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Elliptic blowup equations for 6d SCFTs. Part IV. Matters

Jie Gu,[a] Babak Haghighat,[b] Albrecht Klemm,[c,d] Kaiwen Sun[e,f] and Xin Wang[c,g]

[a] *Département de Physique Théorique, Université de Genève, 24 quai Ernest Ansermet, 1211 Genève 4, Switzerland*
[b] *Yau Mathematical Sciences Center, Tsinghua University, Haidian District, 100084 Beijing, China*
[c] *Bethe Center for Theoretical Physics, Universität Bonn, Nussallee 12, D-53115 Bonn, Germany*
[d] *Hausdorff Center for Mathematics, Endenicher Allee 62, D-53115 Bonn, Germany*
[e] *Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, 34136 Trieste, Italy*
[f] *INFN, Sezione di Trieste, via Valerio 2, 34127 Trieste, Italy*
[g] *Max Planck Institute for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany*

E-mail: jie.gu@unige.ch, babak@math.tsinghua.edu.cn, aklemm@th.physik.uni-bonn.de, ksun@sissa.it, wxin@mpim-bonn.mpg.de

Abstract: Given the recent geometrical classification of 6d $(1,0)$ SCFTs, a major question is how to compute for this large class their elliptic genera. The latter encode the refined BPS spectrum of the SCFTs, which determines geometric invariants of the associated elliptic non-compact Calabi-Yau threefolds. In this paper we establish for *all* 6d $(1,0)$ SCFTs in the atomic classification blowup equations that fix these elliptic genera to large extent. The latter fall into two types: the unity and the vanishing blowup equations. For almost all rank one theories, we find unity blowup equations which determine the elliptic genera completely. We develop several techniques to compute elliptic genera and BPS invariants from the blowup equations, including a recursion formula with respect to the number of strings, a Weyl orbit expansion, a refined BPS expansion and an ϵ_1, ϵ_2 expansion. For higher-rank theories, we propose a gluing rule to obtain all their blowup equations based on those of rank one theories. For example, we explicitly give the elliptic blowup equations for the three higher-rank non-Higgsable clusters, ADE chain of -2 curves and conformal matter theories. We also give the toric construction for many elliptic non-compact Calabi-Yau threefolds which engineer 6d $(1,0)$ SCFTs with various matter representations.

Keywords: Solitons Monopoles and Instantons, Supersymmetric Gauge Theory, Topological Strings, Conformal Field Theory

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Contents

1 Introduction

In the absence of a Lagrangian description, six dimensional theories with $(2,0)$ and $(1,0)$ superconformal symmetries are generally difficult to study. The data of BPS spectra thus become crucial for understanding the properties of this important class of theories and their compactification to lower dimensions. One particularly interesting type of BPS states are those associated to non-critical strings wrapped on a torus. The counting of these BPS states are encoded in the elliptic genera[1] of bound states of the non-critical strings. Among

[1]These are Ramond-Ramond elliptic genera, and they can be related to Neveu-Schwarz-Ramond elliptic genera by spectral flows [1].

the techniques to obtain the elliptic genera are calculations in a dual 2d quiver gauge theory [2], the refined holomorphic anomaly equation and the closely related modular bootstrap approach [1, 3–5], the application of the topological vertex [6] and the elliptic blowup equations [7–9]. The purpose of this paper is to generalise the last approach to generic elliptic non-compact Calabi-Yau spaces, which geometrically engineer six dimensional superconformal field theories with geometrically realisable gauge symmetries and matters in generic flavour representations. Geometrically these theories can be viewed as a generalisation of the theories associated to non-higgsable clusters discussed in [7, 8] to include matter by considering additional singularities of the elliptic fibration over the non-compact directions in the base. In this paper, we write down elliptic blowup equations for all the rank one 6d superconformal theories with one tensor multiplet. These theories are geometrically realised as non-compact elliptic fibrations with an isolated curve of self intersection $-n$ in the base. The BPS states associated to non-critical strings are interpreted as the BPS states of M2-branes wrapped on the base curve.[2] We demonstrate that for almost all rank one theories, with the exception of 12 theories (see the beginning of section 4 for a list of these theories), the BPS spectrum associated to the non-critical strings or equivalently the elliptic genera can be computed from the elliptic blowup equations. Our results go beyond all previous results in the literature, see for instance [1, 2, 10–15], in that firstly, in many cases our methods can be pushed to compute BPS invariants with arbitrary base degrees, or equivalently elliptic genera with arbitrary numbers of strings, and secondly, we can produce results with both gauge and flavor fugacities turned on. In contrast, the existing techniques either produce complete solutions only for a limited set of theories (localisation on quiver construction), or are limited to one-string elliptic genera (analysis of worldsheet theory), or cannot turn on all fugacities (modular bootstrap). We review all known computational methods and previous results on elliptic genera, if they exist, in detail in section 2.2.1. In the case of higher rank theories, we formulate the generic *gluing rules*, by which we can build elliptic blowup equations for them using as ingredients elliptic blowup equations of rank one theories. However for almost all higher rank theories, elliptic blowup equations are not strong enough to solve the corresponding elliptic genera uniquely. Nevertheless these novel equations do give interesting structural constraints on the elliptic genera in particular on their modular properties. We leave it as an open problem how to supplement in particular the theories with only vanishing elliptic blowup equations by further data so that one can determine their elliptic genera and the BPS degeneracies completely.

The original blowup equations were derived in order to confirm the gauge theory instanton corrections to the so called Seiberg-Witten prepotential of $\mathcal{N} = 2$ four dimensional supersymmetric gauge theory. In particular Nakajima and Yoshioka confirmed the Nekrasov partition function, which depends on the ϵ_1, ϵ_2 background parameters, encodes the prepotential in the $\epsilon_1, \epsilon_2 \to 0$ limit [16]. The same authors generalised them to K-theoretic invariants of five dimensional gauge theories and obtained the K-theoretic blowup equations [17], which count K-theoretic Donaldson invariants as discussed in a paper with Göttsche [18].

[2]In F-theory construction for the 6d SCFT, non-critical strings arise from D3-branes wrapped on the base curve. When the non-critical string is put on a torus, the D3-branes become M2-branes wrapped on the base curve in the dual M-theory.

The K-theoretic invariants are closely related to the refined topological string or refined BPS invariants on local Calabi-Yau spaces, which engineer these 5d gauge theories. In [19] it was shown that the five dimensional blowup equations can be used to calculate these refined BPS invariants $N^{\beta}_{j_l,j_r}$ on arbitrary toric Calabi-Yau spaces. Mathematically the $N^{\beta}_{j_l,j_r}$ are defined as refined stable pair invariants [20]. Unrefined stable pair invariants were introduced in [21] and were explicitly calculated using localisation on toric Calabi-Yau manifolds [22]. It is in this context that refined invariants could also be calculated using localisation in [20]. However their definition as invariants is suggested by physical arguments, only when there is a global $\mathfrak{u}(1)_R$ symmetry in five dimensions, which geometrically is induced by a $\mathfrak{u}(1)$ isometry of the Calabi-Yau space. Mathematically one must have an orientation on the moduli space of gauge theory — or here specifically stable pair invariants [23]. Both conditions are naturally realised on non-compact Calabi-Yau spaces.

In [19] a generalisation of the K-theoretic blowup equations of Nakajima and Yoshioka was proposed that calculates the refined BPS invariants also for those non-compact toric Calabi-Yau spaces that do not by themselves define gauge theories like local $\mathbb{P}^2$. Later [4, 19], based on [24, 25], generalised them to elliptic blowup equations for elliptic non-compact Calabi-Yau threefolds associated to 6d SCFTs. In particular, consistency checks of blowup equations with the E-string BPS spectrum, which corresponds to refined BPS invariants on the elliptic non-compact half K3, which is not toric, were made in [4, 19]. In [7] the rank one non-higgsable clusters with A or D type gauge groups were solved using the elliptic blowup equations, while this was extended in [8] to include all rank one non-higgsable clusters except for the E_7 theory with a half hypermultiplet. In [9] the blowup equations for E– and M– string theories and chains of these were discussed. The current paper completes these results by generalising elliptic blowup equations to all 6d superconformal theories covered in the atomic classification [26, 27].

The discussion of elliptic blowup equations falls into three steps. First the formulation of the elliptic blowup equations for each model. For rank one theories, this is neatly summarised in section 3. They also serve as basic building blocks. Together with the "gluing rules" proposed in section 7.1, they can be used to construct elliptic blowup equations of any higher rank 6d SCFT, whose forms are given in section 7.2. It is important to classify elliptic blowup equations into the *unity* type and the *vanishing* type, according to whether integral or fractional gauge flux is summed over, or equivalently whether the right hand side of the equation vanishes identically. Almost all rank one theories have unity blowup equations, except for 12 theories, which have only vanishing blowup equations. On the contrary, the majority of higher rank theories have no unity equation. This fact is important for the solvability of these theories.

Second, the proof that the elliptic genera which satisfy elliptic blowup equations are forms with modular transformation in the elliptic parameter, the Kähler parameter of the fibre class τ, as well as with Jacobi-form transformations with a prescribed *index of the elliptic parameters* given by the two $\epsilon_\pm$ deformations as well as by the masses of Cartan gauge bosons m_{G_i} and flavour masses m_{F_i}. This is discussed in section 3.3 for the rank one cases and in section 7.3 for the higher rank cases as well as commented in more detail in the examples. This *index* can be inferred either from the anomaly polynomial of the 6d chiral SCFT, as reviewed in section 2.1.1, or from the geometric auto-equivalences acting on the central charges of the derived category of quasi coherent sheaves of the Calabi-Yau space [28–31]. Both data are related ultimately to classical intersections on the Calabi-Yau space. It has been also related to the Casimir energy of the 2d $(0,2)$ supersymmetric quiver theory that depends quadratically on the flavour fugacities [5, 32, 33]. On the physics side we explain the interesting structural changes of the blowup equations along the Higgsing trees in section 3.4 and give a direct heuristic physical interpretation and partial derivation of them in section 3.2. We also discuss the relation between the elliptic blowup equations of rank one 6d SCFTs and the K-theoretic five dimensional blowup equations in section 3.5. We point out that the 5d limit $q = \mathrm{e}^{2\pi i\tau} \to 0$ is quite non-trivial. In particular, as explained in section 3.5, not all K-theoretic blowup equations in 5d can be obtained from the 6d elliptic blowup equations.

The last question is the actual solvability of BPS invariants or elliptic genera from the elliptic blowup equations. This depends critically on whether the theory has unity blowup equations or only vanishing blowup equations. We call the latter case class **C**. In the former case, we distinguish further by whether the theory has enough unity blowup equations so that a recursion formula in the spirit of [34] can be written down, or not. We call these two cases classes **A** and **B** respectively. The case of class **A** is clear cut, and we can write down elliptic genera with arbitrary number of strings using the recursion formula presented in section 4.1. In class **B**, using various methods presented in sections 4.2, 4.3, 4.4, in practice elliptic genera or BPS invariants can still be solved order by order from blowup equations. In these two cases, compared with all the other techniques to solve the refined BPS invariants mentioned above, the method of the elliptic blowup equations is generically the most efficient one to obtain results for arbitrary spins and arbitrary curve degrees. In class **C**, the theory cannot be solved completely. Nevertheless we can still get partial results and interesting structural insight from the vanishing blowup equations. Particularly interesting are the leading degree θ-functions identities that follow from the vanishing blowup equations. The prototypical example is given for the $n = 1, G = \mathfrak{su}(3), F = \mathfrak{su}(12)$ theory in (3.25). Many more non-trivial ones are scattered in sections 5, 6 and 7, and summarised in appendix B. Also it is quite notable that we can derive (4.11) general *exact* formulas of the v expansion of 5d one-instanton Nekrasov partition functions with matters. This is a generalization for the pure gauge case — the Hilbert series (4.10) of the reduced moduli spaces of one G instanton [35, 36].

To save readers' precious time, we provide a quick guide for the paper here. Readers merely interested in the elliptic genera of rank-one 6d $(1,0)$ SCFTs can directly go to appendix B, check the list and find the links to the results for individual theories. Readers interested in the structure of elliptic blowup equations can first go to the main equation (3.1) and tables 5, 6, 7, 8, 9, which contain the information for all elliptic blowup equations — both unity and vanishing — for all rank-one theories. Then, if one is also interested in higher-rank theories, one can go to section 6 for the three familiar higher-rank non-Higgsable clusters and section 7 for the gluing rules and general form of elliptic blowup equations for all $(1,0)$ SCFTs in atomic classification, especially the explicit examples in

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section 7.4 for ADE chains of -2 curves, conformal matter theories and blown-up of some $-n$ curves. Readers interested in how to solve blowup equations can go to section 4 where four different methods are provided, each with its own merits and range of application. Then in section 5, details are given on blowup equations and elliptic genera of some of the most interesting rank one theories. Readers interested in toric constructions for elliptic noncompact Calabi-Yau threefolds, exact v expansion formulas for 5d one-instanton Nekrasov partition functions, vanishing theta identities are suggested to check appendix B for a list of results.

2 6d $(1,0)$ SCFTs and Calabi-Yau geometries

This section provides the necessary background for the paper. We first give a quick review of 6d $(1,0)$ SCFTs, including in particular a discussion of anomaly cancellation, which will be useful for the formulation of elliptic blowup equations, and the atomic classification. We then proceed to describe the objects of interest, the elliptic genera of 6d SCFT, and summarise the current status of the computational results. Then we describe the semi-classical pieces of the free energy, which will be crucial for deriving the elliptic blowup equations. We finally discuss the toric construction of the non-compact Calabi-Yau geometries, whose classical data can be used to reaffirm the semi-classical free energy. Together with mirror symmetry one can compute the quantum corrected prepotential, which provides further useful initial data in order to solve all BPS invariants by the elliptic blowup equations.

2.1 Review of 6d $(1,0)$ SCFTs

A (1,0) 6d SCFT has superconformal algebra $\mathfrak{osp}(6,2|1)$, which has a bosonic sub-algebra

$$\mathfrak{osp}(6,2|1) \supset \mathfrak{so}(6,2) \times \mathfrak{sp}(1), \tag{2.1}$$

where $\mathfrak{so}(6,2)$ is the conformal algebra, and $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ the R-symmetry. Massless states are labeled by representations of the sub-algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2) \subset \mathfrak{so}(6,2)$. They can be grouped into the following three types of (1,0) supermultiplets:

- Tensor multiplets: each has an anti-self-dual tensor field B_i of spin $(1,0)$, a scalar field ϕ_i of spin $(0,0)$, and two fermions of spin $(\frac{1}{2},0)$.
- Vector multiplets: each has a vector field of spin $(\frac{1}{2},\frac{1}{2})$, and two fermions of spin $(0,\frac{1}{2})$.
- Hypermultiplets: each has four scalars of spin $(0,0)$, and two fermions of spin $(\frac{1}{2},0)$.

In addition, there are massless BPS strings sourced by the tensor fields. The charges of these strings $n=(n_i)$ can be computed by integrating the flux H_i of tensor fields

$$n_i = \int_{M_4^\perp} \mathrm{d}H_i \in \mathbb{Z}_{\geq 0} \tag{2.2}$$

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over the four dimensional hypersurface $M_4^\perp$ transverse to the worldsheet of the BPS string. The lattice of string charges Λ is equipped with a symmetric pairing

$$\langle n, n' \rangle = \sum_{i,j} A_{ij} n_i n'_j \tag{2.3}$$

analogous to the Dirac pairing in 4d electromagnetism. Dirac quantisation condition requires that A_{ij} is integral.

6d SCFT can be constructed by F-theory compactification on an elliptic Calabi-Yau threefold whose base is an orbifold singularity of the type

$$\mathcal{B}_{\text{sing}} = \mathbb{C}^2/\Gamma_{\mathfrak{u}(2)} , \tag{2.4}$$

where $\Gamma_{\mathfrak{u}(2)}$ is some discrete subgroup of $\mathfrak{u}(2)$ [27]. In order to have a better understanding of the 6d SCFT, it is beneficial to move to the tensor branch of the moduli space. This corresponds in geometry to resolving the base singularity by successive blow-ups. In a generic point of the full tensor branch, the base surface is smooth and the compact curves inside are rational curves which intersect with each other in such a way that the intersection matrix

$$A_{ij} = \Sigma_i \cap \Sigma_j \tag{2.5}$$

is negative definite. This implies that all of them can be blown down by the Theorem of Grauert [37]. In addition, every elliptic fiber over a base curve should be of Kodaira-Tate type. All these geometric conditions allow a systematic classification of 6d SCFTs [26, 27] as we will later review in section 2.1.2.[3]

After resolution of the base singularity, massless fields and tensionless strings have clear geometric origin. Tensor multiplets come from dimensional reduction of type IIB fields on compact base curves. The number of these base curves gives the number of tensor mulipltes, which is also called the rank of a 6d SCFT, while the volumes of these curves are identified with the imaginary parts of the tensor moduli. Vector multiples come from string modes on seven branes wrapping the discriminant loci Δ of elliptic fibration. The irreducible components in Δ can be both compact or non-compact. Correspondingly the associated vector fields are either dynamic or fixed as background fields, and they induce non-Abelian gauge and flavor symmetries respectively. We split $\Delta = \Delta_c \cup \Delta_n$, where $\Delta_{c,n}$ are the unions of compact and non-compact components respectively. There could also be Abelian flavor symmetries, which are not localised but are rather associated to additional sections of the elliptic fibration [44, 45]. Furthermore, charged hypermultiplets are localised at intersection loci of two base curves, at least one of which is a compact curve in Δ. They come from the zero modes of strings stretched between the seven branes wrapping the two base curves. Finally, D3 branes of type IIB can wrap compact base curves and give rise to BPS strings. It is clear that the pairing of strings should be identified with intersection matrix of compact base curves (2.5). Furthermore the tension of strings is proportional to the volumes of base curves, i.e. the tensor moduli. The BPS strings thus become tensionless precisely at the origin of the tensor branch where all compact base curves are blown down.

[3]A handful of 6d SCFTs with the so-called "frozen singularities" do not have valid geometric construction [38–43]. We will treat these special 6d SCFTs with our methods in a later work.

2.1.1 Gauge anomalies

From the field theory point of view, a 6d SCFT in its tensor branch is simply a weakly coupled 6d gauge theory, whose vector multiplets are coupled to tensor multiplets by the following kinetic terms in the Lagrangian

$$\mathcal{L} \supset \phi_i \,\mathrm{tr}\, F_i^{\mu\nu} F_{i,\mu\nu}, \tag{2.6}$$

where the tensor moduli ϕ_i serve as inverse gauge couplings. It is clear that this 6d field theory must be anomaly free. Symmetry anomalies of a 6d field theory are encoded in a closed and gauge invariant eight form I_{tot}. We distinguish two different scenarios. With the absence of a current from the BPS strings, I_{tot} can be written as

$$I_{\text{tot}} = I_{\text{1-loop}} + I_{\text{GS}} \tag{2.7}$$

where $I_{\text{1-loop}}$ are 1-loop contributions from massless fields, while I_{GS} are contributions from Green-Schwarz counter-terms [46–48]. The latter can be written as

$$I_{\text{GS}} = \frac{1}{2}\sum_{i,j}(A^{-1})_{ij} X_i X_j. \tag{2.8}$$

The four-forms X_i read [46–48]

$$X_i = \frac{1}{4} a_i p_1(M_6) + b_i c_2(I) + \sum_{k'} b_{i,k'} c_2(\mathfrak{g}_{k'}) + \frac{1}{2}\sum_{m,n} b_{i,mn} c_1(\mathfrak{u}(1)_m) c_1(\mathfrak{u}(1)_n). \tag{2.9}$$

Here $p_1(M_6)$ is the first Pontryagin class of the tangent bundle of the six dimensional spacetime, $c_2(I)$ and $c_2(\mathfrak{g}_{k'})$ are the second Chern classes of the bundle of the $\mathfrak{su}(2)$ R-symmetry, and the bundles of non-Abelian gauge or flavor symmetries respectively. We also include in the last term the contributions from the first Chern classes of the flavor $\mathfrak{u}(1)$ bundles. The anomaly coefficients actually have a beautiful geometric meanings [46, 49, 50] and determine the index of the elliptic genus [4, 31]. The four-form X_i is related to the compact base curve Σ_i, b_i is conjectured to be minus the dual Coxeter number $h^\vee_{\mathfrak{g}_i}$ of the symmetry algebra supported on Σ_i [51],[4] and we can interpret the coefficients $a_i, b_{i,k'}$ as

$$a_i = \Sigma_i \cap (-K), \quad b_{i,k'} = \Sigma_i \cap \Sigma_{k'} =: A_{ik'}, \quad k' \in \Delta = \Delta_c \cup \Delta_n. \tag{2.10}$$

Here $-K$ is the anti-canonical divisor of the base. Since the base surface is non-compact, we would like to interpret $-K$ as the Poincaré dual of the cohomology class, which is locally meaningful. In practice, the neighborhood in the base of the curve Σ_i with self-intersection number $-n := A_{ii}$ can be locally replaced by the Hirzebruch surface $\mathbb{F}_n$, so that we find readily $a_i = 2 - n = 2 + A_{ii}$. $\Sigma_{k'}$ is an irreducible component of the discriminant $\Delta = \Delta_c \cup \Delta_n$ which supports the symmetry algebra $\mathfrak{g}_{k'}$. In the same formula we have extended the definition of $A_{ik'}$ to accommodate possible intersection with non-compact base curves. Finally $b_{i,mn}$ can also be interpreted as intersection numbers of the

[4] If Σ_i is not part of the discriminant and there is no non-trivial symmetry algebra, $h^\vee_{\mathfrak{g}_i}$ is replaced by 1.

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vertical divisor pulled back from Σ_i with the additional sections that induce the Abelian symmetries [31].

Every term concerning gauge symmetry in I_{tot} must be canceled. This includes not only pure gauge anomaly, but also mixed gauge-flavor and mixed gauge-gravity anomalies in order to preserve superconformal invariance [52, 53]. Let us define the fiducial trace $\operatorname{tr} = \frac{1}{2\,\mathrm{ind}_\square}\operatorname{tr}_\square$, where $\square$ is the defining representation, so that

$$\operatorname{tr} F^2 = \frac{1}{2h^\vee_{\mathfrak{g}}}\operatorname{tr}_{\mathfrak{adj}} F^2, \tag{2.11}$$

and the following Lie algebraic constants[5]

$$\operatorname{tr}_R F^2 = 2\,\mathrm{ind}_R \operatorname{tr} F^2, \quad \operatorname{tr}_R F^4 = x_R \operatorname{tr} F^4 + y_R(\operatorname{tr} F^2)^2, \quad \operatorname{tr}_R F^3 = z_R \operatorname{tr} F^3. \tag{2.13}$$

These constants for common representations of simple Lie algebras in our convention can be found in [1, 49, 50]. We then have the following anomaly cancellation conditions [46–48]:

- Mixed gauge-gravity anomaly cancellation:

$$\mathrm{ind}_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i}\,\mathrm{ind}_{R_i} = -3(A_{ii}+2). \tag{2.14}$$

- Pure gauge anomaly cancellation:

$$x_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i} x_{R_i} = 0, \tag{2.15}$$

$$y_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i} y_{R_i} = -3A_{ii}, \tag{2.16}$$

- Mixed gauge-gauge anomaly cancellation:[6]

$$\sum_{R_i,R_j} n_{R_i,R_j}\,\mathrm{ind}_{R_i}\,\mathrm{ind}_{R_j} = \frac{1}{4}. \tag{2.17}$$

- Mixed gauge-flavor anomaly cancellation:

$$\sum_{R_i,R_{\ell'}} n_{R_i,R_{\ell'}}\,\mathrm{ind}_{R_i}\,\mathrm{ind}_{R_{\ell'}} = \frac{1}{4}b_{i,\ell'}\,, \tag{2.18}$$

$$\sum_{R_i,q_m,q_n} n_{R_i,q_m,q_n} q_m q_n\,\mathrm{ind}_{R_i} = \frac{1}{2}b_{i,mn}\,, \tag{2.19}$$

$$\sum_{R_i,q_m} n_{R_i,q_m} q_m z_{R_i} = 0. \tag{2.20}$$

[5]The second definition implies

$$\operatorname{tr}_R F_1F_2F_3F_4 = B_R \operatorname{tr} F_1F_2F_3F_4 + \frac{1}{3}C_R\left(\operatorname{tr} F_1F_2 \operatorname{tr} F_3F_4 + \operatorname{tr} F_1F_3 \operatorname{tr} F_2F_4 + \operatorname{tr} F_1F_4 \operatorname{tr} F_2F_4\right). \tag{2.12}$$

[6]Implicit in this condition is that two irreducible components of Δ_c can at most intersect once transversely, which is required for a 6d SCFT, but not generally true for a 6d field theory.

Here i,j label gauge symmetries, and ℓ' a non-Abelian flavor symmetry. n_{R_i}, n_{R_i,R_j}, $n_{R_i,R_{\ell'}}$, n_{R_i,q_m}, n_{R_i,q_m,q_n} are the numbers of charged hypermultiplets respectively transforming in symmetry representations with $\mathfrak{u}(1)$ charges q_m, q_n.

On the other hand, if the current of BPS strings is present, it induces additional contribution to I_{tot}, which must be canceled by the anomaly on the world-sheet theory of BPS strings through the anomaly inflow mechanism [54, 55] (see [1] for a nice summary). This determines the 't Hooft anomaly four-form I_4 on the worldsheet theory wrapping the base curve $S = \sum_i d_i \Sigma_i$ as[7]

$$I_4 = -\frac{1}{2}\sum_{i,j} A_{ij} d_i d_j (c_2(L) - c_2(R)) \tag{2.21}$$
$$+\sum_i d_i \left(h^\vee_{\mathfrak{g}_i} c_2(I) - \frac{2+A_{ii}}{4}(p_1(T_2) - 2c_2(L) - 2c_2(R)) - \frac{1}{4} b_{i,k'} \operatorname{tr} F^2_{k'} - \frac{1}{2} b_{i,mn} \operatorname{tr} F_{\mathfrak{u}(1)_m} F_{\mathfrak{u}(1)_n} \right).$$

Here $c_2(L), c_2(R)$ refer to the second Chern classes of the bundles associated to the global $\mathfrak{su}(2)_L, \mathfrak{su}(2)_R$ symmetry of $\mathbb{R}^4$ perpendicular to the string worldsheet M_2 in 6d, and we have used the identity

$$p_1(M_6) = p_1(M_2) - 2c_2(L) - 2c_2(R). \tag{2.22}$$

$F_{k'}$ are the field strength of non-Abelian symmetries and we sum over both gauge and flavor symmetries, while $F_{\mathfrak{u}(1)_m}, F_{\mathfrak{u}(1)_n}$ are the field strength of Abelian flavor symmetries. In the case of flavor symmetries, the coefficients $b_{i,k'}$ and $b_{i,mn}$ are interpreted as the levels[8] of the associated current algebras [1], and are sometimes denoted as k_F.

2.1.2 Classification

The geometric constraints on the elliptic Calabi-Yau threefold associated to 6d SCFTs in the tensor branch discussed in the beginning of this section, as well as the anomaly cancellation conditions discussed previously, allow for a possible classification of 6d SCFTs. This program was systematically carried out in [26, 27][9] (see also [43, 56]), and we quickly review the salient points here.

The classification scheme is divided into two steps. In the first step, all possible configurations of compact base curves are classified. There are three types of basic configurations called "atoms"

- A single -1 curve, i.e. a single rational curve (a $\mathbb{P}^1$) with self-intersection -1.
- Configuration of -2 curves intersecting according to appropriate ADE Dynkin diagrams.[10]

[7] We use the normalisation that $c_2(\mathfrak{g}) = \frac{1}{4}\operatorname{tr} F^2$ and $c_1(\mathfrak{u}(1)) = \operatorname{tr} F_{\mathfrak{u}(1)}$.

[8] The level of $\mathfrak{u}(1)$ is interpreted as the radius of the compact boson that realises the current algebra squared [1].

[9] As mentioned before, this program misses a handful of 6d SCFTs with "frozen singularities", which will not be discussed in this paper.

[10] This class includes a single -2 curve.

-n curves: n	3	4	5	6	7	8	12	3, 2	2, 3, 2	3, 2, 2
algebra	$\mathfrak{su}(3)$	$\mathfrak{so}(8)$	F_4	E_6	E_7	E_7	E_8	$G_2 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$G_2 \oplus \mathfrak{su}(2) \oplus \emptyset$
hypers	$---$	$---$	$---$	$---$	$\frac{1}{2}\mathbf{56}$	$---$	$---$	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1}) \oplus \frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2},\mathbf{1})$

Table 1. All possible non-Higgsible clusters with minus the self-intersection numbers n of curves, the symmetry algebras of the minimal singularities of elliptic fibers, and possible charged hypers.

- Non-Higgsible clusters [57], which include: a single $-n$ curve, i.e. a rational curve with self-intersection $-n$ with $n = 3, \ldots, 8, 12$,[11] and three higher rank cases.

The -1 curve in the first category comes equipped with an E_8 flavor symmetry. The chain of -2 curves of type A in the second category always has an overall $\mathfrak{u}(1)$ flavor symmetry.[12] The non-Higgsible clusters in the last category distinguish themselves in that elliptic fibers over them have minimal non-trivial singularity (hence the name non-Higgsible), and they are tabulated in table 1. A larger configuration of base curves $\{\Sigma_i\}$ is then built by gluing the last two categories of "atomic" configurations using -1 curves subject to certain constraints,[13] the most important of which is the gluing condition that the minimal algebras $\mathfrak{g}_L, \mathfrak{g}_R$ carried by two curves glued by a -1 curve must satisfy $\mathfrak{g}_L \times \mathfrak{g}_R \subset E_8$.

All such configurations can be classified, and they generally fit into the pattern of a generalised quiver [26]. A "node" in such a quiver is a $-n$ curve with $n = 4, 6, 7, 8, 9$ which supports a minimal symmetry algebra of D- or E-type. A "link" is an appropriate configuration of curves which do not involve any nodes. All possible links are listed in [26]. The simplest links are called the minimal conformal matters, some examples of which are listed below

$$[\mathfrak{so}(8)]\ 1\ [\mathfrak{so}(8)] \tag{2.23}$$

$$[E_6]\ 1, 3, 1\ [E_6] \tag{2.24}$$

$$[E_7]\ 1, 2, 3, 2, 1\ [E_7] \tag{2.25}$$

$$[E_8]\ 1, 2, 2, 3, 1, 5, 1, 3, 2, 2, 1\ [E_8] \tag{2.26}$$

Here the symmetry algebras wrapped in square brackets are flavor symmetries, and they are also the symmetry algebras carried by the nodes that can be connected to the links, while the chains of integers n in the middle represent intersecting $-n$-curves. These configurations are so named because they come from resolving the singularity at the intersection of two seven branes that carry D- or E-type symmetry algebras, similar to conventional matters which can be found at the intersection of A-type seven branes. They can also be realised in M-theory as a M5 brane probing D- or E-type singularity $\mathbb{C}^2/\Gamma_{DE}$. A complete list of minimal conformal matters can be found in [58]. Some of the more complicated link configurations can be obtained by joining two minimal conformal matters and gauging the common flavor symmetry, or by performing Higgs branch RG flow [58, 59].

[11]For a single $\mathbb{P}^1$ with self-intersection $-9, -10, -11$, the elliptic fiber is not of Kodaira-Tate type, and additional blow-ups are required.

[12]This overall flavor symmetry is enhanced to $\mathfrak{su}(2)$ when the fibers over the -2 curves are not very singular.

[13]The others are i) three curves cannot intersect in a point, ii) two curves cannot intersect tangentially, iii) intersection graphs contain no loops, iv) -1 curves can intersect at most two other curves, v) two -1 curves Σ, Σ' have $\Sigma \cdot \Sigma' = 0$.

The second step of classification is to assign suitable singular fibers so that the total space of fibration is a Calabi-Yau threefold. In particular, one has to make sure that every elliptic fiber is of the Kodaira-Tate type, which is in general equivalent to the condition of gauge anomaly cancellation discussed in section 2.1.1. This step can also be done in two parts. The first part involves the classification of singular fibers over a single base curve or equivalently the associated symmetry algebra,[14] i.e. the classification of rank one 6d SCFTs. The minimal symmetry algebras have been given in table 1, and they can be enhanced by making worse the singularity of elliptic fibers. At the same time the numbers of charged hypermultiplets increase. Their numbers as well as the representations of symmetry algebras under which they transform are completely determined by the anomaly cancellation conditions (2.14), (2.15), (2.16).

If there are multiple hypermultiplets in the same gauge representation R, they support a non-trivial flavor symmetry F. The type of the flavor symmetry is determined by the number m of hypermultiplets and the nature of R (see for instance [61]). If R is complex, the hypermultiplets transform in the representation $\mathbf{m}$ of the flavor symmetry $\mathfrak{su}(m)$. To be more precise, each hypermultiplet can be regarded as consisting of two half-hypers, and the $2m$ half-hypers transform in the gauge-flavor bi-representation $(R, \overline{\mathbf{m}}) \oplus (\overline{R}, \mathbf{m})$. If R is real, the flavor symmetry is enhanced to the quaternionic representation of $\mathfrak{sp}(m)$;[15] if R is pseudo-real, the flavor symmetry is enhanced to the vector representation of $\mathfrak{so}(2m)$. In the last case, half-hypers can exist by themselves and thus the number m of hypermultipets can be half integers. The flavor symmetry determined in this field theoretic way is expected to hold at the superconformal fixed point as well, except for some rare cases. The eight half-hypers of the $n = 2, G = \mathfrak{su}(2)$ theory were found to be in the spinor representation of $\mathfrak{so}(7)$ instead of the vector representation of $\mathfrak{so}(8)$ [62]. Furthermore, for a handful of $n = 1, 2$ theories, i.e. $n = 2, G = \mathfrak{so}(11)$, $n = 1, G = \mathfrak{so}(11)$, and $n = 1, G = \mathfrak{so}(12)_b$, the flavor symmetries deduced by the field theoretic method do not seem to yield a consistent current algebra on the worldsheet of BPS string [1].

We also comment that a rank one 6d SCFT may also have Abelian flavor symmetry [59], which can be uncovered by either subjecting the candidate Abelian symmetry that accompanies complex representations to the test of the ABJ anomaly cancellation condition (2.20) [45], or by studying the current algebras on the worldsheet theory of BPS strings [1]. These Abelian flavor symmetries are interpreted as the weak coupling limit of Abelian gauge symmetries in supergravity when gravity is turned off [44]. Finally, once the flavor symmetry is known, the associated anomaly coefficients $b_{i,k'}, b_{i,mn}$ can be computed by (2.18), (2.19). With these caveats taken into account, all possible gauge symmetries and flavor symmetries of rank one 6d SCFTs are given in [1, 26], and we reproduce it in tables 2, 3.

The second part of fiber classification is to consider mixed representation of two gauge algebras, which are further constrained by the anomaly cancellation condition (2.17). There are only five possibilities [63]

[14] Some symmetry algebras could be realised by different singular fibers in the tensor branch [60]. Nevertheless it was argued [59] that at the origin of tensor branch they correspond to the same SCFT.

[15] We use the convention that the Lie group $\mathfrak{sp}(m)$ has rank m.

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n	G	F	(R_G, R_F)
12	E_8	$-$	$-$
8	E_7	$-$	$-$
7	E_7	$-$	$(\mathbf{56}, \mathbf{1})$
6	E_6	$-$	$-$
6	E_7	$\mathfrak{so}(2)_{12}$	$(\mathbf{56}, \mathbf{2})$
5	F_4	$-$	$-$
5	E_6	$\mathfrak{u}(1)_6$	$\mathbf{27}_{-1} \oplus c.c.$
5	E_7	$\mathfrak{so}(3)_{12}$	$(\mathbf{56}, \mathbf{3})$
4	$\mathfrak{so}(8)$	$-$	$-$
4	$\mathfrak{so}(N \geq 9)$	$\mathfrak{sp}(N-8)_1$	$(\mathbf{N}, \mathbf{2}(\mathbf{N}-\mathbf{8}))$
4	F_4	$\mathfrak{sp}(1)_3$	$(\mathbf{26}, \mathbf{2})$
4	E_6	$\mathfrak{su}(2)_6 \times \mathfrak{u}(1)_{12}$	$(\mathbf{27}, \overline{\mathbf{2}})_{-1} \oplus c.c.$
4	E_7	$\mathfrak{so}(4)_{12}$	$(\mathbf{56}, \mathbf{2} \oplus \mathbf{2})$
3	$\mathfrak{su}(3)$	$-$	$-$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	$(\mathbf{8}, \mathbf{4})$
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(1)_1^b \times \mathfrak{sp}(1)_1^c$	$(\mathbf{8}_v \oplus \mathbf{8}_c \oplus \mathbf{8}_s, \mathbf{2})$
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(1)_2^b$	$(\mathbf{9}, \mathbf{4}^a) \oplus (\mathbf{16}, \mathbf{2}^b)$
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a \times (\mathfrak{su}(1)_4 \times \mathfrak{u}(1)_4)^b$	$(\mathbf{10}, \mathbf{6}^a) \oplus [(\mathbf{16}_s)_1^b \oplus c.c.]$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a \times \mathrm{Ising}^b$	$(\mathbf{11}, \mathbf{8}^a) \oplus (\mathbf{32}, \mathbf{1}_s^b)$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	$(\mathbf{12}, \mathbf{10}) \oplus (\mathbf{32}_s, \mathbf{1})$
3	G_2	$\mathfrak{sp}(1)_1$	$(\mathbf{7}, \mathbf{2})$
3	F_4	$\mathfrak{sp}(2)_3$	$(\mathbf{26}, \mathbf{4})$
3	E_6	$\mathfrak{su}(3)_6 \times \mathfrak{u}(1)_{18}$	$(\mathbf{27}, \overline{\mathbf{3}})_{-1} \oplus c.c.$
3	E_7	$\mathfrak{so}(5)_{12}$	$(\mathbf{56}, \mathbf{5})$

Table 2. Gauge, flavor symmetries and charged matter contents of rank one 6d SCFTs with $n \geq 3$ [1]. The subscript in a flavor symmetry algebra indicates the level of the associated current algebra. When a flavor symmetry has multiple simple components, superscripts are used to distinguish them and their representations. Matters are presented as the gauge and flavor representations by which the half-hypermultiplets transform. If there is an Abelian flavor symmetry, the Abelian charge is given as subscript.

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n	G	F	(R_G, R_F)
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	–
2	$\mathfrak{su}(2)$	$\mathfrak{so}(7)_1 \times \text{Ising}$	$(\mathbf{2}, \mathbf{8}_s \times \mathbf{1}_s)$
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$(\mathbf{N}, \overline{\mathbf{2N}}) \oplus c.c.$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	$(\mathbf{7}, \mathbf{2}^a) \oplus (\mathbf{8}, \mathbf{8}^b)$
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	$(\mathbf{8}_v, \mathbf{4}^a) \oplus (\mathbf{8}_s, \mathbf{4}^b) \oplus (\mathbf{8}_c, \mathbf{4}^c)$
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	$(\mathbf{9}, \mathbf{6}^a) \oplus (\mathbf{16}, \mathbf{4}^b)$
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	$(\mathbf{10}, \mathbf{8}^a) \oplus [(\mathbf{16}_s, \mathbf{2}^b)_1 \oplus c.c.]$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times ?^b$	$(\mathbf{11}, \mathbf{10}^a) \oplus (\mathbf{32}, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{1}_s^b) \oplus (\mathbf{32}_c, \mathbf{1}_s^c)$
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	$(\mathbf{13}, \mathbf{14}) \oplus (\mathbf{64}, \mathbf{1})$
2	G_2	$\mathfrak{sp}(4)_1$	$(\mathbf{7}, \mathbf{8})$
2	F_4	$\mathfrak{sp}(3)_3$	$(\mathbf{26}, \mathbf{6})$
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	$(\mathbf{27}, \overline{\mathbf{4}})_{-1} \oplus c.c.$
2	E_7	$\mathfrak{so}(6)_{12}$	$(\mathbf{56}, \mathbf{6})$
1	$\mathfrak{sp}(0)$	$(E_8)_1$	–
1	$\mathfrak{sp}(N \geq 1)$	$\mathfrak{so}(4N+16)_1$	$(\mathbf{2N}, \mathbf{4N+16})$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	$(\mathbf{3}, \overline{\mathbf{12}})_1 \oplus c.c.$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a \times \mathfrak{su}(2)_1^b$	$[(\mathbf{4}, \overline{\mathbf{12}}_1^a) \oplus c.c.] \oplus (\mathbf{6}, \mathbf{2}^b)$
1	$\mathfrak{su}(N \geq 5)$	$\mathfrak{su}(N+8)_1 \times \mathfrak{u}(1)_{2N(N-1)(N+8)}$	$[(\mathbf{N}, \overline{\mathbf{N+8}})_{-N+4} \oplus (\mathbf{\Lambda}^2, \mathbf{1})_{N+8}] \oplus c.c.$
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	$[(\mathbf{6}, \overline{\mathbf{15}}) \oplus c.c.] \oplus (\mathbf{20}, \mathbf{1})$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(6)_1^b$	$(\mathbf{7}, \mathbf{4}^a) \oplus (\mathbf{8}, \mathbf{12}^b)$
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(3)_1^b \times \mathfrak{sp}(3)_1^c$	$(\mathbf{8}_v, \mathbf{6}^a) \oplus (\mathbf{8}_s, \mathbf{6}^b) \oplus (\mathbf{8}_c, \mathbf{6}^c)$
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a \times \mathfrak{sp}(3)_2^b$	$(\mathbf{9}, \mathbf{8}^a) \oplus (\mathbf{16}, \mathbf{6}^b)$
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a \times (\mathfrak{su}(3)_4 \times \mathfrak{u}(1)_{12})^b$	$(\mathbf{10}, \mathbf{10}^a) \oplus [(\mathbf{16}_s, \mathbf{3}^b)_1 \oplus c.c.]$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a \times ?^b$	$(\mathbf{11}, \mathbf{12}^a) \oplus (\mathbf{32}, \mathbf{3}^b)$
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a \times \mathfrak{so}(3)_8^b$	$(\mathbf{12}, \mathbf{14}^a) \oplus (\mathbf{32}_s, \mathbf{3}^b)$
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a \times ?^b \times ?^c$	$(\mathbf{12}, \mathbf{14}^a) \oplus (\mathbf{32}_s, \mathbf{2}^b) \oplus (\mathbf{32}_c, \mathbf{1}^c)$
1	G_2	$\mathfrak{sp}(7)_1$	$(\mathbf{7}, \mathbf{14})$
1	F_4	$\mathfrak{sp}(4)_3$	$(\mathbf{26}, \mathbf{8})$
1	E_6	$\mathfrak{su}(5)_6 \times \mathfrak{u}(1)_{30}$	$(\mathbf{27}, \overline{\mathbf{5}})_{-1} \oplus c.c.$
1	E_7	$\mathfrak{so}(7)_{12}$	$(\mathbf{56}, \mathbf{7})$

Table 3. Gauge, flavor symmetries and charged matter contents of rank one 6d SCFTs with $n = 1, 2$ [1]. Λ^2 is the rank-two anti-symmetric representation. ? means the flavor symmetry predicted by field theoretic considerations cannot be realised consistently on the worldsheet of BPS strings [1].

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- $\mathfrak{g}_a = \mathfrak{su}(n_a), \mathfrak{g}_b = \mathfrak{su}(n_b), R = (\mathbf{n}_a, \mathbf{n}_b)$
- $\mathfrak{g}_a = \mathfrak{su}(n_a), \mathfrak{g}_b = \mathfrak{sp}(n_b), R = (\mathbf{n}_a, \mathbf{2n}_b)$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = \mathfrak{so}(n_b), R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{n}_b)$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = \mathfrak{so}(n_b), n_b = 7, 8, R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{8}_{s,c})$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = G_2, R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{7})$

At the end of this subsection, we comment that 6d SCFTs in this classification could be related to each other by RG flows. There are two types of RG flows, the tensor branch and the Higgs branch flows. The former simply corresponds to blowing up or blowing down base curves, while the latter are related to complex structure deformation and they do not change the rank of 6d SCFTs. RG flows of 6d SCFTs have been intensively studied in [58, 59, 64, 65]. In this paper, we will mainly be interested in RG flows of rank one 6d SCFTs. All rank one 6d SCFTs with the same n are connected to each other by Higgs branch RG flows, which are summarised in section 2.4 of [1].

2.2 Elliptic genera

We are interested in the partition function of 6d SCFT on the tensor branch on the 6d Ω background.[16] The latter is a curved spacetime background, which is topologically $T^2 \times \mathbb{R}^4$ with the metric [69]

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} + (\mathrm{d}x^\mu + \Omega^\mu \mathrm{d}z + \bar{\Omega}^\mu \mathrm{d}\bar{z})^2, \quad \mu = 1,2,3,4 \tag{2.27}$$

where $z, \bar{z}$ are coordinates on T^2 and x^μ coordinates on $\mathbb{R}^4$. The Ω^μ satisfy

$$\mathrm{d}\Omega = \epsilon_1\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 - \epsilon_2\, \mathrm{d}x^3 \wedge \mathrm{d}x^4, \tag{2.28}$$

and $\epsilon_{L,R} = (\epsilon_1 \mp \epsilon_2)/2$ are the background field strengths for the spacetime symmetry $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ acting on $\mathbb{R}^4$. The compactification on T^2 allows access to the BPS states on BPS strings, encoded in the Ramond-Ramond elliptic genera, which are the generalised Witten index on the worldsheet theory of BPS strings. The BPS strings wrapped on T^2 would appear as instantons on $\mathbb{R}^4$, and the curvature on $\mathbb{R}^4$ serves as the IR regulator analogous to the 4d Ω background [70]. The partition function of 6d SCFT is then a finite quantity and it splits as follows:

$$Z(\phi,\tau,m_{G,F},\epsilon_{1,2}) = Z^{\mathrm{cls}}(\phi,\tau,m_{G,F},\epsilon_{1,2}) Z^{\text{1-loop}}(\tau,m_{G,F},\epsilon_{1,2}) \left(1 + \sum_d \mathrm{e}^{\mathrm{i}2\pi\phi\cdot d}\mathbb{E}_d(\tau,m_{G,F},\epsilon_{1,2})\right). \tag{2.29}$$

Here Z^{cls}, $Z^{\text{1-loop}}$ are semi-classical contributions, and one-loop contributions from tensor, vector and hypermultiplets respectively. $\mathbb{E}_d$ is the RR elliptic genus of the BPS strings

[16] A 6d SCFT directly compactified on a S^1 is also known as a 5d KK or marginal theory [66, 67]. One can also consider a twisted circle compactification of a 6d SCFT by modding out a discrete symmetry on the string charge lattice Λ or/and the affinised fibral symmetry algebra [68]. This is also one way of realising the "frozen singularity" [40, 41, 68]. We will consider these constructions in a later work.

with string charge $d = (d_i) \in \Lambda$ associated to the base curve $S = \sum_i d_i \Sigma_i$. $\phi = (\phi_i), \tau$ are respectively the tensor moduli and the complex structure of T^2. We have turned on the vevs $m_{G,F}$ of Wilson loops of gauge and flavor vector fields along 1-cycles in T^2, also called the gauge and flavor fugacities. They take value in the complexified Cartan subalgebra of the corresponding symmetry algebra, where a Weyl invariant bilinear form $(\bullet,\bullet)$ is defined. See appendix A for our Lie algebraic convention. We will also use the notation of the *reduced* d-string elliptic genus:

$$\mathbb{E}_d^{\mathrm{red}}(\tau, m_{G,F}, \epsilon_1, \epsilon_2) = \mathbb{E}_d(\tau, m_{G,F}, \epsilon_1, \epsilon_2)/\mathbb{E}_{c.m.}(\tau, \epsilon_1, \epsilon_2), \tag{2.30}$$

where the contribution from the center of mass free hypermultiplet

$$\mathbb{E}_{c.m.}(\tau, \epsilon_1, \epsilon_2) = \frac{\eta(\tau)^2}{\theta_1(\tau, \epsilon_1)\theta_1(\tau, \epsilon_2)} \tag{2.31}$$

is factored out [33]. This brings certain simplification for elliptic genera especially for the one-string case.

The 6d Ω background has the additional advantage of allowing for connection with topological string theory. F-theory compactified on an elliptic Calabi-Yau threefold X and T^2 is dual to M-theory compactified on the same threefold X and the M-theory circle S^1, where the volume of elliptic fiber in X is inversely proportional to the volume of T^2. Turning on Wilson loops of gauge and flavor vector fields amounts to resolving singular elliptic fibers so that the threefold X is smooth. M-theory BPS states are computed in this setup by topological string theory which encodes in particular the numbers of BPS states of M2 branes wrapping 2-cycles in X. Note that the elliptic genera contain BPS states which wrap the base curves non-trivially. One can therefore use topological string theory techniques to get information about the $\mathbb{E}_d$ and in particular initial data for the recursive blow up equations.

One important property of the elliptic genera is how they transform under the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$. Thanks to the non-trivial 't Hooft anomalies, the elliptic genera are not invariant, but transform as meromorphic Jacobi forms of weight zero but non-trivial index, where both the gauge/flavor fugacities and the parameters of the Ω background transform as elliptic parameters.

$$\mathbb{E}_d\left(\frac{a\tau+b}{c\tau+d}, \frac{m_{G,F}}{c\tau+d}, \frac{\epsilon_{1,2}}{c\tau+d}\right) = \mathrm{e}^{\frac{c}{c\tau+d}\,\mathrm{Ind}\,\mathbb{E}_d(m_{G,F},\epsilon_{1,2})}\mathbb{E}_d(\tau, m_{G,F}, \epsilon_{1,2}). \tag{2.32}$$

Here $\mathrm{Ind}\,\mathbb{E}_d(m_{G,F},\epsilon_{1,2})$, called the modular index polynomial, is a quadratic polynomial. The index polynomial can be given by an equivariant integral of the 't Hooft anomaly four-form [32], and it boils down to the following replacement rules [5, 33]

$$p_1(M_2)\to 0, \quad c_2(L)\to -\epsilon_L^2, \quad c_2(R), c_2(I)\to -\epsilon_R^2, \quad \mathrm{tr}F_{k'}^2\to -2(m_{k'},m_{k'}), \quad \mathrm{tr}F_{\mathfrak{u}(1)}\to i m_{\mathfrak{u}(1)}. \tag{2.33}$$

Applying these rules on (2.21) yields the following modular index polynomial

$$\begin{aligned}\mathrm{Ind}\,\mathbb{E}_d(m_{G,F},\epsilon_{1,2}) = &-\frac{1}{4}(\epsilon_1+\epsilon_2)^2\sum_i(2+A_{ii}+h^\vee_{\mathfrak{g}_i})d_i + \frac{1}{2}\epsilon_1\epsilon_2\Big(\sum_i(2+A_{ii})d_i - \sum_{i,j}A_{ij}d_id_j\Big)\\ &+\frac{1}{2}\sum_{i,k'}b_{i,k'}d_i(m_{k'},m_{k'}) + \frac{1}{2}\sum_{i,\ell,n}b_{i,\ell n}m_\ell m_n.\end{aligned} \tag{2.34}$$

2.2.1 Known computational methods

In this section, we summarize all known results on the elliptic genera of 6d $(1,0)$ SCFTs, in particular all rank one theories. For the minimal SCFTs which are the pure gauge rank one theories, a thorough summary on the results from all kinds of approaches has been presented in the introduction of [9]. Here we focus more on theories with matters. Three methods with relatively wide range of application are 2d quiver gauge theories, modular ansatz and refined topological vertex. In the following, we briefly introduce each method, list the theories it can solve and comment on its advantages and disadvantages.

In the spirit of the ADHM construction for 4d/5d instantons, certain 6d $(1,0)$ SCFTs are known to correspond to 2d quiver gauge theories. Once the 2d quiver construction is found, one can use localization — Jeffrey-Kirwan residue to exactly compute the elliptic genera to arbitrary number of strings. However, like in the ADHM construction, such correspondence normally just exists for classical gauge groups, but difficult to generalize to exceptional gauge groups. All rank one $(1,0)$ theories with known 2d quiver construction are listed below:

- $n=1$, $G=\mathfrak{sp}(N)$ [11, 12]
- $n=1$, $G=\mathfrak{su}(N)$ [11]
- $n=2$, $G=\mathfrak{su}(N)$ [10]
- $n=3$, $G=\mathfrak{su}(3), G_2$ and $\mathfrak{so}(7)$ [14]
- $n=4$, $G=\mathfrak{so}(8+N)$ [1, 2]

For all these theories, we use the known elliptic genera from quiver formulas to check against our elliptic blowup equations and find perfect agreement.

The modular ansatz method exploits the Jacobi-form transformations of the elliptic genera as well as their pole structures and can be very constraining sometimes. For the reduced one string elliptic genus with all gauge and flavor fugacities turned off, the modular ansatz has a particularly simple form and was extensively studied in [1]. For example, using the constraints from the spectral flow relation between RR and NSR elliptic genus, such ansatze were fixed in [1] for all rank-one theories except for

- $n=1,2,3,4$, $G=E_7$
- $n=1,2$, $G=E_6$, $\mathfrak{so}(11)$ and $\mathfrak{so}(12)_b$

These results provide an excellent testing ground for our blowup equations. Indeed, for all the theories we have studied where the modular ansatz is fixed in [1], we find agreement for the one-string elliptic genera. Besides, we are able to use blowup equations to further determine the modular ansatz for $n=2,4$ E_7 theories, $n=1,2$ E_6 theories and $n=2$ $\mathfrak{so}(11)$ theory and make cross checks. The modular ansatz method also extends to the situation with gauge and flavor fugacities turned on, where Weyl-invariant Jacobi forms are involved and the computation becomes much more complicated. Still, the ansatz for the one-string elliptic genus with gauge fugacities turned on for $n = 3$ $\mathfrak{su}(3)$ and $n = 4$ $\mathfrak{so}(8)$ theories was determined in [5], and for $n = 1$ $\mathfrak{sp}(1)$, $n = 2$ $\mathfrak{su}(2)$, $n = 3$ $\mathfrak{su}(3)$ and $n = 3$ G_2 theories was determined in [13]. Note the modular ansatz method works even for compact elliptically fibred Calabi-Yau manifolds [71].

The refined topological vertex and the brane-webs can also compute the elliptic genera of some 6d theories with matters. For example, the brane web construction was known for $n = 1, G = \mathfrak{sp}(N)$ theories [72], $n = 1, G = \mathfrak{su}(N)$ theories [72], $n = 1, G = \mathfrak{su}(6)_*$ theory [73], a family of $n = 2, 3$ $\mathfrak{so}(N)$ theories [15], the D-type conformal matter theories [74]. See also [75, 76]. The brane construction for theories with non $\mathfrak{su}$ type gauge symmetry or complicated matter representations typically involves orientifold 7-plane and O5-planes.

It is also worthwhile to point out some relevant 5d results. For example, the 5d Nekrasov partition functions of $n = 2, G = \mathfrak{su}(N)$ theories were well known long time ago, see [35, 77, 78]. The 5d blowup equations with matters were initially studied in [79]. Recently, the 5d unity blowup equations for all possible gauge and matter content were studied in [80]. For a lot of 5d theories, their Nekrasov partition functions can be solved from these blowup equations recursively with respect to the instanton numbers. Such blowup equations can be regarded as the 5d limit of our elliptic blowup equations in this paper. Besides, the brane web construction for 5d G_2 theories with a fundamental matter was also obtained recently in [81]. These results provide consistency checks for the elliptic genera we solved from elliptic blowup equations when taking $q \to 0$ limit.

For higher rank 6d SCFTs, the known results on elliptic genera are only for some special theories. For example, the 2d quiver constructions are known for the three higher-rank non-Higgsable clusters [14], ADE chain of (-2) curves with gauge symmetry [82], and notably (E_6, E_6) conformal matter theory [55]. The modular ansatz has been studied for higher rank E-string and M-string theories in [4]. Beside, the elliptic genera of A-type chain of (-2) curves can be computed by refined topological vertex [10] or from the viewpoint of 2d sigma model [10, 83]. The recently proposed elliptic topological vertex can also compute the partition function of these theories [84, 85].

2.3 Semiclassical free energy

We consider a 6d SCFT as the F-theory compactification on an elliptic-fibered Calabi-Yau threefold X, where the base B is a non-compact two dimensional surface. Because of the duality between M-theory and F-theory, the refined BPS spectrum is captured by refined topological string theory on such a non-compact elliptic-fibered Calabi-Yau threefold X. In order to obtain the elliptic blowup equations for a rank one 6d SCFT, we first write down the blowup equation for refined topological string on X, and then transform it to our preferred form — in terms of elliptic genera, more details can be found in appendix D. As described in [7], we only need the semiclassical pieces of the genus zero and the genus one free energies, and the one-loop contributions from BPS particles. The latter can be readily read off from the vector and hypermultiplet spectrum [7]. There are also recent results on how to compute the BPS particle spectrum from elliptic Calabi-Yau geometry [86, 87]. In this section, we consider the computation of semiclassical free energies. We focus on the case of rank one 6d SCFTs, and relegate some results on higher rank theories to the appendix C.

We start from the results of [68]. The elliptic non-compact Calabi-Yau threefold associated to a 6d SCFT on S^1 is locally the neighborhood of a union of compact surfaces

$$\mathcal{S} = \cup_{i,a}\mathcal{S}_{a,i}\,. \tag{2.35}$$

Here i is the index of base curves, and the compact surfaces $\mathcal{S}_{a,i}$ with fixed i project to the same base curve. They intersect with each other according to the affine Dynkin diagram of a Lie algebra, and we denote the divisor associated to the affine node by $\mathcal{S}_{0,i}$. The Kähler class is then parameterized by

$$J = -\sum_{i,a}\phi_{a,i}\mathcal{S}_{a,i}, \tag{2.36}$$

where the Kähler parameters $\phi_{a,i}$ measure the volumes of the divisors $\mathcal{S}_{a,i}$. The semiclassical prepotential of the 6d SCFT on S^1 is given by the triple intersection $J\cdot J\cdot J/6$ together with Chern-Simons terms coming from the tree-level circle reduction of the Green-Schwarz counter-terms. One finds that the prepotential is

$$F^{\text{cls}}_{(0,0)} = -\frac{1}{12}\left(\sum_{\alpha\in\Delta}|\alpha\cdot\phi|^3 - \sum_{f=1}^{N_f}\sum_{\omega\in R_f}|\omega\cdot\phi+m_f|^3\right) - \frac{1}{2}\sum_{i,j}\Omega_{ij}\phi_{0,i}\sum_{a,b}K^{ab}_j\phi_{a,j}\phi_{b,j}. \tag{2.37}$$

Here Ω_{ij} is the negative base intersection matrix; K^{ab}_j is the Killing form for the Lie algebra associated to the intersecting divisors over the base curve j, so that the last term can be written as $\phi_{,j}\cdot\phi_{,j}$. In the case of a rank one theory, we simply have

$$\Omega_{ii} = n, \quad \Omega_{ij} = -k_{F_j}, \quad \text{for fixed } i. \tag{2.38}$$

In order to incorporate flavor symmetry, we also introduce Kähler parameters formally associated to non-compact vertical divisors, denoted collectively by ϕ'. Then the prepotential can be written as

$$F^{\text{cls}}_{(0,0)} = -\frac{1}{12}\left(\sum_{\alpha\in\Delta}|\alpha\cdot\phi|^3 - \frac{1}{2}\sum_{\omega_{G,F}\in R_{G,F}}|\omega_G\cdot\phi+\omega_F\cdot\phi'|^3\right) - \frac{1}{2}\phi_0\left(n\phi\cdot\phi - \sum_j k_{F_j}\phi'_j\cdot\phi'_j\right). \tag{2.39}$$

This formula is still incomplete, as we are still missing terms from the intersections of the base divisor with associated Kähler parameter ϕ_B. The non-vanishing triple intersection numbers involving the base divisor $\mathcal{S}_B$ are

$$\mathcal{S}_B\cdot\mathcal{S}_B\cdot\mathcal{S}_0 = (B\cdot B)_{\mathcal{S}_0} = n-2, \quad \mathcal{S}_B\cdot\mathcal{S}_0\cdot\mathcal{S}_0 = (B\cdot B)_{\mathcal{S}_B} = -n \tag{2.40}$$

and the corresponding additional terms to prepotential are

$$\frac{n\phi_B\phi_0^2 - (n-2)\phi_B^2\phi_0}{2}. \tag{2.41}$$

Finally, we need to change bases and use the Kähler parameters measuring curve volumes instead. Using the curve-divisor intersection numbers, we find the following identification

$$t_B = -(n-2)\phi_B + n\phi_0, \quad \tau = -\phi_B, \quad m_G = \phi, \quad m_F = \phi'. \tag{2.42}$$

For later convenience, we also define[17]

$$t_{\rm ell} = t_B - \frac{n-2}{2}\tau. \tag{2.43}$$

Therefore, the final form of the semiclassical prepotential for a rank one theory with $-n$ base curve and gauge symmetry G as well as flavor symmetry $F = \otimes_j F_j$ is

$$\begin{aligned} F^{\rm cls}_{(0,0)} = &-\frac{1}{6}\sum_{\alpha\in\Delta^+}(\alpha\cdot m_G)^3 + \frac{1}{12}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F)^3 \\ &+ \frac{t_{\rm ell}-(n-2)\frac{\tau}{2}}{2n}(-n m_G\cdot m_G + k_F m_F\cdot m_F) - \frac{1}{2n}t_{\rm ell}^2\tau + \mathcal{O}(\tau^3). \end{aligned} \tag{2.44}$$

where we have defined

$$R^{m+}_{G,F} = \{\omega_G\in R_G, \omega_F\in R_F\,;\, \omega_G\cdot m_G+\omega_F\cdot m_F\geq 0\}. \tag{2.45}$$

We ignore all terms only in m_G, m_F, τ, as they depend on the embedding of the associated Calabi-Yau in a compact geometry and thus are not inherent properties of the 6d SCFT. In addition, we can also fix the semiclassical pieces of genus one free energies from 5d results [70, 80, 88], as well as modularity of elliptic blowup equations as discussed in section 3.3. We find

$$F^{\rm cls}_{(0,1)} = -\frac{1}{12}\sum_{\alpha\in\Delta^+}\alpha\cdot m_G + \frac{1}{24}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F) + \frac{n-2}{2n}t_{\rm ell}, \tag{2.46}$$

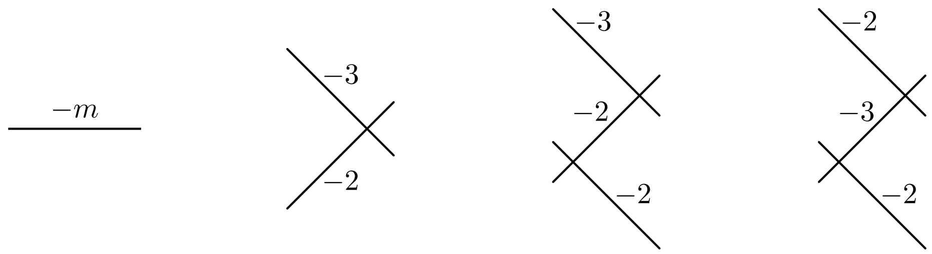
$$F^{\rm cls}_{(1,0)} = \frac{1}{12}\sum_{\alpha\in\Delta^+}\alpha\cdot m_G + \frac{1}{48}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F) + \frac{n-2-h^\vee_G}{4n}t_{\rm ell}. \tag{2.47}$$

Note here all the summations over roots and weights only sum over half sets of them, and the one loop contributions of BPS particles have to have the same half sets of them. The choices of the half weights do not have effects on our final result, since they are the same under analytic continuation. In the language of geometries, different choices of half weights reflect different choices of Calabi-Yau phases, and they are connected by flop transitions.

2.4 Calabi-Yau construction

In this section, we construct the elliptic non-compact Calabi-Yau three-folds directly. Our basic strategy is to first construct a smooth toric base which has the correct intersection numbers, and then add elliptic fibers. We embed the whole geometry into a four dimensional non-compact toric variety, then the non-compact Calabi-Yau three-fold is a hypersurface inside the toric variety, described by a 4d polytope. The mirror construction for compact Calabi-Yau hypersurfaces requires that the polytope is reflexive, and thus necessarily has an unique inner lattice point [89]. However, for our direct construction of non-compact Calabi-Yau threefolds, the polytope does not have any inner lattice point,

[17] t_B is more natural in the description of the geometries, we will always use t_B in the discussion of the geometries.



(a) $\mathfrak{su}(3), \mathfrak{so}(8), F_4$ E_6, E_7, E_8 (b) $G_2 \oplus \mathfrak{su}(2)$ (c) $G_2 \oplus \mathfrak{su}(2)$ (d) $\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$

Figure 1. Intersection numbers of bases of non-Higgsible clusters.

and it can not be reflexive. Here, we relax the reflexive condition by requiring that the dual polytope is still a lattice polytope, and have the same origin point, but it is not necessarily bounded. In fact, the dual polytope always has infinitely many points.

We claim here the mirror construction for compact Calabi-Yau hypersurfaces in [90] still works in this setting. There should be always a compact geometry[18] to start with, which is a hypersurface or a complete intersection Calabi-Yau. After taking the volume of some curves to infinity, we get our non-compact Calabi-Yau. A proper minimal combination of the Mori cone generators of the compact Calabi-Yau gives us the Mori cone of the non-compact geometry. From the perspective of lattice polytopes, this limit is equivalent to removing some lattice points, after which the simplices separated by these lattice points merge to give a triangulation of the new polytope. Alternatively if we start from a non-compact toric variety directly, we can simply triangulate it, and get all the Mori cone generators of the ambient space. We then use the method in [91] to find the Mori cone of the hypersurface. Note that such a naive construction sometimes does not give the correct phase that a 6d SCFT wants. There could be some finite curves flop out of the physical curves, we then have to modify the invariants according to the rules for the flop transitions. This phenomenon happens for the geometries of NHC $3,2$, NHC $3,2,2$ and NHC $2,3,2$. We will point out these degrees in the example sections.

Construction of the bases. The intersections of divisors in base can be realised easily in toric geometry. For a *smooth* toric surface, if we put the ray generators $\{u_i\}$ in clockwise order, the intersection numbers of the divisors $\{D_i\}$ are [92]

$$D_i \cdot D_j = \begin{cases} 0, & |i-j| > 1, \\ 1, & |i-j| = 1, \\ -n_i, & i = j, \end{cases} \tag{2.48}$$

where n_i is defined in

$$u_{i-1} + u_{i+1} = n_i u_i, \tag{2.49}$$

and is minus the self-intersection of the divisor D_i.

[18]The compact geometry can be recovered by completing the rays in the toric base and then resolving all toric singularities.

D	ν_i^*		$l^{(1)}$
D_u	1	0	1
S	0	-1	$-m$
D_v	-1	$-m$	1

(a)

D	ν_i^*		$l^{(1)}$	$l^{(2)}$
D_u	1	0	1	0
S_1	0	-1	-3	1
S_2	-1	-3	1	-2
D_v	-2	-5	0	1

(b)

D	ν_i^*		$l^{(1)}$	$l^{(2)}$	$l^{(3)}$
D_u	1	0	1	0	0
S_1	0	-1	-3	1	0
S_2	-1	-3	1	-2	1
S_3	-2	-5	0	1	-2
D_v	-3	-7	0	0	1

(c)

D	ν_i^*		$l^{(1)}$	$l^{(2)}$	$l^{(3)}$
D_u	1	0	1	0	0
S_1	0	-1	-2	1	0
S_2	-1	-2	1	-3	1
S_3	-3	-5	0	1	-2
D_v	-5	-8	0	0	1

(d)

Table 4. Toric realisation of non-Higgsable clusters.

With these rules, we can write down the toric construction of the A-type bases for non-compact Calabi-Yau threefolds. We list in particular the bases of non-Higgsible clusters in figure 1 and their toric construction in table 4. The other A-type bases can be constructed in a similar way. For bases of D, E-type chain of (-2) curves, there is no toric construction, but they can still be embedded in toric varieties as hypersurfaces.

Adding elliptic fibers. The fiber of an elliptic Calabi-Yau threefold is an elliptic curve, which can be embedded into the weighted projective space $\mathbb{P}^{2,3,1}(x,y,z)$, where the generic form of the elliptic curve is of the Tate form

$$y^2 + x^3 + a_1(s_i)xyz + a_2(s_i)x^2z^2 + a_3(s_i)yz^3 + a_4(s_i)xz^4 + a_6(s_i)z^6 = 0. \tag{2.50}$$

We can promote the coordinates x, y, z and the coefficients $a_j(s_i)$ to sections of line bundles over the base, and the equation then defines the entire elliptic Calabi-Yau threefold. The variables s_i are the local coordinates of the base, and the equation $s_i = 0$ defines a vertical divisor pulled back from the corresponding base curve. In general, this vertical divisor can be singular, signaling the singularity of elliptic fibers supported on the base curve. The type of singularity can be determined by the Tate's algorithm [93, 94]. Depending on whether the supporting base curve is compact or not, the singularity type is identified with either gauge symmetry or flavor symmetry. The two cases can be converted to each other by tuning the volume of base curves, corresponding to gauging flavor symmetry and turning off coupling of gauge symmetry. We can also make the Calabi-Yau smooth by performing the crepant resolution on singular vertical divisors, e.g. [95], corresponding to turning on gauge or flavor fugacities.

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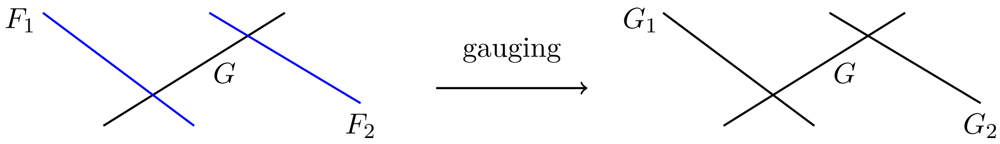

Figure 2. An illustration of gauging flavor symmetries, with two non-compact curves intersecting with one compact curve.

Note that the smooth toric base can not be the correct base for theories with $k_{F_i} > 1$. From (2.39), we know that k_{F_i} has a geometric meaning as the intersection number of compact and non-compact base curves. If $k_{F_i} > 1$, there is no compact smooth base support the intersection number.

In the following we give the example of the toric construction of NHC $3,2$ and its rank one limit. More examples can be found in appendix F.

NHC 3,2.

$$\begin{array}{|c cccc|cccccc|ccccccc|}
 & & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l_{\mathfrak{g}_2}^{(0)} & l_{\mathfrak{g}_2}^{(1)} & l_{\mathfrak{g}_2}^{(2)} & l_{\mathfrak{su}(2)}^{(0)} & l_{\mathfrak{su}(2)}^{(1)} & l_{B_1} & l_{B_2} \\
D_0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 & -2 & -4 & 0 & 0 \\
D_x & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
D_y & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\
D_z & 2 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
D_u & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
S_1 & 2 & 3 & 0 & -1 & 0 & -2 & 1 & -1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & -3 & 1 \\
S_1' & 2 & 3 & 0 & -2 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
S_1'' & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & 3 & -2 & 0 & 0 & 0 & 0 \\
S_2 & 2 & 3 & -1 & -3 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & -2 \\
S_2' & 1 & 2 & -1 & -3 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
D_v & 2 & 3 & -2 & -5 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \tag{2.51}$$

The compact geometry of the NHC $3,2$ was studied in [96]. For our non-compact case, we have the polytope (2.51), and we choose one phase with Mori cone vectors $l^{(i)}, i = 1, \cdots, 6$. The triple intersection numbers behave like the ones of an orbifold, with the intersection ring

$$\begin{aligned}
\mathcal{R} = -\frac{1}{5}(&8J_1^3+8J_3J_1^2+8J_4J_1^2+6J_5J_1^2+4J_6J_1^2+8J_3^2J_1+8J_4^2J_1+12J_5^2J_1+2J_6^2J_1+8J_3J_4J_1 \\
&+6J_3J_5J_1+6J_4J_5J_1+4J_3J_6J_1+4J_4J_6J_1+3J_5J_6J_1+2J_2^3+12J_3^3+24J_5^3+6J_6^3+3J_2J_3^2 \\
&+2J_2J_4^2+4J_3J_4^2+12J_3J_5^2+12J_4J_5^2+2J_3J_6^2+2J_4J_6^2+4J_5J_6^2+J_2^2J_3+2J_2^2J_4+6J_3^2J_4+J_2J_3J_4 \\
&+6J_3^2J_5+6J_4^2J_5+6J_3J_4J_5+4J_3^2J_6+4J_4^2J_6+6J_5^2J_6+4J_3J_4J_6+3J_3J_5J_6+3J_4J_5J_6),
\end{aligned} \tag{2.52}$$

where J_i are the Kähler cone generators. However, the intersection ring does not match the prepotential as we predicted in (2.44). By computing the genus zero Gopakumar-Vafa invariants, we found that there are two irreducible rational curve classes of degrees

$$\beta_1 \cdot t = t_{\mathfrak{g}_2,1} + t_{\mathfrak{g}_2,2} - \frac{1}{2} t_{\mathfrak{su}(2),1} - t_{B_2}, \quad \beta_2 \cdot t = t_{\mathfrak{g}_2,1} + 2t_{\mathfrak{g}_2,2} - \frac{1}{2} t_{\mathfrak{su}(2),1} - t_{B_2}, \tag{2.53}$$

with multiplicity one; in particular, the degree of one base curve is negative. Such a phenomenon means our toric construction is not exactly in the phase corresponding to the rank two SCFT, we have to flop these two curve classes to reach the correct geometric phase. As explained in [9, 97], the flop of a rational curve $\beta \cdot t \to -\beta \cdot t$ with multiplicity one changes the semiclassical prepotential by subtracting a term $\frac{1}{6}(\beta \cdot t)^3$. After the flop, we indeed get the correct semiclassical prepotential.

We can reduce the current geometry to that of the rank one $n = 3$ theory with gauge symmetry G_2 and flavor symmetry $\mathfrak{su}(2)$ by removing the lattice point $(2, 3, -2, -5)$, and the corresponding homogeneous coordinate is sent to zero. In this process, some Kähler parameters are sent to positive infinity, and some sent to negative infinity; their appropriate linear combinations remain finite and become the new Kähler parameters of the reduced geometry. In our current example, the volumes of the curve classes $l^{(2)}, l^{(1)}+l^{(3)}, l^{(4)}, l^{(5)}, l^{(1)}+l^{(6)}$ remain finite, and they are the Mori cone generators of the reduced geometry.

We list the result in (2.54).

	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$
D_0	0	0	0	0	0	−1	0	0	−2
D_x	−1	0	0	0	0	0	0	0	1
D_y	0	−1	0	0	0	0	0	1	0
D_z	2	3	0	0	1	0	0	0	0
D_u	2	3	1	0	0	0	1	0	0
S_1	2	3	0	−1	−2	1	−1	0	0
S_1'	2	3	0	−2	1	−1	−1	1	−1
S_1''	1	1	0	−1	0	1	0	−2	2
S_2	2	3	−1	−3	0	−1	1	0	1
S_2'	1	2	−1	−3	0	1	0	0	−1

(2.54)

The intersection ring is

$$\mathcal{R} = -\frac{1}{3}(J_1^3 + J_3 J_1^2 + J_3^2 J_1 + 2J_2^3 + 16J_4^3 + 2J_2 J_3^2 + 7J_2 J_4^2 + 7J_3 J_4^2 + 2J_2^2 J_3 + 3J_2^2 J_4 \\ + 3J_3^2 J_4 + 3J_2 J_3 J_4 + 2J_2^2 J_5 + 2J_3^2 J_5 + 2J_4^2 J_5 + 2J_2 J_3 J_5 + J_2 J_4 J_5 + J_3 J_4 J_5). \tag{2.55}$$

3 Elliptic blowup equations

We present the elliptic blowup equations for all rank one 6d SCFTs on the 6d Omega background and discuss various properties of these equations in this section. Since the derivation of the elliptic blowup equations from the blowup equations for the topological string theory is extremely similar to those given in [7–9], we refer to those papers for details of the derivation and only provide some important points in appendix D. The additional information required in this process includes the semiclassical free energies, which we have discussed in section 2.3, and the one-loop partition functions coming from vector and hypermultiplets, whose formulas can be found in [7, 76].

We first write down the explicit form of the elliptic blowup equations and list various constraints on and properties of the equations. This is followed by a subsection where we give some physical arguments for the elliptic blowup equations, generalising the arguments presented in [9] for theories with no gauge symmetry. We then discuss in details two of these properties, the modularity and the possibility to transform equations along the Higgsing trees. Finally, we illustrate the relation between the K-theoretic blowup equations and our elliptic blowup equations.

3.1 Elliptic blowup equations for all rank one theories

Consider a rank one 6d SCFT with tensor branch coefficient n, gauge symmetry G, flavor symmetry F, and half-hypermultiplets transforming in the representations (R_G, R_F). The flavor symmetry induces a current algebra of level k_F on the worldsheet of BPS strings. Then the elliptic genera $\mathbb{E}_d(\tau, m_{G,F}, \epsilon_{1,2})$ satisfy the following elliptic blowup equations

$$\begin{aligned}
&\sum_{\lambda_G\in\phi_{\lambda_0}(Q^\vee(G))}^{\frac{1}{2}||\lambda_G||^2+d'+d''=d+\delta} (-1)^{|\phi_{\lambda_0}^{-1}(\lambda_G)|}\\
&\times \theta_i^{[a]}(n\tau, -n\lambda_G\cdot m_G + k_F\lambda_F\cdot m_F + \left(y-\frac{n}{2}||\lambda_G||^2\right)(\epsilon_1+\epsilon_2) - nd'\epsilon_1 - nd''\epsilon_2)\\
&\times A_V(\tau, m_G, \lambda_G) A_H(\tau, m_G, m_F, \lambda_G, \lambda_F)\\
&\times \mathbb{E}_{d'}(\tau, m_G+\epsilon_1\lambda_G, m_F+\epsilon_1\lambda_F, \epsilon_1, \epsilon_2-\epsilon_1)\mathbb{E}_{d''}(\tau, m_G+\epsilon_2\lambda_G, m_F+\epsilon_2\lambda_F, \epsilon_1-\epsilon_2, \epsilon_2)\\
&\quad = \Lambda(\delta)\,\theta_i^{[a]}(n\tau, k_F\lambda_F\cdot m_F + y(\epsilon_1+\epsilon_2))\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2), \qquad d=0,1,2,\dots
\end{aligned}\tag{3.1}$$

where[19]

$$y = \frac{n-2+h^\vee_{\mathfrak{g}}}{4} + \frac{k_F}{2}(\lambda_F\cdot\lambda_F),\tag{3.2}$$

and

$$\Lambda(\delta) = \begin{cases} 1, & \delta = 0,\\ 0, & \delta > 0.\end{cases}\tag{3.3}$$

In the generalised theta function $\theta_i^{[a]}$, the subscript i is 3 if n is even and 4 if n is odd, and the characteristic a of the theta function can be one of the following n numbers

$$a = \frac{1}{2} - \frac{k}{n}, \quad k = 0, 1, \dots, n-1.\tag{3.4}$$

Besides, if there is an Abelian factor in the flavor symmetry, the argument $k_F\lambda_F\cdot m_F$ should be extended to

$$k_F\lambda_F\cdot m_F \to k_F\lambda_F\cdot m_F + k_{\mathfrak{u}(1)}\lambda_{\mathfrak{u}(1)}m_{\mathfrak{u}(1)}.\tag{3.5}$$

[19] We set $h^\vee_{\mathfrak{g}} = 1$ if gauge symmetry is trivial.

The summation index λ_G is a coweight vector[20] of G; to be more precise, it takes value in the shifted coroot lattice defined by the embedding through a coweight vector λ_0

$$\begin{aligned}\phi_{\lambda_0} : Q^\vee &\hookrightarrow P^\vee \\ \alpha^\vee &\to \alpha^\vee + \lambda_0, \quad \lambda_0 \in P^\vee.\end{aligned} \tag{3.6}$$

The index λ_G in fact consists of components of the so-called r-field[21] in the blowup equations of topological string, and different λ_G correspond to r-fields which are equivalent to each other. On the other hand, there can be different embeddings. The number of different embeddings is the index of $Q^\vee$ as an Abelian subgroup of $P^\vee$, which is also the determinant of the Cartan matrix of G. There is a special embedding where the shift λ_0 is a coroot vector. δ is the smallest norm in the shifted coroot lattice; it is zero in the special embedding and positive otherwise. The inverse $\phi_{\lambda_0}^{-1}$ pulls back the coweight λ_G to the coroot lattice, and $|\bullet|$ in the sign factor sums up the coefficients in its decomposition in terms of simple coroots. We say the blowup equation is of the *unity* type if the embedding is the special embedding so that Λ is unity. Otherwise, the r.h.s. of the blowup equation vanishes identically and we say the blowup equations are of the *vanishing* type. Clearly if $P^\vee \cong Q^\vee$, which happens for G_2, F_4 and E_8, there can be no vanishing blowup equation.

The components A_V and A_H are contributions from vector and hypermultiplets respectively. They have the form

$$A_V(\tau, m_G, \lambda_G) = \prod_{\beta\in\Delta_+} \breve{\theta}_V(\beta\cdot m_G, \beta\cdot\lambda_G)\,, \tag{3.7}$$

$$A_H(\tau, m_{G,F}, \lambda_{G,F}) = \prod_{\omega_{G,F}\in R^+_{G,F}} \breve{\theta}_H(\omega_G\cdot m_G + \omega_F\cdot m_F, \omega_G\cdot\lambda_G + \omega_F\cdot\lambda_F)\,. \tag{3.8}$$

Here $R^+_{G,F}$ is half of the total weight space. For unity blowup equations

$$R^+_{G,F} = \{\omega_G \in R_G, \omega_F \in R_F \,|\, \omega_F\cdot\lambda_F = +1/2\}, \tag{3.9}$$

and for vanishing blowup equations

$$R^+_{G,F} = \{\omega_G \in R_G, \omega_F \in R_F \,|\, \omega_G\cdot\lambda_G + \omega_F\cdot\lambda_F > 0\}. \tag{3.10}$$

Furthermore, the $\breve{\theta}$ functions are defined as

$$\breve{\theta}_V(z,R) := \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-1}} \frac{\eta}{\theta_1(z+sm\epsilon_1+sn\epsilon_2)} \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-2}} \frac{\eta}{\theta_1(z+s(m+1)\epsilon_1+s(n+1)\epsilon_2)}, \quad R\in\mathbb{Z}, \tag{3.11}$$

$$\breve{\theta}_H(z,R) := \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-3/2}} \frac{(-1)^R\theta_1(z+s(m+1/2)\epsilon_1+s(n+1/2)\epsilon_2)}{\eta}, \quad R\in\frac{1}{2}+\mathbb{Z}, \tag{3.12}$$

[20] This is sometimes called a magnetic weight vector in the literature, e.g. [98]. See appendix A for our Lie algebraic convention.

[21] Up to a factor of 1/2.

with s the sign of R. For unity blowup equations, the factor $(-1)^R$ in (3.12) is a constant under the choice (3.9), so that we drop it in later computations.

There is still one free parameter λ_F. It is a coweight vector of the flavor symmetry, and in fact also consists of components of the so-called r-field[22] [4, 7, 19]. The value of λ_F can be determined by the following constraints:

- Checker board pattern: the second arguments on the r.h.s. of (3.7), (3.8) are one half of the r-field component associated to the refined BPS states of vector and hypermultipets, and thus they must satisfy the conditions [7]

$$\beta \cdot \lambda_G \in \mathbb{Z}, \qquad \beta \in \Delta, \tag{3.13}$$

$$\omega_G \cdot \lambda_G + \omega_F \cdot \lambda_F \in \frac{1}{2} + \mathbb{Z}, \qquad \omega_G \in R_G, \omega_F \in R_F. \tag{3.14}$$

  The first condition only confirms that λ_G is a coweight vector of G. The second condition constrains that λ_F takes value in a subset of the coweight lattice of F depending on the domain of ϕ_{λ_0}. In the case of unity equations, λ_G is a coroot vector of G and $\omega_G \cdot \lambda_G$ is an integer, (3.14) reduces to

$$\omega_F \cdot \lambda_F \in \frac{1}{2} + \mathbb{Z}, \quad \omega_F \in R_F. \tag{3.15}$$

  The above conditions are also called the B *field condition*.

- Modularity: we observe that the elliptic blowup equations (3.1) are identities of Jacobi forms. An important consistency condition for (3.1) is that every term on the l.h.s. should have the same modular weight and modular index, and when the r.h.s. does not vanish, they coincide with the modular weight and modular index of the r.h.s. as well. The condition on modular weight is trivially satisfied as every term has weight one half. The condition on modular index, on the other hand, is highly nontrivial and very constraining. As we will see in section 3.3, this condition puts strong constraints on λ_F, especially in the case of unity blowup equations.

- Higgsing: rank one 6d SCFTs with the same tensor branch parameter n are related to each other via the Higgs branch RG flows. As we will discuss in section 3.4 unity blowup equations of rank one 6d SCFTs can be transformed to each other along the Higgsing trees, and in addition the Higgsing process puts some mild constraints on λ_F for unity blowup equations as well.

- Leading degree identities: the degree d is the degree of the shifted base curve t_{ell}. In the leading degree with $d = d' = d'' = 0$, the elliptic genera do not contribute and the blowup equations become identities of Jacobi theta functions. For unity blowup equations the leading order identities are trivial, while for vanishing blowup equations the leading order identities are very non-trivial and they can be used to constrain the parameter λ_F.

[22] Up to a factor of 1/2.

We list below the values of the parameter λ_F satisfying all the four constraints for each rank one model and the corresponding y parameter. The coweight vectors λ_F are presented by their Dynkin labels. Note that such a coweight vector can be mapped to a weight vector by the isomorphism φ defined in (A.11). We are sometimes sloppy in the main text and refer to λ_F as weights, by which we actually mean the images of φ. Besides in the main text we often directly write the factor $(-1)^{|\phi_{\lambda_0}^{-1}(\lambda_G)|}$ in (3.1) as $(-1)^{|\lambda_G|}$. We will later test the corresponding elliptic blowup equations in sections 4 and 5 by checking them explicitly with known results of elliptic genera, and by solving unknown elliptic genera as well as refined BPS invariants from them.

There is another convenient form of elliptic blowup equations, in which we replace $(y-\frac{n}{2}||\lambda_G||^2)$ by $(\bar{y}-nd)$ in (3.1). The advantage is that d is always integer for both unity and vanishing cases, and $\bar{y}$ are typically simpler numbers than y. On the other hand, the merit of the current form (3.1) is that the modularity proof of both unity and vanishing blowup equations can be combined together. We will also use the notion of d and $\bar{y}$ in the example sections, where $\bar{y}$ and y are related by $y=\bar{y}+n\delta$ in the vanishing cases and naturally $y=\bar{y}$ in the unity cases.

3.1.1 Unity blowup equations

We tabulate in tables 5, 6 the coweight vectors λ_F and the associated parameter y for unity blowup equations which satisfy the four constraints discussed above. We note that if a coweight vector λ_F is valid, all the vectors in the same Weyl orbit should be valid as well, and we only list in tables 5, 6 the dominant coweight vectors. We commment that for ease of computation, we have used the isomorphism of algebras

$$\mathfrak{so}(2)\cong\mathfrak{sp}(1),\quad \mathfrak{so}(4)\cong\mathfrak{sp}(1)\times\mathfrak{sp}(1). \tag{3.16}$$

Whenever possible, we prefer the notation $\mathfrak{sp}(1)$ instead of $\mathfrak{su}(2)$ as it is more similar to other C-algebras instead of A-algebras.

Note that the following theories have unpaired half-hypers and they do not have unity blowup equations. Technically this is because their flavor weight spaces have zero weight, with which the checkerboard pattern constraint (3.15) cannot be satisfied.

- $n=1$: $G=\mathfrak{su}(6)_*,\mathfrak{so}(11),\mathfrak{so}(12)_{a,b},E_7$;
- $n=2$: $G=\mathfrak{so}(12)_b,\mathfrak{so}(13)$;
- $n=3$: $G=\mathfrak{so}(11),\mathfrak{so}(12),E_7$;
- $n=5,7$: $G=E_7$.

3.1.2 Vanishing blowup equations

We tabulate in tables 7, 8, 9 the values of λ_F and the associated parameter y for vanishing equations that satisfy the constraints discussed in the beginning of this section. In particular, we have tested the leading degree identities for all the vanishing blowup equations up

n	G	F	$\#$	y	λ_F
12	E_8	$-$	1	10	$\emptyset$
8	E_7	$-$	1	6	$\emptyset$
7	E_7	$-$	0	$-$	$-$
6	E_6	$-$	1	4	$\emptyset$
6	E_7	$\mathfrak{so}(2)_{12} = \mathfrak{sp}(1)_6$	2	7	(1)
5	F_4	$-$	1	3	$\emptyset$
5	E_6	$\mathfrak{u}(1)_6$	2	$9/2$	$\pm 1/2$
5	E_7	$\mathfrak{so}(3)_{12}$	0	$-$	$-$
4	$\mathfrak{so}(8)$	$-$	1	2	$\emptyset$
4	$\mathfrak{so}(N \geq 9)$	$\mathfrak{sp}(N-8)_1$	2^{N-8}	$(N-4)/2$	$(0\dots01)$
4	F_4	$\mathfrak{sp}(1)_3$	2	$7/2$	(1)
4	E_6	$\mathfrak{su}(2)_6 \times \mathfrak{u}(1)_{12}$	4	5	$(1)_0$ or $(0)_{\pm\frac{1}{2}}$
4	E_7	$\mathfrak{so}(4)_{12} = \mathfrak{sp}(1)_6 \times \mathfrak{sp}(1)_6$	4	8	(1),(1)
3	$\mathfrak{su}(3)$	$-$	1	1	$\emptyset$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	4	2	(01)
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(1)_1^b \times \mathfrak{sp}(1)_1^c$	8	$5/2$	(1),(1),(1)
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(1)_2^b$	8	3	(01),(1)
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a \times (\mathfrak{su}(1)_4 \times \mathfrak{u}(1)_4)^b$	16	$7/2$	(001),$\pm 1/2$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a \times \text{Ising}^b$	0	$-$	$-$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	0	$-$	$-$
3	G_2	$\mathfrak{sp}(1)_1$	2	$3/2$	(1)
3	F_4	$\mathfrak{sp}(2)_3$	4	4	(01)
3	E_6	$\mathfrak{su}(3)_6 \times \mathfrak{u}(1)_{18}$	8	$11/2$	$\pm(01)_{\frac{1}{6}}$ or $(00)_{\pm\frac{1}{2}}$
3	E_7	$\mathfrak{so}(5)_{12}$	0	$-$	$-$

Table 5. The parameters y, λ_F of unity blowup equations for rank one models with $n \geq 3$. $\#$ is the number of unity equations with fixed characteristic a.

to order 20 in $q = \exp(2\pi \mathrm{i}\tau)$. We find unlike the unity λ_F fields which form Weyl orbits, the admissible vanishing λ_F fields typically form representations rather than just Weyl orbits. To be precise, the admissible vanishing λ_F fields are all coweight vectors inside the representation whose highest coweight is given in Dynkin label in tables 7, 8, 9. Note one representation in general contains many Weyl orbits. Besides, different Weyl orbits inside one representation in general have different associated y which are easily computable with equation (3.2). Thus for the situation where several values of y are involved, we leave $\cdots$ in tables 7, 8, 9.

In the following we discuss some special cases in tables 7, 8, 9 in more detail.

n	G	F	$\#$	y	λ_F
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	2	1/2	(1)
2	$\mathfrak{su}(2)$	$\mathfrak{so}(8)_1 \to \mathfrak{so}(7)_1 \times \text{Ising}$	6	1	(100)
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$\binom{2N}{N}$	N/2	$(0\dots010\dots0)$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	32	5/2	(1),(0001)
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	64	3	(01),(01),(01)
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	32	7/2	(001),(01)
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	64	4	(0001) and $(1)_0$ or $(0)_{\pm\frac{1}{2}}$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times (? \to \mathfrak{so}(2)_8)^b$	64	9/2	(00001),(1)
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8^b$	128	5	(000001),(1)
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	0	–	–
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	0	–	–
2	G_2	$\mathfrak{sp}(4)_1$	16	2	(0001)
2	F_4	$\mathfrak{sp}(3)_3$	8	9/2	(001)
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	16	6	$(010)_0$ or $\pm(001)_{\frac{1}{4}}$ or $(000)_{\pm\frac{1}{2}}$
2	E_7	$\mathfrak{so}(6)_{12}$	8	9	(001) or (010)
1	$\mathfrak{sp}(0)$	$(E_8)_1$	240	1	$(10\dots0)$
1	$\mathfrak{sp}(N \geq 1)$	$\mathfrak{so}(4N+16)_1$	2^{2N+7}	$(N+2)/2$	$(0\dots01)$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	924	2	$(0\dots010\dots0)$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a \times \mathfrak{su}(2)_1^b$	1848	5/2	$(0\dots010\dots0)$, (1)
1	$\mathfrak{su}(N \geq 5)$	$\mathfrak{su}(N+8)_1 \times \mathfrak{u}(1)_{2N(N-1)(N+8)}$	$2\binom{N+8}{6}$	$(N+1)/2$	$(0000010\dots)_{-\frac{1}{2(N+8)}}$ or minus
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	0	–	–
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(6)_1^b$	256	3	(01),(000001)
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(3)_1^b \times \mathfrak{sp}(3)_1^c$	512	7/2	(001),(001),(001)
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a \times \mathfrak{sp}(3)_2^b$	128	4	(0001),(001)
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a \times (\mathfrak{su}(3)_4 \times \mathfrak{u}(1)_{12})^b$	256	9/2	(00001), and $\pm(01)_{\frac{1}{6}}$ or $(00)_{\pm\frac{1}{2}}$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a \times ?^b$	0	–	–
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a \times \mathfrak{so}(3)_8^b$	0	–	–
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a \times ?^b \times ?^c$	0	–	–
1	G_2	$\mathfrak{sp}(7)_1$	128	5/2	$(0\dots01)$
1	F_4	$\mathfrak{sp}(4)_3$	16	5	$(0\dots01)$
1	E_6	$\mathfrak{su}(5)_6 \times \mathfrak{u}(1)_{30}$	32	13/2	$\pm(0001)_{\frac{3}{10}}$ or $\pm(0010)_{\frac{1}{10}}$ or $(0000)_{\pm\frac{1}{2}}$
1	E_7	$\mathfrak{so}(7)_{12}$	0	–	–

Table 6. The parameters y, λ_F of unity blowup equations for rank one models with $n = 1, 2$. $\#$ is the number of unity equations with fixed characteristic a.

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n	G	F	λ_0	y	λ_F
12	E_8	$-$	$-$	$-$	$-$
8	E_7	$-$	(0000010)	6	$\emptyset$
7	E_7	$-$	(0000010)	23/4	$\emptyset$
6	E_6	$-$	(100000)	4	$\emptyset$
			(000010)	4	$\emptyset$
6	E_7	$\mathfrak{so}(2)_{12}=\mathfrak{sp}(1)_6$	(0000010)	11/2	(0)
5	F_4	$-$	$-$	$-$	$-$
5	E_6	$\mathfrak{u}(1)_6$	(100000)	23/6	$-1/6$
			(000010)	23/6	1/6
5	E_7	$\mathfrak{so}(3)_{12}$	(0000010)	21/4	(0)
4	$\mathfrak{so}(8)$	$-$	all three	2	$\emptyset$
4	$\mathfrak{so}(2N), N\geq 5$	$\mathfrak{sp}(2N-8)_1$	$(10\ldots0)$	$N-2$	$(0\ldots01)$
			$(\ldots010)$	$\cdots$	$[N-2,0\ldots00]$
			$(\ldots001)$	$\cdots$	$[N-2,0\ldots00]$
4	$\mathfrak{so}(2N-1), N\geq 5$	$\mathfrak{sp}(2N-9)_1$	$(10\ldots0)$	$(2N-5)/2$	$(0\ldots01)$
4	F_4	$\mathfrak{sp}(1)_3$	$-$	$-$	$-$
4	E_6	$\mathfrak{su}(2)_6\times\mathfrak{u}(1)_{12}$	(100000)	11/3	$(0),-1/6$
			(000010)	11/3	$(0),1/6$
4	E_7	$\mathfrak{so}(4)_{12}=\mathfrak{sp}(1)_6\times\mathfrak{sp}(1)_6$	(0000010)	5	(0),(0)
3	$\mathfrak{su}(3)$	$-$	(10) or (01)	1	$\emptyset$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	(100)	$\cdots$	[10]
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)^a_1\times\mathfrak{sp}(1)^b_1\times\mathfrak{sp}(1)^c_1$	(1000)	$\cdots$	(1),(0),[2] or (1),[2],(0)
			(0010)	$\cdots$	[2],(1),(0) or (0),(1),[2]
			(0001)	$\cdots$	(0),[2],(1) or [2],(0),(1)
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)^a_1\times\mathfrak{sp}(1)^b_2$	(1000)	5/2	(01),(0)
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)^a_1\times(\mathfrak{su}(1)_4\times\mathfrak{u}(1)_4)^b$	(10000)	3	(001),(0),0
			(00010)	$\cdots$	$[j00],(0),-1/4+\ell$: see text
			(00001)	$\cdots$	$[j00],(0),1/4-\ell$: see text
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)^a_1\times\mathrm{Ising}^b$	(10000)	7/2	(0001)
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	(100000)	4	(00001)
			(000001)	$\cdots$	[30000]
			(000010)	$-$	$-$
3	G_2	$\mathfrak{sp}(1)_1$	$-$	$-$	$-$
3	F_4	$\mathfrak{sp}(2)_3$	$-$	$-$	$-$
3	E_6	$\mathfrak{su}(3)_6\times\mathfrak{u}(1)_{18}$	(100000)	7/2	$(00),-1/6$
			(000010)	7/2	$(00),1/6$
3	E_7	$\mathfrak{so}(5)_{12}$	(0000010)	19/4	(00)

Table 7. The parameters y,λ_F of vanishing blowup equations for rank one models with $n\geq 3$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference.

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n	G	F	λ_0	y	λ_F
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	$-$	$-$	$-$
2	$\mathfrak{su}(2)$	$\mathfrak{so}(8)_1 \to \mathfrak{so}(7)_1 \times \text{Ising}$	(1)	1/2	(000)
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	see text		
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	(100)	$\cdots$	(1),[1000]
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	(1000)	$\cdots$	(01),(00),[10] or (01),[10],(00)
			(0010)	$\cdots$	[10],(01),(00) or (00),(01),[10]
			(0001)	$\cdots$	(00),[10],(01) or [10],(00),(01)
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	(1000)	5/2	(001),(00)
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	(10000)	3	(0001),(0),0
			(00010)	$\cdots$	$[j000], (0), -1/4 + \ell$: see text
			(00001)	$\cdots$	$[j000], (0), 1/4 - \ell$: see text
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times (? \to \mathfrak{so}(2)_8)^b$	(10000)	7/2	(00001),(0)
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8^b$	(100000)	4	(000001),(0)
			(000001)	$\cdots$	[300000], (0)
			(000010)	$\cdots$	[200000], (1)
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	(100000)	4	(000001)
			(000001)	$-$	$-$
			(000010)	$-$	$-$
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	(100000)	9/2	(0000001)
2	G_2	$\mathfrak{sp}(4)_1$	$-$	$-$	$-$
2	F_4	$\mathfrak{sp}(3)_3$	$-$	$-$	$-$
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	(100000)	10/3	$(000), -1/6$
			(000010)	10/3	$(000), 1/6$
2	E_7	$\mathfrak{so}(6)_{12}$	(0000010)	9/2	(000)

Table 8. The parameters y, λ_F of vanishing blowup equations for rank one models with $n = 2$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference. See the main text for more discussion.

- $G = \mathfrak{so}(7)$: for $n = 3$, the representation [10] of $\lambda_{\mathfrak{sp}(2)}$ has two Weyl orbits generated by coweights (00) and (10), whose associated y are 3/2 and 5/2 respectively. For $n = 2$, the representation [1000] of $\lambda_{\mathfrak{sp}(4)}$ has two Weyl orbits generated by coweights (0000) and (1000), whose associated y are 3/2 and 5/2 respectively. For $n = 1$, the representation [100000] of $\lambda_{\mathfrak{sp}(6)}$ has two Weyl orbits generated by coweights (000000) and (100000), whose associated y are 3/2 and 5/2 respectively.

- $G = \mathfrak{so}(8)$: for $n = 3$, the representation [2] of $\lambda_{\mathfrak{sp}(1)}$ has two Weyl orbits (0) and (2), whose associated y are 2 and 3 respectively. For $n = 2$, the representation [10] of $\lambda_{\mathfrak{sp}(2)}$ has two Weyl orbits generated by coweights (00) and (10), whose associated

n	G	F	λ_0	y	λ_F
1	$\mathfrak{sp}(0)$	$(E_8)_1$	$\emptyset$	0	$(0\ldots0)$
1	$\mathfrak{sp}(N\geq1)$	$\mathfrak{so}(4N+16)_1$	$(0\ldots01)$	$\cdots$	$[N,0\ldots0]$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	(10) or (01)	$\cdots$	$[\ldots02]$ or $[20\ldots]$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a\times\mathfrak{su}(2)_1^b$	(100) or (001)	$\cdots$	$[\ldots03]$ or $[30\ldots0]$,(0)
			(010)	$\cdots$	$[10\ldots01]$,(1)
1	$\mathfrak{su}(N\geq5)$	$\mathfrak{su}(N{+}8)_1\times\mathfrak{u}(1)_{2N(N-1)(N+8)}$	see text		
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	(10000) or (00001)	$\cdots$	$[\ldots05]$ or $[50\ldots]$
			(00100)	$\cdots$	$[20\ldots02]$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a\times\mathfrak{sp}(6)_1^b$	(100)	$\cdots$	(01),$[100000]$
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a\times\mathfrak{sp}(3)_1^b\times\mathfrak{sp}(3)_1^c$	(1000)	$\cdots$	(001),(000),$[100]$ or (001),$[100]$,(000)
			(0010)	$\cdots$	$[100]$,(001),(000) or (000),(001),$[100]$
			(0001)	$\cdots$	(000),$[100]$,(001) or $[100]$,(000),(001)
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a\times\mathfrak{sp}(3)_2^b$	(1000)	$5/2$	(0001),(000)
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a\times(\mathfrak{su}(3)_4\times\mathfrak{u}(1)_{12})^b$	(10000)	3	(00001),(0),0
			(00010)	$\cdots$	$[20000],(0),-1/4$
			(00001)	$\cdots$	$[20000],(0),1/4$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a\times(?\to\mathfrak{sp}(1)_6)^b$	(10000)	$7/2$	(000001),(0)
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a\times\mathfrak{so}(3)_8^b$	(100000)	4	(0000001),(0)
			(000001)	$\cdots$	$[3000000],(0)$
			(000010)	$-$	$-$
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a\times(?\to\mathfrak{sp}(1)_4)^b{\times}?^c$	(100000)	4	(0000001),(0)
			(000001)	$-$	$-$
			(000010)	$\cdots$	$[2000000],(1)$
1	G_2	$\mathfrak{sp}(7)_1$	$-$	$-$	$-$
1	F_4	$\mathfrak{sp}(4)_3$	$-$	$-$	$-$
1	E_6	$\mathfrak{su}(5)_6\times\mathfrak{u}(1)_{30}$	(100000)	$19/6$	$(0000),-1/6$
			(000010)	$19/6$	$(0000),1/6$
1	E_7	$\mathfrak{so}(7)_{12}$	(0000010)	$17/4$	(000)

Table 9. The parameters y,λ_F of vanishing blowup equations for rank one models with $n=1$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference. We make assumption for flavor symmetries of the $G=\mathfrak{so}(11)$ and $\mathfrak{so}(12)_b$ models which allow for vanishing blowup equations; see the main text for more discussion.

y are 2 and 3 respectively. For $n=1$, the representation $[100]$ of $\lambda_{\mathfrak{sp}(3)}$ has two Weyl orbits generated by coweights (000) and (100), whose associated y are 2 and 3 respectively.

- $G=\mathfrak{so}(10)$: for $n=3$, the possibilities are

$$\begin{aligned}&\ell=0, && j=2,\\ &\ell=-1, && j=0,\\ &\ell=1, && j=1.\end{aligned}\tag{3.17}$$

For $n = 2$, the possibilities are

$$\begin{aligned} \ell &= 0, \quad j = 2, \\ \ell &= 1, \quad j = 0. \end{aligned} \tag{3.18}$$

For $n = 1$, the possibilities are

$$\ell = 0, \quad j = 2. \tag{3.19}$$

- $n = 2, G = \mathfrak{su}(N), N \geq 3$: when $\lambda_0 = \omega_j^{\vee\mathfrak{su}(N)}$, $j = 1, 2, \ldots, N-1$,

$$\lambda_F{}^{\mathfrak{su}(2N)} \in [j-1, 0, \ldots, 0, (N-1-j)]. \tag{3.20}$$

For fixed N and j, this is a very large representation which contains many Weyl orbits, each of which has its own associated y. We do not list all of them since they are easily computable from equation (3.2). Instead, we just point out one a particularly simple Weyl orbit inside such representation. For example, if N is odd,

$$\lambda_0 = \omega_j^{\vee\mathfrak{su}(N)}, \quad \lambda_F \in \mathcal{O}_{N+2j}{}^{\mathfrak{su}(2N)}, \quad y = \frac{N^2 - 2j^2}{2N}, \quad j = 1, \ldots, (N-1)/2, \tag{3.21}$$

and

$$\lambda_0 = \omega_{N-j}^{\vee}{}^{\mathfrak{su}(N)}, \quad \lambda_F \in \mathcal{O}_{N-2j}{}^{\mathfrak{su}(2N)}, \quad y = \frac{N^2 - 2j^2}{2N}, \quad j = 1, \ldots, (N-1)/2, \tag{3.22}$$

where $\mathcal{O}_i$ is the Weyl orbit generated by the i-th fundamental coweight. If N is even, j runs from 1 up to $N/2$ in the equations (3.21) and (3.22). We will explicitly show the leading degree vanishing identities for these Weyl orbits in section (5.3).

- $n = 1, G = \mathfrak{su}(N), N \geq 5$: for $k \leq \lfloor N/2 \rfloor$, we have

$$\lambda_0 = \omega_k^{\vee\mathfrak{su}(N)}, \quad \lambda_F{}^{\mathrm{U}(1)} = \frac{4k - N}{2N(N+8)}, \quad \lambda_F{}^{\mathfrak{su}(N+8)} \in [k-1, 0, \ldots, 0, N+1-2k], \tag{3.23}$$

The associated parameters y are computed by (3.2) as,

$$y = \frac{N-1}{4} + \frac{1}{2}(\lambda_F{}^{\mathfrak{su}(N+8)}, \lambda_F{}^{\mathfrak{su}(N+8)}) + N(N-1)(N+8)(\lambda_F{}^{\mathrm{U}(1)})^2. \tag{3.24}$$

The cases of $k > \lfloor N/2 \rfloor$ can be obtained by complex conjugation.

- $n = 1, G = \mathfrak{so}(11), \mathfrak{so}(12)_b$: the flavor symmetries consistent at the level of worldsheet theory are not known for these two models [1], especially the component governing the three half-hypers in spinor representation of $\mathfrak{so}(11)$ in the first model, and the component governing the two half-hypers in spinor and one half-hyper in conjugate spinor representations of $\mathfrak{so}(12)$ in the second model. We find that if we assume the three half-hypers in the first model transform as **3** of flavor symmetry $\mathfrak{sp}(1)$, and the two half-hypers in the second model transform as **2** of flavor symmetry $\mathfrak{sp}(1)$, we can find λ_F of vanishing equations which satisfy all the constraints discussed in the beginning of this section. In particular, we have checked the leading base degree identities also up to order 20 in q.

Let us give a simple example of the leading base degree identities. Consider $n=1, G=\mathfrak{su}(3), F=\mathfrak{su}(12)$ theory with matter representation $(\mathbf{3},\overline{\mathbf{12}})$. Let us look at the situation with $\lambda_G=(10)_{\mathfrak{su}(3)}=\mathbf{3}$. The admissible λ_F form representation $[\dots 002]_{\mathfrak{su}(12)}$ which has two Weyl orbits $(\dots 010)$ and $(\dots 002)$. The first Weyl orbit itself is a representation $\overline{\mathbf{66}}$. In this case, the leading base degree of the vanishing blowup equations gives the following identity: $\forall\lambda\in\overline{\mathbf{66}}$,

$$\sum_{w\in\mathbf{3}}(-1)^{|w|}\theta_1(-m_w+m_\lambda+2\epsilon_+)\prod_{\beta\in\Delta(\mathfrak{su}(3))}^{w\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}\prod_{\mu\in\overline{\mathbf{12}}}^{\mu\cdot\lambda=5/6}\theta_1(m_w+m_\mu+\epsilon_+)=0. \tag{3.25}$$

We have checked this identity to $\mathcal{O}(q^{20})$. To write it more explicitly, we have

$$\begin{aligned}&\frac{\theta_1(-a_1+x_i+x_j)\theta_1(a_1+x_i)\theta_1(a_1+x_j)}{\theta_1(a_1-a_2)\theta_1(a_1-a_3)}+\frac{\theta_1(-a_2+x_i+x_j)\theta_1(a_2+x_i)\theta_1(a_2+x_j)}{\theta_1(a_2-a_1)\theta_1(a_2-a_3)} \\ &+\frac{\theta_1(-a_3+x_i+x_j)\theta_1(a_3+x_i)\theta_1(a_3+x_j)}{\theta_1(a_3-a_1)\theta_1(a_3-a_2)}=0, \text{for } a_1+a_2+a_3=0,\ 1\le i<j\le 12.\end{aligned} \tag{3.26}$$

Here $a_k, k=1,2,3$ are the $\mathfrak{su}(3)$ fugacities and $x_i=m_i+\epsilon_+, i=1,2,\dots,12$ where m_i are the symmetric $\mathfrak{su}(12)$ fugacities. The modularity here means that each among the three terms in the above equation has the same index $-(a_1^2+a_2^2+a_3^2)/2$. The leading base degree identities from the other set of vanishing blowup equations are just similar. We have tons of vanishing theta identities like this involving the root and weight lattices of Lie algebras from the leading base degree of vanishing blowup equations. We present some of them in section 5 and make a summary in appendix B.

3.2 Physical interpretation

In our previous paper [9], we proposed a physical explanation for the elliptic blowup equations of rank one 6d SCFTs with no gauge symmetry, namely, the E-string and the M-string theories. We first revisit this argument, and then generalise it to cover 6d SCFTs with gauge symmetry as well.

Review of argument for 6d SCFTs with no gauge symmetry. Let us first briefly summarise the salient points of the argument in our previous paper. For 6d SCFTs with no gauge symmetry, the elliptic blowup equations read[23]

$$\begin{aligned}&\sum_{d'+d''=d}\theta_i^{[a]}(n\tau,\lambda_F\cdot m_F+y(\epsilon_1+\epsilon_2)-n(d'\epsilon_1+d''\epsilon_2)) \\ &\cdot\mathbb{E}_{d'}(\tau,m_F+\lambda_F\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d''}(\tau,m_F+\lambda_F\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2) \\ &\quad=\theta^{[a]}(n\tau,\lambda_F+y(\epsilon_1+\epsilon_2))\mathbb{E}_d(\tau,m_F,\epsilon_1,\epsilon_2)\end{aligned} \tag{3.27}$$

where

$$y=\frac{n-1}{4}+\frac{1}{2}(\lambda_F,\lambda_F). \tag{3.28}$$

To explain this equation, the idea is to compute the Nekrasov partition function of the 6d SCFT on $T^2\times\widehat{\mathbb{C}}^2$, first with a finite size of the exceptional divisor $\mathbb{P}^1$ of the blowup

[23] λ_F here equals $\frac{1}{2}r_m$ in [9].

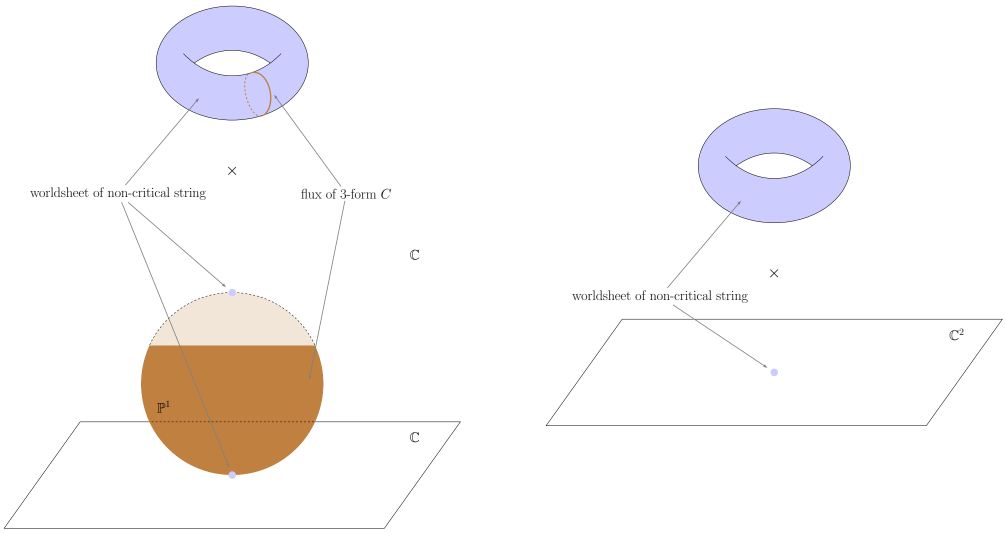

Figure 3. Localised contributions to the partition function of 6d SCFT on $T^2 \times \widehat{\mathbb{C}}^2$ (left) and on $T^2 \times \mathbb{C}^2$ (right).

space $\widehat{\mathbb{C}}^2$ (left hand side of the equation). As illustrated in the left panel of figure 3, non-trivial contributions come from the world-sheet of the non-critical string wrapped around the T^2 and localised at the north pole or the south pole of the $\mathbb{P}^1$ as well as the flux of the field strength of the tensor field through both the $\mathbb{P}^1$ and a one-cycle in T^2. Then we compute the same partition function with the exceptional $\mathbb{P}^1$ blown down (right hand side of the equation). As illustrated in the right panel of figure 3, now the only non-trivial contributions come from the worldsheet of the non-critical string wrapped around the T^2 and localised at the center of the $\mathbb{C}^2$. Since the partition function does not depend on the size of the $\mathbb{P}^1$, we can identify the partition functions on both sides, which yields the blowup equations. In the following, we consider these individual contributions more carefully and derive their explicit forms as appearing in the blowup equation.

We first consider the left hand side of the equation. There are essentially two types of configurations with finite energy. The first type corresponds to the worldsheet of non-critical string wrapping the torus. The strings appear as solitons in $\widehat{\mathbb{C}}^2$ and they are localised at the north pole and the south pole of the $\mathbb{P}^1$, whose neighborhoods locally resemble the Omega background with twist parameters $(\epsilon_1, \epsilon_2 - \epsilon_1)$ and $(\epsilon_1 - \epsilon_2, \epsilon_2)$ respectively. Suppose there are d' and d'' solitons localised at the north pole and the south pole respectively, the one-loop contribution of this saddle point configuration gives rise to the elliptic genera

$$\mathbb{E}_{d'}(\tau, m_F + \lambda_F \epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1)\mathbb{E}_{d''}(\tau, m_F + \lambda_F \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) \tag{3.29}$$

The shifted flavor mass is to account for the embedding of the $\mathrm{SU}(2)_R$-symmetry in the flavor symmetry.

The second type of finite energy configuration corresponds to the flux of the self-dual 3-form C, which is the field strength of the tensor field, through $\mathbb{P}^1 \times S^1$. Either by an argument of computing the partition function of 4d theory resulting from torus reduction of 6d theory, or by a holographic argument, we found [9] that this type of configuration gives rise to a generalised theta function of the form

$$\Theta_{\Omega}^{[a]}(\tau, z) = \sum_{n_i \in \mathbb{Z}} \exp\left(\frac{1}{2}\Omega_{ij}(n_i + a_i)(n_j + a_j)\tau + \Omega_{ij}(n_i + a_i)z_j\right), \tag{3.30}$$

where $\Omega_{ij} = -A_{ij}$ is the opposite of the base intersection matrix. These theta functions with different characteristics a are sections of a line bundle over the torus

$$\mathbb{T} = \mathbb{C}^r/(\Omega\mathbb{Z}^r \oplus \tau\Omega\mathbb{Z}^r), \tag{3.31}$$

where r is the number of base curves, and the total number of independent sections is equal to the determinant of Ω. The elliptic parameter of the theta function is given by

$$z_i = \int_{S^1_A \times \mathbb{P}^1} C_i + \tau \int_{S^1_B \times \mathbb{P}^1} C_i \tag{3.32}$$

where $S^1_{A,B}$ are the two 1-cycles on the torus, and the 4-form fluxes C_i are identified with anti-derivative of the four-forms X_i in the Green-Schwarz counter-term (2.8). In a rank one 6d SCFT, the anomaly four-form reads

$$X = -\frac{n}{4}\operatorname{Tr} F_{\mathfrak{g}}^2 + \frac{k_F}{4}\operatorname{Tr} F_{\mathfrak{f}}^2 + h^\vee_{\mathfrak{g}} c_2(R) - \frac{2-n}{4} p_1(M_6). \tag{3.33}$$

Here $F_{\mathfrak{g}}$ and $h^\vee_{\mathfrak{g}}$ are the field strength and the dual Coxeter number of gauge symmetry. In a theory with trivial gauge symmetry, we suppress $F_{\mathfrak{g}}$ and set $h^\vee_{\mathfrak{g}}$ to 1. $F_{\mathfrak{f}}$ is the flavor symmetry field strength, and k_F is the level of the associated current algebra on the string worldsheet, which is 1 in the E-, M-string theories. We expect therefore the elliptic parameter to be

$$z = \int_{S^1 \times \mathbb{P}^1} \frac{k_F}{4}\omega_{\mathfrak{f}} + \omega_R - \frac{2-n}{4} p_1^{-1}(M_6), \tag{3.34}$$

where

$$\omega_{\mathfrak{f}} = \operatorname{Tr}\left(\frac{2}{3}A_{\mathfrak{f}}^3 + A_{\mathfrak{f}} \wedge \mathrm{d}A_{\mathfrak{f}}\right), \quad \omega_R = \operatorname{Tr}\left(\frac{2}{3}A_R^3 + A_R \wedge \mathrm{d}A_R\right) \tag{3.35}$$

and p_1^{-1} is the anti-derivative of the Pontryagin class. We compute

$$\frac{k_F}{4}\int_{S^1 \times \mathbb{P}^1} \omega_{\mathfrak{f}} = k_F \lambda_F \cdot m_F \tag{3.36}$$

and

$$\int_{S^1 \times \mathbb{P}^1} \omega_R = \epsilon_+/2 \tag{3.37}$$

as well as

$$\int_{S^1 \times \mathbb{P}^1} p_1^{-1}(M_6) = \epsilon_1 + \epsilon_2 = 2\epsilon_+. \tag{3.38}$$

Due to the embedding of the R-symmetry, we shift

$$m_F \to m_F + \lambda_F \epsilon_+. \tag{3.39}$$

Putting all these pieces together, we thus recover the main part of elliptic parameter of theta function in (3.27), (3.28). Finally, the coupling to the number of strings in the elliptic parameter is due to the modification of the three form flux C in the presence of string source, in which case [9]

$$C = X + d'\chi_4(N') + d''\chi_4(N'') \tag{3.40}$$

where $\chi_4(N'), \chi_4(N'')$ are the Euler classes of the normal bundles of the strings localised at the north pole and the south pole of $\mathbb{P}^1$. Integrating the anti-derivative $e_3^{(0)} = \chi_4^{-1}(N)$ of the Euler class through the enveloping 3-manifold of the string worldsheet, which is $\mathbb{P}^1 \times S^1$, gives us the coupling terms $n(\epsilon_1 d' + \epsilon_2 d'')$.

In order to obtain the right hand side of the blowup equation, we simply choose the exceptional $\mathbb{P}^1$ to be very small. Looking from far away, the $d = d' + d''$ solitons of strings are clustered at the origin of $\mathbb{C}^2$. The enveloping 3-manifold of these strings becomes S^3 which does not intersect the exceptional $\mathbb{P}^1$ and hence the integral of $e_3^{(0)}$ vanishes.

Extension of argument to 6d SCFTs with gauge symmetry. Now we would like to generalise this argument to the cases with non-trivial gauge symmetries. For simplicity we focus on unity equations of rank one theories. The presence of gauge symmetry entails three modifications to the argument above. First, now we need to use the proper value of $h_{\mathfrak{g}}^\vee$, and turn on the gauge flux $F_{\mathfrak{g}}$ in (3.33). Following a similar computation as (3.36), we find

$$-\frac{n}{4}\int_{S^1\times\mathbb{P}^1} \omega_{\mathfrak{g}} = -n\left(\frac{1}{4}\int_{\mathbb{P}^1} F_{\mathfrak{g}}\right)\cdot m_G = -n\lambda_G \cdot m_G. \tag{3.41}$$

Since both gauge and flavor symmetries of the 6d SCFT are global symmetries on the string worldsheet, we need to embed $\mathrm{SU}(2)_R$ in the gauge symmetry as well, which leads to a similar shift as (3.39)

$$m_G \to m_G + \lambda_G \epsilon_+. \tag{3.42}$$

These extra ingredients allow us to recover the elliptic parameter of the theta function in (3.1), (3.2). Second, the gauge flux is coupled to the tensor field by the following term in the Lagrangian

$$\mathcal{L} \supset \int \phi \frac{1}{4}\operatorname{Tr} F_{\mathfrak{g}} \wedge \star F_{\mathfrak{g}} \tag{3.43}$$

This means with the gauge flux is present, the tensor modulus is weighted not only by non-critical strings but in addition by $||\lambda_G||^2/2$, which explains the balancing condition

$$||\lambda_G||^2/2 + d' + d'' = d \tag{3.44}$$

in (3.1).

Finally, we would like to explain the appearance of the factors $A_V(\tau, m_G, \lambda_G)$, $A_H(\tau, m_{G,F}, \lambda_{G,F})$ whose expressions are given by (3.7), (3.8), (3.11), (3.12). Let us first look at A_V which is given as a product of theta functions in the denominator. We interpret each such theta function as arising from the oscillator modes of a complex bosonic coordinate or two real such coordinates in the target space of a 2d sigma model with the torus of complex structure τ being the worldsheet [83]. Next, we want to count the bosonic dimension of this target space. To this end, observe that the modular index of each ingredient is

$$\operatorname{Ind} \breve{\theta}_V(z,R) = -R^2\frac{z^2}{2} + \dots \tag{3.45}$$

Thus, the total modular index of A_V becomes

$$\operatorname{Ind} A_V = \sum_{\beta\in\Delta_+} -(\beta\cdot\lambda_G)^2\frac{z^2}{2} + \dots \tag{3.46}$$

Now we know that the modular index of a single theta function in the denominator is

$$\operatorname{Ind} \frac{1}{\theta_1(z)} = -\frac{z^2}{2}. \tag{3.47}$$

Thus we conclude that the total number of theta-functions in A_V is

$$\sum_{\beta\in\Delta_+} R^2 = \sum_{\beta\in\Delta_+}(\beta\cdot\lambda_G)(\beta\cdot\lambda_G) = \frac{1}{2}\sum_{\beta\in\Delta}(\beta\cdot\lambda_G)(\beta\cdot\lambda_G) = h^\vee_G||\lambda_G||^2. \tag{3.48}$$

Using $d_0 = \frac{1}{2}||\lambda_G||^2$, we see that the total bosonic dimension of our target space $\mathcal{M}$ is

$$\dim_{\mathbb{R}}\mathcal{M} = 2||\lambda_G||^2 h^\vee_G = 4h^\vee_G d_0. \tag{3.49}$$

But this is nothing else than the dimension of the moduli space of d_0 G-instantons, denoted by $\mathcal{M}_{G,d_0}$. The arguments in the theta-functions in (3.11) are obtained from equivariant localization on $\mathcal{M}_{G,d_0}$, see [16]. A similar argument can be worked out for A_H describing the fermionic coordinates on the moduli space. Thus we see that the factors A_V and A_H arise from path integrals over collective coordinates of strings moving in the moduli space of d_0 G-instantons! Therefore, the instanton contributions on the left-hand side of (3.1) split into three pieces: strings localized at the north pole of the exceptional $\mathbb{P}^1$, strings localized at the south pole, and non-localized strings on the $\mathbb{P}^1$. This mirrors and generalizes the three contributions for the instanton moduli space on the blowup geometry found in [16].

3.3 Modularity

Here we discuss the modularity constraint. For both unity and vanishing elliptic blowup equations, every term on the l.h.s. of (3.1) should have the same modular index independent of the summation index λ_G, d', d''. The modular index polynomial for the generalised theta function $\theta_i^{[a]}(n\tau,z)$ is

$$\operatorname{Ind} \theta_i(n\tau,z) = \frac{1}{2n}z^2. \tag{3.50}$$

The modular index polynomial for d-string elliptic genus for a rank one model can be deduced from (2.34) [5, 33]

$$\operatorname{Ind}\mathbb{E}_d(\epsilon_1,\epsilon_2,m_G,m_F) = -\left(\frac{\epsilon_1+\epsilon_2}{2}\right)^2(2-n+h^\vee_{\mathfrak{g}})d + \frac{\epsilon_1\epsilon_2}{2}(nd^2+(2-n)d)$$
$$+\frac{d}{2}(-n\, m_G\cdot m_G + k_F\, m_F\cdot m_F)\,.$$

Here if the flavor symmetry has a $\mathfrak{u}(1)$ factor, we should replace

$$k_F m_F \cdot m_F \to k_F m_F \cdot m_F + k_{\mathfrak{u}(1)} m_{\mathfrak{u}(1)}^2. \tag{3.51}$$

Finally the index polynomials of A_V, A_H can be calculated from their definitions (3.7), (3.8), (3.11), (3.12); in particular, the following results are useful

$$\mathrm{Ind}\breve{\theta}_V(z,R) = -\frac{R^2 z^2}{2} - \frac{(R-1)R(R+1)}{3} z(\epsilon_1+\epsilon_2) - \frac{(R-1)R^2(R+1)}{12}(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2), \tag{3.52}$$

$$\begin{aligned}\mathrm{Ind}\breve{\theta}_H(z,R) = {} & \frac{(R+1/2)(R-1/2)}{4} z^2 + \frac{R(R-1/2)(R+1/2)}{6} z(\epsilon_1+\epsilon_2) \\ & + \frac{(R-1/2)(R+1/2)(R^2-3/4)}{24}(\epsilon_1^2+\epsilon_2^2) + \frac{(R-1/2)(R+1/2)(R^2+3/4)}{24}\epsilon_1\epsilon_2.\end{aligned} \tag{3.53}$$

If we compute the modular index polynomial of an arbitary term on the l.h.s. subtracted by that of the r.h.s., we find that the dependence on λ_G, d', d'' cancel completely thanks to the choice of y (3.2) and the constraints on the number of hypermultiplets imposed by the anomaly cancellation conditions (2.14), (2.15), (2.16), (2.18), (2.20), (2.19). What remains is a quadratic polynomial of the following form

$$\mathrm{Ind}(\delta, d, m_G, \epsilon_{1,2}, \lambda_F) = \delta^2 P_2(d, m_G, \epsilon_{1,2}, \lambda_F) + \delta P_1(d, m_G, \epsilon_{1,2}, \lambda_F) + P_0(d, m_G, \epsilon_{1,2}, \lambda_F). \tag{3.54}$$

where

$$\begin{aligned}P_0(d,m_G,\epsilon_{1,2},\lambda_F) = {} & \frac{1}{8} n_{R_G,R_F}(2\mathrm{ind}_{R_G}(m_G\cdot m_G)\mathrm{id}_1 + \dim R_G \mathrm{id}_2) + \frac{1}{12} n_{R_G,R_F} \dim R_G \mathrm{id}_3(\epsilon_1+\epsilon_2) \\ & + \frac{1}{48} n_{R_G,R_F} \dim R_G(\mathrm{id}_4 - \mathrm{id}_1)(\epsilon_1+\epsilon_2)^2 \\ & - \left(\frac{1}{2} n_{R_G,R_F} \mathrm{ind}_{R_G} d\, \mathrm{id}_1 + \frac{1}{96} n_{R_G,R_F} \dim R_G(2\mathrm{id}_4 - 5\mathrm{id}_1)\right)\epsilon_1\epsilon_2\end{aligned} \tag{3.55}$$

and

$$\mathrm{id}_1 = \sum_{\omega\in R_F} \left((\omega\cdot\lambda_F)^2 - \frac{1}{4}\right), \tag{3.56}$$

$$\mathrm{id}_2 = \sum_{\omega\in R_F} \left((\omega\cdot\lambda_F)^2(\omega\cdot m_F)^2 - \frac{1}{4}(\omega\cdot m_F)^2\right), \tag{3.57}$$

$$\mathrm{id}_3 = \sum_{\omega\in R_F} \left((\omega\cdot\lambda_F)^3(\omega\cdot m_F) - \frac{1}{4}(\omega\cdot\lambda_F)(\omega\cdot m_F)\right), \tag{3.58}$$

$$\mathrm{id}_4 = \sum_{\omega\in R_F} \left((\omega\cdot\lambda_F)^4 - \frac{1}{4}(\omega\cdot\lambda_F)^2\right). \tag{3.59}$$

In the case of unity blowup equations with $\delta = 0$, the index polynomial $\mathrm{Ind}(0, d, m_G, \epsilon_{1,2}, \lambda_F)$ should vanish identically for arbitrary values of $d, \epsilon_{1,2}, m_G$. This implies the additional conditions

$$\mathrm{id}_1 = \mathrm{id}_2 = \mathrm{id}_3 = \mathrm{id}_4 = 0. \tag{3.60}$$

The checkerboard pattern condition (3.15) together with the first condition (3.56) above lead to the identity

$$\omega \cdot \lambda_F = \pm\frac{1}{2}, \quad \omega \in R_F, \tag{3.61}$$

with which the other three conditions above (3.57), (3.58), (3.59) are automatically satisfied. The identity (3.61) fixes the norm of λ_F

$$\lambda_F \cdot \lambda_F = \frac{1}{2\,\mathrm{ind}_{R_F}} \sum_{\omega \in R_F} (\omega \cdot \lambda_F)^2 = \frac{h^\vee_{\mathfrak{g}} - 3n + 6}{2k_F}, \tag{3.62}$$

where we have used (2.14), (2.18). The expression (3.2) can then be simplified to

$$y = \frac{h^\vee_{\mathfrak{g}} - n + 2}{2}. \tag{3.63}$$

The identity (3.61) completely characterises the coweight vector λ_F for unity blowup equations. There are only four types of flavor symmetry. The half-hypers transform in either $\mathbf{2n}$ of $\mathfrak{sp}(n)$, or $\mathbf{2n}$ of $\mathfrak{so}(2n)$,[24] or $\mathbf{n} \oplus \overline{\mathbf{n}}$ of $\mathfrak{u}(n)$ or $\mathfrak{su}(n)$. In the case of $F = \mathfrak{sp}(n)$, the representation $\mathbf{2n}$ has weights[25]

$$\pm\,\omega^{(i)}, \quad i = 1, \ldots, n \tag{3.64}$$

where $w^{(i)}$ are all independent. The coweight λ_F is characterised by

$$\lambda_F{}^{(i)} := \omega^{(i)} \cdot \lambda_F = \pm\frac{1}{2}. \tag{3.65}$$

This implies that when presented as a vector in $\mathbb{R}^n$ with e_i^* as the standard basis, the coweight λ_F is

$$\lambda_F = \sum_i \lambda_F{}^{(i)} e_i^* = \pm\frac{1}{2}e_1^* \pm \ldots \pm \frac{1}{2}e_n^* \tag{3.66}$$

and there are 2^n of them. The Weyl group of $\mathfrak{sp}(n)$ permutes the components $\lambda_F{}^{(i)}$ and flips signs of $\lambda_F{}^{(i)}$, and therefore all the coweights λ_F are in a single Weyl orbit whose dominant element is

$$\lambda_F = \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_n^* = (0\ldots01), \tag{3.67}$$

where in the last equality we give the Dynkin labels of λ_F. In the case of $F = \mathfrak{so}(2n), \mathfrak{u}(n)$, it is completely the same with the characterisation (3.65) of λ_F and there are 2^n of them. The difference is that for $F = \mathfrak{so}(2n)$ the Weyl group permutes $\lambda_F{}^{(i)}$ and flips *pairs* of $\lambda_F{}^{(i)}$ and thus all λ_F are in two Weyl orbits whose dominant elements are

$$\lambda_F = \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_{n-1}^* + \frac{1}{2}e_n^* = (0\ldots01), \tag{3.68}$$

and

$$\lambda_F = \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_{n-1}^* - \frac{1}{2}e_n^* = (0\ldots10). \tag{3.69}$$

[24]Flavor symmetries of type $\mathfrak{so}(2n+1)$ only appears in rank one models with no unity blowup equations and thus we do not have to consider them here.

[25]The weights $\omega^{(i)}$ are the standard basis of $\mathbb{R}^n$.

while the flavor group $F = \mathfrak{u}(n)$ is usually presented as the product $F = \mathfrak{su}(n) \times \mathfrak{u}(1)$, and λ_F is presented as a coweight of $\mathfrak{su}(n)$ plus a $\mathfrak{u}(1)$ charge. Finally in the case of $F = \mathfrak{su}(n)$, the representation $\mathbf{n} \oplus \overline{\mathbf{n}}$ has weights (3.64) subject to the constraint $\omega^{(1)} + \ldots + \omega^{(n)} = 0$. The coweight λ_F is then characterised by (3.65) with the constraint

$$\lambda_F{}^{(1)} + \ldots + \lambda_F{}^{(n)} = 0. \tag{3.70}$$

The number of such λ_F is $\binom{n}{n/2}$. They are all in a single Weyl orbit whose dominant element is

$$\lambda_F = (0 \ldots 010 \ldots 0). \tag{3.71}$$

Note that when the flavor symmetry is $\mathfrak{su}(n)$, n is always an even integer and therefore the number $\binom{n}{n/2}$ makes sense. All the λ_F for unity blowup equations in tables 5, 6 can be determined in this way, except for the models $n = 2, G = \mathfrak{su}(2)$ and $n = 1, G = \mathfrak{sp}(N), N \geq 1$, where the Higgsing procedure discussed in section 3.4 imposes further constraints.

3.4 Blowup equations along Higgsing tree

Let us consider a rank one 6d SCFT $\mathcal{T}$ with gauge symmetry G Higgsed to a daughter theory $\mathcal{T}'$ with gauge symmetry G'. We expect that the weight lattice of G is projected by a surjective map f to the weight lattice of G'. In particular, the vector multiplets transform in the adjoint representation of G whose weight space decomposes under the projection by

$$f : \Delta \to \Delta' \oplus R'. \tag{3.72}$$

In addition there is a set of hypermultiplets transforming in a representation R whose weight space decomposes under the projection by

$$f : R \to R' \oplus \mathbf{1}. \tag{3.73}$$

To trigger the Higgsing, we should give non-zero vev to the scalar in the hypermultiplet (H_0) transforming in the singlet after the projection. The hypermultipets transforming in R' then become massive. They get eaten by the vector multiplets transforming in R' and decouple together. Concretely this operation is realised as follows. The vev of a hypermultiplet with gauge and flavor charges ω_G, ω_F is

$$\omega_G \cdot m_G + \omega_F \cdot m_F. \tag{3.74}$$

First of all, the gauge charge carried by the hypermultiplet (H_0) should be in $\mathrm{Ker}(f) \cap R$, and thus we set

$$\omega_G \cdot m_G = 0, \quad \omega_G \in \mathrm{Ker}(f) \cap R. \tag{3.75}$$

Secondly, let the hypermultiplet (H_0) carry certain flavor charge $\omega_F \in R_F$.[26] We set values for the following components of m_F

$$\omega_F \cdot m_F = \mp \frac{\epsilon_1 + \epsilon_1}{2}, \quad \text{for } \omega_F \cdot \lambda_F = \pm \frac{1}{2}. \tag{3.76}$$

[26]More precisely, the hyper H_0 consists of two half-hypers carrying flavor charges $\pm\omega_F$.

It has been shown in the literature (for instance [14]) that elliptic genera reduce properly along the Higgsing tree through the limits of parameters (3.75), (3.76). We find that all the other components of unity elliptic blowup equations in (3.1) also reduce properly and therefore the unity blowup equations transform consistently along the Higgsing trees.

- The contributions to A_V from vector multiplets transforming in R' cancel with the contributions to A_H from hypermultiplets transforming in the same representation. Pick a pair $\pm\omega_G \in R$ which is mapped to R' and let $+\omega_G$ be the weight in Δ_+. If $\omega_G \cdot \lambda_G > 0$, the vector charged with ω_G contributes

$$\prod_{\substack{k,l\geq 0\\ k+l\leq \omega_G\cdot\lambda_G-1}} \frac{\eta}{\theta_1(\omega_G\cdot m_G+k\epsilon_1+l\epsilon_2)} \prod_{\substack{k,l\geq 0\\ k+l\leq \omega_G\cdot\lambda_G-2}} \frac{\eta}{\theta_1(\omega_G\cdot m_G+(k+1)\epsilon_1+(l+1)\epsilon_2)}, \tag{3.77}$$

and the hypers charged with $\pm\omega_G$ and ω_F contribute

$$\begin{aligned}&\prod_{\substack{k,l\geq 0\\ k+l\leq \omega_G\cdot\lambda_G+\omega_F\cdot\lambda_F-\frac{3}{2}}} \frac{\theta_1(\omega_G\cdot m_G+\omega_F\cdot m_F+(k+\frac{1}{2})\epsilon_1+(l+\frac{1}{2})\epsilon_2)}{\eta}\\ \times &\prod_{\substack{k,l\geq 0\\ k+l\leq \omega_G\cdot\lambda_G-\omega_F\cdot\lambda_F-\frac{3}{2}}} \frac{\theta_1(-\omega_G\cdot m_G+\omega_F\cdot m_F-(k+\frac{1}{2})\epsilon_1-(l+\frac{1}{2})\epsilon_2)}{\eta}.\end{aligned} \tag{3.78}$$

If $\omega_G \cdot \lambda_G < 0$, the vector charged with ω_G contributes

$$\prod_{\substack{k,l\geq 0\\ k+l\leq -\omega_G\cdot\lambda_G-1}} \frac{\eta}{\theta_1(\omega_G\cdot m_G-k\epsilon_1-l\epsilon_2)} \prod_{\substack{k,l\geq 0\\ k+l\leq -\omega_G\cdot\lambda_G-2}} \frac{\eta}{\theta_1(\omega_G\cdot m_G-(k+1)\epsilon_1-(l+1)\epsilon_2)}, \tag{3.79}$$

and the hypers charged with $\pm\omega_G$ and ω_F contribute

$$\begin{aligned}&\prod_{\substack{k,l\geq 0\\ k+l\leq -\omega_G\cdot\lambda_G+\omega_F\cdot\lambda_F-\frac{3}{2}}} \frac{\theta_1(-\omega_G\cdot m_G+\omega_F\cdot m_F+(k+\frac{1}{2})\epsilon_1+(l+\frac{1}{2})\epsilon_2)}{\eta}\\ \times &\prod_{\substack{k,l\geq 0\\ k+l\leq -\omega_G\cdot\lambda_G-\omega_F\cdot\lambda_F-\frac{3}{2}}} \frac{\theta_1(\omega_G\cdot m_G+\omega_F\cdot m_F-(k+\frac{1}{2})\epsilon_1-(l+\frac{1}{2})\epsilon_2)}{\eta}\end{aligned} \tag{3.80}$$

In both cases, the contributions of vectors and hypers cancel up to a sign if we set (3.76).

- The coroot lattice $Q^\vee$ over which the summation index λ_G runs reduces properly to the coroot sub-lattice $Q^{\vee,\prime}$ of the daughter theory. As discussed in appendix A.2, the projection f induces an injection f^* from $Q^{\vee,\prime}$ to $Q^\vee$, which preserves the norms of coroot vectors. If λ_G is in the image of f^*, we know from (A.24) that $\omega_G \cdot \lambda_G = 0$ for $\omega_G \in \mathrm{Ker}(f) \cap R$ projected to the singlet by f. The contribution to A_{H} by the hyper in the singlet collapses to 1, and the term corresponding to λ_G survives in the limit (3.75), (3.76). If, however, $\lambda_G \in Q^\vee$ is not in the image of f^*, $\omega_G \cdot \lambda_G \neq 0$ for $\omega_G \in \mathrm{Ker}(f) \cap R$. The hyper in the singlet contributes to A_{H} by at least

$$\frac{\theta_1(m_G \cdot \omega_G + \omega_F \cdot m_F \pm \frac{1}{2}(\epsilon_1+\epsilon_2))}{\eta}, \quad \omega_G \in \mathrm{Ker}(f) \cap R \tag{3.81}$$

which becomes zero in the limit (3.75), (3.76), thus annihilating the corresponding term.

- The only terms in the argument of the generalised theta function affected by Higgsing procedure are

$$\begin{aligned} &k_F \lambda_F \cdot m_F + y(\epsilon_1+\epsilon_2) \\ &= \mathrm{ind}_{R_G} \sum_{\omega_F \in R_F} \left((\omega_F \cdot \lambda_F)(\omega_F \cdot m_F) + \frac{\epsilon_1+\epsilon_2}{2} \left((\omega_F \cdot \lambda_F)(\omega_F \cdot \lambda_F) + \frac{1}{4} \right) \right) \\ &= \mathrm{ind}_{R_G} \sum_{\omega_F \in R_F} \left((\omega_F \cdot \lambda_F)(\omega_F \cdot m_F) + \frac{\epsilon_1+\epsilon_2}{4} \right) \end{aligned} \tag{3.82}$$

which reduces properly to the argument of the daughter under the limit (3.76).

- There could be extra signs coming from the map of $(-1)^{|\lambda_G|}$, from projection of positive roots,[27] and from cancellation of one-loop contributions. We checked with concrete examples that the extra sign is always positive.

Let us comment that the case of vanishing blowup equations is very different: it is in general not possible to transform vanishing blowup equations along Higgsing trees. There are two arguments for this. First, recall that in unity blowup equations the summation index λ_G takes value in the unshifted coroot lattice and it is crucial that in the Higgsing process from a mother theory $\mathcal{T}$ with coroot lattice $Q^\vee$ to a daughter theory $\mathcal{T}'$ with coroot sub-lattice $Q^{\vee,\prime}$ there exists an injection f^* from the coroot lattice $Q^{\vee,\prime}$ to $Q^\vee$ induced by the projection f. In vanishing blowup equations, on the other hand, the summation index λ_G takes value in the coweight lattice, or more precisely in the shifted coroot lattice. Unfortunately, a similar map from $P^{\vee,\prime}$ of a daughter theory to $P^\vee$ of a mother theory does not exist as discussed in appendix A.2. Another way to see this is that the leading order identities of vanishing blowup equations do not transform to each other properly along the Higgsing trees in the limit (3.75), (3.76).

In the remainder of the section, we discuss in detail some typical examples of the Higgsing procedure for unity blowup equations as well as some mild constraints on λ_F this procedure entails.

3.4.1 Examples of Higgsing unity blowup equations

- $n = 6, G = E_7 \to E_6$: representations of gauge symmetry decompose by

$$\begin{aligned} \mathbf{133} &\to \mathbf{78} \oplus \mathbf{27} \oplus \overline{\mathbf{27}} \oplus \mathbf{1} \quad (V) \\ \mathbf{56} &\to \mathbf{27} \oplus \overline{\mathbf{27}} \oplus 2 \cdot \mathbf{1} \quad (H) \end{aligned} \tag{3.83}$$

[27] Even if a positive root of G is mapped to a root of G', it is not necessary still *positive*. We may have to flip its sign.

Here (V) means vectors and (H) means hypers. We mark in red the multiplets whose 1-loop contributions cancel with each other after Higgsing. In these models components ${\lambda_F}^{(i)}$ of λ_F are free. Since only one hyper is used in the Higgsing, one component of λ_F is fixed and the number of different λ_F is reduced by half.
Most other models follow the same procedure of Higgsing.

- $n = 4, G = E_7 \supset E_6$: the gauge symmetry branching rules are the same as (3.83), and naively the number of λ_F is reduced by half from 4 to 2. On the other hand, the remaining hyper in **56** also branches as in the second line of (3.83) and it splits to two hypers with opposite charges, which have opposite flavor masses and λ_F components. In the daughter theory, the two flavor masses can actually be made independent and the flavor symmetry is enhanced. We can use the enhanced Weyl group to generate full Weyl orbits from the reduced λ_F and increase the number of λ_F from 2 to 4. Many other models are Higgsed by a similar procedure including
  - $n = 3, G = \mathfrak{so}(9) \to \mathfrak{so}(8), F_4 \to \mathfrak{so}(8)$,
  - $n = 2, G = E_7 \to E_6, \mathfrak{so}(11) \to \mathfrak{so}(10), \mathfrak{so}(9) \to \mathfrak{so}(8), F_4 \to \mathfrak{so}(8), G_2 \to \mathfrak{su}(3)$,
  - $n = 1, G = \mathfrak{so}(9) \to \mathfrak{so}(8), F_4 \to \mathfrak{so}(8), G_2 \to \mathfrak{su}(3)$.

- $n = 2, G = \mathfrak{su}(N) \to \mathfrak{su}(N-1)$ for $N \geq 3$: the gauge symmetry branching rules for the cancelling vectors and hypers are

$$\begin{aligned} \Delta_N &\to \Delta_{N-1} \oplus (\mathbf{N-1}) \oplus \overline{(\mathbf{N-1})} \oplus \mathbf{1} \quad (V) \\ \mathbf{N} &\to (\mathbf{N-1}) \oplus \mathbf{1} \quad (H) \\ \overline{\mathbf{N}} &\to \overline{(\mathbf{N-1})} \oplus \mathbf{1} \quad (H) \end{aligned} \tag{3.84}$$

while the remaining $2N-2$ hypers also branch by the last two lines of (3.84). In these models, the flavor symmetry is of type $\mathfrak{su}(2N)$ and the components ${\lambda_F}^{(i)}$ are always subject to the constraint (3.70). In the Higgsing procedure, two hypers are decoupled and thus two components of λ_F should be fixed. In order to maintain the condition (3.70), we should fix two components of λ_F as in (3.76) with opposite signs. The number of λ_F is then reduced from $\binom{2N}{N}$ to $\binom{2N-2}{N-1}$.
The Higgsing procedure for $n = 1, G = \mathfrak{su}(N) \to \mathfrak{su}(N-1)$ with $N \geq 3$ is similar.

- $n = 2, G = \mathfrak{su}(2) \to \emptyset$: we follow the same branching rules in the gauge sector as (3.84). The λ_F obtained by Higgsing should still follow (3.70), and the number of λ_F of the daughter theory, which is the M-string theory, is therefore $\binom{2}{1} = 2$.
Note that in the mother theory with $G = \mathfrak{su}(2)$, since the fundamental and the anti-fundamental representations of $\mathfrak{su}(2)$ are isomorphic, one usually expects the flavor symmetry is enhanced from $\mathfrak{su}(4)$ to $\mathfrak{so}(8)$, and consequently the lifting of the constraint (3.70). This however would mean too many λ_F that will be reduced to the M-string theory. Since there are only two λ_F for the latter [9], we conclude the condition (3.70) cannot be lifted for the mother theory.

- $n = 1, G = \mathfrak{sp}(N) \to \mathfrak{sp}(N-1)$ with $(N \geq 1)$: recall that the analysis of modularity constraint in section 3.3 together with checkerboard pattern constraint indicates that λ_F for unity equations should be

$$\lambda_F = \begin{cases} \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_{n-1}^* + \frac{1}{2}e_n^* = (0\ldots01), \\ \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_{n-1}^* - \frac{1}{2}e_n^* = (0\ldots10), \end{cases} \tag{3.85}$$

up to Weyl transformations. For unity blowup equations to be Higgsed properly, one of the two possibilities has to be eliminated. Upon Higgsing, the gauge symmetry branching rules of canceling vectors and hypers are

$$\begin{aligned} &\Delta_N \to \Delta_{N-1} \oplus 2(\mathbf{2N-2}) \oplus 3\cdot\mathbf{1} \quad (V) \\ &\mathbf{2N} \to (\mathbf{2N-2}) \oplus 2\cdot\mathbf{1} \quad (H) \end{aligned} \tag{3.86}$$

Here we need two hypers to cancel with massive vectors, and we set the corresponding components of m_F by (3.76) depending on the value of λ_F components. On the other hand, the reduced one-string elliptic genus of the $n = 1, G = \mathfrak{sp}(N)$ theory reads [12]

$$\mathbb{E}_1^{\text{red}}(v, q, m_G, m_F) = \frac{1}{2}\sum_{j=1,2,3,4}\left(\prod_{i=1}^{8+2N}\frac{\theta_j(m_F^i)}{\eta}\right)\left(\prod_{i=1}^{N}\frac{\eta^2}{\theta_j(\epsilon_+ + m_G^i)\theta_j(\epsilon_+ - m_G^i)}\right). \tag{3.87}$$

Clearly the elliptic genus can only be properly Higgsed if a pair of m_F^i, m_F^j take the limit (3.76) with the same sign, and correspondingly we should fix two components of λ_F with the same value. This indicates the following chain of Higgsing for λ_F

$$\lambda_F : \begin{cases} (\underbrace{0\ldots001}_{2N+8}) \to (\underbrace{0\ldots001}_{2N+6}) \to \ldots \to (\underbrace{0\ldots01}_{8}) = \frac{1}{2}e_1^* + \ldots + \frac{1}{2}e_8^* \\ (\underbrace{0\ldots010}_{2N+8}) \to (\underbrace{0\ldots010}_{2N+6}) \to \ldots \to (\underbrace{0\ldots10}_{8}) = \frac{1}{2}e_1^* + \ldots - \frac{1}{2}e_8^* \end{cases} \tag{3.88}$$

At the end of the Higgsing chain, we find the E-string theory with $G = \mathfrak{sp}(0)$. The flavor symmetry is in fact enhanced from $\mathfrak{so}(16)$ to E_8. The (co)weight lattice of E_8, however, is a sub-lattice with index two of the (co)weight lattice of $\mathfrak{so}(16)$. Between the two coweights at the end of the chain in (3.88) only $\lambda_F = (0\ldots01)$ can be lifted to a coweight vector of E_8, which we can use together with the Weyl group of E_8 to generate the full Weyl orbit $\mathcal{O}_{2,240}$. This means going back the Higgsing chain the λ_F can *never* take the value of $(0\ldots10)$ for $n = 1, G = \mathfrak{sp}(N)$ models.

3.5 K-theoretic blowup equations

When taking the K-theoretic limit $q \to 0$, the elliptic genera of a 6d theory T_{6d} in general reduce to the K-theoretic instanton partition function of the 5d theory T_{5d} with the same gauge, flavor group and the same matter contents on a circle of radius one,[28] and the elliptic blowup equations in general reduce to the K-theoretic blowup equations. For example, it

[28]The radius can be easily recovered to arbitrary β by dimensional analysis.

is easy to find that the unity elliptic blowup equations (3.1) with $n \geq 3$ in the $q \to 0$ limit naturally reduce to the 5d blowup equations with matters proposed in [80]. However, there are several subtle points.

- The elliptic blowup equation with characteristic $a = -1/2$ could split to *two* K-theoretic blowup equations in the $q \to 0$ limit. This was already observed for the minimal $\mathfrak{su}(3)$ and $\mathfrak{so}(8)$ SCFTs in [4]. For other characteristics a, each elliptic blowup equation will reduce to one K-theoretic blowup equation.

- For $n = 2$ theories, the elliptic genera in the $q \to 0$ limit give the 5d Nekrasov partition function with an extra term which are neutral with respect to G. To obtain the precise 5d blowup equations from 6d, one needs to factor out the extra term which possibly contributes to the Λ factor.

- All $n = 1$ theories in the $q \to 0$ limit just reduce to the theory of a free hypermultiplet, whose associated Calabi-Yau space is simply the resolved conifold [1]. The reduced one-string elliptic genera all have leading q order as just q^{-1}, and the 5d gauge theory information are encoded in the q^0 coefficient. It is easy to see this works along well with elliptic blowup equations. In fact, the unity elliptic blowup equations for all $n = 1$ theories at the q leading order just give the blowup equation of the resolved conifold: [19]
$$S(\epsilon_2)S(m+(y_{\mathrm{u}}-1)\epsilon_1+y_{\mathrm{u}}\epsilon_2)-S(\epsilon_1)S(m+y_{\mathrm{u}}\epsilon_1+(y_{\mathrm{u}}-1)\epsilon_2)=S(\epsilon_2-\epsilon_1)S(m+y_{\mathrm{u}}(\epsilon_1+\epsilon_2)), \tag{3.89}$$
where we denote $S(x) = e^{\frac{x}{2}} - e^{-\frac{x}{2}}$ and $m = \lambda_F \cdot m_F$. It is easy to check this identity. After factoring out the q^{-1} term and a gauge natural term similar with the $n = 2$ situation, one can obtain the 5d blowup equations from the order q^0 of 6d ones.

- More importantly, we find in general, *not all* K-theoretic blowup equations are reduced from elliptic blowup equations. In particular, the admissible range of the shifts for the 5d instanton counting parameter $\mathfrak{q}$ can be larger than the admissible range of the shifts for the 6d string number counting parameter Q_{ell}. This makes some $n = 2$ theories such as $G = \mathfrak{su}(N), F = \mathfrak{su}(2N)$ theory not recursively solvable in 6d, but recursively solvable in 5d.

Let us discuss the pure gauge minimal 6d $(1,0)$ SCFTs as examples. In the $q \to 0$ limit, the elliptic genera directly reduce to the 5d Nekrasov-partition functions. We find all possible 5d blowup equations for the pure gauge theory with $G = A_2, D_4, F_4, E_{6,7,8}$. The 5d r fields and Λ factors and their 6d origins are listed in table 10. Note the first three rows were given by Keller-Song's K-theoretic blowup equations [34].

In fact, we further find for all simple Lie groups G, there exist $h^\vee_G + 3 + 2(r_{\mathrm{c}} - 1)$ non-equivalent 5d unity r fields and $(h^\vee_G - 1)(r_{\mathrm{c}} - 1)$ non-equivalent 5d vanishing r fields, where r_{c} is the rank of the center of G with $r_{\mathrm{c}} = |P^\vee/Q^\vee|$. We summarize the corresponding Λ factors in table 11, in which $\delta = (w, w)$. Note the first row was conjectured by Nakajima-Yoshioka [16] and explicitly checked by Keller-Song [34]. The second row is beyond Nakajima-Yoshioka's range for flux d for $\mathfrak{su}(N)$ geometries. The existence of unity

	5d r	5d Λ	from 6d r	6d Λ
$j=1,2,\dots,n-1$	$(0,-n+2j)$	1	$(0,0,-n+2j)$	$\Lambda^{[1/2-j/n]}$
	$(0,\pm n)$	1	$(0,0,n)$	$\Lambda^{[-1/2]}$
$j=1,2,\dots,n-3$	$(0,\pm(n+2j))$	1		
	$(0,\pm(3n-4))$	$1-(-1)^n e^{\pm\frac{3(n-2)}{2}(\epsilon_1+\epsilon_2)}\mathfrak{q}$		
$j=1,2,\dots,n-1$	$(2w,-n+2j)$	0	$(0,2w,-n+2j)$	0
	$(2w,\pm n)$	0	$(0,2w,n)$	0
$j=1,2,\dots,n-4$	$(2w,\pm(n+2j))$	0		
	$(2w,\pm(3n-6))$	$e^{\pm\frac{3(n-2)^2}{2n}(\epsilon_1+\epsilon_2)}\mathfrak{q}^{\frac{n-2}{n}}$		

Table 10. The 5d and 6d blowup equations for pure gauge theories with $G=A_2,D_4,F_4,E_{6,7,8}$ and corresponding $n=3,4,5,6,8,12$. The 5d r fields are denoted as $(r_{m_G},r_{\log\mathfrak{q}})$, with weight $w\in(P^\vee\backslash Q^\vee)_G$ and 6d r fields are denoted as $(r_\tau,r_{m_G},r_{\log Q_{\mathrm{ell}}})$.

	5d r	5d Λ
$j=0,1,\dots,h_G^\vee$	$(0,-h_G^\vee+2j)$	1
	$(0,\pm(h_G^\vee+2))$	$1-(-1)^{h_G^\vee}\exp\left(\pm\frac{1}{2}h_G^\vee(\epsilon_1+\epsilon_2)\right)\mathfrak{q}$
$j=1,2,\dots,h_G^\vee-1$	$(2w,-h_G^\vee+2j)$	0
	$(2w,\pm h_G^\vee)$	$\exp\left(\pm\frac{3}{2}\delta_G h_G^\vee(\epsilon_1+\epsilon_2)\right)\mathfrak{q}^{\delta_G}$

Table 11. 5d blowup equations for all simple Lie group G.

blowup equations of such type was already noticed for local $\mathbb{P}^1\times\mathbb{P}^1$ Calabi-Yau geometries in [19]. The third and fourth rows agree with Nakajima-Yoshioka's K-theoretic blowup equations for $\mathfrak{su}(N)$ gauge group and Chern class $c=1,2,\dots,N-1$ [17].

4 Solving blowup equations

In this section, we discuss how to solve the blowup equations. By solving we mean extracting refined BPS invariants of the local Calabi-Yau threefold, or equivalently computing elliptic genera in the case of 6d theories and instanton partition functions in the case of 5d theories. Although it was believed that for general local Calabi-Yau threefolds the blowup equations always uniquely determine the refined topological string partition function [19], the question remains what is the minimal set of required input data. It was at first conjectured [19] that merely the classical intersection numbers of the Calabi-Yau geometries as input data already allow for a complete solution of the blowup equations, but it turns out not to be the case in some examples such as the massless half K3 associated to the massless E-string theory [9]. This gives a flavor of the complexity of the problem of solving the blowup equations in general.

To discuss solving the blowup equations for 6d $(1,0)$ SCFTs, it is convenient to divide all these theories into three classes according to the difficulty of solving their associated blowup equations:

A These theories have unity blowup equations and possibly vanishing blowup equations as well; there are enough unity equations so that recursion formulas à la [17, 34] can be written down and the blowup equations can be solved immediately.

– In the case of rank one 6d SCFTs, these are the theories with $n \geq 3$ and without unpaired half-hypermultiplets.[29] There are infinitely many theories in this class.

B These theories have unity blowup equations and possibly also vanishing blowup equations; the number of unity blowup equations is not sufficient to allow for recursion formulas. Nevertheless in practise it is still possible to solve blowup equations order by order using other methods.

– In the case of rank one 6d SCFTs, these are the theories with $n = 1, 2$ and without unpaired half-hypermultiplets. There are also infinitely many theories in this class.

C These theories have only vanishing blowup equations but no unity blowup equations. There is currently no algorithm to solve these equations completely.

– In the case of rank one theories, these are the theories with unpaired half-hypermultiplets. There are in total 12 theories in this class which are $n = 1, 3, 5, 7$ $G = E_7$ theories, $n = 1, 3$ $G = \mathfrak{so}(11)$ theories, $n = 3$ $G = \mathfrak{so}(12)$ theory, $n = 2$ $G = \mathfrak{so}(12)_b, \mathfrak{so}(13)$ and $n = 1$ $G = \mathfrak{su}(6)_\star, \mathfrak{so}(12)_{a,b}$ theories.

In this section and the next section of examples, we will focus on rank one theories. In later sections, we will see that all *higher-rank* theories belong to classes **B** or **C**.

We discuss four methods to solve blowup equations, summarised in table 12. The first two methods, the recursion formulas and the Weyl orbit expansion, are designed to compute elliptic genera. Since they are based on elliptic blowup equations, they require implicitly the semiclassical intersection numbers of Calabi-Yau and the one-loop partition function as input data. The recursion formulas, as a generalisation of [34], has the least scope of applicability among the first two methods; but when it applies, it is the most powerful, as it calculates explicitly elliptic genera of arbitrary numbers of strings. The Weyl-orbit expansion, initiated in [9] and fully developed and exploited in this paper, has a wider range of applicability. The last two methods, the refined BPS expansion and the ϵ_1, ϵ_2 expansion, are designed to compute refined BPS invariants or refined free energies. They are in fact applied to general refined topological string theory [19], and therefore require a slightly different set of input data. We comment that although theories in class **C** cannot be solved completely, there are some examples, for instance the $n = 7, G = E_7$ model as we see in section 5.9, where one can use the BPS expansion method to solve the majority of refined BPS invariants below any degree bound. We also need to point out that although the method of ϵ_1, ϵ_2 expansion seems to apply to all three classes, the necessary

[29] The theory of $n = 3, G = \mathfrak{su}(3)$ is a bit special. The recursion formulas do not work for the *one*-string elliptic genus, as the latter enjoys an enhanced symmetry so that the number of independent unity equations is reduced; the recursion formulas, nevertheless, still work for elliptic genera of more than one string [8].

methods	solvable classes	input data	output results
recursion formulas	**A**	semiclassical, one-loop	elliptic genera
Weyl orbit expansion	**A**, **B**	semiclassical, one-loop	elliptic genera
refined BPS expansion	**A**, **B**, partially **C**	semiclassical, prepotential	BPS invariants
ϵ_1, ϵ_2 expansion	**A**, **B**, **C**	depends, see section 4.4	free energies

Table 12. Summary of methods to solve blowup equations.

initial data are sometimes rather difficult to come by. Here it is only used to discuss the solvability of the blowup equations associated to different classes of theories.

We explain individual methods in turn in the following subsections. We give an inventory of all our results in appendix B, and present some of these results explicitly in section 5 and appendix E, G; more results can be found in the supplementary material or on the website [99].

4.1 Recursion formula

From (3.1), the unity blowup equation can be written as

$$\begin{aligned}
&\theta_i^{[a]}(n\tau, k_F\lambda_F\cdot m_F + ny(\epsilon_1+\epsilon_2))\mathbb{E}_d(m_G, m_F, \epsilon_1, \epsilon_2)\\
&-\theta_i^{[a]}(n\tau, k_F\lambda_F\cdot m_F + n(y(\epsilon_1+\epsilon_2)-d\epsilon_1))\mathbb{E}_d(m_G, m_F+\epsilon_1\lambda_F, \epsilon_1, \epsilon_2-\epsilon_1)\\
&-\theta_i^{[a]}(n\tau, k_F\lambda_F\cdot m_F + n(y(\epsilon_1+\epsilon_2)-d\epsilon_2))\mathbb{E}_d(m_G, m_F+\epsilon_2\lambda_F, \epsilon_1-\epsilon_2, \epsilon_2)\\
&=\sum_{\lambda_G,d_1,d_2}{}^{'}(-1)^{|\alpha^\vee|}\theta_i^{[a]}(n\tau, -n\,\lambda_G\cdot m_G + k_F\,\lambda_F\cdot m_F + n((y-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2))\\
&\quad\times A_V(\tau, m_G, \lambda_G)A_H(\tau, m_{G,F}, \lambda_{G,F})\\
&\quad\times \mathbb{E}_{d_1}(\tau, m_{G,F}+\epsilon_1\lambda_{G,F}, \epsilon_1, \epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau, m_{G,F}+\epsilon_2\lambda_{G,F}, \epsilon_1-\epsilon_2, \epsilon_2),
\end{aligned}\tag{4.1}$$

where $\sum'_{\lambda_G,d_1,d_2}$ means the summation over all $\lambda_G \in Q^\vee, d_0 = \frac{1}{2}||\lambda_G||_G^2$ and $0 \le d_{1,2} < d$ with $d_0+d_1+d_2 = d$. All the instances of d-string elliptic genus are collected on the l.h.s., and there are only less than d-string elliptic genera on the r.h.s. With the characteristic a taking value as in (3.4), the number of such equations with fixed d and λ_F is n. For models with $n \ge 3$, we can choose three arbitrary characteristics a_1, a_2, a_3 and solve the d-string elliptic genus from (4.1) as

$$\begin{aligned}
&\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2)\\
&=\sum_{\lambda_G,d_1,d_2}{}^{'}(-1)^{|\lambda_G|}[D/D]^{\text{red}}_{n,(a_1,a_2,a_3)}(m_{G,F}, \epsilon_{1,2}, \lambda_{G,F})\\
&\quad\times\frac{\theta_1((d-d_2)\epsilon_1-d_1\epsilon_2-\lambda_G\cdot m_G)\theta_1((d-d_1)\epsilon_2-d_2\epsilon_1-\lambda_G\cdot m_G)}{\theta_1(d\epsilon_1)\theta_1(d\epsilon_2)}\\
&\quad\times A_V(\tau, m_G, \lambda_G)A_H(\tau, m_{G,F}, \lambda_{G,F})\\
&\quad\times \mathbb{E}_{d_1}(\tau, m_{G,F}+\epsilon_1\lambda_{G,F}, \epsilon_1, \epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau, m_{G,F}+\epsilon_2\lambda_{G,F}, \epsilon_1-\epsilon_2, \epsilon_2)
\end{aligned}\tag{4.2}$$

where we define

$$[D/D]^{\rm red}_{n,(a_1,a_2,a_3)}(m_{G,F},\epsilon_{1,2},\lambda_{G,F}) \\ = \frac{D^{\rm red}_{n,(a_1,a_2,a_3)}(\lambda_G\cdot m_G - d_0(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2,-d\epsilon_1,-d\epsilon_2;k_F\lambda_F\cdot m_F+ny(\epsilon_1+\epsilon_2))}{D^{\rm red}_{n,(a_1,a_2,a_3)}(0,-d\epsilon_1,-d\epsilon_2;k_F\lambda_F\cdot m_F+ny(\epsilon_1+\epsilon_2))} \tag{4.3}$$

with a reduced version of determinant

$$D^{\rm red}_{n,(a_1,a_2,a_3)}(z_1,z_2,z_3;z_n) = \frac{\det\left(\theta_i^{[a_j]}(n\tau,z_n+nz_k)_{j,k=1,2,3}\right)}{\theta_1(\tau,z_1-z_2)\theta_1(\tau,z_2-z_3)\theta_1(\tau,z_3-z_1)}. \tag{4.4}$$

In particular, the one-string elliptic genus can be simply obtained as

$$\mathbb{E}_1(\tau,m_{G,F},\epsilon_{1,2}) = \sum_{||\lambda_G||^2=2}(-1)^{|\lambda_G|}A_V(\tau,m_G,\lambda_G)A_H(\tau,m_{G,F},\lambda_{G,F})\frac{\theta_1(\epsilon_1-\lambda_G\cdot m_G)\theta_1(\epsilon_2-\lambda_G\cdot m_G)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \\ \times\frac{D^{\rm red}_{n,(a_1,a_2,a_3)}(\lambda_G\cdot m_G-(\epsilon_1+\epsilon_2),-\epsilon_1,-\epsilon_2;k_F\lambda_F\cdot m_F+ny(\epsilon_1+\epsilon_2))}{D^{\rm red}_{n,(a_1,a_2,a_3)}(0,-\epsilon_1,-\epsilon_2;k_F\lambda_F\cdot m_F+ny(\epsilon_1+\epsilon_2))}. \tag{4.5}$$

Note that in the definition of $D^{\rm red}_{n,(a_1,a_2,a_3)}$ the determinant has zeros at $z_1-z_2=z_2-z_3=z_3-z_1=0$, which cancel with the zeros of the theta functions in the denominator.[30] Furthermore, as a consistency condition, $\mathbb{E}_d$ solved by (4.2) should not depend on the choice of $a_{1,2,3}$, i.e. any choice of three distinct $a_{1,2,3}$ should yield the same $\mathbb{E}_d$. This is not obvious from (4.2) , but it can be and have been checked by calculations with many examples.

4.2 Weyl orbit expansion

The problem to solve class **B** theories was already encountered in [9] for E-strings and M-strings. Without sufficient number of unity blowup equations with different characteristics, one can not have explicit recursion formulas for $\mathbb{E}_d$. A new method based on Weyl orbit expansion of elliptic genera was initiated in [9] using which the one-string elliptic genera of E-string and M-string were successfully solved. Here we further develop this method for all theories in class **A** and **B**. Although we do not have a proof, explicit computation for many examples shows that the unity blowup equations are sufficient to uniquely determine the elliptic genera.[31] This method is particularly efficient for *one*-string elliptic genus and *small* flavor group such as $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$. The reason is that the reduced one-string elliptic genus only depends on $v=\mathrm{e}^{(\epsilon_1+\epsilon_2)/2}$ but not on $x=\mathrm{e}^{(\epsilon_1-\epsilon_2)/2}$, while reduced higher string elliptic genera do depend on x thus the flavor group is effectively $F\times\mathfrak{su}(2)_x$.

Let us focus on the reduced one-string elliptic genus $\mathbb{E}_1^{\rm red}(v,q,m_G,m_F)$. We can always write it in the following Weyl orbit expansion of F, or v expansion in other words:[32]

$$\mathbb{E}_1^{\rm red}(v,q,m_G,m_F) = \sum c_{n,m,p,k}(m_G)q^nv^m\mathcal{O}^F_{p,k}. \tag{4.6}$$

[30] One can also define the determinant $D_{n,(a_1,a_2,a_3)}$ without the denominator in (4.4), in which case the recursion formulas (4.2) and (4.5) will be cleaner with the line of Jacobi θ_1 functions disappearing. In the meantime, one should be careful about the pole cancellation.

[31] We exclude E-string theory in the following discussion. As there is no gauge symmetry, the unity blowup equations can only determine the elliptic genus up to a free function of q. See more details in our last paper [9].

[32] For a Weyl orbit $\mathcal{O}_{p,k}$, we adopt the common notation that p is the length of its elements and k is the number of its elements.

Here $c_{n,m,p,k}(m_G)$ are some G Weyl-invariant rational functions of $\exp(m_G)$. Note the order n of q has a known *lower* bound, and for each n, the order m of v also has a *lower* bound, while for each (n,m) pair, the lengths of Weyl orbit elements of flavor group p have an *upper* bound, i.e. there are only finitely many different flavor Weyl orbits. One can further decompose $c_{n,m,p,k}(m_G)$ into the Weyl orbit expansion w.r.t. the gauge group. In this case, for fixed (n,m) and flavor Weyl orbit $\mathcal{O}^F_{p,k}$, there could be in general infinitely many gauge Weyl orbits.[33] Our strategy here is to use the unity blowup equations to solve the coefficient functions $c_{n,m,p,k}$.[34] Remember that the full one-string elliptic genus is

$$\mathbb{E}_1(q,m_G,m_F,\epsilon_1,\epsilon_2)=\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\mathbb{E}_1^{\mathrm{red}}(\mathrm{e}^{(\epsilon_1+\epsilon_2)/2},q,m_G,m_F). \tag{4.7}$$

For class **A** and **B** theories, the unity blowup equation for one-string elliptic genus reads

$$\begin{aligned}
&\theta_i^{[a]}(n\tau,k_F\lambda_F\cdot m_F+ny(\epsilon_1+\epsilon_2))\mathbb{E}_1(m_G,m_F,\epsilon_1,\epsilon_2)\\
&-\theta_i^{[a]}(n\tau,k_F\lambda_F\cdot m_F+n(y(\epsilon_1+\epsilon_2)-\epsilon_1))\mathbb{E}_1(m_G,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\\
&-\theta_i^{[a]}(n\tau,k_F\lambda_F\cdot m_F+n(y(\epsilon_1+\epsilon_2)-\epsilon_2))\mathbb{E}_1(m_G,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)\\
&=\sum_{\lambda_G\in Q^\vee_G}(-1)^{|\lambda_G|}\theta_i^{[a]}(n\tau,-n\lambda_G\cdot m_G+k_F\lambda_F\cdot m_F+n((y-1)(\epsilon_1+\epsilon_2)))\\
&\quad\times A_V(\tau,m_G,\lambda_G)A_H(\tau,m_{G,F},\lambda_{G,F}).
\end{aligned} \tag{4.8}$$

By substituting the v expansion ansatz (4.6) into the above equation, it is expected that all the coefficient functions $c_{n,m,p,k}(m_G)$ can be determined. Note the shift such as $m_F+\epsilon_1\lambda_F$ in the elliptic genus will break the flavor Weyl orbits into pieces, and the blowup equations work in a miraculous way such that all such pieces in each term of the l.h.s. are reorganized again into a Weyl invariant on the r.h.s.

The complexity of solving Weyl orbit expansion in blowup equations increases with the complexity of the Weyl orbits of the flavor group F. Therefore, the solution process can be complicated (but still feasible) when the flavor group is large. Practically, one can normally just turn on a subgroup $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$ of the flavor group to make use of the blowup equations and still obtain the elliptic genera with lots of useful information. In particular, if one does not need the gauge fugacities, the computation can be even easier where one can firstly turn off the gauge fugacities in unity blowup equations (4.8) and then solve $c_{*,*,*,*}$ as numbers. In such a situation, we can even withdraw the v expansion and arrive in the following useful ansatz for the reduced one-string elliptic genus of a 6d SCFT from a $-n$ curve:

$$\mathbb{E}_1^{\mathrm{red}}(v,q,m_G=0,m_F)=\delta_{n,1}\,q^{-\frac{1}{3}}+q^{\frac{1}{6}-\frac{n-2}{2}}\sum_{m,p,k}\frac{P_{m,p,k}(v^2)}{(1-v^2)^{2h^\vee_G-2}}q^m\mathcal{O}^F_{p,k}. \tag{4.9}$$

[33]These properties can also be easily seen from the unity elliptic blowup equations. For example, the flavor parameters only appear in the nominators of blowup equations, which determine that there exist only finitely many different flavor Weyl orbits for elliptic genus at each fixed order of q and v.

[34]It is proposed in [1] that one should be able to $\mathbb{E}_1^{\mathrm{red}}(v,q,m_G,m_F)$ in terms of representations of G and F rather than just Weyl orbits. Nevertheless, from the viewpoint of solving blowup equations, the most natural setting is Weyl orbit expansion.

Here the $P_{m,p,k}(v^2)$ are, up to an overall factor like v^N, $N \in \mathbb{Z}$, *palindromical polynomial* functions of v^2 with integral coefficients. We use this ansatz to solve many class **B** theories.

The leading q order of $\mathbb{E}_1^{\text{red}}(v, q, m_G, m_F)$,[35] i.e. the 5d reduced one-instanton partition function is expected to yield *exact* formulae as v expansions. For example, it is well-known that the reduced one G-instanton Nekrasov partition function in 5d, i.e. the Hilbert series of the reduced moduli space of one G-instanton has the following exact formula [35, 36]

$$\sum_{n=0}^{\infty} v^{h_G^\vee - 1 + 2n} \chi_{n\theta}^G, \qquad \text{where } \theta \text{ is the Dynkin label for } \mathfrak{adj}_G. \tag{4.10}$$

For theories with matter and flavor group F, the one-instanton Nekrasov partition function still have an exact but more complicated v expansion formula as follows

$$\sum_{i \in I} \pm v^{k_i} \chi_{a_i}^G \chi_{b_i}^F + \sum_{j \in J} \left(\pm \sum_{n=0}^{\infty} v^{h_j + 2n} \chi_{c_j + n\theta}^G \chi_{d_j}^F \right). \tag{4.11}$$

Here both I and J are finite sets, k_i and h_j are integers, a_i and c_j are Dynkin labels of G, while b_i and d_j are Dynkin labels of F. Besides, $\pm$ means the coefficient can only be either $+1$ or -1. For lots of 6d $(1, 0)$ SCFTs, this type of exact formulae were conjectured or found in [1] and [80]. Benefiting from the results of one-string elliptic genera solved from the blowup equations, we are able to confirm them and obtain such exact formulas for more theories, see section 5 and appendix E. We hope these exact v expansion formulas could have an interpretation from 3d monopole formulas [35] in the future.

4.3 Refined BPS expansion

The one-loop partition function and the elliptic genera of a 6d SCFT together can be identified with the worldsheet instanton partition function Z^{inst} of the topological string theory on the associated non-compact Calabi-Yau threefold, i.e.

$$Z^{\text{inst}} = Z^{\text{1-loop}} \Big(1 + \sum_d \mathrm{e}^{\mathrm{i}2\pi\phi\cdot d} \mathbb{E}_d \Big). \tag{4.12}$$

On the other hand, Z^{inst} can be expressed in terms of refined BPS invariants [100]

$$Z^{\text{inst}} = \exp \left[\sum_{j_l, j_r = 0}^{\infty} \sum_{\beta} \sum_{w=1}^{\infty} \frac{N_{j_l, j_r}^{\beta}}{w} f_{(j_l, j_r)}(q_1^w, q_2^w) Q^{w\beta} \right]. \tag{4.13}$$

Here we define

$$f_{(j_l, j_r)}(q_1, q_2) = \frac{\chi_{j_l}(q_l)\chi_{j_r}(q_r)}{(q_1^{1/2} - q_1^{-1/2})(q_2^{1/2} - q_2^{-1/2})}, \tag{4.14}$$

where $\chi_j(q)$ is the $\mathfrak{su}(2)$ character given by

$$\chi_j(q) = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}}. \tag{4.15}$$

[35] The subleading order for $n = 1$ theories.

Define

$$Bl_{(j_l,j_r,R)}(q_1,q_2)=f_{(j_l,j_r)}(q_1,q_2/q_1)q_1^R+f_{(j_l,j_r)}(q_1/q_2,q_2)q_2^R-f_{(j_l,j_r)}(q_1,q_2), \tag{4.16}$$

where R is the shift of the Kähler parameter t in the blowup equation, the blowup equation of topological string can be reformulated as [9]

$$\sum_{n\in\mathbb{Z}^{b_4^c}}(-1)^{|n|}\mathrm{e}^{f_0(n)(\epsilon_1+\epsilon_2)+\sum_{i=1}^{b_2^c}f_i(n)t_i}\exp\left[-\sum_{j_l,j_r,\beta}\sum_{w=1}^{\infty}N_{j_l,j_r}^{\beta}\frac{Q^{w\beta}}{w}Bl_{(j_l,j_r,R)}(q_1^w,q_2^w)\right]=\Lambda(\epsilon_1,\epsilon_2,m), \tag{4.17}$$

where b_2^c,b_4^c are numbers of linearly independent compact curves and surfaces in the Calabi-Yau geometry, and $f_0(n),f_i(n)$ are respectively some cubic and quadratic polynomials, whose expressions can be found in [9]. For any fixed curve degree β, comparison of coefficients on both sides of the equation gives

$$\sum_{j_l,j_r}N_{j_l,j_r}^{\beta}Bl_{(j_l,j_r,R(\beta,n_0))}(q_1,q_2)=I^{\beta}(q_1,q_2), \tag{4.18}$$

where $I^{\beta}(q_1,q_2)$ consists only of invariants of lower curve degrees. The BPS invariants N_{j_l,j_r}^{β} can be regarded as the coefficients of $Bl_{(j_l,j_r,R(\beta,n_0))}(q_1,q_2)$, and they can be fixed if the decomposition of $I^{\beta}(q_1,q_2)$ in terms of $Bl_{(j_l,j_r,R(\beta,n_0))}(q_1,q_2)$ is unique. It was proved in [9, 19] to be indeed the case except for spin $(0,0)$ and spin $(0,1/2)$ invariants, and the degeneracy can be lifted if genus zero BPS invariants are available. We conclude therefore that if genus zero BPS invariants are provided as initial input, we can always use one unity blowup equation to solve all the refined BPS invariants order by order. In practice, if we have a toric construction of the Calabi-Yau geometry, we can use mirror symmetry techniques [90] to compute the genus zero invariants to furnish the necessary input data. In this paper, we constructed the geometries for $n=3,\mathfrak{so}(7)$ and $n=1,2,3,G_2$ theories to get the genus zero invariants. We list partial results of BPS invariants for these geometries in appendix G.

4.4 ϵ_1,ϵ_2 expansion

One particular useful approach to study blowup equations is the ϵ_1,ϵ_2 expansion. This method applies to general local Calabi-Yau threefolds and has been analyzed in detail in [19]. Expanding both blowup equations and refined free energy

$$F(t,\epsilon_1,\epsilon_2)=\sum_{n,g=0}^{\infty}(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}F_{(n,g)}(t) \tag{4.19}$$

we can obtain lots of algebraic/differential equations for all $F_{n,g}(t)$ which are graded by the "total genus" $n+g$. For example, for total genus one free energies, blowup equation of one r field in general gives

$$\sum_{n\in\mathbb{Z}^g}\Theta(n,r):=\sum_{n\in\mathbb{Z}^g}(-)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}+F_{(0,1)}-F_{(1,0)}\right)=\Lambda_0(r), \tag{4.20}$$

and

$$\sum_{n\in\mathbb{Z}^g}\left(-\frac{1}{6}R^3F^{(3)}_{(0,0)}+R\left(F'_{(0,1)}+F'_{(1,0)}\right)\right)\Theta(n,r)=\Lambda_1(r). \tag{4.21}$$

Here g is the mirror curve genus, $R := C \cdot n + r/2$ where C is the intersection matrix between the b curve classes and the g irreducible compact divisor classes. $R^k f^{(k)}$ is defined as $\sum_{i_1,\ldots,i_k=1}^{b} R_{i_1} \ldots R_{i_k} \partial_{t_{i_1}} \ldots \partial_{t_{i_k}} f(t)$. Moreover, Λ_0, Λ_1 are the leading ϵ_1, ϵ_2 expansion coefficients of the full Λ factor with $\Lambda = \Lambda_0 + (\epsilon_1+\epsilon_2)\Lambda_1 + \ldots$. Consider a local Calabi-Yau with one Kähler parameter and mirror curve of genus one, it is easy to see that if there exists one unity blowup equation, the component equation (4.20) can be solved for $F_{(0,1)} - F_{(1,0)}$, while component equation (4.21) can be solved for $F_{(0,1)} + F_{(1,0)}$, thus we obtain $F_{(0,1)}$ and $F_{(1,0)}$ at the same time.[36] In general, by counting the number of independent equations for $F_{n,g}$, we can find the following conclusions. Let w_u and w_v be the number of unity and vanishing blowup equations respectively.

- For a generic local Calabi-Yau with b Kähler parameters, if $w_{\rm u} \geq 1$ and $w_{\rm u} + w_{\rm v} \geq b$, then given $F_{(0,0)}$, one can solve all $F_{(n,g)}$ with $n, g \geq 0$ from blowup equations.
- For a generic local Calabi-Yau with b Kähler parameters, if $w_{\rm v} \geq b$, then given NS free energy or the self-dual free energy, i.e. all $F_{(n,0)}$ or all $F_{(0,g)}$, one can solve all $F_{(n,g)}$ with $n, g \geq 0$ from the blowup equations.

For example, all rank one theories in class **B** can be solved according to the first conclusion, and all rank one theories in class **C** except for the four $G = E_7$ theories can be solved according to the second conclusion, if the necessary input data are provided. Since the full NS free energy or the full self-dual free energy are themselves difficult to compute, and besides for 6d SCFTs, we are more interested in the elliptic genera and BPS invariants rather than $F_{n,g}$ themselves, we only use this method to discuss the solvability of 6d theories in different classes, but do not further explore this method.

5 Examples

In this section, we choose some of the most interesting rank one theories to explicitly show the λ_F parameter and the elliptic blowup equations. The chosen theories with gauge symmetry of classical type all have known 2d quiver theory correspondence, therefore the elliptic genera are exactly computable via Jeffrey-Kirwan residue of localization. We checked against them our results from blowup equations and found perfect agreement, mostly for one-string elliptic genera and some up to two-string. For theories with exceptional gauge symmetries, we explicitly show our computational results on the elliptic genera for most of them.[37] Sometimes to specify a theory with base curve $-n$ and gauge group G, we denote the reduced k-string elliptic genus as

$$\mathbb{E}_{h_{n,G}^{(k)}}(q, v, x, m_G, m_F) = \frac{\theta_1(\tau,\epsilon_1)\theta_1(\tau,\epsilon_2)}{\eta(\tau)^2}\mathbb{E}_k(\tau, m_G, m_F, \epsilon_1, \epsilon_2). \tag{5.1}$$

[36] We assume the linear coefficients $b_{\rm GV}$ and $b_{\rm NS}$ are already known, which fix the integration constants here.

[37] To shorten the paper, we usually only show the elliptic genera with gauge and flavor fugacities turned off or partially turned on. Readers interested in more detailed results for certain theories are welcome to send requests to us or visit the website [99].

Recall $v = \mathrm{e}^{(\epsilon_1+\epsilon_2)/2} = \mathrm{e}^{\epsilon_+}$, $x = \mathrm{e}^{(\epsilon_1-\epsilon_2)/2}$ and reduced one-string elliptic genus does not depend on x.

We also show some interesting theta identities coming from the leading degree of vanishing blowup equations. Although we have checked the leading degree identities for all the vanishing blowup equations in tables 7, 8, 9, here we only explicitly written down a small part of them, in particular with λ_F in small representations.

5.1 $n = 1$ $\mathfrak{sp}(N)$ theories

The $n = 1$ $G = \mathfrak{sp}(N)$ theories have $8 + 2N$ fundamental hypermultiplets and flavor symmetry $\mathfrak{so}(16+4N)$. For $N = 0$, it specializes to E-string theory, with flavor symmetry $\mathfrak{so}(16)$ enhanced to E_8. The 2d quiver description for these theories was proposed in [11, 12], therefore their elliptic genera can be exactly computed from localization. For example, the reduced one-string elliptic genus has been shown in equation (3.87). The index of d-string elliptic genus of $\mathfrak{sp}(N)$ theory is known to be

$$\mathrm{Ind}_{\mathbb{E}_d} = -\frac{N+2}{4}(\epsilon_1+\epsilon_2)^2 d + \epsilon_1\epsilon_2\frac{d^2+d}{2} - \frac{d}{2}(m,m)_{\mathfrak{sp}(N)} + \frac{d}{2}(m,m)_{\mathfrak{so}(16+4N)}. \tag{5.2}$$

Let us first discuss the vanishing blowup equations. Since for C type Lie algebra $(P^\vee/Q^\vee)_{C_n} \cong \mathbb{Z}_2$, there could exist one vanishing equation when the paramter λ_F and the characteristic a are fixed with λ_G taking value in $(P^\vee\backslash Q^\vee)_{C_n}$. Denote the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{C_n}$ as $\mathcal{O}_{\mathrm{min}}$, which is just $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. Note $|\mathcal{O}_{\mathrm{min}}| = 2^N$. Then for $N \geq 2$, the leading base degree of the vanishing blowup equations with $\lambda_F = 0$ can be universally written as

$$\sum_{w\in\mathcal{O}_{\mathrm{min}}} (-1)^{|w|}\theta_1(\tau, m_w) \times \prod_{\beta\in\Delta(C_N)}^{w\cdot\beta=1} \frac{1}{\theta_1(\tau, m_\beta)} = 0, \qquad N \geq 2. \tag{5.3}$$

We have checked this identity up to $\mathcal{O}(q^{20})$ for $N = 2, 3, 4, 5$. Note there are $(N+1)N/2$ Jacobi θ_1 functions in the denominator.

The $G = \mathfrak{sp}(1), F = \mathfrak{so}(20)$ case is peculiar due to the Lie algebra isomorphism $C_1 \cong A_1$. In fact, it is easy to check (5.3) *does not* hold for $N = 1$. The correct λ_F in this case belongs to vector representation $\mathbf{20_v}$ of $\mathfrak{so}(20)$. The leading base degree of the vanishing blowup equations turn out to be the following trivial identity

$$\theta_1(m_w + \lambda_F\cdot m_F + \epsilon_+)\theta_1(-m_w + \lambda_F\cdot m_F + \epsilon_+) - (w \to -w) = 0. \tag{5.4}$$

Here w is the fundamental weight of $\mathfrak{sp}(1)$, the first θ_1 comes from the contribution of perturbative part, the second θ_1 comes from the contribution of hypermultiplet and we have factored out the contribution from vector multiplet.

Now let us turn to unity blowup equations. All the unity λ_F fields are just the weights of the spinor representation $S = [0, 0, \ldots, 0, 1]$ of $\mathfrak{so}(16+4N)$. There are 2^{7+2N} of them. The matters are in representation $([1, 0, \ldots, 0, 0], [1, 0, \ldots, 0, 0])$ of $\mathfrak{sp}(N) \times \mathfrak{so}(16+4N)$, i.e. $(\mathbf{2N}, \mathbf{16+4N})$. The following fact about the weight system of S is crucial for blowup equations to hold: $\forall w \in [0, 0, \ldots, 0, 1]$, there are precisely $8 + 2N$ weights $w' \in \mathbf{16+4N}$ such that $w\cdot w' = 1/2$, and the rest $8+2N$ weights $w' \in \mathbf{16+4N}$ are such that $w\cdot w' = -1/2$.

Besides, $w\cdot w=2+N/2$. Note the conjugate spinor representation $C=[0,0,\dots,1,0]$ of $\mathfrak{so}(16+4N)$ if serving as λ_F is not correct, although they satisfy the modularity of unity blowup equations!

The unity elliptic blowup equations for $G=\mathfrak{sp}(N)$ theory with $\lambda_F=\lambda\in S_{\mathfrak{so}(16+4N)}$ can be written as

$$\begin{aligned}&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{sp}(N)}}^{d_0+d_1+d_2=d}(-1)^{|\alpha^\vee|}\theta_1\left(\tau,-\alpha^\vee\cdot m_{\mathfrak{sp}(N)}+\lambda\cdot m_{\mathfrak{so}(16+4N)}+\left(\frac{N+2}{2}-d_0\right)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\right)\\&\times A_V^{\mathfrak{sp}(N)}(\alpha^\vee,\tau,m_{\mathfrak{sp}(N)})A_H^{\frac{1}{2}(\mathbf{2N},\mathbf{4N+16})}(\alpha^\vee,\tau,m_{\mathfrak{sp}(N)},m_{\mathfrak{so}(16+4N)},\lambda)\\&\times\mathbb{E}_{d_1}(\tau,m_{\mathfrak{sp}(N)}+\epsilon_1\alpha^\vee,m_{\mathfrak{so}(16+4N)}+\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\&\times\mathbb{E}_{d_2}(\tau,m_{\mathfrak{sp}(N)}+\epsilon_2\alpha^\vee,m_{\mathfrak{so}(16+4N)}+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)\\&=\theta_1\left(\tau,\lambda\cdot m_{\mathfrak{so}(16+4N)}+\frac{N+2}{2}(\epsilon_1+\epsilon_2)\right)\mathbb{E}_d(\tau,m_{\mathfrak{sp}(N)},m_{\mathfrak{so}(16+4N)},\epsilon_1,\epsilon_2).\end{aligned}\tag{5.5}$$

For $N=0$ case, there is no summation over coroots, and the above equation goes back to the unity blowup equations of E-strings given in [9]. For $N=1$ case, using the 2d quiver formula for one-string elliptic genus (3.87), we have verified the unity blowup equations for all possible λ_F up to $\mathcal{O}(q^{10})$.

Let us have a closer look at the $N=1$ case. From the 2d quiver formula (3.87), it is easy to find the following expansion

$$\begin{aligned}\mathbb{E}_{h^{(1)}_{1,\mathfrak{sp}(1)}}=q^{-1/3}+\Bigg(&\frac{\chi^{\mathfrak{sp}(1)}_{(2)}}{v^2}-\frac{\chi^{\mathfrak{sp}(1)}_{(1)}\cdot\mathbf{20}_{\rm v}}{v}+\chi^{\mathfrak{sp}(1)}_{(2)}+1+\mathbf{190}-\chi^{\mathfrak{sp}(1)}_{(1)}\cdot\mathbf{20}_{\rm v}v+\chi^{\mathfrak{sp}(1)}_{(2)}v^2\\&+\sum_{n=0}^{\infty}\left[\chi^{\mathfrak{sp}(1)}_{(2n)}\cdot\mathbf{512}_{\rm s}\,v^{1+2n}-\chi^{\mathfrak{sp}(1)}_{(1+2n)}\cdot\mathbf{512}_{\rm c}\,v^{2+2n}\right]\Bigg)q^{2/3}+\mathcal{O}(q^{5/3}).\end{aligned}\tag{5.6}$$

Here the bold numbers are the representations for flavor symmetry $\mathfrak{so}(20)$ or its character based on the context. We can also check the unity blowup equation using this expansion or use the Weyl orbit expansion method to solve elliptic genus in this form from unity blowup equation (5.5). We summarize some useful information on the intersection distribution relevant to Weyl orbit expansion in table 13. Combining equation (5.6) and table 13, one can already understand why weights in the spinor representation $\mathbf{512_s}$ can serve as λ_F fields while weights in the conjugate spinor representation $\mathbf{512_c}$ cannot. This is because in (5.6) the coefficients of each v^n should have λ_F shifts all even or all odd to preserve the B field condition.

Similarly, from (3.87), the reduced one-string elliptic genus of $G=\mathfrak{sp}(2),F=\mathfrak{so}(24)$ theory has the following expansion

$$\begin{aligned}\mathbb{E}_{h^{(1)}_{1,\mathfrak{sp}(2)}}=q^{-1/3}+\Bigg(&\frac{\chi^{\mathfrak{sp}(2)}_{(20)}}{v^2}-\frac{\chi^{\mathfrak{sp}(2)}_{(10)}\cdot\mathbf{24}_{\rm v}}{v}+\chi^{\mathfrak{sp}(2)}_{(20)}+1+\mathbf{276}-\chi^{\mathfrak{sp}(2)}_{(10)}\cdot\mathbf{24}_{\rm v}v+\chi^{\mathfrak{sp}(2)}_{(20)}v^2\\&+\sum_{n=0}^{\infty}\left[\chi^{\mathfrak{sp}(2)}_{(2n,0)}\cdot\mathbf{2048}_{\rm s}\,v^{2+2n}-\chi^{\mathfrak{sp}(2)}_{(1+2n,0)}\cdot\mathbf{2048}_{\rm c}\,v^{3+2n}\right]\Bigg)q^{2/3}+\mathcal{O}(q^{5/3}),\end{aligned}\tag{5.7}$$

which we also reconfirm by solving the unity blowup equations in Weyl orbit expansion with the fugacity of one subalgebra $\mathfrak{so}(3)$ of the flavor symmetry turned on.

λ	w	$-5/2$	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$
$\mathbf{512_c}$	$\mathbf{512_s}$		10		120		252		120		10	
$\mathbf{512_c}$	$\mathbf{20_v}$					10		10				
$\mathbf{512_c}$	$\mathbf{512_c}$	1		45		210		210		45		1
$\mathbf{512_s}$	$\mathbf{512_s}$	1		45		210		210		45		1
$\mathbf{512_s}$	$\mathbf{20_v}$					10		10				
$\mathbf{512_s}$	$\mathbf{512_c}$		10		120		252		120		10	

Table 13. For any λ in a fixed reprentation, the numbers of weights w in another representation with $\lambda \cdot w$ equal to a given number.

5.2 $n = 1$ $\mathfrak{su}(N)$ theories

All $n = 1$ $\mathfrak{su}(N)$ theories with $N \geq 2$ have known universal 2d quiver gauge constructions in [11], therefore the elliptic genera are exactly computable via Jeffrey-Kirwan residue. For example, the reduced one-string elliptic genus can be universally written as

$$\begin{aligned}&-\sum_{i=1}^{N}\frac{\prod_{j=1}^{N+8}\theta_1(m_i-\epsilon_+-\mu_j)}{\eta^8\theta_1(2m_i-3\epsilon_++\mu_{N+9})}\prod_{1\leq j\leq N}^{j\neq i}\frac{\theta_1(m_i+m_j-\epsilon_++\mu_{N+9})}{\theta_1(m_i-m_j)\theta_1(2\epsilon_+-(m_i-m_j))}\\&-\frac{1}{2\eta^8}\bigg(\frac{\prod_{j=1}^{N+8}\theta_1(\frac{\epsilon_+-\mu_{N+9}}{2}-\mu_j)}{\prod_{i=1}^{N}\theta_1(\frac{3\epsilon_+-\mu_{N+9}}{2}-m_i)}+(-1)^N\sum_{k=2}^{4}\frac{\prod_{j=1}^{N+8}\theta_k(\frac{\epsilon_+-\mu_{N+9}}{2}-\mu_j)}{\prod_{i=1}^{N}\theta_k(\frac{3\epsilon_+-\mu_{N+9}}{2}-m_i)}\bigg).\end{aligned} \tag{5.8}$$

Here $m_i, i = 1, 2, \ldots, N$ are the symmetric $\mathfrak{su}(N)$ fugacities and $\mu_j, j = 1, 2, \ldots, N + 9$ are the symmetric $\mathfrak{su}(N + 9)$ fugacities. Note this formula is from UV 2d gauge theory, where the IR global symmetry i.e. the true flavor symmetry does not manifest itself. One can convert μ_j into the fugacities of the true flavor symmetry according to the matter representations. Note all these theories are on one single branch of the E-string Higgsing tree, which is also easy to see from the above elliptic genus formula.[38] Besides, for the $N = 2$ case, the flavor symmetry is enhanced to $\mathfrak{so}(20)$ which is just the $n = 1, G = \mathfrak{sp}(1)$ theory we have discussed in the last subsection.

One additional case is the $G = \mathfrak{su}(6)_*, F = \mathfrak{su}(15)$ theory, where there is a half hypermultiplet in the 3-antisymmetric representation $\Lambda^3_{\mathfrak{su}(6)} = \mathbf{15}$. This theory does not have known 2d quiver gauge construction, but has a brane web construction, thus the topological string partition function can be computed by refined topological vertex [73]. Due to the presence of half hypermultiplet, this theory does not have unity blowup equation. This is the single case with $\mathfrak{su}$ gauge symmetry where we could not solve elliptic genera from blowup equations.

Let us first discuss the vanishing blowup equations. We have shown some leading degree vanishing identities for $G = \mathfrak{su}(3), F = \mathfrak{su}(15)$ theory in (3.25) and (3.26) with

[38] See some recent discussion on this Higgsing tree using Hasse diagrams in [101].

$\lambda_G = \mathbf{3}$. In fact, the vanishing theta identity (3.26) can be generalized to all $N \geq 2$:

$$\sum_{i=1}^{N} \frac{\theta_1(-a_i + \sum_{k=1}^{N-1} x_k) \prod_{k=1}^{N-1} \theta_1(a_i + x_k)}{\prod_{1\leq j \leq N}^{j\neq i} \theta_1(a_i - a_j)} = 0, \qquad \text{for } \sum_{i=1}^{N} a_i = 0. \tag{5.9}$$

These identities come from the leading base degree of vanishing blowup equations for $G = \mathfrak{su}(N)$ theories with $\lambda_G = \omega_1 \in \mathbf{N}$, i.e. the first fundamental weight that induces the fundamental representation. Similarly, for $\lambda_G = \omega_2 \in \Lambda^2$ i.e. the second fundamental weight that induces the anti-symmetric representation, we find the leading degree of vanishing blowup equations result in the following identities for arbitrary $N \geq 4$,

$$\sum_{1\leq i<j\leq N} \frac{\theta_1(-a_i - a_j + y + \sum_{k=1}^{N-4} x_k)\theta_1(a_i + a_j + y) \prod_{k=1}^{N-4} \theta_1(a_i + x_k)\theta_1(a_j + x_k)}{\prod_{1\leq k\leq N}^{k\neq i,j} \theta_1(a_i - a_k)\theta_1(a_j - a_k)} = 0. \tag{5.10}$$

More generally, for $\lambda_G = \omega_k$, we find the leading degree of vanishing blowup equations result in the following identities for arbitrary $N \geq 3k - 2$,

$$\begin{aligned}\sum_{1\leq i_1<i_2<\cdots<i_k\leq N} &\frac{\theta_1(-\sum_{s=1}^{k} a_{i_s} + (k-1)y + \sum_{h=1}^{N-3k+2} x_h) \prod_{1\leq s<s'\leq k} \theta_1(a_{i_s} + a_{i_{s'}} + y)}{\prod_{1\leq l\leq N}^{l\neq i_1,\ldots,i_k} \prod_{s=1}^{k} \theta_1(a_{i_s} - a_l)} \\ &\times \prod_{h=1}^{N-3k+2} \prod_{s=1}^{k} \theta_1(a_{i_s} + x_h) = 0.\end{aligned} \tag{5.11}$$

Here still $\sum_{i=1}^{N} a_i = 0$ and y and x_k are arbitrary numbers. We have checked this identity for many different (N, k) up to very high order of q. Note the second line in (5.11) comes from the contribution of hypermultiplets in $(\mathbf{N}, \overline{\mathbf{N+8}})_{-N+4}$, while the product $\prod_{1\leq s<s'\leq k} \theta_1(a_{i_s} + a_{i_{s'}} + y)$ in the first line comes from the contribution of hypermultiplets in $(\mathbf{\Lambda}^2, \mathbf{1})_{N+8}$. Besides, we also find the following identity for arbitrary $N \geq 1$:

$$\sum_{1\leq i_1<i_2<\cdots<i_N\leq 2N} \frac{\theta_1(-\sum_{s=1}^{N} a_{i_s} + \epsilon_+)\theta_1(\sum_{s=1}^{N} a_{i_s} + \epsilon_+)}{\prod_{1\leq l\leq 2N}^{l\neq i_1,\ldots,i_N} \prod_{s=1}^{N} \theta_1(a_{i_s} - a_l)} = 0, \qquad \text{for } \sum_{i=1}^{2N} a_i = 0. \tag{5.12}$$

This identity is related to the situation where matters are in the middle representation of gauge group such as the $\mathfrak{su}(6)_*$ theory with 3-antisymmetric representation. For example, taking $N = 2$ it gives the leading base degree of vanishing blowup equation of $n = 1, G = \mathfrak{su}(4), F = \mathfrak{su}(12) \times \mathfrak{su}(2)$ theory with $(\lambda_G, \lambda_F) = (\mathbf{6}, \mathbf{1})$, and taking $N = 3$ gives the one of $n = 1, G = \mathfrak{su}(6)_*, F = \mathfrak{su}(15)$ theory with $(\lambda_G, \lambda_F) = (\mathbf{20}, \mathbf{1})$.

Now we turn to the unity blowup equations for all $\mathfrak{su}(N)$ theories with $N \geq 3$. Since flavor enhancement does not matter here, let us use the symmetric $\mathfrak{su}(N + 9)$ fugacities $\mu_j, j = 1, 2, \ldots, N + 9$ in (5.8) to make the form of blowup equations universal. Consider the flavor decomposition $\mathfrak{su}(N + 8) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(N + 9)$ according to

$$(\nu_1 + \nu_0, \nu_2 + \nu_0, \ldots, \nu_{N+8} + \nu_0, -(N + 8)\nu_0) \tag{5.13}$$

such that $\nu_j, j = 1, 2, \ldots N + 8$ are the symmetric $\mathfrak{su}(N + 8)$ fugacities and ν_0 is the $\mathfrak{u}(1)$ fugacity. Then the unity r fields have two possibilities

$$\lambda = (\lambda^{\mathfrak{su}(N+8)}, \lambda^{\mathfrak{u}(1)}) = \left(\omega_6, -\frac{1}{2(N + 8)}\right) \text{ or } \left(\omega_{N+2}, \frac{1}{2(N + 8)}\right). \tag{5.14}$$

The unity blowup equations can be universally written as

$$
\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{su}(N)}}^{d_0+d_1+d_2=d}(-1)^{|\alpha^\vee|}\theta_1\left(\tau,-\alpha^\vee\cdot m_{\mathfrak{su}(N)}+\lambda\cdot\mu_{\mathfrak{su}(N+9)}+\left(\frac{N+1}{2}-d_0\right)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\right)\\
&\times A_V^{\mathfrak{su}(N)}(\alpha^\vee,\tau,m_{\mathfrak{su}(N)})A_H^{\mathfrak{R}}(\alpha^\vee,\tau,m_{\mathfrak{su}(N)},\mu_{\mathfrak{su}(N+9)},\lambda)\\
&\times\mathbb{E}_{d_1}(\tau,m_{\mathfrak{su}(N)}+\epsilon_1\alpha^\vee,\mu_{\mathfrak{su}(N+9)}+\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times\mathbb{E}_{d_2}(\tau,m_{\mathfrak{su}(N)}+\epsilon_2\alpha^\vee,\mu_{\mathfrak{su}(N+9)}+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)\\
&=\theta_1\left(\tau,\lambda\cdot\mu_{\mathfrak{su}(N+9)}+\frac{N+1}{2}(\epsilon_1+\epsilon_2)\right)\mathbb{E}_d(\tau,m_{\mathfrak{su}(N)},\mu_{\mathfrak{su}(N+9)},\epsilon_1,\epsilon_2).
\end{aligned}\tag{5.15}
$$

For $N=4$, it is easy to find the two copies of λ combined together form the middle representation $\chi_{(0000001000000)}=\mathbf{924}$ of flavor $\mathfrak{su}(12)$. Indeed, for arbitrary one of the 924 λ fields, we have used the quiver formula (5.8) to check the above unity blowup equations up to $\mathcal{O}(q^{10})$. Conversely, we also used blowup equation (5.15) to solve elliptic genus independently. In the following we show two examples. As the quiver formulas are powerful enough for computational purposes in these cases, we only turn on a small subgroup of the flavor to solve blowup equations and only to the subleading q order which contains the information of 5d one-instanton partition functions.

$\boldsymbol{n=1, G=\mathfrak{su}(3), F=\mathfrak{su}(12)}$**.** Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{su}(2)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{1,\mathfrak{su}(3)}}(q,v,m_{\mathfrak{su}(3)}=0,m_{\mathfrak{su}(12)}=0)=q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^4},\tag{5.16}
$$

where

$$
P_0(v)=8(1-5v-11v^2+81v^3+364v^4+81v^5-11v^6-5v^7+v^8).
$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&\chi^{\mathfrak{su}(3)}_{(1,1)}v^{-2}-(\chi^{\mathfrak{su}(3)}_{(1,0)}\chi^{\mathfrak{su}(12)}_{(00000000001)}+c.c.)v^{-1}+\chi^{\mathfrak{su}(3)}_{(1,1)}+1+\chi^{\mathfrak{su}(12)}_{(10000000001)}\\
&+(\chi^{\mathfrak{su}(12)}_{(00100000000)}v-\chi^{\mathfrak{su}(3)}_{(1,0)}\chi^{\mathfrak{su}(12)}_{(01000000000)}v^2+\chi^{\mathfrak{su}(3)}_{(2,0)}\chi^{\mathfrak{su}(12)}_{(10000000000)}v^3-\chi^{\mathfrak{su}(3)}_{(3,0)}v^4+c.c)\\
&+\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{su}(3)}_{(n,n)}\chi^{\mathfrak{su}(12)}_{(00000100000)}v^{2+2n}+\Big(-\chi^{\mathfrak{su}(3)}_{(n+1,n)}\chi^{\mathfrak{su}(12)}_{(00001000000)}v^{3+2n}+\chi^{\mathfrak{su}(3)}_{(n+2,n)}\chi^{\mathfrak{su}(12)}_{(00010000000)}v^{4+2n}\\
&\qquad\qquad\qquad\qquad-\chi^{\mathfrak{su}(3)}_{(n+3,n)}\chi^{\mathfrak{su}(12)}_{(00100000000)}v^{5+2n}+\chi^{\mathfrak{su}(3)}_{(n+4,n)}\chi^{\mathfrak{su}(12)}_{(01000000000)}v^{6+2n}\\
&\qquad\qquad\qquad\qquad-\chi^{\mathfrak{su}(3)}_{(n+5,n)}\chi^{\mathfrak{su}(12)}_{(10000000000)}v^{7+2n}+\chi^{\mathfrak{su}(3)}_{(n+6,n)}v^{8+2n}+c.c.\Big)\Big].
\end{aligned}\tag{5.17}
$$

Here *c.c.* means complex conjugate. We have checked this agrees with the localization formula (5.8) from 2d quiver gauge theory.

$\boldsymbol{n=1,\ G=\mathfrak{su}(4),\ F=\mathfrak{su}(12)_a\times\mathfrak{su}(2)_b}$**.** Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{su}(2)\times\mathfrak{su}(2)$ of the full flavor to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{su}(4)}}(q,v,m_{\mathfrak{su}(4)}=0,m_F=0)=q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^6}, \tag{5.18}$$

where

$$P_0(v)=15-18v-261v^2-72v^3+2934v^4+10676v^5+2934v^6-72v^7-261v^8-18v^9+15v^{10}.$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$\begin{aligned}
&\chi^{\mathfrak{su}(4)}_{(1,0,1)}v^{-2}-(\chi^{\mathfrak{su}(4)}_{(1,0,0)}\chi^F_{(00000000001)_a}+c.c.)v^{-1}-\chi^{\mathfrak{su}(4)}_{(0,1,0)}\chi^F_{(1)_b}v^{-1}+\chi^{\mathfrak{su}(4)}_{(1,0,1)}+\chi^F_{(10000000001)_a\oplus(2)_b}\\
&+1+\chi^F_{(1)_b}(\chi^F_{(01000000000)_a}v-\chi^{\mathfrak{su}(4)}_{(1,0,0)}\chi^F_{(10000000000)_a}v^2+\chi^{\mathfrak{su}(4)}_{(2,0,0)}v^3+c.c.)+\chi^{\mathfrak{su}(4)}_{(0,1,0)}\chi^F_{(1)_b}v\\
&-\chi^{\mathfrak{su}(4)}_{(1,0,1)}v^2+(\chi^F_{(00010000000)_a}v^2-\chi^{\mathfrak{su}(4)}_{(1,0,0)}\chi^F_{(00100000000)_a}v^3+\chi^{\mathfrak{su}(4)}_{(2,0,0)}\chi^F_{(01000000000)_a}v^4\\
&-\chi^{\mathfrak{su}(4)}_{(3,0,0)}\chi^F_{(10000000000)_a}v^5+\chi^{\mathfrak{su}(4)}_{(4,0,0)}v^6+c.c.)\\
&+\sum_{n=0}^{\infty}\Big[\chi^F_{(1)_b}\Big(\chi^{\mathfrak{su}(4)}_{(n,0,n)}\chi^F_{(00000100000)_a}v^{3+2n}+(-\chi^{\mathfrak{su}(4)}_{(n+1,0,n)}\chi^F_{(00001000000)_a}v^{4+2n}\\
&+\chi^{\mathfrak{su}(4)}_{(n+2,0,n)}\chi^F_{(00010000000)_a}v^{5+2n}-\chi^{\mathfrak{su}(4)}_{(n+3,0,n)}\chi^F_{(00100000000)_a}v^{6+2n}+\chi^{\mathfrak{su}(4)}_{(n+4,0,n)}\chi^F_{(01000000000)_a}v^{7+2n}\\
&-\chi^{\mathfrak{su}(4)}_{(n+5,0,n)}\chi^F_{(10000000000)_a}v^{8+2n}+\chi^{\mathfrak{su}(4)}_{(n+6,0,n)}v^{9+2n}+c.c.)\Big)+\Big(-\chi^{\mathfrak{su}(4)}_{(n,1,n)}\chi^F_{(00000100000)_a}v^{4+2n}\\
&+(\chi^{\mathfrak{su}(4)}_{(n+1,1,n)}\chi^F_{(00001000000)_a}v^{5+2n}-\chi^{\mathfrak{su}(4)}_{(n+2,1,n)}\chi^F_{(00010000000)_a}v^{6+2n}+\chi^{\mathfrak{su}(4)}_{(n+3,1,n)}\chi^F_{(00100000000)_a}v^{7+2n}\\
&-\chi^{\mathfrak{su}(4)}_{(n+4,1,n)}\chi^F_{(01000000000)_a}v^{8+2n}+\chi^{\mathfrak{su}(4)}_{(n+5,1,n)}\chi^F_{(10000000000)_a}v^{9+2n}-\chi^{\mathfrak{su}(4)}_{(n+6,1,n)}v^{10+2n}+c.c.)\Big)\Big].
\end{aligned} \tag{5.19}$$

We have checked this agrees with the localization formula (5.8) from 2d quiver gauge theory.

We also used the Weyl orbit expansion to compute the reduced one-string elliptic genus of $G=\mathfrak{su}(5)$, $F=\mathfrak{su}(13)_a\times\mathfrak{u}(1)_b$ theory and found consistency with the modular ansatz in [1].

5.3 $\boldsymbol{n=2}$ $\boldsymbol{\mathfrak{su}(N)}$ theories

The $n=2$ $\mathfrak{su}(N)$ theories with flavor $\mathfrak{su}(2N)$ and matter in bi-representation $(R^G,R^F)=(\mathbf{N},\overline{\mathbf{2N}})$ are the most familiar SCFTs. The theory at $N=2$ is special, as the flavor symmetry is enhanced to $\mathfrak{so}(7)$. Nevertheless, as the flavor enhancement does not affect blowup equations, one can still use $\mathfrak{su}(4)$ effectively. Besides, the $N=1$ case is just the M-string. All these theories are on one single Higgsing tree, and the elliptic genus of $\mathfrak{su}(N)$ theory can be obtained by Higgsing from the elliptic genus of $\mathfrak{su}(N+1)$ theory. The 2d quiver construction is a slight modification of the A_1 string chain with $\mathfrak{su}(N)$ gauge group proposed in [82]. By Jeffrey-Kirwan residue, the n-string elliptic genus can be computed as[39]

$$\mathbb{E}_n^N = \sum_{\sum_{\ell=1}^N |Y_\ell|=n} \left(\prod_{\ell,m=1}^N \prod_{\substack{(x_1,y_1)\in Y_\ell \\ (x_2,y_2)\in Y_m}} \frac{\theta_1(\frac{s_\ell}{s_m} t^{x_1-x_2} d^{y_1-y_2})\theta_1(\frac{s_\ell}{s_m} t^{x_1-x_2+1} d^{y_1-y_2+1})}{\theta_1(\frac{s_\ell}{s_m} t^{x_1-x_2+1} d^{y_1-y_2})\theta_1(\frac{s_\ell}{s_m} t^{x_1-x_2} d^{y_1-y_2+1})} \right)$$
$$\times \left(\prod_{\ell,m=1}^N \prod_{(x,y)\in Y_\ell} \theta_1\left(\frac{s_\ell}{s_m} t^x d^y\right)\theta_1\left(\frac{s_\ell}{s_m} t^{x+1} d^{y+1}\right)\right)^{-1} \left(\prod_{\ell=1}^N \prod_{m=1}^{2N} \prod_{(x,y)\in Y_\ell} \theta_1\left(\frac{s_\ell}{f_m} t^{x+\frac12} d^{y+\frac12}\right)\right). \tag{5.20}$$

Here $s_\ell, \ell = 1,\dots,N$ are gauge parameters for $\mathfrak{su}(N)$, and $f_m, m = 1,\dots,2N$ are the flavor parameters for $\mathfrak{su}(2N)$. Note the products include various factors of $\theta_1(1)$, which however completely cancel against each other. The index of d-string elliptic genera of $\mathfrak{su}(N)$ theory is known to be

$$\mathrm{Ind}_{\mathbb{E}_d} = -\frac{Nd}{4}(\epsilon_1+\epsilon_2)^2 + d^2\epsilon_1\epsilon_2 - d(s,s)_{su(N)} + \frac{d}{2}(f,f)_{su(2N)}. \tag{5.21}$$

For A type Lie algebra $(P^\vee/Q^\vee)_{A_n} \cong \mathbb{Z}_{n+1}$. The zero element, i.e. the coroot lattice $Q^\vee$ is labeled by trivial representation and results in unity blowup equations, while the n other elements each labeled by one of the n fundamental weights ω_i, $i = 1,2,\dots,n$ result in vanishing blowup equations. The checkerboard pattern condition of blowup equations is guaranteed by the following Lie algebra facts. For $\mathfrak{su}(N)$ algebra, i.e. A_{N-1}, we denote by $\mathcal{O}_{\omega_i}, i = 1,2,\dots,N-1$ the Weyl orbit containing the fundamental weight ω_i. Note $|\mathcal{O}_{\omega_i}| = \binom{N}{i}$. Then $\forall w' \in \mathcal{O}_{\omega_i}$, w' intersects with i weights and $(N-i)$ weights of $\mathbf{N} = \mathcal{O}_{\omega_1}$ with intersection numbers $(N-i)/N$ and $-i/N$ respectively. Similarly, w' intersects with i weights and $(N-i)$ weights of $\overline{\mathbf{N}} = \mathcal{O}_{\omega_{N-1}}$ with intersection numbers $-(N-i)/N$ and i/N respectively.

Let us first discuss some vanishing blowup equations. For odd N and $i = 1,2,\dots,(N-1)/2$, the leading base degree of the vanishing blowup equations with λ_F as $\lambda \in \mathcal{O}_{\omega_{N+2i}}(\mathfrak{su}(2N))$ can be universally written as

$$\begin{aligned}&\sum_{w'\in\mathcal{O}_{\omega_i}(\mathfrak{su}(N))} (-1)^{|w'|}\theta_3^{[a]}(2\tau,-2m_{w'}+m_\lambda+(N-2i)\epsilon_+) \times \prod_{\beta\in\Delta(\mathfrak{su}(N))}^{w'\cdot\beta=1} \frac{1}{\theta_1(m_\beta)} \\ &\times \prod_{\mu\in\mathbf{N}}^{\mu\cdot w'=1-\frac{i}{N}} \prod_{\nu\in\overline{\mathbf{2N}}}^{\nu\cdot\lambda=\frac12+\frac{i}{N}} \theta_1(m_\mu+m_\nu+\epsilon_+) = 0, \qquad a=-1/2,0.\end{aligned} \tag{5.22}$$

For $i = (N+1)/2,\dots,N-2,N-1$, the leading base degree of the vanishing blowup equations with λ_F as $\lambda \in \mathcal{O}_{\omega_{2N-1-2i}}(\mathfrak{su}(2N))$ can be universally written as

$$\begin{aligned}&\sum_{w'\in\mathcal{O}_{\omega_i}(\mathfrak{su}(N))} (-1)^{|w'|}\theta_3^{[a]}(2\tau,-2m_{w'}+m_\lambda+(2i-N)\epsilon_+) \times \prod_{\beta\in\Delta(\mathfrak{su}(N))}^{w'\cdot\beta=1} \frac{1}{\theta_1(m_\beta)} \\ &\times \prod_{\mu\in\mathbf{N}}^{\mu\cdot w'=-\frac{i}{N}} \prod_{\nu\in\overline{\mathbf{2N}}}^{\nu\cdot\lambda=-\frac32+\frac{i}{N}} \theta_1(m_\mu+m_\nu-\epsilon_+) = 0, \qquad a=-1/2,0.\end{aligned} \tag{5.23}$$

[39]Here we adopt the same notation as in [82] to make the formula simple. The variable of theta functions are multiplicative. Deformation parameters $t, d = e^{\epsilon_{1,2}}$. The coordinates of the boxes in a Young diagram start from 0 rather than 1.

Note in the denominator there are $i(N-i)$ Jacobi θ_1, while in nominator there are $i(N-2i)$ Jacobi θ_1 if $i \leq N/2$ or $(N-i)(2i-N)$ Jacobi θ_1 if $i \geq N/2$. For even N, the leading base degree of the vanishing blowup equations look almost the same with the above formulas, except the two cases are divided by $i = N/2$. In fact, we find for all integers $N \geq 2$ and $1 \leq i \leq N/2$, the leading base degree of vanishing blowup equations result in the following mathematical identity:

$$\sum_{\substack{\sigma\subset I_N\\ |\sigma|=i}} \frac{\theta_3^{[a]}(2\tau, -2\sum_{j=1}^i m_{\sigma_j} + \sum_{k=1}^{N-2i} y_k)\prod_{j=1}^i \prod_{k=1}^{N-2i}\theta_1(m_{\sigma_j}+y_k)}{\prod_{j=1}^i \prod_{s\in I_N\backslash\sigma}\theta_1(m_{\sigma_j}-m_s)} = 0, \qquad \text{for } \sum_{i=1}^N m_i = 0. \tag{5.24}$$

Here $\sigma = (\sigma_1,\dots,\sigma_i)$ runs over all unordered subsets of size i of $I_N = (1,2,\dots,N)$. Note y_k are arbitrary numbers. We have verified this identity for lots of N and i pair up to $\mathcal{O}(q^{20})$. For example, for $i=1$, the above identity gives

$$\sum_{i=1}^N \frac{\theta_3^{[a]}(2\tau, -2m_i + \sum_{k=1}^{N-2} y_k)\prod_{k=1}^{N-2}\theta_1(m_i+y_k)}{\prod_{j\neq i}\theta_1(m_i-m_j)} = 0, \qquad \text{for } \sum_{i=1}^N m_i = 0. \tag{5.25}$$

All the unity λ_F fields are just the weights of representation $[0,\dots,0,1,0,\dots,0]$ of $\mathfrak{su}(2N)$, which is the largest representation generated by fundamental weights. There are $\binom{2N}{N} = \frac{(2N)!}{N!N!}$ of them, i.e. the sums of arbitrary N fundamental weights among the total $2N$ fundamental weights. Note $\forall w \in \mathbf{2N}$ and $w' \in \chi^{\mathfrak{su}(2N)}_{[0,\dots,0,1,0,\dots,0]}$,

$$w\cdot w' = \begin{cases} 1/2 & \text{if } w \text{ is among the } N \text{ weights that sum up to } w', \\ -1/2 & \text{otherwise.}\end{cases} \tag{5.26}$$

Besides, for $\mathfrak{su}(N)$, any vector $\alpha^\vee$ in the coroot lattice and any fundamental weight w, there always is $\alpha^\vee\cdot w \in \mathbb{Z}$. These two properties are necessary for A_H to have correct R shift.

The unity elliptic blowup equations for $G = \mathfrak{su}(N), F = \mathfrak{su}(2N)$ theory with $\lambda_F \in \chi^{\mathfrak{su}(2N)}_{[0,\dots,0,1,0,\dots,0]}$ can be written as

$$\begin{aligned}&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{su}(N)}}^{d_0+d_1+d_2=d} (-1)^{|\alpha^\vee|}\theta_3^{[a]}\left(2\tau, 2\left(-\alpha^\vee\cdot m_G + \lambda_F\cdot m_F + \left(\frac{N}{4}-d_0\right)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2\right)\right)\\ &\times A_V(\alpha^\vee,\tau,m_G)A_H^{\mathfrak{R}}(\alpha^\vee,\tau,m_G,m_F,\lambda_F)\\ &\times \mathbb{E}_{d_1}(\tau,m_G+\epsilon_1\alpha^\vee,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\cdot\mathbb{E}_{d_2}(\tau,m_G+\epsilon_2\alpha^\vee,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)\\ &= \theta_3^{[a]}\left(2\tau, 2\lambda_F\cdot m_F + \frac{N}{2}(\epsilon_1+\epsilon_2)\right)\mathbb{E}_d(\tau,m_G,m_F,\epsilon_1,\epsilon_2).\end{aligned} \tag{5.27}$$

Using the quiver formula (5.20) for one-string elliptic genus, we have checked the above unity blowup equations hold for $G = \mathfrak{su}(3)$ theory for all fifteen λ_F and $a = -1/2, 0$ up to $\mathcal{O}(q^{10})$. The $G = \mathfrak{su}(2)$ case is more subtle, we leave the check of blowup equations later. Conversely, we also used the Weyl orbit expansion method to solve one-string elliptic genus from above unity blowup equations at $a = 0$ for $G = \mathfrak{su}(2), \mathfrak{su}(3)$ and obtained consistent results with the quiver formulas.

$n = 2$, $G = \mathfrak{su}(2)$, $F = \mathfrak{so}(7)$. The $G = \mathfrak{su}(2)$ case is special because the flavor symmetry $\mathfrak{su}(4)$ is enhanced to $\mathfrak{so}(7)$. In [1], an inspiring exact formula for the reduced one-string elliptic genus was proposed in which it is found the flavor fugacities are even naturally arranged in $\mathfrak{so}(8)$ characters:

$$\begin{aligned}\mathbb{E}_{h^{(1)}_{2,\mathfrak{su}(2)}}(q,v,m_{\mathfrak{su}(2)},m_{\mathfrak{so}(8)}) = {}&\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{0}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{0}}(m_{\mathfrak{su}(2)},v,q)\\ &+\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{c}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{c}}(m_{\mathfrak{su}(2)},v,q)\\ &+\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{v}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{v}}(m_{\mathfrak{su}(2)},v,q),\end{aligned}\tag{5.28}$$

where the affine characters of $\mathfrak{so}(8)$ representations are defined as

$$\begin{aligned}\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{1}}(m_{\mathfrak{so}(8)}) &= \frac{1}{2}\sum_{j=3}^{4}\prod_{i=1}^{4}\frac{\theta_j(m_i)}{\eta}, &\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{v}}(m_{\mathfrak{so}(8)}) &= \frac{1}{2}\sum_{j=3}^{4}(-1)^{j+1}\prod_{i=1}^{4}\frac{\theta_j(m_i)}{\eta},\\ \widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{s}}(m_{\mathfrak{so}(8)}) &= \frac{1}{2}\sum_{j=1}^{2}\prod_{i=1}^{4}\frac{\theta_j(m_i)}{\eta}, &\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{c}}(m_{\mathfrak{so}(8)}) &= \frac{1}{2}\sum_{j=1}^{2}(-1)^{j}\prod_{i=1}^{4}\frac{\theta_j(m_i)}{\eta},\end{aligned}\tag{5.29}$$

and

$$\begin{aligned}\xi^{2,\mathfrak{su}(2)}_{\mathbf{0}} &= \frac{1}{q^{1/6}\prod_{j=1}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{q^{k+1/2}(v^{2k+1}+v^{-2k-1})}{1-q^{2k+1}}\chi^{\mathfrak{su}(2)}_{(2k)}(m_{\mathfrak{su}(2)}),\\ \xi^{2,\mathfrak{su}(2)}_{\mathbf{c}} &= -\frac{1}{q^{1/6}\prod_{j=1}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{v^{2k+1}+q^{2k+1}v^{-2k-1}}{1-q^{2k+1}}\chi^{\mathfrak{su}(2)}_{(2k)}(m_{\mathfrak{su}(2)}),\\ \xi^{2,\mathfrak{su}(2)}_{\mathbf{v}} &= \frac{1}{q^{1/6}\prod_{j=1}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{v^{2k+2}-q^{k+1}v^{-2k-2}}{1+q^{k+1}}\chi^{\mathfrak{su}(2)}_{(2k+1)}(m_{\mathfrak{su}(2)}),\end{aligned}$$

and a modified version of Weyl-Kac determinant

$$\widetilde{\Delta}_G(m_G,q) = \prod_{j=1}^{\infty}(1-q^j)^{\mathrm{rank}(G)}\prod_{\alpha\in\Delta^G_+}(1-q^jm_\alpha)(1-q^jm_\alpha^{-1}).\tag{5.30}$$

Using the above formula for one string elliptic genus, we have checked the unity elliptic blowup equations (5.27) hold only for $F = \mathfrak{so}(7)$ but not $\mathfrak{so}(8)$. For arbitrary $m_{\mathfrak{so}(7)}$, we checked the 6×2 unity blowup equations up to $\mathcal{O}(q^{10})$.

5.4 $n = 3$ $\mathfrak{so}(7)$ theory

The $n = 3$, $G = \mathfrak{so}(7)$ theory has flavor symmetry $F = \mathfrak{sp}(2)$ and matter representation $\mathbf{8}$. This theory is particularly interesting because it has a known 2d quiver description and can be Higgsed to the $n = 3, G = G_2$ theory, making which the first exactly computable exceptional 6d SCFT [14]. The elliptic genera of this theory were computed via Jeffrey-Kirwan residue of localization in [14]. For example, the reduced one-string elliptic genus can be expressed as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(7)}}(\tau,\epsilon_{1,2},m_i,\mu_k) = \sum_{i=1}^{3}\frac{\theta(4\epsilon_+-2m_i)\prod_{k=1}^{2}\theta(\mu_k\pm(m_i-\epsilon_+))}{\prod_{j\neq i}\theta(m_{ij})\theta(2\epsilon_+-m_{ij})\theta(2\epsilon_+-m_i-m_j)},\tag{5.31}$$

where $\theta(z) = \theta_1(\tau,z)/\eta(\tau)$, $m_{ij} \equiv m_i - m_j$, and $m_i, i = 1,2,3$ are the $\mathfrak{so}(7)$ fugacities such that $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)} = 1 + \sum_{i=1}^{3}(m_i + m_i^{-1})$ and $\mu_k, k = 1,2$ are associated to each $\mathfrak{sp}(1)$ in flavor decomposition $\mathfrak{sp}(2) \to \mathfrak{sp}(1) \times \mathfrak{sp}(1)$.

Let us first discuss the vanishing blowup equations. Since $(P^\vee/Q^\vee)_{\mathfrak{so}(7)} \cong \mathbb{Z}_2$, there should exist vanishing blowup equations with λ_G taking value in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$. For flavor fugacities, we find λ_F has five possible values, weights of representation $\mathbf{1}$ or $\mathbf{4}$ of $\mathfrak{sp}(2)$. The checkerboard pattern condition of A_V is guaranteed by the Lie algebra fact $\forall\alpha \in \Delta(\mathfrak{so}(7)), w \in (P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$, the intersection $\alpha\cdot w \in \mathbb{Z}$. On the other hand, the checkerboard pattern condition of A_H is guaranteed by the Lie algebra fact $\forall\omega' \in \mathbf{8}, w \in (P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$, the intersection $\omega'\cdot w \in \mathbb{Z} + 1/2$.

Note the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$ is $\mathcal{O}_{1/2,6}$, which is contained in the weight space of the vector representation $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)} = 1 + \mathcal{O}_{1/2,6}$. We find the leading base degree of the vanishing blowup equations with $\lambda_F = 0$ can be written as

$$\sum_{w\in\mathcal{O}_{1/2,6}} (-1)^{|w|}\theta_4^{[a]}(3\tau, 3m_w) \times \prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1} \frac{1}{\theta_1(\tau, m_\beta)} = 0, \tag{5.32}$$

where $a = -1/2$ and $\pm 1/6$. We have checked this identity up to $\mathcal{O}(q^{20})$. Here the hypermultiplets do not contribute to the leading base degree equation, since $\forall w \in \mathcal{O}_{1/2,6}, w' \in \mathbf{8}$, $w\cdot w' = \pm 1/2$. On the other hand, the leading base degree of the vanishing blowup equations with $\lambda_F \in \mathbf{4}$ is

$$\sum_{w\in\mathcal{O}_{1/2,6}} (-1)^{|w|}\theta_4^{[a]}(3\tau, 3m_w + 2x) \prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1} \frac{1}{\theta_1(m_\beta)} \prod_{\omega'\in\mathbf{8}}^{w\cdot\omega'=-1/2} \theta_1(m_{\omega'} + x) = 0, \tag{5.33}$$

where $x = \pm m_{\mathfrak{sp}(1)} + \epsilon_+$ is an arbitrary number. We also checked this identity up to $\mathcal{O}(q^{20})$. For higher base degrees, we checked all five vanishing blowup equations from the viewpoint of Calabi-Yau.

For unity blowup equations, λ_F has four choices which are just the four short roots of $\mathfrak{sp}(2)$, or explicitly $(\pm 1, \pm 1)$ if we view the effective flavor group as $\mathfrak{sp}(1)_a \times \mathfrak{sp}(1)_b$. Therefore, all the 12 unity blowup equations with $\lambda_F = \lambda \in \mathcal{O}_{[01]}^{\mathfrak{sp}(2)}$ can be written as

$$\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{so}(7)}}^{d_0+d_1+d_2=d} (-1)^{|\alpha^\vee|}\theta_4^{[a]}\left(3\tau, 3\left(-\alpha^\vee\cdot m_{\mathfrak{so}(7)} + \lambda\cdot m_{\mathfrak{sp}(2)} + \left(\frac{2}{3} - d_0\right)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2\right)\right)\\
&\times A_V^{\mathfrak{so}(7)}(\alpha^\vee, \tau, m_{\mathfrak{so}(7)}) A_H^{(\mathbf{8},\frac{1}{2}\mathbf{4})}(\alpha^\vee, \tau, m_{\mathfrak{so}(7)}, m_{\mathfrak{sp}(2)}, \lambda)\\
&\times \mathbb{E}_{d_1}(\tau, m_{\mathfrak{so}(7)} + \epsilon_1\alpha^\vee, m_{\mathfrak{sp}(2)} + \epsilon_1\lambda, \epsilon_1, \epsilon_2 - \epsilon_1)\cdot \mathbb{E}_{d_2}(\tau, m_{\mathfrak{so}(7)} + \epsilon_2\alpha^\vee, m_{\mathfrak{sp}(2)} + \epsilon_2\lambda, \epsilon_1 - \epsilon_2, \epsilon_2)\\
&= \theta_4^{[a]}\left(3\tau, 3\lambda\cdot m_{\mathfrak{sp}(2)} + 2(\epsilon_1+\epsilon_2)\right)\mathbb{E}_d(\tau, m_{\mathfrak{so}(7)}, m_{\mathfrak{sp}(2)}, \epsilon_1, \epsilon_2).
\end{aligned} \tag{5.34}$$

Here $a = -1/2, \pm 1/6$. All four possible λ just give $\lambda\cdot m_{\mathfrak{sp}(2)} = \pm m_{\mathfrak{sp}(1)_a} \pm m_{\mathfrak{sp}(1)_b}$. Fix arbitrary one λ, there are three unity blowup equations with different characteristics from which one can solve elliptic genera recursively. For example, using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q^3)$. Our result agrees precisely with the quiver formula in [14] and the modular ansatz in [1], therefore we just present the first few q orders with all gauge and flavor fugacities turned off. For example, denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(7)}}(q,v,m_{\mathfrak{so}(7)}=0,m_{\mathfrak{sp}(2)}=0)=q^{-1/3}v^4\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^4(1+v)^8}. \tag{5.35}$$

We obtain

$$\begin{aligned}
P_0(v)&=-(5-12v+22v^2-12v^3+5v^4),\\
P_1(v)&=v^{-6}(1+4v+2v^2-12v^3-18v^4+4v^5+158v^6-316v^7+418v^8-\cdots+v^{16}).
\end{aligned}$$

Note that the polynomials in the parentheses are palindromic. The full expression of $P_1(v)$ can be recovered from this property. We also computed the two-string elliptic genus using the recursion formula and find agreement with the quiver formula in [14]. For example,

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(7)}}(q,v,x=1,m_{\mathfrak{so}(7)}=0,m_{\mathfrak{sp}(2)}=0)=-q^{-5/6}v^9\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{10}(1+v+v^2)^9}, \tag{5.36}$$

where

$$\begin{aligned}
P^{(2)}_0(v)=&\,(14+18v-3v^2+69v^3+298v^4+295v^5+175v^6+684v^7+1426v^8+1132v^9\\
&+660v^{10}+\cdots+14v^{20}),\\
P^{(2)}_1(v)=&\,v^{-6}(5+23v+68v^2+135v^3+216v^4+273v^5+649v^6+838v^7-117v^8-407v^9\\
&+3496v^{10}+6341v^{11}+6252v^{12}+12839v^{13}+24595v^{14}+23918v^{15}+19272v^{16}+\cdots+5v^{32}).
\end{aligned}$$

Again the full expressions of the polynomials in the parentheses can be recovered by their palindromic properties.

5.5 $n=4$ $\mathfrak{so}(N+8)$ theories

The $n=4$, $G=\mathfrak{so}(N+8)$ theories have flavor group $F=\mathfrak{sp}(N)$ and matter representation $(R^G,R^F)=(\mathbf{N+8},\mathbf{2N})$. For even $N=2p$, such theories can be realized by type IIB superstring theory with orientfold. The Kodaira elliptic singularity of type $I_p^\star$ here is due to the presence of $4+p$ D7-branes wrapping the base $\mathbb{P}^1$ together with an orientifold 7-plane. This picture results in a quiver gauge theory description which makes the elliptic genera exactly computable via Jeffrey-Kirwan residues [2]. For example, the reduced one-string elliptic genus can be computed as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(8+2p)}}=\frac{1}{2}\sum_{i=1}^{4+p}\left[\frac{\theta(2\epsilon_++2m_i)\theta(4\epsilon_++2m_i)\prod_{j=1}^{2p}\theta(\epsilon_++m_i\pm\mu_j)}{\prod_{j\neq i}\theta(m_i\pm m_j)\theta(2\epsilon_++m_i\pm m_j)}+(m_i\to -m_i)\right]. \tag{5.37}$$

Here $\theta(z)=\theta_1(\tau,z)/\eta(\tau)$, m_i and μ_j are fugacities of gauge $\mathfrak{so}(8{+}2p)$ and flavor $\mathfrak{sp}(2p)$. For odd N cases, the 2d quiver description also exists similarly and was discussed in appendix D of [1]. For example, the reduced one-string elliptic genus for $G=\mathfrak{so}(9{+}2p)$, $F=\mathfrak{sp}(1{+}2p)$

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theory is

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(9+2p)}} = \frac{1}{2}\sum_{i=1}^{4+p}\left[\frac{\theta(2\epsilon_+ + 2m_i)\theta(4\epsilon_+ + 2m_i)\prod_{j=1}^{2p+1}\theta(\epsilon_+ + m_i \pm \mu_j)}{\theta(m_i)\theta(2\epsilon_+ + m_i)\prod_{j\neq i}\theta(m_i \pm m_j)\theta(2\epsilon_+ + m_i \pm m_j)} + (m_i \to -m_i)\right]. \tag{5.38}$$

Still m_i and μ_j are gauge and flavor fugacities respectively.

Let us first discuss the vanishing blowup equations. As is well-known in Lie algebra, $(P^\vee/Q^\vee)_{B_n} \cong \mathbb{Z}_2$ and $(P^\vee/Q^\vee)_{D_n} \cong \mathbb{Z}_4$. Consider the vanishing blowup equations with λ_G taking value in $\mathcal{O}^{\mathfrak{so}(8+N)}_{[10\cdots00]}$, i.e. the Weyl orbit associated to the vector representation. For flavor fugacities, we find λ_F can always take value in Weyl orbit $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. Let us denote the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(8+N)}$ as $\mathcal{O}_{\min}$. It has relation with the vector representation of $\mathfrak{so}(8+N)$ as

$$(\mathbf{8+N})_{\mathbf{v}} = \begin{cases} \mathcal{O}_{\min}, & \text{for even } N, \\ \mathbf{1} + \mathcal{O}_{\min}, & \text{for odd } N, \end{cases} \tag{5.39}$$

Then the leading base degree of the vanishing blowup equations of $G = \mathfrak{so}(8+N)$ theory with $\lambda_F \in \mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$ can be universally written as

$$\sum_{w\in\mathcal{O}_{\min}} (-1)^{|w|}\theta_3^{[a]}(4\tau, 4m_w + Nx)\theta_1(-m_w + x)^N \times \prod_{\beta\in\Delta(\mathfrak{so}(8+N))}^{w\cdot\beta=1} \frac{1}{\theta_1(\tau, m_\beta)} = 0, \qquad N \geq 0. \tag{5.40}$$

Here $a = -1/2, -1/4, 0, 1/4$ and $x = \lambda_F \cdot m_F + \epsilon_+$. We have checked this identity up to $\mathcal{O}(q^{20})$ for several N. Note there are $N+6$ Jacobi θ_1 functions in the denominator.

For even N cases, there exist more vanishing blowup equations with λ_G taking value in $\mathcal{O}^{\mathfrak{so}(8+N)}_{[00\cdots01]}$ and $\mathcal{O}^{\mathfrak{so}(8+N)}_{[00\cdots10]}$, which coincide with the spinor and conjugate spinor representations. For example, the leading base degree of the vanishing blowup equations with $\lambda_F = 0$ can be universally written as

$$\sum_{w\in S}(-1)^{|w|}\theta_3^{[a]}(4\tau, 4m_w) \times \prod_{\beta\in\Delta(\mathfrak{so}(8+N))}^{w\cdot\beta=1} \frac{1}{\theta_1(\tau, m_\beta)} = 0, \qquad N \geq 0,\ N \equiv 0\,(\mathrm{mod}\,2). \tag{5.41}$$

Here S is the spinor representation of $\mathfrak{so}(8+N)$ which can also be replaced by its conjugate representation. We have checked this identity up to $\mathcal{O}(q^{20})$ for several even N. Note there are $(N+6)(N+8)/8$ Jacobi θ_1 functions in the denominator.

The unity λ_F fields of $\mathfrak{so}(N+8)$ theories all take value in the Weyl orbit $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. There are 2^N of them. The unity elliptic blowup equations for $G = \mathfrak{su}(8+N), F = \mathfrak{sp}(N)$ theory with λ short for λ_F can be written as

$$\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{so}(8+N)}}^{d_0+d_1+d_2=d} (-1)^{|\alpha^\vee|}\theta_3^{[a]}\left(4\tau, 4(-\alpha^\vee\cdot m_{\mathfrak{so}(8+N)} + \lambda\cdot m_{\mathfrak{sp}(N)} + \left(\frac{N+4}{8} - d_0\right)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2\right)$$
$$\times A_V^{\mathfrak{so}(8+N)}(\alpha^\vee, \tau, m_{\mathfrak{so}(8+N)}) A_H^{\frac{1}{2}(\mathbf{8+N},\mathbf{2N})}(\alpha^\vee, \tau, m_{\mathfrak{so}(8+N)}, m_{\mathfrak{sp}(N)}, \lambda)$$
$$\times \mathbb{E}_{d_1}\left(\tau, m_{\mathfrak{so}(8+N)} + \epsilon_1\alpha^\vee, m_{\mathfrak{sp}(N)} + \epsilon_1\lambda, \epsilon_1, \epsilon_2 - \epsilon_1\right)$$

$$\times\mathbb{E}_{d_2}\big(\tau,m_{\mathfrak{so}(8+N)}+\epsilon_2\alpha^\vee,m_{\mathfrak{sp}(N)}+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2\big)$$
$$=\theta_3^{[a]}\left(4\tau,4\lambda\cdot m_{\mathfrak{sp}(N)}+\frac{N+4}{2}(\epsilon_1+\epsilon_2)\right)\mathbb{E}_d\big(\tau,m_{\mathfrak{so}(8+N)},m_{\mathfrak{sp}(N)},\epsilon_1,\epsilon_2\big). \tag{5.42}$$

Here $a=-1/2,-1/4,0,1/4$. Fix arbitrary one λ and choose arbitrary three characteristics a, one can use the three unity blowup equations to solve elliptic genera recursively.

In the following, we present some of our computational results on one-string and two-string elliptic genera from recursion formula. To save space, we turn off both gauge and flavor fugacities. For $G=\mathfrak{so}(9)$, $F=\mathfrak{sp}(1)$ theory, let us denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(9)}}(q,v,m_{\mathfrak{so}(9)}=0,m_{\mathfrak{sp}(1)}=0)=q^{-5/6}v^6\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{12}}. \tag{5.43}$$

We obtain

$$P_0(v)=2-5v+36v^2-46v^3+130v^4-90v^5+130v^6-46v^7+36v^8-5v^9+2v^{10},$$
$$P_1(v)=4(19-52v+270v^2-368v^3+815v^4-648v^5+815v^6-368v^7+270v^8-52v^9+19v^{10}).$$

This agrees precisely with the quiver formula (5.38) and the modular ansatz result in [1]. Using recursion formula, we also computed the reduced two-string elliptic genus with all gauge and flavor fugacities turned off. Denote

$$\mathbb{E}_{h^{(2)}_{4,\mathfrak{so}(9)}}(q,v,x=1,m_{\mathfrak{so}(9)}=0,m_{\mathfrak{sp}(1)}=0)=-q^{-11/6}v^{13}\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{22}(1+v)^{16}(1+v+v^2)^{13}},$$

we obtain

$$\begin{aligned}P^{(2)}_0(v)=&\,3+5v+41v^2+184v^3+623v^4+1987v^5+6119v^6+16024v^7+38003v^8+84127v^9\\&+170974v^{10}+315783v^{11}+541464v^{12}+864989v^{13}+1277738v^{14}+1747831v^{15}\\&+2235019v^{16}+2666784v^{17}+2956416v^{18}+3054876v^{19}+\cdots+3v^{38},\\P^{(2)}_1(v)=&\,2(62+193v+1031v^2+4553v^3+16024v^4+49985v^5+146893v^6+383794v^7\\&+904569v^8+1962488v^9+3926557v^{10}+7208099v^{11}+12237790v^{12}+19308839v^{13}\\&+28304443v^{14}+38563232v^{15}+49018799v^{16}+58173759v^{17}+64417144v^{18}\\&+66611780v^{19}+\cdots+62v^{38}).\end{aligned} \tag{5.44}$$

Again, the full polynomials can be recovered from their palindromic properties.

For $G=\mathfrak{so}(10)$, $F=\mathfrak{sp}(2)$ theory, let us denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(10)}}(q,v,m_{\mathfrak{so}(10)}=0,m_{\mathfrak{sp}(2)}=0)=q^{-5/6}v^7\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{14}}. \tag{5.45}$$

We obtain

$$P_0(v)=-(5-20v+99v^2-184v^3+370v^4-360v^5+370v^6-184v^7+99v^8-20v^9+5v^{10}),$$
$$P_1(v)=v^{-2}(1+4v-249v^2+1024v^3-3873v^4+7172v^5-12223v^6+12688v^7-\cdots+v^{14}).$$

This agrees precisely with the quiver formula in (5.37) and the modular ansatz in [1]. Using recursion formula, we also computed the reduced two-string elliptic genus. Denote

$$\mathbb{E}_{h^{(2)}_{4,\mathfrak{so}(10)}}(q,v,x{=}1,m_{\mathfrak{so}(10)}{=}0,m_{\mathfrak{sp}(2)}{=}0) = -q^{-11/6}v^{15}\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{22}(1+v)^{20}(1+v+v^2)^{15}},$$

we obtain

$$\begin{aligned}
P^{(2)}_0(v) =\, & 14+42v+174v^2+840v^3+3180v^4+9606v^5+28723v^6+80545v^7+200547v^8\\
&+453260v^9+967049v^{10}+1923811v^{11}+3524339v^{12}+6005020v^{13}+9637502v^{14}\\
&+14497632v^{15}+20342110v^{16}+26767114v^{17}+33232318v^{18}+38795360v^{19}\\
&+42443836v^{20}+43677620v^{21}+\cdots+14v^{42},\\
P^{(2)}_1(v) =\, & -v^{-2}(5+35v-566v^2-2413v^3-9796v^4-43257v^5-166563v^6-516948v^7\\
&-1493092v^8-4045182v^9-9976992v^{10}-22346950v^{11}-46615056v^{12}-90796062v^{13}\\
&-164272366v^{14}-276641406v^{15}-437103585v^{16}-648567657v^{17}-902450252v^{18}\\
&-1179498629v^{19}-1452843842v^{20}-1686000677v^{21}-1841747735v^{22}-1895883244v^{23}\\
&+\cdots+5v^{46}).
\end{aligned}\tag{5.46}$$

5.6 G_2 theories

$G = G_2$ theories on base curve $(-n)$, $n = 1,2,3$ have flavor group $F = \mathfrak{sp}(10-3n)$ and $n_f = (10-3n)$ hypermultiplets in fundamental representation $\mathbf{7}$ of gauge symmetry. There only exist unity blowup equations but no vanishing due to the Lie algebra fact $Q^\vee \cong P^\vee$ for G_2. The unity λ_F fields are just all the elements of the Weyl orbit $[0,0,\ldots,0,1]$ of $\mathfrak{sp}(10-3n)$ or in other word take value ±1 for each $\mathfrak{sp}(1)$ with decomposition $\mathfrak{sp}(10-3n) \to \mathfrak{sp}(1)^{10-3n}$. There are in total $n \times 2^{10-3n}$ unity blowup equations when different choices of the characteristic are also taken into account.

$n = 3$, $G = G_2$, $F = \mathfrak{sp}(1)$. This theory can be Higgsed from the $n = 3, G = \mathfrak{so}(7)$, $F = \mathfrak{sp}(2)$ theory and to the $n = 3$, $G = \mathfrak{su}(3)$ minimal SCFT. The 2d quiver description was found in [14], therefore the elliptic genus can be computed exactly via localization. For example, the reduced one-string elliptic genus of such theory is given in [14] as

$$\mathbb{E}_{h^{(1)}_{3,G_2}} = \sum_{i=1}^{3}\frac{\theta(2m_i-4\epsilon_+)\theta(m_{\mathfrak{sp}(1)}\pm(m_i-\epsilon_+))}{\theta(m_i-2\epsilon_+)\prod_{j\neq i}\theta(m_{ij})\theta(2\epsilon_+-m_{ij})\theta(2\epsilon_++m_j)},\tag{5.47}$$

where $\theta(z) = \theta_1(\tau,z)/\eta(\tau)$ and $m_{1,2,3}$ are the embedding of G_2 into $\mathfrak{su}(3)$ with $m_1+m_2+m_3=0$ and $m_{ij}=m_i-m_j$.

Using the recursion formula from blowup equations, we computed the one-string elliptic genus to $\mathcal{O}(q^3)$. Our result agrees precisely with the quiver formula in [14] and the modular ansatz in [13] and [1], therefore we just present the first few q orders with all gauge and flavor fugacities turned off. For example, denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,G_2}}(q,v,m_{G_2}=0,m_{\mathfrak{sp}(1)}=0) = q^{-1/3}v^3\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^4(1+v)^6}.\tag{5.48}$$

We obtain

$$P_0(v) = 2 - 3v + 8v^2 - 3v^3 + 2v^4, \tag{5.49}$$
$$P_1(v) = v^{-5}(1 + 2v - 3v^2 - 8v^3 + 2v^4 + 44v^5 - 60v^6 + 92v^7 + \cdots + v^{14}), \tag{5.50}$$
$$\begin{aligned} P_2(v) = v^{-7}(&14 + 14v - 52v^2 - 34v^3 + 85v^4 - 8v^5 - 105v^6 + 396v^7 \\ &- 542v^8 + 728v^9 - \cdots + 14v^{18}). \end{aligned} \tag{5.51}$$

We also computed the two-string elliptic genus using the recursion formula and find perfect agreement with the quiver formula in [14]. For example,

$$\mathbb{E}_{h^{(2)}_{3,G_2}}(q, v, x = 1, m_{G_2} = 0, m_{\mathfrak{sp}(1)} = 0) = -q^{-5/6} \sum_{n=0}^{\infty} q^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^6(1+v+v^2)^7}, \tag{5.52}$$

where

$$\begin{aligned} P_0^{(2)}(v) &= v^7(3 - 3v + 8v^2 + 21v^3 + 17v^4 + 16v^5 + 89v^6 + 71v^7 + 42v^8 + \cdots + 3v^{16}), \\ P_1^{(2)}(v) &= v^2(2 + 3v + 11v^2 + 9v^3 + 20v^4 + 46v^5 - 24v^6 + 19v^7 + 313v^8 + 442v^9 \\ &\qquad + 569v^{10} + 1364v^{11} + 1473v^{12} + 1226v^{13} + \cdots + 2v^{26}). \end{aligned} \tag{5.53}$$

$n = 2$, $G = G_2$, $F = \mathfrak{sp}(4)$. We study this theory from the viewpoint of both Weyl orbit expansion and elliptic non-compact Calabi-Yau. Let us first just turn on the fugacity of a subalgebra $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ of the flavor symmetry $\mathfrak{sp}(4)$. We gave a toric construction for the elliptic non-compact Calabi-Yau threefold corresponding to this configuration, and computed the triple intersection numbers and genus zero Gromov-Witten invariants. Using the unity blowup equations in refined BPS expansion, we computed the refined BPS invariants to very high degrees. We describe our toric construction in appendix F.15 and list some low degree BPS invariants in the geometric bases in appendix G.

On the other hand, using the Weyl orbit expansion method elaborated in section 4.2 and the unity blowup equation with characteristic $a = 0$, we solved the reduced one-string elliptic genus with flavor subalgebra $\mathfrak{su}(2)$ fugacity q_m at leading q order, and found it to be

$$\mathbb{E}_{h^{(1)}_{2,G_2}}(q, v, m_{G_2} = 0, q_m) = q^{1/6} v^{-1} \sum_{n=0}^{\infty} q^n \frac{P_n(v, q_m)}{(1-v^2)^6}, \tag{5.54}$$

where

$$\begin{aligned} P_0(v, q_m) = &-(q_m^{-4} + q_m^4)(v^4 + 8v^6 + 8v^8 + v^{10}) + (q_m^{-3} + q_m^3)(28v^5 + 88v^7 + 28v^9) \\ &- (q_m^{-2} + q_m^2)(10v^4 + 242v^6 + 242v^8 + 10v^{10}) \\ &- (q_m^{-1} + q_m)(4v^3 - 164v^5 - 688v^7 - 164v^9 + 4v^{11}) \\ &+ (1 - 6v^2 + 2v^4 - 627v^6 - 627v^8 + 2v^{10} - 6v^{12} + v^{14}), \end{aligned} \tag{5.55}$$

and

$$
\begin{aligned}
v^2 P_1(v,q_m) = {} & (q_m^{-5}+q_m^5)(28v^7+88v^9+28v^{11}) - (q_m^{-4}+q_m^4)(31v^6+653v^8+653v^{10}+31v^{12}) \\
& +(q_m^{-3}+q_m^3)(4v^3-44v^5+1048v^7+3888v^9+1048v^{11}-44v^{13}+4v^{15}) \\
& +(q_m^{-2}+q_m^2)(10v^2-60v^4-20v^6-7562v^8-7562v^{10}-20v^{12}-60v^{14}+10v^{16}) \\
& -(q_m^{-1}+q_m)(28v-188v^3+692v^5-5132v^7-17008v^9-5132v^{11}+692v^{13} \\
& -188v^{15}+28v^{17})+(14-53v^2+31v^4-85v^6-15531v^8-15531v^{10}-85v^{12} \\
& +31v^{14}-53v^{16}+14v^{18}).
\end{aligned} \tag{5.56}
$$

When the flavor fugacity is turned off, i.e. $q_m = 1$, the above result agrees with the modular ansatz in [1]. Besides, at leading q order, the reduced one-string elliptic genus given by (5.54) and (5.55) has the following expansion

$$
\begin{aligned}
& v^{-1} - 4(q_m + q_m^{-1})v^2 - (q_m^{-4} + 10q_m^{-2} + 13 + 10q_m^2 + q_m^4)v^3 \\
& + (28q_m^{-3} + 140q_m^{-1} + 140q_m + 28q_m^3)v^4 + \mathcal{O}(v^5).
\end{aligned} \tag{5.57}
$$

It is easy to check this agrees with the exact expression for reduced 5d one-instanton partition function proposed in [1]

$$
\begin{aligned}
& v^{-1} - \chi^{\mathfrak{sp}(4)}_{(1000)} v^2 + \chi^{G_2}_{(10)} v^3 + \sum_{n=0}^{\infty} \Big[-\chi^{G_2}_{(0n)} \chi^{\mathfrak{sp}(4)}_{(0001)} v^{3+2n} + \chi^{G_2}_{(1n)} \chi^{\mathfrak{sp}(4)}_{(0010)} v^{4+2n} \qquad (5.58) \\
& \qquad - \chi^{G_2}_{2n} \chi^{\mathfrak{sp}(4)}_{(0100)} v^{5+2n} + \chi^{G_2}_{(3n)} \chi^{\mathfrak{sp}(4)}_{(1000)} v^{6+2n} - \chi^{G_2}_{(4n)} v^{7+2n} \Big] \\
& = v^{-1} - \mathbf{8}^{\mathfrak{sp}(4)} v^2 + (\mathbf{7}^{G_2} - \mathbf{42}^{\mathfrak{sp}(4)}) v^3 + \mathbf{7}^{G_2} \cdot \mathbf{48}^{\mathfrak{sp}(4)} v^4 + \mathcal{O}(v^5), \qquad (5.59)
\end{aligned}
$$

with flavor symmetry $\mathfrak{sp}(4)$ restricted to $\mathfrak{su}(2)_{q_m}$. One can also turn on full flavor fugacity and gauge fugacity and push the computation to higher q orders and higher number of strings. For example, we obtained the subleading q order of the reduced one-string elliptic genus as

$$
\mathbf{14} v^{-3} - \mathbf{7} \cdot \chi^{\mathfrak{sp}(4)}_{(1000)} v^{-2} + (\mathbf{14} + 1 + \chi^{\mathfrak{sp}(4)}_{(2000)}) v^{-1} + \chi^{\mathfrak{sp}(4)}_{(0100)} + \mathbf{7} v + \mathcal{O}(v^2). \tag{5.60}
$$

Here and below bold letters in the v expansion represent characters of representations of gauge symmetry.

$\boldsymbol{n = 1,\ G = G_2,\ F = \mathfrak{sp}(7)}$**.** We study this theory from the viewpoint of both Weyl orbit expansion and elliptic non-compact Calabi-Yau. Let us just turn on a subgroup $\mathfrak{sp}(1)$ of the flavor $\mathfrak{sp}(7)$. We constructed toric embedding of the elliptic non-compact Calabi-Yau corresponding to this configuration, and computed the triple intersection numbers and genus zero Gromov-Witten invariants. Using the unity blowup equations in refined BPS expansion, we computed the refined BPS invariants up to high degrees. We describe our toric construction in appendix F.18 and list some low degree BPS invariants in geometric bases in appendix G.

Using the Weyl orbit expansion method and the unity blowup equation with characteristic $a = 1/2$, we solved the reduced one-string elliptic genus with flavor subgroup $\mathfrak{sp}(1)$

at leading q order as

$$\mathbb{E}_{h^{(1)}_{1,G_2}}(q,v,m_{G_2}=0,q_m)=q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v,q_m)}{(1-v^2)^6}, \tag{5.61}$$

where q_m is the $\mathfrak{sp}(1)$ flavor fugacity and

$$\begin{aligned}P_0(v,q_m)=&(q_m^{-7}+q_m^7)(v^5+8v^7+8v^9+v^{11})-(q_m^{-6}+q_m^6)(49v^6+154v^8+49v^{10})\\&+(q_m^{-5}+q_m^5)(28v^5+791v^7+791v^9+28v^{11})\\&+(q_m^{-4}+q_m^4)(35v^4-994v^6-4634v^8-994v^{10}+35v^{12})\\&+(q_m^{-3}+q_m^3)(35v^3-259v^5+9233v^7+9233v^9-259v^{11}+35v^{13})\\&+(q_m^{-2}+q_m^2)(28v^2+56v^4-3787v^6-28630v^8-3787v^{10}+56v^{12}+28v^{14})\\&-(q_m^{-1}+q_m)(49v-434v^3+2163v^5-28805v^7-28805v^9+\dots+49v^{15})\\&+(14-20v^2+218v^4-5800v^6-50600v^8-5800v^{10}+\dots+14v^{16}).\end{aligned} \tag{5.62}$$

When the flavor fugacity is turned off, i.e. $q_m=1$, the above result agrees with the modular ansatz in [1]. Besides, if turning on both gauge and flavor fugacities, we find the following v expansion for the subleading q order of reduced one-string elliptic genus:

$$\begin{aligned}&\mathbf{14}v^{-2}-\mathbf{7}\cdot\chi^{\mathfrak{sp}(7)}_{(1000000)}v^{-1}+\mathbf{14}+\chi^{\mathfrak{sp}(7)}_{(2000000)}+1+\chi^{\mathfrak{sp}(7)}_{(0010000)}v+\chi^{\mathfrak{sp}(7)}_{(0001000)}v^2\\&+(\chi^{\mathfrak{sp}(7)}_{(0000001)}-\mathbf{7}\cdot\chi^{\mathfrak{sp}(7)}_{(0010000)}-\mathbf{14}\cdot\chi^{\mathfrak{sp}(7)}_{(1000000)})v^3+\mathcal{O}(v^4).\end{aligned} \tag{5.63}$$

In fact, we find the following exact formula of the v expansion:

$$\begin{aligned}&\chi^{\mathfrak{sp}(7)}_{(0001000)}v^2+\chi^{\mathfrak{sp}(7)}_{(0010000)}(v-\chi^{G_2}_{(10)}v^3)+\chi^{G_2}_{(20)}\chi^{\mathfrak{sp}(7)}_{(0100000)}v^4-\chi^{\mathfrak{sp}(7)}_{(1000000)}(\chi^{G_2}_{(10)}v^{-1}\\&+\chi^{G_2}_{(01)}v^3+\chi^{G_2}_{(30)}v^5)+\chi^{G_2}_{(01)}v^{-2}+\chi^{G_2}_{(01)}+\chi^{\mathfrak{sp}(7)}_{(2000000)}+1+\chi^{G_2}_{(11)}v^4+\chi^{G_2}_{(40)}v^6\\&+\sum_{n=0}^{\infty}\Big[\chi^{G_2}_{(0n)}\chi^{\mathfrak{sp}(7)}_{(0000001)}v^{3+2n}-\chi^{G_2}_{(1n)}\chi^{\mathfrak{sp}(7)}_{(0000010)}v^{4+2n}+\chi^{G_2}_{(2n)}\chi^{\mathfrak{sp}(7)}_{(0000100)}v^{5+2n}-\chi^{G_2}_{(3n)}\chi^{\mathfrak{sp}(7)}_{(0001000)}v^{4+2n}\\&\quad+\chi^{G_2}_{(4n)}\chi^{\mathfrak{sp}(7)}_{(0010000)}v^{7+2n}-\chi^{G_2}_{(5n)}\chi^{\mathfrak{sp}(7)}_{(0100000)}v^{8+2n}+\chi^{G_2}_{(6n)}\chi^{\mathfrak{sp}(7)}_{(1000000)}v^{9+2n}-\chi^{G_2}_{(7n)}v^{10+2n}\Big].\end{aligned} \tag{5.64}$$

5.7 F_4 theories

$G=F_4$ theories on base curve $(-n)$, $n=1,2,3,4,5$ have flavor group $F=\mathfrak{sp}(5-n)$ and $n_f=(5-n)$ hypermultiplets in the fundamental representation **26** of gauge symmetry. There only exist unity blowup equations but no vanishing equations due to the Lie algebra fact $Q^\vee\cong P^\vee$ for F_4. The corresponding Calabi-Yau geometries with flavor fugacities turned off were constructed in [2, 86]. The unity λ_F fields of these theories are just all the elements of the Weyl orbit $[0,0,\dots,0,1]$ of $\mathfrak{sp}(5-n)$. For $n=3,4,5$ cases, we can use the recursion formula to exactly compute the elliptic genera to arbitrary numbers of strings. For $n=1,2$ cases, we used the Weyl orbit expansion to compute them. The $n=5$ case belongs to minimal 6d SCFTs and was discussed in detail in our previous paper of this series [8]. In the following, we discuss the $n=1,2,3,4$ cases individually.

$\boldsymbol{n=4,\ G=F_4,\ F=\mathfrak{sp}(1).}$ There exist 8 unity blowup equations in total with $\lambda_F = \pm 1$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q^3)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,F_4}}(q,v,m_{F_4}=0,m_{\mathfrak{sp}(1)}=0) = q^{-5/6}v^7\sum_{n=0}^{\infty} q^n \frac{P_n(v)}{(1-v)^{10}(1+v)^{16}}. \tag{5.65}$$

We obtain

$$P_0(v) = 1+10v-49v^2+266v^3-549v^4+1068v^5-1110v^6+\cdots+v^{12}, \tag{5.66}$$
$$P_1(v) = 2(28+277v-1552v^2+6305v^3-13020v^4+21834v^5-23904v^6+\cdots+28v^{12}). \tag{5.67}$$

One can also keep all flavor and gauge fugacities in the recursion formula to compute the full elliptic genus. Indeed, as the leading q order of elliptic genus, we confirm the conjectural formula of the reduced 5d one-instanton partition function in (H.36) of [1]:

$$v^7+\sum_{n=0}^{\infty}\Big[-\chi^{F_4}_{(n000)}\chi^{\mathfrak{sp}(1)}_{(3)}v^{8+2n}+\chi^{F_4}_{(n001)}\chi^{\mathfrak{sp}(1)}_{(2)}v^{9+2n}-\chi^{F_4}_{(n010)}\chi^{\mathfrak{sp}(1)}_{(1)}v^{10+2n}+\chi^{F_4}_{(n100)}v^{11+2n}\Big].$$

For the subleading q order of the reduced one-string elliptic genus, we obtain the following v expansion

$$\begin{aligned}&(\mathbf{52}+1+\chi^{\mathfrak{sp}(1)}_{(2)})v^7+((\mathbf{52}+2)\chi^{\mathfrak{sp}(1)}_{(3)}+\chi^{\mathfrak{sp}(1)}_{(1)})v^8\\&-(\mathbf{26}\cdot\chi^{\mathfrak{sp}(1)}_{(4)}+(\chi^{F_4}_{(1001)}+\mathbf{273}+3\cdot\mathbf{26})\chi^{\mathfrak{sp}(1)}_{(2)}+\mathbf{324}+\mathbf{26})v^9+\mathcal{O}(v^{10})\end{aligned}$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q. For example, denote the reduced two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{4,F_4}}(q,v,x=1,m_{F_4}=0,m_{\mathfrak{sp}(1)}=0) = -q^{-11/6}v^{15}\sum_{n=0}^{\infty} q^n\frac{P^{(2)}_n(v)}{(1-v)^{22}(1+v)^{16}(1+v+v^2)^{17}},$$

we obtain

$$\begin{aligned}P_0(v) =\ &1+15v+34v^2+97v^3+715v^4+2022v^5+4997v^6+15039v^7+41395v^8+87572v^9\\&+180994v^{10}+376306v^{11}+700157v^{12}+1152469v^{13}+1848360v^{14}+2846743v^{15}\\&+3983439v^{16}+5139498v^{17}+6428973v^{18}+7611291v^{19}+8253543v^{20}\\&+8388168v^{21}+\cdots+v^{42},\\P_1(v) =\ &2(30+480v+1478v^2+4015v^3+20963v^4+63895v^5+157718v^6+414969v^7\\&+1079969v^8+2315076v^9+4619079v^{10}+9059109v^{11}+16530696v^{12}+27157331v^{13}\\&+42451387v^{14}+63499177v^{15}+88251928v^{16}+113833998v^{17}+140332628v^{18}\\&+163891834v^{19}+178266540v^{20}+182276136v^{21}+\cdots+v^{42}).\end{aligned} \tag{5.68}$$

$\boldsymbol{n=3,\ G=F_4,\ F=\mathfrak{sp}(2).}$ Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q^3)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders with all gauge and flavor fugacities turned off. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,F_4}}(q,v,m_{F_4}=0,m_{\mathfrak{sp}(2)}=0)=q^{-1/3}v^6\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^4(1+v)^{16}}, \tag{5.69}$$

we obtain

$$\begin{aligned}P_0(v)=&\,5+80v+268v^2-1232v^3+2142v^4-1232v^5+268v^6+80v^7+5v^8,\\ P_1(v)=&\,v^{-8}(1+12v+62v^2+172v^3+237v^4-20v^5-722v^6-1472v^7-1357v^8\\ &+4812v^9+21908v^{10}-72624v^{11}+101054v^{12}+\cdots+v^{24}).\end{aligned}\tag{5.70}$$

Keeping all flavor and gauge fugacities in the recursion formula to compute the full elliptic genus. Indeed, as the leading q order of elliptic genus, we confirm the conjectural formula of reduced 5d one-instanton partition function in (H.36) of [1]. For example, the first few terms are

$$\begin{aligned}&\chi^{\mathfrak{sp}(2)}_{(01)}v^6+\chi^{\mathfrak{sp}(2)}_{(30)}v^7+(\chi^{\mathfrak{sp}(2)}_{(03)}-\mathbf{52}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(2)}_{(20)})v^8+(\mathbf{273}\cdot\chi^{\mathfrak{sp}(2)}_{(10)}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(2)}_{(12)})v^9\\ &+(\mathbf{52}\cdot\chi^{\mathfrak{sp}(2)}_{(03)}+\mathbf{273}\cdot\chi^{\mathfrak{sp}(2)}_{(21)}+\mathbf{324}\cdot\chi^{\mathfrak{sp}(2)}_{(02)}-\mathbf{1274})v^{10}+\mathcal{O}(v^{11}).\end{aligned}$$

For the subleading q order the reduced one-string elliptic genus, we obtain the following v expansion

$$\begin{aligned}&v^{-2}-\chi^{\mathfrak{sp}(2)}_{(10)}v^3-\chi^{\mathfrak{sp}(2)}_{(20)}v^4+\Big(\chi^{\mathfrak{sp}(2)}_{(21)}+\chi^{\mathfrak{sp}(2)}_{(20)}+(\mathbf{52}+\mathbf{26}+2)\chi^{\mathfrak{sp}(2)}_{(01)}\Big)v^6\\ &-(\chi^{\mathfrak{sp}(2)}_{(31)}+\chi^{\mathfrak{sp}(2)}_{(12)}+\chi^{\mathfrak{sp}(2)}_{(11)}+\chi^{\mathfrak{sp}(2)}_{(10)}+(\mathbf{52}+2)\chi^{\mathfrak{sp}(2)}_{(30)})v^7+\mathcal{O}(v^8)\end{aligned}$$

We also computed the two-string elliptic genus to the subleading order of q. For example, denote the reduced two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{3,F_4}}(q,v,x=1,m_{F_4}=0,m_{\mathfrak{sp}(2)}=0)=-q^{-5/6}v^{13}\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{16}(1+v+v^2)^{17}},$$

we have

$$\begin{aligned}P^{(2)}_0(v)=&\,15+449v+5327v^2+30906v^3+101183v^4+187889v^5+183238v^6+180121v^7\\ &+820970v^8+2527029v^9+3954101v^{10}+3268018v^{11}+2502062v^{12}+6631296v^{13}\\ &+14672455v^{14}+17834663v^{15}+12802905v^{16}+8758778v^{17}+\cdots+15v^{34},\\ P^{(2)}_1(v)=&\,v^{-8}(5+145v+1763v^2+11722v^3+53549v^4+182991v^5+493575v^6+1078556v^7\\ &+1935972v^8+2865208v^9+3665294v^{10}+5010010v^{11}+8956794v^{12}+15093412v^{13}\\ &+14295923v^{14}-2110395v^{15}-13976451v^{16}+18409580v^{17}+78794748v^{18}\\ &+85716318v^{19}+44817687v^{20}+102304199v^{21}+290636920v^{22}+388309453v^{23}\\ &+271239229v^{24}+167708226v^{25}+\cdots+5v^{50}).\end{aligned}\tag{5.71}$$

$\boldsymbol{n=2,\ G=F_4,\ F=\mathfrak{sp}(3).}$ We used the Weyl orbit expansion to solve the one-string elliptic genus. Let us first just turn on the fugacity of a subalgebra $\mathfrak{sp}(1)$ of the flavor symmetry $\mathfrak{sp}(3)$. Using the unity blowup equation with characteristic $a=0$, we solved the reduced one-string elliptic genus with flavor subgroup $\mathfrak{sp}(1)$ fugacity q_m as

$$\mathbb{E}_{h^{(1)}_{2,F_4}}(q,v,m_{F_4}=0,q_m)=q^{1/6}v^{-1}\sum_{n=0}^{\infty}q^n\frac{P_n(v,q_m)}{(1-v^2)^{12}}, \tag{5.72}$$

where

$$\begin{aligned}
&P_0(v,q_m)=-v^9f_9(1+36v^2+341v^4+1208v^6+1820v^8+\cdots+v^{16})\\
&+39v^{10}f_8(1+v^2)(2+47v^2+274v^4+506v^6+\cdots+2v^{12})\\
&-3v^9f_7(2+942v^2+14439v^4+62278v^6+99270v^8+\cdots+2v^{16})\\
&+2v^8f_6(-5+314v^2+29213v^4+264959v^6+723887v^8+\cdots-5v^{18})\\
&-3v^7f_5(2-155v^2+7792v^4+238244v^6+1250686v^8+2098702v^{10}+\cdots+2v^{20})\\
&+3v^6f_4(-1+29v^2-1795v^4+135107v^6+1736758v^8+5258478v^{10}+\cdots-v^{22})\\
&-v^5f_3(1+20v^2-2077v^4+34266v^6+3666667v^8+22762210v^{10}+39749314v^{12}+\cdots+v^{24})\\
&-6v^6f_2(3-138v^2+5599v^4-176466v^6-3072141v^8-9995641v^{10}-\cdots+3v^{22})\\
&-3v^5f_1(2-10v^2-761v^4-8900v^6+2607160v^8+17861126v^{10}+31896078v^{12}+\cdots+2v^{24})\\
&+(1-16v^2+120v^4-588v^6+3293v^8-59309v^{10}+1403134v^{12}+27648874v^{14}+92360011v^{16}\\
&\quad+\cdots+v^{34}).
\end{aligned} \tag{5.73}$$

Here $f_n=q_m^{-n}+q_m^n$. When the flavor fugacity is turned off, i.e. $q_m=1$, the above result agrees with the modular ansatz in [1]. Besides, at leading q order, the reduced one-string elliptic genus given by (5.72) and (5.73) has the following expansion

$$\begin{aligned}
&v^{-1}-v^4(q_m^3+6q_m+6q_m^{-1}+q_m^{-3})-v^5(3q_m^{-4}+18q_m^{-2}+28+18q_m^2+3q_m^4)\\
&-6v^6(q_m^{-5}+6q_m^{-3}+11q_m^{-1}+11q_m+6q_m^3+q_m^5)+\mathcal{O}(v^7).
\end{aligned} \tag{5.74}$$

It is easy to check with flavor symmetry $\mathfrak{sp}(3)$ restricted to $\mathfrak{sp}(1)_{q_m}$ this agrees with the exact formula of reduced 5d one-instanton partition function conjectured in (H.26) of [1]. For example, the first few terms are

$$\begin{aligned}
&v^{-1}-v^4\chi^{\mathfrak{sp}(3)}_{(001)}-v^5\chi^{\mathfrak{sp}(3)}_{(101)}-v^6\chi^{\mathfrak{sp}(3)}_{(201)}+v^7(\mathbf{52}\cdot\chi^{\mathfrak{sp}(3)}_{(010)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(101)}-\chi^{\mathfrak{sp}(3)}_{(030)})\\
&+v^8(\mathbf{52}\cdot\chi^{\mathfrak{sp}(3)}_{(300)}-\mathbf{273}\cdot\chi^{\mathfrak{sp}(3)}_{(001)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(120)})+\mathcal{O}(v^9).
\end{aligned} \tag{5.75}$$

One can also turn on full flavor fugacity and gauge fugacity and push the computation to higher q orders and higher number of strings. For example, for the subleading q order of reduced one-string elliptic genus, we obtain

$$\begin{aligned}
&\mathbf{52}v^{-3}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(100)}v^{-2}+(\mathbf{52}+\chi^{\mathfrak{sp}(3)}_{(200)}+1)v^{-1}+\chi^{\mathfrak{sp}(3)}_{(300)}\\
&+\chi^{\mathfrak{sp}(3)}_{(020)}v+\chi^{\mathfrak{sp}(3)}_{(011)}v^2+(\chi^{\mathfrak{sp}(3)}_{(002)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(010)})v^3+\mathcal{O}(v^4)
\end{aligned} \tag{5.76}$$

$n = 1$, $G = F_4$, $F = \mathfrak{sp}(4)$. Let us first turn on the fugacity of a subalgebra $\mathfrak{sp}(1)$ of the full flavor symmetry $\mathfrak{sp}(4)$. Using the Weyl orbit expansion method and the unity blowup equation with characteristic $a = 1/2$, we solved the reduced one-string elliptic genus with flavor subalgebra $\mathfrak{sp}(1)$ at leading q order as

$$\mathbb{E}_{h^{(1)}_{1,F_4}}(q, v, m_{F_4} = 0, q_m) = q^{-1/3} + q^{2/3}v^{-2}\sum_{n=0}^{\infty} q^n \frac{P_n(v, q_m)}{(1-v^2)^{16}}, \tag{5.77}$$

where q_m is the $\mathfrak{sp}(1)$ flavor fugacity and

$$\begin{aligned}P_0(v,q_m)=&v^{10}(q_m^{-12}+q_m^{12})(1+36v^2+341v^4+1208v^6+1820v^8+\dots+v^{16})\\&-52v^{11}(q_m^{-11}+q_m^{11})(2+49v^2+321v^4+780v^6+\dots+2v^{14})+\cdots\\&-4v(q_m^{-1}+q_m)(26-426v^2+3215v^4-14760v^6+58005v^8-494529v^{10}+2024378v^{12}\\&+306868947v^{14}+1249149000v^{16}+\dots+26v^{34})+(52-763v^2+5256v^4-21590v^6\\&+39900v^8+421246v^{10}-13984964v^{12}+300172490v^{14}+3270987324v^{16}\\&+6383908850v^{18}+\dots+52v^{36}).\end{aligned} \tag{5.78}$$

When the $\mathfrak{sp}(1)$ fugacity is turned off,

$$\begin{aligned}P_0(v,1)=&(1-v)^{16}(52+624v+3001v^2+5704v^3-8932v^4-81464v^5-210244v^6-145256v^7\\&+896624v^8+3964136v^9+7404438v^{10}+\dots+52v^{20}).\end{aligned}$$

We checked this agrees with the modular ansatz in [1]. We also turned on all $\mathfrak{sp}(4)$ flavor fugacities to perform the Weyl orbit expansion, from which we found an exact formula for the Weyl orbit expansion of the subleading q order of the reduced one-string elliptic genus, which will be given in appendix (E.7). For example, the first few terms are

$$\begin{aligned}&\mathbf{52}\,v^{-2}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(4)}_{(1000)}v^{-1}+\mathbf{52}+\chi^{\mathfrak{sp}(4)}_{(2000)}+1+\chi^{\mathfrak{sp}(4)}_{(3000)}v+\chi^{\mathfrak{sp}(4)}_{(0200)}v^2\\&+\chi^{\mathfrak{sp}(4)}_{(0110)}v^3+(\chi^{\mathfrak{sp}(4)}_{(0020)}+\chi^{\mathfrak{sp}(4)}_{(2001)}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(4)}_{(0001)})v^4+\mathcal{O}(v^5).\end{aligned} \tag{5.79}$$

This contains the information of the 5d Nekrasov partition function of the $G = F_4, F = \mathfrak{sp}(4)$ theory.

5.8 E_6 theories

$G = E_6$ theories on base curve $(-n)$ have flavor symmetry $F = \mathfrak{su}(6-n)_6 \times \mathfrak{u}(1)_{6(6-n)}$ and $n_f = (6-n)$ hypermultiplets in the bi-representation $(\mathbf{27}, (\mathbf{6}-\mathbf{n})_1)$. Note $\mathbf{6}-\mathbf{n}$ is the fundamental representation of flavor symmetry $\mathfrak{su}(6-n)$, and $n = 1, 2, 3, 4, 5, 6$. There are $2n$ vanishing blowup equations with $\lambda_{\mathfrak{su}(6-n)} = 0$ and $\lambda_{\mathfrak{u}(1)} = \pm 1/6$.

The reason there are two copies of vanishing equations is that the Dynkin diagram of E_6 is axisymmetric, in particular there exist two fundamental representations of E_6: $\mathbf{27}$ and $\overline{\mathbf{27}}$. For any two weights $w_1, w_2 \in \mathbf{27}$, $w_1 \cdot w_2 = 4/3, 1/3, -2/3$. The same for $\overline{\mathbf{27}}$. While for $w_1 \in \mathbf{27}$ and $w_2 \in \overline{\mathbf{27}}$, one has $w_1 \cdot w_2 = -4/3, -1/3, 2/3$. Since $(P^\vee/Q^\vee)_{E_6} = \mathbb{Z}_3$, accordingly let us denote $P^\vee = Q^\vee \oplus \Lambda \oplus \overline{\Lambda}$, such that $\mathbf{27} \subset \Lambda$ and $\overline{\mathbf{27}} \subset \overline{\Lambda}$. For any $w_1 \in \mathbf{27}$, $w_2 \in \overline{\mathbf{27}}$, $\lambda_1 \in \Lambda$ and $\lambda_2 \in \overline{\Lambda}$, always

$$\begin{aligned} &w_1 \cdot \lambda_1 \in \mathbb{Z} + 1/3, \qquad && w_1 \cdot \lambda_2 \in \mathbb{Z} - 1/3, \\ &w_2 \cdot \lambda_1 \in \mathbb{Z} - 1/3, \qquad && w_2 \cdot \lambda_2 \in \mathbb{Z} + 1/3. \end{aligned} \tag{5.80}$$

It is easy to find the leading base degree of one copy of the vanishing blowup equations

$$\sum_{w\in\mathbf{27}} (-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w^{E_6} + (6-n)\epsilon'_+) \prod_{w'\in\mathbf{6-n}} \theta_1(m_w^{E_6} + m_{w'}^{su(6-n)} - \epsilon'_+) \prod_{\alpha\in\Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha^{E_6})} = 0, \tag{5.81}$$

where we denote $\epsilon'_+ = m_{u(1)} + \epsilon_+$. We have verified this identity up to q^{10} for all $n = 1, 2, \ldots, 6$. Note this identity contains m_{E_6}, $m_{su(6-n)}$, $m_{u(1)}$ and ϵ_+ as free parameters, thus are highly nontrivial. By setting m_F as zero, one obtains

$$\sum_{w\in\mathbf{27}} (-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w + (6-n)\epsilon_+)(\theta_1(m_w - \epsilon_+))^{6-n} \prod_{\alpha\in\Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha)} = 0, \tag{5.82}$$

which may be easier in case interested readers want to give a direct proof.

For the $n = 5$ case where the flavor is just $\mathfrak{u}(1)$ itself, we find there exist two more vanishing r fields with $\lambda_{\mathfrak{u}(1)} = \pm 5/6$. For example, for $(\lambda_{E_6}, \lambda_{\mathfrak{u}(1)}) = (\mathbf{27}, 5/6)$, the leading degree vanishing identities can be written as

$$\sum_{w\in\mathbf{27}} (-1)^{|w|}\theta_4^{[a]}(5\tau, 5m_w^{E_6} + 5\epsilon'_+) \prod_{w'\in\mathbf{27}}^{w'\cdot w=-2/3} \theta_1(m_{w'}^{E_6} - \epsilon'_+) \prod_{\alpha\in\Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha^{E_6})} = 0. \tag{5.83}$$

Here again $\epsilon'_+ = m_{u(1)} + \epsilon_+$. Note the hypermultiplet contribution i.e. the first product contains ten θ_1 functions. Although we do not find vanishing r fields with $\lambda_{\mathfrak{u}(1)} = \pm 5/6$ suitable for $n = 4, 3, 2, 1$ theories, we indeed find one kind of generalization of (5.83) which is

$$\sum_{w\in\mathbf{27}} (-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w^{E_6}) \prod_{w'\in\mathbf{27}}^{w'\cdot w=-2/3} \theta_1(m_{w'}^{E_6})^{6-n} \prod_{\alpha\in\Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha^{E_6})} = 0. \tag{5.84}$$

Here $n = 1, 2, 3, 4, 5, 6$, while i and a are defined accordingly by the general rule of blowup equations.

For unity blowup equations, there are 2^{6-n} choices for λ_F fields. In fact, they form the Weyl orbit $\mathcal{O}_{[00\ldots01]}^{\mathfrak{sp}(6-n)}$ if we embed $\mathfrak{su}(6-n) \times \mathfrak{u}(1)$ into $\mathfrak{sp}(6-n)$. Note there always exist λ_F fields $(\lambda_{su(6-n)}, \lambda_{u(1)}) = (0, \pm 1/2)$. For $n = 3, 4, 5, 6$, one can choose arbitrary one λ_F and three unity blowup equations with different characteristics a to solve elliptic genera recursively. The $n = 6$ case belongs to minimal 6d SCFTs and was discussed in detail in the previous paper of this series [8]. In the following, we discuss the $n = 1, 2, 3, 4, 5$ cases.

$\boldsymbol{n = 5}$**,** $\boldsymbol{G = E_6}$**,** $\boldsymbol{F = \mathfrak{u}(1)}$**.** There exist 5 unity blowup equations with $r_F = 0$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders. For example, denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{5,E_6}}(q,v,m_{E_6}=0,m_{u(1)}=0)=q^{-4/3}v^{10}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{16}(1+v)^{22}}, \tag{5.85}$$

we obtain

$$\begin{aligned}
P_0(v)=&\,1+8v-43v^2+456v^3-1436v^4+5116v^5-9848v^6+19504v^7-24164v^8\\
&+30016v^9+\cdots+v^{18},\\
P_1(v)=&\,2(40+320v-2072v^2+16128v^3-51094v^4+155036v^5-297317v^6+530598v^7\\
&-670889v^8+785764v^9-\cdots+40v^{18}),\\
P_2(v)=&-2v^{-1}(27-1498v-13658v^2+95382v^3-590835v^4+1824915v^5-4912446v^6\\
&+9187979v^7-15230210v^8+19237562v^9-21771556v^{10}+\cdots+27v^{20}).
\end{aligned} \tag{5.86}$$

By keeping the gauge and flavor fugacities in the recursion formula and taking the leading q order, we confirm the conjectural formula of reduced 5d one-instanton partition function in (H.38) of [1]:

$$\begin{aligned}
v^{10}+\sum_{n=0}^{\infty}\Big[&\chi^{E_6}_{(00000n)}\chi^{u(1)}_{(3)\oplus(-3)}v^{11+2n}-(\chi^{E_6}_{(00001n)}\chi^{u(1)}_{(-2)}+\chi^{E_6}_{(10000n)}\chi^{u(1)}_{(2)})v^{12+2n}\\
&+(\chi^{E_6}_{(00010n)}\chi^{u(1)}_{(-1)}+\chi^{E_6}_{(01000n)}\chi^{u(1)}_{(1)})v^{13+2n}-\chi^{E_6}_{(00100n)}v^{14+2n}\Big].
\end{aligned} \tag{5.87}$$

For the subleading q order, we obtain

$$\begin{aligned}
&(\mathbf{78}+2)v^{10}+(\mathbf{78}+2)\chi^{u(1)}_{(3)\oplus(-3)}v^{11}-\Big(\chi^{E_6}_{(100000)}\chi^{u(1)}_{(-4)}+\chi^{E_6}_{(000010)}\chi^{u(1)}_{(4)}\\
&+\chi^{E_6}_{(000011)\oplus(010000)\oplus3(000010)}\chi^{u(1)}_{(-2)}+\chi^{E_6}_{(100001)\oplus(000100)\oplus3(100000)}\chi^{u(1)}_{(2)}+\chi^{E_6}_{(100010)}\Big)v^{12}+\mathcal{O}(v^{13}).
\end{aligned}$$

Using recursion formula, we also computed the two-string elliptic genus to the subleading order of q which will be given in appendix E.

$n=4$, $G=E_6$, $F=\mathfrak{su}(2)\times\mathfrak{u}(1)$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q^2)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{-5/6}v^{9}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{22}}, \tag{5.88}$$

We obtain

$$\begin{aligned}
P_0(v)=&-(3+44v+33v^2-1052v^3+6513v^4-17404v^5+31905v^6-37432v^7+\cdots+3v^{14}),\\
P_1(v)=&\,v^{-2}(3+36v-135v^2-4000v^3-3894v^4+106168v^5-500700v^6+1239080v^7\\
&-2078322v^8+2430488v^9-\cdots+3v^{18}).
\end{aligned}$$

One can also keep all flavor and gauge fugacities in blowup equations to compute the full elliptic genus. In [1], the Weyl orbit expansion of reduced 5d one-instanton partition function was conjectured up to v^{11}. Using the recursion formula from blowup equations, we find the following exact formula where $F=\mathfrak{su}(2)_a\times\mathfrak{u}(1)_b$:

$$
\begin{aligned}
&-v^9\chi^F_{(2)_a}-v^{10}\chi^F_{(3)_a\otimes((3)_b\oplus(-3)_b)}+v^{11}(\chi^{E_6}_{(100000)}\chi^F_{(2)_a\otimes(2)_b}+c.c.)+v^{11}\chi^{E_6}_{(000001)}\\
&-v^{12}(\chi^{E_6}_{(010000)}\chi^F_{(1)_a\otimes(1)_b}+c.c)+v^{13}\chi^{E_6}_{(001000)}\\
&+\sum_{n=0}^{\infty}\Big[-v^{11+2n}\chi^{E_6}_{(00000n)}(\chi^F_{(6)_b\oplus(-6)_b}+\chi^F_{(6)_a})+v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^F_{(1)_a\otimes(5)_b}+\chi^{E_6}_{(00001n)}\chi^F_{(5)_a\otimes(1)_b}+c.c.)\\
&-v^{13+2n}\Big((\chi^{E_6}_{(01000n)}\chi^F_{(2)_a\otimes(4)_b}+\chi^{E_6}_{(00010n)}\chi^F_{(4)_a\otimes(2)_b}+\chi^{E_6}_{(20000n)}\chi^F_{(4)_b}+c.c.)+\chi^G_{(10001n)}\chi^F_{(4)_a}\Big)\\
&+v^{14+2n}\Big(\chi^{E_6}_{(00100n)}\chi^F_{(3)_a\otimes((3)_b\oplus(-3)_b)}+(\chi^{E_6}_{(11000n)}\chi^F_{(1)_a\otimes(3)_b}+\chi^{E_6}_{(10010n)}\chi^F_{(3)_a\otimes(1)_b}+c.c.)\Big)\\
&-v^{15+2n}\Big((\chi^{E_6}_{(10100n)}\chi^F_{(2)_a\otimes(2)_b}+\chi^{E_6}_{(02000n)}\chi^F_{(-2)_b}+c.c.)+\chi^{E_6}_{(01010n)}\chi^F_{(2)_a}\Big)\\
&+v^{16+2n}(\chi^{E_6}_{(01100n)}\chi^F_{(1)_a\otimes(1)_b}+c.c.)-v^{17+2n}\chi^{E_6}_{(00200n)}\Big].
\end{aligned}
\tag{5.89}
$$

This formula can be reconfirmed by the Weyl dimension formula of representation of E_6 and $\mathfrak{su}(2)$, where one can obtain the rational function of v as in (5.88). For the subleading q order of reduced one-string elliptic genus, we obtain

$$
\begin{aligned}
&\chi^F_{(2)_a}v^7-(\chi^F_{(4)_a}+(\mathbf{78}+3)\chi^F_{(2)_a}+\chi^{E_6}_{(000010)}\chi^F_{(-2)_b}+\chi^{E_6}_{(100000)}\chi^F_{(2)_b}+1)v^9\\
&-(\chi^F_{(5)_a}+(\mathbf{78}+3)\chi^F_{(3)_a}+\chi^F_{(1)_a})\chi^F_{(-3)_b\oplus(3)_b}v^{10}+\mathcal{O}(v^{11}).
\end{aligned}
$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q which will be given in appendix E.

$n=3$, $G=E_6$, $F=\mathfrak{su}(3)\times\mathfrak{u}(1)$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q^3)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders. Denote the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{3,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{-1/3}v^7\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^4(1+v)^{12}},
\tag{5.90}
$$

We obtain

$$
\begin{aligned}
P_0(v)=&\,2(1+28v+356v^2+2045v^3+1583v^4-19638v^5+36572v^6-\cdots+v^{12}),\\
P_1(v)=&\,v^{-9}(1+18v+149v^2+744v^3+2454v^4+5412v^5+7230v^6+2216v^7-14256v^8\\
&-39160v^9-61154v^{10}-18988v^{11}+372829v^{12}+642294v^{13}-3309245v^{14}\\
&+4904064v^{15}+\cdots+v^{30}).
\end{aligned}
\tag{5.91}
$$

We can also turn on all gauge and flavor fugacities. Using recursion formula from blowup equations, we find the exact formula for the leading q order of reduced one-string elliptic genus with $F=\mathfrak{su}(3)_a\times\mathfrak{u}(1)_b$, which will be presented in appendix (E.33). The first few terms are

$$\begin{aligned}&v^7\chi^F_{(3)_b\oplus(-3)_b}+v^8(\chi^F_{(30)_a}+\chi^F_{(03)_a})+v^9\chi^F_{(22)_a\otimes((3)_b\oplus(-3)_b)}\\&-v^{10}(\chi^G_{(100000)}\chi^F_{(12)_a\otimes(2)_b}+\chi^G_{(000010)}\chi^F_{(21)_a\otimes(-2)_b}+\chi^G_{(000001)}\chi^F_{(11)_a}\\&-\chi^F_{(30)_a\otimes(-6)_b}-\chi^F_{(03)_a\otimes(6)_b}-\chi^F_{(33)_a})+\mathcal{O}(v^{12}),\end{aligned}\tag{5.92}$$

which were already conjectured in [1]. For the subleading q order of reduced one-string elliptic genus, we obtain

$$v^{-2}-\chi^F_{(11)_a}v^4-\chi^F_{(11)_a\otimes((3)_b\oplus(-3)_b)}v^5-\chi^F_{(22)_a}v^6+(\mathbf{78}+\chi^F_{(11)_a}+2)\chi^F_{(3)_b\oplus(-3)_b}v^7+\mathcal{O}(v^8).$$

Using recursion formula, we also computed the two-string elliptic genus to the leading order of q which will be given in appendix E.

$n=2$, $G=E_6$, $F=\mathfrak{su}(4)\times\mathfrak{u}(1)$. We use Weyl orbit expansion to solve elliptic genus for this theory. Let us first turn off the $\mathfrak{su}(4)$ fugacities and only keep $\mathfrak{u}(1)$ and make use of the unity blowup equations with nonzero λ_F only on $\mathfrak{u}(1)$. Then the reduced one-string elliptic genus can be computed as

$$\mathbb{E}_{h^{(1)}_{2,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{1/6}v^{-1}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{22}},\tag{5.93}$$

where

$$\begin{aligned}P_0(v)=&(1-v)^2(1+24v+278v^2+2072v^3+11181v^4+46624v^5+156660v^6+436728v^7\\&+1030043v^8+2066568v^9+3435967v^{10}+4315392v^{11}+3435967v^{12}+\cdots+v^{22}),\\P_1(v)=&v^{-2}(78+1500v+13361v^2+72354v^3+260839v^4+631520v^5+910434v^6+142972v^7\\&-2884243v^8-7465814v^9-7830327v^{10}+5820340v^{11}+30116822v^{12}+14704216v^{13}\\&-68988104v^{14}+14704216v^{15}+\cdots+78v^{28}).\end{aligned}$$

We have cross-checked our result against the modular ansatz in [1].[40] Let us denote

$$\mathbb{E}_{h^{(1)}_{2,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{1/6}v^{-1}\sum_{i,j}c_{i,j}v^j(q/v^2)^i.\tag{5.95}$$

Then we have the following table 14 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_6 where θ is the adjoint representation. The blue numbers in the second column are eight times of the dimensions of representations $\square+k\theta$ of E_6, where the eight is the double of the dimension of matter representation $\mathbf{4}$ of flavor $\mathfrak{su}(4)$. The orange number 95 in the third column is given by $\dim(E_6)+\dim(\mathfrak{su}(4)\times\mathfrak{u}(1))+1=78+16+1=95$. These are the constraints predicted in [1] by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus.

[40] In [1], the modular ansatz for this theory is determined up to three unfixed parameters. Using our result from blowup equations, we are able to determine their three unfixed parameters as

$$a_1=\frac{6581939}{638959998741245853696},a_2=-\frac{12286901}{5111679989929966829568},a_3=\frac{16984805}{5750639988671212683264}.\tag{5.94}$$

i,j	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	-20	-72	-319	240
1	78	-216	95	40	84	120	195	1248	-2155	-11488
2	2430	-13824	28392	-20520	-1555	-3760	3102	12264	17277	166800
3	43758	-370656	1334745	-2526856	2380950	-587824	-213080	-601120	-339398	510992

Table 14. Series coefficients $c_{i,j}$ for the one-string elliptic genus of $n=2$ E_6 model.

We also computed the elliptic genus with all flavor $\mathfrak{su}(4)_a\times\mathfrak{u}(1)_b$ fugacities turned on and gauge fugacities turned off. For example, the q leading order of reduced one-string elliptic genus has v expansion as

$$\begin{aligned}&\frac{1}{v}-\chi^F_{(020)_a}v^5-(\chi^F_{(102)_a\otimes(3)_b}+c.c.)v^6-\Big((\chi^F_{(200)_a\oplus(6)_b}+27\chi^F_{(4)_b}+c.c.)+\chi^F_{(400)_a\oplus(004)_a\oplus(121)_a}\Big)v^7\\&+(27\chi^F_{(100)_a\otimes(5)_b}+78\chi^F_{(001)_a\otimes(3)_b}-\chi^F_{((130)_a\oplus(203)_a)\otimes(3)_b}+c.c.)v^8\\&-\Big((\chi^F_{(022)_a\otimes(6)_b}+351\chi_{(4)_b}-27\chi^F_{((030)_a\oplus(103)_a)\otimes(2)_b}+c.c)+\chi^F_{(222)_a}-78\chi^F_{(210)_a\oplus(012)_a}\Big)v^9+\mathcal{O}(v^{10}),\end{aligned}$$

or in the descending order of the absolute value of $\mathfrak{u}(1)$ charge as

$$\sum_{n=0}^{\infty}\Big[-\chi^F_{(-12)_b\oplus(12)_b}\chi^{E_6}_{(00000n)}v^{11+2n}+(\chi^F_{(001)a\otimes(11)_b}\chi^{E_6}_{(10000n)}+c.c.)v^{12+2n}+\dots\Big].\qquad(5.96)$$

$\boldsymbol{n=1,\ G=E_6,\ F=\mathfrak{su}(5)\times\mathfrak{u}(1).}$ We use Weyl orbit expansion to solve elliptic genus for this theory. Let us first turn off the $\mathfrak{su}(5)$ fugacities and only keep $\mathfrak{u}(1)$ and make use of the unity blowup equations with nonzero λ_F only on $\mathfrak{u}(1)$. Then the reduced one-string elliptic genus with all gauge and flavor fugacities turned off can be computed as

$$\mathbb{E}_{h^{(1)}_{1,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{22}},\qquad(5.97)$$

where

$$\begin{aligned}P_0(v)=&78+1446v+12182v^2+60108v^3+180534v^4+260152v^5-365242v^6-3157324v^7\\&-9013936v^8-13246110v^9+3729696v^{10}+83186464v^{11}+255829040v^{12}\\&+405233216v^{13}+\dots+78v^{26},\\P_1(v)=&v^{-2}(2430+36180v+222432v^2+630204v^3+69266v^4-5565632v^5-17594496v^6\\&-11700192v^7+74362142v^8+245593684v^9+202313896v^{10}-730064340v^{11}-2618359266v^{12}\\&-2448587624v^{13}+5677163436v^{14}+16560265456v^{15}+\dots+2430v^{30}).\end{aligned}\qquad(5.98)$$

We have cross-checked our result against the modular ansatz in [1].[41] Let us further denote

$$\mathbb{E}_{h^{(1)}_{1,E_6}}(q,v,m_{E_6}=0,m_F=0)=q^{-1/3}\sum_{i,j}c_{i,j}v^j(q/v^2)^i.\qquad(5.100)$$

[41] In [1], the modular ansatz for this theory is determined up to three unfixed parameters. Using our result from blowup equations, we are able to determine their three unfixed parameters as

$$a_1=-\frac{14389465}{35941499929195079270 4},\ a_2=-\frac{227027173}{1150127997734242536652 8},\ a_3=\frac{146734631}{3450383993202727609958 4}.\qquad(5.99)$$

i,j	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	78	-270	104	70	200	420	1124	5220	3468
2	2430	-17280	41262	-28080	-8746	-18640	-10490	7680	35296
3	43758	-463320	1999296	-4254770	3930732	-200322	-14660	-1987042	-3198410

Table 15. Series coefficients $c_{i,j}$ for the one-string elliptic genus of $n=1$ E_6 model.

Then we have the following table 15 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_6 where θ is the adjoint representation. The blue numbers in the second column are 10 times of the dimensions of representations $\square + k\theta$ of E_6, where the 10 is the double of the dimension of matter representation $\mathbf{5}$ of flavor $\mathfrak{su}(5)$. The orange number 104 in the third column is given by $\dim(E_6)+\dim(\mathfrak{su}(5)\times\mathfrak{u}(1))+1 = 78+25+1 = 104$. These are the constraints predicted in [1] by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus.

Let us also show some results with all flavor $\mathfrak{su}(5)_a\times\mathfrak{u}(1)_b$ fugacities turned on. For example, the q subleading order of reduced one-string elliptic genus with $m_{E_6}=0$ is

$$\begin{aligned}&78\,v^{-2} - (27\chi^F_{(1000)_a\oplus(1)_b} + c.c.)v^{-1} + \chi^F_{(1001)_a} + 80 + (\chi^F_{(3000)_a\oplus(3)_b} + c.c.)v\\ &+ \chi^F_{(2002)_a}v^2 + (\chi^F_{(0201)_a\oplus(3)_b} + c.c.)v^3 + \mathcal{O}(v^4),\end{aligned}$$

or in the descending order of the absolute value of $\mathfrak{u}(1)$ charge as

$$\sum_{n=0}^{\infty}\Big[\chi^F_{(-15)_b\oplus(15)_b}\chi^{E_6}_{(00000n)}v^{11+2n} - (\chi^F_{(0001)a\otimes(14)_b}\chi^{E_6}_{(10000n)} + c.c.)v^{12+2n} + \dots\Big]. \quad (5.101)$$

5.9 E_7 theories

$G=E_7$ theories on base curve $(-n)$ have flavor symmetry $F=\mathfrak{so}(8-n)$ and $n_f=(8-n)/2$ hypermultiplets in bi-representation $\frac{1}{2}(\mathbf{56},\mathbf{8-n})$. Note $\mathbf{8-n}$ is the fundamental representation of flavor group and $n=1,2,3,\dots,7,8$. There are n vanishing blowup equations with $\lambda_F=0$. Using the fact that the minimal Weyl orbit of $(P^\vee\backslash Q^\vee)_{E_7}$ consists just of weights of $\mathbf{56}$, it is easy to find the leading base degree of the vanishing blowup equations can be written as

$$\sum_{w\in\mathbf{56}}(-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w^{E_7}+(8-n)\epsilon_+)\prod_{w'\in\mathbf{8-n}}\theta_1(m_w^{E_7}+m_{w'}^{\mathfrak{so}(8-n)}-\epsilon_+)\prod_{\alpha\in\Delta(E_7)}^{w\cdot\alpha=1}\frac{1}{\theta_1(m_\alpha^{E_7})}=0, \quad (5.102)$$

which we have checked to be correct up to q^{20} for all n. Note these identities contain m_{E_7}, $m_{\mathfrak{so}(8-n)}$ and ϵ_+ as free parameters, thus are highly nontrivial. By setting $m_{\mathfrak{so}(8-n)}$ as zero, one obtains

$$\sum_{w\in\mathbf{56}}(-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w+(8-n)\epsilon_+)(\theta_1(m_w-\epsilon_+))^{8-n}\prod_{\alpha\in\Delta(E_7)}^{w\cdot\alpha=1}\frac{1}{\theta_1(m_\alpha)}=0, \quad (5.103)$$

which may be easier in case interested readers want to give a direct proof.

The unity blowup equations for $G = E_7$ theories only exist for even n, because for odd n the theory involves half-hyper. In the following, we discuss the cases $n = 7, 6, 4, 2$ individually.

$n = 7$, $G = E_7$. This theory is a minimal SCFT with a half-hypermultiplet in **56**. The associated Calabi-Yau geometry was constructed in [2] by slightly modifying the $\hat{E}_7$ resolution on $\mathcal{O}(-8) \to \mathbb{P}^1$ as $\mathcal{O}(-7)$. We provide a non-compact toric construction in (F.1), using which we computed the triple intersection numbers and the genus zero Gromov-Witten invariants.

There are in total seven non-equivalent vanishing blowup equations for this theory, which can be written as

$$\sum_{\lambda\in(P^\vee\backslash Q^\vee)_{E_7}}^{d_{1,2}\geq 0}(-1)^{|\lambda|}\theta_4^{[a]}(7\tau,-7m_\lambda-(7d_0-1/2)(\epsilon_1+\epsilon_2)-7d_1\epsilon_1-7d_2\epsilon_2)A_V^{E_7}(m,\lambda)A_H^{\frac{1}{2}\mathbf{56}}(m,\lambda)$$
$$\times\mathbb{E}_{d_1}(\tau,m+\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau,m+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)=0, \tag{5.104}$$

where $\lambda\cdot\lambda/2 = d_0 + 3/4$, $d_0 \in \mathbb{Z}$ and $a = 1/2 - i/7$, $i = 1, 2, \ldots, 7$. In base degree d expansion, the numbers of λ one needs to sum over and the Weyl orbits they lie in are summarized in the following table:

d	0	1	2	3	$\cdots$
Weyl orbit	[0000010]	[0000001]	[1000010]	[0000011]	$\cdots$
$\#\{\lambda\cdot\lambda/2 = d+3/4\}$	56	576	1512	4032	$\cdots$

For example, for the leading base degree, i.e. $d_0 = d_1 = d_2 = 0$, λ are just all the weights of fundamental representation **56**. Thus we have the following nontrivial identity:

$$\sum_{w\in\mathbf{56}}(-1)^{|w|}\theta_4^{[a]}(7\tau,-7m_w+\epsilon_+)\theta_1(m_w+\epsilon_+)\prod_{\alpha\in\Delta(E_7)}^{w\cdot\alpha=1}\frac{1}{\theta_1(m_\alpha)}=0. \tag{5.105}$$

For higher base degrees, we study the vanishing blowup equations from the viewpoint of local Calabi-Yau geometry. We find the seven vanishing blowup equations with the input of prepotential $F_{(0,0)}$ can determine most of the refined BPS invariants, although not all of them. We list some refined BPS invariants solved from blowup equations in table 24 in appendix G.

$n = 6$, $G = E_7$, $F = \mathfrak{so}(2)$. There are 12 unity blowup equations with $\lambda_F = (\pm 1)$. Using the recursion formula, we computed the one-string elliptic genus with flavor fugacities turned off to $\mathcal{O}(q^1)$. Our result agrees precisely with the modular ansatz in [1], therefore we just present the first few q orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{6,E_7}}(q,v,m_{E_7}=0,m_{\mathfrak{so}(2)}=0)=q^{-11/6}v^{15}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{22}(1+v)^{34}}, \tag{5.106}$$

We obtain

$$P_0(v)=-(2+24v-43v^2+52v^3+8027v^4-53360v^5+279039v^6-950972v^7+2698740v^8$$
$$-5898532v^9+10988680v^{10}-16600348v^{11}+21616127v^{12}-23243264v^{13}+\cdots+v^{26}).$$

and

$$
\begin{aligned}
P_1(v) = v^{-2}(&1+12v-226v^2-3284v^3+8157v^4+28752v^5-1098207v^6+6964508v^7\\
&-32103023v^8+103825488v^9-273840598v^{10}+575865704v^{11}-1024745731v^{12}\\
&+1517074676v^{13}-1931373701v^{14}+2077804192v^{15}+\cdots+v^{30}),\\
P_2(v) = v^{-4}(&-1-12v+91v^2+1776v^3-10620v^4-236256v^5+594632v^6+4166640v^7\\
&-76480778v^8+449325704v^9-1870890749v^{10}+5714898268v^{11}-14169525888v^{12}\\
&+28626262964v^{13}-49011331352v^{14}+70988810780v^{15}-88777609823v^{16}\\
&+95280766576v^{17}-88777609823v^{18}+\cdots-v^{34}).
\end{aligned}
$$

With gauge and flavor fugacities turned on, we confirm the conjectural exact formula for the reduced 5d one-instanton partition function in (H.40) of [1]. For example, the leading q order of (5.106) is

$$
\begin{aligned}
&-\chi^F_{(2)\oplus(-2)}v^{15}-(\chi^F_{(6)\oplus(-6)}-\mathbf{133})v^{17}-(\mathbf{912}\cdot\chi^F_{(1)\oplus(-1)}-\mathbf{56}\cdot\chi^F_{(5)\oplus(-5)})v^{18}\\
&+(\mathbf{8645}-\mathbf{133}\cdot\chi^F_{(6)\oplus(-6)}-\mathbf{1539}\cdot\chi^F_{(4)\oplus(-4)})v^{19}+\mathcal{O}(v^{20}),
\end{aligned}
\tag{5.107}
$$

and the subleading q order is

$$
\begin{aligned}
&v^{13}-(\mathbf{133}+2)\chi^F_{(4)\oplus(-4)}v^{15}+(-(\mathbf{133}+2)\chi^F_{(6)\oplus(-6)}+\mathbf{1539}\cdot\chi^F_{(2)\oplus(-2)}\\
&+\mathbf{8645}+\mathbf{7371}+\mathbf{1539}+3\cdot\mathbf{133}+1)v^{17}+\mathcal{O}(v^{18}).
\end{aligned}
\tag{5.108}
$$

$\boldsymbol{n=4,\ G=E_7,\ F=\mathfrak{so}(4).}$ There are 16 unity blowup equations with $\lambda_F=(\pm1,\pm1)$ if we regard $\mathfrak{so}(4)\cong\mathfrak{su}(2)\times\mathfrak{su}(2)$. Using the recursion formula, we computed the one-string elliptic genus with flavor fugacities turned off to $\mathcal{O}(q^4)$. Denote the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{4,E_7}}(q,v,m_{E_7}=0,m_{\mathfrak{so}(4)}=0)=q^{-5/6}v^{11}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{34}},
\tag{5.109}
$$

We obtain

$$
\begin{aligned}
P_0(v) = -(&1+24v+305v^2+2720v^3+14385v^4+10328v^5-213107v^6+227936v^7\\
&+3681535v^8-15349240v^9+32121373v^{10}-40005232v^{11}+32121373v^{12}+\cdots+v^{22}).\\
P_1(v) = v^{-2}(&9+216v+2296v^2+13704v^3+35681v^4-191536v^5-2195202v^6-3469024v^7\\
&+34360924v^8+12656096v^9-543596903v^{10}+1892316824v^{11}-3595032965v^{12}\\
&+4390454000v^{13}+\cdots+9v^{26}).
\end{aligned}
$$

The leading q order exactly agrees with the reduced 5d one-instanton partition function in (A.20) of [80]. We record our higher order results for in appendix E in equations (E.39), (E.40), (E.41). Let us denote

$$
\mathbb{E}_{h^{(1)}_{4,E_7}}(q,v,m_{E_7}=0,m_{\mathfrak{so}(4)}=0)=q^{-5/6}v^{11}\sum_{i,j}c_{i,j}v^j(q/v^2)^i.
\tag{5.110}
$$

i,j	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
0	0	0	0	0	0	0	0	0	0	0	-1	0	-39
1	0	0	0	0	0	0	0	0	0	0	9	0	-98
2	1	0	0	0	0	0	0	0	-9	0	0	0	1330
3	133	-224	140	0	25	0	14	0	-42	224	-1463	0	0
4	7371	-25920	41249	-31360	10010	-2688	3500	0	2050	2688	-7419	31360	-127480

Table 16. Series coefficients $c_{i,j}$ for the one-string elliptic genus of the $n=4$ E_7 model.

Then we have the following table 16 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_7 where θ is the adjoint representation. The blue numbers in the second column are four times the dimensions of representations $\square + k\theta$ of E_7, where the four is the dimension of matter representation $\mathbf{4}$ of flavor $\mathfrak{so}(4)$. The orange number 140 in the third column is given by $\dim(E_7)+\dim(\mathfrak{so}(4))+1 = 133+6+1 = 140$. These are the constraints given in [1] by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus which our result satisfies perfectly. By combining our result and the constraints from NSR elliptic genus at even higher q order, we are able to determine the modular ansatz of $\mathbb{E}_{h^{(1)}_{4,E_7}}(q,v)$, which will be given in the `Mathematica` file in the supplementary material or on the website [99].

If turning on all gauge E_7 and flavor $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ fugacities, we find the leading q order of reduced one-string elliptic genus, i.e. the reduced 5d Nekrasov partition function has the following expansion

$$
\begin{aligned}
&- v^{11} - \chi^F_{(60)\oplus(06)\oplus(44)} v^{13} + (\mathbf{133}\cdot\chi^F_{(42)\oplus(24)} - \chi^F_{(48)\oplus(84)})v^{15}\\
&- (\mathbf{912}\cdot\chi^F_{(33)} - \mathbf{56}\cdot\chi^F_{(73)\oplus(37)})v^{16} + \mathcal{O}(v^{17}),
\end{aligned}
$$

which agrees with the (A.20) of [80]. In fact, we find an exact formula for the reduced 5d Nekrasov partition function which will be given in appendix (E.38). For the subleading q order we obtain the following expansion

$$
\begin{aligned}
&\chi^F_{(22)}v^9 + (\chi^F_{(26)\oplus(62)} - \chi^F_{(02)\oplus(20)} - 1 - \mathbf{133})v^{11} - (\chi^F_{(64)\oplus(46)\oplus(80)\oplus(08)\oplus(62)\oplus(26)\oplus(40)\oplus(04)}\\
&+ (\mathbf{133}+3)\chi^F_{(44)} + (\mathbf{133}+2)\chi^F_{(60)\oplus(06)} + (\mathbf{133}+1)\chi^F_{(42)\oplus(24)})v^{13} + \mathcal{O}(v^{15}).
\end{aligned}
$$

***n* = 2, *G* = E_7, *F* = $\mathfrak{so}(6)$.** There are 16 unity blowup equations with $\lambda_F \in \mathbf{4}$ or $\bar{\mathbf{4}}$. Noticing the flavor symmetry $\mathfrak{so}(6) \cong \mathfrak{su}(4)$, we can turn on the fugacity of a sub-algebra $\mathfrak{su}(2)$ to perform the computation on elliptic genus easily. Using the Weyl orbit expansion method, we computed the one-string elliptic genus with $\mathfrak{su}(2)$ flavor fugacities to $\mathcal{O}(q^2)$. For example, denote the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{2,E_7}}(q,v,m_{E_7}=0,m_{\mathfrak{so}(6)}=0) = q^{1/6}v^{-1}\sum_{n=0}^{\infty} q^n \frac{(1-v)^2 P_n(v)}{(1+v)^{34}}, \tag{5.111}
$$

we obtain

$$
\begin{aligned}
P_0(v)=&(1+36v+632v^2+7212v^3+60168v^4+391380v^5+2067496v^6+9123228v^7+34335094v^8\\
&+111995836v^9+320744719v^{10}+815144896v^{11}+1854166712v^{12}+3796415104v^{13}\\
&+6997399845v^{14}+11475775012v^{15}+16204920073v^{16}+18551114752v^{17}+\cdots+v^{34}),\\
P_1(v)=&v^{-2}(133+4452v+72109v^2+752208v^3+5673385v^4+32915460v^5\\
&+152504980v^6+577794348v^7+1815737068v^8+4761819476v^9+10385374307v^{10}\\
&+18472471608v^{11}+25278998607v^{12}+21455489108v^{13}-5924034231v^{14}\\
&-61899269488v^{15}-122152636908v^{16}-122341883440v^{17}-16307972890v^{18}\\
&+84187540856v^{19}+\cdots+133v^{38}),\\
P_2(v)=&v^{-4}(7371+226476v+3331334v^2+31148940v^3+207151332v^4+1037448756v^5\\
&+4032702373v^6+12310917456v^7+29294737640v^8+52132350336v^9+59789988702v^{10}\\
&+9188063128v^{11}-131878217677v^{12}-303150457484v^{13}-293119312706v^{14}\\
&+124103122340v^{15}+762220055405v^{16}+700512400544v^{17}-879753089280v^{18}\\
&-2366194285936v^{19}+148716225866v^{20}+4344623389448v^{21}+\cdots+7371v^{42}).
\end{aligned}
$$

If we turn on all gauge E_7 and flavor $\mathfrak{su}(4)$ fugacities, we find the leading q order of reduced one-string elliptic genus has the following expansion

$$
\begin{aligned}
&v^{-1}-(\chi^{\mathfrak{su}(4)}_{(400)}+\chi^{\mathfrak{su}(4)}_{(004)})v^7-\chi^{\mathfrak{su}(4)}_{(222)}v^9-\mathbf{56}\cdot\chi^{\mathfrak{su}(4)}_{(030)}v^{10}\\
&-(\chi^{\mathfrak{su}(4)}_{(602)}+\chi^{\mathfrak{su}(4)}_{(206)}+\chi^{\mathfrak{su}(4)}_{(323)}+\chi^{\mathfrak{su}(4)}_{(060)}-\mathbf{133}\cdot\chi^{\mathfrak{su}(4)}_{(121)}+\mathbf{1463})v^{11}+\mathbf{6480}\cdot\chi^{\mathfrak{su}(4)}_{(010)}v^{12}+\mathcal{O}(v^{13}).
\end{aligned}
$$

The subleading q order has expansion as

$$
\begin{aligned}
&\mathbf{133}\,v^{-3}-\mathbf{56}\cdot\chi^{\mathfrak{su}(4)}_{(010)}v^{-2}+(\mathbf{133}+\chi^{\mathfrak{su}(4)}_{(101)}+1)v^{-1}+\chi^{\mathfrak{su}(4)}_{(040)}v\\
&+\chi^{\mathfrak{su}(4)}_{(303)}v^3+(\chi^{\mathfrak{su}(4)}_{(420)}+\chi^{\mathfrak{su}(4)}_{(024)})v^5+\mathcal{O}(v^6).
\end{aligned}
$$

Let us further denote

$$
\mathbb{E}_{h^{(1)}_{2,E_7}}(q,v,m_{E_7}=0,m_{\mathfrak{so}(6)}=0)=q^{1/6}\sum_{i,j}c_{i,j}v^j(q/v^2)^i. \tag{5.112}
$$

Then we have the following table 17 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_7 where θ is the adjoint representation. The blue numbers in the second column are six times the dimensions of representations $\square+k\theta$ of E_7, where the six is the dimension of the matter representation **6** of flavor symmetry $\mathfrak{so}(6)$. The orange number 149 in the third column is given by $\dim(E_7)+\dim(\mathfrak{so}(6))+1=133+15+1=149$. These are the constraints given in [1] by analyzing the spectral flow to NSR elliptic genus, which our result satisfies perfectly. By combining our result and the constraints from NSR elliptic genus at even higher q order, we are able to determine the modular ansatz of $\mathbb{E}_{h^{(1)}_{2,E_7}}(q,v)$, which will be given in the `Mathematica` file in the supplementary material or on the website [99].

i,j	-1	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	-70	0	-729
1	133	-336	149	0	105	0	300	0	720	7840	-20777
2	7371	-38880	72542	-50064	11324	-21504	15645	-30240	47340	-146106	1938800

Table 17. Series coefficients $c_{i,j}$ for the one-string elliptic genus of the $n = 2$ E_7 model.

base	$3,2$	$3,2,2$	$2,3,2$
gauge symmetry	$G_2 \times \mathfrak{su}(2)$	$G_2 \times \mathfrak{su}(2) \times \{\ \}$	$\mathfrak{su}(2) \times \mathfrak{so}(7) \times \mathfrak{su}(2)$
matter	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$

Table 18. Three higher-rank NHCs.

6 Three higher rank non-Higgsable clusters

The three non-Higgsable clusters in table 18 are the simplest higher-rank 6d $(1,0)$ SCFTs and building blocks for more complicated higher-rank theories [57]. The 2d quiver gauge theories corresponding to these three NHCs have been constructed in [14]. Using Jeffrey-Kirwan residue, the elliptic genera can be explicitly computed as formulae involving Jacobi theta functions. It is interesting to see how blowup equations work for these higher rank theories. The most prominent feature here is that there only exist *vanishing* blowup equations for these three NHCs. We study them with two approaches: from the viewpoint of gauge theory, to which end we derive the *higher-rank elliptic blowup equations*, and from the viewpoint of elliptic non-compact Calabi-Yau. In particular, we give the toric constructions for the elliptic non-compact Calabi-Yau threefolds associated with the three NHCs, which to our knowledge are new.

We first introduce a special kind of higher dimension Riemann theta function associated to a $N \times N$ matrix Ω. It turns out that the polynomial part of the higher rank 6d theories contributes to the blowup equation as this type of Riemann theta function after de-affinization. We define

$$\Theta_{\Omega}^{[a]}(\tau,z)=\sum_{k\in\mathbb{Z}^N+a}(-1)^{k\cdot\mathrm{diag}(\Omega)}\exp\left(\frac{1}{2}k\cdot\Omega\cdot k\,\tau+k\cdot\Omega\cdot z\right). \tag{6.1}$$

Here the characteristic $a=(a_1,a_2,\dots,a_N)$ takes the following values

$$a_i=\sum_j(\Omega^{-1})_{ij}\left(\frac{1}{2}\Omega_{jj}+m_j\right),\qquad m_j\in\mathbb{Z}. \tag{6.2}$$

The number of different such characteristics is $\mathrm{Det}(\Omega)$. This kind of Riemann theta function is the proper generalization of $\theta_i^{[a]}(n\tau,nz)$ appearing in rank one elliptic blowup equations. As in the rank one cases, when the characteristic a is trivial, we suppress the superscript $\Theta_\Omega=\Theta_\Omega^{[0]}$.

6.1 NHC 3, 2

Elliptic genera from quiver gauge theory. The elliptic genera of the NHC $3,2$ can be obtained from the elliptic genera $\mathbb{E}_{k_1,k_2,k_3}$ of the NHC $2,3,2$, which are given in [14], via decompactification and Higgsing mechanism. Recall that the NHC $2,3,2$ carries the product of gauge groups $\mathfrak{su}(2)_1\times\mathfrak{so}(7)\times\mathfrak{su}(2)_2$ over the three compact curves, and the matter content is $\frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})+\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1})$. In the formula for $\mathbb{E}_{k_1,k_2,k_3}$ given in [14], the gauge fugacities of $\mathfrak{su}(2)_1,\mathfrak{so}(7),\mathfrak{su}(2)_2$ are respectively $\nu,v_\ell,\tilde{\nu}$ for $\ell=1,\dots,4$ with the constraint $\sum_\ell v_\ell=0$.

We first decompactify the first (-2) curve, and arrive at the $(-3,-2)$ geometry with gauge group $\mathfrak{so}(7)\times\mathfrak{su}(2)_2$ and matter content $\frac{1}{2}(\mathbf{8},\mathbf{2})+(\mathbf{8},\mathbf{1})$. The gauge group $\mathfrak{su}(2)_1$ becomes flavor symmetry after decompactification. The hyper $(\mathbf{8},\mathbf{1})$ is nontrivially charged under this flavor symmetry, and its mass is ν. This step corresponds to setting $k_1=0$ in the elliptic genus $\mathbb{E}_{k_1,k_2,k_3}$. Then following the discussion on page 18 of [14], we can use the hyper $(\mathbf{8},\mathbf{1})$ to Higgs $\mathfrak{so}(7)$ down to G_2. Given the branching rule $\mathbf{8}\to\mathbf{7}+\mathbf{1}$ for $G_2\subset\mathfrak{so}(7)$, we can give vev to the hyper in $\mathbf{1}$, which becomes massive and decouples in the IR. The hypers in $\mathbf{7}$ get eaten by 7 copies of vector multiplets from the adjoint $\mathbf{21}$ of $\mathfrak{so}(7)$, which also become massive and decouple. The remaining 14 copies of vector multiplets form the adjoint of G_2. The hypers in $\frac{1}{2}(\mathbf{8},\mathbf{2})$ decompose to $\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$, which is precisely the matter content of the $(-3,-2)$ NHC. This step of Higgsing is realised by [14] $\nu=\epsilon_+,v_4=0$ in the elliptic genus. In the end, we find the elliptic genus $\mathbb{E}_{k_1,k_2}$ of the NHC $3,2$ should be (k_1 degree on (-3) curve, k_2 degree on (-2) curve)

$$\begin{aligned}
&(-1)^{k_1+k_2}\mathbb{E}_{k_1,k_2}(\tau,v_\ell,\tilde{\nu})\\
&=\sum_{\substack{Y_{(1,2)}\\|Y_{(a)}|=k_a}}\prod_{i=1}^{3}\prod_{s_1\in Y_{(1)_i}}\frac{\theta(2\phi(s_1))\theta(2\phi(s_1)-2\epsilon_+)\theta(\tilde{\nu}\pm\phi(s_1))}{\left(\prod_{j=1}^3\theta(E_{ij}(s_1))\theta(E_{ij}(s_1)-2\epsilon_+)\theta(\epsilon_+-\phi(s_1)-v_j)\right)\theta(\epsilon_+-\phi(s_1))}\\
&\quad\times\prod_{i\le j}^{3}\prod_{\substack{s_1,\tilde{s}_1\in Y_{(1)_{i,j}}\\s_1<\tilde{s}_1}}\frac{\theta(\phi(s_1)+\phi(\tilde{s}_1))\theta(\phi(s_1)+\phi(\tilde{s}_1)-2\epsilon_+)}{\theta(\epsilon_{1,2}-\phi(s_1)-\phi(\tilde{s}_1))}\\
&\quad\times\prod_{i=1}^{3}\prod_{j=1}^{2}\prod_{s_1\in Y_{(1)_i}}\prod_{s_2\in Y_{(2)_j}}\frac{\theta(\epsilon_-\pm(\phi(s_1)-\phi(s_2)))}{\theta(\epsilon_+\pm(\phi(s_1)-\phi(s_2)))}\cdot\prod_{i=1}^{2}\prod_{s_2\in Y_{(2)_i}}\frac{\theta(\phi(s_2))\prod_{\ell=1}^3\theta(v_\ell-\phi(s_2))}{\prod_{j=1}^2\theta(E_{ij}(s_2))\theta(E_{ij}(s_2)-2\epsilon_+)}.
\end{aligned}\tag{6.3}$$

Here $Y_{(1)},Y_{(2)}$ are tuples of three and two Young diagrams respectively, and $|Y_{(a)}|=k_a$ means the total number of boxes in the tuple of Young diagrams is k_a. v_ℓ and $\tilde{\nu}$ are fugacities of G_2 and $\mathfrak{su}(2)$ with the constraint $\sum_{\ell=1}^3 v_\ell=0$. We define that for $s_a=(m_a,n_a)\in Y_{(a)_i}$

$$E_{ij}(s_a)=v_{(a)_i}-v_{(a)_j}-\epsilon_1 h_i(s_a)+\epsilon_2(v_j(s_a)+1)\,,\tag{6.4}$$

$$\phi(s_a)=v_{(a)_i}-\epsilon_+-(n_a-1)\epsilon_1-(m_a-1)\epsilon_2\,.\tag{6.5}$$

Here $h_i(s_a)$ denotes the distance from s_a to the right end of the diagram $Y_{(a)_i}$, and $v_j(s_a)$ denotes the distance from s_a to the bottom of the *diagram* $Y_{(a)_j}$. Concretely

$$h_i(s_a)=Y_{(a)_i}(m)-n\,,\quad v_j(s_a)=Y^t_{(a)j}(n)-m\,,\quad s=(m,n)\,.\tag{6.6}$$

Besides, $s_i<s_j$ means $(i<j)$ or $(i=j,m_i<m_j)$ or $(i=j,m_i=m_j,n_i<n_j)$. We also use the notation $v_{(1)_{1,2,3}}=(v_1,v_2,v_3)$ and $v_{(2)_{1,2}}=\pm\tilde{\nu}$. In the derivation of the Higgsing

formula, we need to use the identity that

$$\prod_{s\in Y_i}\theta(E_{i4}(s))\theta(E_{i4}(s)-2\epsilon_+)=-\prod_{s\in Y_i}\theta(\epsilon_+\pm\phi(s)))\,,\quad \text{with } v_4=0,\ Y_4=\emptyset,\ i=1,2,3\,, \tag{6.7}$$

which can be easily proved by a change of summation order in $s\in Y_i$.

We checked that indeed when $k_2=0$, (6.3) reduces to the elliptic genus $\mathbb{E}_{k_1}$ of G_2 with one hyper in $\mathbf{7}$ over (-3) curve [14], where $\tilde{\nu}$ is identified with the flavor mass. Furthermore the genus zero GV invariants with base degrees $(1,0)$, $(0,1)$, $(1,1)$ agree with mirror symmetry calculations.

Elliptic blowup equations. The geometric construction of this NHC has been given in (2.51). Let us denote the intersection matrix between the two base curves as $-\Omega$, i.e.

$$\Omega=\begin{pmatrix}3 & -1\\ -1 & 2\end{pmatrix}. \tag{6.8}$$

Note $\det\Omega=5$. It turns out there are in total five vanishing λ_F fields and no unity λ_F fields. To see this, one can simply look at the matter representation $(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})$. Note G_2 can only bear unity equations due to the Lie algebra fact $P^\vee\cong Q^\vee$. On the other hand, the unpaired half-hyper on the $\mathfrak{su}(2)$ node indicates only vanishing equations. Thus combining unity and vanishing equations naturally results in vanishing equations. The idea can be roughly expressed as

$$\mathrm{U}\star\mathrm{V}=\mathrm{V}. \tag{6.9}$$

To derive the elliptic blowup equations, we apply the similar *de-affinization* procedure [7] as in the rank one cases. As there are two base curves and each has a nontrivial gauge symmetry, now we have *two* degeneration directions. From the polynomial part given in (2.52), we can compute the coefficients of t_{ell_1} and t_{ell_2} in the blowup equations as

$$P_1(n_0,n_1,n_2)=P_{\widehat{G}_2}(n_0,n_1,n_2)=3n_1^2-3n_1n_2+n_2^2-n_0n_2+n_0^2, \tag{6.10}$$

and

$$P_2(n_3,n_4)=P_{\widehat{\mathfrak{su}(2)}}(n_3,n_4)=n_3^2-2n_3n_4+n_4^2. \tag{6.11}$$

It is important that P_1 and P_2 satisfy the following shift invariance

$$\begin{aligned}P_1(n_0+k,n_1+k,n_2+2k)&=P_1(n_0,n_1,n_2), & k\in\mathbb{Z},\\ P_2(n_3+l,n_4+l)&=P_2(n_3,n_4), & l\in\mathbb{Z}.\end{aligned} \tag{6.12}$$

Besides, the R shifts satisfy

$$R(n_0+k,n_1+k,n_2+2k,n_3+l,n_4+l)-R(n_0,n_1,n_2,n_3,n_4)=(-3k+l,k-2l,0,0,0,0), \tag{6.13}$$

which is crucial for modularity. By careful calculations, we find the polynomial part contributes to blowup equations as a special Riemann theta function defined in (6.1). The contributions from vector and hypermultiplets remain invariant under the shift. Finally, after careful calculations, the five vanishing elliptic blowup equations can be written as

$$\sum_{\substack{\alpha^\vee\in Q^\vee_{G_2},d_{1,2}\\ d_0=\frac{1}{2}||\alpha^\vee||^2}}^{d_0+d_1+d_2=k_1}\ \sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d'_{1,2}\\ d'_0+1/4=\frac{1}{2}||\lambda||^2}}^{d'_0+d'_1+d'_2=k_2}(-1)^{|\alpha^\vee|+|\lambda|}\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}\alpha^\vee\cdot m_{G_2}+(\bar y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \lambda\cdot m_{\mathfrak{su}(2)}+(\bar y_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\end{pmatrix}\right)$$
$$\times A_V^{G_2}(\alpha^\vee,\tau,m_{G_2})A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})}(\alpha^\vee,\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})$$
$$\times\mathbb{E}_{d_1,d'_1}(\tau,m_{G_2}-\epsilon_1\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)$$
$$\times\mathbb{E}_{d_2,d'_2}(\tau,m_{G_2}-\epsilon_2\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)=0,\qquad \text{for fixed}\,k_{1,2}\in\mathbb{Z}_{\geq 0},\tag{6.14}$$

where the summation indices $d_{0,1,2},d'_{0,1,2}\in\mathbb{Z}_{\geq0}$. The parameters $y_{1,2}$ are $\bar y_1=3/5,\bar y_2=3/10$, and

$$a=\begin{pmatrix}a_k\\ a_l\end{pmatrix}=\begin{pmatrix}2j/5\\ -1/2+j/5\end{pmatrix},\qquad j=-2,-1,0,1,2.\tag{6.15}$$

This is our starting point to prove the modularity. Note $\bar y_1,\bar y_2$ satisfy the following relation

$$\Omega\begin{pmatrix}\bar y_1\\ \bar y_2\end{pmatrix}=\begin{pmatrix}3/2\\ 0\end{pmatrix}=\begin{pmatrix}\bar y_{\rm u}\text{ of rank one }n=3\ G_2\text{ theory}\\ \bar y_{\rm v}\text{ of rank one }n=2\ \mathfrak{su}(2)\text{ theory}\end{pmatrix}.\tag{6.16}$$

It can be shown this is necessary to be consistent with the established elliptic blowup equations for rank one theories when decompactifying one of the base curves.

The leading base degree of the vanishing blowup equations, i.e. $d_0=d_1=d_2=d'_0=d'_1=d'_2=0$ can be simply written as[42]

$$\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}6\epsilon_+/5\\ m_{\mathfrak{su}(2)}+3\epsilon_+/5\end{pmatrix}\right)-\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}6\epsilon_+/5\\ -m_{\mathfrak{su}(2)}+3\epsilon_+/5\end{pmatrix}\right)=0.\tag{6.17}$$

It is easy to check that the above identity is correct. For higher base degrees, the vanishing blowup equations (6.14) involve nontrivial elliptic genera. We have checked them from the Calabi-Yau setting to high degrees of Kähler classes. Besides, we find the five vanishing blowup equations are not sufficient to solve all refined BPS invariants. This is not surprising since vanishing blowup equations give less constraints just like in the rank one theories.

Modularity. The index of the elliptic genus $\mathbb{E}_{k_1,k_2}$ is known to be

$$\begin{aligned}\text{Ind}_{\mathbb{E}_{k_1,k_2}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{4}(3k_1+2k_2)+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2+2k_2^2-2k_1k_2-k_1)\\&+(-3k_1+k_2)\frac{(m,m)_{G_2}}{2}+(-2k_2+k_1)\frac{(m,m)_{\mathfrak{su}(2)}}{2}.\end{aligned}\tag{6.18}$$

[42]The $\mathfrak{su}(2)$ vector multiplets do contribute to the blowup equation here. However, the contribution to each of the two terms can be factored out.

Let us use this to prove the modularity of (6.14). First, it is easy to derive from the general theory of Riemann theta functions that the index quadratic form of $\Theta_\Omega^{[a]}(\tau,z)$ under special modular transformation $\tau \to -1/\tau$ is just

$$\frac{1}{2} z\cdot\Omega\cdot z. \tag{6.19}$$

This fact is useful when computing the index of the polynomial contribution. Indeed, the index of polynomial part in (6.14) is

$$\begin{aligned}
&\mathrm{Ind}_{\mathrm{poly}} = \frac{3}{2}(\alpha^\vee\cdot m_{G_2}+(y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2)^2\\
&+(\lambda\cdot m_{\mathfrak{su}(2)}+(y_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2)^2\\
&-(\alpha^\vee\cdot m_{G_2}+(y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2)(\lambda\cdot m_{\mathfrak{su}(2)}+(y_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2).
\end{aligned}$$

The G_2 vector multiplet contributes to the index as

$$\begin{aligned}
\mathrm{Ind}_V^{G_2} &= -\frac{5}{3}\Big((\alpha^\vee\cdot m_{G_2})^2+d_0 m_{G_2}\cdot m_{G_2}\Big)+\frac{2}{3}(5d_0-2)(\epsilon_1+\epsilon_2)(\alpha^\vee\cdot m_{G_2})\\
&\quad-\frac{1}{3}(5d_0^2-2d_0)(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2).
\end{aligned}$$

and the $\mathfrak{su}(2)$ vector multiplet contributes to the index as

$$\begin{aligned}
\mathrm{Ind}_V^{\mathfrak{su}(2)} &= -\frac{4}{3}\left((\lambda\cdot m_{\mathfrak{su}(2)})^2+\left(d_0'+\frac{1}{4}\right)m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\right)+\frac{8}{3}d_0'(\epsilon_1+\epsilon_2)(\lambda\cdot m_{\mathfrak{su}(2)})\\
&\quad-\frac{1}{3}(4{d_0'}^2+d_0')(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2).
\end{aligned}$$

The hypermultiplet in the representation $(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})$ contributes to the index as

$$\begin{aligned}
&\mathrm{Ind}_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})} = \frac{1}{4}\left(\frac{2}{3}\Big((\alpha^\vee\cdot m_{G_2})^2+d_0 m_{G_2}\cdot m_{G_2}\Big)+2d_0 m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}+2\left(d_0'+\frac{1}{4}\right)m_{G_2}\cdot m_{G_2}\right.\\
&\left.+\frac{4}{3}\left((\lambda\cdot m_{\mathfrak{su}(2)})^2+\left(d_0'+\frac{1}{4}\right)m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\right)+4(\alpha^\vee\cdot m_{G_2})(\lambda\cdot m_{\mathfrak{su}(2)})\right)\\
&-\frac{1}{8}(m_{G_2}\cdot m_{G_2}+2m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)})-\frac{1}{6}\left(2d_0\alpha^\vee\cdot m_{G_2}+6\left(d_0'+\frac{1}{4}\right)\alpha^\vee\cdot m_{G_2}+6d_0\lambda\cdot m_{\mathfrak{su}(2)}\right.\\
&\left.+4\left(d_0'+\frac{1}{4}\right)\lambda\cdot m_{\mathfrak{su}(2)}\right)+\frac{1}{12}(\alpha^\vee\cdot m_{G_2}+2\lambda\cdot m_{\mathfrak{su}(2)})+\cdots.
\end{aligned}$$

Using (6.18), we can also easily compute the index of $\mathbb{E}_{d_1,d_1'}(\tau, m_{G_2}-\epsilon_1\alpha^\vee, m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)$ as

$$\begin{aligned}
\mathrm{Ind}_{\mathbb{E}_{d_1,d_1'}} &= -\frac{\epsilon_2^2}{4}(3d_1+2d_1')+\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{2}(3d_1^2+2{d_1'}^2-2d_1d_1'-d_1)\\
&\quad+(-3d_1+d_1')\left(\frac{(m,m)_{G_2}}{2}-\epsilon_1\alpha^\vee\cdot m_{G_2}+d_0\epsilon_1^2\right)\\
&\quad+(d_1-2d_1')\left(\frac{(m,m)_{A_1}}{2}-\epsilon_1\lambda\cdot m_{\mathfrak{su}(2)}+d_0\epsilon_1^2\right).
\end{aligned}$$

and the index of $\mathbb{E}_{d_2,d_2'}(\tau, m_{G_2}-\epsilon_2\alpha^\vee, m_{\mathfrak{su}(2)}-\epsilon_2\lambda, \epsilon_1-\epsilon_2, \epsilon_2)$ as

$$\begin{aligned}\mathrm{Ind}_{\mathbb{E}_{d_2,d_2'}} &= -\frac{\epsilon_1^2}{4}(3d_2+2d_2') + \frac{(\epsilon_1-\epsilon_2)\epsilon_2}{2}(3d_2^2+2{d_2'}^2-2d_2d_2'-d_2) \\ &+(-3d_2+d_2')\left(\frac{(m,m)_{G_2}}{2}-\epsilon_2\alpha^\vee\cdot m_{G_2}+d_0\epsilon_2^2\right) \\ &+(d_1-2d_1')\left(\frac{(m,m)_{\mathfrak{su}(2)}}{2}-\epsilon_2\lambda\cdot m_{\mathfrak{su}(2)}+d_0\epsilon_2^2\right).\end{aligned}$$

Finally, by directly adding all contributions together and using the constraints $d_0+d_1+d_2=k_1$ and $d_0'+d_1'+d_2'=k_2$, we obtain

$$\begin{aligned}&\mathrm{Ind}_{\mathrm{poly}}+\mathrm{Ind}_V^{G_2}+\mathrm{Ind}_V^{\mathfrak{su}(2)}+\mathrm{Ind}_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})}+\mathrm{Ind}_{\mathbb{E}_{d_1,d_1'}}+\mathrm{Ind}_{\mathbb{E}_{d_2,d_2'}} \\ &=-\frac{1}{2}\begin{pmatrix}k_1 & k_2\end{pmatrix}\Omega\begin{pmatrix}m_{G_2}\cdot m_{G_2}\\ m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\end{pmatrix}-\frac{\epsilon_1^2+\epsilon_2^2}{4}(3k_1+2k_2)+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2-2k_1k_2+2k_2^2-4k_1)+\frac{9}{5}\epsilon_+^2.\end{aligned} \tag{6.20}$$

The final sum is *independent* from $\alpha^\vee, \lambda, d_1, d_1', d_2, d_2'$ themselves, but only depends on their combination (k_1,k_2)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

Limit to rank one theories. By taking the node 2 to zero limit, one obtains the $n=3$ G_2 theory with $n_{\mathbf{7}}=1$. The ungauged $\mathfrak{su}(2)$ becomes the $\mathfrak{sp}(1)$ flavor symmetry, thus $t_{\mathfrak{su}(2)}$ becomes the mass m of matter $\mathbf{7}$. As shown in section 5.6, there are six unity elliptic blowup equations for the $n=3$ G_2 theory. In the following, we analyze how they can be obtained from the five vanishing blowup equations of 3, 2 NHC. In fact, it is not hard to find that under the limit $Q_{\mathrm{ell}_2}\to 0$, the vanishing blowup equation (6.14) with characteristic (6.15) labeled with j reduces to

$$\theta_4^{[\frac{1}{6}+\frac{2j}{5}]}(15\tau,3\epsilon_+)\mathcal{U}_{G_2}^{[-\frac{1}{6}]}+\theta_4^{[-\frac{1}{6}+\frac{2j}{5}]}(15\tau,3\epsilon_+)\mathcal{U}_{G_2}^{[\frac{1}{6}]}+\theta_4^{[-\frac{1}{2}+\frac{2j}{5}]}(15\tau,3\epsilon_+)\mathcal{U}_{G_2}^{[\frac{1}{2}]}=0. \tag{6.21}$$

where we define

$$\mathcal{U}_{G_2}^{[a]}=\mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)}=1)-\mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)}=-1), \tag{6.22}$$

and $\mathrm{U}_{G_2}^{[a]}$ denotes the l.h.s. of unity blowup equations of the $n=3$ G_2 theory with characteristic a. Since $j=-2,-1,0,1,2$, clearly, one can conclude

$$\mathcal{U}_{G_2}^{[a]}=0,\quad \text{for}\quad a=-1/6,1/6,1/2, \tag{6.23}$$

which are

$$\mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)}=1)=\mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)}=-1),\quad \text{for}\quad a=-1/6,1/6,1/2. \tag{6.24}$$

By adding the r.h.s. of the unity blowup equations, these give exactly the six unity blowup equations as we already knew.

On the other hand, by taking the node 3 to zero limit, one obtains the $n=2$ $\mathfrak{su}(2)$ theory with 8 half-hypers transforming in $\mathbf{2}$ of $\mathfrak{su}(2)$. There are two vanishing elliptic blowup equations for the $n=2$ $\mathfrak{su}(2)$ theory. In fact, it is not hard to find that under the limit $Q_{\mathrm{ell}_1}\to 0$, the vanishing blowup equation (6.14) with characteristic (6.15) labeled with j reduces to

$$\theta_3^{[\frac{j}{5}]}(10\tau,6\epsilon_+)\mathrm{V}_{\mathfrak{su}(2)}^{[-\frac{1}{2}]}-\theta_3^{[-\frac{1}{2}+\frac{2j}{5}]}(10\tau,6\epsilon_+)\mathrm{V}_{\mathfrak{su}(2)}^{[0]}=0, \tag{6.25}$$

where $\mathrm{V}_{\mathfrak{su}(2)}^{[a]}$ denotes the l.h.s. of vanishing blowup equations of the $n=2$ $\mathfrak{su}(2)$ theory. Since $j=-2,-1,0,1,2$, clearly, one can conclude

$$\mathrm{V}_{\mathfrak{su}(2)}^{[-\frac{1}{2}]}=\mathrm{V}_{\mathfrak{su}(2)}^{[0]}=0. \tag{6.26}$$

These are just the two vanishing blowup equations of the $n=2$ $\mathfrak{su}(2)$ theory as we already knew.

6.2 NHC 3, 2, 2

NHC 3, 2, 2 can be understood as coupling a M-string node 2 to NHC 3, 2 from the right. The 2d quiver construction was conjectured in [14], therefore the elliptic genera are exactly computable. We give a geometric construction for the local Calabi-Yau associated to this theory in appendix F.

Elliptic blowup equations. There are in total seven vanishing blowup equations and no unity blowup equations, which is as expected since the M-string only have unity blowup equations, while the NHC 3, 2 has only vanishing equations. The idea can be roughly expressed as

$$\mathrm{V}\star\mathrm{U}=\mathrm{V}. \tag{6.27}$$

To derive the elliptic blowup equations, we apply the similar de-affinization procedure as in rank one cases. As there are three base curves $(-3,-2,-2)$ and only the first two have nontrivial gauge symmetry, we have *two* degeneration directions. Let us denote the intersection matrix between the three base curves as $-\Omega$, i.e.

$$\Omega=\begin{pmatrix}3&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}. \tag{6.28}$$

Note $\det\Omega=7$ gives the number of non-equivalent vanishing blowup equations. We find the seven vanishing elliptic blowup equations can be written as

$$
\begin{aligned}
0=&\sum_{\substack{\alpha^\vee\in Q^\vee_{G_2},d_{1,2}\\ d_0=\frac{1}{2}||\alpha^\vee||^2}}^{d_0+d_1+d_2=k_1}\;\sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d'_{1,2}\\ d'_0+1/4=\frac{1}{2}||\lambda||^2}}^{d'_0+d'_1+d'_2=k_2}\;\sum_{d''_{1,2}}^{d''_1+d''_2=k_3}(-1)^{|\alpha^\vee|+|\lambda|}\\
&\times\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}\alpha^\vee\cdot m_{G_2}+(\bar y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \lambda\cdot m_{\mathfrak{su}(2)}+(\bar y_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\\ \bar y_3(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{G_2}(\alpha^\vee,\tau,m_{G_2})A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\emptyset)}(\alpha^\vee,\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})\\
&\times\mathbb{E}_{d_1,d'_1,d''_1}(\tau,m_{G_2}-\epsilon_1\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times\mathbb{E}_{d_2,d'_2,d''_2}(\tau,m_{G_2}-\epsilon_2\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2),\qquad\text{for fixed }k_{1,2,3}\in\mathbb{Z}_{\ge0},
\end{aligned}
\tag{6.29}
$$

where the summation indices $d_{0,1,2},d'_{0,1,2},d''_{1,2}\in\mathbb{Z}_{\ge0}$. The parameters $(\bar y_1,\bar y_2,\bar y_3)=(5/7,9/14,4/7)$, and

$$
a=\begin{pmatrix}a_k\\ a_l\\ a_s\end{pmatrix}=\begin{pmatrix}3j/7\\ -1/2+2j/7\\ j/7\end{pmatrix},\qquad j=-3,-2,-1,0,1,2,3. \tag{6.30}
$$

Note $\bar y_1,\bar y_2,\bar y_3$ satisfy the following relation

$$
\Omega\begin{pmatrix}\bar y_1\\ \bar y_2\\ \bar y_3\end{pmatrix}=\begin{pmatrix}3/2\\ 0\\ 1/2\end{pmatrix}=\begin{pmatrix}\bar y_{\mathrm{u}}\text{ of rank one }n=3\ G_2\text{ theory}\\ \bar y_{\mathrm{v}}\text{ of rank one }n=2\ \mathfrak{su}(2)\text{ theory}\\ \bar y_{\mathrm{u}}\text{ of }n=2\text{ M-string theory}\end{pmatrix}. \tag{6.31}
$$

This is necessary to be consistent with the rank one elliptic blowup equations when decompactifying one of the base curves.

The leading base degree of the vanishing blowup equations, i.e. $d_0=d_1=d_2=d'_0=d'_1=d'_2=0$ can be simply written as

$$
\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}10\epsilon_+/7\\ m_{\mathfrak{su}(2)}+9\epsilon_+/7\\ 8\epsilon_+/7\end{pmatrix}\right)-\Theta^{[a]}_\Omega\left(\tau,\begin{pmatrix}10\epsilon_+/7\\ -m_{\mathfrak{su}(2)}+9\epsilon_+/7\\ 8\epsilon_+/7\end{pmatrix}\right)=0. \tag{6.32}
$$

It is easy to check the above identity is correct. For higher base degrees, we have checked the seven vanishing blowup equations from the Calabi-Yau setting to substantial degrees of Kähler classes.

Modularity. The index of the elliptic genus $\mathbb{E}_{k_1,k_2,k_3}$ is known to be

$$
\begin{aligned}
\mathrm{Ind}_{\mathbb{E}_{k_1,k_2,k_3}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{4}(3k_1+2k_2+k_3)+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2+2k_2^2+2k_3^2-2k_1k_2-2k_2k_3-k_1)\\
&+(-3k_1+k_2)\frac{(m,m)_{G_2}}{2}+(-2k_2+k_1+k_3)\frac{(m,m)_{\mathfrak{su}(2)}}{2}.
\end{aligned}
\tag{6.33}
$$

To prove modularity, we need to calculate the index of each term in the elliptic blowup equations (6.29). After lengthy computations similar with the NHC 3, 2 case, by directly adding all contributions together and using the constraints $d_0+d_1+d_2=k_1$ and $d_0'+d_1'+d_2'=k_2$ and $d_1''+d_2''=k_3$, we obtain

$$\begin{aligned}
&\mathrm{Ind}_{\mathrm{poly}}+\mathrm{Ind}_V^{G_2}+\mathrm{Ind}_V^{\mathfrak{su}(2)}+\mathrm{Ind}_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\emptyset)}+\mathrm{Ind}_{\mathbb{E}_{d_1,d_1',d_1''}}+\mathrm{Ind}_{\mathbb{E}_{d_2,d_2',d_2''}}\\
&=-\frac{1}{2}\begin{pmatrix}k_1 & k_2 & k_3\end{pmatrix}\Omega\begin{pmatrix}m_{G_2}\cdot m_{G_2}\\ m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\\ 0\end{pmatrix}-\frac{\epsilon_1^2+\epsilon_2^2}{4}(3k_1+2k_2+k_3)\\
&\quad+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2-2k_1k_2+2k_2^2-2k_2k_3+2k_3^2-4k_1)+\frac{19}{28}(\epsilon_1+\epsilon_2)^2.
\end{aligned}\tag{6.34}$$

This final sum is *independent* from $\alpha^\vee,\lambda,d_1,d_1',d_2,d_2',d_1'',d_2''$ themselves, but only depends on their combination (k_1,k_2,k_3)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

Limits. It is well-known by dropping the last -2 base curve, i.e. taking $k_3=0$, one goes back to the $-3,-2$ NHC. By dropping the left $-3,-2$ base curves, i.e. taking $k_1=k_2=0$, one obtains the M-string theory. By dropping the left -3 base curve, i.e. taking $k_1=0$, one obtains a rank-two Higgsable theory with three vanishing blowup equations. Such theory has

$$\Omega=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}.\tag{6.35}$$

This theory can be obtained in the following way: one can take the $n=2, G=\mathfrak{su}(2)$ theory, restrict the flavor $\mathfrak{so}(7)\to G_2$, and make the gauge $\mathfrak{su}(2)$ coincide with the flavor $\mathfrak{su}(2)$ of an M-string theory. It is easy to write down the three vanishing blowup equations at degree (k_2,k_3) as

$$\begin{aligned}
\sum_{d_0'+1/4=\frac{1}{2}||\lambda||^2}^{d_0'+d_1'+d_2'=k_2}\ \sum_{d_1''+d_2''=k_3}&(-1)^{|\lambda|}\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}\lambda\cdot m_{\mathfrak{su}(2)}+(\bar{y}_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2\\ \bar{y}_3(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\emptyset)}(\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})\\
&\times\mathbb{E}_{d_1',d_1''}(\tau,m_{G_2},m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times\mathbb{E}_{d_2',d_2''}(\tau,m_{G_2},m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)=0.
\end{aligned}\tag{6.36}$$

where $\bar{y}_2=1/6,\bar{y}_3=1/3$.

6.3 NHC 2, 3, 2

NHC 2, 3, 2 can be understood as coupling two $2_{\mathfrak{su}(2)}$ theories to the rank one theory $3_{\mathfrak{so}(7)}$. The 2d quiver construction of this theory was given in [14]. Besides, this model has an orbifold construction [102], where the underlying geometry $T^2\times\mathbb{C}^2/\Gamma$ has discrete action Γ generated by $(\omega^{-6},\omega,\omega^5)$, where ω is a root of unity with $\omega^8=1$. The S^1 compactification to 5d has been studied with topological vertex in [103]. We give a toric geometric construction for the local Calabi-Yau associated to this theory in appendix F.

Elliptic blowup equations. Let us denote the intersection matrix between the three base curves as $-\Omega$, i.e.

$$\Omega=\begin{pmatrix}2&-1&0\\-1&3&-1\\0&-1&2\end{pmatrix}. \tag{6.37}$$

Note $\det\Omega=8$. It turns out there exist in total 16 vanishing blowup equations and no unity blowup equation. These vanishing equations are divided to two types, each consists of eight equations. One type comes from the configuration

$$\mathrm{V}\star\mathrm{U}\star\mathrm{V}=\mathrm{V}, \tag{6.38}$$

which means the unity equations of $3_{\mathfrak{so}(7)}$ theory coupled with the vanishing equations of two $2_{\mathfrak{su}(2)}$ theories. The other comes from the configuration

$$\mathrm{U}\star\mathrm{V}\star\mathrm{U}=\mathrm{V}, \tag{6.39}$$

which means the vanishing equations of $3_{\mathfrak{so}(7)}$ theory coupled with the unity equations of two $2_{\mathfrak{su}(2)}$ theories.

To precisely derive the elliptic blowup equations, we apply the similar de-affinization procedure as in rank one cases. As there are three base curves $(-2,-3,-2)$ and each has a nontrivial gauge symmetry, we have *three* degeneration directions. After careful computations, we find the eight VUV type vanishing elliptic blowup equations can be written as

$$\begin{aligned}0=&\sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d_{1,2}\\ d_0+1/4=\frac12||\lambda||^2}}^{d_0+d_1+d_2=k_1}\ \sum_{\substack{\alpha^\vee\in Q^\vee_{\mathfrak{so}(7)},d'_{1,2}\\ d'_0=\frac12||\alpha^\vee||^2}}^{d'_0+d'_1+d'_2=k_2}\ \sum_{\substack{\lambda'\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d''_{1,2}\\ d''_0+1/4=\frac12||\lambda'||^2}}^{d''_0+d''_1+d''_2=k_3}(-1)^{|\alpha^\vee|+|\lambda|+|\lambda'|}\\
&\times\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}\lambda\cdot m_{\mathfrak{su}(2)}+(\bar y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \alpha^\vee\cdot m_{\mathfrak{so}(7)}+(\bar y_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\\ \lambda'\cdot m'_{\mathfrak{su}(2)}+(\bar y_3-d''_0)(\epsilon_1+\epsilon_2)-d''_1\epsilon_1-d''_2\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{\mathfrak{so}(7)}(\alpha^\vee,\tau,m_{\mathfrak{so}(7)})A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_V^{\mathfrak{su}(2)}(\lambda',\tau,m'_{\mathfrak{su}(2)})A_H^{\mathfrak{R}}(\lambda,\alpha^\vee,\lambda',\tau,m_{\mathfrak{so}(7)},m_{\mathfrak{su}(2)},m'_{\mathfrak{su}(2)})\\
&\times\mathbb{E}_{d_1,d'_1,d''_1}(\tau,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,m_{\mathfrak{so}(7)}-\epsilon_1\alpha^\vee,m'_{\mathfrak{su}(2)}-\epsilon_1\lambda',\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times\mathbb{E}_{d_2,d'_2,d''_2}(\tau,m_{\mathfrak{su}(2)}-\epsilon_2\lambda,m_{\mathfrak{so}(7)}-\epsilon_2\alpha^\vee,m'_{\mathfrak{su}(2)}-\epsilon_2\lambda',\epsilon_1-\epsilon_2,\epsilon_2),\quad \text{fixed}\, k_{1,2,3}\in\mathbb{Z}_{\geq0},\end{aligned} \tag{6.40}$$

where the summation indices $d_{0,1,2},d'_{0,1,2},d''_{0,1,2}\in\mathbb{Z}_{\geq0}$. The parameters $\bar y_{1,2,3}$ are $\bar y_1=1/2,\bar y_2=1,\bar y_3=1/2$, $\mathfrak{R}=(\mathbf{1},\mathbf{8},\frac12\mathbf{2})+(\frac12\mathbf{2},\mathbf{8},\mathbf{1})$, and

$$a=\begin{pmatrix}(2j-1)/8\\(2j-1)/4\\(2j-1)/8\end{pmatrix},\qquad j=-3,-2,-1,0,1,2,3,4. \tag{6.41}$$

Note $\bar{y}_1, \bar{y}_2, \bar{y}_3$ satisfy the following relation

$$\Omega \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{y}_{\mathrm{v}} \text{ of rank one } n=2\ \mathfrak{su}(2) \text{ theory} \\ \bar{y}_{\mathrm{u}} \text{ of rank one } n=3\ \mathfrak{so}(7) \text{ theory} \\ \bar{y}_{\mathrm{v}} \text{ of rank one } n=2\ \mathfrak{su}(2) \text{ theory} \end{pmatrix}. \tag{6.42}$$

This is necessary to be consistent with the established elliptic blowup equations for rank one theories when decompactifying some of the base curves.

The leading base degree of the vanishing blowup equations (6.40), i.e. $d_0 = d_1 = d_2 = d'_0 = d'_1 = d'_2 = 0$ can be simply written as

$$\begin{aligned} &\Theta_\Omega^{[a]}\left(\tau, \begin{pmatrix} m'_{\mathfrak{su}(2)} + \epsilon_+ \\ 2\epsilon_+ \\ m'_{\mathfrak{su}(2)} + \epsilon_+ \end{pmatrix}\right) + \Theta_\Omega^{[a]}\left(\tau, \begin{pmatrix} -m'_{\mathfrak{su}(2)} + \epsilon_+ \\ 2\epsilon_+ \\ -m'_{\mathfrak{su}(2)} + \epsilon_+ \end{pmatrix}\right) \\ -&\Theta_\Omega^{[a]}\left(\tau, \begin{pmatrix} m_{\mathfrak{su}(2)} + \epsilon_+ \\ 2\epsilon_+ \\ -m'_{\mathfrak{su}(2)} + \epsilon_+ \end{pmatrix}\right) - \Theta_\Omega^{[a]}\left(\tau, \begin{pmatrix} -m_{\mathfrak{su}(2)} + \epsilon_+ \\ 2\epsilon_+ \\ m'_{\mathfrak{su}(2)} + \epsilon_+ \end{pmatrix}\right) = 0. \end{aligned} \tag{6.43}$$

It is easy to check the above identity is correct.

By similar de-affinazation procedure, the eight UVU type vanishing elliptic blowup equations can be written as

$$\begin{aligned} 0 = &\sum_{d_0 = \frac{1}{2}||\alpha||^2}^{d_0+d_1+d_2=k_1} \sum_{d'_0+1/2=\frac{1}{2}||\lambda||^2}^{d'_0+d'_1+d'_2=k_2} \sum_{d''_0=\frac{1}{2}||\alpha'||^2}^{d''_0+d''_1+d''_2=k_3} (-1)^{|\alpha|+|\lambda|+|\alpha'|} \\ &\times \Theta_\Omega^{[a]}\left(\tau, \begin{pmatrix} \alpha \cdot m_{\mathfrak{su}(2)} + (\bar{y}_1 - d_0)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2 \\ \lambda \cdot m_{\mathfrak{so}(7)} + (\bar{y}_2 - d'_0)(\epsilon_1+\epsilon_2) - d'_1\epsilon_1 - d'_2\epsilon_2 \\ \alpha' \cdot m'_{\mathfrak{su}(2)} + (\bar{y}_3 - d''_0)(\epsilon_1+\epsilon_2) - d''_1\epsilon_1 - d''_2\epsilon_2 \end{pmatrix}\right) \\ &\times A_V^{\mathfrak{so}(7)}(\lambda,\tau,m_{\mathfrak{so}(7)}) A_V^{\mathfrak{su}(2)}(\alpha,\tau,m_{\mathfrak{su}(2)}) A_V^{\mathfrak{su}(2)}(\alpha',\tau,m'_{\mathfrak{su}(2)}) A_H^{\mathfrak{R}}(\alpha,\lambda,\alpha',\tau,m_{\mathfrak{so}(7)},m_{\mathfrak{su}(2)},m'_{\mathfrak{su}(2)}) \\ &\times \mathbb{E}_{d_1,d'_1,d''_1}(\tau, m_{\mathfrak{su}(2)} - \epsilon_1\alpha, m_{\mathfrak{so}(7)} - \epsilon_1\lambda, m'_{\mathfrak{su}(2)} - \epsilon_1\alpha', \epsilon_1, \epsilon_2 - \epsilon_1) \\ &\times \mathbb{E}_{d_2,d'_2,d''_2}(\tau, m_{\mathfrak{su}(2)} - \epsilon_2\alpha, m_{\mathfrak{so}(7)} - \epsilon_2\lambda, m'_{\mathfrak{su}(2)} - \epsilon_2\alpha', \epsilon_1 - \epsilon_2, \epsilon_2), \quad \text{fixed}\, k_{1,2,3} \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{6.44}$$

where $\lambda \in (P^\vee \backslash Q^\vee)_{\mathfrak{so}(7)}$ and $\alpha, \alpha' \in Q^\vee_{\mathfrak{su}(2)}$, and $\bar{y}_1 = \bar{y}_3 = 3/4, \bar{y}_2 = 1/2$. The characteristics a are still those defined in (6.41). Note $\bar{y}_1, \bar{y}_2, \bar{y}_3$ satisfy the following relation

$$\Omega \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{y}_{\mathrm{u}} \text{ of rank one } n=2\ \mathfrak{su}(2) \text{ theory} \\ \bar{y}_{\mathrm{v}} \text{ of rank one } n=3\ \mathfrak{so}(7) \text{ theory} \\ \bar{y}_{\mathrm{u}} \text{ of rank one } n=2\ \mathfrak{su}(2) \text{ theory} \end{pmatrix}. \tag{6.45}$$

Since the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$ is $\mathcal{O}_6$,[43] the leading base degree of the vanishing blowup equations (6.44) can be simply written as

$$\sum_{w\in\mathcal{O}_6}(-1)^{|w|}\Theta_{\Omega}^{[a]}\left(\tau,\begin{pmatrix}3\epsilon_+/2\\ m_\omega+\epsilon_+\\ 3\epsilon_+/2\end{pmatrix}\right)\times\prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)}=0. \tag{6.46}$$

We have checked this identity up to $\mathcal{O}(q^{10})$. For higher base degrees, we have checked all the 16 vanishing blowup equations from the Calabi-Yau setting to substantial degrees of Kähler classes.

Modularity. The index of the elliptic genus $\mathbb{E}_{k_1,k_2,k_3}$ is known to be

$$\begin{aligned}\mathrm{Ind}_{\mathbb{E}_{k_1,k_2,k_3}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{2}(k_1+2k_2+k_3)+\frac{\epsilon_1\epsilon_2}{2}(2k_1^2+3k_2^2+2k_3^2-2k_1k_2-2k_2k_3-k_2)\\ &+(-2k_1+k_2)\frac{(m_1,m_1)_{\mathfrak{su}(2)}}{2}+(-3k_2+k_1+k_3)\frac{(m_2,m_2)_{\mathfrak{so}(7)}}{2}+(-2k_3+k_2)\frac{(m_3,m_3)_{\mathfrak{su}(2)}}{2}.\end{aligned}$$

Let us just show the modularity of the VUV type equations here. We need to calculate the index of each term in the vanishing elliptic blowup equations (6.40). After lengthy computations similar with the NHC 3, 2 case, by directly adding all contributions together and using the constraints $d_0+d_1+d_2=k_1$ and $d_0'+d_1'+d_2'=k_2$ and $d_0''+d_1''+d_2''=k_3$, we obtain

$$\begin{aligned}&\mathrm{Ind}_{\mathrm{poly}}+\mathrm{Ind}_V^{\mathfrak{so}(7)}+\mathrm{Ind}_V^{\mathfrak{su}(2)}+\mathrm{Ind}_V^{\mathfrak{su}(2)'}+\mathrm{Ind}_H^{\mathfrak{R}}+\mathrm{Ind}_{\mathbb{E}_{d_1,d_1',d_1''}}+\mathrm{Ind}_{\mathbb{E}_{d_2,d_2',d_2''}}\\ &=-\frac{1}{2}\begin{pmatrix}k_1&k_2&k_3\end{pmatrix}\Omega\begin{pmatrix}m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\\ m_{\mathfrak{so}(7)}\cdot m_{\mathfrak{so}(7)}\\ m_{\mathfrak{su}(2)'}\cdot m_{\mathfrak{su}(2)'}\end{pmatrix}-\frac{\epsilon_1^2+\epsilon_2^2}{4}(k_1+2k_2+k_3)\\ &\quad+\frac{\epsilon_1\epsilon_2}{2}(2k_1^2+3k_2^2+2k_3^2-2k_1k_2-2k_2k_3-5k_2)+\frac{7}{8}(\epsilon_1+\epsilon_2)^2.\end{aligned} \tag{6.47}$$

This final sum is *independent* from $\alpha^\vee,\lambda_1,\lambda_2,d_1,d_1',d_2,d_2',d_1'',d_2''$ themselves, but only depends on their combination (k_1,k_2,k_3)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

7 Arbitrary rank

In this section we study the most general 6d $(1,0)$ SCFTs in the atomic classification. Inspired from the structure of elliptic blowup equations for the three higher-rank NHCs given in the last section, we find such structure can be directly generalized to all higher-rank theories by a gluing procedure. The procedure can be divided to two steps:

- First, we propose a simple set of rules to glue together the r fields of rank one theories to the r fields of higher-rank theories.

[43] Note the vector representation $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)}=1+\mathcal{O}_6$.

- Second, with the glued r fields at hand, we can immediately write down the precise form of the elliptic blowup equations based on the requirement that after degenerating to rank-one theories, the higher-rank blowup equations should decompose back to the rank-one blowup equations.

We also prove the modularity of the general higher-rank elliptic blowup equations. We then present the admissible blowup equations for a lot of examples including the ADE chain of -2 curves with gauge symmetry, all conformal matter theories and the blown-ups of some $-n$ curves in particular $-9, -10, -11$ curves. The prominent feature here is that for higher-rank theories, most of their blowup equations are of vanishing type.

7.1 Gluing rules for r fields

One of the key steps to write down the blowup equations for a higher rank theory is to fix the parameters λ_G and λ_F in the gauge and the flavor symmetry sectors. They are in fact both components of the r-field in the blowup equations of topological string theory [4, 19]. Besides constructing higher rank theories from rank one theories involves gauging the flavor symmetry. Therefore we can view λ_G, λ_F on an equal footing, and here we consider them collectively as the r-field (λ_G, λ_F).

Based on the gluing rules of higher rank 6d $(1,0)$ SCFTs [26, 27], we propose the following gluing rules to write down the r fields for higher rank elliptic blowup equations, which are simple criteria to determine which r fields of one node can be coupled to which r fields of the adjacent nodes. These rules essentially are the necessary condition for the higher-rank r fields to be consistent with the rank-one r fields after higher-rank 6d theories degenerate to many rank-one theories.

- A higher rank 6d (1,0) SCFT is represented by a generalised quiver where each node is labeled by a tuple of gauge symmetry and flavor symmetry. It is subject to the consistency condition that any node labeled by (G, F) with adjacent nodes labeled by (G_i, F_i) satisfies [26, 27]
$$F \supset \left(\prod_i G_i\right) \times F_f, \tag{7.1}$$
where F_f is the surviving ungauged flavor symmetry for free hypers.

- The rank-one blowup equation labeled by the r fields (λ, ω) associated to the node with symmetries (G, F) can be glued with the rank-one blowup equations labeled by the r fields (λ_i, ω_i) of the adjacent nodes if and only if the Weyl orbit $\mathcal{O}_w$ containing the flavor r fields w can be decomposed to a product of the Weyl orbits $\mathcal{O}_{\lambda_i}$ containing the gauge r fields λ_i
$$\mathcal{O}_w \supset \left(\prod_i \mathcal{O}_{\lambda_i}\right) \times \mathcal{O}_{w_f}, \tag{7.2}$$
according to (7.1), where w_f is the flavor r fields of the surviving ungauged flavor symmetry F_f.

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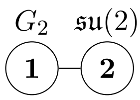
- The admissible blowup equations for a higher-rank theory are such that all its nodes satisfy the above criteria. They are represented by the gauge r fields and the r fields for ungauged flavor symmetry associated to each individual nodes. The glued blowup equation is of the unity type if all the component rank-one blowup equations are of the unity type; otherwise it is of the vanishing type.

A few comments are in order. Note that a node may bear no gauge group such as the E-string theory, in which case $G = \emptyset$ and $\lambda \in \mathbf{1}$. The concept of nodes in the criteria can be generalized to molecules in the atomic classification, which makes it easier to find all admissible blowup equations when lots of molecules are involved. These criteria actually guarantee the consistency with the blowup equations of lower-rank theories when decoupling nodes.

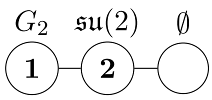
With the gluing rules, we can efficiently write down all admissible r fields for any higher-rank theory once the gauge groups, flavor groups and matter representation are known. We use a simple quiver diagram to denote blowup equations with the following rules:

- We use a circle for a compact base curve and a rectangle for a noncompact one.
- For each base curve with associated gauge/flavor symmetry G, we mark it with a Weyl orbit $\mathbf{n}_p$ (p is often suppressed if it is minimal) of G to denote the r field of G fugacities. If a compact curve has no associated gauge symmetry, we leave the circle blank.

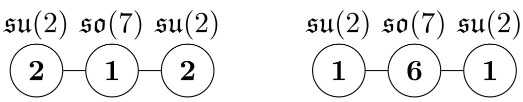
For example, for the $3,2$ NHC with associated gauge symmetry $G_2, \mathfrak{su}(2)$, we denote the five vanishing blowup equations (6.14) simply as



Meanwhile, the vanishing blowup equations (6.29) of $3,2,2$ NHC can be denoted respectively as



and those of $2,3,2$ NHC in equation (6.40) and (6.44) as



(7.3)

Note each quiver diagram above represents $\det(\Omega)$ non-equivalent blowup equations where $-\Omega$ is the intersection matrix of compact base curves. Also note that in this section, we will use the notation $\mathbf{n}_p$ to denote a Weyl orbit consisting of n weights which all have norm square p. Very often we will suppress the subscript p if p is minimal and there is no cause for confusion. We sometimes also use the conjugate bar and subscripts s, c to distinguish orbits of the same length just like in the notation of irreducible representations.

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Now let us demonstrate the gluing rules with the example of the NHC $2,3,2$ theory. We recall the r-fields of the individual nodes from tables 5, 6, 7, 8

$$n=2,(G,F)=(\mathfrak{su}(2),\mathfrak{so}(7)):\begin{cases}\text{unity } r\text{-fields} \in (\mathbf{1}_0,\mathbf{6}_1)\\ \text{vanishing } r\text{-fields} \in (\mathbf{2}_{1/2},\mathbf{1}_0)\end{cases} \tag{7.4}$$

$$n=3,(G,F)=(\mathfrak{so}(7),\mathfrak{sp}(2)):\begin{cases}\text{unity } r\text{-fields} \in (\mathbf{1}_0,\mathbf{4}_1)\\ \text{vanishing } r\text{-fields} \in (\mathbf{6}_1,\mathbf{1}_0),(\mathbf{6}_1,\mathbf{4}_{1/2})\end{cases} \tag{7.5}$$

First, to couple the central node with symmetries $G=\mathfrak{so}(7), F=\mathfrak{sp}(2)$ of the NHC 2,3,2 with the two side nodes with symmetries $G_{1,2}=\mathfrak{su}(2), F_{1,2}=\mathfrak{so}(7)$, the flavor symmetry F must decompose as $\mathfrak{sp}(2)\to\mathfrak{su}(2)\times\mathfrak{su}(2)$. As we have seen, the unity r fields $(\lambda_{\mathfrak{so}(7)},\omega_{\mathfrak{sp}(2)})$ of the central node $3_{\mathfrak{so}(7)}$ are elements of the orbit $(\mathbf{1}_0,\mathbf{4}_1)$. Under the flavor F decomposition, the orbit of flavor r fields decomposes by $\mathbf{4}_1\to(\mathbf{2}_{1/2},\mathbf{2}_{1/2})$. We notice $\mathbf{2}_{1/2}$ is the orbit of the gauge component of the vanishing r fields $(\mathbf{2}_{1/2},\mathbf{1}_0)$ of the side node $2_{\mathfrak{su}(2)}$, besides the flavor component $\mathbf{1}_0$ of the latter is the correct gauge component of the unity r fields of the central node $3_{\mathfrak{so}(7)}$. We could therefore conclude that we have found one set of admissible r fields for the NHC $2,3,2$ with the parameter λ_G of the entire $2,3,2$ chain belonging to $(\mathbf{2}_{1/2},\mathbf{1}_0,\mathbf{2}_{1/2})$, which give rise to vanishing blowup equations (6.40) represented by the left hand side of (7.3).

On the other hand, the vanishing r fields of the central node $3_{\mathfrak{so}(7)}$ are elements of the orbit $(\mathbf{6}_1,\mathbf{1}_0)$ or $(\mathbf{6}_1,\mathbf{4}_{1/2})$. Under flavor F decomposition, the orbits of flavor r fields decompose by $\mathbf{1}_0\to(\mathbf{1}_0,\mathbf{1}_0)$ and $\mathbf{4}_{1/2}\to(\mathbf{2}_{1/2},\mathbf{1}_0)+(\mathbf{1}_0,\mathbf{2}_{1/2})$. $\mathbf{1}_0$ and $\mathbf{2}_{1/2}$ are respectively the orbits of the gauge components of the unity r fields $(\mathbf{1}_0,\mathbf{6}_1)$ and the vanishing r fields $(\mathbf{2}_{1/2},\mathbf{1}_0)$ of the side node, but only the flavor component $\mathbf{6}_1$ of the these r fields contains properly the gauge component of the vanishing r fields of the central node. Thus we find another set of admissible r fields with the overall λ_G parameters belonging to $(\mathbf{1}_0,\mathbf{6}_1,\mathbf{1}_0)$ and they give rise to the other set of vanishing blowup equations (6.44) for NHC $2,3,2$ represented by the right hand side of (7.3) and there is no other possible admissible λ_G. These simple analysis confirms our blowup equations in section 6.3. For more examples with adherent free hypers, we refer to section 7.4.

7.2 Arbitrary rank elliptic blowup equations

Once we have found the r fields for a higher rank 6d SCFT, it is relatively easy to write down the elliptic blowup equation for it. The basic idea is to generalise the form of the elliptic blowup equations of the three higher-rank NHCs, which were derived in section 6 from their geometric constructions using the de-affinization procedure, and make sure that after decomposing to rank-one theories the higher-rank blowup equations break into the known form of rank-one equations (3.1).

Let us first briefly recall the geometric construction for a higher rank 6d SCFT in order to set up the parameters that will appear in the elliptic blowup equations. Consider F-theory compactifications on an elliptic Calabi-Yau threefold, whose non-compact base contains r compact curves with a negative definite intersection matrix $-\Omega_{ij}=A_{ij}$. Recall the symmetry algebras and the massless fields which can arise in this theory. Over the i-th

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compact curve C_i there could be singular elliptic fibers corresponding to a symmetry algebra G_i. In addition C_i could intersect with a non-compact curve N_i with intersection number k_{F_i}, and the latter supports singular elliptic fibers corresponding to symmetry algebra F_i.

The resulting field theory is a 6d SCFT in its r dimensional tensor branch with total gauge symmetry $\prod_i G_i$ and flavor symmetry $\prod_i F_i$. If we compactify the 6d SCFT on a torus, we can also turn on the gauge and flavor fugacities m_{G_i}, m_{F_i}. There are also charged matter fields localised at intersections of curves. At the intersection locus of two compact base curves C_i, C_j there are hypermultiplets charged under both gauge groups G_i, G_j. We also consider hypermultiplets localised at the intersection locus of compact and non-compact curves. Finally BPS strings arise from D3-branes wrapping compact base curves. The number of times a string wraps each base curve is interpreted as the charge of this string. The string charges form a rank r lattice Λ with the negative definite bilinear form defined by $-\Omega_{ij} = A_{ij}$.

Suppose we already know the gauge and flavor r fields $(\lambda_{G_i}, \lambda_{F_i})$ for each individual node of the higher rank SCFT, we propose the following blowup equations for the elliptic genera $\mathbb{E}_{d_i}(\tau, m_{G_i}, m_{F_i}, \epsilon_1, \epsilon_2)$

$$
\begin{aligned}
&\sum_{\alpha_i\in\phi_i(Q^\vee(G_i)),d_i',d_i''\in\mathbb{N}}^{||\alpha_i||^2/2+d_i'+d_i''=d_i+\delta_i/2} (-1)^{\sum_i|\phi_i^{-1}(\alpha_i)|} \\
&\times\Theta_\Omega^{[a_i]}\Big(\tau,-(\alpha_i\cdot m_{G_i})+\sum_j(\Omega^{-1})_{ij}k_{F_j}(\lambda_{Fj}\cdot m_{F_j})+\Big(y_i-\frac{1}{2}(\alpha_i\cdot\alpha_i)\Big)(\epsilon_1+\epsilon_2)-d_i'\epsilon_1-d_i''\epsilon_2\Big) \\
&\times\prod_i A_{V_i}(\tau,m_{G_i},\alpha_i)\prod_{ij}A_{H_{ij}}(\tau,m_{G_i},\mu_j,\alpha_i,\alpha_j)\prod_i A_{H_i}(\tau,m_{G_i},m_{F_i},\alpha_i,\lambda_{Fi}) \\
&\times\mathbb{E}_{d_i'}(\tau,m_{G_i}+\alpha_i\epsilon_1,m_{F_i}+\lambda_{Fi}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d_i''}(\tau,m_{G_i}+\alpha_i\epsilon_2,m_{F_i}+\lambda_{Fi}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2) \\
&=\Lambda(\delta_i)\Theta_\Omega^{[a_i]}\left(\tau,\sum_j(\Omega^{-1})_{ij}k_{F_j}(\lambda_{Fj}\cdot\nu_j)+y_i(\epsilon_1+\epsilon_2)\right)\mathbb{E}_{d_i}(\tau,m_{G_i},m_{F_i},\epsilon_1,\epsilon_2), \qquad (7.6)
\end{aligned}
$$

with

$$
y_i = \sum_j(\Omega^{-1})_{ij}\left(\frac{1}{4}(-2+\Omega_{jj}+h^\vee_{G_j})+\frac{1}{2}k_{F_j}(\lambda_{Fj}\cdot\lambda_{Fj})\right), \qquad (7.7)
$$

and

$$
\Lambda(\delta_i) = \begin{cases} 1, & \forall i, \delta_i = 0, \\ 0, & \exists i, \delta_i > 0\,. \end{cases} \qquad (7.8)
$$

Here $\Theta_\Omega^{[a]}$ is the generalized theta function defined in section 6 in equations (6.1), (6.2). As in the rank one case, ϕ_i is an embedding of the coroot lattice $Q^\vee(G_i)$ in the coweight lattice of G_i by an overall shift of a coweight vector given by the gauge r field λ_{Gi}. δ_i is the smallest norm in the image $\phi_i(Q^\vee(G_i))$; δ_i is zero if the embedding is unshifted so that $\phi_i(Q^\vee(G_i)) = Q^\vee(G_i)$ and positive otherwise. $\phi_i^{-1}(\alpha_i)$ gives back a coroot vector, and $|\bullet|$ is the sum of the coefficients in its decomposition in terms of simple coroots. A_{V_i} is the contribution of vector multiplets transforming in the adjoint representation of G_i, and $A_{H_{ij}}, A_{H_i}$ are respectively the contributions of hypermultiplets charged in the mixed representation of two gauge groups, and in the representation of one gauge group. Their expressions have been given in (3.7), (3.8). Finally the parameters λ_{Fi} are the components of r fields associated to the flavor symmetries, and they take value also in the coweight lattice.

One important consistency condition is that if we turn off all string charges except for the one indexed by i, i.e. we set $d_j = 0$ for $j \neq i$, and consequently $d'_j = d''_j = 0$, $\alpha_j = 0$ for $j \neq i$ as well, (7.6) must reduce to the blowup equations for a rank one 6d SCFT with $n = \Omega_{ii}$, gauge group G_i, and the surviving λ_{Fi} should be the λ_F parameter worked out in section 3.1. We have seen some examples of this decompactification in action in sections 6.1, 6.2, and 6.3.

7.3 Modularity

In this section we illustrate that the elliptic blowup equations (7.6) satisfy the modularity consistency conditions. For ease of presentation, we divide each term on the l.h.s. by the r.h.s. except for the $\Lambda(\delta_i)$ factor. The modularity consistency conditions then require that the modular index of each term on the l.h.s. after division is independent of the summation indices α_i, d'_i, d''_i, and in the case of unity equations where $\Lambda(\delta_i) = 1$ be equal to zero.

We prove the modularity in two steps. First, we isolate an individual string charge indexed i, and turn off all the other string charges by setting $d_j = 0$ for $j \neq i$. As we argued before, (7.6) reduces to those of a rank one 6d SCFT, and by the way we choose λ_i, the modularity condition is automatically satisfied. Second, we consider the modular index of mixed string charges. Let us pick a pair of string charges, say d_1, d_2, whose associated base curves intersect. Then we turn off all the other string charges $d_j = 0$ ($j \neq 1, 2$), and check the modularity conditions for the following components of the modular index for each term

$$\mathrm{Ind}^{\mathrm{mix}}(d_1, d_2) = \mathrm{Ind}(d_1, d_2) - \mathrm{Ind}(0, d_2) - \mathrm{Ind}(d_1, 0) + \mathrm{Ind}(0, 0). \tag{7.9}$$

The modular index polynomial of the generalised theta function $\Theta_\Omega^{[a_i]}(\tau, z_i)$ is

$$\mathrm{Ind}\,\Theta_\Omega(z_i) = \frac{1}{2}\sum_{i,j} z_i \Omega_{ij} z_j. \tag{7.10}$$

Its contribution to the modular index of the blowup equations is

$$\begin{aligned}\mathrm{Ind}\,\Theta(d_i) = &-\sum_i \left((\alpha_i \cdot m_{G_i}) + \frac{1}{2}||\alpha_i||^2(\epsilon_1+\epsilon_2) + d'_i\epsilon_1 + d''_i\epsilon_2 \right)\\ &\times \left(k_{F_i}(\lambda_i \cdot m_{F_i}) + \left(y_i + \frac{1}{2}k_{F_i}||\lambda_i||^2\right)(\epsilon_1+\epsilon_2)\right)\\ &+\frac{1}{2}\sum_{i,j}\left(\alpha_i \cdot m_{G_i} + \frac{1}{2}||\alpha_i||^2(\epsilon_1+\epsilon_2) + d'_i\epsilon_1 + d''_i\epsilon_2\right)\\ &\times \left(\alpha_j \cdot \mu_j + \frac{1}{2}||\alpha_j||^2(\epsilon_1+\epsilon_2) + d'_j\epsilon_1 + d''_j\epsilon_2\right)\Omega_{ij}.\end{aligned} \tag{7.11}$$

The components of mixed string charges d_1, d_2 are

$$\begin{aligned}\mathrm{Ind}^{\mathrm{mix}}\,\Theta(d_1, d_2) = &\left((\alpha_1 - \beta_1)\cdot m_{G_1} + \frac{1}{2}(||\alpha_1||^2 - ||\beta_1||^2)(\epsilon_1+\epsilon_2) + d'_1\epsilon_1 + d''_1\epsilon_2\right)\\ &\times\left((\alpha_2 - \beta_2)\cdot m_{G_2} + \frac{1}{2}(||\alpha_2||^2 - ||\beta_2||^2)(\epsilon_1+\epsilon_2) + d'_2\epsilon_1 + d''_2\epsilon_2\right)\Omega_{12}\end{aligned} \tag{7.12}$$

where β_i is a vector in $\phi_i(Q^\vee(G_i))$ whose norm equals δ_i. The modular index polynomial of the elliptic genus with degrees d_i is given in (2.34), which we reproduce here

$$\begin{aligned}
&\mathrm{Ind}\mathbb{E}_{d_i}(\epsilon_1,\epsilon_2,m_{G_i},m_{F_i},d_i)=\\
&\quad=-\frac{1}{4}(\epsilon_1+\epsilon_2)^2\sum_i(2-\Omega_{ii}+h^\vee_{G_i})d_i+\frac{1}{2}\epsilon_1\epsilon_2\Big(\sum_i(2-\Omega_{ii})d_i+\sum_{ij}d_id_j\Omega_{ij}\Big)\\
&\qquad-\frac{1}{2}\sum_{ij}d_j(m_{G_i},m_{G_i})\Omega_{ij}+\frac{1}{2}\sum_i k_{F_i}d_i(m_{F_i},m_{F_i}).
\end{aligned}\tag{7.13}$$

The components of its contribution associated to mixed string charges are

$$\begin{aligned}
\mathrm{Ind}^{\mathrm{mix}}\,\mathbb{E}(d_1,d_2)=&-\frac{1}{2}d_1'((||\alpha_2||^2-||\beta_2||^2){\epsilon_1}^2+2((\alpha_2-\beta_2)\cdot m_{G_2})\epsilon_1)\Omega_{12}\\
&-\frac{1}{2}d_1''((||\alpha_2||^2-||\beta_2||^2){\epsilon_2}^2+2((\alpha_2-\beta_2)\cdot m_{G_2})\epsilon_2)\Omega_{12}\\
&-\frac{1}{2}d_2'((||\alpha_1||^2-||\beta_1||^2){\epsilon_1}^2+2((\alpha_1-\beta_1)\cdot m_{G_1})\epsilon_1)\Omega_{21}\\
&-\frac{1}{2}d_2''((||\alpha_1||^2-||\beta_1||^2){\epsilon_2}^2+2((\alpha_1-\beta_1)\cdot m_{G_1})\epsilon_2)\Omega_{21}.
\end{aligned}\tag{7.14}$$

Neither A_{V_i} nor A_{H_i} would contribute to the modular index of mixed string charges, and we only have to consider $A_{H_{12}}$, whose modular index polynomial can be read off from (3.8), (3.12). Combining all these ingredients together, the modular index polynomial of an arbitrary term on the l.h.s. after division for mixed string charge d_1, d_2 is

$$\begin{aligned}
\mathrm{Ind}^{\mathrm{mix}}(d_1,d_2)=&\,\frac{1}{4}\Omega_{12}(4\,\mathrm{ind}_{R_{G_1}}\mathrm{ind}_{R_{G_2}}n_{12}-1)\Big(4(\alpha_1-\beta_1)\cdot m_{G_1}\times(1\to 2)\\
&+2(\epsilon_1+\epsilon_2)((||\alpha_1||^2-||\beta_1||^2)(\alpha_2-\beta_2)\cdot m_{G_2}+(1,2\to 2,1))\\
&+({\epsilon_1}^2+\epsilon_1\epsilon_2+{\epsilon_2}^2)(||\alpha_1||^2-||\beta_1||^2)\times(1\to 2)\Big).
\end{aligned}\tag{7.15}$$

This contribution vanishes identically thanks to the anomaly cancellation condition (2.17).

7.4 Examples

In the following, we will show the blowup equations for some most interesting examples of higher-rank theories including ADE chains of -2 curves, conformal matters and the blowups of $-9,-10,-11$ curves.

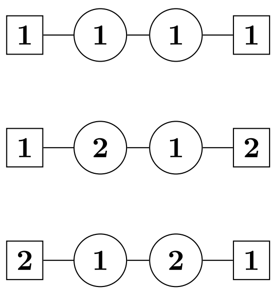
7.4.1 ADE chains of -2 curves

The 2d quiver construction and elliptic genera for this type of 6d $(1,0)$ quiver SCFTs are given in [82], see also another form in [104]. A crucial property of simply-laced Dynkin diagrams is needed in order to achieve admissible gluing of the blowup equations of individual nodes: the mark of each node has to be the average of the marks of all its adjacent nodes. Besides, when a node is at the end, its mark is half of the mark of its adjacent node. The problem of finding all admissible blowup equations then reduces to the decomposition of Weyl orbits of the special unitary algebra to its subalgebras.

In the following we demonstrate the application of gluing rules for some typical examples including $A_{2,3}$, $D_{4,5}$ and $E_{6,7,8}$ quivers. The gluing here often involves decomposition like $\mathrm{SU}(N+M) \to \mathrm{SU}(N) \times \mathrm{SU}(M) \times \mathrm{U}(1)$. In the following, for simplicity, we do not explicitly writ down the U(1) part. But it should be emphasized that the shifts of U(1) indeed play a role during the gluing. For A type quiver, there exist one more global U(1) symmetry, while for D, E type quivers, such U(1) is anomalous [82]. This also makes a difference when counting the total number of admissible blowup equations.

- We first demonstrate the gluing for a simple example which is an A type quiver with gauge group $\mathfrak{su}(2)$. Note when two $n=2$ $\mathfrak{su}(2)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(4)$ (or equivalently $\mathfrak{so}(7)$) breaks down to $\mathfrak{su}(2)\times\mathfrak{su}(2)$. Then one of the flavor symmetry $\mathfrak{su}(2)$ becomes the gauge symmetry $\mathfrak{su}(2)$ for the other theory. For rank one $n=2$ $\mathfrak{su}(2)$ theory, the unity λ_F is in $\mathbf{1}$, while the vanishing λ_F is in $\mathbf{6}$. Under the flavor group splitting, $\mathbf{6} = 2(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$. Note also $\mathbf{2} \subset (P^\vee \backslash Q^\vee)_{\mathfrak{su}(2)}$. This means for a unity -2 node, the adjacent two -2 nodes must be both unity or both vanishing. On the other hand, for a vanishing -2 node, the adjacent two -2 nodes can only be both unity.

  For example, for the A_2 quiver, we find the following structure or the blowup equations
$$
\begin{aligned}
&\mathrm{U} \star \mathrm{U} = \mathrm{U},\\
&\mathrm{V} \star \mathrm{U} = \mathrm{V},\\
&\mathrm{U} \star \mathrm{V} = \mathrm{V}.
\end{aligned}
\tag{7.16}
$$
  Keep in mind there are two $\mathfrak{su}(2)$ fundamental matters at the two ends of the A_2 quiver. Therefore, the above blowup equations can be expressed in quiver diagrams as

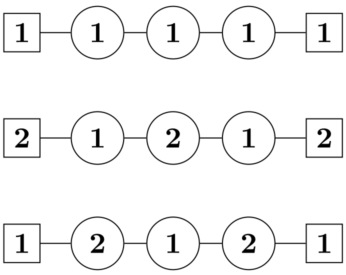


  The first quiver diagram represents unity equations, while the other two represent vanishing equations. The number of equations with fixed characteristic represented by each quiver diagram is the product of numbers in square nodes, while the number of characteristics is the determinant of the Cartan matrix C of the quiver diagram. We find $\det(C_{A_2}) = 3$. The first diagram actually represents two possible shifts of the global U(1) flavor $\lambda_{\mathrm{U}(1)} = \pm 2$, while the second and third diagram have $\lambda_{\mathrm{U}(1)} = 0$. Thus there are in total $3 \times 2 = 6$ unity equations and $3 \times (2+2) = 12$ vanishing equations.

For A_3 quiver, there are following blowup equations

$$
\begin{aligned}
\mathrm{U} \star \mathrm{U} \star \mathrm{U} &= \mathrm{U}, \\
\mathrm{U} \star \mathrm{V} \star \mathrm{U} &= \mathrm{V}, \\
\mathrm{V} \star \mathrm{U} \star \mathrm{V} &= \mathrm{V},
\end{aligned}
\tag{7.17}
$$

or in quiver diagrams as

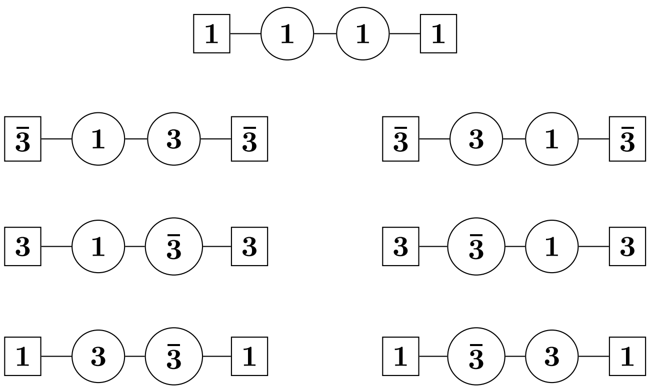


- Consider A type quiver theories with $\mathfrak{su}(3)$ symmetry. When two $n = 2$ $\mathfrak{su}(3)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(6)$ breaks down to $\mathfrak{su}(3) \times \mathfrak{su}(3)$. Then one of the flavor symmetry $\mathfrak{su}(3)$ becomes the gauge symmetry $\mathfrak{su}(3)$ for the other theory. We summarize the r fields behavior of rank one $n = 2$ $\mathfrak{su}(3)$ theory under the flavor group splitting in the following table,

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{su}(3) \times \mathfrak{su}(3)$
unity	$\mathbf{1}$	$\mathcal{O}_{\omega_3} = \mathbf{20}$	$2(\mathbf{1},\mathbf{1}) + (\mathbf{3},\overline{\mathbf{3}}) + (\overline{\mathbf{3}},\mathbf{3})$
vanishing	$\mathbf{3}$	$\mathcal{O}_{\omega_5} = \overline{\mathbf{6}}$	$(\overline{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\overline{\mathbf{3}})$
vanishing	$\overline{\mathbf{3}}$	$\mathcal{O}_{\omega_1} = \mathbf{6}$	$(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$

where $\mathcal{O}_{\omega_i}$ is the Weyl orbit generated by the i-th fundamental coweight.

For example, for A_2 quiver, read from the table above and gluing rules, we find there are following blowup equations



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The first quiver represents unity equations, while all the other quivers represent vanishing equations.

For A_3 quiver, there are following blowup equations

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{1} - \mathbf{3} - \bar{\mathbf{3}} - \boxed{\mathbf{1}} \qquad \boxed{\mathbf{1}} - \bar{\mathbf{3}} - \mathbf{3} - \mathbf{1} - \boxed{\bar{\mathbf{3}}}$$

$$\boxed{\mathbf{3}} - \mathbf{1} - \bar{\mathbf{3}} - \mathbf{3} - \boxed{\mathbf{1}} \qquad \boxed{\mathbf{1}} - \mathbf{3} - \bar{\mathbf{3}} - \mathbf{1} - \boxed{\mathbf{3}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{3} - \mathbf{1} - \bar{\mathbf{3}} - \boxed{\mathbf{3}} \qquad \boxed{\mathbf{3}} - \bar{\mathbf{3}} - \mathbf{1} - \mathbf{3} - \boxed{\bar{\mathbf{3}}}$$

The first quiver represents unity equations, while the remaining quivers represent vanishing equations.

- Consider A type quiver theories with $\mathfrak{su}(4)$ symmetry. When two $n = 2$ $\mathfrak{su}(4)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(8)$ breaks down to $\mathfrak{su}(4) \times \mathfrak{su}(4)$. Then one of the flavor $\mathfrak{su}(4)$ becomes the gauge $\mathfrak{su}(4)$ for the other theory. Note $(P^\vee/Q^\vee)_{A_3} = \mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \times \mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$2(\mathbf{1},\mathbf{1}) + (\mathbf{4},\bar{\mathbf{4}}) + (\bar{\mathbf{4}},\mathbf{4}) + (\mathbf{6},\mathbf{6})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$(\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{6}) + (\bar{\mathbf{4}},\bar{\mathbf{4}})$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$(\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{6}) + (\mathbf{4},\mathbf{4})$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations. For example, for A_2 quiver, there are following blowup equations

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

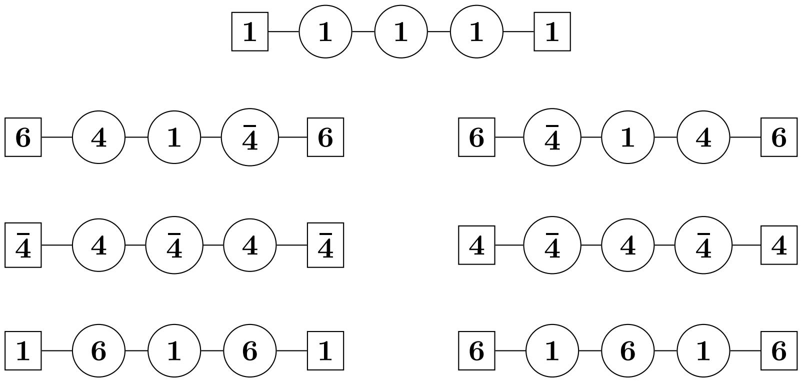
$$\boxed{\bar{\mathbf{4}}} - \mathbf{1} - \mathbf{4} - \boxed{\mathbf{6}} \qquad \boxed{\mathbf{4}} - \mathbf{1} - \bar{\mathbf{4}} - \boxed{\mathbf{6}}$$

$$\boxed{\mathbf{6}} - \mathbf{4} - \mathbf{1} - \boxed{\bar{\mathbf{4}}} \qquad \boxed{\mathbf{6}} - \bar{\mathbf{4}} - \mathbf{1} - \boxed{\mathbf{4}}$$

$$\boxed{\bar{\mathbf{4}}} - \mathbf{4} - \bar{\mathbf{4}} - \boxed{\mathbf{4}} \qquad \boxed{\mathbf{4}} - \bar{\mathbf{4}} - \mathbf{4} - \boxed{\bar{\mathbf{4}}}$$

The first quiver diagram represents unity equations, while the other quiver diagrams represent vanishing equations.

For the A_3 quiver, there are the following blowup equations:

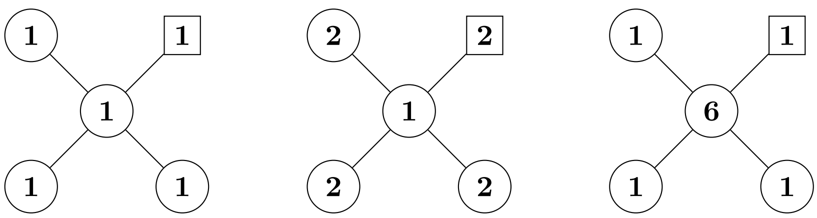


The first quiver diagram represents unity equations, while the other quiver diagrams represent vanishing equations.

- Now consider a D_4 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple a $n=2$ $\mathfrak{su}(4)$ gauge theory with three $n=2$ $\mathfrak{su}(2)$ gauge theories and a extra $\mathfrak{su}(2)$ fundamental. Note the flavor symmetry $\mathfrak{su}(8)$ of the center node breaks down to $\mathfrak{su}(2)^4$. Note $(P^\vee/Q^\vee)_{A_3}=\mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit of the i-th fundamental coweight.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2)^4$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4}=\mathbf{70}$	$6(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})+2((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$)+(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6}=\overline{\mathbf{28}}$	$4(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})+((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$)$
v	$\mathbf{6}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2}=\mathbf{28}$	$4(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})+((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$)$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:



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The first quiver diagram represents unity equations, while the remaining two diagrams represent vanishing equations. Note $\det(C_{D_4}) = 4$. Thus there are in total 4 unity and $4\times(2+1) = 12$ vanishing blowup equations. Let us show the leading base degree identities for the two types of vanishing blowup equations. The intersection matrix among base curves $-\Omega$ is just the negative of the Cartan matrix of D_4, i.e.

$$\Omega = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{7.18}$$

Then we find the first type of vanishing blowup equations has the following leading degree vanishing identities

$$\sum_{\lambda_{a,b,c}=\pm 1/2} (-1)^{\lambda_a+\lambda_b+\lambda_c} \Theta_\Omega^{[a]} \left(\tau, \begin{pmatrix} -\lambda_a m_a + 2(\epsilon_1+\epsilon_2) \\ 4(\epsilon_1+\epsilon_2) \\ -\lambda_b m_b + 2(\epsilon_1+\epsilon_2) \\ -\lambda_c m_c + 2(\epsilon_1+\epsilon_2) \end{pmatrix} \right) = 0. \tag{7.19}$$

where $m_{a,b,c}$ are the fugacities associated to the three $\mathfrak{su}(2)$ gauge node. Here the contributions from vector and hyper multiplets do not depend on the summation indices $\lambda_{a,b,c}$ and thus we have factored them out. The four possible characteristics a are defined according to (6.2). The second type of vanish blowup equation has leading base degree as

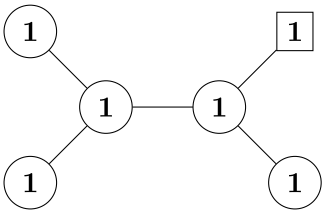
$$\sum_{1\le i<j\le 4} \Theta_\Omega^{[a]} \left(\tau, \begin{pmatrix} 2(\epsilon_1+\epsilon_2) \\ -m_i - m_j + 3(\epsilon_1+\epsilon_2) \\ 2(\epsilon_1+\epsilon_2) \\ 2(\epsilon_1+\epsilon_2) \end{pmatrix} \right) \frac{1}{\prod_{k\neq i,j} \theta_1(m_i-m_k)\theta_1(m_j-m_k)} = 0. \tag{7.20}$$

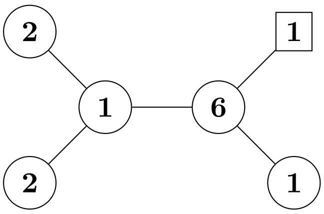
Here $m_i, i = 1,2,3,4$ are the $\mathfrak{su}(4)$ fugacities of the central node with $\sum_{i=1}^4 m_i = 0$. We have checked these identities up to order $\mathcal{O}(q^{10})$.

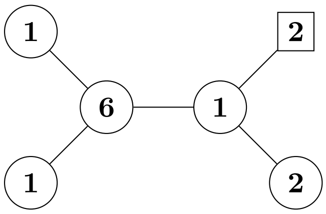
- Consider a D_5 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple two $n = 2$ $\mathfrak{su}(4)$ gauge theories together with three $n = 2$ $\mathfrak{su}(2)$ gauge theories and an extra $\mathfrak{su}(2)$ fundamental. Note the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down to $\mathfrak{su}(4) \times \mathfrak{su}(2)^2$. Note also $(P^\vee/Q^\vee)_{A_3} = \mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit generated by the i-th fundamental coweight.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(4)\times\mathfrak{su}(2)\times\mathfrak{su}(2)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4}=\mathbf{70}$	$2(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{2},\mathbf{2})+2(\mathbf{6},\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{2},\mathbf{1})+(\mathbf{4},\mathbf{1},\mathbf{2})+(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6}=\overline{\mathbf{28}}$	$2(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{1},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{2})$
v	$\mathbf{6}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2}=\mathbf{28}$	$2(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{2},\mathbf{1})+(\mathbf{4},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{2})$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:







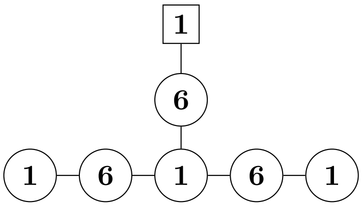
The first quiver diagram represents unity equations, while the remaining two diagrams represent vanishing equations. Note $\det(C_{D_5})=4$. Thus there are in total 4 unity and $4\times(1+2)=12$ vanishing blowup equations.

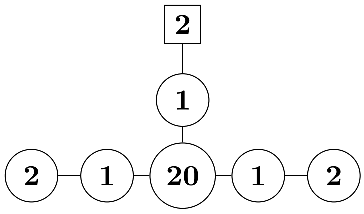
- Consider the E_6 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple an $n=2$ $\mathfrak{su}(6)$ gauge theory with three $n=2$ $\mathfrak{su}(4)$ gauge theories and two of the $\mathfrak{su}(4)$ theories each with an $\mathfrak{su}(2)$ theory and the other $\mathfrak{su}(4)$ theory to an extra $\mathfrak{su}(2)$ fundamental hypermultiplet. All the nodes together then form the Dynkin diagram of affine E_6. Note the flavor symmetry $\mathfrak{su}(12)$ of the center node breaks down to $\mathfrak{su}(4)^3$, and the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down to $\mathfrak{su}(2)\times\mathfrak{su}(6)$. Besides, $(P^\vee/Q^\vee)_{A_5}=\mathbb{Z}_6$. We summarize the r fields behavior under the flavor group splitting in the following tables.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(6)$	$\mathfrak{su}(12)$	$\mathfrak{su}(4)^3$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_6}=\mathbf{924}$	$2(\mathbf{6},\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{4},\mathbf{1})+(\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{1})+(\mathbf{4},\mathbf{6},\bar{\mathbf{4}})+(\mathbf{6},\mathbf{6},\mathbf{6})+$permutations
v	$\mathbf{6}$	$\mathcal{O}_{\omega_8}=\overline{\mathbf{495}}$	$3(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{4},\bar{\mathbf{4}},\mathbf{1})+(\mathbf{6},\mathbf{6},\mathbf{1})+(\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{6})+$permutations
v	$\mathbf{15}$	$\mathcal{O}_{\omega_{10}}=\overline{\mathbf{66}}$	$(\mathbf{6},\mathbf{1},\mathbf{1})+(\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{1})+$permutations
v	$\mathbf{20}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\overline{\mathbf{15}}$	$\mathcal{O}_{\omega_2}=\mathbf{66}$	$(\mathbf{6},\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{4},\mathbf{1})+$permutations
v	$\bar{\mathbf{6}}$	$\mathcal{O}_{\omega_4}=\mathbf{495}$	$3(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{4},\bar{\mathbf{4}},\mathbf{1})+(\mathbf{6},\mathbf{6},\mathbf{1})+(\mathbf{4},\mathbf{4},\mathbf{6})+$permutations

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2)\times\mathfrak{su}(6)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4}=\mathbf{70}$	$(\mathbf{1},\overline{\mathbf{15}})+(\mathbf{2},\mathbf{20})+(\mathbf{1},\mathbf{15})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6}=\overline{\mathbf{28}}$	$(\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{6})+(\mathbf{1},\overline{\mathbf{15}})$
v	$\mathbf{6}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2}=\mathbf{28}$	$(\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{6})+(\mathbf{1},\mathbf{15})$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:





Both quiver diagrams represent vanishing equations. Note $\det(C_{E_6})=3$. Thus there are in total $3\times(1+2)=9$ vanishing blowup equations.

- Consider the E_7 quiver with gauge group $\mathfrak{su}(2d_i)$. In this case, the flavor symmetry $\mathfrak{su}(16)$ of the center node breaks down to $\mathfrak{su}(6)^2\times\mathfrak{su}(4)$, and the flavor symmetry $\mathfrak{su}(12)$ of the $\mathfrak{su}(6)$ node breaks down to $\mathfrak{su}(8)\times\mathfrak{su}(4)$, and the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down to $\mathfrak{su}(6)\times\mathfrak{su}(2)$. Besides, $(P^\vee/Q^\vee)_{A_7}=\mathbb{Z}_8$. We summarize the r fields behavior under the flavor group splitting in the following tables.

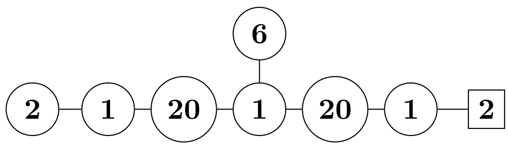
	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(8)$	$\mathfrak{su}(16)$	$\mathfrak{su}(6)\times\mathfrak{su}(6)\times\mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_8}=\mathbf{12870}$	$(\mathbf{1},\mathbf{15},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{20},\mathbf{1})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{1})+(\mathbf{1},\mathbf{6},\mathbf{4})+(\bar{\mathbf{6}},\mathbf{15},\mathbf{4})+(\overline{\mathbf{15}},\mathbf{20},\mathbf{4})$ $+(\mathbf{1},\mathbf{1},\mathbf{6})+(\bar{\mathbf{6}},\mathbf{6},\mathbf{6})+(\overline{\mathbf{15}},\mathbf{15},\mathbf{6})+(\mathbf{20},\mathbf{20},\mathbf{6})+\cdots$
v	$\mathbf{8}$	$\mathcal{O}_{\omega_{10}}=\overline{\mathbf{8008}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{6},\mathbf{1})+(\overline{\mathbf{15}},\mathbf{15},\mathbf{1})+(\mathbf{20},\mathbf{20},\mathbf{1})+(\mathbf{6},\mathbf{1},\bar{\mathbf{4}})+(\mathbf{15},\bar{\mathbf{6}},\bar{\mathbf{4}})$ $+(\mathbf{20},\overline{\mathbf{15}},\bar{\mathbf{4}})+(\mathbf{15},\mathbf{1},\mathbf{6})+(\mathbf{20},\bar{\mathbf{6}},\mathbf{6})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{6})+\cdots$
v	$\mathbf{28}$	$\mathcal{O}_{\omega_{12}}=\overline{\mathbf{1820}}$	$(\mathbf{15},\mathbf{1},\mathbf{1})+(\mathbf{20},\bar{\mathbf{6}},\mathbf{1})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{1})+(\mathbf{20},\mathbf{1},\bar{\mathbf{4}})+(\overline{\mathbf{15}},\bar{\mathbf{6}},\bar{\mathbf{4}})$ $+(\overline{\mathbf{15}},\mathbf{1},\mathbf{6})+(\bar{\mathbf{6}},\bar{\mathbf{6}},\mathbf{6})+\cdots$
v	$\mathbf{56}$	$\mathcal{O}_{\omega_{14}}=\overline{\mathbf{120}}$	$(\overline{\mathbf{15}},\mathbf{1},\mathbf{1})+(\bar{\mathbf{6}},\bar{\mathbf{6}},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{1},\bar{\mathbf{4}})+(\mathbf{1},\mathbf{1},\mathbf{6})+\cdots$
v	$\mathbf{70}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\overline{\mathbf{56}}$	$\mathcal{O}_{\omega_2}=\mathbf{120}$	conjugate
v	$\overline{\mathbf{28}}$	$\mathcal{O}_{\omega_4}=\mathbf{1820}$	conjugate
v	$\bar{\mathbf{8}}$	$\mathcal{O}_{\omega_6}=\mathbf{8008}$	conjugate

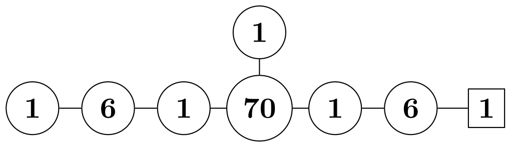
Note here the $\ldots$ means conjugate representations and permutations over the first two $\mathfrak{su}(6)$.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(6)$	$\mathfrak{su}(12)$	$\mathfrak{su}(8) \times \mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_6} = \mathbf{924}$	$(\overline{\mathbf{28}}, \mathbf{1}) + (\overline{\mathbf{56}}, \mathbf{4}) + (\mathbf{70}, \mathbf{6}) + (\mathbf{56}, \overline{\mathbf{4}}) + (\mathbf{28}, \mathbf{1})$
v	$\mathbf{6}$	$\mathcal{O}_{\omega_8} = \overline{\mathbf{495}}$	$(\mathbf{70}, \mathbf{1}) + (\overline{\mathbf{56}}, \overline{\mathbf{4}}) + (\overline{\mathbf{28}}, \overline{\mathbf{6}}) + (\overline{\mathbf{8}}, \mathbf{4}) + (\mathbf{1}, \mathbf{1})$
v	$\mathbf{15}$	$\mathcal{O}_{\omega_{10}} = \overline{\mathbf{66}}$	$(\overline{\mathbf{28}}, \mathbf{1}) + (\overline{\mathbf{8}}, \overline{\mathbf{4}}) + (\mathbf{1}, \overline{\mathbf{6}})$
v	$\mathbf{20}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$
v	$\overline{\mathbf{15}}$	$\mathcal{O}_{\omega_2} = \mathbf{66}$	$(\mathbf{28}, \mathbf{1}) + (\mathbf{8}, \mathbf{4}) + (\mathbf{1}, \mathbf{6})$
v	$\overline{\mathbf{6}}$	$\mathcal{O}_{\omega_4} = \mathbf{495}$	$(\mathbf{70}, \mathbf{1}) + (\mathbf{56}, \mathbf{4}) + (\mathbf{28}, \mathbf{6}) + (\mathbf{8}, \overline{\mathbf{4}}) + (\mathbf{1}, \mathbf{1})$

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2) \times \mathfrak{su}(6)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$(\mathbf{1}, \overline{\mathbf{15}}) + (\mathbf{2}, \mathbf{20}) + (\mathbf{1}, \mathbf{15})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$(\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{6}) + (\mathbf{1}, \overline{\mathbf{15}})$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1}, \mathbf{1})$
v	$\overline{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$(\mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{6}) + (\mathbf{1}, \mathbf{15})$

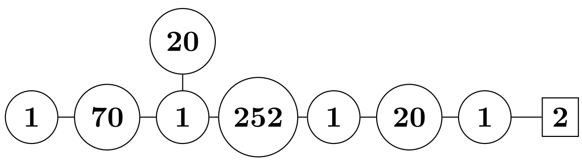
Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:

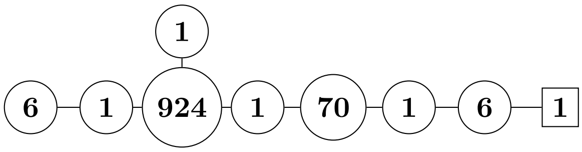




Both quiver diagrams represent vanishing equations. Note $\det(C_{E_7}) = 2$. Thus there are in total $2 \times (2+1) = 6$ vanishing blowup equations.

- Consider the E_8 quiver with gauge group $\mathfrak{su}(2d_i)$. After a long but elementary computation on the representation decomposition like in the cases above, and based on the general gluing procedure, we can directly write down all admissible blowup equations as:



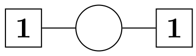


Both quiver diagrams represent vanishing equations. Note $\det(C_{E_8}) = 1$. Thus there are in total $2 + 1 = 3$ vanishing blowup equations.

7.4.2 Conformal matter theories

6d conformal matter theories are interesting SCFTs coming from M5-branes probing an ADE singularity in M-theory or intersecting an ADE singularity with a Horava-Witten M9-wall [58]. The elliptic genera of these theories are rarely known except for a few rank one cases such as the (D_N, D_N) models. In the following, we present the blowup equations for all notable conformal matter theories. Note for all conformal matter theories except for $(\mathfrak{sp}(n)$,$\mathfrak{sp}(n))$ theory, the determinant of the intersection matrix of base curves is $\det(\Omega) = 1$.[44] Therefore the number of non-equivalent blowup equations for each of these theories is just the number of non-equivalent admissible r fields for the nodes.

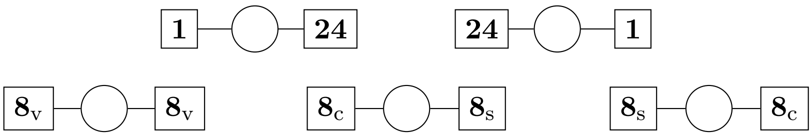
- (D_4, D_4) conformal matter theory is often denoted as $[D_4], 1, [D_4]$. The elliptic genera of this theory can be computed from 2d quiver gauge theory [105]. This model is actually a special case of the E-string theory. The E_8 flavor group of node 1 splits to $\mathfrak{so}(8)\times\mathfrak{so}(8)$. Since the vanishing r field of E-string theory decomposes as $\mathbf{1} \to (\mathbf{1}, \mathbf{1})$, we obtain the following vanishing equation for (D_4, D_4):



  On the other hand, the unity r fields of E-string theory decompose as

$$\mathbf{240}_2 \to (\mathbf{24}_2, \mathbf{1}) + (\mathbf{1}, \mathbf{24}_2) + (\mathbf{8}_{\rm v}, \mathbf{8}_{\rm v}) + (\mathbf{8}_{\rm c}, \mathbf{8}_{\rm s}) + (\mathbf{8}_{\rm s}, \mathbf{8}_{\rm c}). \tag{7.21}$$

  Apply the gluing rules, we find the following five types of unity blowup equations:



- (D_{N+4}, D_{N+4}) theories are often denoted as $[D_{N+4}], 1_{\mathfrak{sp}(N)}, [D_{N+4}]$. For $N \geq 1$, the $D_{2(N+4)}$ flavor group of the $n = 1$ node splits to $D_{N+4} \times D_{N+4}$. Under splitting $D_{2(N+4)} \to D_{N+4} \times D_{N+4}$,

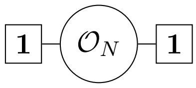
$$S_{D_{2(N+4)}} \to (S_{D_{N+4}}, C_{D_{N+4}}) + (C_{D_{N+4}}, S_{D_{N+4}}). \tag{7.22}$$

  Denote $\mathcal{O}_N$ as the Weyl orbit of $\mathfrak{sp}(N)$ generated by weight $[0, 0, \ldots, 0, 1]$, and V, S, C as the Weyl orbits of $\mathfrak{so}(N+4)$ generated by weights $[1, 0, \ldots, 0, 0]$, $[0, 0, \ldots, 0, 1]$ and $[0, 0, \ldots, 1, 0]$. Apply the gluing rules, we find two types of unity blowup equations

[44] This property can be easily deduced from the fact that all these conformal matter theories can be blown down successively to one single -1 curve.

There also exist numerous vanishing blowup equations. For example, for $N=1$ case, i.e. the $(\mathfrak{so}(10),\mathfrak{so}(10))$ model, the vanishing blowup equations are

For $N\geq 2$, there exist many vanishing blowup equations including



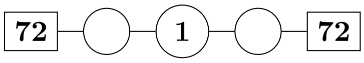
- (E_6,E_6) conformal matter theory is often denoted as $[E_6],1,3_{\mathfrak{su}(3)},1,[E_6]$. The base curve intersection matrix $-\Omega$ has

$$\Omega=\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{7.23}$$

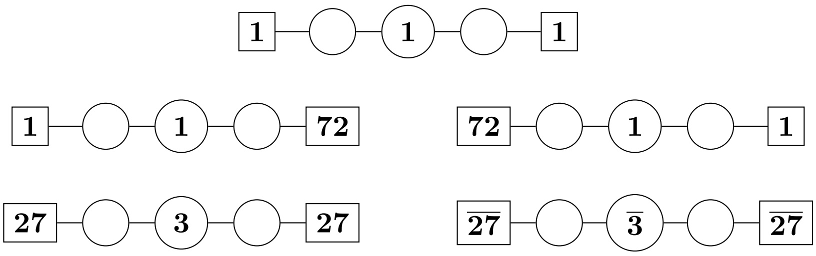
Note $\mathrm{Det}(\Omega)=1$. The E_8 flavor group of node 1 splits to $E_6\times\mathfrak{su}(3)$ when coupled with $n=6$ E_6 gauge theory and $n=3$ $\mathfrak{su}(3)$ gauge theory. Since $\mathbf{1}\to(\mathbf{1},\mathbf{1})$ and

$$\mathbf{240}_2\to(\mathbf{72}_2,\mathbf{1})+(\mathbf{27}_{4/3},\mathbf{3}_{2/3})+(\overline{\mathbf{27}}_{4/3},\overline{\mathbf{3}}_{2/3})+(\mathbf{1},\mathbf{6}_2), \tag{7.24}$$

apply the gluing rule, we find one type of unity blowup equations



and five types of vanishing blowup equations



One can easily check the leading degree vanishing identities. For example, the first vanish blowup equation has leading base degree as

$$\Theta_{\Omega}^{[a]}\left(\tau,\begin{pmatrix}\epsilon_1+\epsilon_2\\ \epsilon_1+\epsilon_2\\ \epsilon_1+\epsilon_2\end{pmatrix}\right)=0, \tag{7.25}$$

while the second vanish blowup equation has leading base degree as

$$\Theta_\Omega^{[a]}\left(\tau,\Omega^{-1}\begin{pmatrix}0\\ \epsilon_1+\epsilon_2\\ m_\alpha^{E_6}+\epsilon_1+\epsilon_2\end{pmatrix}\right)=0, \tag{7.26}$$

and the forth vanish blowup equation has leading base degree as

$$\sum_{i=1,2,3}\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}0\\ -m_i\\ 0\end{pmatrix}+\Omega^{-1}\begin{pmatrix}m_w^{E_6}+\epsilon_1+\epsilon_2\\ 0\\ m_{w'}^{E_6}+\epsilon_1+\epsilon_2\end{pmatrix}\right)\frac{1}{\prod_{j\neq i}\theta_1(m_i-m_j)}=0. \tag{7.27}$$

Here the characteristic $a=(0,1/2,0)$ and α,α' are arbitrary roots of E_6, and w,w' are arbitrary weights of the fundamental representation $\mathbf{27}$. Besides, $m_i, i=1,2,3$ are the $\mathfrak{su}(3)$ fugacities satisfying $m_1+m_2+m_3=0$. It is easy to check these identities are correct.

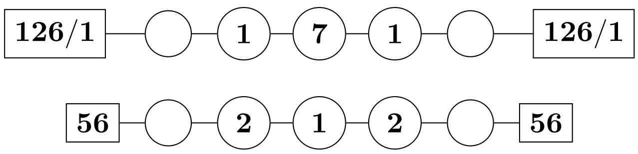
- (E_7,E_7) conformal matter theory is often denoted as $[E_7],1,2_{\mathfrak{su}(2)},3_{\mathfrak{so}(7)},2_{\mathfrak{su}(2)},1,[E_7]$. The base curve intersection matrix $-\Omega$ has

$$\Omega=\begin{pmatrix}1&-1&0&0&0\\ -1&2&-1&0&0\\ 0&-1&3&-1&0\\ 0&0&-1&2&-1\\ 0&0&0&-1&1\end{pmatrix}. \tag{7.28}$$

Note $\mathrm{Det}(\Omega)=1$. The E_8 flavor group of node 1 splits to $E_7\times\mathfrak{su}(2)$ when coupled with $n=8$ E_7 gauge theories and $n=2$ $\mathfrak{su}(2)$ gauge theory. Since

$$\mathbf{240}_2\to(\mathbf{126}_2,\mathbf{1})+(\mathbf{56}_{3/2},\mathbf{2}_{1/2})+(\mathbf{1},\mathbf{3}_2), \tag{7.29}$$

apply the gluing rules, we find the following possible blowup equations which are all vanishing:

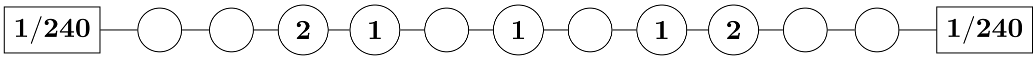


For example, the second type vanish blowup equation has leading base degree as

$$\sum_{\lambda_{a,b}=\pm1/2}(-1)^{\lambda_a+\lambda_b}\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}0\\ -\lambda_a m_a^{\mathfrak{su}(2)}\\ 0\\ -\lambda_b m_b^{\mathfrak{su}(2)}\\ 0\end{pmatrix}+\Omega^{-1}\begin{pmatrix}m_w+\epsilon_1+\epsilon_2\\ 0\\ 2(\epsilon_1+\epsilon_2)\\ 0\\ m_{w'}+\epsilon_1+\epsilon_2\end{pmatrix}\right)=0. \tag{7.30}$$

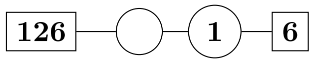
Here the characteristic $a = (1/2, 0, 1/2, 0, 1/2)$, $w, w' \in \mathbf{56}$ of E_7, and $\mathcal{O}_{1/2,6}$ is the Weyl orbit $\mathcal{O}^{\mathfrak{so}(7)}_{(100)}$. We have checked this identity is correct.

- (E_8, E_8) theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, 5_{F_4}, 1, 3_{G_2}, 2_{\mathfrak{su}(2)}, 2, 1, [E_8]$. The base curve intersection matrix $-\Omega$ has $\mathrm{Det}(\Omega) = 1$. Apply the gluing rule, we find the following possible vanishing blowup equations:



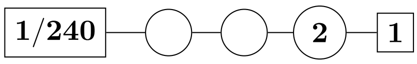
Thus there is no unity and just one type of vanishing blowup equations.

- $(E_7, \mathfrak{so}(7))$ conformal matter theory is often denoted as $[E_7], 1, 2_{\mathfrak{su}(2)}, [\mathfrak{so}(7)]$. Apply the gluing rule, we find the following possible unity blowup equation

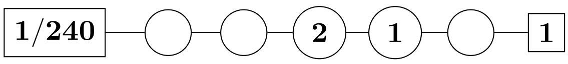


and vanishing blowup equations:

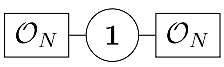
- (E_8, G_2) conformal matter theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, [G_2]$. Apply the gluing rule, we find the following possible vanishing blowup equations:



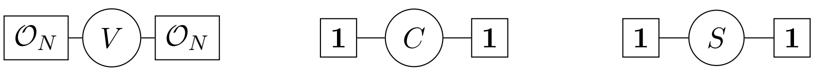
- (E_8, F_4) conformal matter theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, [F_4]$. Apply the gluing rule, we find the following possible blowup equations:



- $(\mathfrak{sp}(N), \mathfrak{sp}(N))$ conformal matter theory is often denoted as $[\mathfrak{sp}(N)], 4_{\mathfrak{so}(2N+8)}, [\mathfrak{sp}(N)]$. The flavor $\mathfrak{sp}(2N)$ of node 4 splits to $\mathfrak{sp}(N) \times \mathfrak{sp}(N)$. Apply the gluing rule, we find the following possible blowup equations: one type of unity equation



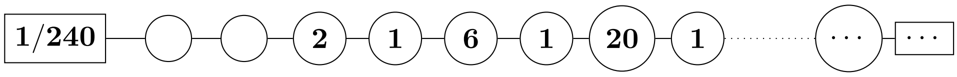
and lots of vanishing ones including



- $(E_8, \mathfrak{su}(N))$ conformal matter theory is often denoted as

$$[E_8]\ 1\ \overset{\mathfrak{su}(1)}{2}\ \overset{\mathfrak{su}(2)}{2}\ \ldots\ \overset{\mathfrak{su}(N-1)}{2}\ [\mathfrak{su}(N)]\,. \tag{7.31}$$

  Apply the gluing rule, we find the following possible blowup equations:

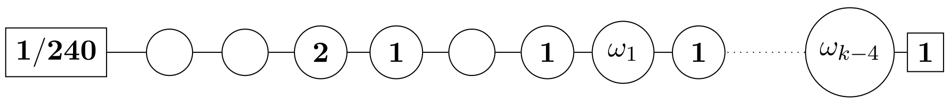


  The λ_G/λ_F field associated to the circular/rectangular node carrying gauge/flavor symmetry $\mathfrak{su}(k)$ $(k = 1, \ldots, N)$ is trivial if k is odd and is a non-trivial weight vector belonging to the Weyl orbit $\mathcal{O}_{k/2}$ if k is even.

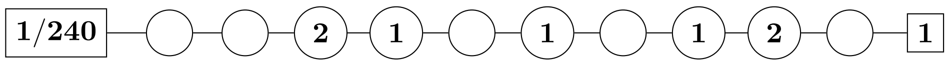
- $(E_8, B_k/D_k)$ conformal matter theory is often denoted as

$$[E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}(9)}{4}\ \overset{\mathfrak{sp}(1)}{1}\ \overset{\mathfrak{so}(11)}{4}\ \ldots\ \overset{\mathfrak{sp}(k-4)}{1}\ [\mathfrak{so}(2k)/\mathfrak{so}(2k+1)]\,. \tag{7.32}$$

  Apply the gluing rule, we find the following possible blowup equations:



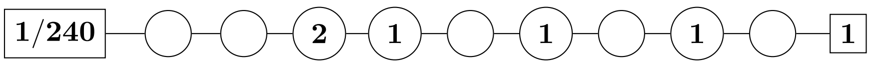
- (E_8, E_7) theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, 5_{F_4}, 1, 3_{G_2}, 2_{\mathfrak{su}(2)}, 1, [E_7]$. Apply the gluing rule, we find the following possible blowup equations:



- (E_8, E_6) conformal matter theory is often denoted as

$$[E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{su}(3)}{3}\ 1\ [E_6]\,. \tag{7.33}$$

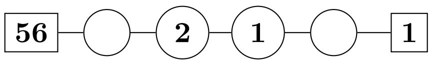
  Apply the gluing rule, we find the following possible blowup equations:



- (E_7, D_4) conformal matter theory is often denoted as

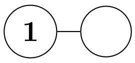
$$[E_7]\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ [\mathfrak{so}(8)]\,. \tag{7.34}$$

  Apply the gluing rule, we find the following possible blowup equations:

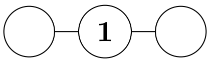


7.4.3 Blowups of $n = 9, 10, 11$ curves

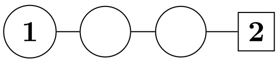
The rank one theories with $n = 9, 10, 11$ do not admit Kodaira-Tate elliptic fibers. One needs to do further blowups which result in higher dimensional tensor branches. There are normally several ways to do this, see for example [63]. The toric construction of some blown-up Calabi-Yau geometries were given in [2]. For $n = 11$ curve, one blows up once and gets theory $12_{E_8}, 1$. It is easy to find the following vanishing blowup equation for it:



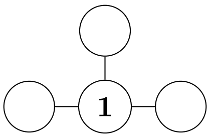
For $n = 10$, one blows up twice and gets $1, 12_{E_8}, 1$ with vanishing blowup equation



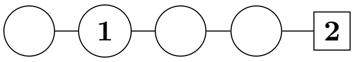
or $12_{E_8}, 1, 2$ with vanishing blowup equations



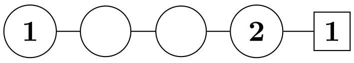
For $n = 9$, one blows up twice and gets $1, \overset{1}{12}_{E_8}, 1$ with vanishing blowup equation



or $1, 12_{E_8}, 1, 2$ with vanishing blowup equations



or $12_{E_8}, 1, 2, 2$ with vanishing blowup equation



Let us now take a closer look at the first example the $12_{E_8}, 1$ theory. The intersection matrix between the two base curves is just

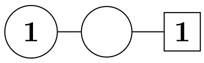
$$\Omega = \begin{pmatrix} 12 & -1 \\ -1 & 1 \end{pmatrix}, \tag{7.35}$$

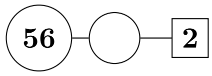
thus we have $\det(\Omega) = 11$ vanishing blowup equations. Since there is only one E_8 vector multiplet and no hypermultiplet, the leading base degree of the vanishing blowup equations can be simply written as

$$\Theta_{\Omega}^{[a]}\left(\tau, \begin{pmatrix} 5/33 \\ 5/33 \end{pmatrix} \epsilon_+\right) = 0. \tag{7.36}$$

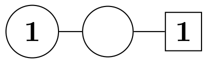
We have checked this identity up to q^{30}. Remember here characteristics a are associated to Ω as defined in (6.2).

As a similar example, we consider the blown-up of a -7 curve, which can be represented as $8_{E_7}, 1, [\mathfrak{su}(2)]$. There are two types of vanishing blowup equations:





In fact, for any of $-2, -3, -4, -5, -7, -11$ curves, one can blowup once and obtain a rank two theory which is the coupling between a pure gauge minimal 6d SCFT and the E-string theory. For these rank two theories, there always exists one type of vanishing blowup equations represented as



The leading base degree of the vanishing equations are due to the following identity:

$$\Theta_{\Omega}^{[a]}\left(\tau, \begin{pmatrix} z \\ z \end{pmatrix}\right) = 0, \tag{7.37}$$

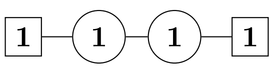
where

$$\Omega = \begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}. \tag{7.38}$$

In fact, this identity holds for arbitrary $n \geq 2$.

7.5 Remarks on solving elliptic genera

For higher rank theories, there in general seems to be no efficient way to solve elliptic genera from elliptic blowup equations. The main reason as mentioned before is that there usually only exist vanishing blowup equations for higher rank theories which do not give enough constraints. Besides, even in the rare cases where exist unity blowup equations, we can hardly make use of the equations to solve elliptic genera.[45] Naively, one may think there could exist some explicit higher dimensional recursion formulas analogous to the rank one cases as long as there exist three or more unity blowup equations. Unfortunately, because any such higher-rank theory involves -2 or -1 curves, the recursion fails when one of these curves is left but all other base curves are decompactified. Therefore, in some sense, *all higher-rank theories with unity blowup equations are in class* **B** *as in section 4, and all those with only vanishing blowup equations are in class* **C**. Let us consider a good example, the A_2 chain with gauge symmetry $\mathfrak{su}(N)$ on each node. For arbitrary N, there always exist unity blowup equations:



Since the intersection matrix between the base classes is

$$\Omega = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{7.39}$$

[45]The higher rank theories with unity blowup equations include for example all A, D type chain of -2 curves with gauge symmetry and (E_6, E_6) conformal matter theory.

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we have in total $\det(\Omega) = 3$ non-equivalent unity blowup equations. To solve elliptic genus say $\mathbb{E}_{2,1}$ by recursion, one need to know $\mathbb{E}_{2,0}, \mathbb{E}_{1,1}, \mathbb{E}_{1,0}, \mathbb{E}_{0,1}$ as initial data. However, all the essentially rank one elliptic genera $\mathbb{E}_{n,0}$ and $\mathbb{E}_{0,n}$ are not possible to solve by recursion as they are in class **B** of rank one theories. In fact, when one decompactifies the right -2 curve, the three unity blowup equations will reduce to just two non-equivalent unity equations of the left -2 curve which are just the two unity equations of the $n = 2$, $G = \mathfrak{su}(N)$ theory. Thus there are not enough unity equations to proceed with the recursion. See the detailed analysis for the degeneration of M-M string chain in section 3.3 of [9]. Nevertheless, from the perspective of the ϵ_1, ϵ_2 expansion, the refined BPS expansion or the Weyl orbit expansion, one can still get some constraints. We do not pursue this direction further since the perfect 2d quiver description were already found for these higher-rank theories.

8 Conclusions and outlook

In this paper, we obtained the elliptic blowup equations for all 6d $(1,0)$ SCFTs in the atomic classification. In particular, we studied extensively all the rank one theories which are labeled by an integer n, a gauge symmetry G and a flavor symmetry F. We divide these theories into three classes: class **A** $(n \geq 3)$ and class **B** $(n = 1, 2)$ contain theories without unpaired half hypermultiplet, which make up the most of the rank one list, while class **C** contains the remaining 12 theories with unpaired half hypermultiplets. We find that for classes **A** and **B**, there always exist unity blowup equations and possibly also vanishing blowup equations, while for class **C**, there only exist vanishing blowup equations. This has the following implications for the solvability of the elliptic genera from the blowup equations: for class **A**, we obtain a recursion formula that determines the elliptic genera completely, i.e. for arbitrary numbers of strings from the unity blowup equations, which is the ideal situation. For class **B**, we can solve the elliptic genera and the refined BPS invariants order by order from the Weyl orbit expansion, the refined BPS expansion or the ϵ_1, ϵ_2 expansion. For class **C**, we do not have a universal description how to solve elliptic genera from vanishing blowup equations. For the classes **A** and **B**, we have checked that our results for elliptic genera agree with all previous partial results from 2d quiver gauge theories, 5d partition functions and the modular ansatz. For class **C**, although we could not solve the full elliptic genera. However we checked theta identities in leading base degree and showed that part of the refined BPS invariants can be determined. We expect that these theories can still be completely solved by combining the constraints from the vanishing blowup equations, the modular ansatz and the Higgsing conditions.

We also propose the elliptic blowup equations for three higher rank non-Higgsable clusters, which only have vanishing equations due to the presence of unpaired half hypermultiplets. We checked our blowup equations by using the elliptic genera computed by localization formulas in the 2d quiver construction in [14] and by analysing the base curve decompactification limits where they reduce correctly to the blowup equations of rank one theories. We further give *gluing rules* which make it easy to write down all admissible blowup equations for any 6d $(1,0)$ SCFTs in the atomic classification. In particular, we explicitly present the blowup equations for ADE chains of -2 curves, conformal matter theories and the blown-ups of $-9, -10, -11$ curves. Most of the higher rank theories only have vanishing blowup equations. One can even consider the blowup equations for little string theories. We leave this for future study.

The solution of the theories with the blowup equations proceeds in two steps: first one establishes the validity of blowup equations by demonstrating that the partition functions or equivalently the elliptic genera satisfy these equations; in the second step one has to develop efficient procedures to solve the blowup equations. Albeit the first step has been very successful for all kinds of theories including all 6d $(1, 0)$ SCFTs in the atomic classification, there is still uncertainty in the second step. In particular we do not know what precise conditions or inputs are needed for a complete solution in general. In practice, we have developed several efficient techniques to extract information from blowup equations, such as recursion formulas, Weyl orbit expansions, refined BPS expansions, and the ϵ_1, ϵ_2 expansion. Each method typically requires different inputs. For the theories of class **C** with only vanishing blowup equations, we do not know in general what the minimal inputs should be. This may cast some doubts on the solvability of blowup equations in the general situation, since even the seemingly simple $n = 7, G = E_7$ theory can not be solved completely. One remedy may be to combine the blowup equations and modular ansatz together. Another possible remedy draws inspiration from the massless E-string theory, which corresponds to a naturally realised elliptic non-compact Calabi-Yau 3-fold with two Kähler parameters. Although the theory itself has only one vanishing blowup equation and is therefore not solvable, once one of the eight possible mass parameters is turned on, there are enough unity blowup equations to allow for the complete solution of the theory. The example suggests that in some cases one can recover the necessary unity blowup equations after deforming the theory with additional natural parameters with a mass scale.

The 6d $(1, 0)$ SCFTs we are considering have to be compactified on a torus in order for elliptic genera to be defined. In the low energy limit, these theories can be equivalently seen as 5d KK/marginal theories [66] compactified on a circle, where the radius of the other circle in the 6d theory is identified with the KK scale. They can then be reduced to 5d SCFTs either by decoupling mass deformed hypermultiplets which corresponds to flopping (-1) curves out of compact surfaces [66, 67, 106–109], or by decoupling a gauge sector, which corresponds to decompactifying the surfaces themselves [110–112]. The simplest 5d SCFTs are the infinite coupling limit of 5d gauge theories, possibly with matter. On the one hand, the blowup equations of 5d gauge theories have been studied for all simple Lie groups in [34] and for all possible matter contents in [80]. On the other hand, we have developed techniques in our previous papers to reduce blowup equations of 6d SCFTs to those of 5d SCFTs through either of the two methods [7, 9]. In this paper we do not go into details about reducing the large collection of 6d blowup equations to 5d equations. Nevertheless, we do point out and present many new blowup equations for 5d gauge theories beyond those found in [34] and [80]. The blowup equations for 5d SCFTs obtained in this way could be helpful for solving the BPS states of these theories [67, 113].

There are many open problems. First of all, we expect blowup equations to exist for many other field theories, for instance 6d SCFTs with "frozen singularity" [39, 42] not covered in the atomic classification, 6d SCFTs with twisted compactification on circle [68],

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and little string theories [40]. Furthermore, in the second paper of this series [8], we studied a surprising conjectural relation [5] between the elliptic genera of pure gauge 6d $(1,0)$ SCFTs and the Schur indices of 4d $\mathcal{N}=2$ H_G SCFTs, and generalized it from one string elliptic genera to higher strings. For theories with matter, it was identified in [1] that the worldsheet $(0,4)$ theories also correspond to some 4d $\mathcal{N}=2$ SCFTs but with some $(0,4)$ surface defects. The Schur indices of such configurations have rarely been studied, see some relevant results in [114]. It is interesting to see if the Schur indices of such 4d SCFTs with $(0,4)$ defects are also related to the elliptic genera of 6d $(1,0)$ SCFTs with matter.

Acknowledgments

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A Lie algebraic convention

We collect some definitions in (affine) Lie algebras and fix our convention used throughout the paper.

A.1 Definitions and convention

Given a simple Lie algebra $\mathfrak{g}$ of rank r, there are four important r-dimensional lattices: the root and coroot lattices $Q, Q^\vee$, as well as the weight and coweight lattices[46] $P, P^\vee$. They are related to each other by

$$Q^\vee \subset P^\vee \subset \mathfrak{h}_\mathbb{C}\,, \tag{A.1}$$

$$Q \subset P \subset \mathfrak{h}^*_\mathbb{C}\,, \tag{A.2}$$

where $\mathfrak{h}_\mathbb{C}, \mathfrak{h}^*_\mathbb{C} \cong \mathbb{C}^r$ denote the complexified Cartan subalgebra and its dual equipped with the natural pairing

$$\langle \bullet, \bullet \rangle : \mathfrak{h}^*_\mathbb{C} \times \mathfrak{h}_\mathbb{C} \to \mathbb{C}\,. \tag{A.3}$$

The root and coroot lattices $Q, Q^\vee$ are spanned by the simple roots α_i and the simple coroots $\alpha_j^\vee$, whose pairings are entries of the Cartan matrix A

$$\left\langle \alpha_i, \alpha_j^\vee \right\rangle = A_{ij}. \tag{A.4}$$

The weight and coweight lattices $P, P^\vee$ are spanned by the fundamental weights ω_i and the fundamental coweights $\omega_i^\vee$, defined through

$$\left\langle \alpha_i, \omega_j^\vee \right\rangle = \left\langle \omega_i, \alpha_j^\vee \right\rangle = \delta_{ij}\,, \tag{A.5}$$

[46] The coweight lattice is sometimes called the magnetic weight lattice in the literature, e.g. [98].

in other words, they are the duals of the coroot and the root lattices respectively. Every weight vector ω can be represented by the coefficients λ_i in its decomposition in terms of the fundamental weights, which are called the Dynkin labels

$$\omega = \sum_i \lambda_i \omega_i. \tag{A.6}$$

A weight vector is said to be *dominant* if all of its Dynkin labels are non-negative integers. Likewise, we can represent a coweight vector $\omega^\vee$ by the coefficients $\lambda_i^\vee$ in its decomposition in terms of the fundamental coweights

$$\omega^\vee = \sum_i \lambda_i^\vee \omega_i^\vee. \tag{A.7}$$

We will also call $\lambda_i^\vee$ the Dynkin labels of the coweight $\omega^\vee$ and say the coweight vector is dominant if all $\lambda_i^\vee$ are non-negative. Dominant (co)weight vectors can be used to label Weyl orbits as each Weyl orbit of (co)weight vectors has one and only one dominant element.

We define the Weyl invariant bilinear form $(\bullet, \bullet)$ on $\mathfrak{h}_\mathbb{C}$ by

$$(k, \ell) := \frac{1}{2h_\mathfrak{g}^\vee} \sum_{\alpha\in\Delta} \langle \alpha, k\rangle \langle \alpha, \ell\rangle\,, \quad k, \ell \in \mathfrak{h}_\mathbb{C}, \tag{A.8}$$

where $h_\mathfrak{g}^\vee$ is the dual Coxeter number of $\mathfrak{g}$. It has the nice property that the norm $||k||^2 = (k,k)$ of any coroot is an even integer, and in particular the norm of the shortest non-zero coroot $\theta^\vee$ is two. Note that the dual Coxeter number $h_\mathfrak{g}^\vee$ can be interpreted as the Dynkin index of the adjoint representation $\mathfrak{adj}$, while for an arbitrary representation R its Dynkin index ind_R is defined by [115]

$$\mathrm{tr}_R(\mathcal{R}(J^a)\mathcal{R}(J^b)) = 2\,\mathrm{ind}_R\,\delta_{ab}, \tag{A.9}$$

where $\mathcal{R}(J^a)$ is the matrix representation of the generator J^a of $\mathfrak{g}$. Consequently the bilinear form (A.8) can be expressed in terms of any representation R of $\mathfrak{g}$ through

$$(k, \ell) = \frac{1}{2\,\mathrm{ind}_R} \sum_{\omega\in R} \langle \omega, k\rangle \langle \omega, \ell\rangle\,, \quad k, \ell \in \mathfrak{h}_\mathbb{C}, \tag{A.10}$$

where we have used the same symbol R for the weight space of the representation.

The bilinear form $(\bullet, \bullet)$ is symmetric and non-degenerate. It then defines an isomorphism from $\mathfrak{h}_\mathbb{C}$ to $\mathfrak{h}_\mathbb{C}^*$ by

$$\begin{aligned} \varphi : \mathfrak{h}_\mathbb{C} &\xrightarrow{\sim} \mathfrak{h}_\mathbb{C}^* \\ k &\mapsto \varphi(k) = (k, \bullet); \end{aligned} \tag{A.11}$$

in other words, we have

$$\langle \varphi(k), \ell\rangle = (k, \ell), \quad \forall \ell \in \mathfrak{h}_\mathbb{C}. \tag{A.12}$$

The isomorphism then induces a Weyl invariant bilinear form on $\mathfrak{h}_\mathbb{C}^*$

$$(\omega, \eta) = \left\langle \omega, \varphi^{-1}(\eta)\right\rangle = (\varphi^{-1}(\omega), \varphi^{-1}(\eta)), \quad \omega, \eta \in \mathfrak{h}_\mathbb{C}^*. \tag{A.13}$$

Concretely we have

$$\varphi(\alpha_i^\vee) = \frac{||\alpha_i^\vee||^2}{2}\alpha_i, \quad \varphi(\omega_i^\vee) = \frac{||\alpha_i^\vee||^2}{2}\omega_i. \tag{A.14}$$

It is easy to see that the Dynkin labels $\lambda_i^\vee$ of a coweight $\omega^\vee$ and the Dynkin labels λ_i of its isomorphic weight vector $\omega = \varphi(\omega^\vee)$ are related by

$$\lambda_i = \lambda_i^\vee \frac{||\alpha_i^\vee||^2}{2}. \tag{A.15}$$

We list below the norms of simple coroots of simple Lie algebras used in this paper.

- $A_n, D_n, E_{6,7,8}$: these are simply laced Lie algebras and all the simple coroots have norm 2.
- $B_n (n \geq 2)$:
$$||\alpha_i^\vee||^2 = 2, \;\; i = 1, \ldots, n-1, \quad ||\alpha_n^\vee||^2 = 4. \tag{A.16}$$
- $C_n (n \geq 2)$:
$$||\alpha_i^\vee||^2 = 4, \;\; i = 1, \ldots, n-1, \quad ||\alpha_n^\vee||^2 = 2. \tag{A.17}$$
- G_2:
$$||\alpha_1^\vee||^2 = 2, \quad ||\alpha_2^\vee||^2 = 6. \tag{A.18}$$
- F_4:
$$||\alpha_1^\vee||^2 = ||\alpha_2^\vee||^2 = 2, \quad ||\alpha_3^\vee||^2 = ||\alpha_4^\vee||^2 = 4. \tag{A.19}$$

We give in figure 4 the affine Dynkin diagrams of simple Lie algebras and the ordering of nodes used in our paper.

In the main text, to lighten notation we use $\cdot$ to represent both the pairing $\langle \bullet, \bullet \rangle$ and the bilinear form $(\bullet, \bullet)$. Hopefully the actual meaning of $\cdot$ will be clear from the context.

A.2 Lie sub-algebra decomposition

Let $\mathfrak{g}'$ be a Lie sub-algebra of $\mathfrak{g}$. The weight lattice P of $\mathfrak{g}$ can be mapped to the weight lattice P' of $\mathfrak{g}'$ by a surjective map

$$f : P \twoheadrightarrow P'. \tag{A.20}$$

This map induces an injective map from $Q^{\vee,'}$ to $Q^\vee$

$$f^* : Q^{\vee,'} \hookrightarrow Q^\vee \tag{A.21}$$

defined by

$$\left\langle \omega, f^*(\alpha^{\vee,'}) \right\rangle = \left\langle f(\omega), \alpha^{\vee,'} \right\rangle \in \mathbb{Z}, \quad \forall \omega \in P, \tag{A.22}$$

for $\alpha^{\vee,'} \in Q^{\vee,'}$. It is easy to see that f^* is indeed an injection. If $f^*(\alpha^{\vee,'}) = f^*(\beta^{\vee,'})$, we have

$$\left\langle f(\omega), \alpha^{\vee,'} - \beta^{\vee,'} \right\rangle = \left\langle w, f^*(\alpha^{\vee,'}) - f^*(\beta^{\vee,'}) \right\rangle = 0, \quad \forall \omega \in P \tag{A.23}$$

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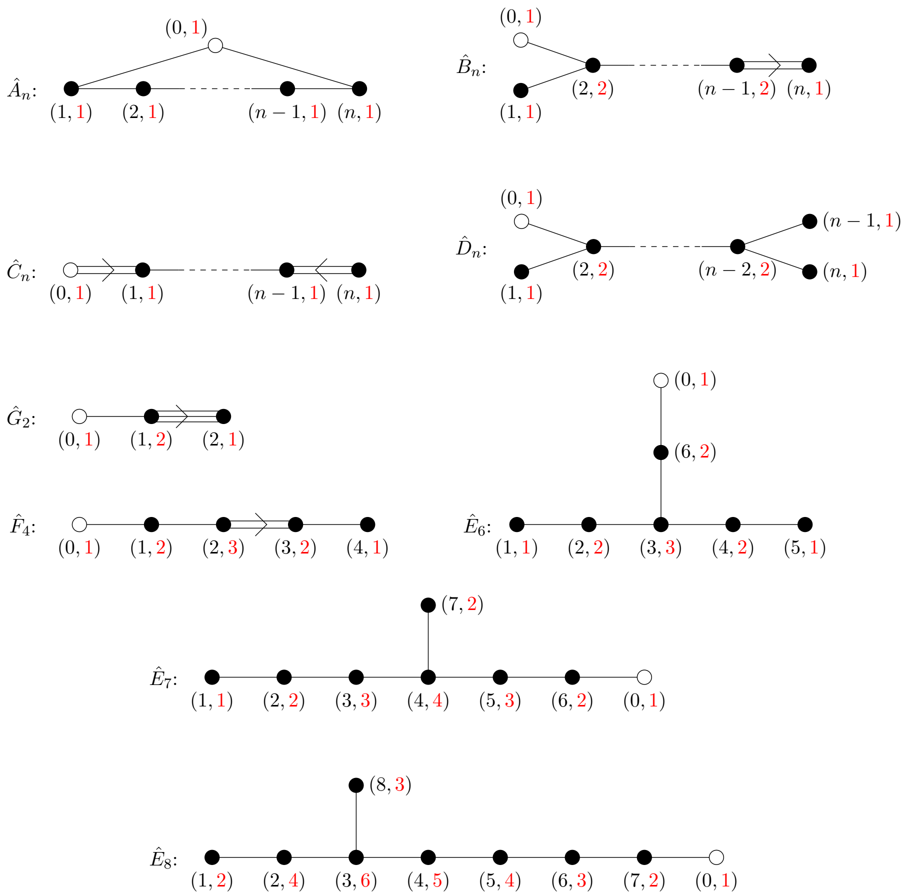

Figure 4. Affine Dynkin diagrams associated to simple Lie algebras. The i-th node with comark m_i is labeled by the pair (i, m_i) where the number m_i is colored in red. In each diagram, the white node is the affine node, and the black nodes are nodes of simple Lie algebra. The arrows point from short coroots to long coroots. We follow the same node order and same representation names as in the `LieART` package [116, 117] of `Mathematica`.

Since $f(\omega)$ runs over P', we must have $\alpha^{\vee,\prime} - \beta^{\vee,\prime} = 0$. We can also show that

$$\mathrm{Im}(f^*) = \{\alpha^\vee \in Q^\vee \mid \langle \omega, \alpha^\vee \rangle = 0,\ \forall \omega \in \mathrm{Ker}\, P\}. \tag{A.24}$$

First of all $\mathrm{Im}(f^*)$ is a subset of the r.h.s., since if $\alpha^\vee \in \mathrm{Im}(f^*)$, there exists $\alpha^{\vee,\prime}$ such that $\forall \omega \in \mathrm{Ker}\, P$

$$\langle \omega, \alpha^\vee \rangle = \left\langle f(\omega), \alpha^{\vee,\prime} \right\rangle = \left\langle 0, \alpha^{\vee,\prime} \right\rangle = 0. \tag{A.25}$$

On the other hand, both the l.h.s. and r.h.s. of (A.24) are linear spaces, and they have the same dimension. Therefore they must be identical. Finally, by definition, the pair of maps f, f^* preserve the natural pairing $\langle \bullet, \bullet \rangle$. We can also deduce that they preserve the bilinear form $(\bullet, \bullet)$ on coroot lattices as well. This is because given the images $f^*(\alpha^{\vee,\prime}), f^*(\beta^{\vee,\prime}) \in Q^\vee$ of two vectors $\alpha^{\vee,\prime}, \beta^{\vee,\prime}$ in $Q^{\vee,\prime}$, we can define their bilinear form by an arbitrary representation R in $\mathfrak{g}$

$$\begin{aligned}(f^*(\alpha^{\vee,\prime}), f^*(\beta^{\vee,\prime})) &= \frac{1}{2\,\mathrm{ind}_R} \sum_{\omega\in R} \left\langle \omega, f^*(\alpha^{\vee,\prime})\right\rangle \left\langle \omega, f^*(\beta^{\vee,\prime})\right\rangle \\ &= \frac{1}{2\,\mathrm{ind}_{f(R)}} \sum_{f(\omega)\in f(R)} \left\langle f(\omega), \alpha^{\vee,\prime}\right\rangle \left\langle f(\omega), \beta^{\vee,\prime}\right\rangle \\ &= (\alpha^{\vee,\prime}, \beta^{\vee,\prime}),\end{aligned} \tag{A.26}$$

where in the second step, we used the definition of the pull-back map f^* and that the representation index does not change under the projection of weight spaces.[47]

We comment that the injective map f^* *cannot* be extended to a map from $P^{\vee,\prime}$ to P. One important reason the map between coroot lattices can be defined is that the pairing between P and $Q^\vee$ always takes value in $\mathbb{Z}$, and thus the definition (A.22) makes sense. On the other hand, the pairing between P and $P^\vee$ takes value in different domains for different Lie algebras and (A.22) will no longer make sense. For instance, it takes value in $\mathbb{Z}/2$ for E_7 and in $\mathbb{Z}/3$ for E_6.

B List of results

For the convenience of readers, we make a list for the computational results in this paper.

- Elliptic genera.
  Although our computation on the elliptic genera of rank one 6d $(1,0)$ SCFTs mostly contain all gauge and flavor fugacities, for some $\mathfrak{so}(N)$ theories we only present the results with all fugacities turned off. For most theories especially the exceptional theories, we not only show the elliptic genera with fugacities turned off, but also the v expansion with gauge and flavor fugacities turned on.
  Class **A**
  - $n=3$, $G=\mathfrak{so}(7)$, $\mathbb{E}_1$ (5.35), $\mathbb{E}_2$ (5.36)
    $G=\mathfrak{so}(8)$, $\mathbb{E}_1$ (E.21), $\mathbb{E}_2$ (E.24)
    $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (E.25), $\mathbb{E}_2$ (E.27)
    $G=\mathfrak{so}(10)$, $\mathbb{E}_1$ (E.29), $\mathbb{E}_2$ (E)
    $G=G_2$, $\mathbb{E}_1$ (5.48), $\mathbb{E}_2$ (5.52)
    $G=F_4$, $\mathbb{E}_1$ (5.69), $\mathbb{E}_2$ (5.7)
    $G=E_6$, $\mathbb{E}_1$ (5.90), $\mathbb{E}_2$ (E)
  - $n=4$, $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (5.43), $\mathbb{E}_2$ (5.5)
    $G=\mathfrak{so}(10)$, $\mathbb{E}_1$ (5.45), $\mathbb{E}_2$ (5.5)
    $G=F_4$, $\mathbb{E}_1$ (5.65), $\mathbb{E}_2$ (5.7)
    $G=E_6$, $\mathbb{E}_1$ (5.88), $\mathbb{E}_2$ (E.35)
    $G=E_7$, $\mathbb{E}_1$ (5.109)

[47] If an irreducible representation decomposes to multiple irreducible representations after the projection of weight space, the index of the composite representation is the sum of the indices of the individual irreducible representations.

- $n=5$, $G=E_6$, $\mathbb{E}_1$ (5.85), $\mathbb{E}_2$ (E.42)
- $n=6$, $G=E_7$, $\mathbb{E}_1$ (5.106)

Class **B**

- $n=1$, $G=\mathfrak{su}(3)$, $\mathbb{E}_1$ (5.16)
  $G=\mathfrak{su}(4)$, $\mathbb{E}_1$ (5.18)
  $G=\mathfrak{so}(7)$, $\mathbb{E}_1$ (E.1)
  $G=\mathfrak{so}(8)$, $\mathbb{E}_1$ (E.3)
  $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (E.5)
  $G=G_2$, $\mathbb{E}_1$ (5.61)
  $G=F_4$, $\mathbb{E}_1$ (5.77)
  $G=E_6$, $\mathbb{E}_1$ (5.97)
- $n=2$, $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (E.8)
  $G=\mathfrak{so}(10)$, $\mathbb{E}_1$ (E.10)
  $G=\mathfrak{so}(11)$, $\mathbb{E}_1$ (E.13)
  $G=\mathfrak{so}(12)_a$, $\mathbb{E}_1$ (E.18)
  $G=G_2$, $\mathbb{E}_1$ (5.54)
  $G=F_4$, $\mathbb{E}_1$ (5.72)
  $G=E_6$, $\mathbb{E}_1$ (5.93)
  $G=E_7$, $\mathbb{E}_1$ (5.111)

- Exact v expansion formulas for 5d one-instanton Nekrasov partition functions.
  For a lot of rank-one theories with matters, the exact formulas for the v expansion of 5d one-instanton partition function have been proposed in [1] and [80]. In this paper, we further obtain the exact formulas for the following new theories

  - $n=1$, $G=\mathfrak{su}(3)$ (5.17), $\mathfrak{su}(4)$ (5.19), $\mathfrak{so}(7)$ (E.2), $\mathfrak{so}(8)$ (E.4), $\mathfrak{so}(9)$ (E.6)
    G_2 (5.64), F_4 (E.7), E_6 (5.101)
  - $n=2$, $G=\mathfrak{so}(9)$ (E.9), $\mathfrak{so}(10)$ (E.11), $\mathfrak{so}(11)$ (E.15), $\mathfrak{so}(12)_a$ (E.19),
    E_6 (5.96), E_7 (E.20)
  - $n=3$, $G=\mathfrak{so}(12)$ (E.32), E_6 (E.33)
  - $n=4$, $G=E_6$ (5.89), E_7 (E.38)

- Modular ansatz.
  Among the ten theories whose modular ansatz for reduced one-string elliptic genus were not fixed in [1], five of them listed below belong to class **A** or **B**. Benefitting from blowup equations, we are able to determine their modular ansatz. See results in the `Mathematica` file `ModularAnsatzAppendix.nb` in the supplementary material or on the website [99].

  - $n=1$, $G=E_6$
  - $n=2$, $G=\mathfrak{so}(11)$, E_6, E_7
  - $n=4$, $G=E_7$

- Calabi-Yau construction and triple intersection numbers.
  We give the polytope, the Mori cone generators and triple intersection ring for the non-compact elliptic Calabi-Yau threefolds associated to the following theories
  - $n=1$, $G=G_2$ (F.18), (F.19)
  - $n=2$, $G=G_2$ (F.15), (F.16)
  - $n=3$, $G=G_2$ (2.54), (2.55)
  - $n=3$, $G=\mathfrak{so}(7)$ (F.12), (F.13)
  - $n=7$, $G=E_7$ (F.1), (F.2)
  - NHC $3,2$ (2.51), (2.52)
  - NHC $3,2,2$ (F.8), (F.9)
  - NHC $2,3,2$ (F.4), (F.5)
- Refined BPS invariants.
  - $n=1$, $G=G_2$ (table 20)
  - $n=2$, $G=G_2$ (table 21)
  - $n=3$, $G=G_2$ (table 22)
  - $n=3$, $G=\mathfrak{so}(7)$ (table 23)
  - $n=7$, $G=E_7$ (table 24)
- Vanishing theta identities.
  We checked the leading degree identities for all the vanishing blowup equations in table 7 and 8 up to $\mathcal{O}(q^{20})$. We write down the explicit form of the vanishing identities for the following theories:
  - $n=1$, $G=\mathfrak{su}(3)$ (3.25), (3.26)
  - $n=1$, $G=\mathfrak{su}(N)$ (5.9), (5.10), (5.11), (5.12)
  - $n=1$, $G=\mathfrak{sp}(N)$ (5.3), (5.4)
  - $n=2$, $G=\mathfrak{su}(N)$ (5.22), (5.23), (5.24), (5.25)
  - $n=3$, $G=\mathfrak{so}(7)$ (5.32), (5.33)
  - $n=4$, $G=\mathfrak{so}(8+N)$ (5.40), (5.41)
  - $n=1,2,\ldots,6$, $G=E_6$ (5.81), (5.82), (5.83), (5.84)
  - $n=1,2,\ldots,8$, $G=E_7$ (5.102), (5.103), (5.105)
  - NHC $3,2$ (6.17)
  - NHC $3,2,2$ (6.32)
  - NHC $2,3,2$ (6.43), (6.46)
  - D_4 quiver of -2 curves (7.19), (7.20)
  - (E_6,E_6) conformal matter (7.25), (7.26), (7.27)
  - (E_7,E_7) conformal matter (7.30)
  - blown-up of $-n$ curve with $n=2,3,4,5,7,9,10,11$ (7.36), (7.37)

C Semi-classical free energy for higher rank theories

In this section, we give the genus zero and genus one free energies at large volume limit for a generic higher rank SCFT from gluing of rank one Calabi-Yau three-folds. When we say gluing Calabi-Yau three-folds, we put their cycles together, which means the B-periods should have the same expression as they were in the rank one theories. Simply integrating over periods in rank one theories, we get the tree level prepotential of higher rank theories. By adding the one loop contributions, for a generic 6d (1,0) SCFT, the classical prepotential at the large volume limit is

$$\begin{aligned} F^{(0,0)} =& -\frac{1}{6}\sum_i \sum_{\alpha\in\Delta_{i,+}} (\alpha\cdot m_{G_i})^3 + \frac{1}{12}\sum_{i,j}\sum_{\omega_{G_i,G_j}\in\mathfrak{R}^+_{G_i,G_j}} (\omega_{G_i}\cdot m_{G_i}+\omega_{G_j}\cdot m_{G_j})^3 \\ & -\frac{1}{2}(t_{\mathrm{ell},i}-(n_i-2)\tau/2)\Omega^{-1}_{ij}\left(-k_{F,j}m_{F_j}\cdot m_{F_j}+\sum_k \Omega_{jk}m_{G_k}\cdot m_{G_k}\right) \\ & -\frac{1}{2}t_{\mathrm{ell},i}\Omega^{-1}_{ij}t_{\mathrm{ell},j}\tau + \mathcal{O}(\tau^3), \end{aligned} \tag{C.1}$$

where $-\Omega_{ij}$ is the intersection matrix of the base, $\Omega^{-1}_{ij} = (\Omega^{-1})_{ij}$. Note here $\mathfrak{R}_{G_i,G_j}$ are all the possible bi-representations, also including cases when one of the group is flavor group F_j.

For genus one part, we have

$$b_i^{(1,0)}t_i = \frac{1}{12}\sum_i\sum_{\alpha\in\Delta_{i,+}}\alpha\cdot t + \frac{1}{48}\sum_{i,j}\sum_{\omega\in\mathfrak{R}^+_{G_i,F_j}}\omega\cdot t + \frac{1}{4}\sum_{i,j}\Omega^{-1}_{ij}(n_j-2-h^\vee_{G_j})t_{\mathrm{ell},i}, \tag{C.2}$$

$$b_i^{(0,1)}t_i = -\frac{1}{12}\sum_i\sum_{\alpha\in\Delta_{i,+}}\alpha\cdot t + \frac{1}{24}\sum_{i,j}\sum_{\omega\in\mathfrak{R}^+_{G_i,F_j}}\omega\cdot t + \frac{1}{2}\sum_{i,j}\Omega^{-1}_{ij}(n_j-2)t_{\mathrm{ell},i}. \tag{C.3}$$

Here, the $b^{(0,1)}_{\mathrm{ell},i} = \frac{1}{24}\int c_2\wedge J_{\mathrm{ell},i}$ has a geometric meaning. For complete intersetion Calabi-Yau threefolds, it is proved that [90, 118]

$$\int c_2\wedge J_a = \frac{1}{2}\sum_{bc}\kappa_{abc}(l_0^b l_0^c - \sum_{i>0} l_i^b l_i^c), \tag{C.4}$$

where κ_{abc} are the triple intersection numbers, and l_i^a the components of the Mori cone vector l^a. It seems this formula is correct for both compact and non-compact Calabi-Yau hypersurfaces [8]. By turning off all the gauge fugacities, the Mori cone vector for $\tau, t_{\mathrm{ell},i}$ can be write down effectively as

$$l_\tau = \{-6;2,3,1,0,\cdots\}, \tag{C.5}$$

$$l_{B,i} = \{0;0,0,n_i-2,1,-n_i,1,0,0,\cdots\}, \tag{C.6}$$

with $l_{B,i} = l_{\mathrm{ell},i} + l_\tau(n_i-2)/2$. Strictly speaking, (C.5) (C.6) are correct only for A-type bases, which are realized by a smooth toric surface with self intersection number $-n_i$. Together with (C.1) and (C.4), if we want to compute $b^{(0,1)}_{\mathrm{ell},i}$, we fix a in (C.4) to the base direction then we only encounter products of l vector between l_τ and $l_{\mathrm{ell},i}$, which means we can always compute the product locally, as a A_1 type base and therefore we conclude that they are effectively true even for D, E type bases when we compute $b^{(0,1)}_{\mathrm{ell},i}$. Then we have

$$b^{(0,1)}_{\mathrm{ell},i}=\frac{1}{24}\int c_2\wedge J_{\mathrm{ell},i}=\frac{1}{24}\left(\sum_j\Omega^{-1}_{ij}(n_j-2)+\frac{1}{2}\sum_j\Omega^{-1}_{ij}(n_j-2)(36-9-4-1)\right)=\frac{1}{2}\sum_j\Omega^{-1}_{ij}(n_j-2), \tag{C.7}$$

as we predicted in (C.3).

D Derivation of the elliptic blowup equations

In this appendix, we derive the elliptic blowup equations from the blowup equations for refined topological strings. This procedure is called de-affinazation which we have elaborated in length in [8]. Therefore we will be very brief here. Remember that in the blowup equations for refined topological strings, there is always a B field shift to the instanton part [19]. This shift is trivial for 6d pure gauge theories, but play a crucial rule in theory with matters.

It is well-known the hypermultiplets in representation $\mathfrak{R}$ contributes to $Z_{\text{1-loop}}$ as

$$Z_H^{\mathfrak{R}}=\mathrm{PE}\left(f_{(0,0)}(q_1,q_2)\left(\sum_{w\in\mathfrak{R}_+(G)}\left(Q^w+\frac{q}{Q^w}\right)\right)\left(\frac{1}{1-q}\right)\right). \tag{D.1}$$

Recall the definition (4.14), here $f_{(0,0)}(q_1,q_2)=((q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2}))^{-1}$, which represents the spin $(0,0)$ nature of hypermultiplets. For general local Calabi-Yau, the spin (j_L,j_R) of the non-vanishing refined BPS invariants at degree d have a checkerboard pattern, i.e. satisfy the B field condition. The B field is defined by

$$2j_L+2j_R+1=B\cdot d\mod 2. \tag{D.2}$$

For spin $(0,0)$, the B fields always belong to $\mathbb{Z}+1/2$. Then recall the definition of Bl function in (4.16), the Z_H contributes to the blowup equations as

$$\begin{aligned}&\mathrm{PE}\left(\left(Bl_{(0,0,R_z)}(q_1,q_2)(-Q_z)+Bl_{(0,0,-R_z)}(q_1,q_2)\frac{q}{-Q_z}\right)\left(\frac{1}{1-q}\right)\right)\\&=\left(\frac{Q_z^{1/2}}{q^{1/12}}\right)^{\frac{(R_z^2-1/4)}{2}}(q_1q_2)^{\frac{(R_z^2-1/4)R_z}{12}}\prod_{\substack{0\le m,n\\ m+n\le R_z-3/2}}\frac{\theta_2(z+(m+1/2)\epsilon_1+(n+1/2)\epsilon_2)}{\eta}.\end{aligned} \tag{D.3}$$

Here $R_z>0$ and PE is the plethystic exponent operator defined as $\mathrm{PE}[f(x)]=\exp[\sum_{n=1}^\infty\frac{1}{n}f(x^n)]$. Note in (D.3), the Jacobi theta function is θ_2 rather than θ_1 because of the B field shift. Shifting back the B field gives a factor $(-1)^{R_z}$ as is already shown in (3.12). On the other hand, the vector multiplet with spin $(0,1/2)$ has B field as integers and contributes to blowup equations as [7, 8]

$$\begin{aligned}&\mathrm{PE}\left(-\left(Bl_{(0,1/2,R_z)}(q_1,q_2)Q_z+Bl_{(0,1/2,-R_z)}(q_1,q_2)\frac{q}{Q_z}\right)\left(\frac{1}{1-q}\right)\right)=(\mathrm{i}q^{1/12}Q_z^{-1/2})^{R_z^2}\\&\times(q_1q_2)^{-\frac{(R_z-1)R_z(R_z+1)}{6}}\prod_{\substack{0\le m,n\\ m+n\le R_z-1}}\frac{\eta}{\theta_1(z+m\epsilon_1+n\epsilon_2)}\prod_{\substack{0\le m,n\\ m+n\le R_z-2}}\frac{\eta}{\theta_1(z+(m+1)\epsilon_1+(n+1)\epsilon_2)}.\end{aligned} \tag{D.4}$$

Here $R_z \geq 0$. The tensor multiplet does contribute to $Z_{\text{1-loop}}$, but decouples from elliptic blowup equations. See more in [8].

Consider only the $R_z^3(\epsilon_1+\epsilon_2)$ terms, it is easy to see they cancel with the summation over the positive roots and half weights in (C.1) if the half weights belong to the same set. The remaining θ_2 part is not sensitive to the choices of half weights, which indicates that different Calabi-Yau phases give the same blowup equation.

In (D.4) and (D.3), there are τ linear part, this part will break the modularity in our blowup equation. The cancellation of the τ linear term gives a constraint on the representations of the theory

$$\sum_i \frac{3(n_i-2)-h^\vee_{Gi}}{h^\vee_{Gi}} \sum_{\alpha\in\Delta_i^+}(\alpha\cdot t)^2 = -\frac{1}{2}\sum_{i,j}\sum_{(\omega_i,\omega_j)\in\mathfrak{R}^+_{G_i,G_j}}(\omega_i\cdot m_{G_i}+\omega_j\cdot m_{G_j})^2 - \frac{1}{2}\sum_{i,j}\sum_{(\omega_i,\omega_j)\in\mathfrak{R}^+_{G_i,F_j}}(\omega_i\cdot m_{G_i})^2. \tag{D.5}$$

In rank one theories, the above constraint can be simplified as

$$\frac{h^\vee_G-3(n-2)}{12h^\vee_G}\sum_{\alpha\in\Delta}(\alpha\cdot t)^2 = \frac{\dim\mathfrak{R}_F}{24}\sum_{\omega\in\mathfrak{R}_G}(\omega\cdot t)^2. \tag{D.6}$$

By basic properties of root lattice and weight lattice, we obtain the following useful constraint for arbitrary rank-one 6d $(1,0)$ SCFTs:

$$2h^\vee_G - 6(n-2) = \dim\mathfrak{R}_F \mathrm{ind}\mathfrak{R}_G. \tag{D.7}$$

With the constraint (D.5), one can check the non-modular part of (D.4) and (D.3) combined with the polynomial contributions indeed give the theta function in (7.6), with

$$y_{u/v,i} = \frac{1}{4}\Omega^{-1}_{ij}\left(n_j - 2 + h^\vee_{Gj} + 2k_{F_j}\lambda_{F_j}\cdot\lambda_{F_j}\right). \tag{D.8}$$

Finally, we shift back the B field in the instanton part, the θ_2 in (D.3) becomes θ_1, and we arrive at the elliptic blowup equations which are functional equations of the conventional RR elliptic genera.

E More on elliptic genera

Here we record more results on the one-string and two-string elliptic genera for certain rank one theories which we obtain from blowup equations. Note all "..." in the polynomial of v means palindromic. More detailed results can be found in the supplementary material or on the website [99].

$n=1$, $G=\mathfrak{so}(7)$, $F=\mathfrak{sp}(2)_a\times\mathfrak{sp}(6)_b$. Using the Weyl orbit expansion, we turn on a diagonal subgroup $\mathfrak{sp}(1)\times\mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(7)}}(q,v,m_{\mathfrak{so}(7)}=0,m_F=0) = q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^8}, \tag{E.1}$$

where

$$P_0(v) = 21 + 44v - 294v^2 - 1156v^3 + 475v^4 + 13400v^5 + 38508v^6 + 13400v^7 + \cdots + 21v^{12}.$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$\begin{aligned}
&\chi^{\mathfrak{so}(7)}_{(010)}v^{-2} - (\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(10)_a} + \chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(100000)_b})v^{-1} + (\chi^{\mathfrak{so}(7)}_{(002)} + \chi^F_{(20)_a} + \chi^F_{(200000)_b} + 1) \\
&+ \chi^F_{(10)_a\otimes(010000)_b}v + (\chi^F_{(000100)_b} + \chi^F_{(01)_a\otimes(010000)_b} + \chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(01)_a})v^2 \\
&+ (\chi^F_{(10)_a\otimes(000100)_b} - \chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(01)_a\otimes(100000)_b} - \chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(100000)_b})v^3 \\
&- (\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(000100)_b} + \chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(10)_a\otimes(001000)_b} + \chi^{\mathfrak{so}(7)}_{(010)}\chi^F_{(010000)_b} - \chi^{\mathfrak{so}(7)}_{(002)}\chi^F_{(01)_a})v^4 \\
&+ (\chi^{\mathfrak{so}(7)}_{(101)}\chi^F_{(001000)_b} + \chi^{\mathfrak{so}(7)}_{(002)}\chi^F_{(10)_a\otimes(010000)_b} + \chi^{\mathfrak{so}(7)}_{(011)}\chi^F_{(100000)_b})v^5 \\
&- (\chi^{\mathfrak{so}(7)}_{(102)}\chi^F_{(010000)_b} + \chi^{\mathfrak{so}(7)}_{(003)}\chi^F_{(10)_a\otimes(100000)_b} + \chi^{\mathfrak{so}(7)}_{(012)})v^6 \\
&+ (\chi^{\mathfrak{so}(7)}_{(103)}\chi^F_{(100000)_b} + \chi^{\mathfrak{so}(7)}_{(004)}\chi^F_{(10)_a})v^7 - \chi^{\mathfrak{so}(7)}_{(104)}v^8 + \\
&+ \sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{so}(7)}_{(0n0)}\chi^F_{(01)_a\otimes(000001)_b}v^{4+2n} - (\chi^{\mathfrak{so}(7)}_{(1n0)}\chi^F_{(10)_a\otimes(000001)_b} + \chi^{\mathfrak{so}(7)}_{(0n1)}\chi^F_{(01)_a\otimes(000010)_b})v^{5+2n} \\
&\qquad + (\chi^{\mathfrak{so}(7)}_{(2n0)}\chi^F_{(000001)_b} + \chi^{\mathfrak{so}(7)}_{(1n1)}\chi^F_{(10)_a\otimes(000010)_b} + \chi^{\mathfrak{so}(7)}_{(0n2)}\chi^F_{(01)_a\otimes(000100)_b})v^{6+2n} \\
&\qquad - (\chi^{\mathfrak{so}(7)}_{(2n1)}\chi^F_{(000010)_b} + \chi^{\mathfrak{so}(7)}_{(1n2)}\chi^F_{(10)_a\otimes(000100)_b} + \chi^{\mathfrak{so}(7)}_{(0n3)}\chi^F_{(01)_a\otimes(001000)_b})v^{7+2n} \\
&\qquad + (\chi^{\mathfrak{so}(7)}_{(2n2)}\chi^F_{(000100)_b} + \chi^{\mathfrak{so}(7)}_{(1n3)}\chi^F_{(10)_a\otimes(001000)_b} + \chi^{\mathfrak{so}(7)}_{(0n4)}\chi^F_{(01)_a\otimes(010000)_b})v^{8+2n} \\
&\qquad - (\chi^{\mathfrak{so}(7)}_{(2n3)}\chi^F_{(001000)_b} + \chi^{\mathfrak{so}(7)}_{(1n4)}\chi^F_{(10)_a\otimes(010000)_b} + \chi^{\mathfrak{so}(7)}_{(0n5)}\chi^F_{(01)_a\otimes(100000)_b})v^{9+2n} \\
&\qquad + (\chi^{\mathfrak{so}(7)}_{(2n4)}\chi^F_{(010000)_b} + \chi^{\mathfrak{so}(7)}_{(1n5)}\chi^F_{(10)_a\otimes(100000)_b} + \chi^{\mathfrak{so}(7)}_{(0n6)}\chi^F_{(01)_a})v^{10+2n} \\
&\qquad - (\chi^{\mathfrak{so}(7)}_{(2n5)}\chi^F_{(100000)_b} + \chi^{\mathfrak{so}(7)}_{(1n6)}\chi^F_{(10)_a})v^{11+2n} + \chi^{\mathfrak{so}(7)}_{(2n6)}v^{12+2n}\Big].
\end{aligned} \tag{E.2}$$

After turning off all gauge and flavor fugacities, this goes back to the rational function of v by Weyl dimension formulas.

$n = 1$, $G = \mathfrak{so}(8)$, $F = \mathfrak{sp}(3)_a \times \mathfrak{sp}(3)_b \times \mathfrak{sp}(3)_c$. Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{sp}(1) \times \mathfrak{sp}(1) \times \mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(8)}}(q, v, m_{\mathfrak{so}(8)} = 0, m_F = 0) = q^{-1/3} + q^{2/3}v^{-2}\sum_{n=0}^{\infty} q^n \frac{P_n(v)}{(1+v)^{10}}, \tag{E.3}$$

where

$$P_0(v) = 4(7+34v-22v^2-496v^3-1128v^4+1326v^5+14327v^6+35392v^7+14327v^8+\cdots+7v^{14}).$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we find the following exact formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&\chi^F_{(010)_a\otimes(010)_b\otimes(010)_c}v^4-(\chi^G_{(1000)}\chi^F_{(100)_a\otimes(010)_b\otimes(010)_c}+\text{tri.})v^5+(\chi^F_{(100)_a\otimes(100)_b\otimes(001)_c}+\text{tri.})v^3\\
&-(\chi^G_{(1000)}\chi^F_{(100)_b\otimes(001)_c}+\text{tri.})v^4+(\chi^F_{(010)_b\otimes(010)_c}+\text{tri.})v^2+(\chi^G_{(2000)}\chi^F_{(010)_b\otimes(010)_c}+\text{tri.})v^6\\
&+(\chi^G_{(0011)}\chi^F_{(010)_a\otimes(100)_b\otimes(100)_c}+\text{tri.})v^6+(v-\chi^G_{(0100)}v^5-\chi^G_{(1011)}v^7)\chi^F_{(100)_a\otimes(100)_b\otimes(100)_c}\\
&-(\chi^G_{(2010)}\chi^F_{(100)_b\otimes(010)_c}+\text{tri.})v^7+(\chi^G_{(1000)}\chi^F_{(001)_a}+\text{tri.})v^3+(\chi^G_{(0011)}\chi^F_{(001)_a}+\text{tri.})v^5\\
&-(\chi^G_{(0100)}\chi^F_{(010)_a}+\text{tri.})v^4+(\chi^G_{(0022)}\chi^F_{(010)_a}+\text{tri.})v^8+(\chi^G_{(1100)}\chi^F_{(100)_b\otimes(100)_c}+\text{tri.})v^6\\
&+(\chi^G_{(2011)}\chi^F_{(100)_b\otimes(100)_c}+\text{tri.})v^8-(\chi^G_{(1000)}\chi^F_{(100)_a}+\text{tri.})v^{-1}-(\chi^G_{(0111)}\chi^F_{(100)_a}+\text{tri.})v^7\\
&-(\chi^G_{(1022)}\chi^F_{(100)_a}+\text{tri.})v^9+\chi^G_{(0100)}v^{-2}+\chi^G_{(0100)}+\chi^F_{(200)_a\oplus(200)_b\oplus(200)_c}+1\\
&+\chi^G_{(0200)}v^6+\chi^G_{(1111)}v^8+\chi^G_{(2022)}v^{10}+\\
&+\sum_{n=0}^{\infty}\Big[\chi^G_{(0n00)}\chi^F_{(001)_a\otimes(001)_b\otimes(001)_c}v^{5+2n}-(\chi^G_{(1n00)}\chi^F_{(010)_a\otimes(001)_b\otimes(001)_c}+\text{tri.})v^{6+2n}\\
&\qquad+(\chi^G_{(1n10)}\chi^F_{(010)_a\otimes(010)_b\otimes(001)_c}+\text{tri.})v^{7+2n}-\chi^G_{(1n11)}\chi^F_{(010)_a\otimes(010)_b\otimes(010)_c}v^{8+2n}\\
&\qquad-(\chi^G_{(2n10)}\chi^F_{(100)_a\otimes(010)_b\otimes(001)_c}+\text{tri.})v^{8+2n}-(\chi^G_{(3n00)}\chi^F_{(001)_b\otimes(001)_c}+\text{tri.})v^{8+2n}\\
&\qquad+(\chi^G_{(2n11)}\chi^F_{(100)_a\otimes(010)_b\otimes(010)_c}+\text{tri.})v^{9+2n}+(\chi^G_{(3n10)}\chi^F_{(010)_b\otimes(001)_c}+\text{tri.})v^{9+2n}\\
&\qquad+(\chi^G_{(2n20)}\chi^F_{(100)_a\otimes(100)_b\otimes(001)_c}+\text{tri.})v^{9+2n}-(\chi^G_{(3n20)}\chi^F_{(100)_b\otimes(001)_c}+\text{tri.})v^{10+2n}\\
&\qquad-(\chi^G_{(3n11)}\chi^F_{(010)_b\otimes(010)_c}+\text{tri.})v^{10+2n}-(\chi^G_{(1n22)}\chi^F_{(010)_a\otimes(100)_b\otimes(100)_c}+\text{tri.})v^{10+2n}\\
&\qquad+\chi^G_{(2n22)}\chi^F_{(100)_a\otimes(100)_b\otimes(100)_c}v^{11+2n}+(\chi^G_{(3n21)}\chi^F_{(100)_b\otimes(010)_c}+\text{tri.})v^{11+2n}\\
&\qquad+(\chi^G_{(0n33)}\chi^F_{(001)_a}+\text{tri.})v^{11+2n}-(\chi^G_{(1n33)}\chi^F_{(010)_a}+\text{tri.})v^{12+2n}\\
&\qquad-(\chi^G_{(3n22)}\chi^F_{(100)_b\otimes(100)_c}+\text{tri.})v^{12+2n}+(\chi^G_{(2n33)}\chi^F_{(100)_a}+\text{tri.})v^{13+2n}-\chi^G_{(3n33)}v^{14+2n}\Big].
\end{aligned}
\tag{E.4}
$$

Here "tri." means the two or five more terms implied by triality of both $\mathfrak{so}(8)$ and the three $\mathfrak{sp}(3)$ flavor groups together. We represent the v expansion terms both inside and outside the infinite summation in a descending order of the flavor representations. By Weyl dimension formulas of $\mathfrak{so}(8)$ and $\mathfrak{sp}(3)$, the above exact formula goes back to the rational function of v after turning off the gauge and flavor fugacities.

$n=1$, $G=\mathfrak{so}(9)$, $F=\mathfrak{sp}(4)_a\times\mathfrak{sp}(3)_b$. Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1)\times\mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(9)}}(q,v,m_{\mathfrak{so}(9)}=0,m_F=0)=q^{-1/3}+q^{2/3}v^{-2}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{12}},\tag{E.5}
$$

where

$$
\begin{aligned}
P_0(v)=2(&18+132v+227v^2-936v^3-5226v^4-7904v^5+17037v^6\\
&+118788v^7+263632v^8+118788v^9+\cdots+18v^{16}).
\end{aligned}
$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$\begin{aligned}
&\chi^{\mathfrak{so}(7)}_{(0100)}\chi^{F}_{(1000)_a}v^{-2}-(\chi^{\mathfrak{so}(7)}_{(1000)}\chi^{F}_{(1000)_a}+\chi^{\mathfrak{so}(7)}_{(0001)}\chi^{F}_{(002)_b})v^{-1}+\chi^{\mathfrak{so}(7)}_{(0100)}+\chi^{F}_{(2000)_a}+\chi^{F}_{(200)_b}+1\\
&+\chi^{F}_{(1000)_a\otimes(200)_b}v+(\chi^{F}_{(020)_b}+\chi^{F}_{(0100)_a\otimes(010)_b})v^2\\
&+(\chi^{F}_{(1000)_a\otimes(101)_b}+\chi^{F}_{(0010)_a\otimes(010)_b}+\chi^{\mathfrak{so}(7)}_{(0001)}\chi^{F}_{(001)_b})v^3+\ldots\\
&+\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{so}(7)}_{(0n00)}\chi^{F}_{(0001)_a\otimes(002)_b}v^{6+2n}-(\chi^{\mathfrak{so}(7)}_{(0n01)}\chi^{F}_{(0001)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(1n00)}\chi^{F}_{(0010)_a\otimes(002)_b})v^{7+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(0n02)}\chi^{F}_{(0001)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(0n10)}\chi^{F}_{(0001)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(1n01)}\chi^{F}_{(0010)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(2n00)}\chi^{F}_{(0100)_a\otimes(002)_b})v^{8+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(0n03)}\chi^{F}_{(0001)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(0n11)}\chi^{F}_{(0001)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(1n02)}\chi^{F}_{(0010)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(1n10)}\chi^{F}_{(0010)_a\otimes(020)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(2n01)}\chi^{F}_{(0100)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(3n00)}\chi^{F}_{(1000)_a\otimes(002)_b})v^{9+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(0n20)}\chi^{F}_{(0001)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(0n12)}\chi^{F}_{(0001)_a\otimes(010)_b}+\chi^{\mathfrak{so}(7)}_{(1n03)}\chi^{F}_{(0010)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(1n11)}\chi^{F}_{(0010)_a\otimes(110)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(2n02)}\chi^{F}_{(0100)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(2n10)}\chi^{F}_{(0100)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(3n01)}\chi^{F}_{(1000)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(4n00)}\chi^{F}_{(002)_b})v^{10+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(0n21)}\chi^{F}_{(0001)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(1n20)}\chi^{F}_{(0010)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(1n12)}\chi^{F}_{(0010)_a\otimes(010)_b}+\chi^{\mathfrak{so}(7)}_{(2n03)}\chi^{F}_{(0100)_a\otimes(001)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(2n11)}\chi^{F}_{(0100)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(3n02)}\chi^{F}_{(1000)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(3n10)}\chi^{F}_{(1000)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(4n01)}\chi^{F}_{(011)_b})v^{11+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(0n30)}\chi^{F}_{(0001)_a}+\chi^{\mathfrak{so}(7)}_{(1n21)}\chi^{F}_{(0010)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(2n20)}\chi^{F}_{(0100)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(2n12)}\chi^{F}_{(0100)_a\otimes(010)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(3n03)}\chi^{F}_{(1000)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(3n11)}\chi^{F}_{(1000)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(4n02)}\chi^{F}_{(101)_b}+\chi^{\mathfrak{so}(7)}_{(4n10)}\chi^{F}_{(020)_b})v^{12+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(1n30)}\chi^{F}_{(0010)_a}+\chi^{\mathfrak{so}(7)}_{(2n21)}\chi^{F}_{(0100)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(3n20)}\chi^{F}_{(1000)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(3n12)}\chi^{F}_{(1000)_a\otimes(010)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(4n03)}\chi^{F}_{(001)_b}+\chi^{\mathfrak{so}(7)}_{(4n11)}\chi^{F}_{(110)_b})v^{13+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(2n30)}\chi^{F}_{(0100)_a}+\chi^{\mathfrak{so}(7)}_{(3n21)}\chi^{F}_{(1000)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(4n20)}\chi^{F}_{(200)_b}+\chi^{\mathfrak{so}(7)}_{(4n12)}\chi^{F}_{(010)_b})v^{14+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(3n30)}\chi^{F}_{(1000)_a}+\chi^{\mathfrak{so}(7)}_{(4n21)}\chi^{F}_{(100)_b})v^{15+2n}+\chi^{\mathfrak{so}(7)}_{(4n30)}v^{16+2n}\Big].
\end{aligned}\tag{E.6}$$

The sporadic terms outside the infinite summations are too long to present, thus here we only present those in a few leading orders.

$n=1$, $G=F_4$, $F=\mathfrak{sp}(4)$. Using v expansion method, we turn on all flavor $\mathfrak{sp}(4)$ fugacities to compute the reduced one-string elliptic genus. The 5d one-instanton Nekrasov partition function is contained in the subleading q order, for which we find the following exact formula

$$\begin{aligned}
&\chi^{\mathfrak{sp}(4)}_{(0030)}v^7+\chi^{\mathfrak{sp}(4)}_{(0201)}v^6-\chi^{F_4}_{(0001)}\chi^{\mathfrak{sp}(4)}_{(0120)}v^8+\chi^{F_4}_{(0002)}\chi^{\mathfrak{sp}(4)}_{(1020)}v^9+\chi^{\mathfrak{sp}(4)}_{(1101)}(v^5-\chi^{F_4}_{(0001)}v^7)\\
&+\chi^{F_4}_{(0010)}\chi^{\mathfrak{sp}(4)}_{(0210)}v^9+\chi^{\mathfrak{sp}(4)}_{(0101)}(-\chi^{F_4}_{(0001)}v^6+\chi^{F_4}_{(0002)}v^8)+\chi^{\mathfrak{sp}(4)}_{(0020)}(v^4-\chi^{F_4}_{(0003)}v^{10})\\
&+\chi^{\mathfrak{sp}(4)}_{(2001)}(v^4+\chi^{F_4}_{(0010)}v^8)-\chi^{F_4}_{(0011)}\chi^{\mathfrak{sp}(4)}_{(1110)}v^{10}-\chi^{\mathfrak{sp}(4)}_{(0300)}(\chi^{F_4}_{(1000)}v^8+\chi^{F_4}_{(0100)}v^{10})\\
&-\chi^{F_4}_{(0011)}\chi^{\mathfrak{sp}(4)}_{(1001)}v^9+\chi^{\mathfrak{sp}(4)}_{(0110)}(v^3-\chi^{F_4}_{(0012)}v^{11})+\chi^{\mathfrak{sp}(4)}_{(2010)}(-\chi^{F_4}_{(1000)}v^7+\chi^{F_4}_{(0020)}v^{11})\\
&+\chi^{\mathfrak{sp}(4)}_{(1200)}(\chi^{F_4}_{(1001)}v^9+\chi^{F_4}_{(0101)}v^{11})+\chi^{\mathfrak{sp}(4)}_{(0001)}(\chi^{F_4}_{(0001)}v^4+\chi^{F_4}_{(0100)}v^8+\chi^{F_4}_{(0020)}v^{10})\\
&-\chi^{\mathfrak{sp}(4)}_{(1010)}(\chi^{F_4}_{(1000)}v^6-\chi^{F_4}_{(1001)}v^8+\chi^{F_4}_{(0021)}v^{12})+\chi^{\mathfrak{sp}(4)}_{(0200)}(v^2-\chi^{F_4}_{(1002)}v^{10}-\chi^{F_4}_{(0102)}v^{12})
\end{aligned}$$

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$$
\begin{aligned}
&-\chi^{\mathfrak{sp}(4)}_{(2100)}(\chi^{F_4}_{(1010)}v^{10}+\chi^{F_4}_{(0110)}v^{12})-\chi^{\mathfrak{sp}(4)}_{(0010)}(\chi^{F_4}_{(1000)}v^{5}+\chi^{F_4}_{(1010)}v^{9}-\chi^{F_4}_{(0030)}v^{13})\\
&+\chi^{\mathfrak{sp}(4)}_{(1100)}(\chi^{F_4}_{(1011)}v^{11}+\chi^{F_4}_{(0111)}v^{13})+\chi^{\mathfrak{sp}(4)}_{(3000)}(v+\chi^{F_4}_{(2000)}v^{9}+\chi^{F_4}_{(1100)}v^{11}+\chi^{F_4}_{(0200)}v^{13})\\
&+\chi^{\mathfrak{sp}(4)}_{(0100)}(\chi^{F_4}_{(2000)}v^{8}-\chi^{F_4}_{(1020)}v^{12}-\chi^{F_4}_{(0120)}v^{14})+\chi^{\mathfrak{sp}(4)}_{(2000)}(1-\chi^{F_4}_{(2001)}v^{10}-\chi^{F_4}_{(1101)}v^{12}-\chi^{F_4}_{(0201)}v^{14})\\
&+\chi^{\mathfrak{sp}(4)}_{(1000)}(-\chi^{F_4}_{(0001)}v^{-1}+\chi^{F_4}_{(2010)}v^{11}+\chi^{F_4}_{(1110)}v^{13}+\chi^{F_4}_{(0210)}v^{15})\\
&+(\chi^{F_4}_{(1000)}v^{-2}+\chi^{F_4}_{(1000)}+1-\chi^{F_4}_{(3000)}v^{10}-\chi^{F_4}_{(2100)}v^{12}-\chi^{F_4}_{(1200)}v^{14}-\chi^{F_4}_{(0300)}v^{16})\\
&+\sum_{n=0}^{\infty}\Big[\chi^{F_4}_{(n000)}\chi^{\mathfrak{sp}(4)}_{(0003)}v^{8+2n}-\chi^{F_4}_{(n001)}\chi^{\mathfrak{sp}(4)}_{(0012)}v^{9+2n}+(\chi^{F_4}_{(n010)}\chi^{\mathfrak{sp}(4)}_{(0021)}+\chi^{F_4}_{(n002)}\chi^{\mathfrak{sp}(4)}_{(0102)})v^{10+2n}\\
&\qquad-(\chi^{F_4}_{(n100)}\chi^{\mathfrak{sp}(4)}_{(0030)}+\chi^{F_4}_{(n011)}\chi^{\mathfrak{sp}(4)}_{(0111)}+\chi^{F_4}_{(n003)}\chi^{\mathfrak{sp}(4)}_{(1002)})v^{11+2n}\\
&\qquad+(\chi^{F_4}_{(n012)}\chi^{\mathfrak{sp}(4)}_{(1011)}+\chi^{F_4}_{(n020)}\chi^{\mathfrak{sp}(4)}_{(0201)}+\chi^{F_4}_{(n101)}\chi^{\mathfrak{sp}(4)}_{(0120)}+\chi^{F_4}_{(n004)}\chi^{\mathfrak{sp}(4)}_{(0002)})v^{12+2n}\\
&\qquad-(\chi^{F_4}_{(n102)}\chi^{\mathfrak{sp}(4)}_{(1020)}+\chi^{F_4}_{(n013)}\chi^{\mathfrak{sp}(4)}_{(0011)}+\chi^{F_4}_{(n021)}\chi^{\mathfrak{sp}(4)}_{(1101)}+\chi^{F_4}_{(n110)}\chi^{\mathfrak{sp}(4)}_{(0210)})v^{13+2n}\\
&\qquad+(\chi^{F_4}_{(n022)}\chi^{\mathfrak{sp}(4)}_{(0101)}+\chi^{F_4}_{(n103)}\chi^{\mathfrak{sp}(4)}_{(0020)}+\chi^{F_4}_{(n030)}\chi^{\mathfrak{sp}(4)}_{(2001)}+\chi^{F_4}_{(n111)}\chi^{\mathfrak{sp}(4)}_{(1110)}+\chi^{F_4}_{(n200)}\chi^{\mathfrak{sp}(4)}_{(0300)})v^{14+2n}\\
&\qquad-(\chi^{F_4}_{(n031)}\chi^{\mathfrak{sp}(4)}_{(1001)}+\chi^{F_4}_{(n112)}\chi^{\mathfrak{sp}(4)}_{(0110)}+\chi^{F_4}_{(n120)}\chi^{\mathfrak{sp}(4)}_{(2010)}+\chi^{F_4}_{(n201)}\chi^{\mathfrak{sp}(4)}_{(1200)})v^{15+2n}\\
&\qquad+(\chi^{F_4}_{(n040)}\chi^{\mathfrak{sp}(4)}_{(0001)}+\chi^{F_4}_{(n121)}\chi^{\mathfrak{sp}(4)}_{(1010)}+\chi^{F_4}_{(n202)}\chi^{\mathfrak{sp}(4)}_{(0200)}+\chi^{F_4}_{(n210)}\chi^{\mathfrak{sp}(4)}_{(2100)})v^{16+2n}\\
&\qquad-(\chi^{F_4}_{(n130)}\chi^{\mathfrak{sp}(4)}_{(0010)}+\chi^{F_4}_{(n211)}\chi^{\mathfrak{sp}(4)}_{(1100)}+\chi^{F_4}_{(n300)}\chi^{\mathfrak{sp}(4)}_{(3000)})v^{17+2n}\\
&\qquad+(\chi^{F_4}_{(n220)}\chi^{\mathfrak{sp}(4)}_{(0100)}+\chi^{F_4}_{(n301)}\chi^{\mathfrak{sp}(4)}_{(2000)})v^{18+2n}-\chi^{F_4}_{(n310)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{19+2n}+\chi^{F_4}_{(n400)}v^{20+2n}\Big].
\end{aligned}
\tag{E.7}
$$

Turning off all F_4 and $\mathfrak{sp}(4)$ fugacities, the above exact formula reduces to the rational function of v in (5.77) by Weyl dimension formulas.

$n = 2$, $G = \mathfrak{so}(9)$, $F = \mathfrak{sp}(3)_a \times \mathfrak{sp}(2)_b$. Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(9)}}(q,v,m_{\mathfrak{so}(9)}=0,m_F=0)=q^{1/6}v^{-1}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{12}},
\tag{E.8}
$$

where

$$
P_0(v)=(1-v)^2(1+14v+93v^2+392v^3+1181v^4+2658v^5+4106v^6+2658v^7+\cdots+v^{12}).
$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&-\chi^{F}_{(001)_a}v^{4}-\chi^{F}_{(010)_a}(\chi^{F}_{(20)_b}v^{5}-\chi^{\mathfrak{so}(9)}_{(0001)}\chi^{F}_{(10)_b}v^{6}+\chi^{\mathfrak{so}(9)}_{(0010)}v^{7})+\chi^{F}_{(100)_a}(-\chi^{F}_{(01)_b}v^{4}\\
&+\chi^{\mathfrak{so}(9)}_{(1000)}\chi^{F}_{(20)_b}v^{6}-\chi^{\mathfrak{so}(9)}_{(1001)}\chi^{F}_{(10)_b}v^{7}+\chi^{\mathfrak{so}(9)}_{(1010)}v^{8}+\chi^{\mathfrak{so}(9)}_{(0100)}v^{6})+\chi^{F}_{(01)_b}(-v^{3}+\chi^{\mathfrak{so}(9)}_{(1000)}v^{5})\\
&-\chi^{\mathfrak{so}(9)}_{(2000)}\chi^{F}_{(20)_b}v^{7}+\chi^{\mathfrak{so}(9)}_{(2001)}\chi^{F}_{(10)_b}v^{8}+v^{-1}-\chi^{\mathfrak{so}(9)}_{(1100)}v^{7}-\chi^{\mathfrak{so}(9)}_{(2010)}v^{9}\\
&+\sum_{n=0}^{\infty}\Big[\chi^{F}_{(001)_a}\Big(-\chi^{\mathfrak{so}(9)}_{(0n00)}\chi^{F}_{(02)_b}v^{6+2n}+\chi^{\mathfrak{so}(9)}_{(0n01)}\chi^{F}_{(11)_b}v^{7+2n}-(\chi^{\mathfrak{so}(9)}_{(0n02)}\chi^{F}_{(01)_b}+\chi^{\mathfrak{so}(9)}_{(0n10)}\chi^{F}_{(20)_b})v^{8+2n}\\
&\qquad+\chi^{\mathfrak{so}(9)}_{(0n11)}\chi^{F}_{(10)_b}v^{9+2n}-\chi^{\mathfrak{so}(9)}_{(0n20)}v^{10+2n}\Big)+\chi^{F}_{(010)_a}\Big(\chi^{\mathfrak{so}(9)}_{(1n00)}\chi^{F}_{(02)_b}v^{7+2n}-\chi^{\mathfrak{so}(9)}_{(1n01)}\chi^{F}_{(11)_b}v^{8+2n}
\end{aligned}
$$

$$
\begin{aligned}
&+(\chi^{\mathfrak{so}(9)}_{(1n02)}\chi^{F}_{(01)_b}+\chi^{\mathfrak{so}(9)}_{(1n10)}\chi^{F}_{(20)_b})v^{9+2n}-\chi^{\mathfrak{so}(9)}_{(1n11)}\chi^{F}_{(10)_b}v^{10+2n}+\chi^{\mathfrak{so}(9)}_{(1n20)}v^{11+2n}\Big)\\
&+\chi^{F}_{(100)_a}\Big(-\chi^{\mathfrak{so}(9)}_{(2n00)}\chi^{F}_{(02)_b}v^{8+2n}+\chi^{\mathfrak{so}(9)}_{(2n01)}\chi^{F}_{(11)_b}v^{9+2n}-(\chi^{\mathfrak{so}(9)}_{(2n02)}\chi^{F}_{(01)_b}+\chi^{\mathfrak{so}(9)}_{(2n10)}\chi^{F}_{(20)_b})v^{10+2n}\\
&+\chi^{\mathfrak{so}(9)}_{(2n11)}\chi^{F}_{(10)_b}v^{11+2n}-\chi^{\mathfrak{so}(9)}_{(2n20)}v^{12+2n}\Big)+\Big(\chi^{\mathfrak{so}(9)}_{(3n00)}\chi^{F}_{(02)_b}v^{9+2n}-\chi^{\mathfrak{so}(9)}_{(3n01)}\chi^{F}_{(11)_b}v^{10+2n}\\
&+(\chi^{\mathfrak{so}(9)}_{(3n02)}\chi^{F}_{(01)_b}+\chi^{\mathfrak{so}(9)}_{(3n10)}\chi^{F}_{(20)_b})v^{11+2n}-\chi^{\mathfrak{so}(9)}_{(3n11)}\chi^{F}_{(10)_b}v^{12+2n}+\chi^{\mathfrak{so}(9)}_{(3n20)}v^{13+2n}\Big)\Big]. \qquad \text{(E.9)}
\end{aligned}
$$

A few leading terms in the v expansion has been determined in (H.20) of [1].

$n = 2$, $G = \mathfrak{so}(10)$, $F = \mathfrak{sp}(4)_a \times \mathfrak{su}(2)_b \times \mathfrak{u}(1)_c$. Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(10)}}(q,v,m_{\mathfrak{so}(10)}=0,m_F=0)=q^{1/6}v^{-1}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{14}}, \qquad \text{(E.10)}
$$

where

$$
\begin{aligned}
P_0(v)=&(1-v)^2(1+16v+122v^2+592v^3+2060v^4+5472v^5+11287v^6+16496v^7\\
&+11287v^8+5472v^9+2060v^{10}+592v^{11}+122v^{12}+16v^{13}+v^{14}).
\end{aligned}
$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&v^{-1}-\chi^{F}_{(2)_b}v^3-\chi^{F}_{(1000)_a\otimes((2)_c\oplus(-2)_c)}v^4+(\chi^{\mathfrak{so}(10)}_{(10000)}\chi^{F}_{(2)_c\oplus(-2)_c}-\chi^{F}_{(0100)_a\otimes(2)_b}-\chi^{F}_{(0001)_a})v^5\\
&+(\chi^{F}_{(0010)_a\otimes((2)_c\oplus(-2)_c)}-\chi^{\mathfrak{so}(10)}_{(10000)}\chi^{F}_{(1000)_a})\chi^{F}_{(2)_b}v^6+\dots\\
&+\sum_{n=0}^{\infty}\Big[(\chi^{\mathfrak{u}(1)}_{(-4)\oplus(4)}+\chi^{\mathfrak{su}(2)}_{(4)})\Big(-\chi^{\mathfrak{so}(10)}_{(0n000)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{7+2n}+\chi^{\mathfrak{so}(10)}_{(1n000)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{8+2n}-\chi^{\mathfrak{so}(10)}_{(2n000)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{9+2n}\\
&\qquad+\chi^{\mathfrak{so}(10)}_{(3n000)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{10+2n}-\chi^{\mathfrak{so}(10)}_{(4n000)}v^{11+2n}\Big)+\Big((\chi^{\mathfrak{u}(1)}_{(-3)}\chi^{\mathfrak{su}(2)}_{(1)}+\chi^{\mathfrak{u}(1)}_{(1)}\chi^{\mathfrak{su}(2)}_{(3)})(\chi^{\mathfrak{so}(10)}_{(0n001)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{8+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(1n001)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{9+2n}+\chi^{\mathfrak{so}(10)}_{(2n001)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{10+2n}-\chi^{\mathfrak{so}(10)}_{(3n001)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(4n001)}v^{12+2n})\\
&\qquad+c.c.\Big)+\chi^{\mathfrak{u}(1)}_{(-2)\oplus(2)}\chi^{\mathfrak{su}(2)}_{(2)}\Big(-\chi^{\mathfrak{so}(10)}_{(0n100)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}+\chi^{\mathfrak{so}(10)}_{(1n100)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(2n100)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(3n100)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}-\chi^{\mathfrak{so}(10)}_{(4n100)}v^{13+2n}\Big)\\
&\qquad-\Big(\chi^{\mathfrak{u}(1)}_{(2)}(\chi^{\mathfrak{so}(10)}_{(0n020)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}-\chi^{\mathfrak{so}(10)}_{(1n020)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}+\chi^{\mathfrak{so}(10)}_{(2n020)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(3n020)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}+\chi^{\mathfrak{so}(10)}_{(4n020)}v^{13+2n})+c.c.\Big)+\Big(\chi^{\mathfrak{u}(1)}_{(-1)}\chi^{\mathfrak{su}(2)}_{(1)}(\chi^{\mathfrak{so}(10)}_{(0n101)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{10+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(1n101)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(2n101)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{12+2n}-\chi^{\mathfrak{so}(10)}_{(3n101)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{13+2n}+\chi^{\mathfrak{so}(10)}_{(4n101)}v^{14+2n})\\
&\qquad+c.c.\Big)-\chi^{\mathfrak{su}(2)}_{(2)}(\chi^{\mathfrak{so}(10)}_{(0n011)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}-\chi^{\mathfrak{so}(10)}_{(1n011)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}+\chi^{\mathfrak{so}(10)}_{(2n011)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(3n011)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}+\chi^{\mathfrak{so}(10)}_{(4n011)}v^{13+2n})+\Big(-\chi^{\mathfrak{so}(10)}_{(0n200)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(1n200)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{12+2n}\\
&\qquad-\chi^{\mathfrak{so}(10)}_{(2n200)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{13+2n}+\chi^{\mathfrak{so}(10)}_{(3n200)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{14+2n}-\chi^{\mathfrak{so}(10)}_{(4n200)}v^{15+2n}\Big)\Big]. \qquad \text{(E.11)}
\end{aligned}
$$

The sporadic terms outside the infinite summation are too long to present, thus here we only show some in leading orders. In general they can be recovered from the terms inside the infinite summation. Note the complex conjugate *c.c.* interchanges the Dynkin labels of spinor and conjugate spinor representations of gauge $\mathfrak{so}(10)$ and reverses the charge of $\mathfrak{u}(1)$ flavor simultaneously. We also checked this expression from 5d blowup equations. A few leading terms in the v expansion has been determined in (H.21) of [1].

$\boldsymbol{n=2,\ G=\mathfrak{so}(11),\ F=\mathfrak{sp}(5)_a\times\mathfrak{so}(2)_b}$**.** There are 128 unity blowup equations in total. Let us regard the flavor subgroup $\mathfrak{so}(2)$ as $\mathfrak{u}(1)$. The r fields $\lambda_{\mathfrak{sp}(5)}$ takes value in $\mathcal{O}^{\mathfrak{sp}(5)}_{[00001]}$, while $\lambda_{\mathfrak{u}(1)}=\pm 1/2$. Using the Weyl orbit expansion method, we turn on a subgroup $\mathfrak{sp}(1)\times\mathfrak{u}(1)$ of the flavor and compute the one-string elliptic genus to $\mathcal{O}(q^2)$. For example, with gauge and flavor fugacities turned off we obtain the reduced one-string elliptic genus as[48]

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(11)}}(q,v,m_{\mathfrak{so}(11)}=0,m_F=0)=q^{1/6}\sum_{n=0}^{\infty}q^n\frac{(1-v)^2P_n(v)}{v(1+v)^{16}}, \tag{E.13}$$

where

$$P_0(v)=1+18v+155v^2+852v^3+3367v^4+10208v^5+24624v^6+47390v^7+66362v^8+\cdots+v^{16},$$

and

$$\begin{aligned}P_1(v)=v^{-2}(&55+816v+5505v^2+21936v^3+55038v^4+79650v^5+18864v^6-193544v^7\\&-427293v^8-245690v^9+410958v^{10}-\cdots+55v^{20}).\end{aligned}$$

If we turn on all gauge and flavor fugacities, we find the leading q order of reduced one-string elliptic genus is

$$\begin{aligned}&v^{-1}-\chi^F_{(-2)_b\oplus(2)_b}v^3-\chi^F_{(10000)_a}v^4+(\mathbf{11}-\chi^F_{(01000)_a})v^5\\&+(\mathbf{11}\cdot\chi^F_{(10000)_a}-\chi^F_{(00100)_a\otimes((-2)_b\oplus(2)_b)}-\chi^F_{(00001)_a})v^6\\&+\Big((\mathbf{11}\cdot\chi^F_{(01000)_a}-\chi^F_{(00010)_a})\chi^F_{(-2)_b\oplus(2)_b}-\chi^F_{(00010)_a}-\mathbf{65}\Big)v^7\\&+\Big(-\chi^F_{(00001)_a\otimes((-4)_b\oplus(4)_b)}+(\mathbf{11}\cdot\chi^F_{(00100)_a}-\mathbf{65}\cdot\chi^F_{(10000)_a})\chi^F_{(-2)_b\oplus(2)_b}\\&\qquad+\mathbf{32}\cdot\chi^F_{(00010)_a\otimes((-1)_b\oplus(1)_b)}-\chi^F_{(00001)_a}+(\mathbf{11}+\mathbf{55})\chi^F_{(00100)_a}\Big)v^8+\mathcal{O}(v^9)\end{aligned} \tag{E.14}$$

[48] In [1], the modular ansatz for the reduced one-string elliptic genus of this theory is determined up to two unfixed parameters. Using our result from blowup equations, we are able to determine their two unfixed parameters as

$$a_1=\frac{16291}{1283918464548864},\ a_2=\frac{9983}{7703510787293184}. \tag{E.12}$$

In fact, we find the following exact formula:

$$
\begin{aligned}
&\chi^F_{(-2)_b\oplus(2)_b}\Big(-v^3-\chi^F_{(00100)_a}v^6+(\chi^G_{(10000)}\chi^F_{(01000)_a}-\chi^F_{(00010)_a})v^7\\
&\qquad\qquad+(\chi^G_{(10000)}\chi^F_{(00100)_a}-\chi^G_{(20000)}\chi^F_{(10000)_a})v^8+(\chi^G_{(30000)}-\chi^G_{(20000)}\chi^F_{(01000)_a})v^9\\
&\qquad\qquad+\chi^G_{(30000)}\chi^F_{(10000)_a}v^{10}-\chi^G_{(40000)}v^{11}\Big)\\
&\chi^F_{(-1)_b\oplus(1)_b}\Big(\chi^G_{(00001)}\chi^F_{(00010)_a}v^8-\chi^G_{(10001)}\chi^F_{(00100)_a}v^9+\chi^G_{(20001)}\chi^F_{(01000)_a}v^{10}\\
&\qquad\qquad-\chi^G_{(30001)}\chi^F_{(10000)_a}v^{11}+\chi^G_{(40001)}v^{12}\Big)\\
&+\Big(v^{-1}-\chi^F_{(10000)_a}v^4+(\chi^G_{(10000)}-\chi^F_{(01000)_a})v^5+(\chi^G_{(10000)}\cdot\chi^F_{(10000)_a}-\chi^F_{(00001)_a})v^6\\
&\quad-(\chi^G_{(20000)}+\chi^F_{(00010)_a})v^7+\chi^G_{(10000)\oplus(01000)}\chi^F_{(00100)_a}v^8\\
&\quad-(\chi^G_{(00100)}\chi^F_{(00010)_a}+\chi^G_{(20000)\oplus(11000)}\chi^F_{(01000)_a})v^9\\
&\quad+(\chi^G_{(10100)}\chi^F_{(00100)_a}+\chi^G_{(30000)\oplus(21000)}\chi^F_{(10000)_a})v^{10}\\
&\quad-(\chi^G_{(20100)}\chi^F_{(01000)_a}+\chi^G_{(40000)\oplus(31000)})v^{11}+\chi^G_{(30100)}\chi^F_{(10000)_a}v^{12}-\chi^G_{(40100)}v^{13}\Big)\\
&+\sum_{n=0}^{\infty}\Big[\chi^F_{(-4)_b\oplus(4)_b}\Big(-v^{8+2n}\chi^G_{(0n000)}\chi^F_{(00001)_a}+v^{9+2n}\chi^G_{(1n000)}\chi^F_{(00010)_a}-v^{10+2n}\chi^G_{(2n000)}\chi^F_{(00100)_a}\\
&+v^{11+2n}\chi^G_{(3n000)}\chi^F_{(01000)_a}-v^{12+2n}\chi^G_{(4n000)}\chi^F_{(10000)_a}+v^{13+2n}\chi^G_{(5n000)}\Big)\\
&+\chi^F_{(-3)_b\oplus(3)_b}\Big(v^{9+2n}\chi^G_{(0n001)}\chi^F_{(00001)_a}-v^{10+2n}\chi^G_{(1n001)}\chi^F_{(00010)_a}+v^{11+2n}\chi^G_{(2n001)}\chi^F_{(00100)_a}\\
&-v^{12+2n}\chi^G_{(3n001)}\chi^F_{(01000)_a}+v^{13+2n}\chi^G_{(4n001)}\chi^F_{(10000)_a}-v^{14+2n}\chi^G_{(5n001)}\Big)\\
&+\chi^F_{(-2)_b\oplus(2)_b}\Big(-v^{10+2n}(\chi^G_{(0n100)}+\chi^G_{(0n010)})\chi^F_{(00001)_a}+v^{11+2n}(\chi^G_{(1n100)}+\chi^G_{(1n010)})\chi^F_{(00010)_a}\\
&-v^{12+2n}(\chi^G_{(2n100)}+\chi^G_{(2n010)})\chi^F_{(00100)_a}+v^{13+2n}(\chi^G_{(3n100)}+\chi^G_{(3n010)})\chi^F_{(01000)_a}\\
&-v^{14+2n}(\chi^G_{(4n100)}+\chi^G_{(4n010)})\chi^F_{(10000)_a}+v^{15+2n}(\chi^G_{(5n100)}+\chi^G_{(5n010)})\Big)\\
&+\chi^F_{(-1)_b\oplus(1)_b}\Big(v^{9+2n}(\chi^G_{(0n001)}+v^2\chi^G_{(0n101)})\chi^F_{(00001)_a}-v^{10+2n}(\chi^G_{(1n001)}+v^2\chi^G_{(1n101)})\chi^F_{(00010)_a}\\
&+v^{11+2n}(\chi^G_{(2n001)}+v^2\chi^G_{(2n101)})\chi^F_{(00100)_a}-v^{12+2n}(\chi^G_{(3n001)}+v^2\chi^G_{(3n101)})\chi^F_{(01000)_a}\\
&+v^{13+2n}(\chi^G_{(4n001)}+v^2\chi^G_{(4n101)})\chi^F_{(10000)_a}-v^{14+2n}(\chi^G_{(5n001)}+v^2\chi^G_{(5n101)})\Big)\\
&+\Big(-v^{8+2n}(\chi^G_{(0n000)}+v^2(\chi^G_{(0n100)}+\chi^G_{(0n002)})+v^4\chi^G_{(0n200)})\chi^F_{(00001)_a}\\
&\quad+v^{9+2n}(\chi^G_{(1n000)}+v^2(\chi^G_{(1n100)}+\chi^G_{(1n002)})+v^4\chi^G_{(1n200)})\chi^F_{(00010)_a}\\
&\quad-v^{10+2n}(\chi^G_{(2n000)}+v^2(\chi^G_{(2n100)}+\chi^G_{(2n002)})+v^4\chi^G_{(2n200)})\chi^F_{(00100)_a}\\
&\quad+v^{11+2n}(\chi^G_{(3n000)}+v^2(\chi^G_{(3n100)}+\chi^G_{(3n002)})+v^4\chi^G_{(3n200)})\chi^F_{(01000)_a}\\
&\quad-v^{12+2n}(\chi^G_{(4n000)}+v^2(\chi^G_{(4n100)}+\chi^G_{(4n002)})+v^4\chi^G_{(4n200)})\chi^F_{(10000)_a}\\
&\quad+v^{13+2n}(\chi^G_{(5n000)}+v^2(\chi^G_{(5n100)}+\chi^G_{(5n002)})+v^4\chi^G_{(5n200)})\Big)\Big].
\end{aligned}
\tag{E.15}
$$

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i,j	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	-2	-10	-33	-242	408	18544	-102190
1	55	-174	112	30	91	174	-150	-686	-651	-33420	21765
2	1144	-7106	17037	-17196	2998	330	6602	15822	-16128	-16234	116549

Table 19. Series coefficients $c_{i,j}$ for the one-string elliptic genus of $n=2$ $\mathfrak{so}(11)$ model.

The subleading q order of reduced one-string elliptic genus is

$$\begin{aligned}&\mathbf{55}v^{-3}-(\mathbf{11}\cdot\chi^F_{(10000)_a}+\mathbf{32}\cdot\chi^F_{(-1)_b\oplus(1)_b})v^{-2}+(\mathbf{55}+\chi^F_{(20000)_a}+2)v^{-1}\\&+\chi^F_{(10000)_a\otimes((-1)_b\oplus(1)_b\oplus(0)_b)}+(\chi^F_{(01000)_a\otimes((-2)_b\oplus(2)_b)}+\chi^F_{(-4)_b\oplus(4)_b}+1)v\\&+(\chi^F_{(00100)_a}+\mathbf{32}\cdot\chi^F_{(-1)_b\oplus(1)_b})v^2+\mathcal{O}(v^3)\end{aligned}\tag{E.16}$$

Let us further denote

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(11)}}(q,v,m_{\mathfrak{so}(11)}=0,m_F=0)=q^{1/6}v^{-1}\sum_{i,j}c_{i,j}v^j(q/v^2)^i.\tag{E.17}$$

Then we have the following table 19 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of $\mathfrak{so}(11)$ where θ is the adjoint representation. The blue numbers in the second column are given by $-10\dim(\chi^{\mathfrak{so}(11)}_{[1n000]})-2\dim(\chi^{\mathfrak{so}(11)}_{[0n001]})$ with $n=i-1$, consistent with the fact that the matter is in representation $(\mathbf{11},\mathbf{10}^a)\oplus(\mathbf{32},\mathbf{2}^b)$. The orange number 112 in the third column is given by $\dim(\mathfrak{so}(11))+\dim(\mathfrak{sp}(5)\times\mathfrak{u}(1))+1=55+55+1+1=112$. These are the constraints given in [1] by analyzing the spectral flow to NSR elliptic genus, which our result satisfies perfectly.

$\boldsymbol{n=2}$**,** $\boldsymbol{G=\mathfrak{so}(12)_a}$**,** $\boldsymbol{F=\mathfrak{sp}(6)_a\times\mathfrak{so}(2)_b}$**.** This is a chiral theory in the sense that the spinor and conjugate spinor representations of $\mathfrak{so}(12)$ are not on an equal footing. The chirality comes from the matter representation $(\mathbf{32_s},\mathbf{2}_b)$. This is reflected in the vanishing r fields in table 6 and also the exact v expansion formula below (E.19). Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1)\times\mathfrak{u}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(12)}}(q,v,m_{\mathfrak{so}(12)}=0,m_F=0)=q^{1/6}v^{-1}\sum_{n=0}^{\infty}q^n\frac{P_n(v)}{(1+v)^{18}},\tag{E.18}$$

where

$$\begin{aligned}P_0(v)=&(1-v)^2(1+20v+192v^2+1180v^3+5226v^4+17804v^5+48575v^6+108512v^7\\&+197370v^8+267144v^9+197370v^{10}+\cdots+v^{18}).\end{aligned}$$

This agrees with the modular ansatz in [1]. Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$\begin{aligned}&v^{-1}-\chi^F_{(2)_b\oplus(-2)_b}v^3-\chi^F_{(010000)_a}v^5+\chi^{\mathfrak{so}(12)}_{(100000)}\chi^F_{(100000)_a}v^6-(\chi^G_{(200000)}\chi^F_{(01)_b}\\&+\chi^F_{(000100)_a\otimes((2)_b\oplus(-2)_b)}+\chi^F_{(000001)_a})v^7+\chi^{\mathfrak{so}(12)}_{(100000)}\chi^F_{(001000)_a\otimes((2)_b\oplus(-2)_b)}v^8\end{aligned}$$

$$
\begin{aligned}
&-(\chi^{\mathfrak{so}(12)}_{(200000)}\chi^{F}_{(010000)_a\otimes((2)_b\oplus(-2)_b)}-\chi^{\mathfrak{so}(12)}_{(000010)}\chi^{F}_{(000010)_a\otimes((1)_b\oplus(-1)_b)}-\chi^{\mathfrak{so}(12)}_{(010000)}\chi^{F}_{(000100)_a})v^9+\ldots\\
&+\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{u}(1)}_{(-4)\oplus(4)}\Big(-\chi^{\mathfrak{so}(12)}_{(0n0000)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{9+2n}+\chi^{\mathfrak{so}(12)}_{(1n0000)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{10+2n}-\chi^{\mathfrak{so}(12)}_{(2n0000)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{11+2n}\\
&\qquad+\chi^{\mathfrak{so}(12)}_{(3n0000)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(4n0000)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(5n0000)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{14+2n}\\
&\qquad-\chi^{\mathfrak{so}(12)}_{(6n0000)}v^{15+2n}\Big)+\chi^{\mathfrak{u}(1)}_{(-3)\oplus(3)}\Big(\chi^{\mathfrak{so}(12)}_{(0n0001)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{10+2n}-\chi^{\mathfrak{so}(12)}_{(1n0001)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{11+2n}\\
&\qquad+\chi^{\mathfrak{so}(12)}_{(2n0001)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(3n0001)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(4n0001)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{14+2n}\\
&\qquad-\chi^{\mathfrak{so}(12)}_{(5n0001)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{15+2n}+\chi^{\mathfrak{so}(12)}_{(6n0001)}v^{16+2n}\Big)-\chi^{\mathfrak{u}(1)}_{(-2)\oplus(2)}\Big(\chi^{\mathfrak{so}(12)}_{(0n0100)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{11+2n}\\
&\qquad-\chi^{\mathfrak{so}(12)}_{(1n0100)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{12+2n}+\chi^{\mathfrak{so}(12)}_{(2n0100)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{13+2n}-\chi^{\mathfrak{so}(12)}_{(3n0100)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{14+2n}\\
&\qquad+\chi^{\mathfrak{so}(12)}_{(4n0100)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{15+2n}-\chi^{\mathfrak{so}(12)}_{(5n0100)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{16+2n}+\chi^{\mathfrak{so}(12)}_{(6n0100)}v^{17+2n}\Big)\\
&\qquad+\chi^{\mathfrak{u}(1)}_{(-1)\oplus(1)}\Big(\chi^{\mathfrak{so}(12)}_{(0n1010)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(1n1010)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(2n1010)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{14+2n}\\
&\qquad-\chi^{\mathfrak{so}(12)}_{(3n1010)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{15+2n}+\chi^{\mathfrak{so}(12)}_{(4n1010)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{16+2n}-\chi^{\mathfrak{so}(12)}_{(5n1010)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{17+2n}\\
&\qquad+\chi^{\mathfrak{so}(12)}_{(6n1010)}v^{18+2n}\Big)-\Big((\chi^{\mathfrak{so}(12)}_{(0n0020)}+\chi^{\mathfrak{so}(12)}_{(0n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000001)}v^{11+2n}\\
&\qquad-(\chi^{\mathfrak{so}(12)}_{(1n0020)}+\chi^{\mathfrak{so}(12)}_{(1n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000010)}v^{12+2n}+(\chi^{\mathfrak{so}(12)}_{(2n0020)}+\chi^{\mathfrak{so}(12)}_{(2n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000100)}v^{13+2n}\\
&\qquad-(\chi^{\mathfrak{so}(12)}_{(3n0020)}+\chi^{\mathfrak{so}(12)}_{(3n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(001000)}v^{14+2n}+(\chi^{\mathfrak{so}(12)}_{(4n0020)}+\chi^{\mathfrak{so}(12)}_{(4n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(010000)}v^{15+2n}\\
&\qquad-(\chi^{\mathfrak{so}(12)}_{(5n0020)}+\chi^{\mathfrak{so}(12)}_{(5n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(100000)}v^{16+2n}+(\chi^{\mathfrak{so}(12)}_{(6n0020)}+\chi^{\mathfrak{so}(12)}_{(6n2000)}v^2)v^{17+2n}\Big)\Big].
\end{aligned}
\tag{E.19}
$$

The sporadic terms outside the infinite summation are too long to present, thus here we only show some in leading orders. In general they can be recovered from the terms inside the infinite summation. We also checked this expression from 5d blowup equations. A few leading terms in the v expansion has been determined in (H.22) of [1].

$n=2$, $G=E_7$, $F=\mathfrak{so}(6)$. Let us regard the flavor group as $\mathfrak{su}(4)$ to present the elliptic genus. We use both the v expansion method and the recursion formula from 5d blowup equations to compute the leading q order of the reduced one-string elliptic genus, and find the following exact formula:

$$
\begin{aligned}
&-\chi^{su(4)}_{(8,0,4)\oplus(4,0,8)}v^{15}-\chi^{su(4)}_{(7,0,5)\oplus(5,0,7)}\chi^{E_7}_{(n000001)}v^{18}+\chi^{su(4)}_{(6,0,6)}(\chi^{E_7}_{(1000000)}v^{17}+\chi^{E_7}_{(0100000)}v^{19})\\
&+\chi^{su(4)}_{(7,1,3)\oplus(3,1,7)}\chi^{E_7}_{(0000010)}v^{16}+\chi^{su(4)}_{(6,1,4)\oplus(4,1,6)}\chi^{E_7}_{(0000011)}v^{19}\\
&-\chi^{su(4)}_{(5,1,5)}(\chi^{E_7}_{(1000010)}v^{18}+\chi^{E_7}_{(0100010)}v^{20})+\ldots\\
&+\sum_{n=0}^{\infty}\Big[-\chi^{su(4)}_{(12,0,0)\oplus(0,0,12)}\chi^{E_7}_{(n000000)}v^{17+2n}+\chi^{su(4)}_{(11,0,1)\oplus(1,0,11)}\chi^{E_7}_{(n000010)}v^{18+2n}\\
&\qquad-\chi^{su(4)}_{(10,0,2)\oplus(2,0,10)}\chi^{E_7}_{(n000100)}v^{19+2n}+\chi^{su(4)}_{(9,0,3)\oplus(3,0,9)}\chi^{E_7}_{(n001000)}v^{20+2n}\\
&\qquad-\chi^{su(4)}_{(8,0,4)\oplus(4,0,8)}\chi^{E_7}_{(n010000)}v^{21+2n}+\chi^{su(4)}_{(7,0,5)\oplus(5,0,7)}\chi^{E_7}_{(n100001)}v^{22+2n}\\
&\qquad-\chi^{su(4)}_{(6,0,6)}(\chi^{E_7}_{(n000002)}v^{21+2n}+\chi^{E_7}_{(n200000)}v^{23+n})-\chi^{su(4)}_{(10,1,0)\oplus(0,1,10)}\chi^{E_7}_{(n000020)}v^{19+2n}\\
&\qquad+\chi^{su(4)}_{(9,1,1)\oplus(1,1,9)}\chi^{E_7}_{(n000110)}v^{20+2n}-\chi^{su(4)}_{(8,1,2)\oplus(2,1,8)}\chi^{E_7}_{(n001010)}v^{21+2n}\\
&\qquad+\chi^{su(4)}_{(7,1,3)\oplus(3,1,7)}\chi^{E_7}_{(n010010)}v^{22+2n}-\chi^{su(4)}_{(6,1,4)\oplus(4,1,6)}\chi^{E_7}_{(n100011)}v^{23+2n}\\
&\qquad+\chi^{su(4)}_{(5,1,5)}(\chi^{E_7}_{(n000012)}v^{22+2n}+\chi^{E_7}_{(n200010)}v^{24+n})+\ldots\Big].
\end{aligned}
\tag{E.20}
$$

The full dependence on flavor representations are too long to present. Here we only show the terms involving the largest representations of $\mathfrak{su}(4)$ with Dynkin label (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 12$ and $b_2 = 0, 1$.

$n = 3$, $G = \mathfrak{so}(8)$, $F = \mathfrak{sp}(1)_a \times \mathfrak{sp}(1)_b \times \mathfrak{sp}(1)_c$. Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(8)}}(q, v, m_{\mathfrak{so}(8)} = 0, m_F = 0) = q^{-1/3} v^4 \sum_{n=0}^{\infty} q^n \frac{P_n(v)}{(1-v)^4(1+v)^{10}}. \tag{E.21}$$

From the recursion formula from blowup equations, we obtain

$$\begin{aligned} P_0(v) &= 1+14v-37v^2+68v^3-37v^4+14v^5+v^6, \\ P_1(v) &= v^{-6}(1+6v+11v^2-4v^3-41v^4-50v^5+43v^6+564v^7-1310v^8+1752v^9-\cdots+v^{18}). \end{aligned} \tag{E.22}$$

These agree with the modular ansatz in [1]. With all gauge and flavor fugacities turned on, we reobtain the exact formula for the leading q order of the reduced one-string elliptic genus in [1] and [80] as

$$\begin{aligned} v^4+\sum_{n=0}^{\infty}\Big[&\chi^{\mathfrak{so}(8)}_{(0n00)}\chi^F_{(1)_a\otimes(1)_b\otimes(1)_c}v^{5+2n}-(\chi^{\mathfrak{so}(8)}_{(1n00)}\chi^F_{(1)_b\otimes(1)_c}+\chi^{\mathfrak{so}(8)}_{(0n10)}\chi^F_{(1)_a\otimes(1)_c}+\chi^{\mathfrak{so}(8)}_{(0n01)}\chi^F_{(1)_a\otimes(1)_b})v^{6+2n} \\ &+(\chi^{\mathfrak{so}(8)}_{(1n10)}\chi^F_{(1)_c}+\chi^G_{(1n01)}\chi^F_{(1)_b}+\chi^{\mathfrak{so}(8)}_{(0n11)}\chi^F_{(1)_a})v^{7+2n}-\chi^{\mathfrak{so}(8)}_{(1n11)}v^{8+2n}\Big]. \end{aligned} \tag{E.23}$$

We also obtain the subleading q order as

$$v^{-2}-2v^2+(\chi^F_{(2)_a\oplus(2)_b\oplus(2)_c}+1+\chi^{\mathfrak{so}(8)}_{(0100)})v^4+\chi^F_{(1)_a\otimes(1)_b\otimes(1)_c}(\chi^{\mathfrak{so}(8)}_{(0100)}+4)v^5+\mathcal{O}(v^6).$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(8)}}(q, v, x = 1, m_{\mathfrak{so}(8)} = 0, m_F = 0) = -q^{-5/6} v^9 \sum_{n=0}^{\infty} q^n \frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{10}(1+v+v^2)^{11}}. \tag{E.24}$$

We obtain

$$\begin{aligned} P^{(2)}_0(v) =& 1+19v+94v^2+77v^3+31v^4+592v^5+1681v^6+1395v^7+942v^8+3775v^9 \\ &+7249v^{10}+5434v^{11}+3008v^{12}+\cdots+v^{24}, \\ P^{(2)}_1(v) =& v^{-6}(1+24v+152v^2+541v^3+1377v^4+2582v^5+3949v^6+5335v^7+9170v^8 \\ &+13009v^9+6362v^{10}-5437v^{11}+23841v^{12}+92713v^{13}+134067v^{14}+169449v^{15} \\ &+309565v^{16}+451272v^{17}+425964v^{18}+359168v^{19}+\cdots+v^{38}). \end{aligned}$$

$n = 3$, $G = \mathfrak{so}(9)$, $F = \mathfrak{sp}(2) \times \mathfrak{sp}(1)$. Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(9)}}(q, v, m_{\mathfrak{so}(9)} = 0, m_F = 0) = q^{-1/3} v^5 \sum_{n=0}^{\infty} q^n \frac{P_n(v)}{(1-v)^4(1+v)^{12}}. \tag{E.25}$$

We obtain

$$\begin{aligned}P_0(v) &= -2(2+19v-62v^2+106v^3-62v^4+19v^5+2v^6), \\ P_1(v) &= -v^{-7}(1+8v+24v^2+24v^3-37v^4-132v^5-144v^6+180v^7+2004v^8 \\ &\quad -5264v^9+7056v^{10}-\cdots+v^{20}).\end{aligned} \tag{E.26}$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(9)}}(q,v,x=1,m_{\mathfrak{so}(9)}=0,m_F=0) = -q^{-5/6}v^{11}\sum_{n=0}^{\infty} q^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{12}(1+v+v^2)^{13}}. \tag{E.27}$$

We obtain

$$\begin{aligned}P_0^{(2)}(v) &= 10+174v+707v^2+851v^3-109v^4+1860v^5+11190v^6+16610v^7+6728v^8 \\ &\quad +7008v^9+43183v^{10}+70861v^{11}+45001v^{12}+18164v^{13}+\cdots+10v^{26}, \\ P_1^{(2)}(v) &= v^{-7}(4+74v+398v^2+1414v^3+3488v^4+6697v^5+9871v^6+12142v^7+18585v^8 \\ &\quad +43069v^9+55702v^{10}-10441v^{11}-73597v^{12}+105935v^{13}+359120v^{14}+239627v^{15} \\ &\quad +114575v^{16}+750264v^{17}+1400325v^{18}+990699v^{19}+470338v^{20}+\cdots+4v^{40}).\end{aligned} \tag{E.28}$$

$\boldsymbol{n=3,\ G=\mathfrak{so}(10),\ F=\mathfrak{sp}(3)\times\mathfrak{u}(1).}$ Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(10)}}(q,v,m_{\mathfrak{so}(10)}=0,m_F=0) = q^{-1/3}v^6\sum_{n=0}^{\infty} q^n \frac{P_n(v)}{(1-v)^4(1+v)^{14}}. \tag{E.29}$$

We obtain

$$\begin{aligned}P_0(v) &= 2(7+54v-210v^2+344v^3-210v^4+54v^5+7v^6), \\ P_1(v) &= v^{-8}(1+10v+41v^2+80v^3+35v^4-178v^5-419v^6-428v^7 \\ &\quad +676v^8+7284v^9-20742v^{10}+28016v^{11}-\cdots+v^{22}).\end{aligned} \tag{E.30}$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(10)}}(q,v,x=1,m_{\mathfrak{so}(10)}=0,m_F=0) = -q^{-5/6}v^{13}\sum_{n=0}^{\infty} q^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{18}(1+v+v^2)^{15}}.$$

We obtain

$$\begin{aligned}P_0^{(2)}(v) &= 2(45+932v+6264v^2+21096v^3+37801v^4+32448v^5+31299v^6+178325v^7 \\ &\quad +549579v^8+838987v^9+682443v^{10}+561148v^{11}+1511348v^{12}+3259788v^{13} \\ &\quad +3952706v^{14}+2932464v^{15}+2106794v^{16}+\cdots+45v^{32}), \\ P_1^{(2)}(v) &= 2v^{-8}(7+159v+1412v^2+7693v^3+29780v^4+87899v^5+205494v^6+388084v^7 \\ &\quad +597939v^8+790211v^9+1104282v^{10}+1974138v^{11}+3342747v^{12}+3399917v^{13} \\ &\quad +355771v^{14}-2250673v^{15}+2821724v^{16}+13232633v^{17}+15679593v^{18}+11581039v^{19} \\ &\quad +25206981v^{20}+61068134v^{21}+81796560v^{22}+66229422v^{23}+50700908v^{24}+\cdots+7v^{48}).\end{aligned} \tag{E.31}$$

$\boldsymbol{n=3,\ G=\mathfrak{so}(12),\ F=\mathfrak{sp}(5).}$ This theory belongs to class **C** which only has vanishing blowup equations. The leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function was partially determined in [1]. Using the vanishing blowup equations, we are able to fix it as

$$
\begin{aligned}
&v^8\chi^{\mathfrak{sp}(5)}_{(00010)}-v^9\chi^{\mathfrak{so}(12)}_{(100000)}\chi^{\mathfrak{sp}(5)}_{(00100)}+v^{10}\chi^{\mathfrak{so}(12)}_{(200000)}\chi^{\mathfrak{sp}(5)}_{(01000)}-v^{11}\chi^{\mathfrak{so}(12)}_{(300000)}\chi^{\mathfrak{sp}(5)}_{(10000)}+v^{12}\chi^{\mathfrak{so}(12)}_{(400000)}\\
&+\sum_{n=0}^{\infty}\Big[-v^{10+2n}\chi^{\mathfrak{so}(12)}_{(0n0010)}\chi^{\mathfrak{sp}(5)}_{(00001)}+v^{11+2n}(\chi^{\mathfrak{so}(12)}_{(1n0010)}\chi^{\mathfrak{sp}(5)}_{(00010)}+\chi^{\mathfrak{so}(12)}_{(0n1000)}\chi^{\mathfrak{sp}(5)}_{(00001)})\\
&-v^{12+2n}(\chi^{\mathfrak{so}(12)}_{(2n0010)}\chi^{\mathfrak{sp}(5)}_{(00100)}+\chi^{\mathfrak{so}(12)}_{(1n1000)}\chi^{\mathfrak{sp}(5)}_{(00010)})+v^{13+2n}(\chi^{\mathfrak{so}(12)}_{(3n0010)}\chi^{\mathfrak{sp}(5)}_{(01000)}+\chi^{\mathfrak{so}(12)}_{(2n1000)}\chi^{\mathfrak{sp}(5)}_{(00100)})\\
&-v^{14+2n}(\chi^{\mathfrak{so}(12)}_{(4n0010)}\chi^{\mathfrak{sp}(5)}_{(10000)}+\chi^{\mathfrak{so}(12)}_{(3n1000)}\chi^{\mathfrak{sp}(5)}_{(01000)})+v^{15+2n}(\chi^{\mathfrak{so}(12)}_{(5n0010)}+\chi^{\mathfrak{so}(12)}_{(4n1000)}\chi^{\mathfrak{sp}(5)}_{(10000)})\\
&-v^{16+2n}\chi^{\mathfrak{so}(12)}_{(5n1000)}\Big].
\end{aligned}
\tag{E.32}
$$

$\boldsymbol{n=3,\ G=E_6,\ F=\mathfrak{su}(3)_a\times\mathfrak{u}(1)_b.}$ From the recursion formula, we obtain the following exact formula for the leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function:

$$
\begin{aligned}
&\Big(v^{10}\chi^F_{(03)_a\oplus(6)_b}-v^{11}\chi^{E_6}_{(100000)}\chi^F_{(12)_a\oplus(5)_b}+v^{12}\chi^{E_6}_{(010000)}\chi^F_{(21)_a\oplus(4)_b}+v^{12}\chi^{E_6}_{(200000)}\chi^F_{(02)_a\oplus(4)_b}\\
&v^9\chi^F_{(06)_a\oplus(3)_b}-v^{11}\chi^{E_6}_{(000001)}\chi^F_{(30)_a\oplus(3)_b}-v^{13}\chi^{E_6}_{(001000)}\chi^F_{(30)_a\oplus(3)_b}v^{11}-\chi^{E_6}_{(110000)}\chi^F_{(11)_a\oplus(3)_b}\\
&+\chi^F_{(3)_b}v^7+\chi^{E_6}_{(000100)}\chi^F_{(31)_a\oplus(2)_b}v^{12}-\chi^{E_6}_{(100000)}\chi^F_{(12)_a\oplus(2)_b}v^{10}+\chi^{E_6}_{(100001)}\chi^F_{(20)_a\oplus(2)_b}v^{12}\\
&+\chi^{E_6}_{(101000)}\chi^F_{(20)_a\oplus(2)_b}v^{14}+\chi^{E_6}_{(020000)}\chi^F_{(01)_a\oplus(2)_b}v^{14}-\chi^{E_6}_{(000010)}\chi^F_{(32)_a\oplus(1)_b}v^{11}\\
&-\chi^{E_6}_{(100100)}\chi^F_{(21)_a\oplus(1)_b}v^{13}+\chi^{E_6}_{(010000)}\chi^F_{(02)_a\oplus(1)_b}v^{11}-\chi^{E_6}_{(010001)}\chi^F_{(10)_a\oplus(1)_b}v^{13}\\
&-\chi^{E_6}_{(011000)}\chi^F_{(10)_a\oplus(1)_b}v^{15}+v^8\chi^F_{(03)_a}+c.c.\Big)+\chi^F_{(33)_a}v^{10}+\chi^{E_6}_{(101010)}\chi^F_{(22)_a}v^{12}\\
&-\chi^{E_6}_{(000001)}\chi^F_{(11)_a}v^{10}+\chi^{E_6}_{(010100)}\chi^F_{(11)_a}v^{14}+\chi^{E_6}_{(000002)}v^{12}+\chi^{E_6}_{(001001)}v^{14}+\chi^{E_6}_{(002000)}v^{16}\\
&\sum_{n=0}^{\infty}\Big[v^{11+2n}\chi^{E_6}_{(00000n)}\chi^F_{(9)_b\oplus(-9)_b}-v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^F_{(01)_a\oplus(8)_b}+c.c.)\\
&+v^{13+2n}(\chi^{E_6}_{(01000n)}\chi^F_{(02)_a\oplus(7)_b}+\chi^{E_6}_{(20000n)}\chi^F_{(10)_a\oplus(7)_b}+c.c.)\\
&-v^{14+2n}(\chi^{E_6}_{(00100n)}\chi^F_{(03)_a\oplus(6)_b}+\chi^{E_6}_{(11000n)}\chi^F_{(11)_a\oplus(6)_b}+\chi^{E_6}_{(30000n)}\chi^F_{(00)_a\oplus(6)_b}+c.c.)\\
&+v^{13+2n}(\chi^{E_6}_{(00010n)}\chi^F_{(04)_a\oplus(5)_b}+c.c.)\\
&+v^{15+2n}(\chi^{E_6}_{(10100n)}\chi^F_{(12)_a\oplus(5)_b}+\chi^{E_6}_{(02000n)}\chi^F_{(20)_a\oplus(5)_b}+\chi^{E_6}_{(21000n)}\chi^F_{(01)_a\oplus(5)_b}+c.c.)\\
&-v^{12+2n}(\chi^{E_6}_{(00001n)}\chi^F_{(05)_a\oplus(4)_b}+c.c.)-v^{14+2n}(\chi^{E_6}_{(10010n)}\chi^F_{(13)_a\oplus(4)_b}+c.c.)\\
&-v^{16+2n}(\chi^{E_6}_{(01100n)}\chi^F_{(21)_a\oplus(4)_b}+\chi^{E_6}_{(20100n)}\chi^F_{(02)_a\oplus(4)_b}+\chi^{E_6}_{(12000n)}\chi^F_{(10)_a\oplus(4)_b}+c.c.)\\
&+v^{11+2n}(\chi^{E_6}_{(00000n)}\chi^F_{(06)_a\oplus(3)_b}+c.c.)+v^{13+2n}(\chi^{E_6}_{(10001n)}\chi^F_{(14)_a\oplus(3)_b}+c.c.)\\
&+v^{15+2n}(\chi^{E_6}_{(01010n)}\chi^F_{(22)_a\oplus(3)_b}+\chi^{E_6}_{(20010n)}\chi^F_{(03)_a\oplus(3)_b}+c.c.)\\
&+v^{17+2n}(\chi^{E_6}_{(00200n)}\chi^F_{(30)_a\oplus(3)_b}+\chi^{E_6}_{(11100n)}\chi^F_{(11)_a\oplus(3)_b}+\chi^{E_6}_{(03000n)}\chi^F_{(3)_b}+c.c.)\\
&-v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^F_{(15)_a\oplus(2)_b}+c.c.)-v^{14+2n}(\chi^{E_6}_{(01001n)}\chi^F_{(23)_a\oplus(2)_b}+\chi^{E_6}_{(20001n)}\chi^F_{(04)_a\oplus(2)_b}+c.c.)\\
&-v^{16+2n}(\chi^{E_6}_{(00110n)}\chi^F_{(31)_a\oplus(2)_b}+\chi^{E_6}_{(11010n)}\chi^F_{(12)_a\oplus(2)_b}+c.c.)
\end{aligned}
$$

$$
\begin{aligned}
&-v^{18+2n}(\chi^{E_6}_{(10200n)}\chi^F_{(20)_a\oplus(2)_b}+\chi^{E_6}_{(02100n)}\chi^F_{(01)_a\oplus(2)_b}+c.c.)\\
&+v^{13+2n}(\chi^{E_6}_{(01000n)}\chi^F_{(24)_a\oplus(1)_b}+\chi^{E_6}_{(20000n)}\chi^F_{(05)_a\oplus(1)_b}+c.c.)\\
&+v^{15+2n}(\chi^{E_6}_{(00101n)}\chi^F_{(32)_a\oplus(1)_b}+\chi^{E_6}_{(11001n)}\chi^F_{(13)_a\oplus(1)_b}+\chi^{E_6}_{(00020n)}\chi^F_{(40)_a\oplus(1)_b}+c.c.)\\
&+v^{17+2n}(\chi^{E_6}_{(10110n)}\chi^F_{(21)_a\oplus(1)_b}+\chi^{E_6}_{(02010n)}\chi^F_{(02)_a\oplus(1)_b}+c.c.)+v^{19+2n}\chi^{E_6}_{(01200n)}(\chi^F_{(10)_a\oplus(1)_b}+c.c.)\\
&-v^{14+2n}\chi^{E_6}_{(00100n)}\chi^F_{(33)_a}-v^{14+2n}(\chi^{E_6}_{(11000n)}\chi^F_{(14)_a}+c.c.)\\
&-v^{16+2n}\chi^{E_6}_{(10101n)}\chi^F_{(22)_a}-v^{16+2n}(\chi^{E_6}_{(02001n)}\chi^F_{(03)_a}+c.c.)\\
&-v^{18+2n}\chi^{E_6}_{(01110n)}\chi^F_{(11)_a}-v^{20+2n}\chi^{E_6}_{(00300n)}\Big]
\end{aligned}
\tag{E.33}
$$

After turning off all E_6 gauge fugacities, the above exact formula reduces to the result (A.17) of [80] by Weyl dimension formula of representations of E_6. Further turning off all flavor fugacities, one obtains the rational function of v in (5.90).

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$
\mathbb{E}_{h^{(2)}_{3,E_6}}(q,v,x=1,m_{E_6}=0,m_F=0)=-q^{-5/6}v^{15}\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{26}(1+v+v^2)^{23}},
$$

we obtain

$$
\begin{aligned}
P^{(2)}_0(v)=\;&3+159v+4245v^2+72622v^3+863819v^4+7446591v^5+47902516v^6+235241313v^7\\
&+896085222v^8+2671738023v^9+6257280290v^{10}+11565342413v^{11}+17441014579v^{12}\\
&+24757146408v^{13}+43167107703v^{14}+92340625269v^{15}+184446978968v^{16}\\
&+297014465909v^{17}+380602273913v^{18}+427769333206v^{19}+533426305310v^{20}\\
&+825794587232v^{21}+1287690035763v^{22}+1693325870657v^{23}+1815742557209v^{24}\\
&+1695462175970v^{25}+1602451245554v^{26}+\cdots+3v^{52}.
\end{aligned}
\tag{E.34}
$$

$n=4$, $G=E_6$, $F=\mathfrak{su}(2)\times\mathfrak{u}(1)$. Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$
\mathbb{E}_{h^{(2)}_{4,E_6}}(q,v,x=1,m_G=0,m_F=0)=-q^{-11/6}v^{19}\sum_{n=0}^{\infty}q^n\frac{P^{(2)}_n(v)}{(1-v)^{22}(1+v)^{28}(1+v+v^2)^{23}},
\tag{E.35}
$$

we obtain

$$
\begin{aligned}
P^{(2)}_0(v)=\;&6+200v+2632v^2+17758v^3+75489v^4+243367v^5+760467v^6+2577888v^7\\
&+8317316v^8+23236506v^9+58513940v^{10}+143767140v^{11}+347390848v^{12}+786032254v^{13}\\
&+1633105895v^{14}+3195818881v^{15}+6041990014v^{16}+10959026237v^{17}+18715741117v^{18}\\
&+30093383834v^{19}+46262367433v^{20}+68471264635v^{21}+96730928747v^{22}\\
&+129436722092v^{23}+164888050451v^{24}+201811431341v^{25}+237209409984v^{26}\\
&+265667738531v^{27}+282914996487v^{28}+288440699594v^{29}+\cdots+6v^{58},
\end{aligned}
\tag{E.36}
$$

and

$$\begin{aligned}P_1^{(2)}(v)=&-v^{-2}(8+262v+2954v^2+6882v^3-125701v^4-1314279v^5-6621327v^6-23006770v^7\\&-69417453v^8-213977845v^9-651520698v^{10}-1773023963v^{11}-4276376371v^{12}\\&-9730496854v^{13}-21781260461v^{14}-46890358519v^{15}-94029008670v^{16}-176640724111v^{17}\\&-318761640562v^{18}-556066823089v^{19}-924340036971v^{20}-1452988495522v^{21}\\&-2179171428592v^{22}-3147790892042v^{23}-4365630688208v^{24}-5770440288994v^{25}\\&-7276423650370v^{26}-8812083976234v^{27}-10262845252021v^{28}-11435602558269v^{29}\\&-12163726096281v^{30}-12402928893114v^{31}+\cdots+20v^{62}).\end{aligned}\tag{E.37}$$

$n=4$, $G=E_7$, $F=\mathfrak{so}(4)$. Regarding the flavor symmetry F as $\mathfrak{su}(2)\times\mathfrak{su}(2)$, we obtain the following exact formula for the leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function:

$$\begin{aligned}&-\chi^F_{(8,4)\oplus(4,8)}v^{15}-\chi^F_{(7,5)\oplus(5,7)}\chi^{E_7}_{(0000001)}v^{18}+\chi^F_{(6,6)}(\chi^{E_7}_{(1000000)}v^{17}+\chi^{E_7}_{(0100000)}v^{19})\\&+\chi^F_{(7,3)\oplus(3,7)}\chi^{E_7}_{(0000010)}v^{16}+\chi^F_{(6,4)\oplus(4,6)}\chi^{E_7}_{(0000011)}v^{19}-\chi^F_{(5,5)}(\chi^{E_7}_{(1000010)}v^{18}+\chi^{E_7}_{(0100010)}v^{20})\\&-\chi^F_{(6,2)\oplus(2,6)}\chi^{E_7}_{(0000100)}v^{17}-\chi^F_{(5,3)\oplus(3,5)}\chi^{E_7}_{(0000101)}v^{20}+\chi^F_{(4,4)}(-v^{13}+\chi^{E_7}_{(10000100)}v^{19}+\chi^{E_7}_{(0100100)}v^{21})\\&-\chi^F_{(6,0)\oplus(0,6)}v^{13}+\chi^F_{(5,1)\oplus(1,5)}\chi^{E_7}_{(0001000)}v^{18}+\chi^F_{(4,2)\oplus(2,4)}(\chi^{E_7}_{(1000000)}v^{15}+\chi^{E_7}_{(0001001)}v^{21})\\&-\chi^F_{(3,3)}(\chi^{E_7}_{(0000001)}v^{16}+\chi^{E_7}_{(1001000)}v^{20}+\chi^{E_7}_{(0101000)}v^{22})-\chi^F_{(4,0)\oplus(0,4)}(\chi^{E_7}_{(0100000)}v^{17}+\chi^{E_7}_{(0010000)}v^{19})\\&+\chi^F_{(3,1)\oplus(1,3)}(\chi^{E_7}_{(0000101)}v^{18}-\chi^{E_7}_{(0010001)}v^{22})+\chi^F_{(2,2)}(-\chi^{E_7}_{(2000000)}v^{17}-\chi^{E_7}_{(0000002)}v^{19}\\&+\chi^{E_7}_{(1010000)}v^{21}+\chi^{E_7}_{(0110000)}v^{23})+\chi^F_{(2,0)\oplus(0,2)}(\chi^{E_7}_{(1000002)}v^{21}+\chi^{E_7}_{(0100002)}v^{23})\\&-\chi^F_{(1,1)}(\chi^{E_7}_{(2000001)}v^{20}+(\chi^{E_7}_{(1100001)}+\chi^{E_7}_{(0000003)})v^{22}+\chi^{E_7}_{(0200001)}v^{24})\\&-v^{11}+\chi^{E_7}_{(3000000)}v^{19}+\chi^{E_7}_{(2100000)}v^{21}+\chi^{E_7}_{(1200000)}v^{23}+\chi^{E_7}_{(0300000)}v^{25}+\\&+\sum_{n=0}^{\infty}\Big[-\chi^F_{(12,0)\oplus(0,12)}\chi^{E_7}_{(n000000)}v^{17+2n}+\chi^F_{(11,1)\oplus(1,11)}\chi^{E_7}_{(n000010)}v^{18+2n}-\chi^F_{(10,2)\oplus(2,10)}\chi^{E_7}_{(n000100)}v^{19+2n}\\&+\chi^F_{(9,3)\oplus(3,9)}\chi^{E_7}_{(n001000)}v^{20+2n}-\chi^F_{(8,4)\oplus(4,8)}\chi^{E_7}_{(n010000)}v^{21+2n}+\chi^F_{(7,5)\oplus(5,7)}\chi^{E_7}_{(n100001)}v^{22+2n}\\&-\chi^F_{(6,6)}v^{21+2n}(\chi^{E_7}_{(n000002)}+v^2\chi^{E_7}_{(n200000)})-\chi^F_{(10,0)\oplus(0,10)}\chi^{E_7}_{(n000020)}v^{19+2n}\\&+\chi^F_{(9,1)\oplus(1,9)}\chi^{E_7}_{(n000110)}v^{20+2n}-\chi^F_{(8,2)\oplus(2,8)}\chi^{E_7}_{(n001010)}v^{21+2n}+\chi^F_{(7,3)\oplus(3,7)}\chi^{E_7}_{(n010010)}v^{22+2n}\\&-\chi^F_{(6,4)\oplus(4,6)}\chi^{E_7}_{(n10011)}v^{23+2n}+\chi^F_{(5,5)}v^{22+2n}(\chi^{E_7}_{(n000012)}+v^2\chi^{E_7}_{(n200010)})\\&-\chi^F_{(8,0)\oplus(0,8)}\chi^{E_7}_{(n000200)}v^{21+2n}+\chi^F_{(7,1)\oplus(1,7)}\chi^{E_7}_{(n001100)}v^{22+2n}-\chi^F_{(6,2)\oplus(2,6)}\chi^{E_7}_{(n010100)}v^{23+2n}\\&+\chi^F_{(5,3)\oplus(3,5)}\chi^{E_7}_{(n100101)}v^{24+2n}-\chi^F_{(4,4)}v^{23+2n}(\chi^{E_7}_{(n000102)}+v^2\chi^{E_7}_{(n200100)})\\&-\chi^F_{(6,0)\oplus(0,6)}\chi^{E_7}_{(n002000)}v^{23+2n}+\chi^F_{(5,1)\oplus(1,5)}\chi^{E_7}_{(n011000)}v^{24+2n}-\chi^F_{(4,2)\oplus(2,4)}\chi^{E_7}_{(n101001)}v^{25+2n}\\&+\chi^F_{(3,3)}v^{24+2n}(\chi^{E_7}_{(n001002)}+v^2\chi^{E_7}_{(n201000)})-\chi^F_{(4,0)\oplus(0,4)}\chi^{E_7}_{(n020000)}v^{25+2n}\\&+\chi^F_{(3,1)\oplus(1,3)}\chi^{E_7}_{(n110001)}v^{26+2n}-\chi^F_{(2,2)}v^{25+2n}(\chi^{E_7}_{(n010002)}+v^2\chi^{E_7}_{(n210000)})\\&-\chi^F_{(2,0)\oplus(0,2)}\chi^{E_7}_{(n200002)}v^{27+2n}+\chi^F_{(1,1)}v^{26+2n}(\chi^{E_7}_{(n100003)}+v^2\chi^{E_7}_{(n300001)})\\&-(\chi^{E_7}_{(n000004)}+v^4\chi^{E_7}_{(n400000)})v^{25+2n}\Big].\end{aligned}\tag{E.38}$$

After turning off all E_7 gauge fugacities, the above exact formula reduces to the result (A.20) of [80] by Weyl dimension formula of representations of E_7. Further turning off all flavor fugacities, one obtains the rational function of v in (5.109).

For higher q order, recall the notation (5.109) for the reduced one-string elliptic genus with all gauge and flavor fugacities turned off, we record some higher order results for the numerators here:

$$\begin{aligned}P_2(v)=v^{-14}(&1+24v+266v^2+1784v^3+7911v^4+23344v^5+40636v^6+9264v^7-165238v^8\\&-478912v^9-556440v^{10}+298240v^{11}+2128520v^{12}+3082560v^{13}+550409v^{14}\\&-3596776v^{15}+700889v^{16}+4057056v^{17}-152281616v^{18}-456584704v^{19}\\&+2336222907v^{20}+3518750120v^{21}-39318682643v^{22}+116568917840v^{23}\\&-205266204842v^{24}+245300442560v^{25}-\cdots+v^{50}).\end{aligned}\tag{E.39}$$

$$\begin{aligned}P_3(v)=v^{-16}(&133+2968v+30142v^2+181048v^3+689812v^4+1583048v^5+1289736v^6\\&-4557256v^7-18160096v^8-24750520v^9+11527121v^{10}+99398192v^{11}\\&+140533368v^{12}-28415888v^{13}-371622426v^{14}-468305664v^{15}+101124822v^{16}\\&+1079768624v^{17}+1758536636v^{18}+1201001616v^{19}-8948098843v^{20}\\&-35689905400v^{21}+94419997502v^{22}+281480553480v^{23}-1852610135319v^{24}\\&+4805915623088v^{25}-7955500120108v^{26}+9332969375888v^{27}-\cdots+v^{54}).\end{aligned}\tag{E.40}$$

$$\begin{aligned}P_4(v)=v^{-18}(&7371+150984v+1379855v^2+7213760v^3+22300305v^4+32499512v^5-32572277v^6\\&-252945648v^7-438572547v^8+80757800v^9+1590185774v^{10}+2182830856v^{11}\\&-1301117169v^{12}-6862074336v^{13}-4256604675v^{14}+10850256216v^{15}\\&+18344377949v^{16}-7908490560v^{17}-46555176815v^{18}-23990305416v^{19}\\&+98055494050v^{20}+216546552760v^{21}-184743607501v^{22}-1788205999184v^{23}\\&+2239774014885v^{24}+13565904866280v^{25}-64302524207535v^{26}\\&+149664342880240v^{27}-235464619399970v^{28}+272014399573792v^{29}-\cdots+v^{58}).\end{aligned}\tag{E.41}$$

$\boldsymbol{n=5,\ G=E_6,\ F=\mathfrak{u}(1).}$ Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{5,E_6}}(q,v,x=1,m_{E_6}=0,m_{\mathfrak{u}(1)}=0)=-q^{-17/6}v^{21}\sum_{n=0}^{\infty}q^n\frac{P_n^{(2)}(v)}{(1-v)^{34}(1+v)^{30}(1+v+v^2)^{23}},\tag{E.42}$$

we obtain

$$\begin{aligned}P_0^{(2)}(v)=1&+21v+153v^2+904v^3+5116v^4+25914v^5+116029v^6+477409v^7+1823569v^8\\&+6443864v^9+21148972v^{10}+64945868v^{11}+187225307v^{12}+507470579v^{13}\\&+1296690701v^{14}+3132384316v^{15}+7167102255v^{16}+15555191149v^{17}\\&+32075501088v^{18}+62937552731v^{19}+117653600727v^{20}+209750655294v^{21}\\&+356983566607v^{22}+580561108791v^{23}+902887841711v^{24}+1343669144748v^{25}\\&+1914685757018v^{26}+2613923784990v^{27}+3420367203355v^{28}+4291402109101v^{29}\\&+5164404456225v^{30}+5962900573462v^{31}+6606847822339v^{32}+7025662161955v^{33}\\&+7170987830896v^{34}+\cdots+v^{68},\end{aligned}\tag{E.43}$$

and

$$
\begin{aligned}
P_1^{(2)}(v) = {} & 84 + 1870v + 15150v^2 + 92382v^3 + 509942v^4 + 2529414v^5 + 11170010v^6 + 45018822v^7 \\
& + 167914134v^8 + 580737756v^9 + 1867913107v^{10} + 5619089721v^{11} + 15872495069v^{12} \\
& + 42199602702v^{13} + 105848677375v^{14} + 251124006621v^{15} + 564703393888v^{16} \\
& + 1205575234175v^{17} + 2447284329306v^{18} + 4730834408879v^{19} + 8719854968064v^{20} \\
& + 15341684421093v^{21} + 25790951006163v^{22} + 41466404452278v^{23} + 63813198389587v^{24} \\
& + 94061792487301v^{25} + 132885858904299v^{26} + 180032677369322v^{27} + 234011514454012v^{28} \\
& + 291950610885280v^{29} + 349716381424128v^{30} + 402326438406440v^{31} \\
& + 444618538975344v^{32} + 472069443334672v^{33} + 481585928612732v^{34} + \cdots + 2v^{68}). \qquad \text{(E.44)}
\end{aligned}
$$

F More on Calabi-Yau construction

In this section, we list the dual polytope ν_i^*, Mori cone generators $l^{(i)}$ and the triple intersection ring $\mathcal{R}$ in terms of Kähler classes J_i for the geometries we have constructed. We also give the relation between the geometric bases and Lie bases.

$n = 7, G = E_7$.

$$
\begin{array}{|rrrr|rrrrrrrrr|}
\hline
 & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l^{(7)} & l^{(8)} & l^{(9)} \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
1 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 2 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
2 & 3 & 0 & -4 & -2 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 3 & 0 & -3 & 1 & -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & -2 & 0 & 1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & -1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & -1 & -7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad \text{(F.1)}
$$

The triple intersection ring corresponding to the above Mori cone generators is

$$
\begin{aligned}
\mathcal{R}=&-\frac{1}{7}(4J_1^3+4J_3J_1^2+5J_6J_1^2+6J_7J_1^2+3J_8J_1^2+2J_9J_1^2+4J_3^2J_1+15J_6^2J_1+30J_7^2J_1+11J_8^2J_1\\
&+8J_9^2J_1+5J_3J_6J_1+6J_3J_7J_1+18J_6J_7J_1+3J_3J_8J_1+9J_6J_8J_1+15J_7J_8J_1+2J_3J_9J_1+6J_6J_9J_1\\
&+10J_7J_9J_1+5J_8J_9J_1+3J_2^3+25J_4^3+18J_5^3+45J_6^3+150J_7^3+52J_8^3+32J_9^3+3J_2J_3^2+5J_2J_4^2\\
&+5J_3J_4^2+6J_2J_5^2+6J_3J_5^2+9J_4J_5^2+15J_3J_6^2+30J_3J_7^2+90J_6J_7^2+11J_3J_8^2+33J_6J_8^2+55J_7J_8^2\\
&+8J_3J_9^2+24J_6J_9^2+40J_7J_9^2+20J_8J_9^2+3J_2^2J_3+J_2^2J_4+J_3^2J_4+J_2J_3J_4+2J_2^2J_5+2J_3^2J_5\\
&+15J_4^2J_5+2J_2J_3J_5+3J_2J_4J_5+3J_3J_4J_5+5J_3^2J_6+6J_3^2J_7+54J_6^2J_7+18J_3J_6J_7+3J_3^2J_8\\
&+27J_6^2J_8+75J_7^2J_8+9J_3J_6J_8+15J_3J_7J_8+45J_6J_7J_8+2J_3^2J_9+18J_6^2J_9+50J_7^2J_9+30J_8^2J_9\\
&+6J_3J_6J_9+10J_3J_7J_9+30J_6J_7J_9+5J_3J_8J_9+15J_6J_8J_9+25J_7J_8J_9).
\end{aligned}
\tag{F.2}
$$

The relation between the above geometric bases and the Lie algebra bases is

$$
\begin{aligned}
&l_{E_7}^{(1)}=l^{(5)},\quad l_{E_7}^{(1)}=l^{(2)},\quad l_{E_7}^{(1)}=l^{(1)},\quad l_{E_7}^{(1)}=l^{(6)},\quad l_{E_7}^{(1)}=l^{(7)}+l^{(9)},\quad l_{E_7}^{(1)}=l^{(8)},\\
&l_{E_7}^{(1)}=l^{(6)}+2l^{(7)}+l^{(8)},\quad l_B=l^{(2)}+l^{(3)}+5l^{(4)}+3l^{(5)},\\
&l_\tau\ =4l^{(1)}+3l^{(2)}+l^{(4)}+2l^{(5)}+5l^{(6)}+6l^{(7)}+3l^{(8)}+2l^{(9)}
\end{aligned}
\tag{F.3}
$$

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ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$
0	0	0	0	0	0	−1	0	0	0	0	0	0
−1	0	0	0	0	0	0	0	0	0	0	0	1
0	−1	0	0	0	0	0	0	0	0	0	1	0
2	3	0	0	0	1	0	0	0	0	0	0	0
2	3	1	0	0	0	−1	0	0	0	1	0	0
2	3	0	−1	−1	0	1	0	0	1	0	0	−1
1	2	0	−1	1	0	1	0	0	0	−2	0	1
2	3	−1	−2	0	−2	0	0	1	−1	0	0	0
1	1	−1	−2	0	0	1	0	0	0	0	−2	0
2	3	−2	−4	0	1	−1	0	0	−1	0	1	1
0	1	−1	−2	−1	0	0	1	0	0	1	0	−2
2	3	−3	−5	1	0	0	0	−2	1	0	0	0
1	2	−3	−5	1	0	0	−2	0	0	0	0	0
2	3	−5	−8	−1	0	0	1	1	0	0	0	0

(F.4)

The triple intersection ring corresponding to the above Mori cone generators is

$$
\begin{aligned}
\mathcal{R}=&-\frac{1}{8}(112J_1^3+20J_2J_1^2+24J_3J_1^2+56J_4J_1^2+72J_5J_1^2+32J_6J_1^2+72J_7J_1^2+12J_8J_1^2+8J_9J_1^2\\
&+4J_2^2J_1+16J_3^2J_1+28J_4^2J_1+48J_5^2J_1+16J_6^2J_1+60J_7^2J_1+12J_8^2J_1+16J_9^2J_1+10J_2J_4J_1\\
&+12J_3J_4J_1+12J_2J_5J_1+24J_3J_5J_1+36J_4J_5J_1+4J_2J_6J_1+24J_3J_6J_1+16J_4J_6J_1+24J_5J_6J_1\\
&+10J_2J_7J_1+28J_3J_7J_1+36J_4J_7J_1+52J_5J_7J_1+32J_6J_7J_1+8J_3J_8J_1+6J_4J_8J_1+12J_5J_8J_1\\
&+12J_6J_8J_1+14J_7J_8J_1+16J_3J_9J_1+4J_4J_9J_1+8J_5J_9J_1+8J_6J_9J_1+20J_7J_9J_1+8J_8J_9J_1\\
&+4J_2^3+32J_3^3+22J_4^3+40J_5^3+114J_7^3+12J_8^3+32J_9^3+5J_2J_4^2+14J_3J_4^2+12J_2J_5^2+24J_3J_5^2\\
&+24J_4J_5^2+4J_2J_6^2+24J_3J_6^2+8J_4J_6^2+8J_5J_6^2+5J_2J_7^2+62J_3J_7^2+30J_4J_7^2+50J_5J_7^2+40J_6J_7^2\\
&+8J_3J_8^2+6J_4J_8^2+12J_5J_8^2+12J_6J_8^2+14J_7J_8^2+32J_3J_9^2+8J_4J_9^2+16J_5J_9^2+16J_6J_9^2+40J_7J_9^2\\
&+16J_8J_9^2+2J_2^2J_4+8J_3^2J_4+4J_2^2J_5+16J_3^2J_5+18J_4^2J_5+6J_2J_4J_5+12J_3J_4J_5+4J_2^2J_6\\
&+16J_3^2J_6+8J_4^2J_6+16J_5^2J_6+2J_2J_4J_6+12J_3J_4J_6+4J_2J_5J_6+24J_3J_5J_6+12J_4J_5J_6+2J_2^2J_7\\
&+40J_3^2J_7+26J_4^2J_7+40J_5^2J_7+24J_6^2J_7+5J_2J_4J_7+14J_3J_4J_7+6J_2J_5J_7+28J_3J_5J_7\\
&+26J_4J_5J_7+2J_2J_6J_7+28J_3J_6J_7+16J_4J_6J_7+28J_5J_6J_7+16J_3^2J_8+7J_4^2J_8+12J_5^2J_8\\
&+12J_6^2J_8+31J_7^2J_8+4J_3J_4J_8+8J_3J_5J_8+6J_4J_5J_8+8J_3J_6J_8+6J_4J_6J_8+12J_5J_6J_8\\
&+20J_3J_7J_8+7J_4J_7J_8+14J_5J_7J_8+14J_6J_7J_8+32J_3^2J_9+10J_4^2J_9+8J_5^2J_9+8J_6^2J_9+58J_7^2J_9\\
&+8J_8^2J_9+8J_3J_4J_9+16J_3J_5J_9+4J_4J_5J_9+16J_3J_6J_9+4J_4J_6J_9+8J_5J_6J_9+40J_3J_7J_9\\
&+10J_4J_7J_9+20J_5J_7J_9+20J_6J_7J_9+16J_3J_8J_9+4J_4J_8J_9+8J_5J_8J_9+8J_6J_8J_9+20J_7J_8J_9).
\end{aligned}
\tag{F.5}
$$

The relation between the above geometric bases and the Lie algebra bases is

$$
\begin{aligned}
&l_{\mathfrak{su}(2)}=4l^{(3)}+4l^{(7)}+2l^{(8)}+2l^{(9)},\quad l_{\mathfrak{su}(2)'}=2l^{(1)}+4l^{(3)}+2l^{(4)}+4l^{(7)}+2l^{(8)}+2l^{(9)}\\
&l^{(1)}_{\mathfrak{so}(7)}=2l^{(3)}+2l^{(7)}+l^{(8)}+2l^{(9)},\quad l^{(2)}_{\mathfrak{so}(7)}=2l^{(1)}+2l^{(3)}+l^{(4)}+l^{(5)}+2l^{(7)},\quad l^{(3)}_{\mathfrak{so}(7)}=l^{(8)},\\
&\quad l_{B_1}=2l^{(1)}+l^{(4)}+l^{(5)}+l^{(7)},\quad l_{B_2}=l^{(2)}+l^{(6)},\quad l_{B_3}=l^{(5)},\\
&\quad l_\tau=4l^{(1)}+l^{(2)}+6l^{(3)}+2l^{(4)}+2l^{(5)}+6l^{(7)}+3l^{(8)}+2l^{(9)}.
\end{aligned}
\tag{F.6}
$$

The instanton partition function of refined topological string on the above local Calabi-Yau threefold differs from the elliptic genera of NHC 2,3,2 by the following four degree flipping of refined BPS invariants with spin $(0,0)$:

$$
\begin{aligned}
&-\frac{t_{\mathfrak{su}(2)}}{2}+\frac{t^{(1)}_{\mathfrak{so}(7)}}{2}+t^{(2)}_{\mathfrak{so}(7)}+\frac{t^{(3)}_{\mathfrak{so}(7)}}{2}-t_{B_1},\quad -\frac{t_{\mathfrak{su}(2)}}{2}+\frac{t^{(1)}_{\mathfrak{so}(7)}}{2}+t^{(2)}_{\mathfrak{so}(7)}+\frac{3t^{(3)}_{\mathfrak{so}(7)}}{2}-t_{B_1},\\
&-\frac{t_{\mathfrak{su}(2)'}}{2}+\frac{t^{(1)}_{\mathfrak{so}(7)}}{2}+t^{(2)}_{\mathfrak{so}(7)}+\frac{t^{(3)}_{\mathfrak{so}(7)}}{2}-t_{B_3},\quad -\frac{t_{\mathfrak{su}(2)'}}{2}+\frac{t^{(1)}_{\mathfrak{so}(7)}}{2}+t^{(2)}_{\mathfrak{so}(7)}+\frac{3t^{(3)}_{\mathfrak{so}(7)}}{2}-t_{B_3}.
\end{aligned}
\tag{F.7}
$$

NHC 3, 2, 2.

$$\begin{array}{cccc|ccccccc} & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l^{(7)} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & -1 & 0 & -2 & 0 & 1 & -1 & 0 & 0 \\ 2 & 3 & 0 & -2 & 0 & 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\ 2 & 3 & -1 & -3 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 2 & -1 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & -3 \\ 2 & 3 & -2 & -5 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & -3 & -7 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array} \tag{F.8}$$

The triple intersection ring corresponding to the above Mori cone generators is

$$\begin{aligned}\mathcal{R}=-\frac{1}{7}(&36J_1^3+5J_2J_1^2+18J_3J_1^2+26J_4J_1^2+16J_5J_1^2+9J_6J_1^2+6J_7J_1^2+J_2^2J_1+18J_3^2J_1+20J_4^2J_1\\&+12J_5^2J_1+15J_6^2J_1+2J_7^2J_1+3J_2J_4J_1+18J_3J_4J_1+J_2J_5J_1+18J_3J_5J_1+14J_4J_5J_1+9J_3J_6J_1\\&+9J_4J_6J_1+9J_5J_6J_1+6J_3J_7J_1+6J_4J_7J_1+6J_5J_7J_1+3J_6J_7J_1+3J_2^3+18J_3^3+22J_4^3+25J_6^3\\&+3J_7^3+6J_2J_4^2+18J_3J_4^2+3J_2J_5^2+18J_3J_5^2+6J_4J_5^2+15J_3J_6^2+15J_4J_6^2+15J_5J_6^2+2J_3J_7^2\\&+2J_4J_7^2+2J_5J_7^2+J_6J_7^2+2J_2^2J_4+18J_3^2J_4+3J_2^2J_5+18J_3^2J_5+10J_4^2J_5+2J_2J_4J_5+18J_3J_4J_5\\&+9J_3^2J_6+9J_4^2J_6+9J_5^2J_6+9J_3J_4J_6+9J_3J_5J_6+9J_4J_5J_6+6J_3^2J_7+6J_4^2J_7+6J_5^2J_7+5J_6^2J_7\\&+6J_3J_4J_7+6J_3J_5J_7+6J_4J_5J_7+3J_3J_6J_7+3J_4J_6J_7+3J_5J_6J_7).\end{aligned} \tag{F.9}$$

The relation between the above geometric bases and the Lie algebra bases is

$$\begin{aligned}&l_{G_2}^{(1)}=2l^{(1)}+3l^{(3)}+l^{(4)}+l^{(7)},\quad l_{G_2}^{(2)}=l^{(6)},\quad l_{\mathfrak{su}(2)}=2l^{(1)}+4l^{(3)}+2l^{(6)}+2l^{(7)},\quad l_{B_1}=l^{(2)}+l^{(5)},\\&l_{B_2}=l^{(4)},\qquad l_{B_3}=l^{(1)},\qquad\qquad l_\tau=4l^{(1)}+l^{(2)}+6l^{(3)}+2l^{(4)}+3l^{(6)}+2l^{(7)}.\end{aligned} \tag{F.10}$$

The instanton partition function of refined topological string on the above local Calabi-Yau threefold differs from the elliptic genera of NHC 3,2,2 by the following five degree flipping of refined BPS invariants with spin $(0,0)$:

$$\begin{aligned}&-\frac{t_{\mathfrak{su}(2)}}{2}-t_{B_2}-t_{B_3}+t_{G_2}^{(1)}+t_{G_2}^{(2)},\quad -\frac{t_{\mathfrak{su}(2)}}{2}-t_{B_2}-t_{B_3}+t_{G_2}^{(1)}+2t_{G_2}^{(2)},\quad \frac{t_{\mathfrak{su}(2)}}{2}-t_{B_3},\\&-\frac{t_{\mathfrak{su}(2)}}{2}-t_{B_2}+t_{G_2}^{(1)}+t_{G_2}^{(2)},\quad -\frac{t_{\mathfrak{su}(2)}}{2}-t_{B_2}+t_{G_2}^{(1)}+2t_{G_2}^{(2)}.\end{aligned} \tag{F.11}$$

$n = 3, G = \mathfrak{so}(7)$.

ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$
0	0	0	0	0	0	0	−1	0	0	0
−1	0	0	0	0	0	0	0	0	0	1
0	−1	0	0	0	0	0	0	0	1	0
2	3	0	0	0	0	1	0	0	0	0
2	3	1	0	−1	−1	0	1	1	0	−1
1	2	1	0	1	1	0	−1	0	0	1
2	3	0	−1	0	1	−2	0	−1	0	0
1	1	0	−1	0	0	0	1	0	−2	0
2	3	0	−2	0	0	1	−1	−1	1	1
0	1	0	−1	0	−1	0	1	0	0	−2
2	3	−1	−3	1	−1	0	0	1	0	0
1	2	−1	−3	−1	1	0	0	0	0	0

(F.12)

The triple intersection ring corresponding to the above Mori cone generators is

$$\begin{aligned}\mathcal{R}=-\frac{1}{3}(&2J_2^3+2J_1J_2^2+6J_4J_2^2+2J_5J_2^2+3J_6J_2^2+2J_7J_2^2+8J_4^2J_2+2J_5^2J_2+5J_6^2J_2+4J_7^2J_2\\&+2J_1J_4J_2+2J_1J_5J_2+6J_4J_5J_2+J_1J_6J_2+4J_4J_6J_2+3J_5J_6J_2+4J_4J_7J_2+2J_5J_7J_2\\&+2J_6J_7J_2+J_3^3+12J_4^3+9J_6^3+8J_7^3+2J_1J_4^2+2J_1J_5^2+J_3J_5^2+6J_4J_5^2+2J_1J_6^2+6J_4J_6^2\\&+5J_5J_6^2+8J_4J_7^2+4J_5J_7^2+4J_6J_7^2+J_3^2J_5+8J_4^2J_5+2J_1J_4J_5+6J_4^2J_6+3J_5^2J_6+J_1J_4J_6\\&+J_1J_5J_6+4J_4J_5J_6+8J_4^2J_7+2J_5^2J_7+2J_6^2J_7+4J_4J_5J_7+4J_4J_6J_7+2J_5J_6J_7).\end{aligned} \tag{F.13}$$

The relation between the above geometric bases and the Lie algebra bases is

$$\begin{aligned}&l_{\mathfrak{su}(2)}=4l^{(4)}+2l^{(6)}+2l^{(7)},\quad l_{\mathfrak{so}(7)}^{(1)}=2l^{(4)}+l^{(6)}+2l^{(7)},\quad l_{\mathfrak{so}(7)}^{(2)}=l^{(1)}+l^{(2)}+2l^{(4)},\quad l_{\mathfrak{so}(7)}^{(3)}=l^{(6)},\\&l_{\mathfrak{su}(2)'}=2l^{(1)}+4l^{(4)}+2l^{(6)}+2l^{(7)},\quad l_B=l^{(3)}+l^{(5)},\quad l_\tau=2l^{(1)}+2l^{(2)}+l^{(3)}+6l^{(4)}+3l^{(6)}+2l^{(7)}.\end{aligned} \tag{F.14}$$

$n = 2, G = G_2$. Here we only turn on a subgroup $\mathfrak{su}(2)$ of the full flavor group $\mathfrak{sp}(4)$.

$$\begin{array}{|cccc|ccccc|}\hline & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & -1 & -2 & 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & -2 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & -2 & 2 \\ 2 & 3 & -1 & -2 & 0 & -1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -2 & 0 & 1 & 0 & 0 & -1 \\ \hline\end{array} \tag{F.15}$$

The triple intersection ring corresponding to the above Mori cone generators is

$$\begin{aligned}\mathcal{R}=-\frac{1}{2}(&8J_2^3+4J_3J_2^2+12J_4J_2^2+8J_5J_2^2+2J_3^2J_2+20J_4^2J_2+4J_5^2J_2+6J_3J_4J_2+4J_3J_5J_2 \\ &+8J_4J_5J_2+32J_4^3+2J_5^3+J_1J_3^2+10J_3J_4^2+2J_3J_5^2+4J_4J_5^2+3J_3^2J_4+2J_3^2J_5+12J_4^2J_5+4J_3J_4J_5)\end{aligned} \tag{F.16}$$

The relation between the above geometric bases and the Lie algebra bases is

$$\begin{aligned} &l_{G_2}^{(1)} = l^{(2)} + l^{(5)}, \quad l_{G_2}^{(2)} = l^{(4)}, \qquad l_{\mathfrak{su}(2)} = 2l^{(4)} + 2l^{(5)}, \\ &l_B = l^{(3)}, \qquad\quad l_\tau = l^{(1)} + 2l^{(2)} + 3l^{(4)} + 2l^{(5)} \end{aligned} \tag{F.17}$$

$n = 1, G = G_2$. Here we only turn on a subgroup $\mathfrak{su}(2)$ of the full flavor group $\mathfrak{sp}(7)$.

$$\begin{array}{|cccc|ccccc|}\hline & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & -1 & -2 & 1 & -1 & 0 & 0 \\ 2 & 3 & 0 & -2 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & -2 & 2 \\ 2 & 3 & -1 & -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -1 & 0 & 1 & 0 & 0 & -1 \\ \hline\end{array} \tag{F.18}$$

The triple intersection ring corresponding to the above Mori cone generators is

$$\mathcal{R}=-(J_1^3+2J_2J_1^2+J_3J_1^2+3J_4J_1^2+2J_5J_1^2+6J_2^2J_1+J_3^2J_1+13J_4^2J_1+4J_5^2J_1+2J_2J_3J_1 +9J_2J_4J_1+3J_3J_4J_1+6J_2J_5J_1+2J_3J_5J_1+7J_4J_5J_1+18J_2^3+54J_4^3+8J_5^3+2J_2J_3^2+39J_2J_4^2 +13J_3J_4^2+12J_2J_5^2+4J_3J_5^2+14J_4J_5^2+6J_2^2J_3+27J_2^2J_4+3J_3^2J_4+9J_2J_3J_4+18J_2^2J_5+2J_3^2J_5 +28J_4^2J_5+6J_2J_3J_5+21J_2J_4J_5+7J_3J_4J_5). \tag{F.19}$$

The relation between the above geometric bases and the Lie algebra bases is

$$\begin{aligned} &l_{G_2}^{(1)}=l^{(2)}+l^{(5)}, \quad l_{G_2}^{(2)}=l^{(4)}, && l_{\mathfrak{su}(2)}=2l^{(4)}+2l^{(5)}, \\ &l_B=l^{(3)}, \qquad\qquad l_\tau=l^{(1)}+2l^{(2)}+3l^{(4)}+2l^{(5)} \end{aligned} \tag{F.20}$$

G Refined BPS invariants

In this appendix, we list the refined BPS invariants solved from the blowup equations up to total degree 7. We drop the base degree zero invariants for all the models as they are exactly known from tensor, vector and hyper multiplets. More results for higher total degrees can be found in the supplementary material or on the website [99].

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,0,1,0,0)$	$(0,0)$	$(1,0,1,0,0)$	$(0,1)$
$(1,1,1,0,0)$	$(0,1/2)$	$(1,1,1,1,0)$	$(0,1/2)$
$(1,2,1,1,0)$	$(0,0)$	$(2,0,1,0,0)$	$(0,2)$
$(2,0,2,0,0)$	$(0,5/2)$	$(2,1,1,0,0)$	$(0,3/2)$
$(2,1,1,1,0)$	$(0,3/2)$	$(2,1,2,0,0)$	$(0,2)$
$(2,1,2,1,0)$	$(0,2)$	$(2,2,1,1,0)$	$(0,1)$
$(2,2,2,1,0)$	$(0,3/2)$	$(3,0,1,0,0)$	$(0,3)$
$(3,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,0,3,0,0)$	$(0,3)\oplus(1/2,9/2)$
$(3,1,1,0,0)$	$(0,5/2)$	$(3,1,1,1,0)$	$(0,5/2)$
$(3,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(3,1,2,1,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$
$(3,1,3,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,2,1,1,0)$	$(0,2)$
$(3,2,2,0,0)$	$(0,5/2)$	$(4,0,1,0,0)$	$(0,4)$
$(4,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(4,0,3,0,0)$	$(0,2)\oplus(0,3)\oplus 2(0,4)\oplus(0,5)\oplus(0,6)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus 2(1/2,11/2)\oplus(1,5)\oplus(1,6)\oplus(3/2,13/2)$
$(4,1,1,0,0)$	$(0,7/2)$	$(4,1,1,1,0)$	$(0,7/2)$
$(4,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$	$(5,0,1,0,0)$	$(0,5)$
$(5,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$	$(5,1,1,0,0)$	$(0,9/2)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(6,0,1,0,0)$	$(0,6)$	$(1,1,1,0,1)$	$(0,0)\oplus(0,1)$
$(1,1,1,1,1)$	$5(0,0)\oplus 5(0,1)$	$(1,1,1,2,1)$	$5(0,0)\oplus 5(0,1)$
$(1,1,1,3,1)$	$(0,0)\oplus(0,1)$	$(1,2,1,0,1)$	$(0,1/2)$
$(1,2,1,1,1)$	$4(0,1/2)$	$(1,2,1,2,1)$	$10(0,1/2)$
$(2,1,1,0,1)$	$(0,1)\oplus(0,2)$	$(2,1,1,1,1)$	$5(0,1)\oplus 5(0,2)$
$(2,1,1,2,1)$	$5(0,1)\oplus 5(0,2)$	$(2,1,2,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,2,1,1)$	$5(0,3/2)\oplus 5(0,5/2)$	$(2,2,1,0,1)$	$(0,1/2)\oplus(0,3/2)$
$(2,2,1,1,1)$	$4(0,1/2)\oplus 4(0,3/2)$	$(2,2,2,0,1)$	$(0,1)\oplus(0,2)$
$(3,1,1,0,1)$	$(0,2)\oplus(0,3)$	$(3,1,1,1,1)$	$5(0,2)\oplus 5(0,3)$
$(3,1,2,0,1)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$	$(3,2,1,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(4,1,1,0,1)$	$(0,3)\oplus(0,4)$	$(1,1,1,2,2)$	$(0,1/2)$
$(1,2,1,0,2)$	$(0,1)$	$(1,2,1,1,2)$	$5(0,0)\oplus 5(0,1)$
$(1,3,1,0,2)$	$(0,3/2)$	$(2,2,1,0,2)$	$(0,0)\oplus(0,1)\oplus(0,2)$

Table 20. Refined BPS invariants of 6d $n=1, G_2$ model, with a $\mathfrak{su}(2)$ mass turned on.

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,0,1,0,0)$	$(0,1/2)$	$(0,1,1,0,0)$	$(0,0)$
$(0,1,1,1,0)$	$(0,0)$	$(1,0,1,0,0)$	$(0,3/2)$
$(1,0,2,0,0)$	$(0,5/2)$	$(1,0,3,0,0)$	$(0,7/2)$
$(1,0,4,0,0)$	$(0,9/2)$	$(1,0,5,0,0)$	$(0,11/2)$
$(1,0,6,0,0)$	$(0,13/2)$	$(1,1,1,0,0)$	$(0,1)$
$(1,1,1,1,0)$	$(0,1)$	$(1,1,2,0,0)$	$(0,2)$
$(1,1,2,1,0)$	$(0,2)$	$(1,1,3,0,0)$	$(0,3)$
$(1,1,3,1,0)$	$(0,3)$	$(1,1,4,0,0)$	$(0,4)$
$(1,1,4,1,0)$	$(0,4)$	$(1,1,5,0,0)$	$(0,5)$
$(1,2,1,1,0)$	$(0,1/2)$	$(1,2,2,1,0)$	$(0,3/2)$
$(1,2,3,1,0)$	$(0,5/2)$	$(2,0,1,0,0)$	$(0,5/2)$
$(2,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(2,0,3,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$
$(2,0,4,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$	$(2,0,5,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus 3(0,13/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus 2(1/2,7)\oplus(1,11/2)\oplus(1,13/2)\oplus 2(1,15/2)\oplus(3/2,7)\oplus(3/2,8)\oplus(2,17/2)$
$(2,1,1,0,0)$	$(0,2)$	$(2,1,1,1,0)$	$(0,2)$
$(2,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(2,1,2,1,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$
$(2,1,3,0,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$	$(2,1,3,1,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$
$(2,1,4,0,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus 4(0,5)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus 3(1/2,11/2)\oplus(1,5)\oplus 2(1,6)\oplus(3/2,13/2)$	$(2,2,1,1,0)$	$(0,3/2)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(2,2,2,0,0)$	$(0,5/2)$	$(2,2,2,1,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus(1/2,3)$
$(2,2,3,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,0,1,0,0)$	$(0,7/2)$
$(3,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(3,0,3,0,0)$	$(0,3/2)\oplus(0,5/2)\oplus 3(0,7/2)\oplus 3(0,9/2)\oplus 4(0,11/2)\oplus(1/2,3)\oplus 2(1/2,4)\oplus 3(1/2,5)\oplus 3(1/2,6)\oplus(1/2,7)\oplus(1,9/2)\oplus 2(1,11/2)\oplus 3(1,13/2)\oplus(3/2,6)\oplus(3/2,7)\oplus(2,15/2)$
$(3,0,4,0,0)$	$(0,1/2)\oplus(0,3/2)\oplus 3(0,5/2)\oplus 4(0,7/2)\oplus 7(0,9/2)\oplus 6(0,11/2)\oplus 7(0,13/2)\oplus(0,15/2)\oplus(0,17/2)\oplus(1/2,2)\oplus 2(1/2,3)\oplus 4(1/2,4)\oplus 6(1/2,5)\oplus 8(1/2,6)\oplus 7(1/2,7)\oplus 2(1/2,8)\oplus(1,7/2)\oplus 2(1,9/2)\oplus 5(1,11/2)\oplus 6(1,13/2)\oplus 7(1,15/2)\oplus(1,17/2)\oplus(3/2,5)\oplus 2(3/2,6)\oplus 4(3/2,7)\oplus 4(3/2,8)\oplus(3/2,9)\oplus(2,13/2)\oplus 2(2,15/2)\oplus 3(2,17/2)\oplus(5/2,8)\oplus(5/2,9)\oplus(3,19/2)$	$(3,1,1,0,0)$	$(0,3)$
$(3,1,1,1,0)$	$(0,3)$	$(3,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$
$(3,1,2,1,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$	$(3,1,3,0,0)$	$(0,1)\oplus 2(0,2)\oplus 5(0,3)\oplus 6(0,4)\oplus 7(0,5)\oplus(0,6)\oplus(1/2,5/2)\oplus 3(1/2,7/2)\oplus 6(1/2,9/2)\oplus 6(1/2,11/2)\oplus(1/2,13/2)\oplus(1,4)\oplus 3(1,5)\oplus 5(1,6)\oplus(3/2,11/2)\oplus 2(3/2,13/2)\oplus(2,7)$
$(3,2,1,1,0)$	$(0,5/2)$	$(3,2,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$
$(4,0,1,0,0)$	$(0,9/2)$	$(4,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$
$(4,0,3,0,0)$	$(0,1/2)\oplus(0,3/2)\oplus 3(0,5/2)\oplus 4(0,7/2)\oplus 7(0,9/2)\oplus 6(0,11/2)\oplus 7(0,13/2)\oplus(0,15/2)\oplus(0,17/2)\oplus(1/2,2)\oplus 2(1/2,3)\oplus 4(1/2,4)\oplus 6(1/2,5)\oplus 8(1/2,6)\oplus 7(1/2,7)\oplus 2(1/2,8)\oplus(1,7/2)\oplus 2(1,9/2)\oplus 5(1,11/2)\oplus 6(1,13/2)\oplus 7(1,15/2)\oplus(1,17/2)\oplus(3/2,5)\oplus 2(3/2,6)\oplus 4(3/2,7)\oplus 4(3/2,8)\oplus(3/2,9)\oplus(2,13/2)\oplus 2(2,15/2)\oplus 3(2,17/2)\oplus(5/2,8)\oplus(5/2,9)\oplus(3,19/2)$	$(4,1,1,0,0)$	$(0,4)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(4,1,1,1,0)$	$(0,4)$	$(4,1,2,0,0)$	$(0,2) \oplus 2(0,3) \oplus 3(0,4) \oplus 4(0,5) \oplus (1/2,7/2) \oplus 2(1/2,9/2) \oplus 3(1/2,11/2) \oplus (1,5) \oplus 2(1,6) \oplus (3/2,13/2)$
$(5,0,1,0,0)$	$(0,11/2)$	$(5,0,2,0,0)$	$(0,5/2) \oplus (0,7/2) \oplus 2(0,9/2) \oplus 2(0,11/2) \oplus 3(0,13/2) \oplus (1/2,4) \oplus (1/2,5) \oplus 2(1/2,6) \oplus 2(1/2,7) \oplus (1,11/2) \oplus (1,13/2) \oplus 2(1,15/2) \oplus (3/2,7) \oplus (3/2,8) \oplus (2,17/2)$
$(5,1,1,0,0)$	$(0,5)$	$(6,0,1,0,0)$	$(0,13/2)$
$(0,1,1,0,1)$	$(0,1/2)$	$(0,1,1,1,1)$	$2(0,1/2)$
$(0,1,1,2,1)$	$2(0,1/2)$	$(0,1,1,3,1)$	$(0,1/2)$
$(0,2,1,0,1)$	$(0,1)$	$(0,2,1,1,1)$	$2(0,0) \oplus 2(0,1)$
$(0,2,1,2,1)$	$3(0,0) \oplus (0,1)$	$(0,2,1,3,1)$	$14(0,0) \oplus 2(0,1)$
$(1,1,1,0,1)$	$(0,1/2) \oplus (0,3/2)$	$(1,1,1,1,1)$	$2(0,1/2) \oplus 2(0,3/2)$
$(1,1,1,2,1)$	$2(0,1/2) \oplus 2(0,3/2)$	$(1,1,1,3,1)$	$(0,1/2) \oplus (0,3/2)$
$(1,1,2,0,1)$	$(0,3/2) \oplus (0,5/2)$	$(1,1,2,1,1)$	$2(0,3/2) \oplus 2(0,5/2)$
$(1,1,2,2,1)$	$2(0,3/2) \oplus 2(0,5/2)$	$(1,1,3,0,1)$	$(0,5/2) \oplus (0,7/2)$
$(1,1,3,1,1)$	$2(0,5/2) \oplus 2(0,7/2)$	$(1,1,4,0,1)$	$(0,7/2) \oplus (0,9/2)$
$(1,2,1,0,1)$	$(0,0) \oplus (0,1)$	$(1,2,1,1,1)$	$(0,0) \oplus (0,1)$
$(1,2,1,2,1)$	$4(0,0) \oplus 4(0,1)$	$(1,2,2,0,1)$	$(0,1) \oplus (0,2)$
$(1,2,2,1,1)$	$(0,1) \oplus (0,2)$	$(1,2,3,0,1)$	$(0,2) \oplus (0,3)$
$(1,3,1,1,1)$	$(0,1/2)$	$(2,1,1,0,1)$	$(0,3/2) \oplus (0,5/2)$
$(2,1,1,1,1)$	$2(0,3/2) \oplus 2(0,5/2)$	$(2,1,1,2,1)$	$2(0,3/2) \oplus 2(0,5/2)$
$(2,1,2,0,1)$	$(0,3/2) \oplus 3(0,5/2) \oplus 2(0,7/2) \oplus (1/2,3) \oplus (1/2,4)$	$(2,1,2,1,1)$	$2(0,3/2) \oplus 6(0,5/2) \oplus 4(0,7/2) \oplus 2(1/2,3) \oplus 2(1/2,4)$
$(2,1,3,0,1)$	$(0,3/2) \oplus 3(0,5/2) \oplus 5(0,7/2) \oplus 3(0,9/2) \oplus (1/2,3) \oplus 3(1/2,4) \oplus 2(1/2,5) \oplus (1,9/2) \oplus (1,11/2)$	$(2,2,1,0,1)$	$(0,1) \oplus (0,2)$
$(2,2,1,1,1)$	$(0,1) \oplus (0,2)$	$(2,2,2,0,1)$	$(0,1) \oplus 4(0,2) \oplus 3(0,3) \oplus (1/2,5/2) \oplus (1/2,7/2)$
$(3,1,1,0,1)$	$(0,5/2) \oplus (0,7/2)$	$(3,1,1,1,1)$	$2(0,5/2) \oplus 2(0,7/2)$
$(3,1,2,0,1)$	$(0,3/2) \oplus 3(0,5/2) \oplus 5(0,7/2) \oplus 3(0,9/2) \oplus (1/2,3) \oplus 3(1/2,4) \oplus 2(1/2,5) \oplus (1,9/2) \oplus (1,11/2)$	$(3,2,1,0,1)$	$(0,2) \oplus (0,3)$
$(4,1,1,0,1)$	$(0,7/2) \oplus (0,9/2)$	$(0,1,1,2,2)$	$(0,0)$
$(0,1,1,3,2)$	$(0,0)$	$(0,2,1,0,2)$	$(0,3/2)$
$(0,2,1,1,2)$	$2(0,1/2) \oplus 2(0,3/2)$	$(0,2,1,2,2)$	$5(0,1/2) \oplus (0,3/2)$
$(0,3,1,0,2)$	$(0,2)$	$(0,3,1,1,2)$	$2(0,1) \oplus 2(0,2)$
$(0,3,2,0,2)$	$(0,2)$	$(1,1,1,2,2)$	$(0,1)$
$(1,2,1,0,2)$	$(0,1/2) \oplus (0,3/2)$	$(1,2,1,1,2)$	$4(0,1/2) \oplus 2(0,3/2)$
$(1,2,2,0,2)$	$(0,1/2) \oplus (0,3/2) \oplus (0,5/2)$	$(1,3,1,0,2)$	$(0,1) \oplus (0,2)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(2,2,1,0,2)$	$(0,1/2)\oplus(0,3/2)\oplus(0,5/2)$	$(0,3,1,0,3)$	$(0,5/2)$

Table 21. Refined BPS invariants of 6d $n=2, G_2$ model, with a $\mathfrak{su}(2)$ mass turned on.

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,0,1,0,0)$	$(0,0)$	$(0,1,1,0,0)$	$(0,1/2)$
$(0,1,1,1,0)$	$(0,1/2)$	$(1,0,1,0,0)$	$(0,1)$
$(1,1,1,0,0)$	$(0,1/2)$	$(1,1,1,1,0)$	$(0,1/2)$
$(1,2,1,1,0)$	$(0,0)$	$(2,0,1,0,0)$	$(0,2)$
$(2,0,2,0,0)$	$(0,5/2)$	$(2,1,1,0,0)$	$(0,3/2)$
$(2,1,1,1,0)$	$(0,3/2)$	$(2,1,2,0,0)$	$(0,2)$
$(2,1,2,1,0)$	$(0,2)$	$(2,2,1,1,0)$	$(0,1)$
$(2,2,2,1,0)$	$(0,3/2)$	$(3,0,1,0,0)$	$(0,3)$
$(3,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,0,3,0,0)$	$(0,3)\oplus(1/2,9/2)$
$(3,1,1,0,0)$	$(0,5/2)$	$(3,1,1,1,0)$	$(0,5/2)$
$(3,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(3,1,2,1,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$
$(3,1,3,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,2,1,1,0)$	$(0,2)$
$(3,2,2,0,0)$	$(0,5/2)$	$(4,0,1,0,0)$	$(0,4)$
$(4,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(4,0,3,0,0)$	$(0,2)\oplus(0,3)\oplus 2(0,4)\oplus(0,5)\oplus(0,6)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus 2(1/2,11/2)\oplus(1,5)\oplus(1,6)\oplus(3/2,13/2)$
$(4,1,1,0,0)$	$(0,7/2)$	$(4,1,1,1,0)$	$(0,7/2)$
$(4,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$	$(5,0,1,0,0)$	$(0,5)$
$(5,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$	$(5,1,1,0,0)$	$(0,9/2)$
$(6,0,1,0,0)$	$(0,6)$	$(0,1,1,0,1)$	$(0,1)$
$(0,1,1,1,1)$	$(0,0)\oplus(0,1)$	$(0,1,1,2,1)$	$(0,0)\oplus(0,1)$
$(0,1,1,3,1)$	$(0,1)$	$(0,2,1,0,1)$	$(0,3/2)$
$(0,2,1,1,1)$	$(0,1/2)\oplus(0,3/2)$	$(0,2,1,2,1)$	$(0,1/2)\oplus(0,3/2)$
$(0,2,1,3,1)$	$(0,1/2)\oplus(0,3/2)$	$(0,2,2,0,1)$	$(0,2)$
$(0,2,2,1,1)$	$(0,1)\oplus(0,2)$	$(0,2,2,2,1)$	$(0,0)\oplus(0,1)\oplus(0,2)$
$(0,3,2,0,1)$	$(0,5/2)$	$(0,3,2,1,1)$	$(0,3/2)\oplus(0,5/2)$
$(0,3,3,0,1)$	$(0,3)$	$(1,1,1,0,1)$	$(0,0)\oplus(0,1)$
$(1,1,1,1,1)$	$(0,0)\oplus(0,1)$	$(1,1,1,2,1)$	$(0,0)\oplus(0,1)$
$(1,1,1,3,1)$	$(0,0)\oplus(0,1)$	$(1,2,1,0,1)$	$(0,1/2)\oplus(0,3/2)$
$(1,2,1,1,1)$	$3(0,1/2)\oplus(0,3/2)$	$(1,2,1,2,1)$	$3(0,1/2)\oplus(0,3/2)$
$(1,2,2,0,1)$	$(0,1)\oplus(0,2)$	$(1,2,2,1,1)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(1,3,1,1,1)$	$(0,1)$	$(1,3,2,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,1,0,1)$	$(0,1)\oplus(0,2)$	$(2,1,1,1,1)$	$(0,1)\oplus(0,2)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(2,1,1,2,1)$	$(0,1)\oplus(0,2)$	$(2,1,2,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,2,1,1)$	$(0,3/2)\oplus(0,5/2)$	$(2,2,1,0,1)$	$(0,1/2)\oplus(0,3/2)$
$(2,2,1,1,1)$	$2(0,1/2)\oplus 2(0,3/2)$	$(2,2,2,0,1)$	$(0,0)\oplus 2(0,1)\oplus 2(0,2)\oplus(0,3)$
$(3,1,1,0,1)$	$(0,2)\oplus(0,3)$	$(3,1,1,1,1)$	$(0,2)\oplus(0,3)$
$(3,1,2,0,1)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$	$(3,2,1,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(4,1,1,0,1)$	$(0,3)\oplus(0,4)$	$(0,1,1,2,2)$	$(0,1/2)$
$(0,1,1,3,2)$	$(0,1/2)$	$(0,2,1,0,2)$	$(0,2)$
$(0,2,1,1,2)$	$(0,1)\oplus(0,2)$	$(0,2,1,2,2)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(0,2,2,0,2)$	$(0,5/2)$	$(0,2,2,1,2)$	$(0,3/2)\oplus(0,5/2)$
$(0,3,1,0,2)$	$(0,5/2)$	$(0,3,1,1,2)$	$(0,3/2)\oplus(0,5/2)$
$(0,3,2,0,2)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(1,1,1,2,2)$	$(0,1/2)$
$(1,2,1,0,2)$	$(0,1)\oplus(0,2)$	$(1,2,1,1,2)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(1,2,2,0,2)$	$(0,3/2)\oplus(0,5/2)$	$(1,3,1,0,2)$	$(0,3/2)\oplus(0,5/2)$
$(2,2,1,0,2)$	$(0,0)\oplus(0,1)\oplus(0,2)$	$(0,3,1,0,3)$	$(0,3)$

Table 22. Refined BPS invariants of 6d $n=3, G_2$ model, with a $\mathfrak{su}(2)$ mass turned on.

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,0,0,0,1,0,0)$	$(0,0)$	$(0,0,1,0,1,0,0)$	$(0,1)$
$(0,0,2,0,1,0,0)$	$(0,2)$	$(0,0,2,0,2,0,0)$	$(0,5/2)$
$(0,0,3,0,1,0,0)$	$(0,3)$	$(0,0,3,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$
$(0,0,3,0,3,0,0)$	$(0,3)\oplus(1/2,9/2)$	$(0,0,4,0,1,0,0)$	$(0,4)$
$(0,0,4,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(0,0,4,0,3,0,0)$	$(0,2)\oplus(0,3)\oplus 2(0,4)\oplus(0,5)\oplus(0,6)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus 2(1/2,11/2)\oplus(1,5)\oplus(1,6)\oplus(3/2,13/2)$
$(0,0,5,0,1,0,0)$	$(0,5)$	$(0,0,5,0,2,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$
$(0,0,6,0,1,0,0)$	$(0,6)$	$(0,1,0,0,1,0,0)$	$(0,0)$
$(0,1,0,1,1,0,0)$	$(0,1/2)$	$(0,1,0,1,1,1,0)$	$(0,1/2)$
$(0,1,1,1,1,0,0)$	$(0,1/2)$	$(0,1,1,1,1,1,0)$	$(0,1/2)$
$(0,1,2,1,1,0,0)$	$(0,3/2)$	$(0,1,2,1,1,1,0)$	$(0,3/2)$
$(0,1,2,1,2,0,0)$	$(0,2)$	$(0,1,2,1,2,1,0)$	$(0,2)$
$(0,1,3,1,1,0,0)$	$(0,5/2)$	$(0,1,3,1,1,1,0)$	$(0,5/2)$
$(0,1,3,1,2,0,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(0,1,4,1,1,0,0)$	$(0,7/2)$
$(0,2,1,2,1,1,0)$	$(0,0)$	$(1,1,0,1,1,0,0)$	$(0,1/2)$
$(1,1,0,1,1,1,0)$	$(0,1/2)$	$(1,1,0,2,1,0,0)$	$(0,1)$
$(1,1,0,2,1,1,0)$	$(0,0)\oplus(0,1)$	$(1,1,0,2,1,2,0)$	$(0,1)$
$(1,1,1,1,1,0,0)$	$(0,1/2)$	$(1,1,1,1,1,1,0)$	$(0,1/2)$
$(1,1,1,2,1,0,0)$	$(0,0)\oplus(0,1)$	$(1,1,1,2,1,1,0)$	$(0,0)\oplus(0,1)$

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β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(1,1,2,1,1,0,0)$	$(0,3/2)$	$(1,1,2,1,1,1,0)$	$(0,3/2)$
$(1,1,2,1,2,0,0)$	$(0,2)$	$(1,1,2,2,1,0,0)$	$(0,1)\oplus(0,2)$
$(1,1,3,1,1,0,0)$	$(0,5/2)$	$(1,2,0,2,1,0,0)$	$(0,1)$
$(1,2,0,2,1,1,0)$	$(0,0)\oplus(0,1)$	$(1,2,0,2,2,0,0)$	$(0,3/2)$
$(1,2,0,3,1,0,0)$	$(0,3/2)$	$(1,2,1,2,1,0,0)$	$(0,0)\oplus(0,1)$
$(0,1,0,2,1,1,2)$	$(0,1)$		

Table 23. Refined BPS invariants of 6d $n=3,\mathfrak{so}(7)$ model, with $\mathfrak{su}(2)\times\mathfrak{su}(2)$ masses turned on.

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,0,1,0,0,0,0,0,0)$	$(0,0)$	$(0,1,1,0,0,0,0,0,0)$	$(0,1)$
$(0,1,1,0,1,0,0,0,0)$	$(0,0)\oplus(0,1)$	$(0,1,1,0,2,0,0,0,0)$	$(0,1)$
$(0,1,1,0,3,0,0,0,0)$	?	$(0,1,1,0,4,0,0,0,0)$	?
$(0,1,1,0,5,0,0,0,0)$	?	$(0,1,1,1,1,0,0,0,0)$	$(0,0)\oplus(0,1)$
$(0,1,1,1,2,0,0,0,0)$	$(0,0)\oplus(0,1)$	$(0,1,1,1,3,0,0,0,0)$	?
$(0,1,1,1,4,0,0,0,0)$	?	$(0,1,1,2,2,0,0,0,0)$	$(0,1)$
$(0,1,1,2,3,0,0,0,0)$	?	$(0,2,1,0,0,0,0,0,0)$	$(0,2)$
$(0,2,1,0,1,0,0,0,0)$	$(0,1)\oplus(0,2)$	$(0,2,1,0,2,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$
$(0,2,1,0,3,0,0,0,0)$	$(0,1)\oplus(0,2)$	$(0,2,1,0,4,0,0,0,0)$	?
$(0,2,1,1,1,0,0,0,0)$	$(0,1)\oplus(0,2)$	$(0,2,1,1,2,0,0,0,0)$	$(0,0)\oplus2(0,1)\oplus(0,2)$
$(0,2,1,1,3,0,0,0,0)$	$(0,0)\oplus2(0,1)\oplus(0,2)$	$(0,2,1,2,2,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$
$(0,2,2,0,0,0,0,0,0)$	$(0,5/2)$	$(0,2,2,0,1,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$
$(0,2,2,0,2,0,0,0,0)$	$(0,1/2)\oplus(0,3/2)\oplus(0,5/2)$	$(0,2,2,0,3,0,0,0,0)$	?
$(0,2,2,1,1,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$	$(0,2,2,1,2,0,0,0,0)$	$(0,1/2)\oplus2(0,3/2)\oplus(0,5/2)$
$(0,3,1,0,0,0,0,0,0)$	$(0,3)$	$(0,3,1,0,1,0,0,0,0)$	$(0,2)\oplus(0,3)$
$(0,3,1,0,2,0,0,0,0)$	$(0,1)\oplus(0,2)\oplus(0,3)$	$(0,3,1,0,3,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)\oplus(0,3)$
$(0,3,1,1,1,0,0,0,0)$	$(0,2)\oplus(0,3)$	$(0,3,1,1,2,0,0,0,0)$	$(0,1)\oplus2(0,2)\oplus(0,3)$
$(0,3,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(0,3,2,0,1,0,0,0,0)$	$(0,3/2)\oplus3(0,5/2)\oplus2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(0,3,2,0,2,0,0,0,0)$	$(0,1/2)\oplus3(0,3/2)\oplus4(0,5/2)\oplus2(0,7/2)\oplus(1/2,2)\oplus(1/2,3)\oplus(1/2,4)$	$(0,3,2,1,1,0,0,0,0)$	$(0,3/2)\oplus3(0,5/2)\oplus2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(0,3,3,0,0,0,0,0,0)$	$(0,3)\oplus(1/2,9/2)$	$(0,3,3,0,1,0,0,0,0)$	?
$(0,4,1,0,0,0,0,0,0)$	$(0,4)$	$(0,4,1,0,1,0,0,0,0)$	$(0,3)\oplus(0,4)$
$(0,4,1,0,2,0,0,0,0)$	$(0,2)\oplus(0,3)\oplus(0,4)$	$(0,4,1,1,1,0,0,0,0)$	$(0,3)\oplus(0,4)$
$(0,4,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(0,4,2,0,1,0,0,0,0)$	$(0,3/2)\oplus3(0,5/2)\oplus5(0,7/2)\oplus3(0,9/2)\oplus(1/2,3)\oplus3(1/2,4)\oplus2(1/2,5)\oplus(1,9/2)\oplus(1,11/2)$
$(0,4,3,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)\oplus2(0,4)\oplus(0,5)\oplus(0,6)\oplus(1/2,7/2)\oplus2(1/2,9/2)\oplus2(1/2,11/2)\oplus(1,5)\oplus(1,6)\oplus(3/2,13/2)$	$(0,5,1,0,0,0,0,0,0)$	$(0,5)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(0,5,1,0,1,0,0,0,0)$	$(0,4)\oplus(0,5)$	$(0,5,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$
$(0,6,1,0,0,0,0,0,0)$	$(0,6)$	$(1,0,1,0,0,0,0,0,0)$	$(0,1)$
$(1,0,1,0,0,1,0,0,0)$	$(0,0)\oplus(0,1)$	$(1,0,1,0,0,1,1,0,0)$	$(0,1/2)$
$(1,0,1,0,0,1,1,1,0)$	$(0,1/2)$	$(1,0,1,0,0,1,2,1,0)$	?
$(1,0,1,0,0,2,0,0,0)$	$(0,1)$	$(1,0,1,0,0,2,1,0,0)$	$(0,1/2)$
$(1,0,1,0,0,2,1,1,0)$	$(0,1/2)$	$(1,0,1,0,0,2,2,1,0)$	?
$(1,0,1,0,0,3,0,0,0)$	$(0,2)$	$(1,0,1,0,0,3,1,0,0)$	$(0,3/2)$
$(1,0,1,0,0,3,1,1,0)$	?	$(1,0,1,0,0,4,0,0,0)$	$(0,3)$
$(1,0,1,0,0,4,1,0,0)$	$(0,5/2)$	$(1,0,1,0,0,5,0,0,0)$	$(0,4)$
$(1,1,1,0,0,0,0,0,0)$	$(0,0)\oplus(0,1)$	$(1,1,1,0,0,1,0,0,0)$	$(0,0)\oplus(0,1)$
$(1,1,1,0,0,1,1,0,0)$	$(0,1/2)$	$(1,1,1,0,0,1,1,1,0)$	$(0,1/2)$
$(1,1,1,0,0,1,2,1,0)$	$(0,0)\oplus(0,1)$	$(1,1,1,0,1,0,0,0,0)$	$(0,0)\oplus(0,1)$
$(1,1,1,0,1,1,0,0,0)$	$(0,0)\oplus(0,1)$	$(1,1,1,0,1,1,1,0,0)$	$(0,1/2)$
$(1,1,1,0,1,1,1,1,0)$	$(0,1/2)$	$(1,1,1,1,1,0,0,0,0)$	$(0,0)\oplus(0,1)$
$(1,1,1,1,1,1,0,0,0)$	$(0,0)\oplus(0,1)$	$(1,1,1,1,1,1,1,0,0)$	$(0,1/2)$
$(1,2,1,0,0,0,0,0,0)$	$(0,1)\oplus(0,2)$	$(1,2,1,0,0,1,0,0,0)$	$(0,1)\oplus(0,2)$
$(1,2,1,0,0,1,1,0,0)$	$(0,3/2)$	$(1,2,1,0,0,1,1,1,0)$	$(0,3/2)$
$(1,2,1,0,1,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(1,2,1,0,1,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(1,2,1,0,1,1,1,0,0)$	$(0,1/2)\oplus(0,3/2)$	$(1,2,1,0,2,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(1,2,1,0,2,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(1,2,1,0,3,0,0,0,0)$	$(0,1)\oplus(0,2)$
$(1,2,1,1,1,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(1,2,1,1,1,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(1,2,1,1,2,0,0,0,0)$	$2(0,0)\oplus 3(0,1)\oplus(0,2)$	$(1,2,2,0,0,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$
$(1,2,2,0,0,1,0,0,0)$	$(0,3/2)\oplus(0,5/2)$	$(1,2,2,0,0,1,1,0,0)$	$(0,2)$
$(1,2,2,0,1,0,0,0,0)$	$(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$	$(1,2,2,0,1,1,0,0,0)$	$(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$
$(1,2,2,0,2,0,0,0,0)$	$2(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$	$(1,2,2,1,1,0,0,0,0)$	$(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$
$(1,3,1,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)$	$(1,3,1,0,0,1,0,0,0)$	$(0,2)\oplus(0,3)$
$(1,3,1,0,0,1,1,0,0)$	$(0,5/2)$	$(1,3,1,0,1,0,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$
$(1,3,1,0,1,1,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$	$(1,3,1,0,2,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus 2(0,2)\oplus(0,3)$
$(1,3,1,1,1,0,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$	$(1,3,2,0,0,0,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(1,3,2,0,0,1,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$	$(1,3,2,0,1,0,0,0,0)$	$(0,1/2)\oplus 5(0,3/2)\oplus 7(0,5/2)\oplus 3(0,7/2)\oplus(1/2,2)\oplus 2(1/2,3)\oplus(1/2,4)$
$(1,3,3,0,0,0,0,0,0)$	$(0,2)\oplus 2(0,3)\oplus(0,4)\oplus(1/2,7/2)\oplus(1/2,9/2)$	$(1,4,1,0,0,0,0,0,0)$	$(0,3)\oplus(0,4)$
$(1,4,1,0,0,1,0,0,0)$	$(0,3)\oplus(0,4)$	$(1,4,1,0,1,0,0,0,0)$	$(0,2)\oplus 2(0,3)\oplus(0,4)$
$(1,4,2,0,0,0,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 5(0,7/2)\oplus 3(0,9/2)\oplus(1/2,3)\oplus 3(1/2,4)\oplus 2(1/2,5)\oplus(1,9/2)\oplus(1,11/2)$	$(1,5,1,0,0,0,0,0,0)$	$(0,4)\oplus(0,5)$

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β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(2,0,1,0,0,0,0,0,0)$	$(0,2)$	$(2,0,1,0,0,1,0,0,0)$	$(0,1)\oplus(0,2)$
$(2,0,1,0,0,1,1,0,0)$	$(0,3/2)$	$(2,0,1,0,0,1,1,1,0)$	$(0,3/2)$
$(2,0,1,0,0,1,2,1,0)$	?	$(2,0,1,0,0,2,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$
$(2,0,1,0,0,2,1,0,0)$	$(0,1/2)\oplus(0,3/2)$	$(2,0,1,0,0,2,1,1,0)$	$(0,1/2)\oplus(0,3/2)$
$(2,0,1,0,0,3,0,0,0)$	$(0,1)\oplus(0,2)$	$(2,0,1,0,0,3,1,0,0)$	$(0,1/2)\oplus(0,3/2)$
$(2,0,1,0,0,4,0,0,0)$	$(0,2)\oplus(0,3)$	$(2,0,2,0,0,0,0,0,0)$	$(0,5/2)$
$(2,0,2,0,0,1,0,0,0)$	$(0,3/2)\oplus(0,5/2)$	$(2,0,2,0,0,1,1,0,0)$	$(0,2)$
$(2,0,2,0,0,1,1,1,0)$	$(0,2)$	$(2,0,2,0,0,2,0,0,0)$	$(0,1/2)\oplus(0,3/2)\oplus(0,5/2)$
$(2,0,2,0,0,2,1,0,0)$	$(0,1)\oplus(0,2)$	$(2,0,2,0,0,3,0,0,0)$	$(0,1/2)\oplus(0,3/2)\oplus(0,5/2)$
$(2,1,1,0,0,0,0,0,0)$	$(0,1)\oplus(0,2)$	$(2,1,1,0,0,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(2,1,1,0,0,1,1,0,0)$	$(0,1/2)\oplus(0,3/2)$	$(2,1,1,0,0,1,1,1,0)$	$(0,1/2)\oplus(0,3/2)$
$(2,1,1,0,0,2,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,1,1,0,0,2,1,0,0)$	$2(0,1/2)\oplus(0,3/2)$
$(2,1,1,0,0,3,0,0,0)$	$(0,1)\oplus(0,2)$	$(2,1,1,0,1,0,0,0,0)$	$(0,1)\oplus(0,2)$
$(2,1,1,0,1,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,1,1,0,1,1,1,0,0)$	$(0,1/2)\oplus(0,3/2)$
$(2,1,1,0,1,2,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,1,1,1,1,0,0,0,0)$	$(0,1)\oplus(0,2)$
$(2,1,1,1,1,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,1,2,0,0,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,2,0,0,1,0,0,0)$	$(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$	$(2,1,2,0,0,1,1,0,0)$	$(0,1)\oplus(0,2)$
$(2,1,2,0,0,2,0,0,0)$	$2(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$	$(2,1,2,0,1,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,2,0,1,1,0,0,0)$	$(0,1/2)\oplus 2(0,3/2)\oplus(0,5/2)$	$(2,1,2,1,1,0,0,0,0)$	$(0,3/2)\oplus(0,5/2)$
$(2,2,1,0,0,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$	$(2,2,1,0,0,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(2,2,1,0,0,1,1,0,0)$	$(0,1/2)\oplus(0,3/2)$	$(2,2,1,0,0,2,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$
$(2,2,1,0,1,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,2,1,0,1,1,0,0,0)$	$2(0,0)\oplus 3(0,1)\oplus(0,2)$
$(2,2,1,0,2,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)$	$(2,2,1,1,1,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$
$(2,2,2,0,0,0,0,0,0)$	$2(0,1/2)\oplus 2(0,3/2)\oplus 2(0,5/2)\oplus(0,7/2)$	$(2,2,2,0,0,1,0,0,0)$	$3(0,1/2)\oplus 4(0,3/2)\oplus 3(0,5/2)\oplus(0,7/2)$
$(2,2,2,0,1,0,0,0,0)$	$3(0,1/2)\oplus 4(0,3/2)\oplus 3(0,5/2)\oplus(0,7/2)$	$(2,2,3,0,0,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)\oplus(0,3)\oplus(0,4)$
$(2,3,1,0,0,0,0,0,0)$	$(0,1)\oplus(0,2)\oplus(0,3)$	$(2,3,1,0,0,1,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$
$(2,3,1,0,1,0,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus 2(0,2)\oplus(0,3)$	$(2,3,2,0,0,0,0,0,0)$	$2(0,1/2)\oplus 4(0,3/2)\oplus 5(0,5/2)\oplus 3(0,7/2)\oplus(0,9/2)\oplus(1/2,2)\oplus(1/2,3)\oplus(1/2,4)$
$(2,4,1,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)\oplus(0,4)$	$(3,0,1,0,0,0,0,0,0)$	$(0,3)$
$(3,0,1,0,0,1,0,0,0)$	$(0,2)\oplus(0,3)$	$(3,0,1,0,0,1,1,0,0)$	$(0,5/2)$
$(3,0,1,0,0,1,1,1,0)$	$(0,5/2)$	$(3,0,1,0,0,2,0,0,0)$	$(0,1)\oplus(0,2)\oplus(0,3)$
$(3,0,1,0,0,2,1,0,0)$	$(0,3/2)\oplus(0,5/2)$	$(3,0,1,0,0,3,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)\oplus(0,3)$
$(3,0,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus(1/2,4)$	$(3,0,2,0,0,1,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(3,0,2,0,0,1,1,0,0)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(3,0,2,0,0,2,0,0,0)$	$(0,1/2)\oplus 3(0,3/2)\oplus 4(0,5/2)\oplus 2(0,7/2)\oplus(1/2,2)\oplus(1/2,3)\oplus(1/2,4)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(3,0,3,0,0,0,0,0,0)$	$(0,3)\oplus(1/2,9/2)$	$(3,0,3,0,0,1,0,0,0)$	$(0,2)\oplus 2(0,3)\oplus(0,4)\oplus(1/2,7/2)\oplus(1/2,9/2)$
$(3,1,1,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)$	$(3,1,1,0,0,1,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$
$(3,1,1,0,0,1,1,0,0)$	$(0,3/2)\oplus(0,5/2)$	$(3,1,1,0,0,2,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus 2(0,2)\oplus(0,3)$
$(3,1,1,0,1,0,0,0,0)$	$(0,2)\oplus(0,3)$	$(3,1,1,0,1,1,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$
$(3,1,1,1,1,0,0,0,0)$	$(0,2)\oplus(0,3)$	$(3,1,2,0,0,0,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(3,1,2,0,0,1,0,0,0)$	$(0,1/2)\oplus 5(0,3/2)\oplus 7(0,5/2)\oplus 3(0,7/2)\oplus(1/2,2)\oplus 2(1/2,3)\oplus(1/2,4)$	$(3,1,2,0,1,0,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 2(0,7/2)\oplus(1/2,3)\oplus(1/2,4)$
$(3,1,3,0,0,0,0,0,0)$	$(0,2)\oplus 2(0,3)\oplus(0,4)\oplus(1/2,7/2)\oplus(1/2,9/2)$	$(3,2,1,0,0,0,0,0,0)$	$(0,1)\oplus(0,2)\oplus(0,3)$
$(3,2,1,0,0,1,0,0,0)$	$(0,0)\oplus 2(0,1)\oplus 2(0,2)\oplus(0,3)$	$(3,2,1,0,1,0,0,0,0)$	$(0,1)\oplus 2(0,2)\oplus(0,3)$
$(3,2,2,0,0,0,0,0,0)$	$2(0,1/2)\oplus 4(0,3/2)\oplus 5(0,5/2)\oplus 3(0,7/2)\oplus(0,9/2)\oplus(1/2,2)\oplus(1/2,3)\oplus(1/2,4)$	$(3,3,1,0,0,0,0,0,0)$	$(0,0)\oplus(0,1)\oplus(0,2)\oplus(0,3)$
$(4,0,1,0,0,0,0,0,0)$	$(0,4)$	$(4,0,1,0,0,1,0,0,0)$	$(0,3)\oplus(0,4)$
$(4,0,1,0,0,1,1,0,0)$	$(0,7/2)$	$(4,0,1,0,0,2,0,0,0)$	$(0,2)\oplus(0,3)\oplus(0,4)$
$(4,0,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus(1/2,4)\oplus(1/2,5)\oplus(1,11/2)$	$(4,0,2,0,0,1,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 5(0,7/2)\oplus 3(0,9/2)\oplus(1/2,3)\oplus 3(1/2,4)\oplus 2(1/2,5)\oplus(1,9/2)\oplus(1,11/2)$
$(4,0,3,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)\oplus 2(0,4)\oplus(0,5)\oplus(0,6)\oplus(1/2,7/2)\oplus 2(1/2,9/2)\oplus 2(1/2,11/2)\oplus(1,5)\oplus(1,6)\oplus(3/2,13/2)$	$(4,1,1,0,0,0,0,0,0)$	$(0,3)\oplus(0,4)$
$(4,1,1,0,0,1,0,0,0)$	$(0,2)\oplus 2(0,3)\oplus(0,4)$	$(4,1,1,0,1,0,0,0,0)$	$(0,3)\oplus(0,4)$
$(4,1,2,0,0,0,0,0,0)$	$(0,3/2)\oplus 3(0,5/2)\oplus 5(0,7/2)\oplus 3(0,9/2)\oplus(1/2,3)\oplus 3(1/2,4)\oplus 2(1/2,5)\oplus(1,9/2)\oplus(1,11/2)$	$(4,2,1,0,0,0,0,0,0)$	$(0,2)\oplus(0,3)\oplus(0,4)$
$(5,0,1,0,0,0,0,0,0)$	$(0,5)$	$(5,0,1,0,0,1,0,0,0)$	$(0,4)\oplus(0,5)$
$(5,0,2,0,0,0,0,0,0)$	$(0,5/2)\oplus(0,7/2)\oplus 2(0,9/2)\oplus 2(0,11/2)\oplus(1/2,4)\oplus(1/2,5)\oplus 2(1/2,6)\oplus(1,11/2)\oplus(1,13/2)\oplus(3/2,7)$	$(5,1,1,0,0,0,0,0,0)$	$(0,4)\oplus(0,5)$
$(6,0,1,0,0,0,0,0,0)$	$(0,6)$	$(1,0,1,0,0,1,1,0,1)$	$(0,0)\oplus(0,1)$
$(1,0,1,0,0,1,1,1,1)$	$(0,0)\oplus(0,1)$	$(1,0,1,0,0,1,2,1,1)$	$(0,1/2)$
$(1,0,1,0,0,2,1,0,1)$	$(0,0)\oplus(0,1)$	$(1,0,1,0,0,2,1,1,1)$	$(0,0)\oplus(0,1)$
$(1,0,1,0,0,2,2,0,1)$	$(0,1/2)$	$(1,0,1,0,0,3,1,0,1)$	$(0,1)\oplus(0,2)$
$(1,1,1,0,0,1,1,0,1)$	$(0,0)\oplus(0,1)$	$(1,1,1,0,0,1,1,1,1)$	$(0,0)\oplus(0,1)$
$(1,1,1,0,1,1,1,0,1)$	$(0,0)\oplus(0,1)$	$(1,2,1,0,0,1,1,0,1)$	$(0,1)\oplus(0,2)$
$(2,0,1,0,0,1,1,0,1)$	$(0,1)\oplus(0,2)$	$(2,0,1,0,0,1,1,1,1)$	$(0,1)\oplus(0,2)$

continued on next page

β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	β	$\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(2,0,1,0,0,2,1,0,1)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(2,0,2,0,0,1,1,0,1)$	$(0,3/2)\oplus(0,5/2)$
$(2,1,1,0,0,1,1,0,1)$	$(0,0)\oplus 2(0,1)\oplus(0,2)$	$(3,0,1,0,0,1,1,0,1)$	$(0,2)\oplus(0,3)$

Table 24. Refined BPS invariants of 6d $n=7, E_7$ model up to total degree 8. For those that are not determined by the vanishing blowup equations, we mark them with "?".

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