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Symplectic capacities of disc cotangent bundles of flat tori

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Abstract

We show that on the unit disc cotangent bundle of flat Riemannian tori, all normalized capacities coincide with twice the systole. The same result holds for flat, reversible Finsler tori and normalized capacities that are greater than or equal to the Hofer–Zehnder capacity.

1 Introduction

Given $n \in \mathbb{N}$, symplectic capacities are invariants of symplectic manifolds (M, ω) of dimension $2n$ used to study symplectic embeddings between such manifolds. As introduced by Ekeland and Hofer in [7], a (normalized) symplectic capacity is a function

$$c: \{\text{Symplectic manifolds of dimension } 2n\} \rightarrow [0, \infty]$$

satisfying the following properties:

Embedding If $\varphi: (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ is any symplectic embedding, then the inequality $c(M_1, \omega_1) \leq c(M_2, \omega_2)$ holds.

Scaling If $r > 0$, then $c(M, r\omega) = rc(M, \omega)$.

Normalization If ω_0 is the standard symplectic form on \mathbb{C}^n , then $c(B_r, \omega_0) = \pi r^2 = c(Z_r, \omega_0)$, when B_r is the open Euclidean ball of radius r and $Z_r = D_r \times \mathbb{C}^{n-1}$ is the infinite symplectic cylinder having the open disc $D_r \subset \mathbb{C}$ of radius r as base.

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The first condition yields obstructions to the existence of symplectic embeddings. In view of this, the third condition encodes the celebrated Gromov’s non-squeezing theorem: B_{r_1} can be symplectically embedded in Z_{r_2} if and only if $r_1 \leq r_2$, see [10].

For this reason, finding a function that satisfies the axioms of a capacity is a difficult problem. Nevertheless, a number of capacities have been discovered in the last forty years, see [6]. For instance, the Hofer–Zehnder capacity c_{HZ} is defined using compactly supported Hamiltonian systems on (M, ω) and has a special dynamical significance: if $c_{\text{HZ}}(M, \omega)$ is finite then every proper Hamiltonian $H: M \rightarrow \mathbb{R}$ admits periodic orbits on almost all energy levels. See [11, Ch. 4, Thm. 1] and [13] for a proof of this fact, and [17] for examples of closed manifolds having infinite Hofer–Zehnder capacity.

Despite these advances, it is a challenging task to compute capacities, even for simple classes of manifolds, or to show that two capacities coincide on these classes. For instance, the strong Viterbo conjecture asserts that all capacities coincide on the class of convex domains in \mathbb{C}^n [14, Section 14.9, Problem 53]. Another remarkable class of examples is given by canonical Finsler disc cotangent bundles $(D_F^*Q, \omega_{\text{can}})$. Here,

- Q is a closed manifold and

$$D_F^*Q = \{(q, p) \in T^*Q \mid F^*(q, p) < 1\},$$

where $F: TQ \rightarrow \mathbb{R}$ is a smooth Finsler norm on Q and $F^*: T^*Q \rightarrow \mathbb{R}$ denotes the dual norm

$$F^*(q, p) := \sup_{\substack{v \in T_q Q \\ F(q, v) < 1}} p \cdot v, \quad \forall (q, p) \in T^*Q$$

- $\omega_{\text{can}} = \sum_{i=1}^n dp_i \wedge dq_i$ is the canonical symplectic form.

Characteristics $(q, p): \mathbb{R} \rightarrow \partial(D_F^*Q)$ project to Finsler geodesics $q: \mathbb{R} \rightarrow Q$, that is, extremals of the Finsler length

$$\ell_F(q) := \int_a^b F(q(t), \dot{q}(t)) dt$$

at fixed endpoints. We denote by $\text{sys}(F)$ the Finsler length of the shortest non-contractible closed geodesic on Q . We call a Finsler norm reversible if $F(q, v) = F(q, -v)$ for all $(q, v) \in TQ$. The most fundamental examples of reversible Finsler norms are induced by Riemannian metrics g , that is, $F(q, v) := \sqrt{g_q(v, v)}$ for all $(q, v) \in TQ$.

Capacities of disc cotangent bundles are known only in some symmetric examples. The ball-capacity (also known as Gromov-width) of codisc bundles of S^2 and $\mathbb{R}P^2$ with respect to the round metric was computed by Ferreira and Ramos [8]. These results were also generalized to spheres of revolution in [9]. Furthermore, the Hofer–Zehnder capacity of disc cotangent bundles of $\mathbb{C}P^n$ with respect to the Fubini–Study metric and of $\mathbb{R}P^n$ with respect to the round metric was computed by the second author in [3]. Finally, in [12] Jiang

computed the value of any normalized capacity for the codisc bundle of the ℓ^1 -norm on the n -torus. It is this last result that we aim to generalize in the present paper.

Indeed, our main theorem computes the capacity of disc cotangent bundles of flat, reversible Finsler norms on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We recall that a Finsler norm is flat if, in the global trivialization $T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ induced by the quotient $\mathbb{R}^n \rightarrow \mathbb{T}^n$, there is $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with $F(q, v) = f(v)$ for all $(q, v) \in \mathbb{T}^n \times \mathbb{R}^n$. In this case $D_F^*\mathbb{T}^n = \mathbb{T}^n \times K^*$, where $K^* = \{p \in (\mathbb{R}^n)^* \mid f^*(p) < 1\}$.

Theorem 1.1. *If F is a flat, reversible Finsler norm on \mathbb{T}^n and c is a (normalized) symplectic capacity, then*

$$c(D_F^*\mathbb{T}^n, \omega_{\text{can}}) \leq 2\text{sys}(F).$$

If, in addition,

1. there is a Finsler norm G induced by a flat Riemannian metric with $F \geq G$ and $\text{sys}(F) = \text{sys}(G)$, or
2. $c \geq c_{\text{HZ}}$, where c_{HZ} denotes the Hofer–Zehnder capacity,

then

$$c(D_F^*\mathbb{T}^n, \omega_{\text{can}}) = 2\text{sys}(F).$$

Remark 1.2. In particular, $c(D_F^*\mathbb{T}^n, \omega_{\text{can}}) = 2\text{sys}(F)$ for all normalized capacities, if F is induced by a Riemannian metric or if F is an ℓ^p -norm with $p \leq 2$. As recalled above, the case $p = 1$ was proved in [12].

Before ending this introduction, let us mention that it would be interesting to extend the computation of capacities to the larger class of twisted disc cotangent bundles $(D_F^*Q, \omega_{\text{can}} + \pi^*\sigma)$, where $\pi: D_F^*Q \rightarrow Q$ is the footpoint projection, and σ is a closed two-form on Q . In this case, characteristics on $\partial(D_F^*Q)$ correspond to magnetic geodesics which describe the motion of a particle in the stationary magnetic field σ . We refer the reader to [4, 3, 2] for computations of the Hofer–Zehnder capacity for twisted disc cotangent bundles of hermitian symmetric spaces, where the norm and the magnetic field are induced by the associated canonical Kähler structure.

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2 Proof of Theorem 1.1

Let us identify \mathbb{T}^n with the quotient of \mathbb{R}^n by the standard lattice \mathbb{Z}^n . Geodesics of a flat Finsler norm F are straight lines and therefore

$$s := \text{sys}(F) = \min\{f(v) \mid v \in \mathbb{Z}^n, v \neq 0\}, \quad (1)$$

where $F(q, v) = f(v)$ for all $(q, v) \in \mathbb{T}^n \times \mathbb{R}^n$. We write $K = \{v \in \mathbb{R}^n \mid f(v) < 1\}$ and $K^* = \{p \in (\mathbb{R}^n)^* \mid f^*(p) < 1\}$, where f^* is the dual Finsler norm, which, by reversibility, has the formula $f^*(p) = \sup_{v \in K} |p \cdot v|$.

2.1 Upper bound

Let e_1, \dots, e_n be the standard basis of \mathbb{Z}^n . Let $u = (k_1, \dots, k_n) \in \mathbb{Z}^n$ be such that

$$f(u) = \min\{f(v) \mid v \in \mathbb{Z}^n, v \neq 0\} = s.$$

In particular, the numbers k_1, \dots, k_n are coprime and, therefore, there exists a linear isomorphism $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ represented by a matrix in $SL(n, \mathbb{Z})$ such that $A(e_1) = u$. This isomorphism induces a diffeomorphism $[A]: \mathbb{T}^n \rightarrow \mathbb{T}^n$ which sends the Finsler metric F_A associated with the function $f_A := f \circ A$ to the Finsler metric F . Thus

$$\text{sys}(F_A) = \text{sys}(F) = s \quad \text{and} \quad f_A(e_1) = f(u) = s.$$

The cotangent lift $T^*[A]: T^*\mathbb{T}^n \rightarrow T^*\mathbb{T}^n$, which has the formula $T^*[A] := [A] \times (A^{-1})^*$ under the identification $T^*\mathbb{T}^n \cong \mathbb{T}^n \times (\mathbb{R}^n)^*$, is a symplectic automorphism of $(T^*\mathbb{T}^n, \omega_{\text{can}})$ and

$$T^*[A](D_{F_A}^* \mathbb{T}^n) = D_F^* \mathbb{T}^n.$$

In particular, $c(D_{F_A}^* \mathbb{T}^n, \omega_{\text{can}}) = c(D_F^* \mathbb{T}^n, \omega_{\text{can}})$. Up to relabeling f_A and F_A with f and F , this shows that it is enough to prove the upper bound under the assumption that $f(e_1) = s$.

Every $p \in (\mathbb{R}^n)^*$ has coordinates $p_1 := p \cdot e_1, \dots, p_n := p \cdot e_n$. Using these coordinates and reordering the factors, we get a symplectomorphism

$$\Psi: (T^*\mathbb{T}^n, \omega_{\text{can}}) \rightarrow (\mathbb{T} \times \mathbb{R}, \omega_{\text{can}})^n, \quad \Psi(q, p) := ((q_1, p_1), \dots, (q_n, p_n)).$$

Since $(f^*)^* = f$, we have for all $k = 1, \dots, n$

$$s_k := f(e_k) = \sup_{p \in K^*} |p \cdot e_k| = \sup_{p \in K^*} |p_k|.$$

Therefore, we obtain

$$\Psi(D_F^* \mathbb{T}^n) \subset (\mathbb{T} \times (-s_1, s_1)) \times \dots \times (\mathbb{T} \times (-s_n, s_n)). \quad (2)$$

For all $k = 1, \dots, n$, let $r_k := \sqrt{\frac{2}{\pi}} s_k$ and observe that the maps

$$\phi_k: (\mathbb{T} \times (-s_k, s_k), \omega_{\text{can}}) \rightarrow (D_{r_k}, \omega_0), \quad \phi_k(q, p) = \sqrt{\frac{p + s_k}{\pi}} e^{2\pi i q}$$

are symplectic embeddings [11, Ch. 4.4, Prop. 4]. By (2), we get a symplectic embedding

$$(\phi_1 \times \dots \times \phi_n) \circ \Psi: (D_F^* \mathbb{T}^n, \omega_{\text{can}}) \rightarrow (Z_{r_1}, \omega_0).$$

Hence,

$$c(D_F^* \mathbb{T}^n, \omega_{\text{can}}) \leq \pi r_1^2 = 2s_1 = 2s,$$

where the last equality follows from the additional assumption $s_1 = f(e_1) = s$.

2.2 Lower bound

Denote by $\pi: \mathbb{R}^n \rightarrow \mathbb{T}^n$ the quotient map. We claim that π is injective on $\frac{s}{2}K$. Indeed, if $u_1, u_2 \in \frac{s}{2}K$ and $\pi(u_1) = \pi(u_2)$, then there is $v \in \mathbb{Z}^n$ such that $u_2 - u_1 = v$. Now $-u_1 \in \frac{s}{2}K$ by reversibility and, thus, $\frac{u_2 + (-u_1)}{2} = \frac{1}{2}v \in \frac{s}{2}K$ by convexity. Thus, $v \in sK \cap \mathbb{Z}^n$, which implies that $v = 0$ by definition (1), and the claim is proved. By the claim, we see that $(\frac{s}{2}K) \times K^* \subset D_F^* \mathbb{T}^n$. We get

$$c(D_F^* \mathbb{T}^n, \omega_{\text{can}}) \geq c((\frac{s}{2}K) \times K^*, \omega_{\text{can}}) = c(K \times K^*, \frac{s}{2}\omega_{\text{can}}) = \frac{s}{2}c(K \times K^*, \omega_{\text{can}}). \quad (3)$$

Case 1

Suppose that there is a Finsler norm G induced by a flat Riemannian metric such that $F \geq G$ and $\text{sys}(G) = \text{sys}(F) = s$. In particular, $D_G^* \mathbb{T}^n \subset D_F^* \mathbb{T}^n$ and, therefore, $c(D_F^* \mathbb{T}^n, \omega_{\text{can}}) \geq c(D_G^* \mathbb{T}^n, \omega_{\text{can}})$. Thus, it is enough to prove the lower bound for G , or in other words, we can assume without loss of generality that F is induced by a flat Riemannian metric. In this case, K is an ellipsoid and there is a linear map $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $K = A(B_1)$. The map $A \times (A^{-1})^*$ is a symplectomorphism of $(\mathbb{R}^n \times (\mathbb{R}^n)^*, \omega_{\text{can}})$ and therefore $c(K \times K^*, \omega_{\text{can}}) = c(B_1 \times B_1^*, \omega_{\text{can}})$. By an unpublished argument of Yaron Ostrover and Felix Schlenk [16, Theorem 12.14(i)], for every $s \in (0, r)$, where $r := \sqrt{\frac{4}{\pi}}$, there is a symplectic embedding

$$\Phi_s: (B_s, \omega_0) \rightarrow (B_1 \times B_1^*, \omega_{\text{can}}).$$

This implies that $c(B_1 \times B_1^*, \omega_{\text{can}}) \geq \pi s^2$. Taking the supremum in s , we get

$$c(B_1 \times B_1^*, \omega_{\text{can}}) \geq \pi r^2 = 4.$$

Using (3), we arrive at the desired lower bound $c(D_F^* \mathbb{T}^n, \omega_{\text{can}}) \geq \frac{s}{2}4 = 2s$.

Remark 2.1. Although we do not need it here, we observe that there is also a symplectic embedding $\Phi_r: (B_r, \omega_0) \rightarrow (B_1 \times B_1^*, \omega_{\text{can}})$, see [5, Lemma 3.7] and, for the case $n = 2$, [15].

Remark 2.2. The inequality $c(B_1 \times B_1^*, \omega_{\text{can}}) \leq 4$ is also true by [1, Remark 4.2].

Case 2

Suppose that $c \geq c_{\text{HZ}}$. We have

$$c(K \times K^*, \omega_{\text{can}}) \geq c_{\text{HZ}}(K \times K^*, \omega_{\text{can}}) = 4$$

by [1, Theorem 1.7]. Using (3), we get $c(D_F^* \mathbb{T}^n, \omega_{\text{can}}) \geq \frac{s}{2}4 = 2s$.

Remark 2.3. Actually, for every normalized capacity c , the inequality $c(K \times K^*, \omega_{\text{can}}) \leq 4$ is also true by [1, Remark 4.2], although we do not need this here.

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