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Soliton synchronization with randomness: rogue waves and universality

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Abstract

We consider an N-soliton solution of the focusing nonlinear Schrödinger equations. We give conditions for the synchronous collision of these N solitons. When the solitons velocities are well separated and the solitons have equal amplitude, we show that the local wave profile at the collision point scales as the $\operatorname{sinc}(x)$ function. We show that this behaviour persists when the amplitudes of the solitons are i.i.d. sub-exponential random variables. Namely the central collision peak exhibits *universality*: its spatial profile converges to the $\operatorname{sinc}(x)$ function, independently of the distribution. We derive Central Limit Theorems for the fluctuations of the profile in the near-field regime (near the collision point) and in the far-field regime.

Keywords: solitons, integrable dispersive PDEs, randomness, rogue waves, universality

Mathematics Subject Classification numbers: 35Q51, 37K15, 60F05

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1. Introduction and main results

We consider the focusing nonlinear Schrödinger (fNLS) equation in $1+1$ dimensions

$$\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \quad (x,t) \in \mathbb{R} \times [0, +\infty) \,. \tag{1.1}$$

This equation has countless applications both in physics and engineering. It serves as a model of nonlinear waves: in particular, water waves of small amplitude over infinite depth [46] and finite depth [5, 24], as well as almost monochromatic waves in a weakly nonlinear dispersive medium [4, 13] and rogue waves [14, 35]. It also appears in the study of the propagation of signal in fibre optics [16, 40, 42], plasma of fluids [47], Bose–Einstein condensations [36], to mention a few. In this manuscript we analyse the behaviour of soliton interactions. In particular

- we identify the soliton parameters that maximizes the amplitude of the wave profile at the interaction time;
- we prove that a train of solitons of equal amplitudes and well separated velocities has a distinguished collision profile given by the *sinc* function;
- we prove the universality of the *sinc* profile by showing that it surprisingly persists when the soliton amplitudes are sampled from a probability distribution, while the velocities are well separated, and deterministic.

We further confirm the well known fact that fast solitons interact linearly at leading order, and we provide an explicit expression of the sub-leading (nonlinear) corrections that is instrumental to obtain our results. Below we explain in detail our results.

1.1. Deterministic soliton solutions

The fNLS equation (1.1) is an example of an integrable equation that admits *soliton solutions*. The simplest of these is the family of one-soliton solutions

$$\begin{cases} \psi(x,t;z,c) = \mu \,\mathrm{sech}\,(\mu(x - x_0 - vt))\, \mathrm{e}^{\mathrm{i}\left(xv - \frac{t}{2}\left(v^2 - \mu^2\right) - \phi_0 - \pi\right)}, \\ z = \frac{1}{2}(-v + \mathrm{i}\mu) \in \mathbb{C}^+, \quad c = \mathrm{i}\mu \mathrm{e}^{\mu x_0 + \mathrm{i}\phi_0} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \,, \end{cases} \tag{1.2}$$

where $\mathbb{C}^+$ denotes the complex upper half-plane. Each such solution describes a localized traveling wave with velocity v and maximum amplitude μ. Given $2N$ complex constants, which we call *reflectionless scattering data*,

$$\{(z_k, c_k)\}_{k=1}^N \in \mathbb{C}^+ \times \mathbb{C}^*, \qquad z_k = \frac{1}{2}(-v_k + \mathrm{i}\mu_k), \quad c_k = \mathrm{i}\mu_k \mathrm{e}^{\mu_k x_k + \mathrm{i}\phi_k}, \tag{1.3}$$

Equation (1.1) also admits an N-soliton solutions, which we denote by $\psi_N(x,t)$, whose absolute value has the following determinantal representation

$$|\psi_N(x,t)|^2 = \partial_{xx} \log\det\left(\boldsymbol{I}_N + \boldsymbol{\Phi}_N(x,t)\,\overline{\boldsymbol{\Phi}_N(x,t)}\right), \tag{1.4}$$

where $\boldsymbol{\Phi}_N(x,t)$ is the $N \times N$ matrix with entries

$$[\boldsymbol{\Phi}_N(x,t)]_{jk} = \frac{\sqrt{c_j \bar{c}_k}\,\mathrm{e}^{2\mathrm{i}(\theta(z_j,x,t) - \theta(\bar{z}_k,x,t))}}{\mathrm{i}(z_j - \bar{z}_k)}, \qquad \theta(z,x,t) = xz + tz^2. \tag{1.5}$$

Formula (1.4) for the wave field of an N-soliton solution can become quite involved, as N gets large. However, the scattering theory for the reflectionless fNLS provides a general upper bound on the wave amplitude in terms of the scattering data:

$$|\psi_N(x,t)| \leqslant \sum_{k=1}^{N} \mu_k. \tag{1.6}$$

In fact, this upper bound is tight, and the following proposition gives a precise description of how it can be realized.

Proposition 1.1. *The N-soliton solution $\psi_N(x,t)$ of (1.1) described by reflectionless scattering data (1.3) with norming constant*

$$c_k = \frac{\mathrm{e}^{-2\mathrm{i}z_k x_0 - 2\mathrm{i}z_k^2 t_0}}{B'(z_k)}, \quad \forall k = 1,\dots,N, \qquad B(z) := \prod_{k=1}^{N} \frac{z - z_k}{z - \bar{z}_k}, \tag{1.7}$$

realizes the upper bound in (1.6) at $x = x_0$ and $t = t_0$, namely,

$$|\psi_N(x_0,t_0)| = \sum_{k=1}^{N} \mu_k. \tag{1.8}$$

Additionally, it is well known, see [12], that whenever the velocities $v_k = -2\,\mathrm{Re}(z_k)$ are distinct, such a solution resolves asymptotically in the large time limit to the sum of one-soliton solutions

$$\psi_N(x,t) = \sum_{k=1}^{N} \psi\left(x - x_k^{\pm}, t; z_k, c_k\right) \mathrm{e}^{\mathrm{i}\phi_k^{\pm}} + \mathcal{O}\left(\mathrm{e}^{-\kappa|t|}\right), \qquad t \to \pm\infty, \tag{1.9}$$

where $\kappa > 0$ depends on the eigenvalues z_k, $k = 1,\dots N$, and the asymptotic phase shifts are given by

$$x_k^{+} - x_k^{-} = \frac{2}{\mu_k} \sum_{j \neq k}^{N} \mathrm{sgn}(v_j - v_k) \log\left|\frac{z_k - z_j}{z_k - \bar{z}_j}\right|, \quad \phi_k^{+} - \phi_k^{-} = 2\sum_{j \neq k}^{N} \mathrm{sgn}(v_j - v_k) \arg\left(\frac{z_k - z_j}{z_k - \bar{z}_j}\right). \tag{1.10}$$

This formula is the justification for the statement that soliton collisions are elastic (see again [12]), in the sense that the physical characteristics (velocity and amplitude) of the solitons are invariant before and after the collision; the only effect of the collision is the induced phase shifts.

On the other hand, for intermediate times the interactions of the 'individual' solitons can generate very large wave peaks as supported by proposition 1.1. In this setting, the solitons interact almost linearly, indeed at the level of (1.10) one sees that

$$\frac{z_k - z_j}{z_k - \bar{z}_j} = 1 + \frac{2\mathrm{i}\mu_j}{v_k - v_j} + \mathcal{O}\left((v_k - v_j)^{-2}\right), \tag{1.11}$$

so that, when the soliton velocities are well separated, the phase shifts due to the pairwise soliton interactions become small. Our first main result gives a precise characterization of the asymptotic linearity of the soliton interactions when their velocities are well-separated. Moreover, it provides an explicit first correction to the leading order linearity with bounds on the higher order corrections.

Theorem 1.2. *Given $N \in \mathbb{N}$, consider the N-soliton solution $\psi_N(x,t)$ of (1.1) corresponding to the reflectionless spectral data (1.3) with distinct velocities. Define*

$$\Delta := \min_{j \neq k} |v_j - v_k| > 0.$$

Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$ be the vector of soliton amplitudes. Then there exists $C_* > 0$ such that for all $\Delta > C_* \|\boldsymbol{\mu}\|_\infty$ it holds that

$$\psi_N(x,t) = \sum_{k=1}^{N} \psi^{(k)}(x,t) + \frac{1}{2\mathrm{i}} \sum_{j=1}^{N} \sum_{k=1 k \neq j}^{N} \left[\frac{\psi^{(k)}(x,t) m^{(j)}(x,t)}{z_j - \bar{z}_k} - \frac{m^{(k)}(x,t) \psi^{(j)}(x,t)}{\bar{z}_j - \bar{z}_k} \right] + \mathcal{O}\left(\frac{\|\boldsymbol{\mu}\|_\infty \|\boldsymbol{\mu}\|_2^2}{\Delta^2} \right), \tag{1.12}$$

where $\psi^{(k)}(x,t) = \psi(x,t;z_k,c_k)$ is the one-soliton solution (1.2) and $m^{(k)}(x,t) = \int_x^\infty |\psi^{(k)}(s,t)|^2 \mathrm{d}s$. The error bounds are uniform for all $(x,t) \in \mathbb{R}^2$ and depend on N only through the norms $\|\boldsymbol{\mu}\|_\infty := \sup_{1 \leqslant k \leqslant N} |\mu_k|$ and $\|\boldsymbol{\mu}\|_2^2 := \sum_{k=1}^{N} |\mu_k|^2$.

Remark 1.1. The proof of theorem 1.2 shows that the first correction term, the double sum in (1.12), is of order $\|\boldsymbol{\mu}\|_2^2/\Delta$. In particular, if the magnitudes μ_k are uniformly (independently of N) bounded above and away from zero, then $\|\boldsymbol{\mu}\|_2^2 \sim N$. In this case, the leading term in (1.12) can be of order N (see (1.8) and (1.15) further below), the first correction term is of order N/Δ, while the remaining error terms are of order N/Δ^2.

Theorem 1.2 shows that for well separated velocities the phase shifts induced by soliton interactions become negligible. In what follows, we tune the discrete spectral data so that: (1) the solitons all collide at a fixed point in space time (x_0, t_0), that in view of the Galilean invariance of the fNLS equation, we assume without loss of generality to be $(0,1)$; and (2) the soliton phases are tuned so that the maximum of the solution at collision time is exactly equal to the sum of the soliton amplitudes, as proven in proposition 1.1.

Corollary 1.3. *Given constants $\alpha \in \mathbb{R}$, and $\mu, V > 0$, consider the N-soliton solution $\psi_N(x,t)$ of (1.1) with reflectionless scattering data (1.3) satisfying*

$$\mu_k = \mu, \quad v_k \in [(\alpha N + k - 1)V, (\alpha N + k)V], \quad \text{and} \quad c_k = \mathrm{i}\mu_k \mathrm{e}^{-2\mathrm{i}z_k^2}, \tag{1.13}$$

for all $k = 1, \ldots, N$, where we also assume that $\Delta = \min_{j \neq k} |v_j - v_k| > 0$ (necessarily $V \geqslant \Delta$). Then, there exists $N_0 \in \mathbb{N}$ and $C_* > 0$ such that whenever $\Delta > C_* \mu$ and $N \geqslant N_0$ it holds that

$$\frac{1}{N} \psi_N\left(\frac{2X}{NV}, 1 + \frac{T}{(NV)^2} \right) = \mu \mathrm{e}^{\mathrm{i}\left(2\alpha X - \alpha^2 \frac{T}{2}\right)} \psi_0\left(X - \frac{\alpha T}{2}, T \right) + \mathcal{O}\left(\max\left\{ \frac{1}{N}, \frac{1}{\Delta} \right\} \right),$$

where the error is locally uniform for $X, T \in \mathbb{R}$ and

$$\psi_0(X,T) = -\int_0^1 \mathrm{e}^{\mathrm{i}\left(2Xs - \frac{T}{2}s^2\right)} \mathrm{d}s. \tag{1.14}$$

In particular, when $T = 0$, it holds that

$$\frac{1}{N} \psi_N\left(\frac{2X}{NV}, 1 \right) = -\mu \mathrm{e}^{\mathrm{i}(2\alpha + 1)X} \frac{\sin(X)}{X} + \mathcal{O}\left(\max\left\{ \frac{1}{N}, \frac{1}{\Delta} \right\} \right). \tag{1.15}$$

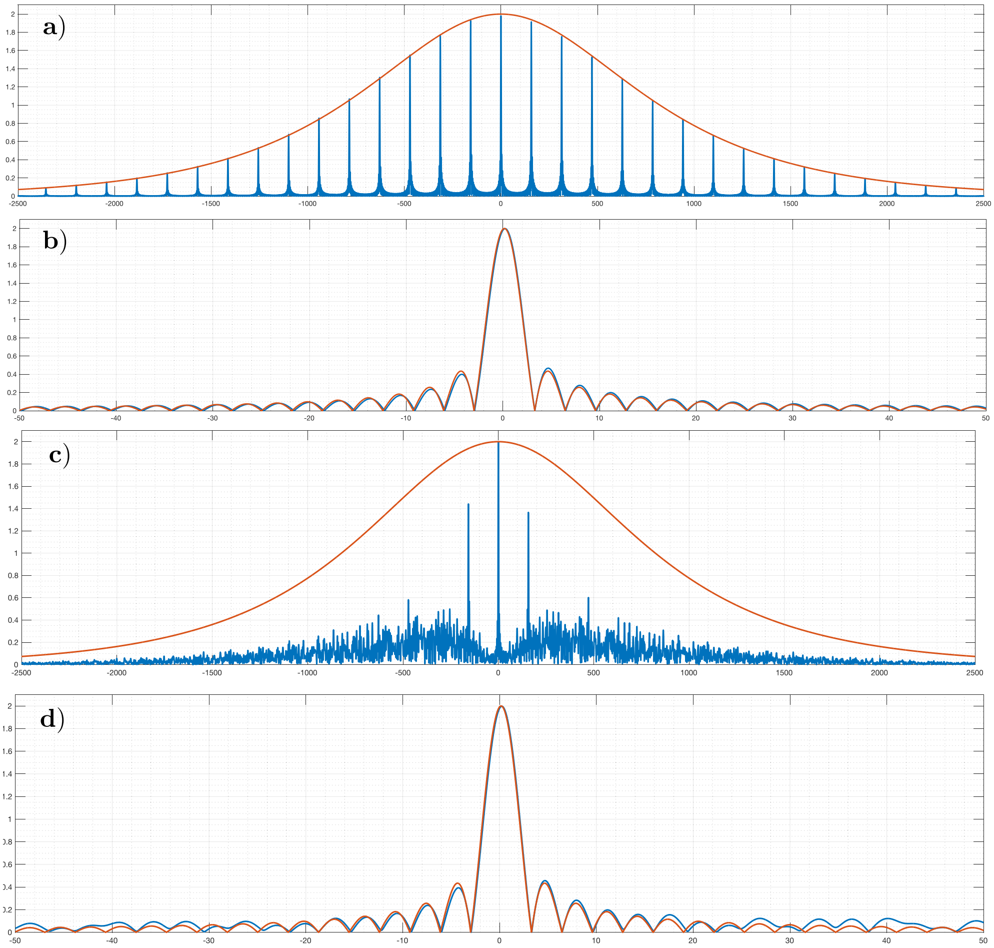

Figure 1. A comparison of equal spaced *vs* randomly spaced velocities in corollary 1.3. (a) plot of the scaled N-soliton solution $\left|\frac{1}{N}\psi_N\left(\frac{2X}{NV},1\right)\right|$ at collision time (blue curve) with parameters $N=50$, $\mu=2$, $V=50$, and equally spaced velocities $v_k = \mathrm{kV}$, $k=1,\ldots,N$, compared with the envelope $\mu\,\mathrm{sech}\left(\frac{2\mu X}{NV}\right)$ (red curve, see (2.37) with $T=0$)) for $|X|<2500$. (b) comparison between the same function (blue curve) and $\frac{2\sin X}{X}$ (red curve) for $|X|<50$. (c) and (d): same setup as in the first two panels, but with perturbed velocities $v_k=(k+\nu_k)V$, where ν_k are i.i.d. random variables uniformly distributed on $[-\frac{1}{5},\frac{1}{5}]$.

In corollary 1.3 both N and Δ act as large parameters and we fixed the norming constants c_k so that all the solitons collide at $(x_0,t_0)=(0,1)$. Moreover, at time $t=0$, the individual solitons are well separated as their individual peaks are located at $x_k=-v_k$, see (1.3). Notice that, in the regime where $N\to\infty$, this choice of norming constants coincides with (1.7) from proposition 1.1, since $\lim_{N\to\infty}(B'(z_k))^{-1}=\mathrm{i}\mu_k$. The results of corollary 1.3 are numerically illustrated in figure 1 in the case of uniformly spaced velocities $v_k=kV$ and with perturbed ones $v_k=k(V+\nu_k)$ where ν_k is a uniform distribution in $\left(-\frac{1}{5},\frac{1}{5}\right)$. In fact, despite our results being valid in the large N limit, figure 2 illustrates a numerically good prediction of the behaviour of the solution near the space-time collision point also for a small number of solitons.

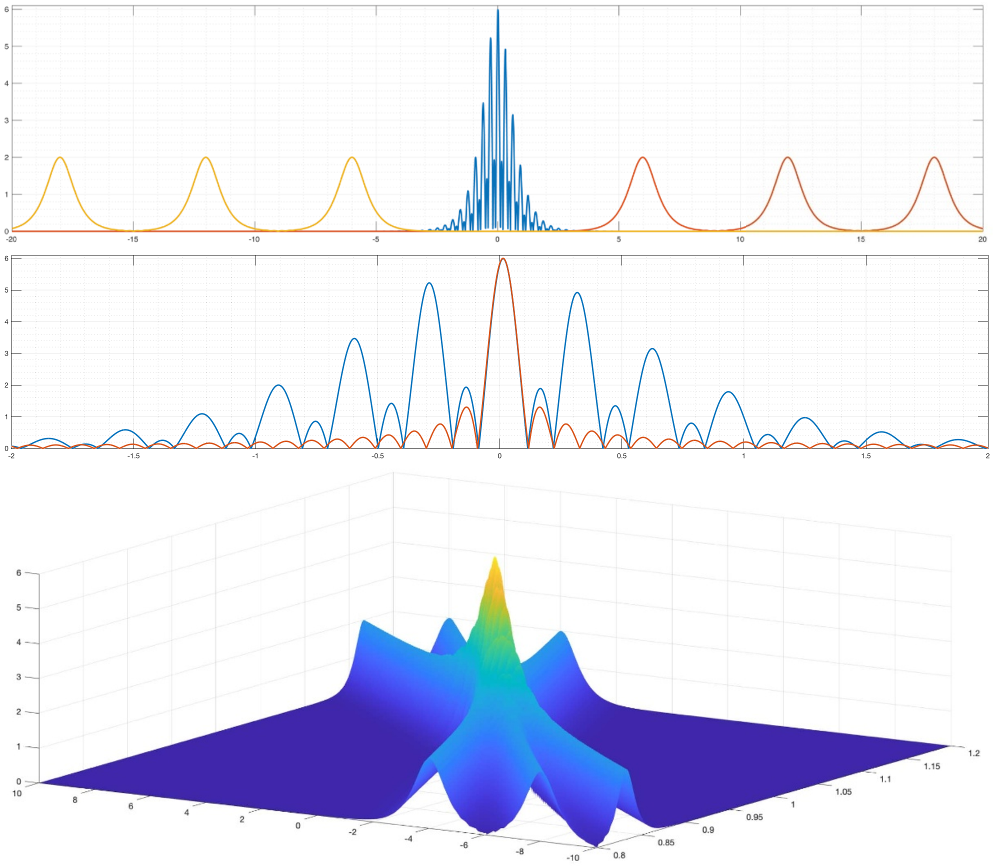

Figure 2. Top: the function $|\psi_3(x,t)|$ with $V = \Delta = 20$ and $\mu = 2$ at times $t = .7$ (yellow), $t = 1$ (blue), and $t = 1.3$ (red) for $|x| < 20$. Middle: comparison between $|\psi_3(x,1)|$ (blue) and a shifted and scaled sinc profile (red) for $|x| < 2$. Bottom: 3D graph of $|\psi_3(x,t)|$ for $|x| < 10$ and $|t-1| < 0.2$. The setting is the one analysed in corollary 1.3 with $v_k = kV$.

1.2. Stochastic soliton solution

We now consider synchronized solitons with *random scattering data*. We assume that the magnitudes μ_k of the scattering data are random, while we keep the velocities v_k deterministic. In order to control probabilistic fluctuations we need sufficiently strong control on the error terms in theorem 1.2. To this end, we make the following assumptions.

Assumption 1.1. In our probabilistic calculations we assume the following:

1. the velocities are deterministic, equally spaced with spacing which grows with the number of solitons:

$$v_k = k\Delta, \quad \Delta = \beta N^\gamma , \qquad \beta > 0, \;\; \gamma > \frac{1}{2}; \tag{1.16}$$

2. the amplitudes μ_k are independent identically distributed (i.i.d.) random variables

$$\mu_k \sim \mathcal{D}, \tag{1.17}$$

where $\mathcal{D}$ is any distribution with positive support, mean $\mu_{\mathcal{D}}$, and sub exponential with parameters (ν, α) (see Definition 3.1)

3. the norming constants are random variables

$$c_k = \mathrm{i}\mu_k \mathrm{e}^{-2\mathrm{i}z_k^2}, \qquad z_k := \frac{1}{2}\left(-k\Delta + \mathrm{i}\mu_k\right). \tag{1.18}$$

Under these assumptions, the N-soliton solution $\psi_N(x,t)$ becomes a random variable, where the solitons have random amplitudes and velocities proportional to N. In this setting, we prove a law of large numbers at the collision point showing that the emerging sinc-profile is universal, independent from the choice of the random distribution (proposition 1.4). This result is consistent with the deterministic case (corollary 1.3). Furthermore, we obtain central limit theorems for the fluctuations of the profile, both in the near-field region, close to the rogue wave peak (theorem 1.5), and in the far-field region (theorem 1.6).

Proposition 1.4 (Universal sinc-profile). *By choosing random soliton amplitudes according to assumption 1.1, it holds for each fixed pair $(X,T) \in \mathbb{R}^2$ that*

$$\frac{1}{\mu_{\mathcal{D}} N} \psi_N\left(\frac{2X}{\Delta N}, 1 + \frac{T}{\Delta^2 N^2}\right) \to \psi_0\left(X,T\right) \quad \text{as } N \to \infty, \text{in probability} \;, \tag{1.19}$$

where $\mu_{\mathcal{D}}$ is the mean value of the distribution $\mathcal{D}$ and $\psi_0(X,T)$ was defined in (1.14). In particular, at collision time ($T = 0$), we have

$$\frac{1}{\mu_{\mathcal{D}} N} \psi_N\left(\frac{2X}{\Delta N}, 1\right) \to -\frac{\sin\left(X\right)}{X} \mathrm{e}^{\mathrm{i}X} \quad \text{as } N \to \infty, \text{in probability}. \tag{1.20}$$

Equation (1.20) shows that a universal macroscopic profile emerges in the large N limit near the collision singularity, which we illustrate numerically in figure 3.

In fact, proposition 1.4 is a consequence of the following more general theorem.

Theorem 1.5 (Central limit theorem in the near-field regime). *Under assumption 1.1, for all $X, T \in \mathbb{R}$, the following holds*

$$\frac{\mathrm{Re}\left(\psi_N\left(\frac{2X}{\Delta N}, 1 + \frac{T}{\Delta^2 N^2}\right) - N\mu_{\mathcal{D}}\,\psi_0\left(X,T\right)\right)}{\sqrt{N \mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma_+\left(X,T\right)\right), \tag{1.21}$$

$$\frac{\mathrm{Im}\left(\psi_N\left(\frac{2X}{\Delta N}, 1 + \frac{T}{\Delta^2 N^2}\right) - N\mu_{\mathcal{D}}\,\psi_0\left(X,T\right)\right)}{\sqrt{N \mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma_-\left(X,T\right)\right), \tag{1.22}$$

$$\frac{\left|\psi_N\left(\frac{2X}{\Delta N}, 1 + \frac{T}{\Delta^2 N^2}\right) - N\mu_{\mathcal{D}}\,\psi_0\left(X,T\right)\right|}{\sqrt{N \mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{H}\left(\varphi\left(X,T\right)\right), \tag{1.23}$$

where $\mathrm{Var}_{\mathcal{D}}$ is the variance of the distribution $\mathcal{D}$, $\mathcal{H}(\varphi)$ is the Hoyt distribution (see lemma 3.3), and

$$\sigma_\pm\left(X,T\right) = \frac{1}{2}\left(1 \pm \int_0^1 \cos\left(4Xs - Ts^2\right) \mathrm{d}s\right).$$

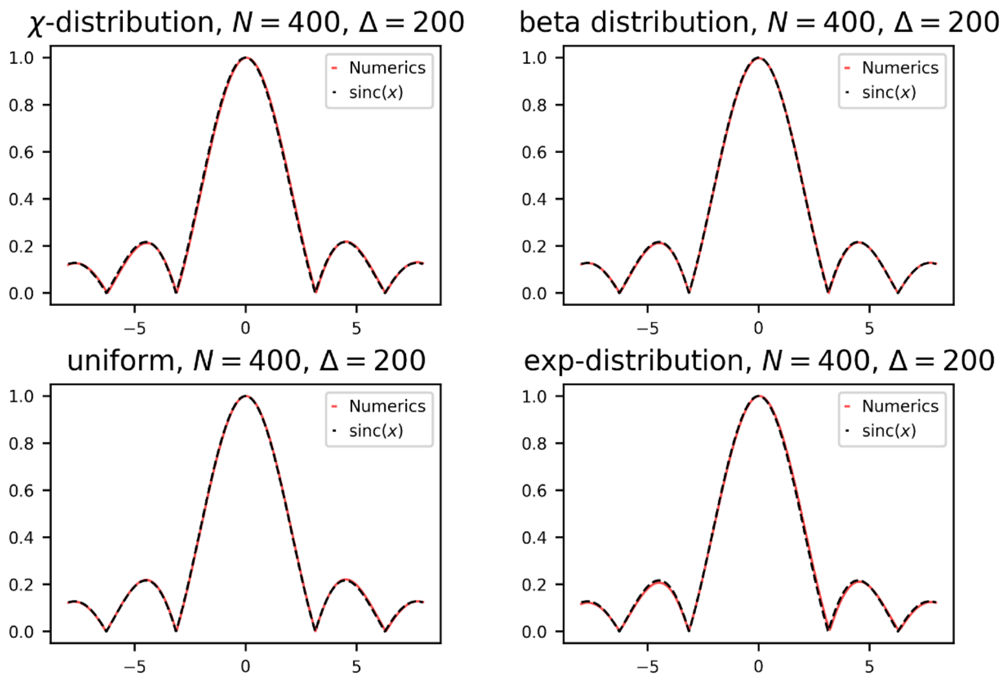

Figure 3. Numerical simulation (red) of the solution of the fNLS equation compared to the theoretical prediction (dashed black) (1.20). The number of solitons $N = 400$ and $\Delta = 200$. Here we are in the near-field, and we have both rescaled the x axis, and divided the solution by $\mu_{\mathcal{D}}N$. The amplitudes μ_j's are sampled according to a $\chi(2)$-distribution (top left), a $\text{Beta}_{2,2}$ distribution (top right), a uniform (0, 1) distribution (bottom left) and an exponential distribution with parameter $\lambda = 1$ (bottom right). To realize this picture, we average over 1000 trials.

Remark 1.2. The variances $\sigma_\pm$ in (1.21)-(1.22) can be expressed as

$$\sigma_\pm(X,T) = \frac{1}{2}\left(1 \pm \sqrt{\frac{\pi}{2T}}\left\{\cos(\xi^2)\left[C\left(\sqrt{\frac{2}{\pi}}(\sqrt{T}-\xi)\right) + C\left(\sqrt{\frac{2}{\pi}}\xi\right)\right]\right.\right.$$
$$\left.\left. + \sin(\xi^2)\left[S\left(\sqrt{\frac{2}{\pi}}(\sqrt{T}-\xi)\right) + S\left(\sqrt{\frac{2}{\pi}}\xi\right)\right]\right\}\right), \tag{1.24}$$

where $\xi(X,T) = \frac{2X}{\sqrt{T}}$, and $C(\cdot)$ and $S(\cdot)$ are the Fresnel integrals [1, formula (7.2.7) and (7.2.8)].

Notice that the limiting variances in (1.21)–(1.23) are independent from the distribution $\mathcal{D}$, thus universal. In figure 4 we show the results for the Beta distribution and the uniform distribution.

Remark 1.3. For $(X,T) = (0,0)$, the variance of the normal distribution in (1.22) vanishes, which implies that the random variable $\text{Im}(\psi_N(0,0))$ is deterministically equal to 0 in the limit as $N \to \infty$. This is consistent with the decomposition in theorem 1.2, where the leading term is real for $(X,T) = (0,0)$, and the remaining terms are asymptotically small with high probability (see lemma 3.1 and (3.9)-(3.10)).

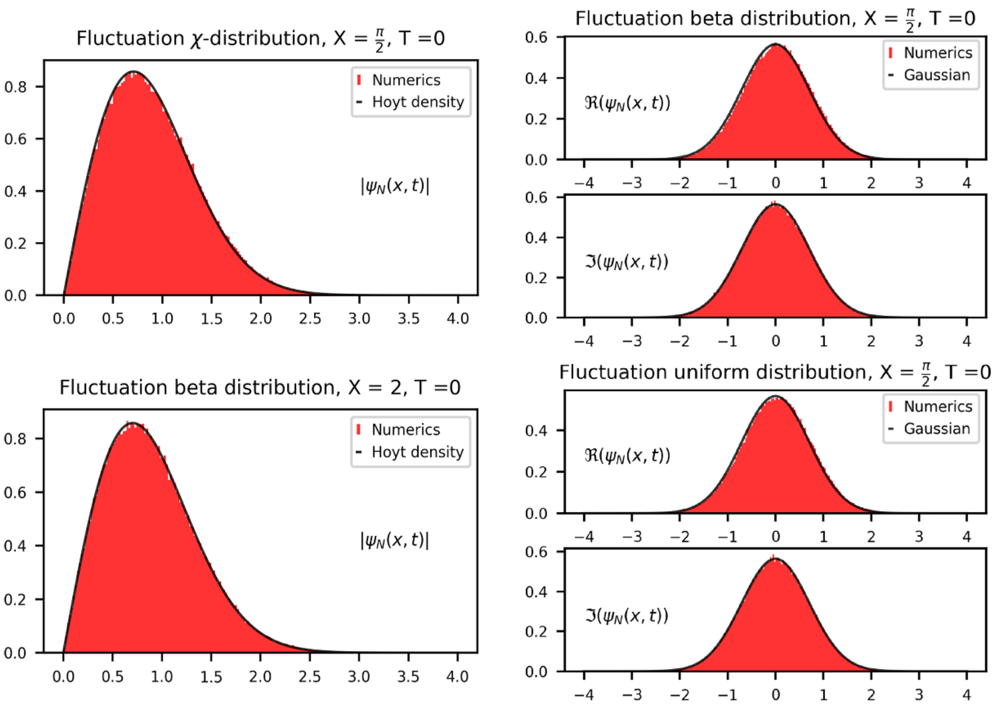

Figure 4. Fluctuation of $\psi_N(x,t)$ with respect to the average solution, $N = 1200$, $\Delta = 500$, 200 000 trials. Top panel: μ_j's are i.d.d. $\mathrm{Beta}_{2,2}$ distribution, left side the fluctuations of $|\psi_N(x,t)|$, right side the one of $\mathrm{Re}(\psi_N(x,t))$ and $\mathrm{Im}(\psi_N(x,t))$. Bottom panel: μ_j's are i.i.d. uniform distribution in $(0,1)$, left side the fluctuations of $|\psi_N(x,t)|$, right side the one of $\mathrm{Re}(\psi_N(x,t))$ and $\mathrm{Im}(\psi_N(x,t))$.

Finally, we analyse the fluctuations of the global profile of the N-soliton solution at collision time $\psi_N(x,1)$ over the whole spatial domain.

Theorem 1.6 (Central Limit theorem for the global profile at collision time). *Let $x \in \mathbb{R}$. Under Assumption 1.1, consider the N-soliton solution $\psi_N(x,t)$ of the fNLS equation (1.1). Then the following holds*

$$\frac{\mathrm{Re}\left(\psi_N(x,1)\right) - \omega_{\mathcal{D}}(x)\cos\left(\frac{x\Delta(N+1)}{2}\right)D_N(x\Delta)}{\sigma_{N,\mathrm{Re}}(x)} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0,1)\,, \tag{1.25}$$

$$\frac{\mathrm{Im}\left(\psi_N(x,1)\right) - \omega_{\mathcal{D}}(x)\sin\left(\frac{x\Delta N}{2}\right)D_{N+1}(x\Delta)}{\sigma_{N,\mathrm{Im}}(x)} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0,1)\,, \quad \text{for } x \neq 0\,, \tag{1.26}$$

where $D_N(x) := \frac{\sin\left(\frac{xN}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ is the Dirichlet kernel,

$$\omega_{\mathcal{D}}(x) = -\mathbb{E}\left[\frac{\xi}{\cosh(x\xi)}\right], \quad \xi \sim \mathcal{D}\,, \tag{1.27}$$

$$\sigma^2_{N,\mathrm{Re}}(x) = \mathrm{Var}\left(\frac{\xi}{\cosh(x\xi)}\right)\left(\frac{N-1}{2} + \frac{1}{2}\cos(x\Delta N)\,D_{N+1}(2x\Delta)\right), \tag{1.28}$$

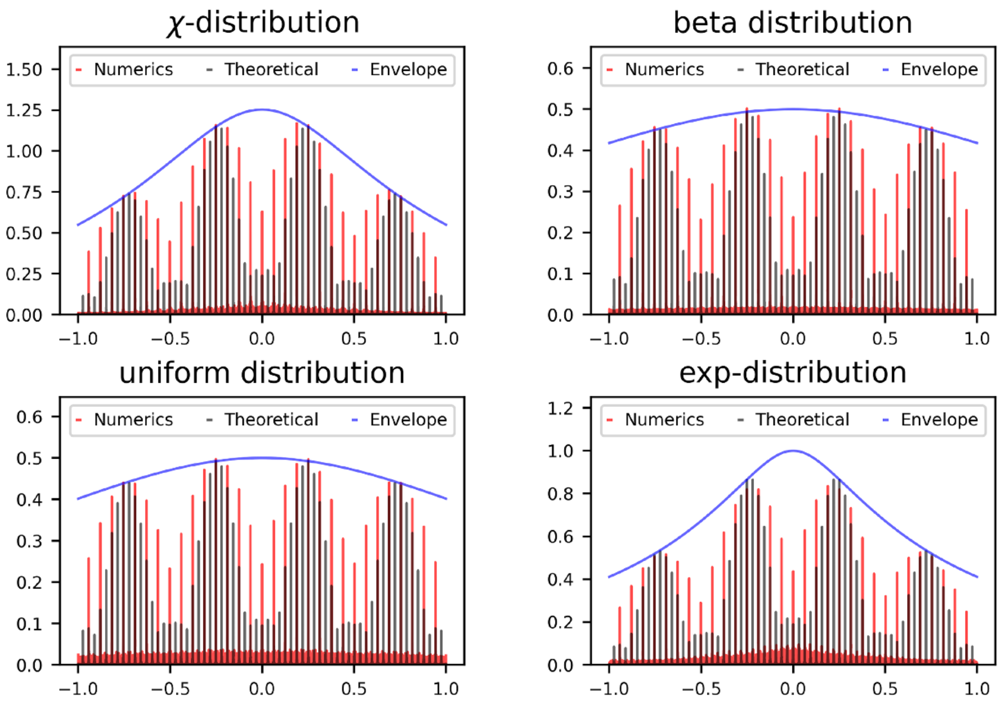

Figure 5. Macroscopic profile: numerical simulation (red) of $|\psi_N(x,1)|$, theoretical prediction (dashed black) (1.30) and the envelope $|\omega_{\mathcal{D}}(x)|$ (blue) (1.27). The number of solitons $N = 400$ and $\Delta = 200$. The μ_j's are sampled according to a $\chi(2)$-distribution (top left), a $\mathrm{Beta}_{2,2}$ distribution (top right), a uniform (0, 1) distribution (bottom left) and an exponential distribution with parameter $\lambda = 1$ (bottom right). To realize this picture, we averaged over 1000 trials.

$$\sigma^2_{N,\mathrm{Im}}(x) = \mathrm{Var}\left(\frac{\xi}{\cosh(x\xi)}\right)\left(\frac{N+1}{2} - \frac{1}{2}\cos(x\Delta N)\, D_{N+1}(2x\Delta)\right), \tag{1.29}$$

and $\mathrm{Var}(\cdot)$ *is the variance of the given random variable. Moreover,*

$$\lim_{N\to\infty}\frac{1}{N}\left(|\psi_N(x,1)| - |\omega_{\mathcal{D}}(x)\, D_N(x\Delta)|\right) \to 0 \tag{1.30}$$

in probability.

We define the function $|\omega_{\mathcal{D}}(x)|$ as the *envelope*, meaning a smooth curve outlining the extremes of an oscillating signal. As an example, if $\mathcal{D} \sim \chi^2(2)$, the envelope is equal to

$$|\omega_{\mathcal{D}}(x)| = \frac{\Psi^{(1)}\left(\frac{|x|+1}{4|x|}\right) - \Psi^{(1)}\left(\frac{1}{4}\left(3 + \frac{1}{|x|}\right)\right)}{8x^2},$$

where $\Psi^{(1)}$ is the 1st poly-gamma function [1, ch 5]. In figure 5, we show the envelope profile $|\omega_{\mathcal{D}}(x)|$, its modulation through the Dirichlet kernel $D_N(x\Delta)$ and the numerical average of the solution for several choice of distributions $\mathcal{D}$.

In recent years, a lot of effort has been put into describing the formation of rogue waves in deep sea and optical fibres. Informally, a rogue wave is a large-amplitude disturbance of the background state. Historically, the first instance of a rogue wave solution was derived by Peregrine [35]. Through the years, many more solutions with similar behavioural patterns have been studied experimentally, numerically and analytically (see for example [14, 15, 17, 43]). Furthermore, numerical and physical experiments have observed that the interaction of suitably prepared solitons also yields rogue waves (see [29, 33, 39] for the NLS and [38] for the modified KdV equations).

In the experiments conducted in [15] the authors studied rogue wave formation in a deep water tank: using the NLS equation as a model, they argue that such events are usually caused by the presence of a Peregrine breather appearing in the dynamics, or a degenerate two-soliton solution. In [43], it has been shown that the Peregrine breather emerges as a universal profile as the compression of the N-soliton solution to the NLS equation; furthermore, at the point of maximal localization, it yields to a peak three times bigger than the background.

Certain types of rogue waves have been extensively studied in [7–11] via a careful Riemann–Hilbert analysis: the authors showed that rogues waves can be constructed from high-order breather solutions [8, 9], high-order soliton solutions [7], or high-order solutions belonging to a one-parameter family which encompasses the previous two classes [10]. In the limit as the order goes to infinity, the solution displays a universal central peak, which is described in terms of a member of the Painlevé III hierarchy [37].

Comparing with the previous literature, the results presented in this paper prove rigorously that the formation of rogue waves can be the result of the constructive interaction of a handful of solitons at one point in space-time (see figure 2), and the resulting peak is universal, as it survives random perturbations of the soliton amplitudes.

Our soliton solution setup is similar to the scenario described in [38]: the authors consider a multi-soliton solution of the modified KdV equation, and tune the scattering data to obtain a rogue wave at a given collision time, so that the height of the peak is equal to the sum of amplitudes. Furthermore, phenomena of coherent soliton pulse trains, modelled by a generalization of the fNLS equation, appear in experimental optics, specifically in (micro)-resonators and dissipative Kerr soliton combs [25, 30].

We finally highlight that the N-soliton configuration studied in this paper is an instance of a *dilute soliton gas* [41]. This model was originally proposed by Zakharov [47] as an infinite collection of KdV solitons with random parameters, which are so sparse that it is possible to distinguish individual soliton-soliton interactions. Zakharov additionally derived a formula to describe the average velocity of a trial soliton as it travels through the diluted KdV gas. A more general kinetic formula for the soliton gas solving the fNLS equation was later derived in [19], however we do not pursue this direction of research here.

2. Deterministic *N*-soliton solutions

The proofs of proposition 1.1, theorem 1.2, and corollary 1.3, which we prove in this section, rely on the integrability of the fNLS equation.

2.1. Darboux (dressing) method

The integrability of fNLS was established by Zakharov and Shabat in [48] where they showed that (1.1) has a Lax Pair structure given by

$$\mathbf{\Phi}_x = \mathcal{L}\mathbf{\Phi}, \quad \mathcal{L} := \begin{bmatrix} -\mathrm{i}z & \psi \\ -\overline{\psi} & \mathrm{i}z \end{bmatrix}, \tag{2.1a}$$

$$\mathbf{\Phi}_t = \mathcal{B}\mathbf{\Phi}, \quad \mathcal{B} := \begin{bmatrix} -\mathrm{i}z^2 + \frac{\mathrm{i}}{2}|\psi|^2 & z\psi + \frac{\mathrm{i}}{2}\psi_x \\ -z\overline{\psi} + \frac{\mathrm{i}}{2}\overline{\psi}_x & \mathrm{i}z^2 - \frac{\mathrm{i}}{2}|\psi|^2 \end{bmatrix}, \tag{2.1b}$$

and established the existence of a simultaneous solution of this overdetermined system of ODEs provided the Lax operators $\mathcal{L}$ and $\mathcal{B}$ satisfy the compatibility condition $\mathcal{L}_t - \mathcal{B}_x + \mathcal{L}\mathcal{B} - \mathcal{B}\mathcal{L} = 0$, which is equivalent to ψ being a solution of (1.1).

The integrable structure allows us to compute solutions via the inverse scattering transform (IST) method [2, 20, 34, 48]. The formulation of the IST method starts by considering the scattering problem for the first operator (2.1*a*) in the fNLS Lax Pair viewed as an eigenvalue problem:

$$\widehat{\mathcal{L}}\mathbf{\Phi} = z\mathbf{\Phi}, \quad \widehat{\mathcal{L}} = \mathrm{i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial x} - \mathrm{i}\begin{bmatrix} 0 & \psi \\ \overline{\psi} & 0 \end{bmatrix}. \tag{2.2}$$

For spatially localized potentials ψ, the spectrum of (2.2) generically[7] consist of a finite number of non-real points (discrete spectrum) and the real line (continuous spectrum). In this setting the scattering data, which is time dependent, consists of a *reflection coefficient* $r : \mathbb{R} \to \mathbb{C}$ defined on the continuous spectrum, the collection of the discrete eigenvalues $z_k \in \mathbb{C}^+$ (the *poles*), and the so-called *norming constants* $C_k \in \mathbb{C}^*$ associated to each discrete eigenvalue in the following sense: for each $z_k \in \mathbb{C}^+$ there exist vector solutions of (2.2)

$$\boldsymbol{\phi}^+(x,t;z_k) \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}z_k x}, \quad x \to +\infty, \qquad \boldsymbol{\phi}^-(x,t;z_k) \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}z_k x}, \quad x \to -\infty,$$

such that $\boldsymbol{\phi}^+(x,t;z_k) = C_k(t)\boldsymbol{\phi}^-(x,t;z_k)$. The key result, which makes the IST effective, is that the time evolution of the scattering data is trivial, i.e. $\mathcal{S}(t) = (\{z_k(t), C_k(t)\}_{k=1}^N, r(z;t))$, the scattering data at time t, is given by

$$\mathcal{S}(t) = \left(\left\{\left(z_k,\ C_k \mathrm{e}^{-2\mathrm{i}z_k^2 t}\right)\right\}_{k=1}^N,\ r(z)\,\mathrm{e}^{2\mathrm{i}z^2 t}\right), \tag{2.3}$$

where $\mathcal{S}(0) = (\{(z_k, C_k\}_{k=1}^N, r(z))$ corresponds to the initial data $\psi_0(x) = \psi(x,0)$. The IST is the process by which one recovers the time-evolved potential $\psi(x,t)$ from the known evolution of the scattering data $\mathcal{S}(t)$. There are several ways to formulate the IST depending on the complexity of the scattering data. In what follows, we are only interested in the reflectionless case $r = 0$. The corresponding solution is the N-soliton solution $\psi_N(x,t)$, and it can be obtained iteratively via the Darboux transform (or dressing method) [21, 22], which we recall here briefly. As usual, we write $z_k = (-v_k + \mathrm{i}\mu_k)/2$. Let $\mathbf{\Phi}_n(z;x,t)$, $n \geqslant 1$, be the solution of the ZS system (2.1*a*)-(2.1*b*). These matrices can be constructed inductively using the so-called dressing matrices $\boldsymbol{\chi}_n(z;x,t)$ via

$$\mathbf{\Phi}_n(z;x,t) = \boldsymbol{\chi}_n(z;x,t)\,\mathbf{\Phi}_{n-1}(z;x,t), \qquad \mathbf{\Phi}_0(z;x,t) = \mathrm{e}^{-\mathrm{i}xz\boldsymbol{\sigma}_3}, \tag{2.4}$$

$$(\boldsymbol{\chi}_n(z;x,t))_{j\ell} = \delta_{j\ell} + \frac{z_n - \overline{z}_n}{z - z_n}\frac{\overline{q_{nj}(x,t)}\,q_{n\ell}(x,t)}{|\boldsymbol{q}_n(x,t)|^2}, \tag{2.5}$$

$$\boldsymbol{q}_n(x,t) = \overline{\mathbf{\Phi}_{n-1}(\overline{z}_n;x,t)}\begin{bmatrix} 1 \\ C_n(t) \end{bmatrix}, \tag{2.6}$$

[7] By 'generic', we refer to potentials belonging to an open dense subset of $L^1(\mathbb{R})$ (see [3]).

where $j,\ell = 1,2$ and $\delta_{j\ell}$ is the Kronecker delta in (2.5), and we write $\boldsymbol{q}_n(x,t) = \begin{bmatrix} q_{n1}(x,t) & q_{n2}(x,t) \end{bmatrix}^\top$. The N-soliton solution $\psi_N(x,t)$ is then given by

$$\psi_N(x,t) = -2\sum_{k=1}^{N} \mu_k \frac{\overline{q_{k1}(x,t)}\, q_{k2}(x,t)}{|\boldsymbol{q}_k(x,t)|^2}. \tag{2.7}$$

Because $2\,|\overline{q_{k1}}\, q_{k2}| \leqslant |\boldsymbol{q}_k|^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, (2.7) readily yields the bound in (1.6).

2.2. Riemann–Hilbert approach

The dressing method is a straightforward method for iteratively computing soliton solutions or, more generally, for adding solitons to a known 'seed' solution. However, it is not well suited to asymptotic analysis. To study the dependence of solutions on external parameters, the IST is more appropriately formulated as a Riemann–Hilbert problem (RHP). Using the nonlinear steepest descent method one can often compute complete asymptotic expansions to the solution of the RHP with explicit error bounds. Below, the Riemann–Hilbert problem for fNLS is given for reflectionless potentials, which is all we need for our purposes. The RHP for generic potentials that decay sufficiently as $|x| \to \infty$ can be found many places. See, for example, [12, 28].

Riemann–Hilbert Problem 2.1. *Given reflectionless scattering data $\{(z_k, C_k)\}_{k=1}^N$, find a matrix function $\boldsymbol{M}(z) \in \mathrm{SL}(2,\mathbb{C})$ such that*

1. *$\boldsymbol{M}(\,\cdot\,;x,t)$ is analytic in $\mathbb{C} \setminus \left(\{z_k, \bar{z}_k\}_{k=1}^N\right)$;*
2. *$\boldsymbol{M}(z;x,t) = \boldsymbol{I} + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$;*
3. *$\boldsymbol{M}(z;x,t)$ has a simple pole at each z_k and $\bar{z}_k$ satisfying the residue relation*

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} \boldsymbol{M}(z;x,t) &= c_k \mathrm{e}^{2\mathrm{i}\theta(z_k;x,t)} \lim_{z\to z_k} \boldsymbol{M}(z;x,t) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
\operatorname*{Res}_{z=\bar{z}_k} \boldsymbol{M}(z;x,t) &= -\bar{c}_k \mathrm{e}^{-2\mathrm{i}\theta(\bar{z}_k;x,t)} \lim_{z\to \bar{z}_k} \boldsymbol{M}(z;x,t) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\end{aligned} \tag{2.8}$$

where $\theta(z;x,t) = tz^2 + xz$ and the residue coefficients c_k are related to the norming constants C_k by (see [21, 22])

$$c_k := \frac{1}{C_k B'(z_k)}, \quad B(z) = \prod_{k=1}^{N} \frac{z - z_k}{z - \bar{z}_k}. \tag{2.9}$$

Expanding the solution of this RHP as $z \to \infty$, it can be shown [12] that

$$\boldsymbol{M}(z;x,t) = \boldsymbol{I} + \frac{1}{2\mathrm{i}z} \begin{bmatrix} -m(x,t) & \psi(x,t) \\ \overline{\psi(x,t)} & m(x,t) \end{bmatrix} + \mathcal{O}\left(z^{-2}\right), \quad z \to \infty, \tag{2.10}$$

where

$$m(x,t) := \int_x^\infty |\psi(s,t)|^2\, \mathrm{d}s. \tag{2.11}$$

It follows that the solution of the fNLS equation (1.1) can be recovered as

$$\psi(x,t) = \lim_{z\to\infty} 2\mathrm{i}z\, [\boldsymbol{M}(z;x,t)]_{1,2}\,. \tag{2.12}$$

2.3. *Proof of proposition 1.1*

According to (2.7), to prove (1.8), it is enough to show that

$$q_{k1}(x_0,t_0) = q_{k2}(x_0,t_0). \tag{2.13}$$

Since $C_k(t_0) = C_k e^{-2iz_k^2 t_0}$, our choice of the constants c_k and (2.9) yields that $C_k(t_0) = e^{2iz_k x_0}$ for each $k = 1,\dots,N$. Hence, we get from (2.6) and (2.4) that

$$\boldsymbol{q}_1(x_0,t_0) = e^{ix_0 z_1}\begin{bmatrix}1\\1\end{bmatrix} \quad \Rightarrow \quad q_{11}(x_0,t_0) = q_{12}(x_0,t_0)$$

as desired. Assume now that (2.13) holds for all $k = 1,\dots,n-1$. Relations (2.6) and (2.4) give

$$\boldsymbol{q}_n(x_0,t_0) = e^{ix_0 z_n}\left(\prod_{k=1}^{n-1}\overline{\chi_k(\overline{z}_n;x_0,t_0)}\right)\begin{bmatrix}1\\1\end{bmatrix}.$$

It follows from (2.13) and (2.5) that

$$\chi_k(z;x_0,t_0)\begin{bmatrix}1\\1\end{bmatrix} = \left(\boldsymbol{I} + \frac{1}{2}\frac{z_k-\overline{z}_k}{z-z_k}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)\begin{bmatrix}1\\1\end{bmatrix} = \left(1+\frac{z_k-\overline{z}_k}{z-z_k}\right)\begin{bmatrix}1\\1\end{bmatrix},$$

which, in turn, readily implies that

$$\boldsymbol{q}_n(x_0,t_0) = e^{ix_0 z_n}\prod_{k=1}^{n-1}\left(1-\frac{z_k-\overline{z}_k}{z_n-\overline{z}_k}\right)\begin{bmatrix}1\\1\end{bmatrix} \quad \Rightarrow \quad q_{n1}(x_0,t_0) = q_{n2}(x_0,t_0)$$

as desired. This clearly finishes the proof of the inductive step and therefore of (1.8).

2.4. *Proof of theorem 1.2*

Given the RHP 2.1, we first perform a rescaling transformation

$$\widehat{\boldsymbol{M}}(\lambda;x,t) := \boldsymbol{M}(\Delta\lambda;x,t)\ .$$

Intuitively, since $\Delta \gg 1$, the matrix $\widehat{\boldsymbol{M}}(\lambda;x,t)$ will have polar singularities very close to the real axis, while still having neighbouring distances (i.e. horizontal spacing) of order $\mathcal{O}(1)$:

$$\lambda_k := \frac{-v_k + i\mu_k}{2\Delta} =: \frac{-\widehat{v}_k + i\widehat{\mu}_k}{2} \quad \text{with} \quad \widehat{v}_k - \widehat{v}_{k-1} \geqslant 1\ .$$

More rigorously, it is easy to verify that $\widehat{\boldsymbol{M}}(\lambda;x,t)$ satisfies the following RHP:

1. $\widehat{\boldsymbol{M}}(\lambda;x,t)$ is analytic for $\lambda \in \mathbb{C}\setminus\{\lambda_k,\overline{\lambda}_k\}_{k=1}^N$;
2. $\widehat{\boldsymbol{M}}(\lambda;x,t) = \boldsymbol{I} + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda\to\infty$;
3. $\widehat{\boldsymbol{M}}(\lambda;x,t)$ has a simple pole at each λ_k and $\overline{\lambda}_k$ satisfying the residue relation

$$\begin{aligned}
\operatorname*{Res}_{\lambda=\lambda_k}\widehat{\boldsymbol{M}}(\lambda;x,t) &= \widehat{\gamma}_k(x,t)\lim_{\lambda\to\lambda_k}\widehat{\boldsymbol{M}}(\lambda;x,t)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\\
\operatorname*{Res}_{\lambda=\overline{\lambda}_k}\widehat{\boldsymbol{M}}(\lambda;x,t) &= -\overline{\widehat{\gamma}_k(x,t)}\lim_{\lambda\to\overline{\lambda}_k}\widehat{\boldsymbol{M}}(\lambda;x,t)\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},
\end{aligned} \tag{2.14}$$

where $\widehat{\gamma}_k(x,t) = \gamma_k(x,t)/\Delta$ with $\gamma_k(x,t) := c_k e^{2i\theta(z_k;x,t)}$.

The solution of fNLS is then given by

$$\psi_N(x,t) = \lim_{\lambda\to\infty} 2\mathrm{i}\Delta\lambda \left[\widehat{\boldsymbol{M}}(\lambda;x,t)\right]_{1,2}. \tag{2.15}$$

In order to consider all values of $N \in \mathbb{N}$ simultaneously, we embed $\widehat{\nu}_1,\ldots,\widehat{\nu}_N$ into a sequence $\{\widehat{\nu}_k\}_{k\in\mathbb{N}}$, $|\widehat{\nu}_k - \widehat{\nu}_j| \geqslant 1$ for any $k \neq j$ and let $c_k = 0$ for all $k > N$ (so that $\widehat{\gamma}_k(x,t) = 0$, $k > N$). Then, for each $k \in \mathbb{N}$ we define

$$\Gamma_k := \left\{\lambda \in \mathbb{C} : |\lambda + \tfrac{1}{2}\widehat{\nu}_k| = \varepsilon_0\right\},$$

a circle centred at $-\frac{1}{2}\widehat{\nu}_k$ and orient it counterclockwise for some $\varepsilon_0 > 0$ sufficiently small, so that $\Gamma_k \cap \Gamma_j = \emptyset$, $\forall j,k \in \mathbb{N}, j \neq k$. We are assuming the parameter Δ is large enough so that the pair of poles λ_k and $\overline{\lambda}_k$ lies in the interior of Γ_k for all $k = 1,\ldots,N$. We set $\Gamma := \bigcup_{k=1}^{\infty}\Gamma_k$.

In the next step we trade polar singularities for jump relations on Γ. Thus, we define

$$\boldsymbol{N}(\lambda;x,t) := \begin{cases} \widehat{\boldsymbol{M}}(\lambda;x,t)\begin{bmatrix} 1 & 0 \\ a_k(\lambda) & 1\end{bmatrix}\begin{bmatrix} 1 & b_k(\lambda) \\ 0 & 1\end{bmatrix}, & \lambda \in \operatorname{int}(\Gamma_k), \\ \widehat{\boldsymbol{M}}(\lambda;x,t), & \text{otherwise}, \end{cases} \tag{2.16}$$

where

$$a_k(\lambda) = \frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k}\frac{\lambda - \overline{\lambda}_k}{\lambda - \lambda_k} = \left(\frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k} - \frac{\widehat{\gamma}_k(x,t)}{\lambda - \lambda_k}\right),$$
$$b_k(\lambda) = \frac{\widehat{\mu}_k^2\overline{\widehat{\gamma}_k(x,t)}}{\widehat{\mu}_k^2 + |\widehat{\gamma}_k(x,t)|^2}\frac{1}{\lambda - \overline{\lambda}_k} = -\frac{1}{2\mathrm{i}\Delta}\frac{\psi^{(k)}(x,t)}{\lambda - \overline{\lambda}_k}. \tag{2.17}$$

Lemma 2.2. *The matrix $\boldsymbol{N}(\lambda;x,t)$ is analytic in $\mathbb{C}\setminus\Gamma$.*

Proof. Write $\widehat{\boldsymbol{M}} = [\widehat{\boldsymbol{M}}_1, \widehat{\boldsymbol{M}}_2]$. Since

$$\widehat{\boldsymbol{M}}\boldsymbol{E}_{21} = \left[\widehat{\boldsymbol{M}}_2, 0\right], \quad \widehat{\boldsymbol{M}}\boldsymbol{E}_{12} = \left[0, \widehat{\boldsymbol{M}}_1\right], \quad \text{and} \quad \widehat{\boldsymbol{M}}\boldsymbol{E}_{22} = \left[0, \widehat{\boldsymbol{M}}_2\right],$$

where $\boldsymbol{E}_{ij}$ is the 2×2 elementary matrix whose entries are zero except for the (i,j)-th entry, which is 1, it holds that

$$\boldsymbol{N} = \widehat{\boldsymbol{M}}(\boldsymbol{I} + a_k\boldsymbol{E}_{21} + b_k\boldsymbol{E}_{12} + a_kb_k\boldsymbol{E}_{22}) = \left[\widehat{\boldsymbol{M}}_1 + a_k\widehat{\boldsymbol{M}}_2, \widehat{\boldsymbol{M}}_2 + b_k\widehat{\boldsymbol{M}}_1 + a_kb_k\widehat{\boldsymbol{M}}_2\right].$$

The residue conditions of $\widehat{\boldsymbol{M}}$ can be rewritten as

$$\begin{cases} \operatorname*{Res}_{\lambda=\lambda_k}\widehat{\boldsymbol{M}} = \left[\operatorname*{Res}_{\lambda=\lambda_k}\widehat{\boldsymbol{M}}_1, 0\right] = \left[\widehat{\gamma}_k(x,t)\widehat{\boldsymbol{M}}_2(\lambda_k), 0\right], \\ \operatorname*{Res}_{\lambda=\overline{\lambda}_k}\widehat{\boldsymbol{M}} = \left[0, \operatorname*{Res}_{\lambda=\overline{\lambda}_k}\widehat{\boldsymbol{M}}_2\right] = \left[0, -\overline{\widehat{\gamma}_k(x,t)}\widehat{\boldsymbol{M}}_1(\overline{\lambda}_k)\right]. \end{cases}$$

Let $\boldsymbol{N}(\lambda) = [\boldsymbol{N}_1(\lambda), \boldsymbol{N}_2(\lambda)]$. We have that

$$\begin{cases} \operatorname*{Res}_{\lambda=\lambda_k}\boldsymbol{N}_1 = \operatorname*{Res}_{\lambda=\lambda_k}\widehat{\boldsymbol{M}}_1 + \operatorname*{Res}_{\lambda=\lambda_k} a_k\,\widehat{\boldsymbol{M}}_2(\lambda_k) = 0, \\ \operatorname*{Res}_{\lambda=\lambda_k}\boldsymbol{N}_2 = b_k(\lambda_k)\operatorname*{Res}_{\lambda=\lambda_k}\left(\widehat{\boldsymbol{M}}_1 + a_k\widehat{\boldsymbol{M}}_2\right) = 0. \end{cases}$$

On the other hand, because $a_k(\overline{\lambda}_k)=0$, $\boldsymbol{N}_1(\lambda)$ does not have a pole at $\overline{\lambda}_k$. Moreover,

$$\begin{aligned}\operatorname*{Res}_{\lambda=\overline{\lambda}_k}\boldsymbol{N}_2&=\left(1+a_k'\left(\overline{\lambda}_k\right)\operatorname*{Res}_{\lambda=\overline{\lambda}_k}b_k\right)\operatorname*{Res}_{\lambda=\overline{\lambda}_k}\widehat{\boldsymbol{M}}_2+\operatorname*{Res}_{\lambda=\overline{\lambda}_k}b_k\,\widehat{\boldsymbol{M}}_1\left(\overline{\lambda}_k\right)\\&=\left(-\overline{\widehat{\gamma}_k\left(x,t\right)}+\left(1+\frac{|\widehat{\gamma}_k\left(x,t\right)|^2}{\widehat{\mu}_k^2}\right)\operatorname*{Res}_{\lambda=\overline{\lambda}_k}b_k\right)\widehat{\boldsymbol{M}}_1\left(\overline{\lambda}_k\right)=0.\end{aligned}$$

Hence, $\boldsymbol{N}$ is analytic in each $\mathrm{int}(\Gamma_k)$, but has a discontinuity across each Γ_k by construction. □

Instead of calculating directly the jump across Γ for the matrix $\boldsymbol{N}$, we first introduce one last transformation. We define

$$\boldsymbol{O}\left(\lambda;x,t\right):=\begin{cases}\boldsymbol{N}\left(\lambda;x,t\right)\begin{bmatrix}1&0\\-\frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k}&1\end{bmatrix}, & z\in\mathrm{int}\left(\Gamma_k\right),\\ \boldsymbol{N}\left(\lambda;x,t\right), & \text{otherwise.}\end{cases}\tag{2.18}$$

Lemma 2.3. *The matrix $\boldsymbol{O}$ solves the following RHP:*

1. $\boldsymbol{O}(\lambda;x,t)$ *is analytic for* $\lambda\in\mathbb{C}\setminus\Gamma$*;*
2. $\boldsymbol{O}(\lambda;x,t)=\boldsymbol{I}+\mathcal{O}\left(\lambda^{-1}\right)$ *as* $\lambda\to\infty$*;*
3. $\boldsymbol{O}(\lambda;x,t)$ *has continuous boundary values on each side of* Γ *satisfying the jump relations (+ and − boundary values are on the right and left side of the contour when traversing it according to its orientation)*

$$\boldsymbol{O}_+\left(\lambda;x,t\right)=\boldsymbol{O}_-\left(\lambda;x,t\right)\boldsymbol{V}_O\left(\lambda;x,t\right)\ ,\qquad \lambda\in\Gamma,\tag{2.19}$$

$$\begin{aligned}\boldsymbol{V}_O\left(\lambda;x,t\right)=\boldsymbol{I}+\frac{1}{2\mathrm{i}\Delta}\sum_{k=1}^{N}&\left(\frac{1}{\lambda-\lambda_k}\begin{bmatrix}0&0\\-\overline{\psi^{(k)}\left(x,t\right)}&-m^{(k)}\left(x,t\right)\end{bmatrix}\right.\\&\left.+\frac{1}{\lambda-\overline{\lambda}_k}\begin{bmatrix}m^{(k)}\left(x,t\right)&-\psi^{(k)}\left(x,t\right)\\0&0\end{bmatrix}\right)\mathbb{1}_k\left(\lambda\right),\end{aligned}\tag{2.20}$$

where $\psi^{(k)}(x,t)$ is the one-soliton solution (1.2) with scattering data (λ_k,c_k), $m^{(k)}(x,t)$ is given by (2.11), and $\mathbb{1}_k$ is the indicator function of Γ_k.

Proof. By construction (see (2.18)), $\boldsymbol{O}$ satisfies points 1. and 2. of the lemma, and the jump condition reads

$$\begin{aligned}&\boldsymbol{O}_+\left(\lambda;x,t\right)=\boldsymbol{O}_-\left(\lambda;x,t\right)\boldsymbol{V}_O\left(\lambda;x,t\right),\\&\boldsymbol{V}_O\left(\lambda;x,t\right)=\left(1-\sum_{k=1}^{N}\mathbb{1}_k\left(\lambda\right)\right)\boldsymbol{I}+\sum_{k=1}^{N}\begin{bmatrix}1&0\\a_k\left(\lambda\right)&1\end{bmatrix}\begin{bmatrix}1&b_k\left(\lambda\right)\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-\frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k}&1\end{bmatrix}\mathbb{1}_k\left(\lambda\right).\end{aligned}\tag{2.21}$$

Consider now the one-soliton problem with data (z_k,c_k). The solution can be found by solving a 2×2 linear system (A.2)–(A.3) which yields

$$\begin{aligned}&\psi^{(k)}\left(x,t\right)=-2\mathrm{i}\overline{\beta^{(k)}\left(x,t\right)}\quad\text{and}\quad m^{(k)}\left(x,t\right)=-2\mathrm{i}\alpha^{(k)}\left(x,t\right)\ ,\\&\text{with }\alpha^{(k)}\left(x,t\right)=\mathrm{i}\Delta\widehat{\mu}_k\frac{|\widehat{\gamma}_k\left(x,t\right)|^2}{\widehat{\mu}_k^2+|\widehat{\gamma}_k\left(x,t\right)|^2}\ ,\quad \beta^{(k)}\left(x,t\right)=\Delta\widehat{\gamma}_k\left(x,t\right)\frac{\widehat{\mu}_k^2}{\widehat{\mu}_k^2+|\widehat{\gamma}_k\left(x,t\right)|^2}\ .\end{aligned}\tag{2.22}$$

Using (2.22), we calculate

$$a_k(\lambda)\,b_k(\lambda) = \mathrm{i}\widehat{\mu}_k \frac{|\widehat{\gamma}_k(x,t)|^2}{\widehat{\mu}_k^2 + |\widehat{\gamma}_k(x,t)|^2}\frac{1}{\lambda-\lambda_k} = -\frac{1}{2\mathrm{i}\Delta}\frac{m_k(x,t)}{\lambda-\lambda_k},$$

$$\frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k} b_k(\lambda) = \mathrm{i}\widehat{\mu}_k \frac{|\widehat{\gamma}_k(x,t)|^2}{\widehat{\mu}_k^2 + |\widehat{\gamma}_k(x,t)|^2}\frac{1}{\lambda-\overline{\lambda}_k} = -\frac{1}{2\mathrm{i}\Delta}\frac{m_k(x,t)}{\lambda-\overline{\lambda}_k},$$

$$a_k(\lambda) - \frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k}\left(1+(a_kb_k)(\lambda)\right) = \frac{\mathrm{i}\widehat{\gamma}_k(x,t)}{\widehat{\mu}_k}\left(\frac{\lambda-\overline{\lambda}_k}{\lambda-\lambda_k} - 1 - (a_kb_k)(\lambda)\right) = -\frac{1}{2\mathrm{i}\Delta}\frac{\overline{\psi^{(k)}(x,t)}}{\lambda-\lambda_k}.$$

These relations readily yield (2.20). □

We are now in a position to use the Small Norm Argument [27] and to derive the asymptotic behaviour of the N-soliton potential in the regime $\Delta \gg 1$ and $\boldsymbol{\mu}$ bounded in ℓ^2- and ℓ^∞-norms. The solution of the RHP $\boldsymbol{O}$ (if it exists) can expressed in the form

$$\boldsymbol{O}(\lambda;x,t) = \boldsymbol{I} + \int_\Gamma \frac{(\boldsymbol{I}+\boldsymbol{\eta}(\xi;x,t))(\boldsymbol{V}_O(\xi;x,t)-\boldsymbol{I})}{\xi-\lambda}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}, \tag{2.23}$$

where $\boldsymbol{\eta}$ is the solution of the integral equation

$$(\mathbb{1}-\mathcal{C}_O)\boldsymbol{\eta} = \mathcal{C}_O[\boldsymbol{I}], \qquad \mathcal{C}_O[\boldsymbol{f}] := \mathcal{C}_-\left[\boldsymbol{f}(\boldsymbol{V}_O-\boldsymbol{I})\right], \tag{2.24}$$

and $\mathcal{C}_- : L^2(\Gamma)\to L^2(\Gamma)$ is the Cauchy projection operator on Γ, namely

$$\mathcal{C}_-[\boldsymbol{f}](\lambda) := \lim_{\substack{z\to\lambda\\ z\in \text{ right side of } \Gamma}} \left(\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\boldsymbol{f}(\xi)}{\xi-z}\mathrm{d}\xi\right). \tag{2.25}$$

It remains to show that the integral operator $\mathbb{1}-\mathcal{C}_O$ in (2.24) is invertible, thus yielding existence (and uniqueness) of the solution $\boldsymbol{O}$.

Lemma 2.4. *The Cauchy operator $\mathcal{C}_O$ defined in (2.24) has bounded norm*

$$\|\mathcal{C}_O\| \leqslant C_*\|\boldsymbol{\mu}\|_\infty \Delta^{-1} \tag{2.26}$$

for some constant $C_*>0$. Then, it follows for $\Delta > C_*\|\boldsymbol{\mu}\|_\infty$ that $\boldsymbol{\eta}$ exists and it can be expanded as a convergent Neumann series

$$\boldsymbol{\eta} = (\mathbb{1}-\mathcal{C}_O)^{-1}\mathcal{C}_O[\boldsymbol{I}] = \sum_{j=1}^\infty \mathcal{C}_O^j[\boldsymbol{I}]. \tag{2.27}$$

Proof. Using the estimates

$$|\psi^{(k)}(x,t)| \leqslant \mu_k \quad \text{and} \quad m^{(k)}(x,t) \leqslant \|\psi^{(k)}(\cdot,t)\|^2_{L^2(\mathbb{R})} = 2\mu_k, \quad \forall (x,t)\in\mathbb{R}^2, \tag{2.28}$$

in the expression (2.20) for the jump matrix $\boldsymbol{V}_O$, it follows that

$$\left\|(\boldsymbol{V}_O(\lambda)-\boldsymbol{I})\big|_{\lambda\in\Gamma_k}\right\| \begin{cases} \leqslant C_0\frac{\mu_k}{\Delta}, & 1\leqslant k\leqslant N,\\ =0, & k>N, \end{cases} \tag{2.29}$$

for some constant $C_0 > 0$, where $\|\boldsymbol{f}\|^2 := \mathrm{Tr}(\boldsymbol{f}^*\boldsymbol{f})$ for a given matrix $\boldsymbol{f}$ (if fact, one can use any matrix norm). Therefore, there exists a constant $C_1 > 0$ such that

$$\|\boldsymbol{V}_O - \boldsymbol{I}\|_{L^\infty(\Gamma)} \leqslant C_1\|\boldsymbol{\mu}\|_\infty \Delta^{-1} \quad \text{and} \quad \|\boldsymbol{V}_O - \boldsymbol{I}\|_{L^2(\Gamma)} \leqslant C_1\|\boldsymbol{\mu}\|_2 \Delta^{-1} \tag{2.30}$$

uniformly for all $(x,t) \in \mathbb{R}^2$, where $\|\boldsymbol{f}\|_{L^\infty(\Gamma)} := \operatorname{ess\,sup}_\Gamma \|\boldsymbol{f}(\xi)\|$ and $\|\boldsymbol{f}\|^2_{L^2(\Gamma)} := \int_\Gamma \|\boldsymbol{f}(\xi)\|^2 |\mathrm{d}\xi|$ for a given 2×2 matrix-functions $\boldsymbol{f}(\xi), \boldsymbol{g}(\xi)$. Since the loops Γ_k have fixed radii and are well separated, the contour Γ is Ahlfors–David regular[8] and it follows [18, 31] that the operator norm $\|\mathcal{C}_-\|$ is finite. The norm estimates (2.30) then yield that

$$\begin{aligned}
&\|\mathcal{C}_O[\boldsymbol{f}]\|_{L^2(\Gamma)} \leqslant \|\mathcal{C}_-\| \, \|(\boldsymbol{V}_O - \boldsymbol{I})\|_{L^\infty(\Gamma)} \|\boldsymbol{f}\|_{L^2(\Gamma)} \leqslant C_*\|\boldsymbol{\mu}\|_\infty \Delta^{-1} \|\boldsymbol{f}\|_{L^2(\Gamma)}, \\
&\|\mathcal{C}_O[\boldsymbol{I}]\|_{L^2(\Gamma)} \leqslant \|\mathcal{C}_-\| \|\boldsymbol{V}_O - \boldsymbol{I}\|_{L^2(\Gamma)} \leqslant C_*\|\boldsymbol{\mu}\|_2 \Delta^{-1},
\end{aligned}$$

where $C_* := \|\mathcal{C}_-\| C_1$. Hence, for any $\Delta > C_*\|\boldsymbol{\mu}\|_\infty$, $(\mathbb{1} - \mathcal{C}_O)^{-1}$ exists and $\boldsymbol{\eta}$ can be expanded as a convergent Neumann series

$$\|\boldsymbol{\eta}\|_{L^2(\Gamma)} = \left\| \sum_{j=1}^\infty \mathcal{C}_O^j[\boldsymbol{I}] \right\|_{L^2(\Gamma)} \leqslant \sum_{j=1}^\infty \|\mathcal{C}_O\|^{j-1} \|\mathcal{C}_O[\boldsymbol{I}]\|_{L^2(\Gamma)} \leqslant \frac{C_*\|\boldsymbol{\mu}\|_2}{\Delta - C_*\|\boldsymbol{\mu}\|_\infty}.$$

□

From (2.20), we can now derive an explicit expression for $\mathcal{C}_O[\boldsymbol{I}]$:

$$\begin{aligned}
\mathcal{C}_O[\boldsymbol{I}](\lambda) &= \sum_{k=1}^N \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_k} (\boldsymbol{V}_O(\xi) - \boldsymbol{I}) \frac{\mathrm{d}\xi}{\xi - \lambda} \\
&= \frac{1}{2\mathrm{i}\Delta} \sum_{k=1}^N \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_k} \left(\frac{1}{\xi - \lambda_k} \begin{bmatrix} 0 & 0 \\ -\overline{\psi^{(k)}(x,t)} & -m^{(k)}(x,t) \end{bmatrix} \right. \\
&\quad \left. + \frac{1}{\xi - \overline{\lambda}_k} \begin{bmatrix} m^{(k)}(x,t) & -\psi^{(k)}(x,t) \\ 0 & 0 \end{bmatrix} \right) \frac{\mathrm{d}\xi}{\xi - \lambda},
\end{aligned} \tag{2.31}$$

where λ is understood to lie outside each of the Γ_k. Evaluating by residues gives

$$\begin{aligned}
\mathcal{C}_O[\boldsymbol{I}](\lambda) = \frac{1}{2\mathrm{i}\Delta} \sum_{k=1}^N & \left(\frac{1}{\lambda - \lambda_k} \begin{bmatrix} 0 & 0 \\ \overline{\psi^{(k)}(x,t)} & m^{(k)}(x,t) \end{bmatrix} \right. \\
& \left. + \frac{1}{\lambda - \overline{\lambda}_k} \begin{bmatrix} -m^{(k)}(x,t) & \psi^{(k)}(x,t) \\ 0 & 0 \end{bmatrix} \right).
\end{aligned} \tag{2.32}$$

Expanding (2.23) for large λ gives

$$\begin{cases}
\boldsymbol{O}(\lambda; x,t) = \boldsymbol{I} + \dfrac{\boldsymbol{O}_1(x,t)}{\lambda} + \mathcal{O}(\lambda^{-2}), \\
\boldsymbol{O}_1(x,t) = -\dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma (\boldsymbol{I} + \boldsymbol{\eta}(\xi; x,t))(\boldsymbol{V}_O(\xi; x,t) - \boldsymbol{I}) \mathrm{d}\xi = \sum_{j=0}^\infty \boldsymbol{O}_1^{(j)}(x,t), \\
\boldsymbol{O}_1^{(j)}(x,t) := -\dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \mathcal{C}_O^j[\boldsymbol{I}](\xi)(\boldsymbol{V}_O(\xi; x,t) - \boldsymbol{I}) \mathrm{d}\xi.
\end{cases}$$

[8] We recall that a set G is Ahlfors–David regular if there exists $c, C > 0$ such that $cr \leqslant \mathcal{H}^1(G \cap B_r(z)) \leqslant Cr$ for any $z \in \mathbb{C}$, $r \in (0, \operatorname{diam} G)$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure and $B_r(z)$ is the open ball centred at z with radius r; see [18].

Computing the first two terms by residues with the help of (2.32) gives

$$\boldsymbol{O}_1^{(0)}(x,t) = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma (\boldsymbol{V}_O(\xi)-\boldsymbol{I})\,\mathrm{d}\xi = \frac{1}{2\mathrm{i}\Delta}\sum_{k=1}^N \begin{bmatrix} -m^{(k)}(x,t) & \psi^{(k)}(x,t) \\ \overline{\psi^{(k)}(x,t)} & m^{(k)}(x,t)\end{bmatrix} \tag{2.33}$$

$$\begin{aligned}\boldsymbol{O}_1^{(1)}(x,t) &= -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathcal{C}_O[\boldsymbol{I}](\xi)(\boldsymbol{V}_O(\xi)-\boldsymbol{I})\,\mathrm{d}\xi \\ &= \frac{1}{(2\mathrm{i}\Delta)^2}\sum_{j=1}^N \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_j}\sum_{k=1}^N \begin{bmatrix} \frac{-m^{(k)}}{\xi-\overline{\lambda}_k} & \frac{\psi^{(k)}}{\xi-\overline{\lambda}_k} \\ \frac{\overline{\psi^{(k)}}}{\xi-\lambda_k} & \frac{m^{(k)}}{\xi-\lambda_k}\end{bmatrix}\begin{bmatrix} \frac{-m^{(j)}}{\xi-\overline{\lambda}_j} & \frac{\psi^{(j)}}{\xi-\overline{\lambda}_j} \\ \frac{\overline{\psi^{(j)}}}{\xi-\lambda_j} & \frac{m^{(j)}}{\xi-\lambda_j}\end{bmatrix}\mathrm{d}\xi \\ &= \frac{1}{(2\mathrm{i}\Delta)^2}\sum_{j=1}^N\sum_{\substack{k=1\\k\neq j}}^N \begin{bmatrix} \frac{\psi^{(k)}\overline{\psi^{(j)}}}{\lambda_j-\overline{\lambda}_k}+\frac{m^{(k)}m^{(j)}}{\overline{\lambda}_j-\overline{\lambda}_k} & \frac{\psi^{(k)}m^{(j)}}{\lambda_j-\overline{\lambda}_k}-\frac{m^{(k)}\psi^{(j)}}{\overline{\lambda}_j-\overline{\lambda}_k} \\ \frac{m^{(k)}\overline{\psi^{(j)}}}{\lambda_j-\lambda_k}-\frac{\overline{\psi^{(k)}}m^{(j)}}{\overline{\lambda}_j-\lambda_k} & \frac{m^{(k)}m^{(j)}}{\lambda_j-\lambda_k}+\frac{\overline{\psi^{(k)}}\psi^{(j)}}{\overline{\lambda}_j-\lambda_k}\end{bmatrix}.\end{aligned} \tag{2.34}$$

Furthermore, using the supremum matrix norm, it holds for any $j \geqslant 1$ that

$$\begin{aligned}\left\|\boldsymbol{O}_1^{(j)}(x,t)\right\| &\leqslant \frac{1}{2\pi}\left\|\int_\Gamma \left|\mathcal{C}_O^j[\boldsymbol{I}](\xi)(\boldsymbol{V}_O(\xi)-\boldsymbol{I})\right|\,|\mathrm{d}\xi|\right\| \leqslant \frac{1}{2\pi}\|\mathcal{C}_O^j[\boldsymbol{I}]\|_{L^2(\Gamma)}\|\boldsymbol{V}_O-\boldsymbol{I}\|_{L^2(\Gamma)} \\ &\leqslant \frac{1}{2\pi}\|\mathcal{C}_O\|^{j-1}\|\boldsymbol{V}_O-\boldsymbol{I}\|^2_{L^2(\Gamma)} \leqslant \left(C_*\frac{\|\boldsymbol{\mu}\|_\infty}{\Delta}\right)^{j-1}\left(C_1\frac{\|\boldsymbol{\mu}\|_2}{\Delta}\right)^2,\end{aligned} \tag{2.35}$$

Finally, undoing all the transformations and using formula (2.15), we have that the N-soliton solution of (1.1) parameterized by scattering data $\{(z_k,c_k)\}_{k=1}^N$ is given by

$$\begin{aligned}\psi_N(x,t) &= \lim_{\lambda\to\infty} 2\mathrm{i}\Delta\lambda\left[\widehat{\boldsymbol{M}}(\lambda;x,t)\right]_{1,2} = \lim_{\lambda\to\infty} 2\mathrm{i}\Delta\lambda\left[\boldsymbol{O}(\lambda;x,t)\right]_{1,2} \\ &= 2\mathrm{i}\Delta\left[\sum_{j=0}^\infty \boldsymbol{O}_1^{(j)}(x,t)\right]_{1,2},\end{aligned} \tag{2.36}$$

which, together with (2.35), gives (1.12).

2.5. *Proof of corollary 1.3*

We get from (1.3) and (1.13) that

$$\mathrm{i}\mu\mathrm{e}^{\mu x_k+\mathrm{i}\phi_k} = c_k = \mathrm{i}\mu\mathrm{e}^{-v_k\mu+\mathrm{i}\left(\mu^2-v_k^2\right)/2}.$$

That is, $x_k = -v_k$ and $\phi_k = (\mu^2-v_k^2)/2$. It follows that

$$\psi^{(k)}(x,t) = -\mu\,\mathrm{sech}\left(\mu(x-v_k(t-1)\right)\mathrm{e}^{\mathrm{i}\left(xv_k-\frac{t-1}{2}(v_k^2-\mu^2)\right)}$$

by (1.2). Hence, using (1.12) and remark 1.1 we get that

$$\begin{aligned}\frac{1}{N}\psi_N\left(\frac{2X}{NV},1+\frac{T}{(NV)^2}\right) &= -\frac{\mu}{N}\sum_{k=1}^N \mathrm{sech}\left(\frac{2\mu}{NV}\left(X-\frac{v_k}{NV}\frac{T}{2}\right)\right)\mathrm{e}^{\mathrm{i}\frac{\mu^2T}{2(NV)^2}}\mathrm{e}^{\mathrm{i}\left(2X\frac{v_k}{NV}-\frac{T}{2}\frac{v_k^2}{(NV)^2}\right)} \\ &\quad+\mathcal{O}\left(\frac{1}{\Delta}\right).\end{aligned} \tag{2.37}$$

Setting $s_k := v_k/(NV) \in [\alpha + \frac{k-1}{N}, \alpha + \frac{k}{N}]$ for each $k = 1, \ldots, N$, it holds locally uniformly with respect to X, T that

$$\frac{1}{N}\psi_N\left(\frac{2X}{NV}, 1 + \frac{T}{(NV)^2}\right) = -\mu \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}\left(2Xs_k - \frac{T}{2}s_k^2\right)} \frac{1}{N} + \mathcal{O}\left(\frac{1}{\Delta}\right),$$

where, for the error bound, we note that $\Delta \leqslant V$. Standard error estimates of Riemann sum approximation now give that

$$\frac{1}{N}\psi_N\left(\frac{2X}{N\Delta}, 1 + \frac{T}{(N\Delta)^2}\right) = -\mu \int_{\alpha}^{\alpha+1} \mathrm{e}^{\mathrm{i}\left(2Xs - \frac{T}{2}s^2\right)} \mathrm{d}s + \mathcal{O}\left(\max\left\{\frac{1}{N}, \frac{1}{\Delta}\right\}\right).$$

It only remains to notice that

$$\int_{\alpha}^{\alpha+1} \mathrm{e}^{\mathrm{i}\left(2Xs - \frac{T}{2}s^2\right)} \mathrm{d}s = \mathrm{e}^{\mathrm{i}\left(2\alpha X - \alpha^2 \frac{T}{2}\right)} \int_0^1 \mathrm{e}^{\mathrm{i}\left(2\left(X - \frac{\alpha T}{2}\right)s - \frac{T}{2}s^2\right)} \mathrm{d}s.$$

3. Stochastic *N*-soliton solutions

We now consider N-soliton solutions whose scattering data are random satisfying Assumption 1.1.

As we are assuming the imaginary part of the poles $\{\mu_k = 2\,\mathrm{Im}(z_k)\}$ to be distributed as a sub-exponential random variable (see [45, definition 2.7]), we recall some of its properties.

Definition 3.1. Given $\alpha, \nu \in \mathbb{R}_+$, a random variable $\mathfrak{X}$ is sub exponential of parameters (ν, α) if

$$\mathbb{E}\left[\mathrm{e}^{\lambda(\mathfrak{X} - \mathbb{E}[\mathfrak{X}])}\right] \leqslant \mathrm{e}^{\frac{\nu^2\lambda^2}{2}}, \quad \forall \lambda < \frac{1}{\alpha}. \tag{3.1}$$

In particular, this implies exponential decay of the tail of the distribution (see [45, proposition 2.10]):

$$\mathbb{P}\left(|\mathfrak{X} - \mathbb{E}[\mathfrak{X}]| \geqslant s\right) \leqslant 2\exp\left(-\frac{1}{2}\min\left\{\frac{s^2}{\nu^2}, \frac{s}{\alpha}\right\}\right), \qquad \forall s > 0. \tag{3.2}$$

As an example, we can consider the amplitudes to be distributed as a chi-squared distribution $\mu_k \sim \chi^2(\beta)$, $\beta \in \mathbb{R}_+$, i.e.

$$\mathbb{E}[f(\mu_k)] = \frac{1}{2^{\frac{\beta}{2}}\Gamma\left(\frac{\beta}{2}\right)} \int f(\mu_k)\, \mu_k^{\frac{\beta}{2}-1} \mathrm{e}^{-\frac{\mu_k}{2}} d\mu_k, \tag{3.3}$$

where $\Gamma(\beta)$ is the Gamma-function [1, ch 5].

The N-soliton setup we are considering is similar to the deterministic case in proposition 1.1 and theorem 1.2: the velocities of the solitons are tuned so that at finite time ($t = 1$) the solution will display a peak of order $\mathcal{O}(N)$, due to all the solitons colliding together.

We recall that provided that $\Delta > C_* \|\boldsymbol{\mu}\|_\infty$, the N-soliton solution can be written as

$$\psi_N(x,t) = \sum_{k=1}^{N} \psi^{(k)}(x,t) + \widetilde{\psi}(x,t) + \mathcal{O}\left(\frac{\|\boldsymbol{\mu}\|_\infty \|\boldsymbol{\mu}\|_2^2}{\Delta^2}\right), \tag{3.4}$$

with

$$\psi^{(k)}(x,t) = -\mu_k \operatorname{sech}\left(\mu_k\left(x - k\Delta\left(t-1\right)\right)\right) \mathrm{e}^{\mathrm{i}\left(k\Delta x + \frac{1}{2}\left(\mu_k^2 - k^2\Delta^2\right)(t-1)\right)}, \tag{3.5}$$

$$\widetilde{\psi}(x,t) := \frac{1}{2\Delta \mathrm{i}} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k\neq j}}^{N} \left[\frac{-m^{(k)}(x,t)\,\psi^{(j)}(x,t)}{\overline{\lambda_j} - \overline{\lambda_k}} + \frac{\psi^{(k)}(x,t)\,m^{(j)}(x,t)}{\lambda_j - \overline{\lambda_k}}\right], \tag{3.6}$$

where we set $z_j = \lambda_j \Delta$ (see theorem 1.2).

3.1. Probability estimates

In order to apply theorem 1.2 in a stochastic setting and obtain a CLT-type result, we will need to control the subleading terms of (3.4) and we will approximate the leading term in an appropriate way.

We start by deriving a probabilistic bound for $\widetilde{\psi}$ (3.6).

Lemma 3.1. *Under assumption 1.1, there exists a constant $K_{\mathcal{D}}$, dependent on the distribution $\mathcal{D}$, but independent of N, such that for any $0 \leqslant \varepsilon < \gamma - \frac{1}{2}$,*

$$\mathbb{P}\left(\left|\widetilde{\psi}(x,t)\right| > N^{\frac{1}{2}-\varepsilon}\right) \leqslant K_{\mathcal{D}} \frac{\ln(N)}{N^{\gamma - \frac{1}{2} - \varepsilon}}, \qquad \mathbb{E}\left[\left|\widetilde{\psi}(x,t)\right|\right] \leqslant K_{\mathcal{D}} \frac{\ln(N)}{N^{\gamma-1}}. \tag{3.7}$$

where $\widetilde{\psi}(x,t)$ has been defined in (8).

Proof. First, we notice that, by exchanging j,k, in (8), we can rewrite the general term of the double sum as

$$m^{(k)}\psi^{(j)}\left(\frac{2\overline{\lambda_j} - \lambda_k - \overline{\lambda_k}}{\left(\overline{\lambda_j} - \overline{\lambda_k}\right)\left(\lambda_k - \overline{\lambda_j}\right)}\right) = m^{(k)}\psi^{(j)}\left(\frac{-2}{\left(\overline{\lambda_j} - \overline{\lambda_k}\right)} + \frac{\lambda_k - \overline{\lambda_k}}{\left(\overline{\lambda_j} - \overline{\lambda_k}\right)\left(\lambda_k - \overline{\lambda_j}\right)}\right). \tag{3.8}$$

Using $|\overline{\lambda_j} - \overline{\lambda_k}| \geqslant |k-j|$ and estimates (2.28), the expected value of the first term is bounded by

$$\mathbb{E}\left[\left|m^{(k)}(x,t)\,\psi^{(j)}(x,t)\,\frac{2}{\left(\overline{\lambda_j} - \overline{\lambda_k}\right)}\right|\right] \leqslant \frac{2\mathbb{E}\left[\left|m^{(k)}(x,t)\right|\,\left|\psi^{(j)}(x,t)\right|\right]}{|k-j|} \leqslant \frac{4\mu_{\mathcal{D}}^2}{|k-j|}, \tag{3.9}$$

where we also used independency of the μ_k's. Analogously, the expected value of the second term can be estimated as

$$\begin{aligned} \mathbb{E}\left[\left|m^{(k)}(x,t)\,\psi^{(j)}(x,t)\,\frac{\lambda_k - \overline{\lambda_k}}{\left(\overline{\lambda_j} - \overline{\lambda_k}\right)\left(\lambda_k - \overline{\lambda_j}\right)}\right|\right] &\leqslant \frac{\mathbb{E}\left[\left|m^{(k)}(x,t)\right|\,\left|\psi^{(j)}(x,t)\right|\mu_k\right]}{\Delta|k-j|^2} \\ &\leqslant \frac{2\mu_{\mathcal{D}}\mathbb{E}\left[\mu_k^2\right]}{\Delta|k-j|^2}. \end{aligned} \tag{3.10}$$

Therefore, from (3.9)-(3.10), there exists a constant $K_{\mathcal{D}}$ independent of N, Δ, but depending on the distribution of the μ_k such that

$$\begin{aligned} \mathbb{E}\left[\left|\widetilde{\psi}(x,t)\right|\right] &\leqslant \frac{1}{\Delta} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k\neq j}}^{N} \left(\frac{2\mu_{\mathcal{D}}^2}{|k-j|} + \frac{\mu_{\mathcal{D}}\mathbb{E}\left[\mu_k^2\right]}{\Delta|k-j|^2}\right) \leqslant \widetilde{K}_{\mathcal{D}} \frac{1}{\Delta} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k\neq j}}^{N} \left(\frac{1}{|k-j|} + \frac{1}{\Delta|k-j|^2}\right) \\ &\leqslant \frac{K_{\mathcal{D}}}{2}\left(\frac{N\ln(N)}{\Delta} + \frac{N}{\Delta^2}\right) \leqslant K_{\mathcal{D}} \frac{N\ln(N)}{\Delta}. \end{aligned} \tag{3.11}$$

The statement of the lemma now follows by applying the Markov inequality that states that for any positive random variable X and positive number λ, $\mathbb{P}(X>\lambda)<\frac{\mathbb{E}[X]}{\lambda}$ and then substituting $\Delta=\beta N^{\gamma}$. □

Our goal is to study the fluctuations of the solutions $\psi_N(x,t)$ in a neighbourhood of the collision singularity (i.e. $x=0$ and $t=1$). Therefore, we rescale the space and time variables

$$x=\frac{2X}{\Delta N}\qquad\text{and}\qquad t-1=\frac{T}{\Delta^2N^2}\ ;$$

note that $T<0$ indicates a time before the collision and $T>0$ indicates a time after the collision.

In a slight abuse of notation, we will express functions of the original variables x and t as functions of X and T. For example, $\psi_N(X,T)$ will be used instead of $\psi_N\left(\frac{2X}{\Delta N},1+\frac{T}{\Delta^2N^2}\right)$, and so on.

In the next proposition we will show that the rescaled profile near the collision singularity

$$\begin{aligned}\psi_N(X,T)=&-\sum_{k=1}^{N}\mu_k\operatorname{sech}\left(\mu_k\left(\frac{2X}{\Delta N}-\frac{kT}{\Delta N^2}\right)\right)\mathrm{e}^{\mathrm{i}\left(2\frac{k}{N}X-\frac{k^2}{2N^2}T+\frac{\mu_k^2T}{2\Delta^2N^2}\right)}\\&+\widetilde{\psi}(X,T)+\mathcal{O}\left(\frac{\|\boldsymbol{\mu}\|_\infty\|\boldsymbol{\mu}\|_2^2}{\Delta^2}\right)\end{aligned}\tag{3.12}$$

can be approximated with high probability by the following function

$$\widehat{\psi}_N(X,T):=-\sum_{k=1}^{N}\mu_k\mathrm{e}^{\mathrm{i}\left(2\frac{k}{N}X-\frac{k^2}{2N^2}T\right)}\ .\tag{3.13}$$

Provided that Δ is large enough (i.e. we are in the small norm setting of theorem 1.2).

To make this statement quantitative, we define the function $f_N(X,T)$ as

$$f_N(X,T)=-\sum_{k=1}^{N}\mu_k\operatorname{sech}\left(\mu_k\left(\frac{2X}{\Delta N}-\frac{kT}{\Delta N^2}\right)\right)\mathrm{e}^{\mathrm{i}\left(2\frac{k}{N}X-\frac{k^2}{2N^2}T+\frac{\mu_k^2T}{2\Delta^2N^2}\right)}+\widetilde{\psi}(X,T)\ ,\tag{3.14}$$

then the following holds:

Proposition 3.2. *Under assumption 1.1, fix (X,T) in a compact set and let $N>2$, then there exist two constants $C_{\mathcal{D}}$, $\hat{C}_{\mathcal{D}}$, depending on the distribution $\mathcal{D}$, such that*

$$\mathbb{E}\left[\left|f_N(X,T)-\widehat{\psi}_N(X,T)\right|\right]\leqslant C_{\mathcal{D}}\frac{\ln(N)}{N^{\gamma-1}}+\hat{C}_{\mathcal{D}}\frac{\left(X^2+T^2+|XT|+|T|\right)}{\beta^2N^{2\gamma+1}}\ ,\tag{3.15}$$

where $f_N(X,T)$ is defined in (3.14) and $\widehat{\psi}_N(X,T)$ is defined in (3.13).

Proof. By linearity of the expected value

$$\mathbb{E}\left[\left|f_N(X,T)-\widehat{\psi}_N(X,T)\right|\right]\leqslant\mathbb{E}\left[\left|\sum_{k=1}^{N}\psi^{(k)}(X,T)-\widehat{\psi}_N(X,T)\right|\right]+\mathbb{E}\left[|\widetilde{\psi}(X,T)|\right].\tag{3.16}$$

From lemma 3.1, we bound the second term as

$$\mathbb{E}\left[|\widetilde{\psi}(X,T)|\right]\leqslant K_{\mathcal{D}}\frac{\ln(N)}{N^{\gamma-1}}\tag{3.17}$$

for some constant $K_{\mathcal{D}} > 0$ depending on the distribution $\mathcal{D}$. Next

$$\mathbb{E}\left[\left|\sum_{k=1}^{N}\psi^{(k)}(X,T) - \widehat{\psi}_N(X,T)\right|\right] \leqslant \left(\mathbb{E}\left[\left|\sum_{k=1}^{N}\psi^{(k)}(X,T) - \sum_{k=1}^{N}\widetilde{\psi^{(k)}}(X,T)\right|\right] + \mathbb{E}\left[\left|\sum_{k=1}^{N}\widetilde{\psi^{(k)}}(X,T) - \widehat{\psi}_N(X,T)\right|\right]\right), \tag{3.18}$$

where

$$\widetilde{\psi^{(k)}}(X,T) := -\mu_k \operatorname{sech}\left(\mu_k\left(\frac{2X}{\Delta N} - \frac{kT}{\Delta N^2}\right)\right) \mathrm{e}^{\mathrm{i}\left(2\frac{k}{N}X - \frac{k^2}{2N^2}T\right)}. \tag{3.19}$$

The first term is estimated as follows

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=1}^{N}\psi^{(k)}(X,T) - \sum_{k=1}^{N}\widetilde{\psi^{(k)}}(X,T)\right|\right] &\leqslant \sum_{k=1}^{N}\mathbb{E}\left[\left|\widetilde{\psi^{(k)}}(X,T)\right|\left|1 - \mathrm{e}^{\mathrm{i}\frac{\mu_k^2 T}{2\Delta^2 N^2}}\right|\right] \\ &\leqslant 2\sum_{k=1}^{N}\mathbb{E}\left[\mu_k^3\right]\frac{|T|}{2\Delta^2 N^2} \leqslant \frac{c_{1,\mathcal{D}}|T|}{\Delta^2 N}\end{aligned} \tag{3.20}$$

for some constant $c_{1,\mathcal{D}}$, where in the second inequality we used $|1-\mathrm{e}^s| \leqslant 2|s|$ for $s \in i\mathbb{R}$. The second term can be easily bounded in a similar way

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=1}^{N}\widetilde{\psi^{(k)}}(X,T) - \widehat{\psi}_N(X,T)\right|\right] &\leqslant \sum_{k=1}^{N}\mathbb{E}\left[\left|\mathrm{i}\mu_k \mathrm{e}^{\mathrm{i}\left(2\frac{k}{N}X - \frac{k^2}{2N^2}T\right)}\right|\left|1 - \operatorname{sech}\left(\mu_k\left(\frac{2X}{\Delta N} - \frac{kT}{\Delta N^2}\right)\right)\right|\right] \\ &\leqslant \frac{c_{2,\mathcal{D}}}{\Delta^2 N}\left(X^2 + T^2 + |XT|\right),\end{aligned} \tag{3.21}$$

for some constant $c_{2,\mathcal{D}}$, where we used the inequality $|1 - \operatorname{sech}(s)| \leqslant s^2$ for $s \in \mathbb{R}$.

Finally, substituting $\Delta = \beta N^{\gamma}$ concludes the proof. □

We can now prove a CLT-type result for the approximating function $\widehat{\psi}_N(X,T)$. We will then be able to extend these results to the original N-soliton solution $\psi_N(x,t)$, using the results from propositions 3.2.

Lemma 3.3. *Fix $X, T \in \mathbb{R}$. Under assumption 1.1, the following convergence results hold*

$$\frac{\operatorname{Re}\left(\widehat{\psi}_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma_+(X,T)\right), \tag{3.22}$$

$$\frac{\operatorname{Im}\left(\widehat{\psi}_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}\left(0, \sigma_-(X,T)\right), \tag{3.23}$$

$$\frac{\left|\widehat{\psi}_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right|}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{H}\left(\varphi(X,T)\right), \tag{3.24}$$

where $\mathrm{Var}_{\mathcal{D}}$ *is the variance of the distribution* $\mathcal{D}$*,*

$$\psi_0(X,T) = -\int_0^1 \mathrm{e}^{\mathrm{i}\left(2Xs - \frac{T}{2}s^2\right)}\mathrm{d}s, \qquad \sigma_{\pm}(X,T) = \frac{1}{2}\left(1 \pm \int_0^1 \cos\left(4Xs - Ts^2\right)\right)\mathrm{d}s, \tag{3.25}$$

$\mathcal{H}(\varphi)$ is a special Hoyt distribution with probability density function

$$\rho(\xi;\varphi) = \frac{2\xi\, \mathrm{e}^{-\frac{\xi^2}{\sin^2(2\varphi)}}}{|\sin(2\varphi)|} I_0\left(\xi^2 \cot(2\varphi)\csc(2\varphi)\right) , \qquad \xi \in \mathbb{R}^+ , \tag{3.26}$$

where I_0 is the modified Bessel function of first kind of order $\nu = 0$ [1, formula (10.32.1)]:

$$I_0(z) = \frac{1}{\pi}\int_0^\pi \mathrm{e}^{\pm z\cos(\theta)}\mathrm{d}\theta = \frac{1}{\pi}\int_0^\pi \cosh(z\cos(\theta))\,\mathrm{d}\theta . \tag{3.27}$$

and $\varphi \in \left[0, \frac{\pi}{2}\right]$ is such that

$$\cos(2\varphi) = \int_0^1 \cos\left(4Xs - Ts^2\right)\mathrm{d}s .$$

Proof. We will resort to the Nagaev–Guivarc'h method, a fundamental technique to prove probabilistic limit theorems for dynamical systems, which we briefly recall in appendix B.

We start by considering the real and imaginary parts of $\widehat{\psi}_N$ (3.13) separately.

It is enough to notice that

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi\,\mathrm{Re}\left(\widehat{\psi}_N\right)}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\left[\sum_{k=1}^N \mu_k \cos\left(2X\frac{k}{N} - \frac{T}{2}\frac{k^2}{N^2}\right)\right]}\right] = \prod_{k=1}^N \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi y\cos\left(2X\frac{k}{N} - \frac{T}{2}\frac{k^2}{N^2}\right)}\right] \\
&= \prod_{k=1}^N \lambda_1\left(\xi;\frac{k}{N}\right) , \qquad (3.28) \\
\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi\,\mathrm{Im}\left(\widehat{\psi}_N\right)}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\left[\sum_{k=1}^N \mu_k \sin\left(2X\frac{k}{N} - \frac{T}{2}\frac{k^2}{N^2}\right)\right]}\right] = \prod_{k=1}^N \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi y\sin\left(2X\frac{k}{N} - \frac{T}{2}\frac{k^2}{N^2}\right)}\right] \\
&= \prod_{k=1}^N \lambda_2\left(\xi;\frac{k}{N}\right) , \qquad (3.29)
\end{aligned}$$

where $y \sim \mathcal{D}$ with mean $\mu_{\mathcal{D}}$ and variance $\mathrm{Var}_{\mathcal{D}}$, and

$$\begin{aligned}
\lambda_1(\xi;s) &= \exp\left\{\log\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi y\cos\left(\left(2Xs - \frac{T}{2}s^2\right)\right)}\right]\right\} \\
&= \exp\left\{\mathrm{i}\cos\left(2Xs - \frac{T}{2}s^2\right)\mu_{\mathcal{D}}\xi - \cos^2\left(2Xs - \frac{T}{2}s^2\right)\mathrm{Var}_{\mathcal{D}}\frac{\xi^2}{2} + o\left(\xi^2\right)\right\} , \qquad (3.30) \\
\lambda_2(\xi;s) &= \exp\left\{\log\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi y\sin\left(\left(2Xs - \frac{T}{2}s^2\right)\right)}\right]\right\} \\
&= \exp\left\{\mathrm{i}\sin\left(2Xs - \frac{T}{2}s^2\right)\mu_{\mathcal{D}}\xi - \sin^2\left(2Xs - \frac{T}{2}s^2\right)\mathrm{Var}_{\mathcal{D}}\frac{\xi^2}{2} + o\left(\xi^2\right)\right\} . \qquad (3.31)
\end{aligned}$$

We define

$$\psi_0(X,T) = -\int_0^1 \mathrm{e}^{\mathrm{i}\left(2Xs - \frac{T}{2}s^2\right)}\mathrm{d}s ,$$

and, by applying [32, theorem 4.2] (see also appendix B), we directly obtain (3.22) and (3.23).

The complex random variable $\hat{\psi}_N(X,T)$ has expected value $N\mu_{\mathcal{D}}\psi_0(X,T)$, which implies

$$\lim_{N\to\infty}\frac{\widehat{\psi}_N(X,T)}{\mu_{\mathcal{D}}N}=\psi_0(X,T) \quad \text{almost surely} . \tag{3.32}$$

which is a *universal profile*. From the previous calculations, it follows that the real random variable

$$\mathcal{X}_N(X,T):=\frac{\left|\widehat{\psi}_N(X,T)-N\mu_{\mathcal{D}}\psi_0(X,T)\right|}{\sqrt{Nm_{2,\mathcal{D}}}}$$

converges to the following probability distribution

$$\mathcal{X}_N(X,T)\xrightarrow{N\to\infty}\mathcal{Z}(X,T) , \qquad \mathcal{Z}(X,T):=\sqrt{\mathcal{U}(X,T)^2+\mathcal{V}(X,T)^2} \tag{3.33}$$

pointwise in $(X,T)\in\mathbb{R}^2$, where $\mathcal{U}(X,T)\sim\mathcal{N}(0,\sigma_+(X,T))$ and $\mathcal{V}(X,T)\sim\mathcal{N}(0,\sigma_-(X,T))$, with

$$\sigma_\pm(X,T)=\frac{1}{2}\left(1\pm\int_0^1\cos\left(4Xs-Ts^2\right)\mathrm{d}s\right) . \tag{3.34}$$

The cumulative distribution function of the modulus of a complex Gaussian is equal to

$$\begin{aligned}\mathbb{P}(|\mathcal{Z}|\leqslant\xi)=\mathbb{P}\left(\mathcal{U}^2+\mathcal{V}^2\leqslant\xi^2\right)&=\frac{1}{2\pi\sqrt{\sigma_+(X,T)\sigma_-(X,T)}}\int_{u^2+v^2\leqslant\xi^2}\mathrm{e}^{-\frac{1}{2}\left(\frac{u^2}{\sigma_+(X,T)}+\frac{v^2}{\sigma_-(X,T)}\right)}\mathrm{d}u\mathrm{d}v\\ &=\frac{1}{|\sin(2\varphi)|\pi}\int_{u^2+v^2\leqslant\xi^2}\mathrm{e}^{-\frac{1}{2}\left(\frac{u^2}{\cos^2(\varphi)}+\frac{v^2}{\sin^2(\varphi)}\right)}\mathrm{d}u\mathrm{d}v, \qquad \forall\,\xi\geqslant0 ,\end{aligned} \tag{3.35}$$

where we defined

$$\cos(2\varphi)=\int_0^1\cos\left(4Xs-Ts^2\right)\mathrm{d}s ,$$

which implies that

$$\sigma_+(X,T)=\cos^2(\varphi) \qquad \text{and} \qquad \sigma_-(X,T)=\sin^2(\varphi) .$$

From this expression, we can compute the probability density function (Hoyt distribution):

$$\rho(\xi;\varphi)=\frac{2\xi\,\mathrm{e}^{-\frac{\xi^2}{\sin^2(2\varphi)}}}{|\sin(2\varphi)|}I_0\left(\xi^2\cot(2\varphi)\csc(2\varphi)\right) , \tag{3.36}$$

where I_0 is the modified Bessel function of first kind of order $\nu=0$ [1, formula (10.32.1)]:

$$I_0(z)=\frac{1}{\pi}\int_0^\pi\mathrm{e}^{\pm z\cos(\theta)}\mathrm{d}\theta=\frac{1}{\pi}\int_0^\pi\cosh(z\cos(\theta))\,\mathrm{d}\theta . \tag{3.37}$$

This proves (3.24).

□

3.2. *Proof of the central limit theorem 1.5 and proposition 1.4*

We will now prove theorem 1.5, and proposition 1.4, obtaining a CLT-type result for the solution $\psi_N(x,t)$ near the collision point $(x_0,t_0) = (0,1)$.

We prove only the first limit (1.21), since the proof in the other cases is analogous. The idea is to show since $\psi_N(X,T)$ behaves like $\widehat{\psi}_N(X,T)$ in the limit as $N \to \infty$, with high probability, it obeys a CLT-type behaviour as well.

Consider the quantity

$$\frac{\mathrm{Re}\left(\psi_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} = \frac{\mathrm{Re}\left(\widehat{\psi}_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} + \frac{\mathrm{Re}\left(\psi_N(X,T) - \widehat{\psi}_N(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}}. \tag{3.38}$$

Thanks to lemma 3.3, the first term will converge to a Gaussian with mean zero and prescribed variance. It remains to prove that there exists a $\delta > 0$ such that

$$\mathbb{P}\left(\frac{\left|\psi_N(X,T) - \widehat{\psi}_N(X,T)\right|}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} > N^{-\delta}\right) \xrightarrow{N\to\infty} 0\,, \tag{3.39}$$

for any $X,T \in \mathbb{R}$ and for any $\Delta > 0$.

Fix $\frac{1}{2} > \delta > 0$ and define the sets Ω and A_δ as

$$\Omega := \left\{\boldsymbol{\mu} \in \mathbb{R}^N_+ \,\middle|\, \|\boldsymbol{\mu}\|_\infty > N^{\frac{1}{3}}\right\}, \tag{3.40}$$

$$A_\delta := \left\{\boldsymbol{\mu} \in \mathbb{R}^N_+ \,\middle|\, \left|\psi_N(X,T) - \widehat{\psi}_N(X,T)\right| > \sqrt{\mathrm{Var}_{\mathcal{D}}}N^{\frac{1}{2}-\delta}\right\} \subseteq \mathbb{R}^N_+. \tag{3.41}$$

Given $\Delta = \beta N^\gamma$ with $\gamma > \frac{1}{2}$, for N big enough, Ω contains the set of vectors $\boldsymbol{\mu}$ for which $\|\boldsymbol{\mu}\|_\infty > \beta N^{\frac{1}{2}}$ (i.e. $\frac{\|\boldsymbol{\mu}\|_\infty}{\Delta} > 1$) by construction; and we can rewrite (3.39) as

$$\mathbb{P}(A_\delta) \xrightarrow{N\to\infty} 0. \tag{3.42}$$

Furthermore

$$\mathbb{P}(A_\delta) = \mathbb{P}(A_\delta \cap \Omega) + \mathbb{P}\left(A_\delta \cap \left(\mathbb{R}^N_+ \setminus \Omega\right)\right). \tag{3.43}$$

One immediately notices (see lemma B.1 in appendix B) that there exists a positive constant c such that

$$\mathbb{P}(A_\delta \cap \Omega) \leqslant \mathbb{P}(\Omega) \leqslant \mathrm{e}^{-cN^{\frac{1}{4}}}. \tag{3.44}$$

Therefore, we just need to estimate $\mathbb{P}(A_\delta \cap (\mathbb{R}^N_+ \setminus \Omega))$. In this set, we can apply theorem 1.2 and proposition 3.2 therefore we conclude that there exist a function $\widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)$ and a constant C, independent of $(\boldsymbol{\mu},\Delta,N)$, such that

$$\begin{aligned} &\psi_N(X,T) = f_N(X,T) + \widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N) \text{ for all } \mu \in \mathbb{R}^+_N \setminus \Omega, \text{ and} \\ &|\widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)| \leqslant C\frac{\|\boldsymbol{\mu}\|_\infty \|\boldsymbol{\mu}\|_2^2}{\Delta^2} \text{ for all } \mu \in \mathbb{R}^N_+. \end{aligned} \tag{3.45}$$

Indeed, one may use

$$\widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N) = \begin{cases} \psi_N(X,T) - f_N(X,T) & \text{if } C^*\|\boldsymbol{\mu}\|_\infty < \Delta \\ \frac{C\|\boldsymbol{\mu}\|_\infty \|\boldsymbol{\mu}\|_2^2}{\Delta^2} & \text{otherwise.} \end{cases} \tag{3.46}$$

Then one can estimate $\mathbb{P}\left(A_\delta \cap (\mathbb{R}_+^N \setminus \Omega)\right)$ as follows.

$$\begin{aligned} \mathbb{P}\left(A_\delta \cap (\mathbb{R}_+^N \setminus \Omega)\right) &= \mathbb{P}\left(\left\{\boldsymbol{\mu} \in \mathbb{R}_+^N \middle| \frac{\left|\widehat{\psi}_N(X,T) - f_N(X,T) - \widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} > N^{-\delta}\right\} \cap (\mathbb{R}_+^N \setminus \Omega)\right) \\ &\leqslant \mathbb{P}\left(\left\{\boldsymbol{\mu} \in \mathbb{R}_+^N \middle| \frac{\left|\widehat{\psi}_N(X,T) - f_N(X,T) - \widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} > N^{-\delta}\right\}\right). \end{aligned} \tag{3.47}$$

By Markov inequality, that states that for a positive random variable $\mathfrak{X}$ and $\lambda > 0$, $\mathbb{P}(\mathfrak{X} > \lambda) \leqslant \frac{\mathbb{E}[\mathfrak{X}]}{\lambda}$, we bound the last term as

$$\begin{aligned} \mathbb{P}\left(\frac{\left|\widehat{\psi}_N(X,T) - f_N(X,T) - \widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} > N^{-\delta}\right) &\leqslant \frac{\mathbb{E}\left[\left|\widehat{\psi}_N(X,T) - f_N(X,T) - \widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|\right]}{N^{\frac{1}{2}-\delta}\sqrt{\mathrm{Var}_{\mathcal{D}}}} \\ &\leqslant \frac{\mathbb{E}\left[\left|\widehat{\psi}_N(X,T) - f_N(X,T)\right|\right] + \mathbb{E}\left[\left|\widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|\right]}{N^{\frac{1}{2}-\delta}\sqrt{\mathrm{Var}_{\mathcal{D}}}}. \end{aligned} \tag{3.48}$$

We recall that for subexponential random variables, there exists a constant c independent of N such that $\mathbb{E}\left[\max_j \mu_j^3\right] \leqslant \mathsf{c}\ln^3(N)$; therefore, by (3.45) one deduces

$$\mathbb{E}\left[\left|\widetilde{f}(X,T,\boldsymbol{\mu},\Delta,N)\right|\right] \leqslant C\frac{\ln^3(N)}{N^{2\gamma-1}}. \tag{3.49}$$

So, applying proposition 3.2 we conclude that (3.39) holds and theorem 1.5 follows. To prove proposition 1.4, one notices that once we have established (1.21), i.e.

$$\frac{\mathrm{Re}\left(\psi_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N\mathrm{Var}_{\mathcal{D}}}} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0,\sigma_+(X,T)) \tag{3.50}$$

it follows that

$$\frac{1}{\sqrt{N}}\frac{\mathrm{Re}\left(\psi_N(X,T) - N\mu_{\mathcal{D}}\psi_0(X,T)\right)}{\sqrt{N}} \xrightarrow{N\to\infty} 0 \qquad \text{in probability,} \tag{3.51}$$

since $\mathcal{N}(0,\sigma_+)$ is a continuous random variable. Finally, convergence in distribution to a constant implies convergence in probability, i.e.

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\mathrm{Re}\left(\frac{1}{N}\psi_N(X,T) - \mu_{\mathcal{D}}\psi_0(X,T)\right)\right| \geqslant \epsilon\right) = 0 \quad \forall \epsilon > 0. \tag{3.52}$$

The same argument holds for the imaginary part of $\psi_N(X,T)$ (see (1.22)), thus implying that

$$\frac{1}{\mu_{\mathcal{D}}N}\psi_N(X,T) \xrightarrow{N\to\infty} \psi_0(X,T) \qquad \text{in probability.} \tag{3.53}$$

3.3. Proof of the central limit theorem 1.6

Finally, we consider the general solution $\psi_N(x,1)$ at the collision time $t=1$ and we prove theorem 1.5. We first obtain a CLT-type result for the leading order term in the expansion (3.4) at collision time $t=1$

$$\varphi_N(x,1) := \sum_{k=1}^{N} \psi^{(k)}(x,1) = \sum_{k=1}^{N} \frac{-\mu_k}{\cosh(\mu_k x)} \mathrm{e}^{\mathrm{i}k\Delta x}. \tag{3.54}$$

Lemma 3.4. *Under Assumption 1.1, for any $x \in \mathbb{R}$ the following convergence results hold*

$$\frac{\mathrm{Re}(\varphi_N(x,1)) - \omega_{\mathcal{D}}(x)\cos\left(x\Delta\frac{N+1}{2}\right)D_N(x\Delta)}{\sigma_{N,\mathrm{Re}}(x)} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0,1), \tag{3.55}$$

$$\frac{\mathrm{Im}(\varphi_N(x,1)) - \omega_{\mathcal{D}}(x)\sin\left(x\Delta\frac{N}{2}\right)D_{N+1}(x\Delta)}{\sigma_{N,\mathrm{Im}}(x)} \xrightarrow[N\to\infty]{\text{law}} \mathcal{N}(0,1), \quad \textit{for } x \neq 0, \tag{3.56}$$

where $D_N(x) := \frac{\sin\left(\frac{xN}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ is the Dirichlet kernel ,

$$\omega_{\mathcal{D}}(x) = -\mathbb{E}\left[\frac{\xi}{\cosh(x\xi)}\right], \quad \xi \sim \mathcal{D}, \tag{3.57}$$

$$\sigma^2_{N,\mathrm{Re}}(x) = \mathrm{Var}\left(\frac{\xi}{\cosh(x\xi)}\right)\left(\frac{N-1}{2} + \frac{1}{2}\cos(x\Delta N)D_{N+1}(2x\Delta)\right), \tag{3.58}$$

$$\sigma^2_{N,\mathrm{Im}}(x) = \mathrm{Var}\left(\frac{\xi}{\cosh(x\xi)}\right)\left(\frac{N+1}{2} - \frac{1}{2}\cos(x\Delta N)D_{N+1}(2x\Delta)\right), \tag{3.59}$$

and $\mathrm{Var}(\cdot)$ is the variance of the given random variable. Moreover,

$$\frac{|\varphi_N(x,1)| - |\omega_{\mathcal{D}}(x)D_N(x\Delta)|}{N} \to 0 \quad \text{as } N\to\infty, \text{in probability.} \tag{3.60}$$

Remark 3.1. Notice that $\mathrm{Im}(\varphi_N(x,1)) = 0$ deterministically, therefore we excluded the value $x=0$ in (3.55).

Proof. Let $x \in \mathbb{R}$. We will show the proof of (3.55) in detail. The proof of (3.56) is analogous. The result easily follows from the classical result of Lyapounov's condition [6], which we recall in appendix B.

Consider the real part of (3.54):

$$\mathrm{Re}(\varphi_N(x,1)) = -\sum_{k=1}^{N} \mathcal{X}_k \cos(kx\Delta),$$

where $\mathcal{X}_k$ is the random variable $\mathcal{X}_k(x) := \frac{\mu_k}{\cosh(\mu_k x)}$, $\mu_k \sim \mathcal{D}$.

Let $\sigma_{N,\mathrm{Re}}(x) := \sqrt{\sum_{k=1}^{N}\mathrm{Var}(\mathcal{X}_k)}$. We compute

$$\sigma^4_{N,\mathrm{Re}}(x) = \left(\sum_{k=1}^{N}\mathrm{Var}(\mathcal{X}_k\cos(kx\Delta))\right)^2 = \mathrm{Var}(\mathcal{X}_1)^2\left(\sum_{k=1}^{N}\cos^2(kx\Delta)\right)^2$$
$$= \mathrm{Var}(\mathcal{X}_1)^2\left(\frac{N}{2} - 1 + \frac{1}{2}\mathrm{Re}\left(\frac{1-\mathrm{e}^{2\mathrm{i}x\Delta(N+1)}}{1-\mathrm{e}^{2\mathrm{i}x\Delta}}\right)\right)^2, \tag{3.61}$$

since the amplitudes μ_k's are i.i.d. and

$$\sum_{k=1}^{N}\mathbb{E}\left[\left(\mathcal{X}_k\cos(kx\Delta)-\mathbb{E}\left[\mathcal{X}_k\cos(kx\Delta)\right]\right)^4\right]=\mathbb{E}\left[\left(\mathcal{X}_1-\mathbb{E}\left[\mathcal{X}_1\right]\right)^4\right]\sum_{k=1}^{N}\cos^4(kx\Delta)$$
$$=\mathbb{E}\left[\left(\mathcal{X}_1-\mathbb{E}\left[\mathcal{X}_1\right]\right)^4\right]\left(\frac{3}{8}N-\frac{5}{8}+\frac{1}{2}\mathrm{Re}\left(\frac{1-\mathrm{e}^{2\mathrm{i}x\Delta(N+1)}}{1-\mathrm{e}^{2\mathrm{i}x\Delta}}\right)+\frac{1}{8}\mathrm{Re}\left(\frac{1-\mathrm{e}^{4\mathrm{i}x\Delta(N+1)}}{1-\mathrm{e}^{4\mathrm{i}x\Delta}}\right)\right). \tag{3.62}$$

Thus, the following Lyapounov's condition (with $\delta=2$) is satisfied

$$\lim_{N\to\infty}\frac{1}{\sigma^4_{N,\mathrm{Re}}(x)}\sum_{k=1}^{N}\mathbb{E}\left[\left(\mathcal{X}_k\cos(kx\Delta)-\mathbb{E}\left[\mathcal{X}_k\cos(kx\Delta)\right]\right)^4\right]=0\,.$$

Therefore, by the Lyapounov's condition theorem B.3, the convergence result (3.56) follows

$$\frac{\mathrm{Re}\left(\varphi_N(x,1)-\mathbb{E}\left[\varphi_N(x,1)\right]\right)}{\sigma_{N,\mathrm{Re}}(x)}\xrightarrow{N\to\infty}\mathcal{N}(0,1)\ , \tag{3.63}$$

where

$$\mathbb{E}\left[\mathrm{Re}\left(\varphi_N(x,1)\right)\right]=-\mathbb{E}\left[\sum_{k=1}^{N}\mathcal{X}_k\cos(kx\Delta)\right]=\omega_{\mathcal{D}}(x)\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}x\Delta\frac{N+1}{2}}D_N(x\Delta)\right). \tag{3.64}$$

Furthermore, since and $\sigma_{N,\mathrm{Re}}(x)$ is of the order $\mathcal{O}\left(N^{1/2}\right)$ for large N ($\forall x\in\mathbb{R}$), Lyapounov's condition theorem implies the following Law of Large Numbers result (see proposition B.4)

$$\frac{1}{N}\left(\mathrm{Re}\left(\varphi_N(x,1)\right)-\omega_{\mathcal{D}}(x)\cos\left(x\Delta\frac{N+1}{2}\right)D_N(x\Delta)\right)\to 0\qquad\text{as } N\to\infty,\text{in probability;} \tag{3.65}$$

similarly for the imaginary part of $\varphi_N(x,1)$. Therefore, we can conclude that

$$\frac{\left|\varphi_N(x,1)\right|-\left|\omega_{\mathcal{D}}(x)D_N(x\Delta)\right|}{N}\to 0\qquad\text{as } N\to\infty,\text{in probability}, \tag{3.66}$$

where we used the fact that $N^{-1}\left(\left|\mathbb{E}\left[\varphi_N(x,1)\right]\right|-\mathbb{E}\left[\left|\varphi_N(x,1)\right|\right]\right)\to 0$, as $N\to\infty$, and

$$\left|\mathbb{E}\left[\varphi_N(x,1)\right]\right|=\sqrt{\mathbb{E}\left[\mathrm{Re}\left(\varphi_N(x,1)\right)\right]^2+\mathbb{E}\left[\mathrm{Im}\left(\varphi_N(x,1)\right)\right]^2}=\left|\omega_{\mathcal{D}}(x)D_N(x\Delta)\right|. \tag{3.67}$$

□

Finally, in order to prove theorem 1.6, we extend the result of the previous lemma to the solution $\psi_N(x)$ in the same way as in the proof of theorem 1.5. We leave the details to the interested reader.

Data availability statement

No new data were created or analysed in this study.

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Most of the figures in the paper were realized using the Python libraries `NumPy` [23], `Scipy` [44] and `Matplolib` [26].

Appendix A. General facts about the *N*-soliton solution of the fNLS equation

We report here some known facts about the RHP for N-soliton solutions, and we present a novel upper bound on the modulus of the solution $\psi_N(x,t)$. Such a bound is suboptimal as compared to (1.6) (see theorem 1.2), but it shows exponential decay of the tails. Given a set of spectral data $\{(z_k,c_k)\}_{k=1}^N$, the solution $\boldsymbol{M}(z;x,t)$ of the RHP 2.1 has the form (see [12, appendix B])

$$\boldsymbol{M}(z;x,t) := \boldsymbol{I} + \sum_{k=1}^{N} \frac{1}{z-z_k}\begin{bmatrix}\alpha_k(x,t) & 0\\ \beta_k(x,t) & 0\end{bmatrix} + \sum_{k=1}^{N}\frac{1}{z-\overline{z}_k}\begin{bmatrix}0 & -\overline{\beta_k(x,t)}\\ 0 & \overline{\alpha_k(x,t)}\end{bmatrix}, \tag{A.1}$$

where $\{\alpha_k(x,t),\beta_k(x,t)\}_{k=1}^N$ solve the following linear system

$$\alpha_k(x,t) = -\gamma_k(x,t)\sum_{j=1}^{N}\frac{\overline{\beta_j(x,t)}}{z_k-\overline{z}_j} \quad \text{and} \quad \beta_k(x,t) = \gamma_k(x,t)\left(1+\sum_{j=1}^{N}\frac{\overline{\alpha_j(x,t)}}{z_k-\overline{z}_j}\right), \tag{A.2}$$

$$\gamma_k(x,t) := c_k e^{2i\theta(z_k;x,t)}, \tag{A.3}$$

which follows directly from the residue conditions satisfied by $\boldsymbol{M}(z;x,t)$.

Proposition A.1. *The system (A.2) is uniquely solvable, and the fNLS solution is given as*

$$\overline{\psi_N(x,t)} = 2i\sum_{k=1}^{N}\beta_k(x,t). \tag{A.4}$$

Proof. Unique solvability of system (A.2) is equivalent to verify that

$$\det\left(\boldsymbol{I}+\boldsymbol{\Phi}_N\overline{\boldsymbol{\Phi}_N}\right)\neq 0, \quad \text{where} \quad \boldsymbol{\Phi}_N := \left[\frac{\sqrt{c_k}\sqrt{\overline{c_n}}}{z_k-\overline{z}_n}e^{i(\theta(z_k;x,t)-\theta(\overline{z}_n;x,t))}\right]_{k,n=1}^{N}, \tag{A.5}$$

$\overline{\mathbf{\Phi}_N}$ is the conjugate matrix of $\mathbf{\Phi}_N$, and we consider the principal value of $\sqrt{c_k}$.

We consider now the matrix $\mathrm{i}\mathbf{\Phi}_N$: since each entry can be viewed as an inner product of linearly independent functions,

$$[\mathrm{i}\mathbf{\Phi}_N]_{k,n} = \int_0^{\infty} \left(\sqrt{c_k}\mathrm{e}^{\mathrm{i}\theta(z_k)}\right) \overline{\left(\sqrt{c_n}\mathrm{e}^{\mathrm{i}\theta(z_n)}\right)} \mathrm{e}^{\mathrm{i}(z_k-\overline{z}_n)s}\mathrm{d}s,$$

$\mathrm{i}\mathbf{\Phi}_N$ is a positive definite matrix. Let $(\mathrm{i}\mathbf{\Phi}_N)^{1/2}$ be the unique positive definite square root of $\mathrm{i}\mathbf{\Phi}_N$. The eigenvalues of $\mathbf{\Phi}_N\overline{\mathbf{\Phi}_N} = (\mathrm{i}\mathbf{\Phi}_N)\overline{(\mathrm{i}\mathbf{\Phi}_N)}$ are the same as the eigenvalues of $(\mathrm{i}\mathbf{\Phi}_N)^{1/2}\overline{(\mathrm{i}\mathbf{\Phi}_N)}(\mathrm{i}\mathbf{\Phi}_N)^{1/2}$, which is also a positive definite matrix. If one labels these eigenvalues by $\lambda_k > 0$, then

$$\det\left(\boldsymbol{I} + \mathbf{\Phi}_N\overline{\mathbf{\Phi}_N}\right) = \prod_{k=1}^{N}(1+\lambda_k) > 0$$

as needed. Finally, from (2.12) and (A.1), it immediately follows that the corresponding solution $\psi_N(x,t)$ of the fNLS equation (1.1) can be expressed as

$$\overline{\psi_N(x,t)} = 2\mathrm{i}\sum_{k=1}^{N}\beta_k(x,t).$$

□

We derive now an alternative expression for the solution $\psi_N(x,t)$, that will be used shortly to derive some estimates on the modulus of the solution.

Lemma A.2. *Let $B(z)$ be the Blaschke product from (1.7). It holds that*

$$\psi_N(x,t) = 2\mathrm{i}\sum_{k=1}^{N}\frac{1}{B'(z_k)}\frac{\alpha_k(x,t)}{\gamma_k(x,t)},$$

where α_k and γ_k are defined in (A.2) and (A.3) respectively.

Proof. It readily follows from (A.2) that

$$2\mathrm{i}\sum_{n=1}^{N}\frac{1}{B'(z_n)}\frac{\alpha_n(x,t)}{\gamma_n(x,t)} = \overline{2\mathrm{i}\sum_{k=1}^{N}\left(\sum_{n=1}^{N}\frac{1}{\overline{B'(z_n)}}\frac{1}{\overline{z}_n - z_k}\right)\beta_k(x,t)}.$$

Since $B(z)$ is a rational function with poles $\overline{z}_1,\ldots,\overline{z}_N$ that is equal to 1 at infinity, it holds that

$$B(z) = 1 + \sum_{n=1}^{N}\frac{1}{\overline{B'(z_n)}}\frac{1}{z-\overline{z}_n}.$$

As $B(z_k) = 0$, it follows that $\sum_{n=1}^{N}\frac{1}{\overline{B'(z_n)}}\frac{1}{z_k-\overline{z}_n} = -1$, and we recover (A.4). □

The modulus of $\psi_N(x,t)$ also admits expressions convenient for analysis. Indeed, it is known (see for example [12, equation (2.3)]) that

$$\partial_x\boldsymbol{M}(z) = -\mathrm{i}z[\sigma_3,\boldsymbol{M}(z)] + \begin{bmatrix} 0 & \psi_N(x,t) \\ -\overline{\psi_N(x,t)} & 0 \end{bmatrix}\boldsymbol{M}(z).$$

Multiplying by z and taking the limit as $z \to \infty$ of the $(2,2)$-entries of the above relation gives after conjugation that

$$|\psi_N(x,t)|^2 = 2\mathrm{i}\partial_x \left(\sum_{k=1}^{N} \alpha_k(x,t) \right). \tag{A.6}$$

We recall that expression (A.6) can further be rewritten using the famous determinantal formula [20]:

$$|\psi_N(x,t)|^2 = \partial_{xx} \log\det\left(\boldsymbol{I} + \boldsymbol{A}\overline{\boldsymbol{A}}\right).$$

We now present a novel, general upper bounds for $|\psi_N(x,t)|$. Despite being suboptimal as compared to (1.6) for finite $x,t \in \mathbb{R}\times\mathbb{R}^+$, it shows exponential decay for $|x|,|t| \gg 1$, which cannot be read from (1.6).

Proposition A.3. *It holds that*

$$\frac{1}{2}|\psi_N(x,t)| \leqslant \min\left\{ \sum_{k=1}^{N} |\gamma_k(x,t)|,\ \sum_{k=1}^{N} |B'(z_k)|^{-1},\ \sum_{k=1}^{N} |B'(z_k)|^{-2}|\gamma_k(x,t)|^{-1} \right\}.$$

Proof. Recall that if $f(z)$ is analytic in a domain D then $|f(z)|^2$ is subharmonic there because $\Delta|f(z)|^2 = 4\partial_z\partial_{\bar z}|f(z)|^2 = 4|f'(z)|^2 \geqslant 0$. Let

$$S_i(z) := \left|[\boldsymbol{M}(z)]_{1,i}\right|^2 + \left|[\boldsymbol{M}(z)]_{2,i}\right|^2, \quad i \in \{1,2\}.$$

Since the second column of $\boldsymbol{M}(z)$ is analytic in $\mathbb{C}_+$, see (A.1), $S_2(z)$ is a subharmonic there. Clearly, $\det\boldsymbol{M}(z)$ is a rational function of z that is equal to 1 at infinity. Since only one column of $\boldsymbol{M}(z)$ can have a pole at a given point, $\det\boldsymbol{M}(z)$ can have at most simple poles. However, it is easy to check that the residue conditions for $\boldsymbol{M}(z)$ imply that the residues of $\det\boldsymbol{M}(z)$ are zero. Thus, $\det\boldsymbol{M}(z) \equiv 1$. Since $S_2(\infty) = 1$ and

$$S_2(z) = [\boldsymbol{M}(z)]_{2,2}(z)[\boldsymbol{M}(z)]_{1,1} - [\boldsymbol{M}(z)]_{1,2}[\boldsymbol{M}(z)]_{2,1} = \det\boldsymbol{M}(z) \equiv 1$$

for z on the real line by (A.1), the maximum principle for subharmonic functions implies that $S_2(z) \leqslant 1$ in $\overline{\mathbb{C}}_+$.

These considerations can also be applied to the matrix $\boldsymbol{M}(z)B(z)^{\boldsymbol{\sigma}_3}$, as it is still meromorphic with unit determinant. However, the roles of the columns are now reversed: the first one is analytic in $\mathbb{C}_+$ while the second one has poles therein. Thus, it now must hold that $(S_1|B|^2)(z) \leqslant 1$ in $\overline{\mathbb{C}_+}$. It readily follows from (A.1) and the residue conditions satisfied by $\boldsymbol{M}$ that

$$\begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = \lim_{z\to z_n}(z - z_n)\begin{bmatrix} [\boldsymbol{M}(z)]_{1,1} \\ [\boldsymbol{M}(z)]_{2,1} \end{bmatrix} = \gamma_n \begin{bmatrix} [\boldsymbol{M}(z_n)]_{1,2} \\ [\boldsymbol{M}(z_n)]_{2,2} \end{bmatrix}.$$

These relations now yield that

$$\sqrt{|\alpha_n|^2 + |\beta_n|^2} = \begin{cases} |\gamma_n|\sqrt{S_2(z_n)} \leqslant |\gamma_n|, \\ \lim_{z\to z_n}|z - z_n|\sqrt{S_1(z)} \leqslant \lim_{z\to z_n}|z - z_n||B(z)|^{-1} = |B'(z_n)|^{-1}. \end{cases}$$

Recalling (A.4), we obtain the first bound of the proposition using the first estimate above, while the second bound follows from the second estimate above. Finally, the last bound is a consequence of lemma A.2 and the second estimate above. □

Appendix B. Results from probability theory

In this appendix we show in details some passages for the proof of theorem 1.5, and we report two results from probability theory that we used for the proof of lemmas 3.3 and 3.4.

Lemma B.1. *Under assumption 1.1 and definition 3.1, the following estimates hold:*

- *If $\alpha > 0$ and $N > \exp\left(\frac{\nu^2}{2\alpha^2}\right)$, for all $s > 0$ it holds that*

$$\mathbb{P}\left(\max_{i=1,\dots,N} |\mu_i - \mu_{\mathcal{D}}| \geqslant 2\alpha\left(\ln(N) + s\right)\right) \leqslant 2\mathrm{e}^{-s}.$$

- *If $\alpha = 0$, for all $s > 0$ it holds that*

$$\mathbb{P}\left(\max_{i=1,\dots,N} |\mu_i - \mu_{\mathcal{D}}| \geqslant \nu\sqrt{2\left(\ln(N) + s\right)}\right) \leqslant 2\mathrm{e}^{-s}.$$

Proof. We prove only the first statement. The proof of the second one is analogous. Let $u = 2\alpha\left(\ln(N) + s\right)$. Then, from standard inequalities,

$$\mathbb{P}\left(\max_{i=1,\dots,N} |\mu_i - \mu_{\mathcal{D}}| \geqslant u\right) \leqslant \sum_{i=1}^{N} \mathbb{P}\left(|\mu_i - \mu_{\mathcal{D}}| \geqslant u\right) \leqslant N\mathbb{P}\left(|\mu_1 - \mu_{\mathcal{D}}| \geqslant u\right).$$

Since $N > \exp\left(\frac{\nu^2}{2\alpha^2}\right)$, we can apply (3.2) to conclude that

$$\mathbb{P}\left(\max_{i=1,\dots,N} |\mu_i - \mu_{\mathcal{D}}| \geqslant u\right) \leqslant 2N\exp\left(-\frac{u}{2\alpha}\right) = 2\mathrm{e}^{-s}.$$

□

From lemma B.1 it follows that the set Ω defined in (3.40) is exponentially small in probability as $N \to \infty$.

The next result is the so-called Nagaev–Guivarc'h method, which is a fundamental technique to prove probabilistic limit theorems for dynamical systems. We used the following theorem to prove lemma 3.3, which is part of the proof of theorem 1.5.

Theorem B.2 (Nagaev–Guivarc'h method, theorem 4.2 in [32]). *Let $X_1, X_2, \dots$ be a sequence of real random variables and let $S_N := \sum_{j=1}^{N} X_j$. Assume that there exists $\delta > 0$ and functions $\lambda(s,\xi) \in C^{1,0}([0,1) \times \mathbb{R})$, $c_N(\xi) \in C^0(\mathbb{R})$, and $h_n(\xi)$ continuous at zero, such that $\forall\, \xi \in [-\delta,\delta]$ and $\forall\, N \in \mathbb{N}$*

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi S_N}\right] = c_N(\xi)\left(\prod_{j=1}^{N}\lambda\left(\xi;\frac{j}{N}\right)\right)\left(1 + h_N(\xi)\right). \tag{B.1}$$

Moreover, assume that

1. *there exist functions $A, \sigma^2 : [0,1] \to \mathbb{C}$ such that*

$$\lambda(\xi;s) = \mathrm{e}^{-\mathrm{i}A(s)\xi - \frac{\sigma^2(s)}{2}\xi^2 + o\left(\xi^2\right)}, \qquad \text{as } \xi \to 0; \tag{B.2}$$

2. $c_N(0) = 1$ *and* $\lim_{N\to\infty} c_N\left(\frac{\xi}{\sqrt{N}}\right) = \lim_{\xi\to\infty} c_N\left(\frac{\xi}{N}\right) = 1$, $\forall \xi \in [-\delta,\delta]$;
3. $h_N \to 0$ *as* $N \to \infty$, *uniformly in* $[-\delta,\delta]$ *and* $h_N(0) = 0$.

Then, $\int_0^1 A(s)\mathrm{d}s \in \mathbb{R}$ *and* $\int_0^1 \sigma^2(s)\mathrm{d}s \geqslant 0$ *and*

$$\frac{S_N - N\int_0^1 A(s)\,\mathrm{d}s}{\sqrt{N}} \to \mathcal{N}\left(0, \int_0^1 \sigma^2(s)\,\mathrm{d}s\right), \qquad \text{as } N\to\infty, \text{in distribution.} \tag{B.3}$$

Finally, we used a classical probability theory result, known as Lyapounov's condition [6], to prove lemma 3.4, which is part of the proof of theorem 1.6.

Theorem B.3 (Lyapounov's condition). *Let* $X_1,\ldots,X_N$ *be independent random variables with means* $\mu_1,\ldots,\mu_N$ *and variances* $\sigma_1^2,\ldots,\sigma_N^2$*. Define* $s_N := \sqrt{\sum_{k=1}^N \sigma_k^2}$ *and assume that there exists a* $\delta > 0$ *such that*

$$\lim_{N\to\infty} \frac{1}{s_N^{2+\delta}} \sum_{k=1}^N \mathbb{E}\left[|X_k - \mu_k|^{2+\delta}\right] = 0\,. \tag{B.4}$$

Then

$$\frac{1}{s_N}\sum_{k=1}^N (X_k - \mu_k) \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as } N\to\infty\,. \tag{B.5}$$

Proposition B.4 (Law of large numbers). *Under the same assumption of theorem B.3, if* s_N *is asymptotically equal to* cN^α *for some* $c > 0$ *and* $0 < \alpha < 1$*, then it implies that*

$$\frac{1}{N}\sum_{k=1}^N X_k - \frac{1}{N}\sum_{k=1}^N \mu_k \xrightarrow{N\to\infty} 0 \tag{B.6}$$

in probability.

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