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Stochastic thermodynamics of a probe in a fluctuating correlated field

Davide Venturelli[1,2(a)(b)], Sarah A. M. Loos[3,4(a)(c)], Benjamin Walter[5,1(a)(d)], Édgar Roldán[4] and Andrea Gambassi[1]

[1] *SISSA – International School for Advanced Studies and INFN - via Bonomea 265, 34136 Trieste, Italy*
[2] *Laboratoire de Physique Théorique de la Matière Condensée, CNRS/Sorbonne Université - 75005 Paris, France*
[3] *DAMTP, Centre for Mathematical Sciences, University of Cambridge - Wilberforce Rd, CB3 0WA Cambridge, UK*
[4] *ICTP – International Centre for Theoretical Physics - Strada Costiera 11, 34151 Trieste, Italy*
[5] *Department of Mathematics, Imperial College London - 180 Queen's Gate, SW7 2AZ London, UK*

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Abstract – We develop a framework for the stochastic thermodynamics of a probe coupled to a fluctuating medium with spatio-temporal correlations, described by a scalar field. For a Brownian particle dragged by a harmonic trap through a fluctuating Gaussian field, we show that near criticality (where the field displays long-range spatial correlations) the spatially-resolved average heat flux develops a dipolar structure, where heat is absorbed in front and dissipated behind the dragged particle. Moreover, a perturbative calculation reveals that the dissipated power displays three distinct dynamical regimes depending on the drag velocity.

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Introduction. – Stochastic thermodynamics provides powerful tools to investigate the entropic and energetic properties of fluctuating systems coupled to media, even far from equilibrium [1–6]. A crucial result in this context is the quantitative relation between the time-reversal asymmetry of the system's fluctuations and the heat dissipated to the environment, associated with entropy production [7–10]. Most of the previous works relied on the assumption that the environment is at all times in equilibrium or in a non-equilibrium steady state, and displays no dynamical spatio-temporal correlations. In various contexts, however, the dynamics crucially hinges on spatio-temporal correlations of the environment – this is the case, *e.g.*, for inclusions in lipid membranes [11–14], microemulsions [15–18], or defects in ferromagnetic systems [19–24]. These correlations become long-ranged and particularly relevant when the environment is close to a critical point, as in the case, *e.g.*, of colloidal particles in binary liquid mixtures [25–31]. Moreover, the simplified assumption of a structureless environment implies that all the information about local energy and entropy flows occurring within the environment is not taken into account [32]. To overcome this paradigm, there is growing interest in extending concepts from stochastic thermodynamics towards systems with spatially-extended correlations [33,34], such as pattern-forming many-body systems [35–38] or critical media [39–42]. Furthermore, a line of recent works discusses the irreversibility of active many-particle systems described by hydrodynamic field theories [38,42–45]. These theoretical advances are driven by state-of-the-art experiments involving optical trapping, ultrafast video-microscopy, or active particles [46–50].

In this work we investigate the stochastic thermodynamics of a system consisting of a mesoscopic, externally-driven probe coupled to a fluctuating medium. The latter is here represented by a scalar field obeying non-conserved or conserved dynamics [51], and the whole system (probe + field) is immersed in a homogeneous heat bath at a fixed temperature T that induces thermal fluctuations. We provide suitable definitions for the heat, work and entropy exchanges between the probe, the field and the thermal bath, which are consistent with the first

(a) These authors contributed equally to this work.
(b) E-mail: davide.venturelli@sissa.it (corresponding author)
(c) E-mail: sl2127@cam.ac.uk
(d) E-mail: b.walter@imperial.ac.uk

and second laws of stochastic thermodynamics. This approach is simpler than addressing a fully microscopic model, while it goes beyond a description based on a generalized Langevin equation (GLE) [52–57], which may capture temporal, but not spatial correlations of the environment. Notably, our framework is particularly powerful when dealing with a fluctuating medium that is close to a critical point, where one can replace a complex microscopic dynamics with the simplest model belonging to the same universality class [51,58]. As a fruit of our approach, one can thereby investigate the interplay between the probe, the spatio-temporal correlated field, and the heat bath under the light of stochastic thermodynamics. We illustrate the theory for the minimal yet insightful example of a probe particle dragged by a harmonic trap through a fluctuating Gaussian field. This mimics the setup commonly used to study friction and viscosity in (active) microrheology experiments [59–61], and is prototypical in stochastic thermodynamics.

The model. – We study a system formed by a mesoscopic probe at position $\mathbf{Y}(t) \in \mathbb{R}^d$ in d spatial dimensions and a scalar field, whose value at $\mathbf{x} \in \mathbb{R}^d$ and time t is denoted by $\phi(\mathbf{x}, t) \in \mathbb{R}$. We assume ϕ to represent the relevant slow degrees of freedom of the medium, after integrating out all other possible faster modes. For example, ϕ may represent the local magnetization in a spin system [22,51], the height of a fluctuating membrane over a reference plane [11–14], the relative concentration of two chemical species in a near-critical binary liquid mixture [62], etc., although in some of these cases the actual dynamics is more complex than the one considered here. The energy of the system is described by the Hamiltonian

$$\mathcal{H}[\phi, \mathbf{Y}, t] = \mathcal{H}_\phi[\phi] + \mathcal{H}^{\text{int}}[\phi, \mathbf{Y}] + \mathcal{U}(\mathbf{Y}, t), \tag{1}$$

where $\mathcal{H}_\phi$ denotes the energy of the field, $\mathcal{U}$ is an external potential acting only on the probe, and $\mathcal{H}^{\text{int}}$ encodes the interaction between the probe and the field. The probe and the scalar field are assumed to obey the following coupled overdamped Langevin equations:

$$\gamma_y \dot{\mathbf{Y}} = -\boldsymbol{\nabla}_{\mathbf{Y}} \mathcal{H} + \mathbf{F}_{\text{ext}} + \boldsymbol{\nu}, \tag{2}$$

$$\gamma_\phi \dot{\phi} = -(-\nabla^2)^{\mathfrak{a}} \frac{\delta \mathcal{H}}{\delta \phi} + \eta^{(\mathfrak{a})}, \tag{3}$$

where $\mathbf{F}_{\text{ext}}(\mathbf{Y}, t)$ in eq. (2) accounts for non-conservative external forces acting on the probe, while in eq. (3) $\mathfrak{a} = 0$ or 1 for locally non-conserved or conserved field dynamics [51,63]. Moreover, $\gamma_{y,\phi}$ are friction coefficients, while $\boldsymbol{\nu}$ and $\eta^{(\mathfrak{a})}$ are independent Gaussian white noises satisfying the fluctuation-dissipation relations $\langle \nu_i(t)\nu_j(t')\rangle = 2\gamma_y T\delta_{ij}\delta(t-t')$ and $\langle \eta^{(\mathfrak{a})}(\mathbf{x}, t)\eta^{(\mathfrak{a})}(\mathbf{x}', t')\rangle = 2\gamma_\phi T(-\boldsymbol{\nabla}^2)^{\mathfrak{a}}\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$, where $\langle \ldots \rangle$ denotes the average over the realizations of the noises (note that we set Boltzmann's constant $k_B = 1$ throughout the paper). Accordingly, in the absence of external driving, *i.e.*, $\mathbf{F}_{\text{ext}} = 0$ and time-independent $\mathcal{U}$, the system reaches a state of thermal equilibrium. Figure 1(a) is a sketch of the model with a harmonic external potential $\mathcal{U}$.

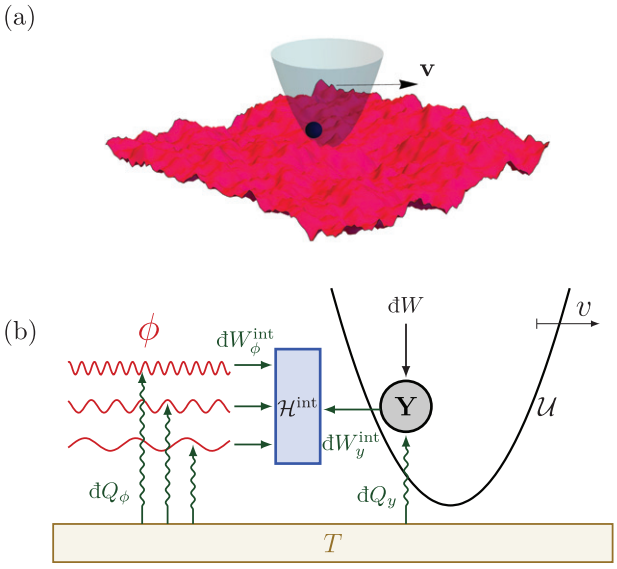

Fig. 1: (a) Sketch of a probe particle dragged by a trap with velocity $\mathbf{v}$ through a correlated field (red surface). (b) Cartoon of the particle at position $\mathbf{Y}$, the modes of the field ϕ (red waves), and the particle-field interaction (blue box), which can store elastic energy $\mathcal{H}^{\text{int}}$. Particle and field exchange heats $đQ_y$ and $đQ_\phi$, respectively, with a bath at temperature T. An external agent exerts work $đW$ on the particle. Particle and field may exchange energies $đW_y^{\text{int}}$ and $đW_\phi^{\text{int}}$, respectively, via their interaction. For clarity we do not indicate work exchanges between field modes. Heat fluxes are considered positive when they are supplied to the system.

First law of stochastic thermodynamics. – To analyze the thermodynamic properties of this system, we generalize ideas from stochastic thermodynamics [1,2]. For the fluxes of energy and entropy between particle, field and thermal bath sketched in fig. 1(b), this yields the expressions below (derived in the Supplementary Material `Supplementarymaterial.pdf` (SM)). First, the only systematic energy input into the system (probe + field) is the work exerted by an external agent on the probe. During a time interval $[t, t + \mathrm{d}t]$, this work is given by

$$đW = \frac{\partial \mathcal{U}}{\partial t}\mathrm{d}t + \mathbf{F}_{\text{ext}} \circ \mathrm{d}\mathbf{Y}, \tag{4}$$

where đ denotes non-exact differentials, and $\circ$ indicates the usage of Stratonovich calculus throughout [1]. Requiring energy conservation for the probe leads to the first law $\mathrm{d}\mathcal{U} = đW + đW_y^{\text{int}} + đQ_y$, according to which any change $\mathrm{d}\mathcal{U}$ in its potential energy can either be due to $đW$, or caused by the elastic coupling to the field,

$$đW_y^{\text{int}} := -\boldsymbol{\nabla}_{\mathbf{Y}}\mathcal{H}^{\text{int}} \circ \mathrm{d}\mathbf{Y}, \tag{5}$$

or else dissipated into the bath in the form of heat $-đQ_y$. From eqs. (2) to (5) it directly follows that the latter is

given by

$$đQ_y = (-\gamma_y \dot{\mathbf{Y}} + \boldsymbol{\nu}) \circ \mathrm{d}\mathbf{Y}, \tag{6}$$

which is identical to the Sekimoto expression [2] for a single particle in a heat bath (*i.e.*, $\mathcal{H}^{\rm int} = 0$). Likewise, requiring local energy conservation for the field leads to the first law for the field (see SM)

$$\mathrm{d}\mathcal{H}_\phi = \int \mathrm{d}^d\mathbf{x} \left[đW_\phi^{\rm int}(\mathbf{x}) + đQ_\phi(\mathbf{x}) \right]. \tag{7}$$

(An analogous, spatially-resolved first law is given in the SM.) Accordingly, all the work done on ϕ is either stored in the field configuration $\mathrm{d}\mathcal{H}_\phi$ (its "bending"), or dissipated in the form of heat $đQ_\phi$. Here,

$$đW_\phi^{\rm int}(\mathbf{x}) := -[\delta\mathcal{H}^{\rm int}/\delta\phi(\mathbf{x})] \circ \mathrm{d}\phi(\mathbf{x}) \tag{8}$$

is the work locally exerted on the field by the elastic coupling. In general, $đW_y^{\rm int} \neq -\int \mathrm{d}^d\mathbf{x} đW_\phi^{\rm int}(\mathbf{x})$, because $\mathcal{H}^{\rm int}$ itself can store energy. From eqs. (3) and (7) we find that the stochastic heat $đQ_\phi$ takes the form

$$\begin{aligned} đQ_\phi(\mathbf{x}) &= \left\{ (-\nabla^2)^{-\mathfrak{a}} \left[\eta^{(\mathfrak{a})}(\mathbf{x}) - \gamma_\phi \dot{\phi}(\mathbf{x}) \right] \right\} \circ \mathrm{d}\phi(\mathbf{x}) \\ &= \left[\frac{\delta\mathcal{H}_\phi}{\delta\phi(\mathbf{x})} + \frac{\delta\mathcal{H}^{\rm int}}{\delta\phi(\mathbf{x})} \right] \circ \mathrm{d}\phi(\mathbf{x}), \end{aligned} \tag{9}$$

generalizing Sekimoto's expression [1]. Importantly, $đQ_\phi$ is a density (or field) of local heat dissipation. For $\mathfrak{a} = 0$, eq. (9) formally resembles the heat exchange of a single particle. The case $\mathfrak{a} = 1$ of conserved dynamics involves the inverse Laplace operator $(\nabla^2)^{-1}$, which is non-local in space, and has thus to be interpreted in terms of its Green's function [64]. In confined geometries, the latter depends on the boundary conditions [65,66], but here we will only consider the system in the bulk.

Second law of stochastic thermodynamics. – Next, we define the irreversibility [1,34,44] associated with an individual, joint (probe + field) trajectory $\{\mathbf{Y}, \phi\}_{t_{\rm i}}^{t_{\rm f}}$ as

$$S_{\rm tot} = \ln \frac{\mathcal{P}[\{\mathbf{Y}, \phi\}_{t_{\rm i}}^{t_{\rm f}}]}{\mathcal{P}^{\rm R}[\{\mathbf{Y}^{\rm R}, \phi^{\rm R}\}_{t_{\rm i}}^{t_{\rm f}}]}. \tag{10}$$

Here $\mathcal{P}$ denotes the path probability of the trajectory, starting with the joint probability density $\rho_{\mathbf{Y},\phi}[\mathbf{Y}(t_{\rm i}), \phi(\mathbf{x}, t_{\rm i})]$, while $\mathcal{P}^{\rm R}$ denotes the path probability of the corresponding time-reversed process[1]. By construction, the second law $\langle S_{\rm tot} \rangle \geq 0$ holds [67]. A central result of this manuscript is that the irreversibility $S_{\rm tot}$ defined in eq. (10) equals the stochastic (total) entropy production (see SM)

$$S_{\rm tot} = -\frac{Q_y}{T} - \int \mathrm{d}^d\mathbf{x} \, \frac{Q_\phi(\mathbf{x})}{T} + \Delta S_{y,\phi}^{\rm sh}, \tag{11}$$

with the heat dissipation Q_y and Q_ϕ defined in eqs. (6) and (9), respectively – indicating the thermodynamic consistency of our framework[2]. Here $\Delta S_{y,\phi}^{\rm sh} = \ln\{\rho_{\mathbf{Y},\phi}[\mathbf{Y}(t_{\rm i}), \phi(t_{\rm i})]/\rho_{\mathbf{Y},\phi}[\mathbf{Y}(t_{\rm f}), \phi(t_{\rm f})]\}$ is the change in the fluctuating Shannon entropy. The first and second terms on the r.h.s. of eq. (11) are the fluxes of entropy to the thermal bath from the probe and from the field, respectively.

If the system reaches a steady state, then $\langle \mathrm{d}\mathcal{H}^{\rm int} \rangle$, $\langle \mathrm{d}\mathcal{H}_\phi \rangle$, and $\langle \mathrm{d}\mathcal{U} \rangle$ vanish. This simplifies the first laws for probe and field, and implies $\langle \int \mathrm{d}^d\mathbf{x} \, đW_\phi^{\rm int}(\mathbf{x}) \rangle = -\langle đW_y^{\rm int} \rangle$. Together with the second law and eq. (11), this implies

$$\langle \dot{W} \rangle = -\langle \dot{Q}_y \rangle - \int \mathrm{d}^d\mathbf{x} \, \langle \dot{Q}_\phi(\mathbf{x}) \rangle = T \langle \dot{S}_{\rm tot} \rangle \geq 0, \tag{12}$$

with $\dot{Q} := đQ/\mathrm{d}t$, $\dot{W} := đW/\mathrm{d}t$, and $\dot{S}_{\rm tot} \simeq S_{\rm tot}/(t_{\rm f} - t_{\rm i})$. Thus, in such steady states, $\langle \dot{S}_{\rm tot} \rangle$ is proportional to the average power $\langle \dot{W} \rangle$ injected into the system – which is partly dissipated into heat directly by the probe, $\langle Q_y \rangle$, and partly through the field, $\langle Q_\phi \rangle$.

Example: particle dragged through a Gaussian field. – Within this framework, we consider the typical setup [59] in which a probe particle is confined in a harmonic trap of stiffness κ moving at constant velocity $\mathbf{v}$. The corresponding potential is $\mathcal{U}(\mathbf{Y}, t) = \kappa \left(\mathbf{Y} - \mathbf{v}t\right)^2/2$, and the dissipation rate follows from eq. (4) as

$$\dot{W} = -\kappa \mathbf{v} \cdot (\mathbf{Y} - \mathbf{v}t). \tag{13}$$

In the long-time limit, the system reaches a steady state in the comoving reference frame with velocity $\mathbf{v}$. As a minimal model for a near-critical medium – and as the simplest approximation of various complex systems displaying long-range spatial correlations – we consider a Gaussian field with Hamiltonian [51]

$$\mathcal{H}_\phi = \frac{1}{2} \int \mathrm{d}^d\mathbf{x} \left[(\nabla\phi)^2 + r\phi^2 \right]. \tag{14}$$

The correlation length $\xi = r^{-1/2} \geq 0$ controls the spatial range of the field correlations at equilibrium, and diverges upon approaching the critical point $r = 0$. We model the particle-field interaction, as in refs. [19–24], by

$$\mathcal{H}_{\rm int} = -\lambda \int \mathrm{d}^d\mathbf{x} \, \phi(\mathbf{x}) V(\mathbf{x} - \mathbf{Y}), \tag{15}$$

where $V(\mathbf{x}) \geq 0$ is a rotationally invariant function, with unit spatial integral and rapidly vanishing as $|\mathbf{x}|$ increases beyond the "radius" R of the particle, while λ sets the

[1] We assume that ϕ and $\mathbf{Y}$ are both even under time reversal. In the time-reversed process, the time-dependent external driving protocols $\mathcal{U}(\mathbf{x}, t) + \mathbf{F}_{\rm ext}(t)$ are reversed in time. The backward trajectories are initialized with $\rho_{\mathbf{Y},\phi}[\mathbf{Y}(t_{\rm f}), \phi(\mathbf{x}, t_{\rm f})]$ (see SM).

[2] The definition of irreversibility given in eq. (10) generalizes those of, *e.g.*, ref. [44] (see eq. (7) therein) and ref. [34] (see eq. (16) therein) in the presence of a probe. Similarly, eq. (11) is compatible with the expression given in eq. (26) of ref. [32] for a generic system strongly coupled to a heat bath, at the level of (total) spatially integrated quantities.

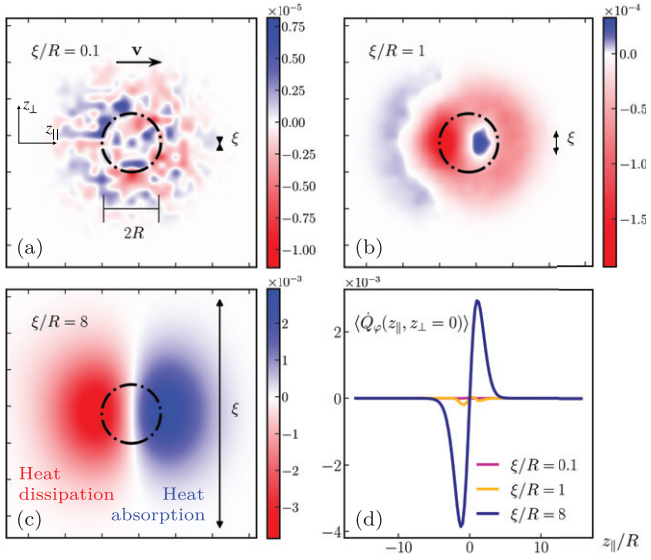

Fig. 2: Average local heat dissipation rate $\langle \dot{Q}_\varphi(\mathbf{z})\rangle$ by the field into the bath, in a frame comoving at velocity $\mathbf{v}$ with the harmonic trap (see fig. 1). These results of numerical simulations (see SM) refer to a Gaussian field in $d = 2$ with non-conserved dynamics and, from (a) to (c), increasing values of ξ/R, where ξ is the correlation length and R the probe size. The dash-dotted circles indicate the size of the probe. Panel (d) shows $\langle \dot{Q}_\varphi(\mathbf{z})\rangle$ along the drag direction $z_\parallel$, for $z_\perp = 0$ and the values of ξ considered in (a)–(c). We used $\lambda = 5$, $v = 5$, lattice size $L = 128$, time step $\Delta t = 10^{-2}$, a Gaussian potential V with variance $R = 4$, and we set all other parameters to unity.

coupling strength. For $\lambda > 0$ and at equilibrium, configurations in which the field is enhanced around the particle are energetically favored. We express the particle position in the comoving frame as $\mathbf{Z} \equiv \mathbf{Y} - \mathbf{v}t + \mathbf{v}\gamma_y/\kappa$, so that $\mathbf{Z} = 0$ is the mean steady-state position for $\lambda = 0$. Similarly, we introduce the field $\varphi(\mathbf{z}, t) \equiv \phi(\mathbf{z} + \mathbf{v}t - \mathbf{v}\gamma_y/\kappa, t)$ in the comoving frame.

Heat dissipation field. – The first thermodynamic quantity we analyze is the spatially-resolved heat dissipation Q_φ in the comoving reference frame (see SM). In fig. 2 we show numerical results for $\langle \dot{Q}_\varphi\rangle$ of a field in $d = 2$ with $\mathfrak{a} = 0$. For small values of ξ (see panel (a)), we observe that $\langle \dot{Q}_\varphi\rangle$ is essentially negligible (within numerical uncertainties) and displays no discernible spatial structure. In contrast, if ξ reaches the order of the particle size R (panel (b)), regions of average heat dissipation ($\langle \dot{Q}_\varphi\rangle < 0$) or absorption ($\langle \dot{Q}_\varphi\rangle > 0$) start developing. Close to criticality, with $\xi \gg R$ (panel (c)), a dissipation dipole forms, with a region of heat absorption in front of the particle, whose spatial extent is approximately given by ξ. Hence, surprisingly, in front of the particle the heat bath supplies net energy as if it was coupled to a cooler object. Note that the second law implies $\langle \dot{Q}_y\rangle + \int \mathrm{d}^d\mathbf{x}\, \langle \dot{Q}_\varphi(\mathbf{x})\rangle < 0$, but it does not preclude local heat absorption, *i.e.*, $\langle \dot{Q}_\varphi(\mathbf{x})\rangle > 0$ for some $\mathbf{x}$. To further elucidate the origin of this effect, below we analytically investigate the statistics of particle and field for various values of $v = |\mathbf{v}|$ and ξ, and the dissipated power.

Particle statistics and bending of the field. – We now assume λ to be small, and use it as a perturbative parameter [65,68–71]. Expressing eqs. (2) and (3) in terms of $\mathbf{Z}$ and φ, we first obtain

$$\gamma_y \dot{\mathbf{Z}} = -\kappa \mathbf{Z} + \lambda \int \frac{\mathrm{d}^d p}{(2\pi)^d} i\mathbf{p}\, V_{-p}\, \varphi_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{Z}} + \boldsymbol{\nu}, \tag{16}$$

$$\gamma_\phi(\partial_t - i\mathbf{p}\cdot\mathbf{v} + \alpha_p)\varphi_{\mathbf{p}} = \lambda p^{2\mathfrak{a}} V_p e^{-i\mathbf{p}\cdot\mathbf{Z}} + \eta^{(\mathfrak{a})}_{\mathbf{p}}, \tag{17}$$

where $\alpha_p \equiv p^{2\mathfrak{a}}(p^2 + r)/\gamma_\phi$, and $\varphi_{\mathbf{p}}$ is the Fourier transform[3] of $\varphi(\mathbf{x}, t)$. From eq. (17) we can calculate the mean stationary profile $\langle\varphi_{\mathbf{p}}\rangle_{\mathrm{st}}$, which is comoving with the particle, and which we call "shadow" (see SM). The latter is shown in fig. 3(a) for a field in $d = 1$ with $\mathfrak{a} = 0$: the field is strongly bent around the particle, while $\langle\varphi(z)\rangle_{\mathrm{st}} \propto \exp(-|z|/\ell_\pm)$ for $z \to \pm\infty$, with $\ell_\pm = \xi[\sqrt{1 + (\xi v\gamma_\phi/2)^2} \mp (\xi v\gamma_\phi/2)]$ (see SM). Far from criticality ($\xi \to 0$, see inset), the shadow vanishes, rationalizing the corresponding vanishing of $\langle \dot{Q}_\varphi(\mathbf{z})\rangle$ in fig. 2(a).

Via a perturbative approach, we can investigate the particle fluctuations analytically. The moment generating function $g(\mathbf{q}) \equiv \langle\exp(-i\mathbf{q}\cdot\mathbf{Z})\rangle$ of the particle position at the lowest non-trivial order in λ reads (see SM)

$$\ln g(\mathbf{q}) = -\frac{q^2 T}{2\kappa} + \frac{i\lambda^2}{\kappa}\int_0^\infty \frac{\mathrm{d}u}{\sigma^2(u)} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(\mathbf{p}\cdot\mathbf{v})}{p^2 + r} |V_p|^2 \times G_{\mathbf{p}}(u) e^{-p^2\sigma^2(u)} [1 - e^{-(\mathbf{p}\cdot\mathbf{q})\sigma^2(u)}] + \mathcal{O}(\lambda^4), \tag{18}$$

where $\sigma^2(u) \equiv T\left(1 - e^{-\kappa u/\gamma_y}\right)/\kappa$, and $G_{\mathbf{p}}(u) \equiv \exp[-\alpha_p u + i\,(\mathbf{p}\cdot\mathbf{v})\,u]$. Interestingly, eq. (18) predicts a non-Gaussian statistics of the particle position. In addition, the variance $\langle Z_l Z_m\rangle_c \propto \delta_{lm}$ changes anisotropically compared to the case $\mathbf{v} = \mathbf{0}$, so that the position distribution is elongated in the direction parallel to $\mathbf{v}$ [72] (see SM). Furthermore, $i\nabla_{\mathbf{q}} g|_{\mathbf{q}=\mathbf{0}}$ gives the average displacement

$$\langle\mathbf{Z}\rangle = -\frac{\lambda^2}{\kappa}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathbf{p}\,(\mathbf{p}\cdot\mathbf{v})}{p^2 + r} |V_p|^2 \int_0^\infty \mathrm{d}u\, G_{\mathbf{p}}(u) e^{-p^2\sigma^2(u)}, \tag{19}$$

which turns out to be directed along $-\mathbf{v}$; hence, the field induces a shift of the probability density of the particle position, which lags behind the average stationary value in the absence of the field (see the inset of fig. 3(b)). Such a lag is the footprint of an underlying additional source of dissipation, which we analyze next.

Power fluctuations. – From the moment generating function in eq. (18), we can access the distribution of the dissipated power. Indeed, rewriting eq. (13) in terms of $\mathbf{Z}$ gives $\dot{W} = \gamma_y v^2 - \kappa\mathbf{v}\cdot\mathbf{Z}$, and, thus,

$$\ln\langle\exp(-i\mu\dot{W})\rangle = -i\mu\gamma_y v^2 + \ln g(-\mu\kappa\mathbf{v}), \tag{20}$$

[3] We adopt here and in the following the Fourier convention $f(\mathbf{x}) = \int[\mathrm{d}^d p/(2\pi)^d]\exp(i\mathbf{p}\cdot\mathbf{x}) f_{\mathbf{p}}$, and we normalize the delta distribution in Fourier space as $\int[\mathrm{d}^d p/(2\pi)^d]\delta^d(\mathbf{p}) = 1$.

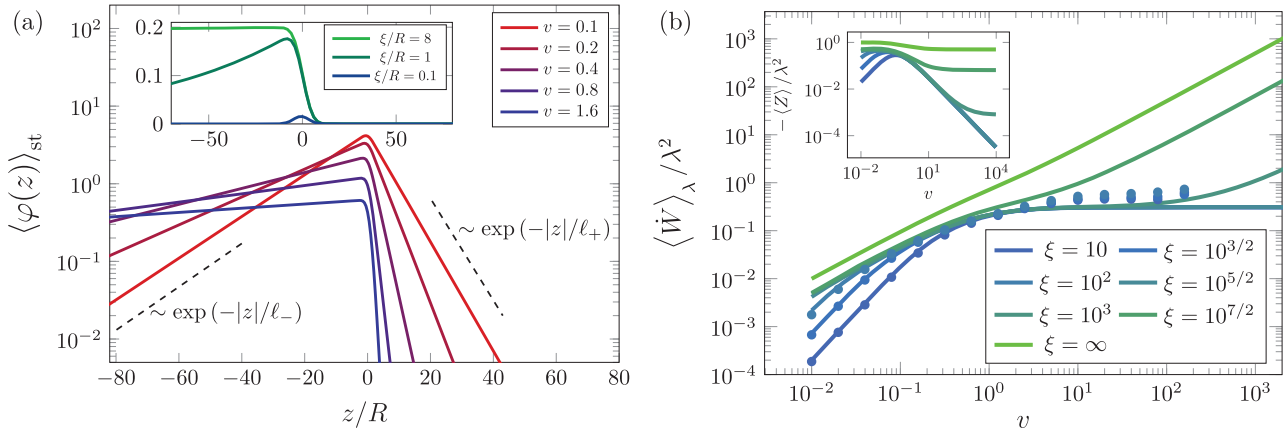

Fig. 3: (a) Steady-state expectation value $\langle\varphi(z)\rangle_{\rm st}$ of the field in the comoving frame, for various drag velocities $v > 0$, in $d = 1$ and with non-conserved dynamics. The *shadow* $\langle\varphi(z)\rangle_{\rm st}$ flattens upon increasing v (at fixed correlation length $\xi = 10$, main plot) or upon decreasing ξ (at fixed $v = 5$, inset). $\langle\varphi(z)\rangle_{\rm st}$ decays exponentially upon increasing $|z|$ with different decay lengths $\ell_\pm$ in front of or behind the particle – see the main text and the SM. We used $V_q = \exp(-q^2R^2/2)$ and $R^2 + T/\kappa = 1$, while the other parameters were set to unity. (b) Additional dissipation rate (see eq. (21)) as a function of v, for various values of ξ (with $T = 0.1$, while the other parameters were set to one). Symbols correspond to simulations. Inset: scaled correction $-\langle Z\rangle/\lambda^2$ to the average particle position (see eq. (19)).

encoding all moments of the dissipated power. To study the impact of the field, we focus on the average power

$$\langle\dot{W}\rangle = \gamma_y v^2 - \kappa\mathbf{v}\cdot\langle\mathbf{Z}\rangle \equiv \langle\dot{W}\rangle_0 + \langle\dot{W}\rangle_\lambda, \qquad (21)$$

where we identify the dissipation rate in the absence of the field $\langle\dot{W}\rangle_0 = \gamma_y v^2 \geq 0$, while $\langle\dot{W}\rangle_\lambda = -\kappa\mathbf{v}\cdot\langle\mathbf{Z}\rangle \geq 0$ encodes the additional dissipation due to the field. According to eq. (12), these are further equal to the entropy production at $\lambda = 0$ and $\lambda > 0$, *i.e.*, $\langle\dot{W}\rangle_0 = T\langle\dot{S}_{\rm tot}\rangle_0$ and $\langle\dot{W}\rangle_\lambda = T\langle\dot{S}_{\rm tot}\rangle_\lambda$, respectively. In fig. 3(b) we show the perturbative prediction for $\langle\dot{W}\rangle_\lambda$ as a function of v, and the corresponding numerical data from simulations (see SM), which are in good agreement. We find that $\langle\dot{W}\rangle_\lambda$ generally displays three distinct regimes upon increasing v: first, $|\langle Z\rangle|$ grows linearly (see fig. 3(b), inset), so that $\langle\dot{W}\rangle_\lambda \propto v^2$, as it would be the case for usual Stokes friction. After a crossover, in the second regime $|\langle Z\rangle| \propto 1/v$, and therefore $\langle\dot{W}\rangle_\lambda$ plateaus at intermediate v, which indicates a constant energetic cost associated with the particle-field interaction. Finally, in the third regime, $|\langle Z\rangle|$ saturates and thus $\langle\dot{W}\rangle_\lambda \propto v$. We remark that the second and third *non-Stokesian* regimes[4] cannot be captured by a linear GLE (see SM).

We present a thorough analysis of these regimes in the SM, for the case $\mathfrak{a} = 0$, and summarize it here. First, by inserting into eq. (16) the formal solution of eq. (17) for $\varphi_{\mathbf{p}}(t)$, an effective equation for $\mathbf{Z}(t)$ can be obtained, which is non-Markovian and nonlinear. However, in the limit of small v, the field is sufficiently fast to equilibrate around the particle at any instant in time [65,70]. Conversely, for large v the evolution of the field is so slow that the particle encounters an effectively static field configuration. Accordingly, the first two regimes can be quantitatively captured by adiabatically replacing the field $\varphi_{\mathbf{p}}(t)$ in eq. (16) with its mean (comoving) profile $\langle\varphi_{\mathbf{p}}\rangle_{\rm st}$, *i.e.*, with the shadow shown in fig. 3(a), resulting in an approximately Markovian evolution of $\mathbf{Z}(t)$. In contrast, at intermediate values of v, the timescales of the particle dynamics are comparable with the relaxation time $\tau_\xi \simeq \gamma_\phi\xi^2$ of the field, and the adiabatic approximation is no longer accurate: the particle dynamics within the crossover between the first two regimes is dominated by the memory effects caused by the mutual influence of the particle and the field (see SM). Finally, in the third regime, the shadow becomes negligible compared to the (critical) fluctuations of the field, and the particle effectively encounters a rough landscape resulting from them. Notably, as the field approaches criticality ($\xi \to \infty$), the amplitude of its fluctuations diverges [63], and thus the last (non-Stokesian) regime extends to low v.

Conclusions. – We developed a thermodynamically consistent framework to study the energetic and entropic flows for a probe in a fluctuating medium with spatio-temporal correlations, modeled here by a scalar field immersed in a heat bath. We showed that the mutual influence of the probe and the correlated environment leads to unusual thermodynamic properties, even for the simple example of a particle dragged by a harmonic trap through a Gaussian field. We showed that, close to criticality or, more generally, when the correlation length ξ of the field exceeds the particle size, a dipolar structure develops in the local heat dissipation field – with systematic heat absorption in front of the particle, and whose extent is determined by ξ. We thus expect such *heat dipoles* to be possibly revealed also in long-range interacting systems of

[4]The first and the third among these regimes are consistent with the scaling of drag forces reported in refs. [19–21], where particles moving with constant velocity (*i.e.*, without positional fluctuations) were studied.

diverse nature [73]. Furthermore, the additional power required to drag the particle in the presence of the field features three regimes with distinct scaling in the drag speed v, among which are two non-Stokesian regimes – a feature that cannot be captured by a linear GLE. Far from criticality ($\xi \to 0$), the medium is only weakly correlated, and indeed both the heat dipole and the additional dissipation vanish. The framework developed here can be readily applied to study more complex scenarios, *e.g.*, non-quadratic Hamiltonians, or extended to systems of multiple particles in a common correlated (active) environment. Moreover, the recent and intense experimental investigations of colloidal particles in correlated fluids – such as viscoelastic media [60,74,75], binary liquid mixtures [25–31], and living cells [76,77] – could provide access to the phenomena described here.

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