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ON EIGENVALUES OF SYMMETRIC MATRICES WITH PSD PRINCIPAL SUBMATRICES

KHAZHGALI KOZHASOV

Abstract. We investigate convexity properties of the set of eigenvalue tuples of $n \times n$ real symmetric matrices, whose all $k \times k$ (where $k \leq n$ is fixed) minors are positive semidefinite. It is proven that the set $\lambda(\mathcal{S}^{n,k})$ of eigenvalue vectors of all such matrices is star-shaped with respect to the nonnegative orthant $\mathbb{R}^n_{\geq 0}$ and not convex already when $(n, k) = (4, 2)$.

1. Introduction and statement of results

Let Sym_n denote the space of $n \times n$ real symmetric matrices. A matrix $X \in \mathrm{Sym}_n$ is called *positive semidefinite* (or, simply, PSD), if $\boldsymbol{v}^T X \boldsymbol{v} \geq 0$ holds for all column vectors $\boldsymbol{v} \in \mathbb{R}^n$. Equivalently, $X \in \mathrm{Sym}_n$ is PSD if all its eigenvalues are nonnegative. For $k \in \{1, \ldots, n\}$ let $\mathcal{S}^{n,k}$ denote the set of those matrices in Sym_n, whose all $k \times k$ principal submatrices are positive semidefinite. Elements of $\mathcal{S}^{n,k}$ are known as *k-locally positive semidefinite matrices*, they have been systematically studied in [1] and [2]. Note that $\mathcal{S}^{n,n}$ consists of all positive semidefinite matrices in Sym_n. The set $\mathcal{S}^{n,k}$ is a closed convex cone containing $\mathcal{S}^{n,n}$ and, moreover, $\mathcal{S}^{n,k} \supsetneq \mathcal{S}^{n,k+1}$ for any $k \in \{1, \ldots, n-1\}$. The dual cone to $\mathcal{S}^{n,k} \subset \mathrm{Sym}_n$ consists of PSD matrices of factor width k, which were introduced and investigated in [3, 7, 5]. Matrices of factor width 2 are also known as scaled diagonally dominant matrices, see [3]. One of the motivations to study properties of locally PSD matrices came from empirical observations that some optimization problems (see [8, 6]), where constraints are given by PSD matrices, have close optimal values to those of their relaxations, where constraints are relaxed to be k-locally PSD.

Given a matrix $X \in \mathrm{Sym}_n$, let $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$, $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$, denote the vector of its ordered eigenvalues and for any permutation on n elements $\sigma \in S_n$ let $\lambda(X)^\sigma = (\lambda_{\sigma_1}(X), \ldots, \lambda_{\sigma_n}(X))$ be the vector of permuted eigenvalues. In this work we are interested in the set

$$\lambda(\mathcal{S}^{n,k}) \;=\; \left\{\lambda(X)^\sigma \,:\, X \in \mathcal{S}^{n,k},\ \sigma \in S_n\right\}$$

of all possible (permuted) eigenvalue vectors of k-locally PSD matrices. As observed in [2], the set $\lambda(\mathcal{S}^{n,k})$ is contained in *the closed hyperbolicity cone*

$$H(e_k^n) \;=\; \{\lambda \in \mathbb{R}^n \,:\, e_k^n(\lambda + t(1, \ldots, 1)) = 0 \Rightarrow t \leq 0\}$$

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of the k-th elementary symmetric polynomial $e_k^n(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdot\ldots\cdot\lambda_{i_k}$. Moreover, the inclusion $\lambda(\mathcal{S}^{n,k}) \subseteq H(e_k^n)$ is strict for $2 < k < n-1$, for $(n,k) = (4,2)$ [2, Cor. 3] and, surprisingly, one has equality $\lambda(\mathcal{S}^{n,n-1}) = H(e_{n-1}^n)$ for $k = n-1$ (see [2, Thm. 2.2]). The authors of [2] pointed out that it is unknown whether the set $\lambda(\mathcal{S}^{n,k})$ of eigenvalue vectors of matrices in $\mathcal{S}^{n,k}$ is convex for any k and n. We give a negative answer to this question, showing that $\lambda(\mathcal{S}^{4,2}) \subsetneq H(e_2^4)$ is not convex.

Theorem 1.1. *The set*

$$\lambda(\mathcal{S}^{4,2}) \;=\; \{(\lambda_{\sigma_1}(X), \lambda_{\sigma_2}(X), \lambda_{\sigma_3}(X), \lambda_{\sigma_4}(X)) \,:\, X \in \mathcal{S}^{4,2},\ \sigma \in S_4\}$$

of eigenvalue vectors of 2*-locally PSD matrices in* Sym_4 *is not convex.*

Remark 1.2. Specifically, we prove that $(4,4,-1,-1)$ is not in $\lambda(\mathcal{S}^{4,2})$, though it lies on the segment joining the vector $(4,4,0,-2) \in \lambda(\mathcal{S}^{4,2})$ with its permuted copy.

Even though the set $\lambda(\mathcal{S}^{n,k})$ is not convex in general, it is star-shaped with respect to the nonnegative orthant $\mathbb{R}^n_{\geq 0} = \lambda(\mathcal{S}^{n,n})$.

Proposition 1.3. *For all $\lambda \in \lambda(\mathcal{S}^{n,k})$, $\lambda^+ \in \mathbb{R}^n_{\geq 0}$ the sum $\lambda + \lambda^+$ is in $\lambda(\mathcal{S}^{n,k})$.*

We now discuss one application of this result. It is shown in [2] that every point on the boundary of $H(e_2^4)$ with exactly one negative entry is an eigenvalue vector of some matrix in $\mathcal{S}^{4,2}$. We observe that the same holds for all points in $H(e_2^4)$ with this property.

Corollary 1.4. *Any $\lambda \in H(e_2^4)$ with at most* 1 *negative entry is an eigenvalue vector of some matrix in $\mathcal{S}^{4,2}$.*

In the rather trivial case $k = 1$ the set $\lambda(\mathcal{S}^{n,1}) = H(e_1^n)$ is a closed half-space.

Theorem 1.5. *For any $n \geq 1$ we have*

$$\lambda(\mathcal{S}^{n,1}) \;=\; \left\{\lambda \in \mathbb{R}^n \,:\, e_1^n(\lambda) = \sum_{j=1}^n \lambda_j \geq 0\right\}$$

2. Some open questions

By its definition, the set $\lambda(\mathcal{S}^{n,k})$ consists of permuted eigenvalue vectors of k-locally PSD matrices. It is also natural to consider the subset

$$\lambda_{\leq}(\mathcal{S}^{n,k}) \;=\; \{\lambda(X) \,:\, X \in \mathcal{S}^{n,k}\}$$

of ordered eigenvalue vectors, that is, the intersection of $\lambda(\mathcal{S}^{n,k})$ with half-spaces cut out by inequalities $\lambda_i \geq \lambda_{i+1}$, $i = 1,\ldots,n-1$. Our proof of Theorem 1.1 does not imply that $\lambda_{\leq}(\mathcal{S}^{4,2})$ is non-convex, see Remark 1.2. It would be interesting to understand whether $\lambda_{\leq}(\mathcal{S}^{n,k})$ is non-convex for some (n,k). When $(n,k) = (n,n-1)$ the set $\lambda_{\leq}(\mathcal{S}^{n,k}))$ is a convex cone by [2, Thm. 2.2] and when $(n,k) = (n,1)$ it is the convex polyhedral cone

$$\lambda_{\leq}(\mathcal{S}^{n,1}) \;=\; \left\{\lambda \in \mathbb{R}^n \,:\, \lambda_1 \geq \cdots \geq \lambda_n,\ \sum_{j=1}^i \lambda_j \geq 0 \text{ for all } i = 1,\ldots,n\right\},$$

see the proof of Theorem 1.5 below.

Nonzero vectors in the hyperbolicity cone $H(e_k^n)$ can have at most $n-k$ negative entries. Generalizing [2, Lemma 2] one can show that if a vector $\lambda \in H(e_k^n)$ has $n-k$ negative entries, then the other k entries must be positive. One interesting question is to understand whether for any (n,k) there exist k-locally PSD matrices with $n-k$ negative eigenvalues. In the cases $(n,n-1)$ and $(n,1)$ [2, Thm. 2.2] and Theorem 1.5 imply a positive answer to this question. Also, one can see that the real symmetric matrix

$$X = \begin{pmatrix} 346 & 84 & -16 & -98 \\ 84 & 240 & 77 & 156 \\ -16 & 77 & 30 & 70 \\ -98 & 156 & 70 & 170 \end{pmatrix}$$

is in $\mathcal{S}^{4,2}$ and has two negative eigenvalues.

See also Remark 4.2 for an interesting polynomial optimization problem.

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4. Proofs

We need the following elementary fact that we prove first.

Lemma 4.1. *If $(\lambda_1,\ldots,\lambda_{n'})$ is in $\lambda(\mathcal{S}^{n',k})$, then $(\lambda_1,\ldots,\lambda_{n'},\underbrace{0,\ldots,0}_{n-n'})$ is in $\lambda(\mathcal{S}^{n,k})$.*

Proof. Let $(\lambda_1,\ldots,\lambda_{n'}) = \lambda(X')$, where $X' \in \mathcal{S}^{n',k}$. Note that the $n\times n$ matrix

$$X = \begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix}$$

is k-locally PSD, since its principal submatrices $X|_S$, $S \subseteq \{1,\ldots,n\}$, $|S| = k$, read

$$X|_S = \begin{pmatrix} X'|_{S'} & 0 \\ 0 & 0 \end{pmatrix},$$

where $S' = S \cap \{1,\ldots,n'\}$. It is easy to see that $(\lambda_1,\ldots,\lambda_{n'},0,\ldots,0) = \lambda(X)$. □

Proof of Theorem 1.1. Let us observe that the vector $\lambda = (4,4,-1,-1)$ belongs to the cone $H(e_2^4)$. Indeed, for any $t \geq 0$ the value

$$\begin{aligned} e_2^4(4+t,4+t,-1+t,-1+t) &= (4+t)^2 + (t-1)^2 + 4(4+t)(t-1) \\ &= 1 + 18t + 6t^2 \end{aligned}$$

is positive and hence the vector $\lambda + t(1,1,1,1)$ lies in $H(e_2^4)$. Furthermore, λ is the midpoint of the segment joining $\lambda' = (4,4,-2,0)$ and $\lambda'' = (4,4,0,-2)$. Note that λ' is obtained by appending a zero to the vector $(4,4,-2) \in H(e_2^3) = \lambda(\mathcal{S}^{3,2})$, where equality of sets is due to [2, Thm. 2.2]. Thus, Lemma 4.1 implies that λ' (and hence also its permuted copy λ'') belongs to $\lambda(\mathcal{S}^{4,2})$. We show that λ is not in $\lambda(\mathcal{S}^{4,2})$. Note that $\lambda = \lambda(X)$ for some matrix $X \in \mathrm{Sym}_4$ if and only if we can write

$$\begin{aligned} X &= V^T \Lambda V = \left(\boldsymbol{v}_1^T \,|\, \boldsymbol{v}_2^T \,|\, \boldsymbol{v}_3^T \,|\, \boldsymbol{v}_4^T\right) \begin{pmatrix} 4 & & & 0 \\ & 4 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \end{pmatrix} \\ &= \begin{pmatrix} \substack{4v_{11}^2+4v_{21}^2\\-v_{31}^2-v_{41}^2} & \substack{4v_{11}v_{12}+4v_{21}v_{22}\\-v_{31}v_{32}-v_{41}v_{42}} & \substack{4v_{11}v_{13}+4v_{21}v_{23}\\-v_{31}v_{33}-v_{41}v_{43}} & \substack{4v_{11}v_{14}+4v_{21}v_{24}\\-v_{31}v_{34}-v_{41}v_{44}} \\ \substack{4v_{11}v_{12}+4v_{21}v_{22}\\-v_{31}v_{32}-v_{41}v_{42}} & \substack{4v_{12}^2+4v_{22}^2\\-v_{32}^2-v_{42}^2} & \substack{4v_{12}v_{13}+4v_{22}v_{23}\\-v_{32}v_{33}-v_{42}v_{43}} & \substack{4v_{12}v_{14}+4v_{22}v_{24}\\-v_{32}v_{34}-v_{42}v_{44}} \\ \substack{4v_{11}v_{13}+4v_{21}v_{23}\\-v_{31}v_{33}-v_{41}v_{43}} & \substack{4v_{12}v_{13}+4v_{22}v_{23}\\-v_{32}v_{33}-v_{42}v_{43}} & \substack{4v_{13}^2+4v_{23}^2\\-v_{33}^2-v_{43}^2} & \substack{4v_{13}v_{14}+4v_{23}v_{24}\\-v_{33}v_{34}-v_{43}v_{44}} \\ \substack{4v_{11}v_{14}+4v_{21}v_{24}\\-v_{31}v_{34}-v_{41}v_{44}} & \substack{4v_{12}v_{14}+4v_{22}v_{24}\\-v_{32}v_{34}-v_{42}v_{44}} & \substack{4v_{13}v_{14}+4v_{23}v_{24}\\-v_{33}v_{34}-v_{43}v_{44}} & \substack{4v_{14}^2+4v_{24}^2\\-v_{34}^2-v_{44}^2} \end{pmatrix}, \end{aligned} \tag{4.1}$$

where $V \in O(4)$ is an orthogonal matrix, whose rows are eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^4$ of X. The representation (4.1) is unique up to the choice of an orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ (respectively, $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$) of the eigenspace of X corresponding to the eigenvalue 4 (respectively, -1). In particular, performing orthogonal changes of bases in $L = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and in $L^\perp = \mathrm{Span}\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$ we can put $v_{12} = v_{34} = 0$. Now, the matrix (4.1) with $v_{12} = v_{34} = 0$ is 2-locally PSD if and only if the polynomial system

$$\begin{aligned}
4v_{11}^2 + 4v_{21}^2 - v_{31}^2 - v_{41}^2 - m_1 &= 0, \\
4v_{22}^2 - v_{32}^2 - v_{42}^2 - m_2 &= 0, \\
4v_{13}^2 + 4v_{23}^2 - v_{33}^2 - v_{43}^2 - m_3 &= 0, \\
4v_{14}^2 + 4v_{24}^2 - v_{44}^2 - m_4 &= 0, \\
(4v_{11}^2 + 4v_{21}^2 - v_{31}^2 - v_{41}^2)(4v_{22}^2 - v_{32}^2 - v_{42}^2) - (4v_{21}v_{22} - v_{31}v_{32} - v_{41}v_{42})^2 - m_{12} &= 0, \\
(4v_{11}^2 + 4v_{21}^2 - v_{31}^2 - v_{41}^2)(4v_{13}^2 + 4v_{23}^2 - v_{33}^2 - v_{43}^2) - (4v_{11}v_{13} + 4v_{21}v_{23} - v_{31}v_{33} - v_{41}v_{43})^2 - m_{13} &= 0, \\
(4v_{11}^2 + 4v_{21}^2 - v_{31}^2 - v_{41}^2)(4v_{14}^2 + 4v_{24}^2 - v_{44}^2) - (4v_{11}v_{14} + 4v_{21}v_{24} - v_{41}v_{44})^2 - m_{14} &= 0, \\
(4v_{22}^2 - v_{32}^2 - v_{42}^2)(4v_{13}^2 + 4v_{23}^2 - v_{33}^2 - v_{43}^2) - (4v_{22}v_{23} - v_{32}v_{33} - v_{42}v_{43})^2 - m_{23} &= 0, \\
(4v_{22}^2 - v_{32}^2 - v_{42}^2)(4v_{14}^2 + 4v_{24}^2 - v_{44}^2) - (4v_{22}v_{24} - v_{42}v_{44})^2 - m_{24} &= 0, \\
(4v_{13}^2 + 4v_{23}^2 - v_{33}^2 - v_{43}^2)(4v_{14}^2 + 4v_{24}^2 - v_{44}^2) - (4v_{13}v_{14} + 4v_{23}v_{24} - v_{43}v_{44})^2 - m_{34} &= 0, \\
v_{11}^2 + v_{21}^2 + v_{31}^2 + v_{41}^2 - 1 &= 0, \\
v_{22}^2 + v_{32}^2 + v_{42}^2 - 1 &= 0, \\
v_{13}^2 + v_{23}^2 + v_{33}^2 + v_{43}^2 - 1 &= 0, \\
v_{14}^2 + v_{24}^2 + v_{44}^2 - 1 &= 0, \\
v_{21}v_{22} + v_{31}v_{32} + v_{41}v_{42} &= 0, \\
v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33} + v_{41}v_{43} &= 0, \\
v_{11}v_{14} + v_{21}v_{24} + v_{41}v_{44} &= 0, \\
v_{22}v_{23} + v_{32}v_{33} + v_{42}v_{43} &= 0, \\
v_{22}v_{24} + v_{42}v_{44} &= 0, \\
v_{13}v_{14} + v_{23}v_{24} + v_{43}v_{44} &= 0,
\end{aligned}$$

has a real solution with nonnegative $m_1, m_2, m_3, m_4, m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}$. These variables represent principal minors of X of sizes 1×1 and 2×2, and the last 10 equations encode orthogonality of the matrix V. Let $I \lhd \mathbb{Q}[v_{ij}, m_i, m_{ij}]$ be the ideal generated by the 20 polynomials of the system. Using SageMath [10] we find out that the elimination ideal $J = I \cap \mathbb{Q}[m_i, m_{ij}]$ of I with respect to the variables v_{ij} is generated by polynomials

$$
\begin{aligned}
L &= m_{12} + m_{13} + m_{14} + m_{23} + m_{24} + m_{34} - 1,\\
f_1 &= 3m_1 + m_{23} + m_{24} + m_{34} - 5,\\
f_2 &= 3m_2 + m_{13} + m_{14} + m_{34} - 5,\\
f_3 &= 3m_3 - m_{13} - m_{23} - m_{34} - 4,\\
f_4 &= 3m_4 - m_{14} - m_{24} - m_{34} - 4,\\
F &= -2816 - 256m_{13} - 256m_{14} - 256m_{23} - 256m_{24} - 224m_{34} - 50m_{34}^3\\
&\quad + 192m_{14}m_{24} + 256m_{13}^2 + 192m_{13}m_{14} + 256m_{14}^2 + 192m_{13}m_{23} + 160m_{14}m_{23} + 256m_{23}^2\\
&\quad + 160m_{13}m_{24} + 192m_{23}m_{24} + 256m_{24}^2 + 96m_{13}m_{34} + 96m_{14}m_{34} + 96m_{23}m_{34} + 96m_{24}m_{34} + 249m_{34}^2\\
&\quad + 64m_{13}^2m_{14} + 64m_{13}m_{14}^2 + 64m_{13}^2m_{23} + 112m_{13}m_{14}m_{23} + 128m_{14}^2m_{23} + 64m_{13}m_{23}^2\\
&\quad + 128m_{14}m_{23}^2 + 128m_{13}^2m_{24} + 112m_{13}m_{14}m_{24} + 64m_{14}^2m_{24} + 112m_{13}m_{23}m_{24} + 112m_{14}m_{23}m_{24}\\
&\quad + 64m_{23}^2m_{24} + 128m_{13}m_{24}^2 + 64m_{14}m_{24}^2 + 64m_{23}m_{24}^2 + 128m_{13}^2m_{34} + 240m_{13}m_{14}m_{34} + 128m_{14}^2m_{34}\\
&\quad + 240m_{13}m_{23}m_{34} + 206m_{14}m_{23}m_{34} + 128m_{23}^2m_{34} + 206m_{13}m_{24}m_{34} + 240m_{14}m_{24}m_{34}\\
&\quad + 240m_{23}m_{24}m_{34} + 128m_{24}^2m_{34} + 78m_{13}m_{34}^2 + 78m_{14}m_{34}^2 + 78m_{23}m_{34}^2 + 78m_{24}m_{34}^2\\
&\quad + 16m_{13}^2m_{14}m_{23} + 16m_{13}m_{14}^2m_{23} + 16m_{13}m_{14}m_{23}^2 + 25m_{14}^2m_{23}^2 + 16m_{13}^2m_{14}m_{24} + 16m_{13}m_{14}^2m_{24}\\
&\quad + 16m_{13}^2m_{23}m_{24} + 14m_{13}m_{14}m_{23}m_{24} + 16m_{14}^2m_{23}m_{24} + 16m_{13}m_{23}^2m_{24} + 16m_{14}m_{23}^2m_{24} + 25m_{13}^2m_{24}^2\\
&\quad + 16m_{13}m_{14}m_{24}^2 + 16m_{13}m_{23}m_{24}^2 + 16m_{14}m_{23}m_{24}^2 + 16m_{13}^2m_{14}m_{34} + 16m_{13}m_{14}^2m_{34}\\
&\quad + 16m_{13}^2m_{23}m_{34} + 82m_{13}m_{14}m_{23}m_{34} + 50m_{14}^2m_{23}m_{34} + 16m_{13}m_{23}^2m_{34} + 50m_{14}m_{23}^2m_{34}\\
&\quad + 50m_{13}^2m_{24}m_{34} + 82m_{13}m_{14}m_{24}m_{34} + 16m_{14}^2m_{24}m_{34} + 82m_{13}m_{23}m_{24}m_{34} + 82m_{14}m_{23}m_{24}m_{34}\\
&\quad + 16m_{23}^2m_{24}m_{34} + 50m_{13}m_{24}^2m_{34} + 16m_{14}m_{24}^2m_{34} + 16m_{23}m_{24}^2m_{34} + 25m_{13}^2m_{34}^2 + 66m_{13}m_{14}m_{34}^2\\
&\quad + 25m_{14}^2m_{34}^2 + 66m_{13}m_{23}m_{34}^2 + 100m_{14}m_{23}m_{34}^2 + 25m_{23}^2m_{34}^2 + 100m_{13}m_{24}m_{34}^2 + 66m_{14}m_{24}m_{34}^2\\
&\quad + 66m_{23}m_{24}m_{34}^2 + 25m_{24}^2m_{34}^2 + 50m_{13}m_{34}^3 + 50m_{14}m_{34}^3 + 50m_{23}m_{34}^3 + 50m_{24}m_{34}^3 + 25m_{34}^4.
\end{aligned}
$$

Note that L and F depend only on the variables m_{ij}. In particular, if (m_i, m_{ij}) is a nonnegative zero of J, then (m_{ij}) is a nonnegative zero of $L = F = 0$. Vice versa, if $L(m_{ij}) = F(m_{ij}) = 0$ for some nonnegative m_{ij}, setting

$$
\begin{aligned}
m_1 &= \frac{1}{3}(m_{12} + m_{13} + m_{14} + 4) \geq 0,\\
m_2 &= \frac{1}{3}(m_{12} + m_{23} + m_{24} + 4) \geq 0,\\
m_3 &= \frac{1}{3}(m_{13} + m_{23} + m_{34} + 4) \geq 0,\\
m_4 &= \frac{1}{3}(m_{14} + m_{24} + m_{34} + 4) \geq 0,
\end{aligned}
$$

we recover a nonnegative zero of J.

Finally, we prove that L and F have no common nonnegative zeros. For this it is enough to show that F (which does not depend on m_{12}) has no zeros in the unit simplex

$$\Delta = \{(m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathbb{R}^5_{\geq 0} : m_{13} + m_{14} + m_{23} + m_{24} + m_{34} \leq 1\}.$$

Since $0 \leq m_{ij}^4 \leq m_{ij}^3 \leq m_{ij}^2 \leq m_{ij}^1 \leq 1$ hold for points in Δ, term-wise estimates give

$$\begin{aligned}
F \;=\;& -2816 - 256(m_{13}+m_{14}+m_{23}+m_{24}+m_{34}) - 50m_{34}^3 - 7m_{34}^2 \\
&+ 32m_{34} + 192(m_{14}m_{24}+m_{13}m_{14}+m_{13}m_{23}+m_{23}m_{24}) + 256(m_{13}^2+m_{14}^2+m_{23}^2+m_{24}^2+m_{34}^2) \\
&+ 160(m_{14}m_{23}+m_{13}m_{24}) + 96(m_{13}m_{34}+m_{14}m_{34}+m_{23}m_{34}+m_{24}m_{34}) \\
&+ 64(m_{13}^2m_{14}+m_{13}m_{14}^2+m_{13}^2m_{23}+m_{13}m_{23}^2+m_{14}^2m_{24}+m_{23}^2m_{24}+m_{14}m_{24}^2+m_{23}m_{24}^2) \\
&+ 112(m_{13}m_{14}m_{23}+m_{13}m_{14}m_{24}+m_{13}m_{23}m_{24}+m_{14}m_{23}m_{24}) \\
&+ 128(m_{14}^2m_{23}+m_{14}m_{23}^2+m_{13}^2m_{24}+m_{13}m_{24}^2+m_{13}^2m_{34}+m_{14}^2m_{34}+m_{23}^2m_{34}+m_{24}^2m_{34}) \\
&+ 240(m_{13}m_{14}m_{34}+m_{13}m_{23}m_{34}+m_{14}m_{24}m_{34}+m_{23}m_{24}m_{34}) \\
&+ 206(m_{14}m_{23}m_{34}+m_{13}m_{24}m_{34}) + 78(m_{13}m_{34}^2+m_{14}m_{34}^2+m_{23}m_{34}^2+m_{24}m_{34}^2) \\
&+ 16(m_{13}^2m_{14}m_{23}+m_{13}m_{14}^2m_{23}+m_{13}m_{14}m_{23}^2+m_{13}^2m_{14}m_{24}+m_{13}m_{14}^2m_{24}+m_{13}^2m_{23}m_{24} \\
&+ m_{14}^2m_{23}m_{24}+m_{13}m_{23}^2m_{24}+m_{14}m_{23}^2m_{24}+m_{13}m_{14}m_{24}^2+m_{13}m_{23}m_{24}^2+m_{14}m_{23}m_{24}^2+m_{13}^2m_{14}m_{34} \\
&+ m_{13}m_{14}^2m_{34}+m_{13}^2m_{23}m_{34}+m_{13}m_{23}^2m_{34}+m_{14}^2m_{24}m_{34}+m_{23}^2m_{24}m_{34}+m_{14}m_{24}^2m_{34}+m_{23}m_{24}^2m_{34}) \\
&+ 25(m_{14}^2m_{23}^2+m_{13}^2m_{24}^2+m_{13}^2m_{34}^2+m_{14}^2m_{34}^2+m_{23}^2m_{34}^2+m_{24}^2m_{34}^2+m_{34}^4) \\
&+ 82(m_{13}m_{14}m_{23}m_{34}+m_{13}m_{14}m_{24}m_{34}+m_{13}m_{23}m_{24}m_{34}+m_{14}m_{23}m_{24}m_{34}) + 14m_{13}m_{14}m_{23}m_{24} \\
&+ 50(m_{14}^2m_{23}m_{34}+m_{14}m_{23}^2m_{34}+m_{13}^2m_{24}m_{34}+m_{13}m_{24}^2m_{34}+m_{13}m_{34}^3+m_{14}m_{34}^3+m_{23}m_{34}^3+m_{24}m_{34}^3) \\
&+ 66(m_{13}m_{14}m_{34}^2+m_{13}m_{23}m_{34}^2+m_{14}m_{24}m_{34}^2+m_{23}m_{24}m_{34}^2) + 100(m_{14}m_{23}m_{34}^2+m_{13}m_{24}m_{34}^2) \\
\leq\;& -2816 - 256m - 50m_{34}^3 - 7m_{34}^2 + 32m_{34} + 192(m_{14}+m_{13}+m_{23}+m_{24}) + 256m \\
&+ 160(m_{14}+m_{13}) + 96(m_{13}+m_{14}+m_{23}+m_{24}) + 64(m_{13}^2+m_{14}^2+m_{23}^2+m_{24}^2) \\
&+ 112(m_{13}+m_{14}+m_{23}+m_{24}) + 128(m_{14}^2+m_{23}^2+m_{13}^2+m_{24}^2) + 240(m_{14}+m_{13}+m_{24}+m_{23}) \\
&+ 206(m_{23}+m_{24}) + 78(m_{13}+m_{14}+m_{23}+m_{24}) + 16(m_{13}m_{14}m_{23}+m_{13}m_{14}m_{24} \\
&+ m_{13}m_{23}m_{24}+m_{14}m_{23}m_{24}+m_{13}m_{14}m_{34}+m_{13}m_{23}m_{34}+m_{14}m_{24}m_{34}+m_{23}m_{24}m_{34}) \\
&+ 25(m_{14}^2+m_{13}^2+m_{23}^2+m_{24}^2+m_{34}^2) + 82(m_{13}+m_{14}+m_{23}+m_{24}) + 14 \\
&+ 50(m_{14}^2+m_{23}^2+m_{13}^2+m_{24}^2+m_{34}^2) + 66(m_{14}+m_{13}+m_{24}+m_{23}) + 100(m_{14}+m_{13}) \\
\leq\;& -2816 + 32 + 192 + 160 + 96 + 64 + 112 + 128 + 240 + 206 + 78 + 16 + 25 + 82 + 14 \\
&+ 50 + 66 + 100 \;=\; -1155 \;<\; 0,
\end{aligned}$$

where $m = m_{13} + m_{14} + m_{23} + m_{24} + m_{34} \leq 1$. It follows that F takes only negative values on Δ. The above implies that no matrix $X = V^T \Lambda V$ as in (4.1) can be 2-locally PSD and hence $\lambda = (4, 4, -1, -1) \notin \lambda(\mathcal{S}^{4,2})$. □

Remark 4.2. Note that the absense of nonnegative roots of $L = F = 0$ implies that the minimum ℓ_* of the linear function $\ell = m_{13} + m_{14} + m_{23} + m_{24} + m_{34}$ on $\{F = 0\} \cap \mathbb{R}^5_{\geq 0}$ is greater than 1. Numerical solving of a bunch of constrained optimization problems suggests that the minimum is $\ell_* = 3.2$ and it is attained at $(m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) = (0, 0, 0, 0, 3.2)$. It would be interesting to find an algebraic certificate of nonnegativity (see e.g. [9] and [4]) of say $\ell - 2$ on the semialgebraic set $\{F = 0\} \cap \mathbb{R}^5_{\geq 0}$. This would give an alternative proof of emptiness of $\{L = F = 0\} \cap \mathbb{R}^5_{\geq 0}$ (cf. the proof of Theorem 1.1).

Proposition 1.3 essentially follows from properties of the PSD cone $\mathcal{S}^{n,n}$.

Proof of Proposition 1.3. Let $X = V^T \Lambda V \in \mathcal{S}^{n,k}$ be a spectral decomposition of a k-locally PSD matrix, where V is an orthogonal matrix and Λ is the diagonal matrix given by $\lambda \in \lambda(\mathcal{S}^{n,k})$. Due to the orthogonal invariance of the PSD cone $\mathcal{S}^{n,n}$, the matrix $X^+ = V^T \Lambda^+ V \in \mathcal{S}^{n,n} \subset \mathcal{S}^{n,k}$ is in particular k-locally PSD, where Λ^+ is the diagonal matrix given by the nonnegative vector $\lambda^+ \in \mathbb{R}^n_{\geq 0} \subset \lambda(\mathcal{S}^{n,k})$. By convexity of $\mathcal{S}^{n,k}$

$$X + X^+ \ = \ V^T(\Lambda + \Lambda^+)V$$

is in $\mathcal{S}^{n,k}$ and hence $\lambda + \lambda^+ \in \lambda(\mathcal{S}^{n,k})$. □

Proof of Corollary 1.4. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in H(e_2^4)$ satisfy $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. If $\lambda_4 \geq 0$, trivially $\lambda \in \mathbb{R}^4_{\geq 0} \subset \lambda(\mathcal{S}^{4,2})$. Let then $\lambda_3 \geq 0 > \lambda_4$. Note that for $t = e_2^4(\lambda)/(\lambda_1 + \lambda_2 + \lambda_3) \geq 0$ the vector $\lambda - t(0,0,0,1)$ lies on the boundary of $H(e_2^4)$ and it has exactly one negative entry $\lambda_4 - t$. By [2, Thm. 8.1] the vector $\lambda - t(0,0,0,1)$ is in $\lambda(\mathcal{S}^{4,2})$. Thus, by Proposition 1.3 the original vector $\lambda = (\lambda - t(0,0,0,1)) + t(0,0,0,1)$ is also in $\lambda(\mathcal{S}^{4,2})$ because $t(0,0,0,1) \in \mathbb{R}^4_{\geq 0}$. □

Our proof of Theorem 1.5 is based on the following classical result.

Theorem 4.3 (Schur-Horn theorem)**.** *Let $x, \lambda \in \mathbb{R}^n$ be two vectors with $x_1 \geq \cdots \geq x_n$ and $\lambda_1 \geq \cdots \geq \lambda_n$. There exists a symmetric matrix $X \in \mathrm{Sym}_n$ with diagonal entries $x_1, \ldots, x_n$ and with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if for all $i = 1, \ldots, n-1$*

$$\sum_{j=1}^{i} x_j \ \leq \ \sum_{j=1}^{i} \lambda_j \quad \text{and} \quad \sum_{j=1}^{n} x_j \ = \ \sum_{j=1}^{n} \lambda_j.$$

Proof of Theorem 1.5. Note that the statement is equivalent to

$$\lambda(\mathcal{S}^{n,1}) \ = \ \left\{ \lambda^\sigma \in \mathbb{R}^n \, : \, \sigma \in S_n, \ \lambda_1 \geq \cdots \geq \lambda_n, \ \sum_{j=1}^{i} \lambda_j \geq 0 \text{ for all } i = 1, \ldots, n \right\}. \tag{4.2}$$

Indeed, if $\sum_{j=1}^{i} \lambda_j < 0$ for some $i < n$, then $\lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_n$ must be negative. Therefore, $\sum_{j=1}^{n} \lambda_j \leq \sum_{j=1}^{i} \lambda_j < 0$ and hence no permuted vector λ^σ, $\sigma \in S_n$, can satisfy $\sum_{j=1}^{n} \lambda_j^\sigma = \sum_{j=1}^{n} \lambda_j \geq 0$. The opposite direction is trivial. Thus, we need to prove that ordered vectors in $\lambda(\mathcal{S}^{n,1})$ are exactly those that satisfy inequalities in (4.2).

Let $\lambda = \lambda(X)$, where the matrix $X \in \mathcal{S}^{n,1}$ has ordered diagonal entries $x_1 \geq \cdots \geq x_n$. By Theorem 4.3 we have that $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} x_j \geq 0$ holds for $i = 1, \ldots, n$.

Let now $\lambda \in \mathbb{R}^n$ be an ordered nonzero vector that satisfies inequalities from (4.2). There exists an index $r \in \{1, \ldots, n\}$ so that $\lambda_1 \geq \cdots \geq \lambda_r > 0 \geq \lambda_{r+1} \geq \cdots \geq \lambda_n$. Let us consider a vector $x \in \mathbb{R}^n$ defined by $x_1 = \cdots = x_r = \frac{1}{r} \sum_{j=1}^{n} \lambda_j \geq 0$ and $x_{r+1} = \cdots =$

$x_n = 0$. For $i = 1, \dots, r-1$ we have

$$\sum_{j=1}^{i} x_j \;=\; \frac{i}{r}\sum_{j=1}^{n}\lambda_j \;\le\; \frac{i}{r}\sum_{j=1}^{r}\lambda_j \;=\; \sum_{j=1}^{i}\lambda_j + \frac{i-r}{r}\sum_{j=1}^{i}\lambda_j + \frac{i}{r}\sum_{j=i+1}^{r}\lambda_j$$
$$\le\; \sum_{j=1}^{i}\lambda_j + \frac{i(i-r)}{r}\lambda_i + \frac{i}{r}\sum_{j=i+1}^{r}\lambda_j \;\le\; \sum_{j=1}^{i}\lambda_j + \frac{i}{r}\sum_{j=i+1}^{r}(\lambda_j - \lambda_i) \;\le \sum_{j=1}^{i}\lambda_j,$$

where ordering $\lambda_1 \ge \cdots \ge \lambda_r > 0 \ge \lambda_{r+1} \ge \cdots \ge \lambda_n$ is used in several places. For $i = r, \dots, n$ we have

$$\sum_{j=1}^{i} x_j \;=\; \sum_{j=1}^{n}\lambda_j \;\le\; \sum_{j=1}^{i}\lambda_j,$$

where we use negativity of $\lambda_{r+1}, \dots, \lambda_n < 0$. These inequalities, the equality $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n}\lambda_j$ and Theorem 4.3 imply that $\lambda_1, \dots, \lambda_n$ are eigenvalues of some matrix $X \in \mathrm{Sym}_n$ with diagonal entries $x_1, \dots, x_n \ge 0$, that is, the vector $\lambda = \lambda(X)$ is in $\lambda(\mathcal{S}^{n,1})$. □

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Khazhgali Kozhasov, Technische Universität Braunschweig, Institut für Analysis und Algebra, Universitätsplatz 2, 38106 Braunschweig, Germany

Email address: k.kozhasov@tu-braunschweig.de