Longtime dynamics for the Landau Hamiltonian with a time dependent magnetic field

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Abstract

We consider a modulated magnetic field, $B(t) = B_0 + \varepsilon f(\omega t)$, perpendicular to a fixed plane, where B_0 is constant, $\varepsilon > 0$ and f a periodic function on the torus $\mathbb{T}^n$. Our aim is to study classical and quantum dynamics for the corresponding Landau Hamiltonian. It turns out that the results depend strongly on the chosen gauge. For the Landau gauge the position observable is unbounded for "almost all" non resonant frequencies ω. On the contrary, for the symmetric gauge we obtain that, for "almost all" non resonant frequencies ω, the Landau Hamiltonian is reducible to a two dimensional harmonic oscillator and thus gives rise to bounded dynamics. The proofs use KAM algorithms for the classical dynamics. Quantum applications are given. In particular, the Floquet spectrum is absolutely continuous in the Landau gauge while it is discrete, of finite multiplicity, in symmetric gauge.

Thank you, Thomas, for sharing your enthusiasm
and your joy of playing with mathematics.

1 Introduction and main results

In this paper we study the dynamics of time dependent perturbations of the Schrödinger equation

$$\mathrm{i}\partial_t \psi = H_{A\#}(t)\psi + V(t)\psi \ , \tag{1.1}$$

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where $H_{A^\#}(t)$ is the magnetic Schrödinger operator in $L^2(\mathbb{R}^3)$:

$$H_{A^\#}(t) := \sum_{1\leq j\leq 3}\left(D_{x_j} - A_j^\#(t,x)\right)^2, \quad D_x := \mathrm{i}^{-1}\frac{\partial}{\partial x}\ , \tag{1.2}$$

with $A^\#(t,x) = (A_1^\#(t,x), A_2^\#(t,x), A_3^\#(t,x))$ a time dependent vector potential and finally $V(t,x)$ is a time dependent scalar potential. We recall that the electric field is given by $\vec{E}(t,x) = -\frac{\partial A^\#}{\partial t}(t,x) - \nabla_x V(t,x)$ and the magnetic field by $\vec{B}(t,x) = \nabla_x \wedge A^\#(t,x)$. We shall assume that the magnetic field has a fixed direction orthogonal to the plane $\{e_1, e_2\}$. Choosing $A_3^\#(t,x) \equiv 0$ then $A_1^\#(t,x)$ and $A_2^\#(t,x)$ depend only on (t, x_1, x_2) and it is enough to consider the 2D magnetic Hamiltonian, with a new simpler notation,

$$H_{A^\#}(t) = \left(D_{x_1} - A_1^\#(t,x)\right)^2 + \left(D_{x_2} - A_2^\#(t,x)\right)^2$$

as an operator in $L^2(\mathbb{R}^2)$.
An important particular case is the constant (in position) magnetic field $\mathbf{B}(t) = (0,0,-B(t))$, for which

$$B(t) = \partial_{x_1} A_2^\# - \partial_{x_2} A_1^\#\ . \tag{1.3}$$

This is usually studied using either the symmetric Gauge or the Landau Gauge, namely

- **the symmetric gauge**: $A_1^\#(t,x) = \frac{B(t)}{2}x_2, \quad A_2^\#(t,x) = -\frac{B(t)}{2}x_1$;

- **the Landau gauge:** $A_1^\#(t,x) = B(t)x_2, \quad A_2^\#(t,x) = 0$.

In this paper we consider the case when B slightly fluctuates around a fix value $B_0 > 0$: $B(t) = B_0 + \varepsilon f(\omega t)$ where $\omega \in \mathbb{R}^n$ is a frequency vector, f is a periodic function real analytic on the torus $\mathbb{T}^n$ and $\varepsilon > 0$ is a small parameter.

Mathematically the source of most of the interesting features of the Landau Hamiltonian rests in the fact that when $\varepsilon = 0$ the Hamiltonian is degenerate, in the sense that it is equivalent (unitary equivalent in the quantum case, canonically equivalent in the classical case) to the Hamiltonian of a 1-d harmonic oscillator. As a result the quantum spectrum of the system is composed just by essential spectrum and coincides with the set $\{\lambda_j = 2B_0(j + 1/2) \mid j \in \mathbb{N}\}$.

The case $\varepsilon \neq 0$ will be discussed in the two different gauges:
(i) the Landau gauge $H_L(t) = (D_{x_1} - B(t)x_2)^2 + D_{x_2}^2$;
(ii) the symmetric gauge $H_{sL}(t) = (D_{x_1} - \frac{B(t)x_2}{2})^2 + (D_{x_2} + \frac{B(t)x_1}{2})^2$.
Notice that, for $\varepsilon = 0$, $B(t) = B_0$ hence H_L and H_{sL} are gauge equivalent, but for $\epsilon \neq 0$ this equivalence is broken.

It turns out that both the main part of the Hamiltonian and the time dependent perturbation are quadratic polynomials in the position and the momentum variables, and this allows to study the problems (both classical and quantum) using the ideas of [3], namely by using classical KAM theory to conjugate the Hamiltonian to a suitable normal form whose dynamics is easy to study.
The results depend drastically of the choice of the gauge:

- In case (i), provided ω is non resonant, a condition which is fulfilled in a set of asymptotically full measure, we get that for $\varepsilon \neq 0$ the position observable is unbounded as $t \to \infty$ as well for the classical motion and the quantum motion. *It may be surprising that for a dynamical system a non-resonance condition generates an instability*[1].
  As a consequence, in the quantum side, we prove that the Floquet spectrum is absolutely continuous.
- In case (ii) we prove that for ω in a set of asymptotically full measure, the dynamics is reducible to a harmonic oscillator with two degrees of freedom, hence with bounded dynamics.
  As a consequence, in the quantum side, we prove that the Floquet spectrum is discrete with finite multiplicity.

Notice that $H_L(t)$ and $H_{sL}(t)$ are gauge equivalent modulo a quadratic scalar potential (see section 1.3). Therefore the two models are not physically equivalent: in the two cases we have the same magnetic field but not the same electric field.

1.1 Main result in the Landau gauge

We consider first the Landau gauge, namely

$$H_L(t) = (D_{x_1} - B(t)x_2)^2 + D_{x_2}^2 \ , \qquad B(t) > 0 \ , \tag{1.4}$$

with $B(t) := B_0 + \varepsilon f(\omega t)$, f is real analytic on the torus $\mathbb{T}^n$, $\hat{f}(0) = 0$ and $\omega \in [0, 2\pi)^n := \mathbb{D}$. Here and below we denote by $\hat{f}(k)$ the k-th Fourier coefficients of f,

$$\hat{f}(k) := (2\pi)^{-n} \int_{\mathbb{T}^n} f(\theta) e^{-\mathrm{i}k\cdot\theta} d\theta \ .$$

We decompose the Hamiltonian $H_L(t)$ in (1.4) as

$$H_L(t) = H_L + R_L(\omega t)$$

where

$$\begin{aligned} H_L &= (D_{x_1} - B_0 x_2)^2 + D_{x_2}^2, \\ R_L(\omega t) &= -2\varepsilon f(\omega t) x_2 (D_{x_1} - B_0 x_2) + \varepsilon^2 f(\omega t)^2 x_2^2. \end{aligned}$$

We denote by $h_L(t)$ the corresponding classical Hamiltonian:

$$h_L(t, x, p) = (p_1 - B(t)x_2)^2 + p_2^2 = h_L(x, p) + r_L(\omega t, x, p). \tag{1.5}$$

[1]But the phenomenon is similar of that encoded in Theorem 3.3 of [3], in which the non resonance condition is used to eliminate from the Hamiltonian as many terms as possible, so that one remains only with the terms actually generating the instability.

We introduce now complex coordinates in which the classical Hamiltonian h_L has the form of a degenerate 2-dimensional Harmonic oscillator. First introduce the symplectic variables

$$Q_1 = \frac{-1}{B_0}(p_1 - B_0 x_2), \quad P_1 = p_2$$

and

$$Q_2 = \frac{-1}{B_0}(p_2 - B_0 x_1), \quad P_2 = p_1.$$

In these variables we have

$$h_L = B_0^2 Q_1^2 + P_1^2.$$

Then we introduce the complex variables

$$z_1 = \frac{B_0 Q_1 + \mathrm{i} P_1}{\sqrt{2B_0}}, \quad z_2 = \frac{B_0 Q_2 + \mathrm{i} P_2}{\sqrt{2B_0}} \ ,$$

fulfilling $\mathrm{i} dz_i \wedge d\bar{z}_i = dQ_i \wedge dP_i$, $i = 1, 2$. In these variables

$$h_L = 2B_0 |z_1|^2. \tag{1.6}$$

The link with the initial coordinates $(x, p) \in \mathbb{R}^4$ is given by $(z_1, z_2) = \tau_0(x, p)$ where τ_0 is the linear symplectic transformation $\tau_0 : \mathbb{R}^4 \to \mathbb{C}^2$ such that

$$\begin{aligned} z_1 &= \frac{B_0}{\sqrt{2B_0}} \left(\frac{B_0 x_2 - p_1}{B_0} \right) + \mathrm{i} \frac{p_2}{\sqrt{2B_0}} \\ z_2 &= \frac{B_0}{\sqrt{2B_0}} \left(\frac{B_0 x_1 - p_2}{B_0} \right) + \mathrm{i} \frac{p_1}{\sqrt{2B_0}}. \end{aligned} \tag{1.7}$$

In order to state the first result we need to define a constant c_ω which is only defined for ω fulfilling a nonresonance condition. To this end we preliminary restrict the set of the allowed frequencies.

Definition 1.1. *The set $\mathbb{D}_0 \subset [0, 2\pi]^n$ is the set of the frequencies ω s.t. there exist positive γ, τ s.t.*

$$|\omega \cdot k + 2B_0| \geq \frac{\gamma}{1 + |k|^\tau} \ , \quad \forall k \in \mathbb{Z}^n \tag{1.8}$$

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau} \ , \quad \forall k \in \mathbb{Z}^n \setminus \{0\} \ . \tag{1.9}$$

Remark that such a set has full measure in $[0, 2\pi]^n$.

We now introduce

$$c_\omega := \int_{\mathbb{T}^n} g_\omega(\theta)^2 d\theta \tag{1.10}$$

where the function

$$g_\omega(\theta) := -\frac{1}{\sqrt{2B_0}} \sum_{k \neq 0} \frac{\omega \cdot k}{\omega \cdot k + 2B_0} \hat{f}(k) e^{\mathrm{i} k \cdot \theta},$$

is well defined for $\omega \in \mathbb{D}_0$ (recall that f is real analytic).

Furthermore, in the following we will say that two polynomials are $O(\epsilon)$ close from each other, if their coefficients are $O(\epsilon)$ close from each other.

Our first result is the following:

Theorem 1.2. *There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, there exist*

- *an asymptotically full measure set of frequencies $\mathcal{C}_\varepsilon \subset \mathbb{D}_0$ satisfying:*
$$\lim_{\varepsilon \to 0} meas(\mathbb{D}_0 \setminus \mathcal{C}_\varepsilon) = 0;$$
- *a linear, symplectic change of variable τ, depending on ω, t, ε, which is close to the identity, namely $\tau = \mathbb{I} + O(\varepsilon)$ uniform in the other parameters,*

such that, for $\omega \in \mathcal{C}_\varepsilon$, the time quasi-periodic Hamiltonian $h_L(t, x, p)$ in (1.5) is conjugated to the constant coefficient quadratic Hamiltonian

$$(h_L(t) \circ \tau_0^{-1}) \circ \tau = b(\varepsilon)|z_1|^2 + c(\varepsilon)(z_2 - \overline{z}_2)^2 \tag{1.11}$$

where

$$b(\varepsilon) = 2B_0 + O(\varepsilon^2), \quad c(\varepsilon) = c_\omega \varepsilon^2 + O(\varepsilon^4) \tag{1.12}$$

and c_ω is given by (1.10).

In the new coordinates (z_1, z_2) the motion is easily computed:

$$\begin{aligned} z_1(t) &= \mathrm{e}^{-\mathrm{i}b(\varepsilon)t} z_1(0) \ , \\ \Im z_2(t) &= \Im z_2(0), \ \ \Re z_2(t) = -4c(\varepsilon)\Im z_2(0)t + \Re z_2(0) \ . \end{aligned}$$

Let us come back to the original coordinates (x, p). First we remark that the Hamiltonian at r.h.s. of (1.11) takes the form

$$\frac{b(\varepsilon)}{2B_0} h_L(x, p) + \alpha c(\varepsilon) p_1^2$$

with h_L the original Landau Hamiltonian and $\alpha \neq 0$ a numerical constant. As a consequence, the corresponding dynamics is just given by the standard circular motion of a particle in a magnetic field with a slightly different frequency and with superimposed a uniform motion in the direction of x_1. This uniform motion is the new effect which gives rise to the growth of the solution. Actually this description holds in the system of coordinates introduced by the KAM procedure. In the true original coordinates this motion is slightly deformed, so that it has superimposed a small oscillation. Precisely the motion for the quadratic Hamiltonian $h_L(t)$ is a linear flow $\Phi_B(t)$ where we have

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \Phi_B(t) \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} .$$

where $\Phi_B(t)$ is a real 4×4 symplectic matrix (it is the classical Hamiltonian flow of the classical Hamiltonian $h_L(t)$). Then we have:

Corollary 1.3. *For $\omega \in \mathcal{C}_\varepsilon$ we have, with $\alpha \neq 0$,*

$$x_1(t) = x_1(0) + \alpha c(\varepsilon) p_1(0) t + \frac{1}{B(t)}(p_2(t) - p_2(0)) + a_{\varepsilon,\omega}(t) \cdot x(0) + b_{\varepsilon,\omega}(t) \cdot p(0) \quad (1.13)$$

where $|a_{\varepsilon,\omega}(t)| + |b_{\varepsilon,\omega}(t)| = O(1)$ for $0 < \varepsilon < 1$ and $t \in \mathbb{R}$.

Moreover, modulo an error term of the form $E_{\tilde{a},\tilde{b}}(t) = \tilde{a}(t) \cdot x(0) + \tilde{b}(t) \cdot p(0)$, such that uniformly for $t \in \mathbb{R}$, $\omega \in \mathcal{C}_\varepsilon$, we have $|\tilde{a}(t)| + |\tilde{b}(t)| = O(\varepsilon)$,

$$\begin{aligned} p_2(t) &= \sqrt{2B_0}\Im(z_1(t)), \quad z_1(t) = \mathrm{e}^{-2ibt} z_1(0) \quad (1.14)\\ p_1(t) &= p_1(0) \\ x_2(t) &= \frac{p_1(0)}{B(t)} + \frac{\sqrt{2}B_0^{3/2}}{B(t)^2}\Re z_1(t). \quad (1.15) \end{aligned}$$

In particular, if both c_ω and $p_1(0)$ are not zero, then the classical flow is not bounded as soon as $\varepsilon \neq 0$ is small enough.

Remark 1.4. *Clearly, in view of (1.10), $c_\omega \neq 0$ holds for ω in a set of asymptotically full measure and for f in a set of codimension 1 (e.g. in L^2). For instance, by a simple calculus one has that if $f(\theta) = \sin(\theta)$ (and thus $n = 1$) then $c_\omega \neq 0$ as soon as $\omega \neq 2B_0$.*

From the result on the classical evolution of $x_1(t)$, we get a direct application to the large time evolution of the quantum position observable $\hat{x}_1(t)$. Let us explicit our notations: we denote by $\hat{x}_j$ and $\hat{p}_j$, $j = 1, 2$, the position and momentum operator

$$\hat{x}_j\psi(x) = x_j\psi(x), \ \hat{p}_j\psi = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x_j}\psi, \ x = (x_1, x_2), \ p = (p_1, p_2), \ \psi \in \mathcal{H}^1_{\mathrm{osc}}.$$

For $r \geq 0$, $\mathcal{H}^r_{\mathrm{osc}}$ is the weighted Sobolev space associated with the harmonic oscillator $H_0 := \hat{p}^2 + \hat{x}^2$,

$$\mathcal{H}^r_{\mathrm{osc}} = \{\psi \in L^2(\mathbb{R}^2) : H_0^{r/2}\psi \in L^2(\mathbb{R}^2)\},$$

endowed with the norm $\|\psi\|_r = \|H_0^{r/2}\psi\|_{L^2(\mathbb{R}^2)}$.
Recall that we are working here with polynomials classical Hamiltonians of degree at most 2 so the correspondence classical-quantum is exact. That means

$$\begin{pmatrix} \hat{x}(t) \\ \hat{p}(t) \end{pmatrix} = \Phi_B(t) \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} ,$$

where $(\hat{x}(t), \hat{p}(t))$ is the solution of the Heisenberg equation.

Our first quantum corollary regards the existence of solutions of the quantum Landau Hamiltonian undergoing unbounded growth of Sobolev norms. Computing $\Phi_B(t)$ and using Corollary 1.3 we get

Corollary 1.5. *Let $\omega \in \mathcal{C}_\varepsilon$. We have, in the operator sense,*

$$\hat{x}_1(t) = \hat{x}_1 + \alpha c_\omega \varepsilon^2 t\, \hat{p}_1 + (1+\varepsilon^4 t)\left(A_{\varepsilon,\omega}(t)\cdot \hat{x} + B_{\varepsilon,\omega}(t)\cdot \hat{p}\right),$$

where $\alpha \neq 0$ and c_ω is given by (1.10), and $|A_{\varepsilon,\omega}(t)| + |B_{\varepsilon,\omega}(t)| = O(1)$. In particular, if $c_\omega \neq 0$, there exists $K \geq 0$ such that for any $\psi \in \mathcal{H}^1_{\mathrm{osc}}$ we have

$$\|\hat{x}_1(t)\psi\|_0 \geq \alpha c_\omega t \varepsilon^2 \|D_{x_1}\psi\|_0 - K(1+\varepsilon^4 t)\|\psi\|_1. \tag{1.16}$$

In particular, if ε is sufficiently small, then $\|\hat{x}_1(t)\psi\|_0 \nearrow +\infty$ as $t \nearrow +\infty$.
We have also a lower bound for the quantum average of the time evolution of the position observable, for $\psi \in \mathcal{H}^{1/2}_{\mathrm{osc}}$:

$$|\langle \psi, \hat{x}_1(t)\psi\rangle| \geq \alpha c_\omega t \varepsilon^2 |\langle \psi, D_{x_1}\psi\rangle| - K(1+\varepsilon^4 t)\|\psi\|_{1/2}. \tag{1.17}$$

Our second corollary regards the Floquet spectrum of the time quasi-periodic Hamiltonian $H_L(t)$.

Corollary 1.6. *The quantum dynamics $\mathcal{U}_{\varepsilon,\omega}(t,0)$ of $H_L(t)$ is conjugated to the quantum dynamics $\mathrm{e}^{-\mathrm{i}t\tilde{H}_{L,\varepsilon,\infty}}$ of the stationary Hamiltonian $\tilde{H}_{L,\varepsilon,\infty} := b(\varepsilon)(D^2_{\tilde{x}_1} + \tilde{x}_1^2) + c(\varepsilon)D^2_{\tilde{x}_2}$. Moreover as far as $b(\varepsilon) > 0$ and $c(\varepsilon) > 0$, the spectrum $\sigma(H_{L,\varepsilon,\infty})$ of $\tilde{H}_{L,\varepsilon,\infty}$ is absolutely continuous, with thresholds at the Landau levels,*

$$\sigma(H_{L,\varepsilon,\infty}) = \bigcup_{j\geq 0} [b(\varepsilon)(j+1/2), +\infty[.$$

Proof. Denote by $h_{L,\varepsilon,\infty} := b(\varepsilon)|z_1|^2 + c(\varepsilon)(z_2 - \overline{z}_2)^2$ the stationary classical Hamiltonian to which $h_L(t)$ is conjugated, see (1.11). In the real coordinates (x,p) we have $h_{L,\varepsilon,\infty}(x,p) = \frac{b(\varepsilon)}{2}(p_2^2 + x_2)^2 + \frac{c(\varepsilon)}{2}(x_1 - p_2)^2$. By the symplectic change of coordinates $\tilde{x}_1 = x_1 - p_2, \tilde{p}_1 = p_1$, $\tilde{x}_2 = x_2 - p_1, \tilde{p}_2 = p_2$, we have

$$\tilde{h}_{L,\varepsilon,\infty}(\tilde{x},\tilde{p}) = b(\varepsilon)(\tilde{p}_2^2 + \tilde{x}_2)^2 + c(\varepsilon)\tilde{p}_1^2.$$

The first part of the Corollary follows.
For the second part notice that we have the following family of generalized eigenfunctions: $\Psi_{j,\xi}(x_1,x_2) = \psi_j(x_1)\mathrm{e}^{\mathrm{i}x_2\xi}$ such that we have

$$\tilde{H}_{L,\varepsilon,\infty}\Psi_{j,\xi} = (b(\varepsilon)(j+1/2) + c(\varepsilon)\xi^2)\Psi_{j,\xi}.$$

Hence we get a description of the spectrum of $\tilde{H}_{L,\varepsilon,\infty}$. □

1.2 Main result in symmetric gauge

We still consider a magnetic Schrödinger operator with a time-quasiperiodic magnetic field, i.e. $B(t) = B_0 + \varepsilon f(\omega t)$ with f real analytic on the torus $\mathbb{T}^n$, and $\hat{f}(0) = 0$, but now in the symmetric gauge, namely

$$H_{sL}(t) = \left(D_{x_1} - \frac{1}{2}B(t)x_2\right)^2 + \left(D_{x_2} + \frac{1}{2}B(t)x_1\right)^2, \tag{1.18}$$

and the frequency vector ω in the set of nonresonant frequencies $\mathbb{D}_0$ of Definition 1.1. We can write

$$H_{sL}(t) = H_{sL} + R_{sL}(\omega t)$$

where

$$H_{sL} = \left(D_{x_1} - \frac{1}{2}B_0 x_2\right)^2 + \left(D_{x_2} + \frac{1}{2}B_0 x_1\right)^2,$$
$$R_{sL}(\omega t) = \varepsilon f(\omega t)\left(x_1(D_{x_2} + \frac{B_0}{2}x_1) - x_2(D_{x_1} - \frac{B_0}{2}x_2)\right) + \frac{1}{4}\varepsilon^2 f(\omega t)^2(x_1^2 + x_2^2).$$

We denote by $h_{sL}(t)$ the corresponding classical Hamiltonian:

$$h_{sL}(t) = \left(p_1 - \frac{B(t)}{2}x_2\right)^2 + \left(p_2 + \frac{B(t)}{2}x_1\right)^2 = h_{sL} + r_{sL}(\omega t). \tag{1.19}$$

We introduce the symplectic variables

$$p_1' = x_1 \ , \ x_2' = x_2 \ \ x_1' = -p_1 \ , \ p_2' = p_2$$

and in these variables h_{sL} reads

$$h_{sL} = \left(x_1' + \frac{B_0}{2}x_2'\right)^2 + \left(p_2' + \frac{B_0}{2}p_1'\right)^2.$$

Then in the new symplectic variables $(y_1, y_2, \eta_1, \eta_2)$ defined by

$$y_1 = \frac{1}{\sqrt{B_0}}x_1' + \frac{\sqrt{B_0}}{2}x_2', \quad \eta_1 = \frac{\sqrt{B_0}}{2}p_1' + \frac{1}{\sqrt{B_0}}p_2'$$

and

$$y_2 = \frac{1}{\sqrt{B_0}}x_1' - \frac{\sqrt{B_0}}{2}x_2', \quad \eta_2 = \frac{\sqrt{B_0}}{2}p_1' - \frac{1}{\sqrt{B_0}}p_2'$$

we obtain that h_{sL} is the *degenerate* two-dimensional Harmonic oscillator

$$h_{sL} = B_0(y_1^2 + \eta_1^2) = 2B_0|z_1|^2$$

where $z_1 = \frac{y_1 + \mathrm{i}\eta_1}{\sqrt{2}}$ and similarly $z_2 = \frac{y_2 + \mathrm{i}\eta_2}{\sqrt{2}}$. We denote by τ_0 the linear symplectic transformation from $\mathbb{R}^4$ to $\mathbb{C}^2$ defined by $(z_1, z_2) = \tau_0(x, p)$. We note that in the complex variables the symplectic form reads:

$$dy_1 \wedge d\eta_1 + dy_2 \wedge d\eta_2 = \mathrm{i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

In order to state the next result we introduce

$$d_\omega = \int_{\mathbb{T}^n} h(\theta)^2 d\theta \tag{1.20}$$

where

$$h(\theta) = -\frac{1}{\sqrt{2B_0}}\sum_{k\neq 0}\frac{\omega\cdot k + 2\mathrm{i}B_0}{\omega\cdot k + 2B_0}\hat{f}(k)e^{\mathrm{i}k\cdot\theta}. \tag{1.21}$$

Remark 1.7. *The number d_ω is well defined and real for $\omega \in \mathbb{D}_0$.*

Our next result shows that, in the symmetric gauge, the perturbed classical Hamiltonian $h_{sL}(t)$ in (1.19) is conjugated to a two dimensional Harmonic oscillator which is *non-degenerate* provided the constant $d_\omega \neq 0$.

Theorem 1.8. *There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, there exist*

- *an asymptotically full measure set of frequencies $\mathcal{C}_\varepsilon \in \mathbb{D}_0$ satisfying:*

$$\lim_{\varepsilon \to 0} meas(\mathbb{D}_0 \setminus \mathcal{C}_\varepsilon) = 0;$$

- *a linear, symplectic change of variable τ, depending on ω, t, ε, which is close to the identity, namely $\tau = \mathbb{I} + O(\varepsilon)$ uniform in the other parameters,*

such that, for $\omega \in \mathcal{C}_\varepsilon$, and provided d_ω in (1.20) does not vanish,

$$(h_{sL}(t) \circ \tau_0^{-1}) \circ \tau = b(\varepsilon)|z_1|^2 + d(\varepsilon)|z_2|^2 \tag{1.22}$$

with

$$b(\varepsilon) = 2B_0 + O(\varepsilon^2), \quad d(\varepsilon) = d_\omega \varepsilon^2 + O(\varepsilon^4)$$

and d_ω is given by (1.20) and does not vanish for $\omega \in \mathcal{C}_\varepsilon$.

In the new coordinates the motion of (1.22) is easily computed:

$$z_1(t) \;\; = \;\; \mathrm{e}^{-\mathrm{i}b(\varepsilon)t} z_1(0)\ , \qquad z_2(t) = \mathrm{e}^{-\mathrm{i}d(\varepsilon)t} z_2(0).$$

In particular, in the symmetric gauge, provided the constant d_ω in (1.20) does not vanish, all the trajectories are bounded, contrary to what happens in the Landau gauge.

Also in this case we are able to describe the quantum flow, which in this case is uniformly bounded in any Sobolev space $\mathcal{H}^r_{\mathrm{osc}}$.
Let $\mathcal{U}_{\varepsilon,\omega}(t,s)$ be the quantum propagator defined by the Hamiltonian $H_{sL}(t)$ in (1.18). So we have $\mathrm{i}\partial_t \mathcal{U}_{\varepsilon,\omega}(t,s) = H_{sL}(t)\mathcal{U}_{\varepsilon,\omega}(t,s)$, $\mathcal{U}_{\varepsilon,\omega}(s,s) = \mathbb{I}$.

Corollary 1.9. *There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| \leq \varepsilon_0$, for any $r > 0$ there exist $0 < c_r \leq C_r$ such that if $\omega \in \mathcal{C}_\varepsilon$ we have for any $\psi_0 \in \mathcal{H}^r_{\mathrm{osc}}$,*

$$c_r \|\psi_0\|_r \leq \|\mathcal{U}_{\varepsilon,\omega}(t,0)\psi_0\|_r \leq C_r \|\psi_0\|_r, \ \forall t \in \mathbb{R}. \tag{1.23}$$

Proof. We follow the proof given in [3], Corollary 1.3. We shall give here only the main steps.
For simpler notation we assume that $B_0 = 1$. Let

$$H_{sL,\varepsilon,\infty} = \frac{b(\varepsilon)}{2}((\hat{p}_1 - \hat{x}_2)^2 + \hat{p}_2^2) + \frac{d(\varepsilon)}{2}((\hat{p}_2 - \hat{x}_1)^2 + \hat{p}_1^2),$$

and $Z_1 = (\hat{p}_1 - \hat{x}_2)^2 + \hat{p}_2^2$, $Z_2 = (\hat{p}_2 - \hat{x}_1)^2 + \hat{p}_1^2$. Notice that $Z_j = \hat{z}_j^* \hat{z}_j$. Moreover $[Z_1, Z_2] = 0$. So $[H_{sL,\varepsilon,\infty}, Z_1 + Z_2] = 0$. But $Z_1 + Z_2$ is a non degenerate harmonic oscillator hence $e^{-itH_{sL,\varepsilon,\infty}}\psi_0$ satisfies the estimates (1.23). Then as in [3], from the classical KAM construction there exists

$$\chi_{\varepsilon,\omega}(t,x,p) = \begin{pmatrix} x \\ p \end{pmatrix} \cdot S_{\varepsilon,\omega}(t) \begin{pmatrix} x \\ p \end{pmatrix},$$

where $S_{\varepsilon,\omega}(t)$ is a symmetric matrix with uniformly bounded entries. Let $U_{\varepsilon,\omega}(t) = e^{i\varepsilon\hat{\chi}_{\varepsilon,\omega}(t)}$ we have

$$\mathcal{U}_{\varepsilon,\omega}(t,0) = U^*_{\varepsilon,\omega}(t) e^{-itH_{\varepsilon,\infty}} U_{\varepsilon,\omega}(t).$$

Using that ([3], Theorem 2.7), uniformly in (t,ε,ω), we have

$$\tilde{c}_r \|\psi\|_r \leq \|U^*_{\varepsilon,\omega}(t)\psi\|_r \leq \tilde{C}_r \|\psi\|_r,$$

we get (1.23). $\square$

Corollary 1.10. *The symmetric Landau Hamiltonian $H_{sL}(t)$ is reducible to a stationary Hamiltonian $H_{sL,\varepsilon,\infty}$ with a discrete spectrum with all eigenvalues of finite multiplicities as far $b(\varepsilon) > 0, d(\varepsilon) \neq 0$. Moreover, up to a linear symplectic transformation, $H_{sL,\varepsilon,\infty}$ is combination of two 1-d harmonic oscillators*

$$H_{sL,\varepsilon,\infty} = b(\varepsilon)(D_{x_1}^2 + x_1^2) + d(\varepsilon)(D_{x_2}^2 + x_2^2).$$

Proof. Keeping the notations of the proof of Corollary 1.9, since $Z_1 + Z_2$ has compact inverse, it has pure point spectrum and there exists a basis of eigenfunctions, which, since $[Z_1, Z_2] = 0$ can be choose in such a way that they are also eigenfunctions of both Z_1 and Z_2. Thus, if one denotes by $\lambda_{i,j} = j + 1/2$, $j \in \mathbb{N}$ the eigenvalues of Z_i, one has that $H_{sL,\varepsilon,\infty}$ is diagonal in the same basis, with eigenvalues $\mu_{j_1,j_2}(\varepsilon) = b(\varepsilon)(j_1+1/2)+d(\varepsilon)(j_2+1/2)$. So the symmetric Landau Hamiltonian $H_{sL}(t)$ is reducible to a stationary Hamiltonian $H_{sL,\varepsilon,\infty}$ with a discrete spectrum with all eigenvalues of finite multiplicities as far $b(\varepsilon) > 0, d(\varepsilon) \neq 0$. $\square$

1.3 About the change of gauge

The fact that one has a completely different behaviour in the case of the Landau gauge and in the case of the symmetric gauge could seem surprising at first sight, however we remark that they correspond to different physical situations. Indeed, the electric and the magnetic field are given by

$$\mathbf{B} = \nabla_x \times A(x,t) \quad \text{and} \quad \mathbf{E} = -\frac{\partial A}{\partial t}(x,t) - \nabla_x V(x,t). \tag{1.24}$$

Thus, in the time dependent case, they differ for the case of the Landau gauge and the case of the symmetric gauge. As it is well known, there exists a gauge transformation which allows to pass from one gauge to the other by keeping the same electromagnetic fields. For example the electric and magnetic fields can be chosen as

$$V(x,t) = 0\ , \quad A(x,t) = B(t)(x_2, 0, 0)$$

or as

$$V(x,t) = -\frac{B'(t)}{2} x_1 x_2 \ , \quad A(x,t) := \frac{B(t)}{2}(x_2, -x_1, 0) \ .$$

The first corresponds to the Landau gauge, while the second corresponds to the symmetric gauge plus a scalar potential.

1.4 Related literature

Most of the literature about the Landau Hamiltonian regards the asymptotic behavior of the perturbed spectrum under time *independent* perturbations, for example scalar potentials in different classes. These works ensure conditions on the perturbation so that the perturbed spectrum is asymptotically localized around the Landau levels $\{2B_0(j+\frac{1}{2})\}_{j\in\mathbb{N}}$, a fact which is not trivial due to the infinite multiplicity of these. We mention for example the works [31, 30, 28, 20, 21, 15] and reference therein.

The case of time *dependent* perturbations, such as (1.1), is less studied. We mention [33, 34, 9] which prove the existence of the quantum flow, and [26, 4, 22] giving time upper bounds on the dynamics. The present paper aims to prove finer properties about the quantum dynamics, and in particular investigates the dicotomy of “existence of solutions with unbounded trajectories” vs “all trajectories are bounded”. This question has received, in the last decade, a lot of attention.

In case of linear, time dependent Schrödinger equations, such as (1.1), the first result about existence of solutions with unbounded paths is due to Bourgain [6] on the torus. Recently, several works have considered non-degenerate Harmonic oscillators on $\mathbb{R}^d$ and constructed time dependent perturbations in the form of pseudodifferential operators [8, 23], polynomial functions [3, 17, 19, 18], or classical potentials [11, 32], that create solutions with unbounded trajectories. We also cite the recent results [24, 25] which prove that generic, time periodic, pseudodifferential perturbations provoke instability phenomena.

On the opposite side, many works prove that, when the perturbation is small in size and quasi-periodic in time with a non resonant frequency ω, all trajectories are bounded in Sobolev spaces. There results are based on KAM reducibility methods ensuring that the linear propagator has operatorial norm (in Sobolev spaces) bounded uniformly in time, in the same spirit of our Corollary 1.9. This is the case, in great generality, for systems in 1-spatial dimensions: limiting ourselves to results considering perturbations of the Harmonic oscillator on $\mathbb{R}$, we cite [7, 35, 14, 2, 1]. In a higher dimensional setting, such as the one of equation (1.1), there are few KAM reducibility results. We cite [10, 29] for the Schrödinger on $\mathbb{T}^d$, [13, 3] for the Harmonic oscillators on $\mathbb{R}^d$, [27] for the wave on $\mathbb{T}^d$, and [12, 5] for transport equations on $\mathbb{T}^d$.

2 Proofs of main Theorems

2.1 Proof of Theorem 1.2

We prefer to work in the extended phase space in which we add the angles $\theta \in \mathbb{T}^n$ as new variables and their conjugated momenta $I \in \mathbb{R}^n$. So our phase space is now $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^4 \ni (\theta, I, z_1, z_2)$, with $\mathbb{C}^2$ considered as a real vector space. The symplectic form is $dI \wedge d\theta + \mathrm{i}dz \wedge d\overline{z}$ and the Hamiltonian equations of a Hamiltonian function $h(\theta, I, z_1, z_2)$ are

$$\dot{I} = -\frac{\partial h}{\partial \theta}\ , \quad \dot{\theta} = \frac{\partial h}{\partial I}\ , \quad \dot{z}_1 = -\mathrm{i}\frac{\partial h}{\partial \overline{z}_1}\ , \quad \dot{z}_2 = -\mathrm{i}\frac{\partial h}{\partial \overline{z}_2}\ . \tag{2.1}$$

In this framework the Hamiltonian equation associated with the classical time dependent Hamiltonian function h_L in (1.5) is equivalent to the autonomous Hamiltonian system (2.1) with

$$h = h_0 + r_1 + r_2$$

and

$$h_0 = \omega \cdot I + 2B_0|z_1|^2, \tag{2.2}$$
$$r_1 = \varepsilon(z_1 + \overline{z}_1)(z_1 + \overline{z}_1 - \mathrm{i}(z_2 - \overline{z}_2))f(\theta), \tag{2.3}$$
$$r_2 = \frac{\varepsilon^2}{2B_0}(z_1 + \overline{z}_1 - \mathrm{i}(z_2 - \overline{z}_2))^2 f(\theta)^2. \tag{2.4}$$

The proof of Theorem 1.2 follows a KAM strategy: we want to eliminate the angles in h by canonical changes of variables. This canonical changes of variables will be constructed as time 1 flows, Φ^1_χ, of some Hamiltonian χ. We begin by computing explicitly the first two KAM steps and then we will be in position to apply a KAM theorem with symmetry, namely Theorem 3.4. First we construct the first change of variables and we begin by solving a so called homological equation.

Lemma 2.1. *Let*

$$\begin{aligned}\chi_1 :=& \mathrm{i}\varepsilon \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1^2}{\omega\cdot k + 4B_0} + \frac{\overline{z}_1^2}{\omega\cdot k - 4B_0} + 2\frac{z_1\overline{z}_1}{\omega\cdot k}\right)\\ &+\varepsilon(z_2 - \overline{z}_2)\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1}{\omega\cdot k + 2B_0} + \frac{\overline{z}_1}{\omega\cdot k - 2B_0}\right)\ ,\end{aligned}$$

then χ_1 solves the following homological equation:

$$\{\chi_1, h_0\} + r_1 = 0. \tag{2.5}$$

Proof. First recall that

$$\{F, G\} := \sum_{j=1}^{n} \frac{\partial F}{\partial \theta_j}\frac{\partial G}{\partial I_j} - \frac{\partial F}{\partial I_j}\frac{\partial G}{\partial \theta_j} + \mathrm{i}\sum_{j=1,2}\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial \overline{z}_j} - \frac{\partial G}{\partial z_j}\frac{\partial F}{\partial \overline{z}_j}. \tag{2.6}$$

Then we introduce

$$N(\theta, z_1) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1}{\omega\cdot k+2B_0} + \frac{\overline{z}_1}{\omega\cdot k-2B_0}\right)$$

and

$$M(\theta, z_1) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1^2}{\omega\cdot k+4B_0} + \frac{\overline{z}_1^2}{\omega\cdot k-4B_0} + 2\frac{z_1\overline{z}_1}{\omega\cdot k}\right)$$

in such way we have

$$\chi_1 = \mathrm{i}\varepsilon M + \varepsilon(z_2 - \overline{z}_2)N.$$

So, since h_0 in (2.2) doesn't depend on z_2, we get

$$\{\chi_1, h_0\} = \mathrm{i}\varepsilon\{M, h_0\} + \varepsilon(z_2 - \overline{z}_2)\{N, h_0\}.$$

Then we compute

$$\begin{aligned}
\{N, h_0\} &= \mathrm{i}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} (k\cdot\omega)\hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1}{\omega\cdot k+2B_0} + \frac{\overline{z}_1}{\omega\cdot k-2B_0}\right)\\
&+\mathrm{i}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{2B_0z_1}{\omega\cdot k+2B_0} + \frac{-2B_0\overline{z}_1}{\omega\cdot k-2B_0}\right)\\
&= \mathrm{i}(z_1+\overline{z}_1)f(\theta),
\end{aligned}$$

and

$$\begin{aligned}
\{M, h_0\} &= \mathrm{i}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} (k\cdot\omega)\hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{z_1^2}{\omega\cdot k+4B_0} + \frac{\overline{z}_1^2}{\omega\cdot k-4B_0} + 2\frac{z_1\overline{z}_1}{\omega\cdot k}\right)\\
&+\mathrm{i}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{4B_0z_1^2}{\omega\cdot k+4B_0} + \frac{-4B_0\overline{z}_1^2}{\omega\cdot k-4B_0}\right)\\
&= \mathrm{i}(z_1+\overline{z}_1)^2f(\theta)\ ,
\end{aligned}$$

from which (2.5) follows. $\square$

Then since,

$$\{\chi_1, h_0\} + r_1 = 0, \quad \{\chi_1, \{\chi_1, h_0\}\} = -\{\chi_1, r_1\},$$

we get

$$h\circ\Phi^1_{\chi_1} = h + \{\chi_1, h\} + \frac{1}{2}\{\chi_1, \{\chi_1, h_0\}\} + O(\varepsilon^3) = h_0 + \frac{1}{2}\{\chi_1, r_1\} + r_2 + O(\varepsilon^3). \quad (2.7)$$

Next we wish to compute explicitly $\{\chi_1, r_1\}$. In view of the expressions of χ_1 and r_1, we first compute

$$\begin{aligned}
\{M,(z_1+\overline{z}_1)^2\} &= 2(z_1+\overline{z}_1)\{M, z_1+\overline{z}_1\},\\
\{M,(z_1+\overline{z}_1)(z_2-\overline{z}_2)\} &= (z_2-\overline{z}_2)\{M, z_1+\overline{z}_1\},\\
\{M, z_1+\overline{z}_1\} &= 2\mathrm{i}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{f}(k)e^{ik\cdot\theta}\left(\frac{z_1}{\omega\cdot k+4B_0}-\frac{\overline{z}_1}{\omega\cdot k-4B_0}+\frac{\overline{z}_1-z_1}{\omega\cdot k}\right)\\
&= -8\mathrm{i}B_0\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{f}(k)e^{ik\cdot\theta}\left(\frac{z_1}{\omega\cdot k(\omega\cdot k+4B_0)}+\frac{\overline{z}_1}{\omega\cdot k(\omega\cdot k-4B_0)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\{N,(z_1+\overline{z}_1)^2\} &= 2(z_1+\overline{z}_1)\{N, z_1+\overline{z}_1\},\\
\{N,(z_1+\overline{z}_1)(z_2-\overline{z}_2)\} &= (z_2-\overline{z}_2)\{N, z_1+\overline{z}_1\},\\
\{N, z_1+\overline{z}_1\} &= -4\mathrm{i}B_0\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{\hat{f}(k)e^{\mathrm{i}k\cdot\theta}}{(\omega\cdot k)^2-4B_0^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\{\chi_1, r_1\} &= \{\mathrm{i}\varepsilon M+\varepsilon(z_2-\overline{z}_2)N, \varepsilon(z_1+\overline{z}_1)(z_1+\overline{z}_1-\mathrm{i}(z_2-\overline{z}_2))f(\theta)\}\\
&= 2\mathrm{i}\varepsilon^2 f(\theta)(z_1+\overline{z}_1)\{M,(z_1+\overline{z}_1)\}+\varepsilon^2(z_2-\overline{z}_2)f(\theta)\{M,(z_1+\overline{z}_1)\}\\
&+ 2\varepsilon^2 f(\theta)(z_2-\overline{z}_2)(z_1+\overline{z}_1)\{N, z_1+\overline{z}_1\}-\mathrm{i}\varepsilon^2(z_2-\overline{z}_2)^2 f(\theta)\{N, z_1+\overline{z}_1\}\\
&= \varepsilon^2 f(\theta)(2\mathrm{i}(z_1+\overline{z}_1)+(z_2-\overline{z}_2))\{M,(z_1+\overline{z}_1)\}\\
&- \mathrm{i}\varepsilon^2 f(\theta)(z_2-\overline{z}_2)(2\mathrm{i}(z_1+\overline{z}_1)+(z_2-\overline{z}_2))\{N, z_1+\overline{z}_1\}\\
&= \varepsilon^2 f(\theta)(2\mathrm{i}(z_1+\overline{z}_1)+(z_2-\overline{z}_2))(\{M,(z_1+\overline{z}_1)\}-\mathrm{i}(z_2-\overline{z}_2)\{N, z_1+\overline{z}_1\})\\
&= -4\mathrm{i}\varepsilon^2 B_0 f(\theta)(2\mathrm{i}(z_1+\overline{z}_1)+(z_2-\overline{z}_2))\times\\
&\times\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{2z_1}{\omega\cdot k(\omega\cdot k+4B_0)}+\frac{2\overline{z}_1}{\omega\cdot k(\omega\cdot k-4B_0)}-\mathrm{i}\frac{z_2-\overline{z}_2}{(\omega\cdot k)^2-4B_0^2}\right).
\end{aligned} \tag{2.8}$$

At the next KAM step we remove from $\frac{1}{2}\{\chi_1, r_1\}+r_2$ all the ε^2 terms except resonant monomials, i.e. in our case, $|z_1|^2$ and $(z_2-\overline{z}_2)^2$. In view of the expressions (2.8), (2.4), we thus obtain

$$h_2 = h\circ\Phi^1_{\chi_1}\circ\Phi^1_{\chi_2} = h_0+a_\omega\varepsilon^2|z_1|^2+c_\omega\varepsilon^2(z_2-\overline{z}_2)^2+O(\varepsilon^3) \tag{2.9}$$

where

$$
\begin{aligned}
c_\omega &= -\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\left(\frac{2B_0}{(\omega\cdot k)^2-4B_0^2}\right)-\frac{1}{2B_0}\langle f^2\rangle\\
&= -\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\left(\frac{2B_0}{(\omega\cdot k)^2-4B_0^2}+\frac{1}{2B_0}\right)\\
&= -\frac{1}{2B_0}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\frac{(\omega\cdot k)^2}{(\omega\cdot k)^2-4B_0^2}
\end{aligned}
$$

as stated in (1.10) while

$$
\begin{aligned}
a_\omega &= 8B_0\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\left(\frac{1}{(\omega\cdot k)((\omega\cdot k)+4B_0)}+\frac{1}{(\omega\cdot k)((\omega\cdot k)-4B_0)}\right)+\frac{1}{B_0}\langle f^2\rangle\\
&= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\left(\frac{16B_0}{(\omega\cdot k)^2-16B_0^2}+\frac{1}{B_0}\right)\\
&= \frac{1}{B_0}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat f(k)\hat f(-k)\frac{(\omega\cdot k)^2}{(\omega\cdot k)^2-16B_0^2}.
\end{aligned}
$$

To end the proof of Theorem 1.2 we just have to iterate this KAM step and to prove the convergence of such a process. In particular we will check that we only remain with resonant monomials, i.e. in our case, $|z_1|^2$ and $(z_2-\overline{z}_2)^2$.
Concretely Theorem 1.2 is obtained by applying Theorem 3.4 to the Hamiltonian (2.9) taking $\nu_1 := 2B_0 + a_\omega\varepsilon^2$, $c_0 := c_\omega\varepsilon^2$ and $\varepsilon_0 := \varepsilon^3$. We stress out that the difference between the estimate (3.8) in Theorem 3.4 and the estimate (1.12) in Theorem 1.2 is due to the specific form of r in (2.3) and the fact that $\hat f(0) = 0$.

2.2 Proof of Theorem 1.8

We still work in the same framework as in section 2.1, i.e. in the extended phase space in which we add the angles $\theta\in\mathbb{T}^n$ as new variables and their conjugated momenta $I\in\mathbb{R}^n$. So our phase space is still $\mathbb{T}^n\times\mathbb{R}^n\times\mathbb{C}^2\ni(\theta,I,z_1,z_2)$ (see (2.1)).

In this framework the Hamiltonian equation associated with the classical time dependent Hamiltonian function (1.19) $h_{sL}(t)$ is equivalent to the autonomous Hamiltonian system (2.1) with $h = h_0 + r$ where

$$h_0 = \omega\cdot I + 2B_0|z_1|^2$$

and $r = r_{sL}(\theta)$. To compute explicitly this term, recall that, in the coordinates introduced in section 1.2,

$$\begin{aligned}
x_2 = x_2' &= \frac{1}{\sqrt{B_0}}(y_1 - y_2) = \frac{1}{\sqrt{2B_0}}(z_1 + \overline{z}_1 - (z_2 + \overline{z}_2)),\\
x_1 = p_1' &= \frac{1}{\sqrt{B_0}}(\eta_1 + \eta_2) = \frac{1}{\mathrm{i}\sqrt{2B_0}}(z_1 - \overline{z}_1 + (z_2 - \overline{z}_2)),\\
p_1 - \frac{B_0}{2}x_2 &= -x_1' - \frac{B_0}{2}x_2' = -\sqrt{B_0}y_1 = -\frac{\sqrt{B_0}}{\sqrt{2}}(z_1 + \overline{z}_1),\\
p_2 + \frac{B_0}{2}x_1 &= p_2 - \frac{B_0}{2}\xi_1 = \sqrt{B_0}\eta_1 = \frac{\sqrt{B_0}}{\mathrm{i}\sqrt{2}}(z_1 - \overline{z}_1).
\end{aligned}$$

Therefore $r(\theta)$ reads

$$\begin{aligned}
r(\theta) &= \varepsilon f(\theta)(y_1(y_1 - y_2) + \eta_1(\eta_1 + \eta_2)) + \frac{\varepsilon^2}{4B_0}f(\theta)^2((y_1 - y_2)^2 + (\eta_1 + \eta_2)^2)\\
&= \frac{\varepsilon}{2}f(\theta)((z_1 + \overline{z}_1)(z_1 + \overline{z}_1 - (z_2 + \overline{z}_2)) - (z_1 - \overline{z}_1)(z_1 - \overline{z}_1 + (z_2 - \overline{z}_2)))\\
&+ \frac{\varepsilon^2}{8B_0}f(\theta)^2((z_1 + \overline{z}_1 - (z_2 + \overline{z}_2))^2 - (z_1 - \overline{z}_1 + (z_2 - \overline{z}_2))^2)\\
&= \varepsilon r_1 + \varepsilon^2 r_2.
\end{aligned}$$

We follow the same strategy than in the previous section and we are interested by the quadratic terms in z_2, $\overline{z}_2$ after the second KAM step. At the first step (at order ε) we don't have such term (in r_1) and thus we eliminate all the terms of order ε by a symplectic change of variables $\Phi^1_{\chi_1}$ where χ_1 is the solution of the homological equation.

$$\{\chi_1, h_{sL}\} = -r_1 \ .$$

Since

$$r_1 = -\varepsilon f(\theta)(z_1 z_2 + \overline{z}_1\overline{z}_2) + \text{ quadratic terms in } z_1,\ \overline{z}_1$$

we take (with $\nu = 2B_0$)

$$\chi_1 = -\mathrm{i}\varepsilon \sum_{k\neq 0} \hat{f}(k)e^{ik\cdot\theta}\left(\frac{z_2 z_1}{k\cdot\omega + \nu} + \frac{\overline{z}_2\overline{z}_1}{k\cdot\omega - \nu}\right) + \text{ quadratic terms in } z_1,\ \overline{z}_1.$$

Then, using (2.7), the quadratic terms in z_2, $\overline{z}_2$ after the second KAM step come from the quadratic terms in z_2, $\overline{z}_2$ in $\frac{1}{2}\{\chi_1, r_1\}$ and in r_2.
From r_2 we get

$$\frac{\varepsilon^2 f(\theta)^2}{4B_0} z_2\overline{z}_2$$

and from $\frac{1}{2}\{\chi_1, r_1\}$ we get

$$-\frac{1}{2}\varepsilon^2 f(\theta)\sum_{k\neq 0}\hat{f}(k)e^{\mathrm{i}k\cdot\theta}\left(\frac{\overline{z}_2 z_2}{k\cdot\omega + \nu} - \frac{z_2\overline{z}_2}{k\cdot\omega - \nu}\right).$$

Now the second KAM step will eliminate all the $e^{ik\cdot\theta}z_2\overline{z}_2$ for $k \neq 0$ and therefore we obtain like in (2.9)

$$h_2 = h \circ \Phi^1_{\chi_1} \circ \Phi^1_{\chi_2} = \omega \cdot I + (2B_0 + a\varepsilon^2)|z_1|^2 + d_\omega \varepsilon^2 |z_2|^2 + O(\varepsilon^3) \tag{2.10}$$

where

$$\begin{aligned} d_\omega &= \frac{1}{4B_0}\sum_{\ell} \hat{f}(\ell)\hat{f}(-\ell) + \sum_{\ell\neq 0} \hat{f}(\ell)\hat{f}(-\ell)\frac{2B_0}{(\ell\cdot\omega)^2 - 4B_0^2} \\ &= \frac{1}{4B_0}\sum_{\ell\neq 0} \hat{f}(\ell)\hat{f}(-\ell)\frac{(\ell\cdot\omega)^2 + 4B_0^2}{(\ell\cdot\omega)^2 - 4B_0^2} \end{aligned}$$

as stated in (1.20).

Then, if $d_\omega \neq 0$, h_2 appears as a perturbation of the non degenerate Hamiltonian $(2B_0 + a\varepsilon^2)|z_1|^2 + d_\omega\varepsilon^2|z_2|^2$ and we can apply Theorem 3.5.

3 Proofs of two reducibility Theorems

In this section we prove the two reducibility theorems that we need to conclude the proofs of Theorem 1.2 and Theorem 1.8.

First, it is usefull to change the notation in order to make clear that the variable $\overline{z}$ is not necessarily the complex conjugate of z: we only have to define the concept of real submanifold of the phase space and it depends on the variables we use (see Definition 3.1 and Remark 3.3 below for more details).

So, first of all we define

$$\xi_j := z_j \ , \quad \eta_j := \overline{z}_j \ , \quad j = 1, 2 \ . \tag{3.1}$$

Definition 3.1. *For $(\xi, \eta) \in C^4$, define the involution*

$$(\xi, \eta) \mapsto I(\xi, \eta) = (\overline{\eta}, \overline{\xi}) \ , \tag{3.2}$$

the states (ξ, η) s.t. $(\xi, \eta) = I(\xi, \eta)$ will be said to be real.

Remark 3.2. *The Hamiltonians we are dealing with are real when (ξ, η) is real.*

In order to deal with the Hamiltonian (2.9) whose expansion up to order ε^2 only depends on z_1, $\overline{z}_1$ and $z_2 - \overline{z}_2$, it is useful to introduce the following canonical change of variables

$$\xi_2' := \xi_2 - \eta_2 \ , \quad \eta_2' := \eta_2 \ . \tag{3.3}$$

In these variables (and omitting primes) (2.9) reads

$$h_2 = \omega \cdot I + (2B_0 + a_\omega\varepsilon^2)\xi_1\eta_1 + c_\omega\varepsilon^2\xi_2^2 + O(\varepsilon^3). \tag{3.4}$$

Remark 3.3. *in these variables the involution I takes the form (omitting the primes)*

$$(\xi_1, \eta_1, \xi_2, \eta_2) \mapsto I(\xi_1, \eta_1, \xi_2, \eta_2) \equiv (\overline{\eta}_1, \overline{\xi}_1, \overline{\xi}_2, \overline{\xi_2 - \eta_2}) \tag{3.5}$$

and the real submanilfold reads: $\eta_1 = \overline{\xi}_1$, ξ_2 is real and $2\Re(\eta_2) = \xi_2$.
In all the situations we will encounter the Hamiltonian is independent of η_2, therefore the fact that the reality condition becomes more complicate will be completely irrelevant. Of course, in the following a Hamiltonian expressed in the variables (3.3) *will be said to be real if it takes real values for real (ξ, η), i.e. for (ξ, η) which are fixed points with respect to the involution* (3.5)*.*

The first theorem we will prove concerns quasi-periodic in time Hamiltonians of the form

$$h_{\varepsilon_0}(t, \xi, \eta) = \nu_1 \xi_1 \eta_1 + c_0 \xi_2^2 + \varepsilon_0 q(\omega t, \xi_1, \eta_1, \xi_2) \tag{3.6}$$

where q is a polynomial in (ξ_1, η_1, ξ_2) homogeneous of degree 2 with coefficients that depend quasi periodically on time. The important point is that q does not depend on η_2.
In the following we denote, for $\sigma > 0$, $\mathbb{T}^n_\sigma := \{x + \mathrm{i}y : x \in \mathbb{T}^n,\ y \in \mathbb{R}^n,\ |y| \le \sigma\}$.

Theorem 3.4. *Assume that $\nu_1 > 0$ and that $\mathbb{T}^n \times \mathbb{C}^4 \ni (\theta, \xi) \mapsto q(\theta, \xi) \in \mathbb{C}$ is a polynomial in (ξ_1, η_1, ξ_2) homogeneous of degree 2, independent of η_2, with coefficients real analytic in $\theta \in \mathbb{T}^n_\sigma$ for some $\sigma > 0$ (i.e. real when (ξ, η) is real and $\theta \in \mathbb{T}^n$)*
Then there exists $\varepsilon_* > 0$ and $C > 0$, such that for $|\varepsilon_0| < \varepsilon_*$:
- there exists a set $\mathcal{E}_{\varepsilon_0} \subset (0, 2\pi]^n$ with $\operatorname{meas}((0, 2\pi]^n \setminus \mathcal{E}_{\varepsilon_0}) \le C \varepsilon_0^{\frac{1}{9}}$,
- for any $\omega \in \mathcal{E}_{\varepsilon_0}$, there exists an analytic map $\theta \mapsto A_\omega(\theta) \in \mathrm{sp}(2)$, such that the change of coordinates

$$(\xi', \eta') = e^{A_\omega(\omega t)}(\xi, \eta) \tag{3.7}$$

conjugates the Hamiltonian equations of (3.6) *to the Hamiltonian equations of a homogeneous polynomial*

$$h_\infty(\xi, \eta) = \nu_1(\varepsilon_0) \xi_1 \eta_1 + c(\varepsilon_0) \xi_2^2$$

with

$$|\nu_1(\varepsilon_0) - \nu_1| \le C\varepsilon_0, \quad |c(\varepsilon_0) - c_0| \le \varepsilon_0. \tag{3.8}$$

Finally A_ω is ε_0-close to zero and $e^{A_\omega(\omega t)}$ leaves invariant the space of real states.

The second reducibility theorem deals with Hamiltonians of the form (in the original variables (3.1))

$$h_\varepsilon(t, \xi, \eta) = \nu_1 \xi_1 \eta_1 + \nu_2 \xi_2 \eta_2 + \varepsilon^3 q(\omega t, \xi_1, \eta_1, \xi_2, \eta_2) \tag{3.9}$$

in which one has that the frequencies depend on $\omega \in \mathbb{D}$ and ε and fulfill

$$c \le |\nu_1(\omega)| \le C, \quad c\varepsilon^2 \le |\nu_2(\omega)| \le C\varepsilon^2, \qquad |\partial_\omega \nu_1|, |\partial_\omega \nu_2| \le C\varepsilon^2, \tag{3.10}$$

with some positive c, C. So it is designed to deal with Hamiltonian (2.10). The difference with the standard KAM context is that the second frequency is of order ε^2 while the perturbation is of order ε^3. The following theorem says that the standard conclusion still holds true, i.e. that the inhomogeneous Hamiltonian system associated with (3.9) is reducible for almost all values of ω:

Theorem 3.5. *Assume* (3.10) *and that* $\mathbb{T}^n \times \mathbb{C}^4 \ni (\theta, \xi, \eta) \mapsto q(\theta, \xi, \eta) \in \mathbb{C}$ *is a polynomial in* $(\xi_1, \eta_1, \xi_2, \eta_2)$ *homogeneous of degree 2, with coefficients analytic in* $\theta \in \mathbb{T}^n_\sigma$ *for some* $\sigma > 0$ *and taking real values when* $\theta \in \mathbb{T}^n$ *and* η *is the complex conjugate of* ξ*.*
Then there exists $\varepsilon_* > 0$ *and* $C > 0$*, such that for* $|\varepsilon| < \varepsilon_*$*:*
- there exists a set $\mathcal{E}_\varepsilon \subset (0, 2\pi]^n$ *with* $\mathrm{meas}((0, 2\pi]^n \setminus \mathcal{E}_\varepsilon) \leq C\varepsilon^{\frac{1}{9}}$*,*
- for any $\omega \in \mathcal{E}_\varepsilon$*, there exists an analytic map* $\theta \mapsto A_\omega(\theta) \in \mathrm{sp}(2)$ *such that the change of coordinates*

$$(\xi', \eta') = e^{A_\omega(\omega t)}(\xi, \eta) \tag{3.11}$$

conjugates the Hamiltonian equations of (3.6) *to the Hamiltonian equations of a homogeneous polynomial*

$$h_\infty(\xi, \eta) = \tilde{\nu}_1(\varepsilon)\xi_1\eta_1 + \tilde{\nu}_2(\varepsilon)\xi_2\eta_2$$

with

$$|\tilde{\nu}_j(\varepsilon) - \nu_j| \leq C\varepsilon^3, \quad j = 1, 2\ . \tag{3.12}$$

Finally A_ω *is* ε*-close to zero.*

In the remainder of this section we give the details of the proof of Theorem 3.4 while we only point out the small changes needed to prove Theorem 3.5.

In fact, the proof of Theorem 3.4 is very standard, but since we're dealing with a degenerate Hamiltonian h_0, we have to be a little careful. The fact that the perturbation q is independent of η_2 is crucial here. If not, the Poisson bracket $\{h, \chi\}$ could generate new quadratic terms in (ξ_2, η_2) and the iteration could diverge.

3.1 General strategy

The canonical change of variables is constructed applying a KAM strategy to the Hamiltonian

$$h(y, \theta, \xi, \eta) = \omega \cdot y + \nu_0\xi_1\eta_1 + \varepsilon q(\omega t, \xi_1, \eta_1, \xi_2)$$

in the extended phase space $\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^2$ endowed with the standard symplectic form $dy \wedge d\theta + \mathrm{i}d\xi \wedge d\eta$.
We will say that the Hamiltonian h is in *normal form* if it reads

$$h(y, \theta, \xi, \eta) = \omega \cdot y + a\xi_1\eta_1 + c\xi_2^2 = \omega \cdot y + N(\xi) \tag{3.13}$$

where a and c are real constants (independent of θ).

Let $q \equiv q_\omega$ be a polynomial Hamiltonian homogeneous of degree 2. We write

$$q(\theta, \xi, \eta) = \sum_{\alpha,\beta} q_{\alpha,\beta}(\theta)\xi^\alpha\eta^\beta \tag{3.14}$$

where the coefficients $q_{\alpha,\beta}(\theta)$ are analytic functions of $\theta \in \mathbb{T}^n_\sigma$, we used the standard notation

$$\xi^\alpha \eta^\beta := \xi_1^{\alpha_1} \xi_2^{\alpha_2} \eta_1^{\beta_1} \eta_2^{\beta_2}$$

and according to the independence on η_2 and the fact that this must be a quadratic polynomial one has the restrictions

$$\beta_2 = 0 \ , \quad |\alpha| + |\beta| = 2 \ . \tag{3.15}$$

The size of such polynomial function depending analytically on $\theta \in \mathbb{T}^n_\sigma$ and C^1 on $\omega \in \mathbb{D} \subseteq (0, 2\pi]^n$ will be controlled by the norm

$$[q]_{\sigma,\mathbb{D}} := \sup_{\substack{|\Im\theta|<\sigma,\ \omega\in\mathbb{D} \\ \alpha,\beta,\ j=0,1}} |\partial_\omega^j q_{\alpha,\beta}(\theta)|$$

and we denote by $\mathcal{Q}(\sigma, \mathbb{D})$ the corresponding class of Hamiltonians of the form (3.14)-(3.15) whose norm $[\cdot]_{\sigma,\mathbb{D}}$ is finite.
Let us assume that $[q]_{\sigma,\mathbb{D}} = O(\varepsilon)$. We search for $\chi \equiv \chi_\omega \in \mathcal{Q}(\sigma, \mathbb{D})$ with $\chi = O(\varepsilon)$ such that its time-one flow $\Phi_\chi \equiv \Phi_\chi^{t=1}$ transforms the Hamiltonian $h + q$ into

$$(h + q(\theta)) \circ \Phi_\chi = h_+ + q_+(\theta), \qquad \forall \omega \in \mathbb{D}_+ \ , \tag{3.16}$$

where $h_+ = \omega \cdot y + N_+(\xi)$ is a new normal form, ε-close to h, and the new perturbation $q_+ \in \mathcal{Q}(\sigma_+, \mathbb{D}_+)$ is of size[2] $O(\varepsilon^{\frac{3}{2}})$, and $\mathbb{D}_+ \subset \mathbb{D}$ is an open set ε^α-close to $\mathbb{D}$ for some $\alpha > 0$. As a consequence of the Hamiltonian structure we have that

$$(h + q(\theta)) \circ \Phi_\chi = h + \{h, \chi\} + q(\theta) + O(\varepsilon^{\frac{3}{2}}).$$

So to achieve the goal above we should solve the *homological equation*:

$$\{h, \chi\} = h_+ - h - q(\theta) + O(\varepsilon^{\frac{3}{2}}), \quad \omega \in \mathbb{D}_+ \ . \tag{3.17}$$

Repeating iteratively the same procedure with h_+ instead of h, we will construct a change of variable Φ such that

$$(h + q(\theta)) \circ \Phi = h_\infty \ , \quad \omega \in \mathbb{D}_\infty$$

with $h_\infty = \omega \cdot y + N(\xi, \eta)$ in normal form and $\mathbb{D}_\infty$ a ε^α-close subset of $\mathbb{D}$. Note that we will be forced to solve the homological equation, not only for the original normal form $h_0 = \omega \cdot y + \nu\xi_1\eta_1$, but for more general normal form Hamiltonians (3.13) with $N(\xi) = a\xi_1\eta_1 + c\xi_2^2$ close to $N_0 = \nu\xi_1\eta_1$. To control this closeness we define a norm on $N = a\xi_1\eta_1 + c\xi_2^2$:

$$\|N\| := \max(|a|, |c|).$$

The key remark is that $\forall q \in \mathcal{Q}(\sigma, \mathbb{D})$ one has $\{\xi_2; q\} \equiv 0$.

[2] Formally we could expect q_+ to be of size $O(\varepsilon^2)$ but the small divisors and the change of analyticity domain will led to $O(\varepsilon^{\frac{3}{2}})$.

3.2 Homological equation

Proposition 3.6. *Let $\mathbb{D} \subset \mathbb{D}_0$. Let $\mathbb{D} \ni \omega \mapsto N(\omega)$ be a C^1 mapping that verifies*

$$\left\|\partial_\omega^j (N(\omega) - N_0)\right\| \leq \frac{\min(1, \nu_0)}{4} \tag{3.18}$$

for $j = 0, 1$ and $\omega \in \mathbb{D}$. Let $h = \omega \cdot y + N(\xi)$, $q \in \mathcal{Q}(\sigma, \mathbb{D})$, $\kappa > 0$ and $K \geq 1$. Then there exists an open subset $\mathbb{D}' = \mathbb{D}'(\kappa, K) \subset \mathbb{D}$, satisfying

$$\text{meas}(\mathbb{D} \setminus \mathbb{D}') \leq 4d^2 K^{2n} \kappa, \tag{3.19}$$

and there exist $\chi, r \in \cap_{0 \leq \sigma' < \sigma} \mathcal{Q}(\sigma', \mathbb{D}')$ and $\tilde{N}$ in normal form such that for all $\omega \in \mathbb{D}'$

$$\{h, \chi\} + q = \tilde{N} + r \ . \tag{3.20}$$

Furthermore for all $0 \leq \sigma' < \sigma$

$$[r]_{\sigma', \mathbb{D}'} \leq C \ \frac{e^{-\frac{1}{2}(\sigma - \sigma')K}}{(\sigma - \sigma')^n} [q]_{\sigma, \mathbb{D}} \ , \tag{3.21}$$

$$[\chi]_{\sigma', \mathbb{D}'} \leq \frac{C}{\kappa^2 (\sigma - \sigma')^n} [q]_{\sigma, \mathbb{D}} \ , \tag{3.22}$$

$$\|\partial_\omega^j \tilde{N}(\omega)\| \leq [q]_{\sigma, \mathbb{D}} \quad j = 0, 1, \ \forall \omega \in \mathbb{D}. \tag{3.23}$$

The constant C depends on n.

Proof. As usual we consider the "homological operator" $\mathcal{L} := \{h; .\}$ and decompose the space $\mathcal{Q}(\sigma, \mathbb{D})$ on the basis of its eigenfunctions. Such a basis is given by the monomials

$$\xi^\alpha \eta^\beta e^{\mathrm{i}k\theta} \ ,$$

where α and β are subject to the restrictions (3.15). The corresponding eigenvalues are

$$\mathrm{i}(\nu_1(\alpha_1 - \beta_1) + \omega \cdot k) \ , \tag{3.24}$$

while $\text{Ker}(\mathcal{L}) = span\left\{\xi_2^2, \xi_1\eta_1\right\}$. So, decomposing q as in (3.14), and expand the coefficients in Fourier series:

$$q_{\alpha,\beta}(\theta) = \sum_{k \in \mathbb{Z}^n} \hat{q}_{\alpha,\beta}(k) e^{\mathrm{i}k\theta} \ ,$$

then one is lead to define

$$\chi_{\alpha,\beta}(\theta) = \sum_{|k| \leq K} \frac{\hat{q}_{\alpha,\beta}(k)}{\mathrm{i}(\nu_1(\alpha_1 - \beta_1) + \omega \cdot k)} e^{ik\theta} \ , \tag{3.25}$$

where, for $\alpha = (1, 0), \beta = (1, 0)$ and for $\alpha = (0, 0)$ and $\beta = (2, 0)$ the sum is restricted to $k \neq 0$. We also define

$$\tilde{N} := \hat{q}_{1,0,1,0}(0)\xi_1\eta_1 + \hat{q}_{0,2,0,0}(0)\xi_2^2 \ , \tag{3.26}$$

$$r(\xi, \eta, \theta) := \sum_{|k| > K, \alpha, \beta} \hat{q}_{\alpha,\beta}(k) \xi^\alpha \eta^\beta e^{ik\theta} \ , \tag{3.27}$$

so that the homological equation is satisfied. Still it remains to prove the estimates of the various terms. We give explicitly the estimate of χ. To this end we have to control the small denominators (3.24) under the restriction

$$|\alpha_1 - \beta_1| + |k| \neq 0\ , \quad |k| \leq K\ , \quad \alpha_1 + \beta_1 \leq 2\ . \tag{3.28}$$

We define $\mathbb{D}'$ to be the set for which the above small denominators are bigger than κ. In order to estimate its measure we recall the following classical lemma:

Lemma 3.7. *Let $f : [0,1] \mapsto \mathbb{R}$ be a C^1-map satisfying $|f'(x)| \geq \delta$ for all $x \in [0,1]$ and let $\kappa > 0$. Then*

$$\text{meas}\{x \in [0,1] \mid |f(x)| \leq \kappa\} \leq \frac{\kappa}{\delta}.$$

Since $|\partial_\omega (k \cdot \omega)(\frac{k}{|k|}))| = |k| \geq 1$, we get, using condition (3.18),

$$|\partial_\omega (k \cdot \omega + \nu_1(\alpha_1 - \beta_1))(\frac{k}{|k|}))| \geq 1/2\,. \tag{3.29}$$

Using (3.29) and Lemma 3.7, we conclude that for any fixed k

$$|k \cdot \omega + \nu_1(\alpha_1 - \beta_1)| > \kappa\ , \tag{3.30}$$

outside a set $F_{k,\alpha,\beta}$ of measure $\leq 2d^2\kappa$ (the case $k = 0$ being evident) so that if F is the union of $F_{k,\alpha,\beta}$ for $|k| < K$ and as α and β vary, we have

$$\text{meas}(F) \leq 4K^{2n} d^2 \kappa\ . \tag{3.31}$$

Thus, defining $\mathbb{D}' \equiv \mathbb{D}'(\kappa, K) = \mathbb{D} \setminus F$, we get for all $\omega \in \mathbb{D}'$, $0 \leq \sigma' < \sigma$ and $\theta \in \mathbb{T}^n_{\sigma'}$

$$|\chi_{\alpha,\beta}(\theta,\omega)| \leq \frac{C}{\kappa(\sigma - \sigma')^n} \sup_{|\Im\theta|<\sigma} |q_{\alpha,\beta}(\theta)|\,.$$

The estimates for the derivatives with respect to ω are obtained by differentiating the definition of χ (for more details see for instance [16]). □

3.3 Iterative lemma

Theorem 3.4 is proved by an iterative KAM procedure. We begin with the initial Hamiltonian $h_0 + q_0$ where

$$h_0(y,\theta,\xi,\eta) = \omega \cdot y + \nu_0 \xi_1 \eta_1 + c_0 \xi_2^2\,, \tag{3.32}$$

and $q_0 = \varepsilon_0 q \in \mathcal{Q}(\sigma_0, \mathbb{D}_0)$, $\mathbb{D}_0 = [\varepsilon, 2\pi]^n$. Then we construct iteratively the change of variables Φ_{χ_m}, the normal form $h_m = \omega \cdot y + \nu_m \xi_1 \eta_1 + c_m \xi_2^2$ and the perturbation $q_m \in \mathcal{Q}(\sigma_m, \mathbb{D}_m)$ as follows: assume that the construction is done up to step $m \geq 0$ then

(i) using Proposition 3.6 we construct $\chi_{m+1}(\omega,\theta)$, $\tilde{N}_m(\omega)$, $r_{m+1}(\omega,\theta)$ and $\mathbb{D}_{m+1} \subset \mathbb{D}_m$ such that

$$\{h, \chi_{m+1}\} = \tilde{N}_m - q_m + r_{m+1}, \quad \omega \in \mathbb{D}_{m+1},\ \theta \in \mathbb{T}^n_{\sigma_{m+1}} \tag{3.33}$$

where $0 < \sigma_{m+1} < \sigma_m$ has to be chosen later;

(ii) we define $h_{m+1} = \omega \cdot y + N_{m+1}$ by

$$N_{m+1} = N_m + \tilde{N}_m \,, \tag{3.34}$$

and

$$q_{m+1} = r_{m+1} + \int_0^1 \{(1-t)(h_{m+1} - h_m + r_{m+1}) + tq_m, \chi_{m+1}\} \circ \Phi^t_{\chi_{m+1}} \mathrm{d}t \,. \tag{3.35}$$

For any regular Hamiltonian f we have, using the Taylor expansion of $g(t) = f \circ \Phi^t_{\chi_{m+1}}$ between $t = 0$ and $t = 1$

$$f \circ \Phi_{\chi_{m+1}} = f + \{f, \chi_{m+1}\} + \int_0^1 (1-t)\{\{f, \chi_{m+1}\}, \chi_{m+1}\} \circ \Phi^t_{\chi_{m+1}} \mathrm{d}t \,.$$

Therefore we get for $\omega \in \mathbb{D}_{m+1}$

$$(h_m + q_m) \circ \Phi_{\chi_{m+1}} = h_{m+1} + q_{m+1}.$$

Following the general scheme above we have for all $m \geq 0$

$$(h_0 + q_0) \circ \Phi^1_{\chi_1} \circ \cdots \circ \Phi^1_{\chi_m} = h_m + q_m.$$

At step m the Fourier series are truncated at order K_m and the small divisors are controlled by κ_m. Now we specify the choice of all the parameters for $m \geq 0$ in term of ε_m which will control $[q_m]_{\mathbb{D}_m, \sigma_m}$.
First we define $\sigma_0 = \sigma$ and for $m \geq 1$ we choose

$$\begin{aligned} \sigma_{m-1} - \sigma_m &= C_* \sigma_0 m^{-2}, \\ K_m &= 2(\sigma_{m-1} - \sigma_m)^{-1} \ln \varepsilon_{m-1}^{-1}, \\ \kappa_m &= \varepsilon_{m-1}^{\frac{1}{8}} \,, \end{aligned}$$

where $(C_*)^{-1} = 2\sum_{j \geq 1} \frac{1}{j^2}$.

Lemma 3.8. *There exists $\varepsilon_* > 0$ depending on d, n such that, for $|\varepsilon_0| \leq \varepsilon_*$ and*

$$\varepsilon_m = \varepsilon_0^{(3/2)^m} \quad , \quad m \geq 0 \,,$$

we have the following:
For all $m \geq 1$ there exist an open set $\mathbb{D}_m \subset \mathbb{D}_{m-1}$ functions $\chi_m, q_m \in \mathcal{Q}(\mathbb{D}_m, \sigma_m)$ and N_m in normal form such that

(i) The mapping

$$\Phi_m(\cdot, \omega, \theta) = \Phi^1_{\chi_m} \ : \ \mathbb{C}^2 \to \mathbb{C}^2, \quad \omega \in \mathbb{D}_m, \ \theta \in \mathbb{T}_{\sigma_m} \tag{3.36}$$

is a linear isomorphism, C^1 in $\omega \in \mathbb{D}_m$, analytic in $\theta \in \mathbb{T}^n_{\sigma_m}$, linking the Hamiltonian at step $m-1$ and the Hamiltonian at step m, i.e.

$$(h_{m-1} + q_{m-1}) \circ \Phi_m = h_m + q_m \ , \quad \forall \omega \in \mathbb{D}_m \ .$$

(ii) we have the estimates

$$\mathrm{meas}(\mathbb{D}_{m-1} \setminus \mathbb{D}_m) \le \varepsilon_{m-1}^{\frac{1}{9}}, \tag{3.37}$$

$$\|\partial_\omega^j (N_m(\omega) - N_{m-1}(\omega))\| \le \varepsilon_{m-1}, \qquad j = 0, 1,\ \omega \in \mathbb{D}_m, \tag{3.38}$$

$$[q_m]_{\sigma_m, \mathbb{D}_m} \le \varepsilon_m, \tag{3.39}$$

$$\|\Phi_m(\cdot, \omega, \theta) - Id\|_{\mathcal{L}(\mathbb{C}^2)} \le \varepsilon_{m-1}^{\frac{1}{2}}, \quad \text{for } \theta \in \mathbb{T}^n_{\sigma_m},\ \omega \in \mathbb{D}_m. \tag{3.40}$$

Proof. At step 1, $h_0 = \omega \cdot y + \nu_0 \xi_1 \eta_1$ and thus hypothesis (3.18) is trivially satisfied and we can apply Proposition 3.6 to construct χ_1, N_1, r_1 and $\mathbb{D}_1$ such that for $\omega \in \mathbb{D}_1$

$$\{h_0, \chi_1\} + q_0 = N_1 - N_0 + r_1 \ .$$

Then, using (3.19), we have

$$\mathrm{meas}(\mathbb{D} \setminus \mathbb{D}_1) \le C K_1^{2n} \kappa_1 \le \varepsilon_0^{\frac{1}{9}}$$

for $\varepsilon = \varepsilon_0$ small enough. Using (3.22) we have for ε_0 small enough

$$[\chi_1]_{\sigma_1, \mathbb{D}_1} \le C \frac{1}{\kappa_1^2 (\sigma_0 - \sigma_1)^n} \varepsilon_0 \le \varepsilon_0^{\frac{1}{2}}.$$

Similarly using (3.21), (3.23) we have

$$\|N_1 - N_0\| \le \varepsilon_0,$$

and

$$[r_1]_{\sigma_1, \mathbb{D}_1} \le C \frac{\varepsilon_0^2}{(\sigma_0 - \sigma_1)^n} \le \varepsilon_0^{\frac{3}{2}}$$

for $\varepsilon = \varepsilon_0$ small enough.

In particular we deduce $\|\Phi_1(\cdot, \omega, \theta) - Id\|_{\mathcal{L}(\mathbb{C}^2)} \le \varepsilon_0^{\frac{1}{2}}$. Thus using (3.35) we get for ε_0 small enough

$$[q_1]_{\sigma_1, \mathbb{D}_1} \le \varepsilon_0^{3/2} = \varepsilon_1.$$

Now assume that we have verified Lemma 3.8 up to step m. We want to perform the step $m+1$. We have $h_m = \omega \cdot y + N_m$ and since

$$\|N_m - N_0\| \le \|N_m - N_{m-1}\| + \cdots + \|N_1 - N_0\| \le \sum_{j=0}^{m-1} \varepsilon_j \le 2\varepsilon_0,$$

hypothesis (3.18) is satisfied and we can apply Proposition 3.6 to construct $\mathbb{D}_{m+1}$, χ_{m+1} and q_{m+1}. Estimates (3.37)-(3.40) at step $m+1$ are proved as we have proved the corresponding estimates at step 1. $\square$

3.4 Transition to the limit and proof of Theorem 3.4

Let

$$\mathcal{E}_\varepsilon := \cap_{m\geq 0}\mathbb{D}_m.$$

In view of (3.37), this is a Borel set satisfying

$$\text{meas}(\mathbb{D}\setminus\mathcal{E}_\varepsilon) \leq \sum_{m\geq 0} \varepsilon_m^{\frac{1}{9}} \leq 2\varepsilon_0^{\frac{1}{9}}.$$

Let us denote $\Psi_N(\cdot,\omega,\theta) = \Phi_1(\cdot,\omega,\theta)\circ\cdots\circ\Phi_N(\cdot,\omega,\theta)$. Due to (3.40) it satisfies for $M\leq N$ and for $\omega\in\mathcal{E}_\varepsilon$, $\theta\in\mathbb{T}^n_{\sigma/2}$

$$\|\Psi_N(\cdot,\omega,\theta)-\Psi_M(\cdot,\omega,\theta)\|_{\mathcal{L}(\mathbb{C}^2)} \leq \sum_{m=M}^{N} \varepsilon_m^{\frac{1}{2}} \leq 2\varepsilon_M^{\frac{1}{2}}.$$

Therefore $(\Psi_N(\cdot,\omega,\theta))_N$ is a Cauchy sequence in $\mathcal{L}(\mathbb{C}^2)$. Thus when $N\to\infty$ the maps $\Psi_N(\cdot,\omega,\theta)$ converge to a limit mapping $\Psi_\infty(\cdot,\omega,\theta)\in\mathcal{L}(\mathbb{C}^2)$. Furthermore since the convergence is uniform on $\omega\in\mathcal{E}_\varepsilon$ and $\theta\in\mathbb{T}^n_{\sigma/2}$, $(\omega,\theta)\to\Psi^1_\infty(\cdot,\omega,\theta)$ is analytic in θ and lipschitzian in ω. Moreover,

$$\|\Psi_\infty(\cdot,\omega,\theta)-Id\|_{\mathcal{L}(\mathbb{C}^2)} \leq \varepsilon_0^{\frac{1}{2}}. \tag{3.41}$$

By construction, the map $\Psi_m(\cdot,\omega,\omega t)$ transforms the original Hamiltonian

$$h_\varepsilon(t,\xi,\eta) = N_0(\xi,\eta)+\varepsilon q(\omega t,\xi,\eta), \quad N_0(\xi,\eta)=\nu_0\xi_1\eta_1$$

into

$$H_m(t,\xi,\eta) = N_m(\xi,\eta)+q_m(\omega t,\xi,\eta).$$

When $m\to\infty$, by (3.39) we get $q_m\to 0$ and by (3.38) we get $N_m\to N$ where

$$N\equiv N(\omega) = N_0+\sum_{k=1}^{+\infty}\tilde{N}_k =: \nu(\omega,\varepsilon)\xi_1\eta_1 + c(\omega,\varepsilon)(\xi_2-\eta_2). \tag{3.42}$$

Further for all $\omega\in\mathcal{E}_\varepsilon$ we have using (3.38)

$$\|N(\omega)-N_0\| \leq \sum_{m=0}^{\infty}\varepsilon^m \leq 2\varepsilon.$$

Let us denote $\Psi_\infty(\theta) = \Psi^1_\infty(\cdot,\omega,\theta)$. Denoting the limiting Hamiltonian $h_\infty(\xi,\eta) = N_\infty(\xi,\eta)$ we have

$$h_\varepsilon(\theta,\Psi_\infty(\theta)(\xi,\eta)) = h_\infty(\xi,\eta), \quad \theta\in\mathbb{T},\ (\xi,\eta)\in\mathbb{C}^{2d},\ \omega\in\mathbb{D}_\varepsilon.$$

Finally we show that the linear symplectomorphism Ψ_∞ can be written as (3.7). To begin with, write each Hamiltonian χ_m constructed in the KAM iteration as

$$\chi_m(\theta, \xi, \eta) = \frac{1}{2} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \cdot E_c \, B_m(\theta) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \ , \quad E_c := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \ , \tag{3.43}$$

where $B_m(\theta)$ is a skew-adjoint matrix of dimension 4×4 of size ε_m. Then Ψ_m has the form

$$\Psi_m(\theta, \xi, \eta) = e^{B_m(\theta)}(\xi, \eta) \ . \tag{3.44}$$

The following lemma is proved analogously to Lemma 3.5 in [3].

Lemma 3.9. *There exists a sequence of Hamiltonian matrices $A_l(\theta)$ such that*

$$\Psi_1 \circ ... \circ \Psi_l(\theta, \xi, \eta) = e^{A_l(\theta)}(\xi, \eta) \quad \forall \xi \in \mathbb{C}^2 \ . \tag{3.45}$$

Furthermore, there exist a Hamiltonian matrix $A_\omega(\theta)$ such that

$$\lim_{l \to +\infty} e^{A_l(\theta)} = e^{A_\infty(\theta)} \ , \qquad \sup_{|\mathrm{Im}\theta| \le \sigma/2} \|A_\omega(\theta)\| \le C\epsilon \ , \tag{3.46}$$

and for each $\theta \in \mathbb{T}^n$,

$$\Psi(\theta, \xi, \eta) = e^{A_\omega(\theta)}(\xi, \eta) \ \ \forall \xi \in \mathbb{C}^2 \ .$$

This concludes the proof of Theorem 3.4.

3.5 Changes for proving Theorem 3.5.

The main change needed for the proof of Theorem 3.5 rests in Proposition 3.6. Indeed one has to assume the r.h.s. of (3.18) to be smaller than a small constant times ε^2 and the conclusion changes in the fact that at the denominator of the r.h.s. (3.22) instead of κ^2, one gets

$$\min\left\{\varepsilon^2, \kappa^2\right\} \ . \tag{3.47}$$

In the next lines we are going to prove this version of Proposition 3.6.

Indeed, the proof of Proposition 3.6 goes excatly in the same way, except that now the eigenvalue (3.24) are substituted by

$$\mathrm{i}(\nu_1(\alpha_1 - \beta_1) + \nu_2(\alpha_2 - \beta_2) + \omega \cdot k) \tag{3.48}$$

with the only selection rule

$$|\alpha - \beta| + |k| \neq 0 \ , \quad \text{and } |k| \le K \ .$$

The estimates of the small divisors are obtained exactly in the same way as far as

$$|\alpha_1 - \beta_1| + |k| \neq 0 \ ,$$

however when such a quantity vanishes, the modulus of the small divisor becomes

$$|2\nu_2| \ge c\varepsilon^2$$

(remark that in this case the normal form contains span$\{\xi_1\eta_1\}$). Concerning contribution of the terms with this denominator to the derivative with respect to ω, remark that the involved terms are

$$\frac{\xi_2^2}{2\mathrm{i}\nu_2}, \quad -\frac{\eta_2^2}{2\mathrm{i}\nu_2}$$

(multiplied by a constant). Thus the derivative with respect to, say ω_i of the coefficient of one of these terms gives, by (3.10),

$$\left| -\frac{1}{2\nu_2^2}\frac{\partial\nu_2}{\partial\omega_i} \right| \leq C\varepsilon^{-2} ,$$

which proves the claimed statement.

Then also the iterative lemma changes. Actually, only few first steps change, for which one has at the first step

$$\min\left\{\varepsilon^2, \kappa_m^2\right\} = \varepsilon^2 ;$$

it is easy to see that with the choices just before the statement of Lemma 3.8, after a finite number of steps one has $\min\left\{\varepsilon^2, \kappa_m^2\right\} = \kappa_m^2$, and therefore after this step the iteration can be repeated exactly in the same way as in the previous case. □

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