Paralinearization and extended lifespan for solutions of the α-SQG sharp front equation

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Abstract

In this paper we paralinearize the contour dynamics equation for sharp-fronts of α-SQG, for any $\alpha \in (0,1)\cup(1,2)$, close to a circular vortex. This turns out to be a quasi-linear Hamiltonian PDE. After deriving the asymptotic expansion of the linear frequencies of oscillations at the vortex disk and verifying the absence of three wave interactions, we prove that, in the most singular cases $\alpha \in (1,2)$, any initial vortex patch which is ε-close to the disk exists for a time interval of size at least $\sim \varepsilon^{-2}$. This quadratic lifespan result relies on a paradifferential Birkhoff normal form reduction and exploits cancellations arising from the Hamiltonian nature of the equation. This is the first normal form long time existence result of sharp fronts.

Keywords: α-SQG equations, vortex patches, paradifferential calculus, Birkhoff normal form.

Contents

1 Introduction and main results

In this paper we consider the generalized surface quasi-geostrophic α-SQG equations

$$\partial_t \theta(t,\zeta) + u(t,\zeta)\cdot\nabla\theta(t,\zeta) = 0\,, \quad (t,\zeta)\in\mathbb{R}\times\mathbb{R}^2\,, \tag{1.1}$$

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with velocity field

$$u := \nabla^{\perp} |D|^{-2+\alpha}\theta\,, \qquad |D| := (-\Delta)^{\frac{1}{2}}\,, \qquad \alpha \in (0,2)\,. \tag{1.2}$$

These class of active scalar equations have been introduced in [28, 68] and, for $\alpha \to 0$, formally reduce to the 2D-Euler equation in vorticity formulation (in this case θ is the vorticity of the fluid). The case $\alpha = 1$ is the surface quasi-geostrophic (SQG) equation in [23] which models the evolution of the temperature θ for atmospheric and oceanic flows.

For the 2D Euler equation global-in-time well-posedness results are well known for either regular initial data, see e.g. [22,65], as well as for $L^1 \cap L^\infty$ initial vorticities, thanks to the celebrated Yudovich Theorem [77]. This result is based on the fact that the vorticity is transported by the particles of fluid along the velocity field, which turns out to be log-Lipschitz, and thus it defines a global flow on the plane. On the other hand, for $\alpha > 0$, an analogous result does not hold because the velocity field u in (1.2) is more singular and does not define a flow. Nevertheless local in time smooth solutions exist thanks to a nonlinear commutator structure of the vector field for $\alpha = 1$, in [24], and for $\alpha \in (1,2)$, in [21]. For $\alpha = 1$, the works [26,27] rule out the possible formation of certain kind of singularities but the question of whether a finite-time singularity may develop from a smooth initial datum remains open. In this context we mention the construction in [45] of solutions that must either exhibit infinite in time growth of derivatives or blow up in finite time.

Existence of global weak L^p solutions has been obtained by energy methods for $\alpha = 1$, if $p > 4/3$, in [66, 70], and for $\alpha \in (1,2)$ if $p = 2$, in [21]. For $\alpha \in (0,1]$ global weak solutions exist also in $L^1 \cap L^2$ as proved in [63]. We also mention that non-unique weak solutions of SQG have been constructed by convex integration techniques in [14, 58].

A particular type of weak solutions are the *vortex patches* -also called *sharp fronts*- which are given by the characteristic function of an evolving domain

$$\theta(t,\zeta) := \begin{cases} 1 & \text{if } \zeta \in D(t)\,, \\ 0 & \text{if } \zeta \notin D(t)\,. \end{cases} \qquad D(t) \subset \mathbb{R}^2\,. \tag{1.3}$$

The vortex patch problem (1.3) can be described by the evolution of the interface $\partial D(t)$ only. The simplest example of a finite energy vortex patch is the circular “Rankine" vortex which is the circular steady solution with $D(t) = D(0) = \{|\zeta| \le 1\}$ at any time t. On the other hand, since for $\alpha \in (0,2)$ there is no analogue of Yudovich theorem, also to establish the local existence theory for sharp fronts nearby is a difficult task. In the last few years special global in time sharp-front solutions of α-SQG close to the Rankine vortex have been constructed: the uniformly rotating V-states in [16, 17, 40, 43], as well as time quasi-periodic solutions in [42] for $\alpha \in \left(0, \frac{1}{2}\right)$, and in [41] for $\alpha \in (1,2)$. We quote further literature after the statement of Theorem 1.1.

In this work we prove the first long-time existence result of sharp fronts of α-SQG, in the more singular cases $\alpha \in (1,2)$, for any initial interface $\partial D(0)$ sufficiently smooth and close to a circular Rankine vortex, see Theorem 1.1. This is achieved thanks to the paralinearization result of the α-SQG sharp front equation in Theorem 4.1 for any $\alpha \in (0,1) \cup (1,2)$, that we consider of independent interest in itself.

Let us present precisely our main results. The evolution of the boundary of the vortex patch is governed by the *Contour Dynamics Equation* for a parametrization $X : \mathbb{T} \to \mathbb{R}^2$, $x \mapsto X(t,x)$, with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, of the boundary $\partial D(t)$ of the vortex patch. The Contour Dynamics Equation for the α-SQG patch –also called sharp-fronts equation- is

$$\partial_t X(t,x) = \frac{c_\alpha}{2\pi} \int \frac{X'(t,x) - X'\left(t,y\right)}{\left|X(t,x) - X\left(t,y\right)\right|^{\alpha}} \, \mathrm{d}y\,, \quad \alpha \in (0,2)\,, \tag{1.4}$$

where $'$ denotes the derivative with respect to x,

$$c_\alpha := \frac{\Gamma\left(\frac{\alpha}{2}\right)}{2^{1-\alpha}\Gamma\left(1 - \frac{\alpha}{2}\right)} \tag{1.5}$$

and $\Gamma(\cdot)$ is the Euler-Gamma function. The local solvability of Equation (1.4) in Sobolev class has been proved in [36] for $\alpha \in (0,1]$, if the initial datum belongs to H^s, $s \ge 3$ and in [37, 38] for less regular initial data (see [71] for C^∞ data). The uniqueness has been established in [25]. For $\alpha \in (1,2)$ the local existence and uniqueness theory has been proved in [21, 38] for initial data in H^s, $s \ge 4$, see also [1, 62]. In the very recent work [59] it is proved that the α-patch problem is ill posed in $W^{k,p}$ if $p \ne 2$.

Very little is known concerning long time existence results. Actually highly unstable dynamical behaviour could emerge. In this context we mention the remarkable work [61] where two smooth patches of opposite sign develop a finite time particle collision. We also quote the numerical study [73] which provides some evidence of the development of filaments, pointing to a possible formation of singularities via a self-similar filament cascade.

In this paper we consider sharp fronts of α-SQG that are a radial perturbation of the unitary circle, i.e.

$$X(x) = (1 + h(x))\,\vec{\gamma}(x)\,, \qquad \vec{\gamma}(x) := (\cos(x), \sin(x))\,. \tag{1.6}$$

Since only the normal component of the velocity field deforms the patch, one derives from (1.4) a scalar evolution equation for $h(x)$. Multiplying (1.4) by the normal vector $n(x) = h'(x)\vec{\gamma}'(x) - (1 + h(x))\,\vec{\gamma}(x)$ to the boundary of the patch at $X(x)$, we deduce that $h(t,x)$ solves the equation

$$\begin{aligned} -(1+h(x))\,\partial_t h(x) = {}& \frac{c_\alpha}{2\pi} \int \frac{\cos(x-y)\left[(1+h(x))\,h'(y) - (1+h(y))\,h'(x)\right]}{\left[(1+h(x))^2 + (1+h(y))^2 - 2(1+h(x))(1+h(y))\cos(x-y)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}y \\ &+ \frac{c_\alpha}{2\pi} \int \frac{\sin(x-y)\left[(1+h(x))(1+h(y)) + h'(x)\,h'(y)\right]}{\left[(1+h(x))^2 + (1+h(y))^2 - 2(1+h(x))(1+h(y))\cos(x-y)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}y\,. \end{aligned} \tag{1.7}$$

In view of [21,38] if $h_0 \in H^s$, for any $s \geq 4$, there exists a unique solution $h \in \mathcal{C}([0,T]; H^s)$ of (1.7) defined up to a time $T > \frac{1}{C_s\,\|h_0\|_{H^s}}$. The following result extends the local-existence result for longer times.

Theorem 1.1 (Quadratic life-span)**.** *Let $\alpha \in (1,2)$. There exists $s_0 > 0$ such that for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $c_{s,\alpha} > 0$, $C_{s,\alpha} > 0$ such that, for any h_0 in $H^s(\mathbb{T};\mathbb{R})$ satisfying $\|h_0\|_{H^s} \leq \varepsilon < \varepsilon_0$, the equation* (1.7) *with initial condition $h(0) = h_0$ has a unique classical solution*

$$h \in \mathcal{C}\left(\left[-T_{s,\alpha}, T_{s,\alpha}\right]; H^s(\mathbb{T};\mathbb{R})\right) \qquad \textit{with} \qquad T_{s,\alpha} > c_{s,\alpha}\varepsilon^{-2}\,, \tag{1.8}$$

satisfying $\|h(t)\|_{H^s} \leq C_{s,\alpha}\,\varepsilon$, for any $t \in \left[-T_{s,\alpha}, T_{s,\alpha}\right]$.

Theorem 1.1 is proved by normal form arguments for quasi-linear Hamiltonian PDEs. The first important step is the *paralinearization* of (1.7) once it has been written in Hamiltonian form, see Theorem 4.1. The paralinearization formula (4.1) of the α-SQG equations holds for any $\alpha \in (0,1) \cup (1,2)$. It is a major result of this paper, that we expect to be used also in other contexts.

In order to prove Theorem 1.1 we reduce the paralinearized equation (4.1), for any $\alpha \in (1,2)$, to Birkhoff normal form up to cubic smoothing terms. This requires to prove the absence of *three wave interactions*, which is verified in Lemma 3.5 by showing the convexity of the linear normal frequencies of the α-SQG equation at the circular vortex patch.

Theorem 1.1 is the first Birkhoff normal form results for sharp fronts equations.

In recent years several advances have been obtained concerning long time existence of solutions for quasi-linear equations in fluids dynamics on $\mathbb{T}$, namely with periodic boundary conditions, as the water waves equations. Quadratic life span of small amplitude solutions have been obtained in [2,6,51,52,55–57, 76], extended to longer times in [5–7,10,12,34,75,78], by either introducing quasi-linear modified energies or using Birkhoff normal form techniques. We also quote the long time existence result [20] for solutions of SQG close to the infinite energy radial solution $|\zeta|$.

Before explaining in detail the main ideas of proof we present further results in literature about α-SQG.

Further literature. Special infinite energy global-in-time sharp front solutions have been constructed in [29, 53,54] if the initial patch is a small perturbation of the half-space, by exploiting dispersive techniques. In [18, 19] special global smooth solutions in the cases $\alpha = 0,1$ are obtained using bifurcation theory. Concerning the possible formation of singularities, we mention that [61,62] constructed special initial sharp fronts in the half-space which develop a splash singularity in finite time if $\alpha \in \left(0, \frac{1}{12}\right)$, later extended in [38] for $\alpha \in \left(0, \frac{1}{3}\right)$. We also mention that [39,60] have proved that, if $\alpha \in (0,1]$, the sharp fronts equation in the whole space does not generate finite-time singularities of splash type.

V-states. The existence of uniformly rotating V-states close to the disk was first numerically investigated in [33] and analytically proved in [15] for the Euler equations, recently extended to global branches in [44].

For α-SQG equations, as already mentioned, local bifurcation results of sharp fronts from the disk have been obtained in [16–19,40,43]. Smooth V-states bifurcating from different steady configurations have been constructed for $\alpha \in [0,2)$ in [30–32, 46–50, 69]. We refer to the introductions in [41, 42] for more references.

Quasi-periodic solutions. As already mentioned global in time quasi-periodic solutions of the α-SQG vortex patch equation close to the circle (1.6) have been recently constructed in [42] for $\alpha \in \left(0, \frac{1}{2}\right)$ and in [41] for $\alpha \in (1,2)$. The result [42] holds for "most" values of $\alpha \in \left(0, \frac{1}{2}\right)$ (used a parameter to impose non-resonance conditions) whereas [41] holds for any $\alpha \in (1,2)$, using the initial conditions as parameters, via a Birkhoff normal form analysis. The $2D$-Euler equation is more degenerate and in this case quasi-periodic solutions have been constructed in [9] close to the family of Kirkhoff ellipses, not only close to the disk (we refer to [9] for a wider introduction to the field and literature about quasi-periodic solutions). These results build on on KAM techniques [3, 4, 8, 13, 35] developed for the water waves equations.

Ideas of the proof

The average-preserving unknown and the Hamiltonian formulation. The equation (1.7) for the unknown $h(x)$ is *not* convenient because its evolution does not preserve the average and it is *not* Hamiltonian. This problem is overcome by reformulating (1.7) in terms of the variable

$$f(x) := h(x) + \tfrac{1}{2} h^2(x) . \tag{1.9}$$

Indeed, symmetrizing in the x, y variables, we get $\int_{\mathbb{T}}$ r.h.s. (1.7) $\mathrm{d}y = 0$ and then $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \left(h(t,x) + \frac{1}{2} h^2(t,x)\right) \mathrm{d}x = 0$. Thus the average of $f(x)$ in (1.9) is preserved along the patch evolution. Note that, inverting (1.9) for small $\|f\|_{L^\infty}$ and $\|h\|_{L^\infty}$, we have $h(x) = \sqrt{1+2f(x)} - 1$ and the Sobolev norms of $f(x)$ and $h(x)$ are equivalent

$$\|f\|_s \sim \|h\|_s , \qquad \forall s > \tfrac{1}{2} . \tag{1.10}$$

Remark 1.2. There is a deep connection between the conservation of the average of $f(x)$ and the incompressibility of the flow generated by the α-SQG patch. Actually the Lebesgue measure $\mathrm{Vol}(t)$ of the finite region of $\mathbb{R}^2$ enclosed by the patch is, passing to polar coordinates,

$$\mathrm{Vol}(t) = \int_{-\pi}^{\pi} \int_0^{1+h(t,x)} \rho \, \mathrm{d}\rho \, \mathrm{d}x = \pi + \int_{-\pi}^{\pi} \left(h(t,x) + \frac{h^2(t,x)}{2} \right) \mathrm{d}x = \pi + \int_{-\pi}^{\pi} f(t,x) \, \mathrm{d}x$$

and therefore the conservation of the average $\int_{\mathbb{T}} f(x) \, \mathrm{d}x$ amounts to the conservation of $\mathrm{Vol}(t)$.

The variable (1.9) has been used in [42] where it is also proved that the evolution equation for f has a Hamiltonian structure, see also [41]. The following result is [42, Proposition 2.1]:

Proposition 1.3 (Hamiltonian formulation of (1.7))**.** *Let $\alpha \in (0,2)$. If h is a solution of Eq.* (1.7) *then the variable f defined in* (1.9) *solves the Hamiltonian equation*

$$\partial_t f = \partial_x \nabla E_\alpha(f) \tag{1.11}$$

where $E_\alpha(f)$ is the pseudo-energy of the patch whose L^2-gradient $\nabla E_\alpha(f)$ is

$$\nabla E_\alpha(f) = \frac{c_\alpha}{2\left(1 - \frac{\alpha}{2}\right)} \int \frac{1 + 2f(y) + \sqrt{1+2f(x)}\, \partial_y \left[\sqrt{1+2f(y)} \sin(x-y)\right]}{\left[1 + 2f(x) + 1 + 2f(y) - 2\sqrt{1+2f(x)}\sqrt{1+2f(y)} \cos(x-y)\right]^{\frac{\alpha}{2}}} \, \mathrm{d}y . \tag{1.12}$$

Note that the evolution equation (1.11) is translation-invariant because $E_\alpha \circ \mathrm{t}_\varsigma = E_\alpha$ for any $\varsigma \in \mathbb{R}$, where $\mathrm{t}_\varsigma f(x) := f(x+\varsigma)$. Moreover, in view of the presence of the Poisson tensor ∂_x in (1.11) it is evident that the space average of f is a prime integral of (1.11). In the sequel we assume the space average of f to be zero.

Paralinearization of (1.11) **for** $\alpha \in (0,1) \cup (1,2)$**.** Section 4 is dedicated to write the Hamiltonian equation (1.11) in paradifferential form and to provide the detailed structure of the principal and subprincipal symbols in the expansion of the paradifferential operator, obtaining

$$\partial_t f + \partial_x \circ \mathrm{Op}^{BW}\left[\left(1 + \nu(f;x)\right) L_\alpha(|\xi|) + V(f;x) + P(f;x,\xi)\right] f = \text{smoothing terms} \tag{1.13}$$

where (see Theorem 4.1 for a detailed statement)

- $\left(1+\nu\left(f;x\right)\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)$ is a real symbol of order $\max\{\alpha-1,0\}$ and $\nu\left(f;x\right)$, $V\left(f;x\right)$ are real functions vanishing for $f=0$;
- the symbol $P\left(f;x,\xi\right)$ has order -1 and vanishes for $f=0$.

We note that in (1.13) the operator $\mathrm{Op}^{BW}[\,]$ is the paradifferential quantization according to Weyl (see Definition 2.10) and thus $\mathrm{Op}^{BW}\left[1+\nu\left(f;x\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)\right]$ is self-adjoint. As a consequence the linear Hamiltonian operator $\partial_x\circ\mathrm{Op}^{BW}\left[1+\nu\left(f;x\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)\right]$ is skew-self-adjoint at positive orders. This is the reason why the unbounded quasi-linear vector field $\partial_x\circ\mathrm{Op}^{BW}\left[1+\nu\left(f;x\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)\right]f$ admits energy estimates in Sobolev spaces H^s via commutator estimates (actually existence and unicity of the solutions of (1.13) would follow as in [11]). We remark the absence in (1.13) of operators like $\partial_x\circ\mathrm{Op}^{BW}\left[\text{symbol of order }(\alpha-2)\right]$. The cancellations of such terms are verified in Appendix A by a direct calculus and it is ultimately a consequence of the Hamiltonian structure of the equation (1.11).

We also note that the equation (1.13) can be written, in homogeneity degrees, as

$$\partial_t f+\omega_\alpha(D)f=\mathcal{O}\left(f^2\right)\qquad\text{where}\qquad\omega_\alpha(D):=\partial_x\circ L_\alpha\left(|D|\right)\tag{1.14}$$

is the unperturbed dispersion relation.

Let us explain how we deduce the paralinearization formula (1.13) in Section 4. The nonlinear term $\nabla E_\alpha\left(f\right)$ in (1.11) can be written as a convolution operator

$$\nabla E_\alpha\left(f\right)(x)=\int_{-\pi}^{\pi}K\left(f;x,z\right)\frac{f(x)-f(x-z)}{|z|^\alpha}\,\mathrm{d}z$$

with a nonlinear real valued convolution kernel $K\left(f;x,z\right)$. By Taylor expanding the kernel $z\mapsto K\left(f;x,z\right)$ at $z=0$ (provided f is sufficiently regular) and expanding in paraproducts the arguments of the above integral, we obtain an expansion of the form

$$\nabla E_\alpha\left(f\right)(x)=\sum_{\mathrm{j}=0}^{\mathrm{J}}\mathrm{Op}^{BW}\left[K_{\mathrm{j}}\left(f,\dots,f^{(\mathrm{j}+1)};x\right)\right]\int_{-\pi}^{\pi}\left(f(x)-f(x-z)\right)\frac{z^{\mathrm{j}}}{|z|^\alpha}\,\mathrm{d}z\tag{1.15a}$$

$$+\int_{-\pi}^{\pi}\mathrm{Op}^{BW}\left[R\left(f,\dots,f^{(\mathrm{j}+1)};x,z\right)\right]\left(f(x)-f(x-z)\right)\mathrm{d}z+\text{ smoothing terms},\tag{1.15b}$$

where $R\left(f,\dots,f^{(\mathrm{j}+1)};x,z\right)=o\left(|z|^{\mathrm{J}-\alpha}\right)$ as $z\to 0$ being the Taylor remainder at order J (here $f^{(\mathrm{j})}(x)$ denotes the j-derivative of $f(x)$). The terms in the finite sum (1.15a) are particularly simple paradifferential operators. Indeed, provided $\alpha<2$,

$$\int_{-\pi}^{\pi}\left(f(x)-f(x-z)\right)\frac{z^{\mathrm{j}}}{|z|^\alpha}\,\mathrm{d}z=\mathbb{V}_{\alpha-\mathrm{j}}f+m_{\alpha-(\mathrm{j}+1)}(D)\,f$$

where $\mathbb{V}_{\alpha-\mathrm{j}}$ is a real constant and $m_{\alpha-(\mathrm{j}+1)}(\xi)$ is a Fourier multiplier of order $\alpha-\left(\mathrm{j}+1\right)$, as follows by standard asymptotics of singular integral operators, see [74]. Thus, by symbolic calculus,

$$\mathrm{Op}^{BW}\left[K_{\mathrm{j}}\right]\int_{-\pi}^{\pi}\left(f(x)-f(x-z)\right)\frac{z^{\mathrm{j}}}{|z|^\alpha}\,\mathrm{d}z=\mathrm{Op}^{BW}\left[V_{\alpha-\mathrm{j}}\left(f,\dots,f^{(\mathrm{j}+1)};x\right)+K_{\mathrm{j}}\left(f,\dots,f^{(\mathrm{j}+1)};x\right)m_{\alpha-(\mathrm{j}+1)}(\xi)\right]f+\text{l.o.t.},$$

where $V_{\alpha-\mathrm{j}}$ are real functions. The unbounded terms $\partial_x\circ\mathrm{Op}^{BW}\left[K_{\mathrm{j}}\left(f,\dots,f^{(\mathrm{j}+1)};x\right)m_{\alpha-(\mathrm{j}+1)}(\xi)\right]f$, $\mathrm{j}=0,1$, would induces a loss of derivatives in the H^s energy estimates if the imaginary part $\mathrm{Im}\,m_{\alpha-(\mathrm{j}+1)}(\xi)\neq 0$. Therefore a detailed analysis of these symbols is essential. The highest order Fourier multiplier $m_{\alpha-1}(\xi)$ turns out to be real. Concerning the subprincipal symbol $K_1\left(f,f';x\right)m_{\alpha-2}(\xi)$, it turns out that $m_{\alpha-2}(\xi)$ has a non-zero imaginary part but a subtle nonlinear cancellation reveals that the corresponding coefficient $K_1\left(f,f';x\right)$ is identically zero, as verified in Appendix A. Such a structure, which ultimately stems by the Hamiltonian nature of (1.11), could be proven up to an arbitrary negative order.

Concerning the first term in (1.15b), we use that $R\left(f,\dots,f^{(\mathrm{j}+1)};x,z\right)$ is $o\left(|z|^{\mathrm{J}-\alpha}\right)$ as $z\to 0$ so that, modulo regularizing operators, it can be expressed as a paradifferential operator of order $\alpha-(\mathrm{J}+1)$, which is a bounded vector field taking $\mathrm{J}\geq 2$, see Proposition 2.36.

Reduction of (1.13) **to Birkhoff normal form up to cubic terms.** In Section 5 we first conjugate the paradifferential equation (1.13) into an equation with constant coefficient symbols, modulo smoothing operators,

$$\partial_t g + \partial_x \circ \mathrm{Op}^{BW}\left[\left(1+\mathtt{c}_0\left(f\right)\right)L_\alpha\left(|\xi|\right)+\mathsf{H}_\alpha\left(f;\xi\right)\right]g = \text{smoothing terms} \tag{1.16}$$

where $\mathtt{c}_0\left(f\right)$ is the average of a real nonlinear function of $\nu(f;x)$ and $\mathsf{H}_\alpha\left(f;\xi\right)$ is a x-independent symbol with imaginary part $\mathrm{Im}\mathsf{H}_\alpha\left(f;\xi\right)$ of order -1, see Proposition 5.2. Thus (1.16) is still Hamiltonian up to order zero and thus it satisfies H^s-energy estimates. The unknowns $g\left(t\right)$ and $f\left(t\right)$ have equivalent Sobolev norms $\left\|g\left(t\right)\right\|_s \sim_{s,\alpha} \left\|f\left(t\right)\right\|_s$. We remark that in (1.16) the constant $\mathtt{c}_0\left(f\right)$ and the symbol $\mathsf{H}_\alpha\left(f;\xi\right)$ vanish quadratically at $f=0$ and thus the only term which can disturb the quadratic life span of the solution $g\left(t\right)$ is the smoothing operator $R_1\left(f\right)$ in the decomposition

$$\text{smoothing terms} = R_1\left(f\right)g + R_{\geq 2}\left(f\right)g\,.$$

Then in Lemma 5.7 we implement a Birkhoff normal form step to cancel $R_1\left(f\right)g$. An algebraic ingredient is to verify the absence of three wave interactions, namely that, for any $n,j,k\in\mathbb{Z}\setminus\{0\}$ satisfying $k=j+n$,

$$\left|\omega_\alpha\left(k\right)-\omega_\alpha\left(j\right)-\omega_\alpha\left(n\right)\right|\geq c>0\,,$$

where $\omega_\alpha\left(j\right)$ are the normal α-SQG frequencies in (1.14). Such a property follows by proving the *convexity* of the the map $\omega_\alpha\left(j\right)$ for $j\in\mathbb{N}$, see Lemma 3.5.

The final outcome is an *energy estimate* for any small enough solution of (1.11) of the form

$$\left\|f\left(t\right)\right\|_{H^s}^2 \lesssim_{s,\alpha} \left\|f\left(0\right)\right\|_{H^s}^2 + \int_0^t \left\|f\left(\tau\right)\right\|_{H^s}^4\,\mathrm{d}\tau\,, \qquad t>0\,,$$

which implies Theorem 1.1.

Structure of the manuscript. Section 2 contains the paradifferential calculus used along the paper. In Section 2.1 we report the main results in [5,10]. Then in Section 2.2 we introduce a z-dependent paradifferential calculus used for the paralinearization of (1.13) in Section 4. Section 3 is dedicated to the linearization of (1.11) at the stationary state $f\equiv 0$. Lemmas 3.1 and 3.6 extend to any $\alpha\in(0,2)$ the asymptotic expansions of the normal frequencies $\omega_\alpha\left(j\right)$ proved in [42] for $\alpha\in(0,1)$. In Section 4 we provide the paralinearization (1.13) of the Hamiltonian equation (1.11) for any $\alpha\in(0,1)\cup(1,2)$. In Section 5 we conjugate the paradifferential equation (1.13) into an equation with constant coefficients, modulo smoothing operators. In Section 5 we perform the Birkhoff normal form step and prove Theorem 1.1.

Notation. We denote with C a positive constant which does not depend on any parameter of the problem. We write $A\lesssim_{c_1,\ldots,c_M} B$ if $A\leq C\left(c_1,\ldots,c_M\right)B$ and $A\sim_{c_1,\ldots,c_M} B$ if $A\lesssim_{c_1,\ldots,c_M} B$ and $B\lesssim_{c_1,\ldots,c_M} A$. We denote with $\mathbb{N}=1,2,\ldots$ the set of natural numbers and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. For any $x\geq 0$ we denote $\lceil x\rceil:=\min\{n\in\mathbb{N}_0 \mid x\leq n\}$. We denote $\mathbb{T}:=\mathbb{R}\setminus(2\pi\mathbb{Z})$ the one-dimensional torus with norm $|x|_{\mathbb{T}}:=\inf_{j\in\mathbb{Z}}\left|x+2\pi j\right|$. We denote $D=-\mathrm{i}\partial_x$ and $[\![A,B]\!]$ the commutator $[\![A,B]\!]=AB-BA=:\mathrm{Ad}_A B$. Given a linear real self-adjoint operator A any operator of the form $\partial_x\circ A$ will be referred as *linear Hamiltonian*. We denote $⨍\bullet\mathrm{d}x=\frac{1}{2\pi}\int_{\mathbb{T}}\bullet\mathrm{d}x$.

2 Functional setting

Along the paper we deal with real parameters

$$s\geq s_0\gg K\gg\rho\gg N\geq 0 \tag{2.1}$$

where $N\in\mathbb{N}$. The values of s,s_0,K and ρ may vary from line to line while still being true the relation (2.1). For the proof of Theorem 1.1 we shall take $N=1$.

We expand a 2π-periodic function $u(x)$ in $L^2(\mathbb{T};\mathbb{C})$ in Fourier series as

$$u(x)=\sum_{j\in\mathbb{Z}}\hat{u}\left(j\right)e^{\mathrm{i}jx}\,,\qquad \hat{u}\left(j\right):=\mathcal{F}_{x\to j}\left(j\right):=u_j:=\frac{1}{2\pi}\int_{\mathbb{T}}u(x)e^{-\mathrm{i}jx}\,\mathrm{d}x\,. \tag{2.2}$$

A function $u(x)$ is real if and only if $\overline{u_j} = u_{-j}$, for any $j \in \mathbb{Z}$. For any $s \in \mathbb{R}$ we define the Sobolev space $H^s := H^s(\mathbb{T};\mathbb{C})$ with norm

$$\|u\|_s := \|u\|_{H^s} = \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} \left|\hat{u}(j)\right|^2\right)^{\frac{1}{2}}, \qquad \langle j\rangle := \max(1,|j|).$$

We define $\Pi_0 u := \hat{u}_0$ the average of u and

$$\Pi_0^{\perp} := \mathrm{Id} - \Pi_0. \tag{2.3}$$

We define H_0^s the subspace of zero average functions of H^s, for which we also denote $\|u\|_s = \|u\|_{H^s} = \|u\|_{H_0^s}$. Clearly $H_0^0(\mathbb{T};\mathbb{C}) = L_0^2(\mathbb{T};\mathbb{C})$ with scalar product, for any $u, v \in L_0^2(\mathbb{T};\mathbb{C})$,

$$\langle u, v\rangle_{L_0^2} = \int_{\mathbb{T}} u(x)\,\overline{v(x)}\,\mathrm{d}x. \tag{2.4}$$

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t = 0$ and $s \in \mathbb{R}$, we define the space

$$C_*^K\left(I; H_0^s(\mathbb{T};\mathbb{X})\right) := \bigcap_{k=0}^{K} C^k\left(I; H_0^{s-\alpha k}(\mathbb{T};\mathbb{X})\right), \qquad \mathbb{X} = \mathbb{R},\, \mathbb{C},$$

resp. $C_*^K(I; H^s(\mathbb{T};\mathbb{X}))$, endowed with the norm

$$\sup_{t\in I} \|u(t,\cdot)\|_{K,s} \qquad \text{where} \qquad \|u(t,\cdot)\|_{K,s} := \sum_{k=0}^{K} \left\|\partial_t^k u(t,\cdot)\right\|_{H^{s-\alpha k}}. \tag{2.5}$$

We denote $B_s^K(I;\epsilon_0)$, resp. $B_{s,\mathbb{R}}^K(I;\epsilon_0)$, the ball of radius $\epsilon_0 > 0$ in $C_*^K(I, H_0^s(\mathbb{T};\mathbb{C}))$, resp. in $C_*^K(I, H_0^s(\mathbb{T};\mathbb{R}))$. We also we define $B_{C_*^K(I,H^s(\mathbb{T};\mathbb{C}))}(0;\epsilon_0)$ the ball of center zero and radius ϵ_0 in $C_*^K(I, H^s(\mathbb{T};\mathbb{C}))$.

Remark 2.1. The parameter s in (2.5) denotes the spatial Sobolev regularity of the solution $u(t,\cdot)$ and K its regularity in the time variable. The α-SQG vector field loses α-derivatives, and therefore, differentiating the solution $u(t)$ for k-times in the time variable, there is a loss of αk-spatial derivatives. The parameter ρ in (2.1) denotes the order where we decide to stop our regularization of the system.

We set some further notation. For $n \in \mathbb{N}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T};\mathbb{C})$ to the linear subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$, $(\Pi_n u)(x) := \hat{u}(n)e^{\mathrm{i}nx} + \hat{u}(-n)e^{-\mathrm{i}nx}$. If $\mathcal{U} = (u_1,\dots,u_p)$ is a p-tuple of functions and $\vec{n} = (n_1,\dots,n_p) \in \mathbb{N}^p$, we set $\Pi_{\vec{n}}\mathcal{U} := \left(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p\right)$ and $\mathrm{t}_\varsigma\mathcal{U} := \left(\mathrm{t}_\varsigma u_1,\dots,\mathrm{t}_\varsigma u_p\right)$, where t_ς is the translation operator

$$\mathrm{t}_\varsigma\colon u(x) \mapsto u(x+\varsigma). \tag{2.6}$$

For $\vec{j}_p = (j_1,\dots,j_p) \in \mathbb{Z}^p$ we denote $|\vec{j}_p| := \max(|j_1|,\dots,|j_p|)$ and $u_{\vec{j}_p} := u_{j_1}\dots u_{j_p}$. Note that the Fourier coefficients of $\mathrm{t}_\varsigma u$ are $(\mathrm{t}_\varsigma u)_j = e^{\mathrm{i}j\varsigma}u_j$.

A vector field $X(u)$ is *translation invariant* if $X \circ \mathrm{t}_\varsigma = \mathrm{t}_\varsigma \circ X$ for any $\varsigma \in \mathbb{R}$.

Given a linear operator $R(u)[\cdot]$ acting on $L_0^2(\mathbb{T};\mathbb{C})$ we associate the linear operator defined by the relation $\overline{R(u)}v := \overline{R(u)\overline{v}}$ for any $v \in L_0^2(\mathbb{T};\mathbb{C})$ An operator $R(u)$ is *real* if $R(u) = \overline{R(u)}$ for any u real.

2.1 Paradifferential calculus

We introduce paradifferential operators (Definition 2.10) following [5], with minor modifications due to the fact that we deal with a scalar equation and not a system, and the fact that we consider operators acting on H_0^s and H^s and not on homogenous spaces $\dot{H}^s$. In this way we will mainly rely on results in [5, 10].

Classes of symbols. Roughly speaking the class $\widetilde{\Gamma}_p^m$ contains symbols of order m and homogeneity p in u, whereas the class $\Gamma_{K,K',p}^m$ contains non-homogeneous symbols of order m that vanish at degree at least p in u and that are $(K-K')$-times differentiable in t. We can think the parameter K' like the number of time derivatives of u that are contained in the symbols. We denote $H_0^\infty(\mathbb{T};\mathbb{C}) := \bigcap_{s\in\mathbb{R}} H_0^s(\mathbb{T};\mathbb{C})$.

Definition 2.2 (Symbols)**.** Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \leq K$, and $\epsilon_0 > 0$.

i) **p-homogeneous symbols.** We denote by $\widetilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $\left(H_0^\infty(\mathbb{T};\mathbb{C})\right)^p$ to the space of C^∞ functions from $\mathbb{T}\times\mathbb{R}$ to $\mathbb{C}$, $(x,\xi)\mapsto a(\mathcal{U};x,\xi)$, satisfying the following: there exist $\mu\ge 0$ and, for any $\gamma,\beta\in\mathbb{N}_0$, there is a constant $C>0$ such that

$$\left|\partial_x^\gamma\partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)\right|\le C|\vec{n}|^{\mu+\gamma}\langle\xi\rangle^{m-\beta}\prod_{j=1}^p\left\|\Pi_{n_j}u_j\right\|_{L^2} \tag{2.7}$$

for any $\mathcal{U}=(u_1,\dots,u_p)\in\left(H_0^\infty(\mathbb{T};\mathbb{C})\right)^p$ and $\vec{n}=(n_1,\dots,n_p)\in\mathbb{N}^p$. Moreover we assume that, if for some $(n_0,\dots,n_p)\in\mathbb{N}_0\times\mathbb{N}^p$, $\Pi_{n_0}a(\Pi_{\vec{n}}\mathcal{U};\cdot)\neq 0$, then there exists a choice of signs $\eta_j\in\{\pm1\}$ such that $\sum_{j=1}^p\eta_j n_j=n_0$. In addition we require the translation invariance property

$$a\left(\mathrm{t}_\varsigma\mathcal{U};x,\xi\right)=a(\mathcal{U};x+\varsigma,\xi),\quad\forall\varsigma\in\mathbb{R}, \tag{2.8}$$

where t_ς is the translation operator in (2.6).

For $p=0$ we denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi\mapsto a(\xi)$ which satisfy (2.7) with $\gamma=0$ and the right hand side replaced by $C\langle\xi\rangle^{m-\beta}$ and we call them Fourier multipliers.

ii) **Non-homogeneous symbols.** We denote by $\Gamma^m_{K,K',p}[\epsilon_0]$ the space of functions $a(u;t,x,\xi)$, defined for $u\in B^{K'}_{s_0}(I;\epsilon_0)$ for some s_0 large enough, with complex values, such that for any $0\le k\le K-K'$, any $s\ge s_0$, there are $C>0$, $0<\epsilon_0(s)<\epsilon_0$ and for any $u\in B^K_{s_0}(I;\epsilon_0(s))\cap C^{k+K'}_*\left(I,H^s_0(\mathbb{T};\mathbb{C})\right)$ and any $\gamma,\beta\in\mathbb{N}_0$, with $\gamma\le s-s_0$ one has the estimate

$$\left|\partial_t^k\partial_x^\gamma\partial_\xi^\beta a(u;t,x,\xi)\right|\le C\langle\xi\rangle^{m-\beta}\|u\|^{p-1}_{k+K',s_0}\|u\|_{k+K',s}. \tag{2.9}$$

If $p=0$ the right hand side has to be replaced by $C\langle\xi\rangle^{m-\beta}$. We say that a non-homogeneous symbol $a(u;x,\xi)$ is *real* if it is real valued for any $u\in B^{K'}_{s_0,\mathbb{R}}(I;\epsilon_0)$.

iii) **Symbols.** We denote by $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ the space of symbols

$$a(u;t,x,\xi)=\sum_{q=p}^N a_q(u,\dots,u;x,\xi)+a_{>N}(u;t,x,\xi)$$

where a_q, $q=p,\dots,N$ are homogeneous symbols in $\widetilde{\Gamma}^m_q$ and $a_{>N}$ is a non-homogeneous symbol in $\Gamma^m_{K,K',N+1}$.

We say that a symbol $a(u;t,x,\xi)$ is *real* if it is real valued for any $u\in B^{K'}_{s_0,\mathbb{R}}(I;\epsilon_0)$.

Notation 2.3. If $a(\mathcal{U};\cdot)$ is a p-homogenous symbol we also denote $a(u):=a(u,\dots,u;\cdot)$ the corresponding polynomial and we identify the p-homogeneous monomial $a(u;\cdot)$ with the p-linear symmetric form $a(\mathcal{U};\cdot)$.

Actually also the non-homogeneous component of the symbols that we will encounter in Section 4 depends on time and space only through u, but since this information is not needed it is not included in Definition 2.2 (as in [5]).

Remark 2.4. If $a(\mathcal{U};\cdot)$ is a homogeneous symbol in $\widetilde{\Gamma}^m_p$ then $a(u,\dots,u;\cdot)$ belongs to $\Gamma^m_{K,0,p}[\epsilon_0]$, for any $\epsilon_0>0$.

Remark 2.5. If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ then $\partial_x a\in\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ and $\partial_\xi a\in\Sigma\Gamma^{m-1}_{K,K',p}[\epsilon_0,N]$. If in addition b is a symbol in $\Sigma\Gamma^{m'}_{K,K',p'}[\epsilon_0,N]$ then $ab\in\Sigma\Gamma^{m+m'}_{K,K',p+p'}[\epsilon_0,N]$.

Remark 2.6 (Fourier representation of symbols)**.** The translation invariance property (2.8) means that the dependence with respect to the variable x of a symbol $a(\mathcal{U};x,\xi)$ enters only through the functions $\mathcal{U}(x)$, implying that a symbol $a_q(u;x,\xi)$ in $\widetilde{\Gamma}^m_q$, $m\in\mathbb{R}$, has the form

$$a_q(u;x,\xi)=\sum_{\vec{\jmath}_q\in(\mathbb{Z}\setminus\{0\})^q}\left(a_q\right)_{\vec{\jmath}_q}(\xi)u_{j_1}\cdots u_{j_q}e^{\mathrm{i}(j_1+\dots+j_q)x} \tag{2.10}$$

where $(a_q)_{\vec{j}}(\xi) \in \mathbb{C}$ are Fourier multipliers of order m satisfying: there exists $\mu \geq 0$, and for any $\beta \in \mathbb{N}_0$, there is $C_\beta > 0$ such that

$$\left|\partial_\xi^\beta \left(a_q\right)_{\vec{j}_q}(\xi)\right| \leq C_\beta \left|\vec{j}_q\right|^\mu \langle\xi\rangle^{m-\beta}, \quad \forall \vec{j}_q \in (\mathbb{Z}\setminus\{0\})^q. \tag{2.11}$$

A symbol $a_q(u;x,\xi)$ as in (2.10) is real if

$$\overline{\left(a_q\right)_{\vec{j}_q}(\xi)} = \left(a_q\right)_{-\vec{j}_q}(\xi) \tag{2.12}$$

By (2.10) a symbol a_1 in $\widetilde{\Gamma}_1^m$ can be written as $a_1(u;x,\xi) = \sum_{j\in\mathbb{Z}\setminus\{0\}}(a_1)_j(\xi)u_j e^{\mathrm{i}jx}$, and therefore, if a_1 is independent of x, it is actually $a_1 \equiv 0$.

We also define classes of functions in analogy with our classes of symbols.

Definition 2.7 (Functions)**.** Let $p, N \in \mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \leq K$, $\epsilon_0 > 0$. We denote by $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[\epsilon_0]$, $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0, N]$, the subspace of $\widetilde{\Gamma}_p^0$, resp. $\Gamma_{K,K',p}^0[\epsilon_0]$, resp. $\Sigma\Gamma_{K,K',p}^0[\epsilon_0,N]$, made of those symbols which are independent of ξ. We write $\widetilde{\mathcal{F}}_p^{\mathbb{R}}$, resp. $\mathcal{F}_{K,K',p}^{\mathbb{R}}[\epsilon_0]$, $\Sigma\mathcal{F}_{K,K',p}^{\mathbb{R}}[\epsilon_0,N]$, to denote functions in $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[\epsilon_0]$, $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0,N]$, which are real valued for any $u \in B_{s_0,\mathbb{R}}^{K'}(I;\epsilon_0)$.

The above class of symbols is closed under composition by a change of variables, see [5, Lemma 3.23].

Lemma 2.8. *Let $K' \leq K \in \mathbb{N}$, $m \in \mathbb{R}$, $p \in \mathbb{N}_0$, $N \in \mathbb{N}$ with $p \leq N$, $\epsilon_0 > 0$ small enough. Consider a symbol a in $\Sigma\Gamma_{K,K',p}^m[\epsilon_0,N]$ and functions b, c in $\Sigma\mathcal{F}_{K,K',1}^{\mathbb{R}}[\epsilon_0,N]$. Then $a\big(v;t,x+b(v;t,x),\xi(1+c(v;t,x))\big)$ is in $\Sigma\Gamma_{K,K',p}^m[\epsilon_0,N]$. In particular, if a is a function in $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0,N]$, then $a(v;t,x+b(v;t,x))$ is in $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0,N]$.*

The following result is [5, Lemma 3.21].

Lemma 2.9 (Inverse diffeomorphism)**.** *Let $0 \leq K' \leq K$ be in $\mathbb{N}$ and $\beta(f;t,x)$ be a real function $\beta(f;t,\cdot)$ in $\Sigma\mathcal{F}_{K,K',1}^{\mathbb{R}}[\epsilon_0,N]$. If s_0 is large enough, and $f \in B_{s_0}^K(I;\epsilon_0)$ then the map $\Phi_f : x \to x+\beta(f;t,x)$ is, for ϵ_0 small enough, a diffeomorphism of $\mathbb{T}^1$, and its inverse diffeomorphism may be written as $\Phi_f^{-1} : y \to y+\breve{\beta}(f;t,y)$ for some $\breve{\beta}$ in $\Sigma\mathcal{F}_{K,K',1}^{\mathbb{R}}[\epsilon_0,N]$.*

Paradifferential quantization. Given $p \in \mathbb{N}$ we consider *admissible cut-off* functions $\psi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R};\mathbb{R})$ and $\psi \in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$, even with respect to each of their arguments, satisfying, for $0 < \delta \ll 1$,

$$\operatorname{supp}\psi_p \subset \left\{(\xi',\xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \leq \delta\langle\xi\rangle\right\}, \qquad \psi_p(\xi',\xi) \equiv 1 \text{ for } |\xi'| \leq \delta\langle\xi\rangle/2, \tag{2.13}$$

$$\operatorname{supp}\psi \subset \left\{(\xi',\xi) \in \mathbb{R}\times\mathbb{R}; |\xi'| \leq \delta\langle\xi\rangle\right\}, \qquad \psi(\xi',\xi) \equiv 1 \text{ for } |\xi'| \leq \delta\langle\xi\rangle/2. \tag{2.14}$$

For $p = 0$ we set $\psi_0 \equiv 1$. We assume moreover that

$$\left|\partial_\xi^\gamma \partial_{\xi'}^\beta \psi_p(\xi',\xi)\right| \leq C_{\gamma,\beta}\langle\xi\rangle^{-\gamma-|\beta|},\ \forall\gamma \in \mathbb{N}_0,\ \beta \in \mathbb{N}_0^p,\ \left|\partial_\xi^\gamma \partial_{\xi'}^\beta \psi(\xi',\xi)\right| \leq C_{\gamma,\beta}\langle\xi\rangle^{-\gamma-\beta},\ \forall\gamma,\beta \in \mathbb{N}_0. \tag{2.15}$$

If $a(x,\xi)$ is a smooth symbol we define its Weyl quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (2.2)) as

$$\operatorname{Op}^W[a]\,u = \sum_{k\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}} \hat{a}\left(k-j, \frac{k+j}{2}\right)\hat{u}\left(j\right)\right)e^{\mathrm{i}kx} \tag{2.16}$$

where $\hat{a}(k,\xi)$ is the k-Fourier coefficient of the $2\pi-$periodic function $x \mapsto a(x,\xi)$.

Definition 2.10 (Bony-Weyl quantization)**.** If a is a symbol in $\widetilde{\Gamma}_p^m$, respectively in $\Gamma_{K,K',p}^m[\epsilon_0]$, we set

$$a_{\psi_p}(\mathcal{U};x,\xi) := \sum_{\vec{n}\in\mathbb{N}^p}\psi_p(\vec{n},\xi)\ a(\Pi_{\vec{n}}\mathcal{U};x,\xi),$$
$$a_\psi(u;t,x,\xi) := \frac{1}{2\pi}\int_{\mathbb{R}}\psi(\xi',\xi)\hat{a}\left(u;t,\xi',\xi\right)e^{\mathrm{i}\xi' x}\mathrm{d}\xi',$$

where $\hat{a}$ stands for the Fourier transform with respect to the x variable, and we define the *Bony-Weyl* quantization of a as

$$\operatorname{Op}^{BW}[a(\mathcal{U};\cdot)] = \operatorname{Op}^W\left[a_{\psi_p}(\mathcal{U};\cdot)\right], \qquad \operatorname{Op}^{BW}[a(u;t,\cdot)] = \operatorname{Op}^W\left[a_\psi(u;t,\cdot)\right]. \tag{2.17}$$

If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0, N]$, we define its *Bony-Weyl* quantization

$$\mathrm{Op}^{BW}[a(u;t,\cdot)] = \sum_{q=p}^{N} \mathrm{Op}^{BW}\left[a_q(u,\dots,u;\cdot)\right] + \mathrm{Op}^{BW}[a_{>N}(u;t,\cdot)] \,.$$

Remark 2.11. • The operator $\mathrm{Op}^{BW}[a]$ maps functions with zero average in functions with zero average, and $\Pi_0^\perp \mathrm{Op}^{BW}[a] = \mathrm{Op}^{BW}[a]\,\Pi_0^\perp$.

• If a is a homogeneous symbol, the two definitions of quantization in (2.17) differ by a smoothing operator according to Definition 2.17 below.

• Definition 2.10 is independent of the cut-off functions ψ_p, ψ, up to smoothing operators (Definition 2.17).

• The action of $\mathrm{Op}^{BW}[a]$ on the spaces H^s_0 only depends on the values of the symbol $a(u;t,x,\xi)$ for $|\xi| \geq 1$. Therefore, we may identify two symbols $a(u;t,x,\xi)$ and $b(u;t,x,\xi)$ if they agree for $|\xi| \geq 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi = 0$, such as, for example, $a = g(x)|\xi|^m$ for $m \in \mathbb{R}\setminus\{0\}$, or $\mathrm{sign}(\xi)$, we will consider its smoothed out version $\chi(\xi)a$, where $\chi \in C^\infty(\mathbb{R};\mathbb{R})$ is an even and positive cut-off function satisfying

$$\chi(\xi) = 0 \ \text{ if } |\xi| \leq \tfrac{1}{8}, \quad \chi(\xi) = 1 \ \text{ if } |\xi| > \tfrac{1}{4}, \quad \partial_\xi \chi(\xi) > 0 \quad \forall \xi \in \left(\tfrac{1}{8}, \tfrac{1}{4}\right). \tag{2.18}$$

Remark 2.12. Given a paradifferential operator $A = \mathrm{Op}^{BW}[a(x,\xi)]$ it results

$$\overline{A} = \mathrm{Op}^{BW}\left[\overline{a(x,-\xi)}\right], \quad A^\intercal = \mathrm{Op}^{BW}[a(x,-\xi)], \quad A^* = \mathrm{Op}^{BW}\left[\overline{a(x,\xi)}\right], \tag{2.19}$$

where $A^\intercal$ is the transposed operator with respect to the real scalar product $\langle u, v\rangle_r = \int_{\mathbb{T}} u(x)\,v(x)\,\mathrm{d}x$, and A^* denotes the adjoint operator with respect to the complex scalar product of L^2_0 in (2.4). It results $A^* = \overline{A}^\intercal$.

• A paradifferential operator $A = \mathrm{Op}^{BW}[a(x,\xi)]$ is *real* (i.e. $A = \overline{A}$) if

$$\overline{a(x,\xi)} = a(x,-\xi)\,. \tag{2.20}$$

It is *symmetric* (i.e. $A = A^\intercal$) if $a(x,\xi) = a(x,-\xi)$. A operator $\partial_x \mathrm{Op}^{BW}[a(x,\xi)]$ is Hamiltonian if and only if

$$a(x,\xi) \in \mathbb{R} \qquad \text{and} \qquad a(x,\xi) = a(x,-\xi) \quad \text{is even in } \xi\,. \tag{2.21}$$

We now provide the action of a paradifferential operator on Sobolev spaces, cf. [5, Prop. 3.8].

Lemma 2.13 (Action of a paradifferential operator)**.** *Let $m \in \mathbb{R}$.*

i) If $p \in \mathbb{N}$, there is $s_0 > 0$ such that for any symbol a in $\tilde{\Gamma}^m_p$, there is a constant $C > 0$, depending only on s and on (2.7) *with $\gamma = \beta = 0$, such that, for any $(u_1,\dots,u_p)$, for $p \geq 1$,*

$$\left\|\mathrm{Op}^{BW}\left[a(u_1,\dots,u_p;\cdot)\right] u_{p+1}\right\|_{H_0^{s-m}} \leq C \|u_1\|_{H_0^{s_0}} \cdots \left\|u_p\right\|_{H_0^{s_0}} \left\|u_{p+1}\right\|_{H_0^s} .$$

If $p = 0$ the above bound holds replacing the right hand side with $C\left\|u_{p+1}\right\|_{H_0^s}$.

ii) Let $\epsilon_0 > 0$, $p \in \mathbb{N}$, $K' \leq K \in \mathbb{N}$, a in $\Gamma^m_{K,K',p}[\epsilon_0]$. There is $s_0 > 0$, and a constant C, depending only on s, ϵ_0, and on (2.9) *with $0 \leq \gamma \leq 2, \beta = 0$, such that, for any t in I, any $0 \leq k \leq K - K'$, any u in $B^K_{s_0}(I;\epsilon_0)$,*

$$\left\|\mathrm{Op}^{BW}\left[\partial_t^k a(u;t,\cdot)\right]\right\|_{\mathcal{L}(H_0^s, H_0^{s-m})} \leq C \|u(t,\cdot)\|^p_{k+K',s_0} ,$$

so that $\|\mathrm{Op}^{BW}[a(u;t,\cdot)]\,v(t)\|_{K-K',s-m} \leq C \|u(t,\cdot)\|^p_{K,s_0} \|v(t)\|_{K-K',s}$.

Classes of m-Operators and smoothing Operators. Given integers $(n_1,\dots,n_{p+1}) \in \mathbb{N}^{p+1}$, we denote by $\max_2(n_1,\dots,n_{p+1})$ the second largest among $n_1,\dots,n_{p+1}$.

We now define m-operators. The class $\widetilde{\mathcal{M}}^m_p$ denotes multilinear operators that lose m derivatives and are p-homogeneous in u, while the class $\mathcal{M}^m_{K,K',p}$ contains non-homogeneous operators which lose m derivatives, vanish at degree at least p in u, satisfy tame estimates and are $(K-K')$-times differentiable in t. The constant μ in (2.23) takes into account possible loss of derivatives in the "low" frequencies. The following definition is a small adaptation of [10, Def. 2.5] as it defines m-operators acting on $H^\infty(\mathbb{T};\mathbb{C})$ and not $\dot H^\infty(\mathbb{T};\mathbb{C}^2)$ (and we state it directly in Fourier series representation).

Definition 2.14 (Classes of m-operators)**.** Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \le K$, and $\epsilon_0 > 0$.

i) **p-homogeneous m-operators.** We denote by $\widetilde{\mathcal{M}}^m_p$ the space of $(p+1)$-linear translation invariant operators from $(H^\infty(\mathbb{T};\mathbb{C}))^p \times H^\infty(\mathbb{T};\mathbb{C})$ to $H^\infty(\mathbb{T};\mathbb{C})$, symmetric in $(u_1,\dots,u_p)$, with Fourier expansion

$$M(u)v := M(u,\dots,u)\,v = \sum_{\substack{(j_1,\dots,j_p,j,k)\in\mathbb{Z}^{p+2}\\ j_1+\dots+j_p+j=k}} M_{j_1,\dots,j_p,j,k}\, u_{j_1}\dots u_{j_p} v_j e^{\mathrm{i}kx}, \tag{2.22}$$

with coefficients $M_{j_1,\dots,j_p,j,k}$ symmetric in $j_1,\dots,j_p$, satisfying the following: there are $\mu \ge 0$, $C>0$ such that, for any $j_1,\dots,j_p,j,k \in \mathbb{Z}^{p+2}$, it results

$$\left|M_{j_1,\dots,j_p,j,k}\right| \le C \max_2\{\langle j_1\rangle,\dots,\langle j_p\rangle,\langle j\rangle\}^\mu \max\{\langle j_1\rangle,\dots,\langle j_p\rangle,\langle j\rangle\}^m, \tag{2.23}$$

and the reality condition holds:

$$\overline{M_{\vec{\jmath}_p,j,k}} = M_{-\vec{\jmath}_p,-j,-k}, \qquad \forall \vec{\jmath}_p = (j_1,\dots,j_p) \in \mathbb{Z}^p, (j,k) \in \mathbb{Z}^2. \tag{2.24}$$

If $p=0$ the right hand side of (2.22) must be substituted with $\sum_{j\in\mathbb{Z}} M_j v_j e^{\mathrm{i}jx}$ with $|M_j| \le C\langle j\rangle^m$.

ii) **Non-homogeneous m-operators.** We denote by $\mathcal{M}^m_{K,K',p}[\epsilon_0]$ the space of operators $(u,t,v) \mapsto M(u;t)v$ defined on $B_{C^{K'}_*(I,H^{s_0}(\mathbb{T};\mathbb{C}))}(0;\epsilon_0) \times I \times C^0_*(I,H^{s_0}(\mathbb{T};\mathbb{C}))$ for some $s_0>0$, which are linear in the variable v and such that the following holds true. For any $s \ge s_0$ there are $C>0$ and $\epsilon_0(s) \in]0,\epsilon_0[$ such that for any $u \in B_{C^{K'}_*(I,H^{s_0}(\mathbb{T};\mathbb{C}))}(0;\epsilon_0) \cap C^K_*(I,H^s(\mathbb{T};\mathbb{C}))$, any $v \in C^{K-K'}_*(I,H^s(\mathbb{T};\mathbb{C}))$, any $0 \le k \le K-K'$, $t \in I$, we have that

$$\left\|\partial_t^k (M(u;t)v)\right\|_{s-\alpha k-m} \le C \sum_{k'+k''=k} \left(\|v\|_{k'',s}\|u\|^p_{k'+K',s_0} + \|v\|_{k'',s_0}\|u\|^{p-1}_{k'+K',s_0}\|u\|_{k'+K',s}\right). \tag{2.25}$$

In case $p=0$ we require the estimate $\|\partial_t^k(M(u;t)v)\|_{s-\alpha k-m} \le C\|v\|_{k,s}$. We say that a non-homogeneous m-operator $M(u;t)$ is *real* if it is real valued for any $u \in B_{C^{K'}_*(I,H^{s_0}(\mathbb{T};\mathbb{R}))}(0;\epsilon_0)$.

iii) **m-Operators.** We denote by $\Sigma\mathcal{M}^m_{K,K',p}[\epsilon_0,N]$ the space of operators

$$M(u;t)v = \sum_{q=p}^N M_q(u,\dots,u)v + M_{>N}(u;t)v \tag{2.26}$$

where M_q are homogeneous m-operators in $\widetilde{\mathcal{M}}^m_q$, $q=p,\dots,N$ and $M_{>N}$ is a non–homogeneous m-operator in $\mathcal{M}^m_{K,K',N+1}[\epsilon_0]$. We say that a m-operator $M(u;t)$ is *real* if it is real valued for any $u \in B_{C^{K'}_*(I,H^{s_0}(\mathbb{T};\mathbb{R}))}(0;\epsilon_0)$.

iv) **Pluri-homogeneous m-Operator.** We denote by $\Sigma_p^N \widetilde{\mathcal{M}}^m_q$ the pluri-homogeneous m-operators of the form (2.26) with $M_{>N}=0$.

We denote with $\dot{\widetilde{\mathcal{M}}}^m_p$, $\dot{\mathcal{M}}^m_{K,K',p}[\epsilon_0]$ and $\Sigma\dot{\mathcal{M}}^m_{K,K',p}[\epsilon_0,N]$ the subspaces of m-operators in $\widetilde{\mathcal{M}}^m_p$, respectively $\mathcal{M}^m_{K,K',p}[\epsilon_0]$ and $\Sigma\mathcal{M}^m_{K,K',p}[\epsilon_0,N]$, defined on zero-average functions taking value $M(u)v$ in zero-average functions.

Remark 2.15. By [10, Lemma 2.8], if $M(u_1,\dots,u_p)$ is a p–homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$ then $M(u) = M(u,\dots,u)$ is a non-homogeneous m-operator in $\mathcal{M}_{K,0,p}^m[\epsilon_0]$ for any $\epsilon_0 > 0$ and $K \in \mathbb{N}_0$. We shall say that $M(u)$ is in $\widetilde{\mathcal{M}}_p^m$.

Remark 2.16. The multiplication operator $v \mapsto \frac{1}{1+2f} v$ belongs to $\Sigma\mathcal{M}_{K,0,0}^0[\epsilon_0, N]$.

If $m \le 0$ the operators in $\Sigma\mathcal{M}_{K,K',p}^m[\epsilon_0, N]$ are referred to as smoothing operators.

Definition 2.17 (Smoothing operators)**.** Let $\rho \ge 0$. A $(-\rho)$-operator $R(u)$ belonging to $\Sigma\mathcal{M}_{K,K',p}^{-\rho}[\epsilon_0, N]$ is called a smoothing operator. We also denote

$$\widetilde{\mathcal{R}}_p^{-\rho} := \widetilde{\mathcal{M}}_p^{-\rho}, \qquad \mathcal{R}_{K,K',p}^{-\rho}[\epsilon_0] := \mathcal{M}_{K,K',p}^{-\rho}[\epsilon_0], \qquad \Sigma\mathcal{R}_{K,K',p}^{-\rho}[\epsilon_0, N] := \Sigma\mathcal{M}_{K,K',p}^{-\rho}[\epsilon_0, N].$$

We define $\dot{\widetilde{\mathcal{R}}}_p^{-\rho} = \dot{\widetilde{\mathcal{M}}}_p^{-\rho}$, $\dot{\mathcal{R}}_{K,K',p}^{-\rho}[\epsilon_0] = \dot{\mathcal{M}}_{K,K',p}^{-\rho}[\epsilon_0]$ and $\Sigma\dot{\mathcal{R}}_{K,K',p}^{-\rho}[\epsilon_0, N] = \Sigma\dot{\mathcal{M}}_{K,K',p}^{-\rho}[\epsilon_0, N]$ as in Definition 2.14.

If $R(u)$ is a homogenous smoothing operator in $\widetilde{\mathcal{R}}_p^{-\rho}$ then $\Pi_0^\perp R(u)$, where $\Pi_0^\perp$ is defined (2.3), restricted to zero average functions u, belongs to $\dot{\widetilde{\mathcal{R}}}_p^{-\rho}$.

Remark 2.18. • Lemma 2.13 implies that, if $a(u;t,\cdot)$ is in $\Sigma\Gamma_{K,K',p}^m[\epsilon_0, N]$, $m \in \mathbb{R}$, then $\mathrm{Op}^{BW}[a(u;t,\cdot)]$ defines a m-operator in $\Sigma\mathcal{M}_{K,K',p}^m[\epsilon_0, N]$.

• The composition of smoothing operators $R_1 \in \Sigma\mathcal{R}_{K,K',p_1}^{-\rho}[\epsilon_0, N]$ and $R_2 \in \Sigma\mathcal{R}_{K,K',p_2}^{-\rho}[\epsilon_0, N]$ is a smoothing operator R_1R_2 in $\Sigma\mathcal{R}_{K,K',p_1+p_2}^{-\rho}[\epsilon_0, N]$. This is a particular case of Proposition 2.23-(i).

Lemma 2.19. *Let $m \in \mathbb{R}$, $\epsilon_0 > 0$, $K, K' \in \mathbb{N}_0$, $K' \le K$, $N, p \in \mathbb{N}_0$, $u \in B_{s,\mathbb{R}}^K(I;\epsilon_0)$ and $M(u;t)$ be a real operator in $\Sigma\mathcal{M}_{K,K',p}^m[\epsilon_0, N]$. Then $M(u;t)u$ is a real function in $\Sigma\mathcal{F}_{K,K',p+1}^{\mathbb{R}}[\epsilon_0, N+1]$ according to Definition 2.7.*

Proof. We decompose $M(u;t) = \sum_{q=p}^N M_q(u) + M_{>N}(u;t)$ in the usual homogeneous and non-homogeneous components. We assume u is in $B_{s,\mathbb{R}}^K(I;\epsilon_0)$ so that u has zero average. We now prove that $M_q(u)u$ is a function in $\widetilde{\mathcal{F}}_{q+1}^{\mathbb{R}}$. For any zero average function u, according to (2.22) we have

$$(M_q(u)u)(x) = \sum_{\substack{(j_1,\dots,j_q,j)\in(\mathbb{Z}\setminus\{0\})^{q+1} \\ j_1+\dots+j_p+j=k}} M_{j_1,\dots,j_p,j,k}\, u_{j_1}\dots u_{j_q} u_j\, e^{\mathrm{i}(j_1+\dots+j_q+j)x}.$$

Moreover, by (2.23), for any $(j_1,\dots,j_p,j) = (\vec{\jmath}_q, j) \in (\mathbb{Z}\setminus\{0\})^{q+1}$, we have

$$\begin{aligned}\left|M_{j_1,\dots,j_p,j,k}\right| &\lesssim \max\nolimits_2\{\langle j_1\rangle,\dots,\langle j_q\rangle,\langle j\rangle\}^\mu \max\{\langle j_1\rangle,\dots,\langle j_q\rangle,\langle j\rangle\}^m \\ &\lesssim \max\{\langle j_1\rangle,\dots,\langle j_q\rangle,\langle j\rangle\}^{2\max\{\mu,m\}} \lesssim \left|(\vec{\jmath}_q, j)\right|^{2\max\{\mu,m\}},\end{aligned}$$

and, in view of (2.11), we thus obtain that $M_q(u)u$ is a function in $\widetilde{\mathcal{F}}_{q+1}$. In view of (2.24) the function $M_q(u)u$ is real.

We now prove that $(M_{>N}(u;t)\,u)(t,x)$ is a function in $\mathcal{F}_{K,K',N+2}^{\mathbb{R}}(\epsilon_0)$. Let $s_0 := 1 + \alpha(K-K') + m$. For any $0 \le k \le K-K'$, for any $s \ge s_0$, and $0 \le \gamma \le s - s_0$ we have that $s - \alpha k - m > \gamma + 1$, and

$$\left|\partial_t^k\partial_x^\gamma(M_{>N}(u;t)\,u)\right| \lesssim \left\|\partial_t^k(M_{>N}(u;t)\,u)\right\|_{\gamma+1} \le \left\|\partial_t^k(M_{>N}(u;t)\,u)\right\|_{s-\alpha k-m} \overset{(2.25)}{\lesssim} \|u\|_{k+K',s_0}^{N+1}\|u\|_{k+K',s}$$

proving, in view of Definitions 2.2 and 2.7, that $M_{>N}(u;t)\,u$ is a function in $\mathcal{F}_{K,K',N+2}(\epsilon_0)$. The reality condition is verified since $M_{>N}$ is a real m-operator per hypothesis. $\square$

Symbolic calculus. Let $\sigma(D_x, D_\xi, D_y, D_\eta) := D_\xi D_y - D_x D_\eta$ where $D_x := \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined. The following is Definition 3.11 in [5].

Definition 2.20 (Asymptotic expansion of composition symbol)**.** Let p, p' in $\mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \le K$, $\rho \ge 0$, $m, m' \in \mathbb{R}$, $\epsilon_0 > 0$. Consider symbols $a \in \Sigma\Gamma^{m}_{K,K',p}[\epsilon_0, N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',p'}[\epsilon_0, N]$. For u in $B^K_\sigma(I;\epsilon_0)$ we define, for $\rho < \sigma - s_0$, the symbol

$$(a\#_\rho b)(u;t,x,\xi) := \sum_{k=0}^{\rho} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\sigma\left(D_x, D_\xi, D_y, D_\eta\right)\right)^k \left[a(u;t,x,\xi)b(u;t,y,\eta)\right]_{|x=y,\xi=\eta} \tag{2.27}$$

modulo symbols in $\Sigma\Gamma^{m+m'-\rho}_{K,K',p+p'}[\epsilon_0, N]$.

The symbol $a\#_\rho b$ belongs to $\Sigma\Gamma^{m+m'}_{K,K',p+p'}[\epsilon_0, N]$. Moreover

$$a\#_\rho b = ab + \frac{1}{2\mathrm{i}}\{a, b\} \tag{2.28}$$

up to a symbol in $\Sigma\Gamma^{m+m'-2}_{K,K',p+p'}[\epsilon_0, N]$, where

$$\{a, b\} := \partial_\xi a\, \partial_x b - \partial_x a\, \partial_\xi b$$

denotes the Poisson bracket. The following result is proved in Proposition 3.12 in [5].

Proposition 2.21 (Composition of Bony-Weyl operators)**.** *Let $p, q, N, K, K' \in \mathbb{N}_0$ with $K' \le K$, $\rho \ge 0$, $m, m' \in \mathbb{R}$, $\epsilon_0 > 0$. Consider symbols $a \in \Sigma\Gamma^{m}_{K,K',p}[\epsilon_0, N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',q}[\epsilon_0, N]$. Then*

$$\mathrm{Op}^{BW}[a(u;t,x,\xi)] \circ \mathrm{Op}^{BW}[b(u;t,x,\xi)] - \mathrm{Op}^{BW}\left[(a\#_\rho b)(u;t,x,\xi)\right] \tag{2.29}$$

is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho+m+m'}_{K,K',p+q}[\epsilon_0, N]$.

We have the following result, see e.g. Lemma 7.2 in [5].

Lemma 2.22 (Bony paraproduct decomposition)**.** *Let u_1, u_2 be functions in $H^\sigma(\mathbb{T};\mathbb{C})$ with $\sigma > \frac{1}{2}$. Then*

$$u_1 u_2 = \mathrm{Op}^{BW}[u_1]\, u_2 + \mathrm{Op}^{BW}[u_2]\, u_1 + R_1(u_1)u_2 + R_2(u_2)u_1 \tag{2.30}$$

where for $\mathrm{j} = 1, 2$, R_{j} *is a homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\rho}_1$ for any $\rho \ge 0$.*

We now state other composition results for m-operators which follow as in [10, Proposition 2.15].

Proposition 2.23 (Compositions of m-operators)**.** *Let $p, p', N, K, K' \in \mathbb{N}_0$ with $K' \le K$ and $\epsilon_0 > 0$. Let $m, m' \in \mathbb{R}$. Then*

i) If $M(u;t)$ is in $\Sigma\mathcal{M}^{m}_{K,K',p}[\epsilon_0, N]$ and $M'(u;t)$ is in $\Sigma\mathcal{M}^{m'}_{K,K',p'}[\epsilon_0, N]$ then the composition $M(u;t) \circ M'(u;t)$ is in $\Sigma\mathcal{M}^{m+\max(m',0)}_{K,K',p+p'}[\epsilon_0, N]$.

ii) If $M(u)$ is a homogeneous m-operator in $\widetilde{\mathcal{M}}^m_p$ and $M^{(\ell)}(u;t)$, $\ell = 1, \dots, p+1$, are matrices of m_ℓ-operators in $\Sigma\mathcal{M}^{m_\ell}_{K,K',q_\ell}[\epsilon_0, N]$ with $m_\ell \in \mathbb{R}$, $q_\ell \in \mathbb{N}_0$, then

$$M\left(M^{(1)}(u;t)u, \dots, M^{(p)}(u;t)u\right) M^{(p+1)}(u;t)$$

belongs to $\Sigma\mathcal{M}^{m+\bar{m}}_{K,K',p+\bar{q}}[\epsilon_0, N]$ with $\bar{m} := \sum_{\ell=1}^{p+1}\max(m_\ell, 0)$ and $\bar{q} := \sum_{\ell=1}^{p+1} q_\ell$.

iii) If $M(u;t)$ is in $\mathcal{M}^m_{K,0,p}[\breve{\epsilon}_0]$ for any $\breve{\epsilon}_0 \in \mathbb{R}^+$ and $\mathbb{M}_0(u;t)$ belongs to $\mathcal{M}^0_{K,K',0}[\epsilon_0]$, then $M(\mathbb{M}_0(u;t)u;t)$ is in $\mathcal{M}^m_{K,K',p}[\epsilon_0]$.

iv) Let a be a symbol in $\Sigma\Gamma^{m}_{K,K',p}[\epsilon_0, N]$ with $m \ge 0$ and R a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',p'}[\epsilon_0, N]$. Then

$$\mathrm{Op}^{BW}[a(u;t,\cdot)] \circ R(u;t), \quad R(u;t) \circ \mathrm{Op}^{BW}[a(u;t,\cdot)],$$

are in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',p+p'}[\epsilon_0, N]$.

Notation 2.24. In the sequel if $K' = 0$ we denote a symbol $a(u;t,x,\xi)$ in $\Gamma^m_{K,0,p}[\epsilon_0]$ simply as $a(u;x,\xi)$, and a smoothing operator in $R(u;t)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[\epsilon_0,N]$ simply as $R(u)$, without writing the t-dependence.

We finally provide the Bony paralinearization formula of the composition operator.

Lemma 2.25 (Bony Paralinearization formula)**.** *Let F be a smooth $\mathbb{C}$-valued function defined on a neighborhood of zero in $\mathbb{C}$, vanishing at zero at order $q \in \mathbb{N}$. Then there is $\epsilon_0 > 0$ and a smoothing operator $R(u)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,q'}[\epsilon_0,N]$, $q' := \max(q-1,1)$, for any ρ, such that*

$$F(u) = \operatorname{Op}^{BW}\left[F'(u)\right]u + R(u)\,u\,. \tag{2.31}$$

Proof. The formula follows by combination of [5, Lemmata 3.19 and 7.2]. □

2.2 z-dependent paradifferential calculus

Along the paralinearization process of the α-SQG equation in Section 4 we shall encounter parameter dependent paradifferential operators depending on a 2π-periodic variable z. The following "Kernel-functions" have to be considered as Taylor remainders of maps of the form $F(u;x,z)$ at $z=0$ which are smooth in u and with finite regularity in x and z. We are interested in the behavior of such functions close to $z=0$.

Definition 2.26 (Kernel functions)**.** Let $n \in \mathbb{R}$, $p,N \in \mathbb{N}_0$, $K \in \mathbb{N}_0$, and $\epsilon_0 > 0$.

i) **p-homogeneous Kernel-functions.** If $p \in \mathbb{N}$ we denote $\widetilde{K\mathcal{F}}^n_p$ the space of z-dependent, p-homogeneous maps from $H^\infty_0(\mathbb{T};\mathbb{C})$ to the space of x-translation invariant real functions $\varrho(u;x,z)$ of class $\mathcal{C}^\infty$ in $(x,z) \in \mathbb{T}^2$ with Fourier expansion

$$\varrho(u;x,z) = \sum_{j_1,\dots,j_p \in \mathbb{Z}\setminus\{0\}} \varrho_{j_1,\dots,j_p}(z)\,u_{j_1}\cdots u_{j_p} e^{\mathrm{i}(j_1+\cdots+j_p)x}\,, \quad z \in \mathbb{T}\setminus\{0\}\,, \tag{2.32}$$

with coefficients $\varrho_{j_1,\dots,j_p}(z)$ of class $\mathcal{C}^\infty(\mathbb{T};\mathbb{C})$, symmetric in $(j_1,\dots,j_p)$, satisfying the reality condition $\overline{\varrho_{j_1,\dots,j_p}}(z) = \varrho_{-j_1,\dots,-j_p}(z)$ and the following: for any $l \in \mathbb{N}_0$, there exist $\mu > 0$ and a constant $C > 0$ such that

$$\left|\partial_z^l \varrho_{j_1,\dots,j_p}(z)\right| \le C\left|\vec{\jmath}\right|^\mu\,|z|_{\mathbb{T}}^{n-l}\,,\quad \forall \vec{\jmath} = (j_1,\dots,j_p) \in (\mathbb{Z}\setminus\{0\})^p\,. \tag{2.33}$$

For $p=0$ we denote by $\widetilde{K\mathcal{F}}^n_0$ the space of maps $z \mapsto \varrho(z)$ which satisfy $\left|\partial_z^l\varrho(z)\right| \le C\,|z|_{\mathbb{T}}^{n-l}$.

ii) **Non-homogeneous Kernel-functions.** We denote by $K\mathcal{F}^n_{K,0,p}[\epsilon_0]$ the space of z-dependent, real functions $\varrho(u;x,z)$, defined for $u \in B^0_{s_0}(I;\epsilon_0)$ for some s_0 large enough, such that for any $0 \le k \le K$ and $l \le \max\{0,\lceil 1+n\rceil\}$, any $s \ge s_0$, there are $C>0$, $0<\epsilon_0(s)<\epsilon_0$ and for any $u \in B^K_{s_0}(I;\epsilon_0(s)) \cap C^k_*\left(I,H^s_0(\mathbb{T};\mathbb{C})\right)$ and any $\gamma \in \mathbb{N}_0$, with $\gamma \le s-s_0$, one has the estimate

$$\left|\partial_t^k\partial_x^\gamma\partial_z^l\varrho(u;x,z)\right| \le C\|u\|^{p-1}_{k,s_0}\|u\|_{k,s}\,|z|_{\mathbb{T}}^{n-l}\,,\qquad z \in \mathbb{T}\setminus\{0\}\,. \tag{2.34}$$

If $p=0$ the right hand side in (2.34) has to be replaced by $|z|_{\mathbb{T}}^{n-l}$.

iii) **Kernel-functions.** We denote by $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ the space of real functions of the form

$$\varrho(u;x,z) = \sum_{q=p}^N \varrho_q(u;x,z) + \varrho_{>N}(u;x,z) \tag{2.35}$$

where $\varrho_q(u;x,z)$, $q=p,\dots,N$ are homogeneous Kernel functions in $\widetilde{K\mathcal{F}}^n_q$, and $\varrho_{>N}(u;x,z)$ is a non-homogeneous Kernel function in $K\mathcal{F}^n_{K,0,N+1}[\epsilon_0]$.

A Kernel function $\varrho(u;x,z)$ is *real* if it is real valued for any $u \in B^0_{s_0,\mathbb{R}}(I;\epsilon_0)$.

In view of Remark 2.4, a homogeneous Kernel function $\varrho(u;x,z)$ in $\widetilde{K\mathcal{F}}^n_p$ defines a non-homogenous Kernel function in $K\mathcal{F}^n_{K,0,p}[\epsilon_0]$ for any $\epsilon_0 > 0$.

Remark 2.27. Let $\varrho(u;x,z)$ be a Kernel function in $\Sigma\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ with $n\ge 0$, which admits a continuous extension in $z=0$. Then its trace $\varrho(u;x,0)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,p}[\epsilon_0,N]$.

Remark 2.28. If $\varrho(u;x,z)$ is a homogeneous Kernel function $\widetilde{K\mathcal{F}}^n_p$, the two definitions of quantization in (2.17) differ by a Kernel smoothing operator in $\widetilde{K\mathcal{R}}^{-\rho,n}_p$, for any $\rho>0$, according to Definition 2.33 below.

Remark 2.29. If $\varrho_1(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{n_1}_{K,0,p_1}[\epsilon_0,N]$ and $\varrho_2(u;x,z)$ in $\Sigma K\mathcal{F}^{n_2}_{K,0,p_2}[\epsilon_0,N]$, then the sum $(\varrho_1+\varrho_2)(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{\min\{n_1,n_2\}}_{K,0,\min\{p_1,p_2\}}[\epsilon_0,N]$ and the product $(\varrho_1\varrho_2)(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{n_1+n_2}_{K,0,p_1+p_2}[\epsilon_0,N]$.

Remark 2.30. Let $\varrho(u;x,z)$ be a Kernel function in $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ with $n>-1$. Then $⨍\varrho(u;x,z)\,\mathrm{d}z$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,p}[\epsilon_0,N]$. This follows directly integrating (2.33) and (2.34) in z.

The m-Kernel-operators defined below are a z-dependent family of m-operators with coefficients small as $|z|^n_{\mathbb{T}}$. They appear for example as smoothing operators in the composition of Bony-Weyl quantizations of Kernel-functions.

Definition 2.31. Let $m,n\in\mathbb{R}$, $p,N\in\mathbb{N}_0$, $K\in\mathbb{N}_0$ with $\epsilon_0>0$.

i) **p-homogeneous m-Kernel-operator.** We denote by $\widetilde{K\mathcal{M}}^{m,n}_p$ the space of z-dependent, x-translation invariant homogeneous m-operators according to Definition 2.14, Item i, in which the constant C is substituted with $C|z|^n_{\mathbb{T}}$, equivalently

$$M(u;z)v(x)=\sum_{\substack{(\vec{\jmath}_p,j,k)\in\mathbb{Z}^{p+2}\\ j_1+\ldots+j_p+j=k}} M_{\vec{\jmath}_p,j,k}(z)\,u_{j_1}\ldots u_{j_p}v_je^{\mathrm{i}kx},\qquad z\in\mathbb{T}\setminus\{0\},\tag{2.36}$$

with coefficients satisfying

$$|M_{\vec{\jmath}_p,j,k}(z)|\le C\max{}_2\{\langle j_1\rangle,\ldots,\langle j_p\rangle,\langle j\rangle\}^{\mu}\max\{\langle j_1\rangle,\ldots,\langle j_p\rangle,\langle j\rangle\}^m|z|^n_{\mathbb{T}}.\tag{2.37}$$

If $p=0$ the right hand side of (2.36) is replaced by $\sum_{j\in\mathbb{Z}}M_j(z)\,v_je^{\mathrm{i}jx}$ with $|M_j(z)|\le C\langle j\rangle^m|z|^n_{\mathbb{T}}$.

ii) **Non-homogeneous m-Kernel-operator.** We denote by $K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0]$ the space of z-dependent, non-homogeneous operators $M(u;z)v$ defined for any $z\in\mathbb{T}\setminus\{0\}$, such that for any $0\le k\le K$

$$\left\|\partial_t^k(M(u;z)v)\right\|_{s-\alpha k-m}\le C|z|^n_{\mathbb{T}}\sum_{k'+k''=k}\left(\|v\|_{k'',s}\|u\|^p_{k',s_0}+\|v\|_{k'',s_0}\|u\|^{p-1}_{k',s_0}\|u\|_{k',s}\right).\tag{2.38}$$

iii) **m-Kernel-Operator.** We denote by $\Sigma K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0,N]$ the space of operators of the form

$$M(u;z)v=\sum_{q=p}^{N}M_q(u,\ldots,u)v+M_{>N}(u;z)v\tag{2.39}$$

where M_q are homogeneous m-Kernel operators in $\widetilde{K\mathcal{M}}^{m,n}_q$, $q=p,\ldots,N$ and $M_{>N}$ is a non–homogeneous m-Kernel-operator in $\mathcal{M}^{m,n}_{K,0,N+1}[\epsilon_0]$.

iv) **Pluri-homogeneous m-Kernel-Operator.** We denote by $\Sigma^N_p\widetilde{\mathcal{M}}^m_q$ the pluri-homogeneous m-operators of the form (2.39) with $M_{>N}=0$.

Remark 2.32. Given $\varrho(u;x,z)\in\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ then $\mathrm{Op}^{BW}\left[\varrho(u;x,z)\right]\in\Sigma K\mathcal{M}^{0,n}_{K,0,p}[\epsilon_0,N]$.

Definition 2.33 (Kernel-smoothing operators)**.** Given $\rho>0$ we define the homogeneous and non-homogeneous Kernel-smoothing operators as

$$\widetilde{K\mathcal{R}}^{-\rho,n}_p:=\widetilde{K\mathcal{M}}^{-\rho,n}_p,\qquad K\mathcal{R}^{-\rho,n}_{K,0,p}[\epsilon_0]:=K\mathcal{M}^{-\rho,n}_{K,0,p}[\epsilon_0],\qquad \Sigma K\mathcal{R}^{-\rho,n}_{K,0,p}[\epsilon_0,N]:=\Sigma K\mathcal{M}^{-\rho,n}_{K,0,p}[\epsilon_0,N].$$

In view of [10, Lemma 2.8], if $M(u,\ldots,u;z)$ is a homogenous m-Kernel operator in $\widetilde{K\mathcal{M}}^{m,n}_p$ then $M(u,\ldots,u;z)$ defines a non-homogenous m-Kernel operator in $K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0]$ for any $\epsilon_0>0$ and $K\in\mathbb{N}_0$.

Proposition 2.34 (Composition of z-dependent operators)**.** *Let $m, n, m', n' \in \mathbb{R}$, and integers $K, p, p', N \in \mathbb{N}_0$ with $p, p' \leq N$.*

1. *Let $\varrho(u;x,z) \in \Sigma K\mathcal{F}^{n}_{K,0,p}[\epsilon_0, N]$ and $\varrho'(u;x,z) \in \Sigma K\mathcal{F}^{n'}_{K,0,p'}[\epsilon_0, N]$ be Kernel functions. Then*
$$\mathrm{Op}^{BW}\left[\varrho(u;x,z)\right] \circ \mathrm{Op}^{BW}\left[\varrho'(u;x,z)\right] = \mathrm{Op}^{BW}\left[\varrho\,\varrho'(u;x,z)\right] + R(u;z)$$
where $R(u;z)$ is a Kernel-smoothing operator in $\Sigma K\mathcal{R}^{-\rho,n+n'}_{K,0,p+p'}[\epsilon_0, N]$ for any $\rho \geq 0$;

2. *Let $M(u;z)$ be a m-operator in $\Sigma K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0, N]$ and $M'(u;z)$ be an m'-operator in $\Sigma K\mathcal{M}^{m',n'}_{K,0,p'}[\epsilon_0, N]$. Then $M(u;z) \circ M'(u;z)$ belongs to $\Sigma K\mathcal{M}^{m+\max(m',0),n+n'}_{K,0,p+p'}[\epsilon_0, N]$;*

3. *Let $\varrho(u;x,z)$ be a Kernel function in $\Sigma K\mathcal{F}^{n}_{K,0,p}[\epsilon_0, N]$ and $R(u;z)$ be a Kernel smoothing operator in $\Sigma K\mathcal{R}^{-\rho,n'}_{K,0,p'}[\epsilon_0, N]$ then $\mathrm{Op}^{BW}\left[\varrho(u;x,z)\right] \circ R(u;z)$ and $R(u;z) \circ \mathrm{Op}^{BW}\left[\varrho(u;x,z)\right]$ are a Kernel smoothing operator in $\Sigma K\mathcal{R}^{-\rho,n+n'}_{K,0,p+p'}[\epsilon_0, N]$;*

4. *Let $M(u;z)$ be an homogeneous m-Kernel operator in $\widetilde{K\mathcal{M}}^{m,n}_{1}$, and $M'(u;z)$ in $\Sigma K\mathcal{M}^{0,0}_{K,0,0}[\epsilon_0, N]$ then $M\left(M'(u;z)\,u;z\right) \in \Sigma K\mathcal{M}^{m,n}_{K,0,1}[\epsilon_0, N]$.*

Proof. The proof of item 1 is performed in [5, Proposition 3.12] keeping track of the dependence in the variable z of the symbols as in (2.33), (2.34) when $\gamma = 0$. More precisely ϱ and ϱ' satisfy z-dependent inequalities (cf. (2.7), (2.9))
$$\left|\partial_x^\alpha \varrho_q(\Pi_{\vec{n}}\mathcal{U};x,z)\right| \leq C|z|^n_{\mathbb{T}}\,|\vec{n}|^{\mu+\alpha}\prod_{j=1}^{p}\left\|\Pi_{n_j}u_j\right\|_{L^2}, \qquad \left|\partial_t^k\partial_x^\alpha\varrho(u;x,z)\right| \leq C|z|^n_{\mathbb{T}}\,\|u\|^{p-1}_{k,s_0}\,\|u\|_{k,s},$$
and, in the proof of [5, Proposition 3.12], the seminorm of the composed symbol always appear as a product of the seminorms of the factor symbols. The proof of item 2 is the same as in [10, Proposition 2.15-i], keeping track of the dependence in z of the m-operators. For item 3, see [10, Proposition 2.19-i] factoring the dependence on z. Item 4 is a consequence of [10, Proposition 2.15-ii] factoring the dependence on z. $\square$

Finally integrating (2.37) and (2.38) in z we deduce the following lemma.

Lemma 2.35. *Let $R(u;z)$ be a Kernel smoothing operator in $\Sigma\mathcal{R}^{-\rho,n}_{K,0,p}[\epsilon_0, N]$ with $n > -1$. Then*
$$\fint R(u;z)\,g(x-z)\,\mathrm{d}z = R_1(u)\,g, \qquad \fint R(u;z)\,\mathrm{d}z = R_2(u),$$
where $R_1(u)$, $R_2(u)$ are smoothing operators in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[\epsilon_0, N]$.

The following proposition will be crucial in Section 4.

Proposition 2.36. *Let $n > -1$ and $\varrho(u;x,z)$ be a Kernel-function in $\Sigma K\mathcal{F}^{n}_{K,0,p}[\epsilon_0, N]$. Let us define the operator, for any $g \in H^s_0(\mathbb{T})$, $s \in \mathbb{R}$,*
$$\left(\mathcal{T}_\varrho g\right)(x) := \fint \mathrm{Op}^{BW}\left[\varrho(u;\bullet,z)\right] g(x-z)\,\mathrm{d}z. \tag{2.40}$$
Then there exists

- *a symbol $a(u;x,\xi)$ in $\Sigma\Gamma^{-(1+n)}_{K,0,p}[\epsilon_0, N]$ satisfying (2.20);*
- *a pluri-homogeneous smoothing operator $R(u)$ in $\Sigma^N_p\widetilde{\mathcal{R}}^{-\rho}_q$ for any $\rho > 0$;*

such that $\mathcal{T}_\varrho g = \mathrm{Op}^{BW}\left[a(u;x,\xi)\right] g + R(u)\,g$.

Proof. In view of Definition 2.10 and Remark 2.28 we have that

$$\mathrm{Op}^{BW}\left[\varrho(u;x,z)\right]-\mathrm{Op}^{W}\left[\varrho_{\psi}(u;x,z)\right]=:R(u;z) \tag{2.41}$$

is a pluri-homogeneous Kernel smoothing operator in $\Sigma_p^N\widetilde{K\mathcal{R}}_q^{-\rho,n}$ for any ρ. Since $n>-1$, integrating in z, we deduce that $\fint R(u;z)\,g(x-z)\mathrm{d}z = R(u)\,g$ where $R(u)$ is a pluri-homogeneous smoothing operator in $\Sigma_p^N\widetilde{\mathcal{R}}_q^{-\rho}$ (cf. Lemma 2.35).

In view of (2.17) and (2.16) we compute for any $\nu\in\mathbb{Z}$

$$\mathcal{F}_{x\to\nu}\left(\fint \mathrm{Op}^{W}\left[\varrho_{\psi}(u;x,z)\right]g(x-z)\,\mathrm{d}z\right)(\nu)=\sum_{k\in\mathbb{Z}}\psi\left(\nu-k,\frac{\nu+k}{2}\right)\fint\hat{\varrho}(u;\nu-k,z)\,e^{-\mathrm{i}kz}\mathrm{d}z\,\hat{g}(k)$$

where $\psi(\xi',\xi)$ is an admissible cut-off function, namely satisfying (2.14)-(2.15). Introducing another admissible cut-off function $\tilde{\psi}(\xi',\xi)$ identically equal to one on the support of $\psi(\xi',\xi)$, and since $\widehat{g}(0)=0$,

$$\begin{aligned}\mathcal{F}_{x\to\nu}&\left(\fint \mathrm{Op}^{W}\left[\varrho_{\psi}(u;x,z)\right]\mathrm{d}z\right)(\nu)\\&=\sum_{k\in\mathbb{Z}}\psi\left(\nu-k,\frac{\nu+k}{2}\right)\tilde{\psi}\left(\nu-k,\frac{\nu+k}{2}\right)\chi(2k)\fint\hat{\varrho}(u;\nu-k,z)\,e^{-\mathrm{i}kz}\mathrm{d}z\,\hat{g}(k)\end{aligned} \tag{2.42}$$

where $\chi(\cdot)$ is the $\mathcal{C}^\infty$ function defined in (2.18). Introducing a $\mathcal{C}^\infty$ function $\eta:\mathbb{R}\to[0,1]$ with compact support such that

$$\eta(z)=1\,,\ \forall|z|\le\frac{\pi}{2}\,,\qquad \eta(z)=0\,,\ \forall|z|\ge\frac{3\pi}{2}\,,\qquad \sum_{j\in\mathbb{Z}}\eta(z+2\pi j)=1\,,\ \forall z\in\mathbb{R}\,, \tag{2.43}$$

we may write the integral on $\mathbb{T}$ as

$$\fint\hat{\varrho}(u;\nu-k,z)\,e^{-\mathrm{i}kz}\mathrm{d}z=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varrho}\left(u;j,z\right)\eta(z)e^{-\mathrm{i}\xi z}\mathrm{d}z\,\Big|_{(j,\xi)=(\nu-k,k)}\,. \tag{2.44}$$

Therefore by (2.41), (2.42) and (2.44) the operator $\mathcal{T}_\varrho$ in (2.40) is equal to

$$\mathcal{T}_\varrho=\mathrm{Op}^{W}\left[a_{\psi}(u;x,\xi)\right]=\mathrm{Op}^{BW}\left[a(u;x,\xi)\right]+R(u)\qquad\text{where}\qquad R(u)\in\Sigma_p^N\widetilde{\mathcal{R}}_q^{-\rho} \tag{2.45}$$

and

$$a(u;x,\xi)=\sum_{j\in\mathbb{Z}}\hat{a}\left(u;j,\xi\right)e^{\mathrm{i}jx}\,,\quad \hat{a}\left(u;j,\xi\right):=\tilde{\psi}\left(j,\xi\right)\chi\left(2\xi-j\right)\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varrho}\left(u;j,z\right)\eta(z)e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z\,. \tag{2.46}$$

In order to prove the lemma, in view of (2.45), it is sufficient to show that $a(u;x,\xi)$ defined in (2.46) is a symbol in $\Sigma\Gamma_{K,0,p}^{-(1+n)}[\epsilon_0,N]$ according to Definition 2.2. Notice that $a(u;x,\xi)$ satisfies the reality condition (2.20). Moreover, in view of the support properties of $\tilde{\psi}\left(j,\xi\right)$ and $\chi\left(2\xi-j\right)$, it results that

$$\hat{a}\left(u;j,\xi\right)\neq0\qquad\Longrightarrow\qquad\left|\xi-\frac{j}{2}\right|\sim|\xi|\,,\quad|\xi|\gtrsim1\,,\quad|\xi|\sim\langle\xi\rangle\,. \tag{2.47}$$

We decompose the Kernel function

$$\varrho(u;x,z)\in\Sigma K\mathcal{F}_{K,0,p}^{n}[\epsilon_0,N]\qquad\text{as}\qquad\varrho(u;x,z)=\sum_{q=p}^{N}\varrho_q(u;x,z)+\varrho_{>N}(u;x,z)\,,$$

where $\varrho_q(u;x,z)$ are homogenous Kernel functions in $\widetilde{K\mathcal{F}}_q^n$ and $\varrho_{>N}(u;x,z)$ is a non-homogenous Kernel function in $K\mathcal{F}_{K,0,N+1}^n[\epsilon_0]$. Accordingly we decompose the symbol $a(u;x,\xi)$ in (2.46) as

$$a(u;x,\xi)=\sum_{q=p}^{N}a_q(u;x,\xi)+a_{>N}(u;x,\xi)$$

where

$$
\begin{aligned}
&a_q(u;x,\xi)=\sum_{j\in\mathbb{Z}}\hat{a}_q(u;j,\xi)\,e^{\mathrm{i}jx}, \quad \hat{a}_q(u;j,\xi):=\tilde{\psi}(j,\xi)\,\chi(2\xi-j)\,\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varrho}_q(u;j,z)\,\eta(z)e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z,\\
&a_{>N}(u;x,\xi)=\sum_{j\in\mathbb{Z}}\hat{a}_{>N}(u;j,\xi)\,e^{\mathrm{i}jx}, \quad \hat{a}_{>N}(u;j,\xi):=\tilde{\psi}(j,\xi)\,\chi(2\xi-j)\,\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varrho}_{>N}(u;j,z)\,\eta(z)e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z.
\end{aligned}
\tag{2.48}
$$

We now prove that, according to Definition 2.2,

$$
a_q\in\widetilde{\Gamma}_q^{-(1+n)}, \quad \forall q=p,\ldots,N, \tag{2.49}
$$

$$
a_{>N}\in\Gamma_{K,0,N+1}^{-(1+n)}[\epsilon_0]. \tag{2.50}
$$

Step 1 (Proof of (2.49))**.** In view of (2.32) the q-homogeneous component $a_q(u;x,\xi)$ in (2.48) has an expansion as in (2.10) (recall the notation $\vec{\jmath}_q=(j_1,\ldots,j_q)$)

$$
a_{\vec{\jmath}_q}(\xi)=\tilde{\psi}(j,\xi)\,\chi(2\xi-j)\,\frac{1}{2\pi}\int_{\mathbb{R}}\varrho_{j_1,\ldots,j_q}(z)\,\eta(z)\,e^{-\mathrm{i}\xi z}\mathrm{d}z, \qquad j=j_1+\ldots+j_q.
$$

Let us prove it satisfies (2.11) with $m=-(1+n)$. Decomposing $1=\chi_1(\cdot)+\chi_2(\cdot)$ where $\chi_1:\mathbb{R}\to[0,1]$ is a smooth cutoff function supported and equal to 1 near 0, we decompose

$$
a_{\vec{\jmath}_q}(\xi)=a^{(1)}_{\vec{\jmath}_q}(\xi)+a^{(2)}_{\vec{\jmath}_q}(\xi)=\sum_{\mathrm{j}=1}^{2}\tilde{\psi}(j,\xi)\,\chi(2\xi-j)\,\frac{1}{2\pi}\int_{\mathbb{R}}\chi_{\mathrm{j}}(\langle\xi\rangle z)\,\varrho_{j_1,\ldots,j_q}(z)\,\eta(z)\,e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z. \tag{2.51}
$$

By (2.33) (with $l=0$) and since $n>-1$ we deduce

$$
\left|a^{(1)}_{\vec{\jmath}_q}(\xi)\right|\lesssim\int_{|z|\lesssim 1/\langle\xi\rangle}\left|\vec{\jmath}_q\right|^{\mu}|z|^{n}\mathrm{d}z\lesssim\left|\vec{\jmath}_q\right|^{\mu}\langle\xi\rangle^{-(1+n)}. \tag{2.52}
$$

We now estimate $a^{(2)}_{\vec{\jmath}}(\xi)$. From $e^{-\mathrm{i}z\left(\xi-\frac{j}{2}\right)}=\left[-\mathrm{i}\left(\xi-\frac{j}{2}\right)\right]^{-l}\partial_z^l\left(e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\right)$ for any $l\in\mathbb{N}_0$, we obtain, by an integration by parts (use (2.43) and that $\chi_2(\langle\xi\rangle z)$ vanishes near zero), that

$$
\begin{aligned}
a^{(2)}_{\vec{\jmath}_q}(\xi)=\left[-\mathrm{i}\left(\xi-\frac{j}{2}\right)\right]^{-l}\tilde{\psi}(j,\xi)\,\chi(2\xi-j)&\sum_{l_1+l_2+l_3=l}c_{l_1,l_2,l_3}\frac{1}{2\pi}\int_{\mathbb{R}}e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\,Y_{l_1,l_2,l_3}(z)\,\mathrm{d}z\\
&\text{where}\quad Y_{l_1,l_2,l_3}(z):=\langle\xi\rangle^{l_1}\left(\partial_z^{l_1}\chi_2\right)(z)\,\partial_z^{l_2}\eta(z)\,\partial_z^{l_3}\varrho_{\vec{\jmath}_q}(z).
\end{aligned}
\tag{2.53}
$$

Since $\varrho_q(u;x,z)$ is a Kernel function in $\widetilde{K\mathcal{F}}^n_p$, using (2.33) and exploiting that $\langle\xi\rangle^{-1}\sim|z|$ on the support of $\chi_2^{(l_1)}(\langle\xi\rangle z)$ for any $l_1\geq 1$, we get

$$
\int_{\mathbb{R}}\left|Y_{l_1,l_2,l_3}(z)\right|\mathrm{d}z\lesssim\left|\vec{\jmath}_q\right|^{\mu}\langle\xi\rangle^{l_1}\int_{\frac{1}{\langle\xi\rangle}\sim|z|}|z|^{n-l_3}\,\mathrm{d}z\lesssim\left|\vec{\jmath}_q\right|^{\mu}\langle\xi\rangle^{l_1+l_3-(n+1)}. \tag{2.54}
$$

When $l_1=0$ we have that

$$
\int_{\mathbb{R}}\left|Y_{0,l_2,l_3}(z)\right|\mathrm{d}z\lesssim\left|\vec{\jmath}_q\right|^{\mu}\int_{\frac{\delta}{\langle\xi\rangle}\leq|z|\leq\frac{3\pi}{2}}|z|^{n-l_3}\,\mathrm{d}z\lesssim\left|\vec{\jmath}_q\right|^{\mu}\left(1+\langle\xi\rangle^{l_3-(n+1)}\right). \tag{2.55}
$$

Then by (2.53), (2.54), (2.55) and (2.47), we deduce, for $l>n+1$,

$$
|a^{(2)}_{\vec{\jmath}_q}(\xi)|\lesssim_l\left|\vec{\jmath}_q\right|^{\mu}\langle\xi\rangle^{-(1+n)}. \tag{2.56}
$$

The bounds (2.52) and (2.56) prove that $a_{\vec{\jmath}_q}(\xi)$ in (2.51) satisfies the estimate (2.11) (for $\beta=0$ and $m=-(1+n)$). Since, for any $\beta\in\mathbb{N}$,

$$
\partial_\xi^\beta a_{\vec{\jmath}_q}(\xi)=\sum_{\beta_1+\beta_2+\beta_3=\beta}C_{\beta_1,\beta_2,\beta_3}\partial_\xi^{\beta_1}\tilde{\psi}(j,\xi)\,\partial_\xi^{\beta_2}\chi(2\xi-j)\int_{\mathbb{R}}\varrho_{\vec{\jmath}_q}(z)\,(-\mathrm{i}z)^{\beta_3}\,e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z, \tag{2.57}
$$

using (2.15), (2.47), the fact that χ is supported near 0, that $z^{\beta_3}\varrho_{\vec{J}_q}(z)$ satisfies (2.33) with (n replaced by $n+\beta_3$, cf. Remark 2.29) and repeating the bound obtained for $\beta_3=0$ for the integral term in Eq. (2.57), we obtain

$$\left|\partial_\xi^\beta a_{\vec{J}_q}(\xi)\right| \lesssim_\beta \sum_{\beta_1+\beta_2+\beta_3=\beta} \langle\xi\rangle^{-\beta_1}\langle\xi\rangle^{-\beta_2}\left|\vec{J}_q\right|^\mu \langle\xi\rangle^{-(1+n+\beta_3)} \lesssim_\beta \left|\vec{J}_q\right|^\mu \langle\xi\rangle^{-(1+n+\beta)}.$$

Note that actually for any $j\in\mathbb{Z}$, $\beta_2\geq 1$, the derivative $\partial_\xi^{\beta_2}\chi\left(2\xi-j\right)=0$, for any $|\xi|\geq 2$. This concludes the proof of (2.49).

Step 2 (Proof of (2.50))**.** We argue similarly to the previous step. Recalling (2.48), for any $0\leq k\leq K$ and $\gamma\in\mathbb{N}_0$, we decompose, with χ_{j}, j = 1,2 defined as in (2.51),

$$\begin{aligned}
&\partial_t^k\partial_x^\gamma a_{>N}(u;x,\xi) = I_1+I_2 \qquad \text{where}\\
&I_{\mathrm{j}} := \sum_{j\in\mathbb{Z}} \tilde{\psi}\left(j,\xi\right)\chi\left(2\xi-j\right)\frac{1}{2\pi}\int_{\mathbb{R}}\chi_{\mathrm{j}}\left(z\langle\xi\rangle\right)\widehat{\partial_t^k\partial_x^\gamma\varrho_{>N}}\left(u;j,z\right)\eta\left(z\right)e^{-\mathrm{i}\left(\xi-\frac{j}{2}\right)z}\mathrm{d}z\, e^{\mathrm{i}jx}.
\end{aligned} \tag{2.58}$$

Fix $\mu_0>1$, let $s_0>0$ associated to $\varrho_{>N}$ as per Definition 2.26, let $\gamma\leq s-\left(s_0+\mu_0\right)$. The term I_1 can be estimated using (2.34), the fact that $\chi_1(z\langle\xi\rangle)$ is supported for $|z|\lesssim 1/\langle\xi\rangle$ and $n>-1$, as

$$|I_1| \lesssim \sum_{j\in\mathbb{Z}}\left\langle j\right\rangle^{-\mu_0}\int_{|z|\lesssim 1/\langle\xi\rangle}\left|\partial_t^k\partial_x^\gamma\widehat{\left(1-\partial_x^2\right)^{\frac{\mu_0}{2}}\varrho_{>N}}\left(u;j,z\right)\right|\mathrm{d}z \lesssim \|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\langle\xi\rangle^{-(1+n)}. \tag{2.59}$$

Next we estimate I_2. After an integration by parts, setting $l=\max\{0,\lceil 1+n\rceil\}$, we have from Eq. (2.58)

$$|I_2| \lesssim \sum_{j\in\mathbb{Z}}\left|\tilde{\psi}\left(j,\xi\right)\chi\left(2\xi-j\right)\right|\left|\xi-\frac{j}{2}\right|^{-l}\sum_{l_1+l_2+l_3=l}\int_{\mathbb{R}}\left|Z_{l_1,l_2,l_3}(z)\right|\mathrm{d}z. \tag{2.60}$$

where

$$Z_{l_1,l_2,l_3}(z) := \langle\xi\rangle^{l_1}\left(\partial_z^{l_1}\chi_2\right)\left(z\langle\xi\rangle\right)\,\partial_z^{l_2}\eta(z)\,\partial_t^k\widehat{\partial_x^\gamma\partial_z^{l_3}\varrho_{>N}}\left(u;j,z\right). \tag{2.61}$$

For any $j\in\mathbb{Z}$

$$\left|\widehat{\partial_t^k\partial_x^\gamma\partial_z^{l_3}\varrho_{>N}}\left(u;j,z\right)\right| \lesssim \left\langle j\right\rangle^{-\mu_0}\sup_{x\in\mathbb{T}}\left|\partial_t^k\partial_x^\gamma\partial_z^{l_3}\left(1-\partial_x^2\right)^{\frac{\mu_0}{2}}\varrho_{>N}(u;x,z)\right| \lesssim \left\langle j\right\rangle^{-\mu_0}\|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}|z|^{n-l_3}. \tag{2.62}$$

With computations analogous to the ones performed in Eqs. (2.54) and (2.55) we obtain using Eqs. (2.61) and (2.62), that

$$\begin{aligned}
&\int_{\mathbb{R}}\left|Z_{l_1,l_2,l_3}(z)\right|\mathrm{d}z \lesssim \left\langle j\right\rangle^{-\mu_0}\|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\langle\xi\rangle^{(l_1+l_3)-(n+1)}, \qquad \text{if } l_1\neq 0,\\
&\int_{\mathbb{R}}\left|Z_{0,l_2,l_3}(z)\right|\mathrm{d}z \lesssim \left\langle j\right\rangle^{-\mu_0}\|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\left(1+\langle\xi\rangle^{l_3-(n+1)}\right).
\end{aligned} \tag{2.63}$$

Since $l=l_1+l_2+l_3>1+n$ and $\mu_0>1$ we obtain, by Eqs. (2.47), (2.60) and (2.63) that

$$|I_2| \lesssim \|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\langle\xi\rangle^{-(1+n)}\sum_{j\in\mathbb{Z}}\left\langle j\right\rangle^{-\mu_0}. \tag{2.64}$$

Inserting Eqs. (2.59) and (2.64) in (2.58) we conclude that

$$\left|\partial_t^k\partial_x^\gamma a_{>N}(u;x,\xi)\right| \lesssim \|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\langle\xi\rangle^{-(1+n)}.$$

Arguing as in (2.57) we thus obtain that, for any $\beta\in\mathbb{N}$, $\left|\partial_t^k\partial_x^\gamma\partial_\xi^\beta a_{>N}(u;x,\xi)\right| \lesssim \|u\|_{k,s_0+\mu_0}^N\|u\|_{k,s}\langle\xi\rangle^{-(1+n+\beta)}$ concluding the proof of (2.50).

□

3 The linearized problem at $f = 0$

The linearized equation (1.11) at $f = 0$ is

$$\partial_t f = \partial_x \, \mathrm{d}\nabla E_\alpha(0) f \,. \tag{3.1}$$

In this section we prove that the linear Hamiltonian operator $\partial_x \, \mathrm{d}\nabla E_\alpha(0)$ is a Fourier multiplier with symbol $-\mathrm{i}\omega_\alpha(j)$ and we provide its asymptotic expansion, see Lemmas 3.1 and 3.6. We also prove a convexity property of the frequency map $j \mapsto \omega_\alpha(j)$ and that $\omega_\alpha(j)$ are positive for any $j \geq 2$, whereas $\omega_\alpha(0) = \omega_\alpha(1) = 0$, cf. Remark 3.3. These latter results do not rely on oscillatory integrals expansions and enable to prove the absence of three-wave resonances in Lemma 3.5.

The following result extends the computations in [42, Section 3], valid for $\alpha \in (0,1)$, to the whole range $\alpha \in (0,2)$.

Lemma 3.1 (Linearization of ∇E_α at zero)**.** *For any $\alpha \in (0,2)$, it results that*

$$\mathrm{d}\nabla E_\alpha(0) = -L_\alpha(|D|) \,, \tag{3.2}$$

where $L_\alpha(|D|)$ is the Fourier multiplier operator

$$L_\alpha(|D|) := \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \left[\mathsf{T}^1_\alpha(|D|) - \mathsf{T}^2_\alpha(|D|) - \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} \right] \tag{3.3}$$

with

$$\mathsf{T}^1_\alpha(|j|) := \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \sum_{k=0}^{|j|-1} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(1-\frac{\alpha}{2}+k\right)} \frac{1}{1-\frac{\alpha}{2}+k} \,, \qquad \mathsf{T}^1_\alpha(0) = 0 \,, \tag{3.4}$$

$$\mathsf{T}^2_\alpha(|\xi|) := \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2}+|\xi|\right)}{\Gamma\left(1-\frac{\alpha}{2}+|\xi|\right)} = \left[|\xi|^2 - \left(1-\frac{\alpha}{2}\right)^2 \right] \mathsf{M}_\alpha(|\xi|) \,, \tag{3.5}$$

$$\mathsf{M}_\alpha(|\xi|) := \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|\xi|^2 - \left(1-\frac{\alpha}{2}\right)^2} \frac{\Gamma\left(\frac{\alpha}{2}+|\xi|\right)}{\Gamma\left(1-\frac{\alpha}{2}+|\xi|\right)} \,. \tag{3.6}$$

The map $j \mapsto \mathsf{T}^1_\alpha(|j|)$ is a Fourier multiplier in $\widetilde{\Gamma}^{\max(0,\alpha-1)}_0$ and $j \mapsto \mathsf{T}^2_\alpha(|j|)$ is a Fourier multiplier in $\widetilde{\Gamma}^{\alpha-1}_0$.

By the previous lemma, in Fourier, the linear equation (3.1) amounts to the decoupled scalar equations

$$\partial_t \hat{f}(j) + \mathrm{i}\omega_\alpha(j) \hat{f}(j) = 0 \,, \qquad j \in \mathbb{Z} \setminus \{0\} \,, \tag{3.7}$$

with linear frequencies of oscillations $\omega_\alpha(j) := j L_\alpha(|j|)$.

Proof of Lemma 3.1. By differentiating $\nabla E_\alpha(f)$ in (1.12) we deduce that

$$\begin{aligned}
\mathrm{d}\nabla E_\alpha(0)\,\phi &= \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{2\phi(y) - (\phi(x)+\phi(y))\cos(x-y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}}} \, \mathrm{d}y \\
&\quad + \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{\phi'(y)\sin(x-y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}}} \, \mathrm{d}y - \frac{\alpha}{4} \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{\phi(x)+\phi(y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}-1}} \, \mathrm{d}y \\
&= -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{\phi(x)-\phi(y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}}} \, \mathrm{d}y + \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \int \frac{\phi'(y)\sin(x-y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}}} \, \mathrm{d}y \\
&\quad + \frac{c_\alpha}{4} \int \frac{\phi(y)}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}-1}} \, \mathrm{d}y + \frac{c_\alpha}{4} \int \frac{\mathrm{d}y}{\left[2\left(1-\cos(x-y)\right)\right]^{\frac{\alpha}{2}-1}} \, \phi(x) =: \sum_{\kappa=1}^{4} L_{\nabla E_\alpha,\kappa} \phi \,.
\end{aligned} \tag{3.8}$$

We now compute these operators.

Step 1 (Evaluation of $L_{\nabla E_\alpha,1}$)**.** We claim that

$$\left(L_{\nabla E_\alpha,1}\phi\right)(x) := -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\fint \frac{\phi(x)-\phi(y)}{\left[2\left(1-\cos\left(x-y\right)\right)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}y = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\mathsf{T}^1_\alpha\left(|D|\right)\phi. \tag{3.9}$$

Indeed, setting $y = x - z$ we have

$$L_{\nabla E_\alpha,1}\phi = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\fint \frac{\phi(x)-\phi(x-z)}{\left[2\left(1-\cos z\right)\right]^{\alpha/2}}\mathrm{d}z = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\fint \frac{\phi(x)-\phi(x-z)}{\left|2\sin\left(\frac{z}{2}\right)\right|^\alpha}\mathrm{d}z. \tag{3.10}$$

We compute the action of $L_{\nabla E_\alpha,1}$ on $\phi(x) = \sum_{j\in\mathbb{Z}}\hat{\phi}\left(j\right)e^{\mathrm{i}jx} = \hat{\phi}(0) + \underbrace{\sum_{j\geq 1}\hat{\phi}\left(j\right)e^{\mathrm{i}jx}}_{=:\phi_+(x)} + \underbrace{\sum_{j\geq 1}\overline{\hat{\phi}\left(j\right)}e^{-\mathrm{i}jx}}_{=:\phi_-(x)}$.

By (3.10) we immediately get $L_{\nabla E_\alpha,1}\hat{\phi}(0) = 0$. Moreover, by (3.10),

$$L_{\nabla E_\alpha,1}\phi_+(x) = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\sum_{j\geq 1}\hat{\phi}\left(j\right)e^{\mathrm{i}jx}\fint \frac{1-e^{-\mathrm{i}jz}}{\left|4\sin^2\left(z/2\right)\right|^{\alpha/2}}\mathrm{d}z. \tag{3.11}$$

We compute

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-e^{-\mathrm{i}jz}}{\left|4\sin^2\left(z/2\right)\right|^{\alpha/2}}\mathrm{d}z &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-e^{-\mathrm{i}jz}}{\left|1-e^{-\mathrm{i}z}\right|^\alpha}\mathrm{d}z\\ &= \frac{1}{\pi}\int_0^{\pi}\frac{1-e^{-2\mathrm{i}jz}}{\left|1-e^{-2\mathrm{i}z}\right|}\left|1-e^{-2\mathrm{i}z}\right|^{1-\alpha}\mathrm{d}z = -\frac{2^{1-\alpha}}{\mathrm{i}\pi}\sum_{k=0}^{j-1}\int_0^\pi e^{-\mathrm{i}z(2k+1)}\left(\sin z\right)^{1-\alpha}\mathrm{d}z\end{aligned} \tag{3.12}$$

having also written $\left|1-e^{-2\mathrm{i}z}\right| = -\mathrm{i}\left(1-e^{-2\mathrm{i}z}\right)e^{\mathrm{i}z}$, for any $z\in[0,\pi]$. We use now the identity (cf. [64, p. 8])

$$\int_0^\pi \sin^X(z)\,e^{\mathrm{i}Yz}\mathrm{d}z = \frac{\pi e^{\mathrm{i}\frac{Y\pi}{2}}\Gamma(X+1)}{2^X\Gamma\left(1+\frac{X+Y}{2}\right)\Gamma\left(1+\frac{X-Y}{2}\right)}, \qquad (X,Y)\in(-1,\infty)\times\mathbb{R}. \tag{3.13}$$

Setting $X = 1-\alpha$ and $Y = -(2k+1)$, and using $e^{-\mathrm{i}(2k+1)\frac{\pi}{2}} = -\mathrm{i}(-1)^k$, we obtain

$$\int_0^\pi e^{-\mathrm{i}z(2k+1)}\left(\sin z\right)^{1-\alpha}\mathrm{d}z = \frac{-\mathrm{i}(-1)^k\pi\Gamma(2-\alpha)}{2^{1-\alpha}\Gamma\left(1-k-\frac{\alpha}{2}\right)\Gamma\left(2+k-\frac{\alpha}{2}\right)}. \tag{3.14}$$

The following consequence of Euler's reflection formula (cf. [67])

$$\Gamma\left(z-j\right) = (-1)^{j-1}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma\left(j+1-z\right)}, \qquad z\in\mathbb{R}\setminus\mathbb{Z}, \quad j\in\mathbb{Z}, \tag{3.15}$$

implies, setting $z = 1-\frac{\alpha}{2}$, $j = k$, and since $\Gamma\left(1+y\right) = y\,\Gamma\left(y\right)$,

$$\Gamma\left(1-k-\frac{\alpha}{2}\right) = (-1)^{k-1}\frac{\Gamma\left(\frac{\alpha}{2}-1\right)\Gamma\left(2-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+k\right)} = (-1)^k\frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+k\right)}. \tag{3.16}$$

By (3.14)-(3.16) we deduce

$$\int_0^\pi e^{-\mathrm{i}z(2k+1)}\left(\sin z\right)^{1-\alpha}\mathrm{d}z = \frac{-\mathrm{i}\pi}{2^{1-\alpha}}\frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(2-\frac{\alpha}{2}+k\right)}. \tag{3.17}$$

Consequently, by (3.12) and (3.17), we conclude that for any $j\geq 1$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-e^{-\mathrm{i}jz}}{\left|4\sin^2\left(z/2\right)\right|^{\alpha/2}}\mathrm{d}z = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\sum_{k=0}^{j-1}\frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(1-\frac{\alpha}{2}+k\right)}\frac{1}{1-\frac{\alpha}{2}+k} = \mathsf{T}^1_\alpha\left(j\right)$$

defined in (3.4), which in turn, recalling (3.11), implies that $L_{\nabla E_\alpha,1}\phi_+(x) = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\sum_{j\geq 1}\mathsf{T}^1_\alpha\left(j\right)\hat{\phi}\left(j\right)e^{\mathrm{i}jx}$. Since $L_{\nabla E_\alpha,1}$ is a real operator $\mathsf{T}^1_\alpha\left(-j\right) = \overline{\mathsf{T}^1_\alpha\left(j\right)} = \mathsf{T}^1_\alpha\left(j\right)$ which, combined with $L_{\nabla E_\alpha,1}\hat{\phi}(0) = 0$, gives us (3.9).

Step 2 (Evaluation of $L_{\nabla E_\alpha,2}$)**.** We claim that

$$\left(L_{\nabla E_\alpha,2}\,\phi\right)(x) = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\,|D|^2\,\left(\mathsf{M}_\alpha(|D|)\,\phi\right)(x)\,. \tag{3.18}$$

Setting $y = x - z$ we have $(L_{\nabla E_\alpha,2}\,\phi)(x) = \frac{c_\alpha}{2(1-\frac{\alpha}{2})} \fint \frac{\phi'(x-z)\sin z}{[2(1-\cos z)]^{\frac{\alpha}{2}}}\,\mathrm{d}z$. As $\frac{\sin z}{[2(1-\cos z)]^{\frac{\alpha}{2}}} = \frac{1}{2(1-\frac{\alpha}{2})}\partial_z\left([2(1-\cos z)]^{1-\frac{\alpha}{2}}\right)$, integrating by parts

$$(L_{\nabla E_\alpha,2}\,\phi)(x) = -\frac{c_\alpha}{\left[2\left(1-\frac{\alpha}{2}\right)\right]^2}\fint \frac{\partial_z\left[\phi'(x-z)\right]}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\mathrm{d}z = \frac{c_\alpha}{\left[2\left(1-\frac{\alpha}{2}\right)\right]^2}\fint \frac{\phi''(x-z)}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\mathrm{d}z. \tag{3.19}$$

For $j \geq 0$ we compute

$$I_j := \fint \frac{e^{-\mathrm{i}jz}}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = \frac{1}{\pi}\int_0^\pi e^{-\mathrm{i}jz}[2(1-\cos z)]^{\frac{\alpha}{2}-1}\,\mathrm{d}z = \frac{2^{2-\alpha}}{\pi}\int_0^\pi e^{-\mathrm{i}\,2jz}(\sin z)^{2-\alpha}\,\mathrm{d}z$$

and applying Eq. (3.13) with $X = 2-\alpha$, $Y = -2j$, and using $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$\fint \frac{e^{-\mathrm{i}jz}}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = (-1)^j\,\frac{(2-\alpha)\Gamma(2-\alpha)}{\Gamma\left(2-j-\frac{\alpha}{2}\right)\Gamma\left(2+j-\frac{\alpha}{2}\right)}\,.$$

We use now the identities

$$\Gamma\left(2-\frac{\alpha}{2}+j\right) = \left(1-\frac{\alpha}{2}+j\right)\Gamma\left(1-\frac{\alpha}{2}+j\right), \qquad \Gamma\left(2-\frac{\alpha}{2}-j\right) = \left(1-\frac{\alpha}{2}-j\right)(-1)^j\,\frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+j\right)}\,,$$

which follows by $\Gamma(1+z) = z\,\Gamma(z)$ and (3.15) (with $z = -\alpha/2$ and $j \rightsquigarrow j-1$), to deduce, for any $j \geq 0$,

$$I_j = \fint \frac{e^{-\mathrm{i}jz}}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\,\frac{2-\alpha}{\left(1-\frac{\alpha}{2}\right)^2-j^2}\,\frac{\Gamma\left(\frac{\alpha}{2}+j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)} = -2\left(1-\frac{\alpha}{2}\right)\mathsf{M}_\alpha(j)$$

with M_α defined in (3.6). Since $I_j = I_{-j}$ we conclude that

$$\fint \frac{e^{-\mathrm{i}jz}}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = -2\left(1-\frac{\alpha}{2}\right)\mathsf{M}_\alpha(|j|)\,. \tag{3.20}$$

By (3.19) and (3.20) we deduce (3.18).

Step 3 (Evaluation of $L_{\nabla E_\alpha,3}$)**.** The action of the operator $L_{\nabla E_\alpha,3}$ in (3.8) on a function $\phi(x) = \sum_{j\in\mathbb{Z}}\hat\phi(j)\,e^{\mathrm{i}jx}$ is, setting $y = x-z$ and using (3.20),

$$\left(L_{\nabla E_\alpha,3}\phi\right)(x) = \frac{c_\alpha}{4}\sum_{j\in\mathbb{Z}}\hat\phi(j)\,e^{\mathrm{i}jx}\fint \frac{e^{-\mathrm{i}jz}}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = -\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\left(1-\frac{\alpha}{2}\right)^2\left(\mathsf{M}_\alpha(|D|)\,\phi\right)(x). \tag{3.21}$$

Step 4 (Evaluation of $L_{\nabla E_\alpha,4}$)**.** By (3.20) with $j=0$ and (3.6) it results that $L_{\nabla E_\alpha,4}$ in (3.8) is

$$\left(L_{\nabla E_\alpha,4}\phi\right)(x) = \frac{c_\alpha}{4}\fint \frac{\phi(x)}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\,\frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\,\phi(x)\,. \tag{3.22}$$

In conclusion, by (3.8), (3.9), (3.18), (3.21), (3.22) we deduce that $\mathrm{d}\nabla E_\alpha(0)$ is equal to $L_\alpha(|D|)$ in (3.3).

Step 5 ($\mathsf{T}^1_\alpha \in \widetilde\Gamma^{\max(0,\alpha-1)}$ and $\mathsf{T}^2_\alpha \in \widetilde\Gamma^{\alpha-1}$)**.** We start with $\mathsf{T}^2_\alpha(|\xi|)$ in (3.5) which is defined on $\mathbb{R}$. We recall the asymptotic expansion for $|\xi| \to \infty$, see [67, Eq. (5.11.13)],

$$\frac{\Gamma(\xi+a)}{\Gamma(\xi+b)} - \xi^{a-b}\left(\sum_{\kappa=0}^N \frac{G_\kappa(a,b)}{\xi^\kappa}\right) = \mathcal{O}\left(|\xi|^{(a-b)-(1+N)}\right)$$

$$G_0(a,b) = 1\,,\ G_1(a,b) = \frac{(a-b)(a+b-1)}{2}\,,\quad \forall N \in \mathbb{N},\ \left|\mathrm{Arg}\,\xi\right| < \pi\,, \tag{3.23}$$

which involves holomorphic functions. We can focus on the case $\mathrm{Re}(\xi) > 0$. We claim that formula (3.23) implies automatically the estimates for the derivatives

$$\left|\partial_\xi^\mu \left(\frac{\Gamma(\xi+a)}{\Gamma(\xi+b)}\right)\right| \lesssim_\mu \xi^{a-b-\mu} \quad \text{for large } \xi > 0 \text{ and for any } \mu \in \mathbb{N}. \tag{3.24}$$

Case $\mu = 0$ of (3.24) follows trivially from (3.23). For any $\mu \in \mathbb{N} \setminus \{0\}$ and for $N_1 \gg \mu \geq 1$, let us set

$$M_1(\xi) := \xi^{a-b} \sum_{\kappa=0}^{N} \frac{G_\kappa(a,b)}{\xi^\kappa}, \qquad M_2(\xi) := \xi^{a-b} \sum_{\kappa=N+1}^{N+N_1} \frac{G_\kappa(a,b)}{\xi^\kappa}, \qquad E(\xi) := \frac{\Gamma(\xi+a)}{\Gamma(\xi+b)} - (M_1(\xi) + M_2(\xi)).$$

Obviously $M_1(\xi) \in \widetilde{\Gamma}_0^{a-b}$ and $M_2(\xi) \in \widetilde{\Gamma}_0^{a-b-(N+1)}$. For $\xi \gg 1$, E is holomorphic in $B(\xi,2)$. Thus $\partial_\xi^\mu E(\xi) = \frac{c_\mu}{2\pi \mathrm{i}} \int_{\partial B(\xi,1)} \frac{E(\zeta)}{(\zeta-\xi)^{1+\mu}}\, \mathrm{d}\zeta$ by the Cauchy formula. Moreover (3.23) is true in $B(\xi,2)$ and so, by $|\zeta| \sim |\xi|$,

$$\left|\partial_\xi^\mu E(\xi)\right| \lesssim_\mu \int_{\partial B(\xi,1)} \frac{|\zeta|^{a-b-(1+N+N_1)}}{|\zeta-\xi|^{1+\mu}} \,|\mathrm{d}\zeta| \lesssim_\mu |\xi|^{a-b-(1+N+N_1)} \lesssim_\mu |\xi|^{a-b-(1+N+\mu)},$$

which implies

$$\left|\partial_\xi^\mu \left(\frac{\Gamma(\xi+a)}{\Gamma(\xi+b)} - M_1(\xi)\right)\right| \leq \left|\partial_\xi^\mu M_2(\xi)\right| + \left|\partial_\xi^\mu E(\xi)\right| \lesssim_\mu |\xi|^{a-b-(1+N+\mu)}.$$

This proves (3.24). From (3.24) we conclude that $\mathsf{T}_\alpha^2(|j|)$ in (3.5) is a Fourier multiplier of order $\alpha - 1$.

We now consider $\mathsf{T}_\alpha^1(|j|)$ defined in (3.4). For any $j \in \mathbb{N}_0$, the discrete derivative of $\mathsf{T}_\alpha^1(j)$ is

$$(\Delta \mathsf{T}_\alpha^1)(j) := \mathsf{T}_\alpha^1(j+1) - \mathsf{T}_\alpha^1(j) = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2}+j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)} \frac{1}{1-\frac{\alpha}{2}+j} = \frac{\mathsf{T}_\alpha^2(j)}{1-\frac{\alpha}{2}+j}.$$

Since T_α^2 is a symbol of order $\alpha-1$ we deduce that $\left|\mathsf{T}_\alpha^1(j)\right| \lesssim 1 + j^{\alpha-1}$ and, for any $\ell \in \mathbb{N}$, the discrete derivatives satisfy $\left|\left(\Delta^\ell \mathsf{T}_\alpha^1\right)(j)\right| \lesssim j^{\alpha-1-\ell}$. By [72, Lemma 7.1.1] there exists a $\mathcal{C}^\infty$ extension of T_α^1 to the whole $\mathbb{R}$ which is a symbol of order $\max(\alpha-1,0)$.

The proof of Lemma 3.1 is complete. □

Remark 3.2. For $\alpha \neq 1$ the Fourier multiplier $\mathsf{T}_\alpha^1(|j|)$ in (3.4) is equal to

$$\mathsf{T}_\alpha^1(|j|) = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{\alpha-1} \left(\frac{\Gamma(\frac{\alpha}{2}+|j|)}{\Gamma(1+|j|-\frac{\alpha}{2})} - \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\right)$$

as follows by induction.

Remark 3.3. The first linear frequency $\omega_\alpha(1) = 0$. This is equivalent to prove that $L_\alpha(1) = 0$, that, in view of (3.3)-(3.5), amounts to show that

$$\mathsf{T}_\alpha^1(1) - \mathsf{T}_\alpha^2(1) - \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)} \left[\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(\frac{\alpha}{2}+1\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)} - \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)}\right] = 0.$$

This holds true because, using the identity $\Gamma(y+1) = y\,\Gamma(y)$,

$$\frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(\frac{\alpha}{2}\right)\frac{\alpha}{2}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)} - \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} = \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)} \left[1 - \frac{\alpha}{2} - \left(1-\frac{\alpha}{2}\right)\right] = 0.$$

The fact that the first frequency $\omega_\alpha(1) = 0$ is zero has a dynamical proof. Indeed, in view of the translation invariance of the problem, the patch equation (1.11) possesses the vector prime integral

$$\int_{\mathbb{T}} \left(\sqrt{1+2f(x)} - 1\right) \vec{\gamma}(x)\, \mathrm{d}x = \int_{\mathbb{T}} f(x)(\cos x, \sin x)\, \mathrm{d}x + O\left(\|f\|^2\right). \tag{3.25}$$

Let us consider a dynamical system $\dot{f} = Y(f)$ with $Y(0) = 0$ and $A := \mathrm{d}Y(0)$. If $b(f)$ is a prime integral then $\nabla b(f) \cdot Y(f) = 0$, $\forall f$. Hence, differentiating and since $Y(0) = 0$, we obtain $\nabla b(0) \cdot Af = 0$, $\forall f$. If A is non singular then $\nabla b(0) = 0$, i.e. the prime integral b is quadratic at $f = 0$. Here the linear operator A (cf. (3.7)) is degenerate in the one-Fourier mode on which (3.25) has a linear component in f.

The other linear frequencies $\omega_\alpha(j)$, $j \neq 0, \pm 1$, are all different from zero.

Lemma 3.4 (Convexity of $\omega_\alpha(j)$)**.** *Let $\alpha \in (0,2)$. The frequency map $j \mapsto \omega_\alpha(j) = j L_\alpha(|j|)$, $j \in \mathbb{Z}$, where L_α is computed in Lemma 3.1, is odd and satisfies the convexity property*

$$\triangle^2 \omega_\alpha(j) := \omega_\alpha(j+1) + \omega_\alpha(j-1) - 2\,\omega_\alpha(j) = \frac{\Gamma(2-\alpha)}{2^{1-\alpha}\Gamma^2\left(1-\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2}-1+j\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)} \alpha j > 0\,, \quad \forall\ j \geq 1\,. \tag{3.26}$$

The linear frequencies $\omega_\alpha(j)$ are different from zero for any $|j| \geq 2$, in particular $\omega_\alpha(j) > 0$ and increasing for any $j \geq 2$.

Proof. In view of Lemma 3.1, for any $j \geq 1$, and the identity $\Gamma(1+y) = y\Gamma(y)$, the second discrete derivative $\triangle^2\omega_\alpha(j)$ is equal to

$$\begin{aligned}\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \Bigg\{ &(j+1)\sum_{k=0}^{j} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(2-\frac{\alpha}{2}+k\right)} + (j-1)\sum_{k=0}^{j-2} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(2-\frac{\alpha}{2}+k\right)} - 2j\sum_{k=0}^{j-1} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(2-\frac{\alpha}{2}+k\right)} \\ &- (j+1)\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\Gamma\left(1-\frac{\alpha}{2}+j+1\right)} - (j-1)\frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(1-\frac{\alpha}{2}+j-1\right)} + 2j\frac{\Gamma\left(\frac{\alpha}{2}+j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)} \Bigg\}\,.\end{aligned} \tag{3.27}$$

The first term inside the above bracket is equal to

$$\begin{aligned} I &= (j+1)\frac{\Gamma\left(\frac{\alpha}{2}+j\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)} - (j-1)\frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j-1\right)} \\ &= \frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\left[(j+1)\left(\frac{\alpha}{2}+j-1\right) - (j-1)\left(1-\frac{\alpha}{2}+j\right)\right] = \frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\,\alpha j\,. \end{aligned} \tag{3.28}$$

Writing the terms in the 2nd line of the bracket in (3.27) as

$$\begin{aligned} -(j+1)\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\Gamma\left(1-\frac{\alpha}{2}+j+1\right)} &= -\frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\,(j+1)\left(\tfrac{\alpha}{2}+j\right)\left(\tfrac{\alpha}{2}+j-1\right), \\ -(j-1)\frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(1-\frac{\alpha}{2}+j-1\right)} &= -\frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\,(j-1)\left(-\tfrac{\alpha}{2}+j\right)\left(1-\tfrac{\alpha}{2}+j\right), \\ 2j\frac{\Gamma\left(\frac{\alpha}{2}+j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)} &= \frac{\Gamma\left(\frac{\alpha}{2}+j-1\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\,2j\left(j^2-\left(1-\tfrac{\alpha}{2}\right)^2\right), \end{aligned} \tag{3.29}$$

we conclude by Eqs. (3.27) to (3.29) and since $c_\alpha = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{2^{1-\alpha}\Gamma\left(1-\frac{\alpha}{2}\right)}$ (cf. (1.5)), that

$$\triangle^2\omega_\alpha(j) = \frac{1}{2^{1-\alpha}(2-\alpha)} \frac{\Gamma(2-\alpha)}{\Gamma^2\left(1-\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha}{2}-1+j\right)}{\Gamma\left(2-\frac{\alpha}{2}+j\right)}\, X_\alpha(j)$$

where

$$X_\alpha(j) := \alpha j - (j+1)\left(\tfrac{\alpha}{2}+j\right)\left(\tfrac{\alpha}{2}-1+j\right) - (j-1)\left(-\tfrac{\alpha}{2}+j\right)\left(1-\tfrac{\alpha}{2}+j\right) + 2j\left(j^2-\left(1-\tfrac{\alpha}{2}\right)^2\right) = (2-\alpha)\alpha j\,.$$

This proves (3.26). The positivity of $\triangle^2\omega_\alpha(j)$ in (3.26) follows because the function Γ is positive on positive numbers. Finally, the convexity property (3.26) and $\omega_\alpha(0) = \omega_\alpha(1) = 0$ (cf. Remark 3.3) imply that $\omega_\alpha(j) > 0$ and increasing for any $j \geq 2$. $\square$

The next lemma is crucial for the normal form construction of Section 5.

Lemma 3.5 (Absence of three wave interactions)**.** *Let $\alpha \in (0,2)$. For any $n, j, k \in \mathbb{Z}\setminus\{0\}$ satisfying $k = j+n$, it results*

$$\left|\omega_\alpha(k) - \omega_\alpha(j) - \omega_\alpha(n)\right| \geq \omega_\alpha(2) > 0\,. \tag{3.30}$$

Proof. Since the map $j \mapsto \omega_\alpha(j)$ is odd and strictly increasing for $j \in \mathbb{N}$, it is sufficient to consider the case $k \geq j \geq n \geq 1$, $k = j+n$. Then, using that $\omega_\alpha(0) = \omega_\alpha(1) = 0$, defining $A_\alpha(\ell) := \omega_\alpha(\ell) - \omega_\alpha(\ell-1)$, we write by a telescoping expansion,

$$\begin{aligned}\omega_\alpha(k) - \omega_\alpha(j) - \omega_\alpha(n) &= \sum_{q=1}^{j+n}\left(\omega_\alpha(q) - \omega_\alpha(q-1)\right) - \sum_{q=1}^{j}\left(\omega_\alpha(q) - \omega_\alpha(q-1)\right) - \sum_{q=1}^{n}\left(\omega_\alpha(q) - \omega_\alpha(q-1)\right)\\ &= \sum_{q=1}^{n}\left(A_\alpha(q+j) - A_\alpha(q)\right) = \sum_{q=1}^{n}\sum_{q'=1}^{j}\left(A_\alpha(q+q') - A_\alpha(q+q'-1)\right)\\ &= \sum_{q=1}^{n}\sum_{q'=1}^{j}\triangle^2\omega_\alpha(q+q'-1) \geq \triangle^2\omega_\alpha(1) = \omega_\alpha(2) > 0\end{aligned}$$

by (3.26) and Lemma 3.4. This proves (3.30). $\square$

We finally prove an asymptotic expansion of the frequencies $\omega_\alpha(j)$. We use the notation $\sum_{j=p_1}^{p_2} a_j \equiv 0$ if $p_2 < p_1$. We denote m_β a real Fourier multiplier of order $\beta \in \mathbb{R}$, and c_α^κ real constants, which may vary from line to line.

Lemma 3.6 (Asymptotic behavior of $L_\alpha(|j|)$)**.** *Let*

$$\mathbb{V}_\alpha := \begin{cases} \dfrac{\alpha c_\alpha}{2-\alpha}\dfrac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} & \alpha \neq 1,\\[2ex] \dfrac{1}{\pi}\left\{\left(\gamma_{\mathrm{EM}} - \dfrac{\pi^2}{12} - 2\right) + \displaystyle\sum_{k=1}^{\infty}\left[\dfrac{1}{\frac{1}{2}+k} - \dfrac{1}{k}\left(1 - \dfrac{1}{2k}\right)\right]\right\} & \alpha = 1,\end{cases} \tag{3.31}$$

where $\gamma_{\mathrm{EM}} := \left(\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k}\right) - \log n$ *is the Euler-Mascheroni constant.*

Then the symbol $L_\alpha(|j|)$ *in Lemma 3.1 has the following asymptotic expansion: for any* $\mathcal{K} \in \mathbb{N}$, $\mathcal{K} \geq 3$,

- *If* $\alpha \in (0,1)\cup(1,2)$ *there exists real constants* c_α^κ, $\kappa \in \{3,\dots,\mathcal{K}-1\}$ *and a Fourier multiplier* $m_{\alpha-\mathcal{K}}$ *of order* $\alpha - \mathcal{K}$ *such that*

$$L_\alpha(|j|) = \mathbb{V}_\alpha + \underbrace{\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\frac{\Gamma(3-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{\alpha-1}}_{:=c_\alpha^1}|j|^{\alpha-1} + \sum_{\kappa=3}^{\mathcal{K}-1}c_\alpha^\kappa|j|^{\alpha-\kappa} + m_{\alpha-\mathcal{K}}(|j|). \tag{3.32}$$

- *If* $\alpha = 1$ *there exists real constants* c_1^κ, $\kappa \in \{3,\dots,\mathcal{K}-1\}$ *and a Fourier multiplier* $m_{1-\mathcal{K}}$ *of order* $1-\mathcal{K}$ *such that*

$$L_1(|j|) = \mathbb{V}_1 + \frac{1}{\pi}\log|j| + \sum_{\kappa=3}^{\mathcal{K}-1}c_1^\kappa|j|^{1-\kappa} + m_{1-\mathcal{K}}(|j|).$$

Note that in the expansion (3.32) there is not a term as $c_\alpha^2|j|^{\alpha-2}$ and that $\frac{1}{\alpha-1}|j|^{\alpha-1}$ is, for $\alpha \in (1,2)$, positive and tends to infinity, whereas, for $\alpha \in (0,1)$, it is negative and tends to zero.

We provide for completeness the expansion also in the cases $\alpha = 1$ and $\alpha \in (0,1)$, although not needed for the proof of Theorem 1.1.

Lemma 3.6 is a direct consequence of (3.5), (3.3) and (1.5) and the following lemma.

Lemma 3.7. *For any* $\mathcal{K} \in \mathbb{N}, \mathcal{K} \geq 3$, *the following holds:*

- *if* $\alpha \in (0,1)\cup(1,2)$, *there exist real constants* c_α^κ, $\kappa \in \{3,\dots,\mathcal{K}-1\}$ *such that*

$$\mathsf{T}_\alpha^1(|j|) = \frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} + \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{\alpha-1}|j|^{\alpha-1} + \sum_{\kappa=3}^{\mathcal{K}-1}c_\alpha^\kappa|j|^{\alpha-\kappa} + m_{\alpha-\mathcal{K}}(|j|). \tag{3.33}$$

- *if* $\alpha = 1$ *there exist real constants* c_1^κ, $\kappa \in \{3,\dots,\mathcal{K}-1\}$ *such that*

$$\mathsf{T}_1^1(|j|) = \frac{1}{\pi}\left\{\log|j| + \left(\gamma_{\mathrm{EM}} - \frac{\pi^2}{12}\right) + \sum_{k=1}^{\infty}\left[\frac{1}{\frac{1}{2}+k} - \frac{1}{k}\left(1-\frac{1}{2k}\right)\right]\right\} + \sum_{\kappa=3}^{\mathcal{K}-1}c_1^\kappa|j|^{-\kappa} + m_{1-\mathcal{K}}(|j|); \tag{3.34}$$

- *if* $\alpha \in (0,2)$ *there exist real constants* c_α^κ, $\kappa \in \{3,\dots,\mathcal{K}-1\}$ *such that*

$$\mathsf{M}_\alpha\left(|j|\right) = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|j|^2 - \left(1-\frac{\alpha}{2}\right)^2} \left[|j|^{\alpha-1} + \sum_{\kappa=3}^{\mathcal{K}-1} c_\alpha^\kappa\, |j|^{\alpha-\kappa} + m_{\alpha-\mathcal{K}}\left(|j|\right)\right]. \tag{3.35}$$

Proof. By the proof of Lemma 3.1 (below (3.23)) we know that

$$\frac{\Gamma(\xi+a)}{\Gamma(\xi+b)} - \xi^{a-b} \sum_{\kappa=0}^{N} \frac{G_\kappa(a,b)}{\xi^\kappa} = m_{a-b-(1+N)}(\xi)\,,$$

where $m_{a-b-(1+N)}(\xi)$ is a Fourier multiplier in $\widetilde{\Gamma}_0^{a-b-(1+N)}$, and therefore, for any $\mathcal{K} \geq 3$,

$$\frac{\Gamma\left(\frac{\alpha}{2}+|j|\right)}{\Gamma\left(1-\frac{\alpha}{2}+|j|\right)} = |j|^{\alpha-1} + \sum_{\kappa=2}^{\mathcal{K}-1} G_\kappa\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right) |j|^{\alpha-(1+\kappa)} + m_{\alpha-\mathcal{K}}\left(|j|\right), \tag{3.36}$$

where we exploited that $G_0(a,b)=1$ and $G_1\left(\frac{\alpha}{2},1-\frac{\alpha}{2}\right)=0$, by (3.23). By Remark 3.2 and (3.36) we deduce (3.33) for $\mathcal{K}=3$. Finally (3.34) for $\alpha=1$ follows by the asymptotic estimate of the harmonic numbers $\sum_{k=1}^{j} k^{-1} = \gamma_{\mathrm{EM}} + \log(j) + \frac{1}{2j} + m_{-2}(j)$. $\square$

4 Paralinearization of the Hamiltonian scalar field

The main result of this section is the following.

Theorem 4.1 (Paralinearization of the α-SQG patch equation)**.** *Let* $\alpha \in (0,1)\cup(1,2)$. *Let* $N \in \mathbb{N}$ *and* $\rho \geq 0$. *For any* $K \in \mathbb{N}_0$, *there exist* $s_0>0$, $\epsilon_0>0$ *such that, if* $f \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ *solves Eq.* (1.11) *then*

$$\partial_t f + \partial_x \circ \mathrm{Op}^{BW}\left[\left(1+\nu\left(f;x\right)\right) L_\alpha(|\xi|) + V\left(f;x\right) + P\left(f;x,\xi\right)\right] f = R\left(f\right) f \tag{4.1}$$

where

- $L_\alpha(|\xi|)$ *is the real valued Fourier multiplier of order* $\max\{0,\alpha-1\}$ *defined in Lemma 3.1;*
- $\nu\left(f;x\right), V\left(f;x\right)$ *are real valued functions in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *(see Definition 2.7);*
- $P\left(f;x,\xi\right)$ *is a symbol in* $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ *(see Definition 2.2) satisfying* (2.20)*;*
- $R\left(f\right)$ *is a real smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,0,1}[\epsilon_0,N]$ *(see Definition 2.17).*

Note that, since the symbol $\left(1+\nu\left(f;x\right)\right)L_\alpha(|\xi|)+V\left(f;x\right)$ is real, the vector field in (4.1) is linearly Hamiltonian up to zero order operators.

4.1 Isolating the integral terms

Notation. In this section we use the following auxiliary functions

$$r = r\left(f;x\right) := \sqrt{1+2f(x)}, \qquad \delta_z f := f(x) - f(x-z), \qquad \Delta_z f := \frac{\delta_z f}{2\sin(z/2)},\ \forall z \in \mathbb{T}\setminus\{0\}. \tag{4.2}$$

We shall denote by $P\left(f;x,\xi\right)$ a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ (see Definition 2.2) by $R\left(f\right)$ a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ (see Definition 2.17) and by $R\left(f;z\right)$ a Kernel-smoothing operator in $\Sigma K\mathcal{R}^{-\rho,0}_{K,0,1}[\epsilon_0,N]$ (see Definition 2.33), whose explicit expression may vary from line to line.

Note that $r\left(f;x\right)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N]$ and, according to Definition 2.31,

$$\delta_z \in \widetilde{K\mathcal{M}}_0^{1,1}. \tag{4.3}$$

In view of (1.12) and performing the change of variable $y = x - z$, the gradient $\nabla E_\alpha(f)$ can be decomposed as

$$
\begin{aligned}
\nabla E_\alpha(f) &= \nabla E_\alpha^{(1)}(f) + \nabla E_\alpha^{(2)}(f),\\
\nabla E_\alpha^{(1)}(f) &:= \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \fint \frac{1+2f(x-z) - \sqrt{1+2f(x)}\,\sqrt{1+2f(x-z)}\cos z}{\left[2\left(1+f(x)+f(x-z)-\sqrt{1+2f(x)}\sqrt{1+2f(x-z)}\cos z\right)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}z,\\
\nabla E_\alpha^{(2)}(f) &:= \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \fint \frac{\sqrt{\frac{1+2f(x)}{1+2f(x-z)}}\, f'(x-z)\sin z}{\left[2\left(1+f(x)+f(x-z)-\sqrt{1+2f(x)}\sqrt{1+2f(x-z)}\cos z\right)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}z.
\end{aligned}
\tag{4.4}
$$

Then, recalling the notation in (4.2), we write

$$
\nabla E_\alpha^{(1)}(f) = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)} \fint \frac{r^2 - 2\delta_z f - r\sqrt{r^2-2\delta_z f}\cos z}{\left[2\left(r^2-\delta_z f - r\sqrt{r^2-2\delta_z f}\cos z\right)\right]^{\frac{\alpha}{2}}}\,\mathrm{d}z = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\, r^{2-\alpha} \fint \mathsf{G}^1_{\alpha,z}\left(\frac{\delta_z f}{r^2}\right)\mathrm{d}z \tag{4.5}
$$

with

$$
\mathsf{G}^1_{\alpha,z}(\mathsf{X}) := \frac{1-2\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z}{\left[2\left(1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z\right)\right]^{\frac{\alpha}{2}}}, \tag{4.6}
$$

and

$$
\nabla E_\alpha^{(2)}(f) = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\,\frac{1}{r^\alpha} \underbrace{\fint \mathsf{G}^2_{\alpha,z}\left(\frac{\delta_z f}{r^2}\right) f'(x-z)\sin z\,\mathrm{d}z}_{=:\mathcal{J}(f)} = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\,\frac{1}{r^\alpha}\,\mathcal{J}(f) \tag{4.7}
$$

with

$$
\mathsf{G}^2_{\alpha,z}(\mathsf{X}) := \frac{\frac{1}{\sqrt{1-2\mathsf{X}}}}{\left[2\left(1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z\right)\right]^{\frac{\alpha}{2}}}. \tag{4.8}
$$

By (4.4), recalling (2.3) and that $\nabla E_\alpha^{(1)}(0)$ is a constant, using (4.5), (4.7), the equation (1.11) can be written as

$$
\begin{aligned}
\partial_t f &= \partial_x\left[\left(\nabla E_\alpha^{(1)}(f) - \nabla E_\alpha^{(1)}(0)\right) + \nabla E_\alpha^{(2)}(f)\right]\\
&= \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\,\partial_x\left[\left(r^{2-\alpha}\Delta I(f)\right) + \fint \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z\,\left(r^{2-\alpha}-1\right) + \frac{1}{r^\alpha}\,\mathcal{J}(f)\right]
\end{aligned}
\tag{4.9}
$$

where

$$
\Delta I(f) := \fint \mathsf{G}^1_{\alpha,z}\left(\frac{\delta_z f}{r^2}\right) - \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z. \tag{4.10}
$$

By (3.22), (4.6) we get

$$
\fint \mathsf{G}^1_z(0)\,\mathrm{d}z = \frac{1}{2}\fint \left[2(1-\cos z)\right]^{1-\frac{\alpha}{2}}\mathrm{d}z = \frac{1}{1-\frac{\alpha}{2}}\,\frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}. \tag{4.11}
$$

The terms $\Delta I(f)$ and $\mathcal{J}(f)$ are yet not in a suitable form to be paralinearized, since the nonlinear convolution kernels need to be desingularized at $z=0$.

Lemma 4.2. *The term $\Delta I(f)$ in* (4.10) *can be written as*

$$
\Delta I(f) = \mathcal{I}(f) + R(f)f \tag{4.12}
$$

where $R(f)$ is a real smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0, N]$, and

$$
\mathcal{I}(f) := \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{r^2\,|2\sin(z/2)|^\alpha}\,\mathrm{d}z \tag{4.13}
$$

with

$$\begin{aligned}\mathsf{K}^1_{\alpha,z}(\mathsf{X}) &:= \left(\mathsf{G}^1_{\alpha,z}\right)'(2\mathsf{X}\sin(z/2))\,|2\sin(z/2)|^\alpha \\ &= \Bigg[-\frac{2-\frac{\cos z}{\sqrt{1-4\mathsf{X}\sin(z/2)}}}{\left[2\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)\right]^{\frac{\alpha}{2}}} \\ &\quad +\alpha\frac{\left(1-\frac{\cos z}{\sqrt{1-4\mathsf{X}\sin(z/2)}}\right)\left(1-4\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)}{\left[2\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)\right]^{\frac{\alpha}{2}+1}}\Bigg]|2\sin(z/2)|^\alpha .\end{aligned} \tag{4.14}$$

The term $\mathcal{J}(f)$ *in* (4.7) *can be written as*

$$\mathcal{J}(f) = \fint \mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) f'(x-z)\frac{\sin z}{|2\sin(z/2)|^\alpha}\,\mathrm{d}z \tag{4.15}$$

where

$$\mathsf{K}^2_{\alpha,z}(\mathsf{X}) := \mathsf{G}^2_{\alpha,z}(\mathsf{X}\,2\sin(z/2))\ |2\sin(z/2)|^\alpha = \frac{\frac{1}{\sqrt{1-4\mathsf{X}\sin(z/2)}}\,|2\sin(z/2)|^\alpha}{\left[2\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)\right]^{\frac{\alpha}{2}}}. \tag{4.16}$$

The functions $z\mapsto \mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$, $\mathsf{j}=1,2$, *are* 2π*-periodic.*

Proof. Applying Lemma 2.25 to (4.10) we get

$$\Delta I(f) = \fint \mathrm{Op}^{BW}\left[\left(\mathsf{G}^1_{\alpha,z}\right)'\left(\frac{\delta_z f}{r^2}\right)\right]\frac{\delta_z f}{r^2}\mathrm{d}z + \fint R\left(\frac{\delta_z f}{r^2}\right)\frac{\delta_z f}{r^2}\mathrm{d}z \tag{4.17}$$

where R is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ and, recalling (4.6),

$$\left(\mathsf{G}^1_{\alpha,z}\right)'(\mathsf{X}) = -\frac{2-\frac{\cos z}{\sqrt{1-2\mathsf{X}}}}{\left[2\left(1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z\right)\right]^{\frac{\alpha}{2}}} + \alpha\frac{\left(1-\frac{\cos z}{\sqrt{1-2\mathsf{X}}}\right)\left(1-2\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z\right)}{\left[2\left(1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z\right)\right]^{\frac{\alpha}{2}+1}}. \tag{4.18}$$

In view of (4.18), (4.14) we have that

$$\mathsf{K}^1_{\alpha,z}(\mathsf{X}) = \left(\mathsf{G}^1_{\alpha,z}\right)'(2\mathsf{X}\sin(z/2))\,|2\sin(z/2)|^\alpha$$

so that the first term on the right hand side of (4.17) is equal to $\mathcal{I}(f)$ in (4.13). Notice that, since $\Delta_{z+2\pi}f = -\Delta_z f$ and $\mathsf{K}^1_{\alpha,z+2\pi}(-\mathsf{X}) = \mathsf{K}^1_{\alpha,z}(\mathsf{X})$ (cf. (4.14)), the map $z\mapsto \mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ is 2π-periodic. Similarly $z\mapsto \mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ is 2π-periodic.

We now prove that

$$\fint R\left(\frac{\delta_z f}{1+2f}\right)\left(\frac{\delta_z f}{1+2f}\right)\mathrm{d}z = R(f)f \qquad \text{where} \qquad R(f)\in\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]. \tag{4.19}$$

We write

$$R\left(\frac{\delta_z f}{1+2f}\right)\left(\frac{\delta_z f}{1+2f}\right) = R\left(M(f;z)f\right)M(f;z)f$$

where

$$M(f;z) = M_1(f)+M_2(f;z), \quad M_1(f) := \frac{1}{1+2f}, \quad M_2(f;z) = -\frac{\mathsf{t}_{-z}}{1+2f}.$$

Remark 2.16 shows that $M_1(f)\in\Sigma\mathcal{M}^{0,0}_{K,0,0}[\epsilon_0,N]$ and Proposition 2.34, Item 2 proves that $M_2(f;z)$ belongs to $\Sigma K\mathcal{M}^{0,0}_{K,0,0}[\epsilon_0,N]$. Thus $M(f;z)\in\Sigma K\mathcal{M}^{0,0}_{K,0,0}[\epsilon_0,N]$ and Proposition 2.34, Items 2 and 4 give that

$$R\left(\frac{\delta_z f}{1+2f}\right)\left(\frac{\delta_z f}{1+2f}\right) = R\left(M(f;z)f\right)M(f;z)f = R(f;z)f$$

for some Kernel-smoothing operator $R(f;z)$ in $\Sigma K\mathcal{R}^{-\rho,0}_{K,0,1}[\epsilon_0,N]$. Finally Lemma 2.35 implies (4.19). □

Plugging (4.12) in (4.9) we obtain

$$\partial_t f = \frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\partial_x\left[r^{2-\alpha}\left(\mathcal{I}(f)+R(f)f\right)+\fint \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z\,\left(r^{2-\alpha}-1\right)+\frac{1}{r^\alpha}\mathcal{J}(f)\right]. \tag{4.20}$$

4.2 Analysis of the nonlinear convolution kernels

The goal of this section is to represent the nonlinear convolution kernels in (4.13) and (4.15) as Kernel-functions according to Definition 2.26. In Section 4.4 we shall consider as well the convolution kernel

$$\begin{aligned}\mathsf{K}^3_{\alpha,z}(\mathsf{X}) := \left(\mathsf{G}^2_{\alpha,z}\right)'(\mathsf{X}\,2\sin(z/2))\ \sin z\,|2\sin(z/2)|^\alpha = &\left[\frac{\frac{1}{(1-4\mathsf{X}\sin(z/2))^{3/2}}}{\left[2\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)\right]^{\frac{\alpha}{2}}}\right.\\ &\left.+\alpha\frac{\left(1-\frac{\cos z}{\sqrt{1-4\mathsf{X}\sin(z/2)}}\right)\frac{1}{\sqrt{1-4\mathsf{X}\sin(z/2)}}}{\left[2\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos z\right)\right]^{\frac{\alpha}{2}+1}}\right]|2\sin(z/2)|^\alpha\ \sin z. \end{aligned}\tag{4.21}$$

Lemma 4.3. *Let* $\mathsf{K}^{\mathsf{j}}_{\alpha,z}(\mathsf{X})$, $\alpha\in(0,2)$, $\mathsf{j}=1,2,3$, *be the functions defined in* (4.14), (4.16) *and* (4.21). *Then*

$$\mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) = \mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\Delta_z f}{1+2f}\right)\in \Sigma K\mathcal{F}^0_{K,0,0}[\epsilon_0,N] \tag{4.22}$$

is a Kernel function, which admits the expansion

$$\mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) = \mathsf{K}^{\mathsf{j},0}_\alpha(f;x)+\mathsf{K}^{\mathsf{j},1}_\alpha(f;x)\ \sin z+\mathsf{K}^{\mathsf{j},2}_\alpha(f;x)\,(2\sin(z/2))^2+\varrho^{\mathsf{j},3}_\alpha(f;x,z), \tag{4.23}$$

where

$$\mathsf{K}^{\mathsf{j},l}_\alpha(f;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,\underline{p}(\mathsf{j},l)}[\epsilon_0,N],\qquad \varrho^{\mathsf{j},3}_\alpha(f;x,z)\in\Sigma K\mathcal{F}^3_{K,0,\underline{q}(\mathsf{j})}[\epsilon_0,N]\qquad \underline{q}(\mathsf{j}):=\begin{cases}1 & \textit{if } \mathsf{j}=1,2,\\ 0 & \textit{if } \mathsf{j}=3,\end{cases} \tag{4.24}$$

with $\underline{p}(\mathsf{j},l)\in\{0,1\}$ *and constant functions*

$$\begin{pmatrix} \mathsf{K}^{1,0}_\alpha(0;x) & \mathsf{K}^{2,0}_\alpha(0;x) & \mathsf{K}^{3,0}_\alpha(0;x)\\ \mathsf{K}^{1,1}_\alpha(0;x) & \mathsf{K}^{2,1}_\alpha(0;x) & \mathsf{K}^{3,1}_\alpha(0;x)\\ \mathsf{K}^{1,2}_\alpha(0;x) & \mathsf{K}^{2,2}_\alpha(0;x) & \mathsf{K}^{3,2}_\alpha(0;x)\end{pmatrix} = \begin{pmatrix} -1 & 1 & 0\\ 0 & 0 & 1+\frac{\alpha}{2}\\ -\frac12\left(1-\frac{\alpha}{2}\right) & 0 & 0\end{pmatrix}. \tag{4.25}$$

Proof. The statement (4.22) follows by (4.23)-(4.24) that we now prove. We first claim that for any $R>0$, there exists $\varepsilon_R>0$ such that the functions

$$\mathsf{J}^{\mathsf{j}}_{\alpha,w}(\mathsf{x},\mathsf{y}) := \mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\mathsf{y}}{1+2\mathsf{x}}\right),\qquad w:=2\sin(z/2), \tag{4.26}$$

where $\mathsf{K}^{\mathsf{j}}_{\alpha,z}(\cdot)$, $\mathsf{j}=1,2,3$ are defined in (4.14), (4.16) and (4.21), are analytic in $(\mathsf{x},\mathsf{y},w)$ in the domain

$$|\mathsf{x}|\le\varepsilon_R,\qquad |\mathsf{y}|\le\varepsilon_R,\qquad |w|\le R, \tag{4.27}$$

and there exists $C_R>0$ such that $\left|\mathsf{J}^{\mathsf{j}}_{\alpha,w}(\mathsf{x},\mathsf{y})\right|\le C_R$ in this domain. Let us prove the analiticity of $\mathsf{J}^1_{\alpha,w}(\mathsf{x},\mathsf{y})$. Substituting $\mathsf{X}=\frac{\mathsf{y}}{1+2\mathsf{x}}$, $w=2\sin(z/2)$ and $\cos(z)=1-\frac{w^2}{2}$ in (4.14) we have

$$\mathsf{J}^1_{\alpha,w}(\mathsf{x},\mathsf{y}) = -\frac{2-\frac{1-\frac{w^2}{2}}{\sqrt{1-2\mathsf{X}w}}}{\left[2\left(1-\mathsf{X}w-\sqrt{1-2\mathsf{X}w}+\sqrt{1-2\mathsf{X}w}\frac{w^2}{2}\right)\right]^{\frac{\alpha}{2}}}|w|^\alpha \tag{4.28a}$$

$$+\alpha\frac{1}{\sqrt{1-2\mathsf{X}w}}\frac{\left(\sqrt{1-2\mathsf{X}w}-1+\frac{w^2}{2}\right)\left(1-2\mathsf{X}w-\sqrt{1-2\mathsf{X}w}+\sqrt{1-2\mathsf{X}w}\frac{w^2}{2}\right)}{\left[2\left(1-\mathsf{X}w-\sqrt{1-2\mathsf{X}w}+\sqrt{1-2\mathsf{X}w}\frac{w^2}{2}\right)\right]^{\frac{\alpha+2}{2}}}|w|^\alpha. \tag{4.28b}$$

Since the function $1-\mathsf{X}w-\sqrt{1-2\mathsf{X}w}=(\mathsf{X}w)^2+O(\mathsf{X}w)^3$ is analytic in $\mathsf{X}w$ small, the function in (4.28a) is analytic in the domain (4.27) for ε_R small enough. Furthermore, noting that the functions $\sqrt{1-2\mathsf{X}w}-1=$

$-\mathsf{X}w+O(\mathsf{X}w)^2$ and $1-2\mathsf{X}w-\sqrt{1-2\mathsf{X}w}=-\mathsf{X}w+O(\mathsf{X}w)^2$ are analytic in $\mathsf{X}w$ small, we deduce that also (4.28b) is analytic in $(\mathsf{x},\mathsf{y},w)$ in the domain (4.27). The analiticity of $\mathsf{J}^2_{\alpha,w}(\mathsf{x},\mathsf{y})$ and $\mathsf{J}^3_{\alpha,w}(\mathsf{x},\mathsf{y})$ follow similarly.

Then by Cauchy integral formula,

$$\mathsf{J}^{\mathsf{j}}_{\alpha,w}(\mathsf{x},\mathsf{y})=\sum_{p_1,p_2=0}^{\infty}\underbrace{\frac{1}{p_1!p_2!}\partial_{\mathsf{x}}^{p_2}\partial_{\mathsf{y}}^{p_1}\mathsf{J}^{\mathsf{j}}_{\alpha,w}(0,0)}_{=:\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z)}\mathsf{x}^{p_2}\mathsf{y}^{p_1}\quad\text{where}\quad\left|\partial_{\mathsf{x}}^{p_2}\partial_{\mathsf{y}}^{p_1}\mathsf{J}^{\mathsf{j}}_{\alpha,w}(0,0)\right|\le p_1!p_2!C_R\varepsilon_R^{-p_1-p_2}. \tag{4.29}$$

In view of (4.26) we have $\mathsf{J}^{\mathsf{j}}_{\alpha,w}(\mathsf{x},0):=\mathsf{K}^{\mathsf{j}}_{\alpha,z}(0)$ for any x, and therefore

$$\mathsf{k}^{\mathsf{j}}_{\alpha;0,0}(z)=\mathsf{K}^{\mathsf{j}}_{\alpha,z}(0),\qquad \mathsf{k}^{\mathsf{j}}_{\alpha;0,p_2}(z)\equiv 0,\ \forall p_2\ge 1, \tag{4.30}$$

and, by (4.14), (4.16), (4.21), one computes that

$$\mathsf{K}^1_{\alpha,z}(0)=-\frac12\left(1-\frac{\alpha}{2}\right)\left(2\sin\left(\frac z2\right)\right)^2-1,\qquad \mathsf{K}^2_{\alpha,z}(0)=1,\qquad \mathsf{K}^3_{\alpha,z}(0)=\left(1+\frac{\alpha}{2}\right)\sin(z). \tag{4.31}$$

Choosing above $R=4$ and $\underline{\varepsilon}>0$ such that $|\sin(z/2)|\le 2$ for $|\mathrm{Im}z|\le\underline{\varepsilon}$, we deduce that each $\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z):=\frac{1}{p_1!p_2!}\partial_{\mathsf{x}}^{p_2}\partial_{\mathsf{y}}^{p_1}\mathsf{J}^{\mathsf{j}}_{\alpha,w}(0,0)$, $\mathsf{j}=1,2,3$, satisfies

$$\left|\partial_z^l\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z)\right|\le l!\underline{\varepsilon}^{-l}C_R\varepsilon_R^{-p_1-p_2},\qquad \forall z\in\mathbb{R},\ l\ge 0,p_1,p_2\ge 0. \tag{4.32}$$

Now, in view of (4.26), (4.29) and (4.30) we expand

$$\begin{aligned}\mathsf{K}^{\mathsf{j}}_{\alpha,z}\left(\frac{\Delta_z f}{1+2f}\right)=\mathsf{J}^{\mathsf{j}}_{\alpha,w}(f,\Delta_z f)&=\mathsf{K}^{\mathsf{j}}_{\alpha,z}(0)+\sum_{p\ge1}\underbrace{\sum_{\substack{p_1\ge1\\p_1+p_2=p}}\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z)\,f^{p_2}(\Delta_z f)^{p_1}}_{=:\widetilde{\mathsf{K}}^{\mathsf{j},p}_{\alpha}(f;x,z)}\\&=\mathsf{K}^{\mathsf{j}}_{\alpha,z}(0)+\sum_{p=1}^{N}\widetilde{\mathsf{K}}^{\mathsf{j},p}_{\alpha}(f;x,z)+\underbrace{\sum_{p>N}\widetilde{\mathsf{K}}^{\mathsf{j},p}_{\alpha}(f;x,z)}_{=:\widetilde{\mathsf{K}}^{\mathsf{j},>N}_{\alpha}(f;x,z)}.\end{aligned} \tag{4.33}$$

We claim that for any $p\in\mathbb{N}$ and $\ell=0,\dots,7$,

$$\partial_z^\ell\widetilde{\mathsf{K}}^{\mathsf{j},p}_{\alpha}(f;x,z)\in\widetilde{K\mathcal{F}}^0_p \tag{4.34}$$

$$\partial_z^\ell\widetilde{\mathsf{K}}^{\mathsf{j},>N}_{\alpha}(f;x,z)\in K\mathcal{F}^0_{K,0,N+1}[\epsilon_0]. \tag{4.35}$$

Step 1 (Proof of (4.34))**.** We expand in Fourier

$$\partial_z^l\widetilde{\mathsf{K}}^{\mathsf{j},p}_{\alpha}(f;x,z)=\sum_{\vec{\jmath}_p\in(\mathbb{Z}\setminus\{0\})^p}\partial_z^l\hat{\mathsf{K}}^{\mathsf{j},p}_{\vec{\jmath}_p}(z)\,f_{j_1}\cdots f_{j_p}e^{\mathrm{i}(j_1+\cdots+j_p)x} \tag{4.36}$$

where, in view of (4.33) and (4.2),

$$\hat{\mathsf{K}}^{\mathsf{j},p}_{\vec{\jmath}_p}(z):=\sum_{\substack{p_1\ge1\\p_1+p_2=p}}\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z)\prod_{q=1}^{p_1}\triangle_{j_q}(z),\qquad \triangle_{j_q}(z):=\frac{1-e^{-\mathrm{i}j_qz}}{2\sin(z/2)}.$$

For any $l\in\mathbb{N}_0$ we have

$$\partial_z^l\hat{\mathsf{K}}^{\mathsf{j},p}_{\vec{\jmath}_p}(z)=\sum_{l_2+l_{1,1}+\cdots+l_{1,p_1}=l}\ \sum_{\substack{p_1\ge1\\p_1+p_2=p}}\partial_z^{l_2}\mathsf{k}^{\mathsf{j}}_{\alpha;p_1,p_2}(z)\prod_{q=1}^{p_1}\partial_z^{l_{1,q}}\triangle_{j_q}(z). \tag{4.37}$$

For any $j\in\mathbb{Z}\setminus\{0\}$, we may write $\triangle_j(z)=\frac{1-e^{-\mathrm{i}jz}}{2\sin(z/2)}=\mathrm{i}\sum_{j'=0}^{|j|-1}e^{-\mathrm{isgn}(j)(j'+\mathrm{sgn}(j)\frac12)z}$ and then, for any $l\in\mathbb{N}_0$

$$\left|\partial_z^l\triangle_j(z)\right|\lesssim\sum_{j'=1}^{|j|}(j')^l\lesssim_l|j|^{l+1},\quad\forall z\in\mathbb{R}. \tag{4.38}$$

By (4.32), (4.38), we estimate (4.37) as (the constant ε_4 is the one in (4.32) for $R=4$)

$$
\begin{aligned}
\left|\partial_z^l \hat{\mathsf{K}}^{\mathrm{j},p}_{\vec{\jmath}_p}(z)\right| &\le \sum_{l_2+l_{1,1}+\cdots+l_{1,p_1}=l} \sum_{\substack{p_1\ge 1\\ p_1+p_2=p}} C_l\, \varepsilon_4^{-p_1-p_2} \prod_{q=1}^{p_1} |j_q|^{l_{1,q}+1} \\
&\lesssim_l p^2 \left(\frac{l}{\varepsilon_4}\right)^p |\vec{\jmath}_p|^l \prod_{q=1}^{p_1} |j_q| \le C_l^p |\vec{\jmath}_p|^l \prod_{q=1}^{p} |j_q|
\end{aligned} \tag{4.39}
$$

for some constant $C_l>0$, for any $z\in\mathbb{T}$. The bound (4.39) implies, recalling Definition 2.26, the claim (4.34).

Step 2 (Proof of (4.35))**.** Recalling (4.33) and (4.36), we have, for any $0\le k\le K$, $l,\gamma\in\mathbb{N}_0$,

$$
\begin{aligned}
\left|\partial_t^k\partial_x^\gamma\partial_z^l \widetilde{\mathsf{K}}^{\mathrm{j},>N}_\alpha(f;x,z)\right| &\le \sum_{p>N}\sum_{\vec{\jmath}_p\in(\mathbb{Z}\setminus\{0\})^p} |\vec{\jmath}_p|^\gamma \left|\partial_z^l \hat{\mathsf{K}}^{\mathrm{j},p}_{\vec{\jmath}_p}(z)\right| \left|\partial_t^k (f_{j_1}\cdots f_{j_p})\right| \\
&\le \sum_{p>N}\sum_{\vec{\jmath}_p\in(\mathbb{Z}\setminus\{0\})^p}\sum_{k_1+\ldots+k_p=k} |\vec{\jmath}_p|^\gamma \left|\partial_z^l \hat{\mathsf{K}}^{\mathrm{j},p}_{\vec{\jmath}_p}(z)\right| \prod_{q=1}^p \left|\partial_t^{k_q} f_{j_q}\right| \\
&\le \sum_{p>N}\sum_{\vec{\jmath}_p\in(\mathbb{Z}\setminus\{0\})^p}\sum_{k_1+\ldots+k_p=k} C_l^p |\vec{\jmath}_p|^{\gamma+l} \prod_{q=1}^p |j_q| \left|\partial_t^{k_q} f_{j_q}\right|
\end{aligned}
$$

using (4.39). Then, assuming with no loss of generality that $|\vec{\jmath}_p|=\max\{|j_1|,\ldots,|j_p|\}=|j_1|$ we have

$$
\begin{aligned}
\left|\partial_t^k\partial_x^\gamma\partial_z^l \widetilde{\mathsf{K}}^{\mathrm{j},>N}_\alpha(f;x,z)\right| &\le \sum_{p>N}\sum_{\vec{\jmath}_p\in(\mathbb{Z}\setminus\{0\})^p}\sum_{k_1+\ldots+k_p=k} C_l^p \left(\prod_{q=2}^p |j_q|^{-2} \left\|\partial_t^{k_q} f\right\|_3\right) |j_1|^{-2} \left\|\partial_t^{k_1} f\right\|_{3+\gamma+l}, \\
&\le \sum_{p>N} C_l^p \sum_{k_1,\ldots,k_p=0}^{k} \left(\prod_{q=2}^p \left\|\partial_t^{k_q} f\right\|_3\right) \left\|\partial_t^{k_1} f\right\|_{3+\gamma+l} \\
&\le \sum_{p>N} (C_l k)^p \, \|f\|_{k,3+\alpha k}^{p-1} \|f\|_{k,3+\gamma+l+\alpha k}
\end{aligned}
$$

recalling (2.5). Summing in p and setting $s_0:=11+\alpha k$, we get, for any $l\le 8$, for any $0\le\gamma\le s-s_0$,

$$
\left|\partial_t^k\partial_x^\gamma\partial_z^l \widetilde{\mathsf{K}}^{\mathrm{j},>N}_\alpha(f;x,z)\right| \le C_k^{N+1} \|f\|_{k,s_0}^N \|f\|_{k,s}, \quad \forall x,z\in\mathbb{T},
$$

which, recalling Definition 2.26, proves the claim in (4.35).

Equations (4.34) and (4.35) thus prove (4.22).

[Proof of (4.23)-(4.25)] In view of (4.33), in order to expand $\mathsf{K}^{\mathrm{j}}_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ as in (4.23), we perform a Taylor expansion in z of the functions $\mathsf{K}^{\mathrm{j}}_{\alpha,z}(0)$ and $\widetilde{\mathsf{K}}^{\mathrm{j},p}_\alpha(f;x,z)$, for any $p\ge1$. By (4.31) we have

$$
\mathsf{K}^{\mathrm{j}}_{\alpha,z}(0) = \mathsf{K}^{\mathrm{j},0}_\alpha(0;x) + \mathsf{K}^{\mathrm{j},1}_\alpha(0;x)\sin z + \mathsf{K}^{\mathrm{j},2}_\alpha(0;x)(2\sin(z/2))^2 + \varrho^{\mathrm{j},3}_\alpha(0;x,z) \tag{4.40}
$$

with $\mathsf{K}^{\mathrm{j},l}_\alpha(0;x)$, $\mathrm{j}=1,2,3$, $l=0,1,2$, are the constants computed in (4.25) and

$$
\varrho^{1,3}_\alpha(0;x,z) = \varrho^{2,3}_\alpha(0;x,z) = 0 \qquad \text{and} \qquad \varrho^{\mathrm{j},3}_\alpha(0;x,z)\in\widetilde{K\mathcal{F}}^3_0 \tag{4.41}
$$

is x-independent. Then, for any $p\ge1$, we expand

$$
\begin{aligned}
\widetilde{\mathsf{K}}^{\mathrm{j},p}_\alpha(f;x,z) &= \sum_{l=0}^2 \widetilde{\mathsf{K}}^{\mathrm{j},p,l}_\alpha(f;x)\, z^l + \mathsf{R}^{\mathrm{j},p,3}_\alpha(f;x,z) \\
&= \widetilde{\mathsf{K}}^{\mathrm{j},p,0}_\alpha(f;x) + \widetilde{\mathsf{K}}^{\mathrm{j},p,1}_\alpha(f;x)\sin z + \widetilde{\mathsf{K}}^{\mathrm{j},p,2}_\alpha(f;x)(2\sin(z/2))^2 + \varrho^{\mathrm{j},p,3}_\alpha(f;x,z),
\end{aligned} \tag{4.42}
$$

where, for $l=0,1,2$,

$$
\widetilde{\mathsf{K}}^{\mathrm{j},p,l}_\alpha(f;x) := \frac{1}{l!}\,\partial_z^l \widetilde{\mathsf{K}}^{\mathrm{j},p}_\alpha(f;x,z)\Big|_{z=0} \tag{4.43}
$$

and

$$
\begin{aligned}
\varrho_\alpha^{\mathrm{j},p,3}(f;x,z) &:= \mathsf{R}_\alpha^{\mathrm{j},p,3}(f;x,z) + \tilde{\mathsf{R}}_\alpha^{\mathrm{j},p,3}(f;x,z)\\
\mathsf{R}_\alpha^{\mathrm{j},p,3}(f;x,z) &:= \frac{1}{2!}\int_0^1 (1-\vartheta)^2\,\partial_z^3\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p}(f;x,\vartheta z)\,\mathrm{d}\vartheta\, z^3 \\
\tilde{\mathsf{R}}_\alpha^{\mathrm{j},p,3}(f;x,z) &:= \widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,1}(f;x)(z-\sin z) + \widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,2}(f;x)\left(z^2 - (2\sin(z/2))^2\right).
\end{aligned}
\tag{4.44}
$$

Notice that $z \mapsto \varrho_\alpha^{\mathrm{j},p,3}(f;x,z)$ is 2π-periodic thanks to (4.42). In view of (4.40) and (4.42), we obtain the expansion (4.23) with, for any $\mathrm{j} = 1,2,3$,

$$
\begin{aligned}
\mathsf{K}_\alpha^{\mathrm{j},l}(f;x) &:= \mathsf{K}_\alpha^{\mathrm{j},l}(0;x) + \sum_{p=1}^{N}\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,l}(f;x) + \widetilde{\mathsf{K}}_\alpha^{\mathrm{j},>N,l}(f;x), \qquad l = 0,1,2,\\
\varrho_\alpha^{\mathrm{j},3}(f;x,z) &:= \varrho_\alpha^{\mathrm{j},3}(0;x,z) + \sum_{p=1}^{N}\varrho_\alpha^{\mathrm{j},p,3}(f;x,z) + \varrho_\alpha^{\mathrm{j},>N,3}(f;x,z),
\end{aligned}
\tag{4.45}
$$

and

$$
\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},>N,l}(f;x) := \sum_{p>N}\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,l}(f;x), \qquad \varrho_\alpha^{\mathrm{j},>N,3}(f;x,z) := \sum_{p>N}\varrho_\alpha^{\mathrm{j},p,3}(f;x,z). \tag{4.46}
$$

Let us prove (4.24). We deduce that each $\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,l}(f;x) = \frac{1}{l!}\partial_z^l\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},p,l}(f;x,0)$, $p \geq 1$, is a homogenous function in $\widetilde{\mathcal{F}}_p^{\mathbb{R}}$ by (4.34) and Remark 2.27. Analogously the non-homogenous term $\widetilde{\mathsf{K}}_\alpha^{\mathrm{j},>N,l}(f;x)$ is in $\mathcal{F}_{K,0,N+1}^{\mathbb{R}}[\epsilon_0]$ by (4.35). Next, by (4.34) an integration in z give that $\varrho_\alpha^{\mathrm{j},p,3}(f;x,z)$, $p \geq 1$, defined in (4.44) is a homogenous Kernel-function in $\widetilde{K\mathcal{F}}_p^3$ and, by (4.35), the non-homogenous term $\varrho_\alpha^{\mathrm{j},>N,3}(f;x,z)$ in (4.46) is a Kernel function in $K\mathcal{F}_{K,0,N+1}^3[\epsilon_0]$.

Finally the zero-homogenous functions $\mathsf{K}_\alpha^{\mathrm{j},l}(0;x)$ are the constants in (4.25) (cf. (4.40)) and the Kernel functions $\varrho_\alpha^{\mathrm{j},3}(0;x,z)$ are in (4.41). □

4.3 Paralinearization of the quasilinear integral term $\mathcal{I}(f)$

In this section we paralinearize $\mathcal{I}(f)$.

Lemma 4.4. *The term $\mathcal{I}(f)$ defined in* (4.13) *can be written as*

$$
\mathcal{I}(f) = \mathrm{Op}^{BW}\left[-\left(1+\nu_{\mathcal{I}}(f;x)\right)L_{\mathcal{I}}(|\xi|) + \mathrm{i}S_{\mathcal{I},\alpha-2}(f;x,\xi) + V[\mathcal{I}](f;x) + P(f;x,\xi)\right]f + R(f)f \tag{4.47}
$$

where

- *$\nu_{\mathcal{I}}(f;x)$ is the real function*

$$
\nu_{\mathcal{I}}(f;x) := -\left(r^{-2}\mathsf{K}_\alpha^{1,0}(f;x)+1\right) \in \Sigma\mathcal{F}_{K,0,1}^{\mathbb{R}}[\epsilon_0,N]; \tag{4.48}
$$

- *$L_{\mathcal{I}}(|\xi|) := \mathsf{T}_\alpha^1(|\xi|) + \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} + \left(1-\frac{\alpha}{2}\right)^2\mathsf{M}_\alpha(|\xi|)$ is a real Fourier multiplier in $\widetilde{\Gamma}_0^{\max\{0,\alpha-1\}}$ (the Fourier multipliers $\mathsf{T}_\alpha^1(|\xi|)$ and $\mathsf{M}_\alpha(|\xi|)$ are defined in Lemma 3.1);*
- *$S_{\mathcal{I},\alpha-2}(f;x,\xi) := -\frac{1}{2}\left(\nu_{\mathcal{I}}(f;x)\right)_x\,\partial_\xi L_{\mathcal{I}}(|\xi|) + r^{-2}\mathsf{K}_\alpha^{1,1}(f;x)\mathsf{M}_\alpha(|\xi|)\,\xi$ is a real symbol in $\Sigma\Gamma_{K,0,1}^{\alpha-2}[\epsilon_0,N]$;*
- *$V[\mathcal{I}](f;x)$ is a function in $\Sigma\mathcal{F}_{K,0,1}^{\mathbb{R}}[\epsilon_0,N]$;*
- *$P(f;x,\xi)$ is a symbol in $\Sigma\Gamma_{K,0,1}^{-1}[\epsilon_0,N]$ satisfying* (2.20)*;*
- *$R(f)$ is a real smoothing operator in $\Sigma\mathcal{R}_{K,0,1}^{-\rho}[\epsilon_0,N]$.*

The rest of this section is devoted to prove Lemma 4.4.

By Lemma 2.22 we have

$$
\frac{\delta_z f}{r^2} = \mathrm{Op}^{BW}\left[r^{-2}\right]\delta_z f + \mathrm{Op}^{BW}\left[\delta_z f\right]\left(r^{-2}-1\right) + R_1\left(r^{-2}-1\right)\delta_z f + R_2\left(\delta_z f\right)\left(r^{-2}-1\right)
$$

with smoothing operators R_1, R_2 in $\widetilde{\mathcal{R}}_1^{-\rho}$ for any $\rho \geq 0$. Hence, recalling the definition of $\mathcal{I}(f)$ in (4.13), we write

$$
\begin{aligned}
\mathcal{I}(f) &= \sum_{\mathrm{j}=1}^{4} \mathcal{I}_{\mathrm{j}}(f)\,,\\
\mathcal{I}_1(f) &:= \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] \mathrm{Op}^{BW}\left[r^{-2}\right] \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}} \mathrm{d}z\,,\\
\mathcal{I}_2(f) &:= \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] \mathrm{Op}^{BW}\left[\delta_z f\right] \frac{1}{|2\sin(z/2)|^{\alpha}} \mathrm{d}z\,\left(r^{-2}-1\right),\\
\mathcal{I}_3(f) &:= \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] R_1\left(r^{-2}-1\right) \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}} \mathrm{d}z\,,\\
\mathcal{I}_4(f) &:= \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] R_2\left(\frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\right) \mathrm{d}z\,\left(r^{-2}-1\right).
\end{aligned}
\tag{4.49}
$$

Step 1 (Paralinearization of $\mathcal{I}_1$ in (4.49))**.** By (4.23) and isolating by (4.25) the zero-order components in f, we write

$$
\begin{aligned}
&\mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] \mathrm{Op}^{BW}\left[r^{-2}\right] =\\
&\mathrm{Op}^{BW}\left[\mathsf{K}^{1,0}_{\alpha}(f;x) + \mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \mathsf{K}^{1,2}_{\alpha}(f;x)(2\sin(z/2))^2 + \varrho^{1,3}_{\alpha}(f;x,z)\right] \mathrm{Op}^{BW}\left[r^{-2}\right]\\
&= -1 - \frac{1}{2}\left(1-\frac{\alpha}{2}\right)(2\sin(z/2))^2\\
&\ + \mathrm{Op}^{BW}\left[\left(\mathsf{K}^{1,0}_{\alpha}(f;x)+1\right) + \mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \left(\mathsf{K}^{1,2}_{\alpha}(f;x) + \frac{1}{2}\left(1-\frac{\alpha}{2}\right)\right)(2\sin(z/2))^2 + \varrho^{1,3}_{\alpha}(f;x,z)\right]\\
&\ + \mathrm{Op}^{BW}\left[\mathsf{K}^{1,0}_{\alpha}(f;x) + \mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \mathsf{K}^{1,2}_{\alpha}(f;x)(2\sin(z/2))^2 + \varrho^{1,3}_{\alpha}(f;x,z)\right] \mathrm{Op}^{BW}\left[r^{-2}-1\right]
\end{aligned}
\tag{4.50}
$$

where $\varrho^{1,3}_{\alpha}(f;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{3}_{K,0,1}[\epsilon_0,N]$ by (4.24). We now expand the last line (4.50). By Proposition 2.21 there exists a smoothing operator $R(f)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ such that

$$
\begin{aligned}
&\mathrm{Op}^{BW}\left[\mathsf{K}^{1,0}_{\alpha}(f;x) + \mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \mathsf{K}^{1,2}_{\alpha}(f;x)(2\sin(z/2))^2\right] \mathrm{Op}^{BW}\left[r^{-2}-1\right]\\
&\quad = \mathrm{Op}^{BW}\left[\left(r^{-2}-1\right)\left(\mathsf{K}^{1,0}_{\alpha}(f;x) + \mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \mathsf{K}^{1,2}_{\alpha}(f;x)(2\sin(z/2))^2\right)\right]\\
&\qquad + R(f) + \underbrace{\left(\sin(z) + (2\sin(z/2))^2\right)R(f)}_{:=R_{(1)}(f;z)\in\Sigma K\mathcal{R}^{-\rho,1}_{K,0,1}[\epsilon_0,N]}.
\end{aligned}
\tag{4.51}
$$

Moreover due to Proposition 2.34, Item 1, there exists a Kernel-smoothing operator $R_1(f;z)$ in $\Sigma\, K\mathcal{R}^{-\rho,3}_{K,0,1}[\epsilon_0,N]$ such that

$$
\mathrm{Op}^{BW}\left[\varrho^{1,3}_{\alpha}(f;x,z)\right] \mathrm{Op}^{BW}\left[r^{-2}-1\right] = \mathrm{Op}^{BW}\left[\left(r^{-2}-1\right)\varrho^{1,3}_{\alpha}(f;x,z)\right] + R_1(f;z)\,.
\tag{4.52}
$$

Plugging (4.51) and (4.52) in (4.50) we get

$$
\begin{aligned}
&\mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] \mathrm{Op}^{BW}\left[r^{-2}\right] = -1 - \frac{1}{2}\left(1-\frac{\alpha}{2}\right)(2\sin(z/2))^2\\
&\ + \mathrm{Op}^{BW}\left[\left(r^{-2}\mathsf{K}^{1,0}_{\alpha}(f;x)+1\right) + r^{-2}\mathsf{K}^{1,1}_{\alpha}(f;x)\sin z + \left(r^{-2}\mathsf{K}^{1,2}_{\alpha}(f;x) + \frac{1}{2}\left(1-\frac{\alpha}{2}\right)\right)(2\sin(z/2))^2\right]\\
&\ + \mathrm{Op}^{BW}\left[r^{-2}\varrho^{1,3}_{\alpha}(f;x,z)\right] + R(f) + R_{(1)}(f;z)
\end{aligned}
\tag{4.53}
$$

where $R_{(1)}(f;z)$ is a Kernel smoothing operator in $\Sigma\, K\mathcal{R}^{-\rho,1}_{K,0,1}[\epsilon_0,N]$. Inserting the decomposition (4.53) in the expression of $\mathcal{I}_1(f)$ in (4.49) we obtain that

$$
\mathcal{I}_1(f) = -\fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}} \mathrm{d}z - \frac{1}{2}\left(1-\frac{\alpha}{2}\right)\fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha-2}}\,\mathrm{d}z + \sum_{\mathrm{j}=1}^{5} \mathcal{I}_{1,\mathrm{j}}(f)
\tag{4.54}
$$

where

$$
\begin{aligned}
\mathcal{I}_{1,1}(f) &:= \mathrm{Op}^{BW}\left[r^{-2}\mathsf{K}_{\alpha}^{1,0}(f;x)+1\right] \fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\mathrm{d}z,\\
\mathcal{I}_{1,2}(f) &:= \mathrm{Op}^{BW}\left[r^{-2}\mathsf{K}_{\alpha}^{1,1}(f;x)\right] \fint \frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\delta_z f\,\mathrm{d}z,\\
\mathcal{I}_{1,3}(f) &:= \mathrm{Op}^{BW}\left[\left(r^{-2}\mathsf{K}_{\alpha}^{1,2}(f;x)+\frac{1}{2}\left(1-\frac{\alpha}{2}\right)\right)\right] \fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha-2}}\,\mathrm{d}z,\\
\mathcal{I}_{1,4}(f) &:= \fint \mathrm{Op}^{BW}\left[r^{-2}\varrho_{\alpha}^{1,3}(f;x,z)\right] \frac{\delta_z f}{|2\sin(z/2)|^{\alpha-2}}\,\mathrm{d}z,\\
\mathcal{I}_{1,5}(f) &:= \fint \left(R(f)+R_{(1)}(f;z)\right) \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z.
\end{aligned}
\tag{4.55}
$$

By recalling (4.2) and (3.9) we have

$$
\fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\mathrm{d}z = \mathrm{Op}^{BW}\left[\mathsf{T}_{\alpha}^{1}(|\xi|)\right] f. \tag{4.56}
$$

Next, by Eqs. (3.21) and (3.22) we deduce that

$$
\frac{1}{2}\left(1-\frac{\alpha}{2}\right)\fint \frac{\delta_z f}{|2\sin(z/2)|^{\alpha-2}}\,\mathrm{d}z = \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} f(x) + \left(1-\frac{\alpha}{2}\right)^2 \mathsf{M}_{\alpha}(|D|)\, f. \tag{4.57}
$$

By (4.56), using also Proposition 2.21 and (2.28), and that $\mathsf{T}_{\alpha}^{1}(|\xi|)$ is a symbol of order $\alpha-1$, we have

$$
\begin{aligned}
\mathcal{I}_{1,1}(f) &= \mathrm{Op}^{BW}\left[r^{-2}\mathsf{K}_{\alpha}^{1,0}(f;x)+1\right]\mathrm{Op}^{BW}\left[\mathsf{T}_{\alpha}^{1}(|\xi|)\right] f \\
&= \mathrm{Op}^{BW}\left[\left(r^{-2}\mathsf{K}_{\alpha}^{1,0}(f;x)+1\right)\mathsf{T}_{\alpha}^{1}(|\xi|) + \frac{\mathrm{i}}{2}\partial_x\left(r^{-2}\mathsf{K}_{\alpha}^{1,0}(f;x)+1\right)\,\partial_{\xi}\mathsf{T}_{\alpha}^{1}(|\xi|) + P(f;x,\xi)\right] f + R(f)\, f
\end{aligned}
\tag{4.58}
$$

where $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma_{K,0,1}^{-1}[\epsilon_0,N]$.

In order to compute $\mathcal{I}_{1,2}(f)$ in (4.55) we need the following lemma.

Lemma 4.5. *We have*

$$
\fint \frac{\sin z}{|2\sin(z/2)|^{\alpha}}\delta_z\phi\,\mathrm{d}z = \mathrm{i}\,\mathsf{M}_{\alpha}(|D|)\,D\,\phi, \tag{4.59}
$$

where $\mathsf{M}_{\alpha}(|\xi|)$ *is defined in* (3.6).

Proof. By oddness $\fint \frac{\sin z}{|2\sin(z/2)|^{\alpha}}\mathrm{d}z = 0$ and thus, integrating by parts,

$$
\begin{aligned}
\fint \frac{\sin z}{|2\sin(z/2)|^{\alpha}}\delta_z\phi\,\mathrm{d}z &= -\fint \frac{\sin z}{[2(1-\cos z)]^{\alpha/2}}\phi(x-z)\,\mathrm{d}z = -\fint \partial_z\left(\frac{[2(1-\cos z)]^{1-\frac{\alpha}{2}}}{2\left(1-\frac{\alpha}{2}\right)}\right)\phi(x-z)\,\mathrm{d}z\\
&= -\frac{1}{2\left(1-\frac{\alpha}{2}\right)}\fint \frac{\phi'(x-z)}{[2(1-\cos z)]^{\frac{\alpha}{2}-1}}\,\mathrm{d}z = \mathrm{i}\,\mathsf{M}_{\alpha}(|D|)\,D\,\phi
\end{aligned}
$$

using (3.20). This proves (4.59). □

Lemma 4.5 and Proposition 2.21 and since $\mathsf{M}_{\alpha}(|\xi|)$ is a symbol of order $\alpha-3$ gives that

$$
\mathcal{I}_{1,2}(f) = \mathrm{Op}^{BW}\left[\mathrm{i}\,r^{-2}\mathsf{K}_{\alpha}^{1,1}(f;x)\,\mathsf{M}_{\alpha}(|\xi|)\,\xi + P(f;x,\xi)\right] f + R(f)\, f \tag{4.60}
$$

where $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma_{K,0,1}^{-1}[\epsilon_0,N]$ satisfying (2.20).

Let us now compute $\mathcal{I}_{1,3}(f)$ in (4.55). Applying Proposition 2.36 we deduce that

$$
\mathcal{I}_{1,3}(f) = \mathrm{Op}^{BW}\left[V[\mathcal{I}_{1,3}](f;x) + P(f;x,\xi)\right] f \tag{4.61}
$$

where

$$
V[\mathcal{I}_{1,3}](f;x) := \left(r^{-2}\mathsf{K}_{\alpha}^{1,2}(f;x)+\frac{1}{2}\left(1-\frac{\alpha}{2}\right)\right)\fint \frac{1}{|2\sin(z/2)|^{\alpha-2}}\,\mathrm{d}z \tag{4.62}
$$

is a function in $\Sigma\mathcal{F}_{K,0,1}^{\mathbb{R}}[\epsilon_0,N]$ and $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma_{K,0,1}^{-1}[\epsilon_0,N]$, being $\alpha\in(0,2)$.

Similarly, applying Proposition 2.36,

$$\mathcal{I}_{1,4}(f) = \mathrm{Op}^{BW}\left[V[\mathcal{I}_{1,4}](f;x) + P(f;x,\xi)\right] f \tag{4.63}$$

where

$$V[\mathcal{I}_{1,4}](f;x) := \fint \frac{r^{-2}\varrho_\alpha^{1,3}(f;x,z)}{|2\sin(z/2)|^{\alpha-2}}\, \mathrm{d}z \tag{4.64}$$

is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ by Remark 2.30, and $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$.

Finally, the last term in (4.55) is, applying Lemma 2.35 since $\frac{R_{(1)}(f;z)}{|2\sin(z/2)|^\alpha} \in \Sigma K\mathcal{R}^{-\rho,1-\alpha}_{K,0,1}[\epsilon_0,N]$,

$$\mathcal{I}_{1,5}(f) = R(f)\,\mathrm{Op}^{BW}\left[\mathsf{T}^1_\alpha(|\xi|)\right] f + \tilde{R}(f) f, \qquad R(f), \tilde{R}(f) \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]. \tag{4.65}$$

We thus plug (4.56), (4.57) (4.58), (4.60), (4.61), (4.63), (4.65), in Equation (4.54) and obtain

$$\begin{aligned}
\mathcal{I}_1(f) = & -\mathsf{T}^1_\alpha(|D|) f - \frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2} f - \left(1-\frac{\alpha}{2}\right)^2 \mathsf{M}_\alpha(|D|) f \\
& + \mathrm{Op}^{BW}\left[\left(r^{-2}\mathsf{K}^{1,0}_\alpha(f;x)+1\right)\mathsf{T}^1_\alpha(|\xi|) + \frac{\mathrm{i}}{2}\partial_x\left(r^{-2}\mathsf{K}^{1,0}_\alpha(f;x)+1\right)\,\partial_\xi \mathsf{T}^1_\alpha(|\xi|) + \mathrm{i}\, r^{-2}\mathsf{K}^{1,1}_\alpha(f;x)\,\mathsf{M}_\alpha(|\xi|)\,\xi\right] f \\
& + \mathrm{Op}^{BW}\left[V[\mathcal{I}_1](f;x) + P(f;x,\xi)\right] f + R(f) f
\end{aligned} \tag{4.66}$$

where $V[\mathcal{I}_1](f;x)$ is the function (cf. Eqs. (4.62) and (4.64))

$$V[\mathcal{I}_1](f;x) := V[\mathcal{I}_{1,3}](f;x) + V[\mathcal{I}_{1,4}](f;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]. \tag{4.67}$$

Step 2 (Paralinearization of $\mathcal{I}_2$ in (4.49))**.** Since $\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) \in \Sigma K\mathcal{F}^0_{K,0,0}[\epsilon_0,N]$ (cf. Lemma 4.3) and $\delta_z f \in \widetilde{K\mathcal{F}}^1_1$ we apply Proposition 2.34, Item 1 and obtain that, for some $R_2(f;z) \in \Sigma\, K\mathcal{R}^{-\rho,1-\alpha}_{K,0,1}[\epsilon_0,N]$

$$\begin{aligned}
\fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] \mathrm{Op}^{BW}[\delta_z f] \frac{\mathrm{d}z}{|2\sin(z/2)|^\alpha} &= \fint \mathrm{Op}^{BW}\left[\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\delta_z f\right] \frac{\mathrm{d}z}{|2\sin(z/2)|^\alpha}\mathrm{d}z + \fint R_2(f;z)\,\mathrm{d}z \\
&= \mathrm{Op}^{BW}\left[\tilde{V}[\mathcal{I}_2]\right] + R(f)
\end{aligned}$$

where $R(f)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ (by Lemma 2.35) and

$$\tilde{V}[\mathcal{I}_2](f;x) := \fint \mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\delta_z f \frac{1}{|2\sin(z/2)|^\alpha}\mathrm{d}z$$

is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$, by Remark 2.30 and since since $\mathsf{K}^1_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\delta_z f\,\frac{1}{|2\sin(z/2)|^\alpha}$ is in $\Sigma\, K\mathcal{F}^{1-\alpha}_{K,0,1}[\epsilon_0,N]$.

Finally, using the identity (cf. Lemma 2.25)

$$r^\beta - 1 = \mathrm{Op}^{BW}\left[\beta r^{\beta-2}\right] f + R(f) f, \qquad \forall \beta \in \mathbb{R}, \tag{4.68}$$

we write the term $\mathcal{I}_2(f)$ in (4.49) as, using Propositions 2.21 and 2.23,

$$\begin{aligned}
\mathcal{I}_2(f) &= \left(\mathrm{Op}^{BW}\left[\tilde{V}[\mathcal{I}_2](f;x)\right] + R(f)\right)\left(\mathrm{Op}^{BW}\left[-2r^{-4}\right] f + R(f) f\right) \\
&= \mathrm{Op}^{BW}\left[V[\mathcal{I}_2](f;x)\right] f + R(f) f
\end{aligned} \tag{4.69}$$

where

$$V[\mathcal{I}_2](f;x) := -2r^{-4}\tilde{V}[\mathcal{I}_2](f;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]. \tag{4.70}$$

Step 3 (Paralinearization of $\mathcal{I}_3$ in (4.49))**.** We first note that, in view of (4.68), the fact that $\mathrm{Op}^{BW}\left[\beta r^{\beta-2}\right]$ and $R(f)$ are 0-operators, and Proposition 2.23-(ii), we deduce that

$$R_1\left(r^{-2}-1\right) = \check{R}(f) \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$$

is a smoothing operator, that we may also regard as a Kernel-smoothing operator in $\Sigma K\mathcal{R}^{-\rho,0}_{K,0,1}[\epsilon_0,N]$. Furthermore by (4.3) $\frac{\delta_z}{|2\sin(z/2)|^\alpha}$ is in $\widetilde{K\mathcal{M}}^{1,1-\alpha}_0$ and $\mathsf{K}^{\mathrm{j}}_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ is a Kernel function in $\Sigma K\mathcal{F}^0_{K,0,0}[\epsilon_0,N]$ by (4.22). Therefore by Proposition 2.34 Items 2 and 3 and Lemma 2.35 we obtain that

$$\mathcal{I}_3(f) = \fint R(f;z) f\,\mathrm{d}z = R(f) f \tag{4.71}$$

where $R(f)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[\epsilon_0,N]$.

Step 4 (Paralinearization of $\mathcal{I}_4$ in (4.49))**.** Reasoning as in the previous step there is a smoothing operator $\breve{R}(f)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[\epsilon_0,N]$ such that

$$\mathcal{I}_4(f) = \breve{R}(f)\left(r^{-2}-1\right) = R(f)f \tag{4.72}$$

(use (4.68)) where $R(f)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,p}[\epsilon_0,N]$.

Step 5 (Conclusion)**.** Inserting Eqs. (4.66), (4.69), (4.71) and (4.72) in Eq. (4.49), recalling the definition of $L_{\mathcal{I}}(|\xi|)$ in Lemma 4.4, and that $v_{\mathcal{I}}(f;x) := -\left(r^{-2}\mathsf{K}^{1,0}_{\alpha}(f;x)+1\right)$ (cf. Eq. (4.48)) we obtain

$$\begin{aligned}\mathcal{I}(f) = -L_{\mathcal{I}}(|\xi|)f + \mathrm{Op}^{BW}\Big[-v_{\mathcal{I}}(f;x)\,\mathsf{T}^1_{\alpha}(|\xi|) - \frac{\mathrm{i}}{2}\left(v_{\mathcal{I}}(f;x)\right)_x\partial_\xi\mathsf{T}^1_{\alpha}(|\xi|) + \mathrm{i}\,r^{-2}\mathsf{K}^{1,1}_{\alpha}(f;x)\,\mathsf{M}_{\alpha}(|\xi|)\,\xi\Big]f \\ + \mathrm{Op}^{BW}\left[V[\mathcal{I}_1](f;x) + V[\mathcal{I}_2](f;x) + P(f;x,\xi)\right]f + R(f)f.\end{aligned} \tag{4.73}$$

Finally, substituting $\mathsf{T}^1_{\alpha}(|\xi|) = L_{\mathcal{I}}(|\xi|) - \frac{\Gamma(2-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} - (1-\frac{\alpha}{2})^2 M_{\alpha}(|\xi|)$, we deduce that (4.73) is the paralinearization (4.47) with (cf. Eqs. (4.67) and (4.70))

$$V[\mathcal{I}](f;x) := V[\mathcal{I}_1](f;x) + V[\mathcal{I}_2](f;x) + v_{\mathcal{I}}(f;x)\frac{\Gamma(2-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N] \tag{4.74}$$

and another symbol $P(f;x,\xi)$ in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ satisfying (2.20).

4.4 Paralinearization of the quasilinear integral term $\mathcal{J}(f)$

In this section we paralinearize $\mathcal{J}(f)$.

Lemma 4.6. *The term $\mathcal{J}(f)$ defined in* (4.15) *can be written as*

$$\mathcal{J}(f) = \mathrm{Op}^{BW}\left[-\left(1+v_{\mathcal{J}}(f;x)\right)L_{\mathcal{J}}(|\xi|) + \mathrm{i}\,S_{\mathcal{J},\alpha-2}(f;x,\xi) + V[\mathcal{J}](f;x) + P(f;x,\xi)\right]f + R(f)f \tag{4.75}$$

where

- $v_{\mathcal{J}}(f;x)$ *is the real function*
$$v_{\mathcal{J}}(f;x) := \left(\mathsf{K}^{2,0}_{\alpha}(f;x)-1\right) + \frac{1}{\alpha-1}\frac{f'(x)}{r^2}\mathsf{K}^{3,0}_{\alpha}(f;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]; \tag{4.76}$$
- $L_{\mathcal{J}}(|\xi|) := -|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|)$ *is a real Fourier multiplier in* $\widetilde{\Gamma}^{\alpha-1}_0$ *(the Fourier multiplier* $\mathsf{M}_{\alpha}(|\xi|)$ *is defined in Lemma 3.1);*
- $S_{\mathcal{J},\alpha-2}(f;x,\xi) := -\frac{\partial_x}{2}\left(v_{\mathcal{J}}(f;x)\right)\partial_\xi L_{\mathcal{J}}(|\xi|) + \left((\alpha-2)\,\mathsf{K}^{2,1}_{\alpha}(f;x) + \frac{1}{r^2}\left(f'\mathsf{K}^{3,1}_{\alpha}(f;x) - f''\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\right)\mathsf{M}_{\alpha}(|\xi|)\,\xi$ *is a real symbol in* $\Sigma\Gamma^{\alpha-2}_{K,0,1}[\epsilon_0,N]$*;*
- $V[\mathcal{J}](f;x)$ *is a real function in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$*;*
- $P(f;x,\xi)$ *is a symbol in* $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ *satisfying* (2.20)*;*
- $R(f)$ *is a real smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$.

The rest of this section is devoted to the proof of Lemma 4.6.
By Lemma 2.22 we obtain that

$$\begin{aligned}\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)f'(x-z) = \mathrm{Op}^{BW}\left[\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]f'(x-z) + \mathrm{Op}^{BW}\left[f'(x-z)\right]\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right) \\ + R_1\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right)f'(x-z) + R_2\left(f'(x-z)\right)\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right)\end{aligned} \tag{4.77}$$

where R_1, R_2 are smoothing operators in $\widetilde{\mathcal{R}}_1^{-\rho}$. Hence, recalling the definition of $\mathcal{J}(f)$ in (4.15), we have

$$
\begin{aligned}
&\mathcal{J}(f) := \sum_{\mathrm{j}=1}^{4} \mathcal{J}_{\mathrm{j}}(f)\,, \\
&\mathcal{J}_1(f) := \fint \mathrm{Op}^{BW}\left[\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right] f'(x-z)\,\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,, \\
&\mathcal{J}_2(f) := \fint \mathrm{Op}^{BW}\left[f'(x-z)\right]\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right)\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,, \\
&\mathcal{J}_3(f) := \fint R_1\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right) f'(x-z)\,\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,, \\
&\mathcal{J}_4(f) := \fint R_2\left(f'(x-z)\right)\left(\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right)\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,.
\end{aligned}
\tag{4.78}
$$

Step 1 (Paralinearization of $\mathcal{J}_1$ in (4.78))**.** By (4.23) and (4.25) we obtain that

$$
\begin{aligned}
&\mathcal{J}_1(f) := \fint f'(x-z)\,\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z + \sum_{\mathrm{j}=1}^{3} \mathcal{J}_{1,\mathrm{j}}(f)\,, \\
&\mathcal{J}_{1,1}(f) := \mathrm{Op}^{BW}\left[\mathsf{K}^{2,0}_{\alpha}(f;x) - 1\right] \fint f'(x-z)\,\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,, \\
&\mathcal{J}_{1,2}(f) := \mathrm{Op}^{BW}\left[\mathsf{K}^{2,1}_{\alpha}(f;x)\right] \fint f'(x-z)\,\frac{\sin^2 z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z\,, \\
&\mathcal{J}_{1,3}(f) := \fint \mathrm{Op}^{BW}\left[\varrho^{[3-\alpha]}_{\alpha}(f;x,z)\right] f'(x-z)\,\mathrm{d}z\,,
\end{aligned}
\tag{4.79}
$$

where, by (4.24), (4.25) and Remark 2.29,

$$
\varrho^{[3-\alpha]}_{\alpha}(f;x,z) := \left(\mathsf{K}^{2,2}_{\alpha}(f;x)\,(2\sin(z/2))^2 + \varrho^{2,3}_{\alpha}(f;x,z)\right)\frac{\sin z}{|2\sin(z/2)|^{\alpha}} \in \Sigma\, K\mathcal{F}^{3-\alpha}_{K,0,1}[\epsilon_0, N]\,. \tag{4.80}
$$

Now, by (3.18), the first term in (4.79) is

$$
\fint f'(x-z)\,\frac{\sin z}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z = |D|^2\ \mathsf{M}_{\alpha}(|D|)\,f \tag{4.81}
$$

and, using Proposition 2.21,

$$
\begin{aligned}
\mathcal{J}_{1,1}(f) =&\ \mathrm{Op}^{BW}\left[\left(\mathsf{K}^{2,0}_{\alpha}(f;x) - 1\right)\right]|D|^2\,\mathsf{M}_{\alpha}(|D|)\,f \\
=&\ \mathrm{Op}^{BW}\left[\left(\mathsf{K}^{2,0}_{\alpha}(f;x) - 1\right)\ |\xi|^2\,\mathsf{M}_{\alpha}(|\xi|) + \frac{\mathrm{i}}{2}\partial_x\left(\mathsf{K}^{2,0}_{\alpha}(f;x)\right)\ \partial_{\xi}\left(|\xi|^2\ \mathsf{M}_{\alpha}(|\xi|) + P(f;x,\xi)\right)\right] f + R(f)\,f
\end{aligned}
\tag{4.82}
$$

where $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$. In order to expand $\mathcal{J}_{1,2}(f)$ in (4.79) we write

$$
\frac{\sin^2 z}{|2\sin(z/2)|^{\alpha}} = \frac{\cos^2(z/2)}{|2\sin(z/2)|^{\alpha-2}} = \frac{1}{|2(1-\cos(z)|^{\frac{\alpha}{2}-1}} + \varrho_{1,2}(z)\,, \quad \varrho_{1,2}(z) \in \widetilde{K\mathcal{F}}^{3-\alpha}_{0}\,. \tag{4.83}
$$

As a consequence of (4.83), using also (3.21), Propositions 2.36 and 2.21, for any $\alpha \in (0,2)$, we get

$$
\begin{aligned}
\mathcal{J}_{1,2}(f) =&\ \mathrm{Op}^{BW}\left[\mathsf{K}^{2,1}_{\alpha}(f;x)\right] \fint f'(x-z)\,\frac{\mathrm{d}z}{|2(1-\cos(z)|^{\frac{\alpha}{2}-1}} + \fint \mathrm{Op}^{BW}\left[\mathsf{K}^{2,1}_{\alpha}(f;x)\,\varrho_{1,2}(z)\right] f'(x-z)\ \mathrm{d}z \\
=&\ \mathrm{Op}^{BW}\left[\mathsf{K}^{2,1}_{\alpha}(f;x)\right]\mathrm{i}(\alpha-2)M_{\alpha}(|D|)Df + \mathrm{Op}^{BW}\left[a(f;x,\xi)\right]\partial_x f \\
=&\ \mathrm{Op}^{BW}\left[\mathrm{i}(\alpha-2)\,\mathsf{K}^{2,1}_{\alpha}(f;x)\ \mathsf{M}_{\alpha}(|\xi|)\,\xi + P(f;x,\xi)\right] f + R(f)\,f
\end{aligned}
\tag{4.84}
$$

where $a(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{4-\alpha}_{K,0,1}[\epsilon_0,N]$ and $P(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ satisfying (2.20). Furthermore, by (4.80), Propositions 2.36 and 2.21, for any $\alpha \in (0,2)$, the last term in (4.79) is

$$
\mathcal{J}_{1,3}(f) = \mathrm{Op}^{BW}\left[P(f;x,\xi)\right] f + R(f)\,f\,. \tag{4.85}
$$

In conclusion, by Eqs. (4.25), (4.81), (4.82), (4.84) and (4.85) defining

$$\nu_2(f;x) := \mathsf{K}^{2,0}_\alpha(f;x) - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]\,, \tag{4.86}$$

the term $\mathcal{J}_1(f)$ in (4.79) is

$$\begin{aligned}\mathcal{J}_1(f) &= \mathrm{Op}^{BW}\left[\left(1+\nu_2(f;x)\right)|\xi|^2\,\mathsf{M}_\alpha(|\xi|)\right]f \\ &\quad + \mathrm{i}\,\mathrm{Op}^{BW}\left[\frac{\partial_x}{2}\left(\nu_2(f;x)\right)\,\partial_\xi\left(|\xi|^2\,\mathsf{M}_\alpha(|\xi|)\right) + (\alpha-2)\,\mathsf{K}^{2,1}_\alpha(f;x)\ \mathsf{M}_\alpha(|\xi|)\,\xi + P(f;x,\xi)\right]f + R(f)f.\end{aligned} \tag{4.87}$$

Step 2 (Paralinearization of $\mathcal{J}_2$ in (4.78))**.** Using (4.16), (4.2), the paralinearization formula (2.25) and (4.21), we write

$$\begin{aligned}\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0) &= \left(\mathsf{G}^2_{\alpha,z}\left(\frac{\delta_z f}{r^2}\right) - \mathsf{G}^2_{\alpha,z}(0)\right)|2\sin(z/2)|^\alpha \\ &= \mathrm{Op}^{BW}\left[\left(\mathsf{G}^2_{\alpha,z}\right)'\left(\frac{\delta_z f}{r^2}\right)|2\sin(z/2)|^\alpha\right]\frac{\delta_z f}{r^2} + R\left(\frac{\delta_z f}{r^2}\right)\frac{\delta_z f}{r^2}\ |2\sin(z/2)|^\alpha \\ &= \mathrm{Op}^{BW}\left[\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{r^2\sin(z)} + R\left(\frac{\delta_z f}{r^2}\right)\frac{\delta_z f}{r^2}\ |2\sin(z/2)|^\alpha\end{aligned} \tag{4.88}$$

where R is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ for any ρ. By Eq. (4.21) it results $\mathsf{K}^3_{\alpha,z}(\mathsf{X}) = \mathsf{K}^3_{\alpha,z+2\pi}(-\mathsf{X})$ and the map $z \mapsto \mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ is 2π-periodic. Therefore, by (4.78) and (4.88) we obtain that

$$\mathcal{J}_2(f) = \fint \mathrm{Op}^{BW}\left[f'(x-z)\right]\mathrm{Op}^{BW}\left[\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\ \frac{\delta_z f}{r^2\,|2\sin(z/2)|^\alpha}\,\mathrm{d}z \tag{4.89a}$$

$$+\fint \mathrm{Op}^{BW}\left[f'(x-z)\right]R\left[\frac{\delta_z f}{r^2}\right]\frac{\delta_z f}{r^2}\ \sin z\,\mathrm{d}z. \tag{4.89b}$$

By (4.3) and Remark 2.16 we deduce that $M_2(f;z) := r^{-2}\delta_z$ is an operator in $\Sigma K\mathcal{M}^{1,1}_{K,0,0}[\epsilon_0,N]$. As a consequence by Proposition 2.34 we obtain that

$$R\left(\frac{\delta_z f}{r^2}\right)\frac{\delta_z f}{r^2} = R(f;z)f \qquad \text{where} \qquad R(f;z) \in \Sigma K\mathcal{R}^{-\rho,1}_{K,0,1}[\epsilon_0,N]\,, \tag{4.90}$$

and, by Proposition 2.34, Item 3, Lemma 2.35, being $\alpha \in (0,2)$, we deduce that the integral (4.89b) is

$$\fint \mathrm{Op}^{BW}\left[f'(x-z)\sin z\right]R\left(\frac{\delta_z f}{r^2}\right)\frac{\delta_z f}{r^2}\mathrm{d}z = R(f)f \tag{4.91}$$

where $R(f)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$.

We now consider the term (4.89a). By Lemma 2.22 we write

$$\frac{\delta_z f}{r^2} = \mathrm{Op}^{BW}\left[r^{-2}\right]\delta_z f + \mathrm{Op}^{BW}\left[\delta_z f\right](r^{-2}-1) + R_1\left(r^{-2}-1\right)\delta_z f + R_2\left(\delta_z f\right)\left[r^{-2}-1\right]$$

where R_1, R_2 are smoothing operators in $\widetilde{\mathcal{R}}^{-\rho}_1$ for any $\rho \geq 0$, and thus

$$\text{(4.89a)} = \fint \mathrm{Op}^{BW}\left[f'(x-z)\right]\mathrm{Op}^{BW}\left[\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\ \mathrm{Op}^{BW}\left[r^{-2}\right]\delta_z f\ \frac{\mathrm{d}z}{|2\sin(z/2)|^\alpha} \tag{4.92a}$$

$$+\fint \mathrm{Op}^{BW}\left[f'(x-z)\right]\mathrm{Op}^{BW}\left[\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\ \mathrm{Op}^{BW}\left[\delta_z f\right](r^{-2}-1)\ \frac{\mathrm{d}z}{|2\sin(z/2)|^\alpha} \tag{4.92b}$$

$$+\fint \mathrm{Op}^{BW}\left[f'(x-z)\right]\mathrm{Op}^{BW}\left[\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\left(R_1\left(r^{-2}-1\right)\delta_z f + R_2\left(\delta_z f\right)\left[r^{-2}-1\right]\right)\ \frac{\mathrm{d}z}{|2\sin(z/2)|^\alpha}. \tag{4.92c}$$

Proposition 2.34 give that

$$\text{(4.92a)} = \fint \mathrm{Op}^{BW}\left[r^{-2}f'(x-z)\,\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{|2\sin(z/2)|^\alpha}\ \mathrm{d}z + \underbrace{\fint R(f;z)f\,\mathrm{d}z}_{=R(f)f}$$

where $R(f;z)$ is a Kernel-smoothing operator in $\Sigma\mathcal{R}^{-\rho,1-\alpha}_{K,0,1}[\epsilon_0,N]$ and, since $\alpha\in(0,2)$, the operator $R(f)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0,N]$ by Lemma 2.35. Then by Proposition 2.34, Eq. (4.68), and Remark 2.30, we get

$$(4.92\text{b}) = \mathrm{Op}^{BW}\left[V_1[\mathcal{J}_2](f;x)\right]f + R(f)f$$

where $V_1[\mathcal{J}_2](f;x)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$. Finally (4.92c) is a smoothing term $R(f)f$ and we deduce that

$$(4.89\text{a}) = \fint \mathrm{Op}^{BW}\left[r^{-2}f'(x-z)\,\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z + \mathrm{Op}^{BW}\left[V_1[\mathcal{J}_2](f;x)\right]f + R(f)f. \tag{4.93}$$

Then we write

$$f'(x-z) = f'(x) - f''(x)\,\sin z + \varrho_1(f;x,z)$$

where $\varrho_1(f;x,z)$ is a homogenous Kernel function in $\widetilde{K\mathcal{F}}^2_1$ and, using (4.23), we deduce that

$$r^{-2}f'(x-z)\,\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) = r^{-2}f'(x)\mathsf{K}^{3,0}_{\alpha}(f;x) + r^{-2}\left(f'(x)\mathsf{K}^{3,1}_{\alpha}(f;x) - f''(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\sin z + \breve{\varrho}^{3,2}_{\alpha}(f;x,z),$$

where $\breve{\varrho}^{3,2}_{\alpha}(f;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^2_{K,0,1}[\epsilon_0,N]$, and, by also (3.9), Lemma 4.5 and Proposition 2.36 and Proposition 2.21, we get

$$\begin{aligned}
&\fint \mathrm{Op}^{BW}\left[r^{-2}f'(x-z)\,\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z = \mathrm{Op}^{BW}\left[r^{-2}f'(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right]\mathrm{Op}^{BW}\left[\mathsf{T}^1_{\alpha}(|\xi|)\right]f\\
&+\mathrm{i}\,\mathrm{Op}^{BW}\left[r^{-2}\left(f'(x)\mathsf{K}^{3,1}_{\alpha}(f;x) - f''(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\right]\mathrm{Op}^{BW}\left[\mathsf{M}_{\alpha}(|\xi|)\,\xi\right]f + \mathrm{Op}^{BW}\left[P(f;x,\xi)\right]f\\
&= \mathrm{Op}^{BW}\left[r^{-2}f'(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\,\mathsf{T}^1_{\alpha}(|\xi|)\right]f\\
&+\mathrm{i}\,\mathrm{Op}^{BW}\left[\frac{1}{2}\partial_x\left(\frac{f'(x)}{r^2}\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\partial_{\xi}\mathsf{T}^1_{\alpha}(|\xi|) + r^{-2}\left(f'(x)\mathsf{K}^{3,1}_{\alpha}(f;x) - f''(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\,\mathsf{M}_{\alpha}(|\xi|)\,\xi\right]f\\
&+\mathrm{Op}^{BW}\left[P(f;x,\xi)\right]f + R(f)f.
\end{aligned}\tag{4.94}$$

By Lemma 3.7 we have $\mathsf{T}^1_{\alpha}(|\xi|) = \frac{1}{\alpha-1}|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|) + \tilde{\mathbb{V}}_{\alpha} + m_{\alpha-3}(|\xi|)$ and so, defining the function

$$v_3(f;x) := \frac{1}{\alpha-1}\frac{f'(x)}{r^2}\mathsf{K}^{3,0}_{\alpha}(f;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \tag{4.95}$$

the equation (4.94) becomes

$$\begin{aligned}
&\fint \mathrm{Op}^{BW}\left[r^{-2}f'(x-z)\,\mathsf{K}^3_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)\right]\frac{\delta_z f}{|2\sin(z/2)|^{\alpha}}\,\mathrm{d}z = \mathrm{Op}^{BW}\left[v_3(f;x)\,|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|)\right]f\\
&\quad+\mathrm{i}\,\mathrm{Op}^{BW}\left[\frac{1}{2}\partial_x\left(v_3(f;x)\right)\partial_{\xi}\left(|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|)\right) + r^{-2}\left(f'(x)\mathsf{K}^{3,1}_{\alpha}(f;x) - f''(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\,\mathsf{M}_{\alpha}(|\xi|)\,\xi\right]f\\
&\qquad+\mathrm{Op}^{BW}\left[V_2[\mathcal{J}_2](f;x) + P(f;x,\xi)\right]f + R(f)f
\end{aligned}\tag{4.96}$$

where $V_2[\mathcal{J}_2](f;x) := \tilde{\mathbb{V}}_{\alpha}\,v_3(f;x)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$. By (4.91), (4.93), (4.96) we deduce that $\mathcal{J}_2(f)$ in (4.89a)-(4.89b) is

$$\begin{aligned}
\mathcal{J}_2(f) &= \mathrm{Op}^{BW}\left[v_3(f;x)\,|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|)\right]f\\
&\quad+\mathrm{i}\,\mathrm{Op}^{BW}\left[\frac{\partial_x}{2}\left(v_3(f;x)\right)\partial_{\xi}\left(|\xi|^2\,\mathsf{M}_{\alpha}(|\xi|)\right) + r^{-2}\left(f'(x)\mathsf{K}^{3,1}_{\alpha}(f;x) - f''(x)\mathsf{K}^{3,0}_{\alpha}(f;x)\right)\,\mathsf{M}_{\alpha}(|\xi|)\,\xi\right]f\\
&\qquad+\mathrm{Op}^{BW}\left[V[\mathcal{J}_2](f;x) + P(f;x,\xi)\right]f + R(f)f
\end{aligned}\tag{4.97}$$

where the real function $V[\mathcal{J}_2] := V_1[\mathcal{J}_2] + V_2[\mathcal{J}_2]$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$.

Step 3 (Paralinearization of $\mathcal{J}_3$ in (4.78))**.** By Eqs. (4.88) and (4.90), Proposition 2.34, and Remark 2.32 and $\frac{\delta_z}{r^2\sin z}\in\Sigma K\mathcal{M}^{1,0}_{K,0,0}[\epsilon_0,N]$ (which follows by Eq. (4.3), Remark 2.16, and Proposition 2.34), and since R_1 is a smoothing operator in $\widetilde{\mathcal{R}}^{-\rho}_1$, we deduce that

$$R_1\left[\mathsf{K}^2_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right) - \mathsf{K}^2_{\alpha,z}(0)\right] \in \Sigma K\mathcal{R}^{-\rho,0}_{K,0,1}[\epsilon_0,N]. \tag{4.98}$$

Furthermore $f'(x-z) = \partial_x \circ \mathrm{t}_{-z} f$ and $\partial_x \circ \mathrm{t}_{-z}$ is in $\widetilde{K\mathcal{M}}_0^{1,0}$. By Remark 2.29 and Proposition 2.34 we obtain (after relabeling ρ) that

$$R_1 \left[\mathsf{K}^2_{\alpha,z} \left(\frac{\Delta_z f}{r^2} \right) - \mathsf{K}^2_{\alpha,z}(0) \right] \frac{\sin z}{|2\sin(z/2)|^\alpha} \partial_x \circ \mathrm{t}_{-z} := R^\star(f;z) \in \Sigma K\mathcal{R}^{-\rho,1-\alpha}_{K,0,1}[\epsilon_0, N] \ .$$

Finally Lemma 2.35 implies that

$$\mathcal{J}_3(f) = \fint R^\star(f;z) f \, \mathrm{d}z = R(f) f \qquad \text{where} \qquad R(f) \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0, 1] \ . \tag{4.99}$$

Step 4 (Paralinearization of $\mathcal{J}_4$ in (4.78))**.** We similar arguments one obtains

$$\mathcal{J}_4(f) = R(f) f \qquad \text{where} \qquad R(f) \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0, 1] \ . \tag{4.100}$$

Step 5 (Conclusion)**.** We plug Equations (4.87), (4.97), (4.99) and (4.100) in Eq. (4.78) and, recalling that $L_{\mathcal{J}}(|\xi|) = -|\xi|^2 \mathsf{M}_\alpha(|\xi|)$, defining the real functions $V[\mathcal{J}] := V[\mathcal{J}_2]$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$ and $\nu_{\mathcal{J}} := \nu_2 + \nu_3$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$ (cf. Eqs. (4.86) and (4.95)) we obtain the paralinearization formula (4.75) stated in Lemma 4.6.

4.5 Proof of Theorem 4.1

We now paralinearize the scalar field in Equation (4.20). We apply Lemma 2.22

$$\begin{aligned} r^{2-\alpha}\mathcal{I}(f) =& \mathrm{Op}^{BW}[r^{2-\alpha}]\mathcal{I}(f) + \mathrm{Op}^{BW}[\mathcal{I}(f)](r^{2-\alpha}-1) + R_1(r^{2-\alpha}-1)\mathcal{I}(f) + R_2(\mathcal{I}(f))(r^{2-\alpha}-1), \\ r^{-\alpha}\mathcal{J}(f) =& \mathrm{Op}^{BW}[r^{-\alpha}]\mathcal{J}(f) + \mathrm{Op}^{BW}[\mathcal{J}(f)](r^{-\alpha}-1) + R_1(r^{-\alpha}-1)\mathcal{J}(f) + R_2(\mathcal{J}(f))(r^{-\alpha}-1). \end{aligned}$$

We thus apply (4.68), Lemmas 2.19, 4.4 and 4.6 and Propositions 2.21 and 2.23 and obtain that there exist real functions $\tilde{V}_{\mathcal{I}}, \tilde{V}_{\mathcal{J}}$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$ such that

$$\begin{aligned} r^{2-\alpha}\mathcal{I}(f) &= \mathrm{Op}^{BW}[r^{2-\alpha}]\mathcal{I}(f) + \mathrm{Op}^{BW}[\tilde{V}_{\mathcal{I}}(f;x)] f + R(f) f, \\ r^{-\alpha}\mathcal{J}(f) &= \mathrm{Op}^{BW}[r^{-\alpha}]\mathcal{J}(f) + \mathrm{Op}^{BW}[\tilde{V}_{\mathcal{J}}(f;x)] f + R(f) f, \end{aligned} \tag{4.101}$$

for some smoothing operator $R(f)$ in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0, N]$.

A key fact proved in the next lemma is that the imaginary part of the symbol in (4.102) has order at most -1. This is actually an effect of the linear Hamiltonian structure, see Remark 4.8.

Lemma 4.7. *It results*

$$\begin{aligned} \mathrm{Op}^{BW}[r^{2-\alpha}]\mathcal{I}(f) + \mathrm{Op}^{BW}[r^{-\alpha}]\mathcal{J}(f) = & -\mathrm{Op}^{BW}\left[(1+\tilde{\nu}_{\mathcal{I}}(f;x)) L_{\mathcal{I}}(|\xi|) + (1+\tilde{\nu}_{\mathcal{J}}(f;x)) L_{\mathcal{J}}(|\xi|)\right] f \\ & + \mathrm{Op}^{BW}\left[V_{\mathcal{I}}(f;x) + V_{\mathcal{J}}(f;x) + P(f;x,\xi)\right] f + R(f) f \end{aligned} \tag{4.102}$$

where

- $L_{\mathcal{I}}(|\xi|)$ *and* $L_{\mathcal{J}}(|\xi|)$ *are the real Fourier multipliers defined in Lemmas 4.4 and 4.6;*
- $\tilde{\nu}_{\mathcal{I}}(f;x)$, $\tilde{\nu}_{\mathcal{J}}(f;x)$, $V_{\mathcal{I}}(f;x)$, $V_{\mathcal{J}}(f;x)$ *are real functions in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$*;*
- $P(f;x,\xi)$ *is a symbol in* $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0, N]$*;*
- $R(f)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[\epsilon_0, N]$.

Proof. Proposition 2.21 and Lemmas 4.4 and 4.6 give that

$$\begin{aligned} \mathrm{Op}^{BW}[r^{2-\alpha}]\mathcal{I}(f) = & \ \mathrm{Op}^{BW}\left[r^{2-\alpha}\left(-(1+\nu_{\mathcal{I}}(f;x)) L_{\mathcal{I}}(|\xi|) + \mathrm{i}\, S_{\mathcal{I},\alpha-2}(f;x,\xi)\right)\right] f \\ & + \mathrm{Op}^{BW}\left[\frac{1}{2\mathrm{i}}(r^{2-\alpha})_x (1+\nu_{\mathcal{I}}(f;x)) \partial_\xi L_{\mathcal{I}}(|\xi|)\right] f \\ & + \mathrm{Op}^{BW}\left[V_{\mathcal{I}}(f;x) + P(f;x,\xi)\right] f + R(f) f, \\ \mathrm{Op}^{BW}[r^{-\alpha}]\mathcal{J}(f) = & \ \mathrm{Op}^{BW}\left[r^{-\alpha}\left(-(1+\nu_{\mathcal{J}}(f;x)) L_{\mathcal{J}}(|\xi|) + \mathrm{i}\, S_{\mathcal{J},\alpha-2}(f;x,\xi)\right)\right] f \end{aligned}$$

$$+\mathrm{Op}^{BW}\left[\frac{1}{2\mathrm{i}}\left(r^{-\alpha}\right)_x\left(1+\nu_{\mathcal{J}}(f;x)\right)\partial_\xi L_{\mathcal{J}}(|\xi|)\right]f$$
$$+\mathrm{Op}^{BW}\left[V_{\mathcal{J}}(f;x)+P(f;x,\xi)\right]f+R(f)f,$$

so that, defining $\tilde{\nu}_{\mathcal{I}}(f;x):=r^{2-\alpha}\left(1+\nu_{\mathcal{I}}(f;x)\right)-1$ and $\tilde{\nu}_{\mathcal{J}}(f;x):=r^{-\alpha}\left(1+\nu_{\mathcal{J}}(f;x)\right)-1$, we get

$$\begin{aligned}
&\mathrm{Op}^{BW}\left[r^{2-\alpha}\right]\mathcal{I}(f)+\mathrm{Op}^{BW}\left[r^{-\alpha}\right]\mathcal{J}(f)\\
&\quad=-\mathrm{Op}^{BW}\left[\left(1+\tilde{\nu}_{\mathcal{I}}(f;x)\right)L_{\mathcal{I}}(|\xi|)+\left(1+\tilde{\nu}_{\mathcal{J}}(f;x)\right)L_{\mathcal{J}}(|\xi|)\right]f\\
&\qquad+\mathrm{i}\,\mathrm{Op}^{BW}\left[r^{2-\alpha}\,S_{\mathcal{I},\alpha-2}(f;x,\xi)+r^{-\alpha}\,S_{\mathcal{J},\alpha-2}(f;x,\xi)\right]f \qquad (4.103a)\\
&\qquad+\frac{1}{2\mathrm{i}}\mathrm{Op}^{BW}\left[\left(r^{2-\alpha}\right)_x\left(1+\nu_{\mathcal{I}}(f;x)\right)\partial_\xi L_{\mathcal{I}}(|\xi|)+\left(r^{-\alpha}\right)_x\left(1+\nu_{\mathcal{J}}(f;x)\right)\partial_\xi L_{\mathcal{J}}(|\xi|)\right]f \qquad (4.103b)\\
&\qquad+\mathrm{Op}^{BW}\left[V_{\mathcal{I}}(f;x)+V_{\mathcal{J}}(f;x)+P(f;x,\xi)\right]f+R(f)f.
\end{aligned}$$

We now prove that the sum of (4.103a) and (4.103b) give a paradifferential term of order -1. We first note that, by Lemma 3.7, we have the asymptotic expansions

$$\begin{aligned}
&|\xi|^2\,\mathsf{M}_\alpha(|\xi|)=\check{c}_\alpha|\xi|^{\alpha-1}+m_{\alpha-3}(|\xi|), &&\xi\mathsf{M}_\alpha(|\xi|)=\check{c}_\alpha|\xi|^{\alpha-3}\xi+m_{\alpha-4}(|\xi|),\\
&\mathsf{T}^1_\alpha(|\xi|)=\frac{1}{\alpha-1}\check{c}_\alpha|\xi|^{\alpha-1}+\tilde{\mathbb{V}}_\alpha+m_{\alpha-3}(|\xi|), &&\text{where}\quad \check{c}_\alpha:=\frac{\Gamma(2-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)},
\end{aligned}\tag{4.104}$$

so that

$$\partial_\xi L_{\mathcal{I}}(|\xi|)=\check{c}_\alpha|\xi|^{\alpha-3}\xi+m_{\alpha-4}(|\xi|),\qquad \partial_\xi L_{\mathcal{J}}(|\xi|)=-(\alpha-1)\,\check{c}_\alpha|\xi|^{\alpha-3}\xi+m_{\alpha-4}(|\xi|).$$

By the explicit definition of the symbols $S_{\mathcal{I},\alpha-2}$ and $S_{\mathcal{J},\alpha-2}$ in Lemmas 4.4 and 4.6 and (4.104) we have the expansion of the symbol in (4.103a)

$$\begin{aligned}
&\mathrm{i}\left(r^{2-\alpha}\,S_{\mathcal{I},\alpha-2}(f;x,\xi)+r^{-\alpha}\,S_{\mathcal{J},\alpha-2}(f;x,\xi)\right) \qquad (4.105)\\
&=\mathrm{i}\left[-r^{2-\alpha}\,\frac{1}{2}(\nu_{\mathcal{I}})_x(f;x)+(\alpha-1)\,r^{-\alpha}\,\frac{1}{2}(\nu_{\mathcal{J}})_x(f;x)+A_{\alpha,1}(f;x)\right]\check{c}_\alpha|\xi|^{\alpha-3}\xi+\mathrm{i}P(f;x,\xi),
\end{aligned}$$

where

$$A_{\alpha,1}(f;x):=\frac{1}{r^\alpha}\left[\mathsf{K}^{1,1}_\alpha(f;x)+(\alpha-2)\,\mathsf{K}^{2,1}_\alpha(f;x)+\frac{1}{r^2}\left(f'\,\mathsf{K}^{3,1}_\alpha(f;x)-f''\,\mathsf{K}^{3,0}_\alpha(f;x)\right)\right]\tag{4.106}$$

is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$, recalling (4.25). Then the sum of (4.103a) and (4.103b) gives

$$\begin{aligned}
&r^{2-\alpha}\,S_{\mathcal{I},\alpha-2}(f;x,\xi)+r^{-\alpha}\,S_{\mathcal{J},\alpha-2}(f;x,\xi)\\
&\qquad-\frac{1}{2}\left[\left(r^{2-\alpha}\right)_x\left(1+\nu_{\mathcal{I}}(f;x)\right)\partial_\xi L_{\mathcal{I}}(|\xi|)+\left(r^{-\alpha}\right)_x\left(1+\nu_{\mathcal{J}}(f;x)\right)\partial_\xi L_{\mathcal{J}}(|\xi|)\right]\\
&=\left\{\frac{1}{2}\underbrace{\left[-r^{2-\alpha}\left(1+\nu_{\mathcal{I}}(f;x)\right)+(\alpha-1)\,r^{-\alpha}\left(1+\nu_{\mathcal{J}}(f;x)\right)\right]_x}_{=\left(A_{\alpha,0}(f;x)\right)_x}+A_{\alpha,1}(f;x)\right\}\check{c}_\alpha|\xi|^{\alpha-3}\xi+P(f;x,\xi), \qquad (4.107)
\end{aligned}$$

where, having substituting the explicit values of $\nu_{\mathcal{I}}$, $\nu_{\mathcal{J}}$ in (4.48), (4.76), we define

$$A_{\alpha,0}(f;x):=\frac{1}{r^\alpha}\left[\mathsf{K}^{1,0}_\alpha(f;x)+(\alpha-1)\,\mathsf{K}^{2,0}_\alpha(f;x)+\frac{f'}{r^2}\,\mathsf{K}^{3,0}_\alpha(f;x)\right]+(2-\alpha)\tag{4.108}$$

which is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$. We finally write

$$(4.107)=\left[\frac{1}{2}\left(A_{\alpha,0}(f;x)\right)_x+A_{\alpha,1}(f;x)\right]\check{c}_\alpha|\xi|^{\alpha-3}\xi+P(f;x,\xi)=P(f;x,\xi)\tag{4.109}$$

in view of the key cancellation

$$A_{\alpha,1}(f;x)+\frac{1}{2}\left(A_{\alpha,0}(f;x)\right)_x=0.\tag{4.110}$$

proved in Appendix A. By (4.109) we deduce that (4.103) has the form (4.102). $\square$

Remark 4.8. The algebraic reason of the cancellation (4.110) is that a symbol of the form $\mathrm{i}g\left(f;x\right)|\xi|^{\alpha-3}\xi$, as in (4.105), with a real function $g\left(f;x\right)$, does not respect the Hamiltonianity condition (2.21).

The next lemma enables to highlight the quasilinear structure of the vector field in (4.20).

Lemma 4.9. *It results*

$$
\begin{aligned}
&\mathrm{Op}^{BW}\left[r^{2-\alpha}\right]\mathcal{I}\left(f\right)+\mathrm{Op}^{BW}\left[r^{-\alpha}\right]\mathcal{J}\left(f\right)+\fint \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z\,\left(r^{2-\alpha}-1\right)\\
&\quad=-\left(\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\right)^{-1}\mathrm{Op}^{BW}\left[\left(1+\nu\left(f;x\right)\right)L_\alpha\left(|\xi|\right)\right]f+\mathrm{Op}^{BW}\left[\tilde{\tilde{V}}\left(f;x\right)+P\left(f;x,\xi\right)\right]f+R\left(f\right)f
\end{aligned}
\tag{4.111}
$$

where $L_\alpha\left(|\xi|\right)$ is the Fourier multiplier defined in Lemma 3.1 and

- $\nu\left(f;x\right),\tilde{\tilde{V}}\left(f;x\right)$ *are real functions in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}\left[\epsilon_0,N\right]$;
- $P\left(f;x,\xi\right)$ *is a symbol in* $\Sigma\Gamma^{-1}_{K,0,1}\left[\epsilon_0,N\right]$;
- $R\left(f\right)$ *is a smoothing operator in* $\Sigma\mathcal{R}^{-\rho}_{K,0,1}\left[\epsilon_0,N\right]$.

Proof. By (4.11) and (4.68) we have

$$
\fint \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z\,\left(r^{2-\alpha}-1\right)=2\,\frac{\Gamma\left(2-\alpha\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\left(f+\mathrm{Op}^{BW}\left[r^{-\alpha}-1\right]f\right)+R\left(f\right)f.
\tag{4.112}
$$

Notice now, from Lemmas 4.4 and 4.6 and Lemma 3.1, that

$$
L_{\mathcal{I}}\left(|\xi|\right)+L_{\mathcal{J}}\left(|\xi|\right)-2\,\frac{\Gamma\left(2-\alpha\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}=\left(\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\right)^{-1}L_\alpha\left(|\xi|\right).
\tag{4.113}
$$

Now we claim that

$$
\tilde{\nu}_{\mathcal{I}}\left(f;x\right)L_{\mathcal{I}}\left(|\xi|\right)+\tilde{\nu}_{\mathcal{J}}\left(f;x\right)L_{\mathcal{J}}\left(|\xi|\right)=\left(\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\right)^{-1}\nu\left(f;x\right)L_\alpha\left(|\xi|\right)+\tilde{V}\left(f;x\right)+P\left(f;x,\xi\right),
\tag{4.114}
$$

for a suitable real functions $\nu,\tilde{V}$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}\left[\epsilon_0,N\right]$ and a symbol P in $\Sigma\Gamma^{-1}_{K,0,1}\left[\epsilon_0,N\right]$. From Lemmas 4.4 and 4.6 and the asymptotic decomposition of T^1_α and M_α in Lemma 3.7, we have that

$$
\begin{aligned}
\text{l.h.s. of (4.114)}&=\tilde{\nu}_{\mathcal{I}}\left(f;x\right)\mathsf{T}^1_\alpha\left(|\xi|\right)-\tilde{\nu}_{\mathcal{J}}\left(f;x\right)|\xi|^2\,\mathsf{M}_\alpha\left(|\xi|\right)+V\left(f;x\right)+P\left(f;x,\xi\right)\\
&=\frac{\Gamma\left(2-\alpha\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{|\xi|^{\alpha-1}}{\alpha-1}\left(\tilde{\nu}_{\mathcal{I}}\left(f;x\right)-\left(\alpha-1\right)\tilde{\nu}_{\mathcal{J}}\left(f;x\right)\right)+V\left(f;x\right)+P\left(f;x,\xi\right).
\end{aligned}
$$

Defining $\nu\left(f;x\right):=\frac{\tilde{\nu}_I\left(f;x\right)-\left(\alpha-1\right)\tilde{\nu}_{\mathcal{J}}\left(f;x\right)}{2-\alpha}$ and using the identity $\Gamma\left(3-\alpha\right)=\left(2-\alpha\right)\Gamma\left(2-\alpha\right)$, we get

$$
\text{l.h.s. of (4.114)}=\left(\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\right)^{-1}\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\frac{\Gamma\left(3-\alpha\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{|n|^{\alpha-1}}{\alpha-1}\,\nu\left(f;x\right)+V\left(f;x\right)+P\left(f;x,\xi\right).
\tag{4.115}
$$

By Lemma 3.6 we have

$$
\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\frac{\Gamma\left(3-\alpha\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\frac{|\xi|^{\alpha-1}}{\alpha-1}\,\nu\left(f;x\right)=\nu\left(f;x\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)+P\left(f;x,\xi\right).
\tag{4.116}
$$

Finally plugging (4.116) in (4.115) we deduce (4.114).

Equations (4.102) and (4.112) to (4.114) give that for suitable $\nu,\tilde{\tilde{V}}\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}\left[\epsilon_0,N\right]$ the desired decomposition provided in Equation (4.111). ☐

We can finally paralinearize Equation (4.20). Using (4.101) and (4.111) we have

$$
\begin{aligned}
&r^{2-\alpha}\left(\mathcal{I}\left(f\right)+R\left(f\right)f\right)+\fint \mathsf{G}^1_{\alpha,z}(0)\,\mathrm{d}z\,\left(r^{2-\alpha}-1\right)+r^{-\alpha}\,\mathcal{J}\left(f\right)\\
&\quad=-\left(\frac{c_\alpha}{2\left(1-\frac{\alpha}{2}\right)}\right)^{-1}\mathrm{Op}^{BW}\left[\left(1+\nu\left(f;x\right)\right)L_\alpha\left(|\xi|\right)+V\left(f;x\right)+P\left(f;x,\xi\right)\right]f+R\left(f\right)f
\end{aligned}
$$

where $V\left(f;x\right)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}\left[\epsilon_0,N\right]$. This, combined with the observation that $\partial_x\circ R\left(f\right)\in\Sigma\dot{\mathcal{R}}^{-\rho+1}_{K,0,1}\left[\epsilon_0,N\right]$, proves that Equation (4.20) has the form (4.1). ☐

5 Birkhoff normal form reduction up to cubic terms

In this section we reduce the equation (4.1) to its Birkhoff normal form up to a cubic smoothing vector field, from which Theorem 1.1 easily follows. From now on we consider $\alpha \in (1,2)$.

Proposition 5.1 (Cubic Birkhoff normal form)**.** *Let $\alpha \in (1,2)$ and $N \in \mathbb{N}$. There exists $\underline{\rho} := \underline{\rho}(N,\alpha)$, such that for any $\rho \geq \underline{\rho}$ there exists $\underline{K'} := \underline{K'}(\rho,\alpha) > 0$ such that for any $K \geq \underline{K'}$ there is $\underline{s_0} > 0$ such that for any $s \geq \underline{s_0}$, there is $\underline{\epsilon_0}(s) > 0$ such that for any $0 < \epsilon_0 \leq \underline{\epsilon_0}(s)$ and any solution $f \in B^K_{\underline{s_0},\mathbb{R}}(I;\epsilon_0) \cap C^K_*\left(I; H^s_0(\mathbb{T};\mathbb{R})\right)$ of the equation (4.1) the following holds:*

- *there exists a real invertible operator $\underline{\Psi}\left(f;t\right)$ on $H^s_0(\mathbb{T},\mathbb{R})$ satisfying the following: for any $s \in \mathbb{R}$ there are $C := C(s,\epsilon_0,K)$ and $\epsilon'_0(s) \in (0,\epsilon_0)$, such that for any $f \in B^K_{\underline{s_0},\mathbb{R}}\left(I;\epsilon'_0(s)\right)$ and $v \in C^{K-\underline{K'}}_*\left(I;H^s_0(\mathbb{T},\mathbb{R})\right)$, for any $0 \leq k \leq K-\underline{K'}$, $t \in I$,*

$$\left\|\partial_t^k\left(\underline{\Psi}\left(f;t\right)v\right)\right\|_{s-k} + \left\|\partial_t^k\left(\underline{\Psi}\left(f;t\right)^{-1}v\right)\right\|_{s-k} \leq C\|v\|_{k,s}; \tag{5.1}$$

- *the variable $y := \underline{\Psi}\left(f;t\right)f$ solves the equation*

$$\partial_t y + \mathrm{i}\,\omega_\alpha(D)\,y + \mathrm{i}\mathrm{Op}^{BW}\left[d\left(f;t,\xi\right)\right]y = R_{\geq 2}\left(f;t\right)y \tag{5.2}$$

where

- *$\omega_\alpha(\xi) = \xi L_\alpha(|\xi|)$, with $L_\alpha(|\xi|)$ defined in Lemma 3.1, is a Fourier multiplier of order α;*
- *$d\left(f;t,\xi\right)$ is a symbol in $\Sigma\Gamma^{\alpha}_{K,\underline{K'},2}[\epsilon_0,N]$ independent of x, satisfying (2.20), with $\mathrm{Im}\,d\left(f;t,\xi\right)$ in the space $\Sigma\Gamma^0_{K,\underline{K'},2}[\epsilon_0,N]$;*
- *$R_{\geq 2}\left(f;t\right)$ is a real smoothing operator in $\Sigma\dot{\mathcal{R}}^{-(\rho-\underline{\rho}-\alpha)}_{K,\underline{K'},2}[\epsilon_0,N]$.*

The bounds (5.1) imply in particular that for any $s \geq s_0$, there exists $C := C_{s,K,\alpha} > 0$ such that

$$C^{-1}\left\|f(t)\right\|_s \leq \left\|y(t)\right\|_s \leq C\left\|f(t)\right\|_s, \quad \forall t \in I. \tag{5.3}$$

Note that the x-independent symbol $d\left(f;t,\xi\right)$ in (5.2) has homogeneity at least 2 by Remark 2.6.

Reduction to constant coefficients up to a smoothing operator. The first step is to reduce the symbol of the paradifferential operator in (4.1) to a constant coefficient one, up to a smoothing operator.

Proposition 5.2 (Reduction to constant coefficients up to smoothing operators)**.** *Let $\alpha \in (1,2)$ and $N \in \mathbb{N}$. There exists $\underline{\rho} := \underline{\rho}(N,\alpha)$, such that for any $\rho \geq \underline{\rho}$ there exists $\underline{K'} := \underline{K'}\left(\rho,\alpha\right) > 0$ such that for any $K \geq \underline{K'}$ there are $s_0 > 0$, $\epsilon_0 > 0$ such that for any solution $f \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ of (4.1) the following holds:*

- *there exists a real invertible operator $\Psi\left(f;t\right)$ on $H^s_0(\mathbb{T},\mathbb{R})$ satisfying (5.1);*
- *the variable $g := \Psi\left(f;t\right)f$ solves the equation*

$$\partial_t g + \partial_x \circ \mathrm{Op}^{BW}\left[\left(1 + \mathtt{c}_0\left(f\right)\right)L_\alpha(|\xi|) + \mathsf{H}_\alpha\left(f;t,\xi\right)\right]g = R\left(f;t\right)g \tag{5.4}$$

where

- *$L_\alpha(|\xi|)$ is the Fourier multiplier of order $\alpha-1$ defined in Lemma 3.1;*
- *$\mathtt{c}_0\left(f\right)$ is a x-independent real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[\epsilon_0,N]$;*
- *$\mathsf{H}_\alpha\left(f;t,\xi\right)$ is an x-independent symbol in $\Sigma\Gamma^0_{K,\underline{K'},2}[\epsilon_0,N]$ satisfying (2.20), with $\mathrm{Im}\mathsf{H}_\alpha\left(f;t,\xi\right)$ in $\Sigma\Gamma^{-1}_{K,\underline{K'},2}[\epsilon_0,N]$;*
- *$R\left(f;t\right)$ is a real smoothing operator in $\Sigma\dot{\mathcal{R}}^{-(\rho-\underline{\rho})}_{K,\underline{K'},1}[\epsilon_0,N]$.*

Proposition 5.2 relies on general results (given in Appendix B) that describe how paradifferential operators are conjugated under the flow generated by a paradifferential operator, which is Hamiltonian up to zero order operators. We shall use repeatedly the following result.

Lemma 5.3 (Flows of Hamiltonian operators up to order zero)**.** *Let $p, N \in \mathbb{N}$, $0 \le K' \le K$ and $\delta \ge 0$. Let us consider a "Hamiltonian operators up to order zero"*

$$\Lambda(f,\tau;t) := \partial_x \circ \mathrm{Op}^{BW}\left[\lambda(f,\tau;t,x,\xi)\right]$$

where $\lambda(f,\tau;t,x,\xi)$ is a symbol in $\Sigma\Gamma^{-\delta}_{K,K',p}[\epsilon_0,N]$, uniformly in $|\tau| \le 1$, with $\mathrm{Im}\,\lambda(f,\tau;t,x,\xi) \in \Sigma\Gamma^{-1}_{K,K',p}[\epsilon_0,N]$ satisfying (2.20)*. Then there exists $s_0 > 0$ such that, for any $f \in B^{K'}_{s_0,\mathbb{R}}(I;\epsilon_0)$, the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\Lambda(f,\tau;t) = \Lambda(f,\tau;t)\ \Phi_\Lambda(f,\tau;t)\,, \qquad \Phi_\Lambda(f,0;t) = \mathrm{Id}\,, \tag{5.5}$$

has a unique solution $\Phi_\Lambda(f,\tau) := \Phi_\Lambda(f,\tau;t)$ satisfying the following properties: for any $s \in \mathbb{R}$ the linear map $\Phi_\Lambda(f,\tau;t)$ is bounded and invertible on $H^s_0(\mathbb{T},\mathbb{R})$ and there are a constant $C := C(s,\epsilon_0,K)$ and $\epsilon_0'(s) \in (0,\epsilon_0)$ such that, for any $f \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0'(s))$, for any $0 \le k \le K-K'$, $v \in C^{K-K'}_*(I;H^s_0(\mathbb{T},\mathbb{R}))$, $t \in I$,

$$\left\|\partial_t^k\left(\Phi_\Lambda(f,\tau;t)\,v\right)\right\|_{s-k} + \left\|\partial_t^k\left(\Phi_\Lambda(f,\tau;t)^{-1}\,v\right)\right\|_{s-k} \le C\|v\|_{k,s} \tag{5.6}$$

uniformly in $|\tau| \le 1$.

Proof. Since the imaginary part of the symbol λ has order -1, the flow Φ_Λ of (5.5) is well-posed and satisfies (5.6) arguing as in [5, Lemma 3.22]. Moreover it preserves the subspace of real functions since $\lambda(f,\tau;t,x,\xi)$ satisfies (2.20). □

In the proof of Proposition 5.2 it is convenient to preserve the linear Hamiltonian structure of (4.1) up to order zero along the reduction which leads to (5.4), since it guarantees that the symbol $\left(1+c_0(f)\right)L_\alpha(|\xi|) + \mathsf{H}_\alpha(f;t,\xi)$, as well as those obtained in the intermediate reduction steps, are real, at least up to order -1.

Reduction to constant coefficients at principal order. We first reduce to constant coefficients the highest order paradifferential operator in (4.1). We conjugate (4.1) via the transformation

$$f^{[1]} := \Phi_B(f,1)\,f \tag{5.7}$$

where $\Phi_B(f,\tau)$ is the flow generated as in Lemma 5.3 by the Hamiltonian operator

$$B(f,\tau) := \partial_x \circ \mathrm{Op}^{BW}\left[b(f,\tau;x)\right]\,, \qquad b(f,\tau;x) := \frac{\beta(f;x)}{1+\tau\,\partial_x\left(\beta(f;x)\right)}\,, \tag{5.8}$$

where $\beta(f;x)$ is a real function to be chosen.

Lemma 5.4 (Reduction to constant coefficients at principal order)**.** *Let $\beta(f;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ be the periodic function of the diffeomorphism $x \mapsto x+\beta(f;x)$ of $\mathbb{T}$ whose inverse diffeomorphism is $y \mapsto y+\breve{\beta}(f;y)$, where*

$$\breve{\beta}(f;y) := \partial_y^{-1}\left[\left(\frac{1+c_0(f)}{1+\nu(f;y)}\right)^{\frac{1}{\alpha}} - 1\right] \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]\,, \qquad c_0(f) := \left(\fint \left(1+\nu(f;y)\right)^{-\frac{1}{\alpha}}\mathrm{d}y\right)^{-\alpha} - 1\,, \tag{5.9}$$

and $\nu(f;y)$ is the real function defined in Theorem 4.1. Then, if f solves (4.1)*, the variable $f^{[1]}$ defined in* (5.7) *satisfies the equation*

$$\partial_t f^{[1]} + \partial_x \circ \mathrm{Op}^{BW}\left[\left(1+c_0(f)\right)\ L_\alpha(|\xi|) + V^1(f;t,x) + P(f;x,\xi)\right] f^{[1]} = R(f;t)\,f^{[1]} \tag{5.10}$$

where

- *$c_0(f)$ is the x-independent function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ defined in* (5.9)*;*

- $V^1(f;t,x)$ *is a real function in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[\epsilon_0,N]$*;*

- $P(f;x,\xi)$ *is a symbol in* $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$ *satisfying* (2.20)*;*

- $R(f;t)$ *is a real smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-(\rho-N)}_{K,1,1}[\epsilon_0,N]$.

Proof. If f solves (4.1) then, the variable $f^{[1]} := \Phi_B(f,1)f := \Phi_B(1)f$ satisfies, using also the expansion $L_\alpha(|\xi|) = \mathbb{V}_\alpha + c^1_\alpha|\xi|^{\alpha-1} + m_{\alpha-3}(|\xi|)$ in (3.32), and $\partial_t\Phi_B(1)\circ\Phi_B(1)^{-1} = -\Phi_B(1)\circ\left(\partial_t\Phi_B(1)^{-1}\right)$, the equation

$$\partial_t f^{[1]} + \Phi_B(1)\circ\partial_x\circ \mathrm{Op}^{BW}\left[\left(1+\nu(f;x)\right)\left(c^1_\alpha|\xi|^{\alpha-1} + \mathbb{V}_\alpha + m_{\alpha-3}(|\xi|)\right) + V(f;x) + P(f;x,\xi)\right]\circ\Phi_B(1)^{-1}\ f^{[1]} \\ + \Phi_B(1)\circ\left(\partial_t\Phi_B(1)^{-1}\right)f^{[1]} = \Phi_B(1)\circ R(f)\circ\Phi_B(1)^{-1}f^{[1]}. \quad (5.11)$$

By (B.1), (B.2) the principal order operator in (5.11) is

$$\begin{aligned}&\Phi_B(1)\circ\partial_x\circ\mathrm{Op}^{BW}\left[\left(1+\nu(f;x)\right)c^1_\alpha|\xi|^{\alpha-1}\right]\circ\Phi_B(1)^{-1} \\ &\quad= \partial_x\circ\mathrm{Op}^{BW}\left[c^1_\alpha\ \left(1+\nu(f;y)\right)\left(1+\partial_y\breve{\beta}(f;y)\right)^\alpha\Big|_{y=x+\beta(f;x)}|\xi|^{\alpha-1} + P_1(f;x,\xi)\right] + R(f)\end{aligned} \quad (5.12)$$

where $y\mapsto y+\breve{\beta}(f;y)$ is the inverse diffeomorphism of $x\mapsto x+\beta(f;y)$ given by Lemma 2.9, $P_1(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{\alpha-3}_{K,0,1}[\epsilon_0,N]$ and $R(f)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-(\rho-N)}_{K,0,1}[\epsilon_0,N]$. By (5.9) we deduce that the symbol of highest order in (5.12) is independent of the variable x, that is

$$\Phi_B(1)\,\partial_x\mathrm{Op}^{BW}\left[\left(1+\nu(f;x)\right)c^1_\alpha|\xi|^{\alpha-1}\right]\Phi_B(1)^{-1} = \partial_x\mathrm{Op}^{BW}\left[c^1_\alpha\left(1+c_0(f)\right)|\xi|^{\alpha-1} + P_1(f;x,\xi)\right] + R_1(f). \quad (5.13)$$

The lower order conjugated operator in (5.11) is, by (B.1) and Lemma 2.8,

$$\begin{aligned}&\Phi_B(1)\circ\partial_x\circ\mathrm{Op}^{BW}\left[\left(1+\nu(f;x)\right)\left(\mathbb{V}_\alpha + m_{\alpha-3}(|\xi|)\right) + V(f;x) + P(f;x,\xi)\right]\circ\Phi_B(1)^{-1} \\ &\qquad\qquad= \partial_x\circ\mathrm{Op}^{BW}\left[\mathbb{V}_\alpha + m_{\alpha-3}(|\xi|) + \tilde{V}^1(f;x) + P_2(f;x,\xi)\right] + R(f)\end{aligned} \quad (5.14)$$

where $\tilde{V}^1(f;x)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$, $P_2(f;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$, since $\alpha<2$ (note that $m_{\alpha-3}\left(|\xi|\left(1+\breve{\beta}(f;y)\right)_{|y=x+\beta(f;x)}\right) - m_{\alpha-3}(|\xi|)$ is a symbol in $\Sigma\Gamma^{\alpha-3}_{K,0,1}[\epsilon_0,N]$) and $R(f)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,0,1}[\epsilon_0,N]$, by renaming ρ. Finally by (B.3) there exists a real function $\mathfrak{V}(f;t,x)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[\epsilon_0,N]$ and a smoothing operator $R(f;t)$ in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,1,1}[\epsilon_0,N]$ such that

$$\Phi_B(1)\circ\left(\partial_t\Phi_B(1)^{-1}\right) = \partial_x\circ\mathrm{Op}^{BW}\left[\mathfrak{V}(f;t,x)\right] + R(f;t). \quad (5.15)$$

Lemma 5.4 follows by (5.11), (5.13), (5.14) and (5.15) with $V^1(f;x) := \tilde{V}^1(f;x) + \mathfrak{V}(f;x) - c_0(f)\mathbb{V}_\alpha$, which belongs to $\mathcal{F}^{\mathbb{R}}_{K,1,1}[\epsilon_0,N]$, and $P(f;x,\xi) := (P_1+P_2)(f;x,\xi) - c_0(f)\,m_{\alpha-3}(|\xi|)$ in $\Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$. ☐

Reduction to constant coefficients at arbitrary-order. We now reduce (5.10) to constant coefficients up to a smoothing operator, implementing an inductive process which, at each step, regularizes the symbol of $\delta := \alpha - 1 > 0$. We distinguish two regimes.

Lemma 5.5 (Reduction to constant coefficients up order 0)**.** *Let* $\delta := \alpha-1$ *and*[1] $\mathsf{j}_* := \lceil 1/\delta\rceil + 1$. *For any* $\mathsf{j}\in\{1,\dots,\mathsf{j}_*-1\}$, *there exist* ρ_{j} *defined inductively as* $\rho_1 := N$ *and* $\rho_{\mathsf{j}+1} := \rho_{\mathsf{j}} + N(1-\mathsf{j}\delta)$ *such that for any* $K\geq\mathsf{j}$ *there exist* $s_0>0$ *and a*

- *symbol* $\mathscr{a}^{[\mathsf{j}]}(f;t,\xi) := \left(1+c_0(f)\right)L_\alpha(|\xi|) + \mathsf{H}^{[\mathsf{j}]}_\alpha(f;t,\xi)$ *where* $\mathsf{H}^{[\mathsf{j}]}_\alpha(f;t,\xi)\in\Sigma\Gamma^0_{K,\mathsf{j}-1,2}[\epsilon_0,N]$, *independent of* x*, real, even in* ξ*;*

- *symbol* $r^{[\mathsf{j}]}(f;t,x,\xi)$ *in* $\Sigma\Gamma^{-(\mathsf{j}-1)\delta}_{K,\mathsf{j},1}[\epsilon_0,N]$, *real and even in* ξ*;*

- *symbol* $P^{[\mathsf{j}]}(f;t,x,\xi)$ *in* $\Sigma\Gamma^{-1}_{K,\mathsf{j}-1,1}[\epsilon_0,N]$*;*

[1] Note that $\mathsf{j}_* = \min\{\mathsf{j}\in\mathbb{N}\mid(\mathsf{j}-1)\delta>1\}$.

- *real smoothing operator* $R^{[j]}(f;t)$ *in* $\Sigma\mathcal{R}^{-(\rho-\rho_j)}_{K,j,1}[\epsilon_0,N]$;
- *Hamiltonian operator* $W^{[j]}(f) := \partial_x \circ \mathrm{Op}^{BW}\left[w^{[j]}(f;t,x,\xi)\right]$ *where* $w^{[j]}$ *is the real and even in* ξ *symbol*

$$w^{[j]}(f;t,x,\xi) := -\partial_x^{-1}\left[\frac{r^{[j]}(f;t,x,\xi) - ⨍ r^{[j]}(f;t,x,\xi)\,\mathrm{d}x}{\left(1+c_0(f)\right)\, c^1_\alpha \alpha |\xi|^{\alpha-1}}\right] \in \Sigma\Gamma^{-j\delta}_{K,j,1}[\epsilon_0,N]; \tag{5.16}$$

such that if $f \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ *is a solution of* (4.1) *then* $f^{[j]} := \prod_{j'=1}^{j-1}\Phi_{W^{[j']}}(f;1)^{-1} \circ \Phi_B(f;1)\, f$ *solves*

$$\partial_t f^{[j]} + \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi) + r^{[j]}(f;t,x,\xi) + P^{[j]}(f;t,x,\xi)\right] f^{[j]} = R^{[j]}(f;t)\, f^{[j]}. \tag{5.17}$$

Proof. Note that (5.10) has the form (5.17) for $j=1$ with $\mathsf{H}^{[1]}_\alpha(f;t,\xi) := 0$, $\mathcal{d}^{[1]}(f;t,\xi) := \left(1+c_0(f)\right)L_\alpha(|\xi|)$, $r^{[1]}(f;t,x,\xi) := V^1(f;t,x)$, $P^{[1]}(f;t,x,\xi) := P(f;t,x,\xi)$ and $R^{[1]}(f;t) := R(f;t)$. We now prove that, if $f^{[j]}$ solves (5.17) then

$$f^{[j+1]} := \Phi_{W^{[j]}}(f,1)^{-1} f^{[j]} \tag{5.18}$$

solves (5.17) with $j+1$ instead of j. By conjugation, from (5.18), setting $\Phi_{W^{[j]}}(1) := \Phi_{W^{[j]}}(f,1)$, we have

$$\begin{aligned}\partial_t f^{[j+1]} + \Phi_{W^{[j]}}(1)^{-1} \circ \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi) + r^{[j]}(f;t,x,\xi) + P^{[j]}(f;t,x,\xi)\right] \circ \Phi_{W^{[j]}}(1)\, f^{[j+1]}\\ - \partial_t \Phi_{W^{[j]}}(1)^{-1} \circ \Phi_{W^{[j]}}(1)\, f^{[j+1]} = \Phi_{W^{[j]}}(1)^{-1} \circ R^{[j]}(f;t) \circ \Phi_{W^{[j]}}(1)\, f^{[j+1]}.\end{aligned} \tag{5.19}$$

Using (B.6) we expand the highest order operator in (5.19) as

$$\begin{aligned}&\Phi_{W^{[j]}}(1)^{-1} \circ \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi)\right] \circ \Phi_{W^{[j]}}(1) = \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi)\right]\\ &- \left[\!\left[\partial_x \circ \mathrm{Op}^{BW}\left[w^{[j]}(f;t,x,\xi)\right], \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi)\right]\right]\!\right] + \partial_x \circ \mathrm{Op}^{BW}\left[Q_{-(2j-1)\delta}(f;t,x,\xi)\right] + R(f;t)\end{aligned} \tag{5.20}$$

where, in view of (5.16), $Q_{-(2j-1)\delta}$ is a real and even in ξ valued symbol in $\Sigma\Gamma^{-(2j-1)\delta}_{K,j,2}[\epsilon_0,N]$ and $R(f;t)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,j,2}[\epsilon_0,N]$. By symbolic calculus, (5.16) and since $\mathcal{d}^{[j]}$ is x-independent we have

$$\begin{aligned}&\left[\!\left[\partial_x \circ \mathrm{Op}^{BW}\left[w^{[j]}(f;t,x,\xi)\right], \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi)\right]\right]\!\right] = \partial_x \circ \left[\!\left[\mathrm{Op}^{BW}\left[w^{[j]}(f;t,x,\xi)\right], \mathrm{Op}^{BW}\left[\mathrm{i}\xi\, \mathcal{d}^{[j]}(f;t,\xi)\right]\right]\!\right]\\ &\qquad = \partial_x \circ \mathrm{Op}^{BW}\left[-w^{[j]}_x(f;t,x,\xi)\ \partial_\xi\left(\xi\, \mathcal{d}^{[j]}(f;t,\xi)\right) + Q_{-2-(j-1)\delta}(f;t,x,\xi)\right] + R(f;t)\end{aligned} \tag{5.21}$$

where $Q_{-2-(j-1)\delta}(f;t,x,\xi)$ is a real and even symbol in $\Sigma\Gamma^{-2-(j-1)\delta}_{K,j,1}[\epsilon_0,N]$ and $R(f;t)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,j,1}[\epsilon_0,N]$. Using the asymptotic expansion (3.32) we have that

$$\partial_\xi\left(\xi\, \mathcal{d}^{[j]}(f;t,\xi)\right) = \left(1+c_0(f)\right)\, c^1_\alpha \alpha |\xi|^{\alpha-1} + \tilde{Q}^{[j]}(f;t,\xi) \qquad \text{where} \qquad \tilde{Q}^{[j]}(f;t,\xi) \in \Sigma\Gamma^0_{K,j-1,0}[\epsilon_0,N]. \tag{5.22}$$

So, by (5.20), (5.21), (5.22), (B.6), the definition of $w^{[j]}(f;t,x,\xi) \in \Sigma\Gamma^{-j\delta}_{K,j,1}[\epsilon_0,N]$ provided in Eq. (5.16), (B.29), we obtain

$$\begin{aligned}&\Phi_{W^{[j]}}(1)^{-1} \circ \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi) + r^{[j]}(f;t,x,\xi)\right] \circ \Phi_{W^{[j]}}(1)\\ &= \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi) + w^{[j]}_x(f;t,x,\xi)\ \left(1+c_0(f)\right)\, c^1_\alpha \alpha |\xi|^{\alpha-1} + Q_{-j\delta}(f;t,x,\xi) + r^{[j]}(f;t,x,\xi)\right] + R(f;t)\\ &= \partial_x \circ \mathrm{Op}^{BW}\left[\mathcal{d}^{[j]}(f;t,\xi) + ⨍ r^{[j]}(f;t,x,\xi)\,\mathrm{d}x + Q_{-j\delta}(f;t,x,\xi)\right] + R(f;t)\end{aligned} \tag{5.23}$$

where $Q_{-j\delta}(f;t,x,\xi)$ is a symbol in $\Sigma\Gamma^{-j\delta}_{K,j,1}[\epsilon_0,N]$. By (B.7),

$$-\partial_t \Phi_{W^{[j]}}(1)^{-1} \circ \Phi_{W^{[j]}}(1) = \partial_x \circ \mathrm{Op}^{BW}\left[T^{[j]}(f;t,x,\xi)\right] + R(f;t) \quad \text{with} \quad T^{[j]}(f;t,x,\xi) \in \Sigma\Gamma^{-j\delta}_{K,j+1,1}[\epsilon_0,N] \tag{5.24}$$

real and even in ξ. Furthermore by (B.6)

$$\Phi_{W^{[j]}}(1)^{-1} \circ \mathrm{Op}^{BW}\left[P^{[j]}(f;t,x,\xi)\right] \circ \Phi_{W^{[j]}}(1) = \mathrm{Op}^{BW}\left[P^{[j+1]}(f;t,x,\xi)\right], \tag{5.25}$$

up to a smoothing operator, with a symbol $P^{[j+1]}(f;t,x,\xi)$ in $\Sigma\Gamma^{-1}_{K,j,1}[\epsilon_0,N]$. By (5.23), (5.24), (5.25) and since $\Phi_{W^{[j]}}(1)^{-1} \circ R^{[j]}(f;t) \circ \Phi_{W^{[j]}}(1)$ is in $\Sigma\dot{\mathcal{R}}^{-(\rho-\rho_j-N(1-j\delta))}_{K,j,1}[\epsilon_0,N]$, we deduce that (5.19) has the form (5.17) with j replaced by $j+1$ where $\mathsf{H}^{[j+1]}_\alpha := \mathsf{H}^{[j]}_\alpha + ⨍ r^{[j]}(f;t,x,\xi)\,\mathrm{d}x$ and $r^{[j+1]} := T^{[j]} + Q_{-j\delta}$. □

Now, implementing an analogous algorithmic procedure for the symbols of order ≤ -1, we reduce the equation (5.17) for $\mathrm{j} = \mathrm{j}_*$ to constant coefficients up to a smoothing operator.

Lemma 5.6 (Reduction to constant coefficients up to smoothing operators)**.** *For any integer* $\mathrm{j} \ge \mathrm{j}_*$*, for any* $K \ge \mathrm{j}$ *there exist a*

- *symbol* $\mathtt{d}^{[\mathrm{j}]}(f;t,\xi) := (1 + \mathtt{c}_0(f)) L_\alpha(|\xi|) + \mathsf{H}_\alpha^{[\mathrm{j}]}(f;t,\xi)$ *with* $\mathsf{H}_\alpha^{[\mathrm{j}]}(f;t,\xi) \in \Sigma\Gamma^0_{K,\mathrm{j}-1,2}[\epsilon_0, N]$ *and* $\mathrm{Im}\,\mathsf{H}_\alpha^{[\mathrm{j}]}(f;t,\xi)$ *in* $\Sigma\Gamma^{-1}_{K,\mathrm{j}-1,2}[\epsilon_0,N]$*, independent of* x *and satisfying* (2.20)*;*
- *symbol* $P^{[\mathrm{j}]}(f;t,x,\xi)$ *in* $\Sigma\Gamma^{-1-(\mathrm{j}-\mathrm{j}_*)\delta}_{K,\mathrm{j},1}[\epsilon_0,N]$ *satisfying* (2.20)*;*
- *a real smoothing operator* $R^{[\mathrm{j}]}(f;t)$ *in* $\Sigma\mathcal{R}^{-(\rho-\rho_{\mathrm{j}_*})}_{K,\mathrm{j},1}[\epsilon_0,N]$*;*
- *bounded linear operators* $W^{[\mathrm{j}]}(f) := \partial_x \circ \mathrm{Op}^{BW}\left[w^{[\mathrm{j}]}(f;t,x,\xi)\right]$ *where*

$$w^{[\mathrm{j}]}(f;t,x,\xi) := -\partial_x^{-1}\left[\frac{P^{[\mathrm{j}]}(f;t,x,\xi) - \fint P^{[\mathrm{j}]}(f;t,x,\xi)\,\mathrm{d}x}{(1+\mathtt{c}_0(f))\; c_\alpha^1 \alpha |\xi|^{\alpha-1}}\right] \in \Sigma\Gamma^{-1-(\mathrm{j}-\mathrm{j}_*+1)\delta}_{K,\mathrm{j},1}[\epsilon_0,N]; \tag{5.26}$$

and $s_0 > 0$*, such that if* $f \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ *is a solution of* (4.1) *then* $f^{[\mathrm{j}]} := \prod_{\mathrm{j}'=1}^{\mathrm{j}-1} \Phi_{W^{[\mathrm{j}']}}(f;1)^{-1} \circ \Phi_B(f;1) f$ *solves*

$$\partial_t f^{[\mathrm{j}]} + \partial_x \circ \mathrm{Op}^{BW}\left[\mathtt{d}^{[\mathrm{j}]}(f;t,\xi) + P^{[\mathrm{j}]}(f;t,x,\xi)\right] f^{[\mathrm{j}]} = R^{[\mathrm{j}]}(f;t) f^{[\mathrm{j}]}. \tag{5.27}$$

We now conclude the proof of Proposition 5.2. Let $\mathrm{j}^* := \mathrm{j}^*(\rho) := \min\left\{\mathrm{j} \in \mathbb{N}_0 \;\middle|\; (\mathrm{j}-\mathrm{j}_*)\delta > \rho - \rho_{\mathrm{j}_*}\right\}$, which is explicitly $\mathrm{j}^* := \lceil \frac{\rho-\rho_{\mathrm{j}_*}}{\alpha-1} \rceil + \mathrm{j}_* = \lceil \frac{\rho-\rho_{\mathrm{j}_*}}{\alpha-1} \rceil + \lceil \frac{1}{\alpha-1} \rceil + 1$, so that $\mathrm{Op}^{BW}\left[P^{[\mathrm{j}^*]}(f;t,x,\xi)\right]$ is a smoothing operator in $\Sigma\mathcal{R}^{-(\rho-\rho_{\mathrm{j}_*})}_{K,\mathrm{j}^*,1}[\epsilon_0,N]$ by Remark 2.18. Then the equation (5.27) with $\mathrm{j} = \mathrm{j}^*$ has the form (5.4) with

$$g = f^{[\mathrm{j}^*]} = \Psi(f;t) f, \qquad \Psi(f;t) := \prod_{\mathrm{j}'=1}^{\mathrm{j}^*-1} \Phi_{W^{[\mathrm{j}']}}(f,1)^{-1} \circ \Phi_B(f,1),$$

symbol $\mathsf{H}_\alpha(f;t,\xi) := \mathsf{H}_\alpha^{[\mathrm{j}^*]}(f;t,\xi)$, smoothing operator $R(f;t) := R^{[\mathrm{j}^*]}(f;t) + \mathrm{Op}^{BW}\left[P^{[\mathrm{j}^*]}(f;t,x,\xi)\right]$, and defining $\underline{\rho}(N,\alpha) := \rho_{\mathrm{j}_*}$ and $\underline{K'}(\rho,\alpha) := \mathrm{j}^*$. □

Birkhoff normal form step. We now perform one step of Birkhoff normal form to cancel out the quadratic term in (5.4) which, since $\mathtt{c}_0(f)$ and $H_\alpha(f;t,\xi)$ vanish quadratically at $f=0$, comes only from $R(f;t)g$.

By Proposition 5.1 and using Proposition 2.21 we first rewrite (5.4) as

$$\partial_t g + \mathrm{i}\omega_\alpha(D) g + \mathrm{i}\,\mathrm{Op}^{BW}\left[d(f;t,\xi)\right] g = R_1(f) g + R_{\ge 2}(f;t) g \tag{5.28}$$

where

i) $d(f;t,\xi) := \mathtt{c}_0(f)\omega_\alpha(\xi) + \xi\,\mathsf{H}_\alpha(f;t,\xi)$ is a symbol in $\Sigma\Gamma^\alpha_{K,\underline{K'},2}[\epsilon_0,N]$ independent of x, with imaginary part $\mathrm{Im}\, d(f;t,\xi)$ in $\Sigma\Gamma^0_{K,\underline{K'},2}[\epsilon_0,N]$;

ii) $R_1(f)$ is a real homogenous smoothing operator in $\widetilde{\dot{\mathcal{R}}}^{-(\rho-\underline{\rho})}_1$, that we expand (cf. (2.22)) as

$$R_1(f) v = \sum_{\substack{n,k,j \in \mathbb{Z}\setminus\{0\},\\ n+j=k}} (r_1)_{n,j,k} f_n v_j e^{\mathrm{i}kx}, \qquad (r_1)_{n,j,k} \in \mathbb{C}, \tag{5.29}$$

and $R_{\ge 2}(f;t)$ is a real smoothing operator in $\Sigma\dot{\mathcal{R}}^{-(\rho-\underline{\rho})}_{K,\underline{K'},2}[\epsilon_0,N]$.

In order to remove $R_1(f)$ we conjugate (5.28) with the flow

$$\partial_\tau \Phi_Q^\tau(f) = Q(f)\Phi_Q^\tau(f), \qquad \Phi_Q^0(f) = \mathrm{Id}, \tag{5.30}$$

generated by the 1-homogenous smoothing operator

$$Q(f)v = \sum_{\substack{n,k,j\in\mathbb{Z}\setminus\{0\},\\ n+j=k}} q_{n,j,k} f_n v_j e^{\mathrm{i}kx}, \quad q_{n,j,k} := \frac{-(r_1)_{n,j,k}}{\mathrm{i}\big(\omega_\alpha(k) - \omega_\alpha(j) - \omega_\alpha(n)\big)}, \tag{5.31}$$

which is well-defined by Lemma 3.5. Note also that by (3.30) and since (cf. (2.24), (2.23))

$$\overline{(r_1)_{n,j,k}} = (r_1)_{-n,-j,-k}, \qquad |(r_1)_{n,j,k}| \le C\frac{\max_2(|n|,|j|)^\mu}{\max(|n|,|j|)^{\rho-\underline{\rho}}}, \tag{5.32}$$

also $Q(f)$ is a real smoothing operator in $\widetilde{\dot{\mathcal{R}}}_1^{-\rho+\underline{\rho}}$ as $R_1(f)$.

Lemma 5.7 (Birkhoff step)**.** *If g solves* (5.28) *then the variable $y := \Phi_Q^1(f)g$ solves the equation* (5.2).

Proof. To conjugate (5.28) we apply a Lie expansion (similarly to Proposition B.2). We have

$$\begin{aligned} -\mathrm{i}\Phi_Q^1(f)\,\omega_\alpha(D)\,(\Phi_Q^1)^{-1} &= -\mathrm{i}\omega_\alpha(D) + [\![Q(f), -\mathrm{i}\omega_\alpha(D)]\!] \\ &\quad + \int_0^1 (1-\tau)\Phi_Q^\tau(f)\,[\![Q(f), [\![Q(f), -\mathrm{i}\omega_\alpha(D)]\!]]\!]\,\big(\Phi_Q^\tau(f)\big)^{-1}\,d\tau. \end{aligned} \tag{5.33}$$

Using that $Q(f)$ belongs to $\widetilde{\dot{\mathcal{R}}}_1^{-\rho+\underline{\rho}}$ the term in (5.33) is a smoothing operator in $\Sigma\dot{\mathcal{R}}_{K,0,2}^{-\rho+\underline{\rho}+\alpha}[\epsilon_0,N]$. Similarly we obtain

$$-\mathrm{i}\Phi_Q^1(f)\,\mathrm{Op}^{BW}[d(f;t,\xi)]\,\big(\Phi_Q^1(f)\big)^{-1} = -\mathrm{i}\mathrm{Op}^{BW}[d(f;t,\xi)] \tag{5.34}$$

up to a smoothing operator in $\Sigma\dot{\mathcal{R}}_{K,K_0',2}^{-\rho+\underline{\rho}+\alpha}[\epsilon_0,N]$, and

$$\Phi_Q^1(f)\big(R_1(f) + R_{\ge 2}(f;t)\big)\big(\Phi_Q^1(f)\big)^{-1} = R_1(f) \tag{5.35}$$

plus a smoothing operator in $\Sigma\dot{\mathcal{R}}_{K,K_0',2}^{-\rho+\underline{\rho}+\alpha}[\epsilon_0,N]$. Next we consider the contribution coming from the conjugation of ∂_t. By a Lie expansion (similarly to Proposition B.2) we get

$$\begin{aligned} \partial_t\Phi_Q^1(f)\big(\Phi_Q^1(f)\big)^{-1} &= \partial_t Q(f) \\ &\quad + \frac{1}{2}[\![Q(f), \partial_t Q(f)]\!] + \frac{1}{2}\int_0^1 (1-\tau)^2\Phi_Q^\tau(f)\,[\![Q(f), [\![Q(f), \partial_t Q(f)]\!]]\!]\,\big(\Phi_Q^\tau(f)\big)^{-1}\,d\tau. \end{aligned} \tag{5.36}$$

Since the Eq. (4.1) can be written as $\partial_t f = -\mathrm{i}\omega_\alpha(D)f + M(f)f$ where $M(f)$ is a real α-operator in $\Sigma\dot{\mathcal{M}}_{K,0,1}^\alpha$ by Remark 2.18 and 2.11, we deduce by Proposition 2.23 that

$$\partial_t Q(f) = Q\big(-\mathrm{i}\omega_\alpha(D)f + M(f)f\big) = Q\big(-\mathrm{i}\omega_\alpha(D)f\big) \tag{5.37}$$

up to a smoothing operator in $\Sigma\dot{\mathcal{R}}_{K,0,2}^{-\rho+\underline{\rho}+\alpha}[\epsilon_0,N]$. Since $Q\big(-\mathrm{i}\omega_\alpha(D)f\big)$ is in $\widetilde{\dot{\mathcal{R}}}_1^{-\rho+\underline{\rho}+\alpha}$ we have that the line (5.36) belongs to $\Sigma\dot{\mathcal{R}}_{K,0,2}^{-\rho+\underline{\rho}+\alpha}[\epsilon_0,N]$.

We now prove that $Q(f)$ solves the homological equation

$$Q\big(-\mathrm{i}\omega_\alpha(D)f\big) + [\![Q(f), -\mathrm{i}\omega_\alpha(D)]\!] + R_1(f) = 0. \tag{5.38}$$

Writing (5.31) as $Q(f)v = \sum_{k,j\in\mathbb{Z}\setminus\{0\}}[Q(f)]_k^j v_j e^{\mathrm{i}kx}$ with $[Q(f)]_k^j := q_{n,j,k}f_n$, we see that the homological equation (5.38) amounts to $[Q(-\mathrm{i}\omega_\alpha(D)f)]_k^j + [Q(f)]_k^j\big(\mathrm{i}\omega_\alpha(k) - \mathrm{i}\omega_\alpha(j)\big) + [R_1(f)]_k^j = 0$, for any $j,k\in\mathbb{Z}\setminus\{0\}$, and then, recalling (5.29), to $q_{n,j,k}\,\mathrm{i}\big(\omega_\alpha(k) - \omega_\alpha(j) - \omega_\alpha(n)\big) + (r_1)_{n,j,k} = 0$. This proves (5.38).

In conclusion, by (5.33), (5.34), (5.35), (5.36), (5.37) and (5.38) we deduce (5.2) (after renaming ρ). The bound (5.3) follows by standard theory of Banach space ODEs for the flow (5.30) and (5.3). □

In view of Lemma 5.7, Proposition 5.1 follows defining $\underline{\Psi}(f;t) := \Phi_Q^1(f)\circ\Psi(f;t)$ where $\Psi(f;t)$ is defined in Proposition 5.2 and $\Phi_Q^1(f)$ is defined in (5.31). We now easily deduce Theorem 1.1.

Proof of Theorem 1.1. The following result, analogous to [10, Lemma 8.2], enables to control the time derivatives $\|\partial_t^k f(t)\|_{s-k\alpha}$ of a solution $f(t)$ of (4.1) via $\|f(t)\|_s$.

Lemma 5.8. *Let $K \in \mathbb{N}$. There exists $s_0 > 0$ such that for any $s \geq s_0$, any $\epsilon \in \left(0, \overline{\epsilon_0}(s)\right)$ small, if f belongs to $B^0_{s_0,\mathbb{R}}(I;\epsilon) \cap C^0_*\left(I; H^s_0(\mathbb{T};\mathbb{R})\right)$ and solves* (4.1) *then $f \in C^K_*\left(I; H^s_0(\mathbb{T};\mathbb{R})\right)$ and there exists $C_1 := C_1(s,\alpha,K) \geq 1$ such that $\left\|f(t)\right\|_s \leq \left\|f(t)\right\|_{K,s} \leq C_1 \left\|f(t)\right\|_s$ for any $t \in I$.*

The first step is to choose the parameters in Proposition 5.1. Let $N := 1$. In the statement of Proposition 5.1 we fix $\rho := \underline{\rho}(1,\alpha) + \alpha$ and $K := \underline{K'}(\rho,\alpha)$. Then Proposition 5.1 gives us $\underline{s_0} > 0$. For any $s \geq \underline{s_0}$ we fix $0 < \epsilon_0 \leq \min\{\overline{\epsilon_0}(s), \underline{\epsilon_0}(s)\}$ where $\underline{\epsilon_0}(s)$ is defined in Proposition 5.1 and $\overline{\epsilon_0}(s)$ in Lemma 5.8.

The key corollary of Proposition 5.1 is the following energy estimate where by the time-reversibility of α-SQG we may restrict to positive times $t > 0$.

Lemma 5.9 (Quartic energy estimate)**.** *Let $f(t)$ be a solution of equation* (4.1) *in $B^K_{\underline{s_0},\mathbb{R}}(I;\epsilon_0) \cap C^K_*\left(I; H^s_0(\mathbb{T};\mathbb{R})\right)$. Then there exists $\bar{C}_2(s,\alpha) > 1$ such that*

$$\left\|f(t)\right\|_s^2 \leq \bar{C}_2(s,\alpha)\left(\left\|f(0)\right\|_s^2 + \int_0^t \left\|f(\tau)\right\|_s^4 \mathrm{d}\tau\right), \quad \forall 0 < t < T. \tag{5.39}$$

Proof. The variable $y := \underline{\Psi}\left(f;t\right) f$ defined in Proposition 5.1 solves the equation (5.2) where $\operatorname{Im} d\left(f;t,\xi\right)$ is a symbol in $\Gamma^0_{K,\underline{K'},2}[\epsilon_0]$ and, being x-independent, $\operatorname{Op}^{BW}\left[d\left(f;t,\xi\right)\right]$ commutes with $\langle D\rangle^s$. Furthermore, for the above choice of ρ it results that $R_{\geq 2}\left(f;t\right)$ is in $\mathcal{R}^0_{K,\underline{K'},2}[\epsilon_0]$. Then by (2.25), Lemmata 2.13 and 5.8 we deduce

$$\left\|y(t)\right\|_s^2 \leq \left\|y(0)\right\|_s^2 + \bar{C}_1(s,\alpha)\int_0^t \left\|y(\tau)\right\|_s^4 \mathrm{d}\tau, \quad \forall 0 < t < T,$$

and, by (5.3), we deduce (5.39). □

The energy estimate (5.39), (1.10) and the local existence result in [21] (which amounts to a local existence result for the equation (4.1)), imply, by a standard bootstrap argument, Theorem 1.1. □

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A Proof of Equation (4.110)

We now prove the identity Eq. (4.110) where the functions $A_{\alpha,0}, A_{\alpha,1}$ are defined in (4.108), (4.106), and $\mathsf{K}^{j,l}_\alpha$, $j = 1,2,3$, $l = 0,1$ are the l-th order Taylor expansion in z of the function $z \mapsto \mathsf{K}^j_{\alpha,z}\left(\frac{\Delta_z f}{r^2}\right)$ where the kernel functions $\mathsf{K}^j_{\alpha,z}(\mathrm{x})$ are defined in (4.14), (4.16), (4.21). The verification of Eq. (4.110) can be *automated*. The next small program in `SageMath`, a Python-based, open-source Computer-Algebra System, verifies Eq. (4.110).

```
x, X, z, a = var('x, X, z, a')
assume(0<a<2)

f(x) = function('f')(x)
Deltaf (x,z) = (f(x)-f(x-z))/(2*sin(z/2))

G1(X,z,a) = (1-2*X-sqrt(1-2*X)*cos(z))/((2*(1-X-sqrt(1-2*X)*cos(z)))^(a/2))
DXG1(X,z,a) = diff(G1(X,z,a),X)
K1(X,z,a)= DXG1(2*X*sin(z/2),z,a) * (2*(1-cos(z)))^(a/2)
```

```
G2(X,z,a) = (1/ (sqrt(1-2*X)) )/((2*(1-X-sqrt(1-2*X)*cos(z)))^(a/2))
K2 (X,z,a) = G2( 2*X*sin(z/2) , z , a) * (2*(1-cos(z)))^(a/2)

DXG2(X,z,a) = diff(G2(X,z,a),X)
K3(X,z,a) = DXG2(X*2*sin(z/2), z, a)* (2*(1-cos(z)))^(a/2) * sin(z)

expansionf_K1(x,z,a) = taylor(K1( Deltaf (x,z) / (1+2*f(x)) , z, a ), z, 0, 1)
expansionf_K2(x,z,a) = taylor(K2( Deltaf (x,z) / (1+2*f(x)) , z, a ), z, 0, 1)
expansionf_K3(x,z,a) = taylor(K3( Deltaf (x,z) / (1+2*f(x)) , z, a ), z, 0, 1)

C10(x,a) = expansionf_K1.coefficient(z, n=0)
C11(x,a) = expansionf_K1.coefficient(z, n=1)
C20(x,a) = expansionf_K2.coefficient(z, n=0)
C21(x,a) = expansionf_K2.coefficient(z, n=1)
C30(x,a) = expansionf_K3.coefficient(z, n=0)
C31(x,a) = expansionf_K3.coefficient(z, n=1)

A0(x,a) = ((1+2*f(x))^(-a/2)) * ( C10(x,a) + (a-1)*C20(x,a) + (diff(f(x),x) /
    (1+2*f(x))) * C30(x,a) )
A1(x,a) = ((1+2*f(x))^(-a/2)) * ( C11(x,a) +(a-2)* C21(x,a) + ( 1 / (1+2*f(x)))
    * ( diff(f(x),x) * C31(x,a) - diff(f(x),x,x) * C30(x,a)))


bool(A1(x,a) + 1/2 * diff(A0(x,a) , x)==0)
```

Here we comment the lines of code above.

1,2 Several variables are defined, so that (`x`, `X`, `z`, `a`) $=(x,\mathrm{x},z,\alpha)$ accordingly to the notation of the present manuscript. The variable `a`, which is the parameter α, is limited to the range $(0,2)$.

3,4 We define `f` as an implicit function depending on the variable `x` only, next we define `Deltaf` as the periodic finite difference $\Delta_z f$ defined in (4.2).

5-7 The function `G1` is the function $\mathsf{G}^1_{\alpha,z}$ defined in (4.6), the function `DXG1` is the function $\left(\mathsf{G}^1_{\alpha,z}\right)'$ defined in (4.18) and finally we define `K1` as the function $\mathsf{K}^1_{\alpha,z}$ defined in Eq. (4.14).

8-11 We perform the same computations as $\mathsf{K}^1_{\alpha,z}$ for the kernels $\mathsf{K}^2_{\alpha,z}$, $\mathsf{K}^3_{\alpha,z}$ defined in Eqs. (4.16) and (4.21).

12-14 The asymptotic expansion in Eq. (4.23) is computed for the three kernels.

15-20 We ask the computer to extract the coefficients of the expansions in Eq. (4.23) so that `C`jl`(x,a)` $=\mathsf{K}^{j,l}_{\alpha}(f;x)$, for any $j=1,2,3,\ l=0,1$.

21,22 We define the functions `A0` and `A1` as in Eqs. (4.106) and (4.108).

23 The last line, line 23, is a statement of truth, which asks the computer whether using algebraic simplifications it can prove that Eq. (4.110) is true.

B Conjugation of paradifferential operators under flows

The main results of this section concern transformation rules of paradifferential operators of the form $\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]$ under the flow generated by paradifferential operators which are Hamiltonian, or Hamiltonian up to order zero.

Proposition B.1. *Let $q\in\mathbb{N}$, $K'\le K$, $N\in\mathbb{N}$ with $q\le N$, $\epsilon_0>0$ and $\rho\gg N$. Let $\beta\left(f;t,x\right)$ be a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[\epsilon_0,N]$ and $\Phi_B\left(f,\tau\right)$ be the flow generated by the Hamiltonian operator $B\left(f,\tau\right)$ defined in* (5.8).

i (Conjugation of a paradifferential operator) *Let $\mathsf{a}\left(f;t,x,\xi\right)$ be a symbol in $\Sigma\Gamma^m_{K,K',q}[\epsilon_0,N]$. Then*

$$\Phi_B\left(f,1\right)\circ\partial_x\circ\mathrm{Op}^{BW}\left[\mathsf{a}\left(f;t,x,\xi\right)\right]\circ\Phi_B\left(f,1\right)^{-1}=\partial_x\circ\mathrm{Op}^{BW}\left[a_0\left(f,1;t,x,\xi\right)+P\left(f;t,x,\xi\right)\right]+R\left(f;t\right)\quad\text{(B.1)}$$

where

$$a_0(f,\tau;t,x,\xi) := \left(1+\partial_y \breve{\beta}(f,\tau;t,y)\right) \mathsf{a}\left(f;\, t,\, y,\, \xi\left(1+\partial_y \breve{\beta}(f,\tau;t,y)\right)\right)\Big|_{y=x+\tau\beta(f;t,x)} \tag{B.2}$$

is a symbol in $\Sigma\Gamma^{m}_{K,K',q}[\epsilon_0,N]$, $P(f;t,x,\xi)$ *is a symbol in* $\Sigma\Gamma^{m-2}_{K,K',q+1}[\epsilon_0,N]$ *and* $R(f;t)$ *is a smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-\rho+m+1+N}_{K,K',q+1}[\epsilon_0,N]$.

ii (Conjugation of ∂_t) *There exists a function* $V(f;t,x)$ *in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K'+1,1}[\epsilon_0,N]$ *and a smoothing operator* $R(f;t)$ *in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K'+1,1}[\epsilon_0,N]$ *such that*

$$\Phi_B(f,1)\circ\left(\partial_t\Phi_B(f,1)^{-1}\right) = \partial_x\circ \mathrm{Op}^{BW}\left[V(f;t,x)\right] + R(f;t). \tag{B.3}$$

iii (Conjugation of a smoothing operator) *If* $R(f;t)$ *is a smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',q}[\epsilon_0,N]$ *then the composed operator* $\Phi_B(f,1)\circ R(f;t)\circ\Phi_B(f,1)^{-1}$ *is in* $\Sigma\dot{\mathcal{R}}^{-\rho+N}_{K,K',q}[\epsilon_0,N]$.

We also prove an analogous result when the paradifferential operator which generates the flow has order strictly less than 1.

Proposition B.2 (Lie expansions)**.** *Let* $q\in\mathbb{N}$, $K'\le K$, $N\in\mathbb{N}$ *with* $q\le N$, $\epsilon_0>0$ *and* $\rho\gg N$. *Given a symbol* $w := w(f;t,x,\xi)$ *satisfying*

$$w(f;t,x,\xi)\in\Sigma\Gamma^{-\mathrm{d}}_{K,K',1}[\epsilon_0,N]\,,\ \mathrm{d}>0,\quad \mathrm{Im}\, w(f;t,x,\xi)\in\Gamma^{-\max\{1,\mathrm{d}\}}_{K,K',1}[\epsilon_0,N]\,, \tag{B.4}$$

and (2.20) *and denote* $\Phi_W(f,\tau)$ *the flow generated by*

$$\partial_\tau\Phi_W(f,\tau) = \partial_x\circ\mathrm{Op}^{BW}\left[w(f;t,x,\xi)\right]\,\Phi_W(f,\tau)\,,\qquad \Phi_W(0)=\mathrm{Id}. \tag{B.5}$$

i (Conjugation of a paradifferential operator) *Let* $\mathsf{a} := \mathsf{a}(f;t,x,\xi)$ *be a symbol in* $\Sigma\Gamma^{m}_{K,K',q}[\epsilon_0,N]$. *Then*

$$\begin{aligned}\Phi_W(f,1)^{-1}\circ\partial_x\circ\mathrm{Op}^{BW}\left[\mathsf{a}(f;t,x,\xi)\right]\circ\Phi_W(f,1) = \\ \partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}] - \left[\!\left[\partial_x\circ\mathrm{Op}^{BW}[w]\,,\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right]\!\right] + \partial_x\circ\mathrm{Op}^{BW}\left[P(f;t,x,\xi)\right] + R(f;t)\end{aligned} \tag{B.6}$$

where $P(f;t,x,\xi)$ *is a symbol in* $\Sigma\Gamma^{m-2\mathrm{d}}_{K,K',q+2}[\epsilon_0,N]$, *and* $R(f;t)$ *is a smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',q+2}[\epsilon_0,N]$. *If* a, w *are real and even in* ξ *then* $\left[\!\left[\partial_x\circ\mathrm{Op}^{BW}[w]\,,\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right]\!\right]$ *is Hamiltonian and* P *is real and even in* ξ.

ii (Conjugation of ∂_t) *There exists a symbol* $T(f;t,x,\xi)$ *in* $\Sigma\Gamma^{-\mathrm{d}}_{K,K'+1,1}[\epsilon_0,N]$ *satisfying* (2.20), *and a smoothing operator* $R(f;t)$ *in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K'+1,2}[\epsilon_0,N]$ *such that*

$$-\partial_t\Phi_W(f,1)^{-1}\circ\Phi_W(f,1) = \partial_x\circ\mathrm{Op}^{BW}\left[T(f;t,x,\xi)\right] + R(f;t)\,. \tag{B.7}$$

If w *is real and even in* ξ *then* $\partial_x\circ\mathrm{Op}^{BW}\left[T(f;t,x,\xi)\right]$ *is Hamiltonian, i.e.* T *is real and even in* ξ.

iii (Conjugation of a smoothing operator) *If* $R(f;t)$ *is a smoothing operator in* $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',q}[\epsilon_0,N]$ *then the composed operator* $\Phi_W(f,1)\circ R(f;t)\circ\Phi_W(f,1)^{-1}$ *is in* $\Sigma\dot{\mathcal{R}}^{-\rho+N\max\{0,(1-\mathrm{d})\}}_{K,K',q}[\epsilon_0,N]$.

The rest of this section is devoted to the proof of Propositions B.1 and B.2.

Proof of Proposition B.1

The proof of Propositions B.1 is inspired by the Egorov type analysis in [5, Section 3.5]. The difference is that we highlight the Hamiltonian structure in (B.1) and (B.3) of the conjugated operators.

For simplicity we avoid to track the dependence of β, b and Φ_B on the variable f, as well as on t, and denote $\beta_x(x) := \partial_x\left(\beta(f;t,x)\right)$, $b_x(\tau;t,x) := \partial_x\left(b(f,\tau;t,x)\right)$ and $\Phi_B(\tau) := \Phi_B(f,\tau)$. In the sequel ∂_x^{-1} is the Fourier multiplier with symbol $(\mathrm{i}\xi)^{-1}$ that maps H^s_0 onto H^{s+1}_0 for any $s\in\mathbb{R}$.

Proof of item i : conjugation of a paradifferential operator

The conjugated operator

$$\mathcal{P}(\tau) := \Phi_B(\tau) \circ \partial_x \circ \mathrm{Op}^{BW}[\mathrm{a}] \circ \Phi_B(\tau)^{-1} \in \mathcal{L}\left(H_0^s; H_0^{s-1-m}\right), \quad \forall s \in \mathbb{R}, \tag{B.8}$$

satisfies $\mathcal{P}(0) = \partial_x \circ \mathrm{Op}^{BW}[\mathrm{a}]$, and using that $\partial_\tau\left(\Phi_B(\tau)^{-1}\right) = -\Phi_B(\tau)^{-1} \circ \partial_\tau \Phi_B(\tau) \circ \Phi_B(\tau)^{-1}$, it solves the Heisenberg equation

$$\partial_\tau \mathcal{P}(\tau) = [\![B(\tau), \mathcal{P}(\tau)]\!], \qquad \mathcal{P}(0) = \partial_x \circ \mathrm{Op}^{BW}[\mathrm{a}]. \tag{B.9}$$

Lemma B.3. *The operator $A(\tau) := \partial_x^{-1} \circ \mathcal{P}(\tau) \in \mathcal{L}\left(H_0^s; H_0^{s-m}\right)$ solves*

$$\begin{cases} \partial_\tau A(\tau) = \mathrm{i}\,[\![\mathrm{Op}^{BW}[b(\tau;x)\xi], A(\tau)]\!] - \frac{1}{2}\left(\mathrm{Op}^{BW}[b_x(\tau;x)]\, A(\tau) + A(\tau)\,\mathrm{Op}^{BW}[b_x(\tau;x)]\right) \\ \qquad\qquad + R'(\tau)\; A(\tau) - A(\tau)\; R(\tau) \\ A(0) = \mathrm{Op}^{BW}[\mathrm{a}] \end{cases} \tag{B.10}$$

where $R(\tau)$, $R'(\tau)$ are smoothing operators in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',1}$, uniformly in $|\tau| \le 1$, preserving the zero-average subspaces.

Proof. By (5.8) and Proposition 2.21 we have

$$B(\tau) - \mathrm{Op}^{BW}\left[\mathrm{i}\, b(\tau;x)\,\xi + \tfrac{1}{2} b_x(\tau;x)\right] = R(\tau) \tag{B.11}$$

$$\partial_x^{-1} \circ B(\tau) \circ \partial_x = \mathrm{Op}^{BW}[b(\tau;x)] \circ \partial_x = \mathrm{Op}^{BW}\left[\mathrm{i} b(\tau;x)\,\xi - \tfrac{1}{2} b_x(\tau;x)\right] + R'(\tau) \tag{B.12}$$

where $R(\tau), R'(\tau)$ are smoothing operators in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',1}$ preserving the zero-average subspaces. Then, by (B.9), (B.11), (B.12) we get

$$\begin{aligned} \partial_\tau A(\tau) &= \partial_x^{-1} \circ B(\tau) \circ \partial_x \circ A(\tau) - A(\tau) \circ B(\tau) \\ &= \left(\mathrm{Op}^{BW}\left[\mathrm{i} b(\tau;x)\,\xi - \tfrac{1}{2} b_x(\tau;x)\right]\right) A(\tau) - A(\tau)\left(\mathrm{Op}^{BW}\left[\mathrm{i} b(\tau;x)\,\xi + \tfrac{1}{2} b_x(\tau;x)\right]\right) + R'(\tau)\, A(\tau) - A(\tau)\, R(\tau) \end{aligned}$$

proving (B.10). $\square$

We now look for an approximate solution of (B.10) of the form

$$A^{(J)}(\tau) = \sum_{\mathrm{j}=0}^{J} \mathrm{Op}^{BW}\left[a_{\mathrm{j}}(\tau)\right], \qquad a_0(\tau) \in \Sigma\Gamma^m_{K,K',q}[\epsilon_0, N], \quad a_{\mathrm{j}}(\tau) \in \Sigma\Gamma^{m-2\mathrm{j}}_{K,K',q+1}[\epsilon_0, N], \ \forall \mathrm{j} = 1, \ldots, J. \tag{B.13}$$

We use the following asymptotic expansions derived by Proposition 2.21 and (2.28).

Lemma B.4. *Let a be a symbol in $\Sigma\Gamma^m_{K,K',q}[\epsilon_0, N]$. Then the commutator*

$$[\![\mathrm{Op}^{BW}[\mathrm{i} b(\tau;x)\xi], \mathrm{Op}^{BW}[a]]\!] = \mathrm{Op}^{BW}\left[\left\{b(\tau;x)\xi,\ a\right\}\right] + \mathrm{Op}^{BW}[r_{-3}(b(\tau), a)] + R(\tau)$$

with symbols

$$\left\{b(\tau;x)\xi,\ a\right\} \in \Sigma\Gamma^m_{K,K',q+1}[\epsilon_0, N], \qquad r_{-3}(b(\tau), a) \in \Sigma\Gamma^{m-2}_{K,K',q+1}[\epsilon_0, N],$$

and a smoothing operator $R(\tau)$ in $\Sigma\dot{\mathcal{R}}^{-\rho+m+1}_{K,K',q+1}[r, N]$, uniformly in τ. Moreover

$$\tfrac{1}{2}\mathrm{Op}^{BW}[b_x(\tau;x)]\,\mathrm{Op}^{BW}[a] + \tfrac{1}{2}\mathrm{Op}^{BW}[a]\,\mathrm{Op}^{BW}[b_x(\tau;x)] = \mathrm{Op}^{BW}[b_x(\tau;x)\, a + r_{-2}(b(\tau), a)] + R(\tau)$$

where $r_{-2}(b(\tau), a)$ is a symbol in $\Sigma\Gamma^{m-2}_{K,K',q+1}[\epsilon_0, N]$, and $R(\tau)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho+m}_{K,K',q+1}[\epsilon_0, N]$.

We shall also use the following lemma concerning solutions of a transport equation.

Lemma B.5. *Let $W(f,\tau;x,\xi)$ by a symbol in $\Sigma\Gamma^m_{K,K',q}[\epsilon_0,N]$ uniformly in $|\tau|\le 1$. Then the unique solution of*

$$
\begin{cases} \partial_\tau Q(f,\tau;x,\xi) = \left\{ b(f,\tau;x)\xi,\ Q(f,\tau;x,\xi)\right\} - b_x(f,\tau;x)\,Q(f,\tau;x,\xi) + W(f,\tau;x,\xi) \\ Q(f,\tau;x,\xi)\big|_{\tau=0} = Q_0(f;x,\xi) \in \Sigma\Gamma^m_{K,K',q}[\epsilon_0,N] \end{cases} \tag{B.14}
$$

has the form

$$
\begin{aligned} Q(f,\tau;x,\xi) = {} & \left(1+\breve\beta_y(f,\tau;\ y)\right) Q_0\left(f;\ y\ ,\ \xi\left(1+\breve\beta_y(f,\tau;\ y)\right)\right)\big|_{y=x+\tau\beta(f;x)} \\ & + \int_0^\tau \frac{1+\breve\beta_y(f,\tau;\ y)}{1+\breve\beta_y(f,\tau';\ y)}\, W\left(f,\tau';\ y+\breve\beta(f,\tau';\ y)\ ,\ \frac{\xi\left(1+\breve\beta_y(f,\tau;\ y)\right)}{1+\breve\beta_y(f,\tau';\ y)}\right) \mathrm{d}\tau'\Bigg|_{y=x+\tau\beta(f;x)}\ , \end{aligned} \tag{B.15}
$$

which is a symbol in $\Sigma\Gamma^m_{K,K',q}[\epsilon_0,N]$, uniformly in $|\tau|\le 1$.

Proof. The solution $(x(\tau),\xi(\tau)) = \phi^{0,\tau}(X,\Xi)$ of the characteristics system

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau) = -b(\tau;x(\tau))\ ,\quad \frac{\mathrm{d}}{\mathrm{d}\tau}\xi(\tau) = b_x(\tau;x(\tau))\ \xi(\tau)\ , \tag{B.16}
$$

with initial condition $(x(\tau),\xi(\tau))|_{\tau=0} = \phi^{0,0}(X,\Xi) = (X,\Xi)$ is (cf. [5, p. 83])

$$
(x(\tau),\xi(\tau)) = \phi^{0,\tau}(X,\Xi) = \left(X+\breve\beta(\tau,X)\,,\,\frac{\Xi}{1+\breve\beta_y(\tau,X)}\right). \tag{B.17}
$$

By (B.16) and (B.14) we get $\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\xi(\tau)Q(\tau;x(\tau),\xi(\tau))\right] = \xi(\tau)W(\tau;x(\tau),\xi(\tau))$ and so, by integration,

$$
\xi(\tau)\ Q(\tau;x(\tau),\xi(\tau)) = \Xi Q(0;X,\Xi) + \int_0^\tau \xi(\tau')W\left(\tau';x(\tau'),\xi(\tau')\right)\mathrm{d}\tau'\,. \tag{B.18}
$$

The inverse flow $\phi^{\tau,0}(x,\xi)$, i.e. $(x,\xi) = \phi^{0,\tau}(X,\Xi)$ if and only if $(X,\Xi) = \phi^{\tau,0}(x,\xi)$ is (cf. [5, p. 83])

$$
(X,\Xi) = \phi^{\tau,0}(x,\xi) = \left(x+\tau\beta(x), \xi\left(1+\breve\beta_y(\tau;y)\right)_{|y=x+\tau\beta(x)}\right). \tag{B.19}
$$

In addition, by (B.17) and (B.19),

$$
\left(x(\tau'),\xi(\tau')\right) = \phi^{\tau,\tau'}(x,\xi) = \phi^{0,\tau'}\left(\phi^{\tau,0}(x,\xi)\right) = \left(y+\breve\beta(\tau' y)\ ,\ \frac{\xi\left(1+\breve\beta_y(\tau;y)\right)}{1+\breve\beta_y(\tau';y)}\right)\Bigg|_{y=x+\tau\beta(x)}. \tag{B.20}
$$

We deduce (B.15) inserting (B.19) and (B.20) in (B.18). Finally $Q(f,\tau;x,\xi)$ is a symbol in $\Sigma\Gamma^m_{K,K',q}[\epsilon_0,N]$, by (B.15) and Lemmata 2.8 and 2.9. ☐

Step i): Determination of the principal symbol a_0. From (B.10), (B.13) and Lemma B.4 the principal symbol a_0 solves the equation

$$
\begin{cases} \partial_\tau a_0(\tau;x,\xi) = \left\{ b(\tau;x)\xi,\ a_0(\tau;x,\xi)\right\} - b_x(\tau;x)\ a_0(\tau;x,\xi) \\ a_0(0;x,\xi) = \mathsf{a}(x,\xi)\,. \end{cases} \tag{B.21}
$$

By Lemma B.5 with $W=0$ and $Q_0 = \mathsf{a}$, the solution of (B.21) is given by (B.2). The operator $A^{(0)} := A^{(0)}(\tau) := \mathrm{Op}^{BW}[a_0(\tau)]$ solves approximately (B.10) in the sense that, by (B.21) and Lemma B.4,

$$
\partial_\tau A^{(0)} = \mathrm{i}\left[\!\left[\mathrm{Op}^{BW}[b(\tau)\xi]\,,A^{(0)}\right]\!\right] - \mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right]A^{(0)} - A^{(0)}\mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right] + \mathrm{Op}^{BW}\left[r^{(0)}(\tau)\right] + R^{(0)}(\tau) \tag{B.22}
$$

where $r^{(0)}(\tau) := -r_{-3}(b,a_0) - r_{-2}(b,a_0)$ is a symbol in $\Sigma\Gamma^{m-2}_{K,K',q+1}[\epsilon_0,N]$ and $R^{(0)}(\tau)$ is a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho+m}_{K,K',q+1}[\epsilon_0,N]$, unifomly in $\tau\in[0,1]$.

Step ii): Determination of the subprincipal symbol $\sum_{\mathrm{j}=1}^J a_{\mathrm{j}}$. We define $a_1(\tau;x,\xi)$ as the solution of the transport equation

$$
\begin{cases} \partial_\tau a_1(\tau;x,\xi) = \left\{ b(\tau;x)\xi,\ a_1(\tau;x,\xi)\right\} - b_x(\tau;x)\ a_1(\tau;x,\xi) - r^{(0)}(\tau;x,\xi) \\ a_1(0;x,\xi) = 0\,. \end{cases} \tag{B.23}
$$

By Lemma B.5 the symbol $a_1(\tau;x,\xi)$ is in $\Sigma\Gamma^{m-2}_{K,K',q+1}$. By Equations (B.22) and (B.23) and Lemma B.4

$$A^{(1)}(\tau) := A^{(0)}(\tau) + \mathrm{Op}^{BW}[a_1(\tau)]$$

is a better approximation of equation (B.10) in the sense that

$$\partial_\tau A^{(1)} = \mathrm{i}\left[\!\left[\mathrm{Op}^{BW}[b(\tau)\xi]\,,\,A^{(1)}\right]\!\right] - \mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right]A^{(1)} - A^{(1)}\mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right] + \mathrm{Op}^{BW}\left[r^{(1)}(\tau)\right] + R^{(1)}(\tau) \tag{B.24}$$

where $r^{(1)} := -r_{-3}(b,a_1) - r_{-2}(b,a_1)$ is a symbol in $\Sigma\Gamma^{m-4}_{K,K',q+1}[\epsilon_0,N]$ and $R^{(1)}(\tau)$ are smoothing operators in $\Sigma\dot{\mathcal{R}}^{-\rho+m}_{K,K',q+1}[\epsilon_0,N]$ uniformly in $|\tau|\le 1$.

Repeating J times ($J\sim\rho/2$) the above procedure, until the new paradifferential term may be incorporated into the smoothing remainder, we obtain an operator $A^{(J)}(\tau) := \sum_{\mathrm{j}=0}^{J}\mathrm{Op}^{BW}\left[a_{\mathrm{j}}(\tau)\right]$ as in (B.13) solving

$$\begin{cases} \partial_\tau A^{(J)}(\tau) = \mathrm{i}\left[\!\left[\mathrm{Op}^{BW}[b(\tau)\xi]\,,\,A^{(J)}(\tau)\right]\!\right] - \mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right]A^{(J)}(\tau) - A^{(J)}(\tau)\,\mathrm{Op}^{BW}\left[\tfrac{b_x(\tau)}{2}\right] + R^{(J)}(\tau) \\ A^{(J)}(0) = \mathrm{Op}^{BW}[\mathsf{a}] \end{cases} \tag{B.25}$$

where $R^{(J)}(\tau)$ are smoothing operators in $\Sigma\dot{\mathcal{R}}^{-\rho+m}_{K,K',q+1}[\epsilon_0,N]$ uniformly in $|\tau|\le 1$.

Step iii) : Analysis of the error. We finally estimate the difference between the conjugated operator $P(\tau)$ in (B.8) and $P^{(J)}(\tau) := \partial_x\circ A^{(J)}(\tau)$.

Lemma B.6. *$P(\tau)-P^{(J)}(\tau)$ is a smoothing operator $R(\tau)$ in $\Sigma\dot{\mathcal{R}}^{-\rho+m+1+N}_{K,K',q+1}[\epsilon_0,N]$ uniformly in $|\tau|\le 1$.*

Proof. In view of Eqs. (B.11), (B.12) and (B.25), the operator $\mathcal{P}^{(J)}(\tau) = \partial_x\circ A^{(J)}(\tau)$ solves an approximated Heisenberg equation (cf. (B.9))

$$\partial_\tau\mathcal{P}^{(J)}(\tau) = \left[\!\left[B\,,\,\mathcal{P}^{(J)}(\tau)\right]\!\right] + R(\tau)\,,\qquad R(\tau)\in\Sigma\dot{\mathcal{R}}^{-\rho+m}_{K,K',q+1}\,. \tag{B.26}$$

Recalling (B.8) we write

$$P^{(J)}(\tau) - P(\tau) = V(\tau)\Phi_B(\tau)^{-1} \quad \text{where} \quad V(\tau) := P^{(J)}(\tau)\,\Phi_B(\tau) - \Phi_B(\tau)\circ\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\,.$$

By (B.26) we have that $\partial_\tau V(\tau) = B(\tau)V(\tau) + R(\tau)\Phi_B(\tau)$, $V(0)=0$, and therefore, by Duhamel and $\partial_\tau\Phi_B = B\Phi_B$ we deduce $V(\tau) = \Phi_B(\tau)\int_0^\tau\Phi_B(\tau')^{-1}R(\tau')\Phi_B(\tau')\,\mathrm{d}\tau'$ and thus

$$P^{(J)}(\tau) - P(\tau) = \int_0^\tau \Phi_B(\tau)\circ\Phi_B(\tau')^{-1}\circ R(\tau')\circ\Phi_B(\tau')\circ\Phi_B(\tau)^{-1}\,\mathrm{d}\tau'\,.$$

This is a smoothing operator in arguing as in [5, Proof of Thm. 3.27]. □

Lemma B.6 implies that $P(\tau) = \partial_x\circ A^{(J)}(\tau) + R(\tau)$ concluding the proof of Proposition B.1-i with symbol $P = \sum_{\mathrm{j}=1}^{J} a_{\mathrm{j}}(1)$. Item ii follows similarly as in [7, Lemma A.5]. Item iii is given in [5, Remark at page 89].

Proof of Proposition B.2

In view of (B.4) and Lemma 5.3 the flow $\Phi_W(\tau) := \Phi_W(f,\tau)$ generated by (B.5) is well posed and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\Phi_W(\tau)^{-1}\circ\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\circ\Phi_W(\tau)\right) = -\Phi_W(\tau)^{-1}\left[\!\left[\partial_x\circ\mathrm{Op}^{BW}[w]\,,\,\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right]\!\right]\Phi_W(\tau) \tag{B.27}$$

and a Taylor expansion gives

$$\begin{aligned} &\Phi_W(1)^{-1}\,\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\circ\Phi_W(1) \\ &\quad= \partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}] - \left[\!\left[\partial_x\circ\mathrm{Op}^{BW}[w]\,,\,\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right]\!\right] + \sum_{\ell=2}^{L}\frac{(-1)^\ell}{\ell!}\mathrm{Ad}^{\ell}_{\partial_x\circ\mathrm{Op}^{BW}[w]}\left(\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right) \\ &\qquad + \frac{(-1)^{L+1}}{L!}\int_0^1(1-\tau)^L\,\Phi_W(\tau)^{-1}\circ\mathrm{Ad}^{L+1}_{\partial_x\circ\mathrm{Op}^{BW}[w]}\left(\partial_x\circ\mathrm{Op}^{BW}[\mathsf{a}]\right)\circ\Phi_W(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{B.28}$$

Since $\partial_x \circ \mathrm{Op}^{BW}[w]$ belongs to $\Sigma\Gamma^{-\mathtt{d}}_{K,K',1}$, $\mathtt{d}>0$, then, by Proposition 2.21 each commutator $[\![\partial_x \circ \mathrm{Op}^{BW}[w], \cdot]\!]$ gains $\mathtt{d}>0$ unit of order and one degree of vanishing in f and (B.28) is an expansion as in (B.6) in operators with decreasing order and increasing degree of homogeneity with a symbol P of order $m-2\mathtt{d}$. Item iii follows as in [5, Remark at page 89], see also [10], by properties of the flow generated by paradifferential operators. Thus, Proposition 2.21, give that the last term of (B.28) belongs to $\Sigma\dot{\mathcal{R}}^{1+m-\mathtt{d}(L+1)+\max\{0,1-\mathtt{d}\}N}_{K,K',q}[\epsilon_0,N]$, hence if $L+1 \geq \frac{\rho+1+m+\max\{0,1-\mathtt{d}\}N}{\mathtt{d}}$ it belongs to $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',q}[\epsilon_0,N]$. If $w, \mathtt{a}$ are real and even in ξ, then the operators $\partial_x \circ \mathrm{Op}^{BW}[w]$ and $\partial_x \circ \mathrm{Op}^{BW}[\mathtt{a}]$ are Hamiltonian (cf. (2.21)). The commutator of two Hamiltonian operators

$$\mathrm{Ad}_{\partial_x \circ \mathrm{Op}^{BW}[w]}\big(\partial_x \circ \mathrm{Op}^{BW}[\mathtt{a}]\big) = \partial_x \circ S, \quad S := \mathrm{Op}^{BW}[w] \circ \partial_x \circ \mathrm{Op}^{BW}[\mathtt{a}] - \mathrm{Op}^{BW}[\mathtt{a}] \circ \partial_x \circ \mathrm{Op}^{BW}[w], \tag{B.29}$$

where $S = S^*$, $S = \overline{S}$, is another Hamiltonian operator where, by Proposition 2.21, the operator $S = \mathrm{Op}^{BW}[s]$ with a real symbol s in $\Sigma\Gamma^{m-\mathtt{d}}_{K,K',q+1}[\epsilon_0,N]$ even in ξ (cf. (2.21)), up to a smoothing operator in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K',q+1}[\epsilon_0,N]$, by renaming ρ. Applying iteratively this result to $\mathrm{Ad}^{\ell}_{\partial_x \circ \mathrm{Op}^{BW}[w]}\big(\partial_x \circ \mathrm{Op}^{BW}[\mathtt{a}]\big)$ the formula (B.6) follows.

Let us prove (B.7). As in (B.27) we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\Phi_W(\tau)^{-1} \circ \partial_t \circ \Phi_W(\tau)\big) = -\Phi_W(\tau)^{-1} \circ [\![\partial_x \circ \mathrm{Op}^{BW}[w], \partial_t]\!] \circ \Phi_W(\tau) = \Phi_W(\tau)^{-1} \circ \partial_x \circ \mathrm{Op}^{BW}[w_t] \circ \Phi_W(\tau)$$

and a Taylor expansion gives

$$\begin{aligned}\Phi_W(1)^{-1} \circ \partial_t \circ \Phi_W(1) = \partial_t + \partial_x \circ \mathrm{Op}^{BW}[w_t] + \sum_{\ell=2}^{L} \frac{(-1)^{\ell-1}}{\ell!} \mathrm{Ad}^{\ell-1}_{\partial_x \circ \mathrm{Op}^{BW}[w]}\big(\partial_x \circ \mathrm{Op}^{BW}[w_t]\big) \\ + \frac{(-1)^L}{L!} \int_0^1 (1-\tau)^L \Phi_W(\tau)^{-1} \circ \mathrm{Ad}^{L}_{\partial_x \circ \mathrm{Op}^{BW}[w]}\Big(\partial_x \circ \mathrm{Op}^{BW}[w_t]\Big) \circ \Phi_W(\tau)\, \mathrm{d}\tau.\end{aligned} \tag{B.30}$$

Since $\Phi_W(1)^{-1} \circ \partial_t \circ \Phi_W(1) = \partial_t + \Phi_W(1)^{-1} \circ (\partial_t \Phi_W(1)) = \partial_t - \big(\partial_t \Phi_W(1)^{-1}\big) \circ \Phi_W(1)$ we deduce by (B.30) that

$$-\partial_t \Phi_W(1)^{-1} \circ \Phi_W(1) = \partial_x \circ \mathrm{Op}^{BW}[w_t] + \sum_{\ell=2}^{L} \frac{(-1)^{\ell-1}}{\ell!} \mathrm{Ad}^{\ell-1}_{\partial_x \circ \mathrm{Op}^{BW}[w]}\big(\partial_x \circ \mathrm{Op}^{BW}[w_t]\big) + R$$

where, if $L \gtrsim_{\mathtt{d},N} \rho$, then R is in $\Sigma\dot{\mathcal{R}}^{-\rho}_{K,K'+1,2}[\epsilon_0,N]$ (renaming ρ). Then (B.7) follows arguing as for (B.6) and if w is real and even in ξ we also deduce that T is real and even in ξ.

References

[1] A. Ai and O.-N. Avadanei. Well-posedness for the surface quasi-geostrophic front equation. Arxiv preprint arXiv:2212.00117.

[2] Albert Ai, Mihaela Ifrim, and Daniel Tataru. Two dimensional gravity waves at low regularity I: Energy estimates. preprint https://arxiv.org/abs/1910.05323.

[3] Pietro Baldi, Massimiliano Berti, Emanuele Haus, and Riccardo Montalto. Time quasi-periodic gravity water waves in finite depth. *Invent. Math.*, 214(2):739–911, 2018.

[4] Pietro Baldi, Massimiliano Berti, and Riccardo Montalto. KAM for autonomous quasi-linear perturbations of KdV. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 33(6):1589–1638, 2016.

[5] Massimiliano Berti and Jean-Marc Delort. *Almost global solutions of capillary-gravity water waves equations on the circle*, volume 24 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Cham; Unione Matematica Italiana, [Bologna], 2018.

[6] Massimiliano Berti, R. Feola, and Luca Franzoi. Quadratic life span of periodic gravity-capillary water waves. *Water Waves*, 3(1):85–115, 2021.

[7] Massimiliano Berti, Roberto Feola, and Fabio Pusateri. Birkhoff normal form and long time existence for periodic gravity water waves. *Comm. Pure Appl. Math.*, 76(7):1416–1494, 2023.

[8] Massimiliano Berti, Luca Franzoi, and Alberto Maspero. Traveling quasi-periodic water waves with constant vorticity. *Arch. Ration. Mech. Anal.*, 240(1):99–202, 2021.

[9] Massimiliano Berti, Zineb Hassainia, and Nader Masmoudi. Time quasi-periodic vortex patches of Euler equation in the plane. *Invent. Math.*, 233(3):1279–1391, 2023.

[10] Massimiliano Berti, Alberto Maspero, and Federico Murgante. Hamiltonian birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence. https://arxiv.org/abs/2212.12255.

[11] Massimiliano Berti, Alberto Maspero, and Federico Murgante. Local well posedness of the Euler-Korteweg equations on $\mathbb{T}^d$. *J. Dynam. Differential Equations*, 33(3):1475–1513, 2021.

[12] Massimiliano Berti, Alberto Maspero, and Federico Murgante. Hamiltonian Paradifferential Birkhoff Normal Form for Water Waves. *Regular and Chaotic Dynamics*, 2023.

[13] Massimiliano Berti and Riccardo Montalto. Quasi-periodic standing wave solutions of gravity-capillary water waves. *Mem. Amer. Math. Soc.*, 263(1273):v+171, 2020.

[14] Tristan Buckmaster, Steve Shkoller, and Vlad Vicol. Nonuniqueness of weak solutions to the SQG equation. *Comm. Pure Appl. Math.*, 72(9):1809–1874, 2019.

[15] Jacob Burbea. Motions of vortex patches. *Lett. Math. Phys.*, 6(1):1–16, 1982.

[16] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations. *Duke Math. J.*, 165(5):935–984, 2016.

[17] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Uniformly rotating analytic global patch solutions for active scalars. *Ann. PDE*, 2(1):Art. 1, 34, 2016.

[18] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Uniformly rotating smooth solutions for the incompressible 2D Euler equations. *Arch. Ration. Mech. Anal.*, 231(2):719–785, 2019.

[19] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Global smooth solutions for the inviscid SQG equation. *Mem. Amer. Math. Soc.*, 266(1292):v+89, 2020.

[20] Ángel Castro, Diego Córdoba, and Fan Zheng. The lifespan of classical solutions for the inviscid surface quasi-geostrophic equation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 38(5):1583–1603, 2021.

[21] Dongho Chae, Peter Constantin, Diego Córdoba, Francisco Gancedo, and Jiahong Wu. Generalized surface quasi-geostrophic equations with singular velocities. *Comm. Pure Appl. Math.*, 65(8):1037–1066, 2012.

[22] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[23] Peter Constantin, Andrew J. Majda, and Esteban Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7(6):1495–1533, 1994.

[24] Peter Constantin, Andrew J. Majda, and Esteban G. Tabak. Singular front formation in a model for quasigeostrophic flow. *Phys. Fluids*, 6(1):9–11, 1994.

[25] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. Uniqueness for SQG patch solutions. *Trans. Amer. Math. Soc. Ser. B*, 5:1–31, 2018.

[26] Diego Cordoba. Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. *Ann. of Math. (2)*, 148(3):1135–1152, 1998.

[27] Diego Cordoba and Charles Fefferman. Growth of solutions for QG and 2D Euler equations. *J. Amer. Math. Soc.*, 15(3):665–670, 2002.

[28] Diego Córdoba, Marco A. Fontelos, Ana M. Mancho, and Jose L. Rodrigo. Evidence of singularities for a family of contour dynamics equations. *Proc. Natl. Acad. Sci. USA*, 102(17):5949–5952, 2005.

[29] Diego Córdoba, Javier Gómez-Serrano, and Alexandru D. Ionescu. Global solutions for the generalized SQG patch equation. *Arch. Ration. Mech. Anal.*, 233(3):1211–1251, 2019.

[30] Francisco de la Hoz, Zineb Hassainia, and Taoufik Hmidi. Doubly connected V-states for the generalized surface quasi-geostrophic equations. *Arch. Ration. Mech. Anal.*, 220(3):1209–1281, 2016.

[31] Francisco de la Hoz, Zineb Hassainia, and Taoufik Hmidi. Doubly connected V-states for the generalized surface quasi-geostrophic equations. In *Evolution equations: long time behavior and control*, volume 439 of *London Math. Soc. Lecture Note Ser.*, pages 90–117. Cambridge Univ. Press, Cambridge, 2018.

[32] Francisco de la Hoz, Taoufik Hmidi, Joan Mateu, and Joan Verdera. Doubly connected V-states for the planar Euler equations. *SIAM J. Math. Anal.*, 48(3):1892–1928, 2016.

[33] Gary S. Deem and Norman J. Zabusky. Vortex waves: Stationary "v states," interactions, recurrence, and breaking. *Phys. Rev. Lett.*, 40:859–862, Mar 1978.

[34] Yu Deng, Alexandru D. Ionescu, and Fabio Pusateri. On the wave turbulence theory of 2D gravity waves, I: deterministic energy estimates. preprint `https://arxiv.org/abs/2211.10826`.

[35] Roberto Feola and Federico Giuliani. Quasi-periodic traveling waves on an infinitely deep fluid under gravity. Memoires American Mathematical Society, in press.

[36] Francisco Gancedo. Existence for the α-patch model and the QG sharp front in Sobolev spaces. *Adv. Math.*, 217(6):2569–2598, 2008.

[37] Francisco Gancedo, Huy Nguyen, and Neel Patel. Well-posedness for SQG sharp fronts with unbounded curvature. `https://arxiv.org/abs/2105.10982`.

[38] Francisco Gancedo and Neel Patel. On the local existence and blow-up for generalized SQG patches. *Ann. PDE*, 7(1):Paper No. 4, 63, 2021.

[39] Francisco Gancedo and Robert M. Strain. Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem. *Proc. Natl. Acad. Sci. USA*, 111(2):635–639, 2014.

[40] Javier Gómez-Serrano. On the existence of stationary patches. *Adv. Math.*, 343:110–140, 2019.

[41] Javier Gómez-Serrano, Alexandru D. Ionescu, and Jaemin Park. Quasiperiodic solutions of the generalized SQG equation. `https://arxiv.org/abs/2303.03992`.

[42] Zineb Hassaina, Toufik Hmidi, and Nader Masmoudi. KAM theory for active scalar equations. `http://arxiv.org/abs/2110.08615`.

[43] Zineb Hassainia and Taoufik Hmidi. On the V-states for the generalized quasi-geostrophic equations. *Comm. Math. Phys.*, 337(1):321–377, 2015.

[44] Zineb Hassainia, Nader Masmoudi, and Miles H. Wheeler. Global bifurcation of rotating vortex patches. *Comm. Pure Appl. Math.*, 73(9):1933–1980, 2020.

[45] Siming He and Alexander Kiselev. Small-scale creation for solutions of the SQG equation. *Duke Math. J.*, 170(5):1027–1041, 2021.

[46] Taoufik Hmidi and Joan Mateu. Bifurcation of rotating patches from Kirchhoff vortices. *Discrete Contin. Dyn. Syst.*, 36(10):5401–5422, 2016.

[47] Taoufik Hmidi and Joan Mateu. Degenerate bifurcation of the rotating patches. *Adv. Math.*, 302:799–850, 2016.

[48] Taoufik Hmidi and Joan Mateu. Existence of corotating and counter-rotating vortex pairs for active scalar equations. *Comm. Math. Phys.*, 350(2):699–747, 2017.

[49] Taoufik Hmidi, Joan Mateu, and Joan Verdera. Boundary regularity of rotating vortex patches. *Arch. Ration. Mech. Anal.*, 209(1):171–208, 2013.

[50] Taoufik Hmidi, Joan Mateu, and Joan Verdera. On rotating doubly connected vortices. *J. Differential Equations*, 258(4):1395–1429, 2015.

[51] John K. Hunter, Mihaela Ifrim, and Daniel Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346(2):483–552, 2016.

[52] John K. Hunter, Mihaela Ifrim, Daniel Tataru, and Tak Kwong Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proc. Amer. Math. Soc.*, 143(8):3407–3412, 2015.

[53] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Global solutions for a family of GSQG front equations. `https://arxiv.org/abs/2005.09154`, 2020.

[54] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Global solutions of a surface quasigeostrophic front equation. *Pure Appl. Anal.*, 3(3):403–472, 2021.

[55] Mihaela Ifrim and Daniel Tataru. The lifespan of small data solutions in two dimensional capillary water waves. *Arch. Ration. Mech. Anal.*, 225(3):1279–1346, 2017.

[56] Alexandru D. Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in 2d. *Invent. Math.*, 199(3):653–804, 2015.

[57] Alexandru D. Ionescu and Fabio Pusateri. Long-time existence for multi-dimensional periodic water waves. *Geom. Funct. Anal.*, 29(3):811–870, 2019.

[58] Philip Isett and Andrew Ma. A direct approach to nonuniqueness and failure of compactness for the SQG equation. *Nonlinearity*, 34(5):3122–3162, 2021.

[59] Alexander Kiselev and Xiaoyutao Luo. The α-sqg patch problem is illposed in $c^{2,\beta}$ and $w^{2,p}$. preprint `https://arxiv.org/abs/2306.04193`.

[60] Alexander Kiselev and Xiaoyutao Luo. On nonexistence of splash singularities for the α-SQG patches. `https://arxiv.org/abs/2111.13794`, 2021.

[61] Alexander Kiselev, Lenya Ryzhik, Yao Yao, and Andrej Zlatoš. Finite time singularity for the modified SQG patch equation. *Ann. of Math. (2)*, 184(3):909–948, 2016.

[62] Alexander Kiselev, Yao Yao, and Andrej Zlatoš. Local regularity for the modified SQG patch equation. *Comm. Pure Appl. Math.*, 70(7):1253–1315, 2017.

[63] Omar Lazar and Liutang Xue. Regularity results for a class of generalized surface quasi-geostrophic equations. *J. Math. Pures Appl. (9)*, 130:200–250, 2019.

[64] Wilhelm Magnus and Fritz Oberhettinger. *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*. Springer-Verlag, Berlin, 1943.

[65] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[66] Fabien Marchand. Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{-1/2}$. *Comm. Math. Phys.*, 277(1):45–67, 2008.

[67] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).

[68] R.T. Pierrehumbert, I.M. Held, and K.L. Swanson. Spectra of local and nonlocal two-dimensional turbulence. *Chaos Solitons Fractals*, (4):1111–1116, 1994.

[69] Coralie Renault. Relative equilibria with holes for the surface quasi-geostrophic equations. *J. Differential Equations*, 263(1):567–614, 2017.

[70] Serge G. Resnick. *Dynamical problems in non-linear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Chicago.

[71] José Luis Rodrigo. On the evolution of sharp fronts for the quasi-geostrophic equation. *Comm. Pure Appl. Math.*, 58(6):821–866, 2005.

[72] Jukka Saranen and Gennadi Vainikko. *Periodic integral and pseudodifferential equations with numerical approximation*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[73] Richard K. Scott. A scenario for finite-time singularity in the quasigeostrophic model. *J. Fluid Mech.*, 687:492–502, 2011.

[74] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[75] Sijue Wu. The quartic integrability and long time existence of steep water waves in 2d. preprint `https://arxiv.org/abs/2010.09117`.

[76] Sijue Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45–135, 2009.

[77] Victor I. Yudovič. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat i Mat. Fiz.*, 3:1032–1066, 1963.

[78] Fan Zheng. Long-term regularity of 2d gravity water waves. preprint `https://arxiv.org/abs/2206.10350`.