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A sign that used to annoy me, and still does

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ABSTRACT

We provide a proof of the following fact: if a complex scheme Y has Behrend function constantly equal to a sign $\sigma \in \{\pm 1\}$, then all of its components $Z \subset Y$ are generically reduced and satisfy $(-1)^{\dim_{\mathbb{C}} T_p Y} = \sigma = (-1)^{\dim Z}$ for $p \in Z$ a general point. Given the recent counterexamples to the parity conjecture for the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{A}^3)$, our argument suggests a possible path to disprove the constancy of the Behrend function of $\mathrm{Hilb}^n(\mathbb{A}^3)$.

1. Introduction

By work of Behrend [4], every scheme Y of finite type over $\mathbb{C}$ carries a canonical constructible function $\nu_Y : Y(\mathbb{C}) \to \mathbb{Z}$, known as the *Behrend function* of Y. It is a subtle invariant of singularities, with a key role in enumerative geometry. Already in the case of schemes with just one point, its computation is a nontrivial task [8]. The ν_Y-weighted Euler characteristic of Y is the global invariant

$$\chi(Y, \nu_Y) = \sum_{m \in \mathbb{Z}} m \chi(\nu_Y^{-1}(m)) \in \mathbb{Z},$$

where χ is the topological Euler characteristic.

Fix $n \in \mathbb{Z}_{\geq 0}$. Let $\mathrm{H}_n = \mathrm{Hilb}^n(\mathbb{A}^3)$ be the Hilbert scheme of n points on affine 3-space, namely the moduli space of ideals $I \subset \mathbb{C}[x, y, z]$ of colength n. Behrend–Fantechi proved that

$$(-1)^n = (-1)^{\dim_{\mathbb{C}} T_I \mathrm{H}_n} = \nu_{\mathrm{H}_n}(I), \tag{1.1}$$

as soon as I is *monomial* [5]. Moreover, the main result of [5] uses the above identities to compute

$$\chi(\mathrm{H}_n, \nu_{\mathrm{H}_n}) = (-1)^n \chi(\mathrm{H}_n). \tag{1.2}$$

In other words, the ν_{H_n}-weighted Euler characteristic of H_n, also known as the n-th degree 0 *Donaldson–Thomas invariant* of $\mathbb{A}^3$, is the same that one would have if ν_{H_n} were constant. One is then led to make the following prediction.

Conjecture A. *The Behrend function of* $\mathrm{Hilb}^n(\mathbb{A}^3)$ *is constantly equal to* $(-1)^n$.

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A proof of this conjecture was proposed by A. Morrison in [15], but a gap in the proof, to the best of our knowledge, has not been fixed since.

On the other hand, the first identity in Formula (1.1) encourages the following conjecture, due to Okounkov–Pandharipande [16].

Conjecture B *(Parity Conjecture). One has* $(-1)^n = (-1)^{\dim_{\mathbb{C}} T_I \mathrm{H}_n}$ *for all* $I \in \mathrm{H}_n$.

The parity conjecture has been confirmed for monomial ideals by Maulik–Nekrasov–Okounkov–Pandharipande [14, Thm. 2] (see also [5, Lemma 4.1 (c)]) and, more generally, for homogeneous ideals by Ramkumar–Sammartano [17, Thm. 1]. However, we have the following recent result.

Theorem 1.1 *(Giovenzana–Giovenzana–Graffeo–Lella [7]). Conjecture B is false for* $n \geq 12$.

We shall prove the following.

Theorem 1.2 *(Theorem 3.1). Let* Y *be a scheme of finite type over* $\mathbb{C}$. *Fix a sign* $\sigma \in \{\pm 1\}$. *If* $\nu_Y \equiv \sigma$, *then every irreducible component* $Z \subset Y$ *is generically reduced and a general point* $p \in Z$ *satisfies* $\dim_{\mathbb{C}} T_p Y = \dim Z$ *and*

$$(-1)^{\dim_{\mathbb{C}} T_p Y} = \sigma = (-1)^{\dim Z}.$$

In particular, when $Y = \mathrm{Hilb}^n(\mathbb{A}^3)$, Conjecture A implies a 'generic' version of Conjecture B. Such link between the two above conjectures was already stated in [15], but the proof of Theorem 1.2 gives full details. Such details are needed to support the point of this paper, which is to provide a theoretical path (starting from the failure of Conjecture B), to *disprove* Conjecture A.

Consider the Quot scheme of points $\mathrm{Q}_{r,n} = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^{\oplus r}, n)$, parametrising isomorphism classes of $\mathbb{C}[x,y,z]$-linear quotients $\mathscr{O}_{\mathbb{A}^3}^{\oplus r} \twoheadrightarrow T$, where $\dim_{\mathbb{C}} T = n$. Equation (1.1) generalises to

$$(-1)^{rn} = (-1)^{\dim_{\mathbb{C}} T_p \mathrm{Q}_{r,n}} = \nu_{\mathrm{Q}_{r,n}}(p), \tag{1.3}$$

for every $\mathbb{G}_m^{3+r}$-fixed point $p \in \mathrm{Q}_{r,n}$, i.e. for p corresponding to a direct sum of monomial ideals [3,6].

If $\mathrm{Q}_{r,n}$ has constant Behrend function — which cannot be excluded just yet — then it cannot contain any generically nonreduced component. Finding such components, or even just nonreduced points, is an active research direction, for which we refer the reader to [9,10,19]. The case of $\mathbb{A}^3$ is in some sense the last mystery in the land of pathologies on Quot schemes of smooth varieties, for it finds itself sandwiched between smooth Hilbert schemes or mildly singular Quot schemes ($\mathbb{A}^d$ case, for $d \leq 2$) and Quot schemes that happen to admit generically nonreduced components ($\mathbb{A}^d$ case, for $d > 3$).

Finally, the author wants to point out that the title of this work is in homage to P. Tingley's paper *A minus sign that used to annoy me but now I know why it is there* (Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday, pp. 415–427, 2017). The ubiquity of signs in Donaldson–Thomas theory, and in particular the one in Equation (1.3), has annoyed (read: fascinated) the author for some time. Even though we know the theoretical reasons *why it is there*, we felt like writing this note to highlight the structural aspects of the theory, still to be understood, surrounding this sign.

2. Local Euler obstruction and the Behrend function

In this section we fix our notation and recall the definition of the Behrend function from [4].

All schemes are of finite type over $\mathbb{C}$. The group of cycles on a scheme Y is denoted $\mathrm{Z}_* Y$. An irreducible scheme Y, with generic point $\eta \in Y$, is *generically reduced* if the local ring $\mathscr{O}_{Y,\eta}$ is reduced. The *multiplicity* $\mathrm{mult}_Z(Y)$ of an irreducible component $Z \subset Y$, with generic point η, is defined as the length of the local artinian ring $\mathscr{O}_{Y,\eta}$. We assume all schemes admit a closed embedding in a smooth $\mathbb{C}$-scheme; this assumption is never necessary, but is satisfied in all applications we have in mind.

2.1. MacPherson's local Euler obstruction

Let Y be a scheme. The abelian group of constructible functions $Y(\mathbb{C}) \to \mathbb{Z}$ on a scheme Y is denoted $\mathbf{F}(Y)$. The *local Euler obstruction* of Y is an isomorphism of abelian groups

$$\mathsf{Eu}\colon \mathrm{Z}_* Y \xrightarrow{\sim} \mathbf{F}(Y),$$

discovered by MacPherson [13, Lemma 2]. We now recall its purely algebraic definition, different from the original trascendental definition by MacPherson; see [11] for a modern survey on this topic. Let $V \hookrightarrow M$ be a closed immersion of a

d-dimensional integral scheme V inside a smooth scheme M. Let $\mathrm{Gr}_d(\mathcal{T}_M) \to M$ be the Grassmann bundle of d-planes in the fibres of the tangent bundle $\mathcal{T}_M$, and denote by $V_{\mathrm{sm}} \subset V$ the smooth locus of V, which is open and nonempty [18, Tag 056V]. We have a canonical section

$$\mathsf{s}\colon V_{\mathrm{sm}} \to \mathrm{Gr}_d(\mathcal{T}_M), \quad y \mapsto T_y V_{\mathrm{sm}}.$$

Let $\widehat{V} \subset \mathrm{Gr}_d(\mathcal{T}_M)$ be the closure of the image of s. The map

$$\mathsf{n}\colon \widehat{V} \to V$$

restricting the projection $\mathrm{Gr}_d(\mathcal{T}_M) \to M$ is called the *Nash blowup* of $V \hookrightarrow M$. Its base change along $V_{\mathrm{sm}} \hookrightarrow V$ is an isomorphism. Let $\mathcal{U}$ denote the universal rank d bundle on $\mathrm{Gr}_d(\mathcal{T}_M)$. The bundle $\widehat{\mathcal{T}}_V = \mathcal{U}|_{\widehat{V}}$ is called the *Nash tangent bundle*.

Now back to defining $\mathsf{Eu}\colon \mathsf{Z}_*Y \to \mathbf{F}(Y)$. Let $V \subset Y$ be a prime cycle on Y, i.e. a generator of Z_*Y. Let $y \in Y$ be a closed point. The integer

$$\mathsf{Eu}(V)(y) = \int_{\mathsf{n}^{-1}(y)} c(\widehat{\mathcal{T}}_V) \cap s(\mathsf{n}^{-1}(y), \widehat{V})$$

is called the local Euler obstruction of V at the point $y \in Y$. Here $s(\mathsf{n}^{-1}(y), \widehat{V})$ denotes the Segre class of the normal cone to the closed immersion $\mathsf{n}^{-1}(y) \hookrightarrow \widehat{V}$. The map Eu is defined by $\mathbb{Z}$-linear extension. By [13, Section 3], $\mathsf{Eu}(V)(y)$ is equal to:

- 0, if $y \in Y \setminus V$,
- 1, if y is a nonsingular point on V,
- $\mathrm{mult}_y Y$, if y is a closed point on an integral curve $Y = V$,
- $d(2-d)$ is $V \subset \mathbb{A}^3$ is the cone over a smooth degree d plane curve $X \hookrightarrow \mathbb{P}^2$ and $y \in V$ is the vertex of the cone.

The last bullet was generalised by Aluffi, who proved that if $X \subset \mathbb{P}^{n-1}$ is a smooth curve of degree d and genus g, and if y is the vertex of the cone $V \subset \mathbb{P}^n$ over X, then $\mathsf{Eu}(V)(y) = 2 - 2g - d$ [1, Ex. 3.19].

2.2. The Behrend function

Behrend proved that any finite type $\mathbb{C}$-scheme Y carries a canonical cycle $\mathfrak{c}_Y \in \mathsf{Z}_*Y$, whose definition we now briefly recall.

First of all, suppose given a scheme U and a closed immersion $U \hookrightarrow M$ inside a smooth scheme M, cut out by the ideal sheaf $\mathscr{I} \subset \mathscr{O}_M$. Consider the normal cone

$$C_{U/M} = \mathbf{Spec}_{\mathscr{O}_U}\left(\bigoplus_{e \geq 0} \mathscr{I}^e / \mathscr{I}^{e+1}\right) \xrightarrow{\pi} U.$$

Note that if D is an irreducible component of $C_{U/M}$, then $\pi(D)$ is an irreducible closed subset of X, and as such it defines a cycle $\pi(D) \in \mathsf{Z}_*U$. The *signed support of the intrinsic normal cone*, introduced by Behrend in [4, Sec. 1.1], is the cycle

$$\mathfrak{c}_{U/M} = \sum_{D \subset C_{U/M}} (-1)^{\dim \pi(D)} \mathrm{mult}_D(C_{U/M}) \cdot \pi(D) \in \mathsf{Z}_*U,$$

the sum being over the irreducible components of the normal cone $C_{U/M}$.

Now back to our scheme Y. The canonical cycle $\mathfrak{c}_Y \in \mathsf{Z}_*Y$ is defined as follows: it is the unique cycle with the property that for any étale map $U \to Y$ and for any closed immersion $U \hookrightarrow M$ inside a smooth scheme, one has $\mathfrak{c}_{U/M} = \mathfrak{c}_Y|_U$. See [4, Prop. 1.1] for the proof that this is a good definition. In particular, if Y itself admits a closed immersion inside a nonsingular scheme M, then $\mathfrak{c}_Y = \mathfrak{c}_{Y/M}$.

Definition 2.1 *([4, Def. 1.4]).* Let Y be a scheme of finite type over $\mathbb{C}$. The *Behrend function* of Y is the constructible function $\nu_Y = \mathsf{Eu}(\mathfrak{c}_Y)$.

Example 2.2. If Y is a smooth connected scheme of dimension d, then $\mathfrak{c}_Y = (-1)^d[Y]$, so that $\nu_Y \equiv (-1)^d$. The Behrend function of a fat point — a scheme with only one point — is also trivially constant, but hard to compute in general (e.g. for embedding dimension higher than 3), see [8].

Remark 2.3. The Behrend function pulls back along étale maps, in particular $\nu_Y(p) = \nu_U(p)$ if U is open in Y and $p \in U$. Let Y be a scheme, $Z \subset Y$ an irreducible component, $p \in Z$ a closed point. Assume there is an open subset $U \subset Z$ containing p, and not intersecting any other irreducible component of Y. Then U is also open in Y, so

$$\nu_Z(p) = \nu_U(p) = \nu_Y(p).$$

Assume Y is an irreducible scheme admitting a closed immersion into a smooth scheme M. Let $C = C_{Y/M}$ be the normal cone, with projection $\pi : C \to Y$, and write $\overline{D} = \pi(D)$ for an irreducible component D of C.

Let $p \in Y$ be a point. Then, since $\nu_Y = \mathsf{Eu}(\mathfrak{c}_{Y/M})$, we have

$$\nu_Y(p) = \sum_{\substack{D \subset C \\ \overline{D} \ni p}} (-1)^{\dim \overline{D}} \mathrm{mult}_D(C) \cdot \mathsf{Eu}(\overline{D})(p). \tag{2.1}$$

There are two possibilities for each D in the sum: either $\overline{D} = Y$, or $\overline{D} \neq Y$. Therefore Formula (2.1) becomes

$$\nu_Y(p) = \sum_{\substack{D \subset C \\ Y \neq \overline{D} \ni p}} (-1)^{\dim \overline{D}} \mathrm{mult}_D(C) \cdot \mathsf{Eu}(\overline{D})(p) + (-1)^{\dim Y} \mathsf{Eu}(Y)(p) \sum_{\substack{D \subset C \\ \overline{D} = Y}} \mathrm{mult}_D(C). \tag{2.2}$$

We conclude this section with a few examples.

Example 2.4. Here we give an example of an irreducible scheme Y which is generically reduced, but whose Behrend function is not constantly equal to the same sign. Take $Y = \operatorname{Spec} \mathbb{C}[x, y]/(y^2, xy) \subset \mathbb{A}^2$. It is smooth everywhere except at the origin $0 \in Y \subset \mathbb{A}^2$, where an embedded point is located. We have $\nu_Y|_{Y \setminus 0} \equiv -1$, but $\nu_Y(0) = 1$. Indeed, as computed in Examples 3.5 and 4.8 of [2], the normal cone $C = C_{Y/\mathbb{A}^2}$ has two irreducible components D_1 and D_2, where D_1 (called C_0 in [2]) has multiplicity 2 and is contracted by $\pi : C \to Y$ onto $0 \in Y$, and D_2 (called L in [2]) has multiplicity 1 and dominates Y. We therefore have

$$\nu_Y(0) = (-1)^0 \mathrm{mult}_{D_1}(C) \mathsf{Eu}(\overline{D}_1)(0) + (-1)^{\dim Y} \mathrm{mult}_{D_2}(C) \mathsf{Eu}(\overline{D}_2)(0) = 1 \cdot 2 \cdot 1 + (-1) \cdot 1 \cdot 1 = 1.$$

Example 2.5. Here we give an example of a singular, reducible and reduced scheme Y with constant Behrend function. Let $Y \subset \mathbb{A}^3$ be the union of the three coordinate axes in $\mathbb{A}^3$, namely $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/J$ where $J = (xy, xz, yz)$. We claim that $\nu_Y \equiv -1$. The only nontrivial value to compute is $\nu_Y(0)$, where 0 denotes the origin of $\mathbb{A}^3$. We call L_1, L_2, L_3 the coordinate axes, and we set $f_0 = xy$, $f_1 = xz$ and $f_2 = yz$. Fix homogeneous coordinates w_0, w_1, w_2 on $\mathbb{P}^2$, and form the blowup

$$\varepsilon : \mathsf{Bl}_Y \mathbb{A}^3 \hookrightarrow \mathbb{A}^3 \times \mathbb{P}^2 \to \mathbb{A}^3,$$

which is cut out by the ideal

$$\mathscr{I} = (w_2 f_0 - w_1 f_1, w_2 f_0 - w_0 f_2) \subset \mathscr{O}_{\mathbb{A}^3 \times \mathbb{P}^2}.$$

Therefore the exceptional divisor $E_Y \mathbb{A}^3 = V(\varepsilon^{-1}(J) \cdot \mathscr{O}_{\mathsf{Bl}_Y \mathbb{A}^3})$ is cut out by the relations

$$w_2 f_0 = w_1 f_1, \quad w_2 f_0 = w_0 f_2, \quad xy = xz = yz = 0.$$

From these, one can check that $E_Y \mathbb{A}^3$ consists of irreducible components $E_1, \ldots, E_4$ with $E_i \cong \mathbb{A}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3$, and $E_4 \cong \mathbb{P}^2$. The multiplicities are $\mathrm{mult}_{E_i} E_Y \mathbb{A}^3 = 1$ for $i = 1, 2, 3$ and $\mathrm{mult}_{E_4} E_Y \mathbb{A}^3 = 2$. The component E_i dominates L_i for $i = 1, 2, 3$, whereas E_4 is collapsed onto $\{0\}$. Therefore

$$\nu_Y(0) = 3 \cdot (-1)^{\dim L_1} \cdot 1 \cdot \mathsf{Eu}(L_1)(0) + (-1)^0 \cdot 2 \cdot \mathsf{Eu}(\{0\})(0) = -3 + 2 = -1.$$

See also Section 4 for one more comment on this example.

Example 2.6. Let $Y = \mathrm{Hilb}^4(\mathbb{A}^3)$ be the Hilbert scheme of 4 points on $\mathbb{A}^3$. It is an irreducible scheme of dimension 12, with singular locus equal to the closed subscheme $\mathbb{A}^3 \subset Y$ parametrising the squares $\mathfrak{m}_x^2$ of the maximal ideals of closed points $x \in \mathbb{A}^3$. This is the 'first' singular Hilbert scheme. Its Behrend function *is* in fact constant,

$$\nu_Y \equiv 1.$$

Since the Behrend function of a smooth irreducible scheme R is constantly equal to $(-1)^{\dim R}$, we only need to check that $\nu_Y(\mathfrak{m}_x^2) = 1$ for all $x \in \mathbb{A}^3$. In fact, each point $\mathfrak{m}_x^2$ is a translation of the monomial point $I = \mathfrak{m}_0^2$ (the only singular monomial ideal), so it is enough to confirm that $\nu_Y(I) = 1$. But this follows from [5, Thm. 3.4], which implies

$$\nu_Y(I) = (-1)^{\dim_{\mathbb{C}} T_I Y} = (-1)^{18} = 1.$$

3. Proof of the main theorem

In this section we prove Theorem 1.2.

Theorem 3.1. *Let Y be a scheme of finite type over $\mathbb{C}$. Fix a sign $\sigma \in \{\pm 1\}$. If $\nu_Y \equiv \sigma$, then every irreducible component $Z \subset Y$ is generically reduced and a general point $p \in Z$ satisfies $\dim_{\mathbb{C}} T_p Y = \dim Z$ and*

$$(-1)^{\dim_{\mathbb{C}} T_p Y} = \sigma = (-1)^{\dim Z}. \tag{3.1}$$

Proof. Let $Z \subset Y$ be an irreducible component. Consider the open subset

$$W = Y \setminus \bigcup_{Z' \neq Z} Z' \subset Y, \tag{3.2}$$

the union being over the irreducible components $Z' \subset Y$ different from Z. Then for $p \in W$, one has

$$\sigma = \nu_Y(p) = \nu_Z(p), \tag{3.3}$$

the second identity being implied by Remark 2.3.

Let $Y \hookrightarrow M$ be a closed immersion inside a smooth scheme M. Let $C = C_{Z/M}$ be the normal cone to Z in M, with projection $\pi \colon C \to Z$. Let $W' \subset Z$ be the (open) complement of the closed subset

$$\bigcup_{\substack{D \subset C \\ \overline{D} \neq Z}} \overline{D} \subset Z,$$

where $\overline{D} = \pi(D)$ as before. Note that W' might equal Z, but cannot be empty. Then, for $p \in W \cap W'$, one has

$$\nu_Z(p) = (-1)^{\dim Z} \mathsf{Eu}(Z)(p) \sum_{\substack{D \subset C \\ \overline{D} = Z}} \operatorname{mult}_D(C) \tag{3.4}$$

by Formula (2.2). However, we have

$$\mathsf{Eu}(Z)(p) = \mathsf{Eu}(Z_{\mathrm{red}})(p),$$

and Z_{red} is a reduced scheme of finite type over $\mathbb{C}$, therefore its smooth locus W'' is a dense open subset. Thus $\mathsf{Eu}(Z)(p) = 1$ for $p \in W''$, and Formula (3.4) becomes

$$\nu_Z(p) = (-1)^{\dim Z} \sum_{\substack{D \subset C \\ \overline{D} = Z}} \operatorname{mult}_D(C), \quad p \in W \cap W' \cap W''. \tag{3.5}$$

Thus, combining Formula (3.5) and Formula (3.3) with one another, we find

$$\sigma = (-1)^{\dim Z} \cdot \mathsf{m}, \tag{3.6}$$

where $\mathsf{m} \in \mathbb{Z}_{>0}$ is the sum appearing in Formula (3.5). But this forces $\mathsf{m} = 1$ and thus $(-1)^{\dim Z} = \sigma$, too. This proves the second identity in Formula (3.1).

Next, we prove that Z is generically reduced. Since $\mathsf{m} = 1$, there is a unique irreducible component $D \subset C$ such that $\overline{D} = Z$, and moreover C is reduced at the generic point $\xi_D \in C$ corresponding to $D \subset C$, since necessarily $\operatorname{mult}_D(C) = 1$. It follows that Z is generically reduced. In a little more detail, consider the composition $\mathrm{id}_Z = \pi \circ \tau \colon Z \to C \to Z$ where $\tau \colon Z \to C$ is the zero section of the cone. Now, π maps $\xi_D \in C$ to the generic point η of Z. Since the composition

$$\mathscr{O}_{Z,\eta} \to \mathscr{O}_{C,\xi_D} \to \mathscr{O}_{Z,\eta}$$

is the identity, the map $\mathscr{O}_{Z,\eta} \to \mathscr{O}_{C,\xi_D}$ is injective. But $\mathscr{O}_{C,\xi_D}$ is reduced, thus $\mathscr{O}_{Z,\eta}$ is reduced, too.

Finally, we prove the first identity in Formula (3.1). By the previous paragraph, there is a nonempty open reduced subscheme $U \subset Z$. In particular, U contains a nonempty smooth open subset $U' \subset Z$. Consider also the open subset W (open in both Z and Y) from Equation (3.2). Then, for any point $p \in U' \cap W$, we have

$$\begin{aligned}
\dim Z &= \dim U' && Z \text{ is irreducible} \\
&= \dim_{\mathbb{C}} T_p U' && U' \text{ is smooth} \\
&= \dim_{\mathbb{C}} T_p Z && U' \text{ is open in } Z \\
&= \dim_{\mathbb{C}} T_p W && W \text{ is open in } Z \\
&= \dim_{\mathbb{C}} T_p Y && W \text{ is open in } Y
\end{aligned}$$

which finishes the proof. □

Fix integers $r, n \in \mathbb{Z}_{>0}$. Consider the Quot scheme of points $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^{\oplus r}, n)$, parametrising (isomorphism classes of) length n quotients $\mathscr{O}_{\mathbb{A}^3}^{\oplus r} \twoheadrightarrow T$. This is the general situation we have in mind. Note that this Quot scheme has a natural embedding in a smooth quasiprojective variety ncQuot_r^n of dimension $2n^2 + rn$, the so-called noncommutative Quot scheme [3, Thm. 2.6]. We obtain the following consequence of Theorem 1.2.

Corollary 3.2. *Let $Z \subset \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^{\oplus r}, n)$ be an irreducible component. If either Z is not generically reduced, or*

$$(-1)^{\dim Z} \neq (-1)^{rn},$$

then the Behrend function of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^{\oplus r}, n)$ is not constant.

4. Conclusions

The falsity of the parity conjecture is unfortunately not enough to disprove the constancy of the Behrend function of $\mathrm{Hilb}^n(\mathbb{A}^3)$. However, general enough counterexamples to the parity conjecture would constitute a good testing ground for disproving Conjecture A. Our main argument says precisely that points as general as in Equation (3.5) would be enough to disprove it.

We believe, nevertheless, that having constant Behrend function equal to a sign should be thought of as a sort of 'regularity property'. In fact, Conjecture A is a pretty subtle statement, precisely because the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{A}^3)$ does exhibit such regularity (or symmetry): it is a global *critical locus*, i.e. the zero scheme of an exact 1-form $\mathrm{d}f_n$ for f_n a regular function on a smooth $\mathbb{C}$-scheme U_n, see e.g. [20, Prop. 1.3.1] or [3, Thm. 2.6]. We take Example 2.5 as a toy situation to further explain this point. In that example, $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy, xz, yz) \subset \mathbb{A}^3$ is a critical locus,

$$Y = \mathrm{crit}(xyz),$$

and as such it has a symmetric perfect obstruction theory in the sense of [5]. Via the perfect obstruction theory machinery, one can compute directly

$$\nu_Y(0) = (-1)^{\dim_{\mathbb{C}} T_0 Y - \dim_{\mathbb{C}} T_0 Y^{\mathbb{C}^\times}} \cdot \nu_{Y^{\mathbb{C}^\times}}(0) = (-1)^{3-0} \cdot 1 = -1,$$

thanks to [12, Thm. C], using that Y has a $\mathbb{C}^\times$-action and the symmetric perfect obstruction theory is $\mathbb{C}^\times$-equivariant. This confirms once more that $\nu_Y \equiv -1$. Now, Example 2.5 shows *directly* how signed multiplicities in the normal cone for this critical locus balance each other and produce a constant Behrend function. It is possible that this behaviour does in fact occur for $\mathrm{Hilb}^n(\mathbb{A}^3)$ as well, but checking this directly seems unfortunately out of reach at the moment.

Data availability

No data was used for the research described in the article.

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