Nonlinear Differ. Equ. Appl. (2025) 32:59

1021-9722/25/040001-26
published online May 17, 2025
https://doi.org/10.1007/s00030-025-01063-5

Nonlinear Differential Equations and Applications NoDEA

A new space of generalised vector-valued functions of bounded variation

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Abstract. In [18], the authors introduced the space of scalar-valued functions $GBV_\star(A)$ to minimise a class of functionals whose study is motivated by fracture mechanics. In this paper, we extend the definition of $GBV_\star(A)$ to the vectorial case, introducing the space $GBV_\star(A;\mathbb{R}^k)$. We study the main properties of $GBV_\star(A;\mathbb{R}^k)$ and prove a lower semicontinuity result useful for minimisation purposes. With the Direct Method in mind, we adapt the arguments of [18] to show that minimising sequences in $GBV_\star(A;\mathbb{R}^k)$ can be modified to obtain a minimising sequence converging $\mathcal{L}^d$-a.e in A.

Mathematics Subject Classification. 26A45, 49J45.

Keywords. Generalised functions of bounded variation, Minimum problems, Lower semicontinuity.

1. Introduction

In his seminal paper [24], Griffith introduced the idea that the stability or growth of the crack path in a material is determined by the competition of two energies acting at two different scales: the bulk energy spent to elastically deform the material and the surface energy spent to widen the pre-existing crack. In [26], Francfort and Marigo revisited Griffith's theory and proposed a model of crack growth based on an energy minimisation principle, renewing the interest in the study of energy functionals of the type

$$\int_A f(x,\nabla u)\,dx + \int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}, \tag{1.1}$$

where $A \subset \mathbb{R}^d$ is an open bounded set, the displacement $u\colon A \to \mathbb{R}^k$ can vary among different spaces of functions with bounded variation, $[u] = u^+ - u^-$ represents the crack opening along the crack surface J_u, ν_u is the unit normal at J_u, and ∇u is the approximate gradient of u.

The most common examples of such functionals satisfy $g \equiv$ constant (see for instance [11]), a choice which for lower semicontinuity reasons (see [3]) forces f to be superlinear and requires to consider $SBV(A;\mathbb{R}^k)$ as underlying space. Many relevant functionals fall in this class. For instance, setting $f(x,\xi) = f(\xi) = |\xi|^p$, for $p > 1$ one obtains the widely studied Mumford-Shah functional.

However, the hypothesis that g is independent of the crack opening $[u]$ is in general not fully satisfactory in describing the fracture of materials, as cohesive forces are observed to be in action along the crack surface (see [6,21]). This last observation suggests the introduction of cohesive models, in which integrands g that are linear around the origin are considered, the most simple of this kind of integrands being $g(x,\zeta,\nu) = |\zeta| \wedge 1$, where for $a, b \in \mathbb{R}$, $a \wedge b$ is the minimum between a and b. If one considers functions g satisfying growth conditions of type

$$c(|\zeta| \wedge 1) \leq g(x,\zeta,\nu) \leq C(|\zeta| \wedge 1), \tag{1.2}$$

for suitable constants $0 < c \leq C$, lower semicontinuity reasons (see [9, Theorem 3.1]) force f to have linear growth and to consider an additional term in functionals (1.1). The object of study then becomes

$$F(u) = \int_A f(x,\nabla u)\,dx + \int_A f^\infty\Big(x, \frac{dD^c u}{d|D^c u|}\Big)\,d|D^c u| + \int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}, \tag{1.3}$$

where f^∞ is the recession function of f, $D^c u$ is the Cantor part of the displacement u, which one may interpret as a term carrying information about the formation of microfractures at a diffuse scale, and $dD^c u/d|D^c u|$ is the Radon-Nikodým derivative of $D^c u$ with respect to $|D^c u|$.

Functionals of type (1.3) with g satisfying (1.2) and approximations of such functionals have been extensively studied in recent years (see, for instance, [2,7,8,13,14,16–19]).

The choice of the function space where to study the functional F is a delicate matter. Since by (1.2) g does not control the full amplitude of the crack openings $|[u]|$, one is tempted to consider as ambient space Ambrosio's $GBV(A)^k$ (see [4, Section 1] and [5, Chapter 4]), where all the quantities appearing in (1.3) are well-defined, including $D^c u$ (see for instance [2, Lemma 2.10]). However, under the growth conditions (1.2), the sublevels of F are not precompact in $GBV(A)^k$, and compactness is satisfied only by adding to F lower order terms.

In the scalar case, a possible choice is the space $GBV_\star(A)$, introduced by Dal Maso and Toader in [18]. In [17,18] the authors are able to adapt the arguments of [23] to prove that for certain minimum problems with Dirichlet boundary conditions, minimising sequences $(u_n)_n \subset GBV_\star(A)$ admit a modification $(y_n)_n \subset GBV_\star(A)$ which is still minimising and which converges $\mathcal{L}^d$-a.e. to $y \in GBV_\star(A)$. This compactness property is achieved without any additional control on lower order terms.

Adopting this last point of view, the aim of the present work is twofold. First, we address the problem of extending the definition of $GBV_\star(A)$ to the vectorial case and describing the main features of $GBV_\star(A;\mathbb{R}^k)$. Then, we show that the compactness results of [17,18] can be extended to $GBV_\star(A;\mathbb{R}^k)$ and give some sufficient condition for lower semicontinuity of integral functionals whose domain is $GBV_\star(A;\mathbb{R}^k)$. These results will be used in [15] to deal with the homogenisation of functionals of cohesive type in the vectorial setting .

The structure of the paper is as follows. In Section 3 we introduce the space $GBV_\star(A;\mathbb{R}^k)$ and we present some of its main properties. We later study the relation between a function $u \in GBV_\star(A;\mathbb{R}^k)$ and $\phi \circ u$, for ϕ a Lipschitz function with compact support. We show in Proposition 3.9 that $u \in GBV_\star(A;\mathbb{R}^k)$ if and only if for every ϕ as above, the function $\phi\circ u$ belongs to $BV(A;\mathbb{R}^k)$ and some relevant quantities associated with the derivatives of $\phi\circ u$ are controlled from above by a constant M, independent of ϕ, and $\mathrm{Lip}(\phi)$, the Lipschitz constant of ϕ. We then analyse in Proposition 3.13 the relation between $D^c(\phi\circ u)$ and $D^c u$ and use this result in Corollary 3.14 to show that $D^c u$ satisfies the Rank-One property.

In Section 4 we investigate the lower semicontinuity of a class of functionals with respect to the convergence in measure. In Theorem 4.5, we prove the lower semicontinuity of certain functionals depending on ∇u and whose domain is the space $GBV_\star(A;\mathbb{R}^k)$.

In the rest of the section we study the compactness properties of the space $GBV_\star(A;\mathbb{R}^k)$. We first show that the following compactness result (Theorem 4.7) holds: if $(u_n)_n \subset GBV_\star(A;\mathbb{R}^k)$ is a minimising sequence for F given by (1.3) with g satisfying (1.2) and f with linear growth, and if

$$\sup_{n\in\mathbb{N}} \int_A h(|u_n|)\,dx < +\infty, \tag{1.4}$$

for some continuous increasing function h with $h(t) \to +\infty$ for $t \to +\infty$, then there exists a subsequence of $(u_n)_n$ converging $\mathcal{L}^d$-a.e. to some function $u \in GBV_\star(A;\mathbb{R}^k)$.

We conclude by showing (Theorem 4.8) that using the arguments of [18, Theorem 5.5] and of [17, Theorem 7.13], for every $\varepsilon_n \to 0^+$ every sequence of functions $(u_n)_n \subset GBV_\star(A;\mathbb{R}^k)$ satisfying some fixed common Dirichlet boundary conditions and such that $\sup_{n\in\mathbb{N}} F(u_n) < +\infty$ admits a modification $(y_n)_n \subset GBV_\star(A;\mathbb{R}^k)$, with $y_n = u_n$ on ∂A, such that $F(y_n) \le F(u_n) + \varepsilon_n$ and satisfying (1.4) with u_n replaced by y_n.

2. Notation and preliminaries

We fix some notation that will be used throughout the paper.

(a) Given $n \in \mathbb{N}$, the symbol $\cdot$ denotes the scalar product of $\mathbb{R}^n$ and $|\ |$ the Euclidean norm of $\mathbb{R}^n$. If $a \in \mathbb{R}^n$, a_i is the i-th component of a. The unit sphere of $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Given $x \in \mathbb{R}^n$ and $\rho > 0$, the open ball of center x and radius ρ is denoted by $B_\rho(x)$.

(b) Given $k, d \in \mathbb{N}$, we identify vectors in $\mathbb{R}^{k\times d}$ with $k \times d$ matrices. Given a matrix $\xi \in \mathbb{R}^{k\times d}$ and a vector $x \in \mathbb{R}^d$, the vector ξx is defined via the usual matrix by vector product. Given $\xi = (\xi_{ij}) \in \mathbb{R}^{k\times d}$ the Frobenius norm of ξ is defined by

$$|\xi| = \Big(\sum_{i=1}^{k}\sum_{j=1}^{d}\xi_{ij}^2\Big)^{\frac{1}{2}},$$

while the operatorial norm of ξ is defined by

$$|\xi|_{\mathrm{op}} := \sup_{\nu\in\mathbb{S}^{d-1}} |\xi\nu|.$$

(c) If $A \subset \mathbb{R}^d$ is an open set, $\mathcal{A}(A)$ is the collection of all open subsets of A. If $A', A'' \in \mathcal{A}(A)$, $A' \subset\subset A''$ means that A' is relatively compact in A''. The symbol $\mathcal{B}(A)$ denotes the σ-algebra of all Borel subsets of A.

(d) If $A \in \mathcal{A}(\mathbb{R}^d)$, the space of all $\mathbb{R}^m$-valued bounded Radon measures on A is denoted by $\mathcal{M}_b(A;\mathbb{R}^m)$, the indication of $\mathbb{R}^m$ being omitted if $m = 1$. Given a positive $\lambda \in \mathcal{M}_b(A)$ and $\mu \in \mathcal{M}_b(A;\mathbb{R}^m)$, the Radon-Nikodým derivative of μ with respect to λ is denoted by $d\mu/d\lambda$. Given $\mu \in \mathcal{M}(A;\mathbb{R}^{k\times d})$ we denote by $|\mu|$ the total variation computed with respect to the Frobenius norm, while $|\mu|_{\mathrm{op}}$ is the total variation computed with respect to the operatorial norm, i.e., the measure defined for every $B \in \mathcal{B}(A)$ by

$$|\mu|_{\mathrm{op}}(B) := \sup\sum_{i=1}^{n} |\mu(B_i)|_{\mathrm{op}},$$

where the supremum is taken all over $n \in \mathbb{N}$ and finite collections $(B_i)_{i=1}^n$ of pairwise disjoint relatively compact subsets of B. The d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$, while the $(d-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{d-1}$.

(e) Given $A \in \mathcal{A}(\mathbb{R}^d)$, the space of all $\mathcal{L}^d$-measurable functions $u\colon A \to \mathbb{R}^k$ is denoted by $L^0(A;\mathbb{R}^k)$, the indication of $\mathbb{R}^k$ being omitted when $k = 1$. We endow this space with the topology induced by the convergence in measure. With this choice of topology, the space $L^0(A;\mathbb{R}^k)$ is metrisable and separable.

(f) For every $\mathcal{L}^d$-measurable set $E \subset \mathbb{R}^d$, $\chi_E\colon \mathbb{R}^d \to \mathbb{R}$ is the characteristic function of the set E, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise. If $E \subset \mathbb{R}^d$ is an $\mathcal{L}^d$-measurable set with locally finite perimeter (see [25] for the general properties of such sets), we denote by ∂^*E its reduced boundary. We recall that the perimeter of a set relative to $A \in \mathcal{A}(\mathbb{R}^d)$ is defined by

$$\mathrm{Per}(u, A) := \mathcal{H}^{d-1}(\partial^*E \cap A).$$

(g) If $E \subset \mathbb{R}^d$ is an $\mathcal{L}^d$-measurable set a point $x \in \mathbb{R}^d$ is a point with positive density, i.e.,

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^d(E \cap B_\rho(x))}{\rho^d} > 0,$$

and if $u\colon E \to \mathbb{R}^d$ is an an $\mathcal{L}^d$-measurable function, we say that $a \in \mathbb{R}^k$ is the approximate limit of u at the point x, in symbols

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in E}} u(y) = a,$$

if for every $\varepsilon > 0$ we have

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^d(\{|u-a|>\varepsilon\}\cap B_\rho(x))}{\rho^d} = 0,$$

where $\{|u-a|>\varepsilon\} := \{y \in E\colon |u(y)-a| > \varepsilon\}$. Given $A \subset \mathbb{R}^d$ open and $u \in L^0(A;\mathbb{R}^k)$, we denote by $S_u \subset A$ the complement in A of the set of points $x \in A$ such that there exists

$$\widetilde{u}(x) := \operatorname*{ap\,lim}_{\substack{y\to x\\ y\in A}} u(y).$$

If $x \notin S(u)$, we say that x is a point of approximate continuity for u and that u is approximately continuous at x.

(l) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and $u \in L^0(A;\mathbb{R}^k)$ the symbol J_u denotes the jump set of u, that is, the set of $x \in A$ such that there exists a triple $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{S}^{d-1}$, with $u^+(x) \neq u^-(x)$, for which, setting

$$H_x^{\pm} := \{y \in A\colon \ \pm (y-x)\cdot \nu_u(x) > 0\},$$

we have

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H_x^+}} u(y) = u^+(x) \text{ and } \operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H_x^-}} u(y) = u^-(x).$$

The triple $(u^+(x), u^-(x), \nu_u(x))$ is unique up to interchanging the roles of $u^+(x)$ and $u^-(x)$ and changing the sign of $\nu_u(x)$. It holds the inclusion $J_u \subset S_u$. Moreover, it was proved in [20] that if $u \in L^1_{\text{loc}}$ then J_u is a $(d-1)$-countably rectifiable set (see [5, Definition 2.57]).

(m) Given $A \in \mathcal{A}(\mathbb{R}^d)$, the space of $\mathbb{R}^k$-valued functions with bounded variation on A is denoted by $BV(A;\mathbb{R}^k)$. If $k = 1$, we simply write $BV(A)$. We refer the reader to [5,22] for a complete introduction to such spaces. If $u \in BV(A;\mathbb{R}^k)$ then $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$, J_u is a $(d-1)$-countably rectifiable set, and for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ the vector $\nu_u(x)$ is a measure theoretical unit normal to J_u. For every $x \in J_u$ we set

$$[u](x) := u^+(x) - u^-(x).$$

Note that a change in the sign $\nu_u(x)$ naturally implies a change of sign in $[u](x)$.

(n) If $u \in BV(A;\mathbb{R}^k)$, Du denotes its distributional derivative, which is by definition an $\mathbb{R}^{k\times d}$-valued Radon measure. The measure Du can be decomposed as

$$Du = \nabla u \mathcal{L}^d + D^c u + [u] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u,$$

where

- $\nabla u \in L^1(A;\mathbb{R}^{k\times d})$ is the approximate gradient of u, i.e., the unique $\mathbb{R}^{k\times d}$-valued function such that for $\mathcal{L}^d$-a.e $x \in A$ we have

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in A}} \frac{u(y)-\widetilde{u}(x)-\nabla u(x)(y-x)}{|y-x|}=0, \tag{2.1}$$

- $D^c u$, the Cantor part of Du, is a measure vanishing on all $B \in \mathcal{B}(A)$ such that $\mathcal{H}^{d-1}(B) < +\infty$ and which is singular with respect to $\mathcal{L}^d$,
- $\otimes$ is the tensor product defined by $(a\otimes b)_{ij} = a_i b_j$ for $a \in \mathbb{R}^k$, $b \in \mathbb{R}^d$, $\mathcal{H}^{d-1} \llcorner J_u$ is the restriction of $\mathcal{H}^{d-1}$ to J_u, i.e., the measure on A defined by $(\mathcal{H}^{d-1} \llcorner J_u)(B) := \mathcal{H}^{d-1}(B \cap J_u)$ for every $B \in \mathcal{B}(A)$; finally, $[u] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u$ is the measure whose density with respect to $\mathcal{H}^{d-1} \llcorner J_u$ is given by $[u] \otimes \nu_u$.

We briefly recall the properties of slicing of functions in $BV(A;\mathbb{R}^k)$ (see [5, Section 3.12]). Given $\nu \in \mathbb{S}^{d-1}$, the hyperplane through the origin orthogonal to ν is denoted by $\Pi_\nu := \{y \in \mathbb{R}^d : y\cdot\nu = 0\}$. For every $y \in \Pi_\nu$, $B \in \mathcal{B}(A)$, we set $B^\nu_y := \{t \in \mathbb{R} : y + t\nu \in B\}$. For every $u \in BV(A;\mathbb{R}^k)$, $\nu \in \mathbb{S}^{d-1}$, and $y \in \Pi_\nu$, $u^\nu_y : A^\nu_y \to \mathbb{R}^k$ is the function defined by

$$u^\nu_y(t) := u(y + t\nu) \quad \text{for every } t \in A^\nu_y. \tag{2.2}$$

The following proposition describes some of the well known connections between BV functions and their one-dimensional slices (see [5, Theorem 3.108]).

Proposition 2.1. *Let $u \in BV(A;\mathbb{R}^k)$ and let $\nu \in \mathbb{S}^{d-1}$. Then*

(a) for $\mathcal{H}^{d-1}$-a.e $y \in \Pi_\nu$ we have $u^\nu_y \in BV(A^\nu_y;\mathbb{R}^k)$;
(b) we have

$$\nabla(u^\nu_y) = (\nabla u)^\nu_y \quad \text{for } \mathcal{H}^{d-1}\text{-a.e.} y \in \Pi_\nu,$$
$$((D^j u)\nu)(B) = \int_{\Pi_\nu} D^j u^\nu_y(B^\nu_y)\, d\mathcal{H}^{d-1}(y),$$
$$((D^c u)\nu)(B) = \int_{\Pi_\nu} D^c u^\nu_y(B^\nu_y)\, d\mathcal{H}^{d-1}(y),$$

for every $B \in \mathcal{B}(A)$;
(c) if $u_n \rightharpoonup u$ weakly in $BV(A;\mathbb{R}^k)$, then for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\nu$ we have $(u_n)^\nu_y \rightharpoonup u^\nu_y$ weakly* in $BV(A^\nu_y;\mathbb{R}^k)$.*

We recall the chain rule in $BV(A;\mathbb{R}^k)$ for compositions with smooth functions (see [5, Theorem 3.99]). For the rest of the section $A \subset \mathbb{R}^d$ will always be a bounded open set.

Theorem 2.2. (Chain Rule in $BV(A;\mathbb{R}^k)$) *Let $\varphi \in C^1(\mathbb{R}^k;\mathbb{R}^n)$ and $u \in BV(A;\mathbb{R}^k)$. Then, setting $v := \varphi\circ u$, we have $v \in BV(A;\mathbb{R}^n)$ and*

$$Dv = \nabla\varphi(u)\nabla u\mathcal{L}^d + \nabla\varphi(\widetilde{u})D^c u + [\varphi(u)] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_{\phi\circ u} \tag{2.3}$$

as Radon measures.

Dal Maso and Toader introduced in [18] the space $GBV_\star(A)$, a subspace of Ambrosio's $GBV(A)$ (see [5, Section 4.5]). For the reader's convenience we briefly recall the definition of the spaces $GBV(A)$ and $GBV_\star(A)$ and some relevant properties of such spaces. For every $t \in \mathbb{R}$ and $m > 0$, we denote by $t^{(m)} := (t \wedge m) \vee (-m)$, where for $a, b \in \mathbb{R}$, the symbols $a \wedge b$ and $a \vee b$ denote the minimum between a and b and the maximum between a and b, respectively.

Definition 2.3. Let $u \in L^0(A)$. Then

(a) $u \in GBV(A)$ if and only if $u^{(m)} \in BV_{\mathrm{loc}}(A)$ for every $m > 0$;
(b) $u \in GBV_\star(A)$ if and only if $u^{(m)} \in BV(A)$ for every $m > 0$ and there exists $M > 0$ such that

$$\sup_{m>0} \int_A |\nabla u^{(m)}|\,dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1\,d\mathcal{H}^{d-1} \leq M. \tag{2.4}$$

The following proposition collects the main properties of the spaces $GBV(A)$ and $GBV_\star(A)$.

Proposition 2.4. *Let $u \in GBV(A)$. Then*

(a) for $\mathcal{H}^{d-1}$-a.e $x \in A \setminus J_u$ there exists finite

$$\widetilde{u}(x) := \operatorname*{ap\,lim}_{y \to x} u(y);$$

moreover u^+ and u^- are finite for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$;

(b) u is approximately differentiable, i.e., there exists a Borel function $\nabla u : A \to \mathbb{R}^d$ such that formula (2.1) holds for $\mathcal{L}^d$-a.e. $x \in A$; moreover, for every $m > 0$ we have

$$\nabla u(x) = \nabla u^{(m)}(x) \quad \text{for } \mathcal{L}^{\mathrm{d}}\text{-a.e. } x \in \{x \in A : |u(x)| \leq m\};$$

(c) there exists a unique positive measure $|D^c u| \in \mathcal{M}_b(A)$ such that

$$|D^c u|(B) = \sup_{m>0} |D^c u^{(m)}|(B) \quad \textit{for every } B \in \mathcal{B}(A);$$

(d) for every $m > 0$ we have $J_{u^{(m)}} \subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set and $|[u^{(m)}]| \leq |[u]|$ $\mathcal{H}^{d-1}$-a.e. in $J_{u^{(m)}} \cap J_u$. Moreover, for $\mathcal{H}^{d-1}$-a.e. x in J_u there exists $m_x \in \mathbb{N}$ such that $x \in J_{u^{(m)}}$ for every $m \in \mathbb{N}$ with $m \geq m_x$, and $[u^{(m)}](x) \to [u](x)$ as $m \to +\infty$.

If, in addition, $u \in GBV_\star(A)$, then

(e) there exists a unique Radon measure $D^c u \in \mathcal{M}_b(A; \mathbb{R}^d)$ such that for every $m > 0$ we have $D^c u(B) = D^c u^{(m)}(B)$ for every $B \subset \{x \in A \colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)| \leq m\}$ and $D^c u(B) = 0$ for every $B \in \mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B \setminus J_u) = 0$; moreover, for every $B \in \mathcal{B}(A)$ we have

$$D^c(B) = \lim_{m \to \infty} D^c u^{(m)}(B),$$
$$|D^c u|(B) = \sup_{m>0} |D^c u^{(m)}|(B).$$

For the proofs of (a)-(d) we refer the reader to [5, Theorem 4.34], while for the proof of (e) we refer to [18, Theorem 2.7, Propositions 2.9, 3.3].

We conclude the current section recalling the vector space properties of $GBV_\star(A)$.

Proposition 2.5. *([18, Theorem 3.9]) $GBV_\star(A)$ is a vector space. Moreover, for every $u, v \in GBV_\star(A)$ and $\lambda \in \mathbb{R}$ we have*

$$\nabla(u+v) = \nabla u + \nabla v, \;\; \nabla(\lambda u) = \lambda \nabla u \;\; \mathcal{L}^d\text{-a.e.in } A, \tag{2.5}$$

$$D^c(u+v) = D^c u + D^c v, \;\; D^c(\lambda u) = \lambda D^c u \;\; \text{on } A, \tag{2.6}$$

$$[u+v] = [u] + [v] \;\; \text{and} \;\; [\lambda u] = \lambda [u] \;\; \mathcal{H}^{d-1}\text{-a.e.in } A. \tag{2.7}$$

3. The space $GBV_\star(A; \mathbb{R}^k)$

In this section we give the Definition of the space $GBV_\star(A; \mathbb{R}^k)$, and discuss some of its properties.

Throughout the rest of the paper $A \subset \mathbb{R}^d$ will be a bounded open set.

Definition 3.1. Let $u \in L^0(A; \mathbb{R}^k)$. We say $u \in GBV_\star(A; \mathbb{R}^k)$ if $u_i \in GBV_\star(A)$ for every $i = 1, ..., k$.

Remark 3.2. The space $GBV(A; \mathbb{R}^k)$ of generalised functions of bounded variation contains the space $GBV(A)^k$ (strictly if $k > 1$, see [5, Remark 4.27]) of $\mathbb{R}^k$-valued functions whose components are in $GBV(A)$. It is easy to see that $GBV_\star(A; \mathbb{R}^k) \subset GBV(A)^k \subset GBV(A; \mathbb{R}^k)$, the inclusions being in general strict.

It follows immediately from Proposition 3.11a that $GBV_\star(A; \mathbb{R}^k)$ is a vector space.

Proposition 3.3. *$GBV_\star(A; \mathbb{R}^k)$ is a vector space.*

If $m > 0$ and $u \in L^0(A; \mathbb{R}^k)$, we set $u^{(m)} := ((u_1 \wedge m) \vee (-m), ..., (u_k \wedge m) \vee (-m))$.

Definition 3.4. Let $u \in GBV_\star(A; \mathbb{R}^k)$. The measure $D^c u \in \mathcal{M}_b(A, \mathbb{R}^{k \times d})$ is the matrix-valued measure whose i-th row is defined for every $B \in \mathcal{B}(A)$ by

$$(D^c u(B))_i := D^c u_i(B).$$

The measure $D^c u$ enjoys the same properties of its scalar counterpart (see Proposition 2.4(e)).

Lemma 3.5. *Let $u \in GBV_\star(A; \mathbb{R}^k)$. Then*

(a) for every $m > 0$ we have $D^c u(B) = D^c u^{(m)}(B)$ for every $B \subset \{x \in A \colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)| \le m\}$;
(b) $D^c u(B) = 0$ for every $B \in \mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B \setminus J_u) = 0$;
(c) setting $\widetilde{A} := \{x \in A : \widetilde{u}(x)$ exists$\}$, we have $\mathcal{H}^{d-1}(\widetilde{A} \setminus J_u) = 0$.

Moreover,

$$D^c u(B) = \lim_{m\to+\infty} D^c u^{(m)}(B), \tag{3.1}$$

$$|D^c u|(B) = \sup_{m>0} |D^c u^{(m)}|(B), \tag{3.2}$$

for every $B \in \mathcal{B}(A)$.

Proof. The proof of (a) and (b) is a simple consequence of Proposition 2.4 and of the definition of $D^c u$. To prove (c), it is enough to note that for every $i \in \{1, ..., k\}$ we have $\widetilde{A} = \cap_{i=1}^k \widetilde{A}_i$, where $\widetilde{A}_i := \{x \in A : \widetilde{u}_i(x) \text{ exists}\}$, and use (a) of Proposition 2.4.

We are left with proving that for any $B \in \mathcal{B}(A)$ equalities (3.1) and (3.2) hold. To this aim, let us fix $B \in \mathcal{B}(A)$. For every $m \in \mathbb{N}$ we set

$$B_m := \bigcap_{i=1}^k \{x \in B\colon \widetilde{u}_i(x) \text{ exists and } |\widetilde{u}_i(x)| \le m\}.$$

By Proposition 2.4 we see that $|D^c u|(B \setminus \bigcup_{m\in\mathbb{N}} B_m) \le \sum_{i=1}^k |D^c u_i|(B \setminus \bigcup_{m\in\mathbb{N}} B_m) = 0$. Hence, for every $\varepsilon > 0$ there exists $m' \in \mathbb{N}$ such that $|D^c u|(B \setminus B_m) \le \varepsilon$ for every $m \ge m'$. Using again Proposition 2.4, by definition of B_m we have $D^c u_i(B_m) = D^c u_i^{(m)}(B_m)$ for every $i \in \{1, ..., k\}$. Thus for every $m \ge m'$ we have

$$|D^c u(B) - D^c u^{(m)}(B)| \le |D^c u|(B \setminus B_m) \le \varepsilon.$$

Letting $\varepsilon \to 0^+$, we obtain (3.1).

The proof of equality (3.2) follows by similar arguments. This concludes the proof. □

We now show that $GBV_\star(A;\mathbb{R}^k)$ is well-behaved under linear changes of coordinates.

Proposition 3.6. *Let* $u \in GBV_\star(A;\mathbb{R}^k)$ *and let* $C \in \mathbb{R}^{k\times k}$ *be an invertible matrix. Then* $v := Cu \in GBV_\star(A;\mathbb{R}^k)$ *and*

$$\nabla v = C\nabla u \ \ \mathcal{L}^d\text{-a.e. in } A, \tag{3.3}$$

$$D^c v(B) = CD^c v(B) \text{ for every } B \in \mathcal{B}(A), \tag{3.4}$$

$$[v] = C[u] \quad \mathcal{H}^{d-1}\text{-a.e. in } A. \tag{3.5}$$

Proof. The fact that $v \in GBV(A;\mathbb{R}^k)$ is a consequence of Proposition 3.3, while (3.3), (3.4), and (3.5) are a consequence of (2.5), (2.6), and (2.7). □

The following result, which generalises [18, Proposition 3.4], characterises $GBV_\star(A;\mathbb{R}^k)$ as a subspace of $GBV(A)^k$.

Proposition 3.7. *Let $u \in GBV(A)^k$. Then u belongs to $GBV_\star(A;\mathbb{R}^k)$ if and only if u satisfies*

$$\nabla u \in L^1(A;\mathbb{R}^{k\times d}), \tag{3.6}$$

$$\sum_{i=1}^k |D^c u_i| \in \mathcal{M}_b(A), \tag{3.7}$$

$$\int_{J_u\setminus J_u^1} |[u]|\, d\mathcal{H}^{d-1} < +\infty \text{ and } \mathcal{H}^{d-1}(J_u^1) < +\infty, \tag{3.8}$$

where $J_u^1 := \{x \in J_u : |[u](x)| \geq 1\}$.

Proof. The only if part is a consequence of Proposition 2.4 and of the equality

$$\int_{J_u} |[u]| \wedge 1\, d\mathcal{H}^{d-1} = \int_{J_u\setminus J_u^1} |[u]|\, d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(J_u^1). \tag{3.9}$$

To prove the if part, let $u \in GBV(A)^k$ be a function satisfying (3.6), (3.7), and (3.8). Let us fix $i \in \{1,...,k\}$. First, we show that for every $m > 0$ we have $u_i^{(m)} \in BV(A)$. Since $\|u_i^{(m)}\|_{L^\infty(A)} \leq m$ and u satisfies (3.6) and (3.7), we only have to show that

$$\int_{J_{u_i^{(m)}}} |[u_i]|\, d\mathcal{H}^{d-1} < +\infty.$$

Taking advantage of (3.8), we see that

$$\begin{aligned}\int_{J_{u_i^{(m)}}} |[u_i^{(m)}]|\, d\mathcal{H}^{d-1} &= \int_{J_{u_i^{(m)}}\setminus J^1_{u_i^{(m)}}} |[u_i^{(m)}]|\, d\mathcal{H}^{d-1} + \int_{J^1_{u_i^{(m)}}} |[u_i^{(m)}]|\, d\mathcal{H}^{d-1} \\ &\leq \int_{J_u\setminus J_u^1} |[u]|\, d\mathcal{H}^{d-1} + (2m+1)\mathcal{H}^{d-1}(J_u^1) < +\infty.\end{aligned}$$

Hence, $u \in BV(A)$. Finally, by (3.9) we see that for every $m > 0$

$$\int_{J_{u_i^{(m)}}} |[u_i^{(m)}]| \wedge 1\, d\mathcal{H}^{d-1} \leq \int_{J_u\setminus J_u^1} |[u]|\, d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(J_u^1).$$

Combining this last inequality with (3.6), (3.7) and (3.8), we deduce that $u_i \in GBV_\star(A)$, so that from the arbitrariness of i, we deduce $u \in GBV_\star(A;\mathbb{R}^k)$, concluding the proof. □

Remark 3.8. It follows immediately from Definition 3.1, Lemma 3.5, and Proposition 3.7 that $u \in GBV_\star(A;\mathbb{R}^k)$ if and only if there exists a constant $M > 0$ such that for every $m > 0$ the function $u^{(m)}$ belongs to $BV(A;\mathbb{R}^k)$ and

$$\sup_{m>0} \int_A |\nabla u^{(m)}|\, dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1\, d\mathcal{H}^{d-1} \leq M.$$

In [2, Lemma 2.10], Alicandro and Focardi are able to define a measure which they call $D^c u$ for every $u \in GBV(A)^k$ such that $|D^c u|(A) < +\infty$. To define such a measure, they deal with *smooth* truncations. In analogy with their approach, we characterise $GBV_\star(A;\mathbb{R}^k)$ by means of composition with smooth functions. To this aim, we introduce the following functions. Given a positive constant $\sigma > 2$, we fix a smooth radial function $\psi \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ satisfying

$$\begin{cases} \psi(y) = y & \text{if } |y| \le 1, \\ \psi(y) = 0 & \text{if } |y| \ge \sigma, \\ |\psi(y)| \le \sigma, \\ \mathrm{Lip}(\psi) = 1. \end{cases}$$

Given $R > 0$, we set

$$\psi_R(y) := R\psi\left(\frac{y}{R}\right) \quad \text{for every } y \in \mathbb{R}^k. \tag{3.10}$$

Observe that the function ψ_R satisfies

$$\begin{cases} \psi_R(y) = y & \text{for } y \in \{|y| \le R\}, & (3.11a) \\ \psi_R(y) = 0 & \text{for } y \in \{|y| \ge \sigma R\}, & (3.11b) \\ |\psi_R(y)| \le \sigma R & & (3.11c) \\ \mathrm{Lip}(\psi_R) \le 1. & & (3.11d) \end{cases}$$

The following proposition characterises $GBV_\star(A;\mathbb{R}^k)$ in terms of composition with smooth functions.

Proposition 3.9. *For every $u \in GBV_\star(A;\mathbb{R}^k)$ there is a constant $C_u > 0$ such that for every Lipschitz function $\phi\colon \mathbb{R}^k \to \mathbb{R}^k$ with compact support the function $v := \phi \circ u$ is in $BV(A;\mathbb{R}^k)$ and satisfies the following inequality*

$$\int_A |\nabla v|\,dx + |D^c v|(A) + \int_{J_v} |[v]| \wedge 1\,d\mathcal{H}^{d-1} \le C_u(\mathrm{Lip}(\phi) \vee 1). \tag{3.12}$$

Conversely, if $u \in L^0(A;\mathbb{R}^k)$ and there is a constant $C_u > 0$ such that for every integer $R > 0$ the function $\psi_R \circ u$ belongs to $BV(A;\mathbb{R}^k)$ and inequality (3.12) holds with $v = \psi_R \circ u$ and $\phi = \psi_R$, then $u \in GBV_\star(A;\mathbb{R}^k)$.

Proof. We begin by proving that if $u \in GBV_\star(A;\mathbb{R}^k)$ there is a constant $C_u > 0$ such that (3.12) is satisfied. Thanks to the elementary inequalities

$$|\xi| \le \sum_{j=1}^k |\xi_j| \quad \text{for every } i \in \{1,...,k\} \text{ and } \xi \in \mathbb{R}^{k\times d},$$

$$|[\zeta]| \wedge 1 \le \sum_{j=1}^k (|[\zeta_j]| \wedge 1) \quad \text{for every } i \in \{1,...,k\} \text{ and } \zeta \in \mathbb{R}^k,$$

proving (3.12) is equivalent to proving that (3.12) holds with v replaced by v_i.

We prove first the result for $\phi \in C_c^1(\mathbb{R}^k;\mathbb{R}^k)$. We set $v = \phi \circ u$ and $K := \mathrm{supp}(\phi)$. Note that if $m > \max_{y\in K}|y|$, then $v = \phi(u^{(m)})$ $\mathcal{L}^d$-a.e. in A. Since $u \in GBV_\star(A;\mathbb{R}^k)$, then $u^{(m)} \in BV(A;\mathbb{R}^k)$. By the chain rule (2.3) we then infer that $v_i \in BV(A)$ for every $i \in \{1,...,k\}$.

We claim that up to an $\mathcal{H}^{d-1}$-negligible set $J_{v_i} \subset J_{u_i^{(m)}}$. Indeed, by [4, Proposition 1.1 (iii)] every point $x \in A$ of approximate continuity for u_i is also a point of approximate continuity for v_i, so that we have the inclusion

$S_{v_i} \subset S_{u_i^{(m)}}$ and by (m) of Section 2 we have $\mathcal{H}^{d-1}(S_{u_i^{(m)}} \setminus J_{u_i^{(m)}}) = \mathcal{H}^{d-1}(S_{v_i} \setminus J_{v_i}) = 0$, whence the claim.

Using the Chain Rule (2.3), together with Proposition 2.4 and inequality

$$|[\phi_i(\zeta)]| \wedge 1 \leq (\mathrm{Lip}(\phi) \vee 1) \sum_{i=1}^{k} (|[\zeta_i]| \wedge 1),$$

we see that

$$\begin{aligned}
&\int_A |\nabla v_i|\, dx + |D^c v_i|(A) + \int_{J_{v_i}} |[v_i]| \wedge 1 \, d\mathcal{H}^{d-1} \\
&= \int_A |\nabla \phi_i(u^{(m)}) \nabla u^{(m)}|\, dx + \int_A |\nabla \phi_i(\widetilde{u^{(m)}})|\, d|D^c u^{(m)}| + \int_{J_{\phi_i(u^{(m)})}} |[\phi_i(u^{(m)})]|\, d\mathcal{H}^{d-1} \\
&\leq (\mathrm{Lip}(\phi) \vee 1) \sum_{i=1}^{k} \Big(\int_A |\nabla u_i^{(m)}|\, dx + |D^c u_i^{(m)}|(A) + \int_{J_{u_i^{(m)}}} |[u_i^{(m)}]| \wedge 1 \, d\mathcal{H}^{d-1} \Big).
\end{aligned}$$

Since $u \in GBV_\star(A;\mathbb{R}^k)$ there exists a constant $M > 0$ such that (2.4) holds for every $i \in \{1,...,k\}$ and from the previous inequality it follows that

$$\int_A |\nabla v_i|\, dx + |D^c v_i|(A) + \int_{J_{v_i}} |[v_i]| \wedge 1 \, d\mathcal{H}^{d-1} \leq kM(\mathrm{Lip}(\phi) \vee 1),$$

so that (3.12) is proved for $\phi \in C_c^1(\mathbb{R}^k;\mathbb{R}^k)$, with $C_u = k^2 M$.

If ϕ is Lipschitz with compact support, then there exists a sequence $(\phi_n)_n \subset C_c^1(\mathbb{R}^k;\mathbb{R}^k)$ such that $\phi_n \to \phi$ uniformly on $\mathbb{R}^k$ and such that $\mathrm{Lip}(\phi_n) \leq \mathrm{Lip}(\phi)$. In particular, $\phi_n \circ u \to \phi \circ u$ in $L^1(A;\mathbb{R}^k)$. We claim that

$$\sup_{n\in\mathbb{N}} |D(\phi_n \circ u)|(A) < +\infty. \tag{3.13}$$

Indeed, from (3.12) applied to $v = \phi_n \circ u$, we deduce that to prove the claim it is enough to check that

$$\sup_{n\in\mathbb{N}} \int_{J_{\phi_n \circ u}} |[\phi_n \circ u]|\, d\mathcal{H}^{d-1} < +\infty.$$

Using again (3.12) with v replaced by $\phi_n \circ u$, and recalling that $\mathrm{Lip}(\phi_n) \leq \mathrm{Lip}(\phi)$ and $\phi_n \to \phi$ uniformly, for n large enough we obtain that

$$\begin{aligned}
\int_{J_{\phi_n \circ u}} |[\phi_n \circ u]|\, d\mathcal{H}^{d-1} &\leq \int_{J_{\phi_n \circ u} \setminus J^1_{\phi_n \circ u}} |[\phi_n \circ u]|\, d\mathcal{H}^{d-1} + 2 \max_{y\in\mathbb{R}^k} |\phi_n(y)| \mathcal{H}^{d-1}(J^1_{\phi_n \circ u}) \\
&\leq (1 + 4 \max_{y\in\mathbb{R}^k} |\phi(y)|) \int_{J_{\phi_n \circ u}} |[\phi_n \circ u]| \wedge 1 \, d\mathcal{H}^{d-1} \\
&\leq (1 + 4 \max_{y\in\mathbb{R}^k} |\phi(y)|) C_u (\mathrm{Lip}(\phi) \vee 1),
\end{aligned} \tag{3.14}$$

where $J^1_{\phi_n \circ u}$ is defined as in the statement of Proposition 3.7. This proves the claim.

By the lower semicontinuity of the total variation with respect to the L^1-convergence it follows that $\phi\circ u \in BV(A;\mathbb{R}^k)$, while from [9, Theorem 2.1] and $\mathrm{Lip}(\phi_n)\le \mathrm{Lip}(\phi)$ it follows that

$$\int_A |\nabla(\phi\circ u)_i|\,dx + |D^c(\phi\circ u)_i|(A) + \int_{J_{(\phi\circ u)_i}} |[(\phi\circ u)_i]|\wedge 1\,d\mathcal{H}^{d-1} \le C_u(\mathrm{Lip}(\phi)\vee 1),$$

so that (3.12) is proved.

We now show that also the converse holds. To this aim, for every $m\in\mathbb{N}$ let $(\phi_n^m)_n$ be a sequence of C^1-functions from $\mathbb{R}^k$ to $\mathbb{R}^k$ converging to $\xi\mapsto\xi^{(m)}$ uniformly and such that $\mathrm{Lip}(\phi_n^m)\le 1$ and $\|\phi_n^m\|_{L^\infty}\le 2m$ for every $n\in\mathbb{N}$. For $R>0$ and $m,n\in\mathbb{N}$ we set $v_{R,n}^m := \phi_n^m\circ(\psi_R\circ(u))$. It follows immediately from (3.11a) that for every $m,n>0$ we have

$$\lim_{R\to+\infty} v_{R,n}^m(x) = (\phi_n^m\circ u)(x) \qquad \text{for } \mathcal{L}^d\text{-a.e. } x\in A. \tag{3.15}$$

Since $\|v_{R,n}^m\|_{L^\infty(A;\mathbb{R}^k)}\le 2m$ for every $n\in\mathbb{N}$, it follows from (3.15) that

$$v_{R,n}^m \text{ converges to } (\phi_n^m\circ u) \text{ in } L^1(A;\mathbb{R}^k) \text{ as } R\to+\infty. \tag{3.16}$$

By definition of ϕ_n^m we also have

$$\phi_n^m\circ u \text{ converges to } u^{(m)} \text{ in } L^1(A;\mathbb{R}^k) \text{ as } n\to+\infty. \tag{3.17}$$

By the Chain Rule (2.3), we have that

$$\begin{gathered}
\nabla v_{R,n}^m = \nabla\phi_n^m(\psi_R\circ u)\nabla(\psi_R\circ u)\ \mathcal{L}^d\text{-a.e. in } A,\\
D^c v_{R,n}^m = \nabla\phi_n^m(\psi_R\circ\widetilde{u})D^c(\psi_R\circ u)\ \text{ as Borel measures in } A,\\
[v_{R,n}^m] = [\phi_n^m(\psi_R(u^+)) - \phi_{R,n}^m(\psi_R(u^+))]\ \mathcal{H}^{d-1}\text{-a.e in } J_{v_{R,n}^m},
\end{gathered}$$

so that, using $\mathrm{Lip}(\phi_n^m)\le 1$, we have

$$\begin{gathered}
|\nabla v_{R,n}^m| \le |\nabla(\psi_R\circ u)|\ \mathcal{L}^d\text{-a.e. in } A,\\
|D^c v_{R,n}^m|(B) \le |D^c(\psi_R\circ u)|(B)\ \text{ for every } B\in\mathcal{B}(A),\\
|[v_{R,n}^m]| \le |[\psi_R\circ u]|\ \ \mathcal{H}^{d-1}\text{-a.e in } J_{v_{R,n}^m}.
\end{gathered}$$

From these inequalities, together with (3.12), for every $R>0$ and $m,n\in\mathbb{N}$ we obtain that

$$\int_A |\nabla v_{R,n}^m|\,dx + |D^c v_{R,n}^m|(A) + \int_{J_{v_{R,n}^m}} |[v_{R,n}^m]|\wedge 1\,d\mathcal{H}^{d-1} \le C_u, \tag{3.18}$$

for a constant $C_u>0$, independent of $R>0$, n, and of m.

We claim that for every $m>0$

$$\sup_{n\in\mathbb{N}}\sup_{R>0} |Dv_{R,n}^m|(A) < +\infty. \tag{3.19}$$

To show this, it is sufficient to prove that

$$\sup_{n\in\mathbb{N}}\sup_{R>0}\int_{J_{v_{R,n}^m}} |[v_{R,n}^m]|\,d\mathcal{H}^{d-1} < (4m+1)C_u.$$

By (3.18) and $\|v^m_{R,n}\|_{L^\infty(A;\mathbb{R}^k)} \le 2m$ we have

$$\int_{J_{v^m_{R,n}}} |[v^m_{R,n}]|\, d\mathcal{H}^{d-1} \le \int_{J_{v^m_{R,n}} \setminus J^1_{v^m_{R,n}}} |[v^m_{R,n}]|\, d\mathcal{H}^{d-1} + 4m\mathcal{H}^{d-1}(J^1_{v^m_{R,n}})$$
$$= (4m+1) \int_{J_{v^m_{R,n}}} |[v^m_{R,n}]| \wedge 1\, d\mathcal{H}^{d-1} \le (4m+1)C_u,$$

where the last inequality follows from (3.18). This proves (3.19) and shows that $v^m_{R,n} \in BV(A;\mathbb{R}^k)$ for every $R > 0$ and $n \in \mathbb{N}$.

Thanks to (3.16), (3.17), and (3.19), the lower semicontinuity of the Total Variation with respect to the L^1-convergence implies that $u^{(m)} \in BV(A;\mathbb{R}^k)$ for every $m > 0$.

Finally, in light of (3.16), (3.17), (3.19), and $\mathrm{Lip}(\phi^m_n) \le 1$, we may apply [9, Theorem 2.1] twice to conclude that for every $i \in \{1, ..., k\}$ we have

$$\int_A |\nabla u_i^{(m)}|\, dx + |D^c u_i^{(m)}|(A) \int_{J_{u_i^{(m)}}} |[u_i^{(m)}]| \wedge 1\, d\mathcal{H}^{d-1}$$
$$\le \liminf_{n\to+\infty} \liminf_{R\to+\infty} \Big(\int_A |\nabla (v^m_R)_i|\, dx + |D^c (v^m_R)_i|(A)$$
$$+ \int_{J_{(v^m_R)_i}} |[(v^m_R)_i]| \wedge 1\, d\mathcal{H}^{d-1} \Big) \le C_u,$$

which implies that $u \in GBV_\star(A;\mathbb{R}^k)$, concluding the proof. □

Remark 3.10. To guarantee that $u \in GBV_\star(\Omega;\mathbb{R}^k)$, it is crucial that the bound on the right-hand side of (3.12) does not depend on the support of ϕ, as it can be seen considering a function suggested by [27, Remark 4.9]. Indeed, Let $A = (-1, 1)$ and $u = \mathrm{sign}(\sin(\frac{\pi}{x}))/x$. It is immediate to check that $J_u = \{1/n\}_{n\in\mathbb{Z}}$ and $|[u](1/n)| = 2n$ for every $n \in \mathbb{Z}$. For every $R > 0$, consider a function $\phi \in C^1_c(\mathbb{R};\mathbb{R})$, with $\mathrm{supp}(\phi) = [-R; R]$ and $\phi(y) = y$ for $|y| \le R/2$. We set $v := \phi \circ u$ and note that $J_v = \{1/n \colon n \in \mathbb{Z} \text{ and } |n| \le \lfloor R \rfloor\}$, where $\lfloor \cdot \rfloor$ denotes the floor function. Then

$$\int_{J_v} |[v]| \wedge 1\, d\mathcal{H}^0 \ge \mathcal{H}^0(\{1/n \colon n \in \mathbb{Z} \text{ and } |n| \le \lfloor R/2 \rfloor\}),$$

so that we immediately get the lower bound

$$\int_A |\nabla v|\, dx + \int_{J_v} |[v]| \wedge 1\, d\mathcal{H}^0 \ge \mathcal{H}^0(\{1/n \colon n \in \mathbb{Z} \text{ and } |n| \le \lfloor R/2 \rfloor\}),$$

which blows up as $R \to +\infty$, so that $u \notin GBV_\star(A)$.

Remark 3.11. Let $n \in \mathbb{N}$. Following the lines of the proof of Proposition 3.9, it can be shown that $u \in GBV_\star(A;\mathbb{R}^k)$ if and only if there exists a constant $C_u > 0$ such that for every $\phi \in C^1_c(\mathbb{R}^k, \mathbb{R}^n)$ the function $\phi \circ u \in BV(A;\mathbb{R}^n)$ and

$$\int_A |\nabla(\phi \circ u)|\, dx + |D^c(\phi \circ u)|(A) + \int_{J_{\phi\circ u}} |[\phi \circ u]|\, d\mathcal{H}^{d-1} \le C_u(\mathrm{Lip}(\phi) \vee 1).$$

In particular, we can use $n = 1$.

The next proposition collects some properties of the approximate gradient and of the jump part of compositions of functions in $GBV_\star(A;\mathbb{R}^k)$ with the function ψ_R defined in (3.10).

Proposition 3.12. *Let $u \in GBV_\star(A;\mathbb{R}^k)$. Then for every $R>0$ we have*

(i) $\nabla(\psi_R \circ u) = \nabla u$ *$\mathcal{L}^d$-a.e. in* $\{x \in A\colon |u(x)| \le R\}$;

(ii) $J_{\psi_R \circ u} \subset J_u$ *up to an $\mathcal{H}^{d-1}$-negligible set and* $|[\psi_R \circ u]| \le |[u]|$ *on* $J_{\psi_R\circ u} \cap J_u$. *Moreover, for $\mathcal{H}^{d-1}$-a.e x in J_u and every $R > |u^+(x)| \vee |u^-(x)|$ we have* $|[\psi_R \circ u](x)| = |[u](x)|$.

Proof. The proof of (i) is a consequence of [4, Proposition 1.2] and the fact that $\nabla\psi_R$ is the identity on $\{x \in A\colon |u(x)| \le R\}$.

The inclusion $J_{\psi_R\circ u} \subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set was already proved in the first part of the proof of Proposition 3.9, while the inequality $|[\psi_R \circ u]| \le |[u]|$ on $J_{\psi_R\circ u} \cap J_u$ is a consequence of (3.11d). The fact that for $\mathcal{H}^{d-1}$-a.e x in J_u and every $R > |u^+(x)| \vee |u^-(x)|$ we have $|[\psi_R \circ u](x)| = |[u](x)|$, follows immediately from (3.11a), and the fact that $[\psi_R \circ u] = \psi_R(u^+) - \psi_R(u^-)$. □

In the following proposition we study the relation between the Cantor part of $u \in GBV_\star(A;\mathbb{R}^k)$ and the Cantor part of smooth truncations of u. As a consequence, we derive the equivalence of our Definition 3.4 and the one given by Alicandro and Focardi in [2].

Proposition 3.13. *Let $u \in GBV_\star(A;\mathbb{R}^k)$ and $\phi \in C^1_c(\mathbb{R}^k;\mathbb{R}^k)$. Then*

(i) $D^c(\phi \circ u) = \nabla\phi(\widetilde{u})D^c u$ *as Radon measures on A;*

(ii) we have

$$D^c u(B) = \lim_{R\to+\infty} D^c(\psi_R \circ u)(B) \quad \text{for every } B \in \mathcal{B}(A), \tag{3.20}$$

$$|D^c u|(B) = \lim_{R\to+\infty} |D^c(\psi_R \circ u)|(B) \quad \text{for every } B \in \mathcal{B}(A). \tag{3.21}$$

(iii) for every $R>0$ we have

$$\frac{dD^c(\psi_R \circ u)}{d|D^c(\psi_R \circ u)|} = \frac{dD^c u}{d|D^c u|} \quad |D^c u|\text{-a.e. in } A_R, \tag{3.22}$$

where $A_R := \{x \in A\colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)| \le R\}$. As a consequence we have

$$\lim_{R\to+\infty} \frac{dD^c(\psi_R \circ u)}{d|D^c(\psi_R \circ u)|} = \frac{dD^c u}{d|D^c u|} \quad |D^c u|\text{-a.e. in } A. \tag{3.23}$$

Proof. To prove (i), we need to show that for every $B \in \mathcal{B}(A)$ we have equality

$$D^c(\phi \circ u)(B) = \nabla\phi(\widetilde{u})D^c u(B). \tag{3.24}$$

We claim that it is enough to prove (3.24) when B is contained in $A_m := \{x \in A\colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)| \le m|\}$, for some $m>0$. Indeed setting $\widetilde{A} := \{x \in A\colon \widetilde{u}(x)$ exists$\}$, by Proposition 2.4 we have that $\mathcal{H}^{d-1}(\widetilde{A} \setminus J_u) = 0$. Since J_u is σ-finite with respect to $\mathcal{H}^{d-1}$ and $\phi \circ u \in BV(A;\mathbb{R}^k)$, from these previous observations and (b) of Lemma 3.5 it follows that

$$D^c(\phi \circ u)(A \setminus \widetilde{A}) = \nabla\phi(\widetilde{u})D^c u(A \setminus \widetilde{A}) = 0.$$

Since $\widetilde{A} = \bigcup_{m>0} A_m$, the claim is proved.

Let us fix $m > \max_{y\in \mathrm{supp}(\phi)} |y|$ and $B \in A_m$. To prove (3.24), we begin by noting that we have the equality $\phi \circ u = \phi \circ u^{(m)}$ $\mathcal{L}^d$-a.e. in A, so that

$$D^c(\phi \circ u)(B) = D^c(\phi \circ u^{(m)})(B).$$

Since $u^{(m)} \in BV(A;\mathbb{R}^k)$, by the Chain Rule (2.3) we have

$$D^c(\phi \circ u^{(m)})(B) = \nabla\phi(\widetilde{u^{(m)}})D^c u^{(m)}(B) = \nabla\phi(\widetilde{u}^{(m)})D^c u^{(m)}(B), \tag{3.25}$$

where in the second equality we have used that $\widetilde{u^{(m)}} = \widetilde{u}^{(m)}$. Recalling that $B \subset A_m$, from Lemma 3.5 it follows that $D^c u(B) = D^c u^{(m)}(B)$. Hence, by (3.25) and $\phi \circ u = \phi \circ u^{(m)}$ $\mathcal{L}^d$-a.e. in A , we get that

$$D^c(\phi \circ u)(B) = D^c(\phi \circ u^{(m)})(B) = \nabla\phi(\widetilde{u}^{(m)})D^c u^{(m)}(B) = \nabla\phi(\widetilde{u})D^c u(B).$$

This concludes the proof of (i).

We now prove (ii). Let $R > 0$. By (i) we have that

$$D^c(\psi_R \circ u) = \nabla\psi_R(\widetilde{u})D^c u$$

as Radon measures on A. Since ψ_R converges pointwise to the identity as $R \to +\infty$, applying the Dominated Convergence Theorem we deduce both (3.20) and (3.21).

Finally, we prove (iii). Let us fix $R > 0$. By (i) and (3.11a), it is immediate to see that

$$\lim_{\rho\to 0^+} \frac{|D^c(\psi_R \circ u)|(B_\rho(x))}{|D^c u|(B_\rho(x))} = 1$$

for every $x \in A_R$. Using this equality and the Besicovitch Derivation Theorem, we see that

$$\frac{dD^c(\psi_R \circ u)}{d|D^c(\psi_R \circ u)|}(x) = \frac{dD^c u}{d|D^c u|}(x) \quad \text{for } |D^c u|\text{-a.e. } x \in A_R,$$

which proves (3.22).

To prove (3.23), we note that since $|D^c u|(A \setminus \bigcup_{R>0} A_R) = 0$, passing to the limit for $R \to +\infty$ in (3.22) we obtain (3.23), concluding the proof. □

A straightforward consequence of this last proposition is that an analogue of the Alberti Rank-One Theorem (see [1, Corollary 4.6] or [5, Theorem 3.94]) holds in $GBV_\star(A, \mathbb{R}^k)$.

Corollary 3.14. *Let $u \in GBV_\star(A;\mathbb{R}^k)$. Then $dD^c u/d|D^c u|$ has rank one $|D^c u|$-a.e. in A.*

Proof. If $u \in GBV_\star(A;\mathbb{R}^k)$, by Proposition 3.9 we have that for every $R > 0$ the function $\psi_R \circ u$ belongs to $BV(A;\mathbb{R}^k)$. The Alberti Rank-One Theorem implies that $dD^c(\psi_R \circ u)/d|D^c(\psi_R \circ u)|$ has rank-one $|D^c(\psi_R \circ u)|$-a.e. in A and thus $|D^c u|$-a.e. in A_R. Exploiting (3.23) of Proposition 3.13, the claim follows from the lower semicontinuity of the rank. □

4. Lower semicontinuity and Compactness in $GBV_\star(A;\mathbb{R}^k)$

In this section we study the lower semicontinuity and coerciveness of some integral functionals defined on $GBV_\star(A;\mathbb{R}^k)$. In what follows, $c_1, ..., c_4 \geq 0$ are positive constants such that

$$0 < c_1 \leq 1 \leq c_3.$$

Given $\xi \in \mathbb{R}^{k\times d}$, we recall that $|\xi|_{\mathrm{op}}$ denotes its operatorial norm.

We will consider functionals whose bulk density $f : A \times \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfies

(f1) f satisfies

$$x \mapsto f(x,\xi) \quad \text{is Borel measurable for every } \xi \in \mathbb{R}^{k\times d},$$
$$\xi \mapsto f(x,\xi) \quad \text{is continuous for every } x \in A,$$

(f2) $c_1|\xi|_{\mathrm{op}} - c_2 \leq f(x,\xi)$ for every $x \in A$ and every $\xi \in \mathbb{R}^{k\times d}$,
(f3) $f(x,\xi) \leq c_3|\xi|_{\mathrm{op}} + c_4$ for every $x \in A$ and every $\xi \in \mathbb{R}^{k\times d}$,

and whose surface integrand $g : A \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfies

(g1) g is a Borel measurable function,
(g2) $c_1(|\zeta| \wedge 1) \leq g(x,\zeta,\nu)$ for every $x \in A$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$,
(g3) $g(x,\zeta,\nu) \leq c_3(|\zeta| \wedge 1)$ for every $x \in A$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$,
(g4) $g(x,-\zeta,-\nu) = g(x,\zeta,\nu)$ for every $x \in A$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$.

We recall the definition of recession function.

Definition 4.1. For every $f\colon A \times \mathbb{R}^{k\times d} \to [0,+\infty)$, the recession function $f^\infty\colon A \times \mathbb{R}^{k\times d} \to [0,+\infty]$ is defined as

$$f^\infty(x,\xi) := \limsup_{t\to+\infty} \frac{f(x,t\xi)}{t}$$

for every $x \in A$ and for every $\xi \in \mathbb{R}^{k\times d}$.

Remark 4.2. The function $\xi \to f^\infty(x,\xi)$ is positively 1-homogeneous. If f satisfies (f1) then f^∞ is Borel measurable and if, in addition, it satisfies (f2) and (f3), then

$$c_1|\xi|_{\mathrm{op}} \leq f^\infty(x,\xi) \leq c_3|\xi|_{\mathrm{op}} \quad \text{for every } x \in A \text{ and } \xi \in \mathbb{R}^{k\times d}.$$

Definition 4.3. Given $f\colon A \times \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying (f1)-(f3) and $g\colon A \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfying (g1)-(g4) we define

$$E^{f,g}(u) := \int_A f(x,\nabla u)\,dx + \int_A f^\infty\Big(x, \frac{dD^c u}{d|D^c u|}\Big)\,d|D^c u| + \int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1} \tag{4.1}$$

for every $u \in GBV_\star(A;\mathbb{R}^k)$. The definition of $E^{f,g}$ can then be extended to $L^0(A;\mathbb{R}^k)$ by setting $E^{f,g}(u) = +\infty$ for every $u \in L^0(A;\mathbb{R}^k) \setminus GBV_\star(A;\mathbb{R}^k)$.

Let $f(x,\xi) = |\xi|_{\mathrm{op}}$ and $g(x,\zeta,\nu) = |\zeta| \wedge 1$. We denote by V the functional $E^{f,g}$ obtained with these choices of f and g. Note that in this case $f^\infty(x,\xi) = |\xi|_{\mathrm{op}}$, so that

$$V(u) = \int_A |\nabla u|_{\mathrm{op}}\, dx + |D^c u|_{\mathrm{op}}(A) + \int_{J_u} |[u]| \wedge 1\, d\mathcal{H}^{d-1}$$

for every $u \in GBV(A;\mathbb{R}^k)$.

In [9, Theorem 2.1], the authors prove through a slicing argument a lower semicontinuity result for functionals on the space $BV(A)$. By carefully revisiting their proof, we are able to prove the lower semicontinuity with respect of the convergence in measure of a subclass of functionals of type $E^{f,g}$, which includes the functional V. We recall that a sequence $(u_n)_n \subset BV(A;\mathbb{R}^k)$ is said to converge weakly* in BV to $u \in BV(A;\mathbb{R}^k)$ if $u_n \to u$ in $L^1(A;\mathbb{R}^k)$ and $Du_n \overset{*}{\rightharpoonup} Du$ weakly* in the sense of $\mathbb{R}^{k\times d}$-valued bounded Radon measures.

Lemma 4.4. *Let $\hat{f}\colon [0,+\infty) \to [0,+\infty)$ be a convex, non-decreasing, lower semicontinuous function such that*

$$\hat{f}^\infty(1) = \lim_{t\to+\infty} \frac{\hat{f}(t)}{t} = C.$$

Let $\hat{g}\colon [0,+\infty) \to [0,+\infty)$ be a lower semicontinuous function satisfying the subadditivity condition

$$\hat{g}(a+b) \le \hat{g}(a) + \hat{g}(b) \quad \text{for every } a,b \in \mathbb{R},$$

and suppose in addition that

$$\hat{g}^0(1) = \lim_{t\to 0^+} \frac{\hat{g}(t)}{t} = C.$$

Set $f := \hat{f}(|\cdot|_{\mathrm{op}})$ and $g := \hat{g}(|\cdot|)$. Then the functional defined for $u \in BV(A;\mathbb{R}^k)$ as

$$E^{f,g}(u) := \int_A \hat{f}(|\nabla u|_{\mathrm{op}})\, dx + C|D^c u|_{\mathrm{op}}(A) + \int_{J_u} \hat{g}(|[u]|)\, d\mathcal{H}^{d-1}$$

is weakly lower semicontinuous in $BV(A;\mathbb{R}^k)$.*

Proof. The proof follows closely that of [9, Theorem 2.1]. For the reader's convenience, we give a sketch of the proof.

For every $\nu \in \mathbb{S}^{d-1}$, we introduce the functional $E^{f,g}_\nu\colon BV(A;\mathbb{R}^k) \times \mathcal{B}(A) \to [0,+\infty)$ defined for every $B \in \mathcal{B}(A)$ and $u \in BV(A;\mathbb{R}^k)$ by

$$E^{f,g}_\nu(u,B) = \int_B \hat{f}(|(\nabla u)\nu|)\, dx + C|(D^c u)\nu|(B) + \int_{J_u\cap B} \hat{g}(|[u]|)|\nu_u \cdot \nu| d\mathcal{H}^{d-1}.$$

We want to prove that for every $U \in \mathcal{A}(A)$, the functional $E^{f,g}_\nu(\cdot,U)$ is lower semicontinuous with respect to the weak* convergence in $BV(A;\mathbb{R}^k)$.

Since for every $u \in BV(A;\mathbb{R}^k)$ the set function $E^{f,g}_\nu(u,\cdot)$ is a non-negative Borel measure such that $E^{f,g}_\nu(u,B) = 0$ whenever $\mathcal{L}^d(B) + |D^c u|(B) + |D^j u|(B) = 0$, it is enough to check that $E^{f,g}_\nu(\cdot,Q)$ is lower semicontinuous when $Q = \Sigma \times I$ for an open $\Sigma \subset \Pi_\nu = \{x\cdot\nu = 0\}$ and $I \in \mathcal{A}(\mathbb{R})$.

Thanks to Proposition 2.1 and arguing exactly as in [9, Lemma 2.3], we deduce that, for every open $\Sigma \subset \Pi_\nu$ and $I \in \mathcal{A}(\mathbb{R})$, setting $Q = \Sigma \times I$, we have

$$E_\nu^{f,g}(u,Q) = \int_\Sigma \Phi(u_y^\nu, I)\, d\mathcal{H}^{d-1}(y) \quad \text{for every } u \in BV(A;\mathbb{R}^k), \tag{4.2}$$

where $\Phi\colon BV(\mathbb{R};\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}) \to [0,+\infty]$ is the functional defined for every $I \in \mathcal{B}(\mathbb{R})$ and $v \in BV(\mathbb{R};\mathbb{R}^k)$ as

$$\Phi(v,I) = \int_I \hat{f}(|\nabla v|_{\mathrm{op}})\, dx + C|D^c v|(I) + \int_{J_v \cap I} \hat{g}(|[v]|)\, d\mathcal{H}^0$$

and where for every $y \in \Pi_\nu$ the function u_y^ν is defined by (2.2).

Let $u \in BV(A;\mathbb{R}^k)$ and $(u_n)_n \subset BV(A;\mathbb{R}^k)$ be such that $u_n \rightharpoonup u$ weakly* in $BV(A;\mathbb{R}^k)$. Thanks to Proposition 2.1(c), we have that for $\mathcal{H}^{d-1}$-a.e. $y \in \Sigma$

$$(u_n)_y^\nu \rightharpoonup u_y^\nu \text{ weakly }^* \text{ in } BV(I;\mathbb{R}^k).$$

Thanks to [10, Theorem 3.3], we know that Φ is weakly* lower semicontinuous on $BV(\mathbb{R};\mathbb{R}^k)$, so that Fatou's Lemma and (4.2) imply that

$$E_\nu^{f,g}(u,Q) \le \liminf_{n\to\infty} E_\nu^{f,g}(u_n,Q),$$

Fix now $(\nu_i)_{i\in\mathbb{N}}$ a dense family of unit vectors in $\mathbb{S}^{d-1}$. To prove that $E^{f,g}$ is weakly* lower semicontinuous in $BV(A;\mathbb{R}^k)$, we show that it can be written as the supremum of finite sums of functionals of the form $E_{\nu_i}^{f,g}$, which we have just shown to be lower semicontinuous. More precisely, repeating the proof of [9, Lemma 2.4] replacing their measure μ by

$$\mu = \mathcal{L}^d + |D^c u|_{\mathrm{op}} + \mathcal{H}^{d-1} \llcorner J_u,$$

their function ψ by

$$\psi(x) = \psi(x) = \begin{cases} \hat{f}(|\nabla u(x)|_{\mathrm{op}}) & \mathcal{L}^d\text{-}a.e \text{ in } A, \\ C & |D^c u|\text{-}a.e. \text{ in } A, \\ \hat{g}(|[u]|) & \mathcal{H}^{d-1}\text{- a.e. in } J_u, \end{cases}$$

and their function ψ_h by

$$\psi_i(x) = \begin{cases} \hat{f}(|(\nabla u)\nu_i|) & \mathcal{L}^d\text{-}a.e \text{ in } A, \\ C|(\frac{dD^c u}{d|D^c u|})\nu_i| & |D^c u|\text{-}a.e. \text{ in } A, \\ \hat{g}(|[u]|)|\nu_u \cdot \nu_i| & \mathcal{H}^{d-1}\text{-}a.e. \text{ in } J_u, \end{cases}$$

we get

$$E^{f,g}(u) = \sup\Big\{ \sum_{i=1}^n E_{\nu_i}^{f,g}(u,A_i) : \quad n \in \mathbb{N} \text{ and } (A_i)_{i=1}^n \text{ pairwise disjoint open subsets of } A \Big\},$$

concluding the proof. □

Theorem 4.5. *Let $\hat{f}, \hat{g}$ be as in the statement of Lemma 4.4. Assume that $f := \hat{f}(|\cdot|_{\mathrm{op}})$ satisfies (f1)-(f3) and that $g := \hat{g}(|\cdot|)$ satisfies (g1)-(g3) and that $\hat{g}$ is non-decreasing. Then the functional $E^{f,g}$ of Definition 4.3 is lower semicontinuous with respect to the topology of $L^0(A;\mathbb{R}^k)$.*

Proof. We consider $u \in L^0(A;\mathbb{R}^k)$ and a sequence $(u_n)_n \subset L^0(A;\mathbb{R}^k)$ converging to u in $L^0(A;\mathbb{R}^k)$. If $\liminf_n E^{f,g}(u_n, A) = +\infty$ there is nothing to prove. Thus, we suppose that $\sup_{n\in\mathbb{N}} E^{f,g}(u_n) \le M$ for some $M > 0$. Since f and g satisfy (f2) and (g2), we have that $\sup_{n\in\mathbb{N}} V(u_n) \le M'$, for M' a constant depending only on M. With arguments similar to those used in the proof of Proposition 3.9, one can show that the previous condition implies that $u \in GBV_\star(A;\mathbb{R}^k)$ as well.

For every $R > 0$ and every $n \in \mathbb{N}$ we set $v_n^R := \psi_R \circ u_n$, where ψ_R is the function defined by (3.10). By Proposition 3.9 we have that $v_n^R \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$. Arguing as in the proof of Proposition 3.9 we see that there exists $K = K(R, M') > 0$ such that

$$|Dv_n^R|(A) \le K. \tag{4.3}$$

The sequence v_n^R converges to $\psi_R \circ u$ in $L^1(A;\mathbb{R}^k)$ as $n \to +\infty$, so that from (4.3) it follows that v_n^R converges to $\psi_R \circ u$ weakly* in $BV(A;\mathbb{R}^k)$ as well. We can then use Lemma 4.4 to obtain

$$E^{f,g}(\psi_R \circ u) \le \liminf_{n\to+\infty} E^{f,g}(v_n^R) \tag{4.4}$$

for every $R > 0$. By the chain rule (2.3), we may estimate

$$\begin{aligned} E^{f,g}(v_n^R) &= \int_A \hat{f}(|\nabla\psi_R(u_n)\nabla u_n|_{\mathrm{op}})\,dx + C|\nabla\psi_R(\widetilde{u}_n)D^c u_n|_{\mathrm{op}}(A) \\ &\quad + \int_{J_u} \hat{g}(|[\psi_R(u_n)]|)\,d\mathcal{H}^{d-1} \\ &\le \int_A \hat{f}(|\nabla u_n|_{\mathrm{op}})\,dx + C|D^c u_n|_{\mathrm{op}}(A) + \int_{J_u} \hat{g}(|[u_n]|)\,d\mathcal{H}^{d-1}, \end{aligned}$$

where in the last inequality we have used that $\hat{f}$ and $\hat{g}$ are non-decreasing, (3.11d), and Proposition 3.12. From this last estimate and (4.4), we get

$$E^{f,g}(\psi_R \circ u) \le \liminf_{n\to+\infty} E^{f,g}(u_n). \tag{4.5}$$

Recalling Proposition 2.4(b), (d) and taking advantage of the lower semicontinuity of $\hat{f}$ and of $\hat{g}$, we get

$$\hat{f}(|\nabla u|_{\mathrm{op}}) \le \liminf_{R\to+\infty} \hat{f}(|\nabla(\psi_R \circ u)|_{\mathrm{op}}) \qquad \mathcal{L}^d\text{-a.e in } A, \tag{4.6}$$

$$\hat{g}(|[u]|) \le \liminf_{R\to+\infty} \hat{g}(|[\psi_R \circ u]|) \qquad \mathcal{H}^{d-1}\text{- a.e in } J_u. \tag{4.7}$$

Finally, (4.6), (4.7), and (3.21), together with Fatou's Lemma and (4.5), imply

$$E^{f,g}(u) \le \liminf_{n\to+\infty} E^{f,g}(u_n),$$

concluding the proof. □

Corollary 4.6. *The functional V is lower semicontinuous with respect to the topology of $L^0(A;\mathbb{R}^k)$.*

In [18, Theorem 3.11] a useful compactness theorem for $GBV_\star(A)$ is proved. We now show that the result readily adapts to $GBV_\star(A;\mathbb{R}^k)$.

Theorem 4.7. *Let $(u_n)_n$ be a sequence in $GBV_\star(A;\mathbb{R}^k)$. Suppose that there exist a constant $M>0$ and an increasing continuous function $h\colon [0,+\infty)\to[0,+\infty)$ with $h(t)\to+\infty$ as $t\to+\infty$, such that*

$$\sup_n V(u_n)\le M,$$
$$\sup_n \int_A h(|u_n|)\,dx<+\infty. \tag{4.8}$$

Then there exists a subsequence, not relabelled, and $u\in GBV_\star(A;\mathbb{R}^k)$ such that $u_n\to u$ $\mathcal{L}^d$-a.e. in A.

Proof. It is enough to apply Theorem [18, Theorem 3.11] to each component of u. □

We now present a result which shows that given $(u_n)_n\subset GBV_\star(A;\mathbb{R}^k)$ which is only bounded in energy and which satisfies some common Dirichlet boundary condition, it is possible to produce a modification y_n of u_n, satisfying the hypotheses of Theorem 4.7. This result is a direct adaptation of [18, Theorem 5.5] and of [17, Theorem 7.13], whose proof is based on the arguments of [23].

In the following $A'\subset\subset A$ is an open set with Lipschitz boundary. We also assume A to have Lipschitz boundary. We fix an additional positive constant $c_5>0$ satisfying $c_5\ge c_3/c_1$ and make the following additional assumption on the integrand g:

(g5) for every $\zeta_1,\zeta_2\in\mathbb{R}^k$ with $c_5|\zeta_1|\le|\zeta_2|$ it holds $g(x,\zeta_1,\nu)\le g(x,\zeta_2,\nu)$ for every $x\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$.

Note that this condition, first considered in [12], is crucial in the proof of the result.

Theorem 4.8. *Let $f\colon A\times\mathbb{R}^{k\times d}\to[0,+\infty)$ and $g\colon A\times\mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ be two functions satisfying* (f1)-(f3) *and* (g1)-(g5)*, respectively, and let $E^{f,g}$ be the functional introduced in Definition 4.3. Let $w\in W^{1,1}(A;\mathbb{R}^k)$ and let $(u_n)_n\subset GBV_\star(A;\mathbb{R}^k)$ with $u_n=w$ $\mathcal{L}^d$-a.e. on $A\backslash\overline{A'}$, and $V(u_n,A)\le M$ for every $n\in\mathbb{N}$. Then for every $\varepsilon_n\to0^+$ there exists a subsequence of $(u_n)_n$, not relabelled, modifications $y_n\in GBV_\star(A;\mathbb{R}^k)$ of u_n, with $y_n=w$ on $A\backslash\overline{A'}$, and a continuous increasing function $h:[0,+\infty)\to[0,+\infty)$ satisfying $h(t)\to+\infty$ for $t\to+\infty$, such that*

$$E^{f,g}(y_n)\le E^{f,g}(u_n)+\varepsilon_n, \tag{4.9}$$
$$\sup_n\int_A h(|y_n|)\,dx<+\infty. \tag{4.10}$$

Proof. The theorem follows from the same lines of proof of [18, Theorem 5.5] outlined in [18, Section 5], performing the modifications suggested in [17, Theorem 7.13] and replacing [18, Lemma 5.1] by our Lemma 4.9 below. □

Given an $\mathcal{L}^d$-measurable set E, we recall that a countable collection $(P^j)_j$ of $\mathcal{L}^d$-measurable subsets of E is said to be a Caccioppoli partition of E if

$$\mathcal{L}^d\Big(E \setminus \bigcup_{j=1}^{\infty} P^j\Big) = 0,$$

$$\sum_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^* P^j \cap E) < +\infty.$$

Lemma 4.9. *For every $M > 0$ and $u \in GBV_\star(A;\mathbb{R}^k)$ satisfying*

$$V(u) \leq M,$$

there exists a Caccioppoli partition $(P^j)_j$ of A and a family of translations $(t^j)_{j\in\mathbb{N}} \subset \mathbb{R}^k$ such that the function

$$v = u - \sum_{j=1}^{\infty} t^j \chi_{P^j}$$

is in $BV(A;\mathbb{R}^k) \cap L^\infty(A,\mathbb{R}^k)$ and the following estimates hold

$$\sum_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^* P^j) \leq k(2 + 2M + \mathcal{H}^{d-1}(\partial A)), \tag{4.11}$$

$$\|v\|_{L^\infty(A;\mathbb{R}^k)} \leq 2M. \tag{4.12}$$

Proof. Thanks to [18, Lemma 5.1], for every $i = 1, ..., k$ we can find a Caccioppoli Partition $(P_i^j)_j$ and a family of translations $(t_i^j)_{j\in\mathbb{N}} \subset \mathbb{R}$ satisfying

$$\sum_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^* P_i^j) \leq 2 + 2M + \mathcal{H}^{d-1}(\partial A), \tag{4.13}$$

$$\|u_i - \sum_{j=1}^{\infty} t_i^j \chi_{P_i^j}\|_{L^\infty(A)} \leq 2M. \tag{4.14}$$

The family

$$\mathcal{P} := \{P \subset A\colon P = \cap_{i=1}^k P_i^{j_i} \text{ for } (j_1, ..., j_k) \in \mathbb{N}^k\}$$

is countable and is still a partition of A. We denote every element of $\mathcal{P}$ as P_ℓ for some $\ell \in \mathbb{N}$. Thanks to (4.13) and to the standard inequality

$$\mathcal{H}^{d-1}(\partial^*(E_1 \cap E_2) \cap A) \leq \mathcal{H}^{d-1}(\partial^* E_1 \cap A) + \mathcal{H}^{d-1}(\partial^* E_2 \cap A),$$

which holds for every E_1 and E_2 of finite perimter, we get that $(P^\ell)_\ell$ satisfies (4.11). In particular, $(P^\ell)_\ell$ is a Caccioppoli partition of A.

For every $\ell \in \mathbb{N}$ let $(j_1^\ell, ..., j_k^\ell)$ be such that $P_\ell = \cap_{i=1}^k P_i^{j_i^\ell}$ and let $(t_1^{j_1^\ell}, ..., t_k^{j_k^\ell})$ be as above. We set $t^\ell = (t_1^{j_1^\ell}, ..., t_k^{j_k^\ell})$ and

$$v := u - \sum_{\ell=1}^{+\infty} t^\ell \chi_{P^\ell}.$$

Recalling (4.14), we infer that v satisfies (4.12), so that Lemma 4.9 is finally proved. $\square$

Remark 4.10. Let $f\colon A \times \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying (f1)-(f3), let $g\colon A \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfying (g1)-(g5), and let $w \in W^{1,1}(A;\mathbb{R}^k)$. Consider $(u_n)_n \subset GBV_\star(A;\mathbb{R}^k)$ a minimising sequence for the problem

$$\inf\{E^{f,g}(u)\colon u \in GBV_\star(A;\mathbb{R}^k),\ u = w\ \mathcal{L}^d\text{-}a.e. \text{ on } A \setminus \overline{A'}\}, \tag{4.15}$$

where $E^{f,g}$ is as in Definition 4.3.

Thanks to (f2)-(f3), (g2)-(g3), and (4.2), it is easily seen that for some $M > 0$ we have $V(u_n) \le M$ for every $n \in \mathbb{N}$. Hence, by Theorem 4.8 there exist a minimising sequence $(y_n)_n$, with $y_n = w$ on $A \setminus \overline{A'}$ and satisfying (4.10). In particular, $(y_n)_n$ satisfies the hypotheses of Theorem 4.7, so that there exists a subsequence of $(y_n)_n$, not relabelled, converging in $L^0(A;\mathbb{R}^k)$ to a function y, with $y = w$ $\mathcal{L}^d$-a.e. in $A \setminus \overline{A'}$. If the functional $E^{f,g}$ is lower semicontinuous with respect to the convergence of $L^0(A;\mathbb{R}^k)$ (for instance if it satisfies the hypoteses of Theorem 4.5), we conclude that y minimises (4.15).

Acknowledgements

The author wishes to thank Professor Gianni Dal Maso for all the useful discussions on the topic. This paper is based on work supported by the National Research Project (PRIN 2022J4FYNJ) "Variational methods for stationary and evolution problems with singularities and interfaces", funded by the Italian Ministry of University and Research. The author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The work was written by the sole author.

Funding Open access funding provided by Scuola Internazionale Superiore di Studi Avanzati - SISSA within the CRUI-CARE Agreement.

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: 29 July 2024.
Revised: 20 March 2025.
Accepted: 23 March 2025.