lambda_\Sigma, D_\Sigma), \quad (2.9)$$

where σ_Σ is a real scalar field.¹ Condition (2.8) implies that the spin 1 field j_i is actually a conserved current:

$$\partial_a j^a = 0, \quad (2.10)$$

so that linear multiplets are the ones containing conserved currents.

All physical states satisfy a Bogomol'nyi–Prasad–Sommerfield (BPS) bound of the form [74]

$$M \geq |Z|, \quad (2.11)$$

where M is the mass (eigenvalue of P^2), and Z is the central charge appearing in (2.1)

An important comment regarding *Abelian* gauge theories in 3d: the photon A_i may be dualized into a scalar γ as

$$\partial_i \gamma = \frac{\pi}{g^2} \varepsilon_{ijk} F^{jk}, \quad (2.12)$$

where g is the gauge coupling and F^{jk} is the field strength of A_i .² The scalar γ is called *dual photon* [75], and is periodic $\gamma \sim \gamma + 2\pi$ as a consequence of the quantization of the magnetic flux. Now, we can define a current

$$J_i = \varepsilon_{ijk} F^{jk}, \quad (2.13)$$

which is identically conserved by the Bianchi identity and generates a “magnetic” global symmetry $U(1)_J$, corresponding to shifts of γ . This symmetry is *exact* only for Abelian gauge groups. The current (2.13) sits in the linear multiplet

$$\Sigma = -\frac{i}{2} \varepsilon^{\alpha\beta} \overline{D}_\alpha D_\beta V, \quad (2.14)$$

where V is the vector multiplet containing A_i . The SUSY version of this duality turns the vector multiplet V into a chiral multiplet, whose lowest component is given by

$$X = \exp \left(\frac{2\pi\sigma}{g^2} + i\gamma \right), \quad (2.15)$$

where σ is the lowest component scalar of V . As we will see, X parametrizes the Coulomb branch of the moduli space. In the presence of charged matter fields it is not known how to dualize the photon.

2.1.2 SUSY lagrangians

We have reviewed the basic ingredients that enter into a theory; now we need to combine them and write down the theory Lagrangian. To do so, we employ the superspace formalism [14, 17, 76–78], which gives us a compact way to write a SUSY invariant Lagrangian.

¹We use the subscript Σ to distinguish the scalar component of the linear multiplet from the real scalar σ appearing in the vector multiplet V . In what follows we will often drop the subscript when the context makes the distinction clear.

²With this normalization, Dirac flux quantization $\int_{S^2} F \in 2\pi\mathbb{Z}$ implies the periodicity $\gamma \sim \gamma + 2\pi$. Other conventions, differing by the rescaling of γ , appear in the literature.

For a generic gauge theory with gauge group G , we can define chiral and anti-chiral field strengths, similar to 4d:

$$W_\alpha = -\frac{1}{4}\overline{D}D e^{-V} D_\alpha e^V \quad \text{and} \quad \overline{W}_\alpha = -\frac{1}{4}D D e^{-V} \overline{D}_\alpha e^V, \quad (2.16)$$

where V is the vector multiplet, containing a real scalar in the adjoint of the group G , as well as two real fermion gauginos. The classical gauge kinetic terms, which include a kinetic term for the scalar ϕ , can be written as

$$\mathcal{L}_{gauge} = \frac{1}{g^2} \int d^2\theta \, \text{Tr} \, W_\alpha^2 + h.c. \quad (2.17)$$

In 3d, the fermionic coordinate θ has dimension $-\frac{1}{2}$, chiral superfields have dimension $+\frac{1}{2}$ and vector superfields have dimension 0, giving a dimension of $+\frac{3}{2}$ for W_α ; since the gauge coupling $1/g^2$ carries mass dimension -1 in three dimensions, the Lagrangian term has the correct dimension 3. In particular, the gauge coupling is a relevant parameter and 3d gauge theories are asymptotically free. Also, the gauge field and real scalar are neutral under the $U(1)_R$ symmetry, while the complex fermion has a charge we will normalize to $+1$.

If we have matter multiplets Q_f in representations R_f of the gauge group, we also have the matter term

$$\mathcal{L}_{matter} = \sum_f \int d^2\theta d^2\bar{\theta} \, Q_f^\dagger e^V Q_f + \left(\int d^2\theta \, \mathcal{W}(Q_f) + h.c. \right) \quad (2.18)$$

where the superpotential $\mathcal{W}$ is a holomorphic function of the matter fields Q_f . Since V includes a term $\sigma\theta\bar{\theta}$ for the real adjoint scalar σ , the Lagrangian includes a potential term for the squarks of the form

$$\sum_f |\sigma Q_f|^2. \quad (2.19)$$

Note that in (2.19), the label “ Q_f ” denotes the lowest scalar field of the superfield Q_f (this will always be the case when we work with potential terms or vacuum expectation values). If σ acquires a vacuum expectation value (VEV) $\langle\sigma\rangle$, this appears as a “real mass” for the matter fields. This is an important point, and we will come back to it later.

Suppose that the gauge group contains an Abelian factor $U(1)$, and call $\mathcal{V}$ the vector multiplet containing the corresponding gauge field $\mathcal{A}$. Since the D-component $\mathcal{D}$ of $\mathcal{V}$ is gauge-invariant, we can add another term called Fayet-Iliopoulos (FI) [79]

$$\mathcal{L}_{FI} = \xi \mathcal{D} = \xi \int d^2\theta d^2\bar{\theta} \, \mathcal{V}, \quad (2.20)$$

where the FI parameter ξ is real.

We can also couple $\mathcal{V}$ to some background vector superfield V_b corresponding to some $U(1)$ global symmetry. We can then add the following term to the Lagrangian

$$\mathcal{L}_{BF} = \int d^2\theta d^2\bar{\theta} \, V_b \Sigma = \int d^2\theta d^2\bar{\theta} \, \Sigma_b \mathcal{V}, \quad (2.21)$$

which is the SUSY completion of a classical BF term [80]. Σ and Σ_b are the corresponding linear multiplets defined as in (2.14).

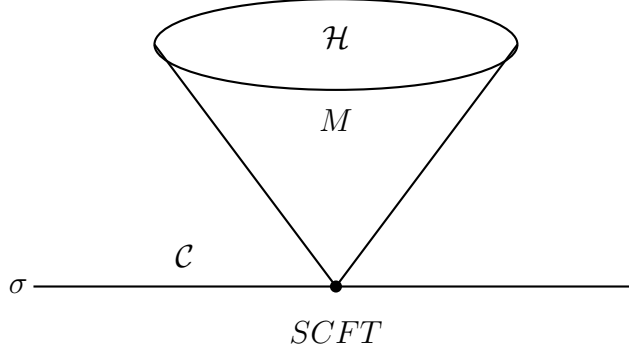


Figure 2.1: Classical moduli space of a $U(1)$ $\mathcal{N} = 2$ gauge theory, parametrized by the real scalar σ and complex mesons $M \sim \tilde{Q}Q$.

Now, remember that in 3d we can define the dual photon as $\partial_i \gamma \sim \varepsilon_{ijk} \mathcal{F}^{jk}$, so it is natural to interpret the FI parameter ξ appearing in (2.20), as the lowest component of the background vector superfield V_J corresponding to the $U(1)_J$ topological symmetry. In 3d the FI term (2.20) is then interpreted as a BF term between the $U(1)$ gauge symmetry, and the $U(1)_J$ topological symmetry.

Finally, we can add the SUSY completion of a Chern–Simons (CS) interaction [31, 81] to the Lagrangian, which takes the following form:

$$\mathcal{L}_{CS} = k \int d^2\theta \, d^2\bar{\theta} \, \Sigma \mathcal{V}. \quad (2.22)$$

A CS interaction also exists for non-Abelian gauge groups; see for example [82, 83]. For clarity, let us also write the $\mathcal{N} = 2$ CS term for gauge group $U(N)$ at level $(k, k + lN)$, with k and l both integers:

$$\begin{aligned} \mathcal{L}_{CS} = & \frac{k}{4\pi} \text{Tr}(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A - 2D\sigma + \bar{\lambda}\lambda) \\ & + \frac{lN}{4\pi} (\hat{A} \wedge d\hat{A} - 2\hat{D}\hat{\sigma} + \bar{\hat{\lambda}}\hat{\lambda}) \end{aligned} \quad (2.23)$$

where the “hatted” variables are the $U(1)$ trace parts: $\hat{A}_\mu = \frac{1}{N} \text{Tr} A_\mu$, $\hat{\sigma} = \frac{1}{N} \text{Tr} \sigma$, $\hat{\lambda} = \frac{1}{N} \text{Tr} \lambda$, $\hat{D} = \frac{1}{N} \text{Tr} D$.

2.1.3 Moduli space of 3d SUSY gauge theories

Let us now turn to the possible vacua of a 3d SUSY gauge theory. These vacua must be described by the VEVs of the scalars in the theory. To have a SUSY-preserving vacuum, the energy must be zero, and in particular the scalar potential must vanish. Because of condition (2.19), the choice $\langle Q_f \rangle \neq 0$ requires $\langle \sigma \rangle = 0$ and vice-versa. If we have matter fields Q_f and $\tilde{Q}_f$ with charges $+1$ and -1 under the $U(1)$, then we can use the gauge-invariant mesons $M \sim \tilde{Q}Q$ to parametrize the vacuum. Therefore, the complete classical moduli space of vacua generally consists of distinct branches:

- A **Coulomb branch** $\mathcal{C}$ with $\langle \sigma \rangle \neq 0$ and $\langle M \rangle = 0$
- A **Higgs branch** $\mathcal{H}$ with $\langle \sigma \rangle = 0$ and $\langle M \rangle \neq 0$

Similarly to 4d, we can have *mixed branches* where both $\langle M \rangle \neq 0$ and $\langle \sigma \rangle \neq 0$, and condition (2.19) is still satisfied. At the origin of the moduli space ($\langle \sigma \rangle = \langle M \rangle = 0$), there is a fixed point of the renormalization group (RG) flow where both SUSY and conformal symmetry are preserved. The resulting interacting theory is called a superconformal field theory (SCFT), as shown in figure 2.1. At such a fixed point, the symmetry algebra is enhanced from the Poincaré superalgebra to the superconformal algebra $\mathfrak{osp}(2|4)$, whose bosonic part is $SO(3, 2) \times U(1)_R$.

Quantum corrections modify the classical picture above. The Coulomb branch receives both perturbative and non-perturbative corrections, and new degrees of freedom are needed to describe the quantum moduli space. More precisely, the Coulomb branch is classically a cylinder $\mathbb{R} \times S^1$ parametrized by the complexified coordinate $\Phi = 2\pi\sigma + ig^2\gamma$, where the dual photon $\gamma \in S^1$ has period 2π . Quantum corrections, which depend on the running gauge coupling g , deform the metric on the Coulomb branch and can change its topology: in particular, one-loop effects generate a non-trivial metric on the Coulomb branch, while monopole-instanton effects can generate a superpotential lifting it entirely (see discussion below). Note that the Higgs branch is protected from quantum corrections in theories with eight or more supercharges ($\mathcal{N} \geq 4$ in 3d) by a non-renormalization theorem [84]: in those theories the gauge coupling cannot appear in the Higgs branch metric because it transforms as a Coulomb branch background field. In $\mathcal{N} = 2$ theories (four supercharges), this argument does not apply and the Higgs branch *metric* does receive quantum corrections; however, the holomorphic data we will be interested in throughout this thesis – the chiral ring and the complex structure of the moduli space – are protected by holomorphy and the standard non-renormalization theorems for the superpotential. The Higgs branch is also invariant under the topological symmetry $U(1)_J$, which acts as shifts of the dual photon γ . As a consequence, the radius of the S^1 factor of the Coulomb branch must vanish at the intersection with the Higgs branch.

The Coulomb branch of a 3d $\mathcal{N} = 4$ theory is a hyper-Kähler manifold of quaternionic dimension r_G (the rank of the gauge group), i.e. complex dimension $2r_G$, since each Cartan direction supports three real scalars and the dual photon [19, 36, 85]; the Coulomb-branch chiral ring is identified with its coordinate ring. In $\mathcal{N} = 2$ language, the coordinate ring is generated by the monopole operators and the scalar σ on the Coulomb branch, subject to the chiral ring relations. This connection, between the algebraic structure of the chiral ring and the geometry of the Coulomb branch, is the reason Hilbert series methods (reviewed in section 2.1.6) are such a powerful probe of the moduli space: counting chiral operators with their quantum numbers is equivalent to computing the coordinate ring of the branch as an algebraic variety. As observed in [86], for large σ one can systematically parametrize the Coulomb branch by the VEV of the chiral superfield $V = e^{\Phi/g^2}$. Thus the Coulomb branch splits into two regions and the quantum moduli space looks like the intersection of three cones, as in figure 2.2. The $\sigma > 0$ half of the Coulomb branch is parametrized by a field V^+ , which semi-classically is given by $V^+ \sim e^{\Phi/g^2}$. The other half is parametrized by another variable V^- , which semi-classically is given by $V^- \sim e^{-\Phi/g^2}$. The topology of the Higgs branch is not changed by quantum corrections, so we can still parametrize it with the mesons M .

For a 3d $\mathcal{N} = 2$ gauge theory, non-perturbative monopole-instanton effects can generate an Affleck-Harvey-Witten (AHW) type superpotential [87]. The prototypical example is pure $SU(2)$ gauge theory: on the Coulomb branch, where $SU(2)$ is broken to $U(1)$, the 't Hooft–Polyakov monopole becomes a finite-action instanton, and it contributes

$$\mathcal{W}_{\text{inst}} \sim e^{-\Phi/g^2}, \quad (2.24)$$

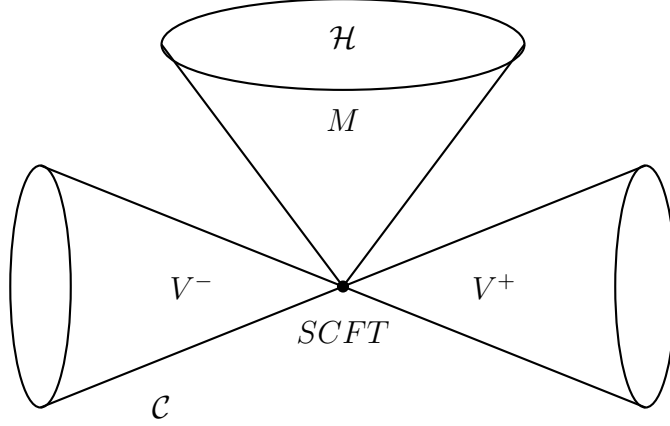


Figure 2.2: Quantum-corrected moduli space of a $U(1)$ $\mathcal{N} = 2$ gauge theory, parametrized by the complex fields $V^\pm$ and complex mesons $M \sim \tilde{Q}Q$.

where $\Phi = 2\pi\sigma + ig^2\gamma$ is the complexified coordinate of the unbroken $U(1)$ (the Weyl symmetry allows us to restrict to $\sigma \geq 0$), and we recognize $e^{-\Phi/g^2}$ as the semi-classical expression of the monopole operator. This runaway superpotential lifts the Coulomb branch completely, and the pure theory has no stable vacuum. Note that the mechanism crucially requires a non-Abelian gauge group: in pure $\mathcal{N} = 2$ $U(1)$ Maxwell theory there are no finite-action monopole-instanton configurations and the photino is neutral, so no superpotential can be generated and the Coulomb branch cylinder is exact (the theory is free).

For a general semisimple gauge group G , as argued in [37], the superpotential at weak coupling is generated by the fundamental monopoles associated with the simple roots α_i of the Lie algebra:

$$\mathcal{W}_{\text{inst}} = \sum_{i=1}^{\text{rank } G} e^{-\alpha_i \cdot \Phi}. \quad (2.25)$$

It is worth stressing that this is the result on $\mathbb{R}^3$. If the 3d theory arises from the compactification of a 4d theory on $\mathbb{R}^3 \times S^1$, there is one additional fundamental monopole-instanton, the Kaluza–Klein (or “affine”) monopole, associated with the affine simple root. For $SU(2)$ this produces the two-term superpotential $\mathcal{W} \sim e^{-\alpha \cdot \Phi} + \eta e^{+\alpha \cdot \Phi}$ (with η the 4d instanton factor), whose two critical points are precisely the two vacua of 4d $\mathcal{N} = 1$ $SU(2)$ gauge theory generated by gaugino condensation.

In the presence of matter fields, the contribution of a monopole background to the superpotential depends on the counting of fermionic zero modes: for a monopole-instanton to contribute to the superpotential, i.e. to saturate the Grassmann integration $\int d^2\theta$, it must support exactly two fermionic zero modes. In a $\frac{1}{2}$ -BPS monopole background two such modes are always present, obtained by acting with the broken supercharges on the bosonic solution; any additional zero mode forces the contribution to vanish, since the corresponding Grassmann integrations cannot be saturated.

The zero modes are counted by the Callias index theorem [88]: in a background of GNO charge $\mathbf{m}$, the gauginos support $2 \sum_{\alpha > 0} |\alpha(\mathbf{m})|$ zero modes, while the fermion in a chiral multiplet in representation R supports $\sum_{\rho \in R} |\rho(\mathbf{m})|$ zero modes. For the fundamental monopole of $SU(2)$, for which $\sum_{\alpha > 0} |\alpha(\mathbf{m})| = 1$, the gauginos provide exactly the two required zero modes, so the AHW superpotential is generated in the pure gauge theory. Adding N_f flavors, i.e. $2N_f$ chiral multiplets in the fundamental representation (each

contributing one zero mode, since the two weights satisfy $|\rho(\mathbf{m})| = \frac{1}{2}$, the same background supports $2 + 2N_f$ zero modes, and the direct contribution to the superpotential vanishes for $N_f > 0$.³

2.1.4 Anomalies and Chern–Simons terms

Unlike in 4d, in three dimensions there are no local gauge anomalies. However, as observed in [32, 34], gauge invariance in odd dimensions can require the introduction of a classical CS term which *breaks parity*. As mentioned before, for a single $U(1)$ gauge group this is

$$\mathcal{L}_{CS} = k \int d^2\theta d^2\bar{\theta} \Sigma V \quad (2.26)$$

If $k \neq 0$, this term breaks parity. This is referred to as a “parity anomaly”. At the quantum level, if we give mass to the fermions and integrate them out, the CS parameter k receives an additional contribution, coming from a one-loop diagram with charged fermions running in the loop:

$$k_{eff} = k + \frac{1}{2} \sum_f q_f^2 \text{sign}(M_f), \quad (2.27)$$

where the sum runs over the fermions, q_f are the charges of the fermions and M_f is the fermion mass. Gauge invariance restricts the CS terms as $k_{eff} \in \mathbb{Z}$.⁴ Then, the bare CS coefficient k must satisfy

$$k + \frac{1}{2} \sum_f q_f^2 \in \mathbb{Z}. \quad (2.28)$$

In particular when $\sum_f q_f^2$ is odd, $k \neq 0$ and parity is necessarily broken. A similar parity anomaly may be defined for non-Abelian gauge theories [33, 34]. In this case, if we call G the non-Abelian gauge group, the condition on the classical CS term k is

$$k + \frac{1}{2} \sum_f d_3(R_f) \in \mathbb{Z}, \quad (2.29)$$

where the sum is over all fermions f in representations R_f of G and $d_3(R_f)$ is the cubic index of R_f , normalized so that the $\mathbf{N}$ of $SU(N)$ has $d_3(\mathbf{N}) = 1$. Again, when $\sum_f d_3(R_f)$ is odd, $k \neq 0$ and parity is necessarily broken. The parity anomaly gives a (weaker) ’t Hooft anomaly matching condition [89] in 3d: whether or not the global symmetries, when weakly gauged, suffer from a parity anomaly must match between the microscopic theory and the IR theory it flows to.

It is also interesting to point out that the CS term gives the photon a mass, lifting the Coulomb branch discussed earlier.

³Physically, these extra zero modes are “soaked up” by the matter fields. In such theories, portions of the Coulomb branch can remain unlifted and become coordinates of the IR effective theory.

⁴The reason integrating out a massive fermion generates a half-integer shift is that the fermion path integral requires a Pauli-Villars (PV) regularization, and the PV regulator field itself carries a CS term at level $\pm 1/2$. When the fermion is integrated out, this regularization-induced contribution survives and shifts k by $\pm 1/2$, yielding the half-integer quantization condition.

2.1.5 Monopole operators

When describing the moduli space, we mentioned that the quantum corrected Coulomb branch may be parametrized in the low-energy effective theory, by the chiral operator $X \sim e^{\Phi/g^2}$, where $\Phi = 2\pi\sigma + ig^2\gamma$. It is natural to ask whether we can also construct local operators in the high-energy gauge theory that would flow to X . Note that the insertion in the path integral of the operator $\exp(i\gamma(x))$ creates one unit of magnetic flux through the S^2 that surrounds the point x . This relates the operator X to a “monopole operator”, which is defined as a disorder operator, obtained by removing the point x and requiring that we sum over gauge field configurations that have one unit of magnetic flux through the S^2 around the point x [26–28, 90, 91]. The field σ should have around x the singular behavior

$$\sigma(y) \simeq \frac{1}{2|x-y|}. \quad (2.30)$$

This definition preserves half of the supersymmetry, so that it defines a chiral operator, and it flows to the Coulomb branch coordinate X at low energies. More generically, for a monopole with magnetic charge m , the singular behavior is

$$\sigma(y) \simeq \frac{m}{2|x-y|}, \quad (2.31)$$

for $y \rightarrow x$. The proportionality between σ and the magnetic charge is fixed by the BPS equation $D_i\sigma = B_i$, which descends from the requirement that the configuration preserves half of supersymmetry; the numerical coefficient $1/2$ then follows from the standard Dirac-monopole solution $B^r = m/(2r^2)$ with the conventional flux normalization $\int_{S^2} F = 2\pi m$ [26, 92].

In a non-Abelian theory, the magnetic flux around x takes values in the Cartan subalgebra $U(1)^{r_G} \subset G$, where r_G is the rank of the gauge group G , and is labeled by a weight of the dual magnetic group $G^\vee$ modulo Weyl transformations. For example, for the gauge group $U(N_c)$, the simplest monopole operator X^+ breaks $U(N_c) \rightarrow U(1) \times U(N_c - 1)$ and carries one unit of flux in the $U(1)$; its supersymmetric version also pushes one of the eigenvalues of σ to $+\infty$. In the effective theory, if we fix σ_1 to be larger than all other eigenvalues, this operator flows to $X^+ \simeq \exp(\sigma_1/g^2 + i\gamma_1)$. Similarly, there is an operator X^- carrying (-1) units of flux in the $U(1)$ that pushes one of the eigenvalues of σ to $-\infty$, and which on the moduli space flows to $X^- \simeq \exp(-\sigma_{N_c}/g^2 - i\gamma_{N_c})$. In the special case of $U(1)$ theories, these operators flow to the Coulomb branch coordinates $V^+ \simeq \exp(+2\pi\sigma/g^2 + i\gamma)$ and $V^- \simeq \exp(-2\pi\sigma/g^2 - i\gamma)$. Classically these obey $V^+V^- = 1$, but quantum mechanically this is modified. In particular, whenever there are massless flavors, it is modified to $V^+V^- = 0$.⁵ Observe that the monopole operators defined in this way only exist in the 3d theory, and are not well-defined at high energies in the 4d theory on $\mathbb{R}^3 \times S^1$. In that theory the σ are compact and cannot be sent to infinity. The 4d theory on S^1 does not have BPS ’t Hooft line operators, and the chiral operators $X^\pm$ only arise in the effective description at low energies. More generally, whenever there

⁵This is a non-trivial quantum result following from the structure of the quantum moduli space: the presence of massless charged matter fields removes the classical identification of the two Coulomb branch components, forcing them to meet only at the origin. Physically, the monopole and anti-monopole cannot simultaneously exist as normalizable states in the Hilbert space when there are massless charged fields, so their product vanishes in the chiral ring. The 3d $\mathcal{N} = 2$ Coulomb-branch chiral ring is discussed in detail in [23, 37, 93]; see also [86] for the closely related compactification analysis.

is a compact region in the moduli space, there is no high-energy operator parameterizing it.

Let us now comment on the charges of the monopole operators. For a theory with connected gauge group G , the continuous topological symmetry is associated with the Abelian factors of G (the conserved currents being $\star F$ for each $U(1)$ inside G); discrete topological symmetries can also arise from $\pi_1(G)$. As mentioned, the magnetic charge m of the monopole is an element of the weight lattice $\Gamma_{G^\vee}$ of the dual magnetic group of G , up to Weyl symmetry. The topological charge of the monopole operator $J(m)$ is the magnetic charge modulo elements of the coroot lattice of G . For example, if $G = U(N)$, the magnetic charge is

$$m = \text{diag}(m_1, \dots, m_N) \in \mathbb{Z}^N / S_N, \quad (2.32)$$

the topological symmetry group is $G_J = U(1)$ and the topological charge is

$$J(m) = \text{Tr}(m) = \sum_i m_i \in \mathbb{Z}. \quad (2.33)$$

If the theory has a CS term, the monopole operators carry also electric charges under both global and gauge symmetries at the classical level. In general, the classical charges of the monopole are given by

$$Q_A^{\text{cl}}(M) = \sum_B k_{AB} M_B, \quad (2.34)$$

where we defined $M_A = \{m_i, \hat{m}_{\hat{i}}, B_i, 0\}$ all the magnetic charges allowed by supersymmetry: m_i for the Cartan subalgebra of the gauge symmetry, $\hat{m}_{\hat{i}}$ for the flavor symmetry, B_i for the topological symmetry and 0 for the R-symmetry. The k_{AB} is the CS matrix, it contains both the CS coefficients as well as the coefficients of the mixed CS interactions between gauge and global symmetries. The formula (2.34) is the statement that magnetic charges, in the presence of CS interactions, lead to electric charges. The charges (2.34) receive quantum corrections of the form [27, 47]

$$Q_A^q(M) = -\frac{1}{2} \sum_{\psi_a} Q_A[\psi_a] |m_a^{\text{eff}}(M)|, \quad (2.35)$$

This formula arises from a one-loop computation in the monopole background: one must evaluate the spectral asymmetry of the Dirac operator for each fermion ψ_a subject to the background $\sigma \sim m/2|x|$. The absolute value of the effective mass counts the net number of modes that shift sign as σ passes through zero, yielding a contribution to the electric charge of the operator that is proportional to the charge of ψ_a and the spectral flow [27, 88]. The sum runs over all fermionic fields ψ_a in matter chiral multiplets and vector multiplets, of charge $Q_A[\psi_a]$ and a fermion “effective mass”⁶

$$m_a^{\text{eff}}(M) = \sum_A Q_A[\psi_a] M_A. \quad (2.36)$$

Note that the previous will give corrections to the gauge and global charges, as well as the R-charge, of the monopole operator. Adding up the contributions, the total electric

⁶Remember from section 2.1.2 that the potential contains an interaction $V \sim |\sigma Q[\psi_a]|^2$. In a 1/2-BPS monopole background the VEV of the real scalar σ is proportional to the magnetic charge as in (2.31), so that the effective real mass is proportional to the effective mass appearing here.

charges of the monopole operators are

$$Q_A^{\text{eff}}(M) = \sum_B k_{AB}^{\text{eff}}(M) M_B , \quad (2.37)$$

where k_{AB} is the effective CS level we already encountered in section (2.1.4):

$$k_{AB}^{\text{eff}}(M) = k_{AB} - \frac{1}{2} \sum_{\psi_a} Q_A[\psi_a] Q_B[\psi_a] \text{sign}(m_a^{\text{eff}}(M)) . \quad (2.38)$$

An important quantity associated with a bare monopole operator is its trial R-charge/conformal dimension, computed from the one-loop spectral asymmetry of the fermions acquiring a mass in the monopole background [26, 94]. With chiral fields assigned R-charges r_i , the general formula reads

$$\Delta(\mathfrak{M}^{\vec{m}}) = - \sum_{\alpha > 0} |\alpha(\vec{m})| + \frac{1}{2} \sum_i (1 - r_i) \sum_{\rho_i \in R_i} |\rho_i(\vec{m})| , \quad (2.39)$$

where the first sum runs over positive roots α of the gauge group (the gaugino contribution, always non-positive) and the second sum runs over all chiral multiplets in representations R_i of the gauge group (the matter contribution). For $\mathcal{N} = 4$ theories the R-charges are fixed by the enhanced supersymmetry: the hypermultiplet chirals have $r = \frac{1}{2}$, so their contribution reduces to $\frac{1}{2} \sum_i \sum_{\rho_i} |\rho_i(\vec{m})|$ with the sum running over the hypermultiplets, while the adjoint chiral in the $\mathcal{N} = 4$ vector multiplet has $r = 1$ and drops out (the gaugino term is unchanged). In $\mathcal{N} = 2$ theories the r_i must instead be determined dynamically (e.g. by Z -extremization) once mixing with global symmetries is taken into account. Equation (2.39) will play a key role in the analysis of the Coulomb branch operators in the following chapters.

A final note: generically, and especially in theories with CS interactions, the monopoles defined so far, which we shall refer to as *bare monopole operators*, are not gauge invariant under the *residual* gauge group G_m (the monopole background has broken the gauge group as $G \rightarrow G_m$). These gauge-variant bare monopole operators may be *dressed* by a polynomial in the residual matter fields to construct an operator that is invariant under G_m . Concretely, consider a $U(2)_k$ theory with a monopole background of GNO charge $(1, 0)$, which breaks $U(2) \rightarrow U(1) \times U(1)$. The residual gauge group is $G_m = U(1) \times U(1)$, and the bare monopole carries non-trivial charge under the first $U(1)$ – equal to k from the CS contribution (2.34) plus a half-integer one-loop shift (2.35) from the charged fermions. Suppose for definiteness that the net charge under the first $U(1)$ is $+1$. To dress it, one multiplies by a residual matter field whose lowest mode in the monopole background carries compensating charge -1 – the available dressing modes at a given magnetic charge are precisely those zero modes of the Dirac operator in the monopole background with the appropriate quantum numbers. The resulting dressed operator is then made fully G -invariant by averaging over the action of the Weyl group $\mathcal{W}_G/\mathcal{W}_{G_m}$.

Charges of SUSY monopoles in quiver gauge theories

Now we briefly comment on how to compute the R-charge, the charges under global symmetries and the representation under gauge symmetries of $\mathcal{N} = 2$ SUSY monopoles in quiver gauge theories.

Consider first the case of a generic $U(N)_{(k, k+lN)}$ gauge theory with n_f fundamental and n_a anti-fundamental chiral fields:

$$\begin{array}{ccc}
\boxed{F} & \xleftarrow{\quad} & \textcircled{N} & \xleftarrow{\quad} & \boxed{A} \\
\vec{Y} & & \textcolor{red}{(k, k+lN)} & & \vec{X}
\end{array}
\tag{2.40}$$

where $\vec{X}$ and $\vec{Y}$ are fugacities related to the $S[U(A) \times U(F)]$ global symmetry, and $\vec{u}$ are the gauge fugacities. The R-charge of a monopole operator with generic magnetic flux $\vec{m}$ follows from (2.39):

$$R[\mathfrak{M}^{\vec{m}}] = \Lambda \sum_{i=1}^N m_i - \frac{1}{2} \left[\sum_{i \neq j} |m_i - m_j| + n_f(r_f - 1) \sum_{i=1}^N |m_i| + n_a(r_a - 1) \sum_{i=1}^N |m_i| \right] \tag{2.41}$$

where Λ parametrizes the mixing between the R-symmetry and the topological symmetry; this is a free parameter at the level of the UV Lagrangian and, in the IR, is fixed by Z -extremization. The quantities $r_{f/a}$ are the R-charges of the fundamentals and anti-fundamentals. From (2.37), the gauge charge of the monopole, written as the polynomial in the gauge fugacities Z_i whose coefficients are the gauge weights of the bare monopole, reads

$$Q_{\text{gauge}}[\mathfrak{M}^{\vec{m}}] = - \sum_{i=1}^N \left(k Z_i + l \sum_{j=1}^N Z_j \right) m_i - \frac{1}{2} \sum_{i=1}^N (n_f - n_a) Z_i |m_i|, \tag{2.42}$$

As noticed in the previous section, one has also to take into account that the magnetic flux $\vec{m}$ generically breaks the $U(N)$ gauge symmetry. The unbroken gauge symmetry is in general $\prod_{k=1}^L U(M_k)$, with $\sum_{k=1}^L M_k = N$ generated by a magnetic flux where the first M_1 entries are equal, then the following M_2 are equal and so on. In this analogy the parameters associated to the Cartans are the $Z_{1, \dots, M_1}$ for $U(M_1)$, then $Z_{M_1+1, \dots, M_1+M_2}$ and so on.

In the present thesis, a crucial role is also played by quiver Abelian gauge theories with mixed CS interactions and matter in bifundamental representation of pairs of gauge groups. We are thus interested in the charges of monopoles in such quiver theories. We will consider monopoles with non-vanishing magnetic fluxes for multiple gauge groups and vanishing flux under global symmetries. In general, given an Abelian quiver theory with the following data:

- n_g Abelian gauge nodes with gauge parameters u_i , $i = 1, \dots, n_g$
- n_f flavor nodes
- Global symmetry $U(1)^r$, not including topological symmetries (this may be the Cartan of a larger non-Abelian symmetry). We denote as y_a , $a = 1, \dots, r$ the corresponding parameters.
- CS levels k_i , BF levels k_{ij} and FI parameters λ_i .
- Bifundamental chirals α_{ij} , which is in the antifundamental representation (charge -1) under the i -th node and fundamental representation (charge $+1$) under the j -th node

- Antifundamentals $\tilde{\beta}_{ia}$ and fundamentals β_{ai}

We consider a monopole with gauge fluxes $\vec{m} = (m_1, \dots, m_{n_g})$. The charges of the monopole under the gauge and global symmetries can be encoded in a polynomial of the fugacities. We adopt here the additive conventions natural to the S_b^3 partition function reviewed in section 2.4.1: $Q \equiv b + b^{-1}$, with b the squashing parameter, and R-charges enter through the imaginary shift $(R - 1)\frac{iQ}{2}$ of the corresponding real mass parameters. The polynomial reads:

$$\begin{aligned}
\mathcal{M}(\vec{m}) = & -\frac{1}{2} \left(\sum_{\alpha_{ij}} |m_i - m_j| \left((R[\alpha_{ij}] - 1) \frac{iQ}{2} - u_i + u_j + \sum_{m=1}^r Q_m[\alpha_{ij}] y_m \right) \right. \\
& + \sum_{\tilde{\beta}_{ia}} |m_i| \left((R[\tilde{\beta}_{ia}] - 1) \frac{iQ}{2} - u_i + \sum_{m=1}^r Q_m[\tilde{\beta}_{ia}] y_m \right) \\
& \left. + \sum_{\beta_{ia}} |m_i| \left((R[\beta_{ia}] - 1) \frac{iQ}{2} + u_i + \sum_{m=1}^r Q_m[\beta_{ia}] y_m \right) \right) \\
& - \sum_{i=1}^{n_g} k_i m_i u_i + \sum_{i=1}^{n_g} \lambda_i m_i - \frac{1}{2} \sum_{i < j} k_{ij} (m_i u_j + m_j u_i)
\end{aligned} \tag{2.43}$$

where Q_m is the charge under a global symmetry $U(1)_m \subset U(1)^r$ and $R[\alpha]$ is the R-charge of (the lowest component of) the chiral multiplet α ; the shift $R[\alpha] - 1$ accounts for the fact that it is the fermion in the multiplet that acquires a mass in the monopole background. The first line in (2.43) corresponds to the contribution coming from the bifundamentals, the second and third lines correspond to fundamentals and antifundamentals, respectively, and the last line corresponds to the contributions from CS terms, FI terms and BF terms. In these conventions the FI parameter λ_i also encodes, through its imaginary part proportional to $\frac{iQ}{2}$, the mixing of the i -th topological symmetry with the R-symmetry. The charges of the monopoles under the various symmetries are given by the corresponding parameters' coefficients, where the R-symmetry is instead the pure number obtained, i.e. the one that does not multiply any parameter.

Chern–Simons fields coupled to fermions

Here and in the following subsections we recall some known results regarding non-supersymmetric monopole operators, which will be vastly employed in Chapter 4. Consider a gauge theory with N_f massless fermions ψ , described by the Lagrangian

$$\sum_{j=1}^{N_f} i \bar{\psi}_j \not{D}_A \psi_j + \frac{k_0}{4\pi} A dA, \tag{2.44}$$

where A is a $U(1)$ gauge field, and k_0 is the *bare* CS level, quantized as an integer. The presence of fermions and the CS term implies an anomaly in the time-reversal symmetry $\mathcal{T}$ [95, 96]. When quantizing the theory, a regulator for the fermion determinant that explicitly breaks time-reversal invariance (e.g., Pauli-Villars regularization, see also footnote 4) introduces a phase $\exp(-\frac{i\pi}{2} \eta(A))$ in the partition function, where η is the APS η -invariant [95]. For our purposes, this phase can be interpreted as a shift in the bare CS level of $U(1)_A$ by $-\frac{N_f}{2}$, accompanied by the addition of a mixed gravitational

CS term $-N_f CS_g$, which avoids Dirac string singularities [96]. We denote this theory as $U(1)_{k_0-N_f/2}$, where the subscript $k = k_0 - N_f/2$ represents the shifted CS level, which can be a half-integer.

Introducing a real mass m for all fermions and integrating them out generates a CS interaction. At low energies, the effective CS level becomes

$$k_{IR} = k_0 - \frac{N_f}{2} + \frac{N_f}{2} \text{sign}(m), \quad (2.45)$$

which is always an integer. Additionally, a gravitational CS term is induced:

$$N_f CS_g \text{sign}(m). \quad (2.46)$$

Of particular interest is the case of vanishing bare level, $k_0 = 0$, i.e. $k = -N_f/2$. When a positive mass ($m > 0$) is turned on for all fermions, the flow leads to a trivial theory with a vanishing Lagrangian ($k_{IR} = 0$ and no gravitational CS term). Conversely, a negative mass ($m < 0$) results in:

$$-\frac{N_f}{4\pi} AdA - 2N_f CS_g. \quad (2.47)$$

Via the level-rank duality $U(1)_{-N_f} \leftrightarrow U(N_f)_1$, this gapped phase admits an equivalent description as a $U(N_f)_1$ TQFT, up to gravitational counterterms [96, 97]. Time reversal acts on a fermion as:

$$\mathcal{T} : i\bar{\psi}\not{D}_A\psi \rightarrow i\bar{\psi}\not{D}_A\psi + \frac{1}{4\pi}AdA + 2CS_g \quad (2.48)$$

In the present thesis a major role is played by quiver-like theories, where fermions can be charged under multiple gauge groups. Consider then a $U(1)^{(a)} \times U(1)^{(b)}$ gauge theory with a fermion ψ of charge $+1$ and -1 under the two gauge groups, bare CS levels $k_{0,a}$ and $k_{0,b}$ and mixed CS level $k_{0,ab}$:

$$\mathcal{L} = i\bar{\psi}\not{D}_{a-b}\psi + \frac{k_{0,a}}{4\pi}ada + \frac{k_{0,b}}{4\pi}bdb + \frac{k_{0,ab}}{4\pi}adb, \quad (2.49)$$

the regularization of the fermion determinant produces a $-\frac{1}{2}$ shift for the gauge field $a-b$:

$$-\frac{1}{8\pi}(a-b)d(a-b) = -\frac{1}{8\pi}ada - \frac{1}{8\pi}bdb + \frac{1}{4\pi}adb \quad (2.50)$$

resulting in the shifted levels:

$$k_a = k_{0,a} - \frac{1}{2}, \quad k_b = k_{0,b} - \frac{1}{2}, \quad k_{ab} = k_{0,ab} + 1. \quad (2.51)$$

Furthermore a mass deformation for the fermion produces an additional $\pm\frac{1}{2}$ shift in the CS levels k_a , k_b and a ∓ 1 shift in the mixed CS level k_{ab} . In the quiver notation used throughout this thesis the theory (2.49) and its mass deformations are denoted as follows⁷:

$$\begin{array}{c} \textcircled{1} \xrightarrow[k_a]{k_{ab}} \textcircled{1} \xrightarrow[k_b]{} \end{array} \xrightarrow{m\bar{\psi}\psi} \begin{cases} \textcircled{1} \xrightarrow[k_a+\frac{1}{2}]{k_{ab}-1} \textcircled{1} \xrightarrow[k_b+\frac{1}{2}]{} & m > 0 \\ \textcircled{1} \xrightarrow[k_a-\frac{1}{2}]{k_{ab}+1} \textcircled{1} \xrightarrow[k_b-\frac{1}{2}]{} & m < 0 \end{cases} \quad (2.52)$$

where here and in the following a red arrow within the quiver will denote a fermion, electrically charged under the nodes it connects. See also Section 2.2 for our quiver gauge theory conventions.

⁷In the quiver notation used in this thesis we do not keep track of background gravitational CS levels CS_g .

Quantum numbers of monopole operators

At the start of this section, we reviewed the definition of a monopole operator of unit topological charge as the integral over field configurations with a Dirac singularity at a point $p \in \mathbb{R}^3$, such that a unit magnetic flux passes through any S^2 surrounding p . This results in an operator $O(x)$ with operator product expansion (OPE) with J^μ :

$$J^\mu(x)O(0) \sim \frac{1}{4\pi} \frac{x^\mu}{|x|^3} O(0) + \text{less singular terms.} \quad (2.53)$$

The quantum numbers of these monopole operators can be determined by analyzing the fermionic zero modes in the radially quantized theory with flux along S^2 [26, 98]. The contribution from fermionic zero modes to the gauge charge of the monopole can be encoded in an effective shift of $\frac{1}{2}$ to the CS level for each charged fermion. Therefore in a $U(1)_k$ theory with N_f fermions the gauge charge of a monopole with GNO flux $+1$ is $k + \frac{N_f}{2}$, which is equal to the *bare* CS level k_0 . Gauge invariant operators can then be constructed by dressing this bare monopole with (derivatives of) fundamental charged fields.

This analysis can be extended to theories with multiple gauge groups, as an example consider the theory with two gauge nodes (2.49):

$$\textcircled{1} \xrightarrow[k_a]{k_{ab}} \textcircled{1} \quad k_b \quad (2.54)$$

The fermion is charged under $a - b$ and its zero mode contributes as a $\frac{1}{2}$ shift to the CS level of $a - b$, which is equivalent to a $\frac{1}{2}$ shift for the CS levels k_a , k_b and a -1 shift to the mixed CS k_{ab} . Consider a generic bare monopole $\mathfrak{M}$ with fluxes n under $U(1)_a$ and m under $U(1)_b$, denoted as $\mathfrak{M}^{(n,m)}$. Gauss' law dictates that the monopole carries gauge charges $(n(k_a + \frac{1}{2}) + m\frac{(k_{ab}-1)}{2}, m(k_b + \frac{1}{2}) + n\frac{(k_{ab}-1)}{2})$ under $U(1)_a \times U(1)_b$. We write:

$$\begin{aligned} Q[\mathfrak{M}^{(n,m)}] &= nQ[\mathfrak{M}^{(+,0)}] + mQ[\mathfrak{M}^{(0,+)}] \\ &= n\left(k_a + \frac{1}{2}, \frac{k_{ab}-1}{2}\right) + m\left(\frac{k_{ab}-1}{2}, k_b + \frac{1}{2}\right) \\ &= \left(n(k_a + \frac{1}{2}) + m\frac{(k_{ab}-1)}{2}, m(k_b + \frac{1}{2}) + n\frac{(k_{ab}-1)}{2}\right) \end{aligned} \quad (2.55)$$

Gauge-invariant states can be constructed by dressing with modes of ψ (or $\bar{\psi}$) in very specific cases. Since ψ has charges $(+1, -1)$, a monopole dressed p times with ψ and q times with $\bar{\psi}$ has charge:

$$Q[\mathfrak{M}^{(n,m)}\psi^p\bar{\psi}^q] = \left(n(k_a + \frac{1}{2}) + m\frac{(k_{ab}-1)}{2} + p - q, m(k_b + \frac{1}{2}) + n\frac{(k_{ab}-1)}{2} + q - p\right) \quad (2.56)$$

As an example, for $k_a = k_b = \frac{1}{2}$ and $k_{ab} = -1$ the monopoles $\mathfrak{M}^{(+,0)}\partial_\bullet\bar{\psi}$ and $\mathfrak{M}^{(0,+)}\partial_\bullet\psi$ are gauge invariant⁸:

$$\textcircled{1} \xrightarrow[\frac{1}{2}]{-1} \textcircled{1} \quad \frac{1}{2} \quad \rightarrow \quad \begin{aligned} Q[\mathfrak{M}^{(+,0)}\partial_\bullet\bar{\psi}] &= (0, 0), \\ Q[\mathfrak{M}^{(0,+)}\partial_\bullet\psi] &= (0, 0) \end{aligned} \quad (2.57)$$

⁸Here $\partial_\bullet$ denotes a generic set of derivatives.

Throughout Chapter 4 of this thesis, we write dressed monopoles by combining the flux assignment and the dressing in the following way:

$$\mathfrak{M}^{(+,0)}\bar{\psi} \equiv \mathfrak{M}^{+ \text{ } \overleftarrow{\text{red}} \text{ } 0} \quad (2.58)$$

$$\mathfrak{M}^{(0,+)}\psi \equiv \mathfrak{M}^{0 \text{ } \overrightarrow{\text{red}} \text{ } +} \quad (2.59)$$

Statistical transmutation via flux Attachment

In the presence of a monopole background, the spin of fundamental fields in the ground state undergoes a shift — a phenomenon known as *statistical transmutation* [99] (see also [100–105]). This shift depends on the GNO charge (or flux) of the monopole, the intrinsic spin of the particle, and its gauge charge.

In what follows, we summarize the effects of statistical transmutation for a particle of spin s , and examine how this behavior is modified in the presence of multiple $U(1)$ gauge factors, particularly in the context of the quiver gauge theories discussed in the main text.

1. Single gauge group

Consider a spin s field φ with charge q under a $U(1)$ gauge group. On the background of a monopole with flux m the ground state of the field φ has the following spin channels:

$$|\frac{qm}{2} - s|, |\frac{qm}{2} - s| + 1, \dots, \frac{qm}{2} + s,$$

For example in the background of a monopole with flux $m = 1$ a fermion has a spin-0 and a spin-1 channel and a scalar has a spin- $\frac{1}{2}$ channel.

2. Bifundamental matter

Consider a spin- s field φ with charge $(+1, -1)q$ under a $U(1) \times U(1)$ gauge symmetry, shown explicitly in Equation (2.60):

$$\textcircled{1} \xrightarrow{\varphi} \textcircled{1} \quad (2.60)$$

The dressed operator $\mathfrak{M}^{(m,n)}\varphi$ will have the following spin channels:

$$|\frac{q}{2}(m - n) - s|, |\frac{q}{2}(m - n) - s| + 1, \dots, \frac{q}{2}(m - n) + s,$$

which depends on the difference of the GNO fluxes of the two gauge groups. In particular in the background of a monopole $\mathfrak{M}^{(+,+)}$ the bifundamental field φ “sees” a vanishing effective flux, and its spin is not transmuted.

3. Absence of transmutation from distant fluxes

Finally, we consider the quiver gauge theory shown in Equation (2.61):

$$\textcircled{1} \xrightarrow{\varphi_1} \textcircled{1} \xrightarrow{\varphi_2} \textcircled{1} \quad (2.61)$$

where φ_i are bifundamental spin- s_i fields. On the background of a monopole $\mathfrak{M}^{[0,0,m]}$ the field φ_1 does not “see” the magnetic flux, and its spin remains s_1 . On the other hand the field φ_2 “sees” an effective flux $-m$, and its spin channels are shifted accordingly:

$$\begin{aligned} \varphi_1 &: s_1 \quad (\text{single channel, no transmutation}) \\ \varphi_2 &: \left\lfloor \frac{|m|}{2} - s_2 \right\rfloor, \left\lfloor \frac{|m|}{2} - s_2 \right\rfloor + 1, \dots, \frac{|m|}{2} + s_2 \end{aligned} \quad \text{on } \mathfrak{M}^{[0,0,m]} \quad (2.62)$$

Generally, consider a field φ of spin- s and charges q_i under some $U(1)_i$ gauge groups, with $i = 1, \dots, M$. Then in the background of a monopole with GNO fluxes m_i the field φ has the same spin channels as a spin- s field with charge 1 in the background of an “effective” GNO flux m_{eff} :

$$m_{eff} = \sum_{i=1}^M q_i m_i \quad (2.63)$$

As an example we consider the following theory, which may be thought of as part of one of the quiver theories considered in the main body of the thesis:

(2.64)

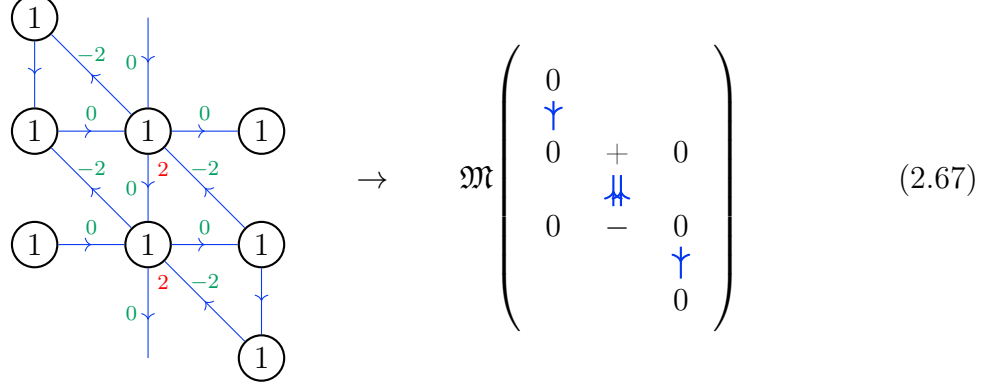
where red arrows correspond to fermions and blue arrows to bosons. We only reported CS and mixed CS terms relevant for the computations that follow. We can consider the monopole with GNO flux $+1/-1$ under the gauge nodes in the central column. Its gauge charges are:

$$Q \left[\mathfrak{M} \begin{pmatrix} 0 \\ 0+0 \\ 0-0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -1 & & & \\ 0 & 4 & -1 & \\ 1 & -4 & 0 & \\ & & & 1 \end{pmatrix} \quad (2.65)$$

therefore it can be made gauge invariant with the following dressing:

$$\mathfrak{M} \left(\begin{array}{ccc} 0 & & \\ 0 & \text{red } \gamma & \\ 0 & \text{red } \gamma & - \text{red } \gamma & 0 \\ 0 & \text{red } \gamma & \text{red } \gamma & 0 \\ & & \text{red } \gamma & 0 \end{array} \right) \quad (2.66)$$

Each fermion involved in this dressing “sees” a GNO flux ± 1 , therefore it has a spin-0 zero mode. Therefore we can dress this monopole to obtain a gauge invariant spin-0 monopole operator. In the theories considered in the main thesis this is one of the monopoles entering the $\mathcal{V}_{monopole}$ interaction. Similarly, we can consider the monopole operators involved in the $\mathcal{V}_{monopole}$ interaction that have GNO flux in the “bosonic” part of the quivers discussed in Section 4.3. There we find that the following monopole can be dressed to be gauge invariant and a Lorentz scalar:



$$\rightarrow \mathfrak{M} \begin{pmatrix} 0 & & \\ \text{blue arrow} & + & 0 \\ 0 & \text{blue double arrow} & \\ 0 & - & 0 \\ & & \text{blue arrow} & 0 \end{pmatrix} \quad (2.67)$$

Notice in particular that the middle scalar involved (twice) in the dressing “sees” a GNO flux 2, while the other two scalars “see” no flux, therefore all scalars have a spin-0 mode in the background of the monopole and we obtain a spin-0 gauge invariant operator. Similarly, one can show that all the monopoles involved in the interactions $\mathcal{V}_{monopole}$ throughout this thesis can be dressed to obtain a gauge invariant operator which is a Lorentz scalar.

2.1.6 Chiral ring and Hilbert series

In SUSY gauge theories with at least four supercharges, closely related to the moduli space of supersymmetric vacua are the concepts of chiral operators and *chiral ring* [106, 107]. Local gauge invariant chiral operators $\mathcal{O}_i(x)$ form a subset of observables in a 4d $\mathcal{N} = 1$, or 3d $\mathcal{N} = 2$, field theory which are protected from quantum corrections. These are $\frac{1}{2}$ -BPS operators that are annihilated by all the supercharges of negative R-charge:

$$\overline{Q}_\alpha \mathcal{O}_i(x) = 0, \quad \alpha = 1, 2. \quad (2.68)$$

An important property of chiral operators is that their spacetime derivatives are $\overline{Q}$ -exact and therefore vanish in expectation values, provided supersymmetry is unbroken:

$$\partial_\mu \mathcal{O}_1(x) = [P_\mu, \mathcal{O}_1(x)] \sim \{\overline{Q}_\alpha, [Q_\beta, \mathcal{O}_1(x)]\} \sim 0 \quad (2.69)$$

It follows that a product of chiral operators is free of short distance divergences and that its expectation value factorizes into the product of constant one-point functions:

$$\langle \mathcal{O}_{i_1}(x_1) \dots \mathcal{O}_{i_n}(x_n) \rangle = \langle \mathcal{O}_{i_1} \rangle \dots \langle \mathcal{O}_{i_n} \rangle. \quad (2.70)$$

Chiral operators form a commutative ring, the *chiral ring* $\mathcal{R}$, with product expansion

$$\mathcal{O}_i \mathcal{O}_j = c_{ij}^k \mathcal{O}_k + \overline{Q}_\alpha (\dots)^\alpha, \quad (2.71)$$

where the only spacetime dependence is in the $\overline{Q}$ -exact term. The latter term is unimportant when studying the moduli space $\mathcal{M}$, since we are interested in expectation values, i.e. we want to work at the level of $\overline{Q}$ -cohomology, so we can just drop off $\overline{Q}$ -exact terms.

The expectation values $\langle \mathcal{O}_i \rangle$ of gauge invariant chiral operators are holomorphic functions on the moduli space of vacua $\mathcal{M}$. It is generally *assumed* that the correspondence between expectation values of chiral operators and holomorphic functions on $\mathcal{M}$ is one-to-one, once relations are taken into account. This is not a theorem and can fail when accidental operators arise in the IR that are not captured by the classical chiral ring; in such cases the Hilbert series computations based on this assumption may miss additional generators. We will rely on this correspondence throughout the rest of the thesis. Then, the chiral ring $\mathcal{R}$ of the supersymmetric field theory is identified with the coordinate ring of its moduli space of vacua $\mathcal{M}$. In general, we want to write the chiral ring as a quotient ring

$$\mathcal{R} = \mathbb{C}[\mathcal{O}_1, \dots, \mathcal{O}_n]/I, \quad (2.72)$$

by determining the generators $\mathcal{O}_1, \dots, \mathcal{O}_n$ of the polynomial ring and the defining relations of the ideal I .

A very useful tool, which exploits as much as possible the symmetries of the theory, is the *Hilbert series* [108–111]. It is a generating function that counts scalar gauge invariant chiral operators, graded by their charges under a Cartan subalgebra of the global symmetry:

$$H(t, \hat{x}) = \text{Tr}_{\mathcal{H}} \left(t^R \prod_i \hat{x}_i^{\widehat{Q}_i} \right), \quad (2.73)$$

where $\widehat{Q}_i$ are the generators of the Cartan subalgebra of flavor symmetry, and $\hat{x}_i$ the corresponding fugacities, while $\mathcal{H}$ denotes the vector space of gauge invariant scalar chiral operators that parametrize the moduli space of SUSY vacua:

$$\mathcal{H} = \{ \mathcal{O}_i | \overline{Q}_\alpha \mathcal{O}_i = 0, M_{\mu\nu} \mathcal{O}_i = 0 \} \quad (2.74)$$

with $M_{\mu\nu}$ being the Lorentz generators for rotations and boosts.

In practice, the Hilbert series is computed using the *Molien-Weyl formula*: one first constructs the single-letter partition function $f(t, \hat{x}, z)$ counting the elementary fields weighted by their fugacities, where z are gauge fugacities. The multi-letter Hilbert series for a freely generated spectrum is then $\text{PE}[f]$ (see equation (2.76)). Gauge invariance is then imposed by integrating over the gauge group G using the Weyl integration formula:

$$H(t, \hat{x}) = \oint_{|z|=1} d\mu_G(z) \text{PE}[f(t, \hat{x}, z)], \quad (2.75)$$

where $d\mu_G$ is the Haar measure of G . For theories with relations among generators (i.e. non-freely generated branches), the PE is replaced by the full gauge-invariant generating function, and the plethystic logarithm is used to extract generators and relations as described below.

Much useful information about the moduli space $\mathcal{M}$ may be extracted from the Hilbert series. For instance, the complex dimension d of $\mathcal{M}$ is the order of the pole at $t = 1$ of the unrefined Hilbert series $H(t, 1)$, and the coefficient of $(1 - t)^{-d}$ is proportional to the volume of the base of the cone $\mathcal{M}$, a real $(2d - 1)$ -dimensional manifold. Most importantly, the charges of the generators and relations can be extracted using plethystic techniques,

if higher syzygies can be disentangled. The key tool is the *plethystic exponential* (PE), defined as [108, 112]

$$\text{PE}[f(t, \hat{x})] = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} f(t^k, \hat{x}^k) \right), \quad (2.76)$$

Physically, the PE is the multi-particle partition function for a bosonic system whose single-particle spectrum is enumerated by f : it freely generates a symmetric (commutative) algebra over the single-particle letters. If $f = t^n$ counts a single generator of dimension n , then $\text{PE}[t^n] = \frac{1}{1-t^n}$, which is indeed the Hilbert series of a freely generated $\mathbb{C}[x]$ with $\deg x = n$. Its functional inverse, the *plethystic logarithm* (PL), is

$$\text{PL}[H(t, \hat{x})] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log H(t^k, \hat{x}^k), \quad (2.77)$$

where $\mu(k)$ is the Möbius function. When applied to a Hilbert series, the plethystic logarithm encodes the generators (positive terms) and relations (negative terms) of the chiral ring: if the PL terminates at finite order in t , the moduli space is a complete intersection; otherwise, higher syzygies must be disentangled.

As a simple illustration, consider the Higgs branch of $\mathcal{N} = 4$ SQED with $F = 2$ electrons (the $T(SU(2))$ theory of Gaiotto-Witten [113]): the branch is the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, realized as the hyperkähler quotient $\mathbb{H}^2//U(1)$. The natural gauge-invariant operators are the components of the meson matrix

$$M^i_j = q^i \tilde{q}_j, \quad i, j = 1, 2, \quad (2.78)$$

where q^i and $\tilde{q}_j$ carry $U(1)$ charges $+1$ and -1 respectively. The F -term arising from the adjoint chiral Φ in the $\mathcal{N} = 4$ vector multiplet, $W = \tilde{q}_i \Phi q^i$, imposes the trace condition

$$\text{tr } M = q^i \tilde{q}_i = 0, \quad (2.79)$$

while the fact that M is built from a single pair of $U(1)$ -charged fields forces $\text{rank } M \leq 1$, equivalently $\det M = 0$ (or $M^2 = 0$ once the trace vanishes). Introducing the three independent components

$$A = M^1_2, \quad B = M^2_1, \quad C = M^1_1 = -M^2_2, \quad (2.80)$$

the chiral ring is generated by $\{A, B, C\}$ subject to the single relation

$$\det M = -AB - C^2 = 0 \iff AB + C^2 = 0, \quad (2.81)$$

which (up to the trivial redefinition $C \rightarrow iC$) is the standard defining equation $AB = C^2$ of the A_1 singularity as a hypersurface in $\mathbb{C}^3$. The Hilbert series, with fugacity t such that each squark $q, \tilde{q}$ contributes t and hence each meson t^2 , reads

$$H(t) = \frac{1 - t^4}{(1 - t^2)^3} = \frac{1 + t^2}{(1 - t^2)^2} = 1 + 3t^2 + 5t^4 + 7t^6 + \dots \quad (2.82)$$

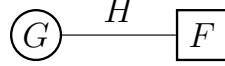
and its plethystic logarithm truncates,

$$\text{PL}[H](t) = 3t^2 - t^4, \quad (2.83)$$

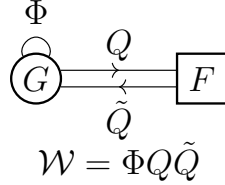
where the positive term $3t^2$ counts the three generators $\{A, B, C\}$ and the negative term $-t^4$ encodes the single relation (2.81). The truncation of the PL at finite order is the usual hallmark of a complete intersection – here in fact a hypersurface in $\mathbb{C}^3$ – confirming the geometric identification $\mathcal{M}_H = \mathbb{C}^2/\mathbb{Z}_2$. More generally, for F flavors the Higgs branch is the closure of the minimal nilpotent orbit of $\mathfrak{sl}_F$, of which the present case $F = 2$ is the smallest non-trivial example.

2.2 Quiver gauge theories notation

A useful way to compactly write down supersymmetric Lagrangian theories is through the use of *quiver diagrams* [114]. Commonly, this technique is used to describe gauge theories with eight supercharges ($\mathcal{N} = 2$ in 4d and $\mathcal{N} = 4$ in 3d), where each node in the diagram represents a gauge group G with the corresponding vector multiplet V , while each edge represents an hypermultiplet H in the bifundamental representation $(\square, \bar{\square})$ of the nodes it is connecting, as in the following example:



where the F inside the square denotes a flavor symmetry. The idea of quiver gauge theories is that supersymmetry is enough to constrain the full Lagrangian theory, once we specify the matter content, and gauge as well as global symmetries. In the present thesis we will focus mostly on 3d theories with four supercharges, so it is convenient to rewrite the previous $\mathcal{N} = 4$ quiver into an $\mathcal{N} = 2$ notation:



Here we wrote the $\mathcal{N} = 4$ vector multiplet as $V = (\tilde{V}, \Phi)$, where $\tilde{V}$ is the $\mathcal{N} = 2$ vector multiplet and Φ is the $\mathcal{N} = 2$ chiral superfield in the adjoint rep. of G (represented in the quiver by an arrow exiting and re-entering in the same node), while the hypermultiplet is decomposed as $H = (Q, \tilde{Q})$ where Q and $\tilde{Q}$ are in the fundamental $(\square)$ and anti-fundamental $(\bar{\square})$ of the gauge group G . Also note that as we decrease the supersymmetry, the quiver by itself is not enough anymore to fully constrain the theory, and we also need to specify the interactions through the superpotential $\mathcal{W} = \Phi Q \tilde{Q}$, which in this case derives from the theory enjoying $\mathcal{N} = 4$ supersymmetry.

Let us now list some conventions that will be used throughout the rest of this thesis.

- A circle $\bigcirc(N)$ will denote a $U(N)$ gauge group (with the corresponding $\mathcal{N} = 2$ vector superfield).
- A double circle $\bigcirc\!\!\bigcirc(N)$ will denote an $SU(N)$ gauge group.
- A square $\square(F)$ will denote F fundamental flavors, corresponding to an $SU(F)$ (or $U(F)$) flavor symmetry. The distinction between $SU(F)$ and $U(F)$ will be clear from context: in $\mathcal{N} = 2$ theories the extra $U(1)$ axial symmetry is present and we write $U(F)$, while for $\mathcal{N} = 4$ theories it is conventional to refer to $SU(F)$.
- An oriented link $\bigcirc(N_1) \rightarrow \bigcirc(N_2)$ will denote an $\mathcal{N} = 2$ chiral superfield Q in the bifundamental representation $(\square, \bar{\square})$ under $SU(N_1) \times SU(N_2)$.

- An (unoriented) loop $\bigcirc(N)$ will denote an $\mathcal{N} = 2$ chiral superfield Φ in the adjoint representation of the gauge group ($U(N)$ in this case).
- A red subscript $\bigcirc(N)_{(k_1, k_2)}$ will denote a CS term for the gauge group⁹.
- An orange label above (or below) the gauge node $\bigcirc(N)^\eta$ will denote an FI parameter η coupled to the vector superfield of the diagonal $U(1)$ inside the $U(N)$.
- A dark green label above a dashed edge $\bigcirc(1) \overset{2k_{12}}{\text{---}} \bigcirc(1)$ will denote the presence of a mixed-CS interaction between the gauge fields of the two $U(1)$ nodes, with no matter fields. Self and mixed-CS interactions for a quiver composed of only $U(1)$ nodes are encoded in a symmetric matrix k_{ab} such that

$$-i \frac{k_{ab}}{4\pi} \int A_a \wedge dA_b + \text{SUSY completion} , \quad (2.84)$$

where A_a is the vector field of the a -th $U(1)$ gauge group. Note that, since the off-diagonal entries appear twice in the sum, the label $2k_{12}$ on the edge is the physical mixed CS level, i.e. the coefficient of $\frac{1}{4\pi} A_1 \wedge dA_2$ once the two contributions are collected.

- Each closed (oriented) loop $\begin{array}{c} \bigcirc(N_1) \xrightarrow{a} \bigcirc(N_2) \\ \quad \quad \quad \searrow \quad \nearrow \\ \quad \quad \quad \bigcirc(N_3) \end{array}$ will correspond to a term (cubic in this case) appearing in the superpotential $\mathcal{W} = a b c$.

- If we have a link $\bigcirc(1) \rightarrow \bigcirc(1)$ and a superpotential term of the form $\mathcal{W} = \mathfrak{M}^{(+ -)}$, it will mean that we are in the background of a $\frac{1}{2}$ -BPS monopole (which from now on we denote as $\mathfrak{M}$), which has $(+1)$ units of flux under the left node and (-1) units of flux under the right node¹⁰. This monopole operator is gauge invariant (it carries no net gauge charge in this balanced configuration), and adding it to the superpotential gives a mass to the diagonal combination of the two topological symmetries.

- A cross above an edge $\bigcirc(N_1) \overset{a}{\times} \bigcirc(N_2)$ will denote a *flipper*: a single scalar field c , gauge invariant under both $U(N_1)$ and $U(N_2)$, which enters in a superpotential term coupled linearly to the gauge invariant built from a and b . We write this as $\mathcal{W} = \text{Flip}[a b] \equiv c \text{tr}_{N_1} \text{tr}_{N_2} a b$, where the two traces run over the color indices of the two gauge groups.

- For the *planar Abelian* quivers, whenever not specified, we adopt the following conventions for CS terms.

⁹For $U(N)$ we will follow the standard notation $U(N)_{(k_1, k_2)} = (SU(N)_{k_1} \times U(1)_{N k_2})/\mathbb{Z}_N$, see for example section 2 of [97]. Also by convention $U(N)_k \equiv U(N)_{k, k}$.

¹⁰Turning on a monopole in the lagrangian means that instanton effects modify the theory dynamics, such that the topological symmetry under which the monopole is charged is broken in the low-energy effective theory.

- For any pair of nodes connected by a down-pointing arrow ($\searrow$, $\swarrow$, $\downarrow$), the mixed CS level is +1, while for upward-pointing arrows ($\swarrow$, $\searrow$, $\uparrow$), it is -1.
- The CS level k_i of each $U(1)_i$ satisfies

$$k_i = -\frac{1}{2} \sum_{j \neq i} k_{ij}, \quad (2.85)$$

where k_{ij} denotes the mixed CS level between $U(1)_i$ and $U(1)_j$. The sum runs over all nodes in the quiver, including both gauge and flavor nodes.

2.3 IR dualities

Dualities are a ubiquitous topic in high energy physics, we can think for example about the electromagnetic (EM) duality, which is based on the symmetry between electric and magnetic charges in Maxwell's equations [115]; or S-duality, which is the exact quantum equivalence of two apparently (classically) different field theories, as appearing for example in the strong-weak coupling duality of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions, under which theories with gauge couplings g^2 and $1/g^2$ are identified [116–118].

A class of dualities of particular interest is given by “Infrared (IR) dualities”. This name refers to the equivalence of two theories at long distances, or low energies, although they may look quite different in the ultraviolet (UV). In particular, it is found that if both theories in the IR-dual pair are asymptotically free (AF), then their strong-coupling scales Λ_1 and Λ_2 will vary inversely [119]:

$$\Lambda_1^{b_1} \Lambda_2^{b_2} \sim \mu^{b_1+b_2}, \quad (2.86)$$

where b_1 and b_2 are specified constants, and μ is an arbitrary matching scale. Interestingly, we can have IR-dual pairs one of which is asymptotically free while the other is IR-free, with different gauge groups and different degrees of freedom. For example, this is the case for the *Seiberg duality* involving $SU(N)$ with F flavors, such that $N + 2 \leq F < \frac{3N}{2}$ (the free-magnetic range below the conformal window, where the electric theory is asymptotically free and the magnetic theory is IR-free). We shall analyze Seiberg duality in detail in the next sub-section, as well as its 3d analog, *Aharony duality*, which will be vastly employed in the main part of this thesis.

2.3.1 Seiberg duality

In his original paper [119–121], Seiberg conjectured the existence of a sort of electromagnetic duality involving 4d $\mathcal{N} = 1$ SQCD with gauge group $SU(N)$ and $F \geq N + 1$ flavors and another $\mathcal{N} = 1$ theory, known as the *magnetic dual* theory, denoted as mSQCD. The duality is summarized in the following quiver notation

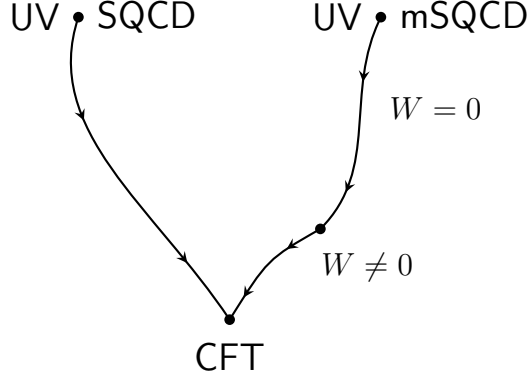
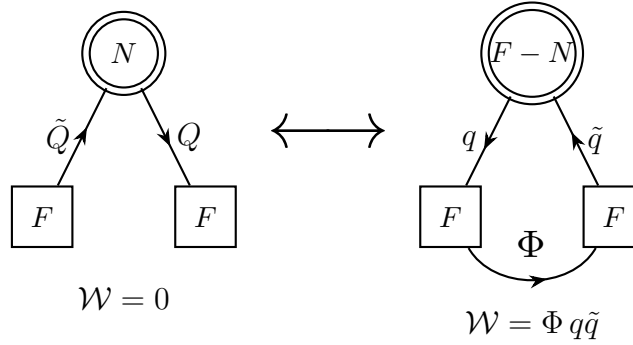


Figure 2.3: What the RG flow looks like for SQCD and mSQCD in the conformal window. In the IR, both theories reach the same fixed point.



as well as through the following mapping of chiral operators

$$\begin{aligned}
M_i^j &= \text{tr}_N Q_i \tilde{Q}^j \longleftrightarrow \Phi_i^j, \\
B &\sim Q_{i_1} \dots Q_{i_N} \longleftrightarrow b \sim q_1 \dots q_{F-N}, \\
\tilde{B} &\sim \tilde{Q}^{i_1} \dots \tilde{Q}^{i_N} \longleftrightarrow \tilde{b} \sim \tilde{q}^1 \dots \tilde{q}^{F-N},
\end{aligned} \tag{2.87}$$

where tr_N is the trace taken over the color indices, Φ_i^j are the collection of gauge singlets of the magnetic theory, which are mapped to the electric mesons M_i^j , while B and $\tilde{B}$ are the baryons of the electric theory and b and $\tilde{b}$ are the baryons of the magnetic theory. The other claim by Seiberg is that SQCD in the range $\frac{3}{2}N < F < 3N$ not only is UV-free, but flows to an interacting IR fixed point as well. The previous range is known as *conformal window* and, as can be easily checked, is invariant under $N \rightarrow F - N$, which means the magnetic dual theory enjoys the same conformal window. The duality claim implies that, within this conformal window, the two theories flow to the same interacting IR-fixed point, as shown in figure 2.3.

There are many non-trivial checks of this duality. From its definition, it follows that mSQCD has the same global symmetry group as the electric theory, $G_F = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$ and, using the mapping (2.87), we can infer the following charges for the magnetic fields:

	$SU(F)_L$	$SU(F)_R$	$U(1)_B$	$U(1)_R$
q_i	$\overline{F}$	$\bullet$	$\frac{N}{F-N}$	$\frac{N}{F}$
$\tilde{q}^j$	$\bullet$	F	$-\frac{N}{F-N}$	$\frac{N}{F}$
Φ_j^i	F	$\overline{F}$	0	$2\frac{F-N}{F}$
$\tilde{\lambda}$	$\bullet$	$\bullet$	0	1

where $\tilde{\lambda}$ are the gauginos of mSQCD. A very non-trivial check given by Seiberg is the matching of 't Hooft anomalies across the two theories [119]. A summary of the anomalies is given in the following table

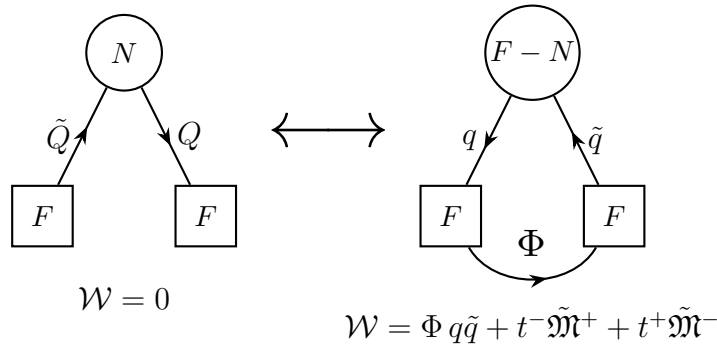
$\text{Tr}(SU(F)^3) :$	N	$\text{Tr}(U(1)_R^3) :$	$-2\frac{N^4}{F^2} + N^2 - 1$
$\text{Tr}(SU(F)^2 U(1)_B) :$	$\frac{N}{2}$	$\text{Tr}(U(1)_R^2 U(1)_B) :$	0
$\text{Tr}(SU(F)^2 U(1)_R) :$	$-\frac{N^2}{2F}$	$\text{Tr}(U(1)_R U(1)_B^2) :$	$-2N^2$
$\text{Tr}(U(1)_B^3) :$	0	$\text{Tr}(U(1)_R) :$	$-N^2 - 1$

Mass terms can also be mapped across the duality, we can then use this mapping to check the consistency of the duality under mass deformations. In particular, a mass for a flavor on the SQCD side is mapped to a VEV to the magnetic side, which Higgses the gauge group down to $F - N - 1$. We then obtain the correct Seiberg dual of $SU(N)$ with $F - 1$ flavors.

The Hilbert series for both theories can also be inspected and verified to be invariant under Seiberg duality [122–124]. Another interesting check is given by the match of the superconformal index (whose definition we will review in a subsequent section) of the two theories [125–127].

2.3.2 Aharony duality

The next IR-duality we review is the 3d $\mathcal{N} = 2$ analog of Seiberg duality, conjectured by Aharony [93]. There are different versions of the dualities for different gauge groups, we will review the one holding for $U(N)$ SQCD with $F \geq N$ flavors (the duality requires at least N flavors for the magnetic theory to be well-defined). The duality in quiver notation is the following



with the following chiral operator mapping:

$$M_i^j = \text{tr}_N Q_i \tilde{Q}^j \longleftrightarrow \Phi_i^j, \quad (2.88)$$

$$\mathfrak{M}^+ \longleftrightarrow t^- \quad (2.89)$$

$$\mathfrak{M}^- \longleftrightarrow t^+ \quad (2.90)$$

where $\mathfrak{M}^\pm$ and $\tilde{\mathfrak{M}}^\pm$ are the monopole operators with unit Abelian fluxes (± 1) under the corresponding gauge group of the electric and magnetic theory, respectively. The *flippers* $t^\mp$ of $\tilde{\mathfrak{M}}^\pm$ appearing in the superpotential are gauge singlets and are charged under the topological symmetry $U(1)_J$. To understand the role of $t^\pm$ it is useful to write down the F-term equations, they read

$$q \tilde{q} = 0, \quad \tilde{\mathfrak{M}}^+ = 0 \quad \text{and} \quad \tilde{\mathfrak{M}}^- = 0, \quad (2.91)$$

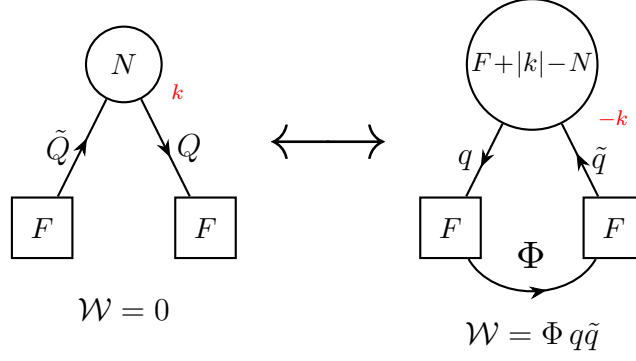
respectively for Φ , t^- and t^+ . We see that the dual mesons $q\tilde{q}$, as well as the dual monopole operators, are set to zero in the chiral ring of the theory. Physically, this means that the dual Coulomb branch (parametrized by $\tilde{\mathfrak{M}}^\pm$) and the dual Higgs branch (parametrized by $q\tilde{q}$) are decoupled at this level. Compared to the 4d theory, 3d $SQCD$ enjoys two extra $U(1)$ global symmetries: the topological $U(1)_J$ symmetry which we already mentioned, and the axial $U(1)_A$ symmetry, which is anomalous in 4d. The charges of the elementary fields under the full global symmetry is summarized below

	$SU(F)_L$	$SU(F)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	$\bullet$	$\overline{F}$	1	0	r_Q
$\tilde{Q}$	F	$\bullet$	1	0	r_Q
q	$\bullet$	F	-1	0	$1 - r_Q$
$\tilde{q}$	$\overline{F}$	$\bullet$	-1	0	$1 - r_Q$
Φ	F	$\overline{F}$	2	0	$2r_Q$
$\tilde{\mathfrak{M}}^+$	$\bullet$	$\bullet$	F	-1	$-F(1 - r_Q) + N + 1$
$\tilde{\mathfrak{M}}^-$	$\bullet$	$\bullet$	F	+1	$-F(1 - r_Q) + N + 1$
t^+	$\bullet$	$\bullet$	$-F$	-1	$F(1 - r_Q) - N + 1$
t^-	$\bullet$	$\bullet$	$-F$	+1	$F(1 - r_Q) - N + 1$

where we assigned an arbitrary R-charge r_Q to (the lowest component of) Q . The charges of the dual monopoles $\tilde{\mathfrak{M}}^\pm$ follow from the quantum correction (2.35) applied to the magnetic theory, whose quarks carry axial charge -1 and R-charge $1 - r_Q$; the charges of the singlets $t^\pm$ are then fixed by requiring the superpotential terms $t^\mp \tilde{\mathfrak{M}}^\pm$ to be neutral, and indeed coincide with those of the electric monopoles $\mathfrak{M}^\pm$, consistently with the operator map. This duality has received many checks as well: while we do not have the strong check of 't Hooft anomaly matching (which is specific to even dimensions), it has been verified at the level of the S^3 partition function [128, 129], as well as through the matching of the superconformal index [94]. Similarly to 4d Seiberg duality, we can verify the consistency of the duality by turning on complex masses for the flavors on the electric side; this maps to VEVs on the magnetic side, which Higgs the gauge group and lead to Aharony duality with a smaller number of flavors. As we discussed when we introduced SUSY gauge theories, in 3d we also have the possibility of turning on *real* masses for the $\mathcal{N} = 2$ chiral fields. This triggers an RG-flow to a fixed point where the number of fundamental and anti-fundamental fields may differ, and a non-zero Chern–Simons term is generated, compatible with charge quantization. Starting from Aharony duality, this procedure allows us to deduce a whole class of 3d $\mathcal{N} = 2$ Seiberg-like dualities, see for example [47–50, 130, 131]. Among these, the most important for the present thesis is the *Giveon-Kutasov duality* [35], which we review in the next subsection.

2.3.3 Giveon-Kutasov duality

Starting from Aharony duality for $U(N)$ with F flavors and turning on equal real masses for $F - F'$ flavors (with $F' < F$), one integrates out the massive matter fields. As discussed in section 2.1.4, integrating out a massive fermion shifts the Chern–Simons level. The resulting duality, known as *Giveon-Kutasov (GK) duality* [35], relates $U(N)_k$ with F flavors (with F now denoting the number of remaining light flavors) to $U(F + |k| - N)_{-k}$ with F flavors, as summarized in the following quiver notation



Compared to Aharony duality, the key difference is that the CS term gives a topological mass to the photon, lifting the Coulomb branch entirely. The monopole operators $\mathfrak{M}^\pm$ acquire a non-zero gauge charge due to the CS interaction (cf. equation (2.34)) and are no longer gauge invariant. As a result, the flippers $t^\pm$ and the superpotential terms involving monopoles that appeared in the Aharony dual are absent. The chiral operator map reduces to the meson identification $M \leftrightarrow \Phi$, and the duality can be checked by matching partition functions on S_b^3 [129] and superconformal indices [132].

2.3.4 Mirror symmetry

Another strong-weak type of duality of 3d $\mathcal{N} = 4$ gauge theories that will play a crucial role in the present thesis is *mirror symmetry* [36, 38, 133]. Mirror symmetry was originally discovered in string theory as a symmetry relating pairs of Calabi-Yau manifolds [134–136]. It was then proved that mirror symmetry in 2d is related to T-duality [137]. After the proposal of 3d mirror symmetry, it has been checked that its compactification on S^1 consistently reduces to 2d mirror symmetry [138, 139].

Let us now review mirror symmetry in three dimensions. SUSY gauge theories in three dimensions with eight supercharges have a global $SO(4) \cong SU(2)_C \times SU(2)_H$ R-symmetry. The content of the elementary multiplets of the 3d $\mathcal{N} = 4$ algebra and their transformations under the R-symmetry is summarized in the following table

3d $\mathcal{N} = 4$ Field Content				
Multiplet	Field	Type	$SU(2)_H$	$SU(2)_C$
Vector	A_μ	Gauge boson	1	1
(Vector multiplet)	ϕ_i ($i = 1, 2, 3$)	Scalars	1	3
	λ_α	Gauginos	2	2
Hyper	q^a ($a = 1, 2$)	Scalars	2	1
(Hypermultiplet)	ψ_α	Hyperinos	1	2

The Higgs branch is a hyper-Kähler manifold parametrized by the mesons, made out of the lowest component scalars in the hypermultiplet, which transform in the adjoint of $SU(2)_H$ and are invariant under $SU(2)_C$. The Coulomb branch is also a hyper-Kähler manifold parametrized by the lowest component scalars in the vector multiplet, which are in the adjoint of $SU(2)_C$ and are invariant under $SU(2)_H$. A few non-renormalization

theorems for $\mathcal{N} = 4$ theories have been proved to hold by promoting some of the coupling constants to background superfields:

1. If we promote the gauge coupling constant to a superfield, the scalars in the resulting superfield transform as $(\mathbf{1} + \mathbf{3}, \mathbf{1})$ of $SU(2)_C \times SU(2)_H$. Therefore, they can only appear in the Coulomb branch. Hence, the Higgs branch is not renormalized by quantum effects [84].
2. Mass terms are background vector superfields [84, 86] which transform as $(\mathbf{3}, \mathbf{1})$ of $SU(2)_C \times SU(2)_H$. As with the gauge couplings, they affect the metric on the Coulomb branch but not the Higgs branch.
3. Fayet-Iliopoulos terms for $U(1)$ factors of the gauge group transform as $(\mathbf{1}, \mathbf{3})$ of $SU(2)_C \times SU(2)_H$. Therefore, they can affect only the metric on the Higgs branch but not the Coulomb branch.

Mirror symmetry is an IR duality which acts through the exchanging of:

1. $SU(2)_C$ and $SU(2)_H$,
2. The Coulomb branch and the Higgs branch,
3. Mass terms and Fayet-Iliopoulos D terms.

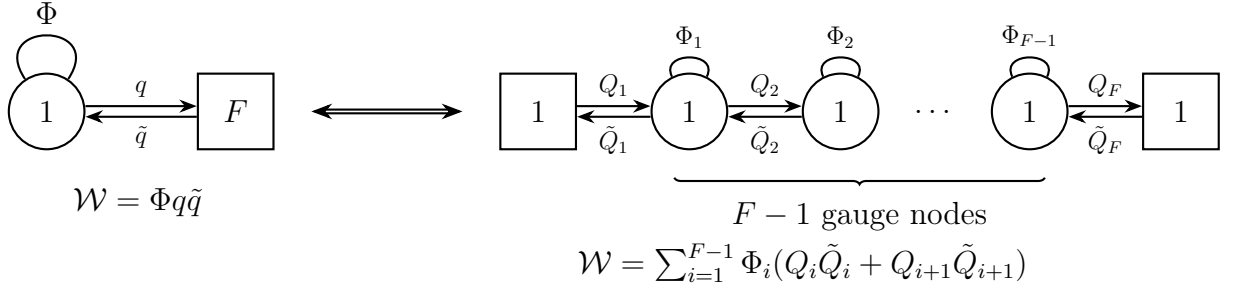
Because the Coulomb branch gets quantum corrections while the Higgs branch does not, the exchange of the two branches under the duality action means that quantum effects in one theory arise classically in the dual and vice-versa.

Another interesting aspect is that manifest symmetries on one side of the duality become hidden symmetries on the other side. To see this, observe that mass terms are related to visible global symmetries¹¹, while FI terms are not associated to any global symmetry. Therefore, the exchange of mass terms with FI terms means that visible global symmetries are exchanged with hidden ones. In particular, if the theory has r $U(1)$ gauge factors, we also have an FI term for each one of them which, as mentioned previously, can be thought as a background vector superfield coupled to the corresponding topological current $*F$. The resulting $U(1)^r$ global symmetry is the Cartan part of, a generally non-Abelian, hidden global symmetry of rank r .

Mirror symmetry for SQED

It is argued in [36] that an $\mathcal{N} = 4$ *linear* quiver gauge theory with $F - 1$ $U(1)$ gauge nodes, two boundary $U(1)$ flavor nodes, and F hypermultiplets flows to the same IR fixed point as $\mathcal{N} = 4$ SQED with F electrons. Concretely, the i -th hypermultiplet Q_i is charged under the i -th and $(i + 1)$ -th nodes (numbering from 0 to F), where nodes 0 and F are flavor nodes. This duality is described by the following quiver in $\mathcal{N} = 2$ notation

¹¹These mass terms can be regarded as VEVs ϕ of background vector superfields transforming in the adjoint of the global symmetry. Because of $\mathcal{N} = 4$ SUSY, the potential must contain a term proportional to $[\phi, \phi^\dagger]^2$, which to be minimized requires the background fields to take values in the Cartan subalgebra.



Monopole operators in the linear quiver are labeled by a vector of integers $\mathfrak{M}^{(n_1, n_2, \dots, n_{F-1})}$, where $n_a \in \mathbb{Z}$ is the magnetic flux under the a -th $U(1)$ gauge node. With this notation the chiral operator mapping is:

$$M_i^j \sim q_i \tilde{q}^j \longleftrightarrow \mathfrak{M}^{0 \dots 0 \overbrace{1}^i 1 \dots 1 \overbrace{1}^{j-1} 0 \dots 0} \quad \text{if } j > i, \quad (2.92)$$

$$\mathfrak{M}^{0 \dots 0 \overbrace{-1}^j -1 \dots -1 \overbrace{-1}^{i-1} 0 \dots 0} \quad \text{if } j < i,$$

$$\mathfrak{M}^+ \longleftrightarrow Q_1 Q_2 \dots Q_F, \quad (2.93)$$

$$\mathfrak{M}^- \longleftrightarrow \tilde{Q}_1 \tilde{Q}_2 \dots \tilde{Q}_F. \quad (2.94)$$

Here the off-diagonal meson M_i^j , $j > i$, maps to the monopole with one unit of flux under the gauge nodes $i, i+1, \dots, j-1$ (the nodes sitting between the i -th and j -th flavor in the linear quiver), and similarly for $j < i$ with negative fluxes. The diagonal mesons M_i^i , which are neutral under all the topological symmetries, are instead mapped to linear combinations of the adjoint chirals Φ_a and of the singlets $Q_a \tilde{Q}_a$, subject to the F-term relations of the quiver superpotential. From the previous mapping we see that mirror symmetry acts as a strong-weak type of duality, as it exchanges mesons or polynomials in chiral fields (perturbative degrees of freedom) with monopole operators (non-perturbative degrees of freedom). The global symmetry on both sides is $G_F = SU(F) \times U(1)_\tau \times U(1)_R \times U(1)_J$ ¹², which is manifest on the SQED side and emergent on the linear quiver side (here $U(1)_\tau$ denotes the axial symmetry). In particular, the $SU(F)$ flavor symmetry on the quiver side arises as the non-Abelian IR enhancement of the $U(1)^{F-1}$ topological symmetries present in the UV. This enhancement, from an Abelian $U(1)^{F-1}$ to the non-Abelian $SU(F)$, is a dramatic infrared phenomenon with no perturbative explanation, and is one of the most striking predictions of mirror symmetry.

The global symmetry charges of the elementary fields, consistent with the mapping proposed above, are summarized in the table below.

	$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_J$		$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_J$	
Φ	1	1	0	0	Φ_a	1	-1	2	0	(2.95)
q	$\overline{\square}$	$-\frac{1}{2}$	1	0	$Q_{2,3,\dots,F}$	1	$\frac{1}{2}$	0	0	
$\tilde{q}$	$\square$	$-\frac{1}{2}$	1	0	$\tilde{Q}_{2,3,\dots,F}$	1	$\frac{1}{2}$	0	0	
$\mathfrak{M}^\pm$	1	$\frac{F}{2}$	0	± 1	$Q_1, \tilde{Q}_1$	1	$\frac{1}{2}$	0	$+1/-1$	

¹²The actual global symmetry is $SU(F) \times SO(4)_R \times U(1)_J$, but since we will be interested in breaking the $\mathcal{N} = 4$ algebra into $\mathcal{N} = 2$, we already picked an $\mathcal{N} = 2$ algebra, such that $SO(4)_R \rightarrow U(1)_R \times U(1)_\tau$, with $U(1)_R$ being the R-symmetry of the chosen $\mathcal{N} = 2$ algebra. See also Sec. 2.2 of [140]

where Φ_a , $a = 1, \dots, F-1$, denotes any of the adjoint chirals of the linear quiver. Note in particular that the axial charge $\tau[\mathfrak{M}^\pm] = F/2$ of the SQED monopoles arises entirely from the quantum correction (2.35), and precisely matches the charge $F \cdot \frac{1}{2}$ of the products $Q_1 \cdots Q_F$ and $\tilde{Q}_1 \cdots \tilde{Q}_F$ on the mirror side.

There are many non-trivial checks for the presented IR duality. In addition to the operator map and the match of global symmetries, there is the identity of partition functions presented in [141], where mirror symmetry for Abelian QED is described as a generalized Fourier transform. Moreover, the partition function on S^3 for $\mathcal{N} \geq 2$ theories can be computed exactly [142], and can be verified to agree on both sides of the duality [143]. Finally, the superconformal indices may also be computed and matched [132].

The string theory point of view

An interpretation of mirror symmetry has been given in terms of the $SL(2, \mathbb{Z})$ duality group of Type IIB string theory in 10 dimensions [40]. The starting point is a supersymmetric configuration in string theory that preserves 8 supercharges, the same as 3d $\mathcal{N} = 4$. The basic ingredients are Dirichlet threebranes (D3) whose world-volume spans the directions x^0, x^1, x^2 and x^6 , Neveu–Schwarz fivebranes (NS5) whose world-volume spans the directions x^0, x^1, x^2, x^3, x^4 and x^5 , and Dirichlet fivebranes (D5) whose world-volume spans the directions x^0, x^1, x^2, x^7, x^8 and x^9 . In other words, the brane configuration is summarized as

$$\begin{array}{cccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{D3} & \circ & \circ & \circ & - & - & - & \circ & - & - & - \\
\text{NS5} & \circ & \circ & \circ & \circ & \circ & \circ & - & - & - & - \\
\text{D5} & \circ & \circ & \circ & - & - & - & - & \circ & \circ & \circ
\end{array} \tag{2.96}$$

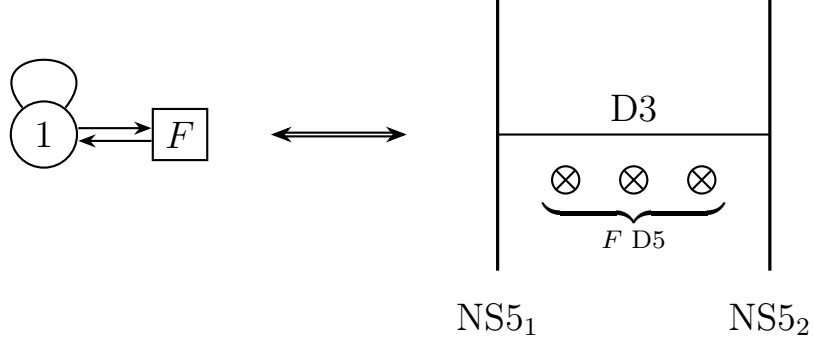
The field theory defined on the world-volume along the common directions x^0, x^1 and x^2 is macroscopically a 2+1-dimensional $\mathcal{N} = 4$ SQFT. The particular low-energy field theory may be identified through the following considerations:

1. If a threebrane has both ends on a NS fivebrane, the effective 2+1 dimensional theory is that of a $U(1)$ vector multiplet. More generally, given N parallel threebranes suspended between the same two NS fivebranes, we get N $U(1)$ vector multiplets; the gauge symmetry is enhanced to a non Abelian $U(N)$ gauge group via Chan-Paton factors when the parallel threebranes become coincident.
2. The world-volume theory for a threebrane stretched between two D fivebranes is that of a massless hypermultiplet. F parallel threebranes suspended between the same two D fivebranes give F massless hypermultiplets which, when their position become coincident, give an enhanced $U(F)$ global symmetry.
3. The last case is that of a threebrane which terminates on a D fivebrane at one end and an NS fivebrane at the other end. Such a threebrane has no moduli since supersymmetry prevents any “moving” of the brane ends along the D5 or the NS5. There are no massless modes on the D3 world-volume, since these are projected out at one end or the other. The infrared theory has a unique vacuum with a mass gap.

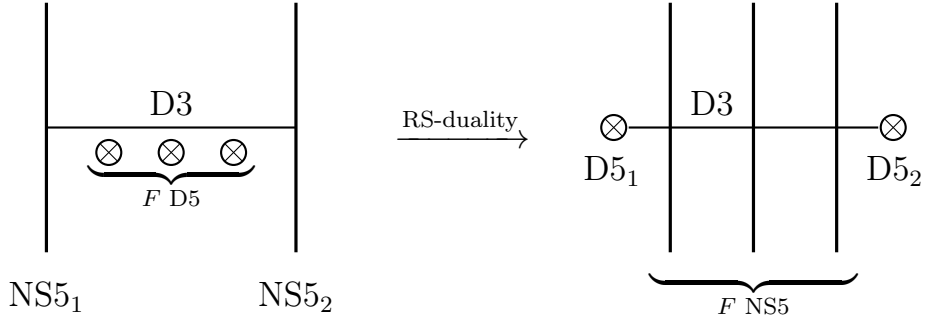
Now, an $SL(2, \mathbb{Z})$ transformation by the matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.97}$$

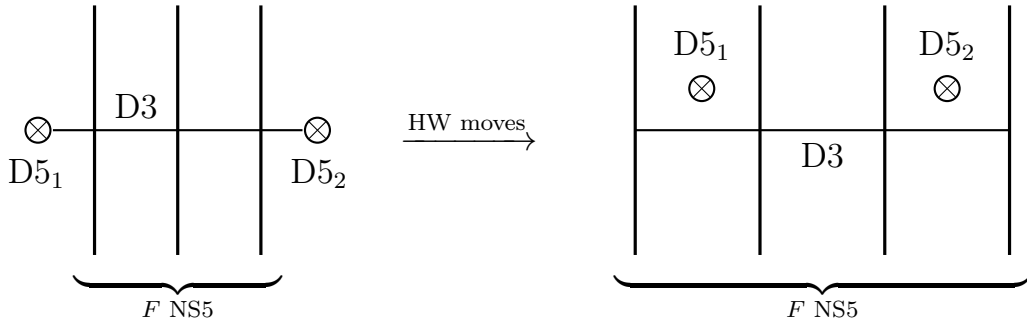
exchanges the two types of fivebranes, while leaving the D3 brane invariant. By combining this with a rotation R , that maps x^j to x^{j+4} and x^{j+4} to $-x^j$ for $j = 3, 4, 5$ and leaves the other coordinates invariant, we obtain a transformation which maps the class of configurations we are considering to itself, while exchanging the two kinds of fivebranes. The duality action represented by RS is what the authors of [40] identify as mirror symmetry from the string theory point of view. To see how this works in practice, let us use the string theory interpretation to derive mirror symmetry for 3d $\mathcal{N} = 4$ Abelian QED. The brane setup corresponding to SQED with F flavors is presented below.



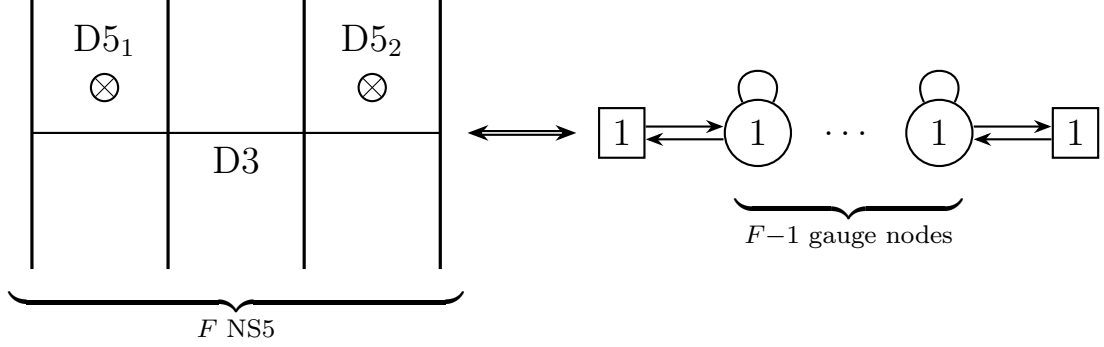
Now we can apply the *mirror* RS transformation to the brane setup, which amounts to the exchange of D5 with NS5 branes, which gives



What is left to do is to identify the low energy theory described by the massless degrees of freedom on the world-volume of the D3 brane. To be able to apply the low energy field theory dictionary defined so far, we need a configuration where all the D3 branes are suspended between NS5. To obtain this configuration, we apply the *Hanany-Witten* (HW) move [40], which amounts to exchange the position of a $D5$ with an $NS5$, while also creating or destroying an $D3$ brane (as required by charge conservation). By applying HW moves we are able to move the two D5 inside, between the NS5, as in the following drawing:



Now we can identify the low energy theory living on the D3 branes world-volume: each segment of the D3 stretched between two adjacent NS5 is interpreted as an $U(1)$ vector multiplet, an NS5 with a D3 segment ending on both its sides is interpreted as a bifundamental hypermultiplet charged under the corresponding $U(1)$ gauge groups, while a D5 with a D3 segment ending on both its sides is interpreted as a single flavor for the corresponding gauge group. In the end we obtain the gauge theory presented below



which is exactly the mirror dual proposed in the previous section for SQED with F flavors. The same discussion can easily be extended to find the mirror dual of $\mathcal{N} = 4$ SQCD with gauge group $U(N)$ and F flavors, or of a generic $\mathcal{N} = 4$ linear quiver gauge theory [40].

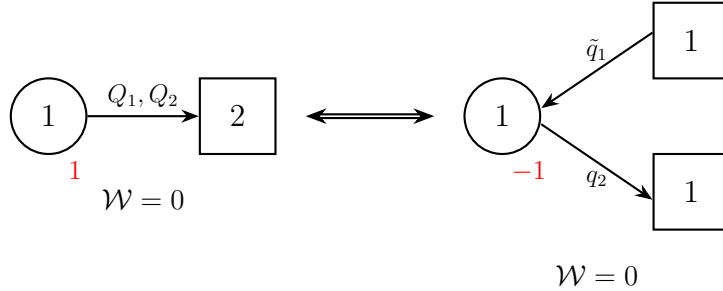
From $\mathcal{N} = 4$ mirror symmetry to $\mathcal{N} = 2$ Chern–Simons theories. The original $\mathcal{N} = 4$ mirror symmetry reviewed above is the conceptual starting point for the $\mathcal{N} = 2$ CS mirror symmetry that forms the main contribution of this thesis. The bridge between the two is provided by real mass deformations: starting from an $\mathcal{N} = 4$ mirror pair, one can deform both sides by turning on real masses for some of the matter fields. As reviewed in section 2.1.4, integrating out the resulting massive fermions generates half-integer shifts in the CS levels. In this way, one obtains $\mathcal{N} = 2$ theories with CS interactions, and the $\mathcal{N} = 4$ mirror map descends to a duality between the CS-deformed theories.

To make this concrete, consider the prototypical $\mathcal{N} = 4$ self-mirror theory: SQED with $F = 2$ electrons (the $T(SU(2))$ theory of [113]), equivalently described by the quiver $[1] - U(1) - [1]$. In $\mathcal{N} = 2$ language, it comprises a $U(1)$ gauge node, two hypermultiplets $(Q_1, \tilde{Q}_1)$ and $(Q_2, \tilde{Q}_2)$, an adjoint chiral Φ (the extra scalar in the $\mathcal{N} = 4$ vector multiplet), and superpotential $\mathcal{W} = \Phi(Q_1\tilde{Q}_1 + Q_2\tilde{Q}_2)$.

We deform by a positive real mass for $\tilde{Q}_1$ and $\tilde{Q}_2$, breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2$. This also implies an opposite (negative) real mass for the adjoint chiral Φ ¹³. The mass deformation must moreover be accompanied by a Fayet–Iliopoulos term $\xi = -m$ for the gauge $U(1)$: the massive fields generate a one-loop linear term in the effective D-term equation, and only if the bare FI parameter is tuned against it does a vacuum survive at the origin of the Coulomb branch, where Q_1 and Q_2 remain massless. This is the Abelian instance, $F = 2$, of the tuning $\eta = -\frac{F}{2}\tau$ derived in section 3.1.2. Integrating out $\tilde{Q}_1$ and $\tilde{Q}_2$ at scales $\mu \ll |m|$ shifts the CS level as $k \rightarrow k + \text{sign}(m)$. Integrating out the adjoint Φ does not shift the CS level since it is neutral under the $U(1)$. Starting from $k = 0$ and $\text{sign}(m) = +1$, the electric theory becomes $\mathcal{N} = 2$ $U(1)_1$ with two fundamentals Q_1 and Q_2 and vanishing superpotential $\mathcal{W} = 0$.

¹³This corresponds to a real mass for the flavor $U(1)_A$ symmetry with charges $(Q_i : 0, \tilde{Q}_i : +1, \Phi : -1)$: invariance of the superpotential $\mathcal{W}$ requires $\text{charge}(\Phi) = -\text{charge}(\tilde{Q}_i) = -1$, so the real mass m for $\tilde{Q}_i$ simultaneously gives Φ a real mass $-m$.

On the mirror side $[1] - U(1) - [1]$, the image of this deformation is dictated by the mirror map: the axial mass maps to an axial mass of opposite sign, while the FI term maps to a real mass for the mirror flavor symmetry. It is this second ingredient that makes the deformation act asymmetrically on the mirror flavors¹⁴: the combined deformation gives a negative real mass to the two $\mathcal{N} = 2$ chirals $q_1, \tilde{q}_2$ and a positive real mass to the adjoint φ , while $\tilde{q}_1, q_2$ remain massless. Integrating out the massive fields yields $\mathcal{N} = 2$ $U(1)_{-1}$ SQED with the two chiral fields $\tilde{q}_1, q_2$; the sign flip $k = +1 \rightarrow -1$ reflects the action of mirror symmetry as a parity transformation on the CS level. The resulting duality between $U(1)_1$ with 2 fundamentals and $U(1)_{-1}$ with a fundamental and an anti-fundamental constitutes the prototypical example of $\mathcal{N} = 2$ CS mirror symmetry studied in this thesis, and is represented by the following quiver identity:



The action of mirror symmetry is also visible from the residual strong-weak mapping of chiral gauge invariant operators: the monopole operator $\mathfrak{M}^+$ of the $U(1)_1$ theory, which is gauge invariant without any dressing since the electric charge induced by the CS term cancels against the contribution of the fermion zero modes, cf. eq. (2.37), is mapped to the single remaining meson $\tilde{q}_1 q_2$ on the right side.

$SL(2, \mathbb{Z})$ action and the S-wall construction. The string theory interpretation of mirror symmetry for Abelian theories can be formulated more systematically using the $SL(2, \mathbb{Z})$ duality group of Type IIB string theory [40]. In the field theory language, this translates into an $SL(2, \mathbb{Z})$ action on the space of 3d Abelian theories [144], generated by two elementary operations. The T -transformation adds a background CS term at level 1 for the gauge field, shifting $k \rightarrow k + 1$. The S -transformation is more subtle: acting on a theory with dynamical gauge field A and background field B for the topological symmetry $U(1)_J$, it *exchanges* the two by coupling them through a BF term and then making the formerly background field dynamical:

$$S : \quad \mathcal{L} \longrightarrow \mathcal{L} + \frac{1}{2\pi} \epsilon^{\mu\nu\rho} B_\mu \partial_\nu A_\rho, \quad \text{then } A \leftrightarrow B. \quad (2.98)$$

This is the field-theory avatar of the exchange of D5 and NS5 branes. Iterating S and T generates the full $SL(2, \mathbb{Z})$ orbit of a given Abelian theory. The resulting algorithmic procedure, sometimes called the *S-wall construction* [113, 145], provides a systematic way to generate dual theories by acting with sequences of S and T transformations. This framework underlies the dualization algorithm used in the original chapters of this thesis, and we refer the reader to [40, 113] for the details of the construction. Moreover, gauge theories with gauge groups $Sp(2N)$ and $SO(N)$ can also be described in string theory by introducing orientifold planes (Op-planes) and placing the D-branes on top of them [146]. Mirror dual theories can be found for the resulting configurations as well [42, 133].

¹⁴The axial mass alone would give the same real mass to all four mirror chirals, and the flow would land on the parity image of the electric theory rather than on a genuinely new dual frame.

2.4 SUSY localization

As mentioned in the previous sections, a very non-trivial check of IR related dualities is given by the matching of the corresponding partition functions. These are RG invariant objects, and thus can be computed where the theory is perturbative (e.g. in the UV for asymptotically free theories) and the resulting values will match also in the IR. The problem is that computing exactly the partition function of a generic QFT is highly non-trivial. Thankfully, in supersymmetry we know that a restricted set of protected operators, the BPS operators, have charges that satisfy an algebra bound and are thus protected against renormalization. We can then reduce the infinite-dimensional partition function to a finite-dimensional BPS locus through a procedure analogous to equivariant localization [147, 148]. The resulting technique, which goes under the name of *supersymmetric localization*, has been successfully applied to many physical theories, for instance to 4d $\mathcal{N} = 2$ theories in the so called Ω -background [149] and on S^4 [150]. We briefly review here the localization argument. In the computations we review below, the relevant supercharge is one of the scalar supercharges that survives on the curved background (e.g. on S_b^3 or $S^2 \times S^1$), chosen so that it squares to a compact bosonic symmetry (a combination of gauge transformations and R-symmetry rotations). The specific choice is described in detail in [142, 151].

Suppose we have a fermionic symmetry $\mathcal{Q}$ of the action:

$$\mathcal{Q} S = 0 \tag{2.99}$$

Since $\mathcal{Q}$ is fermionic, its square is either zero or a bosonic symmetry δ_B of the action. We are interested in computing the path-integral $Z = \int \mathcal{D}\phi e^{-S[\phi]}$. We consider instead the following path-integral

$$Z(t) = \int \mathcal{D}\phi e^{-S[\phi] - t\mathcal{Q}V[\phi]} \quad \text{with} \quad \delta_B V = 0, \tag{2.100}$$

which depends on the parameter t and on some functional V . The dependence on t is

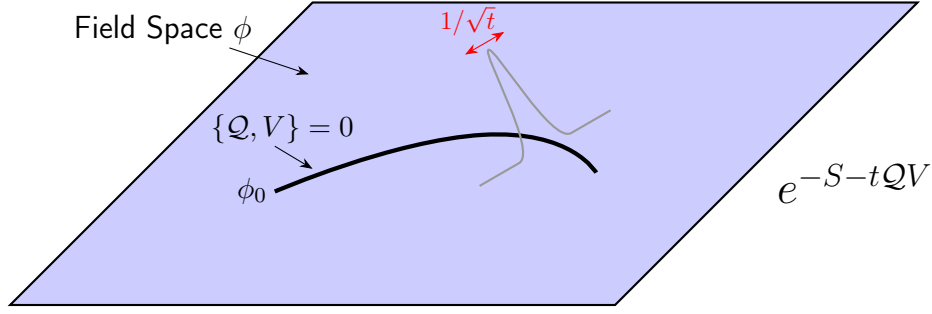
$$\frac{\partial Z}{\partial t} = - \int \mathcal{D}\phi \mathcal{Q}V e^{-S - t\mathcal{Q}V} = - \int \mathcal{D}\phi \mathcal{Q} (V e^{-S - t\mathcal{Q}V}) . \tag{2.101}$$

Under the assumption that the integration measure $\mathcal{D}\phi$ is invariant under the supersymmetry transformation generated by $\mathcal{Q}$, i.e. the fermionic symmetry is non-anomalous, the integral of a total $\mathcal{Q}$ -variation vanishes:

$$\frac{\partial Z}{\partial t} = 0 . \tag{2.102}$$

The argument still holds if we insert $\mathcal{Q}$ -invariant operators, $\mathcal{Q} \mathcal{O}[\phi] = 0$. This argument yields the important result that the partition function or VEV is not modified by the deformation term $\mathcal{Q}V$. Now, at $t = 0$, $Z(0)$ is the original path-integral we want to compute. Suppose we are able to find some functional V such that $\{\mathcal{Q}, V\}$ is positive semi-definite, i.e. the bosonic part satisfies $(\mathcal{Q}V)_{bos} \geq 0$ along the integration contour where the path-integral is defined. Then, in the limit $t \rightarrow +\infty$ all field configurations with $(\mathcal{Q}V)_{bos} > 0$ are infinitely suppressed. Therefore the path-integral *localizes* to the bosonic zeros ϕ_0 of $\mathcal{Q}V$, which are also the saddle points:

$$\phi = \phi_0 + t^{-1/2} \hat{\phi} . \tag{2.103}$$



We can Taylor expand the action around ϕ_0 :

$$S + tQV = S[\phi_0] + (QV)^{(2)}[\hat{\phi}] + \mathcal{O}(t^{-1/2}), \quad (2.104)$$

therefore only the on-shell action S_0 and the quadratic expansion of QV around the fixed point matters. By performing the Gaussian integration, we obtain the following one-loop exact localization formula:

$$Z = \int_{\text{BPS locus } \{\phi_0\}} \mathcal{D}\phi_0 e^{-S[\phi_0]} \frac{1}{\text{SDet}'(QV)_{\phi_0}^{(2)}} \quad (2.105)$$

The superdeterminant is the ratio of the bosonic and fermionic determinants. If $\{\phi_0\}$ is a bosonic moduli space, we remove the zeros of SDet and integrate over the bosonic zero-modes. If $\{\phi_0\}$ has fermion zero-modes, they have to be absorbed, either by operator insertions or by expanding S in fermions.

If the space of fixed points $\{\phi_0\}$ is finite-dimensional, we have then reduced the infinite dimensional path-integral to an ordinary integral which we may be able to solve.

The most common choice for the functional V is

$$V = \sum_{\text{fermions } \psi} (Q\psi)^\dagger \psi, \quad (2.106)$$

from which follows

$$(QV)_{\text{bos}} = \sum_{\text{fermions } \psi} (Q\psi)^\dagger Q\psi \geq 0 \implies \text{localize to } Q\psi = 0 \quad (2.107)$$

which are the BPS equations.

The computation of SDet is the most difficult step of this technique, but because of supersymmetry huge cancellations are expected among the eigenvalues of SDet , so that there are cases where this computation may be carried out.

For example, computation on the round sphere S^3 have been done in [142, 152, 153] and on the squashed three-sphere S_b^3 in [151, 154]. Instead computations on $S^2 \times S^1$, a.k.a. the superconformal index, have been performed in [132, 155–157].

2.4.1 The S_b^3 partition function

Let us review the results for the computation of the partition function on the squashed three-sphere S_b^3 , since it will play a crucial role in the upcoming chapters, see [158] for a review of localization on three-manifolds. There are many ways to deform a three-sphere while preserving supersymmetry, see for example [151]. However, all these may be labeled by a single complex parameter, the “squashing parameter” b , such that the partition

function and supersymmetric observables depend on the background only through b . The general metric we consider is:

$$ds^2 = f(\chi)^2 d\chi^2 + \ell_1^2 \cos^2 \chi d\varphi^2 + \ell_2^2 \sin^2 \chi d\theta^2. \quad (2.108)$$

Here ℓ_i are constants and f is an arbitrary smooth positive function on $[0, \frac{\pi}{2}]$, with the only restriction that $f(\chi = 0) = \ell_1$ and $f(\chi = \frac{\pi}{2}) = \ell_2$, to avoid conical singularities along these circles. The squashing parameter b is defined as

$$b \equiv \sqrt{\ell_1/\ell_2}. \quad (2.109)$$

What we need to list now are the matter multiplet and vector multiplet contributions to the one-loop determinant SDet.

Chiral multiplet We study the matter fields first, focusing on the case with a single chiral multiplet $\Phi = (\phi, \psi, F)$ with R-charge r , coupled to an Abelian vector multiplet $V = (\sigma, \lambda, v_\mu, D)$. The first step is to determine the space of BPS configurations. It can be shown that these correspond to constant values for σ and D , labeled by a Lie algebra element $\hat{\sigma}_0$:

$$\sigma = -D/H \equiv \frac{\hat{\sigma}_0}{\ell} = \text{constant}, \quad (2.110)$$

where $\ell = \sqrt{\ell_1 \ell_2}$, and $H = -i/f$, with f being the smooth function appearing in the general metric (2.108). What we need now is the spectrum of the kinetic operators Δ_ϕ and Δ_ψ for bosons and fermions. Then the ratio of determinants may be written as

$$Z_{chi}^r(\hat{\sigma}) = \prod_{m,n} \frac{mb + nb^{-1} + i\hat{\sigma} + \frac{Q}{2}(2-r)}{mb + nb^{-1} - i\hat{\sigma} + \frac{Q}{2}r} \equiv s_b \left(\frac{iQ}{2}(1-r) - \hat{\sigma} \right), \quad (2.111)$$

where we introduced

$$Q \equiv b + b^{-1}, \quad (2.112)$$

and s_b is the double-sine function, defined for general b as

$$s_b(x) = \prod_{m,n \geq 0} \frac{(m + \frac{1}{2})b + (n + \frac{1}{2})b^{-1} - ix}{(m + \frac{1}{2})b + (n + \frac{1}{2})b^{-1} + ix}. \quad (2.113)$$

The double-sine function satisfies several useful properties that appear throughout the partition function computations in this thesis [159, 160]:

- **Reflection formula:** $s_b(x) s_b(-x) = 1$, which follows directly from the product definition.
- **Poles and zeros:** $s_b(x)$ has poles at $x = i[(m + \frac{1}{2})b + (n + \frac{1}{2})b^{-1}]$ and zeros at $x = -i[(m + \frac{1}{2})b + (n + \frac{1}{2})b^{-1}]$ for $m, n \geq 0$.
- **Functional equations:** $s_b(x \mp ib/2) = 2 \cosh(\pi b x)^{\pm 1} s_b(x \pm ib/2)$, which allow recursive evaluations.

Note that from the reflection formula follows a useful identity:

$$Z_{chi}^r(\hat{\sigma}) Z_{chi}^{2-r}(-\hat{\sigma}) = 1. \quad (2.114)$$

This is a consequence of the fact that a superpotential term $W = XY$ causes the chirals X and Y to gain mass, and so they do not contribute to the low energy theory, and so must not contribute to the partition function.

Vector multiplet Suppose we have a non-Abelian gauge group G , and consider a mode of the vector multiplet corresponding to a root α . On the squashed sphere, since the roots come in positive/negative pairs, we can write:

$$Z_{1-loop}^{gauge}(\hat{\sigma}_0) = \prod_{\alpha \in Ad(G)} Z_{chi}^{r=2}(\alpha(\hat{\sigma}_0)) = \prod_{\alpha > 0} s_b \left(-\frac{iQ}{2} + \alpha(\hat{\sigma}_0) \right) s_b \left(-\frac{iQ}{2} - \alpha(\hat{\sigma}_0) \right) \quad (2.115)$$

Using the functional equations of the double-sine function, this can be rewritten in the perhaps more familiar form

$$Z_{1-loop}^{gauge}(\hat{\sigma}_0) = \prod_{\alpha > 0} 4 \sinh(\pi b \alpha(\hat{\sigma}_0)) \sinh(\pi b^{-1} \alpha(\hat{\sigma}_0)) , \quad (2.116)$$

which vanishes at the fixed points of the Weyl group, as expected from the residual gauge symmetry there.

Classical contribution The only part of the original action S that contributes at the BPS locus is the Chern–Simons term. We find

$$S_{CS}[\hat{\sigma}_0] = \pi i \text{Tr}_{CS}(\sigma_0)^2 \quad (2.117)$$

Background fields An important comment regarding mass deformations is in order. As mentioned, real mass parameters and FI terms may be introduced by turning on BPS configurations for background gauge fields coupled to global symmetries. Denoting as H the flavor symmetry group, a real mass parameter is labeled by an element $\hat{m}$ of the Lie algebra H . By decomposing the chiral multiplet into weights (ρ, ω) of the representation $\tilde{R}$ of $G \times H$ in which they sit, the 1-loop determinant with the real mass turned on is:

$$\prod_{(\rho, \omega) \in \tilde{R}} Z_{chi}^r(\rho(\hat{\sigma}_0) + \omega(\hat{m})) \quad (2.118)$$

Similarly, an FI term is a classical background $U(1)$ gauge multiplet which couples to the dynamical gauge field via a mixed CS term. Thus it modifies the classical contribution via a term

$$S_{FI} = 2\pi i \hat{\zeta}^a \lambda_a(\hat{\sigma}_0) \quad (2.119)$$

The R-charge of a chiral appears in a complex combination with the real mass $\hat{m}$, and a shift of the R-symmetry is implemented as

$$\hat{m} \rightarrow \hat{m} + \frac{iQ}{2} \hat{\mu} \quad (2.120)$$

Full partition function By collecting the above ingredients, we obtain the partition function for the theory by integrating over BPS configurations labeled by $\hat{\sigma}_0$ and by using the Weyl integration formula:

$$\begin{aligned} Z_{S_b^3}(\hat{\zeta}, \hat{m}) &= \frac{1}{|\mathcal{W}|} \int_{\mathfrak{h}} d\hat{\sigma}_0 e^{-\pi i \text{Tr}_{CS}(\hat{\sigma}_0^2) - 2\pi i \hat{\zeta}^a \lambda_a(\hat{\sigma}_0)} \prod_{\alpha > 0} s_b \left(-\frac{iQ}{2} + \alpha(\hat{\sigma}_0) \right) s_b \left(-\frac{iQ}{2} - \alpha(\hat{\sigma}_0) \right) \\ &\quad \times \prod_i \prod_{(\rho, \omega) \in R_i} Z_{chi}^{r_i}(\rho(\hat{\sigma}_0) + \omega(\hat{m})) \end{aligned} \quad (2.121)$$

where $\mathcal{W}$ is the Weyl group of the gauge group G and $\mathfrak{h}$ is the Cartan subalgebra. Note that the partition function depends on the sphere geometry only through the parameter $b = \sqrt{\ell_1/\ell_2}$.

2.4.2 F-theorem and F-maximization

An important application of localization is to define a 3d F -theorem. This may be defined in terms of the S^3 partition function (which is just the $b = 1$ specialization of the S_b^3 partition function computed in the previous section) as follows [161]

$$F = -\log |Z_{S^3}| \quad (2.122)$$

The F -theorem then states that

$$F_{UV} > F_{IR}, \quad (2.123)$$

where F_{UV} is the free energy of a UV CFT, which flows to low energies to another CFT with free energy F_{IR} . This F-theorem was proven in [162], by relating the free energy to the entanglement entropy of a disk in flat space, and then using the strong-subadditivity of entanglement entropy.

To compute the S^3 partition function, we need to pick a $U(1)_R$ symmetry. A general R-symmetry is obtained by mixing the UV R-symmetry with any $U(1)$ flavor symmetries F_i of the theory:

$$R_t = R_{UV} + \sum_i t_i F_i \quad (2.124)$$

So in general we are computing a parametrically dependent partition function $Z_{S^3}(t_i)$. However, in the IR the R-symmetry is constrained by the superconformal algebra, in particular the current of the superconformal R-symmetry sits in the same super-multiplet as the traceless stress energy tensor of the IR CFT. Only for this unique R-symmetry the computed free energy will agree with that of the conformally coupled IR CFT. It was proven in [152] that the superconformal R-symmetry is the one that extremizes F , namely:

$$\frac{\partial}{\partial t_i} F(t_i) = -\frac{\partial}{\partial t_i} \log |Z_{S^3}| = 0 \quad \text{for the superconformal R-symmetry} \quad (2.125)$$

This happens because the derivatives of F are related to 1-point functions of local operators on S^3 , which must vanish in a CFT. Moreover, one can show that the second derivatives with respect to the t_i are related to 2-point functions of the currents, which must be positive by unitarity, and thus F is actually maximized at the superconformal R-charge. This procedure, known as F -maximization, gives an algorithmic way to compute the superconformal R-symmetry of a 3d $\mathcal{N} = 2$ SCFT, provided there are no accidental $U(1)$ flavor symmetries in the IR. It is the 3d analogue of a -maximization in 4d $\mathcal{N} = 1$ theories [163]. Finally, note that the F-theorem is closely related to F-maximization: if we deform the theory by adding a superpotential term which breaks some flavor symmetry, then, provided no accidental symmetries emerge in the IR, this reduces the space of the t_i s, which means that F-maximization lands us on a smaller F than in the UV theory.

2.4.3 $S^2 \times S^1$ partition function

In this section we review another equally important RG invariant: the $S^2 \times S^1$ partition function [132, 155, 164]. The partition function on this space has the interpretation, for

conformal theories, of a *superconformal index*, which counts local operators in the flat space theory. We first review the basic ingredients entering in the partition function.

Localization on $S^2 \times S^1$

The metric of the space is now given by

$$ds^2 = dx^2 + d\theta^2 + \sin^2 \theta d\phi^2 \quad (2.126)$$

with $x \sim x + \tau$.

By writing down the $\mathcal{N} = 2$ Yang-Mills action on this background (see for example [158]) we have:

$$\mathcal{L}_{YM} = \text{Tr} \left(\left(\frac{1}{2} \star F_\mu + D_\mu \sigma + \delta_{\mu 3} \sigma \right)^2 - \frac{1}{2} D^2 - i \tilde{\lambda} \gamma^\mu (D_\mu + \frac{1}{2} \delta_{\mu 3}) \lambda + i \tilde{\lambda} [\sigma, \lambda] \right). \quad (2.127)$$

The BPS locus where this action vanishes is

$$\star F_\mu + D_\mu \sigma + \delta_{\mu 3} \sigma = D = 0, \quad (2.128)$$

where $\star F_\mu$ is the dual field-strength, σ is the lowest component real scalar of the vector multiplet, and D is the auxiliary scalar. These equations are solved by the following BPS configurations:

$$A_\mu = \alpha dx + \mathbf{m} A_{Dir}, \quad \sigma = -\frac{\mathbf{m}}{2}, \quad (2.129)$$

where $\mathbf{m}$ is an element of the coweight lattice Λ_{cw} , labeling the constant flux of $F_{\mu\nu}$ through S^2 . A_{Dir} is the unit-flux Dirac monopole on S^2 , i.e. $dA_{Dir} = \frac{1}{2} \text{vol}(S^2)$, while α lies in the Cartan subalgebra, so that $[\alpha, \mathbf{m}] = 0$.

Chiral multiplet Consider a single chiral multiplet with R-charge r , coupled to a $U(1)$ gauge field holonomy $e^{i\tau\alpha}$ and flux m . After significant cancellations between bosons and fermions, the one-loop determinant reduces to

$$Z_{1-loop}^{chiral} = \prod_{n \in \mathbb{Z}} \prod_{j=0}^{\infty} \frac{j + |m| + 1 - \frac{r}{2} - \frac{1}{\tau}(2\pi i n - i\alpha)}{j + |m| + \frac{r}{2} + \frac{1}{\tau}(2\pi i n - i\alpha)}, \quad (2.130)$$

where the first product is over S^1 angular momenta $2\pi i n / \tau$, $n \in \mathbb{Z}$. This can be performed using $\prod_{n \in \mathbb{Z}} (2\pi i n + z) = e^{-z/2} (1 - e^z)$. Then, after some regularization, we obtain

$$Z_{chi}^r(z, m; q) = e^{-i\pi m|m|/2} (q^{1-r/2} z^{-1})^{|m|/2} \prod_{j=0}^{\infty} \frac{1 - q^{1-r/2+|m|/2+j} z^{-1}}{1 - q^{r/2+|m|/2+j} z}, \quad (2.131)$$

where we defined $q = e^{-\tau}$ and $z = e^{i\tau\alpha}$. By introducing the infinite q -pocharhammer symbol as $(z; q) = \prod_{j=0}^{\infty} (1 - zq^j)$, the previous formula can be rewritten as [145]:

$$Z_{chi}^r(z, m; q) = e^{-i\pi m|m|/2} (q^{1-r/2} z^{-1})^{|m|/2} \frac{(q^{1-r/2+|m|/2} z^{-1}; q)}{(q^{r/2+|m|/2} z; q)}, \quad (2.132)$$

Gauge multiplet Similarly to the S_b^3 partition function, the vector multiplet contribution is the same of an R-charge 2 chiral in the adjoint representation of the gauge group G , which can be written as

$$Z_{1-loop}^{gauge} = \prod_{\alpha \in Ad(G)} q^{-|\alpha(\mathbf{m})|/2} (1 - z^\alpha q^{|\alpha(\mathbf{m})|}) \quad (2.133)$$

Classical contribution and background fields Again, the only classical contribution comes from the Chern–Simons term. The naive computation seems to give

$$S_{CS} = \frac{i}{4\pi} \int_{S^2 \times S^1} \text{Tr}_{CS}(A \wedge dA) = \frac{i}{4\pi} \int_{S^2 \times S^1} \text{Tr}_{CS}(\alpha dx \wedge \mathbf{m} \text{vol}(S^2)) = i\tau \text{Tr}_{CS}(\alpha \mathbf{m}). \quad (2.134)$$

However, as conjectured in [25], the correct contribution should contain an additional phase:

$$e^{-S_{CS}} = e^{-i \text{Tr}_{CS}(\tau \alpha \mathbf{m} + \pi \mathbf{m}^2)} \quad (2.135)$$

We will consider mainly $U(N)$ gauge theories at level k in this thesis, for which the previous phase reads

$$e^{-S_{CS}} = \prod_i (-1)^{km_i} z_i^{-km_i} \quad (2.136)$$

If we have a flavor symmetry H , we can also turn on background gauge fields and put them in fixed BPS configurations. These are labeled by a holonomy $\mu \in H$ along the S^1 and a flux n in the co-weight lattice Λ_H through the S^2 . This modifies the chiral multiplet in the expected way:

$$Z_{chi}^r(z, \rho(m); q) \rightarrow Z_{chi}^r(z \mu^\omega, \rho(m) + \omega(s); q) \quad (2.137)$$

where ω are the weights of the flavor representation. Moreover, a background field with holonomy w and flux n coupled to the topological symmetry $U(1)_J$ of a dynamical $U(1)$ gauge field, with holonomy z and flux m , leads to the insertion:

$$e^{-S_{FI}} = z^{-n} w^{-m} \quad (2.138)$$

Full partition function Putting all the previous ingredients together, we find the full $S^2 \times S^1$ partition function:

$$\begin{aligned} Z_{S^2 \times S^1}(\mu, s, w, n; q) &= \sum_{m \in \Lambda_{cw}} \frac{1}{|\mathcal{W}|} \int_{\mathbb{T}^{r_G}} \prod_{i=1}^{r_G} \frac{dz_i}{z_i} e^{-S_{CS} - S_{FI}} \prod_{\alpha \in Ad(G)} q^{-|\alpha(\mathbf{m})|/2} (1 - z^\alpha q^{|\alpha(\mathbf{m})|}) \\ &\quad \times \prod_i \prod_{(\rho, \omega) \in R_i} Z_{chi}^{r_i}(z^\rho \mu^\omega, \rho(m) + \omega(s); q) \end{aligned} \quad (2.139)$$

where $\mathcal{W}$ is the Weyl group of G and $\mathbb{T}^{r_G}$ is the maximal torus of G .

The superconformal index

There is an alternative definition of the $S^2 \times S^1$ partition function which makes manifest its topological origin, called the *superconformal index* [124, 165, 166]. For superconformal theories in D dimensions, it can be defined as a trace over the space of local operators of

the theory, weighted by global symmetry charges such that it receives contributions only from protected short multiplets.

Under continuous deformations, short multiplets can enter or leave the spectrum only if they can combine to form long multiplets, which by construction do not contribute to the index. Therefore the index is invariant under such continuous deformations. In detail, consider a supercharge $\mathcal{Q}$, and let $\mathcal{F}_i$ be a complete basis of global symmetries commuting with $\mathcal{Q}$. Then we define

$$\mathcal{I}(\mu_i, \beta) = \text{Tr}(-1)^F e^{-\beta\delta} e^{\mu_i F_i} \quad (2.140)$$

where $\delta = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$. The factor of $(-1)^F$ signals that this is a Witten index [21]: states which are not annihilated by δ come in boson/fermion pairs, and cancel out of the trace, so that it only receives contribution from states annihilated by δ : in particular it is independent of β . It is also independent of continuous deformations of the theory, and thus may be deformed to a weakly coupled point, where it can be computed to give us information about the strongly coupled CFT we started with.

Let us specialize to the case of 3d $\mathcal{N} = 2$ SCFTs. The supercharges have the following global symmetry charges:

	Δ	R	j_3	F_a
$\mathcal{Q}_1$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0
$\mathcal{Q}_2$	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$\tilde{\mathcal{Q}}_1$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0
$\tilde{\mathcal{Q}}_2$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0
$\mathcal{Q}_1^\dagger$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0
$\mathcal{Q}_2^\dagger$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0
$\tilde{\mathcal{Q}}_1^\dagger$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0
$\tilde{\mathcal{Q}}_2^\dagger$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0

where Δ is the Hamiltonian on S^2 , R is the $U(1)_R$ charge, and j_3 is the Cartan charge of the $SU(2)$ rotation group of S^2 . One computes

$$\delta = \{\mathcal{Q}_1^\dagger, \mathcal{Q}_1\} = \Delta - R - j_3 \quad (2.141)$$

From the table, we see that the charges δ , F_a and $\Delta + j_3$ all commute with $\mathcal{Q}_1$, so the appropriate index to compute here is

$$\mathcal{I}(q; \mu_a) = \text{Tr}(-1)^F e^{-\beta(\Delta - R - j_3)} e^{-\beta'(\Delta + j_3)} e^{i\rho_a F_a} \quad (2.142)$$

This trace will get contributions only from states with $\delta = \Delta - R - j_3 = 0$. This index can be interpreted as the partition function on $S^2 \times S_\tau^1$, where $\tau = \beta + \beta'$ is the coefficient of Δ in (2.142), with twisted boundary conditions for the fields:

$$\Phi(x + \tau) \sim e^{-\beta(\Delta - R - j_3)} e^{-\beta'(\Delta + j_3)} e^{i\rho_a F_a} \Phi(x) \quad (2.143)$$

However, the localization argument is more general, as it could be defined for non-conformal theories as well. Since the two definitions must agree at the IR fixed point, we can use localization to compute the partition function in the UV description, and thus learn about the spectrum of local operators in the CFT.

Extended supersymmetry It is interesting to study the index in the case of extended SUSY as well. For 3d $\mathcal{N} = 4$ supersymmetry, there is an $SU(2)_H \times SU(2)_C$ R-symmetry group, and the index takes the general form

$$\mathcal{I}(\mu_a; t, q) = \text{Tr}_{\mathcal{H}_{S^2}} (-1)^F q^{\Delta - \frac{1}{2}(R_H + R_C)} t^{R_H - R_C} \mu_a^{F_a} \quad (2.144)$$

where F_a are the flavor symmetries of the theory which commute with the $\mathcal{N} = 4$ algebra, and R_H and R_C are the Cartan generators of the two $SU(2)$ R-symmetry factors. From the $\mathcal{N} = 2$ point of view, t is just an extra flavor fugacity. There are some useful limits of this index, where it simplifies greatly and it allows us to probe interesting information about the moduli spaces of these theories [167, 168]. Consider taking $q, t^{-1} \rightarrow 0$ while holding $x = t q^{1/2}$ finite. In this limit we are computing

$$\mathcal{I}(x) = \text{Tr}_{\mathcal{H}_H} (-1)^F x^{\Delta - R_C} \quad (2.145)$$

where $\mathcal{H}_H$ denotes the subspace of states for which $\Delta = R_H$. The reason this limit computes the Hilbert series of the Higgs branch is the following: the condition $\Delta = R_H$ is precisely the BPS condition for operators that can acquire a VEV on the Higgs branch (since on the Higgs branch $\langle SU(2)_C \rangle = 0$ but $\langle SU(2)_H \rangle \neq 0$). In this subspace, the fermion number $(-1)^F$ does not contribute because the contributing operators are bosonic chiral ring elements. The trace therefore reduces to a weighted count of holomorphic functions on the Higgs branch, i.e. the Hilbert series. This limit is then denoted as the ‘‘Higgs limit’’ of the index. Similarly the ‘‘Coulomb limit’’ may be computed, where $q, t \rightarrow 0$ while $\tilde{x} = t^{-1} q^{1/2}$ is held finite, yielding the Hilbert series of the Coulomb branch. The two limits are exchanged under 3d mirror symmetry.

Chapter 3

Chiral-Planar dualization for 3d $\mathcal{N} = 2$ Chern–Simons-matter theories

The content of this chapter is essentially taken from [2] and [3].

While in the preceding chapter we briefly outlined the string-theoretic origins of 3d mirror symmetry, in this chapter we investigate the phenomenon from a purely field-theoretic perspective. The central object of study is a class of **chiral-planar mirror dualities**: IR equivalences between $\mathcal{N} = 2$ Chern–Simons-matter theories in three dimensions, where one side is a non-Abelian *chiral* quiver (containing both fundamental and anti-fundamental chiral multiplets) and the other is an *Abelian planar* quiver — a two-dimensional array of $U(1)$ gauge nodes arranged on a triangular lattice, with characteristic Chern–Simons interactions and a monopole superpotential.

3.1 A Planar Abelian Dual for CS-SQCD₃ with Fundamental Matter

In this section, we explicitly derive the $\mathcal{N} = 2$ mirror of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2})}$ SQCD with F fundamental chiral multiplets by starting from the $\mathcal{N} = 4$ mirror duality for the $U(N)$ SQCD with F flavors.

Our main tool is the S_b^3 partition function (see section 2.4.1 for a review) which allows us to follow the effect of the real mass deformations efficiently. We verify these results by checking that the resulting vacuum satisfies the F- and D-term equations [24].

3.1.1 The $\mathcal{N} = 4$ Mirror Pair

We start from the mirror duality for the $U(N)$ $\mathcal{N} = 4$ SQCD:

To study possible deformation breaking supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ we work in the so-called $\mathcal{N} = 2^*$ set-up [169] taking $U(1)_R = U(1)_{C+H}$ as the R-symmetry group, where $U(1)_C \in SU(2)_C$ and $U(1)_H \in SU(2)_H$ of the non-Abelian $\mathcal{N} = 4$ R-symmetry $SU(2)_H \times SU(2)_C$. The commutant of $U(1)_R$, which is $U(1)_\tau = U(1)_{C-H}$, is a global symmetry from the point of view of the $\mathcal{N} = 2$ theory.

However, we have the freedom to redefine $U(1)_R$ up to Abelian flavor symmetries, meaning that we can shift the R-charge by $U(1)_\tau$ transformations. Using this freedom we pick a convention in which we assign trial R-charge 1 and $U(1)_\tau$ charge $-1/2$ to the

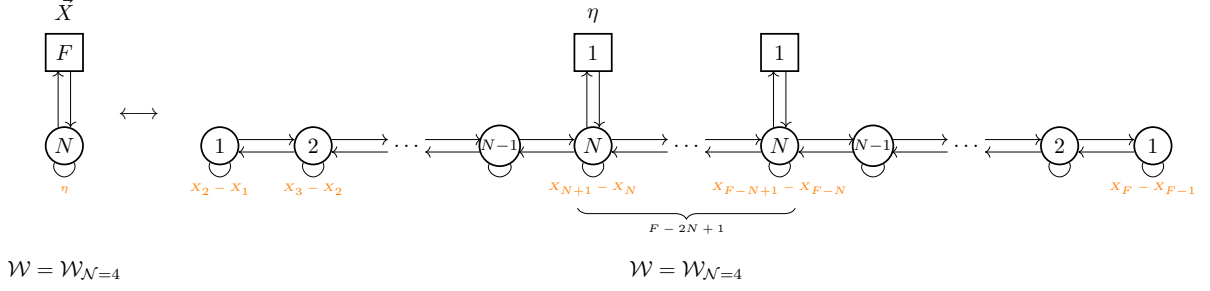


Figure 3.1: Mirror duality for $\mathcal{N} = 4$ SQCD with gauge group $U(N)$ and F flavors. Notice that the theories are depicted in the $\mathcal{N} = 2$ notation. Round nodes denote gauge groups and square nodes denote flavor groups, while arrows denote chiral multiplets charged under the nodes that they connect. Black labels denote fugacities associated to flavor global symmetries while orange labels denote the FI of the corresponding gauge group.

fundamental and anti-fundamental chiral fields while adjoint chiral fields have R-charge 0 and $U(1)_\tau$ charge 1. The charges of the fields under the symmetries of the theories are reported in the tables below where we adopt an $\mathcal{N} = 2$ notation denoting by Φ , Q , $\tilde{Q}$ the adjoint and the fundamental/antifundamental chirals of the SQCD and on the mirror side denoting respectively by Φ_J , $b_{J,J+1}$, $\tilde{b}_{J+1,J}$, the adjoints and the bifundamental chirals and the left and right flavors q , $\tilde{q}$ and p , $\tilde{p}$.

	$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$		$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$	
Φ	1	1	0	0	Φ_J	1	-1	2	0	(3.1)
Q	$\square$	$-\frac{1}{2}$	1	0	$b_{J,J+1}, \tilde{b}_{J,J+1}$	1	$\frac{1}{2}$	0	0	
$\tilde{Q}$	$\square$	$-\frac{1}{2}$	1	0	$q, \tilde{q}$	1	$\frac{1}{2}$	0	$-1/+1$	
					$p, \tilde{p}$	1	$\frac{1}{2}$	0	0	

Notice that the R-charges and the $U(1)_\tau$ charges in the mirror theory are obtained from the definitions given before, taking into account that mirror symmetry swaps $SU(2)_C \leftrightarrow SU(2)_H$.

Global symmetries and Operator Map

The global symmetry of the $U(N)$ SQCD with F flavors, depicted on the l.h.s. of (3.1), is $SU(F) \times U(1)_\eta \times U(1)_\tau$ ¹, where $U(1)_\eta$ is the topological symmetry. On the other hand, the manifest global symmetry of the mirror theory, depicted on the r.h.s. of (3.1), is $\prod_{j=1}^{F-1} U(1)_{x_{j+1}-x_j} \times U(1)_\eta \times U(1)_\tau$, where the first term is the collection of topological symmetries, while $U(1)_\eta$ is a flavor symmetry. In the IR this global symmetry enhances, in particular all the topological symmetries combine to give an enhanced $SU(F)$ global symmetry group. We therefore see that the IR global symmetries of the two theories

¹The actual faithful global symmetry of the theories is $PSU(F) \times U(1)_\eta \times U(1)_\tau$. We will not discuss the features related to the discrete factors of the global symmetry group. Moreover, the $U(1)_\eta$ symmetry enhances to $SU(2)$ in the IR if $F = 2N$. Indeed in the mirror theory for $F = 2N$ there is a manifest $SU(2)$ flavor symmetry instead of $U(1)$. This will not have an effect in the computations that we will perform and we refer to the topological symmetry of the SQCD as being generically $U(1)_\eta$.

are in agreement and topological symmetries are mapped into flavor symmetries and vice-versa, as expected in a mirror duality.

It is also instructive to see how the operators of the electric theory are mapped to those of the mirror theory.

- The fundamental monopoles $\mathfrak{M}^\pm$ of the SQCD are mapped into the long mesons on the mirror quiver side.
- The electric mesons, in the traceless adjoint of $SU(F)$, are mapped to a collection of monopole operators in the mirror theory. The latter is a traceless adjoint matrix of $SU(F)$ constructed with the $F - 1$ traces of the adjoints and with monopoles $(\mathfrak{M}^{\dots,0,\pm 1,0,\dots})$ with flux ± 1 under one of the $F - 1$ gauge nodes, $(F - 2)$ monopoles $(\mathfrak{M}^{\dots,\pm 1,\pm 1,0,\dots})$ carrying GNO flux ± 1 under two adjacent gauge nodes, $(F - 3)$ monopoles $(\mathfrak{M}^{\dots,\pm 1,\pm 1,\pm 1,\dots})$ carrying GNO flux ± 1 under three adjacent gauge nodes, and so on with the final two monopole operators carrying GNO flux ± 1 under all $F - 1$ gauge nodes.

3.1.2 Real Mass Deformation to the $\mathcal{N} = 2$ Chiral-Planar Mirror Pair

We now move on to the SUSY-breaking deformation. We perform a real mass deformation for the $U(1)_\tau$ symmetry, meaning that we turn on a non-zero VEV for the real scalar component of the background vector multiplet associated to the $U(1)_\tau$ symmetry. The VEV induces a mass term for every chiral superfield proportional to its charge under the symmetry. Recalling that $U(1)_\tau$ is a flavor subgroup of the $\mathcal{N} = 4$ R-symmetry, the deformation performed has the effect of breaking the R-symmetry from $SU(2)_C \times SU(2)_H \rightarrow U(1)_R$, indeed this means that supersymmetry is broken from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

As we shall discuss in depth, in the most generic vacua this deformation triggers a flow to a TFT. However, we will try to suitably follow this deformation by moving on special points of the Coulomb branch, so as to reach an interacting vacuum. Both the SQCD and its mirror can have many possible interacting vacua that can be reached and the task of understanding how these are mapped correctly among them is the problem we attempt to solve in the following section. We will start by giving a qualitative description of the deformation of both the SQCD and its mirror in the following subsection, discussing the features of the resulting $\mathcal{N} = 2$ mirror duality. In the following subsections we will show how this duality can be derived from two perspectives: the S_b^3 partition function and the 1-loop corrected EOM.

Let us first discuss the SQCD side, on the l.h.s. of Figure 3.1. As already commented, a real mass for the $U(1)_\tau$ symmetry naively renders every chiral field massive, and no massless fields are left in the deep IR. We then choose to move on the Coulomb branch in such a way to reach an interacting vacuum where all the fundamental chiral fields Q remain massless while the adjoint Φ and the F anti-fundamental chiral fields $\tilde{Q}$ acquire a mass, positive and negative respectively, and are integrated out². The procedure of integrating out chiral fields produces CS interactions due to fermionic modes becoming

²One can also consider different interacting vacua where a certain number of fundamental and anti-fundamental fields remain massless. While this problem is tractable, we will tackle the case of SQCD theories with more general matter content in a different, algorithmic way in Section 3.4.

massive. The resulting theory is the $\mathcal{N} = 2$ $U(N)_{-\frac{F}{2}+N, -\frac{F}{2}}$ SQCD depicted on the l.h.s. of Figure 3.2. We remind the reader that $U(N)_{k, k+lN}$ denotes a $U(N)$ gauge group with CS terms:

$$-i\frac{k}{4\pi} \int \text{tr}(A \wedge dA) - i\frac{l}{4\pi} \int \text{tr}(A) \wedge \text{tr}(dA) + \text{SUSY completion} \quad (3.2)$$

where A is the $U(N)$ gauge field. Each massive antifundamental multiplet, with negative mass, contributes as $(-\frac{1}{2}, -\frac{1}{2})$ to the CS level, and the massive adjoint chiral, with positive mass, contributes as $(N, 0)$. Notice in particular that we have $l = -1$.

We now discuss the result of the deformation in the mirror theory, depicted on the r.h.s. of Figure 3.1. As already pointed out, many possible interacting vacua can be reached with the $U(1)_\tau$ real mass, each lying at different points of the Coulomb branch. We claim that the vacuum dual to the chiral SQCD theory described above is a special Coulomb branch point where each gauge node $U(k)$ of the original $\mathcal{N} = 4$ mirror is Higgsed to its maximal torus $U(1)^k$. Also, some components of the chiral fields in the original theory remain massless. The complete description of the theory is depicted as a quiver on the r.h.s. of Figure 3.2.

A few comments regarding the planar dual on the r.h.s. of Figure 3.2 are in order:

- There is a cubic superpotential term for every triangular loop. There is one term with -1 ($+1$) coefficient for each clock-wise (anti-clock-wise) closed triangle. The cubic terms in the superpotential are a remnant of the cubic $\mathcal{N} = 4$ superpotential that are preserved by the deformation. We denote these superpotential terms in short as $\mathcal{W}_{\text{planar}}$. The diagonal/vertical chiral multiplets have R-charge $0/+2$, compatible with the superpotential.
- $\mathcal{W}_{\text{monopole}}$ are the monopole terms in the superpotential generated by the Polyakov mechanism [75] due to the Higgsing of a $U(k)$ gauge symmetry to $U(1)^k$. The exact collection of terms can be obtained with the following schematic rule. For each vertical arrow, there is a linear superpotential for the monopole with GNO flux -1 and $+1$ under the nodes connected by the arrow, from top to bottom. As a consequence of these monopole superpotential terms, the topological symmetry is broken to $U(1)^{F-1}$ which is expected to enhance in the infrared to $SU(F)$, thus matching the global symmetry on the electric side. The FI terms of the $U(1)$ gauge nodes are compatible with the monopole superpotential terms. With the choice of trial R-symmetry described above, the α^{th} gauge node in the I^{th} column of the planar quiver ($\alpha = 1, \dots, |G^{(I)}|$, and $|G^{(I)}|$ is the rank of the corresponding gauge node in the $\mathcal{N} = 4$ quiver, or equivalently, the number of $U(1)$ gauge nodes in the I^{th} column of the $\mathcal{N} = 2$ quiver) has FI terms equal to $X_{I+1} - X_I + \frac{iQ}{4}(\delta_{\alpha,1} - \delta_{\alpha,|G^{(I)}|})$. In Figure 3.2 we only reported the X_I -dependent part of the FIs.
- CS interactions are present. Self and mixed-interactions can be encoded in a matrix k_{ab} such that:

$$-i\frac{k_{ab}}{4\pi} \int A_a \wedge dA_b + \text{SUSY completion} \quad (3.3)$$

where A_a is the vector field of the a -th $U(1)$ gauge field. Indeed, by construction k_{ab} is symmetric. For the theory in Figure 3.2 the rule is that each gauge node has (self) CS interaction with level 1, denoted by the red labels in Figure 3.2. Nodes connected by diagonal/vertical lines have a mixed-CS interaction with level

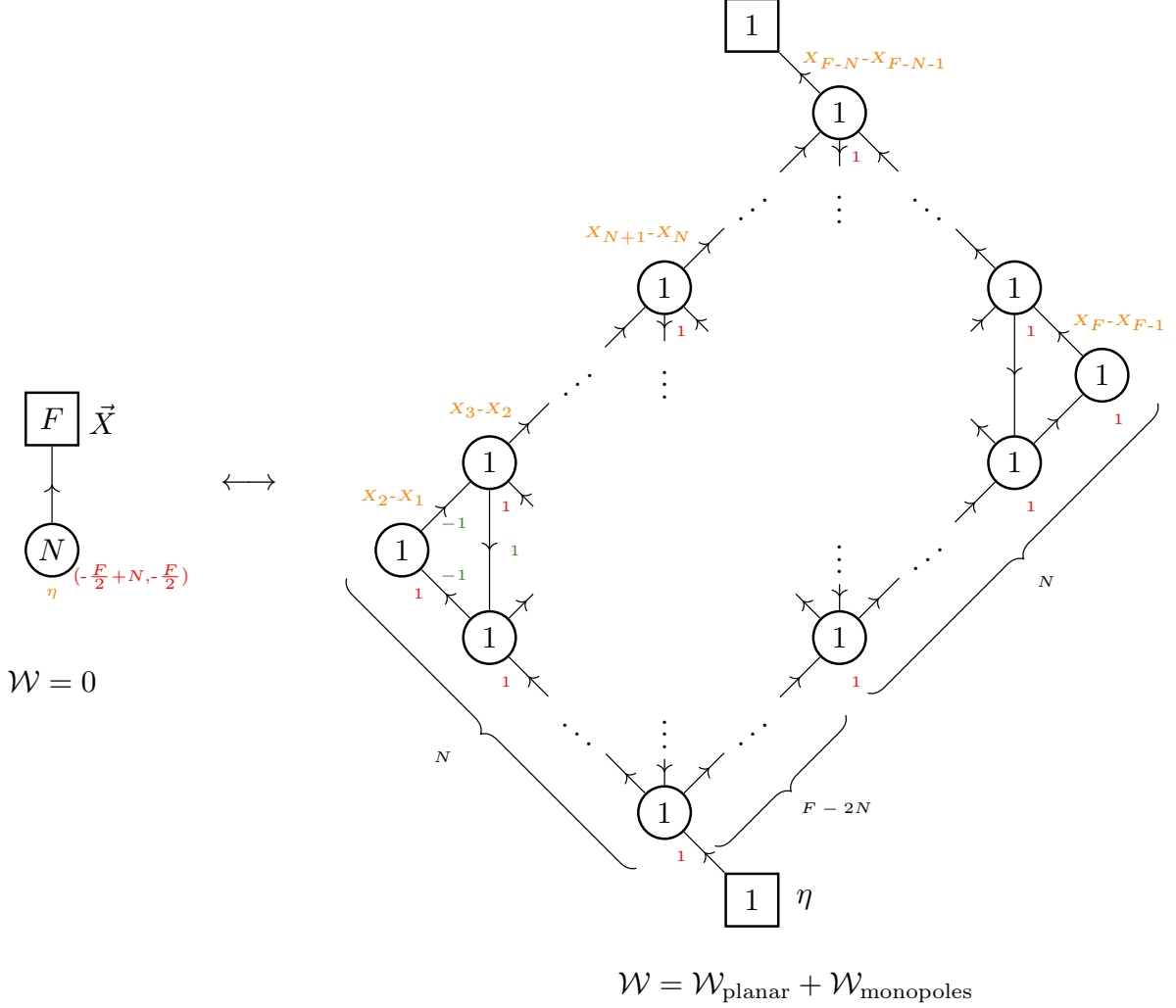


Figure 3.2: On the l.h.s. it is reported the $\mathcal{N} = 2$ $U(N)_{-\frac{F}{2}+N, -\frac{F}{2}}$ SQCD with F fundamental fields obtained from the real mass deformation of the $\mathcal{N} = 4$ SQCD. On the r.h.s. it is reported the planar mirror-like dual of the SQCD. Note that every gauge node has a CS level (red labels) 1 and there are mixed CS terms (green labels) for every pair of nodes connected by a line, with level $-1/+1$ if the line is diagonal/vertical. FI terms are indicated in orange. On the SQCD side we assign trial R-charge 1 to the fundamental chirals and on the planar side the diagonal chiral fields have trial R-charge 0 and the vertical ones 2. The diagonal chirals pointing from SE to NW connect N nodes, while the diagonal chirals pointing from SW to NE connect $F - N$ nodes.

$-1/+1$; in terms of the symmetric matrix, whose off-diagonal entries $k_{ab} = k_{ba}$ are each counted once in the sum over ordered pairs, this corresponds to $k_{ab} = \mp \frac{1}{2}$, so that the total mixed CS coupling for the pair is ∓ 1 . Mixed CS levels are denoted by the green labels along the corresponding line in Figure 3.2.

Superconformal Index, global symmetries and operator map

We now comment on the features of the duality described before, highlighting the tests that this duality passes.

The first test that we performed consists in the matching of the numerical expansion of the Superconformal Index. This guarantees that a number of properties are correctly reproduced by the dual theory such as: the presence of conserved currents associated to the correct global symmetry, the properties of the operators in the chiral ring and more. We performed this test for $N = 2, F = 4, 5, 6$ and $N = 3, F = 6$, the resulting indices are reported in Table 3.1. Notice that in order to perform the matching it is important to have the mirror theory parameterized as in Figure 3.2. In particular the mixing between the R-symmetry and the topological symmetries is crucial for the enhancement of the global symmetry to be seen from the Superconformal Index.

N	F	r	Superconformal Index
2	4	$\frac{1}{5}$	$1 + \eta x^{4/5} + \eta^2 x^{8/5} - 16x^2 + \eta^3 x^{12/5} + \eta^4 x^{16/5} + (\eta^5 + 88)x^4 + \mathcal{O}(x^{21/5})$
2	5	$\frac{1}{5}$	$1 + \eta x + (\eta^2 - 25)x^2 + \eta^3 x^3 + (\eta^4 + 250)x^4 + (\eta^5 - 100\eta)x^5 + \mathcal{O}(x^6)$
2	6	$\frac{1}{5}$	$1 + \eta x^{6/5} - 36x^2 + \eta^2 x^{12/5} + \eta^3 x^{18/5} + 558x^4 + \eta^4 x^{24/5} + \mathcal{O}(x^{26/5})$
3	6	$\frac{1}{7}$	$1 + \eta x^{6/7} + \eta^2 x^{12/7} - 36x^2 + \eta^3 x^{18/7} + \eta^4 x^{24/7} + 558x^4 + \eta^5 x^{30/7} - 225\eta x^{34/7} + \mathcal{O}(x^{36/7})$

Table 3.1: Superconformal index of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2})}$ SQCD with F fundamental chiral multiplets with trial R-charge $1 - r$. The indices match those of the corresponding mirror Abelian quiver in Figure 3.2.

We now proceed by describing the matching of the global symmetries and of the chiral ring generators.

The global symmetry of the SQCD theory is, up to discrete factors, $SU(F) \times U(1)_\eta$. On the mirror side we can match the rank of the global symmetry as follows. In the absence of the monopole superpotential there is a $U(1)$ topological symmetry for each $U(1)$ gauge node. The monopole superpotential breaks the topological symmetries associated to the nodes in a column of the quiver to a diagonal $U(1)$. Therefore there is a $U(1)^{F-1}$ unbroken topological symmetry associated to the $F - 1$ columns of the planar quiver. Furthermore there is a single $U(1)_\eta$ flavor symmetry which is preserved by the planar

superpotential. In total the rank of the global symmetry is F , matching the SQCD side. The mirror-like duality implies that the $U(1)^F$ symmetry of the quiver theory is enhanced to $SU(F) \times U(1)$.

The chiral ring of the planar theory is generated by the meson built out of F diagonal bifundamentals starting from the $U(1)_\eta$ flavor node and ending at the other flavor node, this operator has trial R-charge 0 and $U(1)_\eta$ charge 1. There are many possible paths which are all identified by the F-terms due to the planar superpotential $\mathcal{W}_{\text{planar}}$.

We expect this meson to be mapped to some monopole operator generating the SQCD chiral ring, as observed from the Index. The bare monopole operators $\mathfrak{M}^{\pm 0 \dots 0}$ are gauge variant but can be dressed to obtain gauge-invariant operators. We argue that the relevant operators are dressed with the gaugino as follows. In a monopole background, i.e. when we turn on GNO fluxes, the gauge group is effectively broken and the breaking pattern depends on the magnetic flux. In the case of fundamental monopoles the breaking pattern is $U(N) \rightarrow U(1) \times U(N-1)$. Consequently, all gauge variant operators decompose into representations of the residual group. The branching rules for irreducible representations of $U(N) \supset U(N-1) \times U(1)$ are given below:

$$\begin{aligned}\square_N &= (\square_{N-1}, 1) \oplus (\mathbf{1}, 1-N) \\ \square_N \otimes \bar{\square}_N &= (\square_{N-1} \otimes \bar{\square}_{N-1}, 0) \oplus (\square_{N-1}, N) \oplus (\bar{\square}_{N-1}, -N) \oplus (\mathbf{1}, 0)\end{aligned}\quad (3.4)$$

where $(\bar{\square}_K) \square_K$ denotes the (anti-) fundamental representation of $U(K)$ and $(\cdot, \cdot)$ denotes the representation under the residual group $U(N-1) \times U(1)$. We see, in particular, that the gaugino of $U(N)$ decomposes into the gaugino of $U(N-1)$, a singlet, a fundamental operator and an anti-fundamental operator. In general, for a $U(N)$ gauge theory with F fundamentals and generic CS levels $(k, k+lN)$ the bare monopoles $\mathfrak{M}^{\pm 0 \dots 0}$ transform in a representation of $U(N-1)$ with $(N-1)$ -ality of $\mp l(N-1)$ and with charge $(-\frac{F}{2} \mp k \mp l)$ of $U(1)$ [47]. Dressing with $N-1$ powers of the fundamental component of the gaugino compensates these charges to:

$$Q_{U(1)}(\mathfrak{M}^{\pm 0 \dots 0}) = (N-1) - \left(\frac{F}{2} \pm k \pm l \right). \quad (3.5)$$

In the case studied in this section we have $k = -\frac{F}{2} + N$ and $l = -1$ and the monopole $\mathfrak{M}^{+,0,\dots,0}$ can be dressed $N-1$ times with the gaugino components in the fundamental representation of $U(N-1)$ to obtain a gauge invariant operator³. We expect that this operator is the generator of the chiral ring of $U(N)$ SQCD with levels $(-\frac{F}{2} + N, -\frac{F}{2})$ and F fundamentals, and is mapped on the planar side to the meson discussed above. To support this claim, we compute the R-charge of this dressed monopole. The fundamental monopole $\mathfrak{M}^{+,0,\dots,0}$ has R-charge:

$$R(\mathfrak{M}^{+,0,\dots,0}) = (1 - R_Q) \frac{F}{2} - (N-1) = -(N-1), \quad (3.6)$$

where $R_Q = 1$ is the trial R-charge of the fundamental chirals. The factor $-(N-1)$ is the contribution to the R-charge coming from the zero modes of the gaugini. Dressing $N-1$ times with the gaugino contributes an additional factor of $N-1$ to the R-charge:

$$R(\mathfrak{M}^{+,0,\dots,0} \lambda^{N-1}) = 0, \quad (3.7)$$

³If instead $k = \frac{F}{2} - N$ and $l = 1$ the $\mathfrak{M}^{-,0,\dots,0}$ monopole dressed $N-1$ times with the gaugino is gauge invariant.

thus matching the gauge invariant meson of the planar theory.

It is interesting to notice that by carefully analyzing the SQCD theory, it is possible to find that there is also a monopole with high topological charge which is naturally gauge invariant, without requiring any dressing. This monopole is $\mathfrak{M}^{+,+, \dots, +}$, that has topological charge N and trial R-charge 0. On the other hand in the theories discussed here, namely for $k = -F/2 + N$ and $l = -1$, we find that the dressed fundamental monopole $\mathfrak{M}^{+,0, \dots, 0} \lambda^{N-1}$ is in the chiral ring as well. While this is non-trivial to establish from the SQCD point of view, the presence of this operator is manifest in the mirror dual where it is mapped to the single mesonic operator with $U(1)_\eta$ charge 1. Then the monopole $\mathfrak{M}^{+,+, \dots, +}$ is the N -th power of the fundamental monopole in the chiral ring.

The view from the S_b^3 partition function

We now show how to derive the chiral-planar mirror duality for the SQCD, depicted in Figure 3.2, via an analysis of the S_b^3 partition function following the strategies of [25, 128].

We start from the S_b^3 partition functions of the two theories in Figure 3.1. The partition function of the SQCD is:

$$Z(\vec{X}, \eta, \tau) = \int \prod_{\alpha=1}^N du_\alpha \Delta_{(N)}(\vec{u}, \tau) e^{2\pi i \eta \sum_{\alpha=1}^N u_\alpha} \prod_{j=1}^F s_b\left(\frac{\tau}{2} \pm (u_\alpha - X_j)\right); \quad (3.8)$$

while that of its mirror is:

$$\begin{aligned} \hat{Z}(\eta, \vec{X}, \tau) = & e^{2\pi i \eta \sum_{j=1}^N X_j} \int \prod_{I=1}^{F-1} \left(\prod_{\alpha=1}^{|G^{(I)}|} dz_\alpha^{(I)} e^{2\pi i z_\alpha^{(I)} (X_{I+1} - X_I)} \Delta_{(|G^{(I)}|)}(\vec{z}^{(I)}, iQ - \tau) \right) \\ & \prod_{I=1}^{F-2} \left[\prod_{\alpha=1}^{|G^{(I)}|} \prod_{\beta=1}^{|G^{(I+1)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_\alpha^{(I)} - z_\beta^{(I+1)})\right) \right] \\ & \prod_{\alpha=1}^{|G^{(N)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_\alpha^{(N)} - \eta)\right) \prod_{\alpha=1}^{|G^{(F-N)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm z_\alpha^{(F-N)}\right); \end{aligned} \quad (3.9)$$

where in the partition function of the mirror theory $\vec{z}^{(I)}$ denotes the fugacity of the I^{th} gauge node whose rank is $|G^{(I)}|$. We also use the short notation for the contribution of a $\mathcal{N} = 4$ vector multiplet:

$$\Delta_{(N)}(\vec{z}, \tau) = \frac{1}{N!} \frac{\prod_{\alpha, \beta=1}^N s_b\left(\frac{iQ}{2} - \tau + z_\alpha - z_\beta\right)}{\prod_{\beta < \alpha}^N s_b\left(\frac{iQ}{2} \pm (z_\alpha - z_\beta)\right)} \quad (3.10)$$

We remind the reader that a chiral field of R -charge r and charge $+1$ under an Abelian flavor symmetry $U(1)_x$ contributes to the partition function as $s_b(\frac{iQ}{2}(1-r) - x)$, and that $Q = b + b^{-1}$ is the squashing parameter.

Since the SQCD and its mirror are IR-dual the two partition functions must be equal:

$$Z(\vec{X}, \eta, \tau) = \hat{Z}(\eta, \vec{X}, \tau). \quad (3.11)$$

The two partition functions match as functions of the real mass parameters $\vec{X}, \tau, \eta$. Here $\vec{X}$ is the set of real mass associated to the $SU(F)$ flavor symmetry of the SQCD and they

thus satisfy $\sum_{j=1}^F X_j = 0$. From the mirror point of view the X_j masses parameterize the $F - 1$ topological symmetries in a convenient way so that they reflect how the Cartan subgroup of $SU(F)$ maps across the duality. Also, η is the real mass associated to the $U(1)$ topological/flavor symmetry of the SQCD/mirror⁴. Finally, τ is the real mass of the $U(1)$ flavor subgroup of the $\mathcal{N} = 4$ R-symmetry.

At the level of the partition function, the real mass deformation can be implemented as follows [25, 128]: we perform the $\tau \rightarrow +\infty$ limit combined with a suitable shift in gauge fugacities. Let us start the analysis considering the SQCD. The gauge shift performed is:

$$\vec{u} \rightarrow \vec{u} + \frac{\tau}{2} \quad (3.12)$$

accompanied by a shift of the FI parameter:

$$\eta \rightarrow \eta - \frac{F}{2}\tau. \quad (3.13)$$

Performing the limit $\tau \rightarrow +\infty$ takes us to the non-trivial Coulomb branch vacuum where the fundamental chiral fields remain massless. The limit is implemented using the asymptotic behavior of the double sine function:

$$\lim_{\xi \rightarrow \pm\infty} s_b(x + \xi) \sim e^{\pm \frac{\pi i}{2}(x+\xi)^2}. \quad (3.14)$$

In general, if x is a global parameter then the limit produces highly oscillating phases as well as finite phases corresponding to background terms for global symmetries such as BF couplings. If instead x is a gauge parameter, the limit produces CS interactions and divergent FI parameters together with a highly oscillating phase. The redefinition of η is chosen to cancel exactly every divergent FI parameter produced by the limit. In the end, we obtain a highly oscillating phase $e^{i\Phi(\tau, \eta)}$ independent of the gauge fugacities which therefore factorizes outside the integral:

$$Z(\vec{X}, \eta, \tau) \rightarrow e^{\frac{i\pi}{2}\Phi(\tau, \eta)} Z(\vec{X}, \eta), \quad (3.15)$$

with

$$\Phi(\tau, \eta) = -N^2\tau^2 + 2N\tau \left(\eta + \frac{iQ}{2}N \right), \quad (3.16)$$

and⁵:

$$Z(\vec{X}, \eta) = e^{-\frac{N\pi i}{2} \sum_{j=1}^F X_j^2} \int \prod_{\alpha=1}^N du_{\alpha} \frac{1}{\prod_{\alpha < \beta} s_b\left(\frac{iQ}{2} \pm (u_{\alpha} - u_{\beta})\right)} \quad (3.18)$$

$$e^{2\pi i \eta \sum_{\alpha} u_{\alpha} + \pi i (\sum_{\alpha=1}^N u_{\alpha})^2 + \pi i (\frac{F}{2} - N) \sum_{\alpha} u_{\alpha}^2} \prod_{j=1}^F s_b\left(X_j - u_{\alpha}\right).$$

⁴Indeed, for $F = 2N$ the parameter η becomes the real mass of the $SU(2)$ enhanced topological symmetry. This subtlety does not spoil any of the analysis done in this section

⁵Our sign convention is as follows - a $U(N)$ gauge node with FI parameter η and CS level (k_1, k_2) contributes to the S_b^3 partition function as, with $\vec{u}$ denoting the gauge fugacity:

$$e^{2\pi i \eta \sum_j u_j} e^{-\pi i k_1 \sum_j u_j^2} e^{-\frac{\pi i (k_2 - k_1)}{N} (\sum_j u_j)^2}. \quad (3.17)$$

A mixed CS (BF) level k_{ij} contributes to the S_b^3 partition function as: $e^{-\pi i k_{ij} u_i u_j}$. If $k_1 = k_2 \equiv \kappa$, we specify the CS level as (κ) .

The phase will be crucial in determining the mirror dual theory. Indeed, as explained in [25, 128] the vacua related by duality under the deformation must have the same oscillating phase. In general, different shifts in the gauge fugacities (corresponding to moving to different points on the Coulomb branch) give different oscillating phases so there can be multiple vacua on the electric and magnetic sides and the match of the phases is a necessary criterion to map vacua related by duality.

We now move to the analysis of the partition function of the mirror theory.

We study the limit partition function of the mirror theory $\hat{Z}(\eta, \vec{X}, \tau)$ for large τ and with $\eta \rightarrow -\frac{F}{2}\tau + \eta$. There is a large number of possible $z_\alpha^{(I)}$ shifts that can be performed, each corresponding to different points on the Coulomb branch and each producing (possibly) a different highly oscillating phase times the partition function of an interacting theory describing the vacuum. We look for a vacuum that reproduces the same oscillating prefactor as the SQCD. Schematically this means that:

$$Z(\vec{X}, \eta, \tau) = \hat{Z}(\eta, \vec{X}, \tau) \xrightarrow{\tau \rightarrow +\infty} e^{\frac{i\pi}{2}\Phi(\tau)} Z(\vec{X}, \eta) = e^{\frac{i\pi}{2}\Phi(\tau)} \hat{Z}(\eta, \vec{X}) \quad (3.19)$$

implying the equality of the partition function of the two $\mathcal{N} = 2$ theories living in the vacua. We found that the correct shift for the $z_\alpha^{(I)}$ is such that:

$$\begin{aligned} z_\alpha^{(I)} - z_{\alpha+1}^{(I)} &\rightarrow z_\alpha^{(I)} - z_{\alpha+1}^{(I)} - \tau \\ z_\alpha^{(I)} - z_\alpha^{(I+1)} &\rightarrow z_\alpha^{(I)} - z_\alpha^{(I+1)} + \frac{\tau}{2}, \quad I < F - N \\ z_\alpha^{(I)} - z_\alpha^{(I+1)} &\rightarrow z_\alpha^{(I)} - z_\alpha^{(I+1)} - \frac{\tau}{2}, \quad I \geq F - N \\ z_1^{(N)} &\rightarrow z_1^{(N)} - \frac{F-1}{2}\tau. \end{aligned} \quad (3.20)$$

As a result, each gauge node is fully Higgsed to a Cartan subgroup and the chiral adjoint and bifundamentals decompose accordingly. Most of these fields are massive and are integrated out, resulting in the $\mathcal{N} = 2$ quiver gauge theory shown on the r.h.s. of Figure 3.2.

Notice that when we performed the τ limit, the shift of the gauge fugacities in (3.20) is not invariant under a Weyl transformation S_k associated to any gauge group $U(k)$. Therefore if we take a different shift by reshuffling the k gauge parameters associated to a gauge symmetry we obtain the same result after the limit. Each of these $k!$ possibilities represent a different point of the Coulomb branch where the correct magnetic theory lives. One should then sum over all these vacua, all reproducing the same highly oscillating prefactor, effectively producing a $k!$ factor that cancel that at the denominator in the definition of the $\Delta_{(N)}$ measure in (3.10).

By taking the real mass limit on the partition function on the mirror side we obtain the partition function of the quiver:

$$\begin{aligned}
\hat{Z}(\eta, \vec{X}) = & e^{2\pi i \eta \left(-(2N-1)\frac{iQ}{4} + \sum_{i=1}^N X_i \right)} e^{-\frac{i\pi}{2}\eta^2} \int \prod_{I=1}^{F-1} \prod_{\alpha=1}^{|G^{(I)}|} dz_{\alpha}^{(I)} e^{2\pi i z_{\alpha}^{(I)} (X_{I+1} - X_I)} e^{-i\pi (z_{\alpha}^{(I)})^2} \\
& \prod_{I=1}^{F-1} \prod_{\alpha} s_b \left(-\frac{iQ}{2} + z_{\alpha+1}^{(I)} - z_{\alpha}^{(I)} \right) e^{-i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I)}} \\
& \prod_{I=1}^{F-N-1} \prod_{\alpha} s_b \left(\frac{iQ}{2} + z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} \right) s_b \left(\frac{iQ}{2} - z_{\alpha}^{(I)} + z_{\alpha+1}^{(I+1)} \right) e^{i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I+1)}} \quad (3.21) \\
& \prod_{I=F-N}^{F-2} \prod_{\alpha} s_b \left(\frac{iQ}{2} - z_{\alpha}^{(I)} + z_{\alpha}^{(I+1)} \right) s_b \left(\frac{iQ}{2} + z_{\alpha}^{(I)} - z_{\alpha+1}^{(I+1)} \right) e^{i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I+1)}} \\
& s_b \left(\frac{iQ}{2} - \eta + z_1^{(N)} \right) s_b \left(\frac{iQ}{2} - z_N^{(F-N)} \right) e^{i\pi z_1^{(N)} \eta}
\end{aligned}$$

Where the products run over all the α such that the fugacities z in the corresponding double sine functions are within the ranges described above. Notice that in the first line are encoded the background terms, that are not included in the Figure 3.2.

The view from the semiclassical EOM

We now show that the flow from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ mirror dualities described in Subsection 3.1.2 is consistent with the equation of motion of the $\mathcal{N} = 4$ theories. In particular, we show that the vacua corresponding to the shifts in gauge fugacities satisfies the F-terms and D-terms equations. Furthermore, we show that fluctuations around these vacua reproduce the EOM of the resulting $\mathcal{N} = 2$ theories. This proves to be an important check of our proposal.

We start our analysis by first considering the $\mathcal{N} = 4$ $U(N)$ SQCD with F flavors. Recall that, as an $\mathcal{N} = 2$ supersymmetric theory, it has F (anti-)fundamental fields Q ($\tilde{Q}$) and an adjoint chiral multiplet Φ . Additionally, we also turn on a real FI parameter η . Upon diagonalizing the real scalar σ in the vector multiplet, the equations for the F- and D-terms are as follows (note that each field represents a matrix)⁶ [24]:

$$\begin{aligned}
\sum_{i=1}^F Q_{\alpha}^i \tilde{Q}_i^{\beta} &= 0, \quad \sum_{\alpha=1}^N Q_{\alpha}^i \Phi_{\beta}^{\alpha} = 0, \quad \sum_{\alpha=1}^N \Phi_{\alpha}^{\beta} \tilde{Q}_i^{\alpha} = 0, \\
(\sigma_{\alpha} - X_i - \frac{\tau}{2}) Q_{\alpha}^i &= 0, \quad (X_i - \sigma_{\alpha} - \frac{\tau}{2}) \tilde{Q}_i^{\alpha} = 0, \quad [\sigma, \Phi]_{\alpha}^{\beta} + \tau \Phi_{\alpha}^{\beta} = 0, \\
[\Phi, \Phi^{\dagger}]_{\alpha}^{\beta} + \sum_{i=1}^F (Q_{\alpha}^{i\dagger} Q_{\beta}^i - \tilde{Q}_i^{\alpha} \tilde{Q}_i^{\beta\dagger}) &= \frac{\delta_{\alpha}^{\beta}}{2\pi} \left[-\eta + \frac{1}{2} \sum_{i=1}^F (|\sigma_{\alpha} - X_i - \frac{\tau}{2}| - |X_i - \sigma_{\alpha} - \frac{\tau}{2}|) + \right. \\
&\quad \left. + \frac{1}{2} \sum_{\gamma=1}^N (|\sigma_{\alpha} - \sigma_{\gamma} + \tau| - |\sigma_{\gamma} - \sigma_{\alpha} + \tau|) \right]. \quad (3.22)
\end{aligned}$$

Where the values of $Q_{\alpha}^i, \tilde{Q}_i^{\alpha}, \Phi_{\beta}^{\alpha}$ are the Vacuum Expectation Values (VEVs) of the (scalar component of) chiral fields of the theory. σ_{α} is the VEV for the real scalar in the $\mathcal{N} = 2$

⁶N.B.: Latin letters $i, j = 1, \dots, F$ run over flavor indices, and Greek letters $\alpha, \beta = 1, \dots, N$ run over gauge indices. The sum over indices is never implied.

vector multiplet, noting that in principle it is a matrix in the adjoint representation of the $U(N)$ gauge symmetry but we can always perform a gauge transformation such that it is diagonal and we employ the single index notation for the diagonal components $\sigma_\alpha^\beta = \delta_\alpha^\beta \sigma_\alpha$. Lastly, the real parameters X_i, τ, η are the real masses associated to the $SU(F)/U(1)_\tau/U(1)_\eta$ global symmetries, i.e. VEVs of the real scalars in background $\mathcal{N} = 2$ vector multiplets of the global symmetries. Since X_i parameterizes a $SU(F)$ symmetry they satisfy $\sum_{i=1}^F X_i = 0$.

The set of equations in (3.22) represent the following: In the first line the F-term constraint imposed by the $\mathcal{N} = 4$ superpotential $\mathcal{W} = \sum_{i=1}^F \sum_{\alpha, \beta=1}^N Q_\alpha^i \Phi_\beta^\alpha \tilde{Q}_i^\beta$; in the second line the mass term for the chiral fields; in the third and last line the 1-loop corrected D-term equation.

We now consider a mass deformation for the axial symmetry $U(1)_\tau$, which corresponds to taking a positive large value of τ in (3.22). While it would be interesting to study the most general solution to the vacuum equations under this deformation, we do not attempt to solve this problem here. Instead, we look for a vacuum where the fundamental fields remain massless. In order to do so we consider a large τ real mass, and set the real masses X_i to zero.

The effective mass of the field Q_α^i is given by $(\sigma_\alpha - \frac{\tau}{2})$, therefore we set $\sigma_\alpha = \frac{\tau}{2}$. The first line of equations in (3.22) then requires that $\tilde{Q}_i^\alpha = \Phi_\beta^\alpha = 0$. Also, although the condition $\sigma_\alpha = \frac{\tau}{2}$ is enough to solve the first equation in the first line, we also require Q_α^i to be zero, in order for these fields to not acquire any VEV and to therefore be light degrees of freedom of the theory in the vacuum. The D-term equation therefore reads:

$$0 = \frac{1}{2\pi} \left[-\eta - \frac{F}{2}\tau \right] \longrightarrow \eta = -\frac{F}{2}\tau \quad (3.23)$$

Therefore, we are performing a real mass deformation for both the $U(1)_\tau$ and $U(1)_\eta$ symmetries, keeping fixed the combination $\frac{F}{2}U(1)_\tau + U(1)_\eta$. This corresponds to the deformation considered in the previous subsection.

Now to conclude, we would like to study the theory that lives in the vacuum solution that we found. To do so, we study small fluctuations around the solution:

$$\begin{aligned} Q_\alpha^i &= \tilde{Q}_\alpha^i \\ X_i &= \check{X}_i \\ \sigma_\alpha &= \frac{\tau}{2} + \check{\sigma}_\alpha \\ \eta &= -\frac{F}{2}\tau + \check{\eta} \end{aligned} \quad (3.24)$$

Notice that the equations in the second line of (3.22) impose that the $\tilde{Q}$ and Φ fields do not fluctuate, as expected since they do not describe light degrees of freedom in the chosen vacuum.

Expanding the equations in the light modes we get:

$$\begin{aligned} (\check{\sigma}_\alpha - \check{X}_i) \tilde{Q}_\alpha^i &= 0 \\ \sum_{i=1}^F Q_\alpha^{i,\dagger} Q_i^\beta &= \frac{\delta_{\alpha\beta}}{2\pi} \left[\underbrace{-\check{\eta}}_{=\eta_{eff}} + \underbrace{\left(-\frac{F}{2} + N \right) \check{\sigma}_\alpha - \sum_{\gamma=1}^N \check{\sigma}_\gamma}_{\text{CS interactions}} + \frac{1}{2} \sum_{i=1}^F |\check{\sigma}_\alpha - \check{X}_i| \right] \end{aligned} \quad (3.25)$$

Which can be interpreted as the EOM of a $U(N)$ gauge theory with F massless fundamental chirals and with CS level $(-\frac{F}{2} + N, -\frac{F}{2})$ by comparing with the generic formula:

$$\sum_{i=1}^F Q_{\alpha}^{i,\dagger} Q_i^{\beta} = \frac{\delta_{\alpha\beta}}{2\pi} \left[-\eta_{\text{eff}} + k_{\text{eff}} \sigma_{\alpha} + l_{\text{eff}} \sum_{\gamma=1}^N \sigma_{\gamma} + \frac{1}{2} \sum_{i=1}^F |\sigma_{\alpha} - \tilde{X}_i| \right] \quad (3.26)$$

where the CS level is $(k, k + lN)$.

We now turn to the mirror of the $\mathcal{N} = 4$ $U(N)$ SQCD theory in Figure 3.1. Studying the vacuum of this theory using the equations of motion is a highly non-trivial task and is not attempted here. The difficulty of this task lies clearly in the high number of equations that must be analyzed. Another issue is that for the same problem, i.e. same values of the real mass parameter in input, there can be multiple solutions and therefore multiple possible vacua.

Instead, we employ mirror duality to know how the real mass parameters are mapped across the duality so that we can fix correctly the problem we attempt to study. The system of equations for the mirror theory are⁷:

$$\begin{aligned} (\sigma_i^{(a)} - \sigma_j^{(a+1)} + \frac{\tau}{2}) b_{i,j}^{(a,a+1)} &= 0; & (-\sigma_i^{(a)} + \sigma_j^{(a+1)} + \frac{\tau}{2}) \tilde{b}_{i,j}^{(a,a+1)} &= 0 \\ (\sigma_i^{(N)} - \eta + \frac{\tau}{2}) q_i &= 0; & (-\sigma_i^{(N)} + \eta + \frac{\tau}{2}) \tilde{q}_i &= 0 \\ (\sigma_i^{(F-N)} + \frac{\tau}{2}) p_i &= 0; & (-\sigma_i^{(F-N)} + \frac{\tau}{2}) \tilde{p}_i &= 0 \\ [\sigma^{(a)}, \Phi^{(a)}]_i^j - \tau \Phi^{(a)}_i^j &= 0 \end{aligned} \quad (3.27)$$

All the σ 's are taken to be diagonal after an appropriate gauge fixing. Also, we have the set of F-terms equations, that we do not attempt to report here since they will not be important for the following discussion. And lastly we have the D-terms equations that are:

$$\begin{aligned} [\Phi^{(a)}, \Phi^{(a),\dagger}]_{i,j} + \dots (\text{sum over } b^{(a,a+1)}\text{'s and eventual } q, p) &= \frac{\delta_{i,j}}{2\pi} F_i^{(a)}, \\ F_i^{(a)} &= X_{a+1} - X_a + \sum_{k=1}^{a+1} (|\sigma_i^{(a)} - \sigma_k^{(a+1)} + \frac{\tau}{2}| - |-\sigma_i^{(a)} + \sigma_k^{(a+1)} + \frac{\tau}{2}|) + \\ &+ \sum_{k=1}^{a-1} (-|\sigma_k^{(a-1)} - \sigma_i^{(a)} + \frac{\tau}{2}| + |-\sigma_k^{(a-1)} + \sigma_i^{(a)} + \frac{\tau}{2}|) + \\ &+ \sum_{k=1}^a (|\sigma_i^{(a)} - \sigma_k^{(a)} - \tau| - |-\sigma_i^{(a)} + \sigma_k^{(a)} - \tau|). \end{aligned} \quad (3.28)$$

The solution that we are looking for has a non-zero positive value of τ , the value of η is fixed to $-\frac{F}{2}\tau$ and $X_a = 0$. Notice that the value of the external parameters is fixed by the analysis of the SQCD. We observe that there is a solution for the unknown variables

⁷The index between the round brackets labels the number of the gauge node. The latin indices are used for gauge symmetries.

$\sigma_i^{(a)}$ obtained by taking:

$$\begin{aligned}
\sigma_\alpha^{(a)} - \sigma_{\alpha+1}^{(a)} &= -\tau \\
\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} &= +\frac{\tau}{2}, & a < F - N \\
\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} &= -\frac{\tau}{2}, & a \geq F - N \\
\sigma_1^{(N)} &= -\frac{F-1}{2}\tau
\end{aligned} \tag{3.29}$$

This corresponds to the vacuum considered in the previous section (3.20), where the VEVs for σ 's indicate the pattern of Higgsing. The light DOFs can be found by computing the masses of the various fields from the expression (3.27), which gives the same answer as the analysis of the S_b^3 partition function. One can also try to perform a perturbative expansion of the equation of motion, as we did for the SQCD, to find that this analysis reproduces correctly all the properties of the chiral-planar mirror duality of Figure 3.2, such as the self and mixed-CS interactions. The planar superpotential are the F-term relations surviving in the vacuum. On the other hand, the monopole superpotential can not be inferred easily from the analysis of the EOM.

To summarize, we have shown that the vacua corresponding to the real mass flows considered in the previous Section satisfy the EOM of the $\mathcal{N} = 4$ theory. This, combined with the matching of the S_b^3 partition function and the Superconformal Index discussed above provide a strong check of the chiral-planar mirror duality for $\mathcal{N} = 2$ SQCD.

As a technical aside we would like to comment on the similarities and differences between the analysis of the S_b^3 partition function and the EOM. It can be checked that, under a real mass deformation τ , the requirement that the S_b^3 partition function factorizes into a divergent phase and a finite integral is equivalent to solving the D-term equations up to orders $\mathcal{O}(\tau^0)$. This is due to the fact that large shifts in the gauge fugacities appearing in the S_b^3 partition function correspond to moving on the Coulomb branch far away from the origin. On the other hand, the S_b^3 partition function is not sensitive to motion on the Higgs branch, therefore solving the other EOM provide an additional check of the deformation considered above. When there are multiple vacua the analysis of the EOM does not provide a general prescription to understand the mapping of vacua across the duality. The divergent phase coming from the S_b^3 partition function is sensitive to the matter that becomes massive at a given vacuum, and therefore can be leveraged to constrain the mapping of vacua across the duality.

As an example, both the $\mathcal{N} = 4$ SQCD and its mirror have a *chiral-like* and a *planar-like* vacuum under the axial mass deformation. In [169], where the $\mathcal{N} = 2^*$ framework was introduced, the axial mass deformation appeared to lead to *chiral-chiral* dualities. In the previous Section we showed that the *chiral-like* vacuum of one theory can be mapped to the *planar-like* vacuum of the mirror theory because the divergent prefactors in the S_b^3 partition function cancel between the two. Such a cancellation does not happen for pairs of *chiral-like* vacua or *planar-like* vacua, showing that such a mapping of vacua is not consistent. Therefore, our results represent an example where refined precision tools such as the computation of the S_b^3 partition function can be leveraged to understand qualitative properties of dualities between QFTs.

3.1.3 A Planar Dual for $SU(N)_k$ and $U(N)_{(k,k+\ell N)}$ SQCD

In this chapter, we primarily consider dualities involving $U(N)_{(k,k-\ell N)}$ gauge theories (that is, theories where the Abelian CS level ℓ is -1). One can obtain an $SU(N)_k$ CS theory by introducing a background BF term between the topological symmetry $U(1)_Y$ and a new $U(1)_B$ and then gauging the topological symmetry. This operation is referred to as *ungauging* of the diagonal $U(1) \subset U(N)$. In the notation used throughout this chapter, this ungauging operation consists in

$$\int dY e^{2\pi i Y B N} \begin{array}{c} \textcircled{N}^{Y+\Delta Y} \\ \downarrow (k, k+N\ell) \\ \boxed{F}^{\vec{X}} \\ \xi \end{array} = \begin{array}{c} \textcircled{\textcircled{N}}^k \\ \downarrow \\ \boxed{F}^{\vec{X}} \\ -\Delta Y \frac{N}{F} \left(\xi, \xi + F \left(\frac{Nk+N^2\ell-\xi F}{F^2} \right) \right) \end{array}, \quad \tilde{X}_a = X_a + B \quad (3.30)$$

wherein the double circle denotes a $SU(N)$ gauge symmetry. In the $U(N)$ SQCD the flavor symmetry is $SU(F)$ and thus $\sum_{j=1}^F X_j = 0$ and ξ is a possible background CS for the flavor symmetry. In the $SU(N)$ theory $\sum_{j=1}^F \tilde{X}_j = FB$, where $U(1)_B$ is a baryonic symmetry.

Conversely, starting from SQCD with gauge group $SU(N)$ one can gauge the baryonic symmetry $U(1)_B$ to obtain a unitary gauge group:

$$\int d(BN) e^{-2\pi i \eta B N} \begin{array}{c} \textcircled{\textcircled{N}}^{\eta+\Delta\eta} \\ \downarrow k \\ \boxed{F}^{\vec{X}} \\ -\Delta\eta \frac{N}{F} \left(\xi_1, \xi_1 + F\xi_2 \right) \end{array} = \begin{array}{c} \textcircled{N}^{\eta+\Delta\eta} \\ \downarrow \left(k, k+N \left(\frac{F\xi_1+F^2\xi_2-Nk}{N^2} \right) \right) \\ \boxed{F}^{\vec{X}} \\ \xi_1 \end{array}, \quad \tilde{X}_a = X_a + B \quad (3.31)$$

Where ξ_1 and ξ_2 are possible background CS levels for the $U(F)$ global symmetry of the SQCD on the r.h.s. and $-\Delta\eta \frac{N}{F}$ is a background FI term. Consistently, one can check that by performing both transformations (3.30) and (3.31), a theory is mapped back to itself.

We may apply the ungauging operation (3.30) to the chiral-planar mirror duality of SQCD in Figure 3.2. On the electric side we obtain the $SU(N)_{-F/2+N}$ SQCD. On the mirror side, the topological symmetry corresponds to a flavor symmetry acting on the bottom flavor node. Therefore, the operation described above corresponds to gauging the flavor node itself. In order to correctly perform this gauging it is important to keep track of possible background terms in the duality that are not provided in Figure 3.2, but can be read from the partition function given in (3.21). The resulting chiral-planar duality is given in Figure 3.3 and was first presented in [1]⁸.

⁸Notice that the new gauge node is a $U(1)_{1/2}$ with one chiral and thus it can be confined. This results in a chiral that is not charged under any gauge symmetry and it is coupled to the theory as a flipper of a gauge invariant monopole. This does not trigger a sequential confinement except for the case of $N=2$, see Subsubsection 3.1.3.

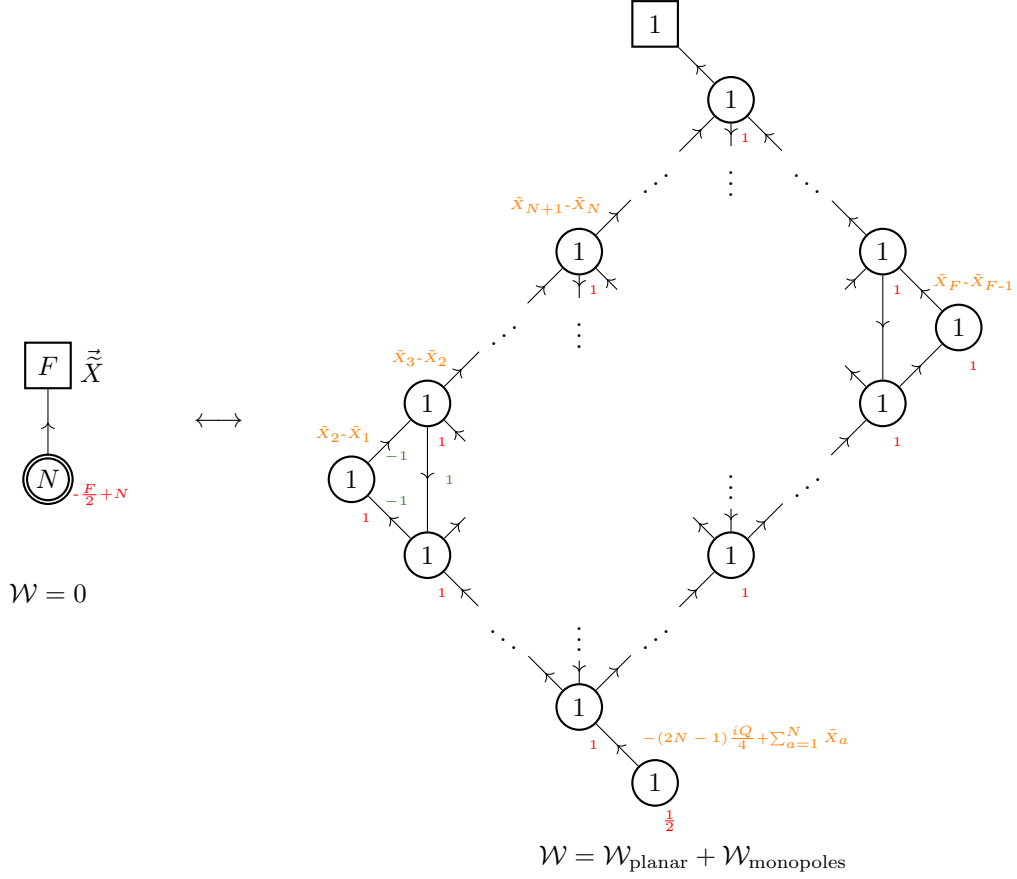


Figure 3.3: The $\mathcal{N} = 2$ planar mirror dual of $SU(N)_{-\frac{F}{2}+N}$ SQCD with F fundamental chiral multiplets.

By gauging the baryonic symmetry in Figure 3.3, using the strategy in (3.31), we revert to the duality for $U(N)_{(k,k-N)}$ SQCD. At this stage, we may also introduce a CS term at level $\Delta\ell$ for the baryonic symmetry prior to gauging, resulting in a $U(N)_{(k,k+N(\Delta\ell-1))}$ gauge theory. The gauging/ungauging operations together with background CS terms correspond to Witten's $SL(2, \mathbb{Z})$ action [144] applied to the topological symmetry of $U(N)$. On the mirror side, this procedure introduces a new $U(1)_{\Delta\ell}$ gauge node coupled to the rest of the quiver by a BF term, the result is the duality in Figure 3.4.

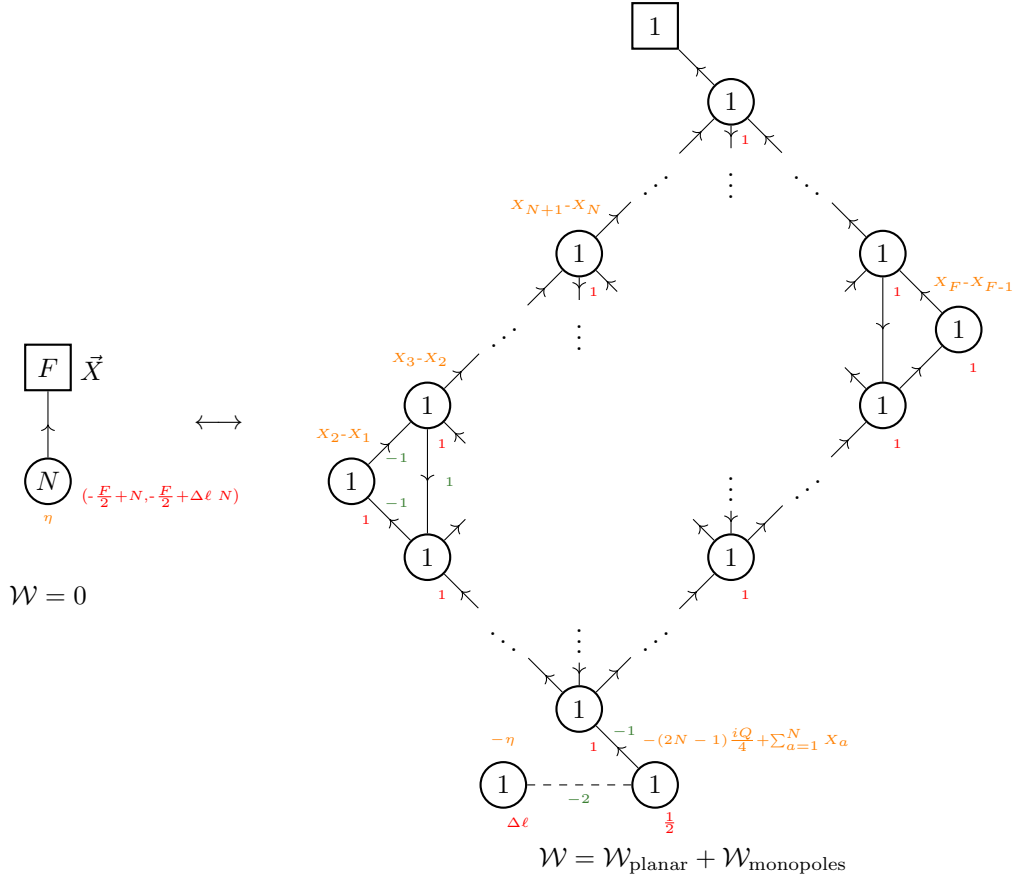


Figure 3.4: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2}+N\Delta\ell)}$ SQCD with F fundamental chiral multiplets.

For $\Delta\ell = 0, \pm 1$ the quiver can be further simplified by explicitly integrating over the additional gauge node. For $\Delta\ell = 0$ the integration results in a delta function which further fixes the gauge field of the node on its right, which becomes a flavor node and we recover the original mirror for $U(N)_{-F/2+N, -F/2}$. For $\Delta\ell = \pm 1$ the additional gauge node is a $U(1)_{\pm 1}$ sector, which is almost trivial and its path integral can be performed exactly as [141, 144]:

$$\begin{array}{c} \lambda_1 \\ \textcircled{1} \\ \pm 1 \end{array} \text{---} \text{---} \begin{array}{c} \lambda_2 \\ \textcircled{1} \\ k_2 \end{array} \cdots = \begin{array}{c} \lambda_2 \mp \frac{\lambda_1 k_{12}}{2} \\ \textcircled{1} \\ k_2 \mp \frac{k_{12}^2}{4} \end{array} \cdots \times e^{\pm i\pi\lambda_1^2} \quad (3.32)$$

In particular, for $\Delta\ell = 1$ we obtain a planar Abelian dual for $U(N)_{(k,k)}$. Integrating out the additional node as above shifts the CS level of the node on its right by -1 . Then the bottom-most node is a $U(1)_{-1/2}$ gauge node with one fundamental, which can be locally dualized to a chiral field, resulting in the duality shown in Figure 3.5.

Consistency-check: S-Confinement of $SU(2)_0$ with 2 flavors

In the previous section we proposed the planar mirror dual for $SU(N)_{(-\frac{F}{2}+N)}$ SQCD with $[F, 0]$ chiral multiplets. For $N = 2$ and $F = 4$ this theory is known to S-confine to a Wess-Zumino model of $6 + 1$ chiral fields, interacting through a cubic superpotential [23]. As a further consistency check of our proposed dualities, we will show that indeed our planar dual theory confines to the expected Wess-Zumino model by applying local Aharony-like

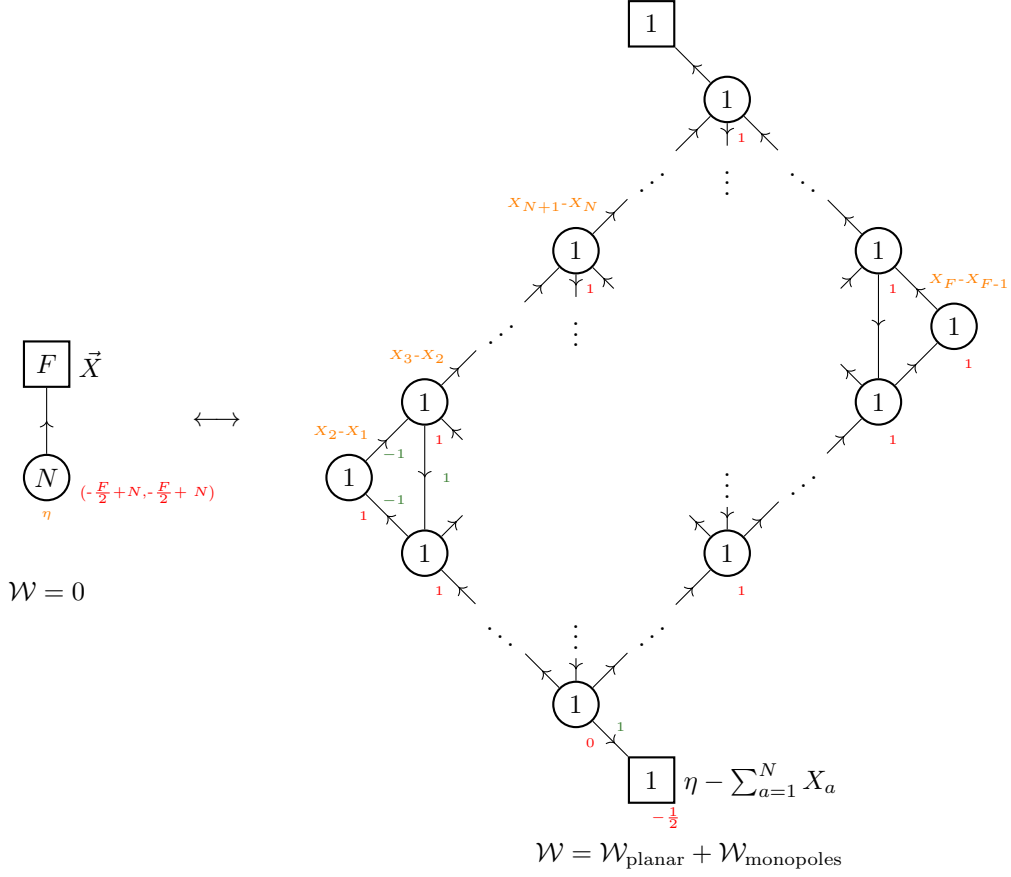


Figure 3.5: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-F/2+N, -F/2+N)}$ SQCD with F fundamental chiral multiplets. In the quiver we omitted the mixing of the topological symmetries with the R-symmetry, in particular the bottom-most gauge node has FI parameter $X_{N+1} - X_N - \frac{iQ}{8}$. The bottom chiral fundamental has trial R-charge $-N + 1$.

dualities [35, 93, 128]. More precisely we make use of the following s-confining dualities: $U(1)_{\frac{1}{2}}$ with one fundamental s-confines into a free chiral and $U(1)_0$ with one flavor is dual to the XYZ model. Furthermore we use the following dualities: $U(1)_1$ with one flavor is dual to $U(1)_{-1}$ with one flavor, modulo flippers and $U(1)_{-\frac{1}{2}}$ with $[1, 2]$ flavors is dual to $U(1)_{\frac{1}{2}}$ with $[2, 1]$ flavors, modulo flippers.

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathfrak{M} \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix} \quad (3.33)$$

Below we give the local dualization sequence where at each step we highlight in blue the gauge node that we dualize.

In the first step we use the confining duality for $U(1)_{1/2}$ with one chiral, which confines to a chiral field. Notice that because of background CS levels produced in the dualization the CS level of the lower node is now $\frac{1}{2}$. We also produce a singlet flipping a monopole.

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathfrak{M} \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix} \quad \mathcal{W} = \mathcal{W}_{\text{planar}} + B_{12} \mathfrak{M} \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix} \quad (3.34)$$

In the second step we use Giveon-Kutasov duality of $U(1)_1$ with a flavor for the left node which removes the vertical chiral and modifies the superpotential.

$$\mathcal{W} = \mathcal{W}_{\text{quartic}} + B_{12} \mathfrak{M} \begin{pmatrix} - & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{M} \begin{pmatrix} 0 & 0 \\ 0 & + \end{pmatrix} \quad \mathcal{W} = B_{12} T_{-} \mathfrak{M} \begin{pmatrix} - & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{W}_{\text{cubic}} + M T_{-} B_{23} \quad (3.35)$$

duality and performing real mass deformations, we obtain two $\mathcal{N} = 2$ dualities: a chiral-chiral duality, relating two chiral quivers and a planar-planar duality, relating two planar quivers.

Notice that starting from the flip-flip duality we do not expect to find a chiral-planar duality, since this is a peculiarity of mirror dualities exchanging flavor and topological symmetries. As shown in [171] the flip-flip duality is not a mirror duality but is more akin to an Aharony duality, and indeed it is possible to prove it by iterative local applications of Aharony duality.

We begin with a quick review of the $FT[U(N)]$ theory. The $FT[U(N)]$ theory is an SCFT with global symmetry $SU(N)_{\vec{X}} \times SU(N)_{\vec{Y}} \times U(1)_\tau$ ¹⁰. We denote this theory with the compact notation:

$$\boxed{N}_{\vec{X}} \text{---} \text{---} \text{---} \boxed{N}_{\vec{Y}} \quad (3.38)$$

Where $\vec{X}, \vec{Y}$ are fugacities for the global symmetries and the label “—” is included for later convenience, see Section 3.3. The $FT[U(N)]$ theory has two operators in its spectrum $A_{\vec{X}}$ and $A_{\vec{Y}}$, which transform in the adjoint representation of $SU(N)_{\vec{X}}$ and $SU(N)_{\vec{Y}}$. The $FT[U(N)]$ SCFT admits a UV completion given by the following UV quiver theory:

$$\begin{array}{ccccccc} Y_1 - Y_2 & Y_2 - Y_3 & Y_3 - Y_4 & Y_{N-1} - Y_N & Y_N & & \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \cdots & \textcircled{N-1} & \textcircled{N} & \vec{X} \\ \uparrow 1-\tau & \uparrow \frac{\tau}{2} & & & & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \end{array} \quad \mathcal{W} = \mathcal{W}_{\mathcal{N}=4} \quad (3.39)$$

The UV global symmetry is $SU(N)_{\vec{X}} \times \prod_{j=1}^{N-1} U(1)_{Y_j - Y_{j+1}} \times U(1)_\tau$, which enhances to $SU(N)_{\vec{X}} \times SU(N)_{\vec{Y}} \times U(1)_\tau$ in the IR. In this UV completion the operator $A_{\vec{X}}$ coincides with the traceless singlet matrix $A_{\vec{X}} := a_N$. The other operator $A_{\vec{Y}}$ is instead constructed by collecting the $N(N-1)$ gauge invariant monopoles with magnetic fluxes $+1$ (-1) under a sequence of gauge nodes and the $N-1$ traces $\text{Tr}(a_i)$. For example, for $N = 4$ we have:

$$A_{\vec{Y}} = \begin{bmatrix} \text{Tr}(a_1) & \mathfrak{M}^{+,0,0} & \mathfrak{M}^{+,+,0} & \mathfrak{M}^{+,+,+} \\ \mathfrak{M}^{-,0,0} & \text{Tr}(a_2) & \mathfrak{M}^{0,+,0} & \mathfrak{M}^{0,+,+} \\ \mathfrak{M}^{-,-,0} & \mathfrak{M}^{0,-,0} & \text{Tr}(a_3) & \mathfrak{M}^{0,0,+} \\ \mathfrak{M}^{-,-,-} & \mathfrak{M}^{0,-,-} & \mathfrak{M}^{0,0,-} & -\sum_{i=1}^3 \text{Tr}(a_i) \end{bmatrix}. \quad (3.40)$$

The $FT[U(N)]$ admits a second UV completion, as shown in Figure 3.6. In each completion only one of the two global $SU(N)$ symmetries is realized manifestly as the flavor symmetry rotating the last bifundamental. The second $SU(N)$ global symmetry instead enhances from the $N-1$ topological symmetries due to all the nodes being balanced. The two UV completions are related by mirror duality, in this case a self-duality.

The S_b^3 partition function of the first UV completion theory is defined recursively as

¹⁰Note that we work in an off-shell parametrization wherein the manifest symmetry is $U(N)$ and not $SU(N)$, as in [57].

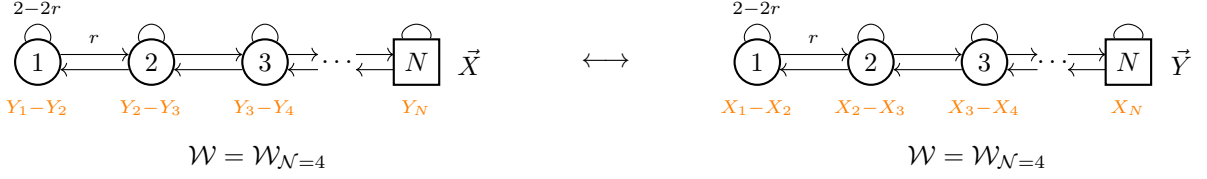


Figure 3.6: The $FT[U(N)]$ quiver gauge theory is self-mirror under the exchange of the manifest and topological symmetries. We also indicate the assignment of trial R-charges for the bifundamental and adjoint chiral fields. Notice that the bifundamental fields have trial R-charge r and adjoint chiral fields have trial R-charge $2 - 2r$ on both sides of the duality. As a consequence, the two moment maps $A_{\vec{X}}, A_{\vec{Y}}$ have both trial R-charge $2 - 2r$.

follows:

$$Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) := e^{2\pi i Y_N \sum_j^N X_j} \prod_{j,k=1}^N s_b \left(\tau - \frac{iQ}{2} (1 - 2r) + X_j - X_k \right) \\ \int \frac{\prod_{\alpha=1}^{N-1} du_{\alpha} e^{-2\pi i Y_N u_{\alpha}}}{(N-1)! \prod_{\alpha < \beta} s_b \left(\frac{iQ}{2} \pm (u_{\alpha} - u_{\beta}) \right)} s_b \left(\frac{iQ}{2} (1 - r) - \frac{\tau}{2} \pm (u_{\alpha} - X_j) \right) \\ Z_{FT[U(N-1)],I}(\vec{u}, \{Y_i\}_{i=1}^{N-1}, \tau). \quad (3.41)$$

with

$$Z_{FT[U(1)],I}(X, Y, \tau) := e^{2\pi i XY} s_b \left(\tau - \frac{iQ}{2} (1 - 2r) \right). \quad (3.42)$$

The partition function of the second UV completion can be obtained by simply swapping $\vec{X} \leftrightarrow \vec{Y}$.

$$Z_{FT[U(N)],II}(\vec{X}, \vec{Y}, \tau) = Z_{FT[U(N)],I}(\vec{Y}, \vec{X}, \tau), \quad (3.43)$$

Notice in the first UV completion the first slot $FT[U(N)](\cdot, \cdot; \tau)$ is for the manifest $U(N)$, the second slot for emergent $U(N)$. On the other hand in the second UV completion the second slot is for the manifest and the first one for the emergent $U(N)$.

Since both the UV completions describe the same SCFT, their partition functions are equal and we can define:

$$Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) = Z_{FT[U(N)],II}(\vec{Y}, \vec{X}, \tau) := Z_{FT[U(N)]}(\vec{X}, \vec{Y}, \tau), \quad (3.44)$$

Furthermore, the partition function satisfies:

$$Z_{FT[U(N)]}(\vec{X}, \vec{Y}, \tau) = Z_{FT[U(N)]}(-\vec{X}, -\vec{Y}, \tau) \quad (3.45)$$

as can be seen through a redefinition of the gauge fields.

For the rest of the chapter, we consider the $FT[U(N)]$ theory with a $U(N)_{\vec{X}} \times U(N)_{\vec{Y}}$ global symmetry instead (in practice, this corresponds to relaxing the constraints $\sum_i X_i = \sum_i Y_i = 0$ on the fugacities in the S_b^3 partition function).

3.2.1 A Chiral-Planar mirror dual

Starting from the self-mirror pair relating the two UV completion of the $FT[U(N)]$ SCFT in Figure 3.6, we will now consider a real mass deformation to produce a new duality relating a chiral quiver to a planar quiver which is depicted in Figure 3.7.

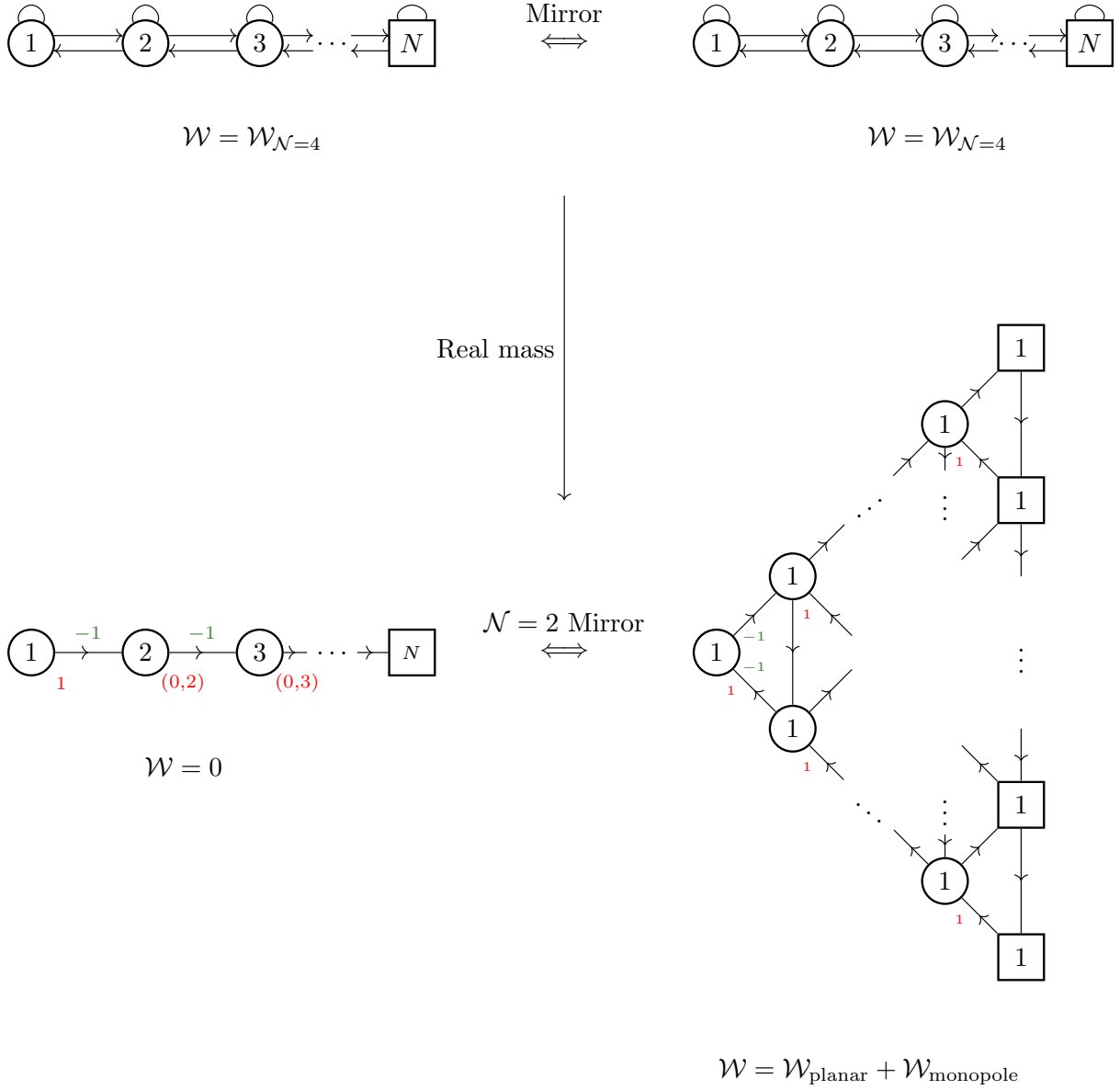


Figure 3.7: On the top row the mirror self-duality relates the two UV completions of the $FT[U(N)]$ theory. The real mass deformation yields a new $\mathcal{N} = 2$ mirror duality relating the two UV completions of the $G[U(N)]$ theory depicted in the second row.

As in the previous section, if on the electric side we implement the real mass deformation landing on a chiral non-Abelian quiver theory, on the mirror dual side, the matching vacuum we flow to, requires to move to a point on the Coulomb branch where all the gauge symmetries are broken to their maximal torus subgroup. The chiral and the planar quivers on the bottom of Figure 3.7 are IR dual and can be regarded as two UV completions of a new SCFT which we name $G[U(N)]$ with $U(1)^{N-1} \times SU(N)$ global symmetry, which we denote in compact form as:

$$G[\vec{X}, \vec{Y}] = \boxed{N}_{\vec{X}} \text{-----} \boxed{1^N}_{\vec{Y}} \quad (3.46)$$

We stress that in our notation the first argument of the $G[\cdot; \cdot]$ theories corresponds to fugacities for an $SU(N)$ global symmetry, while the second argument corresponds to a $U(1)^{N-1}$ global symmetry.

We now describe in detail this deformation beginning from the first UV completion of the $FT[U(N)]$, the electric theory, where the $U(N)_X$ is manifest. We consider the real mass deformation defined at the level of the fugacities by

$$\begin{cases} Y_J \rightarrow Y_J + \frac{\tau}{2}(2J - N - 1), & J = 1, \dots, N \\ \vec{X} \rightarrow \vec{X} + \frac{\tau}{2} \end{cases}, \quad \tau \rightarrow +\infty \quad (3.47)$$

In addition to this we also perform the following shifts in gauge fugacities:

$$\vec{u}^{(J)} \rightarrow \vec{u}^{(J)} - \frac{\tau}{2}(N - 1 - J). \quad (3.48)$$

Here $\vec{u}^{(J)}$ are the fugacities of the J -th gauge group, starting from the left. Integrating out the massive fields we obtain the following $\mathcal{N} = 2$ quiver gauge theory:

$$\begin{array}{c} \textcolor{brown}{Y_1 - Y_2 + (r-1) \frac{iQ}{2}} \quad \textcolor{brown}{Y_2 - Y_3 + (r-1) \frac{iQ}{2}} \quad \textcolor{brown}{Y_N - (r-1) \frac{(N-1)}{4} iQ} \\ \textcolor{red}{1} \xrightarrow{-1} \textcolor{red}{(0,2)} \xrightarrow{-1} \textcolor{red}{(0,3)} \cdots \xrightarrow{-1} \textcolor{red}{(0,N-1)} \xrightarrow{-1} \boxed{N}_{\vec{X}} \\ \textcolor{red}{(-\frac{N+1}{2}, \frac{N-1}{2})} \end{array} \quad \mathcal{W} = 0 \quad (3.49)$$

All the chirals in the theory have trial R-charge r which can, in principle, be fixed by performing F-extremization to determine the superconformal R-charge. However, we will not perform F-extremization in this work and keep r to be generic. Notice also that the FI parameters depend on the value of r . This is the first UV completion of the $G[U(N)]$ theory with UV symmetry $U(1)^{N-1}_Y \times SU(N)_{\vec{X}}$.

The S_b^3 partition function in the $\tau \rightarrow +\infty$ limit is given by:

$$\lim_{\tau \rightarrow +\infty} Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau \rightarrow +\infty} e^{\frac{\pi i}{2} H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau)} Z_{G[U(N)],I}(\vec{X}, \vec{Y}) \quad (3.50)$$

where $Z_{G[U(N)],I}(\vec{X}, \vec{Y})$ is the partition function of the first UV completion of the $G[U(N)]$ theory in (3.49) and:

$$H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) = H_{\text{div}}^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) + H_{\text{res}}^{\text{chiral}} \quad (3.51)$$

$$H_{\text{div}}^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) = \frac{\tau^2}{4} \left(\frac{4}{3}N^3 + 2N^2 + \frac{2}{3}N \right) + \frac{\tau}{2} \left(8 \sum_{J=1}^N JY_J - 4 \sum_{J=1}^N Y_J - iQ(N + N^2) \right. \\ \left. + iQr \left(\frac{2}{3}N^3 + 2N^2 + \frac{4}{3}N \right) \right), \quad (3.52)$$

and $H_{\text{res}}^{\text{chiral}}$ is a finite term independent on the global parameters $\vec{X}$ and $\vec{Y}$, that only depends on the squashing parameter Q .

We now consider the second UV completion of the $FT[U(N)]$ theory, the mirror theory. As in the case of the planar mirror of the chiral SQCD discussed in the previous section, here in addition to the real mass (3.47) to flow to the vacuum dual to the chiral quiver we also need to perform the following shifts in gauge fugacity:

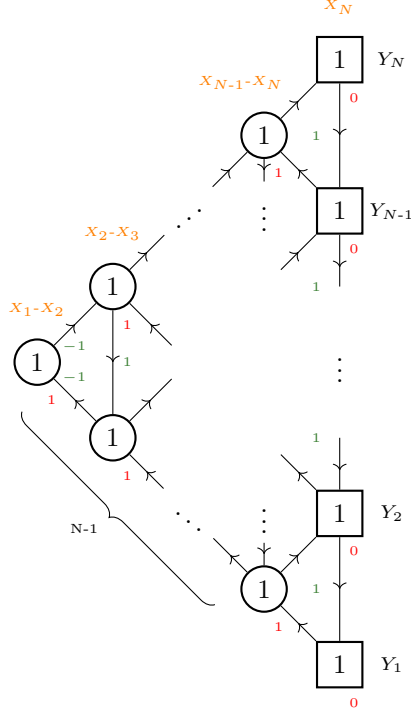
$$\omega_{\alpha}^{(J)} \rightarrow \omega_{\alpha}^{(J)} + \tau \left(\alpha - \frac{J+1}{2} \right) \quad (3.53)$$

and take the $\tau \rightarrow +\infty$ limit. Here $\omega_{\alpha}^{(J)}$ is the α -th fugacity of the J -th gauge group, starting from the left. The resulting $\mathcal{N} = 2$ planar quiver theory in Figure 3.8 is IR dual to the chiral theory in Figure (3.49), and provides the second UV completion of the $G[U(N)]$ SCFT.

A few comments regarding the superpotential of the quiver in Figure 3.8 are in order:

- There is a cubic superpotential term for every closed triangle. This superpotential originates from the cubic $\mathcal{N} = 4$ interactions that are not broken by the real mass deformation. We denote these superpotential terms as $\mathcal{W}_{\text{planar}}$. The assignment of trial R-charges is r for each diagonal chiral field and $2 - 2r$ for each vertical one. Indeed this assignment of R-charges is compatible with the planar superpotential. Notice also that the FI parameters depend on the value of r , encoding a mixing between the topological and R-symmetries as described in Subsection 3.1.2.
- Due to adjoint Higgsing of the gauge nodes, there are monopole terms in the superpotential [75]. There is a linear monopole superpotential for each monopole with charges $+1/-1$ under the upper/lower gauge nodes connected by a vertical line. We denote these linear monopole superpotential terms as $\mathcal{W}_{\text{monopole}}$. These are essential for the topological symmetry to correctly enhance to $SU(N)$, as we will comment later.
- In the planar theory there are self and mixed CS interactions. They are such that each gauge node has CS level $+1$ and each pair of gauge nodes connected by a diagonal/vertical line have a mixed CS interaction with level $-1/+1$. All these interactions arise from the integrated fermionic modes that became massive upon real mass deformation.
- There are shifts in the FIs associated to the mixing of the topological symmetry of the various nodes and the R-symmetry. The FIs of the gauge nodes in the J -th column from the right are:

$$\text{FIs: } X_J - X_{J+1} - \frac{iQr}{4}, X_J - X_{J+1}, \dots, X_J - X_{J+1} + \frac{iQr}{4} \quad (3.54)$$



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$$

Figure 3.8: The $\mathcal{N} = 2$ planar mirror dual of the $G[U(N)]$ theory. CS levels are indicated in red and the level of mixed CS interactions is indicated in green. On top of each column we indicate in orange the FIs of all the gauge nodes of the column, up to shifts that are discussed in the text.

from top to bottom, while the background FIs for the j -th flavor node associated to Y_j are:

$$\text{FIs: } X_N + \frac{iQr}{2}(2j - N - 1) + \delta_{j,1} \frac{iQ}{2} - \delta_{j,N} \frac{iQ}{2} \quad (3.55)$$

where $j = 1$ corresponds to the top flavor node in Figure 3.8 and $j = N$ corresponds to the bottom node. In Figure 3.8 we suppressed these shifts and only report the contribution to the FIs that depends on X_i .

The S_b^3 partition function in the $\tau \rightarrow +\infty$ limit is given by:

$$\lim_{\tau \rightarrow \infty} Z_{FT[U(N)],II}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau \rightarrow \infty} e^{\frac{\pi i}{2} H^{\text{planar}}(\vec{X}, \vec{Y}, \tau)} Z_{G[U(N)],II}(\vec{X}, \vec{Y}) \quad (3.56)$$

where $Z_{G[U(N)],II}(\vec{X}, \vec{Y})$ is the partition function of the second UV completion of the $G[U(N)]$ theory in (3.8) and:

$$H^{\text{planar}}(\vec{X}, \vec{Y}, \tau) = H_{\text{div}}^{\text{planar}}(\vec{X}, \vec{Y}, \tau) + H_{\text{res}}^{\text{planar}} \quad (3.57)$$

Here $H_{\text{res}}^{\text{planar}}$ is a constant phase independent of the global parameters $\vec{X}$ and $\vec{Y}$, that only depends on the squashing parameter Q . $H_{\text{div}}^{\text{planar}}$ is a divergent phase that is equal to $H_{\text{div}}^{\text{chiral}}$ and therefore it cancels against the chiral dual theory, providing a strong check that the theories in Figure 3.8 and (3.49) are dual. We found also that $H_{\text{res}}^{\text{planar}} = H_{\text{res}}^{\text{chiral}}$, therefore taking the limit $\tau \rightarrow +\infty$ we find:

$$Z_{G[U(N)],I}(\vec{X}, \vec{Y}) = Z_{G[U(N)],II}(\vec{X}, \vec{Y}) \quad (3.58)$$

Global Symmetries and Operator Map

We now comment on the duality map between global symmetries and the chiral rings of the two theories.

Let us start from the global symmetries. As already stated at the beginning of the section, the $G[U(N)]$ SCFT has a $U(1)^{N-1} \times SU(N)$ global symmetry. In the chiral UV completion (3.49), the two global symmetries are realized manifestly in the UV. $SU(N)$ is the flavor symmetry while $U(1)^{N-1}$ are the topological symmetries of the $N-1$ nodes. In the planar UV completion, in Figure 3.8, instead the $U(1)^{N-1}$ is realized as the flavor symmetry that is unbroken by the planar superpotential and up to gauge transformations. The $SU(N)$ global symmetry is instead obtained from the enhancement of the $N-1$ topological symmetries. We recall that the presence of $\mathcal{W}_{\text{monopole}}$ has the effect of breaking the $U(1)^k$ topological symmetries in a column with k gauge nodes down to a diagonal $U(1)$, such that effectively each column of gauge nodes contributes with a single $U(1)$ topological symmetry.

Let us now move to the operator map. The chiral ring of the planar $G[U(N)]$ theory in Figure 3.8 is generated by the $N-1$ vertical bifundamental fields connecting adjacent vertical flavor nodes. These can be taken as the only independent chiral ring generators up to F-term relations imposed by $\mathcal{W}_{\text{planar}}$.

We expect these operators to be mapped to monopole operators in the chiral $G[U(N)]$ theory (3.49). This is the case, but the fundamental monopoles are gauge variant and must be appropriately dressed by appropriate powers of fundamental chiral multiplets. Focusing on a section of the chiral $G[U(N)]$ quiver:

$$\begin{array}{ccccccc} \rightarrow & \textcircled{k-1} & \xrightarrow{-1} & \textcircled{k} & \xrightarrow{-1} & \textcircled{k+1} & \rightarrow \\ & (0, k-1) & Q & (0, k) & P & (0, k+1) & \end{array} \quad (3.59)$$

we notice that the fundamental monopole of the $U(k)$ node can be dressed with Q^{k-1} to ensure gauge invariance of the resulting operators. The $N-1$ dressed fundamental monopoles of the chiral $G[U(N)]$ theory generate the chiral ring of the theory. We can also calculate that they have the correct charges to be mapped to the $N-1$ vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror, which can be verified also via the superconformal index. Assume that each bifundamental in the electric theory ($Q, P \dots$) has trial R-charge r . Consequently, the vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror planar theory, that the monopoles map to, have R-charge $2-2r$. By using eq. (2.41), the dressed monopole of the k -th gauge node in the chiral theory $tr(Q^{k-1}\mathfrak{M}^{0,\dots,0,-,0,\dots,0})$ has R-charge:

$$\begin{aligned} R(Q^{k-1}\mathfrak{M}^{0,\dots,0,-,0,\dots,0}) &= -(r-1) + \frac{1}{2}(k-1)(1-r) + \frac{1}{2}(k+1)(1-r) - (k-1) + (k-1)r = \\ &= (1-r) + k(1-r) - (k-1) + (k-1)r = \\ &= 2-2r \end{aligned} \quad (3.60)$$

where the first factor comes from the mixing between the topological symmetry and the R-symmetry¹¹, while the second and third terms in (3.60) are the bifundamental fermions contributions, the $-(k-1)$ is the contribution from the gaugini, and $(k-1)r$ is the

¹¹The mixing can be extracted as the coefficient of $\frac{iQ}{2}$ in the FI of the k -th node: $Y_k - Y_{k+1} + \frac{iQ}{2}(r-1)$. The contribution to the charge is then obtained by multiplying it by the topological charge.

contribution from the dressing with Q^{k-1} . So the $N - 1$ fundamental dressed monopoles map to the $N - 1$ vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror. In the rest of the chapter we denote the gauge-invariant dressed monopoles as $\mathfrak{M}_d$, and we omit the dressing fields.

3.2.2 Flip-Flip chiral-chiral and planar-planar dualities

The $FT[U(N)]$ theory enjoys another UV completion known as the Flip-Flip dual [171]. The Flip-Flip dual quiver $ffFT[U(N)]$ is obtained starting from the $FT[U(N)]$ theory with two extra sets of singlets which flip the operators A_X, A_Y :

$$\mathcal{W}_{ffFT} = \mathcal{W}_{FT} + O_X A_{\vec{X}} + O_Y A_{\vec{Y}} \quad (3.61)$$

Since in the $FT[U(N)]$ theory the meson matrix is already flipped by the singlet a_N , this further flip give mass to a_N . As a result, the spectrum of the Flip-Flip theory is given by the meson matrix, in the adjoint of $SU(N)_X$, and by the flip matrix O_Y , in the adjoint representation of $SU(N)_Y$.

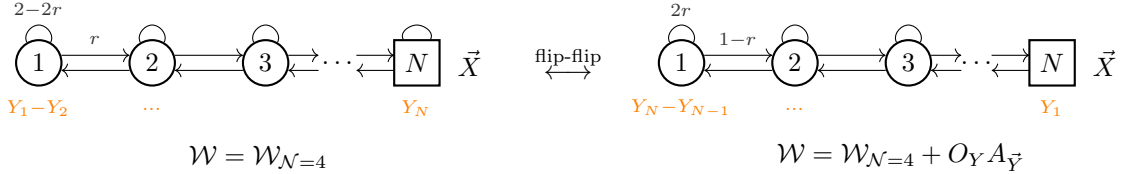


Figure 3.9: The flip-flip duality for the $FT[U(N)]$ theory. On the electric side the bifundamentals have $U(1)_\tau$ charge $\frac{1}{2}$ and trial R-charge r , while on the magnetic side the bifundamentals have $U(1)_\tau$ charge $-\frac{1}{2}$ and trial R-charge $1 - r$, as emphasized by the labels in black.

The Flip-Flip duality is represented by the following equality of S_b^3 partition functions:

$$Z_{ffFT[U(N)]}(\vec{X}, \vec{Y}, \tau) = \prod_{j,k=1}^N s_b \left(\frac{iQ}{2}(2r-1) + \tau + X_j - X_k \right) s_b \left(\frac{iQ}{2}(2r-1) + \tau + Y_j - Y_k \right) Z_{FT[U(N)]}(\vec{X}, \vec{Y}, -\tau) \Big|_{r \rightarrow 1-r} \quad (3.62)$$

As shown in [172] it is possible to prove the flip-flip duality by iterative applications of the Aharony duality. The idea of the proof is the following. Starting from the left-most $U(1)$ node in the $FT[U(N)]$ theory (whose adjoint chiral is just a singlet) we apply Aharony duality [93] which leaves the rank invariant but gives mass to the adjoint chiral field of the adjacent $U(2)$, hence we can apply again the Aharony duality on it. This will remain a $U(2)$ node but the dualization will also give mass to the the adjoint chiral field of the adjacent $U(3)$ node. We continue applying iteratively the Aharony duality until we reach the last $U(N - 1)$ node. Notice that since every $U(k)$ node sees $2k$ flavors, ranks do not change when we apply the Aharony duality. We then perform another sequence of dualizations starting from the leftmost $U(1)$ node and stopping at the second last node $U(N - 2)$. In the third sequence of dualization we start again from the $U(1)$ node and proceed along the tail stopping at the $U(N - 3)$ node. We iterate this procedure for a total of $N - 1$ times, applying Aharony duality $N(N - 1)/2$ times. The singlet fields

flipping the mesons and the monopoles appearing in the Aharony duality reconstruct the singlet matrix O_Y and give mass to the singlet matrix O_X .

We will now show that we can derive similar self-dualities (modulo flips) for the two $G[U(N)]$ UV completions. In analogy with the $\mathcal{N} = 4$ case we denote these dualities as flip-flip dualities. These dualities map chiral theories to chiral theories and planar theories to planar theories, in contrast to the mirror-like dualities described in Section 3.2. This is expected because the $\mathcal{N} = 4$ flip-flip does not exchange flavor and topological symmetries, therefore in the $\mathcal{N} = 2$ case the $SU(N)$ and $U(1)^{N-1}$ factors of the global symmetries of $G[U(N)]$ will not be exchanged.

Chiral-Chiral flip-flip duality

Starting from the flip-flip duality in Figure 3.9, we take the real mass deformation (3.47) that takes the $FT[U(N)]$ on the l.h.s. to the first UV completion of $G[U(N)]$, the chiral UV completion.

Contrary to the case discussed in the previous section, where we started from the two UV completions related by mirror duality, now the same deformation can be taken on the dual side in Figure 3.9. Indeed in this case we find that in the dual the correct vacuum is again a chiral quiver, so we obtain the following new chiral-chiral duality relating the first chiral UV completion of the $G[U(N)]$ theory (3.49):

$$\begin{array}{c}
 \textcolor{brown}{Y_1 - Y_2 + (r-1)\frac{iQ}{2}} \quad \textcolor{brown}{Y_2 - Y_3 + (r-1)\frac{iQ}{2}} \quad \textcolor{brown}{Y_N - (r-1)\frac{(N-1)}{4}iQ} \\
 \textcolor{red}{1} \quad \textcolor{red}{(0,2)} \quad \textcolor{red}{(0,3)} \quad \dots \quad \textcolor{red}{(0,N-1)} \quad \textcolor{red}{(-\frac{N+1}{2}, \frac{N-1}{2})} \\
 \textcircled{1} \xrightarrow{-1} \textcircled{2} \xrightarrow{-1} \textcircled{3} \xrightarrow{\dots} \textcircled{N-1} \xrightarrow{-1} \boxed{N} \quad \vec{X} \\
 \mathcal{W} = 0
 \end{array} \tag{3.63}$$

to its flip-flip chiral dual:

$$\begin{array}{c}
 \textcolor{brown}{Y_N - Y_{N-1} - r\frac{iQ}{2}} \quad \textcolor{brown}{Y_{N-1} - Y_{N-2} - r\frac{iQ}{2}} \quad \textcolor{brown}{Y_1 + r\frac{(N-1)}{4}iQ} \\
 \textcolor{red}{-1} \quad \textcolor{red}{(0,-2)} \quad \textcolor{red}{(0,-3)} \quad \dots \quad \textcolor{red}{(0,-N+1)} \quad \textcolor{red}{(-\frac{N-1}{2}, -\frac{N-1}{2})} \\
 \textcircled{1} \xleftarrow{+1} \textcircled{2} \xleftarrow{+1} \textcircled{3} \xleftarrow{\dots} \textcircled{N-1} \xleftarrow{+1} \boxed{N} \quad \vec{X} \\
 \mathcal{W} = t_1 \mathfrak{M}_d^{-,0,\dots,0} + t_2 \mathfrak{M}_d^{0,-,0,\dots,0} + \dots
 \end{array} \tag{3.64}$$

Notice that in the flip-flip dual theory there is a superpotential containing flippers t_i for all the dressed monopoles $\mathfrak{M}_d^{0,\dots,-,\dots,0}$ with negative GNO flux under a single gauge node. In addition all the arrows and CS coupling signs are flipped and the FIs are reshuffled. The bifundamental fields in the flip-flip theory have trial R-charge $1 - r$. On the flip-flip side there is also a background CS term at level $-\frac{1}{2}$ for each $U(1)$ factor in the $U(1)_Y^{N-1}$ symmetry group.

Under the duality map the dressed monopoles with negative charge under the topological symmetries of the electric theory map to the set of singlets t_i in the flip-flip dual.

Also in this case, similarly as for the $FT[U(N)]$ theory, the duality can be demonstrated by iterative applications of local dualities. Now at each step we need to locally apply the chiral Giveon-Kutasov duality for $U(N)_{1,1}$ with $[N-1, N+1]$ flavors [35, 128, 173].

Planar-Planar flip-flip duality

Let's start again from the duality in Figure 3.9. Now we take the real mass deformation that takes the $FT[U(N)]$ on the l.h.s. to the second UV completion of $G[U(N)]$, the planar UV completion. So we perform the real mass deformation:

$$\begin{cases} X_J \rightarrow X_J + \frac{\tau}{2}(2J - N - 1), & J = 1, \dots, N \\ \vec{Y} \rightarrow \vec{Y} + \frac{\tau}{2}N \end{cases}, \quad \tau \rightarrow +\infty \quad (3.65)$$

and we also perform the following shifts in gauge parameters:

$$\omega_\alpha^{(J)} \rightarrow \omega_\alpha^{(J)} + \tau \left(\alpha - \frac{J+1}{2} \right) \quad (3.66)$$

Also in this case we can implement the same deformation on the dual side and we land on a pair of dual planar Abelian quivers given in Figure 3.10.

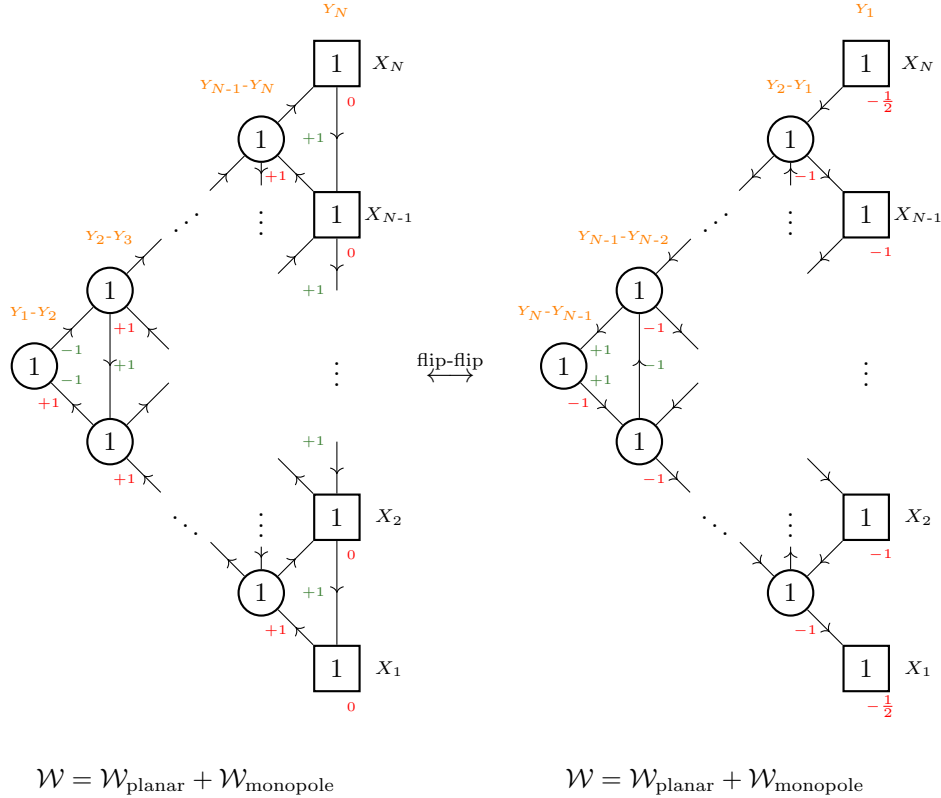


Figure 3.10: Self-duality *modulo flips* displayed by the planar $G[U(N)]$ theory. The background CS terms for the flavor symmetry are different on the two sides of the duality, furthermore on the r.h.s. there is a background CS term at level $-N$ for the emergent $SU(N)_Y$ global symmetry. Flowing from the $\mathcal{N} = 4$ *flip-flip* this is generated from integrating out the monopole flippers O_Y in Figure 3.9. On the r.h.s. there are also shifts in the FI terms, discussed in the main text, which are not reported in the figure.

Notice that in the flip-flip dual theory all the arrows are flipped and the FIs are reshuffled. Furthermore, CS and mixed CS levels have opposite sign with respect to the electric theory and the trial R-charge of the chirals is $1 - r$. To avoid clutter here we set $r = 0$, therefore the electric diagonal bifundamentals have trial R-charge 0 and the

magnetic diagonal bifundamentals have trial R-charge 1. On the r.h.s. there is also a background CS term at level $-N$ for the emergent $SU(N)_Y$ symmetry, as well as shifts for the FIs. On the flip-flip side the FIs for the gauge node of the J -th column from the right are:

$$\text{FI: } Y_J - Y_{J-1} + \frac{iQ}{4}, Y_J - Y_{J-1}, \dots, Y_J - Y_{J-1}, Y_J - Y_{J-1} - \frac{iQ}{4} \quad (3.67)$$

from top to bottom. In Figure 3.10 we only reported the Y -dependent part of the FIs.

The operator map is very simple: the vertical string of $N - 1$ singlets in the last column of the electric theory maps to the $N - 1$ mesons of the flip-flip theory constructed along the shortest path connecting two adjacent $U(1)$ flavor nodes.

Also in this case the duality can be demonstrated by iterative applications of local Aharony-like dualities.

3.3 Towards an Algorithmic Approach: $\mathcal{S}$ -walls, QFT Building Blocks, and Basic Duality Moves

Starting from the $\mathcal{N} = 4$ $U(N)$ SQCD with $F \geq 2N$ hypers we can also consider a more general real mass deformation, w.r.t. the one discussed in the previous section, where we partially break the $SU(F)$ global symmetry to obtain an $\mathcal{N} = 2$ $U(N)$ SQCD with n_f fundamental and n_a anti-fundamental chiral multiplets on the electric side:

$$\begin{array}{ccc} \mathcal{N} = 4 & U(N) & \\ F \text{ hypermultiplets} & \rightarrow & \mathcal{N} = 2 \quad U(N)_{-\frac{F}{2}+N, -\frac{F}{2}} \\ & & [n_f, n_a] \text{ chiral fields, } F = n_f + n_a \end{array} \quad (3.68)$$

Here we consider flows that preserve the maximum amount of chiral matter multiplets $F = n_f + n_a$, while giving mass to the adjoint chiral field. Therefore, the number of (anti-) fundamentals n_f and n_a satisfy $F = n_f + n_a$ but are otherwise unconstrained and depend on the specific mass deformation that we consider. On the other hand, the choice of CS levels is **not arbitrary** and is fixed by the parent $\mathcal{N} = 4$ theory: each massive (anti)fundamental multiplet contributes as $(-\frac{1}{2}, -\frac{1}{2})$ to the CS level and the massive adjoint chiral contributes as $(N, 0)$. In the CS-level notation: $(k, k + lN)$, this corresponds to $k = -\frac{F}{2} + N$ and $l = -1$.

One can generalize to arbitrary values of l by applying Witten's $SL(2, \mathbb{Z})$ action on both sides of the duality as discussed in Subsection 3.1.3.

We can also turn more general real masses and integrate out more chirals to access a more general range of CS levels. Notice that this can also be achieved by first performing the real mass deformation that preserves the maximum number of chirals, discussed here, and then performing additional real masses for the resulting $\mathcal{N} = 2$ theories. We address this generalization in Section 3.5.

One can then consider the same limit on the mirror side and try to identify the dual vacuum. In general, this is a non-trivial exercise as there are multiple potential vacua corresponding to possible ways of moving on the Coulomb branch of the $\mathcal{N} = 4$ mirror (which, in turn, match how we shift gauge fugacities in the S_b^3 partition function).

More generally, one can start on the electric side with a linear unitary $\mathcal{N} = 4$ quiver with gauge group $\prod_{N_i} U(N_i)$. We consider a real mass deformation, paired with a VEV for the scalar in the $\mathcal{N} = 2$ vector multiplets, such that the adjoint chiral fields and only

one of the $\mathcal{N} = 2$ chiral multiplets in each matter hypermultiplet acquires a real mass. Such a deformation preserves the maximum total amount of chiral fields. Integrating out the massive field we obtain a chiral linear quiver:

$(k_{i-1}, k_{i-1} - N_{i-1}) \quad (k_i, k_i - N_i) \quad (k_{i+1}, k_{i+1} - N_{i+1})$

(3.69)

where $f_i \leq F_i$ is the number of fundamental flavors for the i -th node, and $F_i - f_i$ is the number of antifundamental flavors. The CS levels of the i -th gauge node k_i are fixed by the parent theory¹²:

$$k_i = -\frac{N_{i-1} + N_{i+1} + F_i}{2} + N_i. \quad (3.70)$$

One can also consider more general real mass flows such that in the resulting $\mathcal{N} = 2$ quiver some of the bifundamental chiral fields point to the left. As before, one can implement the same limit in the mirror dual quiver theory to reach the dual vacuum. This remains a non-trivial exercise.

Clearly, it is desirable to have a more systematic way to determine the planar mirror dual of a generic flavored $\mathcal{N} = 2$ quiver without the need to resort to S_b^3 partition functions. We do so by adapting the idea of the local dualization algorithm of [45, 46] which we quickly review now.

3.3.1 Lightning Recap: The $\mathcal{N} = 4$ Dualization Algorithm

The dualization algorithm for $\mathcal{N} = 4$ linear quiver theories provides a purely field-theoretical proof of mirror duality assuming only basic Seiberg-like dualities.

It is well known that $\mathcal{N} = 4$ linear quiver theories can be realized on Hanany-Witten brane set-ups and inherit mirror dualities from the S-duality of Type IIB set-ups (see Sec. 2.3.4 for a brief discussion). The idea of the algorithm originates from the observation

¹²The specific mass deformation considered here leads to theories where the gauge nodes have CS parameter $l = -1$, where CS parameters (k, l) refer to the common convention used $U(N)_{(k, k+lN)}$. This arises from giving (anti-)fundamentals a negative mass, thus leading to a positive mass for the $\mathcal{N} = 2$ adjoints in the $\mathcal{N} = 4$ SUSY multiplets. The other possible choice of deformation, arising by giving positive mass to (anti-)fundamentals and negative mass to adjoints, is also a consistent one, and would lead to $\mathcal{N} = 2$ gauge theories with $l = +1$. Both conventions will be used in the following, and the one used will be clear from the context.

[170, 174, 175] that on linear or circular brane setups, $\mathcal{S}$ -duality can act locally on each 5-brane creating an $\mathcal{S}$ -duality wall on its right and an $\mathcal{S}^{-1}$ -duality wall on its left: $D5 = \mathcal{S} \cdot NS5 \cdot \mathcal{S}^{-1}$ and $\overline{NS5} = \mathcal{S} \cdot D5 \cdot \mathcal{S}^{-1}$. The dualization algorithm implements this local action of S-duality in field theory.

In this subsection we briefly review the results presented in [45, 46, 176–179], the reader can look at them for a more comprehensive discussion.

Following [113], we identify the S-duality wall with the $FT[U(N)]$ theory:¹³

$$\begin{aligned} \mathcal{S}_{\mathcal{N}=4}\text{-wall} : \mathcal{S}_{\mathcal{N}=4}[\vec{X}, \vec{Y}] &:= FT[\vec{X}, -\vec{Y}] = \boxed{N}_{\vec{X}} \overset{+}{\text{-----}} \boxed{N}_{\vec{Y}} \\ \mathcal{S}_{\mathcal{N}=4}^{-1}\text{-wall} : \mathcal{S}_{\mathcal{N}=4}^{-1}[\vec{X}, \vec{Y}] &:= FT[\vec{X}, \vec{Y}] = \boxed{N}_{\vec{X}} \overset{-}{\text{-----}} \boxed{N}_{\vec{Y}} \end{aligned} \quad (3.71)$$

As discussed in Section 3.2.1, the $FT[U(N)]$ theory admits a Lagrangian UV completion Figure 3.39 and is self-mirror under the exchange of emergent and manifest $U(N)$ symmetries.

The $\mathcal{S}$ -wall satisfies the fusion to identity encoded in a partition function identity [176]:

$$\int d\vec{W} \Delta_{(N)}(\vec{W}; \tau) Z_{\mathcal{S}_{\mathcal{N}=4}}^{(N)}(\vec{X}, \vec{W}; \tau) Z_{\mathcal{S}_{\mathcal{N}=4}^{-1}}^{(N)}(\vec{W}, \vec{Y}; \tau) =_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau). \quad (3.72)$$

interpretable as $\mathcal{S}\mathcal{S}^{-1} = 1$, where the identity operator on the r.h.s. can be thought of as the identity element, with:

$$_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau) = \frac{1}{N! \Delta_{(N)}(\vec{X}, \tau)} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N \delta(X_j - Y_{\sigma(j)}), \quad (3.73)$$

and the $\mathcal{N} = 4$ measure is defined in (3.10).

This identity can pictorially be represented as follows:

$$\boxed{N}_{\vec{X}} \overset{+}{\text{-----}} \overset{a}{\bigcirc} \overset{-}{\text{-----}} \boxed{N}_{\vec{Y}} \longleftrightarrow {}_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau) \quad (3.74)$$

The adjoint chiral a couples to the moment map operators A_L and A_R of the left and right walls as $\mathcal{W} = a(A_L + A_R)$.

We then identify the QFT blocks associated to the different 5-brane configurations:

$$\begin{array}{c} \boxed{1} \\ \updownarrow \\ \boxed{N} \end{array} \quad ; \quad \begin{array}{c} \text{NS5} \\ \text{N D3} \quad \text{N D3} \\ \vdots \quad \vdots \end{array} \quad ; \quad \boxed{N}_{\vec{X}} \longleftrightarrow \boxed{N}_{\vec{Y}} \quad (3.75)$$

which are denoted the flavor block and the bifundamental block respectively.

We can now define the action of the S-duality wall on the basic QFT blocks:

¹³The $FT[U(N)]$ theory differs from the $T[U(N)]$ theory introduced in [113] for the presence of an extra set of singlets flipping the Higgs branch moment map.

1. The first basic move consists in the $\mathcal{S}$ -dualization of a bifundamental block into a fundamental flavor block (the field-theoretic counterpart of $\mathcal{S} \cdot D5 \cdot \mathcal{S}^{-1} = \overline{NS5}$):

$$\mathcal{W} = a(A_L + A_R) + \tilde{f}(A_L - A_R)f \quad \longleftrightarrow \quad \mathcal{W} = 0 \quad (3.76)$$

wherein we have explicitly reported the trial R -charge and the $U(1)_\tau$ charge of the various fields in the theory. On the r.h.s. the orange fugacities denote the background FIs for the flavor nodes. This corresponds to the partition function identity:

$$\begin{aligned} \int d\vec{W} \Delta_{(N)}(\vec{W}, \tau) Z_{S_{\mathcal{N}=4}}^{(N)}(\vec{X}, \vec{W}, \tau) Z_{S_{\mathcal{N}=4}}^{(N)}(\vec{W}, \vec{Y}, \tau) \prod_{\alpha=1}^N s_b \left(\frac{iQ}{2} - \frac{\tau}{2} \pm (W_\alpha - \eta) \right) = \\ = e^{-2\pi i \eta \sum_{j=1}^N (X_j - Y_j)} \prod_{j,k=1}^N s_b \left(\frac{\tau}{2} \pm (X_j - Y_k) \right). \end{aligned} \quad (3.77)$$

2. The inverse basic move consists of the $\mathcal{S}$ -dualization of a fundamental block into a bifundamental block (which is the field-theoretic counterpart of $\mathcal{S} \cdot NS5 \cdot \mathcal{S}^{-1} = D5$)

$$\mathcal{W} = a(A_L + A_R) + \tilde{f}(A_L - A_R)f \quad \longleftrightarrow \quad \mathcal{W} = 0 \quad (3.78)$$

Notice that the gluing on the l.h.s. does not include the addition of $U(N)$ adjoint multiplets coupling linearly to the moment map of the $\mathcal{S}$ -walls. This inverse basic move is encoded in the following partition function identity:

$$\begin{aligned} \prod_{j=1}^N s_b \left(\frac{\tau}{2} \pm (X_j - \eta) \right) \mathcal{X}_{\vec{Y}}(\tau) = \\ = \int d\vec{U} d\vec{V} \frac{e^{2\pi i \eta \sum_{a=1}^N (U_a - V_a)}}{\prod_{j,k=1}^N s_b \left(\frac{iQ}{2} \pm (U_j - U_k) \right) s_b \left(\frac{iQ}{2} \pm (V_j - V_k) \right)} \\ Z_{S_{\mathcal{N}=4}}^{(N)}(\vec{X}, \vec{U}; \tau) Z_{S_{\mathcal{N}=4}}^{(N)}(\vec{V}, \vec{Y}; \tau) \prod_{a,b=1}^N s_b \left(\frac{iQ}{2} - \frac{\tau}{2} \pm (U_a - V_b) \right). \end{aligned} \quad (3.79)$$

We consider the case of $\mathcal{N} = 4$ $U(N)$ SQCD with $2N + F$ flavors to show how the algorithm works in Figure 3.11. ¹⁴ When decomposing the theory into QFT blocks, it is

¹⁴Here we use a short-cut, by treating $2N$ fundamental hypers as two sets of “frozen” bifundamentals. We could have also proceeded as in [46] by chopping all the $F + 2N$ flavors into fundamental blocks and adding a $(0, N)$ and an $(N, 0)$ bifundamental block.

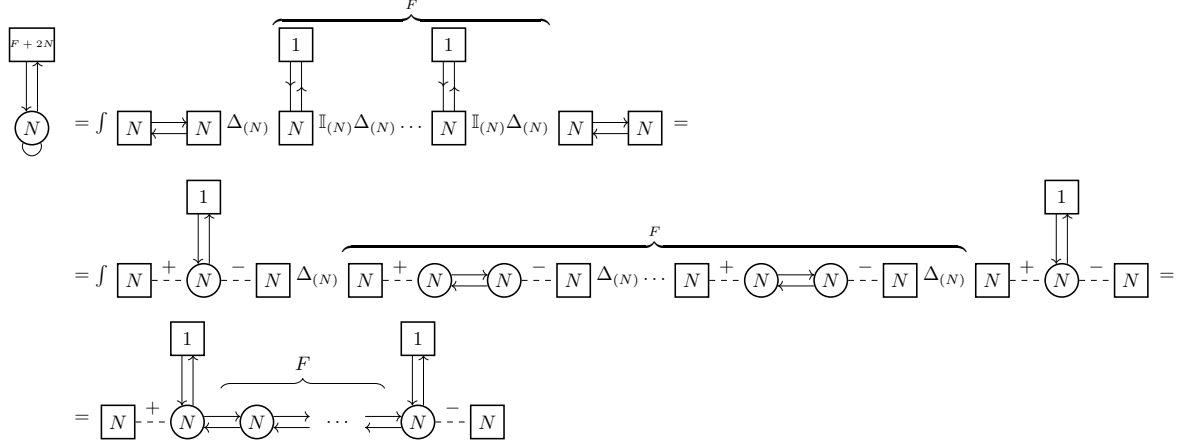


Figure 3.11: Applying the $\mathcal{N} = 4$ dualization to the $\mathcal{N} = 4$ $U(N)$ SQCD with $F + 2N$ fundamental hypermultiplets proceeds as follows - in the first line the quiver is chopped into QFT blocks. In the second line each block is dualized using the basic moves. In the last line we used the fusion to Identity property of the $\mathcal{S}$ -wall to obtain the known mirror dual theory. Notice that the two tails of gauge nodes with increasing ranks are condensed in the $\mathcal{S}$ -wall notation. By carefully performing each step, it is also possible to keep track of all the parameters to find that we land precisely on the duality in Figure 3.1. We have also explicitly kept the integrals and integration measures to emphasize that gauging or ungauging nodes must be performed with the appropriate measure.

important to keep in mind that each block effectively plays the role of a matrix element, with two sets of indices corresponding to the two N -dimensional vectors of parameters associated with the two $U(N)$ global symmetries. In particular, note that flavor blocks are accompanied by an identity wall, reflecting this matrix structure.

In this quick review, we have only discussed QFT blocks and Identity-walls with constant rank N . We can, for example, have quivers with non-constant ranks which are chopped into $U(N) \times U(M)$ *asymmetric* bifundamental blocks. The dualization of such blocks requires the introduction of *asymmetric* Identity-walls. The algorithm in this case includes an addition step, a sequence of Hanany-Witten duality moves to remove all the *asymmetric* walls. For the analysis of these more general cases we refer the reader to [46].

In [57] the local dualization algorithm approach has been extended to a class of generalized $\mathcal{N} = 2$ quiver theories that can be realized in set-ups involving $NS, NS', D5, D5'$ branes preserving four supercharges. Unlike the $\mathcal{N} = 4$ case where brane setups directly inform the structure of the low-energy field theory, here the situation is reversed: the algorithmic field theory approach yields an exact dual description, while the corresponding brane configurations are difficult to interpret directly. This inversion of logic highlights the power of the algorithmic method—not only does it provide precise duals, but it also offers a new lens through which to understand and possibly reconstruct the brane setups themselves. Motivated by this, our aim is now to develop a similar algorithm tailored to chiral-planar $\mathcal{N} = 2$ theories.

The algorithm will allow us to construct mirror duals of $\mathcal{N} = 2$ SQCD theories with both fundamentals and antifundamental chirals, as well as for chiral-planar quiver theories as we will show in Section 3.4. We provide a realization of the $\mathcal{N} = 2$ $\mathcal{S}$ -wall operator in Section 3.3.2 and construct the basic QFT building blocks and the corresponding duality

moves in Section 3.3.3.

3.3.2 The chiral-planar $\mathcal{S}$ -wall, fusion to Identity and gluing rules

In this section we propose, in analogy $\mathcal{N} = 4$ algorithm, to identify $G[U(N)]$ as the “chiral-planar $\mathcal{S}$ -wall” and prove its fusion to Identity properties.

More precisely we define:

$$\begin{aligned}
\mathcal{S}\text{-wall} : \mathcal{S}[\vec{X}, \vec{Y}] = G[-\vec{X}, \vec{Y}] &= \boxed{N}_{\vec{X}} \overset{+}{\rightsquigarrow} \boxed{1^N}_{\vec{Y}} \\
\mathcal{S}^{-1}\text{-wall} : \mathcal{S}^{-}[\vec{X}, \vec{Y}] = G[\vec{X}, \vec{Y}] &= \boxed{N}_{\vec{X}} \overset{-}{\rightsquigarrow} \boxed{1^N}_{\vec{Y}}
\end{aligned} \tag{3.80}$$

Both theories admit (at least) two UV completions whose difference lies in the direction of the arrows and the sign of FIs. For convenience the UV completions of both theories are reported in Figures 3.12 and 3.13.

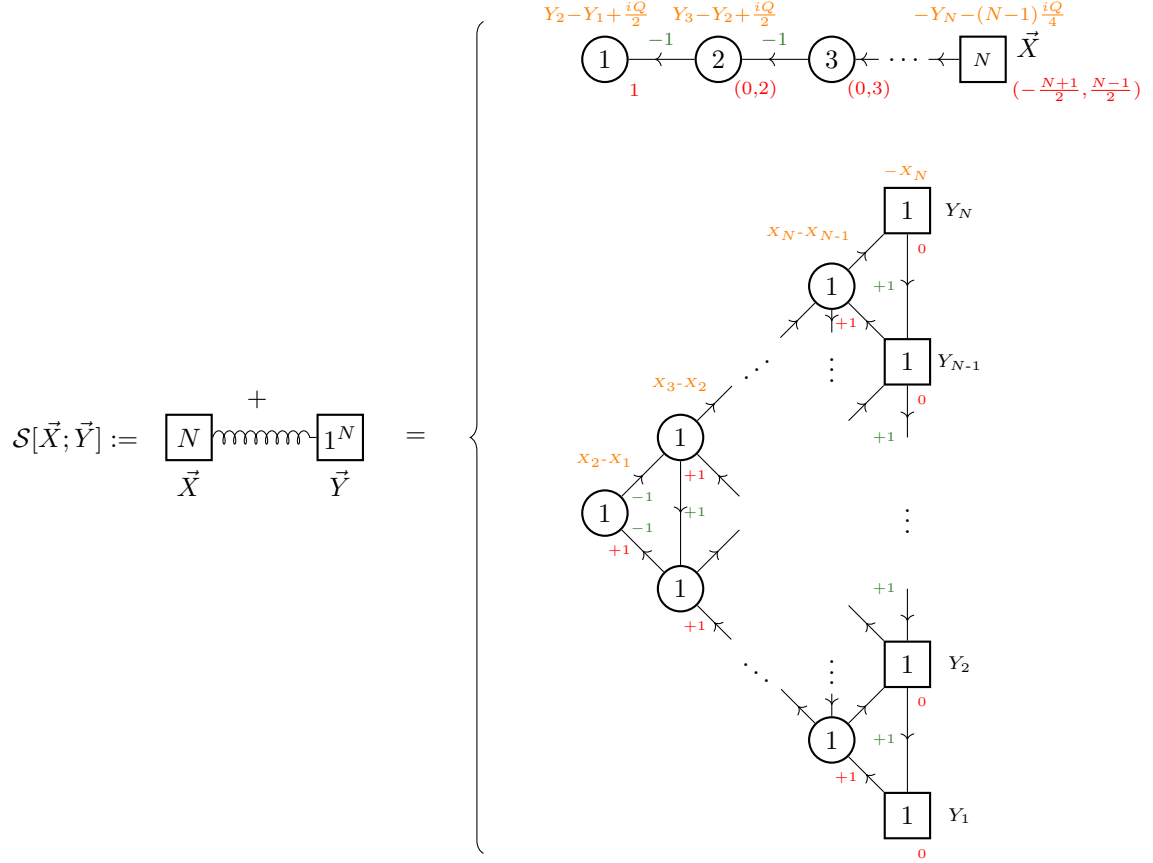


Figure 3.12: The planar and chiral UV completions of the $\mathcal{N} = 2$ chiral-planar $\mathcal{S}$ -wall coinciding with $G[-\vec{X}; \vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity. Here, the trial R-charge r is set to 0. Therefore, in the chiral completion, on top, each bifundamental has trial R-charge 0. In the planar completion the vertical/diagonal chirals have trial R-charge $2/0$.

Notice that in the chiral UV completion of $\mathcal{S}$ the arrow direction is inverted w.r.t. $\mathcal{S}^-$, where it emanates from the manifest $U(N)$, and the FI at each node are opposite: $Y_{i+1} - Y_i$. In the planar UV completion of $\mathcal{S}$ the FI's associated to each string are $X_{i+1} - X_i$ rather than $X_i - X_{i+1}$ as in $\mathcal{S}^-$.

In our notation the first argument of the $Z_{\mathcal{S}^\pm}[\cdot; \cdot]$ theories corresponds to fugacities for an $U(N)$ global symmetry, while the second argument corresponds to a $U(1)^{N-1}$ global symmetry.

We will now prove the fusion to identity properties of the $\mathcal{N} = 2$ $\mathcal{S}$ -wall.

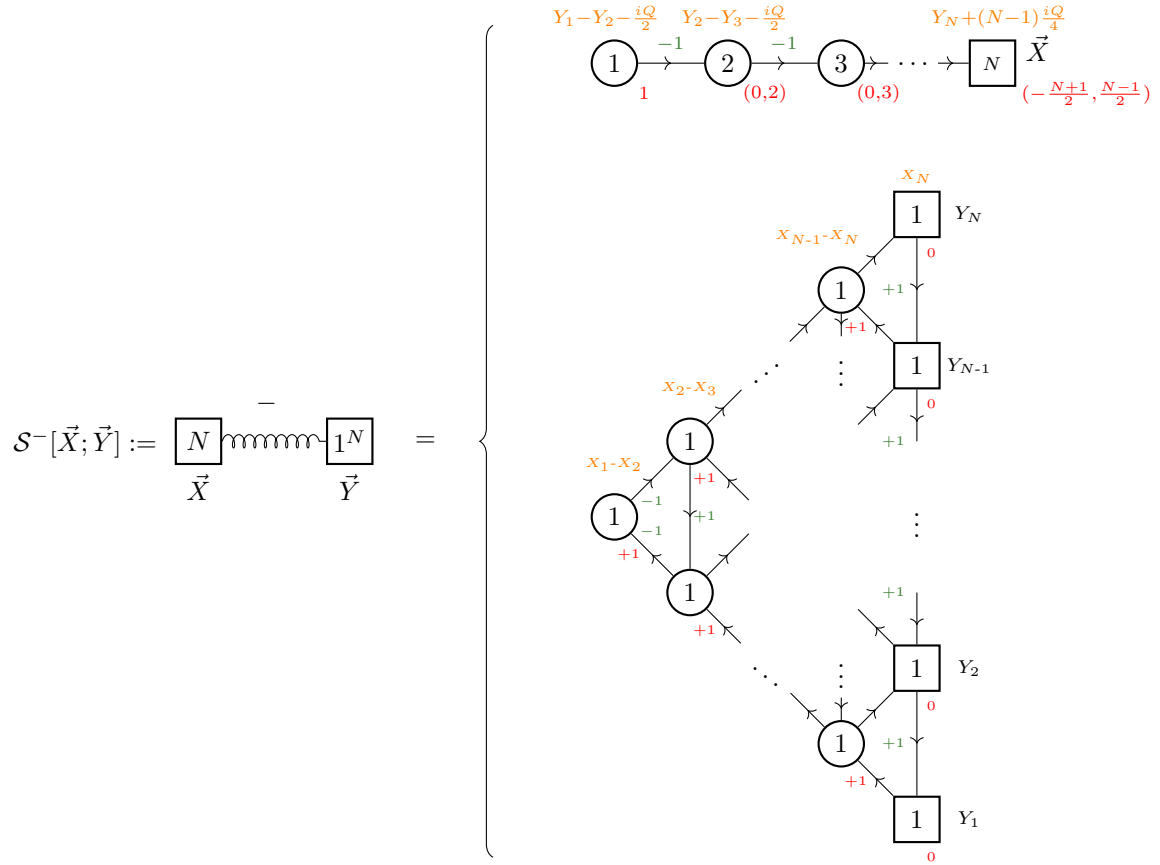


Figure 3.13: The planar and chiral UV completions of the $\mathcal{N} = 2$ $\mathcal{S}^{-1}$ -wall coinciding with $G[\vec{X}; \vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity. Here, the trial R-charge r is set to 0 and the trial R-charges of the chiral fields can be read as in the previous figure.

Planar Fusion to Identity

Our starting point is the $\mathcal{N} = 4$ fusion to identity (3.74) where we consider the UV completion where we gauge the left and right $\mathcal{S}_{\mathcal{N}=4}$ -wall by gauging their manifest $U(N)$ symmetries.

We consider the large mass deformation:

$$\begin{cases} X_j \rightarrow X_j + N\frac{\tau}{2}, \\ Y_j \rightarrow Y_j + N\frac{\tau}{2}, \end{cases} \quad j = 1, \dots, N \quad (3.81)$$

which preserves both the (emergent) $U(N)_{X,Y}$ global symmetries. We also perform the following shifts in gauge fugacities:

$$\begin{cases} u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I - 1), & \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 1), & \text{for } I = N + 1, \dots, 2N - 1 \end{cases} \quad (3.82)$$

with $\tau \rightarrow \infty$, where $u_\alpha^{(I)}$ are the gauge fugacities for the I -th gauge node in the quiver (starting from the left), while $\alpha = 1, \dots, |G^{(I)}|$ with $|G^{(I)}|$ being the rank of the I -th gauge node.

This triggers a flow to the $\mathcal{N} = 2$ planar fusion to identity depicted in Figure 3.14. On the l.h.s. we glue two $\mathcal{S}$ -walls in their planar Abelian UV completion by gauging a diagonal $U(1)^N$ symmetry. On the r.h.s. we find an Identity-wall identifying the $U(N)_X$ and $U(N)_Y$ manifest symmetries.

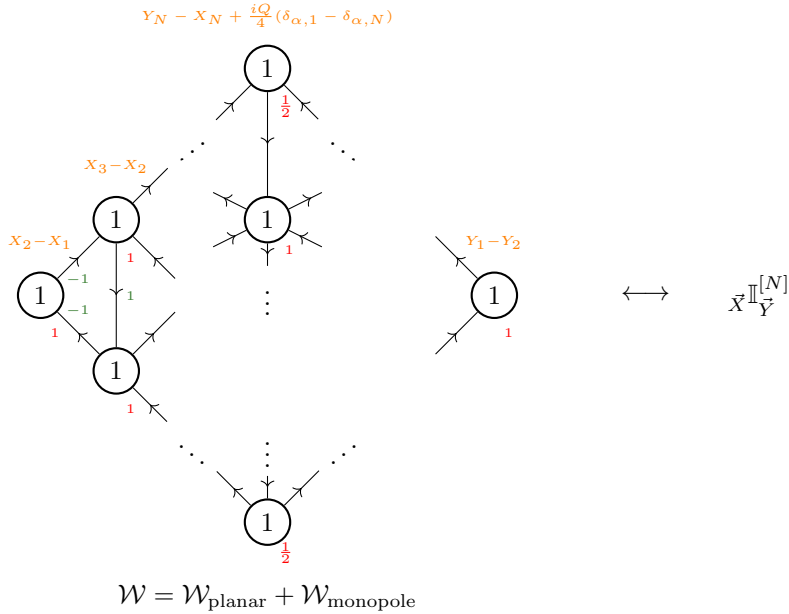


Figure 3.14: The planar fusion to identity observed upon gluing a $\mathcal{S}$ and $\mathcal{S}^-$ wall. Note that the $\delta_{\alpha,j} \frac{iQ}{4}$ indicates the j^{th} node of the middle column has a shifted FI term, counting from below.

We encode the fusion to identity in the following partition function identity:

$$\int d\vec{Z} \Delta^{[1^N]}(\vec{Z}) Z_{\mathcal{S}}(\vec{X}, \vec{Z}) Z_{\mathcal{S}^-}(\vec{Y}, \vec{Z}) = \frac{1}{N! \Delta^{[N]}(\vec{X})} \sum_{\sigma \in S_N} \delta(X_i - Y_{\sigma(i)}) \equiv_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[N]}. \quad (3.83)$$

On the l.h.s. we integrate over the Abelian set of fugacities $\vec{Z}$ entering the second slot of $Z_{\mathcal{S}^\pm}[\cdot; \cdot]$. The **Planar Measure** $\Delta^{[1^N]}(\vec{Z})$ in (3.83) is defined as a limit of the $\mathcal{N} = 4$ gluing measure $\Delta_{(N)}(\vec{Z}, \tau)$ (3.10):

$$\lim_{\tau \rightarrow +\infty} \Delta_{(N)}(\vec{Z}, \tau) \Big|_{Z_i \rightarrow Z_i + \frac{\tau}{2}(2i-N-1)} = \Delta^{[1^N]}(\vec{Z}) \times (\text{divergent phase}). \quad (3.84)$$

Notice the shift on the gauge variables is consistent with eq. (3.82) for $\vec{Z} = \vec{u}^{(I=N)}$. We obtain:

$$\Delta^{[1^N]}(\vec{Z}) = e^{i\pi H^{[1^N]}(\vec{Z})} \prod_{i=1}^{N-1} s_b\left(\frac{iQ}{2} - Z_i + Z_{i+1}\right) \quad (3.85)$$

with

$$H^{[1^N]}(\vec{Z}) := \frac{iQ}{2}(Z_1 - Z_N) - \frac{1}{2}(Z_1^2 + Z_N^2) - \sum_{i=2}^{N-1} Z_i^2 + \sum_{j=1}^{N-1} Z_j Z_{j+1}. \quad (3.86)$$

Note that the contribution from the vector multiplet of $U(N)$ has disappeared, as expected from Higgsing the gauge group down to its maximal torus $U(1)^N$. The limit for the $U(N)$ adjoint gives rise to a string of vertical chiral multiplets connecting adjacent $U(1)^N$ nodes and flowing upward. This string of vertical chiral fields couples to the vertical chiral fields (flowing downward) connecting the $U(1)^N$ nodes of the left and right $\mathcal{S}$ -walls. As a result two strings of vertical bifundamental (flowing in opposite directions) give mass to each other and we are left with a single string of vertical bifundamentals in the middle flowing downward. This is obviously the planar version of the $\mathcal{N} = 4$ gluing $a(A_L + A_R)$ present in (3.74).

The quadratic form $H^{[1^N]}(\vec{Z})$ encodes the CS levels and the BF couplings and FI shifts for the middle column of $U(1)^N$ gauge nodes.

On the r.h.s. of the identity (3.83), we identify the $\mathcal{N} = 2$ **Chiral Identity-wall**:

$$\vec{X} \mathbb{I}_Y^{[N]} \equiv \frac{1}{N! \Delta^{[N]}(\vec{X})} \sum_{\sigma \in S_N} \delta(X_i - Y_{\sigma(i)}) \quad (3.87)$$

identifying the $U(N)_X$ and $U(N)_Y$ global symmetries. The **Chiral Measure** $\Delta^{[N]}(\vec{X})$ is defined by starting again from the definition of the $\mathcal{N} = 4$ gluing measure $\Delta_{(N)}(\vec{X}, \tau)$ in (3.10), then shifting the flavor fugacities $\vec{X}$ as in (3.81) and taking the limit $\tau \rightarrow +\infty$. We obtain (again after suppressing the diverging prefactor):

$$\Delta^{[N]}(\vec{X}) = e^{i\pi H^{[N]}(\vec{X})} \prod_{\alpha=1}^N \prod_{\beta < \alpha} \frac{1}{s_b(\frac{iQ}{2} \pm (X_\alpha - X_\beta))}; \quad (3.88)$$

with

$$\begin{aligned} H^{[N]}(\vec{X}) &:= -(N-1) \left(\sum_{i=1}^N X_i^2 \right) + 2 \sum_{i < j}^N X_i X_j = \\ &= -N \sum_{i=1}^N X_i^2 + \left(\sum_{i=1}^N X_i \right)^2, \end{aligned} \quad (3.89)$$

which corresponds to a CS coupling at level $(N, 0)$.

Observe that in the integrand on the l.h.s. of (3.83) appears the Planar Measure $\Delta^{[1^N]}(\vec{Z})$, while at the denominator of the r.h.s. of the same identity appears the Chiral Measure $\Delta^{[N]}(\vec{X})$. This is consistent with our claim that mirror symmetry sends the $\mathcal{N} = 2$ *chiral* deformation of an $\mathcal{N} = 4$ theory into the $\mathcal{N} = 2$ *planar* deformation of its $\mathcal{N} = 4$ mirror dual and vice versa, and so the same happens for the basic moves.

Although we suppressed the divergent contributions to avoid clutter, we have checked that the divergent pre-factors on l.h.s. and r.h.s. of the partition function identity cancel out.

We schematically depict this planar fusion to identity as:

$$\begin{array}{c} + \\ \boxed{N} \text{-----} \textcircled{1^N} \text{-----} \boxed{N} \\ \vec{X} \qquad \qquad \qquad \vec{Y} \\ \left(\begin{array}{c} \frac{1}{2} \\ 1 \\ \vdots \\ 1 \\ \frac{1}{2} \end{array} \right) \end{array} \longleftrightarrow \vec{X} \mathbb{I}_{\vec{Y}}^{[N]} \quad (3.90)$$

Where the vector in red indicates the CS levels for the middle column of $U(1)^N$ nodes in (3.14). Remember however that there also BF couplings and FI shifts given by the planar measure (3.86).

The Chiral Fusion to Identity

Starting again from the $\mathcal{N} = 4$ fusion to identity (3.74) where we glue the left and right $\mathcal{S}_{\mathcal{N}=4}$ -wall by gauging their manifest $U(N)$ symmetries, we perform a large mass deformations that breaks the $U(N)_{X,Y}$ global symmetries to their Cartan. This is encoded in the following shift in fugacities:

$$\begin{cases} \vec{u}^{(I)} \rightarrow \vec{u}^{(I)} - \frac{\tau}{2} I, & \text{for } I = 1, \dots, 2N - 1 \\ X_j \rightarrow X_j + \frac{\tau}{2}(2j - N), & \text{for } j = 1, \dots, N \\ Y_j \rightarrow Y_j + \frac{\tau}{2}(2j - N), & \text{for } j = 1, \dots, N \end{cases} \quad (3.91)$$

With this limit we flow to the $\mathcal{N} = 2$ operator identity which admits the following Lagrangian UV completion:

$$\begin{array}{c} X_2 - X_1 + \frac{iQ}{2} \qquad \qquad Y_N - X_N \qquad \qquad Y_1 - Y_2 - \frac{iQ}{2} \\ \textcircled{1} \text{-----} \textcircled{2} \text{-----} \dots \text{-----} \textcircled{N} \text{-----} \dots \text{-----} \textcircled{2} \text{-----} \textcircled{1} \\ 1 \qquad \qquad (0,2) \qquad \qquad (-1, N-1) \qquad \qquad (0,2) \qquad \qquad 1 \end{array} \longleftrightarrow \vec{X} \mathbb{I}_{\vec{Y}}^{[1^N]} \quad (3.92)$$

The associated partition function identity reads:

$$\int d\vec{Z} \Delta^{[N]}(\vec{Z}) Z_S(\vec{Z}, \vec{X}) Z_{S^-}(\vec{Z}, \vec{Y}) = \frac{1}{\Delta^{[1^N]}(\vec{X})} \prod_{i=1}^N \delta(X_i - Y_i) \equiv_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]}, \quad (3.93)$$

where we integrate over the first slot of Z_S corresponding to the non-Abelian symmetry. On the l.h.s. we have the chiral integration measure $\Delta^{[N]}(\vec{Z})$ defined in 3.88 which carries

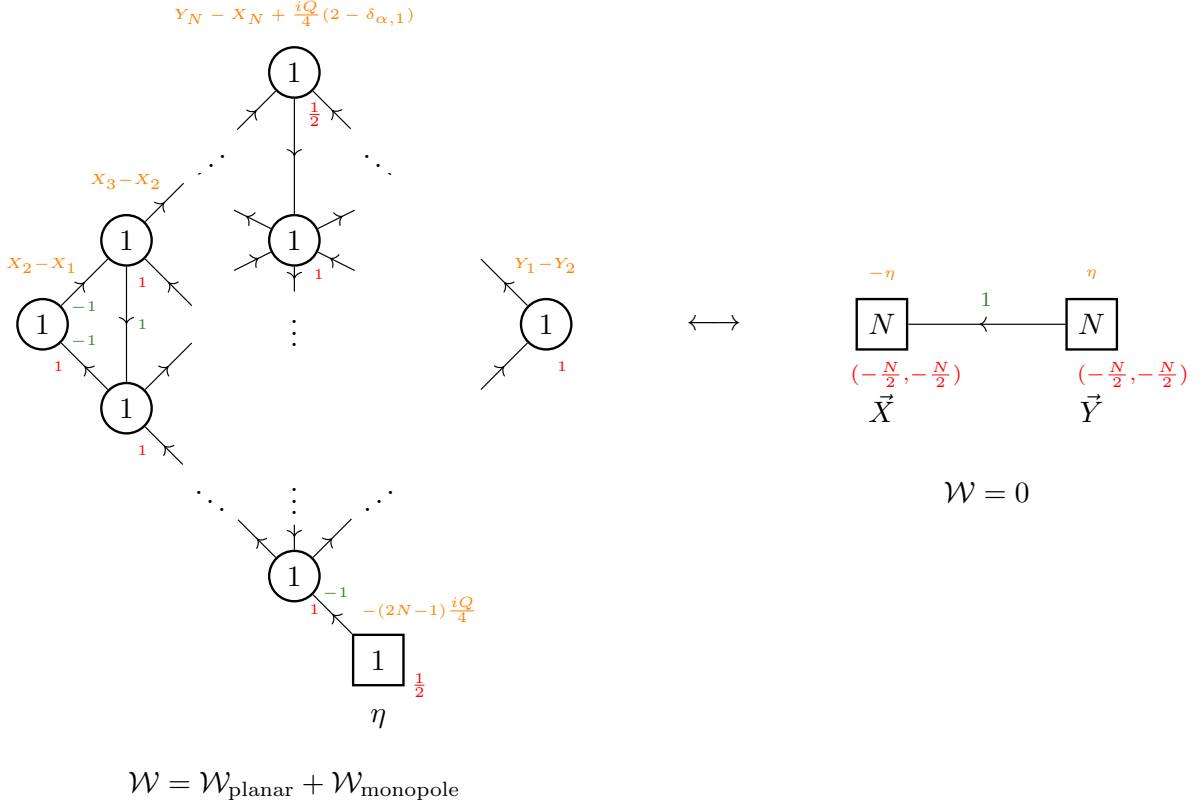


Figure 3.15: A chiral bifundamental on the r.h.s. is $\mathcal{S}$ -dualized into planar flavor sandwiched between two $\mathcal{S}$ -walls. As usual, on the planar side the FIs in orange are those of all the nodes in the corresponding column below. The term $\delta_{\alpha,1}$ indicates that there is a shift valid only for the 1-st node in the middle column, counting from below.

Chiral Bifundamental $\mathcal{S}$ -dualization

Our starting point is the $\mathcal{N} = 4$ basic move in the first line of (3.96). We perform the real mass deformations specified by the following shifts of fugacities, on both sides:

$$\begin{cases} \vec{X} \rightarrow \vec{X} + N\frac{\tau}{2} \\ \vec{Y} \rightarrow \vec{Y} + (N+1)\frac{\tau}{2} \\ u_{\alpha}^{(I)} \rightarrow u_{\alpha}^{(I)} + \frac{\tau}{2}(2\alpha - I - 1), & \text{for } I = 1, \dots, N \\ u_{\alpha}^{(I)} \rightarrow u_{\alpha}^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 1), & \text{for } I = N+1, \dots, 2N-1 \\ \eta \rightarrow \eta - \frac{N}{2}\tau \end{cases} \quad (3.97)$$

where $\vec{X}$ and $\vec{Y}$ are fugacities of the two $U(N)$ global symmetries (which are preserved by the deformations), $u^{(I)}$ are gauge fugacities for the I -th gauge node in the quiver (where we start to count from the left), and η is the fugacity for the flavor. The flow generated by these deformations yields the duality in Figure 3.15. We interpret this duality as a *basic move* relating a *chiral* bifundamental to a *planar* flavor sandwiched between two $\mathcal{S}$ -walls.

bifundamental block with trial R-charge r as:

$$\begin{aligned}
Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) &:= \\
&\begin{array}{c}
\begin{array}{c}
\text{---}\eta\text{---} \\
\text{---}\frac{1}{2}\text{---} \\
\text{---}\frac{1}{2}\text{---} \\
\text{---}\frac{1}{2}\text{---} \\
\text{---}\frac{1}{2}\text{---} \\
\text{---}\frac{1}{2}\text{---}
\end{array}
\begin{array}{c}
X_1 \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{+1} \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_1 \\
X_2 \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{-1} \dots \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_{N-1} \\
X_N \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{-1} \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_N
\end{array}
\end{array}
\end{aligned}$$

(3.107)

In the second line we introduced a short notation wherein we suppressed the specific shift in the background FI of each single node, leaving only the common η dependence manifest. Also the background mixed-CS interaction in green is meant to be present for each bifundamental linking two $U(1)$ symmetries while the background CS levels are given as a column in red. The contribution to the S_b^3 partition function is given below:

$$\begin{aligned}
Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) &:= e^{\frac{\pi i}{2}(X_1^2 + Y_N^2)} e^{\pi i(\sum_{j=2}^N X_j^2 + \sum_{j=1}^{N-1} Y_j^2)} e^{2\pi i \eta \sum_j (Y_j - X_j)} e^{-\pi i \sum_j Y_j (X_j + X_{j+1})} \\
&\quad e^{2\pi i \sum_j X_j (r-1)(N+2-2j)\frac{iQ}{2}} e^{-2\pi i \sum_j Y_j (r-1)(N-2j)\frac{iQ}{2}} e^{2\pi i (Y_N \frac{iQ}{4} - X_1 \frac{iQ}{4})} \\
&\quad \prod_{j=1}^N s_b\left(\frac{iQ}{2} - r\frac{iQ}{2} + X_j - Y_j\right) \prod_{j=1}^{N-1} s_b\left(\frac{iQ}{2} - r\frac{iQ}{2} + Y_j - X_{j+1}\right).
\end{aligned}$$

(3.108)

On the r.h.s. of Figure 3.16 we recognize the left-pointing chiral bifundamental block¹⁶ $Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; 1)$.

The S_b^3 partition function identity corresponding to the duality in Figure 3.16 is:

$$\begin{aligned}
Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; 1) &= \int d\vec{Z} \Delta^{[N]}(\vec{Z}) e^{-\frac{i\pi}{2}[N\eta^2 + iQ(N\eta - \sum_{j=1}^N Z_j) + \sum_{j=1}^N (Z_j^2 - 2\eta Z_j)]} \\
&\quad Z_S(\vec{Z}, \vec{X}) \prod_{j=1}^N s_b\left(\frac{iQ}{2} + Z_j - \eta\right) Z_{S^-}(\vec{Z}, \vec{Y}).
\end{aligned}$$

(3.109)

On the r.h.s. the two $\mathcal{S}$ -walls are glued as in the case of the chiral fusion to identity in (3.92). There is only a slight modification for the CS level of the middle $U(N)$ node to

¹⁶Recall that in this Section we assign trial R-charge 0 to the bifundamentals of the $\mathcal{S}$ -wall theory, which in turn fixed the trial R-charge of the bifundamental on the r.h.s. of Figure 3.16 to 1.

which the flavor is attached, which has now CS level $(-1/2, (2N-1)/2)$ and a shifted FI.

We can write the move in Figure 3.16 in compact form as:

$$\begin{array}{ccc}
\begin{array}{c}
\eta \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{c} \textcolor{red}{\frac{N}{2}} \\ \downarrow \textcolor{green}{-1} \end{array} \\
\begin{array}{c} \textcolor{red}{(-\frac{1}{2}, \frac{2N-1}{2})} \\ \circlearrowleft \end{array} \\
\begin{array}{c} \textcolor{red}{\frac{N}{2}} \\ \downarrow \end{array} \\
\begin{array}{c} \textcolor{red}{(-\frac{1}{2}, \frac{2N-1}{2})} \\ \circlearrowleft \end{array}
\end{array}
\begin{array}{c}
+ \\
-
\end{array}
\begin{array}{c}
\vec{X} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline N \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \vec{Y}
\end{array}
\longleftrightarrow
\begin{array}{c}
\begin{array}{c} \textcolor{brown}{-\eta} \\ \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \begin{array}{c} \textcolor{green}{+1} \\ \downarrow \end{array} \end{array} \\
\begin{array}{c} \textcolor{red}{(-\frac{1}{2}, \frac{2N-1}{2})} \\ \circlearrowleft \end{array} \\
\begin{array}{c} \textcolor{brown}{\eta} \\ \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \begin{array}{c} \textcolor{green}{+1} \\ \downarrow \end{array} \end{array} \\
\begin{array}{c} \textcolor{red}{(-\frac{1}{2}, \frac{2N-1}{2})} \\ \circlearrowleft \end{array}
\end{array}
\begin{array}{c}
\vec{X} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline N \\ \hline \end{array} \text{---} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \vec{Y}
\end{array}
\end{array}
\quad (3.110)$$

Similarly, we can define a right-pointing planar bifundamental block, whose $\mathcal{S}_b^3$ partition function is given by:

$$Z_{\text{bif, right}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) := Z_{\text{bif, left}}^{[1^N]}(\vec{Y}, \vec{X}, -\eta; r) \quad (3.111)$$

and whose basic move in short notation is:

$$\begin{array}{ccc}
\begin{array}{c} \eta \boxed{1} \\ \uparrow -1 \\ \boxed{N} \\ (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} & \begin{array}{c} + \\ - \end{array} & \begin{array}{c} \vec{X} \boxed{1^N} \\ \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \boxed{1^N} \vec{Y} \end{array} \\
& & \longleftrightarrow \\
& & \begin{array}{c} \begin{array}{c} -\eta \\ \nearrow +1 \end{array} \begin{array}{c} \boxed{1^N} \\ (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \begin{array}{c} \boxed{1^N} \vec{Y} \\ (\begin{smallmatrix} -1 \\ \vdots \\ -1 \\ -\frac{1}{2} \end{smallmatrix}) \end{array} \end{array}
\end{array} \quad (3.112)$$

Notice that in short form the right-pointing planar bifundamental has opposite slope. The corresponding partition function identity is:

$$Z_{\text{bif, right}}^{[1N]}(\vec{X}, \vec{Y}, \eta; 1) = \int d\vec{Z} \Delta^{[N]}(\vec{Z}) e^{-\frac{i\pi}{2} [N\eta^2 + iQ(-N\eta + \sum_{j=1}^N Z_j) + \sum_{j=1}^N (Z_j^2 - 2\eta Z_j)]} Z_S(\vec{Z}, \vec{X}) \prod_{j=1}^N s_b \left(\frac{iQ}{2} - Z_j + \eta \right) Z_{S^-}(\vec{Z}, \vec{Y}). \quad (3.113)$$

Chiral fundamental $\mathcal{S}$ -dualization

We now discuss the *inverse* basic moves. A possible way to obtain them is by “multiplying” an $\mathcal{S}$ and $\mathcal{S}^{-1}$ -wall respectively on the right and left sides of the basic moves in Figures 3.15 and 3.16 and using the appropriate *fusion to identity* properties.

However, we will follow a different route which is to start back from the inverse $\mathcal{N} = 4$ basic move in the second line of (3.96) and perform a suitable mass deformation.

Let us start from the basic move for a chiral fundamental. Starting from (3.96) we take the real mass deformation

$$\begin{cases} X_j, Y_j \rightarrow X_j, Y_j \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I), \quad \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 2), \quad \text{for } I = N + 1, \dots, 2N \\ \eta \rightarrow \eta + \frac{\tau}{2} \end{cases} \quad (3.114)$$

$$Z_{\text{flavor, antif}}^{[N]}(\vec{X}, \eta) = e^{\frac{i\pi}{2}[N\eta^2 + \sum_{j=1}^N (X_j^2 - 2\eta X_j)]} \prod_{j=1}^N s_b(-\eta + X_j)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[N]} \quad (3.117)$$

We emphasize the CS level for the $U(N)_{\vec{X}}$ symmetry in (3.115) refers to the contribution coming from the single chiral that has been integrated out. The overall background CS level for this symmetry has to take into account also the shift $(-N, 0)$ encoded in the identity wall, as previously mentioned so that the total CS level for the chiral block is $(-N - \frac{1}{2}, -\frac{1}{2})$.

In the duality in Figure 3.17 we recognize a chiral flavor block on the l.h.s. and the corresponding S_b^3 partition function identity is:

$$Z_{\text{flavor, fund}}^{[N]}(\vec{X}, \eta) = \int d\vec{Z} d\vec{W} Z_{\mathcal{S}}(\vec{X}, \vec{Z}) Z_{\text{bif, right}}^{[1^N]}(\vec{Z}, \vec{W}, -\eta; 0) Z_{\mathcal{S}^-}(\vec{Y}, \vec{W}). \quad (3.118)$$

Where on the r.h.s. we recognize a planar bifundamental sandwiched between two $\mathcal{S}$ -walls. In short notation the duality in Figure 3.17 can be depicted as:

$$\quad (3.119)$$

There is also a variant of this move, corresponding to the $\mathcal{S}$ -dualization of a chiral anti-fundamental.

$$\quad (3.120)$$

Which corresponds to the S_b^3 partition function identity:

$$Z_{\text{flavor, antif}}^{[N]}(\vec{X}, \eta) = \int d\vec{Z} d\vec{W} Z_{\mathcal{S}}(\vec{X}, \vec{Z}) Z_{\text{bif, left}}^{[1^N]}(\vec{Z}, \vec{W}, -\eta; 0) Z_{\mathcal{S}^-}(\vec{Y}, \vec{W}). \quad (3.121)$$

Planar fundamental $\mathcal{S}$ -dualization

Starting again from the $\mathcal{N} = 4$ basic move in the second line of (3.96), we perform the mass deformation defined by the following deformation breaking the $U(N)_{X,Y}$ symmetries:

$$\begin{cases} X_j \rightarrow X_j - (2N - 2j + 1)\frac{\tau}{2} \\ Y_j \rightarrow Y_j - (2N - 2j + 1)\frac{\tau}{2} \\ \vec{u}^{(I)} \rightarrow \vec{u}^{(I)} + (2N - I + 1)\frac{\tau}{2}, \quad \text{for } I = 1, \dots, 2N \\ \eta \rightarrow \eta \end{cases} \quad (3.122)$$

This limit yields the duality in Figure 3.18. On the l.h.s. we identify the anti-fundamental

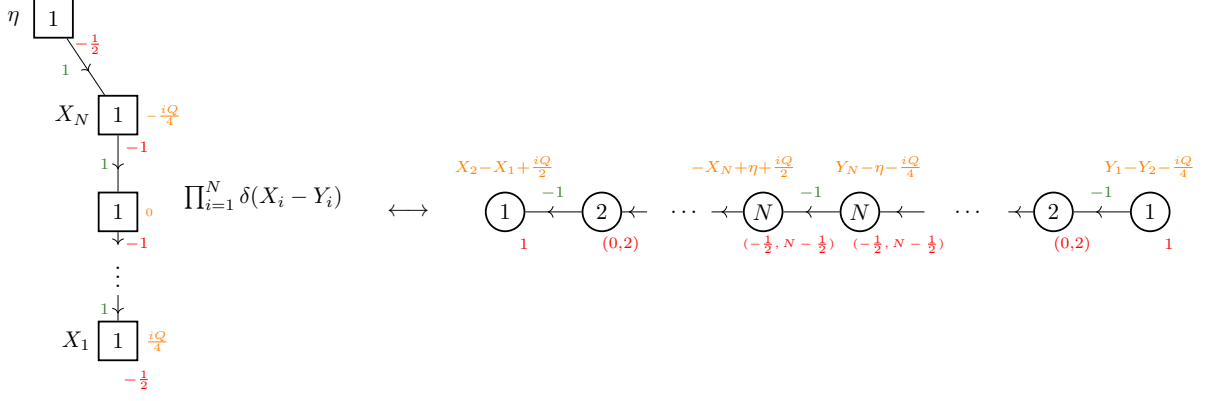


Figure 3.18: The planar mirror dual of the chiral anti-fundamental flavor block is shown here.

Planar flavor block for which we introduce the compact notation:

$$Z_{\text{flavor, antif}}^{[1^N]}(\vec{X}, \eta) := \begin{array}{c} \eta \boxed{1} \\ \swarrow \text{1} \\ \vec{X} \boxed{1^N} \\ \swarrow \vec{\mathbb{I}}_{\vec{Y}}^{[1^N]} \end{array} := \begin{array}{c} \eta \boxed{1} \\ \swarrow \text{1} \\ X_N \boxed{1} \xrightarrow{-iQ/4} \boxed{1} \xrightarrow{0} \boxed{1} \xrightarrow{-1} \dots \xrightarrow{1} X_1 \boxed{1} \xrightarrow{-iQ/4} \boxed{1} \\ \prod_{i=1}^N \delta(X_i - Y_i) \end{array} \quad (3.123)$$

Notice that all the arrows point downward. Similarly we define the planar fundamental block as:

$$Z_{\text{flavor, fund}}^{[1^N]}(\vec{X}; \eta) := \begin{array}{c} \vec{X} \boxed{1^N} \\ \swarrow \text{1} \\ \eta \boxed{1} \end{array} := \begin{array}{c} X_N \boxed{1} \xrightarrow{iQ/4} \boxed{1} \xrightarrow{0} \boxed{1} \xrightarrow{-1} \dots \xrightarrow{1} X_1 \boxed{1} \xrightarrow{-iQ/4} \boxed{1} \\ \prod_{i=1}^N \delta(X_i - Y_i) \\ \swarrow \text{1} \\ \eta \boxed{1} \end{array} \quad (3.124)$$

Notice that for the fundamental planar block the chiral is attached to the bottom of the string of chirals belonging to the Identity-wall. The contribution to the S_b^3 partition function is given below:

$$Z_{\text{flavor, fund}}^{[1^N]}(\vec{X}; \eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_1 + X_1^2)} s_b(\eta - X_1)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]} \quad (3.125)$$

$$Z_{\text{flavor, antif}}^{[1^N]}(\vec{X}; \eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_N + X_N^2)} s_b(X_N - \eta)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]} \quad (3.126)$$

The duality in Figure 3.18 corresponds to the S_b^3 partition function identity:

$$Z_{\text{flavor, antif}}^{[1^N]}(\vec{X}; \eta) = \int d\vec{Z} d\vec{W} Z_S[\vec{Z}; \vec{X}] Z_{\text{bif, left}}^{[N]}(\vec{Z}; \vec{W}; -\eta; 0) Z_{S^-}[\vec{W}; \vec{Y}] \quad (3.127)$$

We represent the previous move in compact form as:

$$\begin{array}{c} \eta \boxed{1} \\ \swarrow \text{red } -\frac{1}{2} \\ \text{green } 1 \\ \searrow \\ \vec{X} \boxed{1^N} \end{array} \quad \vec{X} \mathbb{I}_{\vec{Y}}^{[1^N]} \quad \longleftrightarrow \quad \vec{X} \boxed{1^N} \text{ wavy } \begin{array}{c} + \\ \text{orange } \eta \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \begin{array}{c} \text{green } -1 \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \begin{array}{c} - \\ \text{orange } -\eta \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \text{ wavy } \boxed{1^N} \vec{Y} \quad (3.128)$$

There is also a variant of this move that is the $\mathcal{S}$ -dual of a planar fundamental block.

$$\begin{array}{c} \vec{X} \boxed{1^N} \\ \swarrow \text{green } 1 \\ \text{red } -\frac{1}{2} \\ \searrow \\ \eta \boxed{1} \end{array} \quad \vec{X} \mathbb{I}_{\vec{Y}}^{[1^N]} \quad \longleftrightarrow \quad \vec{X} \boxed{1^N} \text{ wavy } \begin{array}{c} + \\ \text{orange } \eta \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \begin{array}{c} \text{green } -1 \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \begin{array}{c} - \\ \text{orange } -\eta \\ \text{red } (-\frac{1}{2}, \frac{2N-1}{2}) \end{array} \text{ wavy } \boxed{1^N} \vec{Y} \quad (3.129)$$

Which corresponds to the following identity between partition functions:

$$Z_{\text{flavor, fund}}^{[1^N]}(\vec{X}; \eta) = \int d\vec{Z} d\vec{W} Z_S[\vec{Z}; \vec{X}] Z_{\text{bif, right}}^{[N]}(\vec{Z}; \vec{W}; -\eta; 0) Z_{S^-}[\vec{W}; \vec{Y}]. \quad (3.130)$$

3.4 The Chiral-Planar dualization Algorithm at work

In this section we use the algorithm introduced in the previous section to produce new examples of the chiral-planar $\mathcal{N} = 2$ mirror dualities.

Along the lines of [46], we define the steps of the algorithmic dualization of a non-Abelian chiral $\mathcal{N} = 2$ quiver to be the following:

- **Step 1:** The quiver is chopped into its basic QFT blocks that can be either bifundamentals or (anti-)fundamental chiral fields. This step is performed by freezing the gauge interactions that will be reintroduced later.
- **Step 2:** Each QFT block is dualized via the basic duality moves described in Section 3.3.3. These identities, or *basic moves* are summarized for reference in Table 3.2.
- **Step 3:** The dualized blocks are glued back together by turning back on the gauge interactions frozen at the first step. The pairs $\mathcal{S}^{-1}\mathcal{S}$ fuse to Identity-walls as described in Section 3.3.2.

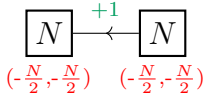
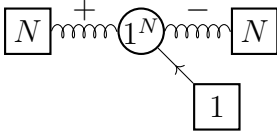
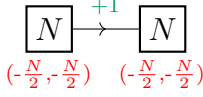
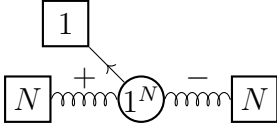
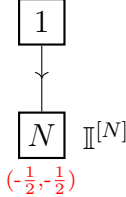
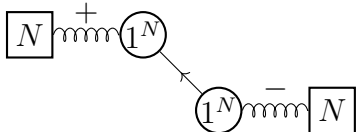
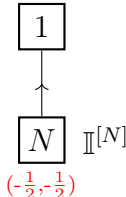
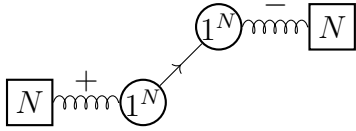
QFT block	$\mathcal{S}$ -dualization	S_b^3 p.f. identity
		Fig. 3.15, eq. (3.100)
		Eq. (3.105)
		Fig. 3.17, eq.(3.118)
		Eq. (3.121)

Table 3.2: Summary of the basic moves that are relevant to run the algorithm. In the first column are given the basic QFT blocks, in the second column their $\mathcal{S}$ -duals. In the last column we give the reference to the S_b^3 partition function identity and to the extended quiver form. We avoided to give all the CS-interactions and FI terms, we only provided those that are dynamical either in the starting chiral theory or in the resulting planar theory. Notice that in order to keep track of all the details more information are needed, see Section 3.3 for a complete discussion.

Each of these steps will be illustrated in more detail in the following sections with explicit examples.

We will use the algorithm to dualize chiral $\mathcal{N} = 2$ quivers into Abelian-planar quivers. Although the same algorithm can be applied in reverse — starting from a planar Abelian quiver to obtain its chiral dual — we do not discuss this case in the present work.

The current version of the algorithm has some limitations that we list below.

- We can dualize chiral quivers with fixed CS levels. A $U(N)$ node with F chirals will have CS level $(k, k + lN)$ with $k = -\frac{F}{2} + N$ and $l = -1$. Moreover, each pair of gauge nodes connected by a bifundamental chiral will be also coupled via a mixed-CS interaction.
- We can dualize only chiral quivers with constant ranks $U(N)$. To algorithmically

dualize quivers with non-constant ranks we would need to introduce asymmetric blocks and the chiral-planar version of the Hanany-Witten duality moves derived in [46].

- We also limit ourselves to cases where each gauge node sees $[n_f, n_a]$ chirals, where $F = n_f + n_a$ with

$$[n_f, n_a] = \begin{cases} [N + F_1, N + F_2] \\ [2N + F_1, F_2] \\ [F_1, 2N + F_2] \end{cases} \quad F_1, F_2 \geq 0 \quad (3.131)$$

The algorithm can be further extended to overcome the current limitations, but we leave this for future work.

Despite these limitations, we can extend the algorithm's reach by applying the strategies outlined in Sections 3.1 and 3.2, as we will demonstrate in Section 3.4.1.

Additionally, starting from the dualities generated by the algorithm, one can overcome the limitations by performing suitable massive deformations; we leave a detailed study of this direction for future work.

It is important to emphasize that, even within its current scope, the algorithm already produces a rich variety of examples that capture essential features of the new type of mirror duality proposed in this chapter. These will be discussed in detail in the following sections.

3.4.1 Chern–Simons SQCD with fundamental and anti-fundamental Matter

As a first example of the application of the algorithm, we consider the case of SQCD with $[n_f, n_a]$ flavors, where $F = n_f + n_a$ satisfying the constraint in eq. (3.131). The second and third cases are analogous, therefore we explicitly present only the first and second case in eq. (3.131).

$U(N)$ CS-SQCD with $[F_1 + N, F_2 + N]$ Chiral Multiplets

We start from the first case in (3.131), therefore we consider $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + N, F_2 + N]$ chiral multiplets, depicted in short as:

$$(3.132)$$

To apply our algorithm we begin by chopping the theory into chiral QFT blocks using the basic ingredients defined in Section 3.3. We do so by considering F_1 chiral fundamental blocks, F_2 chiral anti-fundamental blocks and two chiral bifundamental blocks, in complete analogy with the $\mathcal{N} = 4$ SQCD case done in Figure 3.11, where the two bifundamentals account effectively for $2N$ flavors. The QFT block decomposition is schematically

depicted below.

$$\begin{array}{c}
\begin{array}{c} \boxed{F_1+N} \\ \swarrow \\ \textcircled{N} \\ \searrow \\ \boxed{F_2+N} \end{array} \\
(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)
\end{array}
= \int \begin{array}{c} \boxed{N} \leftarrow \boxed{N} \\ (-\frac{N}{2}, -\frac{N}{2}) \end{array} \Delta^{[N]} \begin{array}{c} \boxed{1} \\ \uparrow \\ \boxed{N} \\ (-\frac{1}{2}, -\frac{1}{2}) \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \dots \begin{array}{c} \boxed{1} \\ \uparrow \\ \boxed{N} \\ (-\frac{1}{2}, -\frac{1}{2}) \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{N} \\ (-\frac{1}{2}, -\frac{1}{2}) \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \dots \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{N} \\ (-\frac{1}{2}, -\frac{1}{2}) \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \begin{array}{c} \boxed{N} \leftarrow \boxed{N} \\ (-\frac{N}{2}, -\frac{N}{2}) \end{array} \quad (3.133)$$

In this schematic picture the act of multiplication between two consecutive blocks is understood. The measure of integration between two blocks is given by a $U(N)$ vector field $\Delta^{[N]}$ which is precisely the measure defined in (3.88). To avoid cluttering we suppressed the labeling of global symmetry parameters.

First, we check that gluing back the QFT blocks we recover the $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD. The gluing procedure consists in identifying and gauging two $U(N)$ global symmetries, one coming from each of the two blocks glued together. This means that there must be a total of $(\# \text{QFT blocks} - 1)$ integrations to account for the gluing. Taking into account the presence of $F_1 + F_2$ Identity-walls coming from flavor blocks, each freezing an integration, we are left with a single dynamical, i.e. non-frozen, $U(N)$ gauge symmetry.

We recall that each QFT block, Identity-wall and measure of integration comes together with CS interactions. In the schematic figure we give explicitly only those coming from the QFT-blocks and leave the remaining implicit. The resulting CS level upon gauging and implementing the Identity-walls can be computed as follows. Each factor of $\Delta^{[N]}$ contributes as $(N, 0)$ to the CS level. Each flavor block consists of a (anti-)fundamental chiral field, with a CS level $(-\frac{1}{2}, -\frac{1}{2})$, and an identity, contributing as $(-N, 0)$, for a total contribution of $(-N - \frac{1}{2}, -\frac{1}{2})$. Lastly, a bifundamental chiral contributes with a CS level of $(-\frac{N}{2}, -\frac{N}{2})$. Therefore the total CS level is:

$$\underbrace{2(-\frac{N}{2}, -\frac{N}{2})}_{\text{bifundamental blocks}} + \underbrace{(F_1 + F_2)(-N - \frac{1}{2}, -\frac{1}{2})}_{\text{flavor blocks}} + \underbrace{(F_1 + F_2 + 1)(N, 0)}_{\text{chiral measure}} = (-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N) \quad (3.134)$$

which matches the CS level of the SQCD theory we are considering.

Now we dualize each QFT block in the decomposition in (3.133) by exploiting the basic moves derived in Section 3.3 and summarized in Table 3.2, we obtain:

$$\begin{array}{c}
\int \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \Delta^{[N]} \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \Delta^{[N]} \dots \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \Delta^{[N]} \times \\
\times \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \Delta^{[N]} \dots \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \Delta^{[N]} \begin{array}{c} \boxed{N} \text{---} \textcircled{1^N} \text{---} \boxed{N} \\ \downarrow \\ \boxed{1} \end{array} \quad (3.135)
\end{array}$$

Now we can use the chiral fusion to identity 3.92 to remove pairs of $\mathcal{S}$ -walls glued through their $U(N)$ symmetry. Schematically this operation can be represented as:

$$\dots \textcircled{1^N} \text{---} \textcircled{N} \text{---} \textcircled{1^N} \dots \rightarrow \dots \textcircled{1^N} \dots \quad (3.136)$$

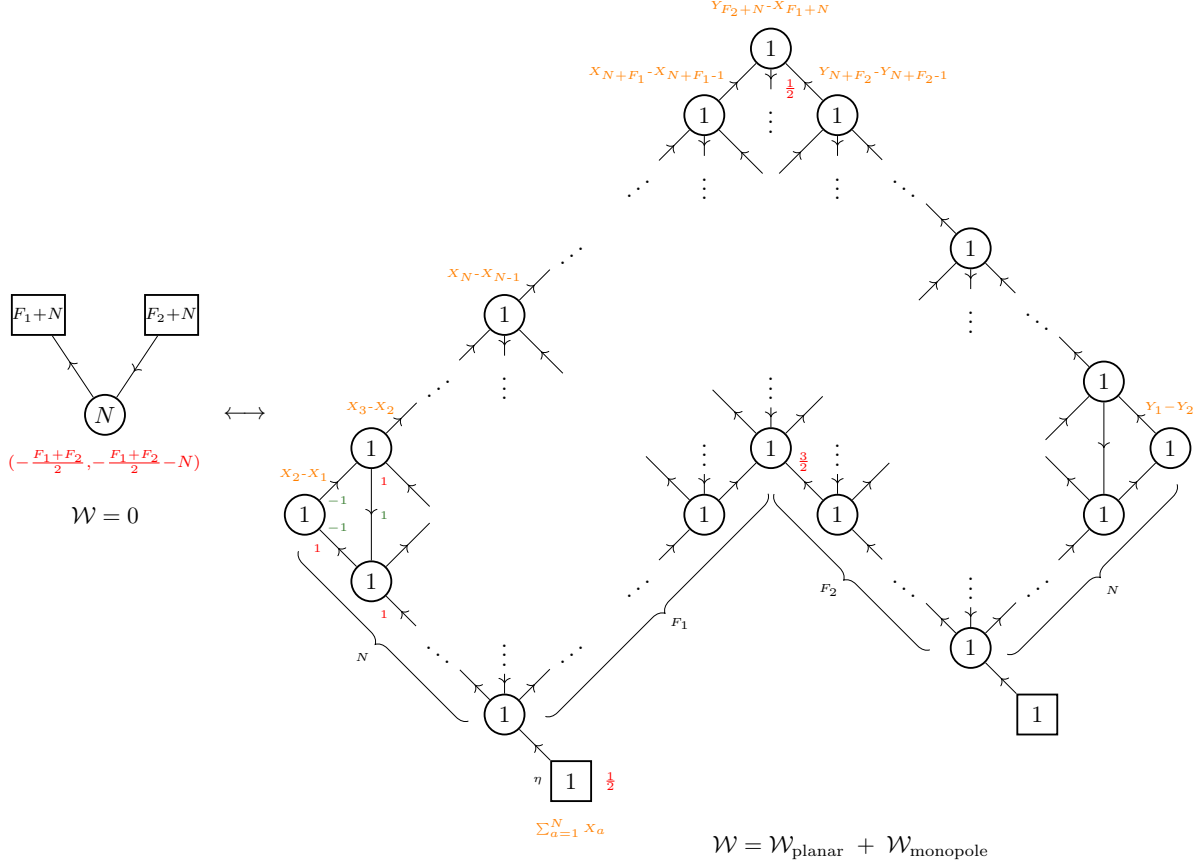
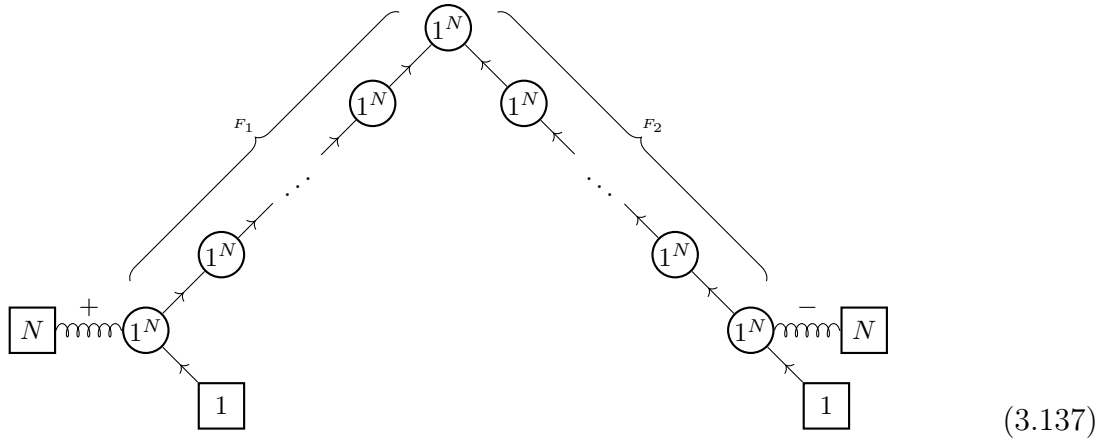


Figure 3.19: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + N, F_2 + N]$ chiral multiplets. The superpotential terms and (mixed) Chern–Simons interactions are as described in Section 3.1.

This allows us to remove all but the first and last $\mathcal{S}$ -walls. The resulting generalized quiver in short notation is:



The generalized quiver in (3.137) can finally be written in the explicit Lagrangian form by using the planar UV completion of the $\mathcal{S}$ -walls and the definition of the planar bifundamental and fundamental blocks. The result is depicted in Figure 3.19.

Let us comment some of the details of the duality in Figure 3.19.

The global symmetry of the SQCD theory is, in general, $S[U(F_1 + N) \times U(F_2 + N)] \times U(1)$, where the first factor is the flavor symmetry while the second one is the topological

symmetry¹⁸. In the mirror dual the $U(1)$ symmetry is the only flavor symmetry, taking into account gauge transformations and superpotential constraints.

The $\mathcal{W}_{\text{monopole}}$ superpotential contains a term that is linear in the monopole with $-1/+1$ magnetic flux under two nodes connected by a vertical line, which reduce to a single $U(1)$ the topological symmetry of each column. So the UV topological symmetry is $U(1)^{F_1+F_2+2N-1}$ which enhances in the IR to $S[U(F_1+N) \times U(F_2+N)]$, so that the global symmetry of the mirror quiver matches that of the SQCD.

In the SQCD the chiral ring is generated by mesonic operators constructed from the fundamentals Q_i and antifundamentals $\tilde{Q}_j$ that form a bifundamental representation of $SU(F_1+N) \times SU(F_2+N)$. These are mapped to monopole operators that are gauge invariant in the planar theory. For example the meson $Q_{F_1+N}\tilde{Q}_{F_2+N}$ is mapped to the monopole with flux $+1$ under the topmost gauge node of the Abelian dual. The other mesons are mapped to monopoles with two strings of $+1$ fluxes starting at the topmost node and propagating along the two upper diagonals of the planar quiver. There are $(F_1+N)(F_2+N)$ such monopoles which reproduce the bifundamental representation of the enhanced $SU(F_1+N) \times SU(F_2+N)$ global symmetry.

Notice that in this case, the planar dual never exhibits a mesonic chiral ring generator. This predicts that, for all values of F_1 and F_2 , there are no dressed gauge-invariant monopoles in the SQCD theory that would parametrize a non-compact moduli space.

$U(N)$ CS-SQCD with $[F_1+2N, F_2]$ Chiral Multiplets

We now consider the second case in (3.131) (which is analogous to the third one), the $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1+2N, F_2]$ chiral multiplets:

$$\begin{array}{c}
 \vec{X} \boxed{F_1+2N} \quad \boxed{F_2} \vec{Y} \\
 \swarrow \quad \searrow \\
 \textcircled{N} \\
 \eta \quad \left(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N \right)
 \end{array} \tag{3.138}$$

We decompose this theory into fundamental blocks according to the rules described in the previous section:

$$\begin{array}{c}
 \boxed{F_1+2N} \quad \boxed{F_2} \\
 \swarrow \quad \searrow \\
 \textcircled{N} \\
 \left(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N \right)
 \end{array}
 = \int \begin{array}{c} \boxed{N} \leftarrow \boxed{N} \end{array} \Delta^{[N]} \begin{array}{c} \boxed{1} \\ \uparrow \\ \boxed{N} \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \dots \begin{array}{c} \boxed{1} \\ \uparrow \\ \boxed{N} \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{N} \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \dots \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{N} \end{array} \mathbb{I}^{[N]} \Delta^{[N]} \begin{array}{c} \boxed{N} \rightarrow \boxed{N} \end{array}
 \tag{3.139}$$

where the diagonal gauging of adjacent $U(N)$ fugacities is understood. The difference w.r.t. (3.133) is that the last bifundamental is right-pointing, so that it contributes as N extra fundamentals instead of anti-fundamentals. We dualize each block using the basic

¹⁸For special cases, namely $F_1 = F_2 = 0$ the topological symmetry enhances to $SU(2)$. This can also be understood by considering the Aharony-like dual for the theory [49], which is an Abelian gauge theory, and then taking the mirror dual to that theory [169]. We will comment on this feature later in Section 3.4.2 and study other chiral theories exhibiting enhancement of the topological symmetry.

[illegible]

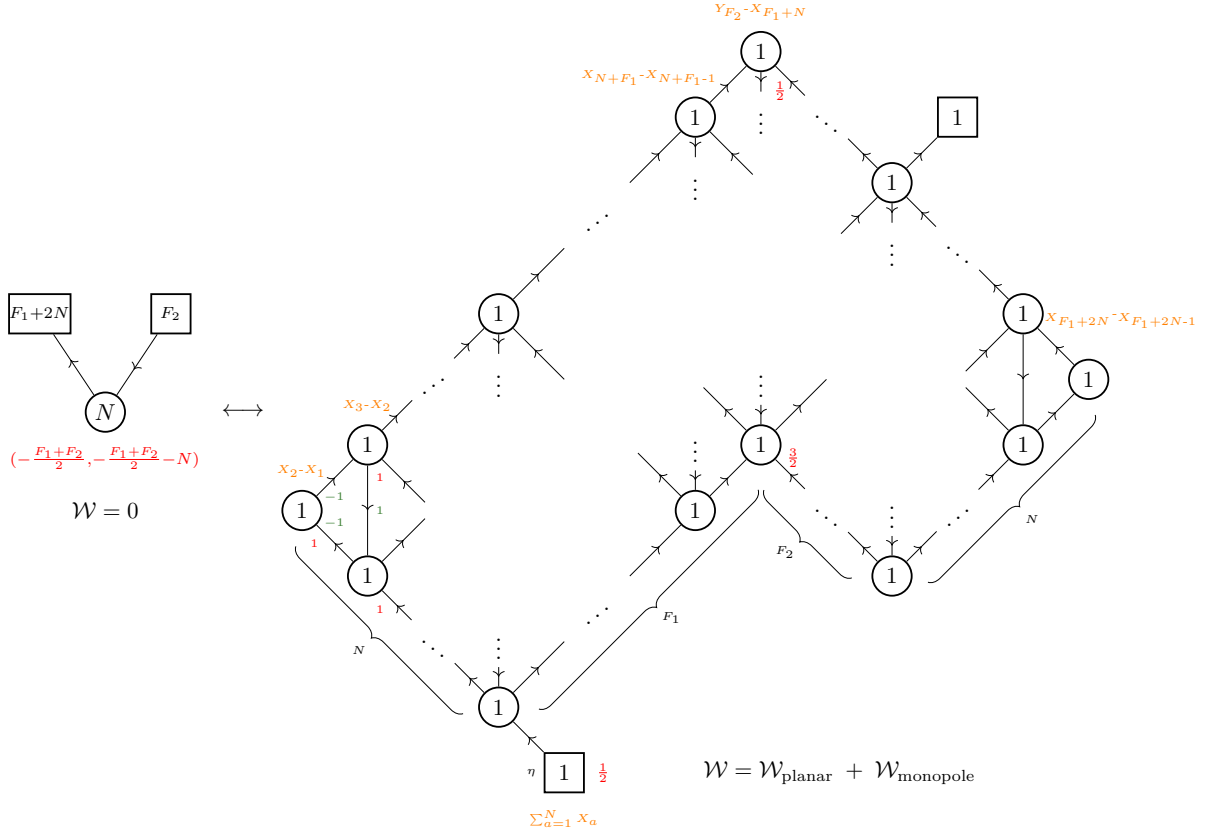


Figure 3.20: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + 2N, F_2]$ chiral multiplets. The superpotential terms and (mixed) Chern–Simons interactions are as described in Section 3.1.

$$(3.143)$$

$$(3.144)$$

Notice that for high enough number of fundamentals and antifundamentals an SQCD theory can admit multiple mirror duals among the ones presented above. As an example, $U(N)$ SQCD with $n_f \geq 2N$ fundamentals and $n_a \geq N$ antifundamentals admits two inequivalent Abelian planar duals, schematically (3.142) and (3.143). It would be interesting to understand whether these two duals, in the range of n_f and n_a mentioned above, can be connected by performing local Aharony-like dualities, but we leave this to future work.

It is possible to derive planar Abelian duals for SQCD theories with $SU(N)$ gauge group or with $U(N)$ gauge group and CS levels $(k, k + lN)$ for any l by applying Witten's $SL(2, \mathbb{Z})$ action [144] on both sides of the dualities described above. This was described in detail for the case of SQCD with only fundamental chirals in Section 3.1.3, the generalization to SQCD with both fundamentals and antifundamentals is analogous and we do not carry it out explicitly.

3.4.2 Topological symmetry enhancement in chiral quivers

A “local” balancing condition

We now move to an example of a quiver theory to show an interesting pattern of topological symmetry enhancement.

We consider the following quiver:

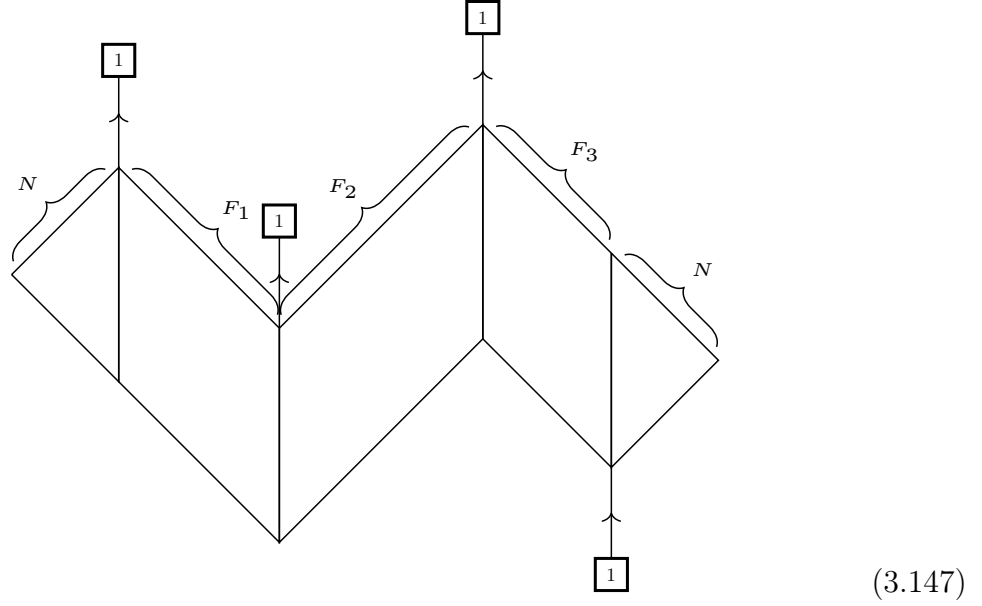
$$\begin{array}{c}
 \begin{array}{c} \boxed{F_1} \\ \downarrow \\ \boxed{N} \end{array} \quad \begin{array}{c} \boxed{F_2} \\ \downarrow \\ \boxed{N} \end{array} \quad \begin{array}{c} \boxed{F_3} \\ \downarrow \\ \boxed{N} \end{array} \\
 \begin{array}{c} \boxed{N} \end{array} \rightarrow \begin{array}{c} \boxed{N} \end{array} \rightarrow \begin{array}{c} \boxed{N} \end{array} \rightarrow \begin{array}{c} \boxed{N} \end{array} \leftarrow \begin{array}{c} \boxed{N} \end{array} \\
 \textcolor{red}{(-\frac{F_1}{2}, -\frac{F_1}{2} - N)} \quad \textcolor{red}{(-\frac{F_2}{2}, -\frac{F_2}{2} - N)} \quad \textcolor{red}{(-\frac{F_3}{2}, -\frac{F_3}{2} - N)}
 \end{array}
 \tag{3.145}$$

Notice that, as emphasized before, the CS levels for each gauge group are not arbitrary and are fixed by the amount of matter fields that each node sees following (3.70). The quiver presented here serves as a specific example, but it is straightforward to modify many of its properties. For instance, the directions of the bifundamental arrows can be flipped, and the flavor content can be generalized by considering different numbers of fundamentals and anti-fundamentals. However, for the sake of concreteness, we focus on this particular choice in the current discussion.

The mirror theory can be obtained by applying the algorithmic procedure described above, resulting in the following planar quiver. Here we only report the compact notation for the mirror quiver for ease of readability:

$$\begin{array}{c}
 \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{1^N} \end{array} \quad \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{1^N} \end{array} \quad \begin{array}{c} \boxed{1} \\ \downarrow \\ \boxed{1^N} \end{array} \\
 \begin{array}{c} \boxed{N} \end{array} \rightsquigarrow \begin{array}{c} \boxed{1^N} \end{array} \quad \begin{array}{c} \boxed{1} \end{array} \rightsquigarrow \begin{array}{c} \boxed{1^N} \end{array} \quad \begin{array}{c} \boxed{1} \end{array} \rightsquigarrow \begin{array}{c} \boxed{1^N} \end{array} \rightsquigarrow \begin{array}{c} \boxed{N} \end{array} \\
 F_1+1 \quad F_2+1 \quad F_3+1
 \end{array}
 \tag{3.146}$$

Therefore the planar quiver has the following structure:



If we slightly modify the starting quiver in (3.145) — for instance, by reversing the direction of a bifundamental arrow or swapping a fundamental flavor with an anti-fundamental — the resulting quiver in (3.146) changes accordingly. For example changing the arrow of a bifundamental changes a fundamental flavor attached to the top node of a column, with an anti-fundamental attached to the bottom node of the same column. Or also, changing a flavor from fundamental to anti-fundamental changes the slope of a planar bifundamental.

It is interesting to consider the special case where each gauge group has exactly $2N$ chiral (anti-)fundamentals, that is the case of $F_1 = F_2 = F_3 = 0$. In this case the mirror planar quiver does not have sequences of planar bifundamentals, and all the flavor nodes are attached to the central column. The result is depicted in the first line of Figure 3.21. This leads to the flavor symmetry of the planar mirror theory to increase, in this case from $U(1)^4/U(1)$ to $SU(3) \times U(1)$.

Changing the orientation of the arrow produces different dualities with possibly different enhancements of the global symmetry. Some other possibilities are depicted in Figure 3.21. This, as far as we know, is a non-trivial prediction regarding the enhancement of topological symmetries in $\mathcal{N} = 2$ quiver theories.

This suggests a rule to predict the enhancement of global symmetries. Whenever a $U(N)$ gauge node sees exactly $[N, N]$ flavors the topological symmetry enhances to at least $SU(2)$. When in a quiver more than one gauge node satisfies this condition then the enhancement is bigger. For example in Figure 3.21, it is shown a collection of linear quivers with $n = 3$ nodes of rank N , with N_R bifundamental chirals pointing to the right and N_L pointing to the left, with $N_R + N_L = n + 1$. Then the chiral-planar mirror duality predicts that the $U(1)^n$ topological symmetry enhances to $S[U(N_R) \times U(N_L)]$. The limiting case with $n = 1$ corresponds to SQCD with $U(N)_{(0, -N)}$ gauge group and $[N, N]$ flavors, where the topological symmetry enhances from $U(1)$ to $SU(2)$. This is a special case of the dualities already discussed in Subsubsection 3.4.1. It is important to notice that the enhancement occurs for any quiver with the same values of N_R and N_L , regardless of the ordering of the bifundamentals since the planar mirror dual is invariant under this choice. This fact implies a duality among all the quivers with the same value of N_L and N_R , one such example is provided in Figure 3.21.

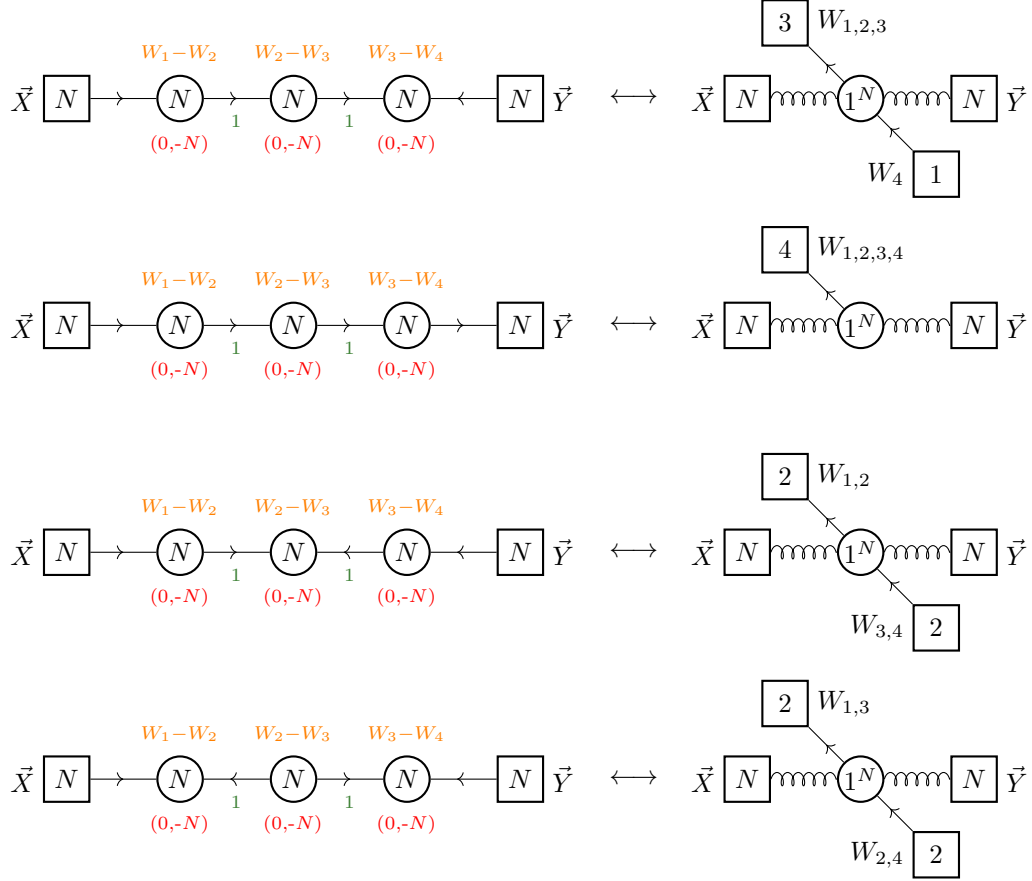


Figure 3.21: Chiral-Planar mirror dualities for a quiver with three gauge nodes. The planar dual of each chiral quiver makes the enhancement of the topological symmetry manifest. In general, the topological symmetry enhances to $S[U(N_L) \times U(N_R)]$, where N_L and N_R are the numbers of left- and right-pointing bifundamentals, respectively. This enhancement occurs independently of whether bifundamentals with the same orientation are consecutive. Furthermore, examining the last two dualities, we observe that all theories with the same values of N_L and N_R are dual to each other.

A “non-local” balancing condition

In Section 3.1, we started from the $\mathcal{N} = 4$ mirror pair of the $U(N)$ SQCD and considered the real mass deformation leading to the $\mathcal{N} = 2$ mirror duality relating the chiral $U(N)$ SQCD to its planar quiver dual pair, depicted in Figure 3.2.

It is interesting to consider the alternative situation where instead we take the chiral limit of the quiver on the dual side, which results in a planar limit for the SQCD. This can indeed be studied using the same strategies outlined in Section 3.1; however, here we will instead use the algorithm to generate the example. We begin with the chiral limit of the mirror of SQCD, namely:

$$(3.148)$$

We decompose this theory into fundamental blocks according to the rules described in the previous section, dualize each of them and then glue-back the result implementing the Identity-walls. These steps are depicted in the following figure:

$$(3.149)$$

Notice that the quiver considered in (3.148) actually does not follow the rule prescribed at the beginning of the section since it contains two tails built with asymmetric $U(N) \times U(M)$ bifundamentals. However we can bypass the problem of dualizing asymmetric bifundamental blocks by noticing that the two tails can be thought simply as a $\mathcal{S}$ and $\mathcal{S}^{-1}$ -walls. These walls do not need to be dualized and in the last step they fuse to Identity-walls with two other walls coming from the dualization of the flavor blocks.

The resulting quiver can be then written explicitly in Lagrangian form, providing the planar mirror description for the original linear quiver gauge theory (3.148), as shown in Figure 3.22.

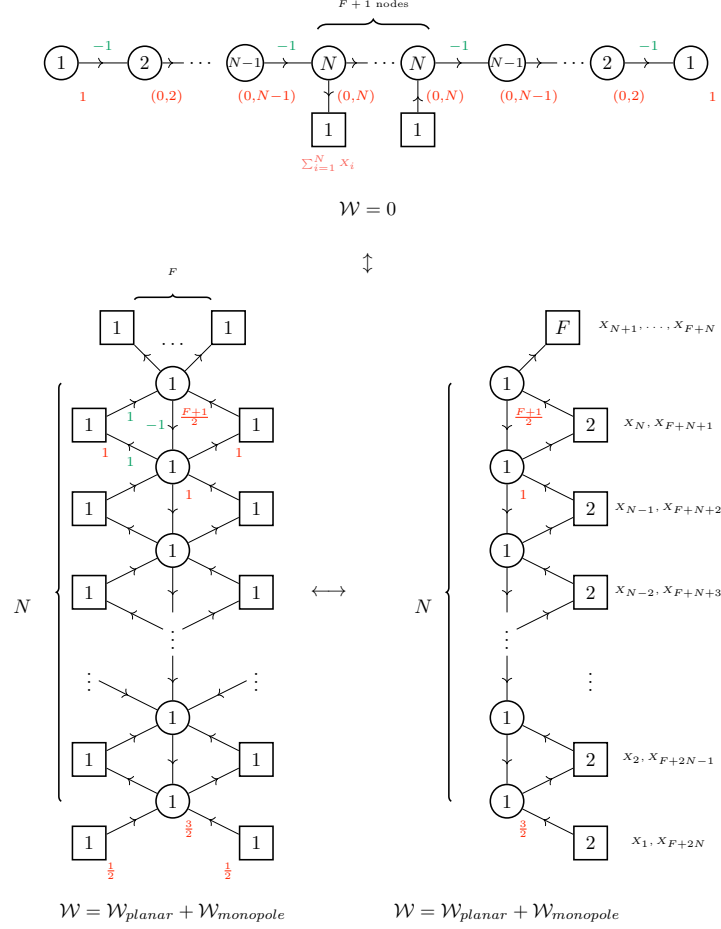


Figure 3.22: On the top, the mirror dual of the planar limit of the $\mathcal{N} = 4$ SQCD, reported from 3.148. On the bottom, the planar limit of the $\mathcal{N} = 4$ SQCD. We have suppressed the FI parameters and all the background terms for brevity. The FI terms for the gauge nodes are $\eta + \frac{iQ}{4}((F-1)\delta_{1,\alpha} - \delta_{N,\alpha})$. On the r.h.s. it is depicted the same theory but after recognizing the manifest flavor symmetry $S[U(F) \times U(2)^N]$ and writing it explicitly.

Interestingly, in the planar dual, the flavor symmetry is not the naive $U(1)^{F+2N}/U(1)$, but is instead enhanced to $S[U(F) \times U(2)^N]$, as shown in fig. 3.22. This reveals an intriguing pattern of topological symmetry enhancement. The appearance of the $U(F)$ factor can be understood using the rule proposed in the previous section: the central part of the quiver consists of a sequence of $U(N)$ nodes, each with $[N, N]$ flavors and bifundamentals oriented in the same direction. However, the emergence of the multiple $U(2)$ factors is a new feature specific to this duality. Notably, this latter enhancement involves topological symmetries associated with non-adjacent nodes — that is, it is non-local — in contrast to the local structure underlying the $U(F)$ enhancement. Nonetheless, both enhancements are manifest in the planar dual.

3.5 The Landscape of $\mathcal{N} = 2$ CS-SQCD and its Planar Mirror Duals

In the following sections we describe the planar Abelian dual of the $U(N)_k$ SQCD with F fundamental flavors for any choice of parameters (N, k, F) . Toward the end we also present examples for the SQCD with both fundamental and anti-fundamental chirals.

3.5.1 The Taxonomy of $\mathcal{N} = 2$ CS-SQCD with Fundamental Matter

Three-dimensional $\mathcal{N} = 2$ $U(N)_k$ SQCD theories with F fundamental chiral multiplets can be organized in a two-dimensional parameter space spanned by (k, F) . Each point in this plane corresponds to a distinct theory. In this chapter, we focus on SQCD with fundamentals, with CS level $k \leq 0$. The case $k > 0$ is analogous.

The results given here can be extended, in principle, to the most general case of SQCD with both fundamental and antifundamental fields by building on the results of the previous sections. The techniques needed for this generalization are analogous to the ones developed here, but the taxonomy is more involved, and we defer this analysis to future work.

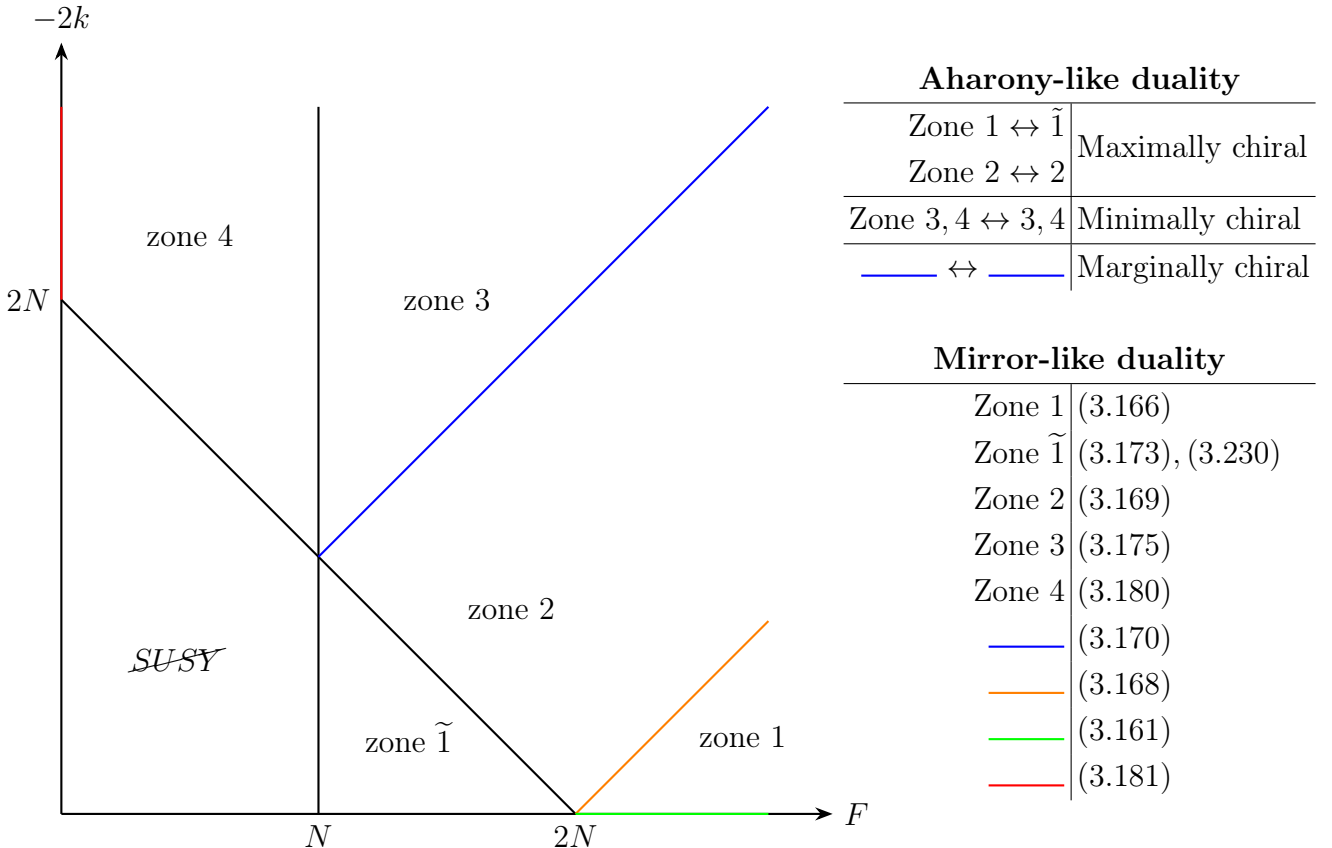


Figure 3.23: Phase diagram of $U(N)_k$ SQCD with F fundamental fields in the (F, k) -plane for fixed N . The diagram is partitioned into various zones, within which the planar Abelian duals take qualitatively distinct forms.

In Figure 3.23, close to the origin, we identify the supersymmetry breaking region

defined by:

$$SUSY : 2|k| < 2N - F \text{ and } F < N. \quad (3.150)$$

On the boundary of this region, namely for $F = N \wedge 2|k| < N$ and $F < N \wedge 2|k| = 2N - F$, the theory is trivial in the IR, with the exception of the cusp in $F = N$ and $2|k| = N$, where the theory reduces to a single free chiral. Indeed, for $F = 0$, the red line in Figure 3.23, the SQCD reduces to a $U(N)_k$ TQFT.

Outside the supersymmetry breaking region and for $F \neq 0$ the theory is interacting. As discussed in [128], the space of $\mathcal{N} = 2$ SQCD theories can be organized in three broad regimes distinguished by the relative magnitude of F , k and N . These three regions are distinguished by a qualitatively different Aharony-like dual²⁰ and are denoted as minimally, marginally, and maximally chiral regions:

$$\begin{aligned} \text{minimally chiral: } F < |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N \\ \text{marginally chiral: } F = |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N = F - N \\ \text{maximally chiral: } F > |2k|, & \quad \tilde{N} = F - N \end{aligned} \quad (3.151)$$

where $\tilde{N}$ is the rank of the Aharony-like dual theory. We point out that marginally chiral theories are characterized by the presence of a single gauge invariant monopole operator parameterizing a non-compact $\mathbb{C}$ moduli space of vacua. In general, for different values of (k, F) the theory does not possess any chiral gauge invariant operator. For the purpose of this thesis we further divide these regions depending on the qualitative form the planar Abelian dual, as we will discuss in the rest of the section.

A special role in the diagram in Figure 3.23 is played by the line $2|k| = F - 2N$, that is highlighted in orange. The SQCD on this line can be derived starting from the $\mathcal{N} = 4$ $U(N)$ SQCD with F flavors and performing a SUSY breaking deformation that preserves the $SU(F)$ flavor symmetry²¹. This is exactly the deformation and planar Abelian mirror studied in Section 3.1.

Starting from this proposal, we can explore the entire (k, F) plane by performing suitable real mass deformations, thereby arriving at a fully general construction of the planar Abelian dual of $U(N)_k$ SQCD with F flavors. This analysis naturally partitions the parameter space into five distinct regions, shown in Figure 3.23, in which the planar dual exhibits qualitatively different features.

For clarity, we adopt a slightly different presentation strategy. We first introduce the planar Abelian dual of $U(N)_0$ SQCD with $F \geq 2N$ fundamental fields, corresponding to the green line in Figure 3.23. We then take this theory, rather than the $\mathcal{N} = 4$ descendant, as the starting point to explore the full (k, F) plane via real mass deformations. In Section 3.7, we explain how the $U(N)_0$ proposal itself can be obtained through a sequence of deformations starting from the $\mathcal{N} = 4$ parent theory.

We emphasize that all dualities presented in this work follow from a systematic analysis of real mass deformations that break supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. These flows are studied in detail using, for instance, the $\mathbf{S}_b^3$ partition function to provide exact

²⁰While the original example of a 3d $\mathcal{N} = 2$ duality for $U(N)$ SQCD was presented in [93], generalizations to $U(N)_k$ theories with chiral matter content were subsequently developed by various authors. We refer to [180] for a comprehensive overview and an extensive list of references.

²¹To be more precise, the $\mathcal{N} = 4$ SUSY breaking deformation results in an $\mathcal{N} = 2$ $U(N)_{N-F/2, -F/2}$ SQCD theory with F fundamental chiral multiplets. By performing an $SL(2, \mathbb{Z})$ transformation [144], implemented via gauging and ungauging the topological symmetry, one can shift the $U(1)$ Chern–Simons level so that it matches that of the $SU(N)$ factor, as described in detail in Section 3.1.3.

mathematical identities between the partition functions of the two dual theories. As an additional non-trivial check, we can verify the dualities, for example, by matching the numerical expansion of the superconformal index for small values of (N, k, F) .

Motion in the (k, F) – Plane

A convenient way to explore the (k, F) parameter space is via real mass deformations of the matter fields. This will be the main tool employed throughout the rest of this chapter. Here we briefly summarize how such deformations allow one to navigate this plane, while a more detailed discussion is deferred to Section 3.6.

For each fundamental chiral multiplet, there are four basic possibilities:

1. **Non-Higgsing Mass Flows:** Assigning a positive or negative real mass $\pm m$ removes one chiral field from the spectrum and shifts the CS level by $\pm \frac{1}{2}$. On the (k, F) –plane these correspond to diagonal moves toward the left:

$$\begin{array}{c}
 -m \\
 \swarrow \\
 \bullet \text{ SQCD} \\
 \searrow \\
 +m
 \end{array} \tag{3.152}$$

which can be stated more succinctly:

$$(N, F, k) \xrightarrow{\pm m} (N, F - 1, k \pm \frac{1}{2}).$$

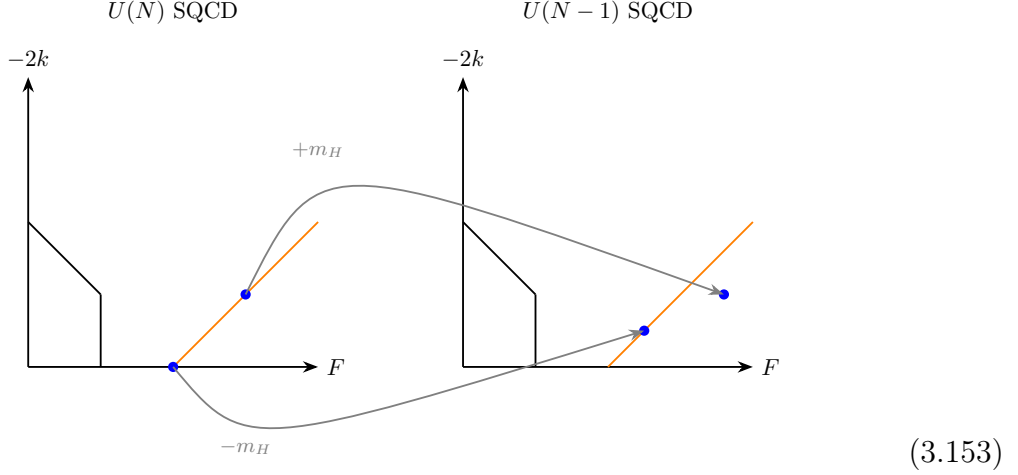
2. **Higgsing Mass Flows:** Assigning a positive or negative real mass $\pm m_H$ removes one chiral field from the spectrum, shifts the CS level by $\pm \frac{1}{2}$ and Higgses the gauge group to $U(N-1)$. These correspond to the following movement across two different (k, F) –planes²²:

$$(N, F, k) \xrightarrow{\pm m_H} (N - 1, F - 1, k \pm \frac{1}{2}).$$

We can interpret this operation as a movement to the right on the (k, F) –plane. As an example, starting from the pseudo-self-dual point $F = 2N$, $k = 0$, the deformation $-m_H$ produces a theory with $F' > 2N'$, which sits on the $\mathcal{N} = 4$ -descendant line to the right of the pseudo-self-dual point $F' = 2N'$. The deformation $+m_H$ reduces a theory sitting on the $\mathcal{N} = 4$ -descendant line to a theory which lies in

²²Notice that the mass deformations with Higgsing $\pm m_H$ are only available in certain regions of the (F, k) –plane; refer to Section 3.6 for more details.

Zone 1:



Therefore, with a slight abuse of notation, we depict the possible deformations in the (k, F) -plane: schematically as:

$$\begin{array}{ccc}
 -m & & -m_H \\
 \swarrow & \bullet & \nearrow \\
 & \text{SQCD} & \\
 \swarrow & & \nearrow \\
 +m & & +m_H
 \end{array} \tag{3.154}$$

Taken together, these deformations allow us to probe the (k, F) -plane completely, starting from a given theory²³. We emphasize that the Higgsing mass deformations $\pm m_H$ also reduce the gauge rank by one; consequently, when including such deformations, one must begin from an SQCD theory with gauge group $U(N+1)$. By systematically mapping these mass flows to the planar mirror description, we construct the planar Abelian dual of every $\mathcal{N} = 2$ $U(N)_k$ SQCD theory with fundamental matter.

In this section, we probe the (k, F) -plane using only $\pm m$ deformations, thereby avoiding Higgsing. Starting from the green line in Figure 3.23, which lies along the k -axis, these deformations allow us to explore the entire plane.

To access the region corresponding to $U(N)_0$ with $F \geq 2N$ flavors, we instead employ massive deformations accompanied by Higgsing. In this case, we begin from the $\mathcal{N} = 4$ descendant line (the orange line in Figure 3.23).

The strategy employed to follow these deformations in the planar Abelian dual consists of studying the large mass asymptotic behavior of the $\mathbf{S}_b^3$ partition function, following the strategies of [25, 128] (see also Section 3.1.2).

3.5.2 A Planar Abelian Dual of $U(N)_k$ SQCD with $F \square$

Having organized $\mathcal{N} = 2$ $U(N)_k$ SQCD theories in the (k, F) -plane, we now turn to the construction of their planar Abelian duals. Our starting point is the case with vanishing

²³A further consequence is that generic Aharony-like dualities for SQCD with only fundamentals can be derived assuming the pseudo-self-duality:

$$U(N)_0 \text{ with } 2N\square \quad \leftrightarrow \quad U(N)_0 \text{ with } 2N\overline{\square} \tag{3.155}$$

for any N .

CS level, $k = 0$, for which the dual description takes a particularly symmetric form. We then show how real mass deformations, corresponding to motion in the (k, F) -plane, map naturally to Higgsing and mass flows in the planar quiver, generating duals for arbitrary k and F . In this way, the planar Abelian dual provides a unifying framework that interpolates continuously between the $\mathcal{N} = 4$ -descendant line and more general $\mathcal{N} = 2$ CS-SQCD theories.

A Planar Abelian Dual of $U(N)_0$ SQCD with $F \geq 2N$ $\square$

In this section, we describe an Abelian theory that is infrared dual to $\mathcal{N} = 2$ $U(N)_0$ SQCD with $F \geq 2N$ fundamental chiral multiplets. Notice that the number of chiral fields F must be even due to the quantization of the effective Chern–Simons level. As mentioned above, the validity of this duality is supported by the fact that it can be derived starting from the mirror dual of $\mathcal{N} = 4$ SQCD, followed by an appropriate supersymmetry-breaking real mass deformation and subsequent real mass deformations.

The Abelian dual exhibits close similarities to the duals discussed in Section 3.1. In particular, in Section 3.8, we show that a sequence of real mass deformations, combined with local Aharony dualities applied to the constructions of Section 3.1, reproduces the duality presented below.

The dual of $U(N)_0$ SQCD with $F \geq 2N$ fundamental chiral multiplets is shown in

(3.156).

$U(N)_0$ with F $\square \longleftrightarrow$

$$\Lambda = \eta - \sum_{i=1}^{F/2} X_i + \frac{F}{2}(3r-1)\omega,$$

$$\sum_{i=1}^N X_i - \frac{\omega}{2}(r-1+2Nr)$$

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

(3.156)

Before discussing the main features of the duality, we present in detail the Abelian dual theory in (3.156). For clarity, we organize its defining data as follows.

Gauge structure and Chern–Simons couplings. The gauge group is $U(1)^{N \times (F-N)}$, with chiral multiplets whose charges are encoded in the quiver. Each $U(1)$ factor carries a CS level, indicated by the red labels on the nodes, together with mixed CS interactions represented by green labels on the links. The rules described in Section 2.2 allow one to reconstruct all CS interactions directly from the quiver.

Superpotential. The superpotential consists of two types of interactions:

- $\mathcal{W}_{\text{planar}}$: it contains a cubic (quartic) term for each triangular (square) face of the quiver, with sign -1 ($+1$) if the arrows run clockwise (anticlockwise) around the face.

- $\mathcal{W}_{\text{monopole}}$: each vertical column of k gauge nodes contributes with $k - 1$ superpotential terms that are linear for the monopole with $-1/+1$ GNO fluxes for the top/bottom nodes connected by a vertical arrow. For the central column, where vertical arrows are absent, we have an analogous superpotential, that is linear for the monopole with $-1/+1$ GNO fluxes under the first/second gauge node of the column, also for the second/third nodes and so on.

Trial R-symmetry. On the SQCD side, we assign trial R-charge r to the fundamental fields. On the quiver side, the chiral multiplets have R-charges:

$$\begin{array}{c} \diagdown, \diagup : r \quad , \quad \updownarrow : 2r \quad , \quad \diagup, \diagdown : 1 - r \quad , \quad \updownarrow : 2 - 2r , \end{array} \quad (3.157)$$

which are compatible with the planar superpotential. The mixing parameter r is to be fixed, in principle, via extremization [152]. Here, we do not perform the extremization and keep r generic instead.

Global symmetries. The Abelian theory has a $U(1)^F$ UV global symmetry. One factor, denoted $U(1)_\Lambda$, is a flavor symmetry compatible with $\mathcal{W}_{\text{planar}}$ and is not gauged. Without loss of generality, we assign charge -1 under $U(1)_\Lambda$ to the bottom-most chiral, with all other fields neutral. This symmetry corresponds to the bottom flavor node of the quiver.

In addition, the theory possesses a $U(1)^{F-1}$ topological symmetry that is preserved by $\mathcal{W}_{\text{monopole}}$. Effectively, each column of nodes carries one such symmetry. We encode the mapping in the FIs for the nodes in the i -th column which are, from top to bottom:

- $i = 1, \dots, \frac{F}{2} - 1$: $\{X_{i+1} - X_i - \frac{\omega}{2}r, X_{i+1} - X_i, \dots, X_{i+1} - X_i, X_{i+1} - X_i + \frac{\omega}{2}r\}$
- $i = \frac{F}{2}$: $\{X_{\frac{F}{2}+1} - X_{\frac{F}{2}} + \frac{\omega}{2}(r-1), X_{\frac{F}{2}+1} - X_{\frac{F}{2}}, \dots, X_{\frac{F}{2}+1} - X_{\frac{F}{2}}, X_{\frac{F}{2}+1} - X_{\frac{F}{2}} + \frac{\omega}{2}r\}$
- $i = \frac{F}{2} + 1, \dots, F - 1$: $\{X_{i+1} - X_i + \frac{\omega}{2}(r-1), X_{i+1} - X_i, \dots, X_{i+1} - X_i, X_{i+1} - X_i - \frac{\omega}{2}(r-1)\}$

where $X_{i+1} - X_i$ parametrizes the corresponding Cartan of $SU(F)$ and the coefficient of $\omega \equiv \frac{iQ}{2}$, which can be thought of as the real mass associated with the R-symmetry, parametrizes the mixing between the topological and the R-symmetry. One of the X_i is redundant, but we keep this overcomplete parametrization for later convenience. One can check that the FIs are compatible with the monopole superpotential described above.

In the IR, the topological symmetry enhances to $SU(F)_X$, with parameters X_i satisfying $\sum_{i=1}^F X_i = 0$. This enhancement can be seen, for instance, via the superconformal index and is also predicted by the duality as we explain in a moment.

Map of symmetries. Across the duality, flavor and topological symmetries are exchanged, as expected in mirror-like dualities. The SQCD side has an $SU(F)_X$ flavor symmetry under which the matter fields transform in the anti-fundamental representation. Its Cartan subgroup, parameterized by real masses X_i with $\sum_i X_i = 0$, maps to linear combinations of the $U(1)^{F-1}$ topological symmetries of the Abelian theory, which enhance in the IR to $SU(F)_X$.

More precisely, the topological $U(1)$ symmetry associated with the i -th column maps to the anti-diagonal combination of $U(1)_{X_i}$ and $U(1)_{X_{i+1}}$.

The topological symmetry $U(1)_\eta$ of SQCD is mapped to a combination of $U(1)_\Lambda$, $U(1)_{X_i}$, and the R-symmetry in the Abelian theory. This map can be expressed at the level of real mass parameters as

$$\Lambda = \eta - \sum_{i=1}^{F/2} X_i + \frac{F}{2}(3r - 1)\omega, \quad (3.158)$$

as shown in (3.156). Replacing real masses by the corresponding generators yields the mapping of the $U(1)_\eta$ symmetry.

Background terms. Finally, the Abelian dual includes background CS terms, conveniently encoded in the $\mathbf{S}_b^3$ partition function as $\exp(\frac{\pi i}{2}\phi^{(\text{mag})})$, with

$$\phi^{(\text{mag})} = N \left(\sum_{i=1}^{\frac{F}{2}} (X_i^2 - 2\omega(1 - 3r)X_i) - \sum_{j=\frac{F}{2}+1}^F (X_j^2 + 2\omega(1 - r)X_j) \right) + \mathcal{O}(\omega^2). \quad (3.159)$$

The quadratic terms encode CS couplings for the global symmetries $U(1)_{X_i}$, while the linear terms correspond to mixed CS couplings between $U(1)_{X_i}$ and the R-symmetry. These contributions are essential for matching the duality in the presence of background fields and for correctly implementing real mass deformations.

Compact Quiver Notation. All the data required to fully specify the planar Abelian theory are encoded in the quiver diagram. Therefore we'll find it useful for the rest of the chapter to adopt a compact notation to describe the duality in (3.156). This notation is summarized in the following table:

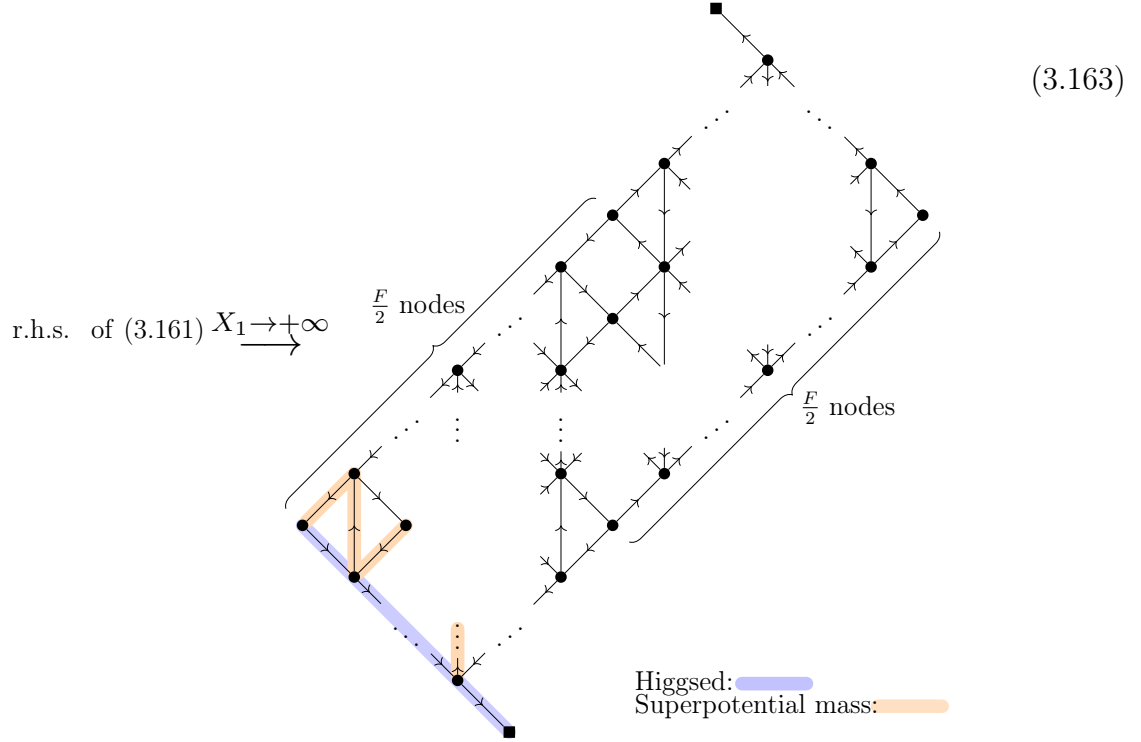
The compact notation used for Abelian planar quivers, together with their contributions to the partition function, is summarized in Table (3.160).

Quiver notation	Field theory
$u \bullet$	$U(1) \mathcal{N} = 2$ vector multiplet
$X \blacksquare$	$U(1)$ Flavor node
$u \bullet \swarrow \searrow v \bullet$, $u \bullet \nwarrow \nearrow v \bullet$	bifundamental chiral with R-charge r and mixed CS level $+1$
$u \bullet \downarrow v \bullet$	bifundamental chiral with R-charge $2r$ and mixed CS level $+1$
$u \bullet \swarrow \searrow v \bullet$, $u \bullet \nwarrow \nearrow v \bullet$	bifundamental chiral with R-charge $1 - r$ and mixed CS level -1
$u \bullet \downarrow v \bullet$	bifundamental chiral with R-charge $2 - 2r$ and mixed CS level -1
$u \bullet \text{---} v \bullet$	Mixed CS term at level -2

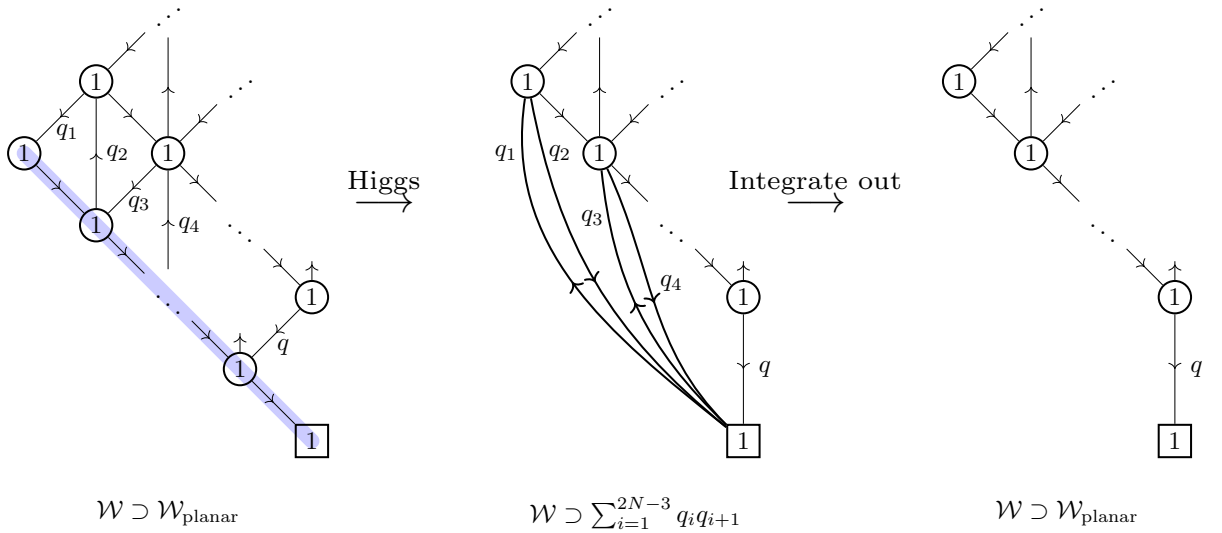
(3.160)

For this shorthand to be effective, we distinguish between diagonal and vertical chirals, as well as their orientation (upward or downward). We remark that this graphical distinction encodes differences in trial R-charges and mixed CS couplings.

terms. Graphically the deformation is the following:



Here, the chiral fields shown in blue are those that acquire Higgs VEVs, consequently the fields in orange obtain complex masses from the planar superpotential as illustrated below:



(3.164)

Integrating out the massive fields yields the planar Abelian dual for $U(N)_{-1/2}$ SQCD

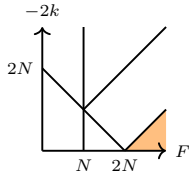
with $F - 1$ fundamentals:

$$\begin{array}{c}
 U(N)_{-\frac{1}{2}} \text{ with } F - 1 \square \longleftrightarrow \\
 \begin{array}{c}
 \text{--- } \frac{F}{2} - 1 \text{ nodes ---} \\
 \text{--- } \frac{F}{2} + 1 \text{ nodes ---}
 \end{array}
 \end{array}
 \quad (3.165)$$

The resulting dual quiver closely resembles the Abelian dual before the mass deformation. The only differences are the number of columns and the position of the “column of squares”. In this case, the column of squares is displaced from the center of the quiver, as indicated by the sizes of the various quiver segments.

Further negative mass deformations are mapped analogously, leading to the following duality for SQCD with $F \geq 2N + 2|k|$ and $k \leq 0$ (**Zone 1** of Figure 3.23):

$$\begin{array}{c}
 U(N)_k \text{ with } F \square \longleftrightarrow \\
 (F \geq 2|k| + 2N) \\
 \begin{array}{c}
 \text{--- } \frac{F}{2} - |k| \text{ nodes ---} \\
 \text{--- } \frac{F}{2} + |k| \text{ nodes ---}
 \end{array}
 \end{array}
 \quad (3.166)$$



The three parameters (N, F, k) are mapped to the sizes of the two edges of the quiver and to the position of the column of squares within it. The (mixed) CS interactions and the superpotential terms follow the same rule as in Section 3.5.2 (see the discussion below (3.156)).

This duality exhibits features very similar to those of (3.156). In particular, the Abelian planar theory contains a superpotential with a cubic (quartic) term for each triangular (square) face of the quiver. In addition, there is a linear monopole superpotential for each monopole operator with $-1/+1$ GNO flux under the top/bottom gauge nodes connected by a vertical arrow, as well as for adjacent vertical gauge nodes not connected by arrows in the column of squares.

Overall, the theory possesses a single unbroken topological symmetry for each vertical column of chiral multiplets, giving a total $U(1)^{F-1}$ symmetry, which enhances in the IR to $SU(F)$. Moreover, there is a single $U(1)$ flavor symmetry that maps to the $U(1)$ topological symmetry of the SQCD²⁴.

We can systematically follow the first $\frac{F}{2} - N$ negative mass deformations in this way. The limiting case corresponds to theories obtained from supersymmetry-breaking real mass deformations of the $\mathcal{N} = 4$ dual pairs, shown in (3.167).

$$\begin{aligned}
 & U(N)_{N-\frac{F}{2}} \text{ with } F \square \longleftrightarrow \\
 & (F \geq 2N)
 \end{aligned}
 \quad
 \begin{aligned}
 & \text{Quiver diagram with } F-N \text{ nodes (left), } N \text{ nodes (right), and } F-N \text{ nodes (bottom).} \\
 & \text{The diagram shows a complex network of nodes and arrows, with labels indicating the number of nodes in different sections.}
 \end{aligned}
 \tag{3.167}$$

This proposal coincides with Figure 3.2 of Section 3.1, as can be shown through a sequence of local dualizations following the procedure described in detail in Appendix B of [2]. In short, starting from the quiver in (3.167), one performs Aharony duality on all the gauge nodes in each column, from left to right, stopping at the $(N-1)$ -th column, then repeats the process up to the $(N-2)$ -th column, and so on. After performing this sequence $N-1$ times, the resulting quiver matches the planar Abelian dual of SQCD on

²⁴More precisely, a combination of the flavor $U(1)$, some of the topological $U(1)$ factors, and the $U(1)_R$ R-symmetry maps to the topological symmetry of the SQCD. We omit the precise combination here for brevity, but it can be determined by carefully tracking the RG flow on the $\mathbf{S}_b^3$ partition function.

the $\mathcal{N} = 4$ -descendant line shown in (3.168).

$$\begin{aligned}
 & U(N)_{N-\frac{F}{2}} \text{ with } F \square \\
 & (F \geq 2N)
 \end{aligned}
 \longleftrightarrow
 \begin{aligned}
 & \text{Diagram showing a network of nodes and edges, with a bracket indicating } F - N \text{ nodes.} \\
 & \text{The diagram consists of a central diamond-like shape formed by nodes (black dots) and edges (lines with arrows). The top and bottom nodes are black squares. The left and right nodes are black dots. The edges are labeled with arrows indicating direction. A bracket on the left side of the diamond indicates } F - N \text{ nodes.}
 \end{aligned}
 \quad (3.168)$$

Duality in Zone 2 ($2|k| > |F - 2N| \wedge F \geq 2|k|$)

We can further probe the region $2|k| > |F - 2N| \wedge F \geq 2|k|$ (**Zone 2** of Figure 3.23) by introducing additional negative real masses in the electric theory and studying the corresponding mirror RG flow trajectory. The associated deformation on the planar Abelian dual is similar to that described at the beginning of Section 3.5.2, with the difference that they now affect fields along the bottom-right diagonal instead of the bottom-left one. A more detailed analysis of this deformation is presented in Section 3.7.1; for now, we focus on describing the resulting theory.

The resulting duality for SQCD with $2N + 2|k| > F \geq 2|k|$ is shown in (3.169).

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \quad \longleftrightarrow \quad \text{Quiver Diagram} \\
 & 2|k| > |F - 2N| \wedge F > 2|k|
 \end{aligned}
 \tag{3.169}$$

The quiver diagram consists of several nodes (black circles and squares) connected by arrows. Brackets indicate the number of nodes in different sections: $F - N$ nodes, N nodes, and $\frac{F}{2} - |k|$ nodes. A small graph is also shown with axes $-2k$ and F , and points N and $2N$ marked on the F -axis.

Turning on exactly N negative real masses results in a planar Abelian dual for marginally chiral SQCD (3.170).

$$\begin{aligned}
 & U(N)_{-\frac{F}{2}} \text{ with } F \square \quad \longleftrightarrow \quad \text{Quiver Diagram}
 \end{aligned}
 \tag{3.170}$$

The quiver diagram is similar to (3.169) but with a different node count for the central section, $F - N$ nodes. A small graph is also shown with axes $-2k$ and F , and points N and $2N$ marked on the F -axis.

In this case, we observe that the planar Abelian quiver contains a long gauge-invariant mesonic operator (highlighted in maroon) in the chiral ring that parametrizes a non-compact moduli space. Consistently, the dual SQCD features a gauge-invariant monopole operator parametrizing a $\mathbb{C}$ moduli space of vacua.

Notice that Zone 2 includes, on its boundary, the case of $U(N)_{-\frac{N}{2}}$ SQCD with N flavors, which corresponds to the cusp of the supersymmetry-breaking region. The planar dual of this theory cannot be obtained as a simple limiting case of the dualities in (3.169)

and (3.170). Indeed, in this limit, certain parameters controlling the size of the quiver degenerate, indicating that the planar dual takes a qualitatively different form.

Nevertheless, the dual description can be accessed by following real mass deformations in the Abelian theory. For instance, starting from the dual of $U(N)_{-\frac{N}{2}+\frac{1}{2}}$ with $N+1$ fundamental chiral multiplets and integrating out one field, we are led to the duality shown in (3.171).

$$\begin{array}{ccc}
 \text{Quiver 1: } U(N)_{-\frac{N}{2}+\frac{1}{2}} \text{ with } N+1 \square & \xrightarrow{-m} & \text{Quiver 2: } U(N)_{-\frac{N}{2}} \text{ with } N \square \\
 \updownarrow & & \updownarrow
 \end{array}
 \tag{3.171}$$

By applying a sequence of Aharony dualities on the gauge nodes of the Abelian quiver, one finds that the theory confines to a single free chiral multiplet. This reproduces the expected infrared duality

$$U(N)_{-\frac{N}{2}} \text{ with } N \square \longleftrightarrow \text{free chiral}, \tag{3.172}$$

where the free chiral is identified with the gauge-invariant monopole operator of the SQCD theory.

Duality in Zone $\tilde{1}$ ($2N - 2|k| > F > N$)

Starting from a theory in Zone 2, we can consider further positive mass deformations to probe the regime $2N - 2|k| > F > N$ (**Zone $\tilde{1}$** of Figure 3.23). For the mirror theory, this deformation is analogous to that described at the beginning of Section 3.5.2, which has simply the effect of shortening the quiver by “removing” the top-right diagonal of the quiver. As in the previous section, we refer to Section 3.7.1 for more details.

The corresponding duality for SQCD is shown in (3.173).

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \quad \longleftrightarrow \quad \text{Quiver Diagram} \\
 & (2N - 2|k| > F > N)
 \end{aligned}
 \tag{3.173}$$

The dual quiver obtained is similar to those considered in (3.169). In Section 3.8, we demonstrate that, through a sequence of local dualizations, the planar Abelian dual of theories in **Zone 1** admits an alternative representation, reported in (3.230), which more closely resembles the planar Abelian dual of **Zone 1** in (3.166). Indeed, theories in **Zone 1** and **Zone 1** are expected to be related by Aharony duality, and the sequence of local dualizations explicitly relates the planar Abelian duals of the two zones, consistent with this expectation.

This duality also closely parallels that of (3.156). The superpotential contains both mesonic and monopole terms, following the rules outlined below (3.156). As a result, the theory has a single $U(1)$ flavor symmetry, while each column of vertical chiral fields contributes one $U(1)$ topological symmetry due to the monopole superpotential. This yields a total $U(1)^{F-1}$ symmetry, which enhances in the IR to $SU(F)$. These symmetries map naturally to the $U(1) \times SU(F)$ topological and flavor symmetries of the original SQCD.

On the boundary between **Zone 1** and the SUSY-breaking region, corresponding to $F = N$ and $2|k| \leq 2N - F$, the theory is expected to become trivial. This behavior cannot be directly inferred from (3.173), since reaching this point would require setting one of the quiver size parameters to zero, leading to a degenerate description.

Nevertheless, this limit can be accessed by considering theories that flow to this boundary via a single real mass deformation. For example, one may start from theories along

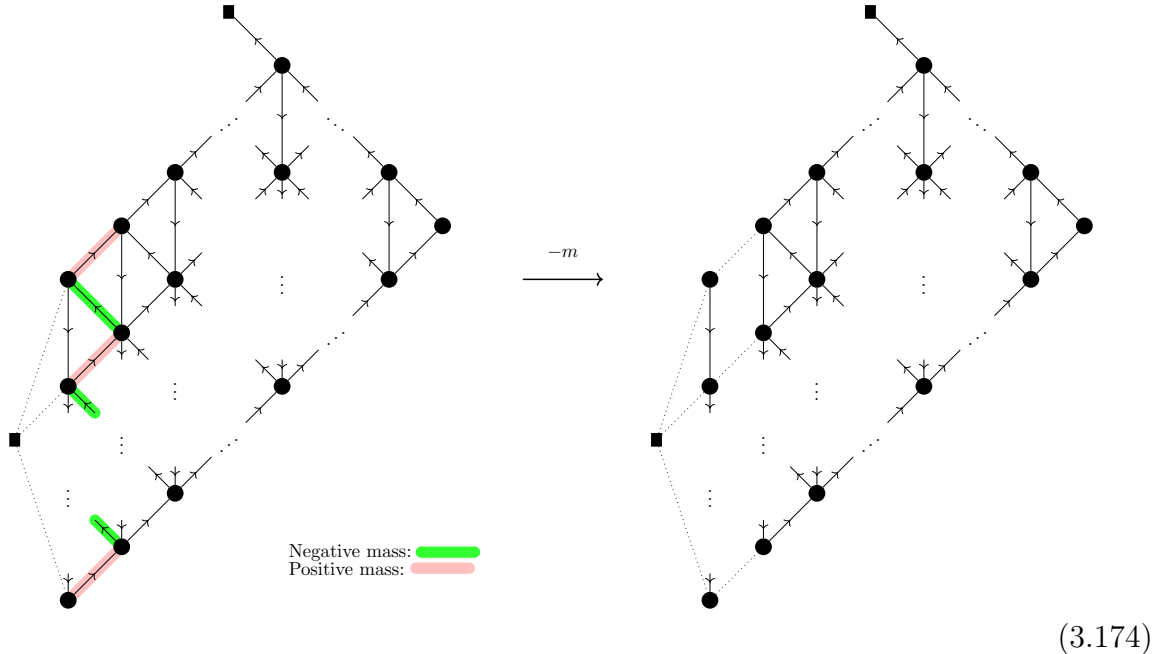
the vertical line with $F = N + 1$ and $2|k| \leq 2N - F$, and follow the deformation. In this way, one finds that the planar Abelian dual reduces to a quiver that fully confines to a $U(1)_1$ theory, which is an almost-trivial theory.

In contrast, the boundary between Zone 1 and Zone 2 can be straightforwardly obtained by setting $2|k| = 2N - F$ in either (3.169) or (3.173), which is a well-defined operation at the level of the quiver.

Similarly, theories along the horizontal line $k = 0$, with $N < F < 2N$, can be obtained by directly setting $k = 0$ in (3.173). The special case $k = 0$ and $F = 2N$ is instead described by (3.161). This point is not captured by a naive limit of (3.173), as expected, since it cannot be reached via positive real mass deformations starting from theories in Zone 2.

Duality in Zone 3 ($2|k| > F \geq N$)

We can also probe the region $2|k| > F \geq N$ (**Zone 3** of Figure 3.23). This is achieved by starting from the dualities on the boundary between Zone 2 and Zone 3 in (3.170) and performing more negative mass deformations. The effect of the first deformation on the planar quiver in (3.170) is to simply give a negative real mass to the left flavor chiral. The following real masses have instead the effect of “integrating out columns”, meaning that a positive/negative real mass is acquired by all the chirals represented as diagonal lines connecting two columns of vertical chirals. In this way, we are left simply by column of vertical chirals of height N that are “connected” only through mixed CS interactions between gauge nodes in the same relative position inside the nearest columns. This effect occurs consecutively on each column, starting from the leftmost one and proceeding to the right, as shown in (3.174).

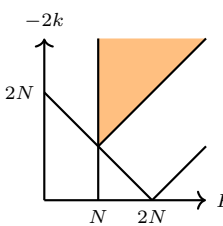


This description is indeed valid so long as we do not cross the boundary between Zone 3 and Zone 4.

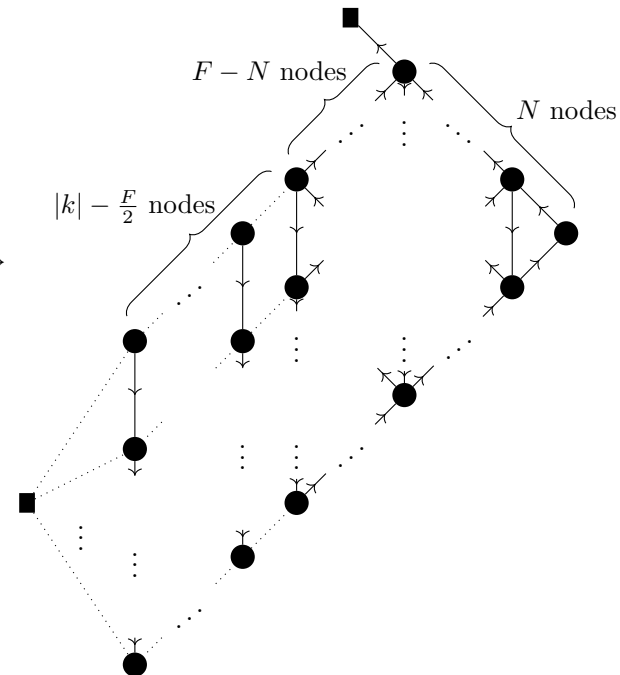
We thus obtain the following duality (3.175):

$U(N)_k$ with $F \square$

$(2|k| > F \geq N)$



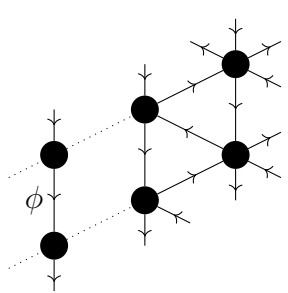
$\longleftrightarrow$



(3.175)

Let us comment on the monopole superpotential in the quiver theories above. There is a monopole superpotential term for each pair of consecutive nodes in a column with GNO flux $-1/+1$ under the top/bottom nodes. In the “bulk” of the quiver, where there are diagonal arrows on both sides of the column, these monopoles are gauge-invariant BPS operators and, hence, enter the superpotential as described below (3.156). When the gauge nodes are missing diagonal chirals on at least one side then the monopole superpotential follows these two new rules.

- In the column that separates the “bulk” of the quiver and the region with BF interactions, as shown in (3.176), these monopoles are dressed with a vertical chiral field ϕ as following:



$$\mathcal{W} \ni \mathfrak{M} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \phi$$

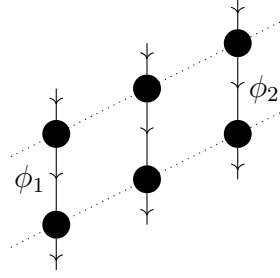
(3.176)

The FIs of the nodes in this type of column are, from top to bottom:

$$\text{FIs: } X_1 - \frac{\omega}{2}, \quad X_1 - r\omega, \quad \dots, X_1 - r\omega, \quad X_1 + \frac{\omega}{2}(1 - 3r) \quad (3.177)$$

where X_1 is a Cartan of the $SU(F)$ flavor symmetry and the coefficient in front of ω encodes the mixing between the topological and the R-symmetry. We remind the reader that we assign trial R-charge r to the fundamental fields of the SQCD. One can check that the FIs above are compatible with the monopole superpotential.

- In the columns with BF interactions on both sides, as shown in (3.178), the monopoles are dressed with two vertical chiral fields ϕ_1 and ϕ_2 :



$$\mathcal{W} \ni \mathfrak{M} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \phi_1 \phi_2 \quad (3.178)$$

The FIs of the nodes in this type of column are, from top to bottom:

$$\text{FIs: } \frac{\omega}{2}(2r-1), \quad 0, \quad \dots, \quad 0, \quad \frac{\omega}{2}(1-2r) \quad (3.179)$$

One can check that the FIs above are compatible with the monopole superpotential.

- We further notice that in a column with no diagonal chirals on both sides, the single unbroken topological symmetry can be reabsorbed by a gauge transformation due to the presence of non-zero CS-levels. Therefore, only the columns in the “bulk” of the quiver and at the edge of the “bulk” contribute to the Cartan of the global symmetry, thus matching the rank of the global symmetry predicted by the dual SQCD.

The same discussion also applies to the planar Abelian quivers in Zone 4 discussed below.

Duality in Zone 4 ($2N - 2|k| < F < N$)

We can continue to perform negative real mass deformations for the SQCD to probe the region $2N - 2|k| < F < N$ (**Zone 4** of Figure 3.23). Starting from the planar quiver in Zone 3 describing $F = N$, in (3.175), the effect of the deformation is now mapped to the chiral fields in the top-right diagonal of the quiver. Qualitatively, the effect of the deformation is the same as the one that we performed to cross the boundary between Zone 1 and Zone 2, but now mapped to the top-right diagonal instead of the bottom-left. By performing multiple such deformations, we can fully explore the entirety of Zone 4.

The resulting dualities are shown in (3.180).

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Diagram of a planar quiver with } N \text{ nodes, } |k| + \frac{F}{2} - N - 1 \text{ nodes, and } F - N \text{ nodes.} \end{array} \\
 & (N > F > 2N - 2|k|)
 \end{aligned}
 \tag{3.180}$$

On the boundary between Zone 4 and the SUSY-breaking region, namely for $2|k| = 2N - F$ with $F < N$, the planar quiver takes a qualitatively different form and cannot be obtained as a well-defined zero-size limit of (3.180).

This duality also describes the line separating Zone 3 and Zone 4, that is, for $F = N$ and $2|k| > N$.

However, using the fact that the theory at $F = |2k| = N$ reduces to a single free chiral multiplet, one can immediately infer the behavior along this boundary. Indeed, a negative real mass deformation of $U(N)_{-N/2}$ SQCD with N flavors maps, in the dual description, to a mass term for the free chiral. Integrating out this field yields a trivial theory, implying that the boundary theory is trivial.

Dualities for $U(N)_k$ Super–Yang–Mills Theory

Starting from Zone 4, we can add more negative massive deformations to integrate out all matter and finally reach the case of $F = 0$. In such a situation, the SQCD reduces to a $U(N)_k$ TQFT. We can follow this sequence of deformations in the planar Abelian theory describing Zone 4, in (3.180), all the way down to $F = 0$. The resulting duality that we obtain is:

$$\begin{array}{c} \mathcal{N} = 2 \quad U(N)_k \\ \updownarrow \\ \mathcal{N} = 0 \quad U(N)_{(k+N,k)} \end{array}$$

$\longleftrightarrow$

(3.181)

Interestingly, the resulting planar dual is not manifestly topological, as it still contains massless charged chiral fields. The duality, therefore, implies that the planar Abelian quiver shown in (3.181) is in fact a gapped theory, flowing in the infrared to a $U(N)$ Chern–Simons theory with levels $(k + N, k)$.

Here we consider two limiting cases corresponding to $k = -N - 1$ and $k = -N$. In these cases the planar Abelian dual does not involve any chiral field and is manifestly a TQFT. In the first case we find the following duality²⁵:

$$\begin{array}{c} \mathcal{N} = 2 \quad U(N)_{-N-1} \\ \downarrow \\ \mathcal{N} = 0 \quad U(N)_{-1, -N-1} \end{array}$$

$\longleftrightarrow$

$N \text{ dotted lines } \left\{ \begin{array}{c} \text{diagram of } N \text{ dotted lines forming a lens shape} \end{array} \right. = U(1)_{N+1}$

(3.182)

In the case $k = -N$ the planar dual does not involve any gauge nodes and is trivially gapped in the IR. In both cases the planar Abelian dual reproduces the generalized level-rank duality discussed in [97, 181, 182] for $U(N)_{k, k-N}$ with $k = -1, 0$.

It would be interesting to study this duality further, in particular its possible connection with level-rank duality in the cases with $|k| > N+1$. We defer this to future work.

To summarize, we find a planar Abelian dual for every SQCD theory with gauge group $U(N)$, CS level k , and F fundamental fields. All these dualities can be obtained from the duality (3.161) by introducing large real mass deformations for some of the fundamental fields. In the following sections, we analyze these real mass deformations in detail, both on the SQCD side and on the mirror side.

²⁵Here and in the rest of the chapter we do not keep track of background gravitational CS terms.

3.5.3 An Example: The Zoology of $U(3)_k$ SQCD with F $\square$

Having established the framework for mapping real mass deformations across mirror duals, we now present a set of illustrative examples for $U(3)_k$ SQCD with F flavors. These examples highlight how the planar Abelian dual realizes qualitatively different structures across the various regions of the (k, F) parameter space.

We begin with the planar Abelian dual of $U(3)_0$ SQCD with $F = 8$, as described by the general proposal in (3.156). In standard quiver notation, the duality takes the form

$U(3)_0$ with 8 $\square \longleftrightarrow$

$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$

(3.183)

where we use the shorthand $X_{ij} := X_i - X_j$ to parametrize the FI terms of the gauge nodes in the planar theory. In the planar theory, there are 14 mesonic superpotential terms, one associated with each face of the quiver, and a total of 8 monopole terms. These correspond to $k - 1$ terms for each vertical column of gauge nodes of height k , as described below (3.156). The CS levels are equal to 1 for all nodes except those in the central column, as indicated by the red labels in the figure. The BF couplings follow the rules described below (3.156).

Starting from this duality, we can turn on a negative real mass for one flavor in the SQCD, flowing to $U(3)_{-1/2}$ with 7 flavors. This theory lies on the $\mathcal{N} = 4$ -descendant line of Figure 3.23. The resulting duality for $U(3)_{-1/2}$ SQCD with 7 fundamental chiral multiplets is shown in (3.184).

$U(3)_{-\frac{1}{2}}$ with 7 $\square \longleftrightarrow$

$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$

(3.184)

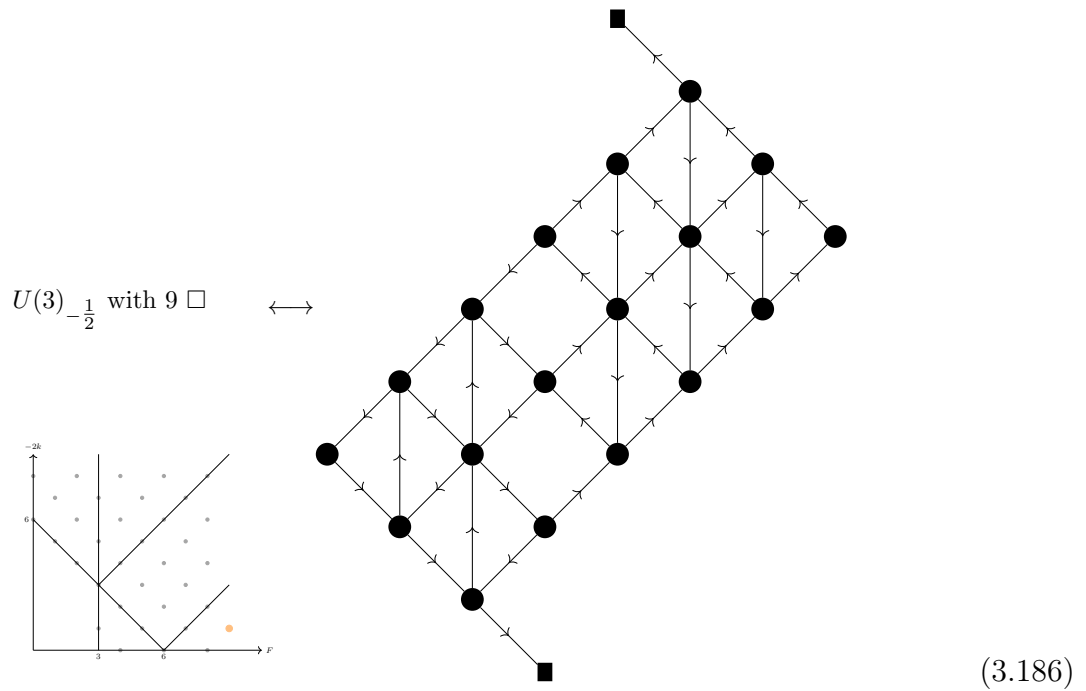
The resulting theory is shown in (3.167). To recast it in the same form as (3.168), we perform a sequence of Aharony dualities on the $U(1)$ gauge nodes, as shown in (3.185).

(3.185)

In a first pass, we apply Aharony duality to all nodes in the first vertical column, and then to those in the second column. In a second pass, we apply Aharony duality only to the left-most gauge node. After this sequence of dualizations, we recover the general proposal in (3.168).

For completeness, we also present the dual of $U(3)_{-1/2}$ with 9 flavors, providing an example of a duality for SQCD in **Zone 1**. From this point onward, we will present

examples using the compact quiver notation summarized in (3.160).



To provide further examples, we now return to the case of $U(3)_{-1/2}$ with 7 flavors and study all theories that can be reached via positive or negative real mass deformations. The resulting dualities are shown in Figures 3.24, 3.25, and 3.26, and exhaustively catalog the theories generated by $\pm m$ real mass deformations.

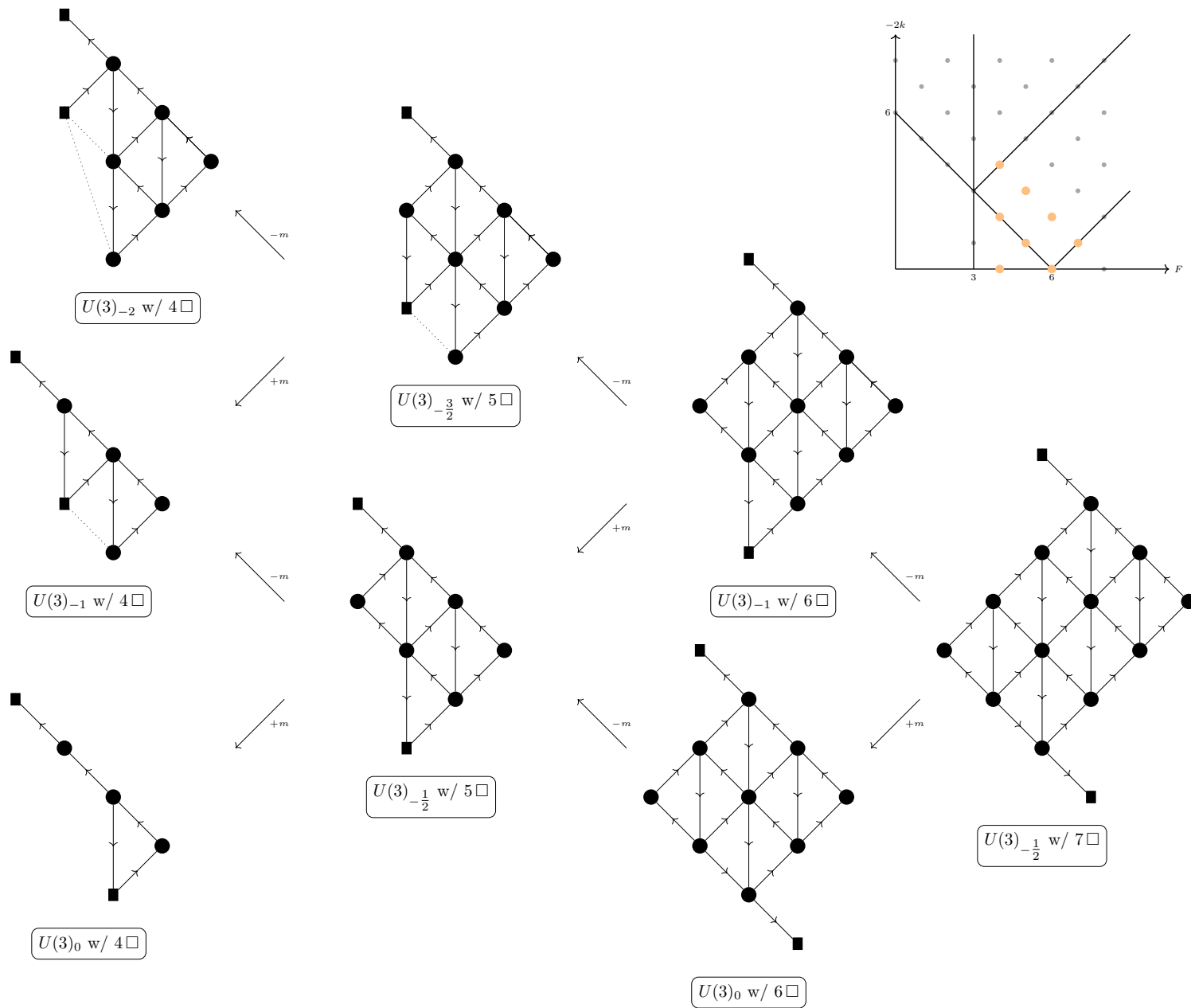


Figure 3.24: The planar mirror dual of $U(3)_{-1/2}$ SQCD with 7 fundamental chiral multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by $\pm m$ real mass deformations. In this figure, we display the first three levels of this deformation sequence. Specifically, we begin with an electric theory containing 7 fundamental chirals and follow the flow to theories with 4 fundamental chirals. We also highlight the location of these theories on the (k, F) -plane in the top-right corner.

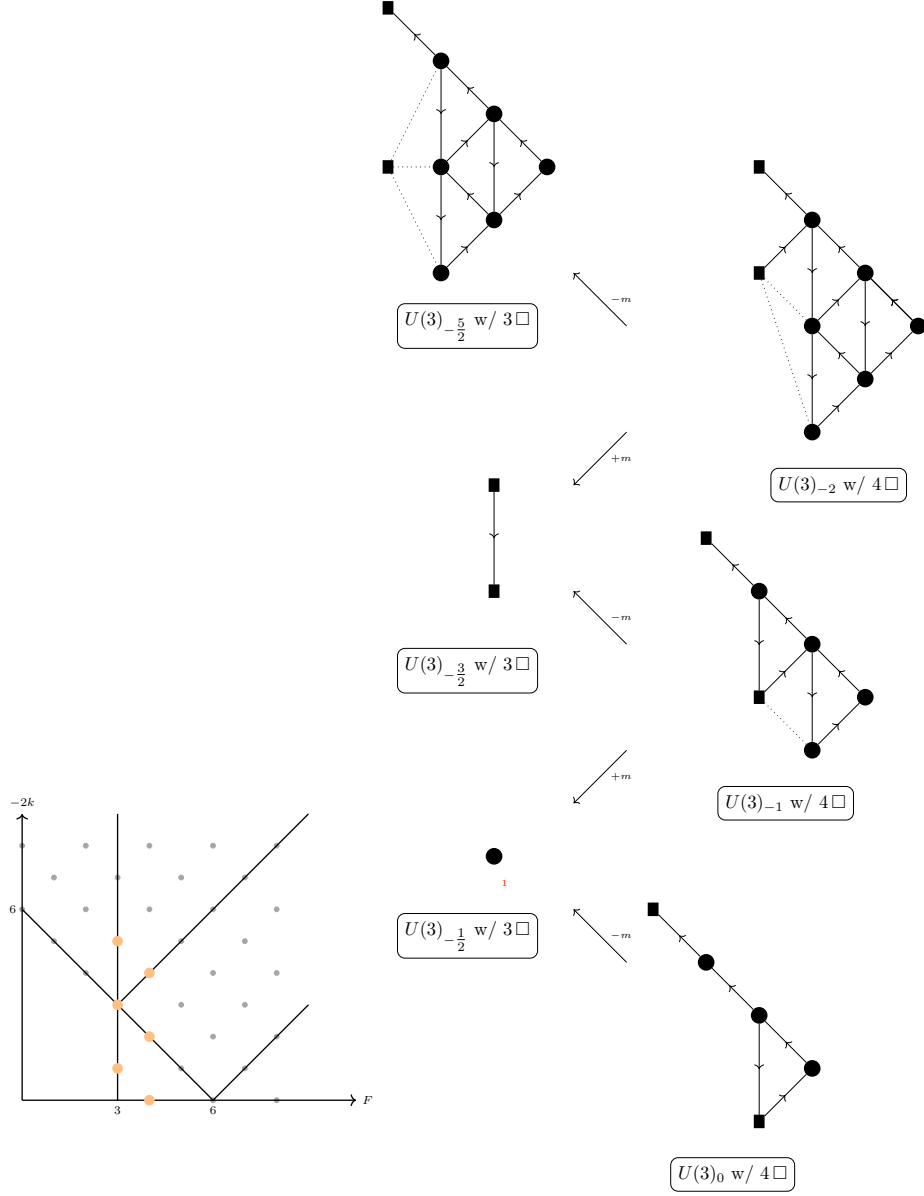


Figure 3.25: We continue analyzing the RG flow illustrated in Figure 3.24. Starting from $U(3)_k$ with 4 fundamental multiplets, we study a single mass deformation down to 3 fundamental multiplets. Note that the dual of $U(3)_{-1/2}$ with 3 flavors is a trivial TQFT, while that of $U(3)_{-3/2}$ with 3 flavors is a free chiral. We highlight the location of these theories in the (k, F) -plane in the bottom-left corner.

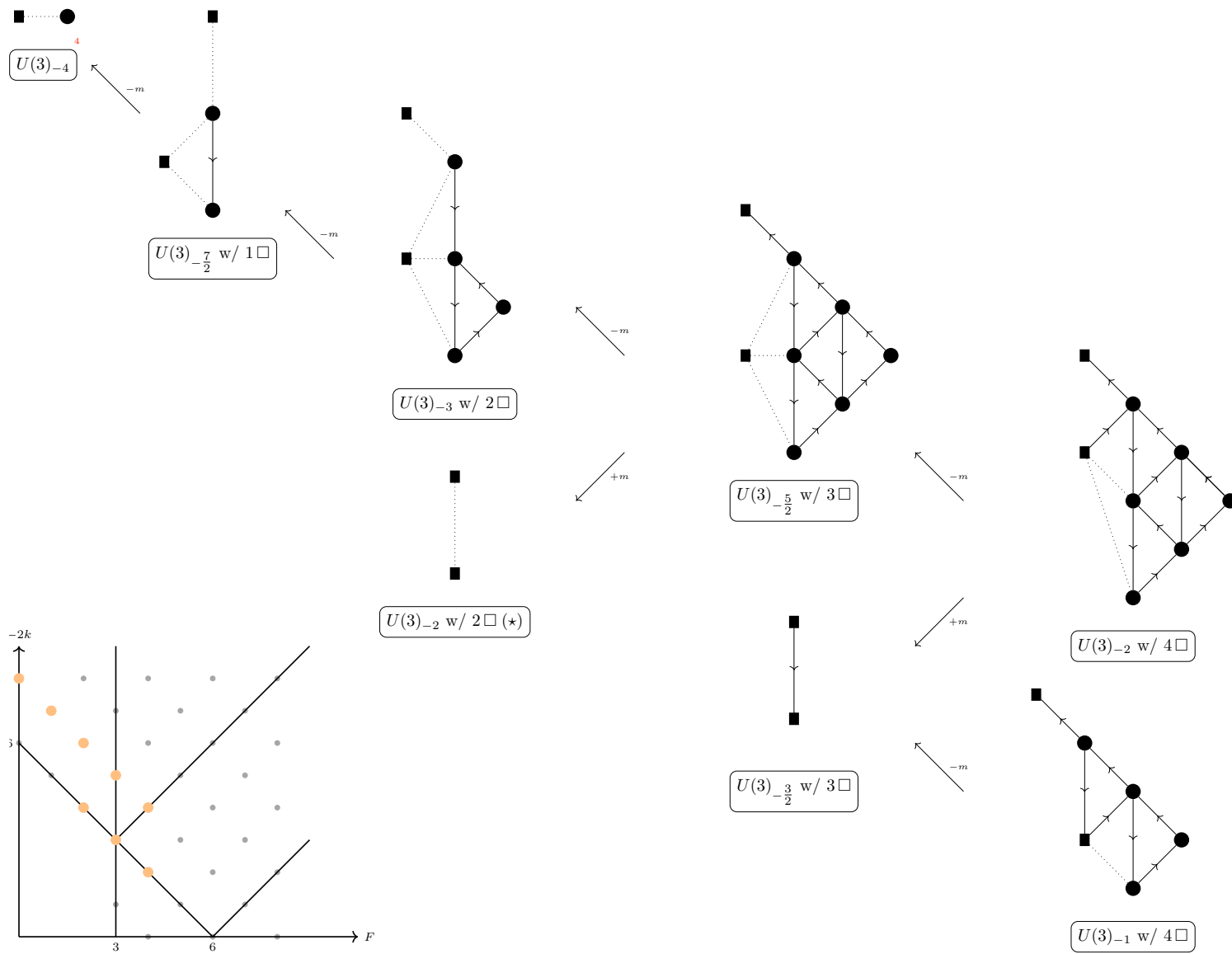


Figure 3.26: We continue the analysis of RG flow trajectories induced by $\pm m$ deformations, extending those considered in Figure 3.25. We focus on the RG flow trajectories of $U(3)_{-5/2}$ and $U(3)_{-3/2}$ SQCD with 3 fundamental chirals and follow these flows until all matter is integrated out. Notice that since the dual of $U(3)_{-3/2}$ with 3 fundamentals is a single free chiral, when we perform a mass deformation, we flow to a trivial theory. The theory with the $(\star)$ label is a gapped theory; in fact, the dual theory is simply a BF coupling. We highlight the location of these theories in the (k, F) -plane in the bottom-left corner.

For completeness, we also present two additional examples. As a representative of **Zone 3**, we display the planar Abelian dual of $U(3)_{-3}$ with 4 fundamental chiral multiplets:

$$\begin{array}{c}
 U(3)_{-3} \text{ with } 4 \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Quiver Diagram} \end{array} \\
 \begin{array}{c} \text{Root System Plot} \end{array}
 \end{array}
 \tag{3.187}$$

The figure shows the duality between a root system plot and a quiver diagram for $U(3)_{-3}$ with 4 fundamental chiral multiplets. The root system plot on the left has axes F and $-2k$, with lines at $F=3$ and $F=6$. A single orange dot is at $(3, 6)$. The quiver diagram on the right is a diamond-shaped grid of 10 black circular nodes connected by solid arrows, with two black square nodes at the top and bottom connected by dotted lines.

As a representative of the dualities along the vertical line $F = 0$, we present the planar Abelian dual of $\mathcal{N} = 2$ $U(3)_{-5}$ SYM.

$$\begin{array}{c}
 U(3)_{-5} \quad \longleftrightarrow \quad \begin{array}{c} \text{Quiver Diagram} \end{array} \\
 \begin{array}{c} \text{Root System Plot} \end{array}
 \end{array}
 \tag{3.188}$$

The figure shows the duality between a root system plot and a quiver diagram for $U(3)_{-5}$ SYM. The root system plot on the left has axes F and $-2k$, with lines at $F=3$ and $F=6$. An orange dot is at $(3, 6)$. The quiver diagram on the right is a diamond-shaped grid of 5 black circular nodes connected by solid arrows, with two black square nodes at the top and bottom connected by dotted lines.

3.5.4 An $\mathcal{N} = 4$ Surprise

From the analysis carried out in this chapter, we uncover that $U(N)_{(-N/2, -3N/2)}$ SQCD with N fundamental chirals exhibits an accidental enhancement to $\mathcal{N} = 4$ supersymmetry. The argument proceeds through a chain of dualities generated by gauging $U(1)$ global symmetries, which we now describe.

We begin with $U(N)_{-N/2}$ SQCD with $F = N$ flavors, which we argued is dual to a single free chiral multiplet, as in (3.171). In this dual frame, the $SU(F)$ global symmetry is unfaithful, as no operator transforms under it; more precisely, it only appears through a background Chern–Simons interaction. The only faithful global symmetry is therefore the $U(1)$ topological symmetry, which acts on the free chiral.

We can then gauge this symmetry, taking into account the background Chern–Simons couplings on both sides of the duality. This leads to

$$SU(N)_{-N/2} \text{ with } N \square \longleftrightarrow U(1)_{1/2} \text{ with one charge } -1 \text{ chiral.} \quad (3.189)$$

On the left-hand side, there is no background Chern–Simons term, so gauging the topological symmetry effectively ungauges the overall $U(1)$ factor.

Starting from this duality, we next gauge the baryonic symmetry on the left-hand side, including an appropriate Chern–Simons level, thereby obtaining a $U(N)_{(-N/2, -3N/2)}$ theory. On the right-hand side, this corresponds to gauging the topological symmetry. Performing this operation carefully yields

$$U(N)_{(-N/2, -3N/2)} \text{ with } N \square \longleftrightarrow \begin{array}{l} U(1) \times U(1) \text{ with one chiral of charge } (-1, 0) \\ \text{and } k = \begin{pmatrix} 1/2 & -1 \\ -1 & -1 \end{pmatrix} \end{array}. \quad (3.190)$$

We can integrate out the $U(1)_{-1}$ topological sector on the right-hand side, which shifts the effective Chern–Simons level of the remaining $U(1)$ gauge field. This leads to

$$U(N)_{(-N/2, -3N/2)} \text{ with } N \square \longleftrightarrow U(1)_{3/2} \text{ with one charge } -1 \text{ chiral.} \quad (3.191)$$

This duality is consistent with a limiting case of Aharony-like dualities (see [180]), and for $N = 1$ it reduces to a trivial identity between theories related by time-reversal.

It was observed in [183] that the theory on the right-hand side exhibits an enhancement to $\mathcal{N} = 4$ supersymmetry, with trivial Higgs and Coulomb branches. By virtue of the duality (3.191), the theory $U(N)_{(-N/2, -3N/2)}$ with N fundamental chirals must therefore also exhibit $\mathcal{N} = 4$ supersymmetry. This enhancement can be further confirmed, for instance, by analyzing the superconformal index using the methods of [184–186].

3.6 Mass Deformations in CS-SQCD with Fundamental Fields

In this section, we study mass deformations in $\mathcal{N} = 2$ $U(N)$ SQCD with CS levels $(k, k + \ell N)$ and F fundamental chiral fields. We introduce a large real mass deformation for one of the fundamentals, implemented through a real mass associated with a $U(1)$ subgroup in the Cartan of the $SU(F)$ flavor symmetry. Our goal is to describe the four mass deformations in (3.154): positive and negative real masses for one of the fundamentals and positive and negative real masses where the gauge group is also Higgsed from $U(N)$ to $U(N - 1)$.

We parameterize the real masses associated to the Cartan of $SU(F)$ as X_i , $i = 1, \dots, F$ subject to the following constraint:

$$\sum_{i=1}^F X_i = 0 \quad (3.192)$$

In the presence of these real mass deformations as well as an FI term η , the equations of motion are [24]:

$$(\sigma_\alpha - X_i)Q_\alpha^i = 0, \quad \alpha = 1, \dots, N, \quad i = 1, \dots, F \quad (3.193)$$

$$\sum_{i=1}^F Q_\alpha^{i\dagger} Q_i^\beta = \frac{\delta_{\alpha\beta}}{2\pi} \left[-\eta + k\sigma_\alpha + \ell \sum_{\gamma=1}^N \sigma_\gamma + \frac{1}{2} \sum_{i=1}^F |\sigma_\alpha - X_i| \right], \quad \alpha, \beta = 1, \dots, N \quad (3.194)$$

where Q_α^i are the quark VEVs, and we use gauge freedom to rotate the VEVs for the scalar σ in the $\mathcal{N} = 2$ vector multiplet to a diagonal matrix.

Mass Flows without Higgsing: We first turn on a real mass deformation parametrized by the shift:

$$\begin{cases} X_1 \rightarrow X_1 \mp (F-1)m, \\ X_i \rightarrow X_i \pm m, \\ \eta \rightarrow \eta + \left(\frac{F}{2} \pm (k + \ell N)\right)m. \end{cases} \quad 2 \leq i \leq F, \quad (3.195)$$

for large m . Since the quarks transform in the antifundamental representation of the $SU(F)$ flavor group, these shifts correspond to a positive mass (upper sign) and a negative mass (lower sign) for the first quark, respectively. We consider a vacuum wherein the gauge group is not Higgsed and the last $F-1$ quarks remain massless in the limit $m \rightarrow \infty$. This is achieved by giving a large VEV to the real scalars σ in the $\mathcal{N} = 2$ vector multiplet:

$$\sigma \rightarrow \pm \text{diag}(m, m, \dots, m) + \sigma. \quad (3.196)$$

The effective mass for the quarks Q_α^i with $i \geq 2$, given by $(\sigma_\alpha - X_i)$ vanish at leading order in m , while the effective mass for the first quark Q_α^1 is:

$$m_1 = \pm mF + \mathcal{O}(m^0) \quad (3.197)$$

rendering this quark massive in the $m \rightarrow \infty$ limit and, hence, can be integrate it out. Up to $\mathcal{O}(m^0)$ terms, the equations of motion (3.193) reduce to the equations of motion for $U(N)$ with $(F-1)$ fundamentals and CS levels $k \pm \frac{1}{2}, k \pm \frac{1}{2} + \ell N$. The real mass deformations (3.195), together with the choice of vacuum (3.196), produce the following RG flows:

$$\begin{array}{ccc} U(N)_{k-\frac{1}{2}, k-\frac{1}{2}+\ell N} & \text{with } F-1 \square & \\ & \swarrow -m & \\ & U(N)_{k, k+\ell N} & \text{with } F \square \\ & \searrow +m & \\ U(N)_{k+\frac{1}{2}, k+\frac{1}{2}+\ell N} & \text{with } F-1 \square & \end{array} \quad (3.198)$$

In the rest of this thesis, we will compactly denote these RG flows as “ $+m$ ” and “ $-m$ ”, respectively.

Mass Flows with Higgsing: We now consider real mass deformations that simultaneously give mass to one quark and Higgs the gauge group to $U(N-1)$. We consider the following shifts of real mass parameters for large m_H :

$$\begin{cases} X_1 \rightarrow X_1 \mp (F-1)m_H, \\ X_i \rightarrow X_i \pm m_H, \\ \eta \rightarrow \eta + \left(\frac{F}{2} \pm (k + \ell N - \ell F)\right) m_H. \end{cases} \quad 2 \leq i \leq F \quad (3.199)$$

and we combine it with a large VEV for the real scalars σ :

$$\sigma \rightarrow \pm \begin{pmatrix} -(F-1) & 0 \\ 0 & \mathbf{1}_{N-1 \times N-1} \end{pmatrix} m_H + \sigma \quad (3.200)$$

which Higgses the gauge group to $U(N-1) \times U(1)$. The fundamentals Q_α^i with $\alpha \geq 2$ and $i \geq 2$ remain massless and form $F-1$ fundamentals for the $U(N-1)$ factor, while the component Q_1^1 is massless as well and carries a charge +1 under the $U(1)$ factor. The other quarks are massive and are integrated out. The D-terms for the $U(N-1)$ sector are satisfied up to $\mathcal{O}(m_H^0)$ order and reduce to the equations for $U(N-1)$ with $F-1$ fundamentals. The D-term for the $U(1)$ gauge factor, namely (3.194) with $\alpha = \beta = 1$, read:

$$\begin{aligned} 2\pi|Q_1^1|^2 &= -\eta - \left(\frac{F}{2} \pm (k + \ell N - \ell F)\right) m_H + k\sigma_1 \mp k(F-1)m_H \\ &\quad + \ell \sum_{\gamma=1}^N \sigma_\gamma \pm \ell(-F+N)m_H + \frac{1}{2} \sum_{i=2}^F |\sigma_1 - X_i \pm (-F)m_H| + \frac{1}{2} |\sigma_1 - X_1| \\ &= \left(\frac{1}{2}F(F-1) - \frac{1}{2}F \pm (-k - \ell N + \ell F - k(F-1) + \ell(N-F))\right) m_H + \mathcal{O}(m_H^0) \\ &= \left(\frac{1}{2}F(F-2) \mp kF\right) m_H + \mathcal{O}(m_H^0) \end{aligned} \quad (3.201)$$

This can be solved by giving a non-zero VEV to Q_1^1 such that²⁶:

$$|Q_1^1|^2 = \frac{m_H}{2\pi} \left(\frac{1}{2}F(F-2) \mp kF\right) \geq 0 \quad (3.202)$$

Positivity of this expression determines the range of validity for the two possible deformations:

$$\begin{aligned} m_H : \quad & F \geq 2(k+1), \\ -m_H : \quad & F \geq 2(-k+1). \end{aligned} \quad (3.203)$$

With a slight abuse of notation, we can depict the movement in the parameter space induced by the Higgsing deformations as illustrated in Figure 3.27. Indeed, since the rank N changes, the deformation actually moves across two planes with different fixed N .

²⁶When $\frac{F}{2} - 1 \mp k = 0$ the D-terms are satisfied with vanishing VEV for the chirals and Q_1^1 remains a massless d.o.f. charged under the $U(1)$ factor of the gauge group. This is the case when the SQCD obtained after the mass deformation is marginally chiral.

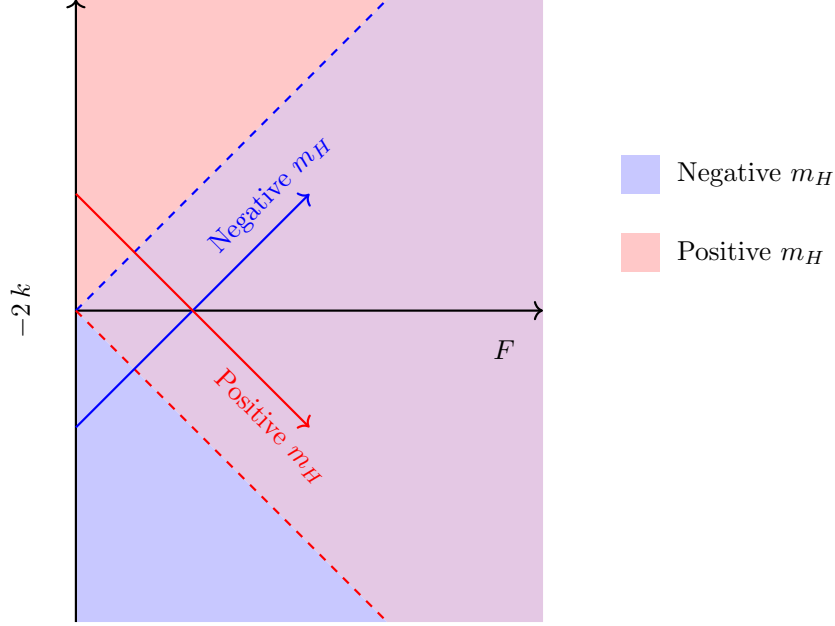


Figure 3.27: RG-flow trajectories induced by real mass deformations accompanied with Higgsing in the (F, k) -plane.

When the deformation is allowed, the $U(1)$ factor is fully Higgsed, and the resulting theory is $U(N-1)$ SQCD with $F-1$ fundamentals and CS levels $(k \pm \frac{1}{2}, k \pm \frac{1}{2} + \ell(N-1))$. Together with the previous non-Higgsing deformations, we obtain four distinct flows from $U(N)$ SQCD with F fundamentals to theories with $F-1$ fundamentals:

$$\begin{array}{ccc}
 U(N)_{k-\frac{1}{2}, k-\frac{1}{2}+N\ell} \text{ with } F-1 \square & & U(N-1)_{k-\frac{1}{2}, k-\frac{1}{2}+(N-1)\ell} \text{ with } F-1 \square \\
 \swarrow -m & & \nearrow -m_H \\
 & U(N)_{k, k+N\ell} \text{ with } F \square & \\
 \swarrow +m & & \searrow +m_H \\
 U(N)_{k+\frac{1}{2}, k+\frac{1}{2}+N\ell} \text{ with } F-1 \square & & U(N-1)_{k+\frac{1}{2}, k+\frac{1}{2}+(N-1)\ell} \text{ with } F-1 \square
 \end{array} \tag{3.204}$$

3.6.1 Mapping Mass Flows across Aharony-like Dualities

We remind the reader that SQCD with $U(N)$ gauge group and CS levels $k, k + \ell N$ and F fundamentals is mapped by Aharony-like duality to $U(\tilde{N})$ SQCD with CS levels $-k, -k + \ell \tilde{N}$ and F antifundamentals [93, 128] (see also [35, 173, 180] for a comprehensive review).

For simplicity, we focus on the case $\ell = 0$, corresponding to vanishing CS level for the diagonal $U(1)$ group. The general case with $\ell \neq 0$ can be obtained by applying Witten's

$SL(2, \mathbb{Z})$ action on both sides of the duality [144].

The dual rank $\tilde{N}$ depends on the relative values of F and k , dividing the parameter space into three regimes — minimally chiral, marginally chiral, and maximally chiral — characterized as follows:

$$\begin{aligned} \text{minimally chiral: } F < |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N \\ \text{marginally chiral: } F = |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N = F - N \\ \text{maximally chiral: } F > |2k|, & \quad \tilde{N} = F - N \end{aligned} \quad (3.205)$$

In the marginally chiral case there is also an additional gauge singlet in the magnetic theory that flips a monopole:

$$\mathcal{W}_{mag}^{\frac{F}{2}=|k|} = T\mathfrak{M}^{\mp} \quad (3.206)$$

where the upper (lower) sign correspond to $k > 0$ ($k < 0$).

One can flow between different Aharony-like dualities by consistently turning on real masses on both sides of the duality. We outline the mapping of mass deformations $\pm m$ and $\pm m_H$ across Aharony-like duality.

In the *maximally chiral case*, all four deformations in (3.204) are available, as can be checked from Figure 3.27. Aharony duality maps positive real masses to negative real masses. Consistency with the value of $\tilde{N}$ implies that mass deformations are mapped as follows:

$$\left\{ \begin{array}{l} +m \\ -m \\ +m_H \\ -m_H \end{array} \right\} \xleftrightarrow[\text{maximally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -\widetilde{m_H} \\ +\widetilde{m_H} \\ -\widetilde{m} \\ +\widetilde{m} \end{array} \right\} \quad (3.207)$$

where $\pm\widetilde{m}$ and $\pm\widetilde{m_H}$ denote mass deformations with or without Higgsing for $U(N)$ SQCD with only antifundamentals, analogous to the deformations discussed above for SQCD with only fundamentals.

In the *minimally* and *marginally chiral* cases, only three of the four deformations are available. The two non-Higgsing deformations, $+m$ and $-m$, are always accessible, whereas only one of the two Higgsing deformations ($\pm m_H$) is allowed, depending on the sign of the Chern–Simons level k . For concreteness, we set $k < 0$, where the electric theory admits the m_H deformation. The Aharony dual theory has CS level $-k > 0$ and one can check that the deformation $-\widetilde{m_H}$ is available. The mapping of mass deformations is given by:

$$k < 0 : \quad \left\{ \begin{array}{l} +m \\ -m \\ +m_H \end{array} \right\} \xleftrightarrow[\text{marginally or minimally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -\widetilde{m_H} \\ +\widetilde{m} \\ -\widetilde{m} \end{array} \right\} \quad (3.208)$$

similarly for $k > 0$:

$$k > 0 : \quad \left\{ \begin{array}{l} +m \\ -m \\ -m_H \end{array} \right\} \xleftrightarrow[\text{marginally or minimally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -m \\ -\widetilde{m_H} \\ +m \end{array} \right\} \quad (3.209)$$

3.7 Mapping Mass Flows to the Planar Mirror Duals

In this section, we analyze how real-mass deformations of $\mathcal{N} = 2$ SQCD are realized on their planar Abelian duals. We do this by comparing the equations of motion on the electric and mirror sides in the large-mass limit. We then focus on a representative example to elaborate on some features of our proposal.

3.7.1 Deformations of the $\mathcal{N} = 4$ -descendant SQCD

As a starting point, we consider the planar Abelian dual of $U(N)_{N-\frac{F}{2}}$ SQCD with F fundamental fields, shown in Figure 3.168 (see Section 3.5.2). We remind the reader that this theory corresponds to the $\mathcal{N} = 4$ -descendant line in Figure 3.23.

These theories lie in the maximally chiral regime, where all four deformations $\pm m$ and $\pm m_H$ are available. Across the duality, a real mass for a quark in SQCD maps to an FI term for the gauge nodes contained in a single column of the planar Abelian quiver, while the FI parameter of the SQCD maps to a real mass for one of the two flavor nodes of the Abelian theory.

Consequently, giving a real mass to the i -th quark corresponds to turning on a large FI parameter for one (or two) adjacent columns of the Abelian quiver. Although different choices of i are equivalent at the level of the IR-SCFT—being related by Weyl transformations of the $SU(F)$ flavor symmetry—this symmetry is not manifest in the planar quiver representation. As a result, the UV analysis of the deformation depends on the specific choice of i , even though the corresponding IR deformations describe the same flow.

For concreteness, we adopt the following conventions for the four possible deformations²⁷:

$$\begin{aligned} +m, +m_H &: \text{large mass for the last quark } Q^F \leftrightarrow X_F \rightarrow -\infty \\ -m_H, -m &: \text{large mass for the first quark } Q^1 \leftrightarrow X_1 \rightarrow +\infty \end{aligned} \tag{3.210}$$

Where the limits indicate that for the deformations $+m, +m_H$, we have to correspondingly take, at the level of the $\mathbf{S}_b^3$ partition function, a large FI limit for the rightmost node in the quiver, while the deformations $-m, -m_H$ correspond to a large FI limit for the leftmost node.

²⁷Notice that in (3.195) and (3.199) all four deformations were assigned to the first quark Q^1 . Here, for the positive-mass deformations, we use the Weyl-equivalent choice acting on the last quark Q^F , which is convenient because the corresponding large-FI limit then acts on the rightmost column of the planar quiver.

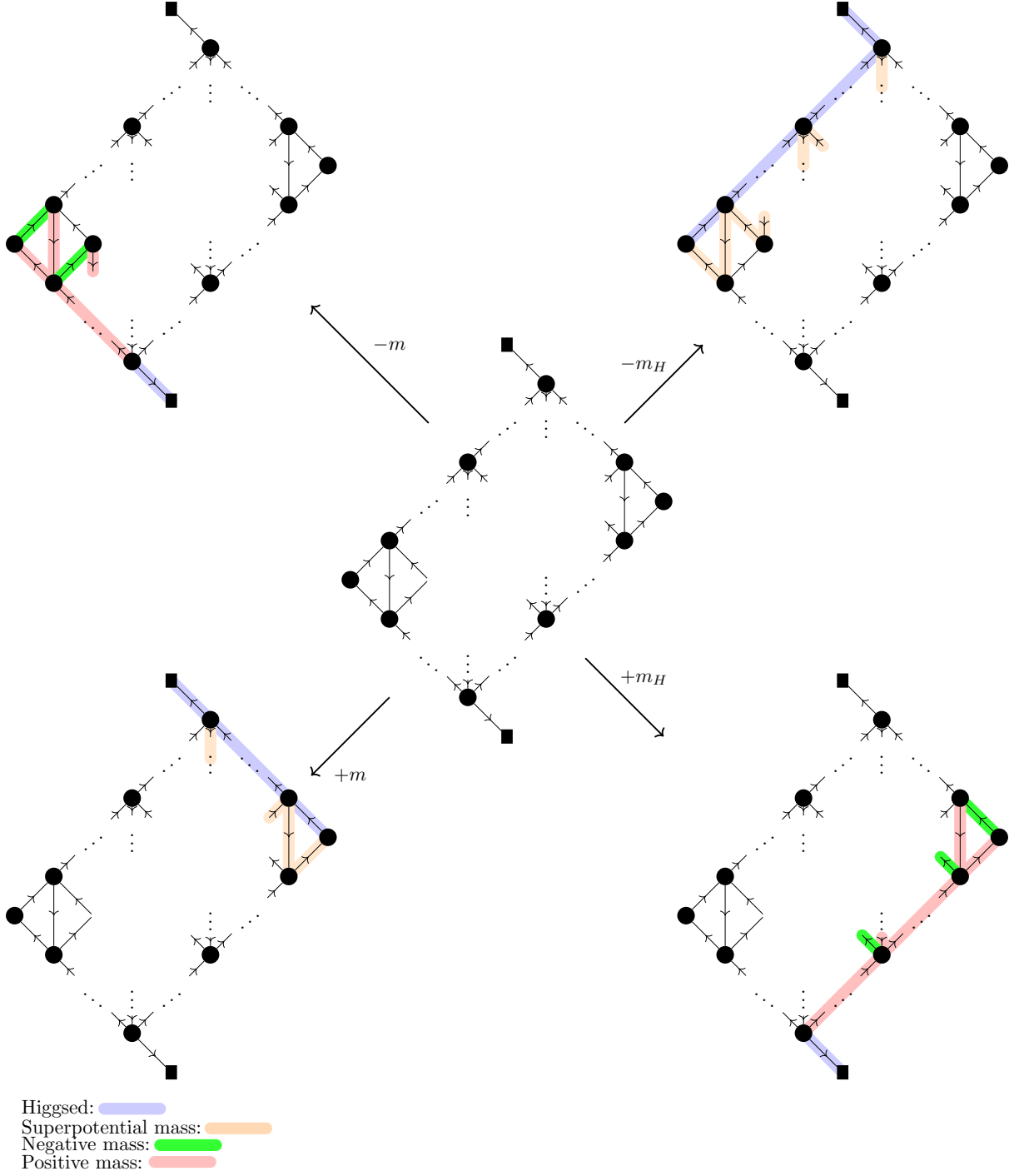


Figure 3.28: Deformations of the planar Abelian quiver dual to $U(N)_{(N-\frac{F}{2})}$ SQCD with F fundamentals. Chiral fields highlighted in pink acquire a large positive mass, fields highlighted in green acquire a large negative mass, and fields highlighted in blue acquire a VEV, thereby Higgsing the nodes that they connect to a diagonal $U(1)$. After the Higgsing the fields in orange acquire a complex superpotential mass. The quivers resulting after integrating out the massive fields and performing the Higgsing are shown in Figure 3.29.

We propose the following vacua for the four mass deformations:

$+m$: In this deformation, the chiral fields along the top-right diagonal and the chiral

field attached to the topmost gauge node acquire non-zero VEVs. These fields are highlighted in blue in the bottom-left panel of Figure 3.28. The Higgsing identifies all gauge nodes along the top-right diagonal with the adjacent flavor node. Consequently, the planar superpotential reduces to a set of mass terms, allowing one to integrate out the $2(N - 1)$ chiral fields highlighted in orange in Figure 3.28, following the same procedure outlined in (3.164).

- m : In this deformation, the real scalars in the vector multiplets associated with the gauge groups along the bottom-left diagonal acquire large VEVs:

$$(\sigma_1^{bottom}, \sigma_2^{bottom}, \dots, \sigma_{N-1}^{bottom}) \propto -((N - 1)Fm, (N - 2)Fm, \dots, Fm) \quad (3.211)$$

where σ_i^{bottom} is the real scalar of the gauge group associated to the bottom-most node in the i -th column. As a consequence, the chiral multiplets connecting these nodes acquire large positive real masses. The chirals connecting these nodes to the rest of the quiver acquire a large real mass as well, which is positive for the vertical arrows and negative for diagonal arrows. These fields are highlighted in pink and green, respectively, in the top-left panel of Figure 3.28. Moreover, the chiral field attached to the bottom flavor node is Higgsed.

Integrating out the massive fields yields the quiver shown below:

$$U(N)_{N-\frac{F}{2}-\frac{1}{2}} \text{ with } F-1 \text{ } \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Quiver Diagram} \\ \text{N nodes} \quad \quad \quad F-N \text{ nodes} \end{array} \quad (3.212)$$

This intermediate quiver contains a tail of $U(1)$ gauge nodes that interact with the rest of the quiver only through mixed CS couplings. Such a tail can be systematically contracted by sequentially integrating out the last node, which corresponds to an almost trivial $U(1)_1$ Chern–Simons theory.

We remind the reader that pure $U(1)_{\pm 1}$ Yang–Mills theory is almost trivial and its path integral can be performed exactly [141, 144]:

$$\begin{array}{c} \lambda_1 \\ \textcircled{1} \\ \pm 1 \end{array} \text{---}^{k_{12}} \text{---} \begin{array}{c} \lambda_2 \\ \textcircled{1} \\ k_2 \end{array} \cdots = \begin{array}{c} \lambda_2 \mp \frac{\lambda_1 k_{12}}{2} \\ \textcircled{1} \\ k_2 \mp \frac{k_{12}^2}{4} \end{array} \cdots \times e^{\pm i\pi \lambda_1^2} \quad (3.213)$$

In the context of Equation 3.212, the contraction of the tail of $U(1)$ gauge nodes proceeds as follows:

$$= \begin{array}{c} \vdots \\ \text{diagram 1} \\ \vdots \\ \text{diagram 2} \\ \vdots \\ \text{diagram 3} \end{array} \quad (3.214)$$

The resulting duality is then represented schematically as:

$$U(N)_{N-\frac{F}{2}-\frac{1}{2}} \text{ with } F-1 \text{ } \square \longleftrightarrow \text{Diagram} \quad (3.215)$$

+ m_H : In this deformation, the real scalars in the vector multiplets of the gauge groups in the right-bottom-most diagonal acquire non-zero VEVs:

$$(\sigma_{F-1}^{bottom}, \sigma_{F-2}^{bottom}, \dots, \sigma_{N+1}^{bottom}) \propto -((F-N-1)Fm, (F-N-2)Fm, \dots, m). \quad (3.216)$$

As a consequence, the chiral multiplets connecting these nodes acquire large positive real masses. The chirals connecting these nodes to the rest of the quiver acquire a large real mass as well, which is positive for the vertical arrows and negative for diagonal arrows. These fields are highlighted in pink and green, respectively, on the bottom-right panel of Figure 3.28. Furthermore, the chiral field connected to the bottom flavor node is Higgsed. Integrating out the massive fields and contracting

the resulting tail, analogous to the $+m$ case, results in the following duality:

$$U(N-1)_{N-\frac{F}{2}+\frac{1}{2}} \text{ with } F-1 \square \longleftrightarrow \text{Quiver Diagram} \quad (3.217)$$

$-m_H$: In this deformation, the chiral fields in the top-left diagonal and the chiral field connected to the top-most gauge node acquire a VEV. These fields are highlighted in blue in the top-right panel of Figure 3.28. As a result, all the gauge nodes in the top-right diagonal are identified with the flavor node. The planar superpotential reduces to mass terms which we use to integrate out $2(N-1)$ chiral fields, analogously to (3.164).

We summarize these mass flows in Figure 3.29.

The proposal of these deformation arises by comparing the large mass asymptotic behavior of the $\mathbf{S}_b^3$ partition functions of the SQCD and of its planar dual, using the strategies of [25, 128].

One may notice also that, after performing either a $+m$ or a $-m_H$ deformation, the resulting quiver retains the same rectangular structure as the original planar Abelian dual. In fact, the resulting theory is precisely the planar Abelian dual of the SQCD model obtained after the corresponding mass deformation on the electric side. These two deformations therefore provide a nontrivial self-consistency check of the duality originally proposed in [1, 2].

In contrast, the deformations $-m$ and $+m_H$ yield quivers with a modified geometry, leading to different planar structures. Successive applications of these deformations require a separate analysis, which we undertake in the next section, where we construct and study the planar Abelian duals of SQCD theories obtained through multiple $-m$ or $+m_H$ mass deformations.

3.7.2 Exploring Zone 1 via Multiple $+m_H$ Mass Deformations

In the previous section, we discussed the mapping of a single positive mass deformation accompanied with Higgsing on the planar Abelian dual of SQCD on the $\mathcal{N} = 4$ -descendant

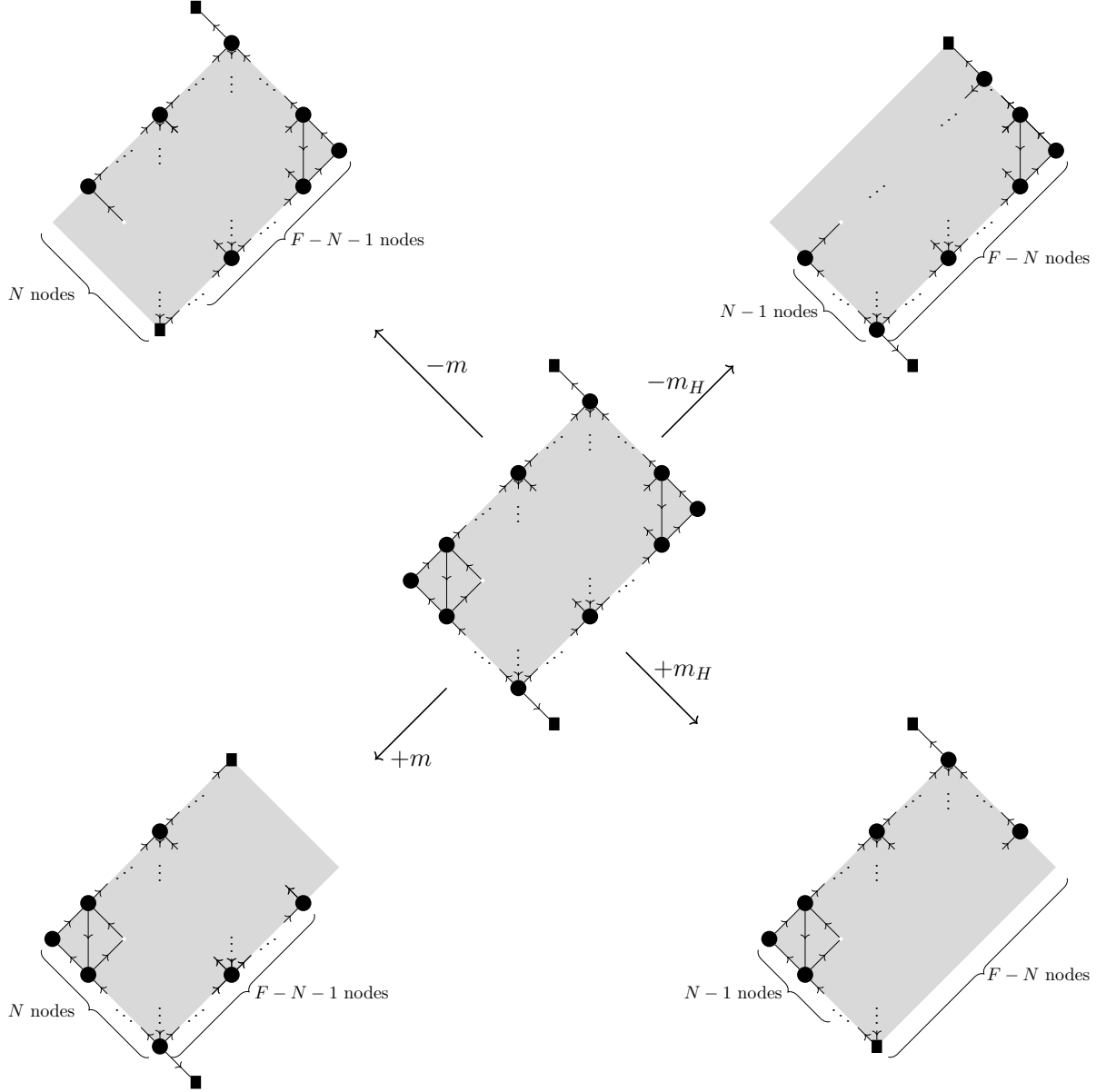


Figure 3.29: Deformations of the planar quivers dual to the SQCD mass deformations in (3.204). The grey shading highlights which parts of the quiver are modified by each mass deformation and makes explicit how the overall quiver geometry changes relative to the initial configuration.

line. This resulted in a duality for SQCD theories with F fundamentals and $k = N - \frac{F}{2} + 1$ CS level, which lie in Zone 1 in the k/F plane (see Figure 3.23), just below the $\mathcal{N} = 4$ descendant (the orange line in Figure 3.23).

To explore theories in Zone 1 more systematically, we must map the effect of adding multiple positive masses accompanied with Higgsing ($+m_H$) on the planar theory.

We perform the $+m_H$ deformation associated with the F -th quark, which is implemented by the real mass deformations in (3.199). We then proceed with a second $+m_H$ deformation, which we find to result in the following.

(3.218)

The corresponding deformation on the magnetic side is the following:

- The vertical bifundamental chiral connected to the bottom-most flavor node acquires a VEV, Higgsing the gauge node under which it is charged.
- The scalars in the vector multiplets of the gauge node in the bottom-right diagonal acquire a VEV:

$$(\sigma_{F-1}^{bottom}, \sigma_{F-2}^{bottom}, \dots, \sigma_{N+1}^{bottom}) \propto -((F - N - 1)Fm, (F - N - 2)m, \dots, m) \quad (3.219)$$

This is a proposal that can be checked on the $\mathbf{S}_b^3$ partition function. Similar to the first mass deformation $+m_H$ considered in Section 3.7.1, the gauge groups in the bottom-right diagonal become a tail of CS theories that only interact via mixed CS levels after integrating out the massive fields. This tail can be sequentially contracted as in (3.214),

resulting in the following duality:

$$U(N-1)_{N-\frac{F}{2}+\frac{3}{2}} \text{ with } F-1 \square \longleftrightarrow \text{Quiver Diagram} \quad (3.220)$$

Further positive mass deformations accompanied by Higgsing are mapped in an analogous manner. In this case, the real scalars associated with the gauge nodes along the bottom-right diagonal acquire non-zero vacuum expectation values, which in turn generate large masses for the fields charged under them. Simultaneously, the vertical chiral multiplet attached to the bottom flavor node develops a Higgs VEV. Performing this sequence of deformations yields the planar Abelian dual quiver corresponding to a generic SQCD theory in Zone 1:

$$U(N)_k \text{ with } F \square \longleftrightarrow \text{Quiver Diagram} \quad (3.221)$$

The quiver on the r.h.s. of (3.221) can be systematically simplified by applying local Aharony-like dualities, similar to the analysis of Appendix B of [2]. Specifically, we apply Aharony duality to all the gauge nodes in a vertical column, proceeding column by column, from left to right and halting at the $(\frac{F}{2} - |k|)$ -th column. The gauge nodes within this column that lie outside the rectangle of dimensions $N \times (F - N)$ confine. We then repeat the process, restarting from the left and stopping one column earlier each

time, until only the leftmost gauge node is dualized. At every iteration, the nodes of the final dualized column that fall outside the $N \times (F - N)$ rectangle again confine.

After performing this complete sequence of local dualizations, we obtain the duality describing SQCD theories with $F > 2N + 2|k|$, namely those lying in Zone 1 of Figure 3.23:

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \quad \longleftrightarrow \\
 & (F \geq 2|k| + 2N)
 \end{aligned}
 \quad
 \begin{aligned}
 & \text{Quiver diagram with } F - N \text{ nodes (top section)} \\
 & \text{Quiver diagram with } N \text{ nodes (bottom left section)} \\
 & \text{Quiver diagram with } \frac{F}{2} - |k| + 1 \text{ nodes (bottom right section)}
 \end{aligned}
 \tag{3.222}$$

The resulting quiver coincides with the dual obtained in Section 3.5.2. Consequently, the duality proposed in Section 3.5.2 for $U(N)_0$ SQCD can be viewed as a descendant of the duality for SQCD on the $\mathcal{N} = 4$ line originally discussed in [1, 2].

3.7.3 Exploring Zones $\tilde{1}, 2, 3$ and 4 via Multiple $-m$ Mass Deformations

In Section 3.5.2, we briefly summarized the effect of negative mass deformations on the $U(N)_0$ SQCD theory with F fundamental fields and studied the corresponding RG flow in the electric and mirror descriptions. We now provide a more systematic justification of these claims. All these deformations are studied from the point of view of the $\mathbf{S}_b^3$ partition function.

Zone 2: The resulting duality for a generic SQCD theory in Zone 2 is shown in (3.169).

At each step, the $-m$ mass deformation is implemented by giving vacuum expectation values to the real scalars in the vector multiplets associated with the gauge nodes along

the bottom-left diagonal:

$$\left(\sigma_1^{\text{bottom}}, \sigma_2^{\text{bottom}}, \dots, \sigma_{\frac{F}{2}-|k|-1}^{\text{bottom}}\right) \propto \left(\left(\frac{F}{2}-|k|-1\right)Fm, \left(\frac{F}{2}-|k|-2\right)Fm, \dots, Fm\right). \quad (3.223)$$

In addition, the vertical chiral multiplet connected to the bottom flavor node acquires a Higgs VEV.

Starting from the $\mathcal{N} = 4$ descendant line, this mapping remains valid for the first N successive negative real mass deformations. This sequence of deformations allows us to fully explore the parameter regions corresponding to Zone 2 in the k/F -plane.

Zone 1: The resulting duality for a generic SQCD theory in Zone 1 is shown in (3.173).

This region of parameter space can be accessed through a sequence of negative real mass deformations starting from **Zone 2**. Similarly to Section 3.7.1, the $+m$ deformations shrink the quiver along the SW-NE directions.

Zone 3: The resulting dualities for a generic SQCD theory in Zone 3 are shown in (3.175).

Starting from **Zone 2**, we perform further $-m$ deformations, each corresponding to a VEV to the real scalars in the $(|k| - \frac{F}{2})$ -th column—namely, the leftmost column still connected to the rest of the quiver by diagonal chiral fields:

$$\sigma_{|k|-\frac{F}{2}}^i = \sigma_{|k|-\frac{F}{2}}^j \quad \forall i, j \quad (3.224)$$

$$\sigma_{|k|-\frac{F}{2}}^i - \sigma_{|k|-\frac{F}{2}+1}^i \propto Fm \quad (3.225)$$

The vertical chirals in the $(|k| - \frac{F}{2})$ -th column remain massless, whereas the diagonal chirals connecting this column to the next one become massive. In particular, the diagonals oriented toward the right acquire positive masses, while those pointing left acquire negative masses, as shown in (3.174). This mapping remains valid as long as the quiver contains columns of diagonal chiral fields, which corresponds to the condition $F \geq N$ (3.175).

Zone 4: In Zone 4, corresponding to $F < N$ (3.180), the negative mass deformation $-m$ instead corresponds to giving a VEV to the vertical chiral at the top of the F^{th} column from the right, together with VEVs for the real scalars in the top-right diagonal:

$$(\sigma_1^{\text{top}}, \dots, \sigma_{F-1}^{\text{top}}) \propto ((F-1)Fm, (F-2)Fm, \dots, Fm) \quad (3.226)$$

where σ_i^{top} is the real scalar associated to the top node of the i -th column from the right.

3.8 Relating Planar Abelian Duals to Aharony Duals

The $\mathcal{N} = 2$ $U(N)_k$ SQCD theory enjoys an Aharony-like duality for arbitrary parameters (N, k, F) (see [180] and references therein). In this section, we focus on the case with only fundamental matter fields, whose dual description involves only anti-fundamental fields. As noted above, the structure of these dualities changes qualitatively as one moves in the (k, F) plane at fixed N (see Figure 3.156).

A natural question is how the planar Abelian dualities proposed in Section 3.5 relate to Aharony duality. Here, we argue that, assuming the validity of the planar Abelian dualities together with certain special instances of Aharony-like duality, specifically those involving Abelian gauge groups in both the electric and magnetic phases, one can derive the general Aharony-like dualities for $U(N)_k$ SQCD with F fundamental chiral fields. Consequently, **Aharony duality for SQCD with fundamentals can be interpreted as a consequence of mirror symmetry.**

Concretely, we consider SQCD in Zone $\tilde{1}$ with $N < F < 2N$ fundamentals and Chern–Simons level k , satisfying $2|k| < 2N - F$, corresponding to a maximally chiral regime [128]. Aharony duality for SQCD theories sitting in the other Zones can be derived from the Zone $\tilde{1}$ /Zone 1 duality via real mass deformations, therefore we do not lose generality by focusing on this case. The Aharony-dual theory is $U(F - N)$ SQCD with F anti-fundamentals and Chern–Simons level $-k$, which lies in (the anti-fundamental analogue of) Zone 1:

$$U(N)_k \text{ with } F < 2N \quad \longleftrightarrow \quad U(F - N)_{-k} \text{ with } F > 2(F - N) \quad (3.227)$$

The topological symmetry of the magnetic theory, $U(1)_{\eta_D}$, is mapped to a linear combination of the R-symmetry and minus the topological symmetry of the electric theory, $U(1)_\eta$. Denoting by Q_{η_D} , R and Q_η the corresponding generators, we have

$$Q_{\eta_D} = k R - Q_\eta. \quad (3.228)$$

The representations of the fields under the $SU(F) \times U(1)_\eta \times U(1)_R$ symmetries are:

	$U(N)$	$U(F - N)$	$SU(F)$	$U(1)_R$	$U(1)_\eta$
Q	$\square$		$\bar{\square}$	r	0
$\tilde{q}$		$\bar{\square}$	$\square$	$1 - r$	0

(3.229)

Here, Q denotes the fundamental chiral fields of the electric theory, and $\tilde{q}$ denotes the anti-fundamental chiral fields of the magnetic theory. No additional gauge-singlet fields are involved in this duality. The only operators charged under $U(1)_\eta$ are the monopole operators in the respective phases.

The corresponding identity between partition functions is encoded in the duality relation (3.227), which also encodes shifts for background CS terms for the global symmetries involved in the duality.

The planar Abelian dual of the electric theory was presented in Section 3.5.2, while the planar Abelian dual of the magnetic theory can be obtained by applying parity and charge conjugation to the dualities discussed in this thesis. Taken together, this produces four UV Lagrangians that are dual to each other: the two SQCD theories related by Aharony duality, and their respective planar Abelian duals. The relations among these theories are summarized schematically in Figure 3.30.

In the remainder of this section, we describe the precise sequence of local dualizations that connects the two planar Abelian quivers shown in Figure 3.30.

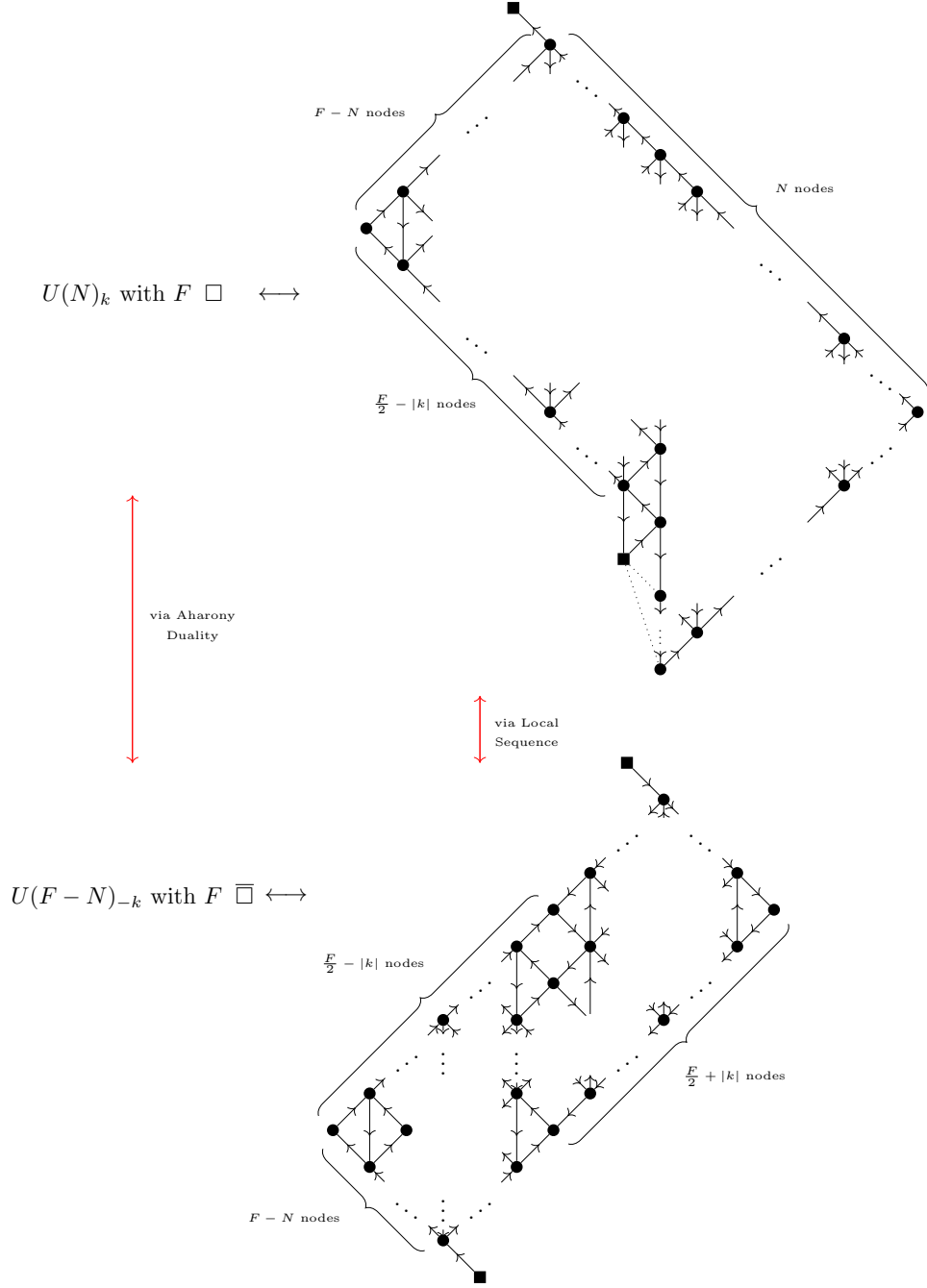


Figure 3.30: Schematic depiction of the duality web between a theory in Zone $\tilde{1}$ and its Aharony dual in Zone 1. The horizontal arrows represent the planar Abelian dualities presented in this thesis. The rightmost vertical arrow corresponds to a sequence of local dualizations discussed in the present chapter. The diagram is commutative, therefore Aharony duality can be thought of as a consequence of mirror symmetry.

Electric Theory. We begin by massaging the planar Abelian dual of the electric theory (3.173), which has gauge group $U(N)$, Chern–Simons level k , FI parameter η , and trial R-charge r . We report the planar Abelian dual of this theory on the left of Figure 3.31. We perform a sequence of dualizations analogous to the planar flip–flip duality discussed in [2].

The dualization proceeds column by column, starting from the rightmost column and

stopping at the column immediately to the right of the bottom flavor node. The result of this first dualization pass is shown in the central quiver of Figure 3.31.

After completing this first dualization pass, a column of square nodes appears at position $\frac{F}{2} - |k|$. Additionally, the flavor node shifts one column to the right, and the bottom-right diagonal of the quiver shortens by one node.

We then perform subsequent dualization passes in a similar manner, starting from the right and ending at the column immediately to the right of the bottom flavor node. Each pass further shortens the bottom-right diagonal by one node and requires one fewer dualization. The result of the second pass is shown on the right of Figure 3.31.

After iterating this procedure, we arrive at the quiver depicted on the left of Figure 3.32, which takes the form of a rectangular lattice of size $N \times (F - N)$. Finally, we perform a flip-flip duality [2] on the rightmost triangular subquiver, resulting in the quiver shown on the right of Figure 3.32.

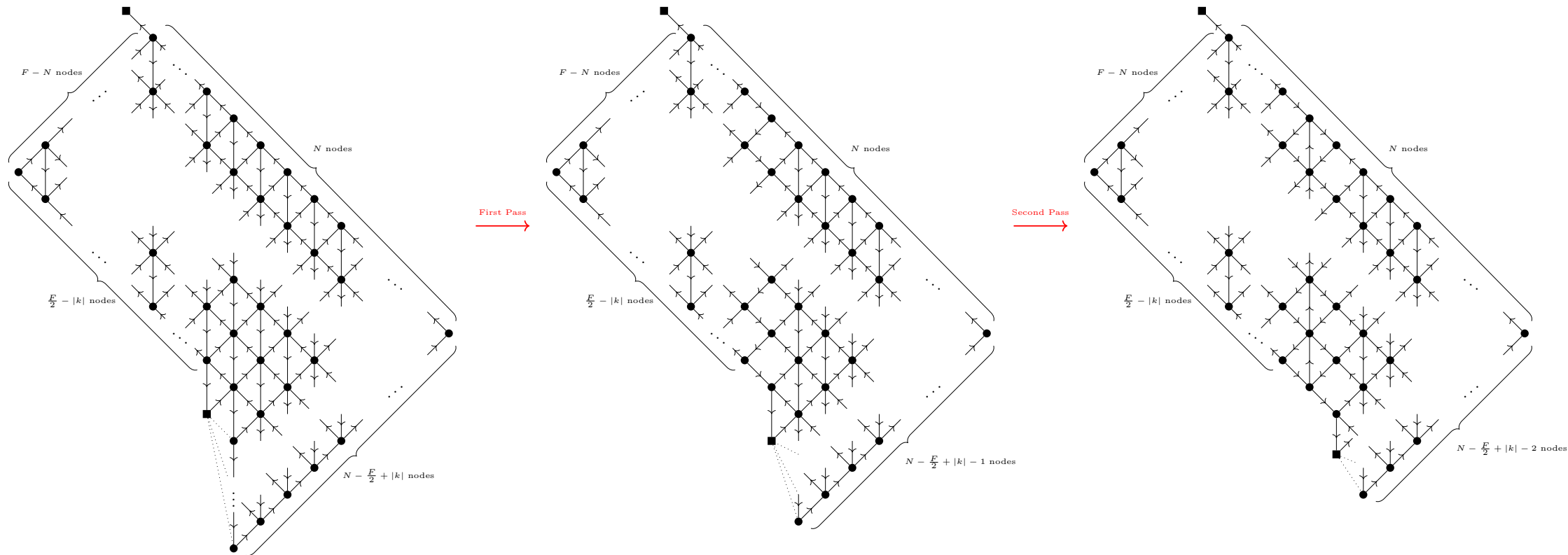


Figure 3.31: The planar Abelian dual for $U(N)_k$ SQCD with F fundamentals in Zone $\tilde{1}$ and the result of local dualizations. Here we show the original planar dual presented in (3.173) on the left. In the center and on the right, we show the quivers obtained after one and two dualization passes as discussed in the main text.

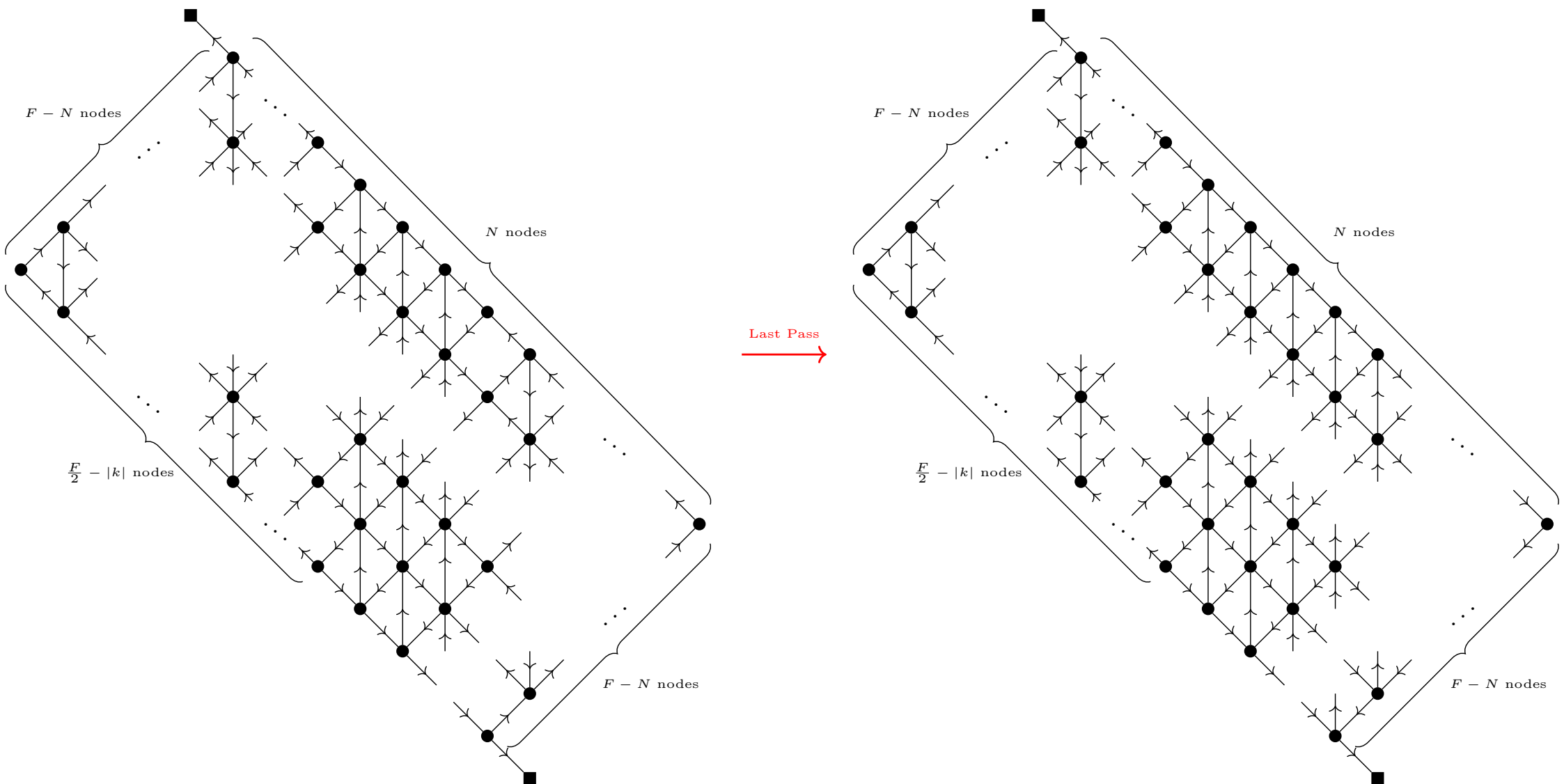


Figure 3.32: Continued from Figure 3.31. The final iterations of the dualization sequence for the Zone $\tilde{1}$ mirror of $U(N)_k$ with F fundamental fields.

This represents our final expression for the electric theory and provides an alternative presentation of the planar Abelian dual in Zone $\tilde{1}$. We remind the reader that we use the shorthand $X_{ij} := X_i - X_j$ to parametrize the FI terms of the gauge nodes in the quiver diagram.

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \iff \\
 & (N < F < 2|k| + 2N)
 \end{aligned}
 \quad \longleftrightarrow \quad
 \begin{aligned}
 & \text{Quiver diagram with nodes and arrows, labeled with } \frac{F}{2} - |k| \text{ nodes and } \frac{F}{2} + |k| \text{ nodes.} \\
 & \text{Specific labels include } X_{21} \text{ and } X_{F, F-1}.
 \end{aligned}
 \tag{3.230}$$

Magnetic Theory. The planar Abelian dual of SQCD with only anti-fundamental fields can be obtained by applying parity (P) and charge conjugation (C) to both sides of the dualities presented in this thesis. In particular, we start from the duality for SQCD in Zone 1 (3.166), with gauge group $U(N_D)$, Chern–Simons level k , FI parameter $\eta - k\omega$, and trial R-charge $1 - r$. This corresponds to the theory appearing on the right-hand side of the Aharony-like duality (3.227), up to background terms.

Applying the combined CP transformation on the left-hand side produces SQCD with gauge group $U(N_D)$, F anti-fundamental fields, Chern–Simons level $-k$, and FI parameter $k\omega - \eta$. On the quiver side, the CP action reverses the direction of all arrows and flips the sign of all Chern–Simons and mixed Chern–Simons terms, including background contributions²⁸.

²⁸FI terms can be interpreted as mixed Chern–Simons couplings between gauge and topological symmetries and are therefore odd under P .

The resulting duality is the following:

$$\begin{aligned}
 & \textcolor{red}{k\omega - \eta} \\
 & U(F - N)_{-k} \text{ with } F \text{ } \square \longleftrightarrow
 \end{aligned}
 \begin{aligned}
 & \text{A quiver diagram consisting of a central chain of nodes connected by double lines. The chain is flanked by two triangular sub-quivers. The left triangular sub-quiver is labeled with } F - N \text{ nodes. The right triangular sub-quiver is labeled with } \frac{F}{2} + |k| \text{ nodes. The central chain is labeled with } \frac{F}{2} - |k| \text{ nodes. The diagram is labeled with } X_{12}, X_{23}, \dots, X_{F-1, F} \text{ and } k\omega - \eta. \end{aligned}
 \tag{3.231}$$

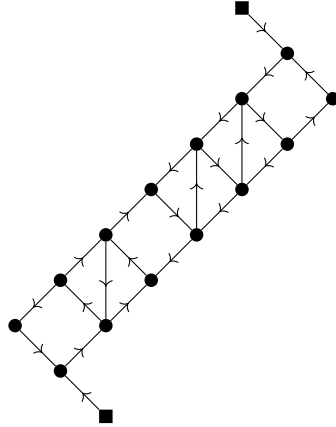
We remind the reader that we use the shorthand $X_{ij} := X_i - X_j$ to parametrize the FI terms of the gauge nodes in the quiver diagram. The Chern–Simons levels, mixed Chern–Simons couplings, and trial R-charges can be determined following the rules outlined in Section 3.5.2.

We now perform a sequence of local dualizations on the quiver on the right-hand side to show that it is equivalent to the right-hand side of (3.231). While these manipulations can be performed for general values of N , F , and k , it is pedagogical to present them through a concrete example. We consider $U(2)_{\frac{1}{2}}$ SQCD with 9 anti-fundamental fields:

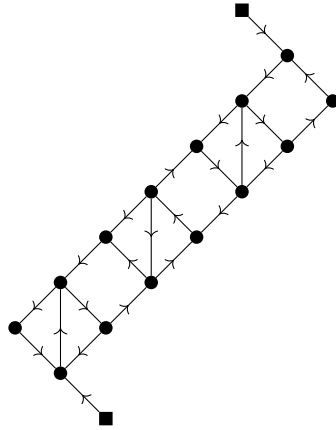
$$U(2)_{\frac{1}{2}} \text{ with } 9 \text{ } \square \longleftrightarrow
 \begin{aligned}
 & \text{A quiver diagram consisting of a chain of nodes connected by double lines. The chain is flanked by two triangular sub-quivers. The left triangular sub-quiver is labeled with } F - N \text{ nodes. The right triangular sub-quiver is labeled with } \frac{F}{2} + |k| \text{ nodes. The central chain is labeled with } \frac{F}{2} - |k| \text{ nodes. The diagram is labeled with } X_{12}, X_{23}, \dots, X_{F-1, F} \text{ and } k\omega - \eta. \end{aligned}
 \tag{3.232}$$

We begin by performing the “flip–flip” dualization sequence [2] on the triangular sub-quivers at the ends of the quiver. In the example above, this corresponds to dualizing

the leftmost and rightmost gauge nodes, yielding the following quiver:

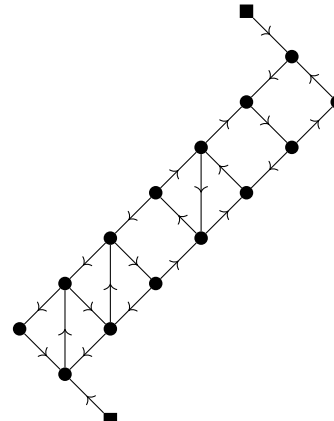
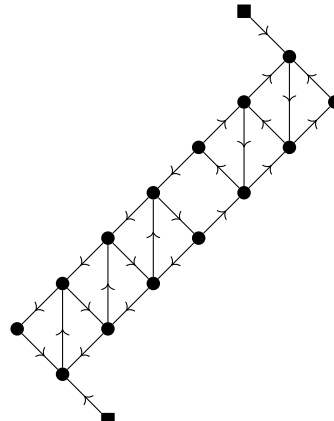
R.H.S. of (3.232) $\rightarrow$  (3.233)

Next, we dualize the column of nodes corresponding to the column of squares in the bulk of the quiver, located at position $\frac{F}{2} - |k|$. We then proceed to dualize the column immediately to its left and continue dualizing columns from right to left until reaching the column attached to the flavor node, thereby completing the first pass. In our example, this corresponds to dualizing column 4, followed by column 3. The resulting quiver is:

R.H.S. of (3.233) $\rightarrow$  (3.234)

We then repeat the procedure, starting from the column at position $\frac{F}{2} - |k| + 1$, which is now a column of squares. We dualize columns from right to left until we reach the opposite column of squares, completing the second pass. This process is iterated until the rightmost flavor node is reached.

In our example, two additional passes are required:

R.H.S. of (3.234) $\rightarrow$  $\rightarrow$  (3.235)

In general, the resulting quiver takes the form:

(3.236)

The final step consists of modifying the two chiral fields connected to the flavor nodes. This manipulation is implemented via a sequence of Witten $SL(2, \mathbb{Z})$ actions [144] on the $U(1)_\eta$ symmetry of the quiver, which together combine into a trivial action. To make this operation nontrivial, we use the fact that the quiver has a single $U(1)$ flavor symmetry, which can be associated either with the left or the right flavor node; the two choices are gauge-equivalent.

Concretely, we perform a $T^{-1}ST^{-1}$ transformation on the left flavor node and a $TS^{-1}T$ transformation on the right flavor node. Although these combine into a trivial $SL(2; \mathbb{Z})$ action, the intermediate steps induce a non-trivial reorganization of the theory, which remains IR dual to the original quiver. The T transformation shifts the Chern–Simons level by $+1$, while the S transformation gauges the symmetry, introducing a new $U(1)$ topological symmetry²⁹.

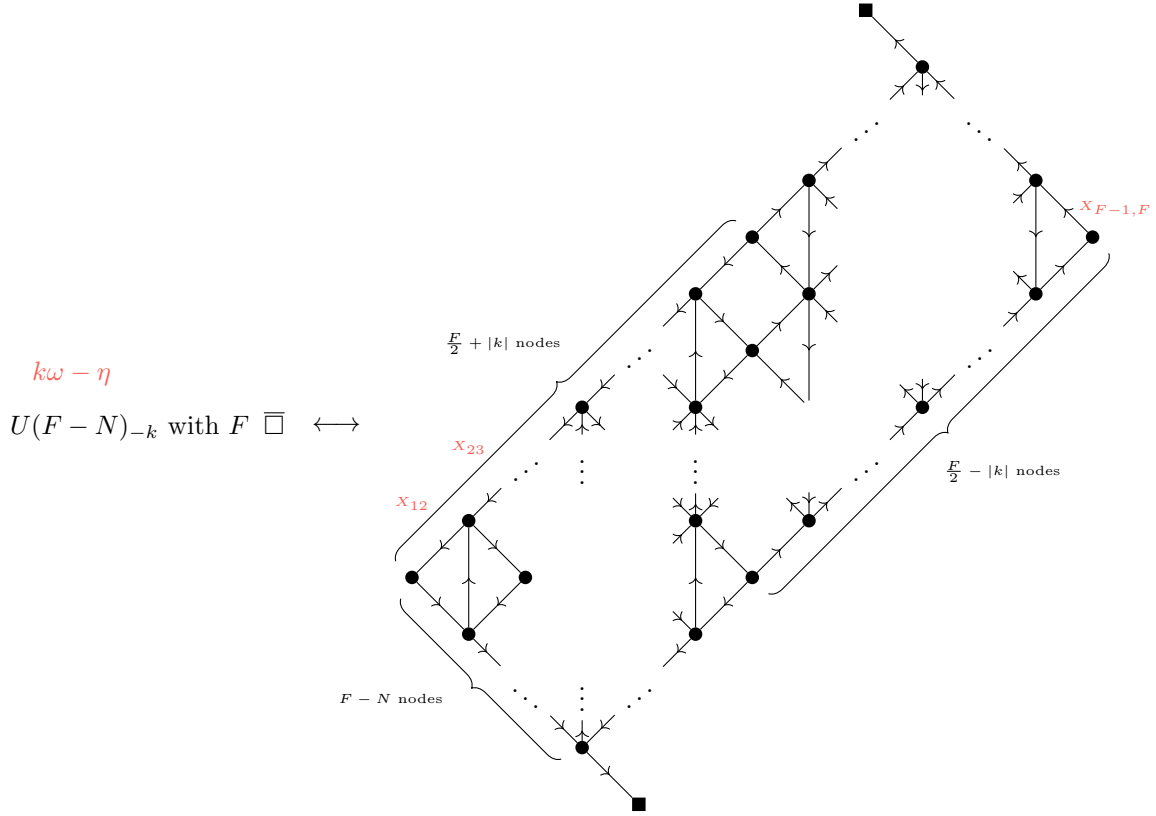
After these transformations, both nodes are confined to a chiral field, corresponding to a monopole operator of the associated gauge node.

The resulting quiver is shown in Equation 3.238. We remind the reader that we use the shorthand $X_{ij} := X_i - X_j$ to parametrize the FI terms of the gauge nodes in the quiver diagram.

²⁹At the level of the $\mathbf{S}_b^3$ partition function, the T and S transformations act as

$$\begin{aligned} T : Z(\mu) &\rightarrow e^{-i\pi\mu^2} Z(\mu), \\ S : Z(\mu) &\rightarrow \int d\mu e^{-2i\pi\mu\lambda} Z(\mu), \end{aligned} \tag{3.237}$$

where μ and λ are the fugacities associated with the original and new $U(1)$ symmetries, respectively.



(3.238)

Notice that the planar quiver obtained this way is equivalent to the planar Abelian dual of Zone $\tilde{1}$ in (3.230), up to a horizontal reflection and a permutation of the X_i fugacities.

This concludes the argument that Aharony duality for $U(N)_k$ SQCD with F fundamental chiral fields follows from the planar Abelian dualities discussed in Section 3.5, together with their massive RG flows and the assumption of Aharony duality for Abelian gauge theories. We expect that a similar argument extends to more general $U(N)_k$ theories with both fundamental and anti-fundamental matter, although we leave this extension for future work.

3.9 Scope and outlook

Scope. This chapter developed a systematic field-theoretic framework for constructing planar Abelian mirror duals of 3d $\mathcal{N} = 2$ CS-SQCD theories, starting from known $\mathcal{N} = 4$ mirror pairs and applying real mass deformations. The key technical contributions were: the identification of the correct $\mathcal{N} = 2$ $\mathcal{S}$ -wall and its fusion rules; the extension of the dualization algorithm of [45, 46] to theories with Chern–Simons interactions; a complete treatment of the (N, k, F) landscape of $\mathcal{N} = 2$ CS-SQCD with fundamental matter, including all mass deformation flows across zones; and a proof that generic Aharony-like dualities are implied by the planar Abelian duals together with basic Abelian duality moves. The algorithmic analysis was deliberately restricted to the CS levels determined by the $\mathcal{N} = 4$ parent, i.e. $\ell = -1$, where the connection to the $\mathcal{N} = 4$ algorithm is most direct and partition-function checks are most transparent; the landscape analysis of Section 3.5 was instead carried out for $\ell = 0$, with other values of ℓ reachable via Witten’s

$SL(2, \mathbb{Z})$ action as described in Section 3.1.3.

Outlook. Several directions remain open. The generalization to arbitrary CS levels has been initiated in [3], but a complete classification of planar Abelian duals for theories with both fundamental and anti-fundamental matter at generic (N, k, F) is still ongoing. Extending the algorithm to orthosymplectic quivers, quivers involving orientifold planes, and circular or non-simply-laced topologies would bring the $\mathcal{N} = 2$ technology to parity with the $\mathcal{N} = 4$ case [177, 178]. Regarding this direction, an extensive investigation of mirror symmetry for symplectic CS-SQCD has been carried out in [187]. It would also be important to clarify the precise interface between the planar Abelian duals and level–rank duality, which controls the endpoint theories in mass deformation flows. Finally, the connection to the non-supersymmetric bosonization dualities of Chapter 4 deserves a more systematic treatment: the SUSY dualities should reduce to those of the next chapter under specific mass deformations, but making this precise at the level of operators, partition functions, and spin-TQFT factors remains an open problem.

Chapter 4

Planar Abelian duals of Chern–Simons QCD

The content of this chapter is essentially taken from [4].

Over the past fifteen years, the study of IR *bosonization* dualities in three-dimensional CS gauge theories has become a vibrant and rapidly developing area of theoretical physics. These developments have extended the framework of supersymmetric dualities deep into the more challenging and intricate domain of non-supersymmetric QFTs [29, 58, 60, 61, 64–66, 188–191].

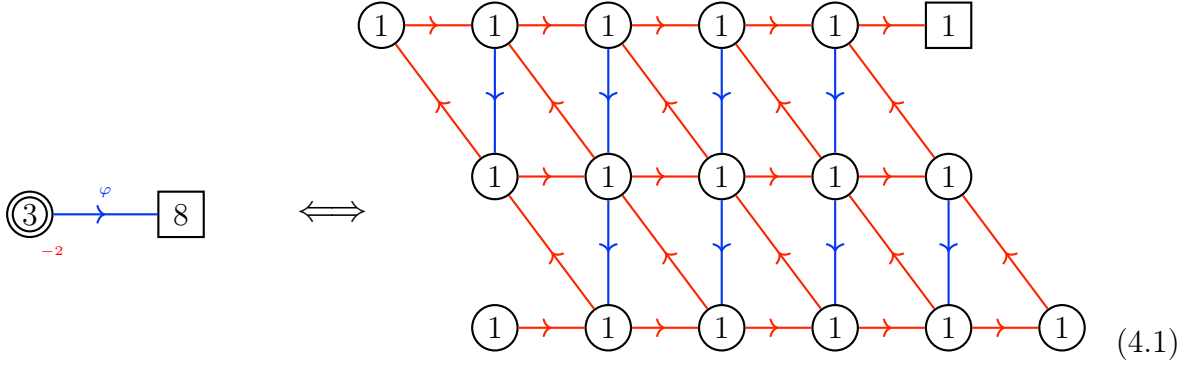
A particularly fruitful idea has been to derive non-supersymmetric dualities as mass deformations of known supersymmetric ones [62, 63, 69, 192–195]. Thus, non-supersymmetric 3d Chern–Simons dualities have become a central tenet within contemporary research on QFTs, with broad implications across particle physics, condensed matter physics, and mathematics.

In this chapter, we explore the new class of dualities between non-Abelian theories such as $SU(N)$ CS-QCD₃ with bosonic and fermionic matter, and Abelian theories featuring multiple $U(1)$ gauge groups, proposed in the paper [4]. A hallmark of the Abelian duals is that gauge and matter content are encoded in a *quiver* diagram¹. In this description, scalar fields are depicted as blue arrows among nodes, while fermionic fields are shown in red. Additionally, these quivers are *planar*, meaning that the diagram can be embedded in the plane such that each face corresponds to an interaction term in the action. The concept of planar quivers originates in the study of the AdS₅/CFT₄ correspondence with minimal supersymmetry [196–201]; here, we generalize this to the non-supersymmetric context.

The dualities we propose bear a strong resemblance to the $\mathcal{N} = 2$ mirror dualities developed in [3]² and reviewed in Chapter 3, which themselves arise from suitably chosen mass deformations of $\mathcal{N} = 4$ mirror pairs [36, 40]. As an illustrative example, Figure (4.1) presents the dual description of $SU(3)_{-2}$ QCD₃ with eight fundamental scalar fields.

¹As previously done in this thesis, we use a circle for unitary groups and a double circle for special unitary groups.

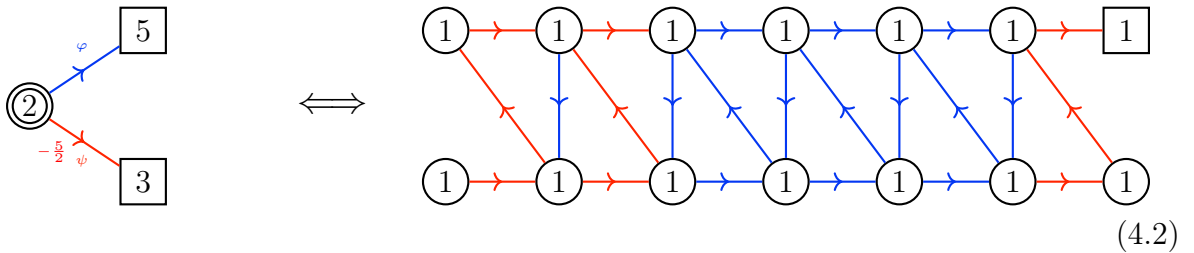
²We actually expect that our dualities arise from supersymmetry breaking mass deformations of the $\mathcal{N} = 2$ dualities of [3], but an in-depth study of the RG flow from the $\mathcal{N} = 2$ to the non-supersymmetric dualities goes beyond the scope of this thesis.



The corresponding planar quiver incorporates additional CS couplings and potential terms not shown in the figure. Section 4.2 presents the general proposal for the planar Abelian dual of $SU(N)_{2N-N_s-k}$ QCD with N_s bosonic flavors and expands on these features³.

The duality displays characteristic features of mirror dualities. The width of the quiver scales with the number of QCD flavors N_s . More precisely, the number of vertical columns of gauge nodes is equal to $N_s - 1$, and there is one additional gauge node at the bottom left of the quiver. On the other hand, the maximum height of these columns is the number of colors N . On the planar side, the UV global symmetry is $U(1)^{N_s}$: each vertical column contributes one topological $U(1)$ symmetry (a single diagonal combination survives the interaction terms, which we write down in the main text and include monopole potentials) and the bottom-left node contributes the remaining factor. This symmetry enhances in the IR to $U(N_s)$.

We further generalize these dualities to theories with both bosonic and fermionic matter, proposing a duality between $SU(N)$ CS-QCD with N_s scalars and N_f fermions,⁴ as detailed in Section 4.3. Figure (4.2) displays the dual for $SU(2)_{-\frac{5}{2}}$ QCD₃ with five scalars and three fermions.



A simple rule governs the quiver structure: adding a bosonic flavor on the QCD side corresponds to inserting a fermionic column in the quiver (red), while adding a fermionic flavor adds a bosonic column (blue). In this way, our dualities can be viewed as generalizations of bosonization dualities [202–204], now extended to non-Abelian theories with richer matter content.

Although we focus extensively on $SU(N)$ gauge theories here, we also demonstrate that these dualities can be extended to $U(N)$ gauge theories easily by gauging the baryonic symmetry and its image under the duality. By setting $N = 1$, one recovers the bosonization dualities of [205], which relate $U(1)$ gauge theories with F bosons (fermions) to a

³Throughout this chapter, we focus on the regime $N_s \geq 2N$ and $k \geq 0$.

⁴The CS level is $2N - N_s - \frac{N_f}{2} - k$. In this work, we consider $N_s \geq 2N$, $N_f \geq 0$, and $k \geq 0$.

linear $U(1)^{F-1}$ quiver with fermionic (bosonic) matter.

In addition, as mentioned before, we observe that our dualities may arise as mass deformations of the $\mathcal{N} = 2$ supersymmetric dualities developed in [1–3], which we deeply reviewed in Chapter 3. The diagrams used here have the same topology as those in the supersymmetric case, with the distinction that each edge now corresponds to a single boson or fermion. Along the RG flow, precisely one field per edge acquires a mass, breaking supersymmetry. Although a detailed study of this flow is beyond the scope of this thesis, the existence of such a flow could provide further checks supporting the dualities proposed here.

Driven by this analogy one can think of our dualities as *non-Abelian mirror symmetry in the absence of supersymmetry*.

4.1 Linear Abelian Duals of CS-QED₃ with Bosons and Fermions

In this section we propose a duality for QED₃ with bosonic and fermionic matter and a Chern–Simons interaction, which is a mild generalization of two *bosonization* dualities of [205, 206]. This duality relates the $U(1)_{-N_s-N_f/2-k}$ QED theory with N_s scalars and N_f fermions to an Abelian linear quiver theory with $N_s + N_f + k - 1$ $U(1)$ nodes. This duality is the $N = 1$ case of the general duality for $U(N)$ or $SU(N)$ QCD with bosons and fermions discussed in this chapter. We discuss the Abelian case as a warm up.

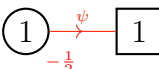
The relevant matter content is complex fermions, usually denoted as ψ , complex scalars, denoted as ϕ and real scalars denoted as σ .

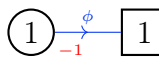
The simplest instances of the duality are the well known bosonization and fermionization dualities, which can be schematically written down as follows:

$$U(1)_{-\frac{1}{2}} + 1\psi \quad \leftrightarrow \quad 1\phi \quad (4.3)$$

$$U(1)_{-1} + 1\phi \quad \leftrightarrow \quad 1\psi \quad (4.4)$$

It is convenient to write the dualities (4.3), (4.4) in a quiver notation in view of the generalization discussed below:

bosonization:  $\leftrightarrow$  $\mathcal{V} = |\phi|^4$ (4.5)

fermionization:  $\leftrightarrow$  $\mathcal{V} = |\phi|^4$ (4.6)

We propose a generalization for the case of QED with N_f fermions ψ , N_s complex scalars ϕ and CS level $-N_s - \frac{N_f}{2} - k$ with $k \geq 0$, shown in Figure 4.1. This proposal is a refinement of the dualities already considered in [62, 205]. The dual is a $U(1)^{N_s+N_f+k-1}$ gauge theory with N_f complex scalars and N_s complex fermions charged under two $U(1)$

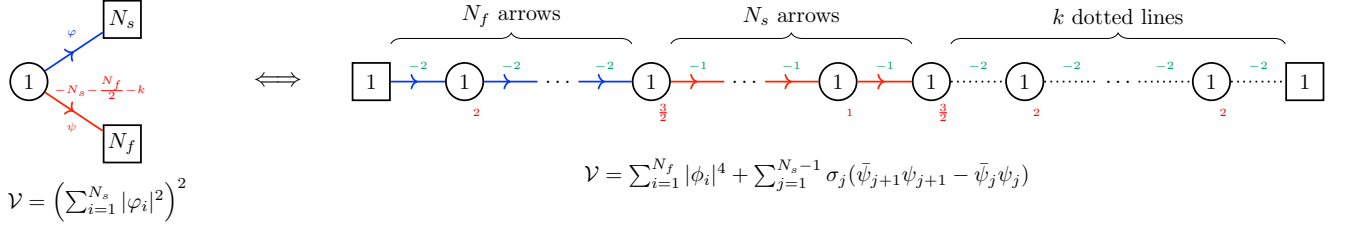


Figure 4.1: The proposed dual for QED₃ with N_s scalars, N_f fermions and CS level $-N_s - \frac{N_f}{2} - k$ with $k > 0$. Round nodes denote $U(1)$ gauge groups, square node denote flavor groups. Red and blue arrows denote fermions and scalars, respectively, charged under the nodes that they connect. Red labels denote CS levels and green labels denote mixed CS levels.

gauge groups with opposite charge. Furthermore for every $U(1)$ gauge group with at least two charged fermions there is a real scalar σ_j that interacts with the two fermions via the Yukawa coupling.

The off-diagonal mesons of the QED theory are mapped to monopole operators in the quiver theory. In particular, consider these spin-0 and spin-1 mesons $\bar{\psi}_i \psi_j$ with $i < j$. The corresponding operator is a monopole with GNO flux +1 under all the gauge groups from the i -th to the $(j-1)$ -th gauge group. As reviewed in Section 2.1.5, this monopole is not gauge invariant and has gauge charges:

$$Q \left[\mathfrak{M}^{(0, \dots, 0, \overset{i}{+}, \dots, \overset{j-1}{+}, 0, \dots)} \right] = (0, \dots, -1, +1, 0, \dots, 0, +1, -1, 0, \dots) \quad (4.7)$$

and can be made gauge invariant by dressing with the scalar ϕ_{i-1} connecting the $(i-1)$ -th and the i -th node and the complex conjugate scalar ϕ_j^* . Therefore we have the mapping:

$$\bar{\psi}_i \psi_j \quad \leftrightarrow \quad \mathfrak{M}^{(0 \dots 0 \xrightarrow{i} + \dots + \xleftarrow{j-1} 0 \dots)} \quad , \quad 1 \leq i < j \leq N_f \quad (4.8)$$

where the two arrows denote the dressing with the corresponding scalars and the left-pointing arrow indicates that the dressing is performed with the complex conjugate scalar. Each boson in the monopole background “sees” a GNO flux +1 and therefore has spin- $\frac{1}{2}$, resulting in a spin-0 and a spin-1 operator. Similarly, the mesons $\varphi_i^* \varphi_j$ and $\bar{\psi}_i \varphi_j$ are mapped to:

$$\varphi_i^* \varphi_j \quad \leftrightarrow \quad \mathfrak{M}^{(0 \dots 0 \xrightarrow{i+N_f} + \dots + \xleftarrow{j-1+N_f} 0 \dots)} \quad , \quad 1 \leq i < j \leq N_s \quad (4.9)$$

$$\bar{\psi}_i \varphi_j \quad \leftrightarrow \quad \mathfrak{M}^{(0 \dots 0 \xrightarrow{i} + \dots + \xleftarrow{j-1+N_f} 0 \dots)} \quad (4.10)$$

which are operators with spin-0 and spin- $\frac{1}{2}$, respectively.

For generic k the monopole operators in the QED transform in non-trivial representations of the flavor symmetry $S(U(N_s) \times U(N_f))$. These monopole operators are expected to map to monopole operators of the dual theory. We will not discuss in detail the mapping of such operators. When $k = 0$ the QED theory has some monopole operators

which are neutral under the flavor symmetry $S(U(N_s) \times U(N_f))$. These are obtained by dressing the bare monopole $\mathfrak{M}^+$, which has gauge charge $-N_s$, with N_s (derivatives of) the scalars φ . The scalars are antisymmetrized in flavor indices, resulting in an operator that is charged under the topological symmetry but is neutral under $S(U(N_s) \times U(N_f))$.

On the mirror side, only for $k = 0$, the quiver has long mesonic operators connecting the two flavor nodes, built out of (derivatives of) all the fields. The long mesons are mapped to the monopoles described above:

$$\mathfrak{M}^+ \overbrace{\partial_{\bullet}\varphi_{\{i_1 \cdots \partial_{\bullet}\varphi_{i_{N_s}}\}}}^{N_s} \leftrightarrow \partial_{\star}\phi_1 \cdots \partial_{\star}\phi_{N_f} \partial_{\star}\psi_1 \cdots \partial_{\star}\psi_{N_s} \quad (4.11)$$

where $\partial_{\bullet}$ and $\partial_{\star}$ indicate generic sets of derivatives. On the l.h.s. Lorentz indices are contracted to obtain a non-vanishing operator. It would be interesting to precisely understand the mapping between the $\partial_{\bullet}$ derivatives on the right and the $\partial_{\star}$ derivatives on the left, but we will not attempt to do so here. While the spin of these operators depend on the sets of derivatives $\partial_{\bullet}$ and $\partial_{\star}$, respectively, the integrality of the spin is given by $\frac{N_s}{2} \bmod 1$ on both sides.

For particular values of the three parameters N_f , N_s and k , which are all non-negative, our proposal 4.1 reproduces known dualities. For $k = 0$ and a single matter species we recover the Abelian dualities for fermionic or bosonic QED discussed in [205]:

$$\begin{array}{c} \text{Diagram 1: } \textcircled{1} \xrightarrow[\textcolor{red}{-N_f/2}]{\textcolor{red}{\psi}} \boxed{N_f} \\ \text{Diagram 2: } \boxed{1} \xrightarrow{\textcolor{blue}{\phi}} \textcircled{1} \xrightarrow[\textcolor{red}{2-2}]{\textcolor{blue}{\phi}} \cdots \xrightarrow[\textcolor{green}{-2}]{\textcolor{blue}{\phi}} \textcircled{1} \xrightarrow[\textcolor{red}{2}]{\textcolor{blue}{\phi}} \boxed{1} \end{array} \quad \leftrightarrow \quad \begin{array}{c} \text{Diagram 3: } \textcircled{1} \xrightarrow[\textcolor{red}{-N_s}]{\textcolor{blue}{\varphi}} \boxed{N_s} \\ \text{Diagram 4: } \boxed{1} \xrightarrow[\textcolor{red}{1-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \cdots \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{1}]{\textcolor{red}{\psi}} \boxed{1} \end{array} \quad (4.12)$$

$$V = \sum_{i=1}^{N_f} |\phi_i|^4$$

$$\begin{array}{c} \text{Diagram 5: } \textcircled{1} \xrightarrow[\textcolor{red}{-N_s}]{\textcolor{blue}{\varphi}} \boxed{N_s} \\ \text{Diagram 6: } \boxed{1} \xrightarrow[\textcolor{red}{1-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \cdots \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{1}]{\textcolor{red}{\psi}} \boxed{1} \end{array} \quad \leftrightarrow \quad \begin{array}{c} \text{Diagram 7: } \textcircled{1} \xrightarrow[\textcolor{red}{-N_s}]{\textcolor{blue}{\varphi}} \boxed{N_s} \\ \text{Diagram 8: } \boxed{1} \xrightarrow[\textcolor{red}{1-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \cdots \xrightarrow[\textcolor{red}{-1}]{\textcolor{red}{\psi}} \textcircled{1} \xrightarrow[\textcolor{red}{1}]{\textcolor{red}{\psi}} \boxed{1} \end{array} \quad (4.13)$$

$$\mathcal{V} = \left(\sum_{i=1}^{N_s} |\varphi_i|^2 \right)^2 \quad V = \sum_{i=1}^{N_s-1} \sigma_i (\bar{\psi}_{i+1} \psi_{i+1} - \bar{\psi}_i \psi_i)$$

In particular, for $k = 0$ and $N_f + N_s = 1$ we reproduce the original bosonization dualities (4.3), (4.4). For $k = 0$ and $N_f + N_s = 2$ the duality 4.1 reduces to dualities between QED theories. For $N_f = N_s = 1$ we recover a duality analyzed in [66] for $U(1)_{-\frac{3}{2}}$ with one scalar and one fermion. For $N_f = 2$, $N_s = 0$ or $N_s = 2$, $N_f = 0$ we recover dualities between scalar and fermionic QED. In the bosonic QED in (4.13) one can turn on a negative mass for all the scalars that preserves the $SU(N_s)$ global symmetry. The scalars condense, Higgsing the gauge group and the theory reduces to a $\mathbb{CP}^{N_s-1}$ NLSM. On the dual side the fermions take negative mass, resulting in a $(U(1)_0)^{N_s-1}$ gauge theory. We conjecture that the N_s-1 dual photons together with the N_s-1 real scalars σ_i combine into a N_s-1 complex dimensional NLSM, reproducing the dual side.

Let us discuss some features of the duality 4.1. The symmetry of the QED theory is, up to discrete factors, $U(1)_{top} \times S(U(N_s) \times U(N_f))$. On the quiver side for $k > 0$ there are $N_f + N_s$ topological symmetries, while for $k = 0$ there are $N_f + N_s - 1$ topological symmetries and a $U(1)$ flavor symmetry rotating the fields. In both cases the rank of the global symmetry matches between the two theories. The duality implies that the $U(1)^{N_f+N_s}$ global symmetry of the quiver theory exhibits non-Abelian enhancement in the IR, matching the global symmetry of the QED theory.

Mass deformations for the matter fields of the QED theory are mapped to masses of the quiver theory, schematically:

$$\text{l.h.s.} \quad \begin{array}{c} |\phi|^2 \\ \bar{\psi}\psi \end{array} \quad \leftrightarrow \quad \begin{array}{c} \bar{\psi}\psi \\ -|\phi|^2 \end{array} \quad \text{r.h.s.} \quad (4.14)$$

One can check that turning on masses for the fermions and positive masses for the scalars one recovers the same duality 4.1 with different values of k, N_f and N_s . This provides a non-trivial consistency check of our proposal. Turning on a negative mass for a scalar Higgses the QED theory. In the dual side one of the fermions is integrated out with a positive mass, triggering a sequential confinement that results in a theory of free fields, matching the electric theory.

4.2 Planar Abelian Duals of Bosonic CS-QCD₃

In this section, we construct a quiver gauge theory dual to non-Abelian QCD₃ with fundamental matter. Our proposal builds upon results presented in [1–3], and reviewed in Section 3, that established a duality between $\mathcal{N} = 2$ supersymmetric non-Abelian linear CS quiver gauge theories and planar Abelian quiver gauge theories with mixed CS interactions. Motivated by these supersymmetric dualities, we conjecture an analogous correspondence in the non-supersymmetric case: non-Abelian CS QCD₃ with fundamental fermionic and bosonic matter should be dual to a planar Abelian quiver theory containing both bosons and fermions. This expectation follows from the observation that our proposed non-supersymmetric dualities should emerge as mass deformations of their supersymmetric counterparts.

4.2.1 General Proposal

We propose the planar Abelian dual for $SU(N)$ QCD₃ with N_s critical scalars at Chern–Simons level $2N - N_s - k$ ($k \geq 0$) shown in Figure 4.2. This duality maps flavor symmetries to topological symmetries and many baryonic and mesonic operators of one theory are mapped to gauge invariant disorder operators (monopoles) of its mirror dual. In this section, we discuss some simple examples explicitly to justify our proposal. The proposed duality is believed to hold in the following range of parameters:

$$N_s \geq 2N, \quad k \geq 0.$$

We note that, in the absence of supersymmetry, the existence of the electric fixed point itself is not guaranteed throughout this range of parameters: when the number of flavors is large compared to the Chern–Simons level, the IR fate of CS-matter theories may involve symmetry breaking rather than a conformal phase [207]. Our proposal should be understood with this caveat in mind: whenever the electric theory does not flow to a CFT, we expect the duality to hold at the level of the IR phase (e.g. as a matching of the symmetry-breaking pattern or of the resulting TQFT). In the planar quiver shown in Figure 4.2, the parameter k counts the number of “empty” columns — columns that are connected to the rest of the quiver solely through horizontal BF couplings.

We now comment on the potential $\mathcal{V}$ shown in Figure 4.2 for the planar mirror theory:

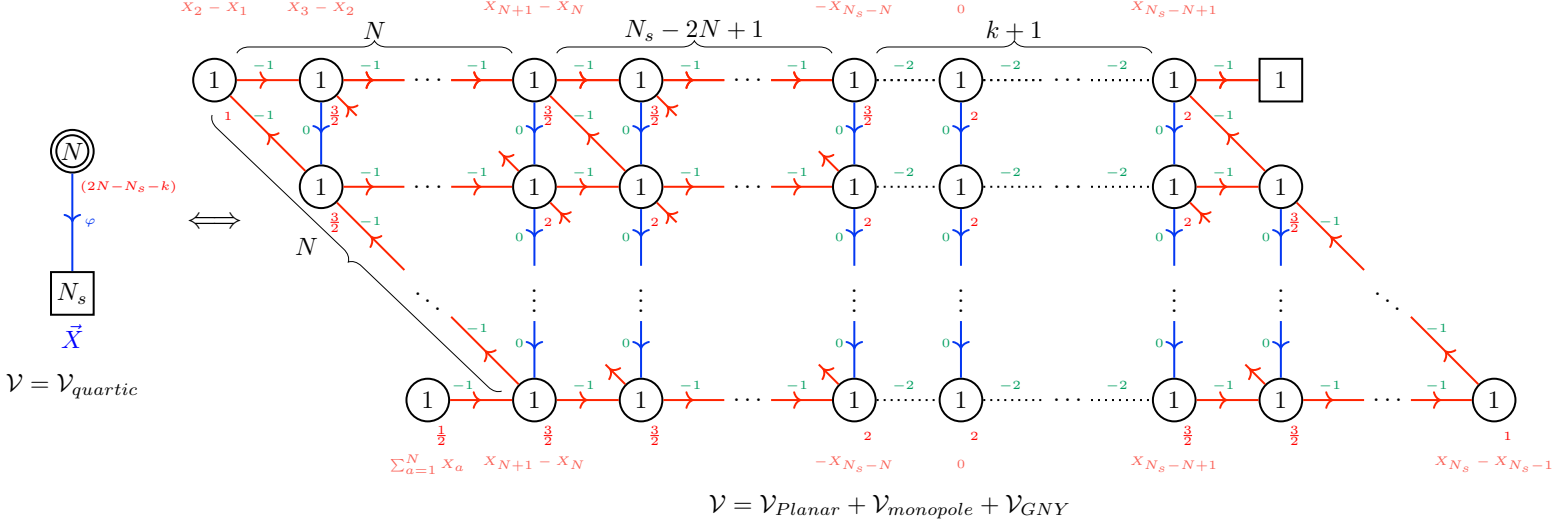


Figure 4.2: The proposed dual for $SU(N)$ QCD₃ with N_s scalars at Chern–Simons level $2N - N_s - k$ is shown here. Single/double circles (squares) correspond to U/SU gauge (flavor) symmetries. We describe scalar (fermion) bifundamental fields in blue (red). Chern–Simons levels of gauge nodes are indicated in maroon, while mixed CS couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. BF couplings between adjacent gauge nodes connected by a fermionic field are -1 , BF couplings between gauge nodes connected by a bosonic field are 0 , while BF couplings between adjacent gauge nodes connected by dotted lines alone are -2 . We also indicate the fugacities of the topological symmetries of each column in orange. The orientation of arrows indicates the representation of the field under the symmetry group, i.e.: an arrow pointing out of (into) a node transforms in the (anti-)fundamental representation of the symmetry group.

- $\mathcal{V}_{Planar}$ contains a cubic Yukawa interaction $(\phi_i \psi_j \psi_k + c.c.)$ for every closed planar triangle. There is a term with a coefficient of $+1$ (-1) for each triangle closed clockwise (anticlockwise)⁵.
- $\mathcal{V}_{monopole}$ captures the monopole terms in the potential. For each vertical bosonic bifundamental field, there is a monopole potential with GNO flux $+1$ and -1 under the nodes connected by the arrow, from top to bottom ($\mathfrak{M}^{(\pm)} + c.c.$). As explained in Section 2.1.5 these monopoles can be dressed to obtain a gauge invariant spin-0 operator. This breaks the topological symmetry of the gauge nodes in a column of the quiver to a single, diagonal combination. We label the unbroken $U(1)$ symmetry with the orange label at the top (or bottom) of the corresponding column. The label encodes the mapping of the $U(1)$ group to a subgroup $U(1) \subset U(N_s)$ of the electric flavor symmetry, as explained below.
- $\mathcal{V}_{GNY}$ captures additional cubic interactions between a real scalar field associated with the α^{th} gauge node in the I^{th} column, and the fermions charged under this gauge group (an arrow pointing out of (into) a node has charge $+1$ (-1) under the

⁵Since the theory is not supersymmetric, the couplings run along the RG flow and only the relative signs of these terms are meaningful; the values ± 1 should be understood as a convenient normalization of the UV Lagrangian.

gauge symmetry). For example, consider a gauge theory represented by the quiver diagram in (4.15).



$\mathcal{V}_{GNY}$ for this theory has the form:

$$\mathcal{V}_{GNY} = \sigma_1(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_4\psi_4) + \sigma_2(\bar{\psi}_3\psi_3 - \bar{\psi}_2\psi_2 - \bar{\psi}_5\psi_5). \quad (4.16)$$

Each gauge node **except** for the bottom-most gauge node has a GNY potential accompanying the Yukawa interactions.

By contrast, if we consider the following gauge theory (the dashed line indicates the absence of any fermionic bifundamental field, and the two nodes may share a bosonic bifundamental field or only mixed CS interactions):



we expect the identification of σ_1 and σ_2 so the resulting $\mathcal{V}_{GNY}$ has the form:

$$\mathcal{V}_{GNY} = \sigma(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_3\psi_3). \quad (4.18)$$

In the quiver theory in Figure 4.2 there are $N(N_s - N)$ real scalars σ .

In the QCD₃ theory, the quartic scalar potential is $\mathcal{V}_{quartic}$ ⁶:

$$\mathcal{V}_{quartic} = \sum_{\alpha, \beta=1}^N \sum_{I, J=1}^{N_s} [\varphi^{\alpha I} \bar{\varphi}_{\alpha J} \varphi^{\beta J} \bar{\varphi}_{\beta I} + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\varphi^{\beta J} \bar{\varphi}_{\beta J})] \quad (4.19)$$

which preserves the full $U(N_s)$ global symmetry.

In the planar mirror theory the presence of the interactions $\mathcal{V}_{monopole}$ and $\mathcal{V}_{Planar}$ are essential to match the rank of the global symmetry on the electric side. This implicitly assumes that the dressed monopole operators entering $\mathcal{V}_{monopole}$ are relevant (or at least dangerously irrelevant) along the RG flow: without supersymmetry their scaling dimensions are not protected, so this is a genuine dynamical assumption of our proposal. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)_{top}^{N_s}$, where each $U(1)$ factor corresponds to the topological symmetries of a column of the quiver, broken

⁶N.B.: Latin indices $i, j = 1, \dots, N_s$ run over flavor indices, and Greek indices $\alpha, \beta = 1, \dots, N$ run over gauge indices.

to a diagonal combination by $\mathcal{V}_{monopole}$ ⁷. We further claim that the symmetry enhances in the IR to:

$$U(1)_{top}^{N_s} \rightarrow SU(N_s) \times U(1) \quad (4.20)$$

reproducing the $SU(N_s) \times U(1)$ flavor and baryonic symmetries of the electric theory respectively. This is most transparent in our choice of parametrization of the fugacities of the topological symmetries of each column in Figure 4.2. Explicitly, denoting by Q_{X_i} the generator of the $U(1)_{X_i}$ factor of the maximal torus $\prod_{i=1}^{N_s} U(1)_{X_i} \subset U(N_s)$, the diagonal combination of the topological symmetries in the i -th column, associated to $X_{i+1} - X_i$, is mapped to the $U(1)$ subgroup generated by $Q_{X_{i+1}} - Q_{X_i}$. Similarly, the topological symmetry of the bottom left node, associated to $\sum_{a=1}^N X_a$, is mapped to the $U(1)$ subgroup generated by $\sum_{i=1}^N Q_{X_i}$. Therefore the fugacities denoted with orange labels in 4.2 encode the explicit embedding of $U(1)_{top}^{N_s}$ in the enhanced $U(N_s)$ global symmetry group.

Moreover, the faithful symmetry of the QCD theory is [66, 208]:

$$\frac{U(N_s)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^C \quad (4.21)$$

where $\mathbb{Z}_2^C$ is charge conjugation, acting as complex conjugation on the scalars for $N \geq 2$. $\mathbb{Z}_N$ is generated by the following element of the maximal torus of $U(N_s)$:

$$\left(e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi i}{N}} \right) \in U(1)_{X_1} \times \dots \times U(1)_{X_{N_s}} \quad (4.23)$$

All the gauge invariant dressed monopoles in the dual quiver transform trivially under this element, and are therefore compatible with being components of representations of the enhanced $\frac{U(N_s)}{\mathbb{Z}_N}$. On the planar side, we expect charge conjugation $\mathbb{Z}_2^C$ to act as a reflection of the quiver diagram combined with charge conjugation on all the Abelian gauge and matter fields, although we do not track this identification in detail.

The fact that there are no mesonic $U(1)$ symmetries in the planar theory is a non-trivial statement and can be checked as follows by looking at the planar quiver:

$$\begin{aligned} \# \text{mesonic } U(1)s &= \# \text{lines} - \# \text{gauge nodes} - \# \text{triangles} = \\ &= \{3N(N_s - N) - 2N_s + 3\} - \{N(N_s - N) + 1\} - \{2N(N_s - N) - 2(N_s - 1)\} \\ &= 0 \end{aligned} \quad (4.24)$$

where we assume that there is a mesonic $U(1)$ rotating each boson/fermion in the mirror, and a cubic potential term associated to each triangle appearing in the quiver⁹.

⁷There are $N_s - 1$ columns which contribute $N_s - 1$ $U(1)$ factors, and the additional $U(1)$ factor is given by the topological symmetry of the bottom-most gauge node. The topological symmetries associated to the columns within the “empty” region of the quiver can be reabsorbed by a gauge transformation and do not contribute to the global symmetry.

⁸For $N = 2$, as $SU(2) \cong USp(2)$, the global symmetry of the electric $SU(2)$ QCD₃ theory is:

$$\frac{USp(2N_s)}{\mathbb{Z}_2}; \quad (4.22)$$

which is broken by $\mathcal{V}_{quartic}$ to $\frac{U(N_s)}{\mathbb{Z}_2} \rtimes \mathbb{Z}_2^C$ [66].

⁹Note that this straightforward computation holds when the parameter $k = 0$. For $k > 0$, matching the $U(1)$ symmetries becomes more subtle. Nonetheless, the matching can be verified in this case as well.

The presence of the Gross-Neveu-Yukawa potential $\mathcal{V}_{GNY}$ also plays an important role in the duality and can be motivated as follows. On adding a negative mass deformation for the N_s scalars in the electric theory, the gauge group is completely Higgsed and the IR phase of the theory is a sigma model with the target manifold identified as the coset space [209, 210]:

$$\frac{SU(N_s)}{SU(N_s - N) \times SU(N)} \quad (4.25)$$

which has $2NN_s - 2N^2 + 1$ real dimensions. Upon mapping the same deformation in the mirror theory, we expect to flow to $N(N_s - N) + 1$ copies of $U(1)_0$, whose dual photons parameterize $(\mathbf{S}^1)^{N(N_s - N) + 1}$ in the deep IR. This can be achieved by giving a negative mass to all the fermions and a positive mass to all the scalars in the quiver. The dual photons are expected to combine with the $N(N_s - N)$ real scalars σ , resulting in a NLSM with real dimension $2N(N_s - N) + 1$, consistent with the QCD theory.

In the rest of this section, we will make these ideas more precise by considering some explicit examples. Through these examples, we illustrate the effects of increasing the Chern-Simons level k and the rank of the gauge symmetries on the respective planar duals.

4.2.2 Example: $SU(2)_{-1}$ QCD₃ with $N_s = 5$ scalars

Let us begin with the example of $SU(2)_{-1}$ with 5 scalars and its planar dual description. Since this is our first example, we are deliberately detailed to illustrate our proposal. We propose the duality shown in Figure 4.3.

Henceforth, we refer to the $SU(2)_{-1}$ QCD₃ with 5 scalars as the “electric” theory and the planar theory as its “mirror”. We are mainly concerned with the mapping of the following gauge-invariant operators in the electric theory:

1. **25 mesons** of the form $\varphi_i \bar{\varphi}_j$ transforming in the adjoint representation of the flavor symmetry group $U(5)$. Hence, with respect to the maximal torus $\prod_{i=1}^5 U(1)_{X_i} \subset U(5)$, they are charged under $Q_{X_i} - Q_{X_j}$. The off-diagonal mesons (those with $i \neq j$) are then mapped to monopole operators of the dual quiver theory. The proposed mapping for the mesonic operators is shown in Table 4.1. The diagonal mesons $|\varphi_i|^2$ are neutral under the $U(1)_{X_i}$ and are mapped to non-monopole operators of the dual theory. The four traceless diagonal mesons are mapped to the real scalars σ and the traceful diagonal meson is mapped to a combination of $\bar{\psi}_i \psi_i$ mesonic operators in the quiver.
2. **25 - 5 = 20 conserved currents** associated to the global symmetry enhancement $U(1)^5 \rightarrow U(5)$. These are spin-1 operators charged under $Q_{X_i} - Q_{X_j}$ with $i \neq j$ and are expected to be mapped to monopole operators, as illustrated below.
3. $\binom{5}{2} = 10$ **baryons** constructed by anti-symmetrizing the color indices with the help of the Levi-Civita tensor $\epsilon_{\alpha\beta} \varphi_i^\alpha \varphi_j^\beta$. Together, these transform in the 2-index antisymmetric representation of the flavor symmetry group $SU(5)$ and are charged under $Q_{X_i} + Q_{X_j}$, with $i \neq j$. The proposed mapping for the baryonic operators is shown in Table 4.2.

The off-diagonal mesons, off-diagonal currents and the baryons are charged under the maximal torus $\prod_{i=1}^5 U(1)_{X_i} \subset U(5)$, therefore they should be mapped to monopole operators in the dual quiver theory. In what follows, we propose a mapping for these

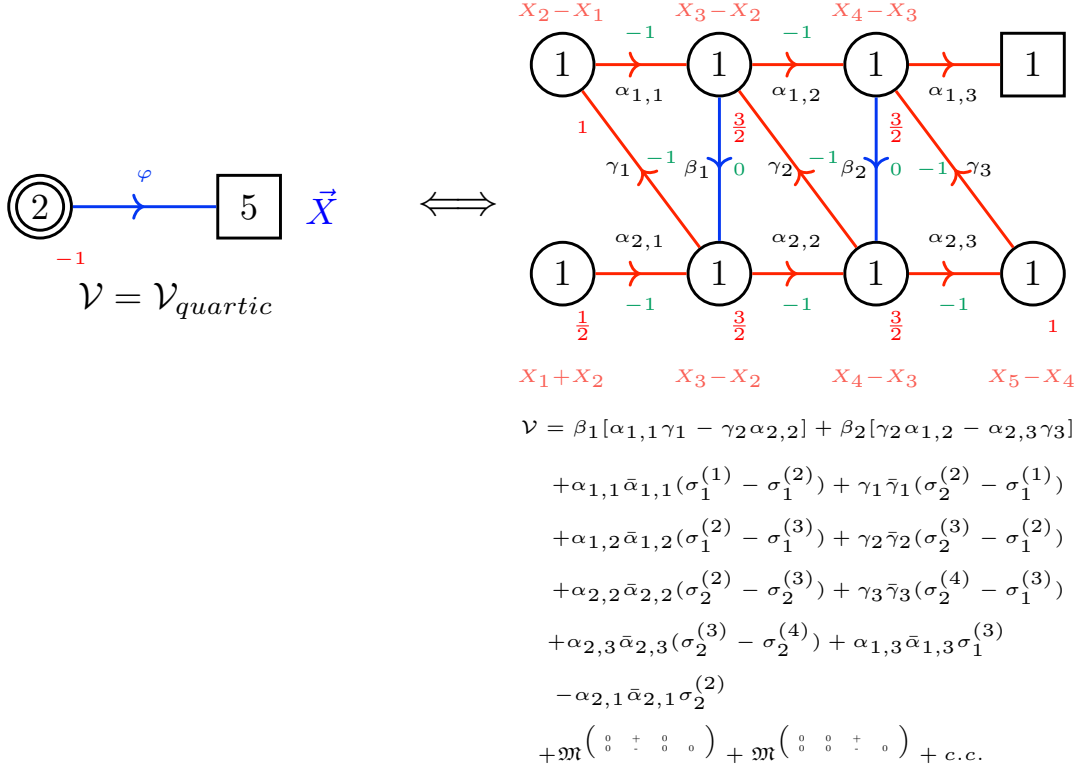


Figure 4.3: The dual for $SU(2)_{-1}$ QCD₃ with 5 scalars is shown here. All labeling conventions follow those of Figure 4.2. The superscript (subscript) for the real scalar σ indicates its column (position in the column), e.g.: $\sigma_1^{(3)}$ corresponds to the 1st gauge node in the 3rd column.

operators, starting from the off-diagonal mesons and the baryons. Thus, the task at hand is to determine gauge invariant monopole operators that are **Lorentz scalars** and are charged under $Q_{X_i} \mp Q_{X_j}$, which are mapped to the topological symmetries of the quiver theory as described above. We make this statement precise in the next paragraphs by considering some examples. The techniques needed to compute the gauge charges of monopole operators, the dressing needed to construct gauge invariant monopoles and the spin of the resulting operator are reviewed in Section 2.1.5.

“Mesonic” Monopole Operators We study monopole operators with the right quantum numbers to be mapped to the mesons of the electric theory here. As an example, we consider the following monopole operator:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.26)$$

This operator has charge +1 under the topological symmetries of the first two gauge nodes; hence, it is charged under $Q_{X_3} - Q_{X_1}$, and transforms as a component of the adjoint representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 3 & -1 \\ 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -1 \end{pmatrix}. \quad (4.27)$$

Clearly, it is not gauge invariant and must be dressed appropriately. We choose the minimal dressing¹⁰:

$$\begin{pmatrix} + & 2 & \text{---} & \\ 0 & \text{---} & \text{---} & 0 \end{pmatrix}, \quad (4.28)$$

hence, we see that the operator:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\gamma_1 \gamma_2 \bar{\alpha}_{1,2}} \quad (4.29)$$

is gauge invariant. We denote this operator by combining the GNO flux assignment and the dressing in the following way:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\gamma_1 \gamma_2 \bar{\alpha}_{1,2}} \equiv \begin{pmatrix} + & + & \text{---} & 0 \\ 0 & \text{---} & \text{---} & 0 \end{pmatrix}. \quad (4.30)$$

As each of the three fermionic fields is quantized in a background of total magnetic flux 1, their spin statistics changes to that of spin-0 bosons¹¹. We conclude that the dressed monopole operator has Lorentz spin 0, and is the right operator to be mapped to $\varphi_3 \bar{\varphi}_1$.

We can readily identify the off-diagonal mesonic operators and summarize their dual descriptions—including necessary dressings—in Table 4.1. The mapping of diagonal mesonic operators $|\varphi_i|^2$ is less straightforward because these operators are neutral under the maximal torus of $U(5)$, visible in the quiver UV Lagrangian. We expect that these operators are mapped to suitable linear combinations of mesonic operators of the quiver, compatible with the mass deformations discussed in Section 4.5.1. We defer the study of the mapping of diagonal mesonic operators to future work.

Table 4.1: Dressed monopole operators of the planar mirror (Figure 4.3) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	“Electric” Meson
$\begin{pmatrix} + & \text{---} & 0 \\ 0 & \text{---} & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & \text{---} & 0 \\ 0 & \text{---} & 0 \end{pmatrix}$	0	$\varphi_2 \bar{\varphi}_1$
$\begin{pmatrix} + & + & \text{---} & 0 \\ 0 & \text{---} & \text{---} & 0 \end{pmatrix}$	$\begin{pmatrix} + & 2 & \text{---} & \\ 0 & \text{---} & \text{---} & 0 \end{pmatrix}$	0	$\varphi_3 \bar{\varphi}_1$
<i>Continued on next page</i>			

¹⁰By this, we mean that we use the minimal number of fields to dress the bare monopole and construct a gauge-invariant operator. A non-minimal dressing—i.e., using additional fields—would yield an operator with a higher scaling dimension. In this work, we focus exclusively on mapping the operators with the lowest scaling dimensions.

¹¹In principle, they also have a spin-1 channel. Unless otherwise stated we dress the monopoles with the zero modes of the fermion fields, resulting in a spin-0 operator [26].

Table 4.1 – continued from previous page

GNO Flux	Gauge Charge	Spin	“Electric” Meson
$\begin{pmatrix} + & + & + & \cancel{+} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & + & 2 & \cancel{-} \\ 0 & - & - & - \end{pmatrix}$	0	$\varphi_4 \bar{\varphi}_1$
$\begin{pmatrix} + & + & + & \cancel{+} \\ 0 & 0 & 0 & \rightarrow + \end{pmatrix}$	$\begin{pmatrix} + & + & + & \cancel{+} \\ 0 & - & - & 2 \rightarrow + \end{pmatrix}$	0	$\varphi_5 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \rightarrow + & \cancel{0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 3 & \cancel{-} & - \\ 0 & 0 & - & 0 \end{pmatrix}$	0	$\varphi_3 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \rightarrow + & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 2 & \cancel{2} & \cancel{-} \\ 0 & 0 & - & - \end{pmatrix}$	0	$\varphi_4 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \rightarrow + & \cancel{0} & \cancel{+} \\ 0 & 0 & 0 & \rightarrow + \end{pmatrix}$	$\begin{pmatrix} - & 2 & \cancel{+} & \cancel{+} \\ 0 & 0 & - & 2 \rightarrow + \end{pmatrix}$	0	$\varphi_5 \bar{\varphi}_2$
$\begin{pmatrix} 0 & 0 & \rightarrow + & \cancel{0} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & - & 3 & \cancel{-} \\ 0 & 0 & 0 & - \end{pmatrix}$	0	$\varphi_4 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & \rightarrow + & \cancel{+} \\ 0 & 0 & 0 & \rightarrow + \end{pmatrix}$	$\begin{pmatrix} 0 & - & 2 & \cancel{+} \\ 0 & 0 & - & \rightarrow + \end{pmatrix}$	0	$\varphi_5 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & 0 & \cancel{+} \\ 0 & 0 & 0 & \rightarrow + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & - & \cancel{2} \\ 0 & 0 & - & \rightarrow 2 \end{pmatrix}$	0	$\varphi_5 \bar{\varphi}_4$

Let us comment on a particular feature of the monopole operators in the quiver theories considered in this chapter, which is a consequence of the presence of the monopole interaction $\mathcal{V}_{monopole}$. When looking for a candidate for the dual of an electric meson, say $\varphi_4 \bar{\varphi}_3$, we may also consider the monopole:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rightarrow + \end{pmatrix}. \quad (4.31)$$

Indeed, this is a spin-0 gauge invariant operator with the right charges to be mapped to $\varphi_4 \bar{\varphi}_3$, similarly to the monopole listed in Table 4.1. This is possible due to the presence of the monopole potential $\mathcal{V}_{monopole}$: even though these two monopoles have different GNO flux assignments they have the same charges under the linear combination of topological symmetries preserved by $\mathcal{V}_{monopole}$. We expect these two monopoles to mix, therefore the electric meson $\varphi_4 \bar{\varphi}_3$ will be mapped to some linear combination of the two:

$$\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rightarrow + \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & \rightarrow + \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \varphi_4 \bar{\varphi}_3, \quad (|\alpha|^2 + |\beta|^2 = 1). \quad (4.32)$$

This phenomenon is ubiquitous in the quiver theories studied in this chapter and the amount of monopoles that can mix between each other increases as N increases due to the higher number of monopole interactions in $\mathcal{V}_{monopole}$. In Table 4.1 and in the rest of this chapter we only report one dressed monopole that has the correct charges to be

mapped to a given operator in the QCD theory¹². It is understood that, in most cases, there are other monopole operators with the same global charges but different GNO flux assignment and dressing, and the mapping will involve a suitable linear combination of these operators. It would be interesting to analyze this mixing systematically, for example by extending the techniques of [26, 98, 211] in the presence of monopole interactions, but we leave this to future work.

Conserved Currents We now examine how the $U(5)$ flavor symmetry currents - including the baryonic $U(1)$ subgroup - map between the electric and dual theories. These spin-1 conserved currents take the form:

$$J_{jk}^\mu = i (\bar{\varphi}_j \partial^\mu \varphi_k - \partial^\mu \bar{\varphi}_j \varphi_k) \quad (4.33)$$

where $j, k = 1, \dots, 5$, transforming in the adjoint representation of $U(5)$. Note that under charge conjugation: $(J_{jk}^\mu)^* = J_{kj}^\mu$.

The off-diagonal currents map to dressed spin-1 monopole operators in the dual theory. For example, J_{21}^μ corresponds to:

$$J_{21}^\mu \leftrightarrow \mathfrak{M} \left(\begin{array}{ccc} + \text{ } \textcolor{red}{\rightarrow} & 0 & 0 \\ 0 & \textcolor{red}{\rightarrow} & 0 & 0 \end{array} \right) \quad (4.34)$$

where we dress with the spin-1 mode of one fermion and the spin-0 mode of the other fermion, resulting in a spin-1 operator. Under charge conjugation, this maps to J_{12}^μ :

$$J_{12}^\mu \leftrightarrow \mathfrak{M} \left(\begin{array}{ccc} - \text{ } \textcolor{red}{\rightarrow} & 0 & 0 \\ 0 & \textcolor{red}{\rightarrow} & 0 & 0 \end{array} \right) \quad (4.35)$$

where, again, we dress with a spin-0 and a spin-1 mode.

The other off-diagonal spin 1 currents map in a similar way. In particular, the (off-diagonal) current J_{jk}^μ is mapped to the same bare monopole as the meson $\varphi_j \bar{\varphi}_k$, the difference being that the former is dressed such that it has spin-1 while the latter has spin-0.

The five diagonal currents J_{jj}^μ map to the topological $U(1)$ currents of the dual theory. Denoting by $A_\mu^{(j,k)}$ the gauge field of the $U(1)$ node in the j -th column at position k (from the top) in quiver (4.3), and by $A_\mu^{(0)}$ the gauge field of the bottom-left node, the fugacity parametrization of Figure 4.3 dictates:

$$J_{j+1,j+1}^\mu - J_{jj}^\mu \longleftrightarrow \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho^{(j)}, \quad j = 1, \dots, 4, \quad J_{11}^\mu + J_{22}^\mu \longleftrightarrow \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho^{(0)} \quad (4.36)$$

with the gauge field combinations:

¹²Notice that also on the QCD side there are in general many operators with the same spin and global charges. For example any operator can be multiplied by a spin-0 flavor singlet or it can be acted on by contracted derivatives.

$$A_\mu^{(j)} = \begin{cases} A_\mu^{(1,1)} & j = 1 \\ A_\mu^{(2,1)} + A_\mu^{(2,2)} & j = 2 \\ A_\mu^{(3,1)} + A_\mu^{(3,2)} & j = 3 \\ A_\mu^{(4,1)} & j = 4 \end{cases} \quad (4.37)$$

That is, $A_\mu^{(j)}$ is the sum of the gauge fields of the j -th column, whose diagonal topological symmetry survives $\mathcal{V}_{monopole}$ and carries fugacity $X_{j+1} - X_j$, while the bottom-left node carries fugacity $X_1 + X_2$. These five combinations are linearly independent, so the mapping (4.36) reproduces the mapping of the maximal torus of the $U(5)$ global symmetry across the duality. Note that the individual currents J_{jj}^μ are recovered only through combinations with half-integer coefficients, consistently with the $\mathbb{Z}_2$ quotient in the faithful global symmetry.

“Baryonic” Monopole Operators We follow the same line of reasoning as before to study monopole operators with the right quantum numbers to be mapped to the baryons of the electric theory here. As an example, we consider the following *bare* monopole operator:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \\ + & + & 0 & 0 \end{pmatrix} \quad (4.38)$$

This operator has charge +1 under the topological symmetries of the bottom two gauge nodes; hence, it is charged under $Q_{X_3} + Q_{X_1}$, and therefore can be identified with a component of the two-index antisymmetric representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix}. \quad (4.39)$$

Clearly, it is not gauge invariant and must be dressed appropriately. We again choose the minimal dressing that cancels the gauge charges of the bare monopole:

$$\begin{pmatrix} - & 0 & 0 \\ 0 & \overset{\text{red}}{2} & \overset{\text{red}}{-} & 0 \end{pmatrix}, \quad (4.40)$$

Therefore the dressed monopole operator:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \\ + & + & 0 & 0 \end{pmatrix} \bar{\gamma}_1 \bar{\alpha}_{2,2} \equiv \begin{pmatrix} 0 & \overset{\text{red}}{0} & 0 \\ + & \overset{\text{red}}{+} & \overset{\text{red}}{0} & 0 \end{pmatrix}; \quad (4.41)$$

is a gauge invariant Lorentz scalar and has the correct charges to be mapped to $\varphi_3 \varphi_2$ (color indices have been suppressed). Similarly, we can identify the 10 baryonic operators, reported with the appropriate dressing in Table 4.2.

Table 4.2: Dressed monopole operators of the planar mirror (Figure 4.3) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	“Electric” Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \leftarrow 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \leftarrow - & 0 & 0 \end{pmatrix}$	0	$\varphi_2 \varphi_1$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \leftarrow 0 & \leftarrow 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 0 & 0 & 0 \\ 0 & \leftarrow 2 & \leftarrow - & 0 \end{pmatrix}$	0	$\varphi_3 \varphi_1$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \leftarrow 0 & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	$\begin{pmatrix} - & - & 0 & 0 \\ 0 & \leftarrow + & \leftarrow 2 & \leftarrow - \end{pmatrix}$	0	$\varphi_4 \varphi_1$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \leftarrow 0 & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	$\begin{pmatrix} - & - & - & 0 \\ 0 & \leftarrow + & \leftarrow + & \leftarrow + \end{pmatrix}$	0	$\varphi_5 \varphi_1$
$\begin{pmatrix} + & 0 & 0 & 0 \\ + & \leftarrow + & \leftarrow 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & - & 0 & 0 \\ 0 & \leftarrow + & \leftarrow - & 0 \end{pmatrix}$	0	$\varphi_3 \varphi_2$
$\begin{pmatrix} + & 0 & 0 & 0 \\ + & \leftarrow + & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	$\begin{pmatrix} + & - & 2 & 0 \\ 0 & \leftarrow 0 & \leftarrow 2 & \leftarrow - \end{pmatrix}$	0	$\varphi_4 \varphi_2$
$\begin{pmatrix} + & 0 & 0 & 0 \\ + & \leftarrow + & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	$\begin{pmatrix} + & - & 2 & - \\ 0 & \leftarrow 0 & \leftarrow + & \leftarrow + \end{pmatrix}$	0	$\varphi_5 \varphi_2$
$\begin{pmatrix} + & + & 0 & 0 \\ + & \leftarrow + & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	$\begin{pmatrix} 0 & + & - & 2 \\ 0 & \leftarrow 0 & \leftarrow 0 & \leftarrow + \end{pmatrix}$	0	$\varphi_5 \varphi_3$
$\begin{pmatrix} + & + & + & 0 \\ + & \leftarrow + & \leftarrow + & \leftarrow + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & + & 0 \\ 0 & \leftarrow 0 & \leftarrow 0 & \leftarrow 0 \end{pmatrix}$	0	$\varphi_5 \varphi_4$

We reiterate that we only consider dressed minimal flux operators in the mapping.

4.2.3 Example: $SU(2)_{-2}$ QCD₃ with $N_s = 5$ scalars

We now consider the example of $SU(2)_{-2}$ with 5 scalars. Notice that it has a lower Chern–Simons level than that of the theory considered in Section 4.2.2; this minor difference will greatly alter the operator map. We propose the duality shown in Figure 4.4.

As before, we are mainly concerned with determining the monopole operators that are dual to the 20 off-diagonal **mesons** and 10 **baryons**. We do not show the explicit calculation and state the results in Tables 4.3 and 4.4 respectively.

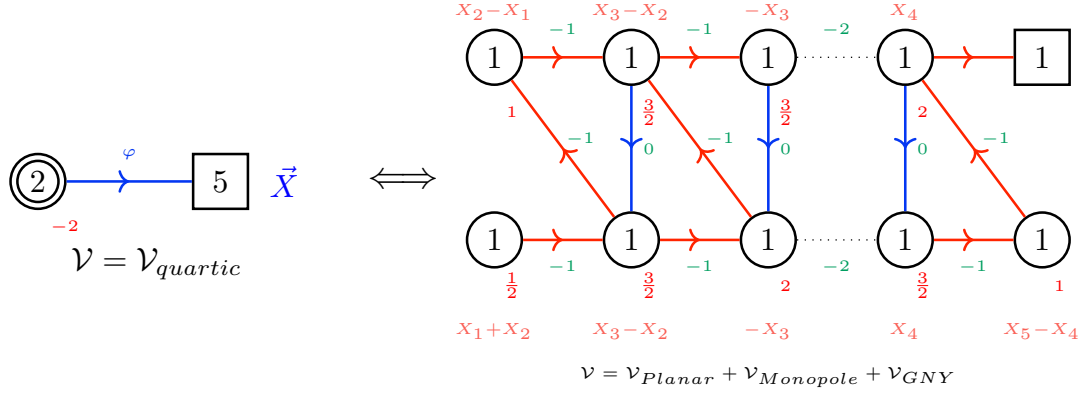


Figure 4.4: The dual for $SU(2)_{-2}$ QCD₃ with 5 scalars is shown here. All labeling conventions follow those of Figure 4.2.

Table 4.3: Dressed monopole operators of the planar mirror (Figure 4.4) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & \leftarrow 0 & 0 & 0 & 0 \\ 0 & \leftarrow 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$	$\begin{pmatrix} + & \leftarrow + & \leftarrow 0 & 0 & 0 \\ 0 & \leftarrow 0 & \leftarrow 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_1$
$\begin{pmatrix} + & \leftarrow + & \leftarrow + & + & \leftarrow 0 \\ 0 & \leftarrow 0 & \leftarrow 0 & 0 & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_1$	$\begin{pmatrix} + & \leftarrow + & \leftarrow + & + & \leftarrow + \\ 0 & \leftarrow 0 & \leftarrow 0 & 0 & \leftarrow + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \leftarrow + & \leftarrow 0 & 0 & 0 \\ 0 & 0 & \leftarrow 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \leftarrow + & \leftarrow + & + & \leftarrow 0 \\ 0 & 0 & \leftarrow 0 & 0 & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \leftarrow + & \leftarrow + & + & \leftarrow + \\ 0 & 0 & \leftarrow 0 & 0 & \leftarrow + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 & \leftarrow + & + & \leftarrow 0 \\ 0 & 0 & 0 & 0 & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & \leftarrow + & + & \leftarrow + \\ 0 & 0 & 0 & 0 & \leftarrow + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \leftarrow + \\ 0 & 0 & 0 & 0 & \leftarrow + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_4$

Table 4.4: Dressed monopole operators of the planar mirror (Figure 4.4) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & \leftarrow 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \varphi_1$	$\begin{pmatrix} 0 & \leftarrow 0 & 0 & 0 & 0 \\ + & \leftarrow + & \leftarrow 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \varphi_1$
$\begin{pmatrix} 0 & \leftarrow 0 & \leftarrow 0 & 0 & 0 \\ + & \leftarrow + & \leftarrow + & + & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \varphi_1$	$\begin{pmatrix} 0 & \leftarrow 0 & \leftarrow 0 & 0 & \leftarrow + \\ + & \leftarrow + & \leftarrow + & + & \leftarrow + \end{pmatrix}$	$\varphi_5 \varphi_1$
<i>Continued on next page</i>			

Table 4.4 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & \leftarrow 0 & 0 & 0 & 0 \\ + & + & \leftarrow 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \varphi_2$	$\begin{pmatrix} + & \leftarrow 0 & 0 & 0 \\ + & + & \leftarrow 0 & + & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \varphi_2$
$\begin{pmatrix} + & \leftarrow 0 & 0 & 0 & + \\ + & + & \leftarrow 0 & + & + \end{pmatrix}$	$\varphi_5 \varphi_2$	$\begin{pmatrix} + & + & \leftarrow 0 & 0 \\ + & + & + & + & \leftarrow 0 \end{pmatrix}$	$\varphi_4 \varphi_3$
$\begin{pmatrix} + & + & \leftarrow 0 & 0 & + \\ + & + & + & + & \leftarrow 0 \end{pmatrix}$	$\varphi_5 \varphi_3$	$\begin{pmatrix} + & + & + & + & \leftarrow 0 \\ + & + & + & + & + \end{pmatrix}$	$\varphi_5 \varphi_4$

4.2.4 Example: $SU(3)_{-1}$ QCD₃ with $N_s = 7$ scalars

We consider the case of $SU(3)_{-1}$ with 7 scalars and its planar dual as our final example of this section. We propose the duality shown in Figure 4.5.

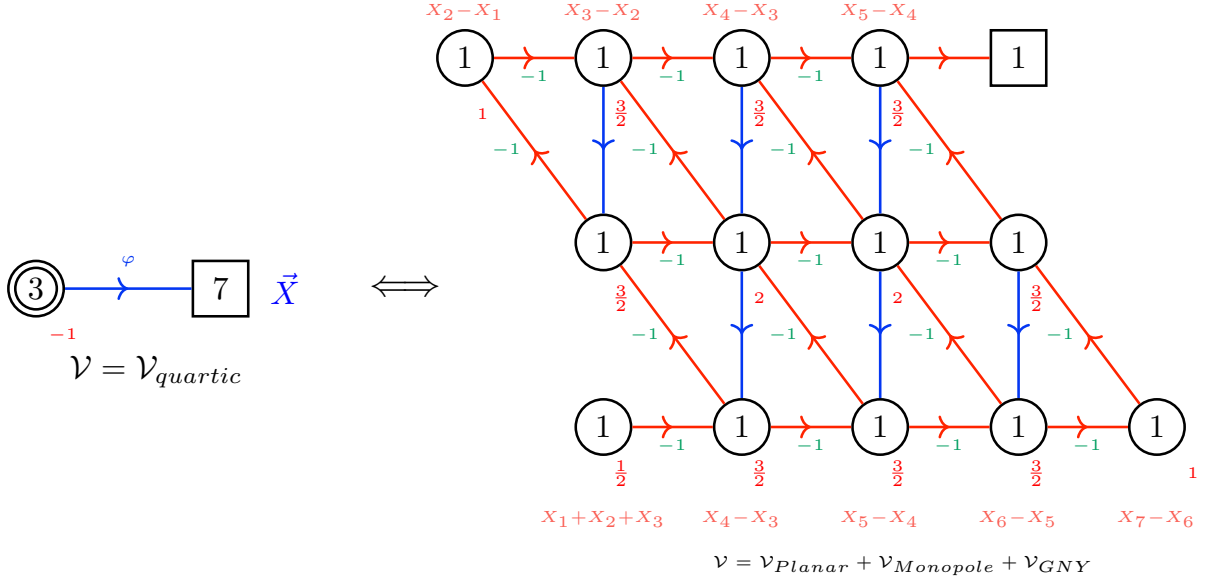


Figure 4.5: The dual for $SU(3)_{-1}$ QCD₃ with 7 scalars is shown here. All labeling conventions follow those of Figure 4.2. Mixed BF couplings between gauge nodes connected by vertical scalar fields have been suppressed for brevity.

We are concerned with the mapping of 42 off-diagonal **mesonic operators** (identified by being charged under $Q_{X_i} - Q_{X_j}$) and $\binom{7}{3} = 35$ **baryonic operators** (identified by being charged under $Q_{X_i} + Q_{X_j} + Q_{X_k}$), all of which are Lorentz scalars. Since the reasoning we follow is the same as before, we leave the complete operator map to the reader and only provide the GNO fluxes of the bare monopoles as a representative example of the general trends of construction and identification of these operators.

We report the monopoles mapped to mesons of the electric theory in Equation (4.42). Note that we only specify the off-diagonal elements of the meson matrix in Equation

(4.42) and the remaining monopoles can be constructed by charge conjugation).

$$\begin{aligned}
\mathbf{M} \leftrightarrow & \left\{ \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\
& \mathfrak{M} \begin{pmatrix} + & + & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} 0 & + & + & + \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & + \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}, \\
& \left. \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & + \end{pmatrix} \right\}
\end{aligned}
\tag{4.42}$$

In order for these monopoles to be gauge invariant they need to be appropriately dressed as explained in Section 2.1.5. The fermions used to dress the monopoles, in the chosen monopole background, have a spin-0 mode. Using this mode we dress the monopoles to obtain spin-0 operators that therefore have the correct spin to map to mesons of the QCD_3 .

We report the monopoles mapped to baryons of the electric theory in Equation (4.43).

Note that monopoles mapped to anti-baryons can be constructed by charge conjugation.

$$\begin{aligned}
\mathbf{B} \leftrightarrow & \left\{ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{pmatrix}, \right. \\
& \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 \\ + & + & + & 0 \\ + & + & + & + & 0 \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & + \end{pmatrix}, \\
& \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 \\ + & + & + & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & + & + \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 \\ + & + & + & + \\ + & + & + & + & + \end{pmatrix}, \\
& \left. \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 \\ + & + & + & + \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 \\ + & + & + & + \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + & + \end{pmatrix} \right\}
\end{aligned}
\tag{4.43}$$

These monopoles are not gauge invariant and hence they should be dressed. The dressing can be done analogously as in Table 4.2 (or 4.4). The resulting operators have Lorentz spin 0, matching that of the baryons.

4.2.5 Mapping Gauge Invariant Operators: The General Pattern

Before we conclude, we briefly comment on the general trend of mapping gauge invariant operators between the dual theories. We remind the reader that the proposed duality exchanges flavor and topological symmetries; hence, mesons, conserved currents and baryons of the electric theory will be mapped to suitably dressed disorder operators (monopoles) of the mirror theory. We ask the reader to keep Figure 4.2 in mind for the rest of the discussion. We also take this opportunity to remind the reader that a specific choice of fugacities of the topological symmetry parametrizes each column.

Mesons and Conserved Currents

Mesons and conserved currents are Lorentz scalars and vectors, respectively, that transform in the adjoint representation of the flavor symmetry group $SU(N_s)$. We now specify the general pattern of mapping these operators to monopoles of the mirror theory (Figure 4.2). We focus on the mapping of the $\frac{1}{2}N_s(N_s - 1)$ “off-diagonal” mesons $\varphi_I \bar{\varphi}_J$ with $J < I$. The case $J > I$ can be obtained by taking the Hermitian conjugate of these operators, which map to monopoles with negative GNO fluxes.

A generic minimal flux mesonic operator **cannot** be charged under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram. This is due to the parameterization of topological fugacities and representation-theoretic constraints. Modulo a subtlety arising from the presence of a monopole interaction in the dual quiver—already discussed in Section 4.2.1—we propose that *off-diagonal* mesonic operators are mapped to monopoles carrying GNO flux under the gauge nodes in the top row and/or along the rightmost diagonal of the quiver.

To describe this mapping systematically, we find it convenient to organize the mesons $\varphi_I \bar{\varphi}_J$ into three distinct classes, based on the values of the flavor indices I and J :

- First, we consider the case where both I and J are less than $N_s - N$.
- Next, we study the regime where $J < N_s - N \leq I$.
- Finally, we address the case in which both I and J are greater than or equal to $N_s - N$.

$\varphi_{I+1} \bar{\varphi}_I$, **where** $J < I < N_s - N$: The simplest case is that of the meson $\varphi_{I+1} \bar{\varphi}_I$ with $(1 \leq I \leq N_s - N - 1)$. In the mirror theory, this is mapped to a monopole with +1 GNO flux under the top node of the I^{th} column:

$$\varphi_{I+1} \bar{\varphi}_I \leftrightarrow \left(\begin{array}{cccccccccccc} & \overbrace{\hspace{1.5cm}}^{N_s - N} & & \overbrace{\hspace{1.5cm}}^k & & & & & & & & \\ 0 & \dots & \boxed{+} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & & \\ & & \uparrow & & & & & & & & & \\ & & I & & 0 \text{ GNO Flux} & & & & & & & \end{array} \begin{array}{c} \nearrow^N \\ 0 \\ \vdots \\ 0 \end{array} \right) \quad (4.44)$$

Thereafter, +1 charges must propagate contiguously from left to right. A meson of the form $\varphi_{I+1+A}\bar{\varphi}_I$ ($A \neq 0$) is dual to a monopole with the following charge assignment:

$$\varphi_{I+1+A}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + \dots + 0}^{N_s - N} \overbrace{0 \dots 0}^k \quad \begin{array}{c} \diagdown^N \\ 0 \end{array} \\ \underbrace{\hspace{10em}}_A \\ 0 \text{ GNO Flux} \end{array} \right) \quad (4.45)$$

$\varphi_I\bar{\varphi}_J$, **where** $J < N_s - N \leq I$: In this case, the dual monopole has a contiguous string of +1 GNO fluxes starting from the J^{th} column and ending on the $(I+k-1)^{\text{th}}$ column. In particular, each of the k nodes inside the region of the quiver connected by BF coupling has GNO flux +1. The first case of this type is the meson $\varphi_{N_s-N+1}\bar{\varphi}_I$, which is mapped to a monopole of the form:

$$\varphi_{N_s-N+1}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + \dots + + + \dots + +}^{N_s - N} \quad \begin{array}{c} \diagdown^N \\ 0 \end{array} \\ 0 \text{ GNO Flux} \end{array} \right) \quad (4.46)$$

Once a charge has been assigned to a column, no additional charges can be assigned to the same column in lower rows. The propagation of fluxes for lower rows follows a strict “staircase” pattern, i.e.: charges in the lower rows can only be assigned to the rightmost column. Thus, a generic mesonic monopole mapped to $\varphi_{N_s-N+1+J}\bar{\varphi}_I$ ($1 \leq J \leq N$) will have the following flux assignment:

$$\varphi_{N_s-N+1+J}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + + + \dots + +}^{N_s - N} \quad \begin{array}{c} \diagdown^J \\ + \end{array} \\ 0 \text{ GNO Flux} \end{array} \right) \quad (4.47)$$

$\varphi_I\bar{\varphi}_J$, **where** $N_s - N \leq J < I$: Mesons corresponding to values of $I \geq N_s - N + 1$ will instead have their +1 flux assigned to the rightmost column of the lower rows, and subsequent charge assignment will follow the usual staircase assignment. Thus, the meson $\varphi_I\bar{\varphi}_J$ ($N_s - N + 1 \leq J < I \leq N_s$) will be mapped to a monopole of the form:

$$\varphi_I \bar{\varphi}_J \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots 0}^{N_s - N} \overbrace{0 \dots 0}^k \\ \hline 0 \text{ GNO Flux} \end{array} \begin{array}{c} + \\ \diagdown \\ \vdots \\ + \\ 0 \end{array} \begin{array}{c} I - J \end{array} \right) \quad (4.48)$$

Generically, all the monopoles considered above will have spin-0 and spin-1 channels after they have been dressed - the spin-0 operators correspond to the off-diagonal elements of the meson matrix, while the spin-1 operators correspond to conserved currents in the original electric theory. We remind the reader that the diagonal elements $|\varphi_i|^2$ are mapped to mass deformations (refer Section 4.5.1 for a more detailed discussion on mass deformations) in the dual theory, while the diagonal currents map to topological $U(1)$ currents (refer Section 4.2.2 for an example of the dressing) in the mirror.

Baryons

Baryons are Lorentz scalars that transform in the rank- N_s antisymmetric representation of the flavor symmetry group $SU(N_s)$. We describe how these operators map to minimal flux operators in the planar mirror theory. We focus on the mapping of $\binom{N_s}{N}$ baryons to monopoles in the dual theory. Anti-baryons can be obtained by taking the Hermitian conjugate of these operators, which would map to monopoles with negative GNO fluxes.

$(\prod_{j=1}^{N-1} \varphi_j) \varphi_{N+I}$, **where** $0 \leq I \leq N_s + k - N - 1$: A generic minimal flux baryonic monopole operator **must** carry charge +1 under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram¹³.

Accordingly, the baryon $\prod_{j=1}^N \varphi_j = \varphi_1 \dots \varphi_N$ is dual to the minimal flux configuration:

$$\prod_{j=1}^N \varphi_j \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline + \end{array} \right) \quad (4.49)$$

Thereafter, the charge assignment must propagate contiguously from left to right in a row. A baryon of the form $\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N+I} = \varphi_1 \dots \varphi_{N-1} \varphi_{N+I}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N+I} \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline \underbrace{+ \ + \ \dots \ +}_I \end{array} \right) \quad (4.50)$$

¹³The topological symmetry associated to this node is parametrized by $\sum_{k=1}^N X_k$, whereas all other nodes have topological symmetries parametrized by $X_i - X_{i+1}$. Thus, the group theoretic properties of a monopole acquiring charge +1 under this specific node follow from representation theory.

As before, care must be taken if there is a BF-coupled region (of length k), and each of the k nodes inside the region of the quiver connected only by BF couplings must have GNO flux $+1$. For instance, the baryon $\varphi_1 \dots \varphi_{N-1} \varphi_{N_s-N+1}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N_s-N+1} \leftrightarrow \left(\begin{array}{c} \text{0 GNO Flux} \\ \hline \underbrace{+ \ + \ \dots \ + \ + \ \dots \ + \ +}_{N_s - N + 1} \end{array} \right) \quad (4.51)$$

$(\prod_{k=1}^{N-J} \varphi_k)(\prod_{k=N-J+2}^{N+1} \varphi_k)$, **where** $1 \leq J \leq N-1$: The flux assignment in higher rows follows a “staircase” pattern. For instance, the baryon $\left(\prod_{j=1}^{N-2} \varphi_j \right) \varphi_N \varphi_{N+1} = \varphi_1 \dots \varphi_{N-2} \varphi_N \varphi_{N+1}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-2} \varphi_j \right) \varphi_N \varphi_{N+1} \leftrightarrow \left(\begin{array}{c} \text{0 GNO Flux} \\ \hline \begin{array}{c} \text{---} \\ + \\ + \end{array} \underbrace{\dots \ 0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 0}_{N_s - N + 1} \end{array} \right) \quad (4.52)$$

Thus, the baryon $\left(\prod_{k=1}^{N-J} \varphi_k \right) \left(\prod_{k=N-J+2}^{N+1} \varphi_k \right) = \varphi_1 \dots \varphi_{N-J} \varphi_{N-J+2} \dots \varphi_{N+1}$ is dual to the flux configuration:

$$\left(\prod_{k=1}^{N-J} \varphi_k \right) \left(\prod_{k=N-J+2}^{N+1} \varphi_k \right) \leftrightarrow \left(\begin{array}{c} 0 \qquad \qquad \qquad \text{0 GNO Flux} \\ \hline \begin{array}{c} \ddots \\ \text{---} \\ + \end{array} \underbrace{\begin{array}{c} \text{---} \\ + \\ + \end{array} \dots \ 0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 0}_{N_s - N + 1} \end{array} \right) \quad (4.53)$$

Mapping a generic baryon: Unlike mesons, baryons allow column reuse: a single column may host fluxes in multiple rows, provided the staircase rule is respected. Thus, a generic baryonic monopole will have the following flux assignment:

$$\left(\prod_{m=1}^{N-J-1} \varphi_m \right) \left(\prod_{l=1}^J \varphi_{N+b_l-l} \right) \varphi_{N+I} \quad \leftrightarrow \quad \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \overbrace{+ \dots +}^{b_J} & 0 & \dots \\ & & \swarrow & & \\ & & & \overbrace{+ \dots +}^{b_1} & 0 & \dots \\ & & & & \underbrace{+ \dots +}_{I} & 0 & \dots \end{pmatrix} \quad (4.54)$$

We note that $k \geq 0$ does not change the counting of the mesonic and baryonic operators.

4.3 Planar Abelian Duals of CS-QCD₃ with Bosons and Fermions

In this section, we generalize the proposal of the previous section to non-Abelian theories with fundamental scalars and fermions. We anticipate that adding a fermionic flavor in the electric theory corresponds to “bosonizing” a column in the planar theory. We provide several examples of this here.

4.3.1 General Proposal

We propose the following planar, Abelian dual for $SU(N)$ QCD₃ with N_s scalars and N_f fermions at Chern–Simons level $2N - N_s - \frac{N_f}{2} - k$ in Figure 4.6. The proposed duality is believed to hold in the following range of parameters¹⁴:

$$N_s \geq 2N, \quad N_f \geq 0, \quad k \geq 0 \quad (4.55)$$

This proposal should be seen as a generalization of the one considered for scalar QCD₃ obtained by replacing N_f scalars with N_f fermions and replacing the corresponding fermionic columns in the planar mirror with purely bosonic columns.

We briefly comment on the potential $\mathcal{V}$ shown in Figure 4.6. In addition to quartic scalar interactions (Equation 4.19), mixed scalar-fermion interactions are also generated radiatively in the electric QCD₃ theory and are captured by the electric potential $\mathcal{V}_{quartic}$:

$$\begin{aligned} \mathcal{V}_{quartic} = & \sum_{\alpha, \beta=1}^N \sum_{I, J=1}^{N_s} \sum_{A=1}^{N_f} \left[\varphi^{\alpha I} \bar{\varphi}_{\alpha J} \varphi^{\beta J} \bar{\varphi}_{\beta I} + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\varphi^{\beta J} \bar{\varphi}_{\beta J}) + \varphi^{\alpha I} \bar{\varphi}_{\beta I} \bar{\psi}_{\alpha A} \psi^{\beta A} \right. \\ & \left. + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\psi^{\beta A} \bar{\psi}_{\beta A}) \right]. \end{aligned} \quad (4.56)$$

On the other hand, in addition to the usual Yukawa interactions, the monopole potential, and the GNY potential, an extra contribution is captured by $\mathcal{V}_{Planar}$ in the mirror

¹⁴In the present chapter we just focus on the case where $N_s \geq 2N$, in [4] we also consider examples with $2N - 2 \leq N_s < 2N$

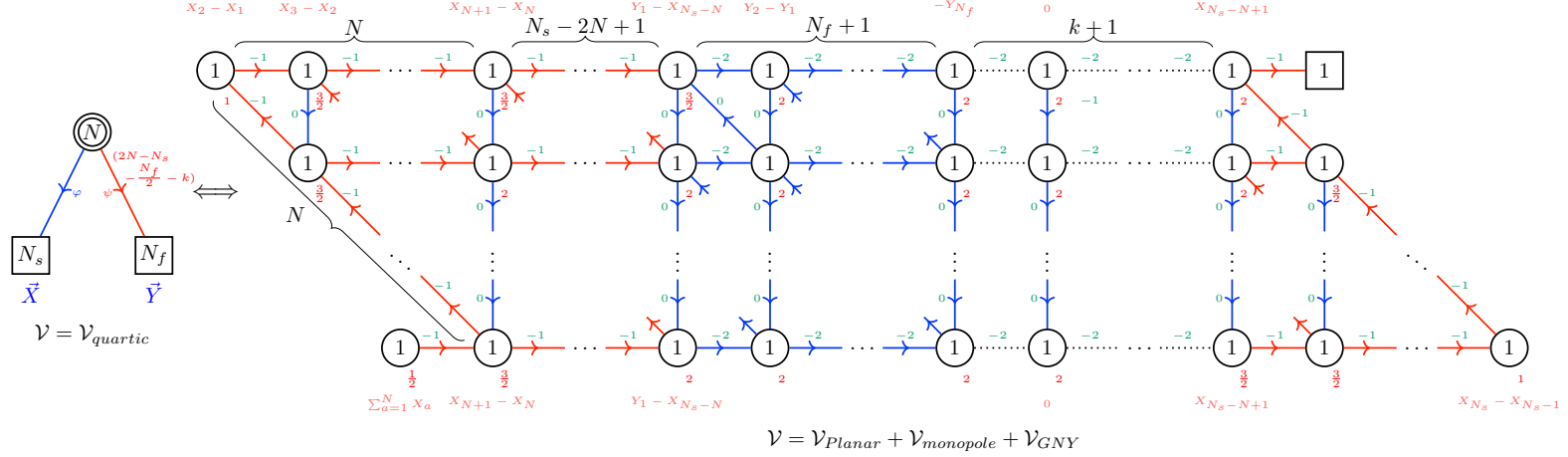


Figure 4.6: The proposed dual for $SU(N)$ QCD₃ with N_s scalars and N_f fermions at Chern–Simons level $2N - N_s - \frac{N_f}{2} - k$ is shown here. As before, single/double circles (squares) correspond to U/SU gauge (flavor) symmetries, scalar (fermion) bifundamental fields are indicated in blue (red), Chern–Simons levels of gauge nodes are indicated in maroon, while mixed BF couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. Note that the Chern–Simons level of each gauge node is equal to minus half the sum of all BF couplings with the surrounding nodes. We also indicate the fugacities of the topological symmetries of each column in orange.

theory. $\mathcal{V}_{Planar}$ contains a cubic scalar interaction ($\phi_i \phi_j \phi_k + c.c.$) for every closed planar triangle. There is a term with a coefficient of $+1$ (-1) for each triangle closed clockwise (anticlockwise). $\mathcal{V}_{monopole}$ involves every monopole with GNO fluxes $+1/-1$ under two nodes connected by a vertical scalar, analogous to the case discussed in Section 4.2, and are dressed as explained in Section 2.1.5. As before, we claim $\mathcal{V}_{monopole}$ and $\mathcal{V}_{Planar}$ are essential to match the rank of the global symmetry on the electric side. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)_{top}^{N_s} \times U(1)_{top}^{N_f}$, where each $U(1)$ factor corresponds to the topological symmetry of the $N_s + N_f - 1$ columns of the quiver, and the additional $U(1)$ factor is the topological symmetry of the bottom-most gauge node. We also claim the symmetry enhances in the IR:

$$U(1)_{top}^{N_s+N_f} \rightarrow U(N_s) \times U(N_f) \quad (4.57)$$

which is mapped precisely to the flavor symmetries of the electric theory. Furthermore, we expect the faithful symmetry to be [66]:

$$\frac{U(N_s) \times U(N_f)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^C \quad (4.58)$$

where $\mathbb{Z}_N$ is the center of $SU(N)$ and $\mathbb{Z}_2^C$ is a charge conjugation symmetry.

In the rest of this section, we make these ideas more precise by considering some examples.

A Note on Higher Spin Baryons and Dual Monopoles We take a moment to comment on higher spin baryonic operators and their dual flux configurations here. Consider

a generic baryon of the electric theory:

$$(\chi_1)^{i_1}(\chi_2)^{i_2} \cdots (\chi_F)^{i_F} \quad (4.59)$$

where each χ_i is either a boson or fermion, and the powers satisfy $\sum_{k=1}^F i_k = N$ ¹⁵. This baryon maps to the following monopole operator in the mirror theory:

$$\begin{pmatrix} \overbrace{+ \ +}^{j_1} \ \overbrace{+ \ +}^{j_2} \ \cdots \ \overbrace{+ \ +}^{j_N} \ \cdots \\ + \ + \ \cdots \ + \ + \\ \quad + \ \cdots \ + \ + \ + \\ \quad \quad \quad + \ + \ \cdots \ \cdots \ 0 \ 0 \\ \quad \quad \quad \overbrace{+ \ +}^{j_{F-1}} \ \cdots \ \cdots \ 0 \ 0 \end{pmatrix} \quad (4.60)$$

Here, j_k denotes the number of $+$ in column k (excluding the base corner $+$ which is always present). The j_k are related to the baryonic charges i_k via:

$$\begin{cases} j_1 &= 1 - i_1 \\ j_2 &= 2 - i_1 - i_2 \\ \vdots & \\ j_N &= N - \sum_{k=1}^N i_k \\ j_{N+1} &= N - \sum_{k=1}^{N+1} i_k \\ \vdots & \\ j_{F-1} &= N - \sum_{k=1}^{F-1} i_k \end{cases} \quad (4.61)$$

some j_k may be negative—for instance, if $i_1 > 1$, then $j_1 < 0$. In such cases, negative fluxes (denoted by “ $-$ ” symbols) must appear in the corresponding column. The fluxes within each column should be distributed across rows in a way that ensures the resulting monopole is *minimally* dressed. We will see some explicit examples of this now.

Finally, we note that the number of spin- J baryons that can be built without derivatives in $SU(N)$ QCD₃ with N_s scalars and N_f fermions is $\binom{N_f+2J-1}{2J} \times \binom{N_s}{N-2J}$, constructed by (anti-)symmetrizing the (bosonic) fermionic fields, where the first factor is the dimension of the $2J$ -index symmetric representation of $SU(N_f)$. Consider in particular the spin- J baryons:

$$\psi_{\{i_1} \psi_{i_2} \cdots \psi_{i_{2J}}\} \varphi_{[j_1} \varphi_{j_2} \cdots \varphi_{j_{N-2J}}] \quad (4.62)$$

where the flavor indices of scalars j_n are antisymmetrized and the flavor indices of fermions i_n are symmetrized. The fermions are contracted to give a spin- J operator. This operator transforms in the $2J$ -index symmetric representation of $SU(N_f)$ and in the $(N-2J)$ -index antisymmetric representation of $SU(N_s)$. The operator in the highest weight:

$$(\psi_1)^{2J} \varphi_1 \varphi_2 \cdots \varphi_{N-2J} \quad (4.63)$$

¹⁵N.B.: If multiple copies of the same scalar appear, derivatives must be inserted; otherwise, the antisymmetric contraction vanishes.

is mapped to the following monopole on the mirror side:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ & 0 & \dots & & \\ & & + & + & + & \dots & + \\ & & & + & + & \dots & + \\ & & & & + & + & \dots & + \end{pmatrix} \left. \vphantom{\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ & 0 & \dots & & \\ & & + & + & + & \dots & + \\ & & & + & + & \dots & + \\ & & & & + & + & \dots & + \end{pmatrix}} \right\}^J \dots \begin{pmatrix} 0 & 0 \\ \dots & 0 & 0 \end{pmatrix}$$

$\underbrace{\hspace{10em}}_{N_s - N}$

(4.64)

4.3.2 Example: $SU(3)_{-\frac{1}{2}}$ QCD₃ with $N_s = 6$ scalars and $N_f = 1$ fermion

Let us begin with the example of $SU(3)_{-\frac{1}{2}}$ with 6 scalars and 1 fermion, and its planar dual description. We propose the duality shown in Figure 4.7.

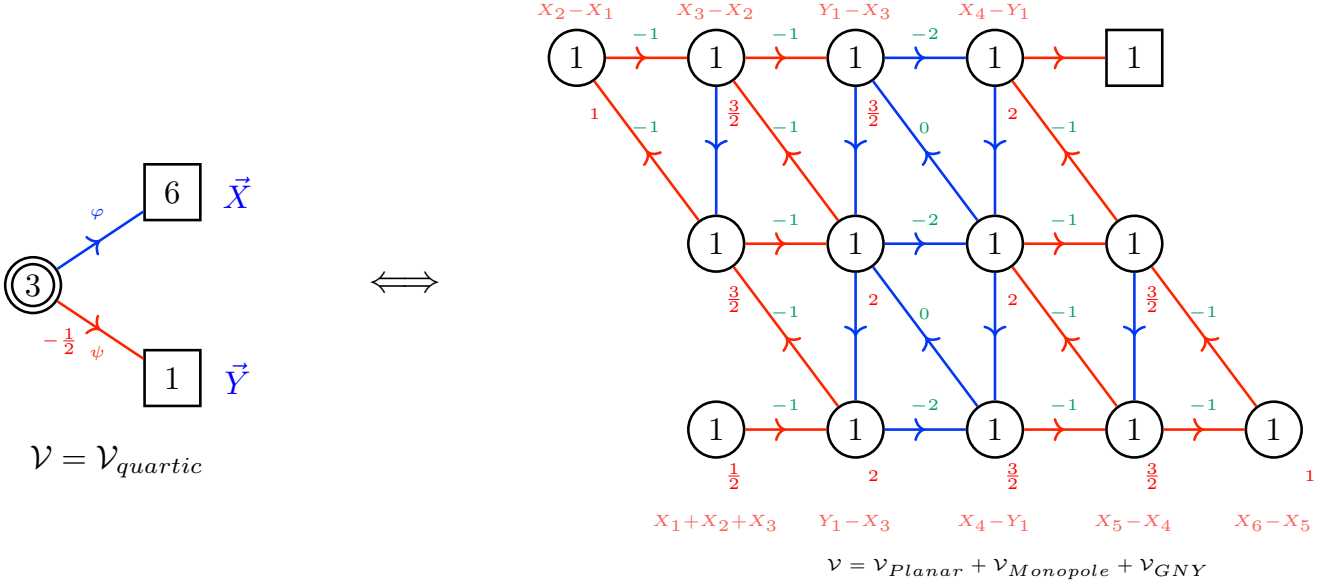


Figure 4.7: The dual for $SU(3)_{-\frac{1}{2}}$ QCD₃ with 6 scalars and 1 fermion is shown here. All labeling conventions are the same as Figure 4.6, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

We are mainly concerned with mapping the following gauge invariant operators of the electric theory:

1. There are 48 **mesonic operators** of which:
 - (a) 35 spin-0 mesons of the form $\varphi_i \bar{\varphi}_j$, identified by $X_i - X_j$ (including their charge conjugates).
 - (b) 12 spin- $\frac{1}{2}$ mesons of the form $\varphi_i \bar{\psi}$, identified by $X_i - Y_1$ (including their charge conjugates).
 - (c) 1 meson with a spin-0 and spin-1 channel $\psi \bar{\psi}$.

As already discussed in the previous cases the mapping of diagonal mesons, which are neutral under the maximal torus of the global symmetry, is less straightforward, and we leave it to future work. The mapping of these off-diagonal mesons is given in Table 4.5.

2. There are many **baryonic operators**:

- (a) $\binom{6}{3} = 20$ spin-0 baryons of the form $\varphi_i \varphi_j \varphi_k$, identified by $X_i + X_j + X_k$.
- (b) 6 baryons with a spin-1 channel of the form $\psi \psi \varphi_i$, identified by $2Y_1 + X_i$. Since there is a single fermion flavor, the antisymmetrization of the color indices forces the two fermions to be symmetrized in their spinor indices, so there is no spin-0 channel.
- (c) $\binom{6}{2} = 15$ spin- $\frac{1}{2}$ baryons of the form $\psi \varphi_i \varphi_j$, identified by $Y_1 + X_i + X_j$.
- (d) 1 baryon with spin- $\frac{3}{2}$ channels, identified by $3Y_1$, constructed by symmetrizing ψ in the flavor and spinor indices.

The mapping of baryons is given in Table 4.6.

Table 4.5: Dressed monopole operators of the planar mirror (Figure 4.7) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & \text{red arrow} & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{red arrow} & + & \text{red arrow} & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_1$
$\begin{pmatrix} + & \text{red arrow} & + & \text{red arrow} & + & \text{blue arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_1$	$\begin{pmatrix} + & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_1$
$\begin{pmatrix} + & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & + \\ & 0 & 0 & 0 & 0 & + & \text{red arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \end{pmatrix}$	$\varphi_5 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & + \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \end{pmatrix}$	$\varphi_6 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \text{red arrow} & + & \text{red arrow} & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{red arrow} & + & \text{red arrow} & + & \text{blue arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_2$
$\begin{pmatrix} 0 & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & + \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & + \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \end{pmatrix}$	$\varphi_5 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \text{red arrow} & + & \text{red arrow} & + & \text{red arrow} & + \\ & 0 & 0 & 0 & 0 & + & \text{red arrow} \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \end{pmatrix}$	$\varphi_6 \bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 & \text{red arrow} & + & \text{blue arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & \text{red arrow} & + & + & \text{red arrow} & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & \text{red arrow} & + & + & \text{red arrow} & 0 \\ & 0 & 0 & 0 & 0 & + & \text{red arrow} \\ & 0 & 0 & 0 & 0 & 0 & + & \text{red arrow} \end{pmatrix}$	$\varphi_5 \bar{\varphi}_3$

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Table 4.5 – continued from previous page

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} 0 & 0 \xrightarrow{+} & + \xrightarrow{-} \\ 0 & 0 & 0 \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \end{pmatrix}$	$\varphi_6 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 \xrightarrow{-} & \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{-} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 & 0 \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \end{pmatrix}$	$\varphi_5 \bar{\varphi}_4$	$\begin{pmatrix} 0 & 0 & 0 & 0 \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \end{pmatrix}$	$\varphi_6 \bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 \xrightarrow{-} & \xrightarrow{+} \\ 0 & 0 & 0 \xrightarrow{-} & \xrightarrow{-} \\ 0 & 0 & 0 & 0 \xrightarrow{-} \end{pmatrix}$	$\psi \bar{\varphi}_5$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \xrightarrow{-} \\ 0 & 0 & 0 & 0 \xrightarrow{+} \end{pmatrix}$	$\varphi_6 \bar{\varphi}_5$
$\begin{pmatrix} 0 & 0 & 0 \xrightarrow{-} & \xrightarrow{+} \\ 0 & 0 & 0 \xrightarrow{-} & \xrightarrow{-} \\ 0 & 0 & 0 & 0 \xrightarrow{-} \end{pmatrix}$	$\psi \bar{\varphi}_6$		

Table 4.6: Dressed monopole operators of the planar mirror (Figure 4.7) and their putative mapping to baryonic operators of the electric theory (all indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + \xrightarrow{-} 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \varphi_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 \xrightarrow{-} & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \psi$
$\begin{pmatrix} 0 \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_3 \psi$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \varphi_4$
$\begin{pmatrix} 0 \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_3 \varphi_4$	$\begin{pmatrix} 0 \xrightarrow{-} 0 \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_4 \psi$
$\begin{pmatrix} + \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_2 \varphi_3 \varphi_4$	$\begin{pmatrix} + \xrightarrow{-} 0 \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_2 \varphi_4 \psi$
$\begin{pmatrix} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \end{pmatrix}$	$\varphi_3 \varphi_4 \psi$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 \xrightarrow{-} 0 & 0 & 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \varphi_5$
$\begin{pmatrix} 0 \xrightarrow{-} 0 & 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} 0 & 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_1 \varphi_3 \varphi_5$	$\begin{pmatrix} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_2 \varphi_3 \varphi_5$
$\begin{pmatrix} 0 \xrightarrow{-} 0 \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_1 \varphi_5 \psi$	$\begin{pmatrix} + \xrightarrow{-} 0 \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_2 \varphi_5 \psi$
$\begin{pmatrix} + \xrightarrow{-} + \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_3 \varphi_5 \psi$	$\begin{pmatrix} 0 \xrightarrow{-} 0 \xrightarrow{-} 0 & 0 & 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \\ + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} + \xrightarrow{-} 0 \end{pmatrix}$	$\varphi_1 \varphi_4 \varphi_5$

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Table 4.6 – continued from previous page

[illegible]

4.3.3 Example: $SU(2)_{-\frac{5}{2}}$ with $N_s = 5$ scalars and $N_f = 1$ fermion

We now consider the example of $SU(2)_{-\frac{5}{2}}$ with 5 scalars and 1 fermion. Notice that this theory has $k = 1$. We propose the duality shown in Figure 4.8.

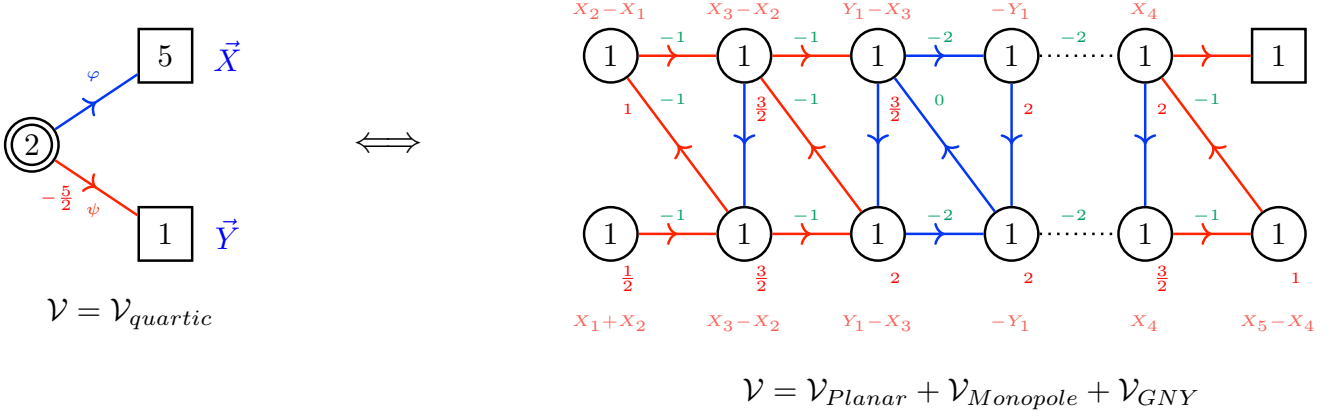


Figure 4.8: The dual for $SU(2)_{-\frac{5}{2}}$ QCD₃ with 5 scalars and 1 fermion is shown here. All labeling conventions are the same as Figure 4.6, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

We are mainly concerned with the mapping of $\binom{5}{2} = 10$ spin-0 mesons, 5 spin- $\frac{1}{2}$ mesons, and $\binom{5}{2} = 10$ spin-0 baryons, 5 spin- $\frac{1}{2}$ baryons and 1 baryon with spin-1 channel. Since the details of the dressings are identical to the discussions preceding this section, we leave the complete operator map to the reader and provide the GNO fluxes of the bare monopoles as a representative example of the general trends of identifying these operators.

The mapping of the off-diagonal spin-0 mesons is provided in Equation 4.65 (the remaining mesons can be identified by charge conjugation):

$$\mathbf{M}_{|spin-0} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\mathfrak{M} \begin{pmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{M} \begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \left. \right\}. \quad (4.65)$$

A generic spin-0 mesonic monopole operator will be dressed as follows¹⁶:

$$\mathfrak{M} \begin{pmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.66)$$

The mapping of the off-diagonal spin- $\frac{1}{2}$ mesons is provided in Equation 4.67 (the remaining mesons can be identified by charge conjugation):

¹⁶We indicate the presence of only BF couplings in the second example by keeping a dotted line explicit between the two nodes.

$$\mathbf{M}|_{\text{spin-}\frac{1}{2}} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} + + + 0 0 \\ 0 0 0 0 0 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 + + 0 0 \\ 0 0 0 0 0 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 0 + 0 0 \\ 0 0 0 0 0 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 0 0 + + \\ 0 0 0 0 0 0 \end{pmatrix}, \right. \\ \left. \mathfrak{M} \begin{pmatrix} 0 0 0 + + \\ 0 0 0 0 0 + \end{pmatrix} \right\}. \quad (4.67)$$

A generic spin- $\frac{1}{2}$ meson will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 \text{ } \textcolor{red}{\rightarrow} + \textcolor{blue}{\leftarrow} 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \text{ } \textcolor{blue}{\rightarrow} + \cdots + \textcolor{red}{\leftarrow} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.68)$$

The mapping of the spin-0 baryons is provided in Equation 4.69 (the remaining antibaryons can be identified by charge conjugation):

$$\mathbf{B}|_{\text{spin-}0} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \right. \\ \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ + & + & + & + & + \end{pmatrix}, \\ \left. \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & + \end{pmatrix} \right\}. \quad (4.69)$$

A generic spin-0 baryon will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} + \textcolor{red}{\leftarrow} 0 & 0 & 0 & 0 \\ + & + \textcolor{red}{\leftarrow} 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 \textcolor{red}{\leftarrow} 0 \textcolor{red}{\leftarrow} 0 & 0 & 0 & 0 \\ + & + & + & + \textcolor{red}{\leftarrow} 0 \end{pmatrix} \quad (4.70)$$

The mapping of the spin- $\frac{1}{2}$ baryons is provided in Equation 4.71 (the remaining antibaryons can be identified by charge conjugation):

$$\mathbf{B}|_{\text{spin-}\frac{1}{2}} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & 0 \end{pmatrix}, \right. \\ \left. \mathfrak{M} \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & + \end{pmatrix} \right\}. \quad (4.71)$$

A generic spin- $\frac{1}{2}$ baryon will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} + \textcolor{red}{\leftarrow} 0 & 0 & 0 & 0 \\ + & + & + \textcolor{blue}{\leftarrow} 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + \textcolor{blue}{\leftarrow} 0 & 0 \\ + & + & + & + \cdots + \textcolor{red}{\leftarrow} 0 \end{pmatrix} \quad (4.72)$$

Finally, the baryon ψ^2 obtained by symmetrizing the flavor and spin indices is mapped to the dressed monopole:

$$\mathfrak{M} \begin{pmatrix} + & + & + \textcolor{blue}{\leftarrow} 0 & 0 \\ + & + & + \textcolor{blue}{\leftarrow} 0 & 0 & 0 \end{pmatrix} \quad (4.73)$$

4.4 Planar Abelian Duals of Unitary CS-QCD₃ with Bosons and Fermions

We have so far focused on constructing duals of $SU(N)$ QCD₃ with mixed matter content. These results, however, extend naturally to the case of *unitary* QCD₃ by gauging the baryonic symmetry. From the discussion in Section 4.2, we know that the baryonic symmetry of the $SU(N)$ QCD₃ maps to the topological symmetry of the bottom-left gauge node of the planar Abelian quiver (Figure 4.2). Gauging of this topological symmetry with zero CS level introduces a $U(1)_0$ sector with no charged matter whose path integral results in a delta-function, which effectively ungauges the $U(1)$ bottom-left gauge symmetry. While we did not keep track of background CS levels when presenting the duality in Figure 4.2, we expect the duality to involve a non-zero CS level for the baryonic symmetry. In analogy with the SUSY result of [1, 2], we expect a background CS at level -2 for the baryonic symmetry on the QCD side of 4.2. Therefore we propose that the dual description for $U(N)$ QCD₃ with N_s critical scalars and N_f fermions, where the theory is defined at Chern–Simons level $(2N - N_s - \frac{N_f}{2} - k, -N_s - \frac{N_f}{2} - k)$ ¹⁷ is the planar Abelian quiver depicted in Figure 4.9.

A particularly interesting special case arises when $k = 0$, shown in Figure 4.10. In this setting, the planar mirror theory contains a gauge-invariant mesonic operator built from bifundamental fields that connect the two $U(1)$ flavor nodes. This meson must map to a monopole operator in the electric $U(N)$ QCD₃ theory that is neutral under the $S(U(N_s) \times U(N_f))$ global symmetry. Consider the bare monopole $\mathfrak{M}$ with GNO fluxes $(+, 0, \dots, 0)$ in the $U(N)_{2N - N_s - \frac{N_f}{2}, -N_s - \frac{N_f}{2}}$ QCD₃ that transforms as the highest weight of the $(N_s - 2N)$ -index symmetric conjugate representation of $SU(N)$ and has charge N_s under the diagonal $U(1)$. This monopole can be made gauge invariant by dressing with N_s

¹⁷We adopt the standard notation:

$$U(N)_{(k, k + \ell N)} \cong \frac{SU(N)_k \times U(1)_{N(k + \ell N)}}{\mathbb{Z}_N},$$

which corresponds to the Chern–Simons action

$$-\frac{ik}{4\pi} \int \text{tr}(A \wedge dA) - \frac{i\ell}{4\pi} \int \text{tr}(A) \wedge \text{tr}(dA).$$

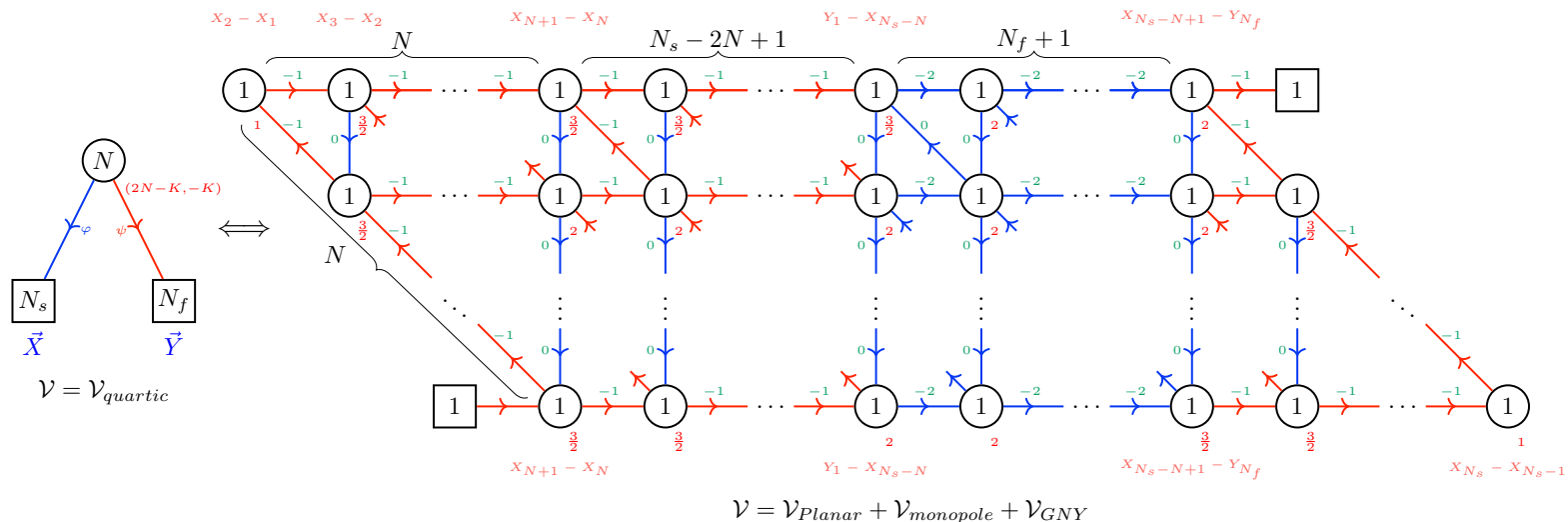


Figure 4.10: The dual of $U(N)$ QCD_3 with N_s scalars and N_f fermions at Chern–Simons level $(2N - K, -K)$, with $K = N_s + \frac{N_f}{2}$, is shown here. All labeling conventions follow those of Figure 4.6.

4.5 Mass Deformations and RG Flow Across Dualities

In this section, we focus on studying the mapping of mass deformations across dualities, which serves as an important consistency check for our proposed dualities. We consider mass deformations for the fundamental fields in the QCD theory and propose a corresponding deformation in the Abelian dual that is partly inspired by the structure of the SUSY dualities reviewed in Chapter 3, and is a natural generalization of the Abelian case.

We also investigate the topological phases obtained by giving mass to all the charged fields in the QCD_3 theory and provide a consistent mapping of these deformations for some values of N , k and the number of charged fields.

4.5.1 Mass for a Single Field

A simple deformation that can be studied in the QCD_3 theory on the l.h.s. of Figure 4.6, consists of giving a mass to a boson or to a fermion that can be either positive or negative. One could think of it as a linear potential term for a mesonic operator of the form $|\varphi|^2$ or $\bar{\psi}\psi$. In principle, knowing the map of these operators across the duality 4.6 gives the knowledge of the deformation that is turned on in the dual theory. However, as we partially comment in Subsection 4.2.5, the map for these operators is not trivial¹⁸. Partially inspired by the SUSY ancestor of the duality, for which the deformations are analyzed in Section 3.5, we consider examples to discuss massive deformations of the duality 4.6. We stress that, since the map of the singlet operators $|\varphi_i|^2$ and $\bar{\psi}_i\psi_i$ is not derived independently, the checks of this section should be understood as consistency checks of the endpoints of the RG flows: the deformation on the mirror side is proposed,

¹⁸We recall that the reason behind the difficulty of this map is that these operators are not charged under the Cartan subgroup of the global symmetry. Indeed, in the dual theory, there are a large number of operators, carrying the correct spin, that are not charged under the global symmetry, and in principle any linear combination of them is a valid candidate as the target of the map.

and the non-trivial statement is that the two theories obtained at the end of the flow are again dual according to our general proposal.

An Electric Scalar with Positive Mass

Concretely, consider $SU(2)$ at CS level $4 - F$ with F scalars and its planar mirror dual (top row of Figure 4.11). Suppose we turn on a positive mass term $m^2|\varphi_3|^2$ in the electric QCD_3 theory for one of the scalar fields (φ_3). We claim that this deformation maps to real mass terms for the fermions in one of the columns of the dual theory as:

$$m^2|\varphi_3|^2 \quad \leftrightarrow \quad m(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3), \quad m > 0 \quad (4.75)$$

where ψ_1 and ψ_3 are the horizontal fermions in the second column and ψ_2 is the diagonal fermion in the second column. Which column of fermions acquires a mass is dictated by the position of the massive flavor in the fugacity parametrization of Figure 4.2: the deformation must preserve the topological symmetries dual to the Cartan generators of the unbroken $U(F-1) \times U(1)$ flavor symmetry. Upon integrating out the massive scalar field, the electric theory flows to $SU(2)$ QCD_3 with $F-1$ scalars at CS level $4 - F$. On the mirror side, we integrate out the fermions in the second column, leaving behind only BF couplings between the horizontal gauge nodes, as shown in the bottom row of Figure 4.11. The resulting duality is exactly the one we propose for $SU(N)$ with $F-1$ scalars at CS level $2N - k - (F-1)$, where $N=2$ and $k=1$ in this case. Notice that this is a natural generalization of the Abelian case in Figure 4.1, where giving a positive mass to a boson on the QED side corresponds to giving a positive mass to a fermion in the quiver side. The choice of signs for the fermionic masses in (4.75) reproduces the BF couplings expected from the general proposal 4.2.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric scalar with a positive mass results in the integration of a column of fermions in the bulk of its planar mirror, leaving behind only horizontal BF couplings between gauge nodes.

One can immediately realize that the number of columns composed of fermions in the bulk of the dual quiver is $N_s - 2N$, where N_s is the number of scalars. Therefore we can use this mechanism — which trades a bulk fermionic column for an empty, BF-coupled column — to integrate out at most $N_s - 2N$ bosons out of N_s , and it cannot by itself produce a dual for QCD_3 with $N_s < 2N$, which indeed is beyond the range of validity of the duality 4.6. Positive masses for additional scalars are instead expected to map to masses for the fermions in the outer (triangular) columns of the quiver: integrating these out removes entire columns of gauge nodes, rather than leaving empty ones, and produces quivers outside the general template of Figure 4.6. We encounter an explicit instance of this second mechanism in Subsection 4.5.2. A systematic study of these further massive deformations is a quite non-trivial exercise and we leave it for a future work.

One could also consider a negative mass for an electric scalar, which would condense and Higgs the gauge group $SU(N)_{2N-N_s-k} \rightarrow SU(N-1)_{2(N-1)-(N_s-1)-k'}$, where $k' = k-1$. We stress that the map of the deformation in (4.75) works only for $m > 0$. It would be interesting to see how this may be seen in the planar mirror theory, but it is beyond the scope of this chapter.

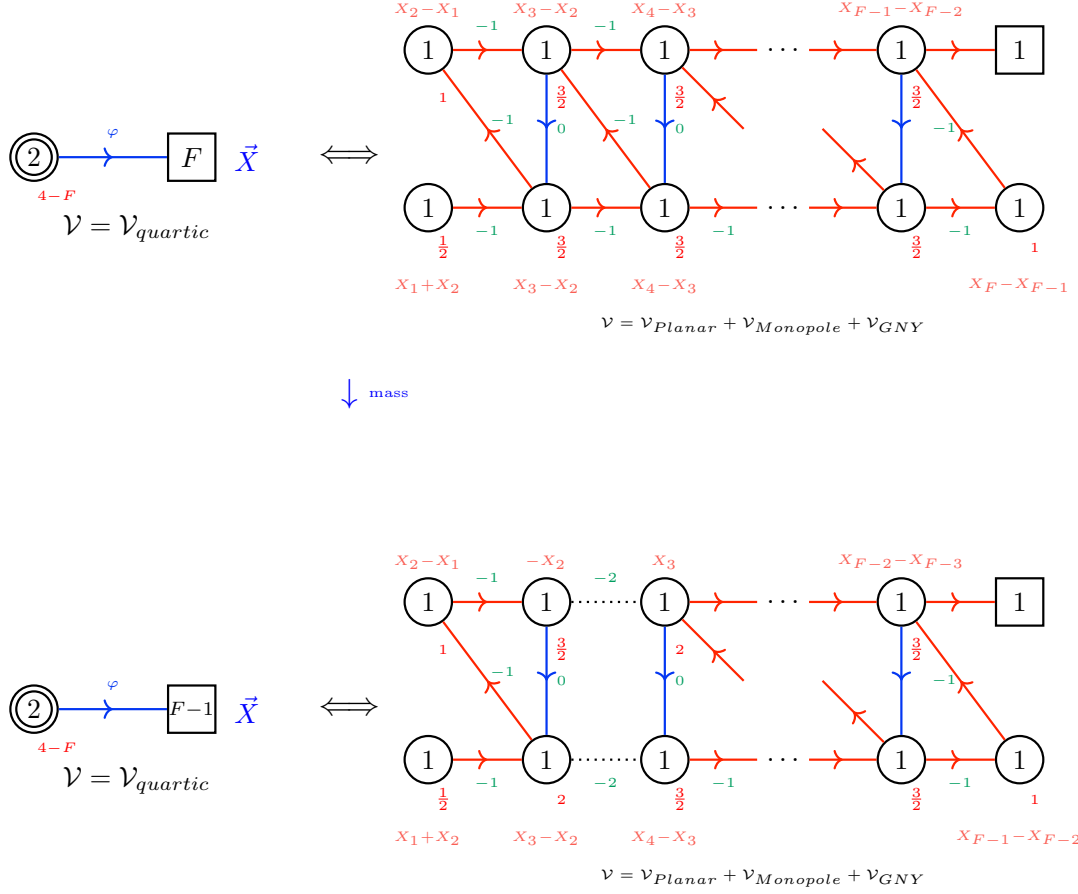


Figure 4.11: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single scalar field in $SU(2)_{4-F}$ QCD₃ with F scalar fields and the corresponding fermionic column in its planar dual (top row). Notice that CS and BF interactions have been generated according to the sign of the corresponding fermion masses. The resulting duality is exactly what we propose for $SU(2)$ with $F - 1$ scalars at CS level $4 - F$.

An Electric Fermion with Positive Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 4.12). Suppose we turn on a positive mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a negative mass term for the horizontal bosonic fields in a column of the mirror theory:

$$m\bar{\psi}_{N_f}\psi_{N_f} \leftrightarrow -m^2(|\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2). \quad (4.76)$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD₃ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$. On the mirror side, the deformation causes the horizontal scalars to take a negative mass and thus a VEV, Higgsing pairs of $U(1)$ gauge groups connected by a horizontal scalar to the diagonal $U(1)$. This effectively identifies the two columns of $U(1)$ gauge groups to the left and right of the column of scalars. Furthermore, some of the cubic interactions for the scalars turn into effective masses, which allow one to integrate out the 2 diagonal scalars and 2 of the vertical scalars. The resulting quiver is shown in the bottom row of Figure 4.12.

The resulting duality coincides with our proposed dual for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2}$, where $N = 3$ in this case.

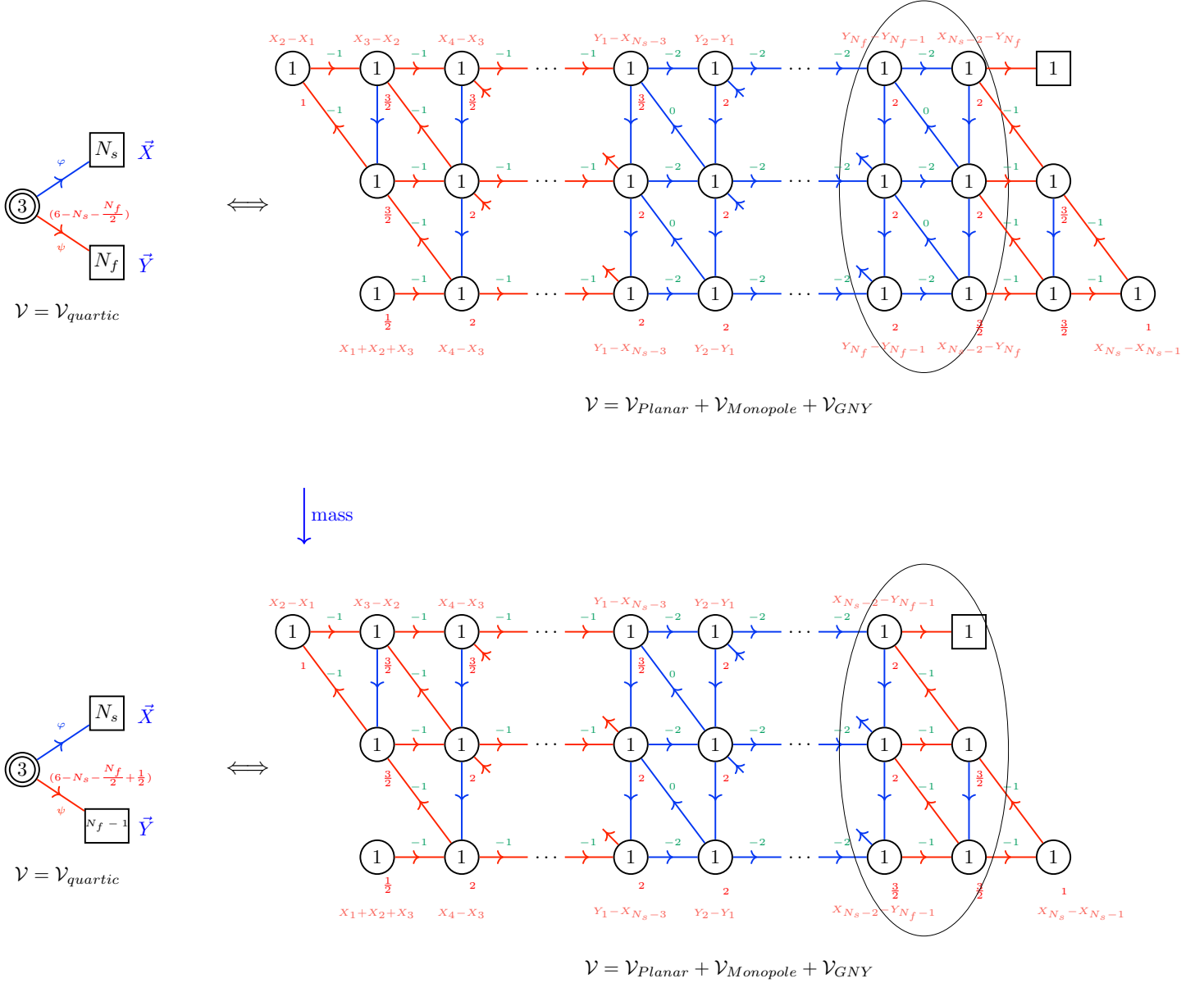


Figure 4.12: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with a positive mass in $SU(3)_{6-N_s-\frac{N_f}{2}}$ QCD₃ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a positive mass results in Higgsing the horizontal bosons in a column in the bulk of its planar mirror, thereby shortening the quiver. One can check that CS and BF coupling of the resulting theory agree with our proposal (Figure 4.6), providing a consistency check of our results.

An Electric Fermion with Negative Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 4.13). Suppose we turn on a negative mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a positive mass term for the bosonic fields in a column of the mirror theory:

$$-m\bar{\psi}_{N_f}\psi_{N_f} \quad \leftrightarrow \quad m^2(|\phi_{diag}^{(1)}|^2 + |\phi_{diag}^{(2)}|^2 + |\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2). \quad (4.77)$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD₃ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$. On the mirror side, the same deformation leads to the integration of a column of bosons, leaving behind only BF couplings between horizontal gauge nodes, as shown in the bottom row of Figure 4.13. The resulting duality is exactly what we propose for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2} - k$, where $N = 3$ and $k = 1$ in this case.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a negative mass results in the integration of a column of bosons in the bulk of its planar mirror, leaving behind only BF couplings between horizontal gauge nodes.

We conclude by mentioning that in the duality 4.6, each fermion in the QCD₃ corresponds to a column of bosons in the dual theory. Therefore, the results summarized in Figure 4.12 and 4.13 can be used to study the effect on the dual theory of a mass for any number of fermions in the QCD₃.

4.5.2 A consistency check via particle-vortex

Consider $SU(2)$ at CS level 0 with 4 scalars and its planar mirror dual (top row of Figure 4.14). Suppose we turn on a positive mass term $m^2|\varphi_1|^2$ in the electric QCD₃ theory for the scalar field φ_1 . Note that here $N_s = 2N$, so the mirror quiver has no fermionic columns in the bulk: the deformation is mapped to masses for the fermions of an outer (triangular) column, whose integration removes a full column of gauge nodes, as discussed at the end of Subsection 4.5.1. This results in the duality shown in the middle row of Figure 4.14, which lies outside the general template of Figure 4.6.

We can further deform the resulting QCD₃ by giving a mass also to the φ_3 scalar field, resulting in a $SU(2)_0$ QCD₃ with only two scalars. We claim that this deformation corresponds to giving mass to the remaining fermions, leading to the duality in the last row of Figure 4.14. Notice that in this last step we actually remain with one scalar and two $U(1)$ gauge nodes, the diagonal combination decouples as a $U(1)_1$ which is an almost trivial theory.

At this point the $U(1)_0$ theory with one scalar can be also dualized using particle-vortex [212–215], resulting in a $O(2)$ Wilson-Fisher scalar. This in turn is also related, via bosonization, to a $U(1)_{\frac{1}{2}}$ theory with a fundamental fermion (4.3). The duality between $SU(2)_0$ with two scalars and $U(1)_{\frac{1}{2}}$ with one fermion, is consistent with a duality proposed in [66].

We can generalize this result further to $SU(N)_0$ with N scalars by starting from $SU(N)_0$ with $2N$ scalars and integrating out N of them, and find that $SU(N)_0$ with N scalars is also dual to an $O(2)$ Wilson-Fisher scalar.

4.5.3 TQFT Dualities from Mass Deformations

Instead of considering mass deformations for a single boson or fermion, we can consider a mass deformation for all of them. In particular we could consider a positive mass for all the scalars and a homogeneous mass for the fermions in the QCD_3 , which always leads to a TQFT, and track the effect in the dual theory that should flow to a dual TQFT to match the IR phase. A general analysis shows that the resulting TQFT duality obtained is quite complicated. Throughout this subsection, TQFT matchings are understood up to invertible (SPT) phases, which we do not track systematically. A possible strategy is to consider gapped phases of the $\mathcal{N} = 2$ theories involved in the ancestor duality [2]. In particular one can study the flow from the ancestor $\mathcal{N} = 2$ duality to a TQFT duality which we believe is equivalent to the TQFT duality we would reach starting from our proposed dual pair. We look at two special cases, consisting of the SQCD theories whose Abelian planar dual contains one or zero “bulk” columns.

We start by discussing the TQFT duality obtained starting from the duality for $SU(N)_{-1}$ with $N_s = 2N$ scalars and zero fermions 4.2. Turning on positive masses for all the scalars results in $SU(N)_{-1}$ CS theory. The planar Abelian dual theory has a single empty bulk column, see Figure 4.2¹⁹.

As explained above we can consider a chiral-planar SUSY duality (as introduced in Chapter 3) where the electric theory flows to the same $SU(N)_{-1}$ TQFT. There the mass deformation can be followed on the planar Abelian side (see in particular Section 3.5.2), resulting in a $U(1)^2$ CS theory with CS matrix:

$$\mathbf{K} = \begin{pmatrix} N & -N \\ -N & N+1 \end{pmatrix}; \quad (4.78)$$

This is the TQFT duality reported in quiver notation in Figure 4.15. We expect that this is the same TQFT duality one would obtain by deforming the non-SUSY duality for $SU(N)_{-1}$ with $N_s = 2N$ scalars and zero fermions. In the remainder of this Section we show that this duality can be proved via known TQFT dualities, providing a check for the flows discussed above.

The 1-form symmetry group of the Abelian CS theory associated to $\mathbf{K}$ is $\mathbb{Z}_N^{(1)}$, as can be deduced from the Smith normal form of $\mathbf{K}$ [216], matching the $SU(N)_{-1}$ on the electric side. The duality between the Abelian CS theory associated to $\mathbf{K}$ and $SU(N)_{-1}$ can be proved by first applying level/rank duality (red arrow in 4.15), and then relating the resulting Abelian theories (double arrow in 4.15). The last step requires us to find two 2×2 matrices $\mathbf{Q}$ and $\mathbf{P}$ with integer coefficients, as described in [216], satisfying the condition

$$\mathbf{Q}^t \mathbf{K}_L^{-1} \mathbf{Q} - \mathbf{K}^{-1} = \mathbf{P}, \quad (4.79)$$

where $\mathbf{K}_L^{-1} = \text{diag}(1, \frac{1}{N})$ and $\mathbf{K}^{-1}$ are the inverse Chern–Simons matrices of the two dual

¹⁹The analysis discussed in this Section extends naturally to other QCD theories whose Abelian dual has a single “bulk” column.

theories, respectively. The following matrices satisfy these requirements:

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad (4.80)$$

where the diagonal elements of $\mathbf{P}$ are even, as required for spin preservation. The matrix $\mathbf{Q}$ is the (left-)invertible matrix specifying the bijective mapping of the $2|\det(\mathbf{K})|$ independent Wilson lines, from the dual side to the electric side, i.e., $\vec{\alpha}_L = \mathbf{Q}\vec{\alpha}$, where $\vec{\alpha}_L$ and $\vec{\alpha}$ are the charge vectors of the Wilson line on the electric and dual side, respectively. Notice that the sequence of dualities depicted in 4.15 requires an additional SPT $U(N-1)_1$ on the l.h.s.. It would be interesting to keep track of the invertible backgrounds generated by the deformation of the quiver theory considered here in order to determine the precise SPT factors involved in the duality for $SU(N)_{-1}$ with $2N$ scalars, but we leave this analysis to future work.

Similarly one can consider the case of $SU(N)_0$ QCD with $N_s = 2N$ scalars. By giving positive masses to all the scalars the theory flows to $SU(N)_0$ YM. The analysis of the flow to the TQFT from the corresponding SUSY theory, together with its planar Abelian dual, predicts that the dual flows to $U(1)_1$, which is an almost trivial TQFT matching the electric side.

4.6 Scope and outlook

Scope. This chapter proposed and tested planar Abelian duals for a family of non-supersymmetric 3d CS-QCD theories with bosonic and fermionic matter simultaneously, for both $SU(N)$ and $U(N)$ gauge groups. The analysis focused on the well-controlled regime $N_s \geq 2N$, where the quiver structure of the Abelian dual is most transparent. In this range, extensive checks were performed: matching of gauge-invariant operators across the duality, consistency with mass deformations and RG flows, derivation of known pure-TQFT limits, and reduction to established Abelian bosonization dualities in special cases. The dualities proposed here are expected to arise as non-supersymmetric limits of the $\mathcal{N} = 2$ mirror pairs of Chapter 3, though establishing this connection precisely lies beyond the scope of the present chapter.

Outlook. Three directions are most immediate. First, a systematic treatment of the small-flavor regime $N_s < 2N$ is needed (examples with $2N - 2 \leq N_s < 2N$ are presented in [4], but a systematic study is lacking): this range exhibits qualitatively new features that cannot be read off directly from the well-controlled quiver construction. Second, the SUSY origin of these dualities should be made precise by tracking SUSY-breaking real mass deformations of the $\mathcal{N} = 2$ mirror pairs of Chapter 3 at the level of operators, partition functions, and the spin-TQFT factors generated along the flow. Third, an algorithmic formulation analogous to the chiral-planar algorithm of Chapter 3 would allow a systematic derivation of planar Abelian bosonization duals for arbitrary non-Abelian CS-QCD theories, moving beyond the case-by-case analysis of this chapter.

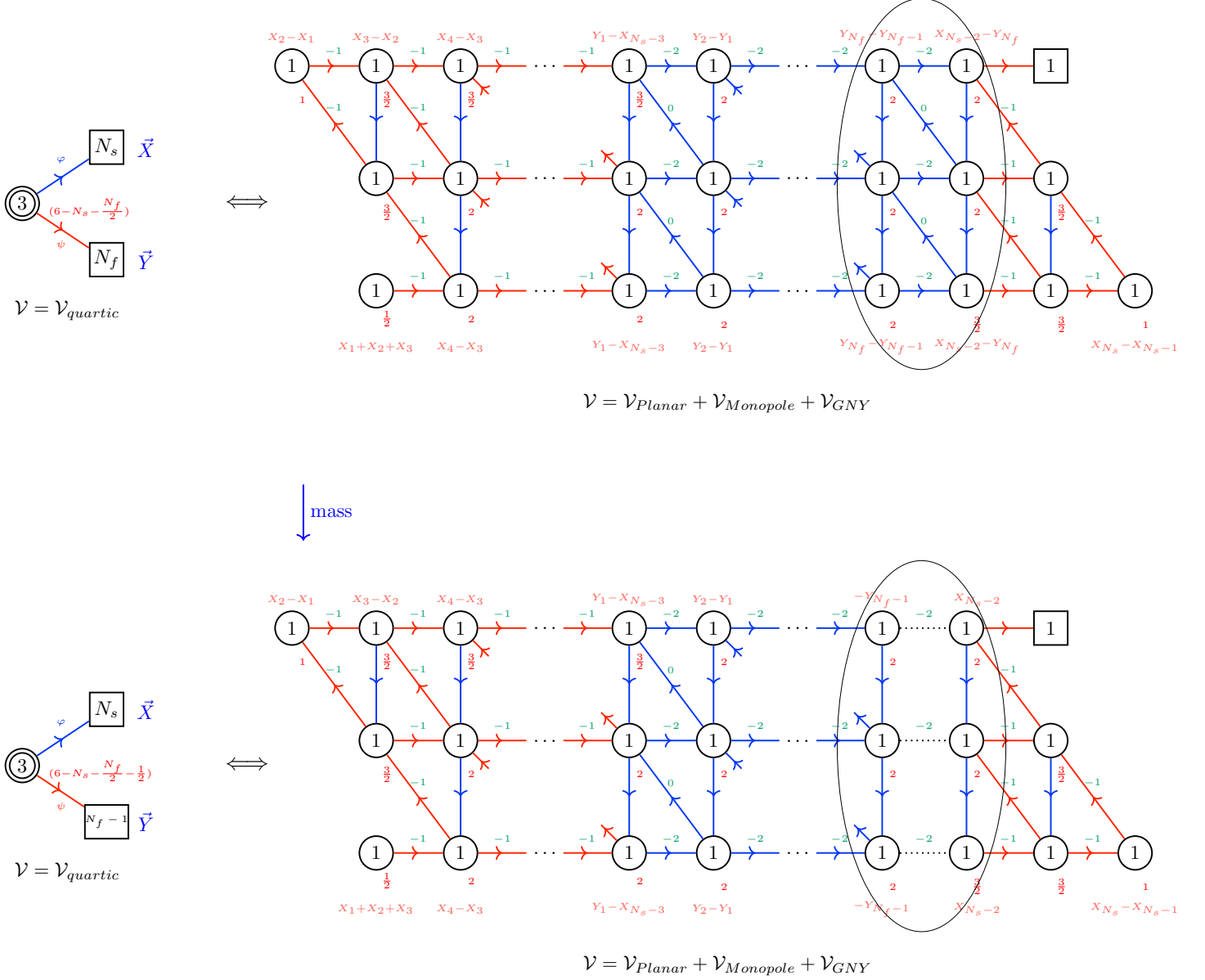


Figure 4.13: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with negative mass in $SU(3)_{6-N_s-\frac{N_f}{2}}$ QCD₃ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$.

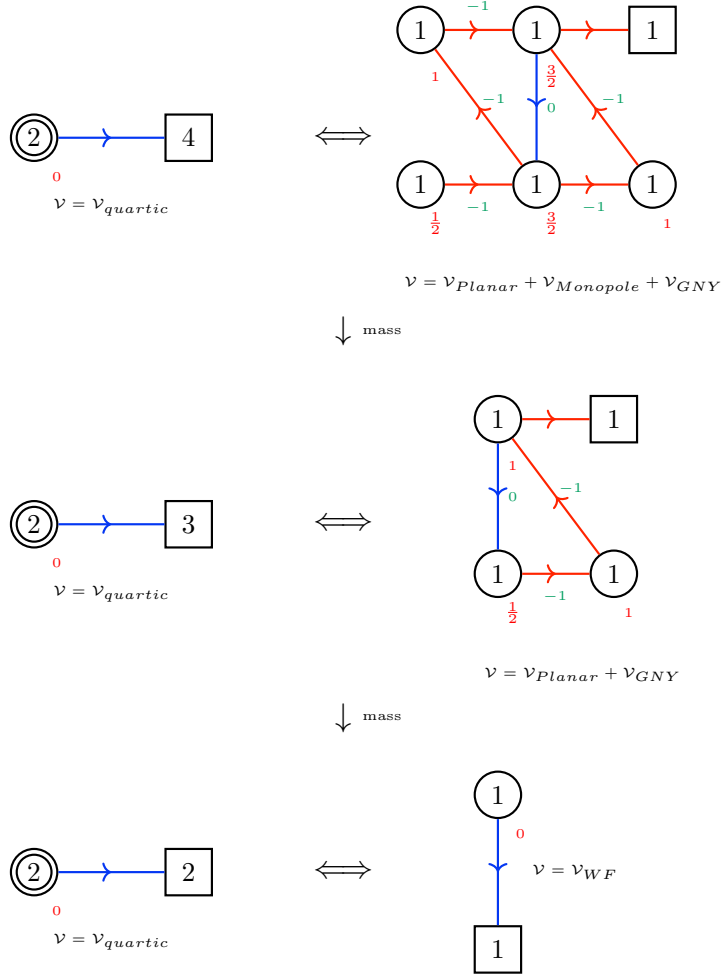


Figure 4.14: Schematic depiction of the duality conjectured for $SU(2)_0$ with 2 scalars. We omit writing topological fugacities for brevity. As per our conventions, fermionic (bosonic) fields are shown in red (blue), and (mixed) Chern–Simons interactions in maroon (green).

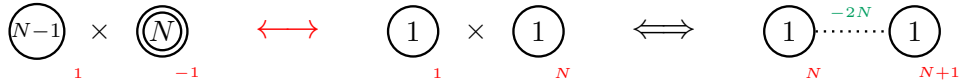


Figure 4.15: Duality between $SU(N)_{-1} \times U(N-1)_1$ CS theory and the Abelian CS theory associated to the CS matrix (4.78). (Mixed) Chern–Simons interactions are indicated in (green) red. The red arrow indicates level/rank duality for $SU(N)_{-1}$, while the double arrow indicates a duality between Abelian TQFTs described in the main body.

Chapter 5

Hilbert series and SCI of the improved bifundamentals

The content of this chapter is essentially taken from [5].

In this chapter we take a detour and we investigate a novel class of SCFTs with four supercharges, termed *Improved Bifundamentals* [56, 171, 217, 218]. Among them, the FE_N series exists in both 4d and 3d, while the FM_N , FH_N , FC_N , and FT_N families are purely three-dimensional. These Improved Bifundamentals exhibit intriguing star-triangle dualities [56] and serve as fundamental building blocks for constructing quivers relevant to various generalizations of 3d mirror symmetry [45, 46, 57], the compactification of 6d E-strings on Riemann surfaces [217], and Hanany-Witten brane setups with four supercharges [57].

In the following sections we aim to elucidate the structure of the moduli space of vacua for the Improved Bifundamentals. Our analysis reveals that the moduli spaces are irreducible algebraic varieties, exhibiting a single branch rather than the multiple intersecting branches typically found in SCFT moduli spaces. Specifically, for FT_N , FC_N , and FH_N , the moduli space consists of a single branch generated by operators in rank-2 representations of the IR non-Abelian global symmetry. In contrast, for FM_N and FE_N , in addition to this *main branch*, we identify supplementary simple branches associated with vacuum expectation values of the B_{ij} operators, which are singlets under the non-Abelian global symmetry.

To explore the moduli space of vacua, we employ the *Plethystic Programme* [108, 112] (see also section 2.1.6)¹. In this context, the HS enumerates holomorphic functions on the moduli space, which are in one-to-one correspondence with chiral ring operators. Consequently, the HS also encodes the relations among these operators, which can be either classical or quantum in nature. In three dimensions, the presence of local monopole operators [26, 27] complicates the computation of the Hilbert Series, generally requiring a summation over monopole sectors. While this problem has been solved for 3d $\mathcal{N} = 4$ gauge theories [85, 226], it remains largely open for 3d $\mathcal{N} = 2$ theories (see [168, 224]).

To compute the HS, including contributions from monopole operators and quantum relations, we begin by evaluating the Superconformal Index (see section 2.4.3 for a review).

¹The plethystic program has notably been applied to study the moduli space of supersymmetric QCD [219] and instanton moduli spaces [220, 221]. Its versatility has also led to various applications in both 4d [222] and 3d [168, 223–225] supersymmetric gauge theories.

The HS can then be extracted from the SCI as follows: the moduli space is an algebraic variety that is polynomially generated by elements of the chiral ring. These elements are $\frac{1}{2}$ -BPS, gauge-invariant operators that belong to short multiplets of the superconformal algebra and are, therefore, captured by the index. By expanding and carefully analyzing these contributions, we claim that, for the class of theories considered in the present thesis, we are able to reconstruct the full Hilbert Series of the moduli space order by order. In what follows, we demonstrate several applications of this approach.

In general, obtaining the complete Hilbert Series as a direct limit of the Superconformal Index is not always feasible. For $3d \mathcal{N} = 4$ theories, it was proposed in [167] that there exist two distinct limits of the SCI which precisely reproduce the Higgs and Coulomb branches of the moduli space (see also last paragraph of section 2.4.3). However, even for theories with 8 supercharges it is difficult to unambiguously extract mixed-branch operators from the index, see [227]. Moreover, as pointed out in [228], for class $\mathcal{S}$ theories, the Hilbert series for the Higgs branch of the resulting $4d \mathcal{N} = 2$ theory agrees with the index prediction only for $6d \mathcal{N} = (2, 0)$ theories compactified on genus zero Riemann surfaces. For $3d \mathcal{N} = 2$ theories, it was shown in [168] that, whenever the theory possesses global $U(1)$ symmetries, a similar analysis to that of $\mathcal{N} = 4$ theories can be performed. In particular, one can attempt to compute the Hilbert Series for individual branches of the moduli space by taking specific limits of the index. However, it has also been observed that, in general, due to accidental cancellations between fermionic and bosonic contributions, the index may fail to reproduce the Hilbert Series of the full moduli space. For the class of theories studied in this chapter, we provide extensive order-by-order evidence that such accidental cancellations do not occur. As a result, we are able to reproduce the Hilbert Series either for the full moduli space or for its main branch.

Summary of the main results

For the FT_N theory of [171], which is the only series whose moduli spaces are complete intersections², we produce a simple formula for the unrefined HS for any N , analogous to the formula for the Higgs branch of $T(SU(N))$ [223]:

$$\mathcal{HS}_{FT_N}(t) = \text{PE} \left[\chi_{[1,0,\dots,0,1]_{SU(N)_L}} t + \chi_{[1,0,\dots,0,1]_{SU(N)_R}} t - \sum_{j=2}^N t^j \right] \stackrel{\text{unr}}{=} \frac{\prod_{j=1}^{N-1} \sum_{h=0}^j t^h}{(1-t)^{2N^2-N-1}}.$$

Note that the full moduli space of $T(SU(N))$ is a union of multiple branches, whereas the moduli space of FT_N , obtained by flipping the adjoint of $T(SU(N))$ generating the Higgs or Coulomb branch, possesses a single branch.

For the other improved bifundamentals we are not able to write the fully unrefined HS in closed form at generic N ; instead, for $N \leq 3$, we compute the HS expressed in the so-called *Euler form* (in a single fugacity):

$$\mathcal{HS}(t; \mathcal{M}) = \frac{h_0 + h_1 t + \dots + h_d t^d}{(1-t)^{\dim \mathcal{M}}}. \quad (5.1)$$

The improved bifundamentals at $N = 1$ are simple free theories³, rendering the computation of the HS trivial. For $N = 2$, thanks to non generic symmetry enhancements, we manage to write the fully unrefined HS as sums over characters.

²For background on Hilbert functions and h-vectors of Gorenstein algebras and complete intersections, see [229–232].

³In this thesis, we consider a version of the FM_N and FE_N theories without the B_{11} singlet, which

We observe that since the number of chiral ring generators grows quadratically with N , the dimension of the moduli space and the degree of the numerator also exhibit quadratic growth with N . This fact allows us to extrapolate these quantities for generic N based on the $N = 1, 2, 3$ results. Let us write the unrefined HS for $N = 1, 2, 3$ in the following table:

	$N = 1$	$N = 2$	$N = 3$
FT_N	1	$\frac{1+t}{(1-t)^5}$	$\frac{(1+t)(1+t+t^2)}{(1-t)^{14}}$
FC_N	$\frac{1}{1-t}$	$\frac{1+3t+t^2}{(1-t)^7}$	$\frac{1+8t+26t^2+39t^3+26t^4+8t^5+t^6}{(1-t)^{17}}$
FH_N	$\frac{1}{(1-t)^2}$	$\frac{(1+t)(1+4t+t^2)}{(1-t)^{11}}$	$\frac{(1+t)^3(1+3t+t^2)(1+8t+22t^2+8t^3+t^4)}{(1-t)^{26}}$
FM_N	$\frac{1}{(1-t)^2}$	$\frac{(1+t)(1+5t+t^2)}{(1-t)^8}$	$\frac{(1+t)^2(1+14t+72t^2+133t^3+72t^4+14t^5+t^6)}{(1-t)^{18}}$
FE_N	$\frac{1}{(1-t)^4}$	$\frac{(1+t)(1+9t+19t^2+9t^3+t^4)}{(1-t)^{16}}$	$\frac{(1+t)^2(1+26t+288t^2+1716t^3+5970t^4+12545t^5+16071t^6+\dots+t^{12})}{(1-t)^{36}}$

The extrapolation at generic N yields the following results for the complex dimension of the moduli space, the number of generators and the degree of the numerator of the HS:

Ref.	Theory	$\dim_{\mathcal{M}}$	Generators	Numerator degree d	Global symmetry ($N \geq 3$)
(5.5)	FT_N	$2N^2 - N - 1$	$2N^2 - 2$	$N(N-1)/2$	$S(U(N) \times U(N))$
(5.8)	FC_N	$2N^2 - 1$	$3N^2 - 2$	$N(N-1)$	$S(U(N) \times U(N)) \times U(1)$
(5.9)	FH_N	$3N^2 - 1$	$5N^2 - N - 2$	$3N(N-1)/2$	$SU(N) \times USp(2N) \times U(1)^2$
(5.6)	FM_N	$2N^2$	$4N^2 - 2$	$N^2 - 1$	$S(U(N) \times U(N)) \times U(1)^2$
(5.7)	FE_N	$4N^2$	$8N^2 - 2N - 2$	$(2N+1)(N-1)$	$USp(2N) \times USp(2N) \times U(1)^2$

The RG flows relating the five families, together with the data just summarized, are collected in Figure 5.1.

Further symmetry enhancements at $N = 2$

The improved bifundamentals at $N = 2$ exhibit an IR symmetry larger than the generic- N IR global symmetry, as discussed for FE_2 in [233]. One way to argue for this enhanced symmetry is through the *exceptional* 3d dualities of [234, 235]. In Section 5.3 we derive simple formulae for the fully unrefined Hilbert series at $N = 2$ as a sum over characters of representations of the enhanced global symmetry group, generalizing the Hilbert series for FT_2 theory found in [236].

On top of the FE_2 and FM_2 , we consider the $\widetilde{FE}_2$ and $\widetilde{FM}_2$ theories, which we define as the FE_2 and FM_2 where the singlet $B_{2,1}$ has been flipped. This makes the dimension of the moduli space larger by one unit. It turns out that the moduli space of the 5 theories $\widetilde{FE}_2$, $\widetilde{FM}_2$, FH_2 , FC_2 and FT_2 are complex cones over *irreducible Hermitian symmetric spaces*⁴ [237, 238].

is appropriate for studying the isolated FM_N and FE_N theories. This implies that FM_1 and FE_1 are equivalent to 2 and 4 free chiral superfields, instead of cubic Wess-Zumino models with 3 and 5 chiral superfields.

⁴We are grateful to Richard Eager for useful correspondence about this point.

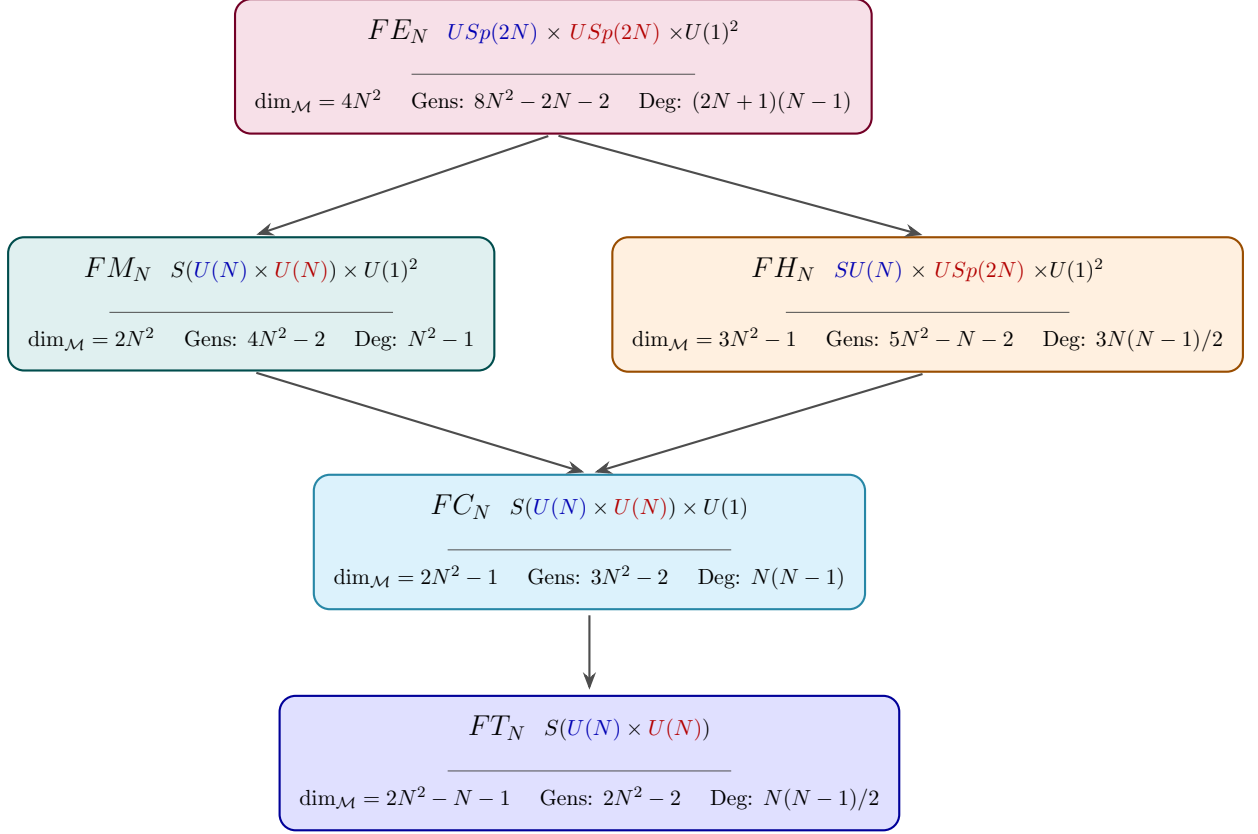


Figure 5.1: Diagram with the relations among the different theories. The arrows denote RG flows: those out of FC_N are triggered by real mass deformations, while those out of FE_N combine a Coulomb-branch vacuum expectation value with a real mass deformation. Each block shows the global symmetry (for $N \geq 3$), the complex dimension of the moduli space, the number of generators and the degree of the numerator of the HS in the Euler form.

A *Hermitian symmetric space* (HSS) of compact type is a compact complex manifold $B = H/K$, where H is a compact simple Lie group and K a closed subgroup of maximal rank whose center contains a $U(1)$ (equivalently, the centraliser of a $U(1) \subset H$), such that B carries an H -invariant Hermitian metric and each point is a fixed point of a global involutive isometry. The central $U(1) \subset K$ endows the tangent space with a complex structure, making B a Kähler manifold. These spaces were classified by Cartan [239–241] into four infinite families and two exceptional cases:

$$\frac{SU(p+q)}{S(U(p) \times U(q))}, \quad \frac{SO(2n)}{U(n)}, \quad \frac{USp(2n)}{U(n)}, \quad \frac{SO(p+2)}{SO(p) \times U(1)}, \quad \frac{E_6}{SO(10) \times U(1)}, \quad \frac{E_7}{E_6 \times U(1)}. \quad (5.2)$$

What makes these spaces especially natural as moduli spaces is their representation-theoretic rigidity: the coordinate ring of the affine cone $\mathcal{M} = \mathcal{C}[H/K]$ decomposes as a *multiplicity-free* sum of irreducible H -representations,

$$\mathbb{C}[\mathcal{M}] = \bigoplus_{k=0}^{\infty} V_{k\lambda}, \quad (5.3)$$

where $V_{k\lambda}$ denotes the irreducible H -representation of highest weight $k\lambda$, and $R = V_{\lambda}$ is the *minimal* one appearing in the decomposition of functions on B . This is precisely

the structure captured by a Hilbert series of the form $\text{HS}(t) = \sum_{k=0}^{\infty} \chi_{V_{k\lambda}} t^k$, which is what we observe for the five theories listed above. The generators of the chiral ring are identified with the elements of R , and the relations they satisfy cut out $\mathcal{C}[H/K]$ inside the free ring $\text{Sym}^\bullet(R)$, whose graded pieces $\text{Sym}^k(R)$ contain $V_{k\lambda}$ together with additional representations that are precisely removed by the relations (Cartan multiplication). For the five theories at hand, the relevant spaces and their generator representations are⁵:

Ref.	Theory	$B = H/K$	R
(5.86)	FT_2	$SU(4)/S(U(2) \times U(2))$	6
(5.115)	FC_2	$SU(5)/S(U(2) \times U(3))$	10
(5.143)	FH_2	$SO(10)/U(5)$	16
(5.167)	$\widetilde{FM}_2$	$SU(6)/S(U(2) \times U(4))$	15
(5.193)	$\widetilde{FE}_2$	$E_6/(SO(10) \times U(1))$	27

(5.4)

Let us summarize the results in diagram 5.2, where the symmetric space is denoted in orange, the HS in blue and $\dim_{\mathcal{M}}$ stands for the complex dimension of the conical moduli space.

5.1 Short review of the *Improved Bifundamental* theories

The *Improved Bifundamentals* are a class of superconformal field theories (SCFTs) with 4 supercharges, introduced in [56, 171, 217, 218], which present quiver UV lagrangian descriptions. They share interesting properties like symmetry enhancement in the IR, (self-)duality of the UV completions under 3d $\mathcal{N} = 2$ mirror symmetry [23, 36] and a rich spectrum of protected operators. In the following sections, we present the theories in the chronological order they were discovered.

5.1.1 The FT_N theory

We begin by considering the $FT_N \equiv FT[SU(N)]$ theory [171], i.e. the flipped version of the $T[SU(N)]$ theory, which arises from the study of S-duality and Dirichlet boundary conditions in 4d $\mathcal{N} = 4$ SYM [113]. We remind the reader that the theory has the following Lagrangian UV completion:

⁵Three of the five spaces carry a well-established magic-square signature: curved $\beta\gamma$ systems on the cones over $\text{Gr}(2, 4)$, $SO(10)/U(5)$, and $E_6/(SO(10) \times U(1))$ exhibit hidden affine $\mathfrak{so}(8)$, $\mathfrak{e}_6$, and $\mathfrak{e}_7$ symmetry respectively, where the last case is precisely the pure spinor sector of the superstring [238]. This fits the general pattern of Landsberg–Manivel [242] projective geometries associated to the Freudenthal–Tits magic square [243–246] arising as moduli spaces of supersymmetric gauge theories [237]. The remaining spaces $\text{Gr}(2, 5)$ and $\text{Gr}(2, 6)$ belong to the same family of rank-two cominuscule Hermitian symmetric spaces and complete the Grassmannian chain $\text{Gr}(2, 4) \subset \text{Gr}(2, 5) \subset \text{Gr}(2, 6)$ and the branching $SU(5) \subset SO(10)$.

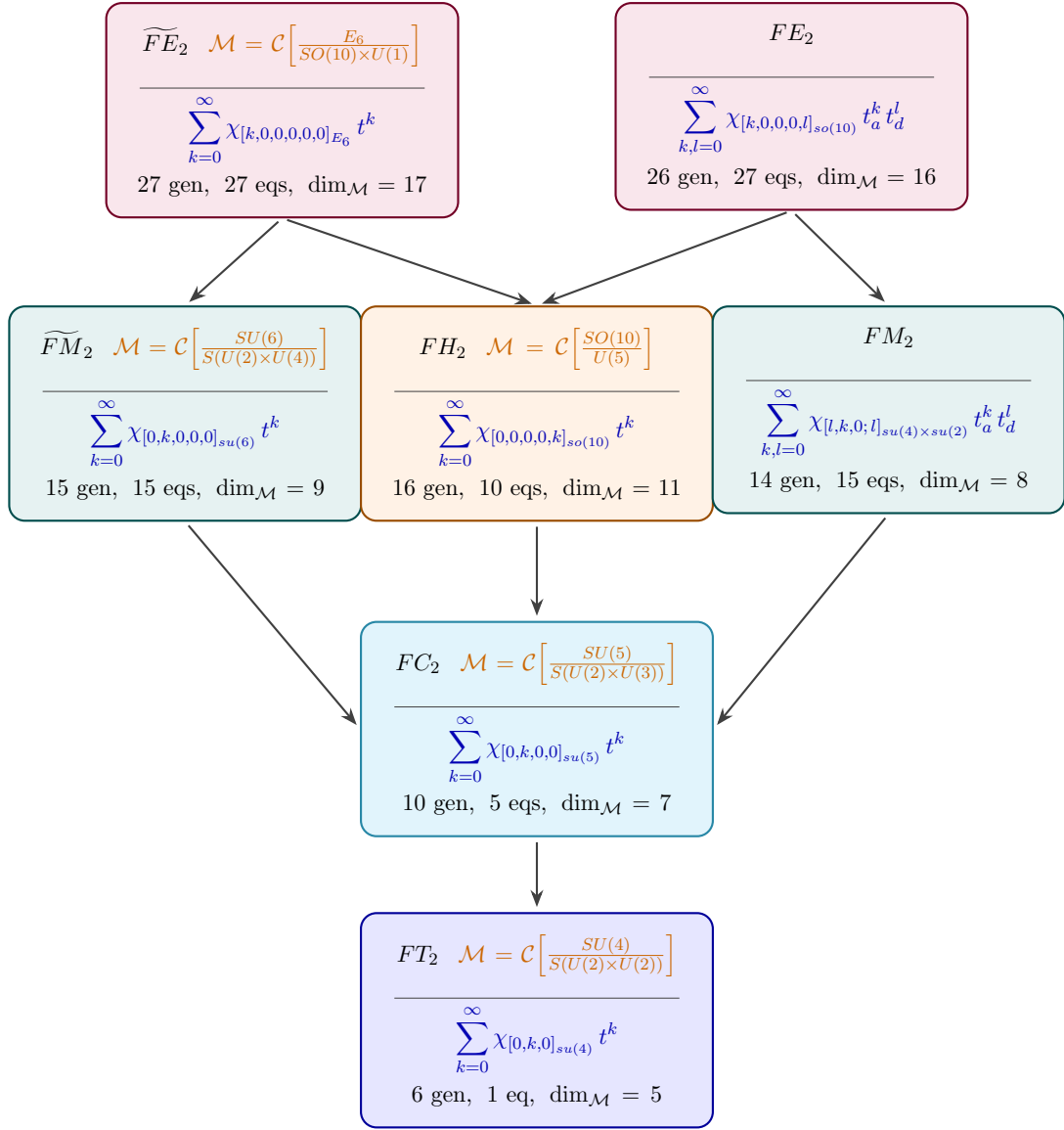


Figure 5.2: Hasse diagram of the $N = 2$ theories. **Orange** labels indicate the associated moduli space as a coset, and **blue** expressions give the corresponding Hilbert series. Arrows denote RG-flow/specialization between theories.

$$\begin{array}{c}
 \begin{array}{c}
 \begin{array}{c}
 \text{Diagram: A chain of nodes } 1 \rightarrow 2 \rightarrow \dots \rightarrow N-1 \rightarrow N. \text{ Nodes 1-} N-1 \text{ are circles, node } N \text{ is a blue square. Links have crosses. Labels } a_2, a_{N-1}, a_N \text{ are above nodes 2, } N-1, N \text{ respectively.}
 \end{array}
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \hline
 R = R_0 + \tau Q_\tau \\
 \hline
 \begin{array}{cc}
 b_i, \tilde{b}_i & \tau/2 \\
 a_i & 2 - \tau
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \mathcal{W} = a_2 b_1 \tilde{b}_1 + \sum_{i=2}^{N-1} b_i (a_i + a_{i+1}) \tilde{b}_i + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i]
 \end{array}
 \quad
 (5.5)$$

In this diagram, a circular node with label k represents a $U(k)$ gauge group, the square blue node represents a global $SU(N)$ symmetry, an outgoing (ingoing) arrow represents an (anti-)fundamental field, and the crosses on the links between the nodes represent flipping fields, i.e., gauge singlets that couple linearly to the gauge-invariant products

$b_i \tilde{b}_i$. The field b_i ($\tilde{b}_i$) transforms in the bifundamental representation $\square \otimes \bar{\square}$ ($\bar{\square} \otimes \square$ resp.) under $U(i) \times U(i+1)$ (where $\square$ ($\bar{\square}$) denotes the (anti-)fundamental representation), while a_i transforms in the adjoint representation of the corresponding $U(i)$ gauge node. The global symmetries of the theory include an $SU(N)$ flavor symmetry, $N-1$ $U(1)$ topological symmetries (one for each gauge node), as well as a $U(1)_\tau$ global symmetry and a $U(1)_R$ R-symmetry. In the table on the right side in (5.5) we list the R charges R of the UV fields, given by a sum of the trial R charge R_0 and mixing with the $U(1)_\tau$ symmetry, where Q_τ denotes the field charge under the symmetry while τ parametrises the mixing of $U(1)_\tau$ with the R-symmetry (the associated fugacity is denoted ψ below).

A notable feature of the theory is that at the IR fixed point, the global symmetry is enhanced to:

$$\textcolor{blue}{SU(N)} \times \textcolor{red}{SU(N)} \times U(1)_\tau \times U(1)_R.$$

This enhancement is related to the self-mirror [36] property of the $\mathcal{N} = 4$ theory, first studied in [113], which implies that both the Coulomb and Higgs branches exhibit an $SU(N)$ global symmetry.

Moreover, at the strongly coupled IR fixed point, the matter content consists of two fields, A_L and A_R , in the adjoint representation of each of the $SU(N)$ global symmetries, respectively. The first operator is the UV traceless matrix $a_N = A_L$, in the adjoint representation of the manifest $\textcolor{blue}{SU(N)}$, while the second operator is constructed by collecting the $N(N-1)$ gauge-invariant monopoles with Goddard-Nuyts-Olive (GNO) [247] charges $\mathfrak{M}^{+0\dots}, \mathfrak{M}^{++0\dots}, \dots$, along with their conjugates with negative GNO charges, and $N-1$ gauge-singlet operators σ_i of R-charge $2-\tau$, given by linear combinations of the traces $\text{Tr}(a_i)$, $i = 1, \dots, N-1$, and of the flipping fields, which fill the Cartan directions.

For example, for $N = 4$, we have:

$$A_R = \begin{bmatrix} 0 & \mathfrak{M}^{+00} & \mathfrak{M}^{++0} & \mathfrak{M}^{+++} \\ \mathfrak{M}^{-00} & 0 & \mathfrak{M}^{0+0} & \mathfrak{M}^{0++} \\ \mathfrak{M}^{--0} & \mathfrak{M}^{0-0} & 0 & \mathfrak{M}^{00+} \\ \mathfrak{M}^{---} & \mathfrak{M}^{0--} & \mathfrak{M}^{00-} & 0 \end{bmatrix} + \sum_{i=1}^3 \sigma_i \mathcal{D}_i$$

where $\mathcal{D}_i$ are the traceless diagonal generators of $SU(4)$.

5.1.2 The FM_N theory

Next we turn to the first “improved” bifundamental discovered, the $FM_N \equiv FM[SU(N)]$ theory [217]. The theory has the following Lagrangian UV completion:

$R = R_0 + \tau Q_\tau + \Delta Q_\Delta$	
$b_i, \tilde{b}_i$	$\tau/2$
a_i	$2 - \tau$
$v_i, \tilde{v}_i$	$2 - \frac{i-N+2}{2}\tau - \Delta$
$d_i, \tilde{d}_i$	$\Delta + \frac{i-N}{2}\tau$

$$\begin{aligned}
\mathcal{W} = & \sum_{j=1}^{N-1} \left[v_j b_j d_{j+1} + \tilde{v}_j \tilde{b}_j \tilde{d}_{j+1} \right] + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] \\
& + (\mathfrak{M}^{\pm 00 \dots} + \mathfrak{M}^{0 \pm 0 \dots} + \dots) + \sum_{i=1}^N \text{Flip}[d_i \tilde{d}_i]
\end{aligned}
\tag{5.6}$$

where the blue node denotes an $SU(N)$ flavor symmetry, while the N red nodes denote $U(1)$ flavor symmetries. This theory, similarly to the FT_N theory, possesses a self-mirror property, which exchanges the manifest $SU(N)$ flavor symmetry with the emergent $SU(N)$ one. The full global symmetry of the theory at the IR fixed point is:

$$S(U(N) \times U(N)) \times U(1)_\tau \times U(1)_\Delta \times U(1)_R.$$

The operator spectrum (chiral ring generators) at the IR fixed point consists of two fields, A_L and A_R , which transform under the representations $\mathbf{Adj} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathbf{Adj}$ of the $SU(N) \times SU(N)$ global symmetries, respectively. Additionally, there are two fields, Π and $\tilde{\Pi}$, in the representations $\square \otimes \bar{\square}$ and $\bar{\square} \otimes \square$, and a collection of singlets, denoted by $B_{n,m}$, with R-charges given by:

$$R(B_{n,m}) = 2n - 2\Delta + (m - n)\tau.$$

In particular, the singlets $B_{1,m}$ correspond to the *flippers* of $d_{N-m+1} \tilde{d}_{N-m+1}$ for $m = 1, 2, \dots, N$, while the singlets $B_{n,m}$ correspond to the gauge singlets $v_{N-m} a_{N-m}^{n-2} \tilde{v}_{N-m}$, for $n = 2, 3, \dots, N$ (reducing to $v_{N-m} \tilde{v}_{N-m}$ for $n = 2$), whose R-charges match the formula above.

This theory is often referred to as an “improved” or “generalized” bifundamental because, compared to the usual bifundamentals of $SU(N) \times SU(N)$, it includes two bifundamental fields in the spectrum and an additional $U(1)$ global symmetry. The usual bifundamentals, on the other hand, only have the global symmetry $S(U(N) \times U(N)) \times U(1)_A \times U(1)_R$. The improved bifundamental becomes particularly relevant when studying mirror symmetry for $\mathcal{N} = 2$ theories, such as $U(N)$ adjoint SQCD with flavors and vanishing superpotential [57]. We mention that the singlet $B_{1,1}$, flipping $d_N \tilde{d}_N$, should be flipped from the theory because its scaling dimension would violate the unitarity bound, signalling that it decouples as a free field along the RG flow (as we checked explicitly for $N = 2$, and we claim holds for $N > 2$ as well). Thus, in the theories that we study below, we will always remove this singlet, and write the correct superpotential by dropping the term $\text{Flip}[d_N \tilde{d}_N] = B_{1,1} d_N \tilde{d}_N$. This is true in the isolated theory, but may not be true when the theory is part of a larger quiver, where the scaling dimension of $B_{1,1}$ should be computed through F -maximization.

5.1.3 The FE_N theory

The next theory we consider is the FE_N theory, also called $FE[USp(2N)]$ in the literature. This theory was originally constructed as a 4d $\mathcal{N} = 1$ class $\mathcal{S}$ theory, resulting from the compactification of the 6d $(1,0)$ rank N E-string SCFT on a torus with flux turned on [218]. If we further compactify the resulting $\mathcal{N} = 1$ theory on $\mathbf{S}^1$, we obtain the FE_N theory in 3d, which has the following Lagrangian UV completion:

$$\mathcal{W} = b_1^2 a_2 + \sum_{i=2}^{N-1} b_i^2 (a_i + a_{i+1}) + \sum_{i=1}^{N-1} [v_i b_i d_{i+1} + \text{Flip}[b_i^2]] + \sum_{j=1}^N \text{Flip}[d_j^2] + (\mathfrak{M}^{100\dots 0} + \mathfrak{M}^{010\dots 0} + \dots + \mathfrak{M}^{00\dots 01})$$

$R = R_0 + \tau Q_\tau + \Delta Q_\Delta$	
b_i	$\tau/2$
a_i	$2 - \tau$
v_i	$2 - \Delta - \frac{i-N+2}{2}\tau$
d_i	$\Delta + \frac{i-N}{2}\tau$

(5.7)

where C_k denotes an $USp(2k)$ symmetry.

Compared to its 4d version, the 3d theory has additional linear monopole terms in the superpotential which necessarily arise after the 3d compactification [234, 248]. The UV global symmetry is given by:

$$USp(2N) \times USp(2)^N \times U(1)_\Delta \times U(1)_\tau \times U(1)_R,$$

which is enhanced in the IR to:

$$USp(2N) \times USp(2N) \times U(1)^2 \times U(1)_R.$$

This enhancement can be understood through a self-duality, which acts by exchanging the manifest $USp(2N)$ symmetry with the emergent one. It is worth noting that the FM_N theory can be reached from FE_N by combining Coulomb branch operators that break $USp(2N)$ to $U(N)$, along with real mass deformations, as described in [176].

In the IR spectrum of the theory, we observe two fields, A_L and A_R , which transform under the antisymmetric traceless representation of each $USp(2N)$ global symmetry. Additionally, there is a field Π in the representation $\mathbf{2N} \otimes \mathbf{2N}$ under $USp(2N) \times USp(2N)$ (where bold numbers label the representation of the corresponding dimension), along with a collection of singlets $B_{n,m}$ with R-charges given by:

$$R(B_{n,m}) = 2n - 2\Delta + (m - n)\tau.$$

In particular, the singlets $B_{1,m}$ correspond to the flippers of d_{N-m+1}^2 for $m = 1, 2, \dots, N$, while the singlets $B_{n,m}$ correspond to the dressed gauge singlets $v_{N-m} a_{N-m}^{n-2} v_{N-m}$, for $n = 2, 3, \dots, N$ (reducing to v_{N-m}^2 for $n = 2$), whose R-charges match the formula above. Again, as for the FM_N theory, the flipper $B_{1,1}$ should be flipped since its scaling dimension is below the unitarity bound (as we checked through F -maximization for $N = 2$). Since

the FE_N is really a 4d $\mathcal{N} = 1$ theory, we are able to perform 4d a -maximization as well, on the corresponding 4d theory FE_N^{4d} , and we find that $R(B_{1,1}) \leq 2/3$ with the equality holding only for $N = 1$ (in this case the theory is a Wess-Zumino model). Moreover, we find that the R-charges of the other singlets $B_{n,m}$ decrease with N , so they eventually hit the 4d unitarity bound ($\Delta = \frac{3}{2}R \geq 1$) and should be flipped to obtain a consistent theory. For the rest of the chapter, whenever we consider the FE_N and FM_N theories, we will consider the version of these theories with the singlet $B_{1,1}$ flipped, and with the correct superpotential where the term $\text{Flip}[d_N^2] = B_{1,1}d_N^2$ (resp. $\text{Flip}[d_N\tilde{d}_N] = B_{1,1}d_N\tilde{d}_N$ for the FM_N theory) has been dropped.

5.1.4 The FC_N theory

Another interesting theory we consider is the FC_N theory, which arises from a specific real mass deformation of the FM_N theory [56]. This deformation gives mass to half of the bifundamentals present in the saw of the FM_N theory. A possible UV completion of this theory is given below:

$R = R_0 + \tau Q_\tau + \Delta Q_\Delta$	
$b_i, \tilde{b}_i$	$\tau/2$
a_i	$2 - \tau$
v_i	$2 - \frac{i-N+2}{2}\tau - \Delta$
d_i	$\Delta + \frac{i-N}{2}\tau$

$$\mathcal{W} = \sum_{j=1}^{N-1} v_j b_j d_{j+1} + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] + (\mathfrak{M}^{+00\cdots} + \mathfrak{M}^{0+0\cdots} + \cdots)$$

(5.8)

Note that, as a result of the deformation, the superpotential contains only $(N - 1)$ cubic terms from the closed triangles and $(N - 1)$ monopole terms. In total, we have integrated out $3N - 1$ chiral fields (the $\tilde{v}_j$, the $\tilde{d}_j$ and the N flippers of $d_j \tilde{d}_j$) while losing $3N - 2$ superpotential terms (the $N - 1$ cubic terms $\tilde{v}_j \tilde{b}_j \tilde{d}_{j+1}$, the $N - 1$ negatively-charged monopole terms and the N terms $\text{Flip}[d_j \tilde{d}_j]$), resulting in one less global $U(1)$ symmetry compared to the FM_N theory.

The FC_N theory also exhibits interesting duality properties derived from the self-duality of FM_N . A consequence of these dualities is the claim that the global symmetry is enhanced in the IR to $SU(N) \times SU(N) \times U(1)_\Delta \times U(1)_\tau \times U(1)_R$. In the IR spectrum, we still have two fields, A_L and A_R , in the representations $\mathbf{Adj} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathbf{Adj}$ of the $SU(N) \times SU(N)$ global symmetries, but a single bifundamental field, Π , in the representation $\square \otimes \bar{\square}$, and *no* additional non-Abelian singlets.

5.1.5 The FH_N theory

Finally, the last theory we wish to study is the FH_N theory, which can be derived starting from the FE_N theory and then performing a combination of Coulomb branch

vacuum expectation value, breaking the gauge USp to U , and real mass deformations, in such a way that one of the two global $USp(2N)$ is broken to $U(N)$, while the other is preserved [56]. A possible Lagrangian UV completion, where the global symmetry is $SU(2)^N \times U(N) \times U(1)_\tau \times U(1)_R$, is as follows:

$R = R_0 + \tau Q_\tau + \Delta Q_\Delta$	
$b_i, \tilde{b}_i$	$\tau/2$
a_i	$2 - \tau$
v_i	$2 - \frac{i-N+2}{2}\tau - \Delta$
d_i	$\Delta + \frac{i-N}{2}\tau$

$$\begin{aligned}
\mathcal{W} = & \sum_{j=1}^{N-1} v_j b_j d_{j+1} + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] \\
& + (\mathfrak{M}^{\pm 00 \dots} + \mathfrak{M}^{0 \pm 0 \dots} + \dots)
\end{aligned}
\tag{5.9}$$

An interesting observation is that from the self-duality of the FE_N theory arises a duality for the FH_N theory exchanging the manifest $U(N)$ global symmetry with the emergent $USp(2N)$. As a consequence, we are able to write a dual frame where the UV symmetry is $U(1)^N \times USp(2N) \times U(1)_\tau \times U(1)_R$, which looks as follows:

$R = R_0 + \tau Q_\tau + \Delta Q_\Delta$	
b_i	$\tau/2$
a_i	$2 - \tau$
v_i	$2 - \frac{i-N+2}{2}\tau - \Delta$
d_i	$\Delta + \frac{i-N}{2}\tau$

$$\begin{aligned}
\mathcal{W} = & \sum_{j=1}^{N-1} v_j b_j d_{j+1} + a_2 b_1^2 + \sum_{j=2}^{N-1} b_j^2 (a_j + a_{j+1}) + \sum_{i=1}^{N-1} \text{Flip}[b_i^2] \\
& + (\mathfrak{M}^{100 \dots} + \mathfrak{M}^{010 \dots} + \dots)
\end{aligned}
\tag{5.10}$$

As a consequence of this duality, the global symmetry of the theory must enhance in the IR to $U(N) \times USp(2N) \times U(1)_\tau \times U(1)_R$, where $U(1)_\Delta \subset U(N)$. In the IR spectrum we now have a field A_L in the adjoint representation of $SU(N) \subset U(N)$, a field A_R in the antisymmetric traceless representation of $USp(2N)$ and a bifundamental Π in the representation $\mathbf{2N} \otimes \bar{\square}$ of $USp(2N) \times U(N)$, and *no* additional non-Abelian singlets.

5.2 Moduli spaces of improved bifundamentals at generic N

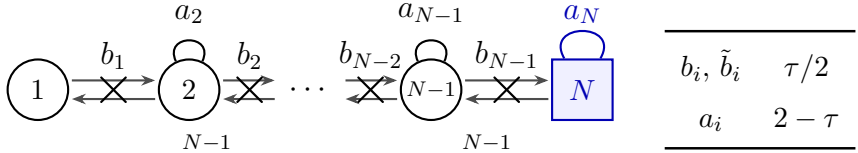
In this section, we study the properties of the Hilbert series for our theories of interest at generic N . Even though the HS are extracted from the superconformal indices, in this section we only discuss the HS; a detailed analysis of the structure of the indices is performed for the $N = 2$ theories in Section 5.3.

We examine the case of $N = 2$ separately, in Section 5.3, highlighting the enhanced symmetry in the IR that this special case enjoys.

The last two theories, FM_N and FE_N , on top of the rank-2 generators (adjoints/antisymmetric and bifundamentals), also have generators which are singlets under the non-Abelian global symmetries. These singlets generate simple branches in the moduli space of vacua. In this section we disregard such branches, and focus on the rank-2 generators, which generate an algebraic variety which we call *main branch*.

5.2.1 FT_N

Let us start from the simplest set of theories, namely FT_N .



$b_i, \tilde{b}_i$	$\tau/2$
a_i	$2 - \tau$

$$\mathcal{W} = a_2 b_1 \tilde{b}_1 + \sum_{i=2}^{N-1} b_i (a_i + a_{i+1}) \tilde{b}_i + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i]$$
(5.11)

For FT_N , the refined Hilbert series is given by

$$\mathcal{HS}_{FT_N} = \text{PE}[\chi_{(\text{adj}_L, \mathbf{1})} t + \chi_{(\mathbf{1}, \text{adj}_R)} t - \sum_{j=2}^N t^j] \quad (5.12)$$

where $\chi_{(\text{adj}_L, \mathbf{1})}$ and $\chi_{(\mathbf{1}, \text{adj}_R)}$ represent the characters of the adjoint representations of $SU(N)_L$ and $SU(N)_R$, respectively. The function PE denotes the Plethystic Exponential.

The unrefined Hilbert series. Setting all flavor fugacities to unity, the refined Hilbert series (5.12) collapses to

$$\mathcal{HS}_{FT_N}(t) = \frac{\prod_{j=1}^{N-1} \sum_{h=0}^j t^h}{(1-t)^{2N^2-N-1}}. \quad (5.13)$$

The numerator is palindromic, which confirms that the moduli space is a Calabi–Yau cone of complex dimension $2N^2 - N - 1$.

Generators and relations from the plethystic logarithm. The main structural result of this subsection is obtained by taking the Plethystic Logarithm (PL) of the refined series (5.12):

$$\text{PL}[\mathcal{HS}_{FT_N}] = \chi_{(\text{adj}_L, \mathbf{1})} t + \chi_{(\mathbf{1}, \text{adj}_R)} t - \sum_{j=2}^N t^j. \quad (5.14)$$

This expression admits a transparent reading: the moduli space is generated by the $2(N^2 - 1)$ components of two adjoint-valued matrices A_L and A_R , transforming in the adjoint of $SU(N)_L$ and $SU(N)_R$ respectively, subject to the $N - 1$ relations

$$\mathrm{tr}_L A_L^k = \mathrm{tr}_R A_R^k, \quad k = 2, \dots, N, \quad (5.15)$$

where tr_L (tr_R) denotes the trace over $SU(N)_L$ ($SU(N)_R$) indices. The fact that (5.14) terminates confirms that no higher syzygies appear: the moduli space is therefore a *complete intersection* of degree $N!$. Analogous results were obtained in [223] for the Higgs branch of the $\mathcal{N} = 4$ $T[SU(N)]$ theory.

The h -sequence and a recursive pattern. Expanding the numerator of (5.13) as

$$\prod_{j=1}^{N-1} \sum_{h=0}^j t^h = \sum_{i=0}^{N(N-1)/2} h_i t^i, \quad (5.16)$$

one reads off the h -sequence $h = (h_0, h_1, \dots, h_d)$ of the moduli space, with $d = N(N - 1)/2$. The first few entries are

$$h = (1, N - 1, \tfrac{1}{2}(N + 1)(N - 2), \tfrac{1}{6}N(N^2 - 7), \dots). \quad (5.17)$$

Remark (First-difference recursion). *We observe that the first-difference sequence of the first half of h ,*

$$\begin{aligned} \hat{h} &= (h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1}) \\ &= (1, N - 2, \tfrac{1}{2}N(N - 3), \tfrac{1}{6}(N - 1)(N^2 - 2N - 6), \dots), \end{aligned} \quad (5.18)$$

coincides, for $k < N$, with the h -sequence of $FT[SU(N - 1)]$. This is in fact an exact consequence of the factorization $[N]!_t = [N - 1]!_t (1 + t + \dots + t^{N-1})$, which implies $h_k(N) - h_{k-1}(N) = h_k(N - 1) - h_{k-N}(N - 1)$: the identity holds exactly for $k < N$ and fails for $k \geq N$ (the first violation inside the first half of the sequence occurs at $k = N = 5$).

Mahonian numbers. The coefficients h_i of the numerator admit a classical combinatorial interpretation. Since $\prod_{j=1}^{N-1} \sum_{h=0}^j t^h = [N]!_t$, where the q -factorial is defined by

$$[n]!_q = \prod_{k=1}^{n-1} (1 + q + q^2 + \dots + q^k), \quad (5.19)$$

one has

$$h_i = \frac{1}{i!} \frac{d^i}{dt^i} [N]!_t \Big|_{t=0}. \quad (5.20)$$

The h_i are the *Mahonian numbers*, first studied by MacMahon [249]; they count permutations of N elements by major index (equivalently, by number of inversions).

Counting BPS operators. The coefficient of t^n in the unrefined Hilbert series (5.13) equals the number of independent gauge-invariant operators of $U(1)_\tau$ charge n . For low N one finds the closed-form expressions

$$N = 1 : \quad \delta_{n,0}, \quad (5.21)$$

$$N = 2 : \quad \frac{1}{12} (n+1)(n+2)^2(n+3), \quad (5.22)$$

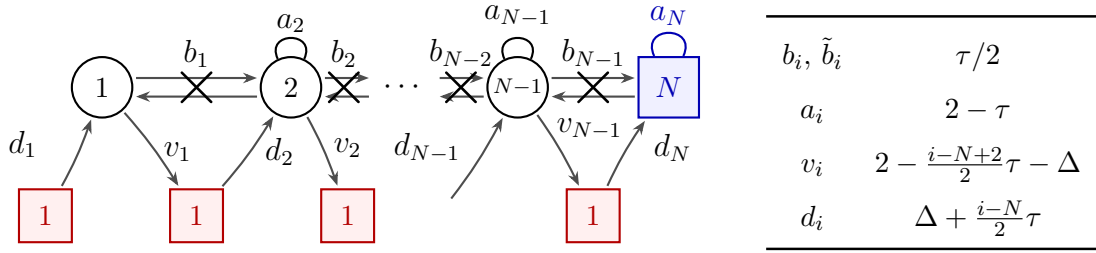
$$N = 3 : \quad \frac{1}{2075673600} (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) \\ \times (n+8)(n+9)(n+10)(2n+11)(n(n+11)+52), \quad (5.23)$$

...

The $N = 1$ case reflects the fact that FT_1 is trivial, with a zero-dimensional moduli space consisting of a single point. The jump in complexity from $N = 2$ to $N = 3$ is notable: while $N = 2$ gives a fully factorized quartic, $N = 3$ already involves an irreducible quadratic factor $n(n+11)+52$, signalling that the Hilbert polynomial ceases to split into linear factors for $N \geq 3$.

5.2.2 FC_N

We now move to a theory which actually contains a bifundamental in the spectrum.



$$\mathcal{W} = \sum_{j=1}^{N-1} v_j b_j d_{j+1} + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] \\ + (\mathfrak{M}^{+00\cdots} + \mathfrak{M}^{0+0\cdots} + \cdots) \quad (5.24)$$

Bigraded structure of the unrefined HS. Unrefining the non-Abelian fugacities but retaining the two Abelian gradings t_a (associated to the adjoints) and t_d (associated to the bifundamental), the Hilbert series of FC_N takes the form

$$\mathcal{HS}_{FC_N}(t_a, t_d) = \frac{\text{Num}(t_a, t_d)}{(1-t_d)^{N^2} (1-t_a)^{2N^2-N-1}}, \quad \text{Num}(t_a, t_d) \equiv \sum_{i,j=0}^{N(N-1)} a_{i,j} t_a^i t_d^j. \quad (5.25)$$

The two exponents in the denominator reflect the structure of the moduli space on the two “pure” branches:

- *Bifundamental branch* ($t_a = 0$). The subring generated by Π is freely generated, so

$$\mathcal{HS}_{FC_N}(0, t_d) = \frac{1}{(1-t_d)^{N^2}}, \quad a_{0,j} = \delta_{j,0}. \quad (5.26)$$

- *Adjoint branch* ($t_d = 0$). The subring generated by (A_L, A_R) reproduces the full chiral ring of FT_N ,

$$\mathcal{HS}_{FC_N}(t_a, 0) = \mathcal{HS}_{FT_N}(t_a) = \frac{\prod_{j=1}^{N-1} \sum_{h=0}^j t_a^h}{(1 - t_a)^{2N^2 - N - 1}}, \quad (5.27)$$

from which one reads off $a_{i,0} = \frac{1}{i!} \partial_t^i [N]!|_{t=0}$.

Degree of the numerator. Explicit SCI computations yield $d[2] = 2$ and $d[3] = 6$ for the total degree of $\text{Num}(t_a, t_d)$. Together with $d[1] = 0$ and the expectation that $d[N]$ is quadratic in N , these values fix

$$d[N] = N(N - 1) \quad (5.28)$$

for generic N .

Constraints from diagonal and boundary specialisations. The numerator satisfies three useful identities, which we have verified explicitly for $N = 2, 3$ and expect to hold for all N . On the diagonal $t_a = t_d = t$, one finds

$$\text{Num}(t, t) = \sum_{i,j} a_{i,j} t^{i+j} = (1 - t)^{N(N-1)} \sum_{k=0}^{N^2-N} c_k t^k, \quad c_k = c_{N^2-N-k}. \quad (5.29)$$

On the two boundary specialisations,

$$\text{Num}(1, t_d) = N! (1 - t_d)^{N(N-1)}, \quad \text{Num}(t_a, 1) = (1 - t_a)^{N(N-1)}, \quad (5.30)$$

which translate into the following row and column sum rules for the coefficient matrix $\{a_{i,j}\}$:

$$\sum_i a_{i,j} = (-1)^j N! \binom{N(N-1)}{j}, \quad \sum_j a_{i,j} = (-1)^i \binom{N(N-1)}{i}. \quad (5.31)$$

Palindromic symmetry of columns. The coefficient matrix exhibits a column-wise palindromic symmetry,

$$a_{j,k} = a_{k-j+N(N-1)/2, k}, \quad a_{j,k} = 0 \quad \text{for } k > j + N(N-1)/2 \quad \text{or } j > k + N(N-1)/2, \quad (5.32)$$

which lifts to the functional identity

$$\text{Num}(t_a, t_d) = t_a^{N(N-1)/2} \text{Num}(1/t_a, t_a t_d). \quad (5.33)$$

Explicit coefficient matrices. For low N the coefficient matrices are

$$N = 1: \begin{pmatrix} 1 \end{pmatrix}, \quad N = 2: \begin{pmatrix} 1 & 0 & 0 \\ 1 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.34)$$

$$N = 3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -9 & 0 & 1 & 0 & 0 & 0 \\ 2 & -18 & 45 & -14 & 0 & 0 & 0 \\ 1 & -9 & 45 & -94 & 45 & -9 & 1 \\ 0 & 0 & 0 & -14 & 45 & -18 & 2 \\ 0 & 0 & 0 & 1 & 0 & -9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.35)$$

Inspection of the upper-left corner for general N reveals the pattern

$$\{a_{j,k}\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ N-1 & -N^2 & 0 & 0 & \dots \\ \frac{1}{2}(N-2)(N+1) & -N^2(N-1) & \dots & & \\ \frac{1}{6}N(N^2-7) & -\frac{N^2}{2}(N-2)(N+1) & \dots & & \\ \vdots & & & \ddots & \end{pmatrix}, \quad (5.36)$$

exhibiting two notable regularities:

- The second column equals $-N^2$ times the first column, with the central entry of the palindrome omitted; that is, $a_{i,1} = -N^2 \hat{h}_{i-1}$, where $\hat{h}$ is the Mahonian sequence of the first column truncated in this way. The closed forms displayed above reproduce $-N^2 h_{i-1}$ and are therefore valid only above the truncation: already at $N = 3$ the entry $a_{3,1}$ equals -9 , not the $-\frac{N^2}{2}(N-2)(N+1) = -18$ that the untruncated expression would give.
- The second row reads $(N-1, -N^2, 0, \dots, 0, 1, 0, \dots, 0)$, where the isolated 1 sits in position N . This entry corresponds precisely to the syzygy $t_a t_d^N$ that appears in the plethystic logarithm below.

The fully unrefined Hilbert series. Setting $t_a = t_d = t$ in (5.25) and using (5.29), one obtains

$$\mathcal{HS}_{FC_N}(t, t) = \frac{\sum_{k=0}^{N^2-N} c_k t^k}{(1-t)^{2N^2-1}}, \quad (5.37)$$

with numerators for low N given by

$$\begin{aligned} N = 1 : & \quad 1, \\ N = 2 : & \quad 1 + 3t + t^2, \\ N = 3 : & \quad 1 + 8t + 26t^2 + 39t^3 + 26t^4 + 8t^5 + t^6. \end{aligned} \quad (5.38)$$

Expanding to the first few orders,

$$\mathcal{HS}_{FC_N}(t, t) = 1 + (3N^2 - 2)t + \left[\binom{3N^2 - 1}{2} - (N^2 + 1) \right] t^2 + \dots. \quad (5.39)$$

Generators and relations. The leading terms in the plethystic logarithm of the refined Hilbert series read

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FC_N}](t_a, t_d) &= \chi_{(\mathbf{N}, \overline{\mathbf{N}})} t_d + (\chi_{(\text{adj}_L, \mathbf{1})} + \chi_{(\mathbf{1}, \text{adj}_R)}) t_a - \sum_{k=2}^N t_a^k \\ &\quad - \chi_{(\mathbf{N}, \overline{\mathbf{N}})} t_a t_d + t_a t_d^N + \chi_{(\mathbf{N}, \overline{\mathbf{N}})} t_a^N t_d + \dots, \end{aligned} \quad (5.40)$$

where χ_R denotes the $SU(N)_L \times SU(N)_R$ character of the representation R . From (5.40) we read off the following chiral-ring data.

Generators ($3N^2 - 2$ in total):

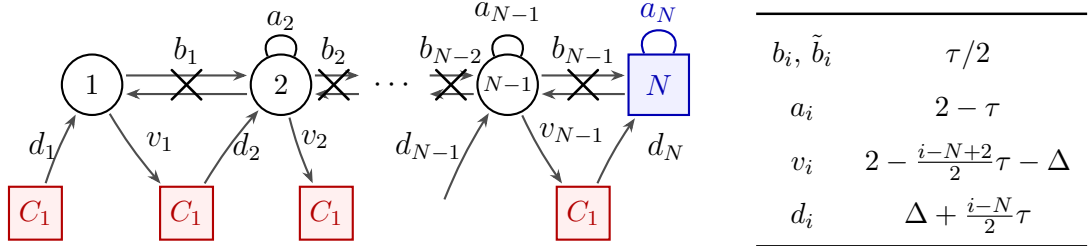
- the bifundamental Π at order t_d ,
- the two adjoints A_L, A_R at order t_a .

Relations:

$$\begin{aligned} A_L \Pi &= \Pi A_R, \quad (N^2 \text{ relations at order } t_a t_d), \\ \text{tr}_L A_L^k &= \text{tr}_R A_R^k, \quad (\text{at order } t_a^k, k = 2, \dots, N). \end{aligned} \quad (5.41)$$

Unlike the case of FT_N , the plethystic logarithm of $\mathcal{HS}_{FC_N}$ is *not* a finite sum: higher-order syzygies (such as those at orders $t_a t_d^N$ and $t_a^N t_d$, appearing with positive coefficients in (5.40)) persist indefinitely. The moduli space of FC_N is therefore not a complete intersection. The set of generators and the leading relations obtained here agree with the results of [56].

5.2.3 FH_N



$$\begin{aligned} \mathcal{W} &= \sum_{j=1}^{N-1} v_j b_j d_{j+1} + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] \\ &\quad + (\mathfrak{M}^{\pm 00 \dots} + \mathfrak{M}^{0 \pm 0 \dots} + \dots) \end{aligned} \quad (5.42)$$

Unrefining the $SU(N)_L \times USp(2N)_R$ fugacities and just keeping the pair (t_a, t_d) of $U(1)_\tau \times U(1)_\Delta$ fugacities that keep track, respectively, of the adjoint-like and bifundamental-like generators, the Hilbert series of FH_N takes the form

$$\mathcal{HS}_{FH_N}(t_a, t_d) = \frac{\sum_{i,j=0}^{\frac{3}{2}N(N-1)} a_{i,j} t_a^i t_d^j}{(1 - t_d)^{2N^2 - \frac{1}{2}N(N-1)} (1 - t_a)^{\dim[SU(N)] + \dim\{\text{asym}[USp(2N)]\} - (N-1)}}. \quad (5.43)$$

The numerator is a polynomial of bidegree $(\frac{3}{2}N(N-1), \frac{3}{2}N(N-1))$ whose coefficients $a_{i,j}$ we now analyze.

The two complete-intersection slices

Setting $t_a = 0$ isolates the bifundamental sector. The resulting series is a complete intersection generated by $2N^2$ operators $\Pi^{a,A}$ (with a, b a $USp(2N)$ index and A, B an $SU(N)$ index), subject to the $\frac{1}{2}N(N-1)$ quadratic relations $\Omega_{ab} \Pi^{a,A} \Pi^{b,B} = 0$:

$$\mathcal{HS}_{FH_N}(0, t_d) = \frac{\sum_{j=0}^{\frac{3}{2}N(N-1)} a_{0,j} t_d^j}{(1-t_d)^{2N^2-\frac{1}{2}N(N-1)}} = \frac{(1-t_d^2)^{\frac{1}{2}N(N-1)}}{(1-t_d)^{2N^2}} = \frac{(1+t_d)^{\frac{1}{2}N(N-1)}}{(1-t_d)^{2N^2-\frac{1}{2}N(N-1)}}. \quad (5.44)$$

In particular $a_{0,j} = \binom{N(N-1)/2}{j}$.

Dually, setting $t_d = 0$ isolates the “adjoint” sector. It is again a complete intersection, with $3N^2 - N - 2$ generators – the $SU(N)$ adjoint A_L and the traceless antisymmetric of the emergent $USp(2N)$, A_R – subject to the $N-1$ trace relations $\text{tr}_L A_L^j = \text{tr}_R A_R^j$, $j = 2, \dots, N$:

$$\mathcal{HS}_{FH_N}(t_a, 0) = \frac{\prod_{j=2}^N (1-t_a^j)}{(1-t_a)^{3N^2-N-2}} = \frac{\prod_{j=1}^{N-1} \sum_{h=0}^j t_a^h}{(1-t_a)^{3N^2-2N-1}}. \quad (5.45)$$

Explicit results for small N

For $N = 1, 2, 3$, the matrices of numerator coefficients $(a_{i,j})$ in (5.43) are

$$\begin{aligned} N = 1 : & \quad (1), \\ N = 2 : & \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -7 & 0 & 0 \\ 0 & 0 & 7 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \\ N = 3 : & \quad \begin{pmatrix} 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -12 & -42 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -30 & 105 & 205 & 9 & -3 & 0 & 0 & 0 & 0 \\ 1 & -15 & 153 & -547 & -318 & 54 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -202 & 1191 & 75 & -59 & 3 & 0 & 0 \\ 0 & 0 & -3 & 59 & -75 & -1191 & 202 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -54 & 318 & 547 & -153 & 15 & -1 \\ 0 & 0 & 0 & 0 & 3 & -9 & -205 & -105 & 30 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 42 & 12 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -3 & -1 \end{pmatrix}. \end{aligned} \quad (5.46)$$

All numerators are anti-palindromic. On the diagonal $t_a = t_d \equiv t$ the numerator factorises as $\text{Num}(t, t) = (1-t)^{\frac{3}{2}N(N-1)} \sum_k c_k t^k$, with the reduced numerators $\sum_k c_k t^k$ given in factored form by

$$\begin{aligned} N = 1 : & \quad 1, \\ N = 2 : & \quad (1+t)(1+4t+t^2), \\ N = 3 : & \quad (1+t)^3(1+3t+t^2)(1+8t+22t^2+8t^3+t^4). \end{aligned} \quad (5.47)$$

Structure of the coefficient matrix

The $a_{i,j}$ fill a $\left(\frac{3}{2}N(N-1)+1\right) \times \left(\frac{3}{2}N(N-1)+1\right)$ matrix. Its upper-left corner has the closed form

$$\{a_{i,j}\} = \begin{pmatrix} 1 & \frac{1}{2}N(N-1) & \binom{N(N-1)/2}{2} & \binom{N(N-1)/2}{3} & \dots \\ N-1 & \frac{1}{2}N(N^2-6N+1) & \dots & & \\ \frac{1}{2}(N-2)(N+1) & \dots & & & \\ \frac{1}{6}N(N^2-7) & \dots & & & \\ \vdots & & & & \end{pmatrix}. \quad (5.48)$$

Sum rules

We have identified four sum rules obeyed by the numerator. The specialisations $t_a = 1$, $t_d = 1$, and $t_a = t_d = t$ give

$$\text{Num}(1, t) = N! (1 - t)^{\frac{3}{2}N(N-1)}, \quad (5.49)$$

$$\text{Num}(t, 1) = 2^{N(N-1)/2} (1 - t)^{\frac{3}{2}N(N-1)}, \quad (5.50)$$

$$\text{Num}(t, t) = \left(\sum_{k=0}^{\frac{3}{2}N(N-1)} c_k t^k \right) (1 - t)^{\frac{3}{2}N(N-1)}, \quad c_k = c_{\frac{3}{2}N(N-1)-k}. \quad (5.51)$$

Finally, along every k -diagonal of $\{a_{i,j}\}$ (with $k = 0$ the main diagonal and $k > 0$, $k < 0$ denoting the diagonals above, resp. below), the coefficients sum to zero.

Generators and relations

The moduli space has $5N^2 - N - 2$ generators and $\frac{1}{2}N(N-1) + 2N^2 + 1$ quadratic relations, coming from

$$\Omega_{ab} \Pi^{a,A} \Pi^{b,B} = 0, \quad \Pi A_R = A_L \Pi, \quad \text{tr}_L A_L^2 = \text{tr}_R A_R^2.$$

Accordingly, the low-order expansion of the Hilbert series reads

$$\mathcal{HS}_{FH_N}(t) = 1 + (5N^2 - N - 2)t + \left[\binom{5N^2 - N - 1}{2} - \left(\frac{1}{2}N(N-1) + 2N^2 + 1 \right) \right] t^2 + \dots. \quad (5.52)$$

The diagonal-specialized numerator coefficients c_k defined in (5.51) (i.e. those appearing in $\text{Num}(t, t) = (\sum_k c_k t^k)(1 - t)^{\frac{3}{2}N(N-1)}$, not the coefficients of the HS expansion above) take the closed forms, for $N > 1$,

$$c_0 = 1, \quad c_1 = 2N^2 - N - 1, \quad c_2 = 2N^4 - 2N^3 - 3N^2 + N - 1.$$

The fully refined plethystic logarithm is

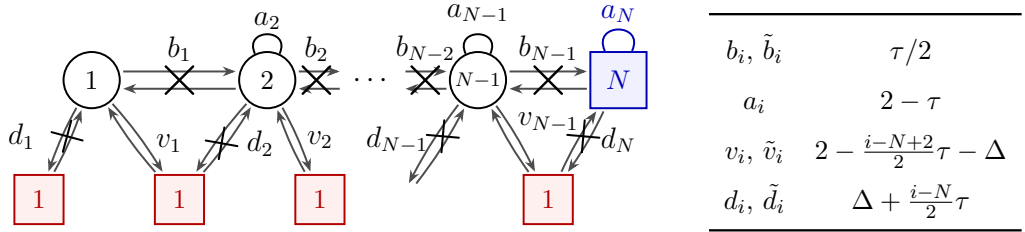
$$\begin{aligned} \text{PL}[\mathcal{HS}_{FH_N}](t_a, t_d) &= \chi(\square, \mathbf{2N}) t_d + \left(\chi(\text{adj}_L, \mathbf{1}) + \chi(\mathbf{1}, \text{asym}_R) \right) t_a - \sum_{k=2}^N t_a^k \\ &\quad - \chi(\text{asym}_L, \mathbf{1}) t_d^2 - \chi(\square, \mathbf{2N}) t_a t_d + \chi(\text{sym}_L, \mathbf{1}) t_a t_d^2 + \dots, \end{aligned} \quad (5.53)$$

where χ_R is the character of the representation R under $SU(N)_L \times USp(2N)_R$. Reading (5.53) off order by order, one recognises: at order t_d the bifundamental $\Pi^{a,A}$; at order t_a the $SU(N)$ adjoint $(A_L)^A_B$ together with the traceless antisymmetric A_R^{ab} of $USp(2N)$. These fields satisfy

$$\begin{aligned}\Omega_{ab} \Pi^{a,A} \Pi^{b,B} &= 0 & (\tfrac{1}{2}N(N-1) \text{ relations at order } t_d^2), \\ \text{tr}_L A_L^k &= \text{tr}_R A_R^k & (\text{at order } t_a^k, k = 2, \dots, N), \\ \Pi^{a,B} (A_L)^A_B &= \Pi^{b,A} A_R^{ba} & (2N^2 \text{ relations at order } t_a t_d).\end{aligned}\tag{5.54}$$

The chiral-ring generators and relations identified above agree with those of [56]. As in the previous cases, the moduli space is *not* a complete intersection: the relations in (5.54) are not independent, but obey higher-order syzygies.

5.2.4 FM_N



$$\begin{aligned}\mathcal{W} &= \sum_{j=1}^{N-1} \left[v_j b_j d_{j+1} + \tilde{v}_j \tilde{b}_j \tilde{d}_{j+1} \right] + a_2 b_1 \tilde{b}_1 + \sum_{j=2}^{N-1} b_j (a_j + a_{j+1}) \tilde{b}_j + \sum_{i=1}^{N-1} \text{Flip}[b_i \tilde{b}_i] \\ &+ (\mathfrak{M}^{\pm 00 \dots} + \mathfrak{M}^{0 \pm 0 \dots} + \dots) + \sum_{i=1}^N \text{Flip}[d_i \tilde{d}_i]\end{aligned}\tag{5.55}$$

The Hilbert series of the branch of FM_N generated by the bifundamentals $\Pi, \tilde{\Pi}$ and the adjoints A_L, A_R , after unrefining the $SU(N)_L \times SU(N)_R \times U(1)_{\text{baryonic}}$ fugacities and keeping the pair (t_a, t_d) of $U(1)_\tau \times U(1)_\Delta$ fugacities, takes the form

$$\mathcal{HS}_{FM_N}(t_a, t_d) = \frac{\sum_{i=0}^{N(N-1)} \sum_{j=0}^{N^2-1} a_{i,j} t_a^i t_d^j}{(1-t_d)^{N^2+1} (1-t_a)^{2N^2-N-1}}.\tag{5.56}$$

The two limiting sectors

The sector at $t_a = 0$ is generated by the $2N^2$ bifundamental components of Π and $\tilde{\Pi}$, subject to the $2(N^2 - 1)$ quadratic relations setting to zero the traceless parts of $\Pi\tilde{\Pi}$ and $\tilde{\Pi}\Pi$:

$$\mathcal{HS}_{FM_N}(0, t_d) = \frac{\sum_{j=0}^{(N-1)^2} a_{0,j} t_d^j}{(1-t_d)^{N^2+1}}.\tag{5.57}$$

This branch is *not* Calabi–Yau: already for $N = 2, 3$ the numerator fails to be palindromic.

Dually, the sector at $t_d = 0$ is generated by the $2N^2 - 2$ components of the adjoints A_L and A_R and satisfies

$$\mathcal{HS}_{FM_N}(t_a, 0) = \frac{\sum_{j=0}^{N(N-1)/2} a_{j,0} t_a^j}{(1 - t_a)^{2N^2 - N - 1}}, \quad (5.58)$$

where the $a_{j,0}$ are Mahonian numbers (as for FT_N). This branch is of Calabi–Yau type.

Explicit results for small N

For $N = 1, 2, 3$, the numerator coefficients $a_{i,j}$ of (5.56) are

$$\begin{aligned} N = 1 : & \quad (1), \\ N = 2 : & \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -5 & -5 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}, \\ N = 3 : & \quad \begin{pmatrix} 1 & 8 & 20 & 10 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & -86 & -140 & -16 & 2 & 0 & 0 & 0 \\ 2 & -20 & 23 & 360 & 272 & -38 & 1 & 0 & 0 \\ 1 & -10 & 66 & -170 & -574 & -170 & 66 & -10 & 1 \\ 0 & 0 & 1 & -38 & 272 & 360 & 23 & -20 & 2 \\ 0 & 0 & 0 & 2 & -16 & -140 & -86 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 10 & 20 & 8 & 1 \end{pmatrix}. \end{aligned} \quad (5.59)$$

In every case the numerator is palindromic, $a_{i,j} = a_{N(N-1)-i, N^2-1-j}$, and exhibits triangular regions of zeros in the bottom-left and top-right corners.

Sum rules

Setting the variables equal or to unity, the numerator factorises:

$$\text{Num}(t, t) = (1 - t)^{N(N-1)} (1 + t)^{N-1} \sum_{j=0}^{N(N-1)} b_j t^j, \quad (5.60)$$

$$\text{Num}(1, t) = N! (1 - t)^{N(N-1)} (1 + t)^{N-1}, \quad (5.61)$$

$$\text{Num}(t, 1) = C_N (t - 1)^{N(N-1)}, \quad (5.62)$$

where $C_N = \sum_j a_{0,j}$ and the auxiliary coefficients b_j are themselves palindromic.

Fully unrefined Hilbert series

Setting $t_a = t_d = t$, the fully unrefined Hilbert series reads

$$\mathcal{HS}_{FM_N}(t, t) = \frac{\sum_{j=0}^{N^2-1} c_j t^j}{(1 - t)^{2N^2}}, \quad (5.63)$$

so that the denominator has degree $2N^2$ and the numerator degree $N^2 - 1$. For $N = 1, 2, 3$,

$$\begin{aligned} N = 1 : & \quad 1, \\ N = 2 : & \quad (1 + t)(1 + 5t + t^2), \\ N = 3 : & \quad (1 + t)^2(1 + 14t + 72t^2 + 133t^3 + 72t^4 + 14t^5 + t^6). \end{aligned} \quad (5.64)$$

Generators and relations

The structure of the chiral ring is most transparent from the plethystic logarithm of (5.56):

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FM_N}](t_a, t_d) = & (\chi_{(\square, \bar{\square})} + \chi_{(\bar{\square}, \square)}) t_d + (\chi_{(\text{adj}_L, \mathbf{1})} + \chi_{(\mathbf{1}, \text{adj}_R)}) t_a - \sum_{k=2}^N t_a^k \\ & - (\chi_{(\text{adj}_L, \mathbf{1})} + \chi_{(\mathbf{1}, \text{adj}_R)}) t_d^2 - (\chi_{(\square, \bar{\square})} + \chi_{(\bar{\square}, \square)}) t_a t_d \\ & + \mathcal{O}(t_a t_d (t_a + t_d)), \end{aligned} \quad (5.65)$$

where χ_R is the character of the representation R under $SU(N)_L \times SU(N)_R$. Reading (5.65) off order by order, the generators of the main branch of the moduli space are:

- the two bifundamentals Π and $\tilde{\Pi}$ ($2N^2$ components at order t_d);
- the two adjoints A_L and A_R , transforming in the adjoint of $SU(N)_L$ and $SU(N)_R$ respectively ($2N^2 - 2$ components at order t_a).

These satisfy the relations

$$\begin{aligned} \Pi \tilde{\Pi} - \frac{1}{N} \text{tr}(\Pi \tilde{\Pi}) \mathbb{I}_L = 0 = \tilde{\Pi} \Pi - \frac{1}{N} \text{tr}(\tilde{\Pi} \Pi) \mathbb{I}_R, & \quad (2N^2 - 2 \text{ relations at order } t_d^2), \\ \text{tr}_L A_L^k = \text{tr}_R A_R^k, & \quad (\text{at order } t_a^k, \ k = 2, \dots, N), \\ A_L \Pi = \Pi A_R, & \quad (N^2 \text{ relations at order } t_a t_d), \\ \tilde{\Pi} A_L = A_R \tilde{\Pi}, & \quad (N^2 \text{ relations at order } t_a t_d), \end{aligned} \quad (5.66)$$

where tr_L (tr_R) denotes the trace over the $SU(N)_L$ ($SU(N)_R$) indices, while $\mathbb{I}_L$ and $\mathbb{I}_R$ are the $SU(N)_L$ and $SU(N)_R$ invariants needed to project onto the adjoint. The chiral-ring generators and relations identified above agree with the proposal of [217].

5.2.5 FE_N

b_i	$\tau/2$
a_i	$2 - \tau$
v_i	$2 - \Delta - \frac{i-N+2}{2}\tau$
d_i	$\Delta + \frac{i-N}{2}\tau$

$$\begin{aligned} \mathcal{W} = & b_1^2 a_2 + \sum_{i=2}^{N-1} b_i^2 (a_i + a_{i+1}) + \sum_{i=1}^{N-1} [v_i b_i d_{i+1} + \text{Flip}[b_i^2]] \\ & + \sum_{j=1}^N \text{Flip}[d_j^2] + (\mathfrak{m}^{100\dots 0} + \mathfrak{m}^{010\dots 0} + \dots + \mathfrak{m}^{00\dots 01}) \end{aligned} \quad (5.67)$$

For the branch of the moduli space of FE_N generated by the bifundamental Π and by the two traceless antisymmetric fields A_L, A_R , unrefining the $USp(2N)_L \times USp(2N)_R$ fugacities to the pair (t_a, t_d) , the Hilbert series takes the form

$$\mathcal{HS}_{FE_N}(t_a, t_d) = \frac{\sum_{i=0}^{2N(N-1)} \sum_{j=0}^{(2N+1)(N-1)} a_{i,j} t_a^i t_d^j}{(1 - t_d)^{2N^2+N+1} (1 - t_a)^{4N^2-3N-1}}. \quad (5.68)$$

The two limiting sectors

The sector at $t_a = 0$ is generated by the $4N^2$ components of the bifundamental $\Pi^{a,A}$, subject to the $2N(2N-1)-2$ quadratic relations given by the vanishing of the Ω -traceless parts of

$$\Omega_{ab} \Pi^{a,A} \Pi^{b,B} \quad \text{and} \quad \Omega_{AB} \Pi^{a,A} \Pi^{b,B},$$

where a, b and A, B are $USp(2N)_L$ and $USp(2N)_R$ indices, respectively. Its Hilbert series is

$$\mathcal{HS}_{FEN}(0, t_d) = \frac{\sum_{j=0}^{N^2-1} a_{0,j} t_d^j}{(1-t_d)^{2N^2+N+1}} = \frac{1 + (2N^2 - N - 1)t_d + \dots}{(1-t_d)^{2N^2+N+1}}. \quad (5.69)$$

At least for $N < 4$ the numerator is palindromic, so this branch is of Calabi–Yau type.

Dually, the sector at $t_d = 0$ is generated by the two traceless antisymmetrics A_L and A_R , subject to the $N-1$ trace relations $\text{tr}_L A_L^j = \text{tr}_R A_R^j$ for $j = 2, \dots, N$. The Hilbert series reads

$$\begin{aligned} \mathcal{HS}_{FEN}(t_a, 0) &= \frac{\prod_{j=2}^N (1-t_a^j)}{(1-t_a)^{4N^2-2N-2}} \\ &= 1 + (4N^2 - 2N - 2)t_a + N(8N^3 - 8N^2 - 4N + 3)t_a^2 + \dots \end{aligned} \quad (5.70)$$

The order- t_a term counts the generators A_L, A_R ; the order- t_a^2 term counts their symmetric product minus the single quadratic relation $A_L^2 = A_R^2$.

Explicit results for small N

The numerator coefficients $a_{i,j}$ of (5.68) fill a $(2N(N-1)+1) \times ((2N+1)(N-1)+1)$ matrix, which we find to be palindromic under $a_{i,j} = a_{2N(N-1)-i, (2N+1)(N-1)-j}$. For $N=3$ the matrix is completely fixed by expanding the Hilbert series up to total degree 13 in (t_a, t_d) , which determines all entries with $i+j \leq 13$, the remaining ones following from

the palindromic symmetry $a_{i,j} = a_{12-i, 14-j}$. For $N = 1, 2, 3$ the coefficients are

$N = 1 :$ (1),

$$N = 2 : \begin{pmatrix} 1 & 5 & 5 & 1 & 0 & 0 \\ 1 & -11 & -29 & -9 & 0 & 0 \\ 0 & 0 & 36 & 36 & 0 & 0 \\ 0 & 0 & -9 & -29 & -11 & 1 \\ 0 & 0 & 1 & 5 & 5 & 1 \end{pmatrix},$$

$$N = 3 : \begin{pmatrix} 1 & 14 & 77 & 204 & 280 & 204 & 77 & 14 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -8 & -307 & -1602 & -3360 & -3294 & -1541 & -328 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -44 & 59 & 3654 & 14589 & 21648 & 13705 & 3614 & 325 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -22 & 459 & -1520 & -26473 & -70030 & -67060 & -24490 & -2720 & 14 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & -2080 & 16723 & 112070 & 183050 & 104326 & 17747 & -160 & -26 & 0 & 0 & 0 & 0 \\ 0 & 0 & -24 & 796 & 967 & -82390 & -267988 & -261506 & -78952 & -2062 & 556 & -22 & 1 & 0 & 0 \\ 0 & 0 & 3 & -30 & -2531 & 23868 & 202392 & 358324 & 202392 & 23868 & -2531 & -30 & 3 & 0 & 0 \\ 0 & 0 & 1 & -22 & 556 & -2062 & -78952 & -261506 & -267988 & -82390 & 967 & 796 & -24 & 0 & 0 \\ 0 & 0 & 0 & 0 & -26 & -160 & 17747 & 104326 & 183050 & 112070 & 16723 & -2080 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 14 & -2720 & -24490 & -67060 & -70030 & -26473 & -1520 & 459 & -22 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 325 & 3614 & 13705 & 21648 & 14589 & 3654 & 59 & -44 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -26 & -328 & -1541 & -3294 & -3360 & -1602 & -307 & -8 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 14 & 77 & 204 & 280 & 204 & 77 & 14 & 1 \end{pmatrix}. \quad (5.71)$$

Sum rules

As for FM_N , the numerator factorises on the diagonal and on the two single-variable specialisations. Writing $\text{Num}(t_a, t_d) = \sum_{i,j} a_{i,j} t_a^i t_d^j$ for the numerator of (5.68), we find

$$\text{Num}(t, t) = (1 - t)^{2N(N-1)} (1 + t)^{N-1} \sum_{j=0}^{2N(N-1)} b_j t^j, \quad (5.72)$$

$$\text{Num}(1, t) = N! (1 - t)^{2N(N-1)} (1 + t)^{N-1}, \quad (5.73)$$

$$\text{Num}(t, 1) = C_N (1 - t)^{2N(N-1)}, \quad (5.74)$$

where $C_N = \sum_j a_{0,j}$ and the auxiliary coefficients b_j are themselves palindromic; $C_1 = 1$, $C_2 = 12$, $C_3 = 872$. Equivalently, in terms of column and row sums of the coefficient matrix,

$$\sum_i a_{i,j} = N! \sum_{m=0}^{N-1} (-1)^{j-m} \binom{2N(N-1)}{j-m} \binom{N-1}{m}, \quad \sum_j a_{i,j} = C_N (-1)^i \binom{2N(N-1)}{i}. \quad (5.75)$$

We have verified (5.72)–(5.75) for $N = 2, 3$; the $N = 2$ case reproduces (5.201) and (5.202). Note that $2N(N-1) > 0$ for $N \geq 2$ implies $\sum_{i,j} a_{i,j} = \text{Num}(1, 1) = 0$. The b_j in (5.72) are precisely the coefficients c_j of (5.76) stripped of their $(1 + t)^{N-1}$ factor. As linear conditions on the $a_{i,j}$, (5.73) fixes each of the $(2N+1)(N-1)+1$ column sums and (5.74) each of the $2N(N-1)+1$ row sums, while the divisibility in (5.72) imposes a further

$2N(N-1)$ conditions on the anti-diagonal sums; for $N = 3$ this is $15 + 13 + 12$ constraints, all of which (5.71) satisfies.

Fully unrefined Hilbert series

Setting $t_a = t_d = t$, the fully unrefined Hilbert series reads

$$\mathcal{HS}_{FE_N}(t, t) = \frac{\sum_{j=0}^{(N-1)(2N+1)} c_j t^j}{(1-t)^{4N^2}}, \quad (5.76)$$

describing a complex cone of dimension $4N^2$ whose numerator has degree $(2N+1)(N-1)$. For $N = 1, 2, 3$,

$$\begin{aligned} N = 1 : & \quad 1, \\ N = 2 : & \quad (1+t)(1+9t+19t^2+9t^3+t^4), \\ N = 3 : & \quad (1+t)^2(1+26t+288t^2+1716t^3+5970t^4+12545t^5+16071t^6 \\ & \quad + 12545t^7+5970t^8+1716t^9+288t^{10}+26t^{11}+t^{12}). \end{aligned} \quad (5.77)$$

Equation (5.77) provides a stringent consistency check on the matrix (5.71). Setting $t_a = t_d = t$ collapses the 13×15 array onto the 27 anti-diagonal sums $\sum_{i+j=n} a_{i,j}$, and the result must be divisible by $(1-t)^{2N(N-1)} = (1-t)^{12}$, because the denominator of (5.68) carries $(2N^2+N+1)+(4N^2-3N-1) = 48$ powers while the cone has dimension $4N^2 = 36$. We have checked that the division is exact, with vanishing remainder, and that the quotient is the palindromic degree-14 polynomial displayed above. Neither the vanishing of the 12 remainder coefficients nor the 7 independent palindromicity constraints on the quotient involves any residual freedom in (5.71), so together they furnish 19 nontrivial tests of the entries.

Generators and relations

The structure of the chiral ring is most transparent from the plethystic logarithm of (5.68):

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FE_N}](t_a, t_d) = & \left(\chi_{(\text{asym}_L, \mathbf{1})} + \chi_{(\mathbf{1}, \text{asym}_R)} \right) t_a + \chi_{(\mathbf{2N}, \mathbf{2N})} t_d - \sum_{k=2}^N t_a^k \\ & - \left(\chi_{(\text{asym}_L, \mathbf{1})} + \chi_{(\mathbf{1}, \text{asym}_R)} \right) t_d^2 - \chi_{(\mathbf{2N}, \mathbf{2N})} t_a t_d \\ & + \chi_{(\mathbf{2N}, \mathbf{2N})} t_d^3 + (\text{further terms of total degree} \geq 3), \end{aligned} \quad (5.78)$$

where χ_R is the character of the representation R under $USp(2N)_L \times USp(2N)_R$. Reading (5.78) off order by order, the generators of the (main branch of the) moduli space are:

- the bifundamental $\Pi^{a,A}$, in the $(\mathbf{2N}, \mathbf{2N})$ of $USp(2N)_L \times USp(2N)_R$ (at order t_d);
- the two traceless antisymmetrics A_L and A_R , in the antisymmetric of $USp(2N)_L$ and $USp(2N)_R$ respectively (at order t_a).

These generators satisfy the relations

$$\begin{aligned} [\Omega_{ab} \Pi^{a,A} \Pi^{b,B}]_{\Omega\text{-traceless}} = 0 &= [\Omega_{AB} \Pi^{a,A} \Pi^{b,B}]_{\Omega\text{-traceless}}, & (4N^2 - 2N - 2 \text{ relations at order } t_d^2), \\ \text{tr}_L A_L^k &= \text{tr}_R A_R^k, & (\text{at order } t_a^k, k = 2, \dots, N), \\ \Omega_{ac} A_L^{ab} \Pi^{c,A} &= \Omega_{BC} A_R^{AB} \Pi^{b,C}, & (4N^2 \text{ relations at order } t_a t_d). \end{aligned} \quad (5.79)$$

The plethystic logarithm does *not* terminate: the cubic term $+\chi_{(\mathbf{2N}, \mathbf{2N})} t_d^3$ is positive, and is the first syzygy among the $4N^2 - 2N - 2$ quadratic relations of order t_d^2 , obtained by contracting them back with Π . We have verified its multiplicity, $4N^2 = 16, 36$, unrefined for $N = 2, 3$.

The chiral-ring generators agree with those of [218], and their quantum relations agree with the proposal of [56]. Since the number of relations greatly exceeds the codimension, and since the syzygies do not stop at cubic order, the moduli space is manifestly not a complete intersection.

5.3 Moduli spaces of $N = 2$ improved bifundamentals: symmetry enhancements

As already mentioned, the case $N = 2$ is special and deserves a separate section. For these theories, the IR fixed point enjoys, as we will show, a further global symmetry enhancement. We are able to find relatively simple formulae for the Hilbert series in terms of characters of the larger global symmetry.

5.3.1 Index resummation for $N = 2$

We observe that, for the five theories of this section with a single Abelian global symmetry in the IR (FT_2 , FC_2 , FH_2 , $\widetilde{FM}_2$ and $\widetilde{FE}_2$), the index may be rewritten as an infinite series over even powers of the fugacity x as follows:

$$\mathcal{I}(x, t) = \sum_{n=0}^{\infty} \mathcal{I}^{(n)}(t) x^{2n} = \mathcal{I}^{(0)}(t) + \mathcal{I}^{(1)}(t) x^2 + \mathcal{O}(x^4) \quad (5.80)$$

where t is a second fugacity defined as $t = \psi x^{2-\tau}$, with ψ being the fugacity for the single Abelian $U(1)_\tau$ global symmetry present in the IR and τ the corresponding mixing with the R -charge, to be determined through F -maximization.

Some comments are in order.

The leading coefficient: Hilbert series of the moduli space. At zeroth order in x , the trace defining the index collapses onto operators saturating $\Delta = R_{\text{IR}}$ with $j_3 = 0$, i.e. chiral-ring generators. Their generating function is, by definition, the Hilbert series of the moduli space $\mathcal{M}$ of vacua of the theory:

$$\mathcal{I}^{(0)}(t) = \text{HS}_{\mathcal{M}}(t) = \sum_{k \geq 0} \dim[\mathbb{C}[\mathcal{M}]]_k t^k, \quad (5.81)$$

where $\mathbb{C}[\mathcal{M}]_k$ denotes the space of chiral-ring elements of $U(1)_\tau$ charge k .

The first subleading coefficient: currents dressed by the chiral ring. The net content of the subleading coefficient $\mathcal{I}^{(1)}(t)$, present at order x^2 , may be written schematically as

$$\mathcal{I}^{(1)}(t) = -\chi_{\text{adj}}(t) \cdot \mathcal{I}^{(0)}(t) + \Delta_{\text{Koszul}}(t), \quad (5.82)$$

where $\chi_{\text{adj}}(t)$ is the character of the fermionic component of the flavor-current multiplet in the adjoint of the enhanced global symmetry. The two terms admit the following interpretation:

- The first term is the naive “current $\times$ chiral ring” tower, built by multiplying the flavor-current fermion with arbitrary chiral-ring generators as if the latter were free polynomial variables.
- The Koszul correction $\Delta_{\text{Koszul}}(t)$ removes the over-counting produced by the chiral-ring relations [108, 250]: if the generators satisfy relations (e.g. the Plücker relations for a Grassmannian), then products of the current fermion with those relations must be subtracted, and so on through the syzygies of the chiral ring. In the five theories we study, at order x^2 only the leading Koszul correction contributes; higher syzygies of the cominuscule (i.e. HSS) BGG complex are pushed to $O(x^4)$ and beyond.⁶

In practice we see that for our theories we may write this coefficient as

$$\mathcal{I}^{(1)}(t) = - \sum_{k=0}^{\infty} C_k^{(1)} t^k, \quad (5.83)$$

where the coefficients $C_k^{(1)}$ are specified by the following table:

Theory	$C_k^{(1)} = [\chi_{\text{adj}}(t) \cdot \mathcal{I}^{(0)}(t)]_k - [\Delta_{\text{Koszul}}(t)]_k$	
FT_2	$\chi[1, k, 1]_{su(4)} - \chi[1, k-2, 1]_{su(4)}$	
FC_2	$\chi[1, k, 0, 1]_{su(5)} - \chi[1, k-2, 1, 0]_{su(5)}$	
FH_2	$\chi[0, 1, 0, 0, k]_{so(10)} - \chi[0, 0, 1, 0, k-2]_{so(10)}$	
$\widetilde{FM}_2$	$\chi[1, k, 0, 0, 1]_{su(6)} - \chi[1, k-2, 1, 0, 0]_{su(6)}$	
$\widetilde{FE}_2$	$\chi[k, 0, 0, 0, 0, 1]_{e_6} - \chi[k-2, 1, 0, 0, 0, 0]_{e_6}$	(5.84)

Higher-order structure and plethystic organization. Iterating the analysis, $\mathcal{I}^{(n)}(t)$ collects all BPS operators of $\Delta + j_3 = R_{\text{IR}} + n$, organized into symmetric products of “single-letter” contributions. A more compact way to phrase this is that (5.80) admits a plethystic exponential representation

$$\mathcal{I}(x, t) = \text{PE}[f(x, t)] \equiv \exp\left(\sum_{k \geq 1} \frac{1}{k} f(x^k, t^k)\right), \quad f(x, t) = f_0(t) + f_1(t) x^2 + \mathcal{O}(x^4), \quad (5.85)$$

with $f_0(t)$ the generating function of single-letter chiral-ring generators (so that $\text{PE}[f_0] = \mathcal{I}^{(0)}$ reproduces the moduli-space Hilbert series) and $f_1(t)$ encoding currents and recombination data.

5.3.2 FT_2

This theory has a UV description as $U(1)$ with $(2_q, 2_{\bar{q}})$ flavors and a superpotential $\mathcal{W} = \text{Flip}[q_i \tilde{q}_j]$:

⁶Explicit BGG resolutions and closed-form Hilbert series for the coordinate rings of Hermitian symmetric spaces were constructed in [251].

$$\mathcal{W} = a_2 q \tilde{q} + \text{Flip}[q \tilde{q}] \quad (5.86)$$

Duality chain and IR symmetry enhancement

Applying Aharony duality [93] to (5.86) maps it onto a $U(1)$ gauge theory with $(2_Q, 2_{\tilde{Q}})$ flavors and superpotential $\mathcal{W} = \text{Flip}[\mathfrak{M}^\pm]$. By eq. (5.7) of [235], the latter is in turn dual to $SU(2)$ with four fundamental flavors and $\mathcal{W} = 0$. It follows that the IR global symmetry of the Abelian theory in (5.86) is $SU(4) \times U(1)$, as was already established by a different route in [236].

Refined and unrefined Hilbert series

The refined Hilbert series was proposed in [236] in the compact form

$$\mathcal{HS}_{FT_2}(t) = \text{PE} \left[\chi_{[0,1,0]_{su(4)}} t - t^2 \right] = \sum_{k=0}^{\infty} \chi_{[0,k,0]_{su(4)}} t^k, \quad t \equiv x^{2-\tau}, \quad (5.87)$$

with χ_R the character of the $SU(4)$ representation R specified by its Dynkin labels. The representation $[0, k, 0]$ has dimension $\binom{k+2}{2} \binom{k+3}{2} / 3 = (k+1)(k+2)^2(k+3)/12$.

Setting all $SU(4)$ fugacities to unity, the unrefined series resums to

$$\mathcal{HS}_{FT_2}(t) = \text{PE}[6t - t^2] = \frac{1+t}{(1-t)^5}, \quad (5.88)$$

and the plethystic logarithm follows immediately:

$$\text{PL}[\mathcal{HS}_{FT_2}](t) = \chi_{[0,1,0]_{su(4)}} t - t^2. \quad (5.89)$$

Generators, relations and geometry

From (5.89) the moduli space is seen to be a complete intersection. Its six generators, appearing at order t , are the four flippers of the mesons together with the two monopoles; they transform in the $\mathbf{6} = [0, 1, 0]$ of $SU(4)$. The single quadratic relation at order t^2 projects out the singlet in the symmetric square of the $\mathbf{6}$, and can be written as

$$\text{tr } A_L^2 = \text{tr } A_R^2,$$

where $A_{L/R}$ are the two adjoints of FT_2 . Because only $SU(4)$ representations of even quadrality appear, the faithful IR symmetry is in fact $SO(6) \times U(1)$ [236].

Geometrically, the moduli space is the complex cone over an irreducible Hermitian symmetric space of compact type $\mathcal{B} = H/K$ with $H = SU(4)$ and $K = S(U(2) \times U(2))$, i.e. the complex Grassmannian $\text{Gr}(2, 4)$.

Decomposition under the generic- N symmetry

Under the generic- N global symmetry $SU(2) \times SU(2) \times U(1)_\tau$, the fundamental and the antisymmetric of $SU(4)$ branch as

$$\mathbf{4} = [1, 0, 0]_{SU(4)} = [1; 1]_{SU(2) \times SU(2)}, \quad (5.90)$$

$$\mathbf{6} = [0, 1, 0]_{SU(4)} = ([2; 0] + [0; 2])_{SU(2) \times SU(2)}. \quad (5.91)$$

In particular the Hilbert series (5.87) can be rewritten as

$$\mathcal{HS}_{FT_2}(t) = \text{PE} \left[\left(\chi_{[2;0]_{su(2) \times su(2)}} + \chi_{[0;2]_{su(2) \times su(2)}} \right) t - t^2 \right]. \quad (5.92)$$

The general branching rule for the chiral-ring representations is

$$[0, k, 0]_{su(4)} = \sum_{k_1, k_2=0}^{\infty} [2k_1; 2k_2]_{su(2) \times su(2)} \sum_{s=0}^{\lfloor k/2 \rfloor} \delta_{k_1+k_2, k-2s}, \quad (5.93)$$

which, substituted into (5.87), yields⁷

$$\mathcal{HS}_{FT_2}(t) = \sum_{k_1, k_2, m=0}^{\infty} \chi_{[2k_1; 2k_2]_{su(2) \times su(2)}} t^{k_1+k_2+2m}. \quad (5.94)$$

We have checked this expression against the superconformal index of (5.96) up to order x^{10} .

Superconformal R-charge and index

Z -extremization [129, 152] fixes the superconformal R-charge of the two adjoints to high precision,

$$\tau = 1.339192 \dots \quad (5.95)$$

⁷Using (5.93) in (5.87) and noting that, for any $k_1, k_2 \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \sum_{s=0}^{\lfloor k/2 \rfloor} \delta_{k_1+k_2, k-2s} t^k = \sum_{k, s=0}^{\infty} \delta_{k_1+k_2, k-2s} t^k = \sum_{s=0}^{\infty} t^{k_1+k_2+2s}.$$

For the index computation we rounded this value to $\tau = 134/100$, obtaining

$$\begin{aligned}
\mathcal{I}_{FT_2} = & 1 + 6\psi x^{33/50} + 20\psi^2 x^{33/25} + 50\psi^3 x^{99/50} - 16x^2 + 105\psi^4 x^{66/25} \\
& - 64\psi x^{133/50} - \psi^{-2} x^{67/25} + 196\psi^5 x^{33/10} - 160\psi^2 x^{83/25} + 20\psi^{-1} x^{167/50} \\
& + 336\psi^6 x^{99/25} - 320\psi^3 x^{199/50} + 88x^4 + 540\psi^7 x^{231/50} - 560\psi^4 x^{116/25} \\
& + 156\psi x^{233/50} - 16\psi^{-2} x^{117/25} + 825\psi^8 x^{132/25} - 896\psi^5 x^{53/10} + 192\psi^2 x^{133/25} \\
& - 64\psi^{-1} x^{267/50} + 1210\psi^9 x^{297/50} - 1344\psi^6 x^{149/25} + 164\psi^3 x^{299/50} + 64x^6 \\
& + 6\psi^{-3} x^{301/50} + 1716\psi^{10} x^{33/5} - 1920\psi^7 x^{331/50} + 40\psi^4 x^{166/25} \\
& + 448\psi x^{333/50} + 20\psi^{-2} x^{167/25} + 2366\psi^{11} x^{363/50} - 2640\psi^8 x^{182/25} \\
& - 212\psi^5 x^{73/10} + 992\psi^2 x^{183/25} - 276\psi^{-1} x^{367/50} + 3185\psi^{12} x^{198/25} \\
& - 3520\psi^9 x^{397/50} - 624\psi^6 x^{199/25} + 1664\psi^3 x^{399/50} - 785x^8 \\
& + 4200\psi^{13} x^{429/50} - 4576\psi^{10} x^{43/5} - 1228\psi^7 x^{431/50} + 2432\psi^4 x^{216/25} \\
& - 824\psi x^{433/50} + 256\psi^{-2} x^{217/25} + 5440\psi^{14} x^{231/25} - 5824\psi^{11} x^{463/50} \\
& - 2056\psi^8 x^{232/25} + 3264\psi^5 x^{93/10} - 320\psi^2 x^{233/25} + 288\psi^{-1} x^{467/50} \\
& + 6936\psi^{15} x^{99/10} - 7280\psi^{12} x^{248/25} - 3140\psi^9 x^{497/50} + 4128\psi^6 x^{249/25} \\
& + 504\psi^3 x^{499/50} - 1072x^{10} + \mathcal{O}(x^{501/50}).
\end{aligned} \tag{5.96}$$

Here ψ is the fugacity for the global $U(1)_\tau$, introduced to track the τ -dependence.

Reading the index

The index (5.96) admits a clean physical decomposition. The chiral-ring operators contribute as $\chi_{[0,k,0]_{SU(4)}} x^{k(2-\tau)}$, while marginal operators and the fermionic components of conserved currents sit at even powers of x and are independent of ψ .

The $-16x^2$ term is the hallmark of the symmetry enhancement to $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$. Its origin is the following. The spin- $\frac{1}{2}$ components of the conserved currents supply

$$(q_i)_S(\bar{q}^j)_F, \quad (\tilde{q}^i)_S(\bar{\tilde{q}}_j)_F \quad (8 \text{ components}), \quad (a_i)_S(\bar{a}^j)_F \quad (16 \text{ components}),$$

with $i, j = 1, 2$ and a_1 the gauge singlet flipping the trace $\text{Tr}(q\tilde{q})$ in the superpotential. The monopole sectors with GNO flux ± 1 add $\mathfrak{M}^\pm(\bar{a}_i)_F$ (8 further gauge-invariant fermionic components) at order x^2 . Sixteen of these current components recombine with the sixteen scalars of the superpotential terms, leaving the sixteen fermionic components that assemble into the adjoint of $SO(6)$ together with the $U(1)$ current – precisely the spin- $\frac{1}{2}$, $\Delta = \frac{3}{2}$, $R = 1$ components of the flavor current [184], which are the only ones that contribute to the index.

One also reads off $\frac{1}{4}$ -BPS operators of schematic form [current] $\times$ [chiral] at orders $x^{2n+k(2-\tau)}$ with $n, k \in \mathbb{N}_+$, as well as further $\frac{1}{4}$ -BPS operators (outside the chiral ring) at orders $x^{2n-k(2-\tau)}$ with $n > 1$. As an example, the fermionic term $-\psi^{-2}x^{67/25}$ arises as follows. Recombination at order $x^{67/50}$ between $(q_i\tilde{q}^j)_S$ and $(a_i)_F$, driven by the cubic superpotential, annihilates those contributions in the index; squaring this relation (with due care for statistics and product-representation decomposition) tracks the residual pieces at order $x^{67/25}$. Concretely, the 16 scalar terms

$$(a_1)_F(a_2)_F [(\mathbf{1}, \mathbf{3})], \quad (a_2)_F^2 [(\mathbf{1}, \mathbf{3})], \quad (q_i\tilde{q}^j)_S^2 [(\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{3})],$$

are subject to the F -term relation

$$\frac{\partial \mathcal{W}}{\partial a_1} = \text{Tr}(q\tilde{q}) = 0.$$

Since a_1 flips only the trace of the meson, this relation removes precisely the $(\mathbf{1}, \mathbf{1})$ component of $q_i \tilde{q}^j$, so the scalar contributions reduce from $3 + 3 + (1 + 9) = 16$ to $3 + 3 + 9 = 15$ independent terms. These recombine with the 16 fermionic terms

$$(a_1)_F(q_i \tilde{q}^j)_S [(\mathbf{2}, \mathbf{2})], \quad (a_2)_F(q_i \tilde{q}^j)_S [(\mathbf{2}, \mathbf{2}) + (\mathbf{2}, \mathbf{4})],$$

the representations referring to $SU(2)_L \times SU(2)_R$. The net imbalance is therefore a single surviving fermionic mode, which is what appears in the index.

Index resummation

First, we can rewrite the index, defining $t = \psi x^{33/50}$, as follows:

$$\mathcal{I}_{FT_2}(x, t) = \mathcal{I}^{(0)}(t) + x^2 \mathcal{I}^{(1)}(t) + x^4 \mathcal{I}^{(2)}(t) + x^6 \mathcal{I}^{(3)}(t) + \mathcal{O}(x^8) \quad (5.97)$$

where the functions $\mathcal{I}^{(k)}(t)$ can be resummed as

$$\mathcal{I}^{(0)}(t) = \frac{1+t}{(1-t)^5} = \mathcal{HS}_{FT_2}(t) \quad (5.98)$$

$$\mathcal{I}^{(1)}(t) = -\frac{16}{(1-t)^4} \quad (5.99)$$

$$\mathcal{I}^{(2)}(t) = -\frac{1-24t-2t^2+72t^3-15t^4}{t^2(1-t)^4} \quad (5.100)$$

$$\mathcal{I}^{(3)}(t) = \frac{-16+224t^2-128t^3-176t^4+64t^5}{t^2(1-t)^4} \quad (5.101)$$

In terms of the $SU(4)$ representations, the functions $\mathcal{I}^{(k)}(t)$ can be written as an exact sum over irreducible representations $\chi_{[q_1, p, q_2]}$ as follows:

$$\mathcal{I}^{(0)}(t) = \sum_{k=0}^{\infty} \chi_{[0, k, 0]} t^k \quad (5.102)$$

$$\mathcal{I}^{(1)}(t) = -\sum_{k=0}^{\infty} (\chi_{[1, k, 1]} - \chi_{[1, k-2, 1]}) t^k \quad (5.103)$$

$$\mathcal{I}^{(2)}(t) = -\frac{\chi_{[0, 0, 0]}}{t^2} + \sum_{k=-1}^{\infty} C_k^{(2)} t^k \quad (5.104)$$

$$\mathcal{I}^{(3)}(t) = -\frac{\chi_{[1, 0, 1]} + \chi_{[0, 0, 0]}}{t^2} - \frac{\chi_{[1, 1, 1]}}{t} + \sum_{k=0}^{\infty} C_k^{(3)} t^k \quad (5.105)$$

When evaluating the sum for $\mathcal{I}^{(1)}(t)$ at $k=0$ and $k=1$, we formally encounter characters with negative Dynkin labels, namely $\chi_{[1, -2, 1]}$ and $\chi_{[1, -1, 1]}$. According to the Weyl character formula, these are resolved via Weyl reflections outside the fundamental chamber. If a

weight lies on the boundary of a Weyl chamber, its character vanishes ($\chi_{[1,-1,1]} = 0$). If it can be reflected back into the fundamental chamber by a single Weyl reflection, it acquires a minus sign, yielding $\chi_{[1,-2,1]} = -\chi_{[0,0,0]} = -1$.

The coefficients $C_k^{(2)}$ and $C_k^{(3)}$ are finite combinations of characters encoding the higher-order syzygies between the chiral ring and the conserved current multiplets. Defining the discrete difference operators:

$$\Delta_1(k) = \chi_{[1,k,1]} - \chi_{[1,k-2,1]} \quad (5.106)$$

$$\Delta_3(k) = \chi_{[0,k+2,0]} - 3\chi_{[0,k+1,0]} + 3\chi_{[0,k,0]} - \chi_{[0,k-1,0]} \quad (5.107)$$

the exact character combinations are explicitly given by:

$$C_k^{(2)} = 4\Delta_1(k+1) - 6\Delta_1(k) - 14\Delta_3(k) - 2\chi_{[0,0,0]} \quad (5.108)$$

$$C_k^{(3)} = 4\Delta_1(k) + 12\Delta_1(k-1) - 26\Delta_1(k-2) + 8\Delta_1(k-3) \quad (5.109)$$

The T_2 theory

It is interesting to see what happens when we flip one of the two adjoints in the IR spectrum. We obtain the $\mathcal{N} = 4$ theory $T_2 \equiv T(SU(2))$ [113]. The first few orders in the expansion of the resulting index read:

$$\begin{aligned} \mathcal{I}_{T_2} = & 1 + 3\psi x^{33/50} + 5\psi^2 x^{33/25} + 3\psi^{-1} x^{67/50} + 7\psi^3 x^{99/50} - 7x^2 + 9\psi^4 x^{66/25} - 4\psi x^{133/50} \\ & + 5\psi^{-2} x^{67/25} + 11\psi^5 x^{33/10} - 4\psi^2 x^{83/25} - 4\psi^{-1} x^{167/50} + 13\psi^6 x^{99/25} - 4\psi^3 x^{199/50} + 7x^4 \\ & + 7\psi^{-3} x^{201/50} + 15\psi^7 x^{231/50} - 4\psi^4 x^{116/25} - 20\psi x^{233/50} \\ & - 4\psi^{-2} x^{117/25} + O(x^5) \end{aligned} \quad (5.110)$$

where again we turned on the fugacity ψ for the single $U(1)$ global symmetry, in order to keep track of the charges of short multiplets appearing in the index. As the index suggests, it presents two identical branches, a Higgs branch and a Coulomb branch, which are orthogonal to each other and are exchanged by mirror symmetry. The Hilbert series for each of the two branches reads [223]:

$$\mathcal{HS}_{T_2}^{CB} = \mathcal{HS}_{T_2}^{HB} = \text{PE}[(\chi_{[2]_{su(2)}})x^a - x^{2a}], \quad (5.111)$$

with $a = \tau$ for one branch and $a = 2 - \tau$ for the other. The unrefined series gives

$$\mathcal{HS}_{T_2}^{unr} = \text{PE}[3x^a - x^{2a}] = 1 + 3x^a + 5x^{2a} + 7x^{3a} + \dots, \quad (5.112)$$

Let us note that both the Coulomb branch and the Higgs branch Hilbert series can be obtained starting from the index (5.110) and then performing the following limits:

$$\begin{aligned} \mathcal{HS}_{T_2}^{CB} &= \lim_{y \rightarrow 0} \mathcal{I}_{T_2}(x \rightarrow xy, \psi \rightarrow \psi y^{\tau-2}) = 1 + 3\psi x^{33/50} + 5\psi^2 x^{33/25} + 7\psi^3 x^{99/50} + \dots \\ \mathcal{HS}_{T_2}^{HB} &= \lim_{y \rightarrow 0} \mathcal{I}_{T_2}(x \rightarrow xy, \psi \rightarrow \psi y^\tau) = 1 + \frac{3x^{67/50}}{\psi} + \frac{5x^{67/25}}{\psi^2} + \frac{7x^{201/50}}{\psi^3} + \dots \end{aligned} \quad (5.113)$$

In this case we are able to reconstruct just a single branch of the full moduli space but, as we will see below, in some special cases we are able to build the HS for the full moduli space out of this procedure.

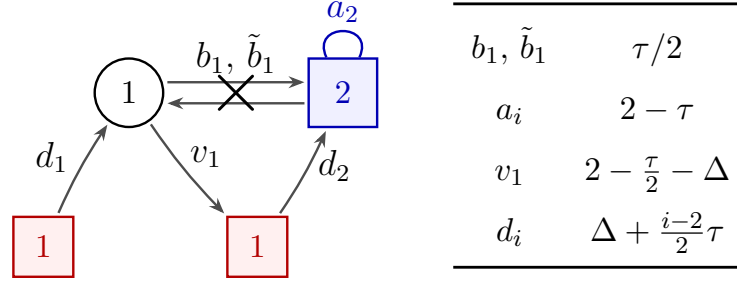
Exact Hilbert series as a limit of the index

We are able to find the Hilbert series by taking an appropriate limit of the index, as noted for 3d $\mathcal{N} = 4$ in [167] and for $\mathcal{N} = 2$ in [168]. This limit does not give the full Hilbert series, but just its localization either on the Higgs or the Coulomb branch. But if we consider theories with a single branch, like the FT_N theory, we are able to obtain the HS for the full moduli space. Consider the FT_2 index (5.96), and take the shifts $x \rightarrow xy$ and $\psi \rightarrow \psi y^{\tau-2}$. Finally, taking the limit $y \rightarrow 0$, yields the HS for the full moduli space:

$$\begin{aligned} \lim_{y \rightarrow 0} \mathcal{I}_{FT_2}(xy, \psi y^{\tau-2}) &= 1 + 6\psi x^{33/50} + 20\psi^2 x^{33/25} + 50\psi^3 x^{99/50} + 105\psi^4 x^{66/25} + O(x^{151/50}) \\ &= \mathcal{HS}_{FT_2}(\psi x^{2-\tau}) \end{aligned} \quad (5.114)$$

5.3.3 FC_2

We now consider the FC_2 , which has the following Abelian theory as a possible UV completion:



$$\mathcal{W} = v_1 b_1 d_2 + a_2 b_1 \tilde{b}_1 + \mathfrak{M}^+ + \text{Flip}[b_1 \tilde{b}_1] \quad (5.115)$$

We can rewrite the FC_2 theory as $U(1)$ with $(3_q, 2_{\tilde{q}} + 1_{\bar{p}})$ flavors and $\mathcal{W} = \mathfrak{M}^+ + \text{Flip}[q_i \tilde{q}_j]$ as follows:

$$\mathcal{W} = W_{triangles} + \mathfrak{M}^+ \quad (5.116)$$

where the first step can be reached by observing that the flipper of $b_1 \tilde{b}_1$, together with the adjoint a_2 of $SU(2)$, can be seen as a bifundamental of $SU(2) \times SU(2)$, from which the second equality follows. In this way we see that the UV global symmetry is actually $SU(3) \times SU(2) \times U(1)^2$.

Duality chain and IR symmetry enhancement

Combining eqs. (5.2) and (5.5) of [235], one finds that $U(1)$ with $(3_q, 2_{\bar{q}} + 1_{\bar{p}})$ flavors and $\mathcal{W} = \mathfrak{M}^+$ is dual to $SU(2)_{1/2}$ with $3_Q + 2_P$ fundamentals and $\mathcal{W} = \text{Flip}[Q_i P_j]$, under the mapping

$$\{q\tilde{q}, q\tilde{p}, \mathfrak{M}^-\} \longleftrightarrow \{\mathcal{F}[QP], QQ, PP\}.$$

Moving the six flippers to the left-hand side, the FC_2 theory is dual to $SU(2)_{1/2}$ with $3_Q + 2_P$ fundamentals and $\mathcal{W} = 0$. Consequently, the IR global symmetry of the Abelian gauge theory is $\mathfrak{su}(5) \oplus \mathfrak{u}(1)$.

Generators, relations and geometry

The chiral-ring generators

$$\{\mathfrak{M}^-, d_2, d_1 \tilde{b}_1, d_1 v_1, a_1, a_2\},$$

with a_1 the flipper of $b_1 \tilde{b}_1$, transform in the **10** of $SU(5)$. In FC_2 language, they decompose into two $\mathfrak{su}(2)$ adjoints and one bifundamental:

$$A_L = \{\mathfrak{M}^-, d_1 v_1, a_1\}, \quad A_R = \{a_2\}, \quad (5.117)$$

$$\Pi = \{d_2, d_1 \tilde{b}_1\}. \quad (5.118)$$

The fact that these operators lie in a single irreducible representation forces $\Delta + \tau = 2$, as can also be checked through Z -extremization.

The ten generators satisfy five quadratic relations of two types,

$$\text{tr}_L A_L^2 = \text{tr}_R A_R^2 \quad (\text{one relation}), \quad A_L \Pi = \Pi A_R \quad (\text{four relations}). \quad (5.119)$$

Geometrically, the moduli space is the complex cone over the Grassmannian $\text{Gr}(2, 5)$, realized as the Hermitian symmetric space of compact type $SU(5)/S(U(3) \times U(2))$, of rank 2.

Refined and unrefined Hilbert series

From the superconformal index we extrapolate the compact form

$$\mathcal{HS}_{FC_2}(t) = \sum_{k=0}^{\infty} \chi_{[0,k,0,0]_{su(5)}} t^k \quad \stackrel{\text{unr}}{=} \quad \frac{1 + 3t + t^2}{(1 - t)^7}, \quad (5.120)$$

which we have verified against the SCI expansion (with $t \equiv x^\Delta$) up to the first 12 terms. The representation $[0, k, 0, 0]_{su(5)}$ has dimension $\binom{k+4}{3} \binom{k+3}{3} / 4$, which grows as k^6 , so the moduli space has dimension 7. It cannot be a complete intersection: accordingly, the plethystic logarithm of (5.120) is an infinite series,

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FC_2}](t) &= \chi_{[0,1,0,0]} t - \chi_{[0,0,0,1]} t^2 + \chi_{[1,0,0,0]} t^3 - \chi_{[0,0,1,0]} t^4 \\ &\quad + \chi_{[1,0,0,1]} t^5 - (\chi_{[0,1,0,1]} + \chi_{[0,1,0,0]}) t^6 \\ &\quad + (\chi_{[1,0,1,0]} + \chi_{[1,0,0,2]} + \chi_{[0,0,0,1]}) t^7 \\ &\quad - 270 t^8 + 640 t^9 - 1524 t^{10} + \mathcal{O}(t^{11}), \end{aligned} \quad (5.121)$$

where from order t^8 onwards we display only the total dimension of the contributing representations.

A clean pattern emerges: representations enter with a plus sign at odd powers of t and with a minus sign at even powers, the latter being conjugate representations of the form $\overline{X}$. From the leading terms one recognises the ten baryons B_{ij} , sitting in the two-index antisymmetric of $SU(5)$ at order t , subject to the five quadratic relations

$$\epsilon^{i_1 i_2 i_3 i_4 i_5} B_{i_1 i_2} B_{i_3 i_4} = 0$$

at order t^2 .

Branching to the generic- N symmetry

Under $\mathfrak{su}(5) \rightarrow \mathfrak{su}(4)$ one has

$$[0, k, 0, 0]_{\mathfrak{su}(5)} = \sum_{h=0}^k [k-h, h, 0]_{\mathfrak{su}(4)}. \quad (5.122)$$

Using $\Delta + \tau = 2$ and $t \equiv x^\Delta$, eq. (5.120) may be rewritten as

$$\mathcal{HS}_{FC_2} = \sum_{m,k=0}^{\infty} \chi_{[m,k,0]_{\mathfrak{su}(4)}} x^{m\Delta+k(2-\tau)}. \quad (5.123)$$

Unrefining the $\mathfrak{su}(4)$ fugacities and setting $t_d = x^\Delta$, $t_a = x^{2-\tau}$, the series resums to the closed form⁸

$$\mathcal{HS}_{FC_2}(t_a, t_d) = \sum_{m,k=0}^{\infty} \dim[m, k, 0]_{\mathfrak{su}(4)} t_d^m t_a^k = \frac{1 + t_a - 4t_a t_d + t_a t_d^2 + t_a^2 t_d^2}{(1 - t_a)^5 (1 - t_d)^4}. \quad (5.124)$$

Writing the numerator of (5.124) as $N(t_a, t_d) = \sum_{j,k} a_{j,k} t_a^j t_d^k$, the matrix of coefficients is

$$\{a_{j,k}\} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.125)$$

and satisfies the sum rules

$$\begin{aligned} N(t, t) &= 1 + t - 4t^2 + t^3 + t^4 = (1 + 3t + t^2)(1 - t)^2, \\ N(1, t) &= 2(1 - t)^2, \\ N(t, 1) &= (1 - t)^2. \end{aligned} \quad (5.126)$$

The general branching rule for $\mathfrak{su}(4) \rightarrow \mathfrak{su}(2)^2$ reads

$$\begin{aligned} [m, k, 0]_{\mathfrak{su}(4)} &= \sum_{k_1, k_2=0}^{\infty} \sum_{h=0}^{\lfloor m/2 \rfloor} [2k_1 + m - 2h; 2k_2 + m - 2h]_{\mathfrak{su}(2) \times \mathfrak{su}(2)} \\ &\times \left(\theta(m - 2h) \sum_{s=0}^k \delta_{k_1+k_2, k-s} + \delta_{m, 2h} \sum_{s=0}^{\lfloor k/2 \rfloor} \delta_{k_1+k_2, k-2s} \right), \end{aligned} \quad (5.127)$$

⁸If the unrefined Hilbert series is written as $\mathcal{HS}(t_a, t_d)$, switching off one of the two fugacities yields

$$\mathcal{HS}(0, t_d) = \frac{1}{(1 - t_d)^{N^2}}, \quad \mathcal{HS}(t_a, 0) = \frac{\prod_{k=2}^N (1 - t_a^k)}{(1 - t_a)^{2(N^2-1)}}.$$

with $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ otherwise. Using this, the Hilbert series can finally be expressed in $SU(2) \times SU(2) \times U(1)_\Delta \times U(1)_\tau$ representations as

$$\mathcal{HS}_{FC_2}(t_a, t_d) = \text{PE}[(\chi_{[2;0]} + \chi_{[0;2]}) t_a + \chi_{[1;1]} t_d - t_a^2 - \chi_{[1;1]} t_a t_d + \dots]. \quad (5.128)$$

Superconformal R-charge and index

Z -extremization gives

$$\tau = 2 - \Delta = 1.312\dots \quad (5.129)$$

hinting that a combination of the two $U(1)$ symmetries is spurious. For the index computation we rounded this value to $\tau = 131/100$. The first few orders of the refined index read

$$\mathcal{I}_{FC_2} = 1 + x^{69/100} \chi_{[0,1,0,0]_{su(5)}} + x^{69/50} \chi_{[0,2,0,0]_{su(5)}} - x^2 (\chi_{[1,0,0,1]_{su(5)}} + 1) + \mathcal{O}(x^{101/50}), \quad (5.130)$$

where the baryon contributions appear at orders $x^{k\Delta}$, $k = 1, 2, 3, \dots$, and the conserved currents of $\mathfrak{su}(5) \oplus \mathfrak{u}(1)$ enter at order x^2 .

Turning on the fugacity ψ for $U(1)_\tau$ to track the $\Delta = 2 - \tau$ dependence, the unrefined index expands as

$$\begin{aligned} \mathcal{I}_{FC_2}^{\text{unr}} = & 1 + 10\psi x^{69/100} + 50\psi^2 x^{69/50} - 25x^2 + 175\psi^3 x^{207/100} - 5\psi^{-2} x^{131/50} \\ & - 175\psi x^{269/100} + 490\psi^4 x^{69/25} + 15\psi^{-1} x^{331/100} - 675\psi^2 x^{169/50} \\ & + 1176\psi^5 x^{69/20} + 5\psi^{-3} x^{393/100} + 250x^4 - 1925\psi^3 x^{407/100} \\ & + 2520\psi^6 x^{207/50} + 1010\psi x^{469/100} - 4550\psi^4 x^{119/25} + 4950\psi^7 x^{483/100} \\ & - 250\psi^{-1} x^{531/100} + 2600\psi^2 x^{269/50} - 9450\psi^5 x^{109/20} + 9075\psi^8 x^{138/25} \\ & - 775x^6 + 5225\psi^3 x^{607/100} - 17850\psi^6 x^{307/50} + 15730\psi^9 x^{621/100} \\ & - \psi^{-5} x^{131/20} + 245\psi^{-2} x^{331/50} - 640\psi x^{669/100} + 8890\psi^4 x^{169/25} \\ & - 31350\psi^7 x^{683/100} + 26026\psi^{10} x^{69/10} + 15\psi^{-4} x^{181/25} \\ & + 300\psi^{-1} x^{731/100} + 1925\psi^2 x^{369/50} + 13300\psi^5 x^{149/20} \\ & - 51975\psi^8 x^{188/25} + 41405\psi^{11} x^{759/100} - 170\psi^{-3} x^{793/100} \\ & - 2450x^8 + \mathcal{O}(x^{801/100}). \end{aligned} \quad (5.131)$$

Chiral-ring residue

As a consistency check that (5.120) fully captures the chiral ring, we subtract it from (5.131); what remains is

$$\begin{aligned} \mathcal{I}_{FC_2}^{\text{red}} = & 1 - 25x^2 - 5\psi^{-2} x^{131/50} - 175\psi x^{269/100} + 15\psi^{-1} x^{331/100} - 675\psi^2 x^{169/50} \\ & + 5\psi^{-3} x^{393/100} + 250x^4 - 1925\psi^3 x^{407/100} + 1010\psi x^{469/100} - 4550\psi^4 x^{119/25} \\ & - 250\psi^{-1} x^{531/100} + 2600\psi^2 x^{269/50} - 9450\psi^5 x^{109/20} - 775x^6 \\ & + 5225\psi^3 x^{607/100} - 17850\psi^6 x^{307/50} - \psi^{-5} x^{131/20} + 245\psi^{-2} x^{331/50} \\ & - 640\psi x^{669/100} + 8890\psi^4 x^{169/25} - 31350\psi^7 x^{683/100} + 15\psi^{-4} x^{181/25} \\ & + 300\psi^{-1} x^{731/100} + 1925\psi^2 x^{369/50} + 13300\psi^5 x^{149/20} - 51975\psi^8 x^{188/25} \\ & - 170\psi^{-3} x^{793/100} - 2450x^8 + \mathcal{O}(x^{801/100}). \end{aligned} \quad (5.132)$$

The surviving terms fall into three physical classes: ψ -independent contributions from conserved currents and marginal operators; $\frac{1}{4}$ -BPS operators of schematic form [current] $\times$ [chiral] at orders $x^{2n+k(2-\tau)}$; and further $\frac{1}{4}$ -BPS operators outside the chiral ring, signalled by negative powers of ψ .

Index resummation

First, we can rewrite the index, defining $t = \psi x^{69/100}$, to absorb all the fractional powers. The series organizes exactly into even powers of x as follows:

$$\mathcal{I}_{FC_2}^{unr}(x, t) = \mathcal{I}^{(0)}(t) + x^2 \mathcal{I}^{(1)}(t) + x^4 \mathcal{I}^{(2)}(t) + \mathcal{O}(x^6) \quad (5.133)$$

where the functions $\mathcal{I}^{(k)}(t)$ can be resummed into closed rational forms:

$$\mathcal{I}^{(0)}(t) = \frac{1 + 3t + t^2}{(1 - t)^7} = \mathcal{HS}_{FC_2}(t, t) \quad (5.134)$$

$$\mathcal{I}^{(1)}(t) = -\frac{25(1 + t)}{(1 - t)^6} \quad (5.135)$$

$$\mathcal{I}^{(2)}(t) = \frac{-5 + 45t + 85t^2 - 165t^3 - 85t^4 + 30t^5 - 5t^6}{t^2(1 - t)^6} \quad (5.136)$$

In terms of the global $SU(5)$ symmetry, the functions $\mathcal{I}^{(k)}(t)$ can be written as an exact sum over irreducible representations (denoted by their Dynkin labels $\chi_{[q_1, q_2, q_3, q_4]}$). The chiral ring is generated by symmetric insertions $\chi_{[0, k, 0, 0]}$, while the current multiplet transforms in the adjoint representation **24** ($\chi_{[1, 0, 0, 1]}$). The exact character combinations encoding the syzygies can be written as:

$$\mathcal{I}^{(0)}(t) = \sum_{k=0}^{\infty} \chi_{[0, k, 0, 0]} t^k \quad (5.137)$$

$$\mathcal{I}^{(1)}(t) = -\sum_{k=0}^{\infty} (\chi_{[1, k, 0, 1]} - \chi_{[1, k-2, 1, 0]}) t^k \equiv -\sum_{k=0}^{\infty} C_k^{(1)} t^k \quad (5.138)$$

$$\mathcal{I}^{(2)}(t) = -\frac{\chi_{[1, 0, 0, 0]}}{t^2} + \frac{\chi_{[2, 0, 0, 0]}}{t} + \sum_{k=0}^{\infty} C_k^{(2)} t^k \quad (5.139)$$

The formula for $\mathcal{I}^{(1)}(t)$ incorporates the $U(1)_\tau$ singlet constraint through the Weyl character formula. When $k = 1$, the subtracted representation is $\chi_{[1, -1, 1, 0]}$, which lies on the boundary of a Weyl chamber and vanishes identically, leaving 175. When $k = 0$, we encounter $\chi_{[1, -2, 1, 0]}$. Reflecting this weight across the second simple root s_2 maps it exactly to the trivial representation $[0, 0, 0, 0]$ with a minus sign. Therefore:

$$C_0^{(1)} = \chi_{[1, 0, 0, 1]} - (-\chi_{[0, 0, 0, 0]}) = \mathbf{24} + \mathbf{1} = 25 \quad (5.140)$$

which reproduces the exact dimension of the conserved current multiplet.

In the unrefined limit, the dimensions of the combinations $C_k^{(k)}$ follow exact polynomials in k derived from the rational generating functions:

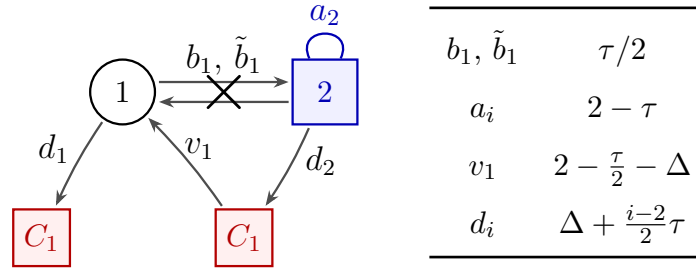
$$\dim C_k^{(1)} = \frac{5}{24}(k+1)(k+2)(k+3)(k+4)(2k+5) \quad (5.141)$$

$$\dim C_k^{(2)} = \frac{5}{6}(k+2)(k+3)(50+47k+5k^2-k^3) \quad (5.142)$$

The negative leading coefficient in $\dim C_k^{(2)}$ ($-\frac{5}{6}k^5$) confirms that this tower consists entirely of higher-order syzygies removing redundant degrees of freedom.

5.3.4 FH_2

Consider the following UV completion of FH_2 with manifest $SU(2) \times USp(2)^2$ global symmetry



$$\mathcal{W} = v_1 b_1 d_2 + a_2 b_1 \tilde{b}_1 + \mathfrak{M}^+ + \mathfrak{M}^- + \text{Flip}[b_1 \tilde{b}_1] \quad (5.143)$$

UV symmetry and self-dual frames

The UV completion (5.143) makes manifest an $\mathfrak{su}(2)^3 \oplus \mathfrak{u}(1)^2$ global symmetry, which enhances in the IR to $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$. Part of the enhancement is already visible in the UV: an extra $\mathfrak{su}(2)$ factor appears once one realises that the $SU(2)$ adjoint a_2 , together with the flipper of $b_1 \tilde{b}_1$, can be reorganized into an $SU(2) \times SU(2)$ bifundamental, as shown below:

$$\mathcal{W} = W_{\text{triangles}} + \mathfrak{M}^\pm \quad (5.144)$$

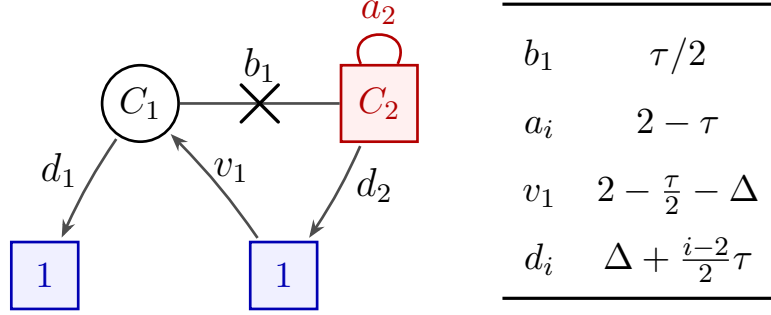
It follows that the true UV symmetry is $\mathfrak{su}(4) \oplus \mathfrak{su}(2)^2 \oplus \mathfrak{u}(1)$. In analogy with the observation of [233] for $FE_2 \equiv FE[USp(4)]$, we expect FH_2 to admit 19 further self-dual frames, since

$$20 = \frac{|W(\mathfrak{so}(10))|}{|W(\mathfrak{su}(4) \oplus \mathfrak{su}(2)^2)|},$$

where the numerator and denominator are the orders of the Weyl groups of the IR and UV global symmetries, respectively.

Mirror UV completion

As anticipated in Section 5.1.5, FH_2 admits another UV completion with gauge group $SU(2)$, descending from the mirror self-duality of FE_2 . For completeness we record the mirror quiver:



$$\mathcal{W} = v_1 b_1 d_2 + a_2 b_1^2 + \mathfrak{M} + \text{Flip}[b_1^2] \quad (5.145)$$

This mirror duality was discussed, up to flippers, in [235].

Superconformal R-charge and index

Z -extremization fixes

$$\tau = 1.326, \quad \Delta = 0.674, \quad (5.146)$$

so that $\Delta + \tau = 2$, confirming that one combination of the two $U(1)$ fugacities is spurious and the theory carries a single effective $U(1)$ in the IR. Rounding to $\tau = 133/100$ for the numerical computation, the refined index reads

$$\begin{aligned} \mathcal{I}_{FH_2} = & 1 + \chi_{[0,0,0,0,1]_{so(10)}} \psi x^{67/100} + \chi_{[0,0,0,0,2]_{so(10)}} \psi^2 x^{67/50} \\ & - (\chi_{[0,1,0,0,0]_{so(10)}} + 1) x^2 + \chi_{[0,0,0,0,3]_{so(10)}} \psi^3 x^{201/100} + \mathcal{O}(x^{133/50}). \end{aligned} \quad (5.147)$$

Unrefining the $\mathfrak{so}(10)$ fugacities, the index expands up to order x^6 as

$$\begin{aligned} \mathcal{I}_{FH_2}^{\text{unr}} = & 1 + 16\psi x^{67/100} + 126\psi^2 x^{67/50} - 46x^2 + 672\psi^3 x^{201/100} \\ & - 10\psi^{-2} x^{133/50} - 560\psi x^{267/100} + 2772\psi^4 x^{67/25} - 3576\psi^2 x^{167/50} \\ & + 9504\psi^5 x^{67/20} + 16\psi^{-3} x^{399/100} + 943x^4 - 16080\psi^3 x^{401/100} \\ & + 28314\psi^6 x^{201/50} + 120\psi^{-2} x^{233/50} + 8064\psi x^{467/100} - 57420\psi^4 x^{117/25} \\ & + 75504\psi^7 x^{469/100} - 864\psi^{-1} x^{533/100} + 39604\psi^2 x^{267/50} - 173712\psi^5 x^{107/20} \\ & + 184041\psi^8 x^{134/25} - 144\psi^{-3} x^{599/100} - 11086x^6 + \mathcal{O}(x^{601/100}). \end{aligned} \quad (5.148)$$

The contributions proportional to $\psi^k x^{k\Delta}$ come from the fields in the $[0, 0, 0, 0, k]$ of $\mathfrak{so}(10)$, $k = 1, 2, 3, \dots$. The $-46x^2$ piece is the conserved-current signature of the symmetry enhancement

$$SU(4) \times SU(2)^2 \times U(1) \longrightarrow Spin(10) \times U(1), \quad \dim = 45 + 1 = 46.$$

Refined and unrefined Hilbert series

From the index one extrapolates the compact form of the full refined Hilbert series,

$$\mathcal{HS}_{FH_2}(t) = \sum_{k=0}^{\infty} \chi_{[0,0,0,0,k]_{so(10)}} t^k, \quad t \equiv x^\Delta. \quad (5.149)$$

The unrefined series resums to a rational function,

$$\begin{aligned} \mathcal{HS}_{FH_2}^{\text{unr}}(t) &= \frac{1 + 5t + 5t^2 + t^3}{(1-t)^{11}} \\ &= 1 + 16t + 126t^2 + 672t^3 + 2772t^4 + 9504t^5 + 28314t^6 \\ &\quad + 75504t^7 + 184041t^8 + \mathcal{O}(t^9) \\ &= \sum_{n=4}^{\infty} \frac{1}{1!2!3!4!} \sum_{1 \leq i,j,k,l \leq n} |\det V(i,j,k,l)| t^{n-4}, \end{aligned} \quad (5.150)$$

where $V(i,j,k,l)$ is the 4×4 Vandermonde matrix.

Generators, relations and geometry

The Euler form of (5.150) identifies the moduli space as a cone of complex dimension 11 (the exponent of the denominator). It is the affine cone over the ten-dimensional orthogonal Grassmannian $OG(5, 10)_+$ (see e.g. [252]), whose base is the Hermitian symmetric space of compact type $SO(10)/U(5)$, of rank 2.

The same conclusion follows from the large- k asymptotics of (5.149): the representations $[0, 0, 0, 0, k]_{so(10)}$ have dimension $\binom{k+5}{5} \binom{k+7}{5} / 21 \sim k^{10}$, giving a moduli-space dimension $10 + 1 = 11$.

The structure of the chiral ring is encoded in the plethystic logarithm,

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FH_2}](t) &= \chi_{[0,0,0,0,1]_{so(10)}} t - \chi_{[1,0,0,0,0]_{so(10)}} t^2 + \chi_{[0,0,0,0,1]_{so(10)}} t^3 \\ &\quad - \chi_{[0,1,0,0,0]_{so(10)}} t^4 + \chi_{[1,0,0,0,1]_{so(10)}} t^5 + \mathcal{O}(t^6). \end{aligned} \quad (5.151)$$

Reading off (5.151): there are 16 chiral generators, in the spinorial of $SO(10)$ (at order t), subject to 10 quadratic relations in the fundamental of $SO(10)$ (at order t^2). Since the series does not terminate, the relations are not independent and the moduli space is not a complete intersection. This is also consistent with the simple count

$$\#\text{relations} = 10 \neq 5 = \text{codim},$$

forbidding a complete-intersection structure.

Branching to the generic- N symmetry

To connect with the generic- N analysis, we break $\mathfrak{so}(10)$ along the chain

$$\mathfrak{so}(10) \oplus \mathfrak{u}(1)_\Delta \longrightarrow \mathfrak{so}(8) \oplus \mathfrak{u}(1)_\tau \oplus \mathfrak{u}(1)_\Delta \longrightarrow \mathfrak{usp}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_\tau \oplus \mathfrak{u}(1)_\Delta. \quad (5.152)$$

Under the chain (5.152), the spinor of $\mathfrak{so}(10)$ decomposes as

$$\begin{aligned} [0, 0, 0, 0, 1]_{so(10)} x^\Delta &= [1, 0; 1]_{usp(4) \times su(2)} x^\Delta \\ &\quad + ([0, 1; 0]_{usp(4) \times su(2)} + [0, 0; 2]_{usp(4) \times su(2)}) x^{2-\tau}, \end{aligned} \quad (5.153)$$

matching the chiral-ring content of FH_2 at generic N : the bifundamental $\Pi \in [1, 0; 1]$ at order x^Δ , together with the traceless antisymmetric $A_R \in [0, 1; 0]$ and the $\mathfrak{su}(2)$ adjoint $A_L \in [0, 0; 2]$ at order $x^{2-\tau}$. More generally,

$$[0, 0, 0, 0, k]_{so(10)} x^{k\Delta} = \sum_{k_1, k_2=0}^{\infty} [0, 0, k_1, k_2]_{so(8)} x^{k_1\Delta+k_2(2-\tau)} \delta_{k_1+k_2, k}. \quad (5.154)$$

Substituting (5.154) into (5.149) and unrefining the $\mathfrak{so}(8)$ fugacities yields a resummed two-variable Hilbert series in the Abelian fugacities $t_a \equiv x^{2-\tau}$ and $t_d \equiv x^\Delta$:

$$\sum_{k_1, k_2=0}^{\infty} \dim[0, 0, k_1, k_2]_{so(8)} t_d^{k_1} t_a^{k_2} = \frac{1 + t_a + t_d - 7t_a t_d + 7t_a^2 t_d^2 - t_a^3 t_d^2 - t_a^2 t_d^3 - t_a^3 t_d^3}{(1 - t_a)^7 (1 - t_d)^7}. \quad (5.155)$$

Structure of the two-variable numerator

The numerator $N(t_a, t_d) = \sum_{j,k} a_{j,k} t_a^j t_d^k$ of (5.155) is *anti-palindromic*, $a_{j,k} = -a_{3-j, 3-k}$, giving the symmetric, block-diagonal coefficient matrix

$$\{a_{j,k}\} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -7 & 0 & 0 \\ 0 & 0 & 7 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}. \quad (5.156)$$

Two immediate consequences are $\sum_{j,k} a_{j,k} = 0$ and the symmetry $t_a \leftrightarrow t_d$ of the Hilbert series – the latter reflecting the triality of $\mathfrak{so}(8)$. Setting $t_a = t_d \equiv t$, the numerator factorises as

$$-t^6 - 2t^5 + 7t^4 - 7t^2 + 2t + 1 = (1 - t)^3 (1 + 5t + 5t^2 + t^3), \quad (5.157)$$

which recovers the unrefined Hilbert series (5.150). Both single-variable specialisations yield the same result,

$$N(1, t) = N(t, 1) = 2(1 - t)^3. \quad (5.158)$$

Index resummation

First let us write $t = \psi x^{67/100}$, then the unrefined index may be written in the form:

$$\mathcal{I}_{FH_2}^{unr}(x, t) = \mathcal{I}^{(0)}(t) + x^2 \mathcal{I}^{(1)}(t) + x^4 \mathcal{I}^{(2)}(t) + \mathcal{O}(x^6) \quad (5.159)$$

where the functions $\mathcal{I}^{(k)}(t)$ can be resummed into closed rational forms:

$$\mathcal{I}^{(0)}(t) = \frac{1 + 5t + 5t^2 + t^3}{(1 - t)^{11}} = \mathcal{HS}_{FH_2} \quad (5.160)$$

$$\mathcal{I}^{(1)}(t) = -\frac{46 + 100t + 46t^2}{(1 - t)^{10}} \quad (5.161)$$

$$\mathcal{I}^{(2)}(t) = \frac{t^8 - 10t^7 + 45t^6 - 136t^5 - 701t^4 - 166t^3 + 493t^2 + 100t - 10}{(1 - t)^{10} t^2} \quad (5.162)$$

In terms of the global $SO(10)$ symmetry, the functions $I^{(k)}(t)$ can be written as an exact sum over irreducible representations (denoted by their Dynkin labels $\chi_{[a1,a2,a3,a4,a5]}$). In this theory, the chiral ring $I^{(0)}(t)$ is generated by the spinor representation **16** ($\chi_{[0,0,0,0,1]}$), while the current multiplet transforms in the adjoint **45** ($\chi_{[0,1,0,0,0]}$). The exact character combinations are:

$$\mathcal{I}^{(0)}(t) = \sum_{k=0}^{\infty} \chi_{[0,0,0,0,k]} t^k \quad (5.163)$$

$$\mathcal{I}^{(1)}(t) = - \sum_{k=0}^{\infty} (\chi_{[0,1,0,0,k]} - \chi_{[0,0,1,0,k-2]}) t^k \equiv - \sum_{k=0}^{\infty} C_k^{(1)} t^k \quad (5.164)$$

$$\mathcal{I}^{(2)}(t) = - \frac{\chi_{[1,0,0,0,0]}}{t^2} + \sum_{k=0}^{\infty} C_k^{(2)} t^k \quad (5.165)$$

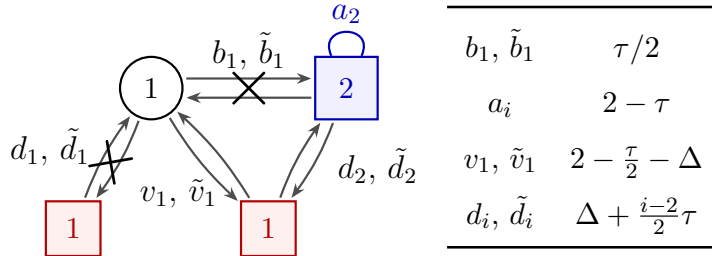
The formula for $\mathcal{I}^{(1)}(t)$ incorporates the $U(1)_\tau$ singlet constraint through the Weyl character formula. When $k = 1$, the subtracted representation is $\chi_{[0,0,1,0,-1]}$, whose ρ -shifted weight $[0,0,1,0,-1] + \rho = [1,1,2,1,0]$ has a vanishing fifth Dynkin label. Since this places it on the wall of the fundamental Weyl chamber associated with the simple root α_5 , the character vanishes identically, leaving $C_1^{(1)} = \chi_{[0,1,0,0,1]}$ of dimension 560. When $k = 0$, we encounter $\chi_{[0,0,1,0,-2]}$. The shifted weight $[0,0,1,0,-2] + \rho = [1,1,2,1,-1]$ has a negative fifth label. A single reflection s_5 across the fifth simple root maps it to $[1,1,1,1,1] = \rho$, the weight of the trivial representation, with a minus sign. Therefore:

$$C_0^{(1)} = \chi_{[0,1,0,0,0]} - (-\chi_{[0,0,0,0,0]}) = \mathbf{45} + \mathbf{1} = 46 \quad (5.166)$$

which reproduces the exact dimension of the conserved current multiplet of the $SO(10)$ theory.

5.3.5 FM_2

Consider the FM_2 theory, having the following possible lagrangian UV completion:



$$\mathcal{W} = v_1 b_1 d_2 + \tilde{v}_1 \tilde{b}_1 \tilde{d}_2 + a_2 b_1 \tilde{b}_1 + \mathfrak{M}^+ + \mathfrak{M}^- + \text{Flip}[b_1 \tilde{b}_1] + \text{Flip}[d_1 \tilde{d}_1] \quad (5.167)$$

Duality chain and IR symmetry enhancement

Naively, the generic- N analysis would predict an IR symmetry $S(U(2)_L \times U(2)_R) \times U(1)_\Delta \times U(1)_\tau$ for this theory. However, by eq. (5.3) of [235], a suitably flipped version of FM_2 is

dual to $SU(2)$ with $4_q + 2_p$ fundamentals and $\mathcal{W} = \text{Flip}[pp]$, with the R-charges of q and p fixed to $1 - \tau/2$ and $\Delta - 1 + \tau/2$, respectively. The resulting R-charge assignments for the gauge-invariant bilinears are

$$R[qq] = 2 - \tau, \quad R[\mathcal{F}[pp]] = 4 - 2\Delta - \tau, \quad R[pq] = \Delta, \quad R[\mathfrak{M}] = 2 - 2\Delta + \tau. \quad (5.168)$$

Consequently, the IR global symmetry of the Abelian theory is $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)^2$, rather than the generic- N expectation above.

Chiral-ring generators

The 16 chiral-ring generators (the 8 mesons together with the 8 flippers of mesons) decompose under $SU(4) \times SU(2)$ as

$$(\mathbf{6}, \mathbf{1})_{2-\tau} \oplus (\mathbf{4}, \mathbf{2})_{\Delta} \oplus (\mathbf{1}, \mathbf{1})_{4-2\Delta-\tau} \oplus (\mathbf{1}, \mathbf{1})_{2-2\Delta+\tau}, \quad (5.169)$$

with the subscripts denoting the $U(1)_{\Delta} \times U(1)_{\tau}$ charges. In FM_2 language, the IR spectrum contains two $SU(2)_L \times SU(2)_R$ bifundamentals Π and $\tilde{\Pi}$, two adjoints A_L and A_R of $SU(2)_L$ and $SU(2)_R$ respectively, and the two singlets $B_{1,2}$ (the flipper of $d_1 \tilde{d}_1$) and $B_{2,1} \equiv v_1 \tilde{v}_1$.

Superconformal R-charges and index

Z -extremization locates the minimum of the S^3 partition function at

$$\Delta = 0.969, \quad \tau = 1.277. \quad (5.170)$$

For the numerical computation we adopt the rational values $\Delta = 82/100$ and $\tau = 131/100$, and turn on the fugacities $\psi_{\tau}, \chi_{\Delta}$ for the two $U(1)$'s. The unrefined index then expands as

$$\begin{aligned} \mathcal{I}_{FM_2}(x) = & 1 + 6\psi_{\tau} x^{69/100} + 8\chi_{\Delta} x^{41/50} + \psi_{\tau} \chi_{\Delta}^{-2} x^{21/20} \\ & + 20\psi_{\tau}^2 x^{69/50} + 40\chi_{\Delta} \psi_{\tau} x^{151/100} + 30\chi_{\Delta}^2 x^{41/25} + \chi_{\Delta}^{-2} \psi_{\tau}^{-1} x^{167/100} \\ & + 6\psi_{\tau}^2 \chi_{\Delta}^{-2} x^{87/50} - 20x^2 + 50\psi_{\tau}^3 x^{207/100} + \psi_{\tau}^2 \chi_{\Delta}^{-4} x^{21/10} \\ & - 8\chi_{\Delta} \psi_{\tau}^{-1} x^{213/100} + 120\chi_{\Delta} \psi_{\tau}^2 x^{11/5} + 135\chi_{\Delta}^2 \psi_{\tau} x^{233/100} + 20\psi_{\tau}^3 \chi_{\Delta}^{-2} x^{243/100} \\ & + 80\chi_{\Delta}^3 x^{123/50} - \psi_{\tau}^{-2} x^{131/50} - 98\psi_{\tau} x^{269/100} + \chi_{\Delta}^{-4} x^{68/25} + 105\psi_{\tau}^4 x^{69/25} \\ & + 6\psi_{\tau}^3 \chi_{\Delta}^{-4} x^{279/100} - 136\chi_{\Delta} x^{141/50} + 280\chi_{\Delta} \psi_{\tau}^3 x^{289/100} \\ & - 45\chi_{\Delta}^2 \psi_{\tau}^{-1} x^{59/20} + 378\chi_{\Delta}^2 \psi_{\tau}^2 x^{151/50} - 16\psi_{\tau} \chi_{\Delta}^{-2} x^{61/20} \\ & + 50\psi_{\tau}^4 \chi_{\Delta}^{-2} x^{78/25} + x^{63/20} (336\chi_{\Delta}^3 \psi_{\tau} + \psi_{\tau}^3 \chi_{\Delta}^{-6}) \\ & + 175\chi_{\Delta}^4 x^{82/25} + 10\psi_{\tau}^{-1} x^{331/100} + \chi_{\Delta}^{-4} \psi_{\tau}^{-2} x^{167/50} - 285\psi_{\tau}^2 x^{169/50} \\ & + 196\psi_{\tau}^5 x^{69/20} + 20\psi_{\tau}^4 \chi_{\Delta}^{-4} x^{87/25} - 512\chi_{\Delta} \psi_{\tau} x^{351/100} + 560\chi_{\Delta} \psi_{\tau}^4 x^{179/50} \\ & - 454\chi_{\Delta}^2 x^{91/25} - \chi_{\Delta}^{-2} \psi_{\tau}^{-1} x^{367/100} + 840\chi_{\Delta}^2 \psi_{\tau}^3 x^{371/100} - 64\psi_{\tau}^2 \chi_{\Delta}^{-2} x^{187/50} \\ & + x^{377/100} (\psi_{\tau} \chi_{\Delta}^{-6} - 144\chi_{\Delta}^3 \psi_{\tau}^{-1}) + 105\psi_{\tau}^5 \chi_{\Delta}^{-2} x^{381/100} \\ & + x^{96/25} (6\psi_{\tau}^4 \chi_{\Delta}^{-6} + 896\chi_{\Delta}^3 \psi_{\tau}^2) + 16\psi_{\tau} \chi_{\Delta}^{-1} x^{387/100} \\ & + 700\chi_{\Delta}^4 \psi_{\tau} x^{397/100} + 166x^4 + \mathcal{O}(x^{401/100}). \end{aligned} \quad (5.171)$$

The $-20x^2$ conserved-current contribution witnesses the symmetry enhancement

$$SU(2) \times U(1)^4 \longrightarrow SU(4) \times SU(2) \times U(1)^2, \quad \dim = 15 + 3 + 1 + 1 = 20.$$

Reorganizing the refined index into $SU(4) \times SU(2)$ characters,

$$\begin{aligned} \mathcal{I}_{FM_2}(x) = & -1 + \sum_{n,k=0}^{\infty} \chi_{[n,k,0;n]_{su(4) \times su(2)}} x^{n\Delta+k(2-\tau)} + \sum_{l,m=0}^{\infty} x^{l(4-2\Delta-\tau)+m(2-2\Delta+\tau)} \\ & + \sum_{k,l=1}^{\infty} \chi_{[0,k,0;0]_{su(4) \times su(2)}} x^{l(4-2\Delta-\tau)+k(2-\tau)} \\ & - (\chi_{[1,0,1;0]_{su(4) \times su(2)}} + \chi_{[0,0,0;2]_{su(4) \times su(2)}} + 2) x^2 + \dots, \end{aligned} \quad (5.172)$$

with χ_R the character of the representation R under $SU(4) \times SU(2)$.

Hilbert series and branch structure

From (5.172) one extrapolates the all-order Hilbert series of the full moduli space:

$$\begin{aligned} \mathcal{HS}_{FM_2} = & \sum_{n,k=0}^{\infty} \chi_{[n,k,0;n]_{su(4) \times su(2)}} x^{n\Delta+k(2-\tau)} + \sum_{l,m=0}^{\infty} x^{l(4-2\Delta-\tau)+m(2-2\Delta+\tau)} \\ & + \sum_{k,l=0}^{\infty} \chi_{[0,k,0;0]_{su(4) \times su(2)}} x^{l(4-2\Delta-\tau)+k(2-\tau)} \\ & - \sum_{k=0}^{\infty} \chi_{[0,k,0;0]_{su(4) \times su(2)}} x^{k(2-\tau)} - \sum_{l=0}^{\infty} x^{l(4-2\Delta-\tau)}. \end{aligned} \quad (5.173)$$

The moduli space is therefore the union of three branches:

- a *main branch*, generated by $\{A_L, A_R, \Pi, \tilde{\Pi}\}$;
- a *second branch*, generated by $\{B_{1,2}, B_{2,1}\}$;
- a *third branch*, generated by $\{A_L, A_R, B_{2,1}\}$.

The first two branches intersect only at the origin; the third has nontrivial intersection with both. Specifically, the third branch is isomorphic to $\mathbb{C}$ times the moduli space of FT_2 . The unrefined Hilbert series of the second and third branches read, respectively,

$$\frac{1}{(1-x^{4-2\Delta-\tau})(1-x^{2-2\Delta+\tau})}, \quad \frac{1+x^{2-\tau}}{(1-x^{4-2\Delta-\tau})(1-x^{2-\tau})^5}, \quad (5.174)$$

while the resummed intersection contributions – the last two terms of (5.173) – give

$$-\frac{1+x^{2-\tau}}{(1-x^{2-\tau})^5} \quad \text{and} \quad -\frac{1}{1-x^{4-2\Delta-\tau}}. \quad (5.175)$$

The main branch

Setting $t_a \equiv x^{2-\tau}$, $t_d \equiv x^\Delta$, the unrefined Hilbert series of the main branch is

$$\mathcal{HS}_{FM_2}^{\text{main}}(t_a, t_d) = \frac{1 + 3t_d + t_a(1 - 5t_d - 5t_d^2 + t_d^3) + t_a^2 t_d^2(3 + t_d)}{(1 - t_a)^5 (1 - t_d)^5}. \quad (5.176)$$

Writing the numerator as $N(t_a, t_d) = \sum_{j,k} a_{j,k} t_a^j t_d^k$, the coefficient matrix is

$$\{a_{j,k}\} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & -5 & -5 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}, \quad (5.177)$$

and obeys the sum rules

$$\begin{aligned} N(t, t) &= (t-1)^2(1 + 6t + 6t^2 + t^3), \\ N(1, t) &= 2(t-1)^2(t+1), \\ N(t, 1) &= 4(t-1)^2. \end{aligned} \quad (5.178)$$

The last two specialisations translate into the row/column sum rules

$$\sum_j a_{j,k} = 2(-1)^k \left[\binom{2}{k} - \binom{2}{k-1} \right], \quad \sum_k a_{j,k} = 4(-1)^j \binom{2}{j}, \quad (5.179)$$

with the convention $\binom{2}{-1} = \binom{2}{3} = 0$.

Generators and relations of the main branch

The plethystic logarithm of the main-branch Hilbert series in (5.173) reads

$$\begin{aligned} \text{PL} \left[\sum_{n,k=0}^{\infty} \chi_{[n,k,0;n]_{su(4) \times su(2)}} x^{n\Delta+k(2-\tau)} \right] &= \chi_{[0,1,0;0]} x^{2-\tau} + \chi_{[1,0,0;1]} x^\Delta - x^{4-2\tau} \\ &\quad - \chi_{[0,0,1;1]} x^{2-\tau+\Delta} - \chi_{[0,1,0;0]} x^{2\Delta} \\ &\quad + \chi_{[1,0,0;1]} x^{\Delta+4-2\tau} + \mathcal{O}(x^{2\Delta+2-\tau}). \end{aligned} \quad (5.180)$$

Packaging the adjoints A_L, A_R into a two-index antisymmetric Φ_{ab} of $SU(4)$ (singlet under $SU(2)$), and the bifundamentals $\Pi, \tilde{\Pi}$ into $\Psi_{a;A}$ in the $(\square, \mathbf{2})$ of $SU(4) \times SU(2)$ (a, b $SU(4)$ indices, A an $SU(2)$ index), one reads off Φ_{ab} at order $x^{2-\tau}$ and $\Psi_{a;A}$ at order x^Δ . They satisfy the relations

$$\begin{aligned} \epsilon^{abcd} \Phi_{ab} \Phi_{cd} &= 0, & (1 \text{ relation at order } x^{4-2\tau}), \\ \epsilon^{AB} \Psi_{a;A} \Psi_{b;B} &= 0, & (6 \text{ relations at order } x^{2\Delta}), \\ \epsilon^{abcd} \Phi_{ab} \Psi_{c;A} &= 0, & (8 \text{ relations at order } x^{2-\tau+\Delta}). \end{aligned} \quad (5.181)$$

Since (5.180) does not terminate, these relations are not independent: the main branch is not a complete intersection.

A flipped theory: $\widetilde{FM}_2$

Consider the theory obtained from FM_2 by flipping the singlet $B_{2,1} \equiv v_1 \tilde{v}_1$; we denote it $\widetilde{FM}_2$. F -maximization returns

$$\Delta = 2 - \tau \approx 0.7374, \quad (5.182)$$

and we claim that the IR global symmetry enhances to $U(6)$. As a check, fixing $\Delta = 37/50$, the first orders of the refined index read

$$\begin{aligned} \mathcal{I}_{\widetilde{FM}_2} = & 1 + \mathbf{15} \chi x^{37/50} + \overline{\mathbf{105}'} \chi^2 x^{37/25} + \chi^{-3} x^{89/50} - (\mathbf{35} + 1) x^2 \\ & + \mathbf{490} \chi^3 x^{111/50} - \mathbf{384} \chi x^{137/50} + \mathbf{1764}' \chi^4 x^{74/25} + \mathcal{O}(x^{151/50}), \end{aligned} \quad (5.183)$$

where χ is the fugacity for the single $U(1)$ of the IR, and the bold numbers denote $SU(6)$ characters. The conserved currents contributing with a minus sign at order x^2 fill the adjoint of $SU(6) \times U(1)$.

The moduli space of $\widetilde{FM}_2$ consists of two branches with trivial intersection:

- a copy of $\mathbb{C}$, freely generated by the remaining singlet $B_{1,2}$ (contributing at order $x^{4-3\Delta}$);
- a main branch generated by the $\mathbf{15}$ of $SU(6)$ (order x^Δ), subject to 15 quadratic relations.

The Hilbert series of the main branch takes the compact form

$$\mathcal{HS}_{\widetilde{FM}_2} = \sum_{k=0}^{\infty} \chi_{[0,k,0,0,0]_{SU(6)}} t^k \stackrel{\text{unr}}{=} \frac{1 + 6t + 6t^2 + t^3}{(1-t)^9}, \quad (5.184)$$

which is exactly the Hilbert series of the complex cone over the Grassmannian $\text{Gr}(2, 6)$ [253], realized geometrically as the Hermitian symmetric space of compact type $SU(6)/S(U(4) \times U(2))$.

Unrefining with respect to the $SU(6)$ fugacities, the index expands up to x^8 as

$$\begin{aligned} \mathcal{I}_{\widetilde{FM}_2}^{\text{unr}}(x; \chi) = & 1 + 15\chi x^{37/50} + 105\chi^2 x^{37/25} + \chi^{-3} x^{89/50} - 36x^2 \\ & + 490\chi^3 x^{111/50} - 384\chi x^{137/50} + 1764\chi^4 x^{74/25} \\ & + 21\chi^{-1} x^{163/50} - 2100\chi^2 x^{87/25} + \chi^{-6} x^{89/25} + 5292\chi^5 x^{37/10} \\ & + 558x^4 - 8064\chi^3 x^{211/50} + 13860\chi^6 x^{111/25} + 36\chi^{-2} x^{113/25} \\ & + 3690\chi x^{237/50} - 24696\chi^4 x^{124/25} + 32670\chi^7 x^{259/50} \\ & - 384\chi^{-1} x^{263/50} + \chi^{-9} x^{267/50} + 14526\chi^2 x^{137/25} \\ & - 64512\chi^5 x^{57/10} - 71\chi^{-3} x^{289/50} + 70785\chi^8 x^{148/25} \\ & - 3536x^6 + 42651\chi^3 x^{311/50} - 149688\chi^6 x^{161/25} \\ & + 135\chi^{-2} x^{163/25} + 143143\chi^9 x^{333/50} - 12216\chi x^{337/50} \\ & + 103236\chi^4 x^{174/25} + 36\chi^{-4} x^{176/25} + \chi^{-12} x^{178/25} \\ & - 316800\chi^7 x^{359/50} + 2562\chi^{-1} x^{363/50} + 273273\chi^{10} x^{37/5} \\ & - 24576\chi^2 x^{187/25} + 217476\chi^5 x^{77/10} - 72\chi^{-3} x^{389/50} \\ & - 622908\chi^8 x^{198/25} + 3465x^8 + \mathcal{O}(x^{401/50}). \end{aligned} \quad (5.185)$$

Index resummation for $\widetilde{FM}_2$

First let us write $t = \chi x^{37/50}$, then the unrefined index may be written in the form:

$$\mathcal{I}_{\widetilde{FM}_2}^{unr}(x, t) = \mathcal{I}^{(0)}(t) + x^2 \mathcal{I}^{(1)}(t) + x^4 \mathcal{I}^{(2)}(t) + \mathcal{O}(x^6) \quad (5.186)$$

where the functions $\mathcal{I}^{(k)}(t)$ can be resummed into closed rational forms:

$$\mathcal{I}^{(0)}(t) = \frac{1 + 6t + 6t^2 + t^3}{(1 - t)^9} = \mathcal{HS}_{\widetilde{FM}_2} \quad (5.187)$$

$$\mathcal{I}^{(1)}(t) = -12 \frac{3 + 8t + 3t^2}{(1 - t)^8} \quad (5.188)$$

$$\mathcal{I}^{(2)}(t) = \frac{1 - 8t + 49t^2 + 334t^3 - 116t^4 - 602t^5 + 13t^6 - 8t^7 + t^8}{(1 - t)^8 t^3} \quad (5.189)$$

In terms of the global $SU(6)$ symmetry, the functions $\mathcal{I}^{(k)}(t)$ can be written as an exact sum over irreducible representations (denoted by their Dynkin labels $\chi[a1, a2, a3, a4, a5]$). In this theory, the chiral ring $\mathcal{I}^{(0)}(t)$ is generated by the representation **15** ($\chi_{[0,1,0,0,0]}$), while the current multiplet transforms in the adjoint **35** ($\chi_{[1,0,0,0,1]}$). The exact character combinations are:

$$\mathcal{I}^{(0)}(t) = \sum_{k=0}^{\infty} \chi_{[0,k,0,0,0]} t^k \quad (5.190)$$

$$\mathcal{I}^{(1)}(t) = - \sum_{k=0}^{\infty} (\chi_{[1,k,0,0,1]} - \chi_{[1,k-2,1,0,0]}) t^k \equiv - \sum_{k=0}^{\infty} C_k^{(1)} t^k \quad (5.191)$$

$$\mathcal{I}^{(2)}(t) = \frac{\chi_{[0,0,0,0,0]}}{t^3} + \frac{\chi_{[2,0,0,0,0]}}{t} + \sum_{k=0}^{\infty} C_k^{(2)} t^k \quad (5.192)$$

The formula for $\mathcal{I}^{(1)}(t)$ incorporates the $U(1)_\tau$ singlet constraint through the Weyl character formula.

For $SU(6)$, $\rho = [1, 1, 1, 1, 1]$. The Weyl character formula gives $\chi_\lambda = 0$ whenever $\lambda + \rho$ lies on a wall of the Weyl chamber, i.e., when any Dynkin label of λ is -1 (since then $\lambda + \rho$ has a zero label, placing it on a wall and the character vanishes after antisymmetrization).

When $k = 1$: the subtracted term is $\chi_{[1,-1,1,0,0]}$. The second Dynkin label is -1 , so $(\lambda + \rho)_2 = 0$, placing it on the wall of the Weyl chamber associated with simple root α_2 . The character vanishes identically, leaving

$$C_1^{(1)} = \chi_{[1,1,0,0,1]}$$

of dimension **384**.

When $k = 0$: the subtracted term is $\chi_{[1,-2,1,0,0]}$. The shifted weight is $[1, -2, 1, 0, 0] + \rho = [2, -1, 2, 1, 1]$, which has a negative second label. A single reflection s_2 across the second simple root maps

$$s_2([2, -1, 2, 1, 1]) = [2, -1, 2, 1, 1] + (-1) \cdot (-\alpha_2) = [1, 1, 1, 1, 1] = \rho,$$

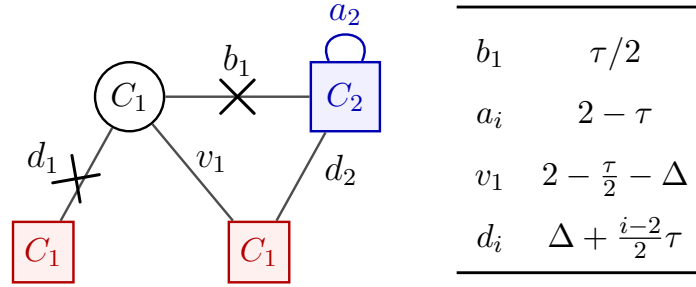
the Weyl vector, which is the shifted weight of the trivial representation $\chi_{[0,0,0,0,0]}$. Since s_2 introduces a sign of -1 , the Weyl formula gives $\chi_{[1,-2,1,0,0]} = -\chi_{[0,0,0,0,0]} = -\mathbf{1}$. Therefore:

$$C_0^{(1)} = \chi_{[1,0,0,0,1]} - (-\chi_{[0,0,0,0,0]}) = \mathbf{35} + \mathbf{1} = 36,$$

consistent with the dimension of the conserved current multiplet of the $SU(6)$ theory together with a singlet.

5.3.6 FE_2

Consider now the FE_2 theory, having the following UV completion:



$$\mathcal{W} = v_1 b_1 d_2 + b_1^2 a_2 + \text{Flip}[b_1^2] + \text{Flip}[d_1^2] + \mathfrak{M} \quad (5.193)$$

Mirror self-duality and IR symmetry enhancement

The FE_2 theory enjoys a self-duality under mirror symmetry, which exchanges the manifest C_2 global symmetry with an emergent C_2 one. The naive IR global symmetry is therefore $USp(4)^2 \times U(1)_\tau \times U(1)_\Delta$. As we shall see, at the strongly-coupled fixed point the symmetry actually enhances to

$$Spin(10) \times U(1)_\tau \times U(1)_\Delta,$$

i.e. an extra $U(1)$ Cartan emerges along the RG flow – a phenomenon already observed for the corresponding $4d$ $\mathcal{N} = 1$ theory in [233], and a general feature of the $N = 2$ theories considered here.

Chiral-ring generators

The IR spectrum contains a bifundamental Π in the $(\mathbf{4}, \mathbf{4})$ of $USp(4)_L \times USp(4)_R$, two traceless antisymmetric fields A_L, A_R of $USp(4)_L$ and $USp(4)_R$, and two singlets:

$$B_{1,2} \text{ (flipping } d_1^2), \quad B_{2,1} \equiv v_1^2.$$

Superconformal R-charges and index

Z -extremization gives

$$\tau = 1.215 \dots, \quad \Delta = 0.874 \dots \quad (5.194)$$

For the numerical index we adopt the rational values $\Delta = 85/100$ and $\tau = 12/10$, and turn on fugacities ψ_τ, χ_Δ for the two $U(1)$ symmetries. Unrefining the non-Abelian $USp(4)$ fugacities yields

$$\begin{aligned}
\mathcal{I}_{FE_2} = & 1 + 10\psi_\tau x^{4/5} + 16\chi_\Delta x^{17/20} + \psi_\tau \chi_\Delta^{-2} x^{11/10} + \chi_\Delta^{-2} \psi_\tau^{-1} x^{3/2} \\
& + 54\psi_\tau^2 x^{8/5} + 144\chi_\Delta \psi_\tau x^{33/20} + 126\chi_\Delta^2 x^{17/10} + 10\psi_\tau^2 \chi_\Delta^{-2} x^{19/10} \\
& - 47x^2 - 16\chi_\Delta \psi_\tau^{-1} x^{41/20} + \psi_\tau^2 \chi_\Delta^{-4} x^{11/5} \\
& + x^{12/5} (210\psi_\tau^3 - \psi_\tau^{-2}) + 720\chi_\Delta \psi_\tau^2 x^{49/20} + 1050\chi_\Delta^2 \psi_\tau x^{5/2} \\
& + 672\chi_\Delta^3 x^{51/20} + \chi_\Delta^{-4} x^{13/5} + 54\psi_\tau^3 \chi_\Delta^{-2} x^{27/10} - 450\psi_\tau x^{14/5} \\
& - 720\chi_\Delta x^{57/20} - 210\chi_\Delta^2 \psi_\tau^{-1} x^{29/10} + x^3 (10\psi_\tau^3 \chi_\Delta^{-4} + \chi_\Delta^{-4} \psi_\tau^{-2}) \\
& - 46\psi_\tau \chi_\Delta^{-2} x^{31/10} - 16\chi_\Delta^{-1} x^{63/20} + 660\psi_\tau^4 x^{16/5} + 2640\chi_\Delta \psi_\tau^3 x^{13/4} \\
& + x^{33/10} (\psi_\tau^3 \chi_\Delta^{-6} + 4950\chi_\Delta^2 \psi_\tau^2) + 5280\chi_\Delta^3 \psi_\tau x^{67/20} + 2772\chi_\Delta^4 x^{17/5} \\
& + x^{7/2} (210\psi_\tau^4 \chi_\Delta^{-2} - \chi_\Delta^{-2} \psi_\tau^{-1}) - 2340\psi_\tau^2 x^{18/5} - 5600\chi_\Delta \psi_\tau x^{73/20} \\
& + x^{37/10} (\psi_\tau \chi_\Delta^{-6} - 5430\chi_\Delta^2) - 1440\chi_\Delta^3 \psi_\tau^{-1} x^{15/4} + 54\psi_\tau^4 \chi_\Delta^{-4} x^{19/5} \\
& - x^{39/10} (320\psi_\tau^2 \chi_\Delta^{-2} + \chi_\Delta^{-2} \psi_\tau^{-3}) + 16\psi_\tau \chi_\Delta^{-1} x^{79/20} \\
& + x^4 (1782\psi_\tau^5 + 1033) + \mathcal{O}(x^{81/20}).
\end{aligned} \tag{5.195}$$

The $-47x^2$ conserved-current contribution signals the enhancement

$$USp(4) \times USp(2)^2 \times U(1)^2 \longrightarrow Spin(10) \times U(1)^2, \quad \dim = 45 + 1 + 1 = 47.$$

Reading the index directly, one identifies A_L, A_R as $\chi_{[1,0,0,0,0]_{so(10)}} x^{2-\tau}$; Π as $\chi_{[0,0,0,0,1]_{so(10)}} x^\Delta$; and the two singlets $B_{2,1}, B_{1,2}$ as $x^{4-2\Delta-\tau}$ and $x^{2-2\Delta+\tau}$, respectively.

Hilbert series and branch structure

From the index one extrapolates the all-order Hilbert series

$$\begin{aligned}
\mathcal{HS}_{FE_2} = & \sum_{k,l=0}^{\infty} \chi_{[k,0,0,0,l]_{so(10)}} x^{k(2-\tau)+l\Delta} + \sum_{k,m=0}^{\infty} \chi_{[k,0,0,0,0]_{so(10)}} x^{k(2-\tau)+m(4-2\Delta-\tau)} \\
& + \sum_{n,m=0}^{\infty} x^{m(4-2\Delta-\tau)+n(2-2\Delta+\tau)} - \sum_{k=0}^{\infty} \chi_{[k,0,0,0,0]_{so(10)}} x^{k(2-\tau)} \\
& - \sum_{m=0}^{\infty} x^{m(4-2\Delta-\tau)}.
\end{aligned} \tag{5.196}$$

The moduli space therefore consists of three branches:

- a *main branch*, generated by $\{A_L, A_R, \Pi\}$;
- a *second branch*, generated by $\{A_L, A_R, B_{2,1}\}$;
- a *third branch*, generated by $\{B_{1,2}, B_{2,1}\}$.

The first two branches meet along the subvariety cut out by $\{A_L, A_R\}$; the second and third branches meet along the locus cut out by $\{B_{2,1}\}$. Defining $t_a = x^{2-\tau}$, $t_d = x^\Delta$, $t_c = x^{4-2\Delta-\tau}$, $t_e = x^{2-2\Delta+\tau}$, the unrefined Hilbert series of the second and third branches are

$$\mathcal{HS}^{(2)} = \frac{1+t_a}{(1-t_a)^9(1-t_c)}, \quad \mathcal{HS}^{(3)} = \frac{1}{(1-t_c)(1-t_e)}, \tag{5.197}$$

so the third branch is simply $\mathbb{C}^2$, while the second is $\mathbb{C}$ times the polynomial ring generated by $\{A_L, A_R\}$ modulo the single relation $\text{tr}_L A_L^2 = \text{tr}_R A_R^2$.

The main branch

For the main branch, generated by $\{A_L, A_R, \Pi\}$, the unrefined Hilbert series takes the closed form

$$\mathcal{HS}_{FE_2}^{\text{main}}(t_a, t_d) = \frac{\sum_{j,k} a_{j,k} t_a^j t_d^k}{(1-t_a)^9 (1-t_d)^{11}}, \quad (5.198)$$

with numerator coefficient matrix

$$\{a_{j,k}\} = \begin{pmatrix} 1 & 5 & 5 & 1 & 0 & 0 \\ 1 & -11 & -29 & -9 & 0 & 0 \\ 0 & 0 & 36 & 36 & 0 & 0 \\ 0 & 0 & -9 & -29 & -11 & 1 \\ 0 & 0 & 1 & 5 & 5 & 1 \end{pmatrix}. \quad (5.199)$$

The matrix is palindromic, $a_{j,k} = a_{4-j, 5-k}$, and satisfies $\sum_{j,k} a_{j,k} = 0$. The diagonal specialization $t_a = t_d = t$ factorises as

$$N(t, t) = (1-t)^4 (1+t) (1+9t+19t^2+9t^3+t^4), \quad (5.200)$$

while the two single-variable specialisations give

$$N(1, t) = 2(1-t)^4(1+t), \quad N(t, 1) = 12(1-t)^4. \quad (5.201)$$

These translate into the row and column sum rules

$$\sum_j a_{j,k} = 2(-1)^k \left[\binom{4}{k} - \binom{4}{k-1} \right], \quad \sum_k a_{j,k} = 12(-1)^j \binom{4}{j}, \quad (5.202)$$

with $\binom{4}{-1} = \binom{4}{5} = 0$.

Generators and relations of the main branch

The structure of the chiral ring on the main branch is encoded in the plethystic logarithm of (5.198):

$$\begin{aligned} \text{PL}[\mathcal{HS}_{FE_2}^{\text{main}}](t_a, t_d) &= \chi_{[1,0,0,0,0]_{\mathfrak{so}(10)}} t_a + \chi_{[0,0,0,0,1]_{\mathfrak{so}(10)}} t_d - t_a^2 \\ &\quad - \chi_{[0,0,0,1,0]_{\mathfrak{so}(10)}} t_a t_d - \chi_{[1,0,0,0,0]_{\mathfrak{so}(10)}} t_d^2 \\ &\quad + \mathcal{O}(t_a t_d (t_a + t_d)). \end{aligned} \quad (5.203)$$

At order t_a one reads off $\{A_L, A_R\}$, jointly transforming in the vector of $\mathfrak{so}(10)$; at order t_d the bifundamental Π , in the spinor. These generators satisfy the quadratic relations

$$\begin{aligned} \Omega_{ab} \Omega_{cd} A_L^{ac} A_L^{bd} &= \Omega_{AB} \Omega_{CD} A_R^{AC} A_R^{BD}, & (1 \text{ relation at order } t_a^2), \\ \Omega_{ac} A_L^{ab} \Pi^{cA} &= \Omega_{BC} A_R^{AB} \Pi^{Cb}, & (16 \text{ relations at order } t_a t_d), \\ [\Omega_{ab} \Pi^{aA} \Pi^{bB}]_{\Omega\text{-traceless}} &= 0 = [\Omega_{AB} \Pi^{aA} \Pi^{bB}]_{\Omega\text{-traceless}}, & (10 \text{ relations at order } t_d^2), \end{aligned} \quad (5.204)$$

with $a, b, \dots$ and $A, B, \dots$ denoting $USp(4)_L$ and $USp(4)_R$ indices respectively. Since (5.203) does not terminate, the relations are not independent and the main branch is not a complete intersection.

An exceptional model: $\widetilde{FE}_2$

Something interesting happens when we flip the singlet $B_{2,1} \equiv v_1^2$, we call the resulting theory $\widetilde{FE}_2$. Now the theory enjoys an $SU(2) \times SU(6) \times U(1)$ UV global symmetry, with two fields in the $(\mathbf{2}, \mathbf{6})$ and $(\mathbf{1}, \mathbf{15})$ of the global symmetries, having the same $U(1)$ charge, together with the field $B_{1,2}$ in the singlet rep $(\mathbf{1}, \mathbf{1})$. This can also be verified performing Z-extremization, which gives $\Delta = 0.834$ and $\tau = 2 - \Delta = 1.166$. We claim that this theory flows to an IR fixed point with enhanced symmetry $E_6 \times U(1)$, which can be checked for example looking at the index. The index for this flipped theory reads (computed with the rational values $\tau = 11/10$, $\Delta = 9/10$):

$$\mathcal{I}_{\widetilde{FE}_2} = 1 + \mathbf{27}\chi x^{9/10} + \frac{x^{13/10}}{\chi^3} + \mathbf{351}'\chi^2 x^{9/5} - (\mathbf{78} + 1)x^2 + \frac{x^{13/5}}{\chi^6} + \mathbf{3003}\chi^3 x^{27/10} + O(x^{29/10}) \quad (5.205)$$

where we turned on a fugacity χ for the single $U(1)$ global symmetry. The bold numbers are characters of the E_6 global symmetry. We observe, for example, at order x^2 the adjoint rep $\mathbf{78}$ of E_6 given by the conserved currents. We also observe the contributions from 1/2-BPS operators $\chi_{[k,0,0,0,0]_{E_6}} x^{k\Delta}$ for $k \geq 1$. The moduli space of the theory is composed by two branches with trivial intersection: we have a branch $\mathbb{C}$ freely generated by $B_{1,2}$ (appearing at order $x^{4-3\Delta}$ in the index), and a main branch generated by 27 generators satisfying 27 quadratic relations. This main branch can be geometrically represented as an irreducible Hermitian symmetric space of compact type $E_6/(SO(10) \times U(1))$, which is of rank 2. We also write down the HS for the main branch, which can be extrapolated from the index. It reads:

$$\mathcal{HS}_{\widetilde{FE}_2}(t \equiv x^\Delta) = \sum_{k=0}^{\infty} \chi_{[k,0,0,0,0]_{E_6}} t^k \quad (5.206)$$

$$\stackrel{unr}{=} \frac{t^5 + 10t^4 + 28t^3 + 28t^2 + 10t + 1}{(1-t)^{17}}$$

This is exactly the HS of the cone over the complex Cayley space, studied for example in [185].

Up to order x^{10} , the unrefined index (with respect to the global E_6 symmetry) reads:

$$\begin{aligned}
\mathcal{I}_{\widetilde{FE_2}}(x; \chi) = & 1 + 27\chi x^{9/10} + \chi^{-3}x^{13/10} + 351\chi^2x^{9/5} - 79x^2 + \chi^{-6}x^{13/5} + 3003\chi^3x^{27/10} \\
& - 1728\chi x^{29/10} + 19305\chi^4x^{18/5} - 18954\chi^2x^{19/5} + \chi^{-9}x^{39/10} + 2923x^4 \\
& + 378\chi^{-2}x^{21/5} + 100386\chi^5x^{9/2} - 140608\chi^3x^{47/10} + 49842\chi x^{49/10} + \chi^{-12}x^{26/5} \\
& - 729\chi^{-3}x^{53/10} + 442442\chi^6x^{27/5} - 799227\chi^4x^{28/5} + 449658\chi^2x^{29/5} - 66834x^6 \\
& - 7317\chi^{-2}x^{31/5} + 1706562\chi^7x^{63/10} + 351\chi^{-4}x^{32/5} + (\chi^{-15} - 3729024\chi^5)x^{13/2} \\
& + 2835508\chi^3x^{67/10} - 835434\chi x^{69/10} + 42120\chi^{-1}x^{71/10} + 5895396\chi^8x^{36/5} \\
& + 11490\chi^{-3}x^{73/10} - 14916616\chi^6x^{37/5} + 729\chi^{-5}x^{15/2} + 14009463\chi^4 \\
& + x^{38/5}(\chi^{-18} - 5964867\chi^2) + x^{39/5} + 1046513x^8 + 18559580\chi^9x^{81/10}(5.207) \\
& - 2754\chi^{-2}x^{41/5} - 52700544\chi^7x^{83/10} - 7371\chi^{-4}x^{42/5} + 57741255\chi^5x^{17/2} \\
& - 1379\chi^{-6}x^{43/5} - 30976062\chi^3x^{87/10} + 8544987\chi x^{89/10} + 53965548\chi^{10}x^9 \\
& + (\chi^{-21} - 1062126\chi^{-1})x^{91/10} - 168018786\chi^8x^{46/5} - 12873\chi^{-3}x^{93/10} \\
& + 206545352\chi^6x^{47/5} + 756\chi^{-5}x^{19/2} - 129114594\chi^4x^{48/5} + 702\chi^{-7}x^{97/10} \\
& + 45121482\chi^2x^{49/5} + 146477916\chi^{11}x^{99/10} - 9449098x^{10} + O(x^{101/10})
\end{aligned}$$

Index resummation for $\widetilde{FE_2}$

First let us write $t = \chi x^{9/10}$, then the unrefined index may be written in the form:

$$\mathcal{I}_{\widetilde{FE_2}}^{unr}(x, t) = \mathcal{I}^{(0)}(t) + x^2\mathcal{I}^{(1)}(t) + x^4\mathcal{I}^{(2)}(t) + \mathcal{O}(x^6) \quad (5.208)$$

where the functions $\mathcal{I}^{(k)}(t)$ can be resummed into closed rational forms:

$$\mathcal{I}^{(0)}(t) = \frac{t^5 + 10t^4 + 28t^3 + 28t^2 + 10t + 1}{(1-t)^{17}} = \mathcal{HS}_{\widetilde{FE_2}}(t) \quad (5.209)$$

$$\mathcal{I}^{(1)}(t) = \frac{-79t^6 - 306t^5 + 63t^4 + 644t^3 + 63t^2 - 306t - 79}{(1-t)^{18}} \quad (5.210)$$

$$\begin{aligned}
\mathcal{I}^{(2)}(t) = & \frac{1}{(1-t)^{18}t^3} \left(-27t^{10} - 2751t^9 - 1269t^8 + 9477t^7 + 886t^6 - 8847t^5 \right. \\
& \left. + 288t^4 + 2107t^3 + 153t^2 - 18t + 1 \right) \quad (5.211)
\end{aligned}$$

In terms of the global E_6 symmetry, the functions $\mathcal{I}^{(k)}(t)$ can be written as an exact sum over irreducible representations (denoted by their Dynkin labels $\chi[a1, a2, a3, a4, a5, a6]$). In this theory, the chiral ring $\mathcal{I}^{(0)}(t)$ is generated by the representation **27** ($\chi_{[1,0,0,0,0,0]}$), while the current multiplet transforms in the adjoint **78** ($\chi_{[0,0,0,0,0,1]}$). The exact character combinations are:

$$\mathcal{I}^{(0)}(t) = \sum_{k=0}^{\infty} \chi_{[k,0,0,0,0,0]} t^k \quad (5.212)$$

$$\mathcal{I}^{(1)}(t) = - \sum_{k=0}^{\infty} (\chi_{[k,0,0,0,0,1]} - \chi_{[k-2,1,0,0,0,0]}) t^k \equiv - \sum_{k=0}^{\infty} C_k^{(1)} t^k \quad (5.213)$$

$$\mathcal{I}^{(2)}(t) = \frac{\chi_{[0,0,0,0,0,0]}}{t^3} + \sum_{k=0}^{\infty} C_k^{(2)} t^k \quad (5.214)$$

The formula for $\mathcal{I}^{(1)}(t)$ incorporates the $U(1)_\tau$ singlet constraint through the Weyl character formula. When $k < 2$, the subtracted representation $\chi_{[k-2,1,0,0,0,0]}$ has a negative or vanishing first Dynkin label, and its fate is determined by the Weyl alternation formula applied to E_6 .

When $k = 1$: the subtracted term is $\chi_{[-1,1,0,0,0,0]}$. The shifted weight is $[-1, 1, 0, 0, 0, 0] + \rho = [0, 2, 1, 1, 1, 1]$, whose first Dynkin label vanishes, placing it on the wall of the Weyl chamber associated with the simple root α_1 . The character vanishes identically, leaving

$$C_1^{(1)} = \chi_{[1,0,0,0,0,1]} \quad (5.215)$$

of dimension **1728**.

When $k = 0$: the subtracted term is $\chi_{[-2,1,0,0,0,0]}$. The shifted weight is $[-2, 1, 0, 0, 0, 0] + \rho = [-1, 2, 1, 1, 1, 1]$, which has a negative first label. A single reflection s_1 across the first simple root acts by

$$[-1, 2, 1, 1, 1, 1] \xrightarrow{s_1} [1, 1, 1, 1, 1, 1] = \rho, \quad (5.216)$$

the Weyl vector, which is the shifted weight of the trivial representation $\chi_{[0,0,0,0,0,0]}$. Since s_1 introduces a sign of -1 , the Weyl formula gives $\chi_{[-2,1,0,0,0,0]} = -\chi_{[0,0,0,0,0,0]} = -\mathbf{1}$. Therefore:

$$C_0^{(1)} = \chi_{[0,0,0,0,0,1]} - (-\chi_{[0,0,0,0,0,0]}) = \mathbf{78} + \mathbf{1} = \mathbf{79} \quad (5.217)$$

which reproduces the exact coefficient 79 appearing in the numerator of $\mathcal{I}^{(1)}(t)$ at order t^0 , consistent with the dimension of the conserved current multiplet of the E_6 theory together with a singlet.

5.4 Scope and outlook

Scope. This chapter investigated the moduli space geometry of the five families of improved bifundamentals — FT_N , FM_N , FE_N , FC_N , and FH_N — using the superconformal index and Hilbert series as primary tools. For each family, the Hilbert series was extracted from the SCI, the generators and relations of the chiral ring were identified, and the branch structure of the moduli space was characterized. A closed-form unrefined Hilbert series was obtained for FT_N at all N ; for the remaining theories, Euler-form Hilbert series were computed through $N = 3$ and extrapolated to generic N . For the special case $N = 2$, the IR symmetry enhancement was characterized explicitly for all five theories: their moduli spaces were identified as complex cones over irreducible Hermitian symmetric spaces, and fully unrefined Hilbert series were derived as sums over characters of the enhanced global symmetry group.

Outlook. Several directions remain open. On the algebraic side, it would be valuable to derive closed-form or recursive formulae for the Hilbert-series numerators at generic N , connecting the growth patterns observed at low N to the representation theory of the IR global symmetry. The computation should be extended to asymmetric improved bifundamentals and to the asymmetric $\mathcal{S}$ -walls that arise in the chiral-planar algorithm of Chapter 3, where these theories appear as elementary building blocks [46, 56]. Whether the Calabi–Yau properties observed at $N = 2$ persist at higher N is a natural target for future investigation. More broadly, the emerging connections between chiral-ring relations, Hilbert series, and Hermitian symmetric spaces suggest a wider geometric picture that may extend to 2d $\mathcal{N} = (0, 2)$ compactifications and sigma-model target spaces [237, 238].

Chapter 6

Conclusion

The guiding thread of this thesis has been the interplay between three-dimensional mirror symmetry and Chern–Simons–matter theories, explored across three different amounts of supersymmetry: theories with four supercharges, theories with no supersymmetry at all, and a distinguished class of $\mathcal{N} = 2$ SCFTs, the improved bifundamentals, which are themselves natural building blocks of mirror symmetry with four supercharges. In each case, the same underlying philosophy was at work: dualities, when suitably organized, are not merely curiosities but precision tools that give access to non-perturbative information about strongly coupled quantum field theories. In this final chapter we summarize what has been achieved, we point out the limitations of the present analysis, and we indicate the directions that we consider most promising for future work.

Mirror symmetry with four supercharges

We started in Chapter 3, where we reviewed how the mirror dual of a generic $\mathcal{N} = 4$ linear quiver gauge theory can be derived purely within field theory, without any recourse to string theory, thanks to the $\mathcal{N} = 4$ dualization algorithm of [45, 46]. The algorithm works by chopping the quiver into elementary QFT blocks, dualizing each block via the $\mathcal{S}$ -wall, and gluing the results back together using a small set of basic duality moves. We then extended this procedure to theories with four supercharges [1, 2], identifying the correct $\mathcal{N} = 2$ analog of the $\mathcal{S}$ -wall, its fusion-to-identity property, and the associated gluing rules.

The resulting *chiral-planar* dualization algorithm applies to all 3d $\mathcal{N} = 2$ theories that descend from $\mathcal{N} = 4$ ones through a real mass deformation that explicitly breaks part of the R -symmetry. Integrating out the massive fields generates Chern–Simons interactions whose levels are completely fixed by the amount of surviving massless matter: for instance, for $U(N)$ SQCD with F fundamental chiral multiplets, the CS levels in our conventions are $(-\frac{F}{2} + N, -\frac{F}{2})$. In its present form the algorithm covers $U(N)$ and $SU(N)$ CS-SQCD with $[n_f, n_a]$ fundamental and anti-fundamental chirals in the three families

$$[n_f, n_a] = [N + F_1, N + F_2], \quad [2N + F_1, F_2], \quad [F_1, 2N + F_2], \quad F_1, F_2 \geq 0, \quad (6.1)$$

as well as several examples of linear quiver gauge theories, either with constant-rank gauge nodes or involving the $\mathcal{N} = 2$ $\mathcal{S}$ -wall as an ingredient. On the other side of the duality one always finds an *Abelian planar* quiver: a two-dimensional array of $U(1)$ gauge nodes arranged on a triangular lattice, with characteristic mixed CS interactions and a

monopole superpotential whose faces are in one-to-one correspondence with the superpotential terms. Every step of the derivation was tracked at the level of the S_b^3 partition function, so that the map of real masses, FI parameters, and R -charges across the duality is under complete control. An important spin-off of this analysis was the observation that certain chiral quivers exhibit an enhancement of the topological symmetry, invisible in the electric description but manifest in the planar dual.

We then used the intuition gained from the algorithm to explore the full landscape of $\mathcal{N} = 2$ $U(N)_k$ SQCD with F fundamental chirals [3]. Starting from the $\mathcal{N} = 4$ -descendant theory and following real mass deformations for the fundamental fields, we mapped the flows to the planar Abelian side and obtained a taxonomy of the (N, k, F) plane in terms of zones, each with its own proposal for a planar Abelian mirror dual. This program led to a number of non-trivial consistency checks and predictions: the planar duals correctly reproduce the known Aharony-like and Giveon–Kutasov-like dualities [35, 93, 180], and in fact we argued that generic Aharony-like dualities for $U(N)_k$ SQCD are *implied* by the planar Abelian duals together with a handful of basic Abelian duality moves. As a by-product, we uncovered an accidental enhancement to $\mathcal{N} = 4$ supersymmetry for $U(N)_{(-N/2, -3N/2)}$ SQCD with N fundamental chirals, a statement that would have been very hard to guess from the electric description alone.

Several aspects of this picture remain incomplete. The algorithmic analysis is currently restricted to the CS levels inherited from the $\mathcal{N} = 4$ parent, and a generalization involving the *asymmetric* $\mathcal{S}$ -wall, needed to dualize quivers with non-constant ranks in a fully algorithmic fashion, is still lacking. Future work should also aim at including in the landscape analysis theories with both fundamental and anti-fundamental chirals at generic (N, k, F) , which we expect to require only a modest extension of the techniques developed here, as well as theories with symplectic and orthogonal gauge groups; a first extensive investigation of mirror symmetry for symplectic CS-SQCD has recently appeared in [187], and the $\mathcal{N} = 4$ technology for orthosymplectic and circular quivers [177, 178] provides a natural blueprint. Finally, the precise interface between the planar Abelian duals and level-rank duality, which controls the endpoints of the mass deformation flows, deserves to be clarified. Taken together, these developments would bring us closer to a complete understanding of the role of mirror symmetry for theories with four supercharges, which is still far from being achieved.

Beyond supersymmetry

With the new intuition gained from the mirror-like dualities of Chapter 3, in Chapter 4 we turned to the much less charted territory of dualities for theories with no supersymmetry at all [4]. Here the toolbox is very limited: without supersymmetry there are no protected quantities, no non-renormalization theorems, and, most importantly, no localization techniques that would allow us to compute exact quantities to be matched across the duality. Nevertheless, guided by the expectation that non-supersymmetric dualities should arise as supersymmetry-breaking mass deformations of supersymmetric ones [62, 63, 69, 192], we conjectured the form that planar Abelian dualities should take in this setting.

Concretely, we proposed that $SU(N)_k$ and $U(N)_k$ CS-QCD₃ with N_s fundamental scalars and N_f fundamental fermions are dual, in the regime $N_s \geq 2N$, to planar Abelian quivers with both bosonic and fermionic matter, where the quiver topology is the same as

in the supersymmetric case but each edge now carries a single boson or fermion. The shape of these dualities was argued for through a set of independent consistency checks: the matching of the lowest-dimension gauge-invariant operators (monopoles with minimal flux on the Abelian side mapped to mesons and baryons on the non-Abelian side), the pattern of symmetry enhancement that the Abelian quiver must exhibit to reproduce the manifest $U(N_s) \times U(N_f)$ symmetry of the electric theory, the consistency of mass deformations and of the resulting RG flows, and the reduction to results already established in the literature in special limits, namely the Abelian bosonization web [61, 66–68], particle-vortex duality, and the level-rank dualities of pure CS theories [97] reached at the end of the mass flows. In all the cases we studied these checks were passed.

We regard these results as a solid conjecture rather than a derivation. New tools are needed to increase the plausibility of our proposals, and three directions seem most immediate. First, the small-flavor regime $N_s < 2N$ exhibits qualitatively new features and calls for a systematic treatment, possibly in connection with the symmetry-breaking scenarios of [207]. Second, the supersymmetric origin of the dualities should be made precise by tracking the SUSY-breaking real mass deformations of the $\mathcal{N} = 2$ mirror pairs at the level of operators, partition functions and the spin-TQFT factors generated along the flow. Third, an algorithmic formulation analogous to the chiral-planar algorithm of Chapter 3 would replace the present case-by-case analysis with a systematic derivation of planar Abelian bosonization duals for arbitrary non-Abelian CS-QCD theories.

Moduli spaces of improved bifundamentals

Finally, in Chapter 5 we focused on a class of 3d $\mathcal{N} = 2$ SCFTs with remarkable properties, the improved bifundamentals FT_N , FM_N , FE_N , FC_N and FH_N [56, 171, 217, 218]. These theories have appeared in the recent literature in connection with 3d and 4d mirror-like dualities and with the compactification of 6d theories on Riemann surfaces, they satisfy star-triangle relations, and they behave nicely under mirror symmetry: FT_N , FE_N and FM_N are self-mirror, while FH_N and FC_N are mapped into each other. Despite their interesting properties, a systematic study of their moduli spaces of vacua was missing [5].

We investigated this question in detail using the superconformal index and the Hilbert series as our main tools, within the framework of the plethystic program [108, 112]. Extracting the Hilbert series from the index is a delicate operation for $\mathcal{N} = 2$ theories, since accidental cancellations between bosonic and fermionic contributions can obscure the chiral ring [168, 224]; for the class of theories at hand we provided extensive order-by-order evidence that such cancellations do not occur, so that the Hilbert series of the full moduli space, or of its main branch, can be reliably reconstructed. This gave us access to the generators of the chiral ring, the relations among them, and the pattern of symmetry enhancement along the RG flow. We found that the moduli spaces of FT_N , FC_N and FH_N consist of a single irreducible branch generated by operators in rank-2 representations of the IR global symmetry, while FM_N and FE_N possess additional simple branches associated with the singlets B_{ij} . For FT_N , whose moduli space is a complete intersection, we obtained a closed-form unrefined Hilbert series valid for all N ; for the other families we computed the Hilbert series in Euler form up to $N = 3$ and extrapolated the dimension of the moduli space and the degree of the numerator to generic N .

The lowest non-trivial rank, $N = 2$, displayed a particularly elegant pattern of symme-

try enhancements, generalizing the observations of [233, 236]. The moduli spaces of FT_2 , FC_2 , FH_2 , $\widetilde{FM}_2$ and $\widetilde{FE}_2$ turned out to be complex cones over irreducible Hermitian symmetric spaces [237–241], namely Grassmannians, the pure spinor space $SO(10)/U(5)$, and the complex Cayley plane $E_6/(SO(10) \times U(1))$, some of which had already appeared in the physics literature in a different guise [185]. In each case the unrefined Hilbert series could be written as an exact sum over characters of the enhanced global symmetry group, in agreement with the exceptional dualities of [234, 235].

Natural continuations of this work include deriving closed-form or recursive expressions for the Hilbert series numerators at generic N , extending the analysis to the asymmetric improved bifundamentals and asymmetric $\mathcal{S}$ -walls that enter the chiral-planar algorithm, and understanding whether the Calabi–Yau property observed at $N = 2$ persists at higher rank. More broadly, the connection between chiral ring relations, Hilbert series and Hermitian symmetric spaces suggests a geometric picture that may extend to 2d $\mathcal{N} = (0, 2)$ compactifications and sigma-model target spaces [237, 238].

Closing remarks

The three parts of this thesis are complementary facets of a single program. The chiral-planar algorithm shows that mirror symmetry with four supercharges is a generative principle rather than a collection of isolated examples; the non-supersymmetric planar dualities show that the same structures survive, at least conjecturally, once supersymmetry is broken; and the study of the improved bifundamentals shows that the elementary blocks entering these constructions have moduli spaces of unexpected geometric richness. All in all, we have obtained a considerably better understanding of the relation between mirror symmetry and Chern–Simons-matter theories, which was the whole reason we embarked on this journey. Much remains to be understood, and we hope that the tools and results collected here will prove useful for the next steps.

Declaration on the Use of AI Tools

During the preparation of this thesis I made use of AI assistants, specifically Anthropic's Claude and OpenAI's ChatGPT, for the following purposes: proofreading and improving the language of the manuscript, checking computations and consistency of conventions, preparing and formatting figures, and assisting with literature research.

All results, arguments and conclusions presented in this thesis were conceived, carried out and verified by the author. Every AI-assisted output was reviewed and, where relevant, independently checked. I take full responsibility for the entire content of this thesis.

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